MATHEMATICAL CENTRE TRACTS

33

MATHEMATICAL CENTRE TRACTS
33

PRESERVATION OF INFINITE DIVISIBILITY UNDER MIXING

and related topics

BY

F.W. STEUTEL

MATHEMATISCH CENTRUM AMSTERDAM
1970

PREFACE

The question, which distributions are infinitely divisible, is answered completely - in a certain sense - by the Lévy-Khintchine representation theorem for infinitely divisible characteristic functions. This theorem does not, in general, provide an answer to the question whether a given distribution function, or probability density, is infinitely divisible. One of the purposes of this tract is to give some simple conditions for the infinite divisibility of probability densities. Special attention is given to infinitely divisible mixtures.

Chapter 1 is introductory. Chapter 2 is devoted to mixtures of Gamma distributions. There, among other things, it is proved that mixtures of Gamma distributions of degrees not exceeding one are infinitely divisible, or, equivalently, that completely monotone densities are infinitely divisible. In chapter 3 a more general class of infinitely divisible mixtures is considered. The final chapter treats some related subjects: waiting-time and renewal distributions, log-convex densities, zeros of infinitely divisible densities, and moment inequalities.

I am indebted to Dr. J. FABIUS and Professor J.Th. RUNNENBURG for advice and encouragement. My thanks are due to Mrs. S. HILLEBRAND, for typing and illustrations, and to Mr. D. ZWARST and Mr. J. SUIKER for printing and binding.

CONTENTS

Chapter 1

INTRODUCTION

1.1 Introduction and summary.

Infinite divisibility is a theoretical concept: one can hardly imagine a practical situation where it would be of interest to know whether a given random variable is infinitely divisible or not. The concept of infinite divisibility has its roots in the theory of stochastic processes with independent increments, its most important applications are in the theory of limit distributions of sums of independent random variables (cf. LÉVY [18] and LOÈVE [20]). Recently much attention has been focused on factorization problems and the study of stable distributions. For extensive bibliographies on infinitely divisible distributions we refer to LINNIK [19] and FISZ [6].

In this thesis infinite divisibility is studied for its own sake, regardless of the applications it may have in theory or practice.

Though the infinitely divisible distributions have been characterized by Lévy, Khintchine and others (see e.g. LUKACS [22]), in general it is very difficult to determine, whether a given distribution is infinitely divisible or not. This thesis is intended to provide some new methods of constructing infinitely divisible distributions, and to give some necessary and (or) sufficient conditions for infinite divisibility. As a relatively novel feature, a number of these conditions are in terms of properties of probability density functions.

In chapters 2 and 3 we investigate classes of infinitely divisible distributions, mixtures of which are also infinitely divisible. The starting point here is the study of mixtures of exponential distributions, initiated by GOLDIE [8]. In chapter 4 we consider a number of more or less related subjects: renewal distributions and monotone densities (both continuous and discrete), waiting times, zeros of infinitely divisible densities, log-convex densities and moment inequalities.

A large part of the investigations was inspired by results in waiting-time theory, where infinite divisibility seems to be the rule rather than the exception. An exception to this rule is noted in chapter 4.

1.2 Notations and conventions.

Throughout this thesis we shall use the following notation, often without further specification.

Random variables will be denoted by underlined lower case letters: $\underline{x}$, $\underline{y}$, ...; their distribution functions by capitals: F, G, ...; their density functions (in case of absolute continuity) by the corresponding lower case letters: f, g, ...; their characteristic functions by the corresponding Greek letters: ϕ, γ, ..., where ϕ is defined by

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)^{1)} \quad (-\infty < t < \infty);$$

their Laplace transforms (in case of non-negative random variables) are denoted by $\check{F}$, $\check{G}$, ..., $\check{F}$ being defined by

$$\check{F}(\tau) = \int_{0}^{\infty} e^{-\tau x}\, dF(x) \quad (\text{Re } \tau \geq 0).$$

In case of distributions on the non-negative integers, which will be called lattice distributions, the probabilities are denoted by p_n, q_n ...; their generating functions by P, Q, ..., where by definition

$$p_n = P[\,\underline{x} = n\,] \qquad (n = 0, 1, 2, \ldots),$$

$$P(u) = \sum_{n=0}^{\infty} p_n u^n \qquad (|u| \leq 1).$$

In many cases it is possible, and desirable, to extend the domains of the functions ϕ, $\check{F}$ and P beyond the values indicated above by analytic continuation.

We use the following abbreviations:

inf div	infinitely divisible (infinite divisibility)
d.f.	distribution function
p.d.f.	probability density function
c.f.	characteristic function
L.T.	Laplace transform (of a d.f.)
p.g.f.	probability generating function
c.m.	completely monotone (complete monotonicity).

1) All integrals are to be interpreted as Lebesgue-Stieltjes integrals.

When dealing with non-negative random variables, we shall sometimes prefer the use of c.f.'s, sometimes the use of L.T.'s. This preference may either be arbitrary or a matter of convenience; it seldom is essential.

The functions Θ, θ, K and k occurring in the various forms of the canonical representation (cf. (1.3.1) and (1.3.4)), will sometimes be used (with or without suffix) without explicit reference to these representations.

If F and F_n are functions of finite total variation on $(-\infty,\infty)$, then we say that F_n tends weakly to F and we write

$$F_n \overset{w}{\rightarrow} F,$$

if $\lim_{n\to\infty} F_n(x) = F(x)$ for all continuity points of F. If in addition $\lim_{n\to\infty} F_n(-\infty) = F(-\infty)$ and $\lim_{n\to\infty} F_n(+\infty) = F(+\infty)$ we say that F_n tends to completely and we write

$$F_n \overset{c}{\rightarrow} F.$$

F will be called the weak limit or the complete limit of the sequence $\{F_n\}$ respectively. For theorems on weak convergence we refer to [20] and [22].

If a function is obviously zero for negative values of the argument it is often defined explicitly for non-negative values only.

The term infinitely divisible is applied, not only to random variables, but also to their d.f.'s, p.d.f.'s, c.f.'s, L.T.'s and p.g.f.'s, and sometimes, somewhat loosely, to their "distributions".

1.3 Definition and basic properties of inf div distributions

DEFINITION 1.3.1

A random variable $\underline{x}$ is called infinitely divisible if and only if for every positive integer n there exist independent and identically distributed random variables $\underline{x}_{n,1}, \ldots, \underline{x}_{n,n}$, such that $\underline{x}$ and $\underline{x}_{n,1} + \ldots + \underline{x}_{n,n}$ have the same distribution.

Equivalently we have

DEFINITION 1.3.2

A characteristic function ϕ is called inf div if and only if for every positive integer n there exists a characteristic function ϕ_n, with the property that $\phi = (\phi_n)^n$.

As we shall need logarithms and non-integral powers of c.f.'s, we formally state (see e.g. TUCKER [35]).

DEFINITION 1.3.3

If ϕ is a c.f. and if $\phi(t) \neq 0$ for $|t| \leq T$, then by log ϕ we denote the function $\psi(t)$, uniquely defined for $|t| \leq T$ by the conditions

$$\exp \psi(t) = \phi(t) \qquad (|t| \leq T)$$

$$\psi(0) = 0$$

$$\psi(t) \text{ is continuous for } |t| \leq T.$$

As every c.f. ϕ is continuous with $\phi(0) = 1$, it follows that for every c.f. ϕ there is an interval $[-T,T]$ where log ϕ is uniquely defined. If $\phi(t) \neq 0$ for all t, then log ϕ is defined for all t. The function ϕ^p is defined for real p by $\phi^p = \exp(p \log \phi)$ for all values of t for which log ϕ is defined.

DEFINITION 1.3.4

If F is a d.f. with c.f. ϕ and if ϕ^p is a c.f., then the d.f. corresponding to ϕ^p is denoted by F^{*p} ($F^{*1} = F$). The corresponding densities are denoted by f^{*p}.

We now list a series of, mostly well-known, theorems about inf div distributions that we need in the following chapters. Their proofs may be found in the references indicated. For general theorems concerning c.f.'s and L.T.'s we refer to LUKACS [22] and FELLER [5] respectively.

THEOREM 1.3.1 [22]

If $\underline{x}_1$ and $\underline{x}_2$ are independent and inf div, then $\underline{x}_1 + \underline{x}_2$ is inf div. Accordingly $\phi_1\phi_2$, $\check{F}_1\check{F}_2$ or P_1P_2 are inf div.

THEOREM 1.3.2 [22]

An inf div c.f. has no real zeros

THEOREM 1.3.3 [22]

An inf div c.f. which is analytic has no zeros in the interior of its strip of regularity.

COROLLARY 1.3.1 [22]

An inf div c.f. which is an entire function, has no zeros.

COROLLARY 1.3.2

The c.f. of a non-negative inf div random variable has no zeros in the upper half-plane (Im $t \geqq 0$).

COROLLARY 1.3.3

The p.g.f. of an inf div distribution on the non-negative integers with $p_0 > 0$, has no zeros in the closed unit disk.

THEOREM 1.3.4 [22]

If $\underline{x}$ is non-degenerate and bounded, then $\underline{x}$ is not inf div.

THEOREM 1.3.5 (closure theorem) [22]

A c.f. which is the limit of a sequence of inf div c.f.'s is inf div.

REMARK: All theorems concerning c.f.'s imply obvious analogues for L.T.'s and p.g.f.'s. Not all of these will be stated, even if they are used.

THEOREM 1.3.6 [22]

A non-vanishing c.f. ϕ is inf div if and only if ϕ^p is a c.f. for all $p > 0$ (or for all $p = \frac{1}{n}$, with $n = 1, 2, \ldots$).

THEOREM 1.3.7 (Lévy-Khintchine canonical representation) [22]

A c.f. ϕ is inf div if and only if $\log \phi$ can be written as

(1.3.1) $$\log \phi(t) = ait + \int_{-\infty}^{\infty} \left(e^{itx} - 1 - \frac{itx}{1+x^2}\right) \frac{1+x^2}{x^2}\, d\Theta(x) \quad (-\infty < t < \infty),$$

where a is a real constant and $\Theta(x)$ is a bounded non-decreasing function with $\Theta(-\infty) = 0$. For $x = 0$ the integrand in (1.3.1) is defined by continuity. The representation is unique.

COROLLARY 1.3.4 [7]

If a c.f. ϕ has a representation of the form (1.3.1), where $\Theta(x)$ is of bounded variation but not non-decreasing, then ϕ is not inf div.

If $\Theta(x)$ is absolutely continuous, we define $\theta(x)$ (almost everywhere) by

$$(1.3.2) \qquad \theta(x)dx = (1 + x^2)x^{-1}\, d\Theta(x).$$

In most applications one tries to write log ϕ in the form (1.3.1) and then to prove that Θ is non-decreasing, or that θ is non-negative.

THEOREM 1.3.8 [22]

$$(1.3.3) \qquad a = \lim_{n\to\infty} n \int_{-\infty}^{\infty} \frac{y}{1+y^2}\, dF^{*1/n}(y); \quad \Theta(x) \overset{1)}{=} \lim_{n\to\infty} n \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, dF^{*1/n}(y).$$

THEOREM 1.3.9 [22]

If $\phi, \phi_1, \phi_2, \ldots$ are inf div c.f.'s then $\phi_n \to \phi$ if and only if $a_n \to a$ and $\Theta_n \overset{c}{\to} \Theta$.

Sometimes simpler representations are possible, for instance

THEOREM 1.3.10 [5]

A L.T. $\check{F}$ is inf div if and only if log $\check{F}$ has the representation

$$(1.3.4) \qquad \log \check{F}(\tau) = \int_0^{\infty} (e^{-\tau x} - 1)\, x^{-1} dK(x) \qquad (\mathrm{Re}\ \tau \geq 0),$$

where $K(x)$ is non-decreasing. It follows that $\int_1^{\infty} x^{-1} dK(x)$ is finite. The representation is unique, if $K(x)$ is defined to be zero for negative x.

Equivalently we have

COROLLARY 1.3.5 [5]

$\check{F}$ is inf div if and only if

---

1) In all continuity points of Θ.

$$(1.3.5) \qquad -\check{F}'/\check{F} = \int_0^\infty e^{-\tau x} dK(x) \qquad (\text{Re } \tau > 0),$$

where K satisfies the conditions of theorem 1.3.10.

COROLLARY 1.3.6 [5]

$\check{F}$ is inf div if and only if $-\check{F}'/\check{F}$ is completely monotone, i.e. if and only if

$$(1.3.6) \qquad \left(-\frac{d}{d\tau}\right)^n [-\check{F}'(\tau)/\check{F}(\tau)] \geq 0 \quad (n = 0,1,2,\ldots; \tau > 0).$$

For properties of c.m. functions we refer to FELLER [5].

If K is absolutely continuous, then we define k(x) (almost everywhere) by

$$(1.3.7) \qquad k(x)dx = dK(x).$$

COROLLARY 1.3.7

If $\check{F}$ is an inf div L.T. with $\mu = -\check{F}'(0) < \infty$, then $-\mu^{-1}\check{F}'(\tau)$ is the L.T. of a d.f. having $\check{F}$ as an inf div factor.

PROOF: $\check{F} = e^{-\psi}$. Hence $\check{F}' = -\psi'\check{F}$, where ψ' is c.m. and $\psi'(0) = \mu$.

THEOREM 1.3.11 [4]

A p.g.f. P with P(0) > 0 is inf div if and only if log P can be written as

$$(1.3.8) \qquad \log P(u) = \lambda\{Q(u) - 1\},$$

where Q(u) is the p.g.f. of a distribution on the non-negative integers and $\lambda > 0$. The representation is unique.

COROLLARY 1.3.8 [4]

A p.g.f. P is inf div if and only if P'/P is absolutely monotone, i.e. if and only if

$$(1.3.9) \qquad \left(\frac{d}{du}\right)^n [P'(u)/P(u)] \geq 0 \qquad (n = 0,1,\ldots; 0 \leq u < 1),$$

or, equivalently, if and only if P'/P is representable in the form

$$(1.3.10) \qquad P'(u)/P(u) = \sum_{0}^{\infty} r_n u^n \qquad (|u| < 1),$$

where $r_n \geq 0$ and ,necessarily, $\sum_{1}^{\infty} n^{-1} r_n < \infty$.

REMARK 1: Theorems 1.3.9 and 1.3.11 and their corollaries have corollaries analogous to and following from corollary 1.3.4. These will be used.

REMARK 2: The condition $p_0 > 0$ is not an essential restriction. It ensures that the distribution with c.f. $[P(e^{it})]^{1/n}$ is again a distribution on the non-negative integers. Of course, if $P(e^{it})$ is an inf div c.f., then the same is true for $e^{ikt}P(e^{it})$ for every real k.

1.4. Mixtures of distributions.

If G is a d.f. with support Λ and F_λ, $\lambda \in \Lambda$, is a family of d.f.'s, such that $F_\lambda(x)$ is Borel-measurable as a function of λ for all x, then the function F defined by

$$(1.4.1) \qquad F = \int_{-\infty}^{\infty} F_\lambda dG(\lambda)$$

is again a d.f., with c.f.

$$\phi = \int_{-\infty}^{\infty} \phi_\lambda dG(\lambda),$$

and (if F_λ is concentrated on $[0,\infty)$ for all $\lambda \in \Lambda$) with L.T.

$$\check{F} = \int_{-\infty}^{\infty} \check{F}_\lambda dG(\lambda).$$

In (1.4.1) we call F a mixture of the d.f.'s F_λ, G is called the mixing (distribution) function. For F in (1.4.1) to be a d.f., the function G, apart from being of bounded variation, must satisfy the condition $G(\infty) - G(-\infty) = 1$; G is not necessarily a d.f. Sometimes we consider these more general mixing functions; however, unless otherwise stated G will be a d.f.

From (1.3.1) it is clear that the set of inf div c.f.'s cannot be expected to be convex, i.e. that one cannot expect arbitrary mixtures of inf div c.f.'s to be again inf div. Even mixtures of distributions which are notoriously inf div, like the normal and Poisson distributions, are in general not inf div. For example, the c.f.

$$\frac{1}{2}\,(e^{-t^2} + e^{-2t^2})$$

is zero for $t^2 = \pi i$ and therefore by corollary 1.3.1 not inf div. On the other hand there are, as we shall see in chapters 2 and 3, surprisingly large classes of distributions which preserve inf div when being mixed. Whereas a mixture of two inf div distributions need not be inf div, a mixture of two distributions, neither of which is inf div, may be inf div. Formally, every c.f. is a mixture of inf div c.f.'s (in fact of degenerate ones) as it is of the form $\int_{-\infty}^{\infty} e^{it\lambda} dG(\lambda)$.

1.5 Rational L.T.'s and c.f.'s

In chapter 2 we shall study mixtures of Γ-distributions. As a start we consider here rational L.T.'s and c.f.'s.

If $\underline{x}$ is exponentially distributed with mean λ^{-1}, then its L.T. is given by

$$(1.5.1) \qquad \check{F}(\tau) = \frac{\lambda}{\lambda+\tau} \qquad (\tau \neq -\lambda).$$

$\check{F}$ is inf div and the function k (or θ, see (1.3.7) and (1.3.2)) can be taken

$$(1.5.2) \qquad k(x) = \theta(x) = \begin{cases} 0 & (x < 0) \\ e^{-\lambda x} & (x \geq 0), \end{cases}$$

as follows from the equality

$$\log \frac{\lambda}{\lambda+\tau} = \int_0^\infty \{e^{-\lambda x} - e^{-(\lambda+\tau)x}\}\, x^{-1} dx \qquad (\text{Re } \tau > -\lambda).$$

We now consider L.T.'s of d.f's of the form

(1.5.3) $$\check{F}(\tau) = \prod_1^n \{\lambda_k / (\lambda_k + \tau)\} / \prod_1^m \{\mu_j / (\mu_j + \tau)\}$$

$$(\tau \neq -\lambda_k,\ k = 1,2,\ldots,n),$$

where $m \leq n$ and the λ's (and μ's) are positive, or occur in complex conjugate pairs with positive real parts. From the correspondence between (1.5.1) and (1.5.2) it follows that $\check{F}$ in (1.5.3) has a representation of the form (1.3.4) with

(1.5.4) $$k(x) = \sum_1^n e^{-\lambda_k x} - \sum_1^m e^{-\mu_j x} \qquad (x \geq 0).$$

Complex conjugate numbers λ and $\bar{\lambda}$ give rise to a term of the form

(1.5.5) $$2 \exp(- x\mathrm{Re}\, \lambda) \cos(x\mathrm{Im}\, \lambda).$$

As an example we take

(1.5.6) $$2\, \check{F}(\tau) = 1 + (1 + \tau)^{-3} = (1 + \tau)^{-3} \prod_1^3 (\mu_j + \tau),$$

where $\mu_1 = 2$ and $\mu_{2,3} = \frac{1}{2}(1 \pm i\sqrt{3})$. It follows that

$$k(x) = 3\, e^{-x} - e^{-2x} - 2\, e^{-\frac{1}{2}x} \cos \frac{1}{2} x\sqrt{3},$$

which is negative for large values of x satisfying $\cos \frac{1}{2} x\sqrt{3} = 1$. Therefore the L.T. $\frac{1}{2}\{1 + (1 + \tau)^{-3}\}$ is not inf div.

From (1.5.4) one can easily obtain sufficient conditions for $\check{F}$ in (1.5.3) to be inf div. A very simple condition is that $\lambda_k \leqq \mu_k$ for $k = 1,2,\ldots,m$. We shall meet this and more sophisticated conditions in chapter 2.

If in (1.5.3) we put $\tau = -it$ and allow some of the λ's and μ's to be negative, we obtain c.f.'s of a more general class of distributions, having representations of the form (1.3.1), with θ of the form

$$(1.5.7) \qquad \theta(x) = \begin{cases} \sum_1^{n^{(1)}} e^{\lambda_k^{(1)}x} - \sum_1^{m^{(1)}} e^{\mu_j^{(1)}x} & (x < 0) \\ \sum_1^{n^{(2)}} e^{-\lambda_k^{(2)}x} - \sum_1^{m^{(2)}} e^{-\mu_j^{(2)}x} & (x > 0) \end{cases} .$$

A special case of this type is considered in chapter 2.

In (1.5.6) the function $2\Pi(\mu_j + \tau)^{-1}$ is not the L.T. of a d.f.; therefore $\check{F}$ is not the quotient of two L.T.'s in an obvious way. However, even the quotient of two inf div rational c.f.'s need not be inf div, if it is a c.f. (see e.g. [22] p.94 and p.187). Some conditions for rational functions to be c.f.'s are given in LUKACS and SZAZ [24].

One may consider the following partial ordering of inf div c.f.'s. If ϕ_1 and ϕ_2 are inf div, then ϕ_1 precedes ϕ_2 if ϕ_2/ϕ_1 is inf div. This is easily seen to be an order relation.[1)] In this way the Γ-distributions with c.f.'s of the form

$$\gamma_{\lambda,\alpha}(t) = \left(\frac{\lambda}{\lambda+it}\right)^{\alpha} \qquad (\lambda > 0;\ \alpha > 0)$$

can be ordered: $\gamma_{\lambda,\alpha_1}$ precedes $\gamma_{\lambda,\alpha_2}$ if $\alpha_2 > \alpha_1$, and $\gamma_{\lambda_1,\alpha}$ precedes $\gamma_{\lambda_2,\alpha}$ if $\lambda_2 < \lambda_1$. Clearly the $\gamma_{\lambda,\alpha}$ cannot be completely ordered. There is, for instance, no order relation between $\gamma_{1,1}$ and $\gamma_{2,2}$: the function $\theta(x)$ corresponding to $\gamma_{1,1}/\gamma_{2,2}$ equals $\theta(x) = e^{-x} - 2e^{-2x}$, and is not of constant sign.

1.6 Some known classes of inf div distributions

Except for the Normal, Poisson and Gamma distributions only relatively few inf div distributions are known. In this section we list some classes of distributions that are known to be inf div.

We shall often use the following theorem

THEOREM 1.6.1 [22]

If γ is a c.f. and if $\lambda > 0$, then ϕ_λ defined by

1) If ϕ_1 and ϕ_2 are considered equal if $|\phi_1\phi_2^{-1}| \equiv 1$.

(1.6.1) $$\phi_\lambda(t) = \frac{\lambda}{\lambda+1-\gamma(t)}$$

is an inf div c.f.

PROOF: Using theorem 1.3.6 we only need to prove that $\phi_\lambda^{1/n}$ is a c.f. for every positive integer n. We have

$$\phi_\lambda^{1/n} = (\frac{\lambda}{\lambda+1})^{1/n} (1 - \frac{1}{\lambda+1}\gamma)^{-1/n} =$$

$$= (\frac{\lambda}{\lambda+1})^{1/n} + \sum_{k=1}^{\infty} c_k^{(n)}\gamma^k,$$

where

(1.6.2) $$c_k^{(n)} = \frac{1}{n} (1 + \frac{1}{n}) \dots (k - 1 + \frac{1}{n}) \frac{1}{k!} \lambda^{1/n} (1 + \lambda)^{-k-1/n}$$

is positive. Therefore $\phi_\lambda^{1/n}$ is a mixture of c.f.'s and hence a c.f. □ □ □ [1)]

A more general class (compare remark on p.55) is the class of compound Poisson distributions, with c.f.'s of the form

(1.6.3) $$\exp\{\lambda(\gamma(t)-1)\}.$$

The corresponding random variables can be represented as

$$\underline{x}_1 + \underline{x}_2 + \dots + \underline{x}_{\underline{n}},$$

where the $\underline{x}_k$ are independent and identically distributed with c.f. γ and $\underline{n}$ has a Poisson distribution with parameter λ. In the same way, the random variable corresponding to (1.6.1) is of the form $\underline{x}_1 + \underline{x}_2 + \dots + \underline{x}_{\underline{m}}$, where $\underline{m}$ has a geometric distribution.

In LUKACS [22] the following class of inf div c.f.'s is mentioned.

$$\phi(t) = \exp\{ - \int_0^t \int_0^u \gamma(v)dv\, du\},$$

where γ is any c.f. Taking $\gamma(v) = e^{iv}$ we obtain $\phi(t) = \exp\{e^{it} - 1 - it\}$,

1) □ □ □ indicates the end of a proof.

a translated Poisson distribution.

DWASS [3] considers the following transformation: if $P(u)$ is a p.g.f. with $P(0) > 0$ and $P'(1) < 1$, then the function u defined by

$$u(z) = z/P(z) \qquad (|z| \leq 1) \tag{1.6.4}$$

has a unique inverse $z(u)$, which is the p.g.f. of an inf div distribution on the non-negative integers. His proof is based on one of his theorems on random walks. The result can be obtained directly by use of Lagrange's theorem (theorem 1.8.1). From (1.6.4) it follows that

$$\log(z/u) = \log P(z) = \log p_0 + \sum_{n=1}^{\infty} \frac{u^n}{n!} [(\frac{d}{dx})^{n-1} P'(x)P^{n-1}(x)]_{x=0} =$$

$$= \log p_0 + \sum_{n=1}^{\infty} p_n^{*n} n^{-1} u^n,$$

where p_k^{*n} is the coefficient of z^k in $P^n(z)$. Hence

$$z(u) = p_0 u \exp (\sum_1^{\infty} p^{*n} n^{-1} u^n), \tag{1.6.5}$$

as obtained by Dwass. From (1.6.5) the inf div of $z(u)$ follows by theorem 1.3.11 (compare remark 2 on p. 8). One can also express $z(u)$ directly as a function of u by Lagrange's theorem:

$$z(u) = \sum_1^{\infty} p_{n-1}^{*n} n^{-1} u^n. \tag{1.6.6}$$

As an example we take $P(z) = 1 - p + pz$, and from (1.6.6) we obtain $p_{n-1}^{*n} n^{-1} = (1-p) p^{n-1}$, i.e. the geometric distribution.

1.7 Waiting times

As a large part of the work in the following chapters was prompted by results in waiting-time theory, in this section we give a brief description of the classical one-counter waiting-time situation. For details we refer to KENDALL [14].

Customers numbered 1, 2, ... arrive at a counter at times $0, \underline{y}_1, \underline{y}_1 + \underline{y}_2$, ...; the time needed to serve the n-th customer is denoted by $\underline{s}_n$; all $\underline{y}$'s have d.f. A(y), all $\underline{s}$'s have d.f. B(s); all $\underline{y}$'s and $\underline{s}$'s are independent. The time elapsing between the arrival of the n-th customer and the moment he starts being served is called his waiting time, and denoted by $\underline{w}_n$, with d.f. C_n. This general situation is labeled GI/G/1, expressing the fact that both A and B are general (unspecified) d.f.'s and that there is only one counter; the I stands for "Independent". If A or B are exponential d.f.'s then GI or G is replaced by M (Markov). The description is completed by the condition that the server is never idle in the presence of customers and by specifying the "queue discipline". Three of the best-known queue disciplines are "first-come-first-served", "random service", where each of the waiting customers has the same probability of being served first, and "last-come-first-served". We shall only be concerned with situations where $C = \lim_{n\to\infty} C_n$ exists. This limiting d.f. will be examined for inf div.

In the M/G/1 first-come-first-served case the L.T. of C is given by the Pollaczek-Khintchine formula:

$$\check{C}(\tau) = \frac{(1-\rho)\tau}{\tau - \lambda + \lambda\check{B}(\tau)}, \tag{1.7.1}$$

where $\lambda^{-1} = E\underline{y}_n$ and $\rho = \lambda E\underline{s}_n$ with $0 < \rho < 1$(see e.g.[16], p.312). Writing $\check{R} = (1 - \check{B})(\tau E\underline{s}_n)^{-1}$ and $\alpha = \rho^{-1} - 1$ we have

$$\check{C} = \frac{\alpha}{\alpha + 1 - \check{R}} \tag{1.7.2}$$

where $\check{R}$ is the L.T. of a d.f. (cf. section 4.1.1). It follows from theorem 1.6.1 that $\check{C}$ is inf div.

In the case GI/G/1 with first-come-first-served queue discipline the L.T. of the waiting-wime is also of the form (1.7.2), where $\check{R}$ is of a more complicated structure (see KINGMAN [17]).

In the case M/G/1 and last-come-first-served queue discipline $\check{C}$ is given by (cf WISHART [38])

$$\check{C} = 1 - \rho + \lambda \frac{1 - \check{G}}{\tau + \lambda - \lambda\check{G}} \tag{1.7.3}$$

where $\check{G}$ is the L.T. of the busy period, i.e. the period during which the server is uninterruptedly busy (c.f. [5]). In section 4.1.4 it is shown that in this case $\check{C}$ need not be inf div.

The limiting case ($\rho \uparrow 1$) of the case M/G/1 with random service gave rise to the investigation of mixtures of exponential distributions by GOLDIE [8]. This case will be discussed in section 2.1.

1.8 Two theorems of analysis

Here we state (without proof) two not too well-known theorems that will be used repeatedly.

THEOREM 1.8.1 (Lagrange's theorem) [36]

If ϕ is a function of the complex variable z, which is regular and $\neq 0$ on and inside a closed contour C around $z = 0$, and if u is such that $|u\phi(z)| < |z|$ on C, then the equation

$$u\phi(z) = z$$

has exactly one solution $z = z(u)$ with z inside C. Furthermore every function f, which is regular on and inside C can be expanded in a power series in u as follows

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{u^n}{n!} [(\frac{d}{dx})^{n-1} f'(x)\phi^n(x)]_{x=0}.$$

THEOREM 1.8.2 (Karamata's inequality) [2]

If f is a real and convex function on $(-\infty,\infty)$ and if $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are real numbers, satisfying

$$x_1 \geq x_2 \geq \cdots \geq x_n; \qquad y_1 \geq y_2 \geq \cdots \geq y_n$$

$$\sum_1^m x_k \geq \sum_1^m y_k \quad (m = 1, 2, \ldots, n-1)$$

$$\sum_1^n x_k = \sum_1^n y_k,$$

then

$$\sum_1^n f(x_k) \geq \sum_1^n f(y_k). \tag{1.8.1}$$

<u>REMARK</u>: If f is non-decreasing (non-increasing) then the inequality (1.8.1) remains true if the condition

$\sum_1^n x_k = \sum_1^n y_k$ is replaced by $\sum_1^n x_k \geq \sum_1^n y_k$ $(\sum_1^n x_k \leq \sum_1^n y_k)$.

Chapter 2

MIXTURES OF Γ-DISTRIBUTIONS

2.1 Products of random variables

In the queueing model M/G/1 with random service (see section 1.7) the distribution of $(1-\rho)\underline{w}$ for $\rho \uparrow 1$ tends to the distribution of $\underline{x}\,\underline{y}$, where $\underline{x}$ and $\underline{y}$ are independent and exponentially distributed (cf. [17]). Runnenburg (cf. [17]) raised the question whether this distribution is inf div. GOLDIE [8] proved that the product of two independent non-negative random variables is inf div if at least one of the two is exponentially distributed. This result is the starting point of this chapter.

Goldie's proof uses a certain type of renewal sequences and is not very well suited for generalization. In section 2.2 we give an altogether different proof of his result, which (as we shall see in this chapter and the next) can be generalized in more than one direction.

In this chapter we consider products of independent non-negative random variables, at least one of which has a Γ-distribution.

2.2 Mixtures of exponential distributions

If $\underline{x}$ and $\underline{y}$ are independent, if $\underline{x}$ is non-negative with d.f. G_1 and $\underline{y}$ exponentially distributed with mean μ^{-1}, then the c.f. ϕ of $\underline{x}\,\underline{y}$ is given by

(2.2.1) $$\phi(t) = \int_0^\infty \frac{\mu}{\mu - itx}\, dG_1(x).$$

Putting $x = \mu\lambda^{-1}$ we obtain

(2.2.2) $$\phi(t) = \int_0^\infty \frac{\lambda}{\lambda - it}\, dG(\lambda) + p = \int_{(0,\infty]} \frac{\lambda}{\lambda - it}\, dG(\lambda),$$

where $G(\lambda) = 1 - G_1(\mu(\lambda+0)^{-1})$. It follows that $G(0) = 0$, and that

(2.2.3) $$p = G_1(0) = 1 - \lim_{\lambda\to\infty} G(\lambda),$$

i.e. G may have an atom at infinity. Though (2.2.1) is in some respects more natural than (2.2.2), we shall often prefer (2.2.2) for reasons that will become clear in the sequel.

Clearly, the distribution of $\underline{x}\,\underline{y}$ is a mixture of exponential distributions. In order to prove that such mixtures are inf div we first consider finite mixtures, i.e. p.d.f.'s of the form

$$(2.2.4) \qquad f(x) = \sum_1^n p_k \lambda_k e^{-\lambda_k x} = \int_0^\infty \lambda e^{-\lambda x}\, dG(\lambda) \qquad (x > 0),$$

where $p_k \neq 0$, $\sum_1^n p_k = 1$ and without loss of generality

$$(2.2.5) \qquad 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n.$$

We do not require all p_k to be positive. As f(x) in (2.2.4) is a p.d.f. the p_k are (here and in the sequel) supposed to be such that $f(x) \geq 0$ for all $x > 0$. Some conditions to this effect are given in BARTHOLOMEW [1]. Sufficient conditions can be obtained by putting $dA = \lambda dG$ (cf.(2.2.4)) and observing that

$$(2.2.6) \qquad \int_0^\infty e^{-\lambda x}\, dA(\lambda) = \int_0^\infty x e^{-\lambda x} A(\lambda) d\lambda,$$

as can be proved by integration by parts. The right-hand side of (2.2.6) is a mixture of $\Gamma(2)$ - densities, whichis a p.d.f. if $A(\lambda) \geq 0$ for all $\lambda > 0$. In the same way, integrating by parts repeatedly, less restrictive conditions can be obtained, e.g.

$$\int_0^\lambda A(u)du \geq 0 \qquad (\lambda > 0).$$

By taking $x \to \infty$ and $x \to 0$ respectively in (2.2.4) it is seen that we must have

$$(2.2.7) \qquad p_1 > 0 \quad ; \quad \sum_1^n p_k \lambda_k \geq 0.$$

The L.T. corresponding to the p.d.f. (2.2.4) is equal to

$$\check{F}(\tau) = \sum_1^n p_k \frac{\lambda_k}{\lambda_k+\tau} \quad . \tag{2.2.8}$$

In (2.2.8) we will sometimes allow λ_n to be infinite. The n-th term in (2.2.8) is then to be read as p_n. In this case p_n cannot be negative, as this would cause $F(x)$ to have an atom $p_n < 0$ at $x = 0$.

We prove the following theorem.

THEOREM 2.2.1

If $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \leq \infty$ and in the sequence $p_1, p_2, \ldots, p_n$ there is not more than one change of sign, then the L.T. (2.2.8) is inf div.

PROOF: As a start we take all $p_k > 0$. Clearly $\check{F}$ can be put in the form

$$\check{F}(\tau) = P(\tau) \Big/ \prod_1^n (\lambda_k+\tau), \tag{2.2.9}$$

where $P(\tau)$ is a polynomial of degree at most $n - 1$. From (2.2.8) it follows that $\check{F} > 0$ if $\tau \downarrow -\lambda_{k+1}$ and $\check{F} < 0$ if $\tau \uparrow -\lambda_k$. As $\check{F}$ is continuous on $(-\lambda_{k+1},-\lambda_k)$ $(k = 1, 2,\ldots,n-1)$, $\check{F}$ has n-1 zeros $-\mu_1, -\mu_2, \ldots, -\mu_{n-1}$ satisfying

$$0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \ldots < \lambda_{n-1} < \mu_{n-1} < \lambda_n. \tag{2.2.10}$$

It follows that $\check{F}$ is of the form

$$\check{F}(\tau) = \prod_1^n \frac{\lambda_k}{\lambda_k+\tau} \Big/ \prod_1^{n-1} \frac{\mu_j}{\mu_j+\tau} \quad , \tag{2.2.11}$$

with the μ_j satisfying (2.2.10). Consequently $\check{F}$ satisfies (1.3.4) with (compare (1.5.1) and (1.5.2))

$$k(x) = \sum_1^n e^{-\lambda_k x} - \sum_1^{n-1} e^{-\mu_j x} \qquad (x > 0) \ , \tag{2.2.12}$$

which is positive by (2.2.10). Hence $\check{F}$ is inf div in this case.

Turning to the general case we assume the p_k to have the following signs:

$$(2.2.13) \qquad p_1 > 0,\ p_2 > 0,\ \dots,\ p_{m-1} > 0,\ p_m < 0,\ p_{m+1} < 0,\ \dots,\ p_n < 0,$$

for some m with $1 < m \leq n$.
Now the preceding argument, leading to a zero $-\mu_k$ with $\lambda_k < \mu_k < \lambda_{k+1}$, can be repeated for all $k \neq m-1$. If $\Sigma p_k \lambda_k = 0$, then P in (2.2.9) is of degree n-2, and all zeros of P are accounted for, and the inf div of $\check{F}$ follows as above. If $\Sigma p_k \lambda_k > 0$ (see (2.2.7)) then $\check{F}(\tau)$ is negative for large negative values of τ. As $\check{F} \to \infty$ for $\tau \uparrow -\lambda_n$ (which in this case is finite) the n-1 st. zero, $-\mu_{n-1}$ satisfies $\mu_{n-1} > \lambda_n$ and it follows that in this case too $\check{F}$ is inf div. □ □ □

<u>COROLLARY 2.2.1</u>

Let G be a function of bounded variation satisfying the conditions

$$G(0) = 0; \int_{(0,\infty]} dG(\lambda) = 1; \int_0^\infty \lambda e^{-\lambda x}\, dG(\lambda) \geq 0 \qquad (x > 0),$$

and having no negative mass at infinity. If, furthermore, there is a λ_0 with $0 < \lambda_0 \leqq \infty$ such that G is non-decreasing for $\lambda < \lambda_0$ and non-increasing for $\lambda > \lambda_0$, then the L.T.

$$(2.2.14) \qquad \check{F}(\tau) = \int_{(0,\infty]} \frac{\lambda}{\lambda+\tau}\, dG(\lambda)$$

is inf div.

<u>PROOF</u>: G is the weak limit of a sequence of functions of the form

$$G_n(\lambda) = \sum_1^{n-1} p_k\, \iota(\lambda-\lambda_k),$$

where $\iota(x)$ is defined by

$$\iota(x) = \begin{cases} 0 & (x < 0) \\ 1 & (x \geq 0), \end{cases}$$

and the p_k and λ_k satisfy the conditions of theorem 2.2.1, with (cf. (2.2.13))

$\lambda_{m-1} < \lambda_0 < \lambda_m$. Applying Helly's second theorem (cf. [22]) on the intervals $(0,\lambda_0]$ and (λ_0,∞) it follows that $\check{F}$ is the limit of a sequence of L.T.'s of the form

$$p_n + \int_0^\infty \frac{\lambda}{\lambda+\tau}\, dG_n(\lambda) = \sum_{k=0}^{n} p_k \frac{\lambda_k}{\lambda_k+\tau} ,$$

with $\lambda_n = \infty$. Hence, by theorems 2.2.1 and 1.3.5 the L.T. $\check{F}$ in (2.2.14) is inf div. □ □ □

COROLLARY 2.2.2

All mixtures of two exponential distributions are ind div.

COROLLARY 2.2.3

If $\underline{x}$ and $\underline{y}$ are independent, $\underline{x} \geq 0$ and $\underline{y}$ exponentially distributed, then $\underline{x}\,\underline{y}$ is inf div.

Corollary 2.2.3 was obtained by GOLDIE [8] as a corollary to a theorem that is more general in some respects and less general in others. His theorem does not allow for negative p_k in (2.2.8).

In LORENTZ [21] (p.12) it is proved that every continuous function $a(y)$ on $[0,1]$ is the uniform limit of a sequence of polynomials $b_n(y)$, in such a way that $b_n(y)$ is non-increasing on $[0,1]$ if $a(y)$ is non-increasing on $[0,1]$. It follows that every bounded continuous function $A(x)$ on $[0,\infty]$ is the uniform limit of a sequence $B_n(e^{-x})$, where B_n is a polynomial, and $B_n(e^{-x})$ is non-decreasing, if $A(x)$ is non-decreasing. Hence, every d.f. is the weak limit of a sequence of d.f.'s with p.d.f.'s of the form (2.2.4). (see also KINGMAN [16], p. 317). As not all L.T.'s are inf div, it follows that not all exponential mixtures are inf div (compare theorem 1.3.5). According to theorem 2.2.1 the simplest counter examples should be looked for in the class of three-component mixtures with $p_1 > 0$, $p_2 < 0$ and $p_3 > 0$. An example of this kind is provided by the p.d.f.

$$f_1(x) = 2e^{-x} - 6e^{-3x} + 5e^{-5x} \qquad (x > 0),$$

with c.f.

$$\phi_1(t) = (15 - t^2)\ \{(1 - it)(3 - it)(5 - it)\}^{-1}.$$

As $\phi(t)$ has real zeros $\pm\sqrt{15}$ this c.f. is not inf div by theorem 1.3.2.

An inf div example of the same class is

$$f_2(x) = \frac{2}{5}\,(e^{-x} - e^{-2x} + 6e^{-3x}) \qquad (x > 0).$$

It's L.T. is

$$\check{F}_2(\tau) = \frac{2}{5}\,(2\tau + 3)(3\tau + 5)\{(1+\tau)(2+\tau)(3+\tau)\}^{-1},$$

with (cf. 1.5.4)

$$k(x) = e^{-x} + e^{-2x} + e^{-3x} - e^{-\frac{3}{2}x} - e^{-\frac{5}{3}x},$$

which is non-negative by Karamata's inequality (theorem 1.8.2).

In the same way Karamata's inequality yields more generally

THEOREM 2.2.2

A L.T. of the form

$$(2.2.15) \qquad \prod_1^n \frac{\lambda_k}{\lambda_k+\tau} \cdot \prod_1^m \frac{\mu_j+\tau}{\mu_j} \qquad (m \le n)$$

with real λ's and μ's, is inf div if

$$(2.2.16) \qquad \sum_1^{\ell} \lambda_k \le \sum_1^{\ell} \mu_k \quad (\ell = 1, 2, \ldots, m).$$

REMARK: Though theorem 2.2.2 is in a sense more general than theorem 2.2.1, it is less interesting for the following reason. There seems to be no other natural way to generate L.T.'s of the form (2.2.15) than by mixing exponential distributions. However, the p_k (cf. (2.2.8)) resulting from partial fraction expansion of (2.2.15) are not very clearly characterized by the conditions (2.2.16) (compare lemma 2.12.1). That is, it is not clear what class of exponential mixtures corresponds to the conditions (2.2.16). This is demonstrated by the two examples above. If $m = n-1$ in (2.2.15), then the inequalities (2.2.10) hold if and only if in (2.2.8) all p_k are positive (see lemma 2.12.1).

2.3 Generalizations

The results of the preceding section can be generalized in several ways. In terms of c.f.'s we have proved that mixtures (with positive weights) of functions of the form

$$\text{(2.3.1)} \qquad \frac{\lambda}{\lambda - it}$$

are inf div c.f.'s. More generally, instead of mixtures of c.f.'s (2.3.1) one may consider mixtures of

$$\text{(2.3.2)} \qquad \gamma_{\lambda,\alpha}(t) = \left(\frac{\lambda}{\lambda - it}\right)^{\alpha} \qquad (\alpha > 0),$$

with p.d.f.

$$\text{(2.3.3)} \qquad g_{\lambda,\alpha}(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x} \qquad (x > 0).$$

Distributions with p.d.f.'s of the form (2.3.3) will be referred to as $\Gamma(\alpha)$ - distributions.

On the other hand (2.3.1) can be generalized to

$$\text{(2.3.4)} \qquad \frac{\lambda}{\lambda - h(t)}$$

with a suitable class of functions h, and then again to

$$\text{(2.3.5)} \qquad \left(\frac{\lambda}{\lambda - h(t)}\right)^{\alpha} .$$

In the remaining sections of this chapter we shall consider mixtures of c.f.'s of the form (2.3.2), where mixing can take place with respect to λ or α or both. In chapter 3 the cases (2.3.4) and to some extent (2.3.5) will be treated.

We conclude this section by considering an other type of generalization of theorem 2.2.1. Let f be the p.d.f. defined by

$$(2.3.6) \qquad f(x) = \begin{cases} \sum_{j=1}^{n'} p_j' \lambda_j' \; e^{\lambda_j' x} & (x < 0) \\ \sum_{k=1}^{n} p_k \lambda_k \, e^{-\lambda_k x} & (x > 0), \end{cases}$$

where

$$(2.3.7) \qquad \begin{array}{l} p_j' > 0 \ (j = 1, 2, \ldots, n'); \ p_k > 0 \ (k = 1, 2, \ldots, n) \\ \sum_1^{n'} p_j' + \sum_1^{n} p_k = 1 \\ 0 < \lambda_1' < \ldots < \lambda_{n'}'; \ 0 < \lambda_1 < \ldots < \lambda_n. \end{array}$$

The c.f. of f is given by

$$(2.3.8) \qquad \phi(t) = \sum_1^{n'} p_j' \frac{\lambda_j'}{\lambda_j'+it} + \sum_1^{n} p_k \frac{\lambda_k}{\lambda_k-it} \; .$$

We prove

<u>THEOREM 2.3.1</u>

The c.f. (2.3.8) with p_j', λ_j', p_k and λ_k satisfying the conditions (2.3.7) is inf div.

<u>PROOF</u>: By analytic continuation we have for $z \neq \lambda_j'$ $(j = 1, 2, \ldots n')$ and $z \neq -\lambda_k$ $(k = 1, 2, \ldots, n)$

$$\phi(iz) = \sum_1^{n'} p_j' \frac{\lambda_j'}{\lambda_j'-z} + \sum_1^{n} p_k \frac{\lambda_k}{\lambda_k+z} \; .$$

Here ϕ is of the form P/Q, where P and Q are polynomials. P is of degree $n' + n - 2$ if $\Sigma p_j' \lambda_j' - \Sigma p_k \lambda_k = 0$ and of degree $n' + n - 1$ otherwise. As in the proof of theorem 2.2.1 it is easily shown that $\phi(iz)$ has $n' + n - 2$ zeros $\mu_1', \ldots, \mu_{n'-1}'$ and $-\mu_1, \ldots, -\mu_{n-1}$ satisfying

$$(2.3.9) \qquad \lambda_1' < \mu_1' < \ldots < \mu_{n'-1}' < \lambda_{n'}' \; ; \; \lambda_1 < \mu_1 < \ldots < \mu_{n-1} < \lambda_n \; .$$

If $\Sigma p_j' \lambda_j' - \Sigma p_k \lambda_k = 0$, then this accounts for all zeros. Otherwise by the argument used in the proof of theorem 2.2.1 there is one more zero $\mu_{n'}'$, with $\mu_{n'}' > \lambda_{n'}'$, or $-\mu_n$ with $\mu_n > \lambda_n$, depending on the sign of $\Sigma p_j' \lambda_j' - \Sigma p_k \lambda_k$. It follows that $\phi(t)$ has a representation of the form (1.3.1) with $\theta(x)$ given by

$$\theta(x) = \begin{cases} \sum_{j=1}^{n'} e^{\lambda'_j x} - \sum_{p=1}^{m'} e^{\mu'_j x} & (x < 0) \\ \sum_{k=1}^{n} e^{-\lambda_k x} - \sum_{k=1}^{m} e^{-\mu_k x} & (x > 0), \end{cases}$$

where m' equals n' or n'-1 and m equals n-1 or n depending on the sign of $\Sigma p_j' \lambda_j' - \Sigma p_k \lambda_k$. The inequalities (2.3.9) and the inequality for the remaining zero ensure that $\theta(x) > 0$ for all x. Hence ϕ is inf div. □ □ □

2.4 The case $0 < \alpha \leq 1$

For the time being we restrict ourselves to mixtures with positive weights. From theorem 2.2.1 we deduce

THEOREM 2.4.1

All mixtures of $\Gamma(\alpha)$ - distributions with $0 < \alpha \leq 1$ are inf div.

PROOF: We have (cf. [36] p. 261) for all $\alpha < \beta$

$$(2.4.1) \qquad (\frac{\lambda}{\lambda+\tau})^{\alpha} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_1^{\infty} (\frac{\lambda u}{\lambda u+\tau})^{\beta} u^{-\beta} (u-1)^{\beta-\alpha-1} du.$$

The equality (2.4.1) expresses the fact that every $\Gamma(\alpha)$ - distribution can be regarded as a mixture of $\Gamma(\beta)$ - distributions with $\beta > \alpha$. In particular every $\Gamma(\alpha)$ - distribution with $0 < \alpha \leq 1$ is a mixture of exponential ($\Gamma(1)$) distributions. Now any finite mixture

$$(2.4.2) \qquad \check{F}(\tau) = \Sigma p_k (\frac{\lambda_k}{\lambda_k+\tau})^{\alpha_k}$$

with $0 < \alpha_k < 1$, by (2.4.1) is a mixture of exponential distributions and therefore is inf div by theorem 2.2.1. The inf div of general mixtures is obtained by regarding these as limits of mixtures of type (2.4.2) and using the closure property (theorem 1.3.5). □ □ □

As an example we take a mixture of $\Gamma(\frac{1}{2})$ - distributions:

$$\frac{2}{1+(1+\tau)^{1/2}} = \int_0^1 (1+\tau x)^{-1/2}\, dx$$

is an inf div L.T.

REMARK: As a generalization of theorem 2.2.1, one would expect the L.T. $\Sigma p_k\ \lambda_k^{\alpha}\ (\lambda_k+\tau)^{-\alpha}$ to be inf div if $0 < \alpha < 1$ and the p_k do not change sign more than once. In general, however, it is not possible to represent these L.T.'s in the form (2.2.14) with G satisfying the conditions of corollary 2.2.1.

2.5 The case $\alpha > 2$

As we saw in section 1.5, the L.T.

$$\frac{1}{2}\{1 + (1+\tau)^{-3}\}$$

is not inf div. As all mixtures of $\Gamma(\alpha)$ - distributions can be regarded as mixtures of $\Gamma(\beta)$ - distributions with $\beta > \alpha$ (cf. (2.4.1)), it follows that mixtures of $\Gamma(\alpha)$ - distributions with $\alpha > 3$ are in general not inf div. It also follows (from the closure theorem) that not all mixtures of $\Gamma(\alpha)$ - distributions with $\alpha < 3$ are inf div. It turns out that mixtures of $\Gamma(\alpha)$ - distributions are in general not inf div for any $\alpha > 2$. To prove this we consider the graphs in the complex plane of $(1-it)^{-\alpha}$, for real t and for $\alpha = 1$, $1 < \alpha \leq 2$ and $2 < \alpha < 3$ respectively. These graphs are sketched in Fig. 2.5.1. Clearly these graphs are symmetric with respect to the real axis, the upper halfs representing the values for $t > 0$, the lower halfs those for $t < 0$.

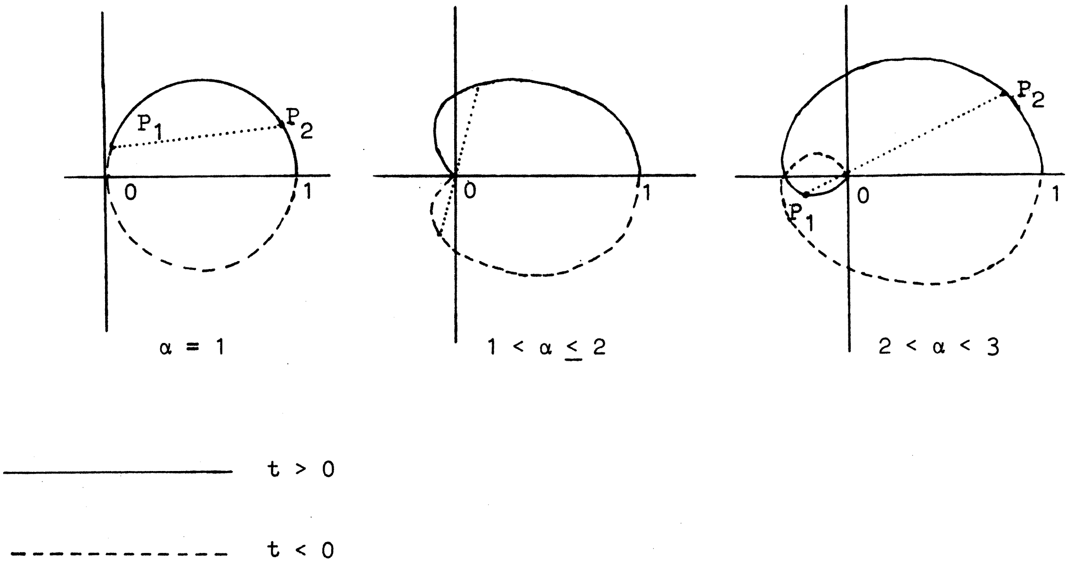

Fig. 2.5.1

Every two different points P_1 and P_2 on the solid part of each of these graphs can be regarded as the images of two functions $(1 - ix_1t)^{-\alpha}$ and $(1 - ix_2t)^{-\alpha}$, with $x_1 \geq 0$ and $x_2 \geq 0$, at the same value $t = t_0 > 0$. The images of mixtures of these functions at $t = t_0$ form the dotted line segment P_1P_2. If this line segment passes through the origin 0, then there is a mixture of the two functions with a zero at $t = t_0$, and therefore one that is not inf div. As follows from Fig. 2.5.1, this is possible only in the case $\alpha > 2$. To obtain concrete examples of such mixtures we consider the c.f.

$$(2.5.1) \qquad 1 - p + p(1 - it)^{-\alpha} \qquad (0 < p < 1),$$

i.e. $x_1 = 0$ and $x_2 = 1$. In this case a zero can occur for $t = t_0$, satisfying $t_0 > 0$ (without restriction) and

$$\text{Im}(1 - it_0)^{-\alpha} = 0 \quad ; \text{Re}(1 - it_0)^{-\alpha} < 0,$$

i.e.

$$\arg(1 - it_0)^{-\alpha} = -\pi,$$

and hence

$$t_0 = \operatorname{tg} \frac{\pi}{\alpha} .$$

It is easily verified that these requirements can be met if (and only if) $\alpha > 2$. If we now choose p such that

$$1 - p + p \operatorname{Re}(1 - it_0)^{-\alpha} = 0,$$

then the c.f. (2.5.1) is not inf div.

REMARK: It follows in the same way (see Fig. 2.5.1, $1 < \alpha \leq 2$) that mixtures of the form

$$p(1 - ix_1t)^{-\alpha} + (1 - p)(1 + ix_2t)^{-\alpha}$$

are in general not inf div for $\alpha > 1$ (compare theorem 2.3.1).

2.6 A conjecture

The results in the foregoing sections are summarized in

THEOREM 2.6.1

The set

$$\{a > 0 \mid \int_0^\infty (1 - itx)^{-\alpha} dG(x) \text{ is inf div for all d.f.'s } G \text{ on } [0,\infty)\}$$

has a maximum α_0 satisfying $1 \leq \alpha_0 \leq 2$.

PROOF: The existence of a maximum and the inequalities for α_0 follow from theorem 1.3.5, corollary 2.2.1 and the considerations in section 2.5. □ □ □

In view of the results in section 2.2 and 2.5 it seems reasonable to conjecture that $\alpha_0 = 2$. Unfortunately we have not been able to prove or disprove this. In the next three sections, however, we shall supply some evidence for the conjecture $\alpha_0 = 2$. How convincing this evidence is seems to be a matter of faith.

2.7 Finite mixtures of $\Gamma(2)$-distributions

In this section we consider L.T.'s of the form

$$(2.7.1)\qquad \check{F}(\tau) = \sum_1^n p_k \left(\frac{\lambda_k}{\lambda_k+\tau}\right)^2 \qquad (\tau \neq -\lambda_k;\ k = 1, 2, \ldots, n),$$

with $p_k > 0$, $\sum_1^n p_k = 1$ and $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$.
$\check{F}(\tau)$ in (2.7.1) van be rewritten as

$$(2.7.2)\qquad \check{F}(\tau) = P(\tau) \prod_1^n (\lambda_k + \tau)^{-2},$$

where $P(\tau)$ is a polynomial of degree $2n - 2$. The zeros τ_j of P occur in complex conjugate pairs:

$$(2.7.3)\qquad \tau_j = -\mu_j \pm i\nu_j \qquad (j = 1, 2, \ldots, n-1),$$

where the μ_j are supposed to be ordered:

$$(2.7.4)\qquad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}.$$

It is easily seen (cf. (1.5.4) and (1.5.5)) that $\check{F}$ has a canonical representation (1.3.4) with

$$(2.7.5)\qquad k(x) = 2 \sum_{k=1}^n e^{-\lambda_k x} - 2 \sum_{j=1}^{n-1} e^{-\mu_j x} \cos \nu_j x \qquad (x > 0).$$

Therefore a sufficient condition for inf div of $\check{F}$ is

$$(2.7.6)\qquad \sum_{k=1}^n e^{-\lambda_k x} \geq \sum_{j=1}^{n-1} e^{-\mu_j x} \qquad (x > 0).$$

As a first attempt to prove (2.7.6) one may try to prove that $\lambda_j \leq \mu_j$ $(j = 1, 2, \ldots, n-1)$, as in the case $\alpha = 1$. As it turns out, these inequalities hold generally only in the case $n = 2$. A counterexample with $n = 3$ will be given in section 2.10. As a second attempt one may try to use Karamata's inequality (see also remark following theorem 1.8.2),by which the inequalities

$$(2.7.7) \qquad \sum_1^m \lambda_k \leq \sum_1^m \mu_k \qquad (m = 1, 2, \ldots, n-1)$$

imply (2.7.6).

To obtain information about the zeros of $\check{F}$, we write

$$(2.7.8) \qquad S(\tau) = \check{F}(\tau) \left(\sum_1^n p_k \lambda_k^2\right)^{-1} = \sum_1^n \frac{A_j}{(\lambda_j+\tau)^2} ,$$

with

$$(2.7.9) \qquad A_j = p_j \lambda_j^2 \left(\sum_1^n p_k \lambda_k^2\right)^{-1} ,$$

and hence

$$\sum_1^n A_j = 1.$$

Putting $\tau = -\mu + i\nu$ it follows that

$$\text{Re } S(\tau) = \sum_1^n A_j \frac{(\lambda_j-\mu)^2-\nu^2}{\{(\lambda_j-\mu)^2+\nu^2\}^2} ,$$

$$(2.7.10) \qquad \text{Im } S(\tau) = -2\nu \sum_1^n A_j \frac{\lambda_j-\mu}{\{(\lambda_j-\mu)^2+\nu^2\}^2} .$$

As S has no real zeros, S = 0 implies that $\nu \neq 0$. From (2.7.10) it then follows that S cannot be zero for values of μ for which $\lambda_j - \mu$ is of constant sign. Hence we have

$$(2.7.11) \qquad \lambda_1 < \mu_j < \lambda_n \qquad (j = 1, 2, \ldots, n-1).$$

For n = 2 the inequality (2.7.11) implies (2.7.6) and the inf div of (2.7.1).

Taking n = 3 we can write

$$\prod_1^3 (\lambda_j+\tau)^{-2} S(\tau) = A_1(\lambda_2+\tau)^2(\lambda_3+\tau)^2 + A_2(\lambda_1+\tau)^2(\lambda_3+\tau)^2 + A_3(\lambda_1+\tau)^2(\lambda_2+\tau)^2 = N(\tau),$$

where on the one hand $N(\tau)$ is of the form

(2.7.12)
$$N = \tau^4 + 2\,\{A_1(\lambda_2+\lambda_3) + A_2(\lambda_1+\lambda_3) + A_3(\lambda_1+\lambda_2)\}\tau^3 + \\ + c_2\tau^2 + c_1\tau + c_0,$$

and on the other hand

(2.7.13)
$$N = (\tau^2+2\mu_1\tau+a_1)(\tau^2+2\mu_2\tau+a_2),$$

where the μ's are defined in (2.7.3) and the a's are constants. Equating the coefficients of τ^3 in (2.7.12) and (2.7.13) we obtain

$$\mu_1 + \mu_2 = A_1(\lambda_2+\lambda_3) + A_2(\lambda_1+\lambda_3) + A_3(\lambda_1+\lambda_2) =$$
$$= \lambda_1 + \lambda_2 + A_1(\lambda_3-\lambda_1) + A_2(\lambda_3-\lambda_2) > \lambda_1 + \lambda_2.$$

As by (2.7.11) we also have $\mu_1 > \lambda_1$, Karamata's inequality yields (2.7.6) and hence,by (2.7.5), the inf div of (2.7.1) for n = 3.

By the same method we have for all n

(2.7.14)
$$\mu_1 > \lambda_1 \quad ; \quad \sum_1^{n-1} \mu_k > \sum_1^{n-1} \lambda_k \;.$$

To be able to apply Karamata's inequality in the case n = 4, we only need the additional inequality $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$. This will be proved by a geometric method suggested by Runnenburg. We shall discuss this method in slightly more detail than we need it to prove the special case n = 4.

First, it is noted that the sum of n complex numbers, all situated at one side of a straight line through the origin, cannot be zero (see e.g. POLYA and SZEGÖ [27], p. 89). From this it follows that the zeros of

$$S(\tau) = \sum_1^n \frac{A_j}{(\lambda_j+\tau)^2}$$

with positive imaginary parts are confined to the un-hatched area in Fig. 2.7.1. This area is bounded on one side by the half-circle with diameter

$\lambda_n - \lambda_1$ passing through $-\lambda_n$ and $-\lambda_1$, on the other side by the half-circles with diameters $\lambda_{k+1} - \lambda_k$ and passing through $-\lambda_{k+1}$ and $-\lambda_k$ $(k = 1, 2, \ldots, n-1)$.

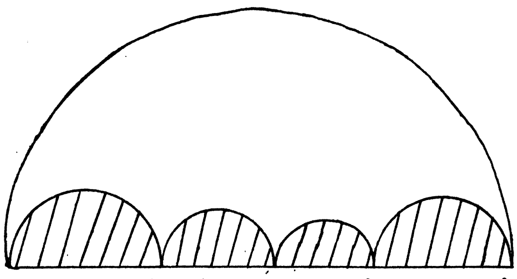

Fig. 2.7.1

Using the fact that $S(\tau)$ (see (2.7.8)) has a partial fraction expansion of the form

$$S(\tau) = \sum_1^n \frac{A_k}{(\tau+\lambda_k)^2} + \sum_1^n \frac{B_k}{\tau+\lambda_k} ,$$

with

$$B_k = 0 \qquad (k = 1, 2, \ldots, n),$$

Runnenburg obtained the relations

$$\text{(2.7.15)} \qquad \sum_{k=1}^{n-1} \frac{1}{\ell_{j,k}} = \sum_{k\neq j} \frac{1}{\lambda_j-\lambda_k} \qquad (j = 1, 2, \ldots, n),$$

where

$$\text{(2.7.16)} \qquad \ell_{j,k} = \frac{(\lambda_j-\mu_k)^2 + \nu_k^2}{\lambda_j-\mu_k} .$$

The quantities $|\ell_{j,k}|$ can be interpreted geometrically as the diameters of the circles through the points $-\lambda_j$ and $\tau_k = -\mu_k + i\nu_k$ and with centres on the real axis. We shall only use (2.7.15) with $j = n$, and we write

$$\ell_k = \ell_{n,k} .$$

For n = 2,(2.7.15) says that the zero $-\mu_1 + i\nu_1$ is situated on the half-circle through $-\lambda_1$ and $-\lambda_2$.

For n = 3 we have the situation as sketched in Fig. 2.7.2.

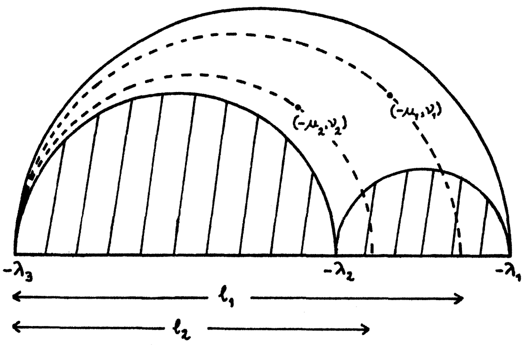

Fig. 2.7.2

It follows that $\ell_1 > \lambda_3 - \lambda_2$ and $\ell_2 > \lambda_3 - \lambda_2$. Using (2.7.15) we therefore have the relations (not knowing whether $\ell_1 > \ell_2$ or $\ell_1 \leq \ell_2$)

$$\frac{1}{\lambda_3-\lambda_2} > \frac{1}{\ell_1} \quad ; \quad \frac{1}{\lambda_3-\lambda_2} > \frac{1}{\ell_2}$$

$$\frac{1}{\lambda_3-\lambda_1} + \frac{1}{\lambda_3-\lambda_2} = \frac{1}{\ell_1} + \frac{1}{\ell_2} .$$

From this, by Karamata's inequality (with $f(x) = \frac{1}{x}$, or otherwise), we obtain (cf. (2.7.16))

$$\lambda_3 - \lambda_2 + \lambda_3 - \lambda_1 \geq \ell_1 + \ell_2 \geq \lambda_3 - \mu_2 + \lambda_3 - \mu_1,$$

and hence

$$\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$$

as we proved earlier.

In the case n = 4 we only have to prove the missing inequality $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$. The only case of interest is sketched in Fig. 2.7.3, where $\ell_1 > \lambda_4 - \lambda_2$ and $\ell_2 > \lambda_4 - \lambda_2$. In the other cases we trivially have $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$: if $\ell_1 < \lambda_4 - \lambda_2$, then $\mu_1 > \lambda_2$ and hence $\mu_2 > \lambda_2$; if $\ell_2 < \lambda_4 - \lambda_2$, then $\mu_2 > \lambda_2$ and (as always) $\lambda_1 > \mu_1$.

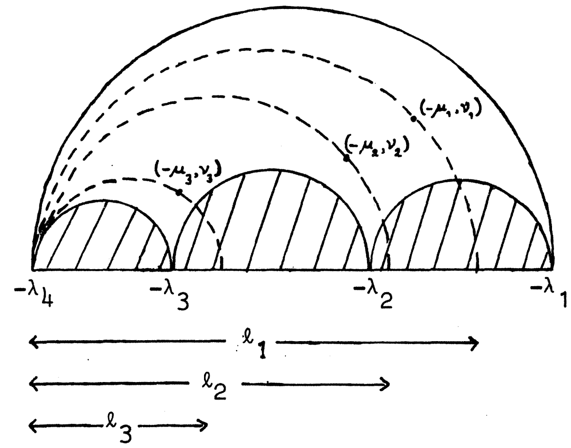

Fig. 2.7.3

We therefore have the inequalities

$$\frac{1}{\lambda_4-\lambda_2} \geq \frac{1}{\ell_1} \quad ; \quad \frac{1}{\lambda_4-\lambda_2} \geq \frac{1}{\ell_2}$$

$$\frac{1}{\lambda_4-\lambda_1} + \frac{1}{\lambda_4-\lambda_2} \geq \frac{1}{\ell_1} + \frac{1}{\ell_2} \quad ,$$

where the last inequality follows from (2.7.15) and the fact that $\ell_3 > \lambda_4 - \lambda_3$. Karamata's inequality (see remark following theorem 1.8.2) yields the inequality $\mu_1 + \mu_2 \geq \lambda_1 + \lambda_2$ as in the case $n = 3$. This inequality, the inequalities (2.7.14) and another appeal to Karamata's inequality establishes (2.7.6) and the inf div of (2.7.1) for $n = 4$.

In order to prove the inf div of (2.7.1) for larger n by this method, one would evidently need more than only one of the equalities (2.7.15). However, I did not succeed in extending this method to values of n larger than 4.

We summarize the foregoing results in

<u>THEOREM 2.7.1</u>

The L.T.

$$\sum_1^n p_k \left(\frac{\lambda_k}{\lambda_k+\tau}\right)^2 \quad ,$$

where $p_k > 0$, $\lambda_k > 0$ $(k = 1, 2, \ldots, n)$ and $\sum_1^n p_k = 1$, is inf div for $n = 1, 2, 3, 4$.

2.8 Other mixtures of $\Gamma(2)$-distributions

As discrete mixtures of $\Gamma(2)$-distributions are rather hard to handle, one is led to consider continuous mixtures. Some of these are trivially inf div, because they are mixtures (with positive weights) of exponential distributions. For example,

$$\frac{1}{b}\int_0^b \frac{1}{(1+\tau x)^2}\,dx = \frac{1}{1+b\tau} \quad .$$

Slightly more general we have

$$\frac{1}{b-a}\int_a^b \frac{1}{(1+\tau x)^2}\,dx = \frac{1}{(1+a\tau)(1+b\tau)} \quad , \tag{2.8.1}$$

which is inf div as it is the product of two L.T.'s of exponential distributions. On the other hand, writing $p = a(b-a)^{-1}$, we have

$$\frac{1}{(1+a\tau)(1+b\tau)} = (1+p)\,\frac{1}{1+b\tau} - p\,\frac{1}{1+a\tau} \quad ,$$

i.e. the L.T. of a mixture of exponential distributions with one negative weight. Generalizing this idea we prove the following theorem about unimodal d.f.'s.

A d.f. G is called unimodal if there exists an x_0 such that $G(x)$ is convex for $x < x_0$ and concave for $x > x_0$.

THEOREM 2.8.1

If G is a unimodal d.f. on $[0,\infty)$, then the L.T.

$$\check{F}(\tau) = \int_0^\infty \frac{1}{(1+\tau x)^2}\,dG(x) \tag{2.8.2}$$

is an inf div mixture of L.T.'s of exponential distributions.

PROOF: First we assume that G has continuous first and second derivatives, g and g', where g satisfies $\lim_{x\downarrow 0} x\,g(x) = \lim_{x\to\infty} x\,g(x) = 0$. Integrating by parts we then have

$$\check{F}(\tau) = \int_0^\infty \frac{1}{(1+\tau x)^2}\, g(x)dx = \frac{x}{1+\tau x}\, g(x) \Big|_0^\infty - \int_0^\infty \frac{x}{1+\tau x}\, g'(x)\, dx =$$

(2.8.3) $$= \int_0^\infty \frac{1}{1+\tau x}\, (-\, xg'(x))dx.$$

As G is unimodal, g' changes sign only once, and it easily follows from corollary 2.2.1 that the L.T. (2.8.3) is inf div. Now every unimodal d.f. on $[0,\infty)$ can be obtained as the limit of a sequence unimodal d.f.'s satisfying the assumptions above. Application of Helly's second theorem and the closure theorem concludes the proof. □ □ □

<u>REMARK 1</u>: A slightly more direct proof of theorem 2.8.1 may be given if one uses Khintchine's integral representation for unimodal distribution functions (see GNEDENKO and KOLMOGOROV [7] p. 157 ff.).

<u>REMARK 2</u>: The case where G is concave for all $x > 0$ can be regarded as a limiting case of unimodality. Then $g'(x) < 0$ and (2.8.3) is a mixture with positive weights.

A special case of theorem 2.8.1 is of interest in view of the starting point of this chapter, the inf div of products of exponentially distributed random variables. We return to this in the next section.

<u>COROLLARY 2.8.1</u>

If

$$g_{\lambda,\alpha}(x) = \frac{1}{\Gamma(\alpha)}\, \lambda^\alpha\, x^{\alpha-1}\, e^{-\lambda x} \qquad (x > 0),$$

then

$$\int_0^\infty \frac{1}{(1+\tau x)^2}\, g_{\lambda,\alpha}(x)dx$$

is an inf div L.T.

As a discrete analogue of theorem 2.8.1 we consider

(2.8.4) $$\check{F}(\tau) = \sum_1^\infty p_k\, \frac{1}{(1+\tau k)(1+\tau k+\tau)}\,,$$

where we take $(1+\tau k)^{-1}(1+\tau k+\tau)^{-1}$ instead of $(1+\tau k)^{-2}$ to make summation by parts possible. We have

$$\check{F}(\tau) = \sum_{1}^{\infty} p_k \left\{\frac{k+1}{1+(k+1)\tau} - \frac{k}{1+k\tau}\right\} = \sum_{1}^{\infty} k(p_{k-1}-p_k)\frac{1}{1+\tau k} \quad (p_0 = 0),$$

which is inf div by theorem 2.2.1 if $p_{k-1} - p_k$ does not change sign more than once. However, (2.8.4) can also be considered a special case of (2.8.2) as we have

$$\check{F}(\tau) = \sum_{1}^{\infty} p_k \int_{k}^{k+1} \frac{1}{(1+\tau x)^2}\, dx = \int_0^{\infty} \frac{1}{(1+\tau x)^2}\, dG(x),$$

where $G(x)$ is unimodal if $p_{k-1} - p_k$ changes sign only once.

2.9 Products of Γ-variates

The starting point of this chapter was the inf div of the product of two independent, exponentially distributed random variables. Denoting by $\underline{x}_\alpha$ the random variable with p.d.f.

$$\frac{x^{\alpha-1}\, e^{-x}}{\Gamma(\alpha)} \qquad (x > 0;\ \alpha > 0),$$

we prove much more generally

THEOREM 2.9.1

If $\underline{x}_\alpha$ and $\underline{x}_\beta$ are independent, then $\underline{x}_\alpha\, \underline{x}_\beta$ is inf div if

$$0 < \min(\alpha,\beta) \leq 2.$$

PROOF: The case $\alpha = 2$ is covered by corollary 2.8.1, and because of symmetry we only have to prove the theorem for $0 < \alpha < 2$ and $\beta > 0$.

For the L.T. of $\underline{x}_\alpha\, \underline{x}_\beta$ we have

$$(2.9.1) \qquad E\, \exp(-\tau \underline{x}_\alpha \underline{x}_\beta) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} (1+\tau y)^{-\alpha}\, y^{\beta-1}\, e^{-y}\, dy.$$

Using (2.4.1) with $\beta = 2$ and $\lambda = y^{-1}$ we obtain a double integral. Inverting the order of integration, substituting ux for y, and inverting the order of integration again, we get

$$E \exp(-\tau \underline{x}_\alpha \underline{x}_\beta) = C \int_0^\infty (1+\tau x)^{-2} f_{\alpha,\beta}(x)\, dx,$$

where C is a constant and

$$(2.9.2) \qquad f_{\alpha,\beta}(x) = x^{\beta-1} \int_1^\infty (u-1)^{1-\alpha} u^{\beta-2} e^{-ux}\, du \qquad (x > 0).$$

We shall prove that $f_{\alpha,\beta}$ is unimodal (or decreasing) on $(0,\infty)$ for all α and β with $0 < \alpha < 2$ and $\beta > 0$.

First, note that $f'_{\alpha,\beta}(x) = 0$ if and only if

$$(2.9.3) \qquad (\beta-1) \int_1^\infty (u-1)^{1-\alpha} u^{\beta-2} e^{-ux}\, du = x \int_1^\infty (u-1)^{1-\alpha} u^{\beta-1} e^{-ux}\, du.$$

Substituting $v = x(u-1)$ we see that $f'_{\alpha,\beta}(x) = 0$ if and only if

$$(2.9.4) \qquad L_{\alpha,\beta}(x) = R_{\alpha,\beta}(x),$$

where

$$(2.9.5) \qquad \begin{aligned} L_{\alpha,\beta}(x) &= (\beta-1) \int_0^\infty v^{1-\alpha}(v+x)^{\beta-2} e^{-v}\, dv, \\ R_{\alpha,\beta}(x) &= \int_0^\infty v^{1-\alpha}(v+x)^{\beta-1} e^{-v}\, dv. \end{aligned}$$

As the cases $\alpha \leq 1$ or $\beta \leq 1$ are covered by theorem 2.4.1, for the time being we assume $1 < \alpha < 2$ and $\beta > 1$. Then we have

$$(2.9.6) \qquad L_{\alpha,\beta}(0) > R_{\alpha,\beta}(0),$$

and, for $x > \beta-1$

$$L_{\alpha,\beta}(x) < R_{\alpha,\beta}(x).$$

It follows that $L_{\alpha,\beta} = R_{\alpha,\beta}$ for at least one value of x. We shall prove that there is exactly one.

If $1 < \beta \leq 2$, then $L_{\alpha,\beta}$ is non-increasing and $R_{\alpha,\beta}$ is increasing. If $2 < \beta \leq 3$, then $L_{\alpha,\beta}$ is concave and $R_{\alpha,\beta}$ is convex. Hence, by (2.9.6), in these cases there is exactly one solution of $L_{\alpha,\beta}(x) = R_{\alpha,\beta}(x)$.

Suppose now that for some integer $k \geq 3$ there is only one solution of $L_{\alpha,\beta}(x) = R_{\alpha,\beta}(x)$ if $k-1 < \beta \leq k$. We prove that this implies that there is also only one solution for $k < \beta \leq k+1$.

Assuming that $L_{\alpha,\beta+1}(x) = R_{\alpha,\beta+1}(x)$ has two solutions, x_0 and x_1, with $x_0 < x_1$, it follows that there are two solutions, x_0' and x_1' of $L'_{\alpha,\beta+1}(x) = R'_{\alpha,\beta+1}(x)$, with $0 < x_0' < x_0$ and $x_0 < x_1' < x_1$: The solution x_1' follows directly from Rolle's theorem, the solution x_0' is a consequence of the fact that $L_{\alpha,\beta+1}(0) > R_{\alpha,\beta+1}(0)$ and also $L'_{\alpha,\beta+1}(0) > R'_{\alpha,\beta+1}(0)$. The latter inequality follows from (2.9.6) and the equalities

$$L'_{\alpha,\beta+1}(x) = (\beta-1)\ L_{\alpha,\beta}(x)$$

(2.9.7)

$$R'_{\alpha,\beta+1}(x) = (\beta-1)\ R_{\alpha,\beta}(x).$$

But, from the equalities (2.9.7) it now follows that there are two solutions, x_0' and x_1', for $L_{\alpha,\beta}(x) = R_{\alpha,\beta}(x)$. As this contradicts our assumption, we have proved that $f'_{\alpha,\beta}(x) = 0$ has exactly one solution if $1 < \alpha < 2$ and $\beta > 1$. In the same way it can be proved that $f'_{\alpha,\beta}(x) = 0$ has no solution if $\alpha \leq 1$ or $\beta \leq 1$ (the latter case follows trivially from (2.9.5)). It follows that $f_{\alpha,\beta}$ is unimodal (or decreasing) for all α and β with $0 < \alpha < 2$ and $\beta > 0$. □□□

As a similar application of theorem 2.8.1 one may consider quotients of Γ-variates. It is easily verified that the p.d.f. of $\underline{x}_\beta^{-1}$ is unimodal, which implies that $\underline{x}_\alpha / \underline{x}_\beta$ is inf div if $\alpha \leq 1$ or $\alpha = 2$. A special case of this is provided by the F-statistic:

$\underline{F}_{m,n} = \sum_1^m \underline{u}_j^2 / \sum_{m+1}^{m+n} \underline{u}_k^2$, where $\underline{u}_1, \ldots, \underline{u}_{m+n}$ are independent random variables having normal distributions with mean zero and variance σ^2. It is easily seen that $\underline{F}_{m,n}$ is distributed as $\underline{x}_{\frac{m}{2}} / \underline{x}_{\frac{n}{2}}$, and hence that $\underline{F}_{m,n}$ is inf div if m = 1, m = 2 or m = 4. We made no attempt to prove the inf div of $\underline{x}_\alpha/\underline{x}_\beta$ for all α with $0 < \alpha \leq 2$.

2.10 Numerical results

Unable to prove the infinite divisibility of

$$\sum_1^n p_k \left(\frac{\lambda_k}{\lambda_k+\tau}\right)^2 \tag{2.10.1}$$

for $n \geq 5$, we looked for counterexamples to the various sufficient conditions, by computing the zeros of (2.10.1) numerically.[1)]

An example contradicting the inequalities (cf. (2.7.3))

$$\lambda_j < \mu_j < \lambda_{j+1} \qquad (j = 1, 2, \ldots, n-1)$$

is obtained by taking

$$n = 3 \; ; \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3 \; ; p_1 : p_2 : p_3 = 2 : 0.9 : 8.1.$$

One finds (with accuracy as indicated)

$$\mu_1 = 1.800 \;, \mu_2 = 1.909.$$

We examined some 80 cases with $5 \leq n \leq 10$, and λ_k and p_k such as seemed to be most promising. The computing time needed per case made a systematic investigation unfeasible. We found no counterexamples to the inequalities

$$\sum_1^m \lambda_j \leq \sum_1^m \mu_j \qquad (m = 1, 2, \ldots, n-1),$$

and even the stronger inequalities (cf. (2.7.16))

$$\sum_1^m \lambda_j \leq \sum_1^m (\lambda_n - \ell_j) \qquad (m = 1, 2, \ldots, n-1)$$

were satisfied in all cases. A difficulty showing up here is the fact that

1) My thanks are due to drs. J.A. van Hulzen for advice and extensive programming.

an increasing order in the μ_j need not result in a decreasing order in the ℓ_j. This, however, would not seriously impair the possibility of giving a proof along the lines of section 2.7.

2.11 Completely monotone densities

If G is any d.f. on $(0,\infty)$ then by corollary 2.2.1 the L.T.

$$\int_0^\infty \frac{\lambda}{\lambda+\tau} \, dG(\lambda)$$

is inf div. Equivalently, all p.d.f.'s of the form

$$(2.11.1) \qquad \int_0^\infty \lambda \, e^{-\lambda x} \, dG(\lambda)$$

are inf div. Clearly (see FELLER [5], p. 416) (2.11.1) is completely monotone on $(0,\infty)$. On the other hand every c.m. p.d.f. $f(x)$ on $(0,\infty)$ can be represented as

$$(2.11.2) \qquad f(x) = \int_0^\infty e^{-\lambda x} d\nu(\lambda),$$

where ν is a measure on $[0,\infty)$. Using the fact that

$$\int_0^\infty f(x) \, dx = 1,$$

by Fubini's theorem we have

$$\int_0^\infty \left(\int_0^\infty e^{-\lambda x} \, d\nu(\lambda)\right) dx = \int_0^\infty \lambda^{-1} \, d\nu(\lambda) = 1.$$

It follows that $f(x)$ in (2.11.2) is of the form (2.11.1) with

$$G(\lambda) = \int_0^\lambda u^{-1} \, d\nu(u),$$

i.e. $G(\lambda)$ is a d.f. on $(0,\infty)$. As mixtures (with positive weights) of c.m. functions are again c.m. we have proved

THEOREM 2.11.1

All (mixtures of) completely monotone densities are inf div.

REMARK: We may restrict ourselves to distributions on $(0,\infty)$ as the c.m. implies that the support of the distribution is of the form (a,∞) with $a > -\infty$, and a shift does not affect the inf div.

COROLLARY 2.11.1

If $\underline{x}$ and $\underline{y}$ are non-negative and independent, and if $\underline{x}$ has a c.m. p.d.f., then $\underline{x}\,\underline{y}$ is inf div.

PROOF: The p.d.f. of $\underline{x}\,\underline{y}$ is a mixture of c.m. p.d.f.'s. □ □ □

The c.m. condition is useful, because it is usually easier to verify this condition directly by verifying that a p.d.f. has alternating derivatives, then to represent it explicitly as a mixture of the form (2.11.1). Examples of p.d.f.'s which are inf div by this criterion are provided by the following functions (p.d.f's up to a multiplicative constant):

$$(1+x)^{-p} \qquad (p > 1 \; ; \; x > 0)$$

(hence Pareto's distributions are inf div),

$$x^{\alpha-1}\exp(-x^{\beta}) \qquad (0 < \alpha \leq 1 \; ; \; 0 < \beta \leq 1; \; x > 0),$$

$$\exp\{-(x-e^{-x})\} \qquad (x > 0)$$

(compare the criteria for c.m. given in [5], p. 417).

In view of corollary 1.3.6 theorem 2.11.1 can be reformulated as follows:

THEOREM 2.11.2

If $f(x)$ is c.m. on $(0,\infty)$, then

$$-\frac{d}{d\tau}\log\int_0^\infty e^{-\tau x} f(x)dx$$

is c.m. on $(0,\infty)$.

In chapter 4 we shall give discrete analogues of theorem 2.11.1 and 2.11.2. In that chapter we shall also prove a theorem somewhat stronger than theorem 2.11.1.

Analogous to the equivalence of c.m. p.d.f.'s and mixtures of exponential distributions, our conjecture that all L.T.'s of the form

$$\int_0^\infty \left(\frac{\lambda}{\lambda+\tau}\right)^2 dG(\lambda)$$

are inf div, is equivalent to the conjecture that all p.d.f.'s of the form $xf(x)$, with $f(x)$ c.m., are inf div. From the counterexamples indicated in section 2.5 it follows that densities of the form $x^{\alpha-1}f(x)$, with $f(x)$ c.m., are in general not inf div if $\alpha > 2$.

2.12 A representation theorem

In this section we prove a representation theorem for the L.T.'s of mixtures of exponential distributions.

From (1.3.4), (1.3.7) and (2.2.12) it follows that, if $\check{F}(\tau)$ is of the form

$$\check{F}(\tau) = \sum_{k=1}^{n} p_k \frac{\lambda_k}{\lambda_k+\tau} , \tag{2.12.1}$$

then $\check{F}(\tau)$ can be written as

$$\check{F}(\tau) = \exp \{-\int_0^\infty \frac{1-e^{-\tau x}}{x} \sum_{k=1}^{n} (e^{-\lambda_k x} - e^{-\mu_k x})\, dx\}. \tag{2.12.2}$$

Here $-\mu_1, \ldots, -\mu_{n-1}$ are the zeros of (2.12.1) and (for notational convenience) $\mu_n = \infty$. We may rewrite (2.12.2) as follows:

$$\check{F}(\tau) = \exp \{- \int_0^\infty (1-e^{-\tau x})(\sum_{k=1}^{n} \int_{\lambda_k}^{\mu_k} e^{-\lambda x} d\lambda)\, dx\} =$$

$$= \exp \{- \int_0^\infty (1-e^{-\tau x}) \int_0^\infty e^{-\lambda x} dm(\lambda) dx\} , \tag{2.12.2'}$$

where $m(\lambda)$ is defined by $m(0) = 0$ and

$$(2.12.3) \qquad m'(\lambda) = \begin{cases} 1 & (\lambda_k < \lambda < \mu_k) \quad (k = 1, 2, \ldots, n) \\ 0 & \text{otherwise} \end{cases}$$

Changing the order of integration we obtain

$$\check{F}(\tau) = \exp\{-\int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda)\}.$$

We prove the following theorem:

<u>THEOREM 2.12.1</u>

A L.T. $\check{F}(\tau)$ is of the form

$$\check{F}(\tau) = \int_{(0,\infty]} \frac{\lambda}{\lambda+\tau}\, dG(\lambda),$$

with $G(x)$ a d.f. on $(0,\infty]$, if and only if $\check{F}(\tau)$ can be represented as

$$(2.12.4) \qquad \check{F}(\tau) = \exp\{-\int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda)\},$$

where m is a measure bounded by Lebesgue measure (i.e. if ℓ denotes Lebesgue measure then $m(A) \leq \ell(A)$ for all ℓ-measurable sets A). Both G and m are uniquely determined by $\check{F}$.

First we prove three lemmas.

<u>LEMMA 2.12.1</u>

If $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_{n-1}$ are given, satisfying

$$(2.12.5) \qquad 0 < \lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_{n-1} < \lambda_n,$$

then there exist unique $p_1 > 0, \ldots, p_n > 0$ such that

$$\sum_1^n p_k \frac{\lambda_k}{\lambda_k+\tau}$$

has zeros $-\mu_1, \ldots, -\mu_{n-1}$, and $\sum_1^n p_k = 1$.

PROOF: Expanding

$$\prod_1^n \frac{\lambda_k}{\lambda_k+\tau} \prod_1^{n-1} \frac{\mu_j+\tau}{\mu_j}$$

in partial fractions we obtain

$$\prod_1^n \frac{\lambda_k}{\lambda_k+\tau} \prod_1^{n-1} \frac{\mu_j+\tau}{\mu_j} = \sum_{k=1}^{n} p_k \frac{\lambda_k}{\lambda_k+\tau} , \tag{2.12.6}$$

where

$$p_k = \lambda_k^{-1} \prod_{\ell\neq k} \frac{\lambda_\ell}{\lambda_\ell-\lambda_k} \prod_{j=1}^{n-1} \frac{\mu_j-\lambda_k}{\mu_j} . \tag{2.12.7}$$

The inequalities (2.12.5) imply that $p_k > 0$, and putting $\tau = 0$ in (2.12.6) yields $\sum_1^n p_k = 1$. The uniqueness of the p_k is a consequence of the uniqueness of both the factorization of polynomials and the partial fraction expansion.

LEMMA 2.12.2

If $\check{F}$ satisfies (2.12.4) then

$$\int_0^1 \lambda^{-1} dm(\lambda) \leq -2 \log \check{F}(1) < \infty.$$

PROOF: $\int_0^1 \lambda^{-1} dm(\lambda) \leq 2\int_0^1 \frac{1}{\lambda(\lambda+1)} dm(\lambda) \leq 2\int_0^\infty \frac{1}{\lambda(\lambda+1)} dm(\lambda) = -2 \log \check{F}(1)$,

which is finite, as $\check{F}(\tau) > 0$ for $\tau < \infty$. □ □ □

The third lemma is put in to enable us to use Helly's second theorem as given in LUKACS [22] (compare [35], p. 85).

LEMMA 2.12.3

If $m_n(\lambda)$ is a sequence of bounded non-decreasing functions on $[0,\infty)$, converging weakly to a bounded function $m(\lambda)$, and such that

$$m_n(\Lambda) \to m_n(\infty)$$

as $\Lambda \to \infty$, uniformly in n, then

$$\lim_{n\to\infty} m_n(\infty) = m(\infty).$$

PROOF: $|m(\infty) - m_n(\infty)| \leq |m(\infty) - m(\Lambda)| + |m(\Lambda) - m_n(\Lambda)| + |m_n(\Lambda) - m_n(\infty)| =$

$$= T_1 + T_2 + T_3,$$

where by definition $T_1 < \epsilon$ for $\Lambda > \Lambda_1$ and $T_3 < \epsilon$ for $\Lambda > \Lambda_2$ and for all n. If we now take $\Lambda_0 \geq \max(\Lambda_1,\Lambda_2)$ and such that Λ_0 is a contuinity point of m, then $T_2 < \epsilon$ if $n > n(\Lambda_0)$. □ □ □

We are now in a position to prove theorem 2.12.1.

PROOF: First, let $\check{F}(\tau)$ be given by (cf. (2.2.2))

$$\check{F}(\tau) = \int_{(0,\infty]} \frac{\lambda}{\lambda+\tau}\, dG(\lambda).$$

It is easily verified that $\check{F}$ can be obtained as

$$\check{F}(\tau) = \lim_{n\to\infty} \check{F}_n(\tau) = \lim_{n\to\infty} \sum_{k=1}^{n} p_{k,n} \frac{\lambda_{k,n}}{\lambda_{k,n}+\tau} =$$

$$= \lim_{n\to\infty} \exp\{-\int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm_n(\lambda)\},$$

with m_n defined in the same way as m (cf. 2.12.3). We split up the integral in two parts:

$$\int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm_n(\lambda) = \int_0^1 \frac{\tau}{\lambda+\tau}\, dm_{n,1}(\lambda) + \int_1^\infty \frac{\tau\lambda}{\lambda+\tau}\, dm_{n,2}(\lambda),$$

where

$$m_{n,1}(\lambda) = \int_0^\lambda u^{-1}\, dm_n(u) \qquad (0 \leq \lambda < 1)$$

and

$$m_{n,2}(\lambda) = \int_1^\lambda u^{-2}\, dm_n(u) \qquad (1 \leq \lambda < \infty).$$

By lemma 2.12.2 we have $m_{n,1}(1) \leq - 2 \log \check{F}_n(1)$, and hence, as $\check{F}_n(1) \to \check{F}(1)$, the measures $m_{n,1}(\lambda)$ are uniformly bounded. The sequence

$m_{n,1}$ therefore contains a subsequence $m_{n_k,1}$ converging weakly to a bounded measure $\tilde{m}_1$. By Helly's second theorem we then have

$$\lim_{k\to\infty} \int_0^1 \frac{\tau}{\lambda+\tau}\, dm_{n_k,1}(\lambda) = \int_0^1 \frac{\tau}{\lambda+\tau}\, d\tilde{m}_1(\lambda). \tag{2.12.8}$$

As $m_n(\lambda)$ is bounded by Lebesgue measure, the sequence $m_{n_k,2}$ is uniformly bounded and contains a subsequence $m_{n'_k,2}$, converging weakly to a bounded measure $\tilde{m}_2$, and satisfying the conditions of lemma 2.12.3. By Helly's second theorem it follows that

$$\lim_{k\to\infty} \int_1^\infty \frac{\lambda\tau}{\lambda+\tau}\, dm_{n'_k,2}(\lambda) = \int_1^\infty \frac{\tau}{\lambda+\tau}\, \tilde{m}_2(\lambda). \tag{2.12.9}$$

If we now take

$$m(\lambda) = \begin{cases} \int_0^\lambda u \; d\, \tilde{m}_1(u) & (0 \le \lambda < 1) \\ \int_0^1 u \; d\, \tilde{m}_1(u) + \int_1^\lambda u^2 d\, \tilde{m}_2(u) & (1 \le \lambda < \infty), \end{cases} \tag{2.12.10}$$

then the subsequence $m_{n'_k}$ of m_n converges weakly to $m(\lambda)$, and we have by (2.12.8) and (2.12.9)

$$\lim_{k\to\infty} \int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm_{n'_k}(\lambda) = \int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda).$$

As

$$\int_0^\infty \frac{1}{\lambda(\lambda+\tau)}\, dm(\lambda) = -\frac{1}{\tau} \log \check{F}(\tau),$$

it follows from the uniqueness theorem for Stieltjes transforms (WIDDER [37], p. 336) that every convergent subsequence of m_n converges weakly to m. Hence m_n converges weakly to m, and we have

$$\check{F}(\tau) = \exp\{-\int_0^\infty \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda)\}, \tag{2.12.4}$$

with m bounded by Lebesgue measure, **and** uniquely determined by $\check{F}$.

On the other hand, suppose that the Laplace transform $\check{F}(\tau)$ is given by (2.12.4) and that m is bounded by Lebesgue measure. By lemma 2.12.2 m satisfies

$$(2.12.11) \qquad \int_0^1 \lambda^{-1}\, dm(\lambda) < \infty.$$

We assume that m also satisfies

$$(2.12.12) \qquad 0 < m(b) - m(a) < b - a$$

for all a and b with a < b. This is not an essential restriction, as every $\check{F}$ of the form (2.12.4) can be obtained as the limit of a sequence of functions of this form and with m satisfying (2.12.12).

We can now approximate m as follows. Define $\lambda_{k,n}$ by

$$\lambda_{k,n} = kn/N_n \qquad\qquad (k = 0, 1, \dots, N_n),$$

where $n/N_n \to 0$ if $n \to \infty$, and $m_n(\lambda)$ by $m_n(0) = 0$ and (cf. (2.12.3))

$$m_n'(\lambda) = \begin{cases} 1 & (\lambda_{k,n} < \lambda < \mu_{k,n} \;;\; k = 1, 2, \dots, N_n - 1) \\ 0 & (\mu_{k,n} < \lambda < \lambda_{k+1,n} \;;\; k = 0, 1, \dots, N_n - 1) \\ 1 & (\lambda > n)\,. \end{cases}$$

Here $\mu_{0,n} = 0$, and $\mu_{k,n}$ is defined by

$$\mu_{k,n} = \lambda_{k,n} + m\,(\lambda_{k+1,n}) - m(\lambda_{k,n}) \quad (k = 1, 2, \dots, N_n - 1),$$

where by (2.12.12)

$$\lambda_{k,n} < \mu_{k,n} < \lambda_{k+1,n} \quad \text{for } k = 1, 2, \dots, N_n - 1.$$

The function $m_n(\lambda)$ is non-decreasing and satisfies

$$m_n(\lambda_{k,n}) = m(\lambda_{k,n}) - m(\lambda_{1,n}) \qquad (k = 1, 2, \ldots, N_n),$$

and

$$|m_n(\lambda) - m(\lambda)| < n/N_n + m(\lambda_{1,n}) \qquad (0 \leq \lambda \leq n).$$

As furthermore n/N_n, $m(\lambda_{1,n})$, $\int_0^{\lambda_{1,n}} \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda)$, $\int_n^{\infty} \frac{\tau}{\lambda(\lambda+\tau)}\, dm(\lambda)$ and $\int_n^{\infty} \frac{\tau}{\lambda(\lambda+\tau)}\, d\lambda$ tend to zero as $n \to \infty$, using Helly's second theorem we have (cf.(1.12.2) and (2.12.2'))

$$\check{F}(\tau) = \lim_{n\to\infty} \exp\{-\int_0^{\infty} \frac{\tau}{\lambda(\lambda+\tau)}\, dm_n(\lambda)\} =$$

$$= \lim_{n\to\infty} \exp\{-\int_0^{\infty} \frac{1-e^{-\tau x}}{x} \sum_{k=1}^{N_n} (e^{-\lambda_{k,n}x} - e^{-\mu_{k,n}x})dx\} =$$

$$= \lim_{n\to\infty} \sum_{k=1}^{N_n} p_{k,n} \frac{\lambda_{k,n}}{\lambda_{k,n}+\tau} = \lim_{n\to\infty} \int_0^{\infty} \frac{\lambda}{\lambda+\tau}\, dG_n(\lambda).$$

Here $\mu_{N_n,n} = \infty$, and the $p_{k,n}$ are uniquely determined by lemma 2.12.1. As every sequence of d.f.'s contains a convergent subsequence, there is a subsequence G_{n_k} of G_n converging weakly to a function G. If

$$\lim_{\lambda\to\infty} G(\lambda) = 1 - p_0 < 1,$$

then $\check{F}(\infty) = p_0$, and F has mass p_0 in zero. Defining $G(\infty) = 1$, we can apply Helly's theorem on $[0,\infty]$, because $\lambda(\lambda+\tau)^{-1}$ is continuous at $\lambda = \infty$. We obtain

$$\check{F}(\tau) = \int_{(0,\infty]} \frac{\lambda}{\lambda+\tau}\, dG(\lambda) = p_0 + \int_0^{\infty} \frac{\lambda}{\lambda+\tau}\, dG(\lambda).$$

The uniqueness of G follows in the same way as above, from the uniqueness of $p_0 = \check{F}(\infty)$ and the uniqueness theorem for Stieltjes transforms. □ □ □

Except for the case where G is a stepfunction, there seem to be few examples where the triple $(\check{F},G,m)$ as occurring in theorem 2.12.1 can be obtained explicitly. We give the following examples:

a. $\check{F}(\tau) = (1+\tau)^{-1/2}$; $G'(\lambda) = \lambda^{-1/2}(\lambda-1)^{-1/2}$ $(\lambda > 1)$; $m'(\lambda) = \begin{cases} 0 & (\lambda < 1) \\ 1/2 & (\lambda > 1) \end{cases}$.

b. $\check{F}(\tau) = 1/2 + 1/2\ (1+\tau)^{-1}$; $G(\lambda) = \begin{cases} 0 & (\lambda < 1) \\ 1/2 & (1 \leq \lambda < \infty) \\ 1 & (\lambda = \infty) \end{cases}$; $m(\lambda) = \begin{cases} 0 & (\lambda < 1) \\ \lambda-1 & (1 \leq \lambda < 2) \\ 1 & (\lambda \geq 2) \end{cases}$.

REMARK: One could avoid distributions G with mass at infinity by putting $\lambda = u^{-1}$. However, this would spoil the direct correspondence with Stieltjes transforms; also, exponential p.d.f.'s are usually written in the form $\lambda \exp(-\lambda x)$ rather than in the form $a^{-1} \exp(-xa^{-1})$.

As a direct consequence of theorem 2.12.1 we have

COROLLARY 2.12.1:

If $0 < \alpha < 1$, then

$$\left\{\int_0^\infty \frac{\lambda}{\lambda+\tau}\, dG(\lambda)\right\}^\alpha = \int_0^\infty \frac{\lambda}{\lambda+\tau}\, dG_\alpha(\lambda),$$

where $G_\alpha(\lambda)$ is a uniquely determined d.f. on $(0,\infty]$.

As a second corollary we prove

COROLLARY 2.12.2:

If n is a positive integer, then

$$\left\{\int_0^\infty \left(\frac{\lambda}{\lambda+\tau}\right)^{1/n} dG(\lambda)\right\}^n = \int_0^\infty \frac{\lambda}{\lambda+\tau}\, d\tilde{G}_n(\lambda),$$

where $\tilde{G}_n(\lambda)$ is a uniquely determined d.f. on $(0,\infty]$.

PROOF: In view of the closure theorem we only have to prove the theorem for the case that G is a stepfunction. We consider

$$\check{F}(\tau) = \left\{\sum_{k=1}^{N} p_k \left(\frac{\lambda_k}{\lambda_k+\tau}\right)^{1/n}\right\}^n,$$

which can be rewritten in the form

$$\check{F}(\tau) = \sum C_{k_1, \ldots, k_n} \check{F}_{k_1} \ldots \check{F}_{k_n} ,$$

where $C_{k_1, \ldots, k_n}$ are positive constants and

$$\check{F}_{k_j}(\tau) = \left(\frac{\lambda_{k_j}}{\lambda_{k_j}+\tau} \right)^{1/n} \qquad (j = 1, 2, \ldots, n).$$

It follows from theorem 2.12.1 that $\check{F}_{k_1} \ldots \check{F}_{k_n}$ is the L.T. of a mixture of exponential distributions. Hence $\check{F}$ also is the L.T. of a mixture of exponential distributions. □ □ □

Theorem 2.12.1 provides two curious analogues to theorem 1.3.10. We write (using a different notation to stress the analogy)

(a) $$\check{F}_1(\tau) = \int_0^\infty e^{-\tau x}\, dF_1(x)$$

(b) $$F_2^*(\tau) = \int_0^\infty \frac{x}{x+\tau}\, dF_2(x)$$

(c) $$F_3^{**}(\tau) = \int_0^\infty \left(\frac{x}{x+\tau}\right)^2 dF_3(x),$$

where F_1 is inf div, F_2 arbitrary on $(0,\infty]$, and F_3 on $(0,\infty]$ such that $1 - F_3(x^{-1} - 0)$ is unimodal. Then we have

(a') $$\check{F}_1(\tau) = \exp\{-\int_0^\infty (1 - e^{-\tau x})x^{-1} dK_1(x)\}$$

(b') $$F_2^*(\tau) = \exp\{-\int_0^\infty (1 - \frac{x}{x+\tau})x^{-1} dK_2(x)\}$$

(c') $$F_3^{**}(\tau) = \exp\{-\int_0^\infty [1 - (\frac{x}{x+\tau})^2]\, x^{-2} dK_3(x)\},$$

where K_1 satisfies the conditions for K in theorem 1.3.10, K_2 satisfies the conditions for m in theorem 2.12.1, and $K_3'/2$ satisfies the conditions for m in theorem 2.12.1. Relation (b') follows immediately from theorem

2.12.1, whereas (c') is obtained by writing (c) in the form (b) with F_2 satisfying the conditions for G in corollary 2.2.1. Then (b) is put in the form (b') where $K_2/2$ now satisfies the conditions for m in theorem 2.12.1 (compare (2.12.2)). Finally (c') is obtained from (b') by integration by parts.

As (c) can be written in the form (b') with $K_2/2$ bounded by Lebesgue measure, it follows that F_3^{**} is the product of two L.T.'s of type (b). Counter examples show that general mixtures of Γ(2)-distributions cannot be obtained as convolutions of exponential mixtures.

Considering mixtures of Γ(2)-distributions, with L.T.'s of the form

$$F^{**}(\tau) = \int_0^\infty (\frac{\lambda}{\lambda+\tau})^2 \, dF(\lambda), \tag{2.12.13}$$

where

$$\int_0^\infty \lambda \, dF(\lambda) = \alpha < \infty, \tag{2.12.14}$$

using corollary 1.3.7 and theorem 2.12.1, we obtain

$$F^{**} = F^* \, F_1^{**}. \tag{2.12.15}$$

Here

$$F^*(\tau) = \alpha^{-1} \int_0^\infty \frac{\lambda}{\lambda+\tau} \; \lambda \, dF \, (\lambda),$$

and

$$F_1^{**} = \alpha \int_0^\infty (\frac{\lambda}{\lambda+\tau})^2 \, \lambda^{-2} \, dm(\lambda),$$

with $m(\lambda)$ corresponding to $F^*(\tau)$ as in (2.12.4). That is, F^{**} not only has F^* as an inf div factor, but the remaining factor is again of the form (2.12.13). As condition (2.12.14) will generally not hold for F_1^{**}, the procedure cannot be repeated. This e.g. is the case if F in (2.12.13) is a finite stepfunction. As we have seen in section 2.7, this does not prevent F^{**} from being inf div. The decomposition (2.12.15) seems to support the conjecture that F^{**} is inf div.

Chapter 3

A MORE GENERAL CLASS OF INF DIV MIXTURES

3.1. Introduction

In chapter 2 we studied mixtures of c.f.'s of the form

(3.1.1) $$\left(\frac{\lambda}{\lambda-it}\right)^{\alpha} \qquad (\lambda > 0 \ ; \ \alpha > 0).$$

In this chapter we consider mixtures of c.f.'s of the more general form

(3.1.2) $$\left(\frac{\lambda}{\lambda-h(t)}\right)^{\alpha} \qquad (\lambda > 0 \ ; \ \alpha > 0),$$

for a suitable class of functions h. The distributions corresponding to c.f.'s of the form (3.1.2), unlike those corresponding to (3.1.1) and its mixtures, are not restricted to $[0,\infty)$. We shall mainly be concerned with mixtures of (3.1.2) in the case $\alpha = 1$.

The inf div of these mixtures is proved by obtaining the canonical representation (1.3.1) explicitly, and showing that Θ is non-decreasing. Some of our results can be proved more easily by use of a theorem of Feller (our theorem 3.5.1). However, we shall give the proofs as presented in [33], as by doing so we obtain some additional results.

For the time being we characterize the class H of functions h, admissible in (3.1.2) by

(3.1.3) $$H = \{h \mid \frac{\lambda}{\lambda-h} \text{ is a c.f. for every } \lambda > 0\}.$$

Another characterization of H will be given in section 3.4.

3.2 The case $h = \gamma - 1$

It follows from theorem 1.6.1 that $\gamma - 1 \in H$ for every c.f. γ. Putting

(3.2.1) $$\phi_{\lambda} = \frac{\lambda}{\lambda+1-\gamma}$$

and using theorem 1.3.8, for the function Θ_λ (cf.(1.3.1)) we have

$$\Theta_\lambda = \lim_{n\to\infty} n \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, dF_n(y), \tag{3.2.2}$$

where (see the proof of theorem 1.6.1)

$$\begin{aligned} F_n(y) &= \left(\frac{\lambda}{\lambda+1}\right)^{1/n} \iota(y) + \sum_{k=1}^{\infty} c_k^{(n)}\, G^{*k}(y) = \\ &= \left(\frac{\lambda}{\lambda+1}\right)^{1/n} \iota(y) + \tilde{F}_n(y). \end{aligned} \tag{3.2.3}$$

Here $\iota(y)$ denotes the unit-step function, and

$$\tilde{F}_n(y) = \sum_{k=1}^{\infty} c_k^{(n)}\, G^{*k}(y), \tag{3.2.4}$$

with

$$c_k^{(n)} = \frac{\lambda^{1/n}}{k!} (1+\lambda)^{-k} \prod_{j=0}^{k-1} \left(j + \frac{1}{n}\right).$$

From (3.2.2) and (3.2.3) we obtain

$$\Theta_\lambda(x) = \lim_{n\to\infty} n \int_{-\infty}^{x} \frac{y^2}{1+y^2}\, d\tilde{F}_n(y).$$

As for $k \geq 1$

$$\lim_{n\to\infty} n\, c_k^{(n)} = k^{-1} (1+\lambda)^{-k},$$

by the uniform convergence (in n and y, for fixed λ) of the sum in (3.2.4), for $M_\lambda(y)$, defined by

$$M_\lambda(y) = \lim_{n\to\infty} n\, \tilde{F}_n(y),$$

we have

$$M_\lambda(y) = \sum_{k=1}^{\infty} k^{-1} (1+\lambda)^{-k}\, G^{*k}(y), \tag{3.2.5}$$

with $M_\lambda(-\infty) = 0$ and

$$(3.2.6) \qquad M_\lambda(\infty) = \sum_1^\infty k^{-1}(1+\lambda)^{-k} = \log\frac{\lambda+1}{\lambda} = \lim_{n\to\infty} n\,\tilde{F}_n(\infty).$$

By Helly's second theorem, for all continuity points x of M_λ, $\Theta_\lambda(x)$ is given by

$$(3.2.7) \qquad \Theta_\lambda(x) = \int_{-\infty}^{x}\frac{y^2}{1+y^2}\,d\,M_\lambda(y) = \sum_{k=1}^{\infty} k^{-1}(1+\lambda)^{-k}\int_{-\infty}^{x}\frac{y^2}{1+y^2}\,d\,G^{*k}(y).$$

If $\lambda < \mu$, then by (3.2.7) for any two continuity points x_1 and x_2 of M_λ and M_μ, with $x_1 < x_2$, we have $\Theta_\lambda(x_1) - \Theta_\mu(x_1) \leq \Theta_\lambda(x_2) - \Theta_\mu(x_2)$, and hence

LEMMA 3.2.1

If $\lambda < \mu$, then $\Theta_\lambda - \Theta_\mu$ is non-decreasing, and therefore ϕ_λ/ϕ_μ is inf div.

REMARK: Taking Fourier-Stieltjes transforms in (3.2.5), by absolute convergence it follows that

$$\int_{-\infty}^{\infty} e^{ity}\,dM_\lambda(y) = -\log\left(1 - \frac{\gamma}{\lambda+1}\right) = \log\frac{\lambda+1}{\lambda+1-\gamma} = \log\frac{\lambda+1}{\lambda} + \log\phi_\lambda.$$

Therefore (cf.(3.2.6)), ϕ_λ has the following canonical representation:

$$(3.2.8) \qquad \log\phi_\lambda(t) = \int_{-\infty}^{\infty}(e^{ity} - 1)\,d\,M_\lambda(y),$$

where by (3.2.5) and (3.2.6) $M_\lambda(y)$ is non-decreasing and bounded. From this it follows that ϕ_λ is of the form $\phi_\lambda = \exp\{c(\gamma-1)\}$ (cf. (1.6.3)). Formula (3.2.8) can also be obtained directly from (3.2.1). The infinite divisibility of ϕ_λ/ϕ_μ for $\lambda < \mu$ can now also be obtained from (3.2.8) and the fact that $M_\lambda - M_\mu$ is non-decreasing.

3.3 Mixtures of $\lambda/(\lambda+1-\gamma)$

We now prove

THEOREM 3.3.1

If γ is a c.f. and if A is a d.f. on $(0,\infty]$, then

$$(3.3.1) \qquad \phi(t) = \int_{(0,\infty]} \frac{\lambda}{\lambda+1-\gamma(t)}\, dA(\lambda)$$

is an inf div c.f.

PROOF: Restricting ourselves to finite mixtures, we have (cf. (2.2.8) and (2.2.11))

$$\phi = \sum_{k=1}^{n} p_k \frac{\lambda_k}{\lambda_k+1-\gamma} = \prod_{1}^{n} \frac{\lambda_k}{\lambda_k+1-\gamma} \Big/ \prod_{1}^{n-1} \frac{\mu_j}{\mu_j+1-\gamma} ,$$

where $\lambda_1 < \mu_1 < \lambda_2 < \ldots < \lambda_{n-1} < \mu_{n-1} < \lambda_n$. By lemma 3.2.1 (cf. (3.2.1)) it now follows that ϕ is infinitely divisible. The inf div of general mixtures again follows from the closure theorem. □ □ □

REMARK: Though $\lambda(\lambda-it)^{-1}$ is not of the form (3.2.1) it is possible to derive theorem 2.2.1 (with positive p_k) from theorem 3.3.1. Writing

$$(3.3.2) \qquad \frac{\lambda}{\lambda-it} = \frac{\mu}{\mu-it}\,\frac{\alpha}{\alpha+1-\mu/(\mu-it)} ,$$

where $\mu > \lambda$ and

$$(3.3.3) \qquad \alpha = \frac{\lambda}{\mu-\lambda} ,$$

we have for $\mu > \max \lambda_k$

$$(3.3.4) \qquad \sum_{k=1}^{n} p_k \frac{\lambda_k}{\lambda_k-it} = \gamma(t) \sum_{k=1}^{n} p_k \frac{\alpha_k}{\alpha_k+1-\gamma(t)} .$$

Here $\gamma(t) = \mu/(\mu-it)$ and α_k is defined as α in (3.3.3). As $\gamma(t)$ is an inf div c.f., by theorem 3.3.1 the right-hand side of (3.3.4) is the product of two inf div c.f.'s, and hence an inf div c.f.

3.4 Mixtures of $\lambda/(\lambda-h)$

First we derive another characterization of the class H defined by (3.1.3). We shall need

<u>LEMMA 3.4.1</u>

If ϕ_λ, defined by

(3.4.1) $$\phi_\lambda = \frac{\lambda}{\lambda-h},$$

is a c.f. for all $\lambda > 0$, then ϕ_λ is inf div for all $\lambda > 0$.

<u>PROOF</u>: Repeated use of (3.3.2), with $\mu = 2\lambda$, i.e. $\alpha = 1$, and t replaced by -ih, yields

(3.4.2) $$\phi_\lambda = \phi_{2\lambda} \frac{1}{2-\phi_{2\lambda}} = \dots = \phi_{2^N\lambda} \prod_1^N \frac{1}{2-\phi_{2^k\lambda}},$$

where $1/(2-\phi_{2^k\lambda})$ is inf div as it is of the form (3.2.1). As $\phi_{2^N\lambda} \to 1$ if $N \to \infty$, for all t, we have

$$\phi_\lambda = \lim_{N\to\infty} \phi_\lambda/\phi_{2^N\lambda} = \lim_{N\to\infty} \prod_1^N \frac{1}{2-\phi_{2^k\lambda}} .$$

Hence, ϕ_λ is the limit of a sequence of inf div c.f.'s and therefore inf div by theorem 1.3.5. □ □ □

It also follows from (3.3.2), that we have (for h not necessarily in H)

<u>COROLLARY 3.4.1</u>

If $\lambda/(\lambda-h)$ is a c.f. for $\lambda = \lambda_0 > 0$, then it is a c.f. for all λ with $0 < \lambda \leq \lambda_0$. If it is inf div for $\lambda = \lambda_0$, then it is inf div for all λ with $0 < \lambda \leq \lambda_0$.

As an interesting special case we have

<u>COROLLARY 3.4.2</u>

If ϕ_1 is an arbitrary c.f., then

(3.4.3) $$\phi_\lambda = \frac{\lambda}{\lambda-1+\phi_1^{-1}}$$

is a c.f. for all λ with $0 < \lambda \leq 1$. If ϕ_1 is inf div, then ϕ_λ is inf div for all λ with $0 < \lambda \leq 1$.

Corollary 3.4.2 can also be obtained by rewriting ϕ_λ in (3.4.3) as

(3.4.4) $$\phi_\lambda = \phi_1 \frac{\lambda}{\lambda+(1-\lambda)(1-\phi_1)} = \phi_1 \frac{\mu}{\mu+1-\phi_1} \quad ,$$

where $\mu = \lambda/(1-\lambda)$.

The set H is characterized in the following lemma.

LEMMA 3.4.2

$\lambda/(\lambda-h)$ is a c.f. for all $\lambda > 0$ if and only if h is continuous, $h(0) = 0$ and $\exp h(t)$ is an inf div c.f.

PROOF: First let $\lambda/(\lambda-h)$ be a c.f. for all $\lambda > 0$. Then by lemma 3.4.1 $\lambda/(\lambda-h)$ is inf div, and therefore $\neq 0$. It follows that h is continuous, and $h(0) = 0$. Now for all $n > 0$, then function ψ_n, defined by

$$\psi_n = \left(\frac{n}{n-h}\right)^n ,$$

is an inf div c.f. By the continuity theorem

$$\exp h(t) = \lim_{n\to\infty} \psi_n(t)$$

is a c.f., which is inf div by the closure property.

Conversely, if h is continuous, $h(0) = 0$ and exp h is an inf div c.f., then h is uniquely determined by exp h (cf. definition 1.3.3), and

$$\frac{\lambda}{\lambda-h} = \int_0^\infty \exp\{-\frac{\lambda-h}{\lambda} s\}ds = \int_0^\infty e^{hs/\lambda}e^{-s}\, ds.$$

That is, $\lambda/(\lambda-h)$ is a mixture of c.f.'s (of the form $(\exp h)^p$, with $p > 0$), and therefore a c.f. for all $\lambda > 0$. □ □ □

From lemma 3.4.2 and the definition of H (cf. (3.1.3)) we obtain (compare definition 1.3.3)

(3.4.5) $\qquad H = \{h \mid h = \log \phi$, where ϕ is an inf div c.f.$\}$.

Corollary 2.2.1 can now be generalized as follows.

THEOREM 3.4.1

If (for h not necessarily in H) ϕ_λ, defined by

$$\phi_\lambda = \frac{\lambda}{\lambda - h},$$

is an inf div c.f. for all λ with $0 < \lambda \leq \lambda_0 \leq \infty$, then

$$\int_{(0,\lambda_0]} \frac{\lambda}{\lambda - h}\, dF(\lambda) \tag{3.4.6}$$

is in inf div c.f. for every d.f. F on $(0,\lambda_0]$.

PROOF: It suffices to prove the theorem for finite mixtures with $\lambda_0 < \infty$ and $\lambda < \lambda_0$. As in (3.3.4) we write

$$\sum_{k=1}^{n} p_k\, \phi_{\lambda_k} = \phi_\mu \sum_{k=1}^{n} p_k \frac{\alpha_k}{\alpha_k + 1 - \phi_\mu}, \tag{3.4.7}$$

with $\max \lambda_k < \mu \leq \lambda_0$. In (3.4.7) ϕ_μ is inf div by hypothesis, and $\sum p_k\, \alpha_k \,/\, (\alpha_k + 1 - \phi_\mu)$ by theorem 3.3.1. Hence $\sum p_k\, \phi_{\lambda_k}$ is inf div. □ □ □

3.5 A theorem of Feller

In [5] (p. 538), as an example, the following theorem is given, in a slightly different form.

THEOREM 3.5.1

If G is an inf div d.f. on $[0,\infty)$, and if $h \in H$, then

$$\check{G}(-h(t)) = \int_0^\infty \exp\{xh(t)\}\, dG(x) \tag{3.5.1}$$

is an inf div c.f.

PROOF: If $\check{F}$ is a L.T., then

$$\check{F}(-h) = \int_0^\infty e^{hx}\, dF(x)$$

is a mixture of c.f.'s and hence a c.f. Taking

$$\check{F} = (\check{G})^p,$$

for any $p > 0$, it follows that $\{\check{G}(-h)\}^p$ is a c.f. for every $p > 0$, and therefore that $\check{G}(-h)$ is inf div (cf. theorem 1.3.6). □ □ □

If in (3.5.1) we take $\check{G}(\tau) = \lambda(\lambda+1-e^{-\tau})^{-1}$ and $h = \log \gamma$ we obtain the c.f. (1.6.1). In the same way taking $\check{G}(\tau) = \exp\{\lambda(e^{-\tau}-1)\}$ yields (1.6.3). The representation $\phi = \check{F}(-\log \gamma)$ is not unique; in fact, for every inf div c.f. we have $\phi = \exp\{-(-\log \phi)\}$.

If we restrict ourselves to $h \in H$, and if for G we take a mixture of exponential d.f.'s, then, by theorem 2.2.1, theorem 3.4.1 can be obtained as an application of theorem 3.5.1. In fact, I conjectured theorem 3.5.1 as a generalization of theorem 3.4.1, and proved it in a way analogous to the proof of theorem 1.3.7 as given in [22], before I found it in [5]. It follows from lemma 3.4.2 and the definition of H that theorem 3.5.1 can be reversed in the following sense.

COROLLARY 3.5.1

If $\check{G}(-h(t))$ is an inf div c.f. for every d.f. G on $[0,\infty)$, then $h \in H$.

From theorem 3.5.1, using the results of chapter 2, we obtain

THEOREM 3.5.2

If $h \in H$, and G is a d.f. on $[0,\infty)$, then

$$\int_0^\infty (1-xh(t))^{-\alpha}\, dG(x)$$

is an inf div c.f. if one of the following conditions holds.

(i) $\alpha \leq 1$

(ii) $\alpha = 2$ and G is unimodal

(iii) $\alpha = 2$ and G has at most four points of increase.

3.6 Examples

As examples of inf div c.f.'s are comparatively scarce it may be useful to list some explicitly.

I Mixtures of the following c.f.'s are inf div:

a. $\dfrac{\lambda}{\lambda-it}$ with p.d.f. $\lambda e^{-\lambda x}$ $(x > 0)$

b. $\dfrac{\lambda^2}{\lambda^2+t^2}$ with p.d.f. $\dfrac{\lambda}{2}\, e^{-\lambda|x|}$

c. $\dfrac{\lambda}{\lambda+1-\exp it}$ with $p_n = \dfrac{\lambda}{(1+\lambda)^{n+1}}$ $(n = 0, 1, \ldots)$

d. $\dfrac{\lambda}{\lambda+\sin^2 t}$

e. $\dfrac{\lambda}{\lambda+\log(1-it)}$ (cf. definition 1.3.3)

f. $\dfrac{\lambda}{\lambda-it+\sqrt{(1-it)^2-1}}$,

where $\sqrt{(1-it)^2-1}$ is defined such that it is positive for $t = i\tau$ with $\tau > 0$.

Denoting the function in f. by ϕ_λ we shall prove that $\phi_\lambda = \lambda/(\lambda-h)$ with $h \in H$. For $\lambda = 1$ we have

$$\phi_1 = 1-it-\sqrt{(1-it)^2-1} ,$$

which is a c.f. (compare [5] p. 414). It follows that we have $\phi_1-1 \in H$, and therefore $h = \phi_1-1+2it = it - \sqrt{(1-it)^2-1} \in H$, and such that $\phi_\lambda = \lambda/(\lambda-h)$.

The density function corresponding to ϕ_λ for $0 < \lambda < 2$ is given by

(3.6.1) $$f_\lambda(x) = \lambda x^{-1}\, e^{-x} \sum_1^\infty n(1-\lambda)^{n-1}\, I_n(x) \qquad (x > 0),$$

where $I_n(x)$ denotes the modified Bessel function of the first kind of order n. For $\lambda < 1$ we have

(3.6.2) $$\phi_\lambda(t) = \phi_1 \frac{\lambda/(1-\lambda)}{\lambda/(1-\lambda)+1-\phi_1} ,$$

and for $\lambda > 1$

(3.6.3) $$\phi_\lambda(t) = \{(\lambda-1)\lambda^{-1} + \lambda^{-1}\, \phi_1\} \frac{1}{1-2(\lambda-1)it/\lambda^2} .$$

In both expressions the inf div of ϕ_λ follows from the special form of ϕ_1, the latter expression being a product of a two-component mixture and the c.f. of an exponential distribution. For $\lambda = 2$ we obtain from (3.6.3)

$$\phi_2(t) = (\tfrac{1}{2} + \tfrac{1}{2}\, \phi_1) \frac{2}{2-it} ,$$

with

$$f_2(x) = e^{-2x} \{1 + \int_0^x e^u\, u^{-1}\, I_1(u)\, du\} \qquad (x > 0).$$

II Examples of inf div mixtures are the following c.f.'s:

a. $$\int_0^1 \frac{1}{1-xh}\, dx = -h^{-1} \log(1-h) \qquad (h \in H)$$

b. $$\frac{6}{\pi^2} \sum_1^\infty \frac{1}{n^2+t^2} = \frac{6}{\pi^2} \sum_1^\infty \frac{1}{n^2} \frac{n^2}{n^2+t^2} = \frac{6}{\pi t} \left\{ \frac{1}{e^{2\pi t}-1} - \frac{1}{2\pi t} + \frac{1}{2} \right\}$$

(compare example I b. and [34] p. 113).
The corresponding p.d.f. equals

$$-\frac{3}{\pi^2}\log\left(1-e^{-|x|}\right),$$

as obtained by term-by-term inversion.

c. $$\int_1^\infty \left(\frac{1}{1-ity}\right)^2 \frac{1}{y^2}\,dy = 1 - \frac{it}{1-it} - 2it\log\frac{it}{it-1},$$

where the logarithm tends to zero for $|t| \to \infty$. The corresponding p.d.f. equals

$$x^{-2}\left(2 - 2e^{-x} - 2xe^{-x} - x^2 e^{-x}\right) \qquad (x > 0)$$

(this example is an application of theorem 2.8.1).

III Examples of type $\lambda/(\lambda-1+\phi^{-1})$ are:

a. $$\phi_\lambda = \frac{\lambda}{\lambda-1+\exp(-it)} \qquad (0 < \lambda \leq 1),$$

with

$$p_n = \lambda(1-\lambda)^{n-1} \qquad (n = 1, 2, \ldots).$$

For $\lambda > 1$ the function ϕ_λ is not a c.f. as then $|\phi_\lambda| > 1$.

b. If $\phi(t) = (1-it)^{-\alpha}$ with $(0 < \alpha \leq 1)$, then $1 - \phi^{-1} \in H$, as $(1+\tau)^\alpha$ has a completely monotone derevative (compare (1.3.6) and (3.4.5)).

Hence

$$\frac{\lambda}{\lambda-1+(1-it)^{\alpha}}$$

is an inf div c.f. for all $\lambda > 0$. For $0 < \lambda < 2$ the corresponding p.d.f. equals

$$\lambda x^{\alpha-1}\, e^{-x} \sum_{0}^{\infty} (1-\lambda)^n \frac{x^{\alpha n}}{\Gamma(\alpha n+\alpha)} \qquad (x > 0).$$

c. The example I f. can be rewritten as $\lambda/(\lambda-1+\phi_1^{-1})$.

3.7 Continued fractions and birth-death processes

In this section we briefly discuss a connection between a class of inf div distributions (especially mixtures of exponential distributions), continued fractions, Stieltjes transforms and birth-death processes. The last-mentioned connection is due to Vervaat, and is also contained implicitly in KARLIN and McGREGOR [11], where the relation between birth-death processes and the Stieltjes moment problem is discussed in detail.

I was led to consider continued fractions as follows. When γ_1 is a c.f., then for $\mu_1 > 0$ the function

$$\frac{\mu_1}{\mu_1+1-\gamma_1}$$

is an inf div c.f. (compare theorem 1.6.1). Therefore, for $\mu_2 > 0$, $0 < \alpha_1 < 1$ and any c.f. γ_2

$$\frac{\mu_1}{\mu_1+1-\alpha_1\gamma_1 - \dfrac{(1-\alpha_1)\mu_2}{\mu_2+1-\gamma_2}}$$

is an inf div c.f. Continuing in this way, for c.f.'s $\gamma_1, \ldots, \gamma_n$ and constants $\mu_1 > 0, \ldots, \mu_n > 0$ and $0 < \alpha_1 < 1, \ldots, 0 < \alpha_{n-1} < 1$, the n-term continued fraction

$$\frac{\mu_1}{\mu_1+1-\alpha_1\gamma_1-}\ \frac{\mu_2(1-\alpha_1)}{\mu_2+1-\alpha_2\gamma_2-}\ \cdots\ -\ \frac{\mu_n(1-\alpha_{n-1})}{\mu_n+1-\gamma_n}$$

is an inf div c.f. Putting $\alpha_j(\gamma_j-1) = h_j$, $1-\alpha_j = \lambda_j$ $(j = 1, 2, \ldots, n-1)$ and $\gamma_n-1 = h_n$ we obtain

(3.7.1) $$\frac{\mu_1}{\mu_1+\lambda_1-h_1-}\ \frac{\lambda_1\mu_2}{\mu_2+\lambda_2-h_2-}\ \cdots\ -\ \frac{\lambda_{n-1}\mu_n}{\mu_n-h_n}.$$

It is not difficult to show that (3.7.1) is an inf div c.f. if $\lambda_j > 0$, $\mu_j > 0$ and $h_j \in H$ (not necessarily of the form $\alpha_j(\gamma_j-1)$) for $j = 1, 2, \ldots, n$. This can be done by repeated reduction of (3.7.1) to a continued fraction having one term less. In this way it is seen that (3.7.1) is of the form $\mu_1/(\mu_1-h_1^*)$ with $h_1^* \in H$, i.e. (3.7.1) is an inf div c.f.

If $h_j = h$ $(j = 1, 2, \ldots, n)$, then by theorem 3.5.1 the inf div of (3.7.1) is equivalent to the inf div of the L.T.

(3.7.2) $$\frac{\mu_1}{\mu_1+\lambda_1+\tau-}\ \frac{\lambda_1\mu_2}{\mu_2+\lambda_2+\tau-}\ \cdots\ -\ \frac{\lambda_{n-1}\mu_n}{\mu_n+\tau}.$$

If for $n \to \infty$ (3.7.1) (or (3.7.2)) converges to a continuous function, then by the continuity theorem this function is an inf div. c.f. (L.T.). If $\lambda_n > 0$ and $\mu_n > 0$ $(n = 1, 2, \ldots)$ then for $\tau \geq 0$ the L.T.'s (3.7.2) form a bounded, non-increasing, and therefore convergent sequence. However, this sequence does not necessarily convergence to a L.T. (compare the interpretation given below).

From PERRON [26] we take the following theorem about continued fractions of a similar type.

<u>THEOREM 3.7.1</u>

If $\lambda_n > 0$ and $\mu_n > 0$ $(n = 1, 2, \ldots)$, then

(3.7.3) $$\check{F}(\tau) = \frac{\mu_1}{\mu_1+\lambda_1+\tau-}\ \frac{\lambda_1\mu_2}{\mu_2+\lambda_2+\tau-}\ \frac{\lambda_2\mu_3}{\mu_3+\lambda_3+\tau-}\ \cdots$$

is convergent for all values of τ with Re $\tau > 0$. If in addition $\check{F}(0+) = 1$, then $\check{F}$ is an inf div L.T.

PROOF: $\check{F}$ can be shown to be of the form

$$\check{F}(\tau) = \frac{a_1}{a_2+\tau-}\ \frac{a_2a_3}{a_3+a_4+\tau-}\ \frac{a_4a_5}{a_5+a_6+\tau-}\ \cdots ,$$

with $a_n > 0$ $(n = 1, 2, \ldots)$. Therefore (cf [26], p. 193) the convergents of $\check{F}$ are the even convergents of the continued fraction

$$\frac{a_1}{\tau+}\ \frac{a_2}{1+}\ \frac{a_3}{\tau+}\ \frac{a_4}{1+}\ \cdots = \frac{1}{b_1\tau+}\ \frac{1}{b_2+}\ \frac{1}{b_3\tau+}\ \frac{1}{b_4+}\ \cdots ,$$

with

$$b_1 = a_1^{-1};\ b_{2n+1} = a_2a_4\ldots a_{2n}/(a_1a_3\ldots a_{2n+1});\ b_{2n} = a_1a_3\ldots a_{2n-1}/(a_2a_4\ldots a_{2n}).$$

It now follows from [26], Satz 4.9 (I and III), that for all complex τ, for which Re $\tau > 0$, the continued fraction in (3.7.3) is convergent, and that we have

$$(3.7.4)\qquad \check{F}(\tau) = \int_0^\infty \frac{1}{u+\tau}\, d\psi(u),$$

where ψ is non-decreasing and $\psi(0-) = 0$. In (3.7.3) $\check{F}(\tau)$ is bounded and, similar to (3.7.2), convergent for $\tau \geq 0$. It follows that $\psi(0+) = 0$. If $\check{F}(0+)=1$, then $\check{F}$ is of the form

$$(3.7.5)\qquad \check{F}(\tau) = \int_0^\infty \frac{u}{u+\tau}\, dG(u),$$

where G is a d.f. on $(0,\infty)$. Therefore $\check{F}$ is an inf div L.T. by corollary 2.2.1. □ □ □

As an interesting alternative approach we mention two lemmas, one of which is due to Vervaat. These lemmas establish a direct connection between finite mixtures of exponential distributions, birth-death processes and terminating continued fractions. We state both lemmas without proof. For a description of birth-death processes we refer to [11].

LEMMA 3.7.1 (Vervaat):

If $\lambda_0 \geq 0, \lambda_1 > 0, \ldots, \lambda_{n-1} > 0, \lambda_n = 0$ and $\mu_0 = 0, \mu_1 > 0, \mu_2 > 0, \ldots, \mu_n > 0$ are the parameters of a birth-death process on 0, 1, ..., n, then the L.T. $\check{F}_{10}$ of the first-passage time from 1 to 0 is of the form

$$(3.7.6) \qquad \check{F}_{10}(\tau) = \sum_1^n p_k \frac{c_k}{c_k+\tau} \quad ,$$

with $p_k > 0$, $\sum p_k = 1$ and $c_k > 0$. On the other hand, for every L.T. of the form (3.7.6) there is a uniquely determined birth-death process such that $\check{F}_{10}$ has the interpretation given above.

Using the same notation it is easily verified that we have

LEMMA 3.7.2:

$$(3.7.7) \qquad \check{F}_{10}(\tau) = \frac{\mu_1}{\mu_1+\lambda_1+\tau-} \; \frac{\lambda_1\mu_2}{\mu_2+\lambda_2+\tau-} \; \cdots \; \frac{\lambda_{n-1}\mu_n}{-\;\mu_n+\tau} \quad .$$

REMARK: The notation is rather awkward because the λ's and μ's in the birth-death process have an interpretation that is different from the interpretation of the λ's and μ's in e.g. theorem 2.2.1. However, the notation used for birth-death processes is so well established that we choose not to change it here.

Lemma 3.7.1 provides an interesting way of generating mixtures of exponential distributions, whereas together the lemmas 3.7.1 and 3.7.2 give an interpretation of the relation between (3.7.3) and (3.7.5). It is probably possible to prove theorem 3.7.1, using both lemmas and some results in [11], which also contains lemma 3.7.1 as a special case. Also, the inf div of (3.7.2) now follows from (3.7.7) and (3.7.6) by theorem 2.2.1. On the other hand, (3.7.6) can be obtained from (3.7.7) as a special case of theorem 3.7.1. Finally, theorem 2.2.1 (with all p's positive) is a consequence of lemma 3.7.1 and a theorem of MILLER [25], where the inf div of a larger class of first-passage times is proved.

We conclude this chapter with three examples of the correspondence between (3.7.3) and (3.7.4) (or (3.7.5)).

a. $\int_0^\infty \frac{1}{u+\tau} \frac{u^\alpha e^{-u}}{\Gamma(\alpha)} du = \frac{\alpha}{\alpha+1+\tau-} \frac{\alpha+1}{\alpha+3+\tau-} \cdots \frac{n(\alpha+n)}{\alpha+2n+1+\tau-} \cdots,$

for $\alpha > 1$ (see [26] p. 219).

b. $$1 + \tau - \sqrt{(1+\tau)^2-1} = \frac{1}{\pi}\int_0^2 \frac{1}{u+\tau} u^{1/2}(2-u)^{1/2} du =$$

$$= \frac{1/2}{1+\tau-} \frac{1/4}{1+\tau-} \frac{1/4}{1+\tau-} \cdots .$$

This is known to be the L.T. of the first-passage time from 0 to 1 in a birth-death process with $\lambda_n = \mu_n = 1/2$ (cf. [5] p. 414, see also (3.7.7) and example I f. in section 3.6).

c. $$\frac{A}{\sqrt{(\tau+a)^2-4}} = \frac{A}{a+\tau-} \frac{2}{a+\tau-} \frac{1}{a+\tau-} \frac{1}{a+\tau-} \cdots ,$$

where $a > 2$ and $A = \sqrt{a^2-4}$. This is the L.T. of a d.f. with p.d.f.

$$A\ e^{-ax}\ I_0(2x) \qquad (x > 0).$$

On the other hand we have (cf. [36] p. 261, 28.)

$$\{(\tau+a)^2 - 4\}^{-1/2} = \frac{1}{\pi}\int_{a-2}^{a+2} \frac{1}{u+\tau} \{4 - (a-u)^2\}^{-1/2} du.$$

If we denote by $\check{G}$ the L.T. in example b., then we have

$$\{(\tau+a)^2 - 4\}^{1/2} = a + \tau - 2\check{G}(\frac{a+\tau-2}{2})$$

This shows that the c.f. $A\{(\alpha-it)^2 - 4\}^{-1/2}$ is of the form $\lambda/(\lambda-h)$ with $h \in \mathcal{H}$.

<u>REMARK</u>: In all three examples the function ψ (see (3.7.4)) is such that

$$\int_0^\infty u^{k-1}\ d\psi < \infty \qquad (k = 0, 1, 2, \ldots).$$

Though in Satz 4.9 ([26], p. 216) the moments of ψ are not mentioned, in converse statements (e.g. Satz 4.10) the existence of all moments of ψ is required. From T.J. Stieltjes, Recherches sur les fractions continues, Oeuvres Complètes II, it appears that ψ has finite moments of all non-negative orders. It follows that the exponential mixtures generated by continued fractions of the form (3.7.3) all have mixing functions possessing all moments. This also seems to be consistent with the results in [11] (see also [30]).

Chapter 4

RELATED INFINITELY DIVISIBLE DISTRIBUTIONS

4.1 Renewal distributions and monotone densities

In this section criteria are derived for the inf div of renewal distributions, i.e. distributions on $(0,\infty)$ with p.d.f.'s of the form

$$(4.1.1) \qquad g(x) = \frac{1-F(x)}{\mu} ,$$

where F is a d.f. on $(0,\infty)$ with finite mean μ. The distribution with p.d.f. g will be called the renewal distribution corresponding to the distribution with distribution function F. We may restrict ourselves to d.f.'s F on $(0,\infty)$, as F and $p+(1-p)F$ $(0 < p < 1)$ have the same renewal distribution. For non-lattice d.f.'s F, the renewal distribution has the following interpretation. Renewals occur at random time intervals, which are independent and all have d.f. F, the first interval having left end point at $t = 0$. One observes the renewal process from time t onward and notes the random time interval elapsing until the next renewal. For $t \to \infty$ the d.f. of this interval tends to a d.f. with p.d.f. g as given by (4.1.1). For additional information we refer to SMITH [31].

Clearly, all bounded non-increasing p.d.f.'s on $(0,\infty)$ can be written in the form (4.1.1). Criteria for the inf div of such p.d.f.'s are given in section 4.1.2. In section 4.1.3 we consider the discrete analogues of renewal distributions and monotone densities. An example of a waiting-time distribution that is not inf div is given in section 4.1.4.

4.1.1 Renewal distributions

It is well known that many waiting-time distributions are inf div. As waiting-time distributions are related to renewal distributions, one might expect that many renewal distributions, are inf div. It turns out, however, that the renewal distributions corresponding to a given distribution does not tend to be "more inf div" than the original one. It may happen that in (4.1.1) F is inf div and g is not, but also the other way around. It is possible, however, to obtain explicit criteria, in terms of F, for g to be inf div.

The Laplace transform of g is given by

$$(4.1.2) \qquad \check{G}(\tau) = \frac{1-\check{F}(\tau)}{\mu\tau} .$$

Examples of d.f.'s F for which G is inf div are provided by the mixtures of exponential d.f.'s. If F is a d.f. with finite mean μ, having L.T.

$$F(\tau) = \int_0^\infty \frac{\lambda}{\lambda+\tau}\, dU(\lambda),$$

then $\mu = \int_0^\infty \lambda^{-1}\, dU(\lambda) < \infty$, and $\check{G}$ satisfies

$$\check{G}(\tau) = \int_0^\infty \frac{\lambda}{\lambda+\tau}\ \frac{1}{\lambda\mu}\, dU(\lambda),$$

i.e. $\check{G}$ is again the L.T. of a mixture of exponential distributions and hence inf div.

Denoting by F^{*k} the k-th convolution of F with itself we define

$$(4.1.3) \qquad L(x) = \sum_{k=1}^{\infty} k^{-1}\, F^{*k}(x),$$

which is finite for finite x and such that

$$(4.1.4) \qquad L(x) \sim \log x \qquad (x \to \infty)$$

(c.f. SMITH [32]). We now prove

THEOREM 4.1.1

The L.T. (4.1.2) is inf div if and only if for all $x > 0$

$$(4.1.5) \qquad \log x - L(x) \text{ is non-decreasing}.$$

PROOF: By corollary 1.3.6 $\check{G}$ is inf div if and only if $-\frac{d}{d\tau}\log\check{G}(\tau)$ is completely monotone. Using Helly's second theorem we have for all $\tau > 0$

$$-\frac{d}{d\tau}\log\check{G}(\tau) = \frac{1}{1-\check{F}(\tau)}\frac{d}{d\tau}\check{F}(\tau) + \tau^{-1} = \sum_{k=1}^{\infty}\{\check{F}(\tau)\}^{k-1}\frac{d}{d\tau}\check{F}(\tau) + \tau^{-1} =$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k}\int_0^{\infty} e^{-\tau x}\, x dF^{*k}(x) + \int_0^{\infty} e^{-x\tau}dx =$$

$$= \int_0^{\infty} e^{-\tau x}\, x d(\log x - L(x)).$$

By the uniqueness theorem for Laplace-Stieltjes transforms and the representation theorem for c.m. functions (see [5] p. 416) it follows that $-\frac{d}{d\tau}\log\check{G}(\tau)$ is c.m. if and only if (4.1.5) holds. □ □ □

REMARK: An alternative proof can be given by writing

(4.1.6) $$\check{G}(\tau) = \frac{1-\check{F}(\tau)}{\tau\mu} = \lim_{\lambda\downarrow 0}\frac{\lambda}{\lambda+\tau\mu}\cdot\frac{\lambda+1-\check{F}(\tau)}{\lambda} = \lim_{\lambda\downarrow 0}\check{F}_{1,\lambda}(\tau)/\check{F}_{2,\lambda}(\tau),$$

where both $\check{F}_{1,\lambda}$ and $\check{F}_{2,\lambda}$ are inf div L.T.'s. Theorem 4.1.1 can now be obtained from (4.1.6) by theorem 1.3.10 (see also theorem 1.6.1).

If $\check{F}$ is the L.T. of a lattice distribution, then (cf. [22], p. 25) $\check{F}(it_0) = 1$ for some real $t_0 \neq 0$. It follows that $\check{G}(it_0) = 0$, and hence, by theorem 1.3.2, $\check{G}$ is not inf div. As $\log x - L(x)$ jumps downward where $F(x)$ is discontinuous, theorem 1.4.1 implies that $F(x)$ is continuous if G is inf div (compare theorem 1.3.10). But theorem 4.1.1 even implies the absolute continuity of F. We have

COROLLARY 4.1.1:

$\{1-\check{F}(\tau)\}/(\mu\tau)$ is inf div if and only if $F(x)$ is absolutely continuous and if the inequality

(4.1.7) $$\sum_{k=1}^{\infty} k^{-1} f^{*k}(x) \leq x^{-1}$$

holds for almost all $x > 0$.

PROOF: If $(1-\check{F})/(\mu\tau)$ is inf div, i.e. if $\log x - L(x)$ is non-decreasing, then certainly $\log x - F(x)$ is non-decreasing. As $F(x)$ itself is non-decreasing, it follows that $F(x)$ is absolutely continuous with respect to the measure $x^{-1}dx$. As furthermore $F(x)$ is supposed to be continuous at $x = 0$, it is absolutely continuous with respect to Lebesgue measure on $[0,\infty)$. In the same way we obtain the absolute continuity of $L(x)$. The inequality (4.1.7) now follows from (4.1.3) and (4.1.5). Conversely, if F is absolutely continuous and (4.1.7) holds almost everywhere, then $\{1-\check{F}(\tau)\}(\tau\mu)^{-1}$ is inf div by theorem 4.1.1. □ □ □

As examples we consider the renewal distributions corresponding to the Γ-densities with mean 1,

(4.1.8) $$f_n(x) = \frac{n^n}{(n-1)!} x^{n-1} e^{-nx} \qquad (x > 0).$$

Defining

$$S_n(x) = x \sum_{k=1}^{\infty} k^{-1} f_n^{*k}(x),$$

we have

$$S_n(x) = n\, e^{-nx} \sum_{k=1}^{\infty} \frac{(nx)^{nk}}{(nk)!} = e^{-nx} \{\sum_{k=0}^{n-1} \exp(nz_k x) - n\},$$

where $z_k = \exp(2k\pi i/n)$. By (4.1.7) for inf div we must have $S_n(x) \leq 1$ for all $x > 0$. We obtain

$$S_1(x) = 1-e^{-x}$$

$$S_2(x) = (1-e^{-2x})^2,$$

and one easily verifies that $S_3 \leq 1$ and $S_4 \leq 1$. However, for $n \geq 5$ we have $S_n(x) > 1$ for suitably chosen large values of x, as Re $z_1 > 0$ for $n \geq 5$.

From (4.1.8) one obtains the asymptotic relation

$$f_n(1) \sim (\frac{n}{2\pi})^{1/2}, \qquad (n \to \infty),$$

contradicting the necessary condition $f_n(x) \leq 1$. It can also be seen without any computation that the renewal distributions corresponding to the densities (4.1.8) cannot be all inf div. For $n \to \infty$ the distribution with density f_n tends to the degenerate distribution concentrated at $x = 1$. The corresponding renewal distribution therefore tends to the uniform distribution on (0,1), which, having bounded support, is not inf div by theorem 1.3.4.

The preceding distributions (for $n \geq 5$) provide examples of inf div distributions with corresponding renewal distributions, that are not inf div. We now give an example of a distribution that is not inf div, but has an inf div renewal distribution. The L.T.

$$\check{F}(\tau) = \frac{15+\tau^2}{(1+\tau)(3+\tau)(5+\tau)}$$

is not inf div as was shown in section 2.2 on p. 22 . The corresponding renewal distribution has L.T.

$$\check{G}(\tau) = \frac{\tau^2+8\tau+23}{23/15(1+\tau)(3+\tau)(5+\tau)}$$

which is inf div, as it has a canonical representation of the form (1.3.4) with (cf. 1.3.7)

$$k(x) = e^{-x} + e^{-3x} + e^{-5x} - 2e^{-4x} \cos x\sqrt{7},$$

which is positive for all $x > 0$.

<u>REMARK</u>: As we have seen, renewal distributions are in general not inf div. However, SHANTARAM [28] has shown that repeated application of the transformation (4.1.1), modified by suitable normalization leads (in case of convergence) to d.f.'s $H(x)$ satisfying the relation

$$(4.1.9) \qquad H(x) = b \int_0^{lx} \{1-H(y)\}dy \qquad (x > 0).$$

Here $b^{-1} = \int_0^\infty \{1-H(y)\}dy$ and $l \geq 1$. It easily follows from (4.1.9) that $H(x)$ has derivatives of all orders, and that $H'(x)$ is completely monotone. Hence, by theorem 2.11.1, $H(x)$ is inf div.

4.1.2 Monotone densities

If $g(x)$ is a p.d.f. on $(0,\infty)$ with the properties

(a) $g(x)$ is non-increasing on $(0,\infty)$

(b) $g(0+) < \infty$,

then $g(x)$ can be written in the form (4.1.1) with

(4.1.10) $$\mu = 1/g(0+); \; F(x) = 1-g(x)/g(0+).$$

From corollary 4.1.1 we deduce

THEOREM 4.1.2

If $g(x)$ is a p.d.f. satisfying the conditions (a) and (b), then

(i) $g(x)$ is not inf div if $g(x)$ is not absolutely continuous on $(0,\infty)$

(ii) if $g(x)$ has a derivative $g'(x)$, then a necessary condition for inf div is

(4.1.11) $$-g'(x) \leq g(0+)x^{-1}$$

for almost all $x > 0$.

Examples of p.d.f.'s, which by this criterion are not inf div are the functions

$$g_n(x) = c_n \exp(-x^n) \qquad (x > 0),$$

where $c_n = n^{-1}/\Gamma(n^{-1})$, for $n > e$. We have

$$-xg'_n(x) = nx^n g_n(x),$$

which reduces condition (4.1.11) to

$$nx^n \leq \exp (x^n).$$

This condition is not satisfied for $x^n = \log n$ and $\log n > 1$, i.e. $c_n \exp (-x^n)$ is not inf div for $n > e$. [1)]

Generally, condition (4.1.11) says that $g(x)$ should not decrease too sharply, and in particular that $g'(x)$ should be bounded in every interval $[\delta,\infty)$ with $\delta > 0$.

REMARK: if $g(x)$ is convex on $(0,\infty)$, then (see (4.1.10)) $f(x)$ is non-increasing on $(0,\infty)$. Taking convolutions, by induction we find

$$f^{*k}(x) \geq \frac{x^{k-1}}{(k-1)!} \{f(x)\}^k.$$

This implies that

$$x \sum_{k=1}^{\infty} k^{-1} f^{*k}(x) \geq \exp (xf(x)) - 1,$$

and therefore (compare 4.1.7)) a necessary condition for the inf div of g is $xf(x) \leq \log 2$, or in terms of g

$$-g'(x) \leq x^{-1} g(0+) \log 2,$$

which is slightly sharper than (4.1.11).

4.1.3 Lattice distributions

If p_k $(k = 0, 1, 2, \ldots)$ is a distribution on the non-negative integers with finite, positive mean μ, then the distribution

$$(4.1.12) \qquad q_k = \frac{1-P_k}{\mu}$$

with $P_k = \sum_0^k p_j$, is the analogue of the renewal distribution given by (4.1.1).

---

1) By a different method it can be proved that this is not the case for any $n > 1$.

Using the interpretation of (4.1.1) with t replaced by n, denoting a positive integer, (4.1.12) can be obtained as an exercise in Markov chains. As in the continuous case, it is no restriction to assume that $p_0 = 0$.

We shall now prove the analogues to corollary 4.1.1 and theorem 4.1.2. Denoting by $p_j^{(k)}$ the k-th convolution of the distribution p_j, we have

THEOREM 4.1.3

A distribution on the non-negative integers of the form (4.1.12) is inf div if and only if

$$\sum_{k=1}^{\infty} \frac{1}{k} p_j^{(k)} \le \frac{1}{j} \qquad (j = 1, 2, \ldots). \tag{4.1.13}$$

PROOF: Taking generating functions in (4.1.12) we obtain

$$Q(u) = \frac{1-P(u)}{\mu(1-u)}. \tag{4.1.14}$$

By corollary 1.3.8, Q(u) is inf div if and only if

$$-\frac{P'(u)}{1-P(u)} + \frac{1}{1-u}$$

has a power series expansion with non-negative coefficients. From this, in a way completely analogous to the proof of theorem 4.1.1, it follows that Q(u) is inf div if and only if (4.1.13) holds. □ □ □

Analogous to theorem 4.1.2, we have, retaining only the first term in the left-hand side of (4.1.13),

COROLLARY 4.1.2

If q_j is a non-increasing lattice distribution, then a necessary condition for the inf div of q_j is

$$q_{j-1} - q_j \le q_0/j \qquad (j = 1, 2, \ldots).$$

This again requires a certain smoothness of the q_j.

We conclude this section by considering the discrete analogue of theorem 2.11.1. We need the following lemma, due to Hausdorff (cf. [5], p. 223).

LEMMA 4.1.1

A distribution p_n is completely monotone if and only if there exists a finite measure μ on $[0,1]$, such that

$$p_n = \int_0^1 p^n \, d\mu(p) \qquad (n = 0, 1, \dots). \tag{4.1.15}$$

Clearly μ cannot have an atom at $p = 1$, as this would imply that $\sum p_n$ is divergent.

In section 3.6 (Example I c.) we have seen that c.f.'s of the form

$$\int_{(0,\infty]} \frac{\lambda}{\lambda+1-\exp it} \, dG(\lambda), \tag{4.1.16}$$

where G is a d.f. on $(0,\infty]$, are inf div. Consequently, lattice distributions p_n, satisfying

$$p_n = \int_0^1 (1-p)\, p^n \, dA(p) \qquad (n = 0, 1, \dots), \tag{4.1.17}$$

where A is a d.f. on $[0,1)$, are inf div. That is, mixtures of geometric distributions are inf div.

As in (4.1.15) we have

$$\sum_0^\infty p_n = \int_0^1 (1-p)^{-1} \, d\mu(p) = 1,$$

(4.1.15) is of the form (4.1.17) with $dA = (1-p)^{-1} d\mu$. Hence we have proved

THEOREM 4.1.4

All completely monotone lattice distributions are inf div.

REMARK: The addition "lattice" in theorem 4.1.4 is essential. The inf div of all c.m. distributions on an arbitrary, ordered, countable set of real

numbers, would imply the inf div of some distributions with finite support, contrary to theorem 1.3.4.
Some examples of c.m. lattice distributions are given in section 4.2.1.

4.1.4 Waiting times

In this section we give an example of a waiting-time distribution that is not inf div. For definitions regarding the waiting-time process we refer to section 1.7.

In the case M/G/1 with last-come-first-served queue discipline, the L.T. of the waiting-time d.f. is of the form

$$\check{C}(\tau) = 1 - \rho + \rho\, \check{C}_1(\tau), \tag{4.1.18}$$

with (see (1.7.3))

$$\check{C}_1(\tau) = \mu^{-1} \frac{1-\check{G}(\tau)}{\tau+\lambda-\lambda\check{G}(\tau)} .$$

Here μ is the mean service time, and $\check{G}$ is uniquely defined by

$$\check{G}(\tau) = \check{B}(\tau+\lambda-\lambda\check{G}(\tau)) \tag{4.1.19}$$

see [5] p. 448). If the renewal d.f. R, corresponding to B, is inf div, then $\check{C}_1$ is inf div: we have

$$\check{C}_1(\tau) = \check{R}(\tau+\lambda-\lambda\check{G}(\tau)),$$

which is an inf div L.T. by theorem 3.5.1, because $\check{C}_1(-it) = \check{R}(-it+\lambda(1-\gamma(t))) = \check{R}(-h)$, with $h \in H$ (cf. (1.6.1) and (3.4.5)). On the other hand, if B is a lattice d.f., then so is G, and (see p.72) $\check{G}(it_0) = 1$ for some real $t_0 \neq 0$ and therefore $\check{C}_1(it_0) = 0$. It follows from theorem 1.3.2 that $\check{C}_1$ is not inf div. For $\mu = 1$ and $\lambda \uparrow 1$ (i.e. $\rho \uparrow 1$) the L.T. $\check{C}_1$, and therefore $\check{C}$, tends to the $\check{C}_1^*$ given by

$$\check{C}_1^* = \frac{1-\check{G}(\tau)}{\tau+1-\check{G}(\tau)} , \tag{4.1.20}$$

which is not inf div for the same reason. An explicit example of such a L.T. is obtained by taking $\check{B} = \exp(-\tau)$. In this case $\check{C}_1^*$ is of the form (4.1.20) with

$$\check{G}(\tau) = \sum_{k=1}^{\infty} \frac{k^{k-1}}{k!} e^{-k(\tau+1)}$$

(see (4.1.19) and theorem 1.8.1).

We now return to (4.1.18), the L.T. of the d.f. of the waiting time proper. As for $\rho \uparrow 1$, $\check{C}$ tends to a L.T. that is not inf div, it follows from the closure property that $\check{C}$ cannot be inf div for all ρ with $0 < \rho < 1$. This proves the existence of waiting time distributions that are not inf div.

For comparison we consider the case M/G/1 with queue discipline first-come-first-served. Now the L.T. of the waiting-time d.f. equals

$$\check{C}_0(\tau) = 1 - \rho + \rho\, \check{C}_2(\tau),$$

with (see (1.7.1) and 1.7.2))

$$\check{C}_2(\tau) = \check{R}(\tau)\, \frac{1-\rho}{1-\rho\check{R}(\tau)}\ .$$

$\check{C}_2$ is inf div if $\check{R}$ is inf div by theorem 1.6.1. Again, if B is a lattice d.f., then $\check{R}(it_0) = 0$ for a real $t_0 \neq 0$ and $\check{C}_2(\tau)$ is not inf div. However, this does not lead to a L.T. $\check{C}_0$ that is not inf div, because in this case $\check{C}_2$ does not tend to the L.T. of a d.f. if $\rho \uparrow 1$. This, of course, is consistent with the fact that $\check{C}_0$ always is inf div, as we saw in section 1.7.

REMARK: It follows from theorem 3.5.1 that $\check{G}$ in (4.1.19) (i.e., the L.T. of the busy period) is inf div whenever $\check{B}$ is inf div.

4.2 Representation theorems for inf div distributions on $[0,\infty)$

In sections 4.2.1 and 4.2.2 representation theorems are derived for inf div distributions on $[0,\infty)$. In the case of lattice distributions the representation is in terms of probabilities, in the absolutely continuous case in terms of p.d.f.'s. The representation in the former case is essentially due to KATTI [12].

We shall use two lemmas, which are implicit in GOLDIE [8]. First we introduce some notation. If $\check{F}$ is the L.T. of a d.f. on $[0,\infty)$, then by corollary 1.3.6 $\check{F}$ is inf div if and only if $-\check{F}'/\check{F}$ is completely monotone. It follows that $\check{F}$ is inf div if and only if $r(\tau;\theta)$, defined by

$$r(\tau;\theta) = -\check{F}'(\tau+\theta)/\check{F}(\tau+\theta) \qquad (\theta > 0), \tag{4.2.1}$$

as a function of τ, has alternating derivatives for $0 < \tau < \theta$ and all $\theta > 0$. Now for all τ with $|\tau| < \theta$ we have

$$\check{F}(\tau+\theta) = \sum_{k=0}^{\infty} (-1)^k \, b_k(\theta)\, \tau^k , \tag{4.2.2}$$

where

$$b_k(\theta) = \frac{1}{k!}\int_0^{\infty} x^k e^{-\theta x} dF(x) \qquad (k = 0, 1, \ldots). \tag{4.2.3}$$

From the equations (4.2.1), (4.2.2) and (4.2.3) we obtain, for $|\tau| < \theta$,

$$r(\tau;\theta) = \sum_{k=0}^{\infty} (-1)^k \, a_k(\theta)\, \tau^k , \tag{4.2.4}$$

where the a_k are determined by

$$(n+1)b_{n+1} = \sum_{k=0}^{n} b_k \, a_{n-k} \qquad (n = 0, 1, \ldots). \tag{4.2.5}$$

We have now proved

LEMMA 4.2.1 (Goldie)

$\check{F}$ is inf div if and only if the quantities a_k defined by (4.2.5) and (4.2.3) are non-negative for all $\theta > 0$.

The following lemma also is implicit in [8] (see also KALUZA [10] for the concluding statement), and may be proved by induction.

LEMMA 4.2.2

The a_k determined by (4.2.5) are non-negative if

(i) the quantities f_k determined by the relations

(4.2.6) $$b_{n+1} = \sum_{k=0}^{n} b_k f_{n-k} \qquad (n = 0, 1, \ldots),$$

are non-negative,

or if

(ii) the b_n satisfy

(4.2.7) $$b_{n+1} b_{n-1} \geq b_n^2 \qquad (n = 1, 2, \ldots).$$

Condition (ii) implies condition (i).

4.2.1 Lattice distributions

We prove the following theorem, which is given in [12] with a different proof.

THEOREM 4.2.1 (Katti)

If p_n is a distribution on the non-negative integers, with $p_0 > 0$, then p_n is inf div if and only if

(4.2.8) $$D_n(p_0, \ldots, p_n) = \begin{vmatrix} p_0 & 0 & \cdot & \cdot & \cdot & 0 & p_1 \\ p_1 & p_0 & 0 & \cdot & \cdot & 0 & 2p_2 \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ p_{n-1} & p_{n-2} & \cdot & \cdot & \cdot & p_1 & np_n \end{vmatrix} \geq 0 \quad (n = 2, 3, \ldots).$$

PROOF: Taking generating functions, by corollary 1.3.8 we know that p_n is inf div if and only if

(4.2.9) $$P'(u)/P(u) = \sum_{n=0}^{\infty} r_n u^n, \text{ with } r_n \geq 0 \qquad (n = 0, 1, \ldots),$$

for $|u| < 1$. Multiplying both sides of (4.2.9) by $P(u)$ (which is $\neq 0$ for $|u| < 1$ by corollary 1.3.3), expanding both sides in power series and equating the coefficients, we obtain

(4.2.10) $$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \qquad (n = 0, 1, \ldots).$$

We use the first n equations of (4.2.10) to solve for r_{n-1}. For $n = 1$ we have $r_0 = p_1/p_0$, which is trivially non-negative. Cramer's rule for $r_1, r_2, \ldots$, together with the condition that $r_n \geq 0$ for $n = 1, 2, \ldots$, yields(4.2.8).□ □ □

Theorem 4.2.1 may be read as a representation theorem. We have

COROLLARY 4.2.1

A lattice distribution p_n with $p_0 > 0$ is inf div if an only if their exist r_n $(n = 0, 1, \ldots)$, such that

(4.2.10) $$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \qquad (n = 0, 1, \ldots),$$

with $r_n \geq 0$ and, necessarily, $\sum_{1}^{\infty} n^{-1} r_n < \infty$.

Lemma 4.2.2 allows us to draw some interesting conclusions from corollary 4.2.1:

THEOREM 4.2.2

If p_n is a lattice distribution with $p_0 > 0$, then p_n is inf div if

(i) there exist $q_n \geq 0$, such that

(4.2.11) $$p_{n+1} = \sum_{k=0}^{n} p_k q_{n-k} \qquad (n = 0, 1, \ldots),$$

or if, especially,

(ii) the p_n satisfy

(4.2.12) $$p_{n+1}\, p_{n-1} \geq p_n^2 \qquad (n = 1, 2, \ldots),$$

or if, (by Schwarz's inequality) still more specially,

(iii) p_n is completely monotone.

Taking generating functions, from (4.2.11) we obtain

$$P(z) = \frac{p_0}{1-zQ(z)} ,$$

and putting $z = e^{-\tau}$

$$P(e^{-\tau}) = \frac{\lambda}{\lambda+1-\check{G}(\tau)} ,$$

with $\lambda = p_0/(1-p_0)$ and $\check{G}(\tau) = e^{-\tau}Q(e^{-\tau})/(1-p_0)$. That is, lattice distributions satisfying (4.2.11) are special cases of distributions with c.f.'s of type (1.6.1).

Of the four examples given by Katti as applications of theorem 4.2.1, at least three have essentially c.m. distributions (c.f. Lemma 4.1.1). His example (i):

$$p_n = c_1 \frac{\theta^{n+1}}{n+1} = c_1 \int_0^\theta p^n \, dp \qquad (0 < \theta < 1;\ n = 0, 1, \ldots).$$

His example (iii):

$$p_n = c_2 \frac{\rho^{n+1}}{1-\rho^{n+1}} = c_2 \sum_{k=1}^{\infty} (\rho^k)^n \rho^k (0 < \rho < 1;\ n = 0, 1, \ldots),$$

which is also of the form $\int_0^1 p^n d\mu(p)$.

His example (iv):

$$p_n^* = c_3 \frac{\rho^{n+1}}{(1-\rho)^n(1-\rho^{n+1})} = c_4(1-\rho)^{-n} p_n ,$$

with p_n as in example (iii). It follows (see (4.2.10)), and the remark on p. 88) that p_n^* is inf div.

In Katti's example (ii) $p_0 = a/(1+a)$ and

$$p_n = \frac{\theta^n}{-n(1+a)\log(1-\theta)} \quad (a > 0;\ 0 < \theta < 1;\ n = 1, 2, \ldots).$$

For $a \geq -2/\log(1-\theta)$ we have $p_{n+1}\, p_{n-1} \geq p_n^2$ for $n = 1, 2, \ldots$. For $-1/\log(1-\theta) \leq a < -2/\log(1-\theta)$ p_n is inf div by direct application of theorem 4.2.1. If $a < -1/\log(1-\theta)$ then p_n is not inf div.

We conclude this section with a theorem that can also be obtained from corollary 4.2.1. Its, more interesting, counterpart in the absolutely continuous case will be given in the next section.

THEOREM 4.2.3

If p_n is an inf div lattice distribution with $p_0 > 0$, then the following implications hold

$$\left.\begin{array}{l} p_m > 0 \\ p_n > 0 \end{array}\right\} \Longrightarrow p_{m+n} > 0 \qquad (m = 1, 2, \ldots;\ n = 1, 2, \ldots).$$

Consequently we have

$$p_m > 0 \quad \Longrightarrow p_{km} > 0 \qquad (k = 2, 3, \ldots),$$

and

$$p_1 > 0 \quad \Longrightarrow p_k > 0 \qquad (k = 2, 3, \ldots).$$

PROOF: This theorem may be proved by induction, using (4.2.10). The proof we give here was suggested by Fabius. By definition 1.3.2 we have

$$p_0 + p_1 z + p_2 z^2 + \ldots = (q_0 + q_1 z + q_2 z^2 + \ldots)^{m+n},$$

where q_n is a lattice distribution with $q_0 > 0$. As $p_m > 0$, there exist $l_j \geq 0$ $(j = 0, 1, \ldots, m)$ with $\sum_1^m j l_j = m$ and such that

$$p_m \geq q_0^{l_0} \ldots q_m^{l_m} > 0 .$$

In the same way

$$p_n \geq q_0^{k_0} \ldots q_n^{k_n} > 0,$$

with $k_j \geq 0$ $(j = 0, 1, \ldots, n)$ and $\sum_1^n jk_j = n$. As, similarly,

$$p_{m+n} = \sum q_0^{i_0} \ldots q_{m+n}^{i_{m+n}} ,$$

with $i_j \geq 0$ $(j = 0, 1, \ldots, n+m)$ and $\sum_1^{m+n} ji_j = m+n$, it follows that

$$p_{m+n} \geq q_0^{l_0} \ldots q_m^{l_m} q_0^{k_0} \ldots q_n^{k_n} > 0. \ \square\square\square$$

4.2.2 Absolutely continuous distributions

By corollary 1.3.5 a L.T. $\check{F}$ is inf div if and only if

(4.2.13) $$-\check{F}'(\tau)/\check{F}(\tau) = \int_0^\infty e^{-\tau x}\, dK(x) \qquad (\text{Re } \tau > 0),$$

where $\check{F}(\tau) \neq 0$ (cf. corollary(1.3.2)), and K(x) is non-decreasing with

(4.2.14) $$\int_1^\infty x^{-1}\, dK(x) < \infty.$$

Multiplying both sides of (4.2.13) with $\check{F}$ and inverting we obtain

THEOREM 4.2.4:

If F is a d.f. on $[0,\infty)$, then F is inf div and only if

(4.2.15) $$\int_0^x u dF(u) = \int_0^x F(x-u) dK(u),$$

where K is non-decreasing and satisfies (4.2.14).

If F is absolutely continuous, then, writing $F(u) = \int_0^u f(t)dt$,

we have, as an analogue to corollary 4.2.1.

COROLLARY 4.2.2

The p.d.f. of a distribution on $(0,\infty)$ is inf div if and only if

$$x\, f(x) = \int_0^x f(x-u)\, dK(u) \qquad (x > 0), \tag{4.2.16}$$

where K is non-decreasing and satisfies (4.2.14).

We now prove the analogue of theorem 4.2.3.

THEOREM 4.2.5

If $f(x)$ is a continuous and inf div p.d.f. on $(0,\infty)$, then the following implication holds

$$f(x_0) = 0 \Rightarrow [f(x) = 0 \text{ for all } x \leq x_0].$$

PROOF: It is no restriction (this can be achieved by a shift) to assume that for every $\delta > 0$ there exists an $x_1 < \delta$ such that $f(x_1) > 0$. We then have to prove that $f(x) \neq 0$ for all $x > 0$. Now suppose that $f(x_1) > 0$ and that x_0 is the smallest number satisfying $f(x_0) = 0$ and $x_0 > x_1$. By (4.2.16) we have

$$0 = x_0 f(x_0) = \int_0^{x_0} f(x_0-u)dK(u).$$

As $f(x) > 0$ for all x with $x_1 \leq x < x_0$, it follows that

$$\int_{0+}^{x_0-x_1} dK(u) = 0,$$

and hence that

$$\int_{0+}^{x} f(x-u)dK(u) = 0$$

for all $x < x_0 - x_1$. Therefore $xf(x) = f(x)\, K(0)$ for all $x < x_0 - x_1$. It follows from the continuity of $f(x)$, that $f(x) = 0$ for all $x < x_0 - x_1$. As this contradicts our assumptions, it follows that x_0 does not exist and hence that $f(x) \neq 0$ for all $x > 0$. □ □ □

REMARK: The conclusion above can also be obtained from the fact that $K(0) = 0$ if $F(x) > 0$ for all $x > 0$. By (4.2.15) we have

$$xF(x) \geq \int_0^x u\,dF(u) \geq F(x)K(0).$$

Taking $x \downarrow 0$, it follows that $K(0) = 0$.

As a slightly weaker statement we have

COROLLARY 4.2.3

A continuous and inf div p.d.f. on $(0,\infty)$, which is positive on $(0,\delta)$ for some $\delta > 0$, has no zeros on the positive half line.

In SHARPE [29] a theorem similar to theorem 4.2.5 is proved for a p.d.f. on $(-\infty,\infty)$, under the condition that all positive powers of its c.f. are integrable.

Examples of p.d.f.'s, that are not inf div by corollary 4.2.3, are

a. $f(x) = \text{const.}\ (e^{-x} - 2e^{-2x})^{2k}$ $\quad (x > 0;\ k = 1, 2, \ldots)$.

b. $f_\alpha(x) = \frac{1}{24} \exp(-x^{1/4})(1 - \alpha \sin x^{1/4})$ $\quad (x > 0)$,

for $\alpha = 1$. It follows that f_α cannot be inf div for all α with $0 \leq \alpha < 1$. For $\alpha = 0$ we have inf div, as f_0 is c.m. (cf. theorem 2.11.1).

REMARK: It easily follows from the corollaries 4.2.1 and 4.2.2 (or otherwise) that if p_n is inf div then const. $q^n p_n$ is inf div, and if $f(x)$ is infinitely divisible then the p.d.f. const. $e^{\lambda x} f(x)$ is inf div. Here $\sum q^n p_n$ and $\int e^{\lambda x} f(x)dx$ are supposed to be finite. Compare example (iv) on p. 84.

We now turn to the non-lattice counterpart of theorem 4.2.2. The analogue of condition (i) would be

$$F(x) - F(0) = \int_0^x F(x-u)dQ(u),$$

where Q is non-decreasing, or,taking Laplace transforms,

$$\check{F}(\tau) = \frac{F(0)}{1 - \check{Q}(\tau)},$$

where $0 < F(0) < 1$, and $\check{Q}(\tau)/(1 - F(0))$ is the L.T. of a d.f. It follows that the c.f. of F is of the form $\lambda/(1 + \lambda - \gamma)$ (cf. theorem 1.6.1). As $F(0) > 0$, in this case there is no analogue for p.d.f.'s.

The obvious analogue of condition (ii) for p.d.f.'s is that log f(x) is convex. Every d.f. F with a log-convex derivative f can be approximated by a lattice d.f. satisfying condition (ii) of theorem 4.2.2, as follows.

$$F(x) = \lim_{h\downarrow 0} \sum_{\substack{n\geq 0 \\ hn\leq x}} hf(\frac{2n+1}{2} h) = \lim_{h\downarrow 0} \sum_{nh\leq x} p_n,$$

with $p_{n+1}p_{n-1} \geq p_n^2$. As mixtures (with positive weights) of log-convex functions are again log-convex, we have proved

THEOREM 4.2.6

All (mixtures of) log-convex p.d.f.'s on $(0,\infty)$ are inf div.

By Schwarz's inequality this theorem is slightly stronger than theorem 2.11.1, which also follows directly from theorem 4.2.2, by use of condition (iii).

An example of a p.d.f. which is log-convex, but not completely monotone, is (compare [13])

$$f(x) = \frac{9}{14}\,(e^{-2x} + 3e^{-3x})^{1/2} \qquad (x > 0).$$

REMARK: It is well known that p.d.f.'s and c.f.'s (being Fourier transforms of one another) have several properties in common. It appears that inf div p.d.f.'s and (real) inf div c.f.'s also share some special properties. One of these is that they seem to have no zeros (this has not been proved generally for inf div p.d.f.'s; see, however, corollary 4.2.3 and [29]).

Another property concerns logarithmic convexity. According to theorem 4.2.6, log-convex p.d.f.'s are inf div. It is easily verified (cf. [5], p. 482 and [22], p. 70) that real c.f.'s, that are log-convex on $(0,\infty)$, also are inf div. This was first observed by HORN [9]. Log-convex c.f.'s are considered in some detail in [13].

4.3. Moment inequalities

The function $\log \check{F}$ is the cumulant generating function, i.e.

$$\kappa_j = \left[\left(-\frac{d}{d\tau} \right)^j \log \check{F}(\tau) \right]_{\tau=0} ,$$

as far as the cumulants exist. If $\check{F}$ is inf div, then by corollary 1.3.6 the κ_j are non-negative, i.e. we have

THEOREM 4.3.1

The cumulants of an inf div distribution on $[0,\infty)$, as far as they exist, are non-negative.

The cumulants of a distribution can be expressed, in terms of its moments, in the form of determinants (cf. KENDALL and STUART [15], p. 90). Doing so we obtain

COROLLARY 4.3.1

If $\mu_1, \mu_2, \ldots, \mu_n$ are the first n (finite) moments of an inf div distribution on $[0,\infty)$ then these moments satisfy the inequalities

$$(4.3.1) \qquad \kappa_j = \begin{vmatrix} 1 & 0 & . & . & 0 & \mu_1 \\ \mu_1 & 1 & 0 & . & 0 & \mu_2 \\ \mu_2 & \binom{2}{1}\mu_1 & 1 & . & 0 & \mu_3 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ \mu_{j-1} & \binom{j-1}{1}\mu_{j-2} & \binom{j-1}{2}\mu_{j-3} & . & \mu_1 & \mu_j \end{vmatrix} \geq 0 \quad (j=2,3,\ldots,n)$$

Using elementary operation on rows and columns, the inequalities (4.3.1) can be rewritten in the form

(4.3.2) $$D_j(1, \frac{\mu_1}{1!}, \ldots, \frac{\mu_j}{j!}) \geq 0 \qquad (j = 2, 3, \ldots, n),$$

where the function D_j is defined by

(4.3.3) $$D_j(x_0,x_1,\ldots,x_j) = \begin{vmatrix} x_0 & 0 & . & . & . & 0 & x_1 \\ x_1 & x_0 & 0 & . & . & 0 & 2x_2 \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ x_{j-1} & x_{j-2} & . & . & . & x_1 & jx_j \end{vmatrix}$$

(see also (4.2.8)). The inequalities (4.3.2) also follow from lemma 4.2.1, by solving the equations (4.2.5) for a_n and letting $\theta \downarrow 0$.

REMARK: It does not seem possible to generalize theorem 4.3.1 to arbitrary distributions on $(-\infty,\infty)$. The inf div c.f.

$$\phi(t) = (1+a_1it)^{-1}(1+a_2it)^{-1}(1+a_3it)^{-1},$$

where one or more of the a's are negative, can have odd cumulants, that are not all of the same sign. For real c.f.'s, however, inf div implies that its cumulants are non-negative (cf. LUKACS [23]).

Unlike the conditions (4.2.8) for lattice distributions, the conditions (4.3.1) are not sufficient for inf div, even if all moments are finite. In view of lemma 4.2.1 this could not be expected. A counter example is provided by the L.T.

$$\check{F}(\tau) = \frac{\tau^2+15}{(1+\tau)(3+\tau)(5+\tau)},$$

which is not inf div (cf. p. 22). The function $-\frac{d}{d\tau}\log \check{F}(\tau)$ has a convergent power series expansion for $|\tau| < 1$, with alternating coefficients, i.e., all cumulants are positive. However, $-\frac{d}{d\tau}\log \check{F}(\tau)$ is not completely monotone.

In example b. on p.88 we consider the density f_α, which is inf div for

$\alpha = 0$, but not for all α with $0 < \alpha < 1$. As f_α is known to have the same (finite) moments for all α (see [5]), this provides another counter example.

Though the conditions (4.3.1) are not sufficient, their necessity can be used to obtain moment inequalities. For instance, from the fact that

$$(4.3.4) \qquad \check{F}(\tau) = \int_0^\infty \frac{1}{1+\tau x}\, dG(x)$$

is an inf div L.T. for all d.f.'s on $[0,\infty)$, we derive, denoting the moments of G by $m_1, m_2, \ldots$,

<u>THEOREM 4.3.2</u>

If $m_1, m_2, \ldots, m_N$ are the finite moments of a distribution on $[0,\infty)$, then (cf. (4.3.3))

$$(4.3.5) \qquad D_n(1, m_1, \ldots, m_n) \geq 0 \qquad (n = 2, 3, \ldots, N).$$

<u>PROOF</u>: By differentiation of both sides of (4.3.4), and putting $\tau = 0$, it follows that

$$\mu_k = k!\, m_k,$$

where μ_k denotes the k-th moment of F. The inequalities (4.3.5) now follow from (4.3.2). □ □ □

Theorem 4.3.2 also provides an alternative proof of theorem 4.1.4, i.e. of the inf div of all c.m. lattice distributions (or mixtures of geometric distributions): $D_n(1, m_1, \ldots, m_n) \geq 0$ implies $D_n(p_0, p_1, \ldots, p_n) \geq 0$ [1], if p_n is a c.m. sequence (see (4.1.15)). On the other hand, the inf div of all mixtures of geometric distributions (i.e. of B(-1)-distributions: negative binomial distributions of degree 1), implies the inf div of all mixtures of exponential distributions. We have for all d.f.'s G on $(0,\infty)$

$$(4.3.6) \qquad \int_0^\infty \frac{\lambda}{\lambda+1-e^{it}}\, dG(\lambda) \qquad \text{is inf div,}$$

and hence for all G

1) See also theorem 4.2.1.

$$\int_0^\infty \frac{\lambda a}{\lambda a+1-e^{it\,a}}\, dG(\lambda) \qquad \text{is inf div,}$$

and, taking $a \downarrow 0$,

$$(4.3.7) \qquad \int_0^\infty \frac{\lambda}{\lambda-it}\, dG(\lambda) \qquad \text{is inf div.}$$

All this means that the inf div of exponential mixtures (with positive weights) follows from theorems 4.3.2 and 4.2.1.

The correspondence between (4.3.6) and (4.3.7) is not one-to-one, that is (4.3.6) for an individual mixing function G does not imply (4.3.7) for the same G. This is illustrated by the following example, where G is not a d.f. Consider the c.f.

$$\int_0^\infty \frac{\lambda}{\lambda-it}\, dG(\lambda) = \frac{15-t^2}{(1-it)(3-it)(5-it)} ,$$

which is not inf div by theorem 1.3.2. If we replace t by $i(1-e^{it})$ we obtain a c.f., which is inf div. Both c.f.'s are mixtures (of $\Gamma(1)$-distributions and B(-1)-distributions, respectively) with the same mixing function.

We now consider mixtures of $\Gamma(2)$-distributions and B(-2)-distributions (i.e. of negative binomial distributions of degree 2, which are convolutions of two identical B(-1)-distributions). B(-2)-distributions have probabilities of the form

$$(4.3.8) \qquad (n+1)(1-p)^2\, p^n \qquad (n = 0, 1, 2, \ldots; \ 0 < p < 1).$$

In the same way as theorem 4.3.2 we have (cf. theorem 2.8.1)

<u>THEOREM 4.3.3</u>

If $m_1, m_2, \ldots, m_N$ are the moments of a unimodal d.f. on $[0,\infty)$, then

$$(4.3.9) \qquad D_n(1, 2m_1, \ldots, (n+1)m_n) \geq 0 \qquad (n = 2, 3, \ldots, N).$$

According to the conjecture in section 2.6 the inequalities (4.3.9) should hold for all distributions on $[0,\infty)$. For comparison we list the first few inequalities in (4.3.5) and (4.3.9).

$2m_2 - m_1 \geq 0$	$3m_2 - 2m_1 \geq 0$
$3m_3 - 3m_2m_1 + m_1^3 \geq 0$	$6m_3 - 9m_2m_1 + 4m_1^3 \geq 0$
$4m_4 - 4m_3m_1 - 2m_2^2 + 4m_2m_1^2 - m_1^4 \geq 0$	$10m_4 - 16m_3m_1 - 9m_2^2 + 24m_2m_1^2 - 8m_1^4 \geq 0$

A direct proof of (4.3.5) has been given by Tijdeman. This proof does not seem to work for (4.3.9). Neither of the sets of inequalities (4.3.5) and (4.3.9) seems to follow directly from known moment inequalities.

If (4.3.9) should hold for all distributions on $[0,\infty)$, then by theorem 4.2.1 this would imply the inf div of all lattice distributions of the form $(n+1)C_n$, with C_n completely monotone, or equivalently (see (4.3.8)) of all mixtures of B(-2)-distributions. But, in the same way as in (4.3.6) and (4.3.7), the inf div of all B(-2)-distributions implies the inf div of all $\Gamma(2)$-distributions. So, concluding we have the equivalence of two conjectures (cf. section 2.6).

<u>THEOREM 4.3.4</u>

The validity of the inequalities (4.3.9) for all d.f.'s on $[0,\infty)$ is equivalent to the inf div of all mixtures of $\Gamma(2)$-distributions.

We conclude this section by considering the generalization of (4.3.5) and (4.3.9). If a mixture of the form

$$(4.3.10) \qquad \int_0^\infty \left(\frac{1}{1+\tau x}\right)^k dG(x),$$

where G has finite moments $m_1, m_2, \ldots, m_N$, is inf div, then these moments satisfy the inequalities

$$(4.3.11) \qquad D_n\left(1, \binom{k}{1}m_1, \ldots, \binom{k+n-1}{n}m_n\right) \geq 0 \quad (n = 2, 3, \ldots, N).$$

If (4.3.10) is inf div for all positive k, then (4.3.11) holds for all positive k. It is easily verified, using elementary operations, that for $k \to \infty$ the inequalities (4.3.11) reduce to

$$D_n(1, \frac{m_1}{1!}, \ldots, \frac{m_n}{n!}) \geq 0 \qquad (n = 2, 3, \ldots, N),$$

that is to (see (4.3.1) and (4.3.2))

$$\kappa_n \geq 0 \qquad (n = 2, 3, \ldots, N).$$

An example of this situation is provided by c.m. p.d.f.'s. If g is a c.m. p.d.f., then

$$\int_0^\infty (\frac{1}{1+\tau x})^{k+1} g(x)dx = \frac{(-1)^k}{k!} \int_0^\infty \frac{1}{1+\tau x} x^k g^{(k)}(x)dx, \tag{4.3.12}$$

which is the L.T, of a mixture of exponential distributions. It follows that in this case (4.3.10) is inf div for all k, and hence that g(x) has non-negative cumulants. This, in fact, is not surprising, as g(x) is inf div by theorem 2.11.1 and therefore has non-negative cumulants by theorem 4.3.1.

Finally, as every d.f. on $[0,\infty)$ can be approximated arbitrarily closely by a mixture of Γ-distributions, from (4.3.12) we obtain again corollary 2.11.1.

REFERENCES

[1] BARTHOLOMEW, D.J. (1969), Conditions for mixtures of exponential densities to be probability densities, Ann. Math. Stat. 40, 2183-2188.

[2] BECKENBACH, E.F. and R. BELLMAN (1961), Inequalities, Springer, Berlin.

[3] DWASS, M. (1967), A theorem about infinitely divisible characteristic functions, J. Appl. Prob. 4, 605-608.

[4] FELLER, W. (1957), An introduction to probability theory and its applications, Vol. 1 (2nd. ed.), Wiley, New York.

[5] FELLER, W. (1966), An introduction to probability theory and its applications, Vol. 2, Wiley, New York.

[6] FISZ, M. (1962), Infinitely divisible distributions: recent results and applications, Ann. Math. Stat. 33, 68-84.

[7] GNEDENKO, B.V. and A.N. KOLMOGOROV (1954), Limit distributions for sums of independent random variables, Addison-Wesley, Cambridge, Mass.

[8] GOLDIE, C.M. (1967), A class of infinitely divisible distributions, Proc. Cambridge Phil. Soc. 63, 1141-1143.

[9] HORN, R.A. (1969), On certain power series and sequences, J. London Math. Soc. (2nd series), Vol. 2, part 1, 160-162.

[10] KALUZA, T. (1928), Über die Koeffizienten reziproker Potenzreihen, Math. Z. 28, 161-170.

[11] KARLIN, S. and J.L. McGREGOR (1957), The differential equations of birth-and-death processes, and the Stieltjes moment problem, Trans. Amer. Math. Soc. 85, 489-546;
The classification of birth and death processes, Trans. Amer. Math. Soc. 86, 366-400.

[12] KATTI, S.K. (1967), Infinite divisibility of integer valued random variables, Ann. Math. Stat. 38, 1306-1308.

[13] KEILSON, J. and F.W. STEUTEL (1970), Families of infinitely divisible distributions closed under mixing and convolution, Center for System Science reprint series, Univ. of Rochester, Rochester, N.Y. (to be published).

[14] KENDALL, D.G. (1953), Stochastic processes occurring in the theory of queues and their analysis by the method of the embedded Markov chain, Ann. Math. Stat. 24, 338-354.

[15] KENDALL, M.G. and A. STUART (1958), The advanced theory of statistics, Vol. 1, Griffin, London.

[16] KINGMAN, J.F.C. (1966), On the algebra of queues, J. Appl. Prob. 3, 285-326.

[17] KINGMAN, J.F.C. (1966), Chapter 6 of W.L. Smith and W.E. Wilkinson (editors), Proceedings of the symposium on congestion theory, Chapel Hill.

[18] LÉVY, P. (1937), Théorie de l'addition des variables aléatoires, Gauthier-Villars, Paris.

[19] LINNIK, Yu. V. (1964), Decompositions of probability distributions, Oliver and Boyd, London.

[20] LOÈVE, M. (1960), Probability theory (2nd. ed.), Van Nostrand, Princeton.

[21] LORENTZ, G.G. (1953), Bernstein polynomials, Univ. of Toronto Press, Toronto.

[22] LUKACS, E. (1960), Characteristic functions, Griffin, London.

[23] LUKACS, E. (1968), Contributions to a problem of van Dantzig, Theory of Prob. and its Appl. 13, 114-125.

[24] LUKACS, E. and O. SZASZ (1954), Non-negative polynomials and certain rational characteristic functions, J. Research Nat. Bureau of Standards 52, 153-160.

[25] MILLER,H.D. (1967), A note on passage times and infinitely divisible distributions, J. Appl. Prob. 4, 402-405.

[26] PERRON, O. (1957), Die Lehre von den Kettenbrüchen, Band II (dritte Aufl.), Teubner, Stuttgart.

[27] POLYA, G. and G. SZEGÖ (1964), Aufgaben und Lehrsätze der Analysis I (dritte Aufl.), Springer, Berlin.

[28] SHANTARAM, R. and W.L. HARKNESS (1969), Convergence of a sequence of transformations of distribution functions, Pacif. J. of Math. 31, 403-415.

[29] SHARPE, M. (1969), Zeroes of infinitely divisible densities, Ann. Math. Stat. 40, 1503-1505.

[30] SHOHAT, J.A. and J.D. TAMARKIN (1963), The problem of moments, Mathematical Surveys No.1 (revised ed.), Amer. Math. Soc., Providence, R.I.

[31] SMITH, W.L. (1958), Renewal theory and its ramifications, J. Roy. Stat. Soc. Ser. B 20, 243-302.

[32] SMITH, W.L. (1967), A theorem on functions of characteristic functions and its applications to some renewal theoretic random walk problems, Proc. 5th. Berkeley Symp. on Math. Stat. and Prob. II, part 2, 265-310.

[33] STEUTEL, F.W. (1968), A class of infinitely divisible mixtures, Ann. Math. Stat. 39, 1153-1157.

[34] TITCHMARSH, E.C. (1939), The theory of functions (2nd. ed.), Oxford University Press, London.

[35] TUCKER, H.G. (1967), A graduate course in probability, Academic Press, New York.

[36] WHITTAKER, E.T. and G.N. WATSON (1958), Modern analysis (4th. ed.), Cambridge Univ. Press, London.

[37] WIDDER, D.V. (1946), The Laplace transform, Princeton Univ. Press, Princeton.

[38] WISHART, D.M.G. (1960), Queuing systems in which the discipline is "last-come, first-served". Operations Res. 8, 591-599.