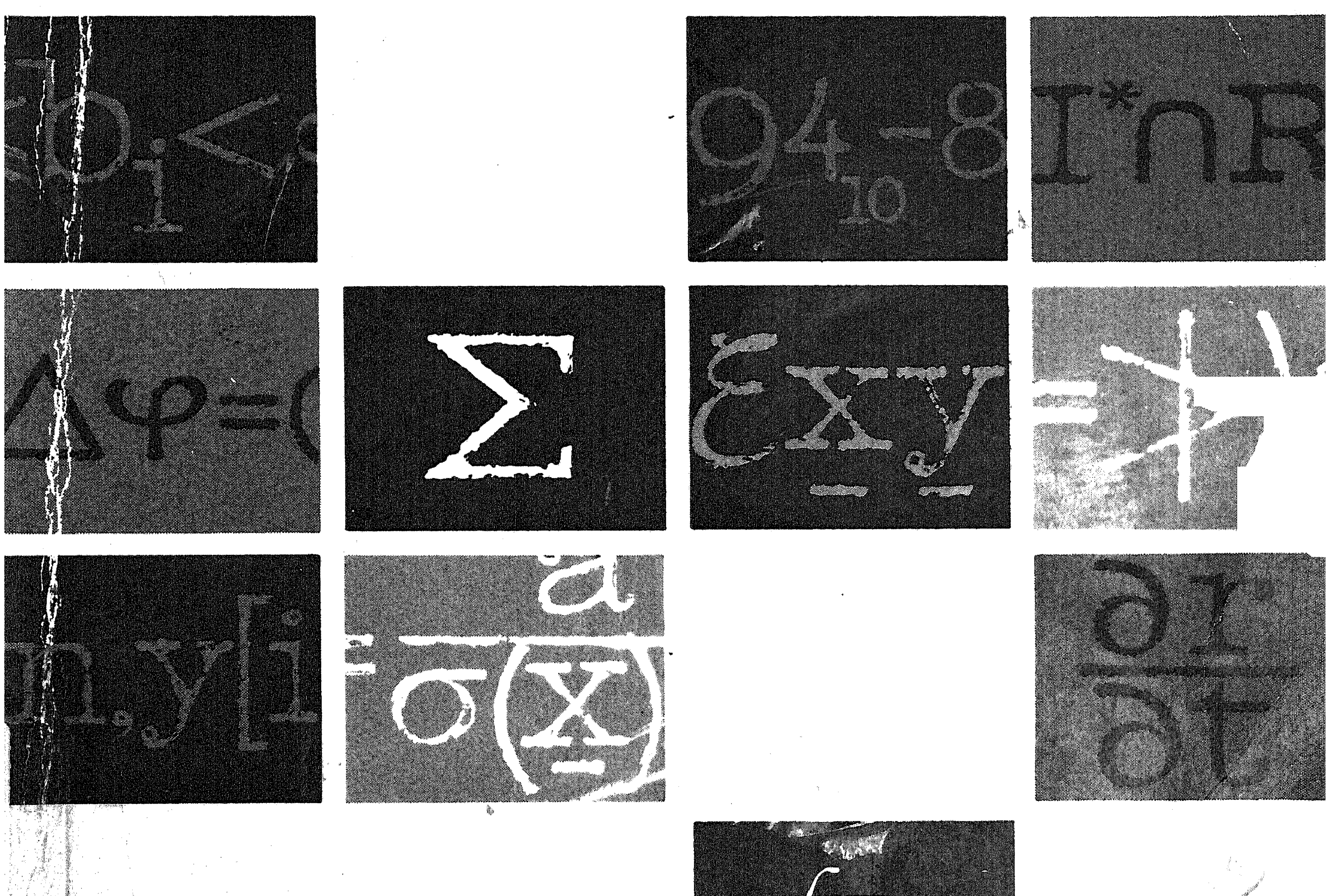
ON REGULAR VARIATION AND ITS APPLICATION TO THE WEAK CONVERGENCE OF SAMPLE EXTREMES

L. DE HAAN



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CONTENTS

The symbol $\square$ is used to indicate the end of a proof.

Introduction

In 1930 J. Karamata has developed the theory of regular variation for positive functions of a positive argument ([11] and [12]). The regularly varying functions of exponent ρ form a class of functions which behave in many respects like x^{ρ} near infinity. Karamata used his theory for an extension of certain Tauberian theorems.

Gradually it has become clear that the regularly varying functions play an important role in probability theory. W. Feller for example has exploited Karamata's results in the theory of stable distributions and their domains of attraction ([2] ch. XVII section 5; cf. [8] p. 175).

In the present work we use Karamata's ideas to derive well-known and new results in extreme value theory. Whereas the theory of regular variation as developed in [11] and [12] is sufficient for the theory of stable distributions, for extreme value theory we need a non-trivial extension of these results.

Our first chapter starts with a complete exposition of the basic theorems for regularly varying functions. After that several extensions are given which are needed for extreme value theory. These ought to lead to interesting results in other domains of application as well.

The second chapter deals with extreme value theory proper. We only consider the simplest model: the sequence of the partial maxima of a sequence of independent, identically distributed random variables. The possible non-degenerate limit distributions for these maxima belong to one of three classes: Φ_{α}, Ψ_{α} and Λ (see theorem 2.2.1). The first two classes were discovered by M. Fréchet [4]; the third one by R.A. Fischer and L.H.C. Tippett [3]. Sufficient conditions for the domains of attraction (see definition 2.2.1) of the limit distributions were given by R. von Mises [15]. B.V. Gnedenko [6] has established necessary and sufficient conditions for the domains of attraction of Φ_{α}, Ψ_{α} and Λ. Our second chapter starts with an exposition of these results, taken mainly from Gnedenko's work. As Gnedenko himself states, in the case of Λ these conditions can neither be regarded as final from a theoretical nor very satisfactory from a practical point of view. The main goal of our research was to find other necessary and sufficient conditions for the domain of attraction of Λ. To do so we had to develop the extension of regular variation described in chapter I. Our main result is contained in section 2.6 (theorems 2.6.1 and 2.6.2), where a unifying approach is given to the domains of attraction of all limit distributions.

The author is indebted to Professor Dr. J.Th. Runnenburg and Dr. H. Jager for their valuable advice and criticism.

CHAPTER I REGULAR VARIATION AND RELATED CONCEPTS

1.0 INTRODUCTION

This chapter is of a purely analytic character and serves as an introduction to chapter II where a problem in probability theory is considered.

First J. Karamata's theory of regularly varying functions is presented (sections 1.1 and 1.2). Detailed proofs are given, which in essence are due to Karamata. Different proofs of the main theorems can be found in W. Feller [2] p. 268-276. In our opinion they lack the elegance and simplicity of Karamata's method. We follow Karamata's second paper [12] but avoid an error in his treatment, due to the application of an incorrect theorem of Cauchy.

Next, after presenting two extensions of Karamata's main theorem (section 1.3), we study a subclass of the class of regularly varying functions with exponent 0 (section 1.4). By methods similar to those of Karamata we show the equivalence of a number of properties. The results (to be applied in chapter II) may be considered as a second order regular variation theory. In the last section (1.5) a complementary theory is given for functions which are inverses of those treated in section 1.4.

(added in proof) It should be noted that recently M. Marcus and M. Pinsky (J. Math. Anal. and Appl. 28 (1969) 440-449) have published results coïnciding with the part a) $\Leftrightarrow$ d) of theorem 1.4.1, corollary 1.4.1 a) and corollary 1.4.2 b) in the present work.

1.1 REGULAR, SLOW AND RAPID VARIATION

For $x > 0$ we adopt the following convention

$$x^{\infty} = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ \infty & \text{for } x > 1 \end{cases}$$

$$x^{-\infty} = \begin{cases} \infty & \text{for } x < 1 \\ 1 & \text{for } x = 1 \\ 0 & \text{for } x > 1 \; . \end{cases}$$

Definition 1.1.1 A function $U : \mathbb{R}^+ \to \mathbb{R}^+$ *) varies regularly at infinity if there exists a $\rho \in \mathbb{R}$ such that for all $x \in \mathbb{R}^+$

(1.1.1) $$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho}.$$

This number ρ is called the exponent of regular variation for U. In the particular case $\rho = 0$, U is often called slowly varying at infinity.

Definition 1.1.2 A function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies rapidly at infinity if for all $x \in \mathbb{R}^+$

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho},$$

where $\rho = +\infty$ or $\rho = -\infty$.

For brevity we shall also use the expression ρ-varying (at infinity) for functions satisfying definition 1.1.1 or definition 1.1.2 (hence $\rho \in \bar{\mathbb{R}}$).

Examples For all real ρ the functions

$$x^{\rho},\; x^{\rho}\log(1+x),\; (x\log(1+x))^{\rho},\; x^{\rho}\log\log(e+x), \ldots$$

are ρ-varying at infinity. The function

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*) $\mathbb{R}^+$ is the set of positive real numbers, $\mathbb{R}$ is the set of all real numbers and $\bar{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$.

$$\operatorname{arctg} x$$

is slowly varying at infinity and the functions

$$e^x, e^{-x}$$

are rapidly varying at infinity ($\rho = +\infty$ and $-\infty$ respectively).
The functions

$$2 + \sin x, \exp\{[\log x]\}$$

(where $[a]$ is the largest integer $\leq a$) are not ρ-varying at infinity.

Karamata [12] and Feller [2] have remarked that for monotonic functions and even for measurable functions definition 1.1.1 is unnecessarily restrictive for applications. The following theorems provide some alternatives.

Theorem 1.1.1 A Lebesgue-measurable function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies regularly if there exists a function $h : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive x

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = h(x). \tag{1.1.2}$$

Proof Obviously h is measurable as a pointwise limit of measurable functions. Writing

$$\frac{U(txy)}{U(t)} = \frac{U(txy)}{U(ty)} \cdot \frac{U(ty)}{U(t)}$$

and using (1.1.2) we see that for $x,y \in \mathbb{R}^+$

$$h(xy) = h(x).h(y). \tag{1.1.3}$$

It is well-known (see e.g. [9] p.116-118) that the only measurable, positive and finite-valued solutions of (1.1.3) on $\mathbb{R}^+$ are

$$h(x) = x^\rho$$

for some real ρ. □

For monotone functions the conditions of definition 1.1.1 can be replaced by the apparently weaker conditions of the next theorem.

Theorem 1.1.2 A monotone function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies regularly if there exist two positive integers m_1 and m_2 for which $\log m_1 (\log m_2)^{-1}$ is irrational and

$$(1.1.4) \qquad \lim_{n\to\infty} \frac{U(nm)}{U(n)} = m^\rho$$

for $m = m_1$ and $m = m_2$ where ρ is a real number; in (1.1.4) n runs through the positive integers.

Proof Suppose U is non-decreasing, then $\rho \geq 0$.

It is not difficult to see that the assumptions of the theorem imply that equation (1.1.4) holds for every integer m from the set

$$V = \{m_1^p m_2^q \mid p,q \text{ non-negative integers}\}.$$

Further it is well-known (and essentially stated in Kronecker's theorem, see [10] chapter XXIII) that if $\log m_1 (\log m_2)^{-1}$ is irrational, the set

$$\{m_1^r m_2^s \mid r,s \text{ integers}\}$$

is dense in $\mathbb{R}^+$. This means that for any pair of positive numbers x and ε we can find integers v_1, v_2 and v_3 from V such that

$$(1.1.5) \qquad x-\varepsilon < \frac{v_1}{v_2} < x < \frac{v_3}{v_2} < x+\varepsilon.$$

a) First we prove

$$(1.1.6) \qquad \lim_{n\to\infty} \frac{U(n+1)}{U(n)} = 1.$$

Suppose (1.1.6) is false. Then we can select a sequence $\{k_r\}$ of positive integers tending to infinity such that

$$\lim_{r\to\infty} \frac{U(k_r+1)}{U(k_r)} = c$$

with $1 < c \leq \infty$. In virtue of (1.1.5) with $x = 1$ and $\varepsilon \leq c^{1/\rho} - 1$ there exist integers v_4 and v_5 in V such that

$$1 < \frac{v_4}{v_5} < c^{1/\rho}.$$

Now take

$$n_r = [k_r . v_5^{-1}]$$

(the largest integer not exceeding $k_r . v_5^{-1}$), then for $n_r > v_5$

$$n_r v_5 \leq k_r < (n_r+1)\ v_5 < n_r(v_5+1) \leq n_r v_4 .$$

Since (1.1.4) holds for every element $m \in V$, we obtain the contradiction

$$c > \left(\frac{v_4}{v_5}\right)^\rho = \lim_{r\to\infty} \frac{U(n_r v_4)}{U(n_r v_5)} \geq \lim_{r\to\infty} \frac{U(k_r+1)}{U(k_r)} = c,$$

hence (1.1.6) is true.

b) In order to prove the assertion of the theorem, we use (1.1.5) for arbitrary x and $\varepsilon \in \mathbb{R}^+$ and define for $t > 0$

$$n_t = [t . v_2^{-1}]. \tag{1.1.7}$$

For all positive x and ε we have

$$\frac{U(n_t)}{U(n_t+1)} \cdot \frac{U(n_t v_1)}{U(n_t)} \cdot \frac{U(n_t+1)}{U((n_t+1)v_2)} \leq \frac{U(tx)}{U(t)} \leq$$
$$\leq \frac{U((n_t+1)v_3)}{U(n_t+1)} \cdot \frac{U(n_t)}{U(n_t v_2)} \cdot \frac{U(n_t+1)}{U(n_t)} . \tag{1.1.8}$$

Combining (1.1.4) (for arbitrary $m \in V$), (1.1.6), (1.1.7) and (1.1.8) we find

$$(x-\varepsilon)^\rho \leq \liminf_{t\to\infty} \frac{U(tx)}{U(t)} \leq \limsup_{t\to\infty} \frac{U(tx)}{U(t)} \leq (x+\varepsilon)^\rho .$$

Hence (1.1.1) holds.

For non-increasing U the proof is completely analogous.□

Theorem 1.1.2 can be applied to improve the formulation of a Tauberian theorem for power series (see [2] p. 423). An application to sequences of coefficients of attraction for partial sums of samples is given in [8].

In chapter 2 we shall need the following theorem on monotone functions (which is mainly lemma 2 p.270 of [2]). This theorem provides a criterion both for regular and for rapid variation.

Theorem 1.1.3 a) A monotone function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies regularly if and only if there exist two sequences $\{\lambda_n\}$ and $\{a_n\}$ of positive numbers with

$$(1.1.9)\qquad \begin{cases} \lim_{n\to\infty} \dfrac{\lambda_{n+1}}{\lambda_n} = 1 \\ \\ \lim_{n\to\infty} a_n = \infty \end{cases}$$

such that for all positive x

$$(1.1.10)\qquad \lim_{n\to\infty} \lambda_n\, U(a_n x) \text{ exists, is positive and finite.}$$

Moreover, if we define the function χ by

$$(1.1.11)\qquad \chi(x) = \lim_{n\to\infty} \lambda_n . U(a_n x) \quad \text{for } x > 0,$$

then we have

$$(1.1.12)\qquad \frac{\chi(x)}{\chi(1)} = x^\rho,$$

where ρ is the exponent of regular variation of U.

b) A monotone function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies rapidly with $\rho = \infty$ (or $\rho = -\infty$) if and only if there exist two sequences $\{\lambda_n\}$ and $\{a_n\}$ of positive numbers satisfying (1.1.9) and a $c \in \mathbb{R}^+$ such that

$$(1.1.13)\qquad \lim_{n\to\infty} \lambda_n\, U(a_n x) = \left(\frac{x}{c}\right)^\infty \text{ (or } \left(\frac{x}{c}\right)^{-\infty} \text{ respectively).}$$

<u>Proof</u> The assertions for a non-decreasing function U are equivalent to the same assertions for the non-increasing function $\frac{1}{U}$. Hence we may restrict ourselves to the case of a non-increasing U.

Furthermore we exclude the trivial case $U(\infty) > 0$ (then the assertions under a) hold with $\rho = 0$, $\lambda_n = 1$ and $a_n = n$).

(i) We first prove both "if" statements of the theorem. Define for $t > 0$ the integer $n = n(t)$ by

$$n = \min \{m \mid a_{m+1} > t\}.$$

Then

$$a_n \leq t < a_{n+1}$$

and for all $x,y \in \mathbb{R}^+$

$$\frac{U(a_{n+1}x)}{U(a_n y)} \leq \frac{U(tx)}{U(ty)} \leq \frac{U(a_n x)}{U(a_{n+1}y)} \; . \tag{1.1.14}$$

a) Suppose (1.1.10) holds. Using (1.1.9), (1.1.10) and (1.1.14) we see

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = \frac{\chi(x)}{\chi(1)}$$

for all positive x. Application of theorem 1.1.1 gives (1.1.12).

b) Suppose (1.1.13) holds. As U is non-increasing, we have $\rho = -\infty$. For $x > 1$ we choose b and d such that

$$0 < b < c < d < \infty$$

and

$$x = \frac{d}{b} \, ,$$

then by (1.1.9), (1.1.10) and (1.1.14)

$$0 \leq \lim_{t\to\infty} \frac{U(tx)}{U(t)} = \lim_{t\to\infty} \frac{U(tb^{-1}d)}{U(t)} = \lim_{t\to\infty} \frac{U(td)}{U(tb)} \leq$$

$$\leq \lim_{n\to\infty} \frac{\lambda_{n+1}}{\lambda_n} \cdot \frac{\lambda_n}{\lambda_{n+1}} \frac{U(a_n d)}{U(a_{n+1}b)} = 0 = x^{-\infty}.$$

For $0 < x < 1$ we proceed in an analogous way.

(ij) Next we prove both "only if" statements. Suppose U is regularly or rapidly varying at infinity. We define for positive y

$$V(y) = \inf \{x \mid U(x) \leq y\},$$

then

(1.1.15) $$U(V(y)+0) \leq y \leq U(V(y)-0)$$

and

(1.1.16) $$\frac{U(V(y)+0)}{U(V(y))} \leq \frac{y}{U(V(y))} \leq \frac{U(V(y)-0)}{U(V(y)} .$$

Furthermore we have

(1.1.17) $$\lim_{y \downarrow 0} V(y) = \infty.$$

a) Suppose U varies regularly with exponent ρ. As U is non-increasing, we have $\rho \leq 0$. For each $\varepsilon > 0$ and $x < 1$ there exists a $t(x,\varepsilon)$ with for $t \geq t(x,\varepsilon)$

$$1 \leq \frac{U(t-0)}{U(t)} \leq \frac{U(tx)}{U(t)} < x^{\rho} + \varepsilon,$$

hence

$$\lim_{t\to\infty} \frac{U(t-0)}{U(t)} = 1.$$

In an analogous way we see that

$$\lim_{t\to\infty} \frac{U(t+0)}{U(t)} = 1,$$

hence by (1.1.16) and (1.1.17)

$$\lim_{y \downarrow 0} \frac{U(V(y))}{y} = 1.$$

Choosing

$$(1.1.18)\qquad \begin{cases} \lambda_n = n \\ a_n = V(\frac{1}{n}) \end{cases}$$

we get for $x > 0$

$$\lim_{n\to\infty} \lambda_n . U(a_n x) = \lim_{n\to\infty} \lambda_n\, U(a_n).\frac{U(a_n x)}{U(a_n)} = x^{\rho}.$$

b) Suppose U varies rapidly, i.e. we now have $\rho = -\infty$. By (1.1.15) we find for $x > 1$

$$\liminf_{y\downarrow 0} \frac{y}{U(xV(y))} \geq \liminf_{y\downarrow 0} \frac{U(V(y)+0)}{U(xV(y))} \geq \liminf_{y\downarrow 0} \frac{U(x^{\frac{1}{2}}V(y))}{U(xV(y))} = \infty$$

and for $0 < x < 1$

$$\limsup_{y\downarrow 0} \frac{y}{U(xV(y))} \leq \limsup_{y\downarrow 0} \frac{U(V(y)-0)}{U(xV(y))} \leq \limsup_{y\downarrow 0} \frac{U(x^{\frac{1}{2}}V(y))}{U(xV(y))} = 0,$$

so that with the choice (1.1.18) we obtain for all positive $x \neq 1$

$$\lim_{n\to\infty} \lambda_n\, U(a_n x) = x^{-\infty}. \;\square$$

<u>Remark 1.1.1</u> The proof shows that for non-increasing ρ-varying functions (1.1.10) holds with the special choice

$$(1.1.19)\qquad \begin{cases} \lambda_n = n \\ a_n = \inf\{x \mid U(x) \leq \frac{1}{n}\}\ . \end{cases}$$

In this case

$$(1.1.20)\qquad \chi(x) = x^{\rho}.$$

<u>Remark 1.1.2</u> The proof also shows that for non-increasing functions the following weaker version of the "if" part of the theorem holds: A non-in-

creasing function $U : \mathbb{R}^+ \to \mathbb{R}^+$ varies regularly if for each $\varepsilon > 0$ there exist sequences $\{\lambda_n(\varepsilon)\}$ and $\{a_n(\varepsilon)\}$ of positive numbers with

$$(1.1.21) \qquad \begin{cases} \liminf\limits_{n\to\infty} \dfrac{\lambda_n(\varepsilon)}{\lambda_{n+1}(\varepsilon)} > 1-\varepsilon \\ \lim\limits_{n\to\infty} a_n(\varepsilon) = \infty, \end{cases}$$

such that for all positive x the expression

$$\lambda_n(\varepsilon).U(a_n(\varepsilon).x)$$

tends for $n \to \infty$ to a positive and finite value $\chi(x)$ not depending on ε.

1.2 KARAMATA'S THEOREM AND SOME CONSEQUENCES

In this section we shall assume all functions to be Lebesgue-summable on finite intervals unless otherwise stated. For this class of functions definition 1.1.1 can be put in two different but equivalent forms, due to Karamata. To do so, we first formulate three lemmas.

Lemma 1.2.1 a) Suppose that for positive functions f and g on $\mathbb{R}$

$$(1.2.1) \qquad \lim_{t\to\infty} \int_0^t f(s)ds = \lim_{t\to\infty} \int_0^t g(s)ds = \infty$$

and

$$(1.2.2) \qquad \lim_{t\to\infty} \frac{f(t)}{g(t)} = c \qquad \text{with } 0 \leq c \leq \infty.$$

Then
$$\lim_{t\to\infty} \frac{\int_0^t f(s)ds}{\int_0^t g(s)ds} = c.$$

b) Suppose that for positive functions f and g on $\mathbb{R}$ both

$$\int_0^\infty f(t)dt \quad \text{and} \quad \int_0^\infty g(t)dt$$

are finite and

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = c \qquad \text{with } 0 \le c \le \infty.$$

Then

$$\lim_{t\to\infty} \frac{\int_t^\infty f(s)ds}{\int_t^\infty g(s)ds} = c.$$

<u>Proof</u> a) Suppose first $0 \le c < \infty$. For each $\varepsilon > 0$ there exists a t_0 such that for $t \ge t_0$

$$(c-\varepsilon)\ g(t) < f(t) < (c+\varepsilon)\ g(t).$$

Then for $t > t_0$

$$c-\varepsilon = (c-\varepsilon)\frac{\int_{t_0}^t g(s)ds}{\int_{t_0}^t g(s)ds} < \frac{\int_{t_0}^t f(s)ds}{\int_{t_0}^t g(s)ds} < (c+\varepsilon)\frac{\int_{t_0}^t g(s)ds}{\int_{t_0}^t g(s)ds} = c+\varepsilon. \tag{1.2.3}$$

From (1.2.1) and (1.2.3) the statement of the lemma follows easily. For $c = \infty$ we only have to interchange the roles of f and g.

b) This part is proved in a similar way as part a). □

<u>Lemma 1.2.2</u> Suppose the function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is ρ-varying ($-\infty \le \rho \le \infty$).

a) If $\rho \ge -1$, the function

$$U_*(x) = \int_0^x U(t)dt \tag{1.2.4}$$

is $(\rho+1)$-varying.

b) If $\rho < -1$ or $\rho = -1$ and $\int_0^\infty U(t)dt < \infty$, the function

(1.2.5) $$U^*(x) = \int_x^\infty U(t)dt$$

is well-defined and $(\rho+1)$-varying.

<u>Proof</u> a) First we prove that for $\rho > -1$

(1.2.6) $$\lim_{x\to\infty} U_*(x) = \infty.$$

By definition 1.1.1 there exists an s_0 such that for $s \geq s_0$

$$U(2s) > 2^{-1} \,.\, U(s),$$

hence for $n > n_0$ with $2^{n_0} \geq s_0$

$$\int_{2^n}^{2^{n+1}} U(s)ds = 2\int_{2^{n-1}}^{2^n} U(2s)ds > \int_{2^{n-1}}^{2^n} U(s)ds.$$

And so

$$\int_{s_0}^\infty U(s)ds \geq \sum_{n=n_0+1}^\infty \int_{2^n}^{2^{n+1}} U(s)ds \geq \sum_{n=n_0+1}^\infty \int_{2^{n_0}}^{2^{n_0+1}} U(s)ds = \infty,$$

which proves (1.2.6).

Using (1.2.6) and lemma 1.2.1 a) (with $f(t) = x.U(tx)$ and $g(t) = U(t)$) we have for $\rho > -1$ and all $x > 0$

$$\lim_{t\to\infty} \frac{U_*(tx)}{U_*(t)} = \lim_{t\to\infty} \frac{\int_0^{tx} U(s)ds}{\int_0^t U(s)ds} = \lim_{t\to\infty} \frac{xU(tx)}{U(t)} = x^{\rho+1}.$$

If $\rho = -1$ and (1.2.6) holds, the same proof applies; if $\rho = -1$ and (1.2.6) does not hold, the function U_* is a trivial example of a slowly varying function and so the lemma holds in this case too.

b) We first prove for $\rho < -1$

(1.2.7) $$\int_0^\infty U(t)dt < \infty.$$

Choose $\varepsilon > 0$ such that $\rho+\varepsilon < -1$. By definition 1.1.1 there exists an s_0 such that for $s \geq s_0$

$$U(2s) < 2^{-1-\varepsilon}.U(s),$$

hence for $n \geq n_0$

$$\int_{2^n}^{2^{n+1}} U(s)ds = 2\int_{2^{n-1}}^{2^n} U(2s)ds < 2^{-\varepsilon}\int_{2^{n-1}}^{2^n} U(s)ds.$$

We now have (1.2.7), as

$$\int_{s_0}^{\infty} U(s)ds \leq \sum_{n=n_0}^{\infty}\int_{2^n}^{2^{n+1}} U(s)ds \leq \sum_{n=n_0}^{\infty} 2^{-\varepsilon(n-n_0)}\int_{2^{n_0}}^{2^{n_0+1}} U(s)ds < \infty.$$

Thus $U^*(x)$ is well-defined.

Applying lemma 1.2.1 b) with $f(t) = x.U(tx)$ and $g(t) = U(t)$ we see that U^* is $(\rho+1)$-varying at infinity. Finally, if $\rho = -1$ and U^* is well-defined, application of lemma 1.2.1 b) gives the result of the lemma. □

Lemma 1.2.3 If f is positive and absolutely continuous on $[a,b]$ $(0<a<b<\infty)$ then $\log f$ is absolutely continuous on $[a,b]$.

Proof For all $a \leq x_1 < y_1 \leq x_2 < y_2 \leq \ldots \leq x_n < y_n \leq b$ we have

$$\sum_{i=1}^{n} |\log f(y_i) - \log f(x_i)| = \sum_{i=1}^{n} \log\{1 + \frac{|f(y_i) - f(x_i)|}{\min(f(y_i),f(x_i))}\} \leq$$

$$\leq \sum_{i=1}^{n} \frac{|f(y_i) - f(x_i)|}{\min(f(y_i),f(x_i))} \leq \frac{\sum_{i=1}^{n} |f(y_i) - f(x_i)|}{\min\{f(x) \mid x\in[a,b]\}} . \quad \square$$

Now we are able to prove Karamata's main theorem.

Theorem 1.2.1 a) Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals and varies regularly with exponent ρ. If $\rho \geq -1$, then

$$\lim_{x\to\infty} \frac{x.U(x)}{\int_0^x U(t)dt} = \rho+1 \; ; \tag{1.2.8}$$

if $\rho < -1$, or $\rho = -1$ and $\int_0^{\infty} U(t)dt < \infty$, then

$$\lim_{x\to\infty} \frac{x.U(x)}{\int_x^{\infty} U(t)dt} = -\rho-1. \tag{1.2.9}$$

b) Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals. If

$$\lim_{x\to\infty} \frac{x.U(x)}{\int_0^x U(t)dt} = \lambda \tag{1.2.10}$$

with $0 < \lambda < \infty$, then U is $(\lambda-1)$-varying at infinity; if

$$\lim_{x\to\infty} \frac{x.U(x)}{\int_x^\infty U(t)dt} = \lambda \tag{1.2.11}$$

with $0 < \lambda < \infty$, then U is $(-\lambda-1)$-varying at infinity.

Proof We define for $x > 0$ the function b by

$$b(x) = \frac{x.U(x)}{\int_0^x U(t)dt} . \tag{1.2.12}$$

Integrating both sides of

$$\frac{b(x)}{x} = \frac{U(x)}{\int_0^x U(t)dt} ,$$

we find by lemma 1.2.3 for some real c_2 and all positive x

$$\int_1^x \frac{b(t)}{t} dt = \log \{\int_0^x U(t)dt\} + c_1$$

(as the derivatives of the two parts exist and are equal a.e.), hence

$$\int_0^x U(t)dt = c. \exp \{\int_1^x \frac{b(t)}{t} dt\} \tag{1.2.13}$$

where $c = e^{c_1}$ is a positive number. In view of (1.2.12) this yields

(1.2.14) $$U(x) = c \cdot \frac{b(x)}{x} \cdot \exp\left\{\int_1^x \frac{b(t)}{t}\,dt\right\} \qquad \text{for all } x > 0.$$

a) Let U be ρ-varying. First suppose $\rho \geq -1$. Using lemma 1.2.2 and the fact that products and quotients of regularly varying functions are regularly varying, we see that the function b varies slowly at infinity; we have to prove

(1.2.15) $$\lim_{x\to\infty} b(x) = \rho+1.$$

By Fatou's lemma

(1.2.16) $$\liminf_{x\to\infty} \{b(x)\}^{-1} = \liminf_{x\to\infty} \int_0^1 \frac{U(xt)}{U(x)}\,dt \geq \int_0^1 \liminf_{x\to\infty} \frac{U(xt)}{U(x)}\,dt = (\rho+1)^{-1}.$$

For $\rho = -1$ this gives (1.2.15). For $\rho > -1$ we proceed as follows. From (1.2.16) we see that there exists an x_0 such that $b(x)$ is bounded on $[x_0,\infty)$. From the slow variation of b we have for $t > 0$

$$\lim_{x\to\infty} \frac{b(xt) - b(x)}{b(x)} = 0$$

and by the boundedness of b

$$\lim_{x\to\infty} \{ b(xt) - b(x)\} = 0.$$

Applying Lebesgue's theorem on dominated convergence we get

(1.2.17) $$\lim_{x\to\infty}\left\{\int_1^s \frac{b(xt)}{t}\,dt - b(x).\log s\right\} = \lim_{x\to\infty} \int_1^s \frac{b(xt)-b(x)}{t}\,dt = 0$$

for $s > 0$.

On the other hand we see from (1.2.13) and lemma 1.2.2, that $\exp\{\int_1^x \frac{b(t)}{t}\,dt\}$ is $(\rho+1)$-varying. Hence for $s > 0$

$$(1.2.18) \qquad \lim_{x\to\infty} \int_1^s \frac{b(xt)}{t}\,dt = \lim_{x\to\infty} \log \left\{ \frac{\exp\left(\int_1^{sx} \frac{b(t)}{t}\,dt\right)}{\exp\left(\int_1^{x} \frac{b(t)}{t}\,dt\right)} \right\} = (\rho+1).\log s.$$

Combining (1.2.17) and (1.2.18), we obtain (1.2.15).

Next suppose $\rho < -1$ (or $\rho = -1$ and $\int_0^\infty U(t)dt < \infty$). Define for $x > 0$

$$b_1(x) = \frac{x.U(x)}{\int_x^\infty U(t)dt} .$$

In an analogous way as before we find

$$(1.2.19) \qquad U(x) = c.\,\frac{b_1(x)}{x}\ \exp\left\{ -\int_1^x \frac{b_1(t)}{t}\,dt\right\} \qquad \text{for all } x > 0.$$

The rest of the proof is practically the same as for $\rho > -1$.

b) Suppose (1.2.10) holds. For each $\varepsilon > 0$ and $s_0 > 0$ there exists a t_0 such that for $t \geq t_0$ and $s \geq s_0$

$$(1.2.20) \qquad \lambda-\varepsilon < b(ts) < \lambda+\varepsilon,$$

where b is defined by (1.2.12). Using the representation (1.2.14) we have for $x > 0$

$$\frac{U(tx)}{U(t)} = \frac{b(tx)}{x.b(t)}\ \exp\left\{\int_1^x \frac{b(ts)}{s}\,ds\right\}.$$

Using (1.2.20) we see that this quantity tends to $x^{\lambda-1}$.

Finally suppose (1.2.11) holds. With the aid of (1.2.19) we can prove in a similar way that U is $(-\lambda-1)$-varying. □

Remark 1.2.1 If U(x) is ρ-varying, theorem 1.2.1 can be applied to $U_1(x) = x^\alpha U(x)$ and we obtain assertions of the type

$$\lim_{x\to\infty} \frac{x^{\alpha+1}U(x)}{\int_0^x t^\alpha U(t)dt} = \alpha+\rho+1 \qquad \text{if } \alpha \geq -\rho-1.$$

Lemma 1.2.2 can also be rewritten in this way.

The proof of theorem 1.2.1 yields the following representation theorem, due to Karamata.

Theorem 1.2.2 a) If a function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is Lebesgue-summable on finite intervals and regularly varying with exponent ρ, then there exist functions $a : \mathbb{R}^+ \to \mathbb{R}$ and $c : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$(1.2.21) \qquad \begin{cases} \lim_{x\to\infty} c(x) = c_0 & (0 < c_0 < \infty) \\ \lim_{x\to\infty} a(x) = \rho, & \end{cases}$$

such that for all positive x

$$(1.2.22) \qquad U(x) = c(x) \exp \{ \int_1^x \frac{a(t)}{t} dt \} .$$

b) Every function of the form (1.2.20) where the auxiliary functions c and a satisfy (1.2.19) with finite or infinite ρ, is ρ-varying.

Proof a) Using (1.2.14) we have for $\rho > -1$

$$U(x) = c.b(x) \exp \{ \int_1^x \frac{b(t)-1}{t} dt \} \qquad \text{for all } x > 0,$$

which in connection with (1.2.15) gives the desired representation. For $\rho < -1$ we use (1.2.19) to get

$$U(x) = c.b_1(x) \exp \{ \int_1^x \frac{-b_1(t)-1}{t} dt \} \qquad \text{for all } x > 0.$$

If $\rho = -1$, $xU(x)$ is a slowly varying function for which (1.2.22) holds. Then

$$U(x) = \frac{c(x)}{x} \exp \{ \int_1^x \frac{a(t)}{t} dt \} = c(x) \exp \{ \int_1^x \frac{a(t)-1}{t} dt \}$$

for all $x > 0$.

b) Suppose (1.2.22) holds with (1.2.21). In a similar way as in the proof of part b) of theorem 1.2.1, we can prove that U is ρ-varying. □

Remark 1.2.2 Theorem 1.2.2 is still true if we replace the condition that U is summable on finite intervals by the condition that U is measurable (see [1]). Theorem 1.2.1 then holds with (1.2.8) replaced by: "there exists an x_0 such that U is summable on finite subintervals of $[x_0,\infty)$ and

$$\lim_{x\to\infty} \frac{xU(x)}{\int_{x_0}^x U(t)dt} = \rho + 1".$$

Remark 1.2.3 A slight adaption of the proof of theorem 1.2.1 shows that in relation (1.2.20) we may take for $\rho > 0$

$$a(x) = c_0 . c(x) = \frac{U(x)}{\int_0^x \frac{U(t)}{t} dt} \qquad \text{for all } x > 0$$

(provided the integral converges at $t = 0$) and for $\rho < 0$

$$- a(x) = c_0 . c(x) = \frac{U(x)}{\int_x^\infty \frac{U(t)}{t} dt} \qquad \text{for all } x > 0.$$

We conclude this section by formulating 8 properties of regularly varying functions, which are then proved in the same order. It will become clear from the proofs, that most of these properties are consequences of theorem 1.2.2. Most of the properties are taken from Karamata's first paper [11]. We recall that all functions are $\mathbb{R}^+ \to \mathbb{R}^+$ and summable on finite intervals.

Corollary 1.2.1

1. If U is ρ-varying at infinity ($-\infty<\rho<\infty$), then

$$\lim_{x\to\infty} \frac{\log U(x)}{\log x} = \rho$$

and hence

$$\lim_{x\to\infty} U(x) = \begin{cases} 0 & \text{if } \rho < 0 \\ \infty & \text{if } \rho > 0. \end{cases}$$

2. If U is ρ-varying at infinity ($-\infty<\rho<\infty$), then for all sequences $\{a_n\}$ and $\{a'_n\}$ of positive numbers with

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} a'_n = \infty \text{ and}$$

$$\lim_{n\to\infty} \frac{a_n}{a'_n} = c \quad (0<c<\infty),$$

we have

$$\lim_{n\to\infty} \frac{U(a_n)}{U(a'_n)} = c^{\rho}.$$

If $\rho \neq 0$ ($-\infty<\rho<\infty$) the conclusion is also true for $c = 0$ and $c = \infty$. If $\rho = \pm\infty$, the conclusion is true for monotone functions U and $c \neq 1$ ($0\leq c\leq\infty$).

3. If U_1 and U_2 vary regularly at infinity with exponents ρ_1 and ρ_2 respectively and

$$\lim_{x\to\infty} U_2(x) = \infty,$$

then

$$U(x) = U_1(U_2(x))$$

is $(\rho_1 . \rho_2)$-varying at infinity.

4. If U is ρ-varying at infinity ($-\infty<\rho<\infty$), the relation

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\rho}$$

holds uniformly on intervals of the form (x_0, x_1) with $0 < x_0 < x_1 < \infty$. If $\rho < 0$, the restriction $x_1 < \infty$ can be dropped. If $\rho > 0$ and U is bounded on bounded intervals, we may take $x_0 = 0$.

5. If U is non-decreasing and ρ-varying at infinity $(0 \leq \rho \leq \infty)$ and if $U(\infty) = \infty$, the function U^* defined by

$$U^*(x) = \inf \{y \mid U(y) \geq x\} \tag{1.2.23}$$

is (ρ^{-1})-varying at infinity. If U is non-increasing and ρ-varying at infinity $(-\infty \leq \rho \leq 0)$ and if $U(\infty) = 0$, the function U^{**} defined by

$$U^{**}(x) = \inf \{y \mid U(y) \leq \frac{1}{x}\}$$

is $(-\rho^{-1})$-varying at infinity.

6. Suppose U_1 and U_2 are non-decreasing and ρ-varying at infinity $(0<\rho<\infty)$. For $0 \leq c \leq \infty$ we have

$$\lim_{x\to\infty} \frac{U_1(x)}{U_2(x)} = c \tag{1.2.24}$$

if and only if

$$\lim_{x\to\infty} \frac{U_1^*(x)}{U_2^*(x)} = c^{-1/\rho} \tag{1.2.25}$$

where U_1^* and U_2^* are defined as in (1.2.23). For $-\infty < \rho < 0$ an analogous result applies for non-increasing functions.

7. Every regularly varying function with exponent $\rho \neq 0$ is asymptotic to a strictly monotone regularly varying function with the same exponent.

8. Suppose that U is ρ-varying at infinity $(-\infty<\rho<\infty)$ and that there exists a monotone function u such that for all positive x

$$U(x) = \int_0^x u(t)\, dt,$$

then

$$\lim_{x\to\infty} \frac{x.u(x)}{U(x)} = \rho.$$

Hence for $\rho \neq 0$ the function $(\mathrm{sgn}\ \rho).u(x)$ is $(\rho-1)$-varying at infinity.

Proof of corollary 1.2.1

1. Property 1 follows directly from the representation (1.2.22).

2. The statements concerning ρ-varying functions with finite ρ can be verified directly with the aid of the representation (1.2.22). The assertion concerning monotone rapidly varying functions is proved in the following way. We consider only $\rho = \infty$ and $1 < c \leq \infty$. Take c_1 such that $1 < c_1 < c$, then for $n \geq n_0$ we have

$$a_n \geq c_1.a_n',$$

thus as U is non-decreasing,

$$\liminf_{n\to\infty} \frac{U(a_n)}{U(a_n')} \geq \liminf_{n\to\infty} \frac{U(a_n'.c^{\frac{1}{2}})}{U(a_n')} = \infty.$$

3. We have to prove that for each $x > 0$ and each sequence $\{t_n\}$ with $t_n \to \infty$ for $n \to \infty$

$$\lim_{n\to\infty} \frac{U_1(U_2(t_n x))}{U_1(U_2(t_n))} = x^{\rho_1\rho_2}.$$

This is easily done using property 2 with $a_n = U_2(t_n x)$ and $a_n' = U_2(t_n)$.

4. The statement to be proved is equivalent to the statement that for all sequences $t_n \to \infty$ and $x_n \to x$ $(x_0 \leq x \leq x_1)$

$$\lim_{n\to\infty} \frac{U(t_n x_n)}{U(t_n)} = x^\rho.$$

Take first $x_0 > 0$ and $x_1 < \infty$. Application of property 2 (take $a_n = t_n x_n$ and $a_n' = t_n$) provides the proof.

Next take $\rho < 0$, $x_0 > 0$ and $x_1 = \infty$. As for $\rho \neq 0$ property 2 also holds with $c = \infty$, the same proof goes through.

Finally take $\rho > 0$, $x_0 = 0$ and $x_1 < \infty$. We may assume that the sequence $\{t_n x_n\}$ converges to some value a with $0 \leq a \leq \infty$. For $a = \infty$ again application of property 2 provides the proof. For $a < \infty$ we have $x = 0$; in this case the sequence $\{U(t_n x_n)\}$ is bounded and hence by property 1

$$0 \leq \limsup_{n\to\infty} \frac{U(t_n x_n)}{U(t_n)} = 0 = x^\rho .$$

5.*) We give the proof of the first statement concerning non-decreasing functions U ($0 \leq \rho \leq \infty$); the second assertion can easily be reduced to the first one.

Suppose the statement is not true. Then there exist a positive x ($x \neq 1$) and a sequence $t_n \to \infty$ such that for a certain $c \neq x^{1/\rho}$ ($0 \leq c \leq \infty$)

$$\lim_{n\to\infty} \frac{U^*(t_n x)}{U^*(t_n)} = c.$$

From (1.2.23) we have for $y > 0$

$$U(U^*(y)-0) \leq y \leq U(U^*(y)+0),$$

hence

$$\frac{U(U^*(t_n x)-1)}{U(U^*(t_n)+1)} \leq \frac{U(U^*(t_n x)-0)}{U(U^*(t_n)+0)} \leq \frac{t_n x}{t_n} \leq \frac{U(U^*(t_n x)+0)}{U(U^*(t_n)-0)} \leq \frac{U(U^*(t_n x)+1)}{U(U^*(t_n)-1)} \tag{1.2.26}$$

For any ρ we may apply property 2 with $a_n = U^*(t_n x) + 1$ and $a'_n = U^*(t_n) - 1$ and also with $a_n = U^*(t_n x) - 1$ and $a'_n = U^*(t_n) + 1$ (as $\rho = 0$ leads to $x^{1/\rho} = 0$ or ∞ and hence $0 < c < \infty$, and $\rho = \infty$ leads to $x^{1/\rho} = 1$ and hence $c \neq 1$). Doing so we find

$$\lim_{n\to\infty} \frac{U(U^*(t_n x)+1)}{U(U^*(t_n)-1)} = \lim_{n\to\infty} \frac{U(U^*(t_n x)-1)}{U(U^*(t_n)+1)} = \lim_{n\to\infty} \frac{U(U^*(t_n x))}{U(U^*(t_n))} = c^\rho .$$

With the inequalities (1.2.26) this gives

$$x = c^\rho ,$$

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*) W. Vervaat: personal communication.

a contradiction.

6. We only consider non-decreasing functions U_1 and U_2.

a) Suppose (1.2.24) is true and (1.2.25) does not hold. Then there exists a sequence $x_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{U_1^*(x_n)}{U_2^*(x_n)} = b \neq c^{-1/\rho}.$$

By property 2 then

$$\lim_{n\to\infty} \frac{U_1(U_1^*(x_n))}{U_1(U_2^*(x_n))} = b^\rho \neq c^{-1};$$

but on the other hand by assumption

$$\lim_{n\to\infty} \frac{U_1(U_1^*(x_n))}{U_1(U_2^*(x_n))} = \lim_{n\to\infty} c^{-1} \cdot \frac{U_1(U_1^*(x_n))}{U_2(U_2^*(x_n))} =$$

$$= c^{-1} \frac{\lim_{n\to\infty} U_1(U_1^*(x_n))/x_n}{\lim_{n\to\infty} U_2(U_2^*(x_n))/x_n} = c^{-1}$$

(the last limits are equal to 1, see part 2a) of the proof of theorem 1.1.3). Hence by contradiction the first part of the proof is complete.

b) The converse statement is proved in an analogous way using the fact that

$$U(x) \sim \inf \{y \mid U^*(y) \geq x\} \qquad \text{for } x \to \infty.$$

7. This follows immediately from the representation (1.2.22), as $a(x)$ is positive for sufficiently large x.

8. This property will be proved in section 2.7 (theorem 2.7.1) in a more convenient context. □

1.3 RELATED RESULTS

In this section two extensions of the main theorem of the previous section (theorem 1.2.1) are presented.

First we show that in the case of monotonic functions U the part of theorem 1.2.1 with $-\infty < \rho < -1$ also holds for $\rho = -\infty$. In view of an application in chapter II (section 2.9) we prove the (now stronger) result of remark 1.2.1.

<u>Theorem 1.3.1</u> Suppose the function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is non-increasing.

a) Let U be $-\infty$-varying at infinity. Then for all real α

$$\int_1^\infty t^\alpha \, U(t) \, dt < \infty$$

and

$$\lim_{x\to\infty} \frac{x^{\alpha+1} \, U(x)}{\int_x^\infty t^\alpha U(t) dt} = \infty. \tag{1.3.1}$$

b) If for some real α the integral $\int_1^\infty t^\alpha \, U(t) \, dt$ converges and (1.3.1) holds, U is $-\infty$-varying at infinity.

<u>Proof</u> a) It suffices to prove the statements for larger α, say $\alpha > -1$ because

$$\frac{x^{\alpha+1} . U(x)}{\int_x^\infty t^\alpha U(t) dt} = \{ \int_1^\infty t^\alpha \frac{U(tx)}{U(x)} \, dt \}^{-1}$$

is a non-decreasing function of α for fixed $x > 0$.

Choose $\varepsilon > 0$ and $\lambda(\varepsilon)$ such that $1 < \lambda < 2^{\frac{1}{\alpha+1}}$ and

$$\frac{1}{\alpha+1} \, \frac{\lambda^{\alpha+1}-1}{1-\frac{1}{2}\lambda^{\alpha+1}} < \varepsilon.$$

We write

$$(1.3.2) \qquad \int_x^\infty t^\alpha\, U(t)\, dt = \sum_{n=0}^{\infty} \int_{\lambda^n . x}^{\lambda^{n+1} . x} t^\alpha\, U(t)\, dt \leq$$

$$\leq \sum_{n=0}^{\infty} U(\lambda^n x)\, (\alpha+1)^{-1}\, x^{\alpha+1}\, \lambda^{n(\alpha+1)}\, (\lambda^{\alpha+1}-1).$$

There exists an $x_0(\lambda)$ such that for $x \geq x_0$

$$U(\lambda x) < 2^{-1} \,.\, U(x),$$

hence also for $n = 1, 2, \ldots$

$$U(\lambda^n x) < 2^{-1} \,.\, U(\lambda^{n-1} x)$$

and (by repeated application)

$$U(\lambda^n x) < 2^{-n} \,.\, U(x).$$

With the aid of this inequality, (1.3.2) takes the form (as $2^{-1}\, \lambda^{\alpha+1} < 1$)

$$\int_x^\infty t^\alpha\, U(t) dt \leq x^{\alpha+1}\, U(x).(\alpha+1)^{-1}\, (\lambda^{\alpha+1}-1) \sum_{n=0}^{\infty} (2^{-1}\lambda^{\alpha+1})^n =$$

$$= x^{\alpha+1}\, U(x)\, (\alpha+1)^{-1}\, \frac{\lambda^{\alpha+1}-1}{1-2^{-1}\lambda^{\alpha+1}}.$$

So the convergence of the integral is ensured and

$$\frac{\int_x^\infty t^\alpha U(t) dt}{x^{\alpha+1} . U(x)} < \varepsilon.$$

b) Suppose that (1.3.1) holds for some real α and U is not $-\infty$-varying. Then there exists a positive number $x \neq 1$, a sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = \infty$ and a certain c ($0 \leq c \leq \infty$) with $c \neq x^{-\infty}$ such that

$$\lim_{n\to\infty} \frac{U(t_n x)}{U(t_n)} = c;$$

we may take the sequence t_n such that $1 < x < \infty$, then $0 < c \leq \infty$.

By Fatou's lemma we have in view of the monotonicity of $U(t_n s)$. $\{U(t_n)\}^{-1}$ as a function of s

$$\liminf_{n\to\infty} \int_1^\infty s^\alpha \frac{U(t_n s)}{U(t_n)} ds \geq \int_1^\infty \liminf_{n\to\infty} s^\alpha \frac{U(t_n s)}{U(t_n)} ds \geq$$

$$\geq \int_1^x s^\alpha \liminf_{n\to\infty} \frac{U(t_n x)}{U(t_n)} ds = c \, . \int_1^x s^\alpha ds > 0.$$

This contradicts (1.3.1). □

For completeness sake we quote without proof the corresponding result for ∞-varying functions.

Theorem 1.3.2 Suppose the function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is non-decreasing.

a) Let U be ∞-varying at infinity. Then for all real α for which the integral $\int_0^1 t^\alpha U(t) dt$ converges, we have

(1.3.3) $$\lim_{x\to\infty} \frac{x^{\alpha+1}.U(x)}{\int_0^x t^\alpha U(t)dt} = \infty.$$

b) If for some real α the integral $\int_0^1 t^\alpha U(t) dt$ is finite and (1.3.3) holds, then U is ∞-varying at infinity.

Next we turn back to regularly varying functions and treat a property which (as the properties of theorem 1.2.1 and theorem 1.2.2) is equivalent to regular variation. An alternative formulation given afterwards, serves as an introduction to the theory of section 1.4.

Theorem 1.2.1 states that (take e.g. $\rho > -1$)

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho \qquad \text{for all } x > 0$$

if and only if

$$\lim_{t\to\infty} \int_0^1 \frac{U(tx)}{U(t)} dx = \int_0^1 x^\rho dx.$$

The next theorem contains a similar property.

Theorem 1.3.3 Suppose $U : \mathbb{R}^+ \to \mathbb{R}^+$ is such that U and log U are summable on finite intervals. U is ρ-varying at infinity ($\rho \in \mathbb{R}$) if and only if

$$\lim_{t\to\infty} \int_0^1 \log \{\frac{U(tx)}{U(t)}\} \, dx = \int_0^1 \log x^\rho \, dx.$$

This theorem is contained in the next one. We only prove the latter theorem.

Theorem 1.3.4 Suppose the function $V : \mathbb{R}^+ \to \mathbb{R}$ is such that V and exp(V) are summable on finite intervals. Let ρ be a real constant. Then the following assertions are equivalent.

a) For every $x > 0$

(1.3.4) $$\lim_{t\to\infty} \{V(tx) - V(t)\} = \rho . \log x.$$

b)

(1.3.5) $$\lim_{x\to\infty} \{V(x) - \frac{1}{x} \int_1^x V(t) \, dt\} = \rho.$$

c) There exist real functions c and a and a real constant c_0 with

(1.3.6) $$\begin{cases} \lim_{x\to\infty} c(x) = c_0 \\ \lim_{x\to\infty} a(x) = \rho \end{cases}$$

such that

(1.3.7) $$V(x) = c(x) + \int_1^x \frac{a(t)}{t} \, dt.$$

Proof Relation (1.3.4) holds if and only if the function U defined by

$$U(x) = \exp \{V(x)\}$$

is ρ-varying at infinity. Hence the equivalence of a) and c) is contained in theorem 1.2.2.

c) $\Rightarrow$ b): Suppose c) is true, then

$$V(x) - \frac{1}{x}\int_1^x V(t)\,dt = c(x) - \frac{1}{x}\int_1^x c(t)\,dt + \int_1^x \frac{a(t)}{t}\,dt +$$

$$-\frac{1}{x}\int_1^x \int_1^t \frac{a(s)}{s}\,ds\,dt.$$

Using Fubini's theorem the last term becomes

$$\int_1^x \frac{a(s)}{s}\,ds - \frac{1}{x}\int_1^x a(s)\,ds.$$

As (1.3.6) implies

$$\begin{cases} \lim\limits_{x\to\infty} \frac{1}{x}\int_1^x c(t)\,dt = c_0 \\ \lim\limits_{x\to\infty} \frac{1}{x}\int_1^x a(t)\,dt = \rho, \end{cases}$$

the assertion (1.3.5) follows easily.

b) $\Rightarrow$ c): For positive x we define

(1.3.8) $$g(x) = V(x) - \frac{1}{x}\int_1^x V(t)\,dt.$$

Then

$$\int_1^x \frac{g(t)}{t}\,dt = \int_1^x \frac{V(t)}{t}\,dt - \int_1^x \int_1^t \frac{V(s)}{t^2}\,ds\,dt.$$

Using Fubini's theorem the latter term becomes

$$-\frac{1}{x}\int_1^x V(s)\,ds + \int_1^x \frac{V(s)}{s}\,ds,$$

so

$$\int_1^x \frac{g(t)}{t}\,dt = \frac{1}{x}\int_1^x V(s)\,ds = V(x) - g(x)$$

or

$$V(x) = g(x) + \int_1^x \frac{g(t)}{t}\,dt. \tag{1.3.9}$$

As $\lim_{x\to\infty} g(x) = \rho$, the representation (1.3.7) is established. □

Remark 1.3.1 The transformations (1.3.8) and (1.3.9) provide a linear one-to-one correspondence between functions V satisfying (1.3.4) and functions g with a limit ρ for $x \to \infty$.

1.4 A SUBCLASS OF THE SLOWLY VARYING FUNCTIONS

In the previous sections on the theory of regularly varying functions we were interested in functions U for which the behaviour of $U(tx)$ and $U(ty)$ (with finite positive x and y) for $t \to \infty$ does not differ too much, i.e. for which $U(tx)\,\{U(ty)\}^{-1}$ tends to a finite and positive limit.

The definition of slow variation for a function U can be written as

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{U(t)} = 0 \qquad \text{for each positive } x.$$

Now we confine our considerations to strictly increasing functions and ask for properties of the class of functions for which the behaviour of

$$\frac{U(tx)-U(t)}{U(t)}$$

and

$$\frac{U(ty)-U(t)}{U(t)}$$

$(x,y > 1)$ does not differ too much for $t \to \infty$, i.e. for which

$$\frac{U(tx)-U(t)}{U(t)} \cdot \left\{\frac{U(ty)-U(t)}{U(t)}\right\}^{-1} = \frac{U(tx)-U(t)}{U(ty)-U(t)}$$

tends to a finite and positive limit $\psi(x,y)$. Obviously $\psi(x,y)$ is non-decreasing in x and non-increasing in y.

We ask which functions ψ can occur. In order to give a definite answer we found it necessary to impose the restriction that ψ is strictly increasing in x and strictly decreasing in y.

Take x_1, x_2, $y > 1$, then

$$\frac{U(tx_1x_2)-U(tx_2)}{U(tx_1)-U(t)} = \left\{\frac{U(tx_1)-U(t)}{U(ty)-U(t)}\right\}^{-1} \cdot \left\{\frac{U(tx_1x_2)-U(t)}{U(ty)-U(t)} - \frac{U(tx_2)-U(t)}{U(ty)-U(t)}\right\}.$$

Taking the limit $t \to \infty$ on both sides we obtain

$$\text{(1.4.1)} \qquad \lim_{t\to\infty} \frac{U(tx_1x_2)-U(tx_2)}{U(tx_1)-U(t)} = \frac{\psi(x_1x_2,y)-\psi(x_2,y)}{\psi(x_1,y)} .$$

As the righthand member is positive and finite for all x_1, x_2, $y > 1$, by theorem 1.1.1 it follows that the function $f_{x_1}(t)$ defined by

$$f_{x_1}(t) = U(tx_1) - U(t)$$

varies regularly at infinity for all $x_1 > 1$. Hence for some real ρ

$$\text{(1.4.2)} \qquad \psi(x_1x_2,y) = x_2^\rho\, \psi(x_1,y) + \psi(x_2,y).$$

Suppose first $\rho \neq 0$. For reasons of symmetry we also have

$$\text{(1.4.3)} \qquad \psi(x_1x_2,y) = \psi(x_1,y) + x_1^\rho\, \psi(x_2,y).$$

From (1.4.2) and (1.4.3) we get that the function $(1-x^\rho)^{-1}\, \psi(x,y)$ does not depend on x. As clearly

$$\text{(1.4.4)} \qquad \psi(x,y) = \{\psi(y,x)\}^{-1},$$

we have $\quad \psi(x,y) = (1-x^\rho)\,(1-y^\rho)^{-1}$, i.e.

$$(1.4.5)\qquad \lim_{t\to\infty} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{1-x^{\rho}}{1-y^{\rho}} \qquad \text{for } x, y > 1.$$

It can be proved that for positive ρ (1.4.5) is equivalent to ρ-variation of U. For negative ρ (1.4.5) implies that U(x) tends to a finite limit $U(\infty)$ for $x \to \infty$; then (1.4.5) is equivalent to ρ-variation of the function $U(\infty) - U(x)$. These observations show that under the present conditions we are dealing with functions we have met before.

Now suppose (1.4.2) holds with $\rho = 0$, i.e. for $x_1, x_2, y > 1$

$$(1.4.6)\qquad \psi(x_1x_2,y) = \psi(x_1,y) + \psi(x_2,y).$$

It is not difficult to see that this relation holds for all positive x_1, x_2 and y ($y\neq 1$). For each fixed $y \neq 1$ the only measurable and finite-valued solution of (1.4.6) on $\mathbb{R}^+$ is (see [9] p. 116-118)

$$\psi(x,y) = c \,.\, \log x$$

for some real constant c depending on y. Using (1.4.4) we easily obtain

$$\psi(x,y) = \frac{\log x}{\log y} .$$

For this function ψ we have the following theorem (see also theorem 1.3.4).

<u>Theorem 1.4.1</u> For a strictly increasing function $U: \mathbb{R}^+ \to \mathbb{R}$ the following assertions are equivalent.

a) For every positive x and y ($y\neq 1$)

$$(1.4.7)\qquad \lim_{t\to\infty} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{\log x}{\log y} .$$

b) The function

$$(1.4.8)\qquad U(x) - \frac{1}{x}\int_1^x U(t)dt$$

is slowly varying at infinity.

c)

$$\lim_{x\to\infty} \frac{x \int_1^x U(t)dt - 2 \int_1^x \left(\int_1^y U(t)dt\right)dy}{x^2\, U(x) - x \cdot \int_1^x U(t)dt} = \tfrac{1}{2} .$$

d) There exist a slowly varying function g and a real constant c such that

$$U(x) = c + g(x) + \int_1^x \frac{g(t)}{t}\, dt. \tag{1.4.9}$$

e) For each positive x

$$\lim_{x\to\infty} \frac{U(tx) - U(t)}{U(t) - \frac{1}{t} \int_1^t U(s)ds} = \log x. \tag{1.4.10}$$

<u>Proof</u> a) $\Rightarrow$ b): Writing $(y>0,\ 0<x<1)$

$$\frac{U(ty)-U(txy)}{U(t)-U(tx)} = \frac{U(t)-U(txy)}{U(t)-U(tx)} - \frac{U(t)-U(ty)}{U(t)-U(tx)}$$

and using (1.4.7) we see that the function

$$h(t) = U(t) - U(tx)$$

is slowly varying at infinity for each $0 < x < 1$. By theorem 1.2.1 this implies (we may take the integral from 1 instead of from 0)

$$\lim_{t\to\infty} \frac{\frac{1}{t} \int_1^t U(s)ds - \frac{1}{tx} \int_1^{tx} U(s)ds}{U(t) - U(tx)} = 1,$$

hence

$$\lim_{t\to\infty} \left\{ \frac{U(t) - \frac{1}{t} \int_1^t U(s)ds}{U(t) - U(tx)} - \frac{U(tx) - \frac{1}{tx} \int_1^{tx} U(s)ds}{U(t) - U(tx)} \right\} = 0. \tag{1.4.11}$$

On the other hand as for $t > 2$

$$\frac{U(t) - \frac{1}{t}\int_1^t U(s)ds}{U(t) - U(tx)} = \frac{\frac{1}{2}U(t) - \int_{1/t}^{\frac{1}{2}} U(ts)ds}{U(t) - U(tx)} +$$

$$+ \int_{\frac{1}{2}}^{1} \frac{U(t) - U(ts)}{U(t) - U(tx)}\, ds \geq \frac{1}{2} \cdot \frac{U(t) - U(\frac{t}{2})}{U(t) - U(tx)} + \int_{\frac{1}{2}}^{1} \frac{U(t) - U(ts)}{U(t) - U(tx)}\, ds,$$

we find by (1.4.7) and Fatou's lemma

$$\liminf_{t\to\infty} \frac{U(t) - \frac{1}{t}\int_1^t U(s)ds}{U(t) - U(tx)} \geq \frac{1}{2}\,\frac{\log \frac{1}{2}}{\log x} + \int_{\frac{1}{2}}^{1} \frac{\log s}{\log x}\, ds = \tag{1.4.12}$$

$$= \frac{-1}{2 \log x} > 0.$$

Dividing the appropriate fractions from (1.4.11) and (1.4.12) we get

$$\lim_{t\to\infty} \frac{\{U(t) - \frac{1}{t}\int_1^t U(s)ds\} - \{U(tx) - \frac{1}{tx}\int_1^{tx} U(s)ds\}}{U(t) - \frac{1}{t}\int_1^t U(s)ds} = 0$$

As it suffices to verify definition 1.1.1 for $0 < x < 1$, part b) is proved.

b) $\Leftrightarrow$ c): Define the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$h(x) = xU(x) - \int_1^x U(t)dt.$$

Property b) is equivalent with 1-variation of h at infinity. This by theorem 1.2.1 is equivalent with

$$\lim_{x\to\infty} \frac{x\, h(x)}{\int_1^x h(t)dt} = 2.$$

As by partial integration

$$\int_1^x h(t)dt = \int_1^x tU(t)dt - \int_1^x (\int_1^y U(t)dt)dy =$$

$$= x \int_1^x U(t)dt - 2 \int_1^x (\int_1^y U(t)dt)dy,$$

the equivalence of b) and c) is established.

c) $\Rightarrow$ d): Defining the function $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

(1.4.13) $$g(x) = U(x) - \frac{1}{x} \int_1^x U(t)dt$$

we have (see (1.3.9))

(1.4.14) $$U(x) = g(x) + \int_1^x \frac{g(t)}{t} dt.$$

d) $\Rightarrow$ e): By partial integration we have (using (1.4.9))

$$U(t) - \frac{1}{t} \int_1^t U(s)ds = g(t) - \frac{1}{t} \int_1^t g(s)ds +$$

$$+ \int_1^t \frac{g(s)}{s} ds - \frac{1}{t} \int_1^t (\int_1^s \frac{g(u)}{u} du)ds = g(t).$$

Hence for $x > 0$

(1.4.15) $$\frac{U(tx) - U(t)}{U(t) - \frac{1}{t} \int_1^t U(s)ds} = \frac{U(tx)-U(t)}{g(t)} = \frac{g(tx)}{g(t)} - 1 + \int_1^x \frac{g(ts)}{g(t)} \cdot \frac{1}{s} ds.$$

Since the relation

$$\lim_{t\to\infty} \frac{g(ts)}{g(t)} = 1$$

holds uniformly for all s between 1 and x (corollary 1.2.1 property 4), we obtain (1.4.10) letting $t \to \infty$ in (1.4.15).

e) $\Rightarrow$ a): Trivial. □

Now we give a formal definition.

Definition 1.4.1 A function $U : \mathbb{R}^+ \to \mathbb{R}$ belongs to the class Π (notation $U \in \Pi$) if U is strictly increasing and satisfies one of the equivalent properties a), b), c), d) or e) of theorem 1.4.1.

The next corollary gives a justification of the title of this section. We write $U(\infty)$ for the (finite or infinite) limit of $U(x)$ for $x \to \infty$.

Corollary 1.4.1 Suppose U belongs to the class Π.

a) If $U(\infty) = \infty$, then $U(x)$ is slowly varying at infinity.

b) If $U(\infty) < \infty$, then $U(\infty) - U(x)$ is slowly varying at infinity.

Proof a) Using the representation (1.4.9) we write

$$\frac{U(x) - c}{\int_1^x \frac{g(t)}{t}\, dt} = 1 + \frac{g(x)}{\int_1^x \frac{g(t)}{t}\, dt}. \tag{1.4.16}$$

By theorem 1.2.1 a) the righthand member of (1.4.16) tends to 1 as x tends to infinity. Hence

$$U(x) \sim \int_1^x \frac{g(t)}{t}\, dt \qquad \text{for } x \to \infty$$

and the latter function by lemma 1.2.2 a) is slowly varying at infinity.

b) If $U(\infty) < \infty$, then by (1.4.16)

$$U(\infty) = c + \int_1^\infty \frac{g(t)}{t}\, dt,$$

hence

$$U(\infty) - U(x) = - g(x) + \int_x^\infty \frac{g(t)}{t}\, dt$$

and

$$\frac{U(\infty) - U(x)}{\int_x^\infty \frac{g(t)}{t}\, dt} = \left\{ 1 - \frac{g(x)}{\int_x^\infty \frac{g(t)}{t}\, dt} \right\}. \tag{1.4.17}$$

By theorem 1.2.1 b) the righthand member of (1.4.17) tends to 1 as x tends to infinity. Hence

$$\{U(\infty) - U(x)\} \sim \int_x^\infty \frac{g(t)}{t}\,dt \qquad \text{for } x \to \infty$$

and the latter function by lemma 1.2.2 b) is slowly varying at infinity.

Corollary 1.4.2

a) If $U_1 \in \Pi$ and $U_2 \in \Pi$, then $U_1 + U_2 \in \Pi$.

b) If $\int_0^x U(t)dt$ is 1-varying at infinity, then

$$\int_1^x \frac{U(t)}{t}\,dt \in \Pi.$$

In particular: if U is slowly varying, the same conclusion holds.

c) The transformations (1.4.13) and (1.4.14) provide a one-to-one correspondence between the functions U in Π and the slowly varying functions g.

d) Part d) of theorem 1.4.1 can be replaced by: there exist slowly varying functions f and g with $f(x) \sim c_1.g(x)$ for $x \to \infty$ for a positive c_1 and a real constant c, such that

$$U(x) = c + g(x) + \int_1^x \frac{f(t)}{t}\,dt. \tag{1.4.18}$$

Proof a) As the sum of two slowly varying functions is slowly varying, by theorem 1.4.1 b) the statement is proved.

b) If we define U_1 by

$$U_1(x) = \int_1^x \frac{U(t)}{t}\,dt,$$

then (with Fubini's theorem)

$$U_1(x) - \frac{1}{x}\int_1^x U_1(t)dt = \frac{1}{x}\int_1^x U(t)dt,$$

hence by theorem 1.4.1 b) we have $U_1 \in \Pi$.

c) Obvious (see theorem 1.4.1 b) and theorem 1.4.1 d)).

d) Suppose (1.4.18) holds, then

$$U(x) - \frac{1}{x}\int_1^x U(t)dt = g(x) - \frac{1}{x}\int_1^x g(t)dt + \frac{1}{x}\int_1^x f(t)dt$$

$$= g(x) \cdot \{1 - \frac{\int_1^x g(t)dt}{x\, g(x)} + \frac{f(x)}{g(x)} \cdot \frac{\int_1^x f(t)dt}{x\, f(x)}\}.$$

Hence

$$\frac{U(x) - \frac{1}{x}\int_1^x U(t)dt}{g(x)} \to c_1 \qquad \text{for } x \to \infty$$

and by theorem 1.4.1 b) this gives $U \in \Pi$.
The converse is trivial. □

<u>Examples and counterexamples</u>.

a) Define the sequence $\{e_r\}_{r=1}^{\infty}$ by

$$e_1 = e$$
$$e_{r+1} = e^{e_r} \qquad \text{for } r = 1, 2, 3, \ldots$$

Then we may define the functions $\log_r^+ x$ by

$$\log_1^+ x = \begin{cases} 0 & \text{for } x < 1 \\ \log x & \text{for } x \geq 1 \end{cases}$$

and for $r = 1, 2, 3, \ldots$

$$\log_{r+1}^+ x = \begin{cases} 0 & \text{for } x < e_r \\ \log(\log_r^+ x) & \text{for } x \geq e_r. \end{cases}$$

Because of

$$\frac{d}{dx}\log_r^+ x = \{x \log_1^+ x \log_2^+ x \ldots \log_{r-1}^+ x\}^{-1} \quad \text{for } x > e_{r-1},$$

the function $\frac{d}{dx} \log_r^+ x$ is -1-varying. By corollary 1.4.2 b) we then have $\log_r^+ x \in \Pi$ for r = 1, 2, 3, ...

b) The converse of corollary 1.4.1 is not true: take

$$U(x) = 2 \log x + \sin(\log x);$$

then U is strictly increasing without bound and slowly varying at infinity, but $U \notin \Pi$ (as (1.4.7) does not hold).

c) In corollary 1.4.2 d) the condition $f(x) \sim c_1 . g(x)$ may not be omitted: take $f(x) \equiv 1$ and $g(x) = \log x + \sin(\log x)$, then f and g are slowly varying but U defined by (1.4.18) with an arbitrary $c \in \mathbb{R}$ does not belong to Π.

The next theorem is a version of theorem 1.4.1 for monotone but not necessarily strictly monotone functions.

Theorem 1.4.2 For a non-decreasing function $U : \mathbb{R}^+ \to \mathbb{R}$ the following assertions are equivalent.

a) For all $z > 1$ the function

$$U(tz) - U(t)$$

is positive for sufficiently large t and for every positive x and y ($y \neq 1$)

$$\lim_{t\to\infty} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{\log x}{\log y} .$$

b) The function

$$U(x) - \frac{1}{x} \int_1^x U(t)dt$$

is positive for sufficiently large x and slowly varying at infinity.

c) The function

$$U(x) - \frac{1}{x} \int_1^x U(t)dt$$

is positive for sufficiently large x and

$$\lim_{x\to\infty} \frac{x \int_1^x U(t)dt - 2 \int_1^x \int_1^y U(t)dtdy}{x^2 U(x) - x \int_1^x U(t)dt} = \tfrac{1}{2}.$$

d) There exist a real constant c and a function $g : \mathbb{R}^+ \to \mathbb{R}$ which is positive for $x \geq x_0 > 0$ and slowly varying at infinity, such that for all $x \in \mathbb{R}^+$

$$U(x) = c + g(x) + \int_{x_0}^x \frac{g(t)}{t}\, dt.$$

e) The function

$$U(t) - \frac{1}{t} \int_1^t U(s)ds$$

is positive for sufficiently large t and for all positive x

$$\lim_{t\to\infty} \frac{U(tx) - U(t)}{U(t) - \frac{1}{t} \int_1^t U(s)ds} = \log x.$$

Proof The proof is similar to that of theorem 1.4.1. □

In theorem 1.4.1 the behaviour of functions U near infinity is studied. Now we present a similar theorem concerning the behaviour near zero. This is the version which we shall apply in chapter II.

Theorem 1.4.3 For a strictly decreasing function $U : \mathbb{R}^+ \to \mathbb{R}$ the following assertions are equivalent.

a) For every positive x and y ($y\neq 1$)

$$\lim_{t\downarrow 0} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{\log x}{\log y} .$$

b) The function

$$\frac{1}{x} \int_0^x U(t)dt - U(x)$$

is well-defined for $x > 0$ and slowly varying at $x = 0$.

c) The integral $\int_0^1 U(t)dt$ is finite and

$$\lim_{x \downarrow 0} \frac{2\int_0^x \int_0^y U(t)dtdy - x\int_0^x U(t)dt}{x\int_0^x U(t)dt - x^2 U(x)} = \tfrac{1}{2}.$$

d) There exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ which is slowly varying at $x = 0$ and a real constant c such that

$$U(x) = c - g(x) + \int_x^1 \frac{g(t)}{t}\, dt \qquad \text{for all } x \in \mathbb{R}^+.$$

e) The integral $\int_0^1 U(t)dt$ is finite and for all $x > 0$

$$\lim_{t \downarrow 0} \frac{U(tx) - U(t)}{\frac{1}{t}\int_0^t U(s)ds - U(t)} = -\log x.$$

Remark Here we use an obvious extension of definition 1.1.1: a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is slowly varying at zero if for all $x > 0$

$$\lim_{t \downarrow 0} \frac{f(tx)}{f(t)} = 1.$$

Proof a) $\Rightarrow$ b): First we prove

$$\int_0^1 U(t)dt < \infty.$$

By assumption the function V defined by

$$V(x) = U(\tfrac{1}{x})$$

satisfies (1.4.7) of theorem 1.4.1. Hence by corollary 1.4.1 this function is slowly varying at infinity and by lemma 1.2.2 b)

$$\int_0^1 U(t)dt = \int_1^\infty \frac{V(t)}{t^2}\, dt < \infty.$$

Further as in the proof of theorem 1.4.1 we can show that for fixed $x > 1$ the function h defined by

$$h(t) = U(t) - U(tx)$$

is slowly varying at $t = 0$. Then $x^{-2}.h(x^{-1})$ is -2-varying at infinity and by theorem 1.2.1 a)

$$\lim_{t\downarrow 0} \frac{t \,.\, h(t)}{\int_0^t h(s)ds} = \lim_{x\to\infty} \frac{x^{-1}.h(x^{-1})}{\int_x^\infty \frac{h(s^{-1})}{s^2}\,ds} = 1.$$

Application of this relation gives the implication a) $\Rightarrow$ b).

The rest of the proof is similar to that of theorem 1.4.1 and is omitted. □

1.5 A SUBCLASS OF THE RAPIDLY VARYING FUNCTIONS

According to definition 1.1.1 a function $U : \mathbb{R}^+ \to \mathbb{R}^+$ is ρ-varying at infinity if for all positive x

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^\rho.$$

To extend this notion we consider the class of functions $U : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property: there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ and a real constant ρ such that

$$(1.5.1) \qquad \lim_{t\to\infty} \frac{U(t.x^{f(t)})}{U(t)} = x^\rho \quad \text{for all positive } x.$$

We confine our considerations to non-decreasing functions U (see section 2.11 for some remarks concerning this restriction) and ask for a characterization of the class of functions for which (1.5.1) holds with $\rho > 0$. Without loss of generality we may take $\rho = 1$ (this only involves a trivial change in f). It turns out to be more convenient to start with the following definition which is a mere transformation of the one just given.

Definition 1.5.1 A non-decreasing function $U : \mathbb{R} \to \mathbb{R}^+$ belongs to the class Γ (notation $U \in \Gamma$) if there exists a function $f : \mathbb{R} \to \mathbb{R}^+$ such that for all $x \in \mathbb{R}$

$$(1.5.2) \qquad \lim_{t\to\infty} \frac{U(t+xf(t))}{U(t)} = e^x.$$

Examples The following functions satisfy (1.5.2) with the given auxiliary functions f:

$$\exp(x^\alpha) \quad \text{for fixed } \alpha > 0 \text{ with } f(t) = \begin{cases} 1 & \text{for } t \le 0 \\ \alpha^{-1} \cdot t^{1-\alpha} & \text{for } t > 0, \end{cases}$$

$$\exp(x \log_+ x) \qquad \text{with } f(t) = \begin{cases} 1 & \text{for } x \le 1 \\ (\log t)^{-1} & \text{for } x > 1, \end{cases}$$

$$\exp(e^x) \qquad \text{with } f(t) = e^{-t}.$$

In chapter II we turn to our main object: the weak convergence of sample extremes. There we need a number of theorems of a purely analytic nature, which can be seen as a natural extension of the earlier sections in this chapter. Therefore on the one hand we could continue the present development but on the other hand later on we would have to derive afresh a number of theorems formulated somewhat differently. To round off the present discussion and for reasons of reference we formulate in this section a number of theorems of which we only prove those which do not reappear in chapter II. For the other ones detailed references to the proofs of the analogous theorems in chapter II are given.

Lemma 1.5.1 If U belongs to Γ, the function f of (1.5.2) satisfies

$$(1.5.3) \qquad \lim_{x\to\infty} \frac{f(x)}{x} = 0.$$

Proof See corollary 2.4.2. □

A justification of the title of this section is given in the next theorem.

<u>Theorem 1.5.1</u> If U belongs to Γ, U is ∞-varying at infinity.

<u>Proof</u> For fixed $x > 1$ and M we choose $t_0(x,M)$ such that for $t \geq t_0$

$$\frac{f(t)}{t} \leq \frac{x-1}{M} \text{ or } tx \geq t + M.f(t)$$

and

$$\frac{U(t+M.f(t))}{U(t)} > e^{\frac{1}{2}M},$$

then

$$\frac{U(tx)}{U(t)} \geq \frac{U(t+M.f(t))}{U(t)} > e^{\frac{1}{2}M}. \square$$

<u>Theorem 1.5.2</u> If U belongs to Γ, then (1.5.2) holds for each f with

$$f(x) \sim \frac{\int_0^x U(t)dt}{U(x)} \qquad \text{for } x \to \infty.$$

<u>Proof</u> See theorem 2.5.1. □

<u>Lemma 1.5.2</u> a) If U belongs to Γ, then $\int_0^x U(t)dt \in \Gamma$.

b) If U belongs to Γ and U has a non-decreasing derivative U', then U' ∈ Γ.

<u>Proof</u> See lemma 2.5.1 and lemma 2.7.1. □

The next theorem provides a characterization of the class Γ.

<u>Theorem 1.5.3</u> For a non-decreasing function $U : \mathbb{R} \to \mathbb{R}^+$ the following assertions are equivalent.

a) U belongs to Γ.

b)

$$(1.5.4) \qquad \lim_{x\to\infty} \frac{U(x) . \{\int_0^x \int_0^t U(s)dsdt\}}{\{\int_0^x U(t)dt\}^2} = 1.$$

c) There exist functions c, a and b : $\mathbb{R} \to \mathbb{R}$ and real constants c_1 and c_2 with

$$
(1.5.5)\quad \begin{cases} c(x) > 0, \ \lim_{x\to\infty} c(x) = c_1 > 0, \\ \lim_{x\to\infty} a(x) = 1, \\ c_2 + \int_1^x b(t)dt > 0 & \text{for all } x \in \mathbb{R} \\ \text{and} \qquad \lim_{x\to\infty} b(x) = 0, \end{cases}
$$

such that for all $x \in \mathbb{R}$

$$
(1.5.6)\qquad U(x) = c(x)\ .\ \exp\ \{\int_0^x \frac{dt}{g(t)}\},
$$

where

$$
g(x) = c_2 + \int_1^x b(t)dt.
$$

Proof See theorem 2.5.2. □

Remark 1.5.1 For functions of the form (1.5.6) (with (1.5.5)) relation (1.5.2) holds with $f(x) = g(x)$ (see the proof of c) ⟹ a) of theorem 2.5.2).

Remark 1.5.2 The condition $b(x) = 0$ for sufficiently large x in (1.5.5), is equivalent to the condition

$$
\lim_{t\to\infty} \frac{U(t+x)}{U(t)} = \exp\ (\rho x) \qquad \text{for all real } x
$$

and some constant $\rho > 0$, i.e. $U(\log x)$ is ρ-varying at infinity (cf. theorem 1.2.2).

Remark 1.5.3 A property connecting Γ with the class R of regularly varying functions is the following: a non-decreasing function U belongs to $\Gamma \cup R$ if and only if the function

$$\{\int_0^x U(t)dt\}^{-2} \cdot U(x) \cdot \int_0^x \int_0^t U(s)dsdt$$

tends to a limit c for $x \to \infty$. Under these conditions we necessarily have $\frac{1}{2} \leq c \leq 1$. If $c = 1$, then $U \in \Gamma$. If $c < 1$ then U is $\{(1-c)^{-1}-2\}$-varying at infinity (see theorem 2.6.2).

The next theorem provides another characterization of Γ.

<u>Theorem 1.5.4</u> a) If $U \in \Gamma$, then for all positive α

$$\lim_{x\to\infty} \frac{\int_0^x \{U(t)\}^\alpha \, dt}{\{U(x)\}^{\alpha-1} \cdot \int_0^x U(t)dt} = \frac{1}{\alpha}. \tag{1.5.7}$$

b) If a positive non-decreasing function U satisfies (1.5.7) for some positive $\alpha \neq 1$, then $U \in \Gamma$.

<u>Proof</u> See theorem 2.8.1. □

The results of section 1.4 and the present section are strongly connected. This is shown by the following theorem.

<u>Theorem 1.5.5</u> a) If U belongs to Γ, the function

$$U^*(x) = \inf \{y \mid U(y) \geq x\}$$

satisfies a), b), c), d) and e) of theorem 1.4.2.

b) If U satisfies one of the equivalent conditions a), b), c), d) or e) of theorem 1.4.2 and

$$\lim_{x\to\infty} U(x) = \infty,$$

the function

$$U^*(x) = \inf \{y \mid U(y) \geq x\}$$

belongs to Γ.

Proof   See theorem 2.4.1. □

For the construction of functions $U \in \Gamma$ we can use the representation given in theorem 1.5.3 c) and also the next theorem describing three rather different constructions of functions in Γ.

Theorem 1.5.6 a) If U_1 is monotone and ρ-varying at infinity ($0<\rho<\infty$) and $U_2 \in \Gamma$, the function U with

$$U(x) = U_1(U_2(x)) \qquad \text{for } x > 0$$

belongs to Γ.

b)   If $U_1 \in \Gamma$ and U_2 has a ρ-varying derivative ($-1<\rho<\infty$), then U with

$$U(x) = U_1(U_2(x)) \qquad \text{for } x > 0$$

belongs to Γ.

c)   If U_1 belongs to Γ and U_2 has a derivative belonging to Γ, then U with

$$U(x) = U_1(U_2(x)) \qquad \text{for } x \in \mathbb{R}$$

belongs to Γ.

Proof   a) By corollary 1.2.1 part 2 we have

$$\lim_{t\to\infty} \frac{U_1\{U_2(t+x\,\frac{f(t)}{\rho})\}}{U_1\{U_2(t)\}} = \left\{ \lim_{t\to\infty} \frac{U_2(t+x\,\frac{f(t)}{\rho})}{U_2(t)} \right\}^{\rho} = e^x .$$

b)   Using the representation (1.5.6) (with (1.5.5)) for U_1 we may write

$$U(x) = c(U_2(x))\ .$$
$$.\ \exp\left\{\int_{U_2^{-1}(0)}^{x} a(U_2(t))\,\{c_2 + \int_1^{U_2(t)} b(s)ds\}^{-1}\, U_2'(t)dt\right\}.$$

By theorem 1.2.1 a)

$$U_2'(x) = b^*(x) \,.\, (\rho+1) \,.\, x^{-1} \,.\, U_2(x)$$

where $b^*(x) \to 1$ for $x \to \infty$. Defining

$$\begin{cases} c^*(x) = c(U_2(x)) \\ a^*(x) = a(U_2(x)) \,.\, b^*(x) \\ g^*(x) = \{c_2 + \int_1^{U_2(x)} b(t)dt\} \,.\, x \{(\rho+1) \,.\, U_2(x)\}^{-1} \end{cases}$$

we get

$$U(x) = c^*(x) \exp \{\int_{U_2^{-1}(0)}^{x} \frac{a^*(t)}{g^*(t)} dt\}.$$

For almost all x we have

$$\frac{d}{dx} g^*(x) = b^*(x) \,.\, b(U_2(x)) + \{\frac{1}{\rho+1} - b^*(x)\} \{U_2(x)\}^{-1} \,.$$

$$\cdot \{c_2 + \int_1^{U_2(x)} b(t)dt\}.$$

As $b(x) \to 0$ and $b^*(x) \to 1$ for $x \to \infty$, the righthand side tends to zero for $x \to \infty$. By theorem 1.5.3 c) the proof is now complete.

c) The proof is analogous to that of part b), this time using

$$U_2'(x) = b^*(x) \,.\, \{U_2(x)\}^2 \,.\, \{\int_0^x U_2(t)dt\}^{-1}$$

(where $b^*(x) \to 1$ for $x \to \infty$) and the auxiliary functions

$$\begin{cases} c^*(x) = c(U_2(x)) \\ a^*(x) = a(U_2(x)) \,.\, b^*(x) \\ g^*(x) = \{c_2 + \int_1^{U_2(x)} b(t)dt\} \,.\, \{U_2(x)\}^{-2} \,.\, \{\int_0^x U_2(t)dt\}. \square \end{cases}$$

Remark 1.5.4 In part c) of theorem 1.5.6 it is supposed that $U_2' \in \Gamma$. By lemma 1.5.2. then also $U_2 \in \Gamma$. It is not true that the latter condition is sufficient, i.e. that the compound function of two functions from Γ necessarily belongs to Γ (a counterexample is given by $U_1(x) = e^x$ and $U_2(x) = e^x + \sin(e^x)$).

Examples As

$$U(x) = e^x$$

satisfies (1.5.2) with $f(t) = 1$, theorem 1.5.6 provides us with a set of examples of functions belonging to Γ:

$$\exp x^\alpha,\ \exp(\exp x^\alpha),\ \exp(\exp(\exp x^\alpha)),\ \ldots \text{ for all } \alpha > 0.$$

Proof Repeated application of part c) of theorem 1.5.6 gives the mentioned functions with $\alpha = 1$. Application of part b) of the theorem then gives the functions for general positive α. $\square$

CHAPTER II EXTREME VALUE THEORY

2.0 INTRODUCTION

This chapter deals with the classical subject of extreme value theory: limit distributions of maxima of independent, identically distributed (real-valued) random variables. Section 2.1 is of a more general character and treats the problem of choosing sequences of stabilizing coefficients for an arbitrary sequence of distribution functions.

In section 2.2 the possible types of limit distributions Φ_α, Ψ_α and Λ for sequences of maxima are derived. In section 2.3 the domains of attraction (see definition 2.2.1) for the types Φ_α and Ψ_α are characterized. The results of the sections 2.2 and 2.3 are due to Gnedenko [6]; the theory is given with full proofs to preserve the continuity of the present work. The sections 2.4 and 2.5 contain a new characterization of the domain of attraction of Λ. In section 2.6 a single criterion is obtained for the convergence of a sequence of maxima to any of the possible limit distributions. In section 2.7 a connection with von Mises' results [15] is given. Section 2.8 presents an intriguing alternative characterization of the domain of attraction of Λ. In section 2.9 the results of Gnedenko concerning the weak law of large numbers and the relative stability of a sequence of maxima are given, together with some new results. Section 2.10 contains two open problems.

2.1 DOMAINS OF ATTRACTION AND CHOICE OF COEFFICIENTS OF ATTRACTION

Suppose we have a sequence of (one-dimensional) distribution functions *) $\{F_n\}$ and a distribution function G. We say that $\{F_n\}$ converges weakly to G or

$$(2.1.1) \qquad F_n(x) \overset{w}{\to} G(x),$$

if

$$(2.1.2) \qquad \lim_{n\to\infty} F_n(x) = G(x) \text{ for all continuity points } x \text{ of } G.$$

A typical situation is that

$$\lim_{n\to\infty} F_n(x) = 0 \qquad \text{for all } x \in \mathbb{R}.$$

To avoid this uninteresting behaviour it is often sufficient to perform a linear transformation of the argument, i.e. it is often possible to choose sequences of real numbers $\{a_n\}$ and $\{b_n\}$ with $a_n > 0$ for $n = 1, 2, \ldots$, such that for a non-degenerate distribution function G

$$F_n(a_n x+b_n) \overset{w}{\to} G(x).$$

Definition 2.1.1 A sequence of distribution functions $\{F_n\}$ is said to belong to the domain of attraction of a non-degenerate distribution function G (notation $\{F_n\} \in \mathcal{D}(G)$) when it is possible to choose two sequences of real numbers $\{a_n\}$ ($a_n > 0$ for $n = 1, 2, 3, \ldots$) and $\{b_n\}$ such that

$$(2.1.3) \qquad F_n(a_n x+b_n) \overset{w}{\to} G(x).$$

The numbers $\{a_n\}$ and $\{b_n\}$ are called stabilizing constants.

A well-known theorem of Gnedenko [6] states to which extent we may change the sequence of stabilizing constants. We quote the theorem in its

---

*) Distribution functions are taken right continuous.

extended form as given by Feller ([2], p. 246).

Theorem 2.1.1 Suppose (2.1.3) holds. We have

$$F_n(\alpha_n x+\beta_n) \xrightarrow{w} G^*(x) \tag{2.1.4}$$

with non-degenerate G^* if and only if there exist real constants a and b ($a>0$) such that

$$\frac{\alpha_n}{a_n} \to a \quad \text{and} \quad \frac{\beta_n - b_n}{a_n} \to b \qquad (n\to\infty) \tag{2.1.5}$$

and

$$G^*(x) = G(ax+b) \qquad \text{for all real } x.$$

This theorem leads to the following definition.

Definition 2.1.2 The distribution function F_1 is said to be of the same type as the distribution function F_2, if there exist two constants a and b ($a>0$), such that

$$F_2(x) = F_1(ax+b) \qquad \text{for all real } x.$$

Clearly this relation between F_1 and F_2 is symmetric, reflexive and transitive. Hence it gives rise to equivalence classes of distribution functions. These classes are called types. Sometimes we shall indicate a type by one representative of the equivalence class.

Theorem 2.1.1 states that the domains of attraction of distribution functions of the same type are identical and that the domains of attraction of distribution functions of different types are disjoint.

In this section we shall give sequences $\{a_n\}$ and $\{b_n\}$ defined in a simple way in terms of quantiles of $\{F_n\}$ which can be used as stabilizing constants when the limit function G is continuous on the whole real line and strictly increasing on $\{x \mid 0 < G(x) < 1\}$. In particular we shall consider the case $F_n = F^n$, the n-th power of a given distribution function F; then Gnedenko's expression for stabilizing constants of a sequence of maxima is seen to be a special case of theorem 2.1.2 (cf. corollary 2.1.1). We shall also give an application concerning stabilization by moments (corollary 2.1.2 and corollary 2.1.3). A connection between quantiles and centering constants used with the weak law of large numbers has been given by J. Geffroy [5].

For a sequence of distribution functions $\{F_n\}$ satisfying (2.1.3) it is not true in general that this relation holds with the standard deviation and the mean as stabilizing constants, i.e. if for b_n we use μ_n defined by

$$\mu_n = \int_{-\infty}^{\infty} x dF_n(x) \tag{2.1.6}$$

and if for a_n we take σ_n defined by

$$\sigma_n^2 = \int_{-\infty}^{\infty} x^2 dF_n(x) - \mu_n^2, \tag{2.1.7}$$

even if μ_n and σ_n^2 exist for every n. This can be seen from the following example: take *)

$$F_n(x) = (1-\frac{1}{n})\ F(x) + \frac{1}{n}\ \iota(x-n^2),$$

where F(x) is an arbitrary distribution function with $\mu = 0$ and $\sigma^2 = 1$. We have here

$$F_n(x) \overset{w}{\to} F(x),$$

so (2.1.3) holds with $a_n = 1$ and $b_n = 0$ for all n whereas $\mu_n = n$ and $\sigma_n^2 = (n^3+1)\ (1-\frac{1}{n})$, hence for $n \to \infty$

---

*) Here is

$$\iota(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0. \end{cases}$$

$$\begin{cases} \dfrac{\mu_n - b_n}{a_n} = n \to \infty \\ \dfrac{\sigma_n}{a_n} = \sqrt{(n^3+1)(1-\frac{1}{n})} \to \infty. \end{cases}$$

Therefore by theorem 2.1.1 we can neither use μ_n for b_n nor σ_n for a_n.

In the next theorem we use quantiles of distribution functions. For each α $(0<\alpha<1)$ and distribution function F_n we define the α-quantile $\xi_\alpha^{(n)}$ by

(2.1.8) $$\xi_\alpha^{(n)} = \inf\{x \mid F_n(x) \geq \alpha\}.$$

Then we have

(2.1.9) $$F_n(\xi_\alpha^{(n)}-0) \leq \alpha \leq F_n(\xi_\alpha^{(n)}).$$

<u>Theorem 2.1.2</u> Let the distribution function G be continuous on the whole real line and strictly increasing on $\{x \mid 0 < G(x) < 1\}$ *) and suppose $\{F_n\} \in \mathcal{D}(G)$. Take arbitrary α and β with $0 < \alpha < \beta < 1$. If we define

(2.1.10) $$\begin{cases} b_n = \xi_\alpha^{(n)}, & b = G^{-1}(\alpha) \\ a_n = \xi_\beta^{(n)} - b_n, & a = G^{-1}(\beta) - b, \end{cases}$$

then

(2.1.11) $$F_n(a_n x + b_n) \overset{w}{\to} G(ax+b).$$

<u>Remark</u> The $\xi_\alpha^{(n)}$ and $\xi_\beta^{(n)}$ need only satisfy (2.1.6) in order that we can prove theorem 2.1.2. Hence with any definition of α-quantiles for which (2.1.9) holds, theorem 2.1.2 is true.

---

*) Hence the inverse function $G^{-1}(y)$ is uniquely defined for $0 < y < 1$.

<u>Proof</u> By assumption there exist two sequences $\{a_n'\}$ $(a_n'>0)$ and $\{b_n'\}$ such that

$$(2.1.12) \qquad F_n(a_n'x+b_n') \overset{w}{\to} G(x).$$

Hence for two arbitrary positive numbers ε_1 and ε_2 there exists an $n_0 = n_0(\varepsilon_1,\varepsilon_2)$ such that for $n \geq n_0$

$$(2.1.13) \qquad \begin{cases} F_n(a_n'(b-\varepsilon_1)+b_n') < G(b-\varepsilon_1) + \varepsilon_2 \\ F_n(a_n'(b+\varepsilon_1)+b_n') > G(b+\varepsilon_1) - \varepsilon_2, \end{cases}$$

where b is defined by (2.1.10). Now if we take $\varepsilon_1 > 0$ arbitrarily and if we choose

$$\varepsilon_2 = \min\{G(b) - G(b-\varepsilon_1),\ G(b+\varepsilon_1) - G(b)\} > 0,$$

then we have

$$(2.1.14) \qquad F_n(a_n'(b-\varepsilon_1)+b_n') < G(b) < F_n(a_n'(b+\varepsilon_1)+b_n') \quad \text{for } n \geq n_0(\varepsilon_1).$$

From (2.1.9) it follows that

$$(2.1.15) \qquad F_n(b_n-0) \leq G(b) \leq F_n(b_n) \qquad \text{for } n = 1, 2, 3, \ldots .$$

Combining (2.1.14) and (2.1.15) we obtain

$$a_n'(b-\varepsilon_1) + b_n' < b_n \leq a_n'(b+\varepsilon_1) + b_n' \qquad \text{for } n \geq n_0(\varepsilon_1),$$

so

$$\frac{b_n-b_n'}{a_n'} \to b \qquad \text{for } n \to \infty.$$

Starting in (2.1.13) with a + b instead of b we obtain

$$\frac{a_n}{a'_n} + \frac{b_n - b'_n}{a'_n} \to a + b \qquad \text{for } n \to \infty.$$

Application of theorem 2.1.1 gives the statement of the theorem. □

In order to avoid cumbersome details theorem 2.1.2 is given in its present form. However the same method of proof leads to theorem 2.1.2* and theorem 2.1.2**; in theorem 2.1.2* the α and β may depend on n, in theorem 2.1.2** the restrictions on G are weakened (the latter theorem is not used in the sequel). We omit the details.

<u>Theorem 2.1.2*</u> Suppose that the distribution function G is continuous on the whole real line and strictly increasing on $\{x \mid 0 < G(x) < 1\}$ and suppose $\{F_n\} \in \mathcal{D}(G)$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers tending for $n \to \infty$ to limits α and β respectively, where $0 < \alpha < \beta < 1$. If we define

$$\begin{cases} b_n = \xi^{(n)}_{\alpha_n}, & b = G^{-1}(\alpha) \\ a_n = \xi^{(n)}_{\beta_n} - b_n, & a = G^{-1}(\beta) - b, \end{cases}$$

then

$$F_n(a_n x + b_n) \overset{w}{\to} G(ax+b).$$

<u>Theorem 2.1.2**</u> Let α and β be real constants with $0 < \alpha < \beta < 1$. Suppose that for the distribution function G the following conditions are fulfilled:

a) $\xi_\alpha < \xi_\beta$ (where $\xi_p = \inf\{x \mid G(x) \geq p\}$ for $0 < p < 1$).

b) For all $\varepsilon > 0$ we have $G(\xi_\alpha + \varepsilon) > G(\xi_\alpha)$ and $G(\xi_\beta + \varepsilon) > G(\xi_\beta)$.

Then (2.1.11) holds with a_n, b_n, a and b defined by (2.1.10).

As an application we prove a well-known result due to Gnedenko [6] (cf. Mezjler [14]).

<u>Corollary 2.1.1</u> If for a distribution F the sequence $\{F^n\}$ is in the domain of attraction of G with

$$G(x) = \exp(-e^{-x}) \qquad \text{for all } x \in \mathbb{R},$$

then for all $a > 1$

$$F^n(a_n x + b_n) \xrightarrow{w} G(x \log a)$$

with

$$\begin{cases} b_n = \inf\{x \mid F(x) \geq 1 - \frac{1}{n}\} \\ a_n = \inf\{x \mid F(x) \geq 1 - \frac{1}{na}\} - b_n \qquad \text{for } n = 1, 2, 3, \ldots . \end{cases}$$

<u>Proof</u> Application of theorem 2.1.2* with $F_n = F^n$, $\alpha_n = (1 - \frac{1}{n})^n$ and $\beta_n = (1 - \frac{1}{na})^n$. □

In the next two applications we consider stabilization by moments.

<u>Corollary 2.1.2</u> Suppose that for a sequence of distribution functions $\{F_n\}$ the integrals μ_n and σ_n defined by (2.1.6) and (2.1.7) converge. If G satisfies the conditions of theorem 2.1.2 and if for some sequence of real constants $\{d_n\}$

$$F_n(\sigma_n x + d_n) \xrightarrow{w} G(x), \tag{2.1.16}$$

then there exists a real constant b such that

$$F_n(\sigma_n x + \mu_n) \xrightarrow{w} G(x+b). \tag{2.1.17}$$

<u>Proof</u> Let for all n $\underline{x}_n$ be a real-valued random variable with distribution function F_n. Take arbitrary α and β satisfying $0 < \alpha < \beta < 1$ and put

$$\underline{y}_n = \frac{\underline{x}_n - b_n}{a_n},$$

where b_n and a_n are defined by (2.1.10). Then the expectation and standard deviation of $\underline{y}_n$ are given by

$$\mu(\underline{y}_n) = \frac{\mu_n - b_n}{a_n} \quad \text{and} \quad \sigma(\underline{y}_n) = \frac{\sigma_n}{a_n} .$$

From (2.1.16), theorem 2.1.2 and theorem 2.1.1 it follows that for some positive a

(2.1.18) $\qquad \sigma(\underline{y}_n) \to a \qquad\qquad$ for $n \to \infty$.

In an analogous way as in Loève ([13] 17.1.a p. 244) one can prove

$$-(1-\alpha)^{-\frac{1}{2}} \leq \frac{\mu_n - \xi_\alpha^{(n)}}{\sigma_n} \leq \alpha^{-\frac{1}{2}} .$$

Hence the sequence $\mu(\underline{y}_n)$ is bounded and by (2.1.18) the sequence $\mu((\underline{y}_n)^2)$ is bounded as well. The latter is a well-known condition (see e.g. [2] p. 245) for the convergence of the lefthand side of

$$\mu(\underline{y}_n) = \frac{\mu_n - b_n}{a_n} .$$

Applying the theorems 2.1.2 and 2.1.1 we now have proved (2.1.17). □

Corollary 2.1.3 Stabilization by moments (mean and variance) is possible if and only if (in the notation of (2.1.7) and (2.1.8)) for some α and β ($0<\alpha<\beta<1$)

(2.1.19) $\qquad \dfrac{\sigma_n}{\xi_\beta^{(n)} - \xi_\alpha^{(n)}} \to c \qquad\qquad$ for $n \to \infty$

where c is some finite positive constant.

Proof By the theorems 2.1.1 and 2.1.2 the condition (2.1.19) is necessary and sufficient for (2.1.16). Application of corollary 2.1.2 completes the proof. □

2.2 THE POSSIBLE LIMIT DISTRIBUTIONS FOR MAXIMA

In this and the next sections we are concerned with the following problem.

Suppose $\underline{x}_1, \underline{x}_2, \underline{x}_3, \ldots$ are independent real-valued random variables with common distribution function F. We define

$$\underline{y}_n = \max(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_n), \quad n = 1, 2, \ldots .$$

It follows from the independence of the $\underline{x}_n$ that

$$P\{\underline{y}_n \leq x\} = F^n(x).$$

We ask for conditions in order that it is possible to choose sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ ($a_n > 0$ for $n = 1, 2, \ldots$) and $\{b_n\}_{n=1}^{\infty}$ such that

$$F^n(a_n x + b_n) \tag{2.2.1}$$

tends weakly to a non-degenerate distribution function for $n \to \infty$. We first investigate which types (see definition 2.1.2) of distribution functions actually occur as the limit of a sequence (2.2.1).

In the sequel we shall frequently use the concept of domain of attraction which is introduced in section 2.1 (definition 2.1.1). According to this definition an element of a domain of attraction is a sequence of distribution functions. As in the following we only deal with sequences which are powers F^n of some given distribution function F, in the sequel we shall use the more restrictive notion of domain of attraction given in the following definition.

<u>Definition 2.2.1</u> . A distribution function F is said to belong to the <u>domain of attraction</u> of a non-degenerate distribution function G (notation $F \in \mathcal{D}(G)$) if there exist sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ ($a_n > 0$, $n = 1, 2, 3, \ldots$) and $\{b_n\}_{n=1}^{\infty}$ such that

$$F^n(a_n x + b_n) \xrightarrow{w} G(x). \tag{2.2.2}$$

Our problem can thus be formulated in this way: find the types of distribution functions with non-empty domains of attraction. In the remainder of this section we follow Gnedenko's paper [6] but give simplified proofs.

Lemma 2.2.1 A non-degenerate distribution function G has a non-empty domain of attraction if and only if there exist two sequences of real numbers $\{A_m\}_{m=1}^{\infty}$ ($A_m > 0$, $m = 1, 2, 3, \ldots$) and $\{B_m\}_{m=1}^{\infty}$ such that

$$G^m(A_m x+B_m) = G(x) \tag{2.2.3}$$

for all $x \in R$ and $m = 1, 2, 3, \ldots$.

Proof If G is a non-degenerate distribution function for which (2.2.3) holds, then by definition $G \in \mathcal{D}(G)$, i.e. $\mathcal{D}(G)$ is non-empty.

Conversely, let $\mathcal{D}(G)$ be non-empty. Then there exists a distribution function F and two sequences $\{a_n\}_{n=1}^{\infty}$ ($a_n > 0$, $n = 1, 2, \ldots$) and $\{b_n\}_{n=1}^{\infty}$ of real numbers such that

$$F^n(a_n x+b_n) \overset{w}{\to} G(x). \tag{2.2.4}$$

Let m be a fixed positive integer. It follows from (2.2.4) that

$$F^{nm}(a_{nm}x+b_{nm}) \overset{w}{\to} G(x)$$

and hence that

$$F^n(a_{nm}x+b_{nm}) \overset{w}{\to} \{G(x)\}^{1/m}.$$

As $\{G(x)\}^{1/m}$ is also a non-degenerate distribution function, we may apply theorem 2.1.1 with $\alpha_n = a_{nm}$, $\beta_n = b_{nm}$. This yields that there exist two constants $A_m > 0$ and B_m such that

$$G(x) = \{G(A_m x+B_m)\}^{1/m},$$

which proves the theorem. □

Corollary 2.2.1 Relation (2.1.5) of theorem 2.1.1 gives in addition for $m = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} \frac{a_{nm}}{a_n} = A_m \quad \text{and} \quad \lim_{n\to\infty} \frac{b_{nm}-b_n}{a_n} = B_m$$

and a slight extension of the argument (using e.g. lemma 2.2.2) shows that for all real s > 1

$$\lim_{n\to\infty} \frac{a_{[ns]}}{a_n} = A_s \;, \; \lim_{n\to\infty} \frac{b_{[ns]}-b_n}{a_n} = B_s \tag{2.2.5}$$

(where [x] is the largest integer not exceeding x) and

$$G^s(A_s x+B_s) = G(x) \qquad \text{for all } x \in \mathbb{R}. \tag{2.2.6}$$

The lemma suggests the following definition.

Definition 2.2.2 A non-degenerate distribution function G is called stable if there exist sequences of real numbers $\{A_m\}_{m=1}^{\infty}$ ($A_m > 0$ for m = 1, 2, ...) and $\{B_m\}_{m=1}^{\infty}$ such that (2.2.3) is true for m = 1, 2, 3, ...

Note that stability is a property of a type of distribution functions. Our definition of stability differs from the usual one (see e.g. Loève [13] p. 326).

Lemma 2.2.1 shows that the problem of identifying all limit laws for sequences of maxima is identical with the problem of identifying all stable distribution functions. To this purpose we start by proving two further lemmas.

Lemma 2.2.2 Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers and $a_n > 0$ for n = 1, 2, 3, ... For distribution functions F and G we have for a fixed real x with $0 < G(x) < 1$

$$\lim_{n\to\infty} F^n(a_n x+b_n) = G(x) \tag{2.2.7}$$

if and only if

$$\lim_{n\to\infty} n\{1 - F(a_n x+b_n)\} = -\log G(x). \tag{2.2.8}$$

Proof As both (2.2.7) and (2.2.8) imply

$$F(a_n x+b_n) < 1 \qquad \text{for large } n$$

and

$$\lim_{n\to\infty} F(a_n x+b_n) = 1,$$

we have

$$\lim_{n\to\infty} \frac{-\log F^n(a_n x+b_n)}{n\{1-F(a_n x+b_n)\}} = \lim_{n\to\infty} \frac{-n\log(1-\{1-F(a_n x+b_n)\})}{n\{1-F(a_n x+b_n)\}} = 1.$$

Hence the equivalence of (2.2.7) and (2.2.8) is established. □

Lemma 2.2.3 Suppose G is a non-degenerate distribution function with $G(0-) = 0$. If there exists a sequence of positive constants $\{A_m\}$ such that

(2.2.9) $$G^m(A_m x) = G(x), \quad x \in \mathbb{R}^+, \quad m = 1, 2, 3, \ldots,$$

then there exist positive constants α and c such that

(2.2.10) $$G(x) = \exp\{-(x/c)^{-\alpha}\} \qquad \text{for all positive } x.$$

Proof First we remark that $0 < G(x) < 1$ for all $x > 0$ because $G(x_0) = 0$ or $G(x_0) = 1$ for some positive x_0 would imply that $A_m = 1$ for $m = 1, 2, 3, \ldots$ and G degenerate. By lemma 2.2.2 relation (2.2.9) implies

$$\lim_{m\to\infty} m\{1 - G(A_m x)\} = -\log G(x) \text{ for all positive } x.$$

This is relation (1.1.10) of theorem 1.1.3 if we take $U(x) = 1 - G(x)$, $\lambda_n = n$ and $a_n = A_n$. Hence by this theorem we have

$$\frac{-\log G(x)}{-\log G(1)} = x^\rho \qquad \text{for all } x \in \mathbb{R}^+$$

where ρ is a negative number as $1 - G$ is non-increasing and non-degenerate. Now (2.2.10) is an easy consequence. □

The next theorem gives a complete description of the class of stable laws. The theorem has been formulated in this form for the first time in 1928 by Fisher and Tippett [3]; the proof given here is essentially Gnedenko's proof.

Theorem 2.2.1 Every stable distribution is of one of the following types:

$$\Phi_\alpha(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ \exp(-x^{-\alpha}) & \text{for } x > 0 \end{cases} \tag{2.2.11}$$

$$\Psi_\alpha(x) = \begin{cases} \exp\{-(-x)^{\alpha}\} & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases} \tag{2.2.12}$$

$$\Lambda(x) = \exp(-e^{-x}). \tag{2.2.13}$$

In (2.2.11) and (2.2.12) α is a positive constant.

Proof We start with the identities (2.2.3) for a non-degenerate G and distinguish three possibilities:

1) Suppose $A_m = 1$ for all m. We define

$$C_m = \exp(B_m) \qquad \text{for } m = 1, 2, 3, \ldots$$

and

$$H(y) = \begin{cases} 0 & \text{for } y \leq 0 \\ G(\log y) & \text{for } y > 0, \end{cases}$$

then for $y > 0$ and $m = 1, 2, 3, \ldots$

$$H^m(C_m y) = H(y),$$

hence by lemma 2.2.3 the function G is of type Λ.

2) Suppose there exists an $m_0 > 1$ such that $A_{m_0} < 1$. We shall prove that G is of type Ψ_α for some positive α. The proof is given in four steps.

a) First we prove

$$G(x) = 1 \qquad \text{for } x \geq \frac{B_{m_0}}{1-A_{m_0}}. \tag{2.2.14}$$

Obviously

$$x \geq \frac{B_{m_0}}{1-A_{m_0}} \Rightarrow A_{m_0}x + B_{m_0} \leq x \Rightarrow G(A_{m_0}x+B_{m_0}) \leq G(x).$$

With (2.2.3) this gives for these values of x

$$G^{m_0}(x) \leq G(x) = G^{m_0}(A_{m_0}x+B_{m_0}) \leq G^{m_0}(x).$$

This is true only if $G(x) = 0$ or 1. As G is non-degenerate, we must have (2.2.14).

b) Now we prove

$$0 < G(x) < 1 \qquad \text{for } x < \frac{B_{m_0}}{1-A_{m_0}}. \tag{2.2.15}$$

Suppose there exists an $x_0 < B_{m_0}(1-A_{m_0})^{-1}$ with

$$\begin{cases} G(x) = 1 & \text{for } x \geq x_0, \\ G(x) < 1 & \text{for all } x < x_0. \end{cases} \tag{2.2.16}$$

Take

$$x_1 = x_0 - \frac{B_{m_0} - x_0(1-A_{m_0})}{2\,A_{m_0}},$$

then we have

$$x_1 < x_0 < A_{m_0}x_1 + B_{m_0}$$

so that

$$G(x_1) = G^{m_0}(A_{m_0} x_1 + B_{m_0}) \geq G^{m_0}(x_0) = 1$$

which contradicts $G(x_1) < 1$. Hence $G(x) < 1$ for $x < B_{m_0}(1-A_{m_0})^{-1}$. In an analogous way the positivity of G for all x is established.

c) Next we prove $A_m < 1$ for all m. Suppose there exists an m_1 with $A_{m_1} = 1$, then for all x

$$G^{m_1}(x+B_{m_1}) = G(x). \tag{2.2.17}$$

Take in (2.2.17) $x = x_2$ with $0 < G(x_2) < 1$, then it follows that $x_2 + B_{m_1} > x_2$ or $B_{m_1} > 0$. Substitution of

$$\frac{B_{m_0}}{1-A_{m_0}} - \frac{B_{m_1}}{2}$$

for x in (2.2.17) gives (see (2.2.14) and (2.2.15))

$$1 = G^{m_1}\left(\frac{B_{m_0}}{1-A_{m_0}} + \frac{B_{m_1}}{2}\right) = G\left(\frac{B_{m_0}}{1-A_{m_0}} - \frac{B_{m_1}}{2}\right) < 1.$$

Hence $A_{m_1} = 1$ is impossible.

Finally suppose there exists an m_2 with $A_{m_2} > 1$, then an analogous reasoning as in a) and b) shows

$$\begin{cases} G(x) = 0 & \text{for } x < \dfrac{B_{m_2}}{1-A_{m_2}} \\ 0 < G(x) < 1 & \text{for } x > \dfrac{B_{m_2}}{1-A_{m_2}}. \end{cases}$$

This contradicts (2.2.14).

d) As (2.2.14) and (2.2.15) hold not only for m_0 but for arbitrary $m > 1$, we have

$$\frac{B_m}{1-A_m} = \frac{B_2}{1-A_2}, \quad m = 2, 3, \ldots .$$

We define

$$\tilde{G}(z) = \begin{cases} 0 & \text{for } z \leq 0 \\ G\left(\frac{B_2}{1-A_2} - \frac{1}{z}\right) & \text{for } z > 0. \end{cases}$$

Take

$$z = \left\{\frac{B_2}{1-A_2} - x\right\}^{-1} \qquad \text{for } x < \frac{B_2}{1-A_2},$$

then we have for all $z > 0$

$$\tilde{G}^m(A_m^{-1}z) = G^m\left(\frac{B_m}{1-A_m} - \frac{A_m}{z}\right) = G^m(A_m x+B_m) = G(x) =$$

$$= G\left(\frac{B_m}{1-A_m} - \frac{1}{z}\right) = \tilde{G}(z).$$

Lemma 2.2.3 then shows that G is of type Ψ_α for some positive α.

3) Suppose there exists an $m_0 > 1$ such that $A_{m_0} > 1$. In an analogous way as in 2) one finds that G is of type Φ_α for some positive α. □

2.3 THE DOMAINS OF ATTRACTION OF Φ_α AND Ψ_α

Necessary and sufficient conditions for a distribution function to belong to the domain of attraction of Φ_α or that of Ψ_α have been given by Gnedenko [6]. We follow Gnedenko's proofs which however seem to contain an error (take e.g. $F(x) = 1-(x \log x)^{-1}$ for sufficiently large x; then for all $\beta > 1$ it is impossible to choose a sequence $\{a_n\}$ such that both (40) and (43) of [6] are fulfilled though $1 - F$ is -1-varying at infinity). We need the following lemma to complement Gnedenko's proofs.

Lemma 2.3.1 For sequences $\{a_n\}$ and $\{b_n\}$ of real numbers with $a_n > 0$ for all n the following implications hold.

a)

$$\left.\begin{aligned}&\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\gamma_1>1\\&\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_n}=0\end{aligned}\right\}\Rightarrow\quad\lim_{n\to\infty}\frac{b_n}{a_n}=0.$$

b)

$$\left.\begin{aligned}&\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\gamma_2,\ 0<\gamma_2<1\\&\lim_{n\to\infty}\frac{b_{n+1}-b_n}{a_n}=0\end{aligned}\right\}\Rightarrow\left\{\begin{aligned}&\lim_{n\to\infty}b_n=b\text{ exists }(-\infty<b<\infty)\\&\text{and }\lim_{n\to\infty}\frac{b-b_n}{a_n}=0.\end{aligned}\right.$$

Proof

a) We define

$$c_1=b_1,\ c_n=b_n-b_{n-1}\qquad\text{for } n=2,3,\ldots,$$

$$d_1=1,\ d_n=a_n^{-1}\cdot a_{n-1}\qquad\text{for } n=2,3,\ldots\ .$$

We have to prove

$$\sum_{k=1}^{n}\frac{c_k}{a_n}=\sum_{k=1}^{n}\frac{c_k}{a_k}\cdot\frac{a_k}{a_n}\to 0\qquad\text{for } n\to\infty. \tag{2.3.1}$$

By assumption we have

$$\lim_{n\to\infty}\frac{c_n}{a_n}=0. \tag{2.3.2}$$

Further, as $d_n\to\gamma_1^{-1}<1$ for $n\to\infty$, it follows that

$$\limsup_{n\to\infty}\sum_{k=1}^{n}\frac{a_k}{a_n}=\limsup_{n\to\infty}\sum_{k=1}^{n}d_{k+1}\cdot d_{k+2}\cdots d_n<\infty. \tag{2.3.3}$$

Now (2.3.1) is an easy consequence of (2.3.2) and (2.3.3).

b) We define

$$c_1=b_1,\ c_n=b_n-b_{n-1}\qquad\text{for } n=2,3,\ldots,$$

$$d_1=1,\ d_n=a_{n-1}^{-1}\cdot a_n\qquad\text{for } n=2,3,\ldots\ .$$

We first have to prove

$$(2.3.4) \qquad b_m - b_n = \sum_{k=n+1}^{m} a_k \cdot \frac{c_k}{a_k} \to 0 \quad \text{for } n, m \to \infty.$$

By assumption we have

$$(2.3.5) \qquad \lim_{n\to\infty} \frac{c_n}{a_n} = 0.$$

Further, as $d_n \to \gamma_2 < 1$ for $n \to \infty$, it follows that

$$(2.3.6) \qquad \limsup_{n\to\infty} \frac{1}{a_n} \sum_{k=n}^{\infty} a_k = \limsup_{n\to\infty} \sum_{k=n}^{\infty} d_k \cdot d_{k-1} \cdots d_n < \infty.$$

As $a_n \to 0$ for $n \to \infty$, we get from (2.3.6)

$$(2.3.7) \qquad \lim_{n,m\to\infty} \sum_{k=n+1}^{m} a_k = 0.$$

Now (2.3.4) is an easy consequence of (2.3.5) and (2.3.7). Finally we have to prove

$$\frac{b-b_n}{a_n} = \frac{1}{a_n} \sum_{k=n+1}^{\infty} a_k \cdot \frac{c_k}{a_k} \to 0 \quad \text{for } n \to \infty.$$

This follows easily from (2.3.5) and (2.3.6). □

The next theorem characterizes the distribution functions which belong to $\mathcal{D}(\Phi_\alpha)$.

Theorem 2.3.1 A distribution function F belongs to the domain of attraction of Φ_α if and only if $1 - F$ is $(-\alpha)$-varying at infinity.

Proof a) Suppose $1 - F$ is $(-\alpha)$-varying at infinity. By theorem 1.1.3 and remark 1.1.1

$$\lim_{n\to\infty} n\{1 - F(a_n x)\} = x^{-\alpha}$$

for all positive x with

(2.3.8) $$a_n = \inf\{x \mid 1 - F(x) \le \frac{1}{n}\}.$$

By lemma 2.2.2 this is equivalent to

$$\lim_{n\to\infty} F^n(a_n x) = \Phi_\alpha(x) \qquad \text{for } x \in \mathbb{R}^+.$$

For non-positive x this relation is true because of the monotonicity of F, hence $F \in \mathcal{D}(\Phi_\alpha)$.

b) Suppose for some sequences $\{a_n\}$ ($a_n > 0$ for all n) and $\{b_n\}$ we have

$$F^n(a_n x + b_n) \overset{w}{\to} \Phi_\alpha(x).$$

Corollary 2.2.1 states that for all real s > 1

(2.3.9) $$\begin{cases} \lim_{n\to\infty} \dfrac{a_{[ns]}}{a_n} = A_s \\ \lim_{n\to\infty} \dfrac{b_{[ns]} - b_n}{a_n} = B_s, \end{cases}$$

where A_s and B_s satisfy

(2.3.10) $$\{\Phi_\alpha(x)\}^{1/s} = \Phi_\alpha(A_s x + B_s)$$

for all x. Relation (2.3.10) gives

(2.3.11) $$A_s = s^{1/\alpha} \quad \text{and} \quad B_s = 0.$$

For fixed s > 1 we define a sequence $\{n(i)\}_{i=1}^{\infty}$ of integers by

$$n(1) = [\tfrac{s}{s-1}]$$

$$n(i+1) = [n(i).s] \qquad \text{for } i = 1, 2, 3, \ldots .$$

It is not difficult to see that

$$(2.3.12)\qquad \begin{cases} \lim_{i\to\infty} n(i) = \infty \\ \\ \lim_{i\to\infty} \dfrac{n(i+1)}{n(i)} = s. \end{cases}$$

From (2.3.9), (2.3.11) and (2.3.12) we get

$$(2.3.13)\qquad \begin{cases} \lim_{i\to\infty} \dfrac{a_{n(i+1)}}{a_{n(i)}} = s^{1/\alpha} > 1 \\ \\ \lim_{i\to\infty} \dfrac{b_{n(i+1)}-b_{n(i)}}{a_{n(i)}} = 0 \,. \end{cases}$$

Application of lemma 2.3.1 a) gives

$$\lim_{i\to\infty} \frac{b_{n(i)}}{a_{n(i)}} = 0,$$

hence by theorem 2.1.1

$$F^{n(i)}(a_{n(i)}x) \overset{w}{\to} \Phi_\alpha(x)$$

or, by lemma 2.2.2,

$$(2.3.14)\qquad \lim_{i\to\infty} n(i)\,\{1 - F(a_{n(i)}x)\} = x^{-\alpha}$$

for all positive x. By (2.3.13) we have

$$(2.3.15)\qquad \lim_{i\to\infty} a_{n(i)} = \infty.$$

Application of theorem 1.1.3 and remark 1.1.2 completes the proof. □

<u>Remark 2.3.1</u> The proof shows that if $F \in \mathcal{D}(\Phi_\alpha)$, then

$$F^n(a_n x) \overset{w}{\to} \Phi_\alpha(x)$$

with

$$a_n = \inf\{x \mid 1 - F(x) \le \tfrac{1}{n}\},\ n = 1, 2, 3, \dots .$$

Corollary 2.3.1 A distribution function F belongs to the domain of attraction of Φ_α if and only if there exist a function $a : \mathbb{R}^+ \to \mathbb{R}$, a function $c : \mathbb{R}^+ \to \mathbb{R}^+$ and a positive constant c_0 with

$$(2.3.16) \qquad \begin{cases} \lim_{x\to\infty} a(x) = \alpha \\ \lim_{x\to\infty} c(x) = c_0 \end{cases}$$

such that for all positive x

$$(2.3.17) \qquad F(x) = 1 - c(x) \exp\{- \int_1^x \frac{a(t)}{t}\, dt\}.$$

Proof Theorem 1.2.2. □

Remark 2.3.2 Remark 1.2.4 shows that in the representation (2.3.17) we may take (with c_1 a positive constant)

$$a(x) = c_0\ c(x) = \frac{1-F(x)}{\int_x^\infty \frac{1-F(t)}{t}\, dt} \text{ for all } x \in \mathbb{R}^+.$$

The next theorem gives necessary and sufficient conditions for a distribution function to belong to the domain of attraction of Ψ_α. In the sequel we use the notation

$$x_0 = \sup\{x \mid F(x) < 1\}.$$

If necessary we write $x_0(F)$ instead of x_0. Note that $x_0 \le \infty$. The number x_0 is called the endpoint of the distribution function F.

Theorem 2.3.2 A distribution function F belongs to the domain of attraction of Ψ_α if and only if F has a finite endpoint x_0 and the function U defined by $U(x) = 1 - F(x_0 - x^{-1})$ for all $x \in \mathbb{R}^+$ is $-\alpha$-varying at infinity.

<u>Proof</u> a) Suppose $x_0(F) < \infty$ and $1 - F(x_0-x^{-1})$ is $-\alpha$-varying at infinity. Define

$$F_*(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ F(x_0-x^{-1}) & \text{for } x > 0. \end{cases}$$

By theorem 2.3.1 we have $F_* \in \mathcal{D}(\Phi_\alpha)$, i.e.

$$F_*^n(a_n x) \overset{w}{\to} \Phi_\alpha(x) \tag{2.3.18}$$

with

$$a_n = \inf\{x \mid 1 - F_*(x) \leq \tfrac{1}{n}\} = (x_0-\inf\{x \mid 1-F(x)\leq\tfrac{1}{n}\})^{-1}. \tag{2.3.19}$$

From (2.3.18) we see

$$F^n(x_0 - \frac{1}{a_n x}) \overset{w}{\to} \Phi_\alpha(x),$$

or (with $y = -x^{-1}$)

$$F^n(a_n^{-1}y+x_0) \overset{w}{\to} \Psi_\alpha(y). \tag{2.3.20}$$

Hence $F \in \mathcal{D}(\Psi_\alpha)$.

b) Suppose for sequences $\{a_n\}$ ($a_n > 0$ for all n) and $\{b_n\}$ we have

$$F^n(a_n x+b_n) \overset{w}{\to} \Psi_\alpha(x).$$

Corollary 2.2.1 states that for all real $s > 1$

$$\begin{cases} \lim\limits_{n\to\infty} \dfrac{a_{[ns]}}{a_n} = A_s = s^{-1/\alpha} \\ \lim\limits_{n\to\infty} \dfrac{b_{[ns]}-b_n}{a_n} = B_s = 0. \end{cases} \tag{2.3.21}$$

With the sequence n(i) from the proof of theorem 2.3.1 we have

$$(2.3.22)\qquad \begin{cases} \lim\limits_{i\to\infty} \dfrac{a_{n(i+1)}}{a_{n(i)}} = s^{-1/\alpha} < 1 \\[2ex] \lim\limits_{i\to\infty} \dfrac{b_{n(i+1)}-b_{n(i)}}{a_{n(i)}} = 0. \end{cases}$$

Application of lemma 2.3.1 b) gives

$$\lim_{i\to\infty} b_{n(i)} = b$$

$(-\infty<b<\infty)$ and

$$\lim_{i\to\infty} \frac{b-b_{n(i)}}{a_{n(i)}} = 0,$$

hence by theorem 2.1.1

$$(2.3.23)\qquad F^{n(i)}(a_{n(i)}x+b) \overset{w}{\to} \Psi_\alpha(x)$$

for all x. By taking x = 0 in (2.3.23) we see

$$F(b) = 1.$$

On the other hand taking x = -1 in (2.3.23) we have

$$F(-a_{n(i)}+b) < 1$$

for sufficiently large i. Hence, as by (2.3.22)

$$\lim_{i\to\infty} a_{n(i)} = 0,$$

we have

$$x_0(F) = b < \infty.$$

With lemma 2.2.2 relation (2.3.23) becomes

$$\lim_{i\to\infty} n(i)\,\{1 - F(x_0 - \frac{a_{n(i)}}{x})\} = x^{-\alpha} \text{ for all } x > 0. \tag{2.3.24}$$

Application of theorem 1.1.3 and remark 1.1.2 completes the proof. □

<u>Remark 2.3.3</u> Part a) of the proof shows that if $F \in \mathcal{D}(\Psi_\alpha)$,

$$F^n(\{x_0-a_n\}x+x_0) \overset{w}{\to} \Psi_\alpha(x)$$

with

$$a_n = \inf\{x \mid 1 - F(x) \le \frac{1}{n}\}, \quad n = 1, 2, 3, \ldots .$$

<u>Corollary 2.3.2</u> A distribution function F belongs to the domain of attraction of Ψ_α if and only if $x_0(F) < \infty$ and there exist a function $a : \mathbb{R}^+ \to \mathbb{R}$, a function $c : \mathbb{R}^+ \to \mathbb{R}^+$ and a positive constant c_0 with

∠ $\mathbb{R}\to\mathbb{R}$
∠ $\mathbb{R}\to\mathbb{R}^+$

$$\begin{cases} \lim\limits_{x\to\infty} a(x) = \alpha \\ \lim\limits_{x\to\infty} c(x) = c_0 \end{cases} \tag{2.3.25}$$

such that for all $x < x_0$

$$F(x) = 1 - c(x) \,.\, \exp\{-\int_{x_0-1}^{x} \frac{a(t)}{x_0-t}\,dt\}. \tag{2.3.26}$$

<u>Proof</u> Theorem 1.2.2 and some obvious calculations. □

<u>Remark 2.3.4</u> Remark 1.2.4 shows that in the representation (2.3.26) we may take (with c_1 a positive constant)

$$a(x) = c_1 \,.\, c(x) = \frac{1-F(x)}{\int_x^{x_0} \frac{1-F(t)}{x_0-t}\,dt} \quad \text{for all } x < x_0.$$

2.4 THE DOMAIN OF ATTRACTION OF Λ: PRELIMINARIES

In section 2.3 we have seen that if $F \in \mathcal{D}(\Phi_\alpha)$, for the distribution function F the endpoint x_0 defined by

$$x_0 = x_0(F) = \sup\{x \mid F(x) < 1\}$$

equals infinity and if $F \in \mathcal{D}(\Psi_\alpha)$, the distribution function F has a finite endpoint. For distribution functions from $\mathcal{D}(\Lambda)$ both possibilities can occur.

In this section we derive some preliminary results concerning the domain of attraction of Λ. First we state a result due to D. Mejzler [14] (in a slightly different form). We also give the proof as the paper is not easily available.

<u>Theorem 2.4.1</u> A distribution function F belongs to the domain of attraction of Λ if and only if

$$\lim_{t \downarrow 0} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{\log x}{\log y} \tag{2.4.1}$$

for all positive x and y ($y \neq 1$), where $U : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$U(x) = \inf\{y \mid 1 - F(y) \leq x\}. \tag{2.4.2}$$

<u>Proof</u> a) Suppose $F \in \mathcal{D}(\Lambda)$. Take a_1 and a_2 greater than 1. Application of corollary 2.1.1 and theorem 2.1.1 (with $a_n = U(\frac{1}{na_2}) - U(\frac{1}{n})$ and $\alpha_n = U(\frac{1}{na_1}) - U(\frac{1}{n})$) gives

$$\lim_{n\to\infty} \frac{U(\frac{1}{na_1})-U(\frac{1}{n})}{U(\frac{1}{na_2})-U(\frac{1}{n})} = \frac{\log a_1}{\log a_2} . \tag{2.4.3}$$

As U is non-increasing, we have for fixed $a_1 > 1$, $a_2 > 1$ and sufficiently large n

$$0 \le \frac{U(\frac{1}{n+1})-U(\frac{1}{n})}{U(\frac{1}{na_2})-U(\frac{1}{n})} \le \frac{U(\frac{1}{na_1})-U(\frac{1}{n})}{U(\frac{1}{na_2})-U(\frac{1}{n})} .$$

Hence for all $a > 1$

$$\lim_{n\to\infty} \frac{U(\frac{1}{n+1})-U(\frac{1}{n})}{U(\frac{1}{na}) - U(\frac{1}{n})} = 0. \tag{2.4.4}$$

For positive t we define $n_t = [t^{-1}]$ (the largest integer not exceeding t^{-1}). For all $\varepsilon > 0$ and sufficiently small t we have

$$a(n_t+1) < (a+\varepsilon)n_t$$

and hence for all $0 < x < 1$ (writing $a = x^{-1}$)

$$1 - \frac{U(\frac{1}{n_t+1})-U(\frac{1}{n_t})}{U(\frac{1}{n_t a}) - U(\frac{1}{n_t})} = \frac{U(\frac{1}{n_t a})-U(\frac{1}{n_t+1})}{U(\frac{1}{n_t a})-U(\frac{1}{n_t})} \le \tag{2.4.5}$$

$$\le \frac{U(tx)-U(t)}{U(n_t^{-1}x)-U(n_t^{-1})} \le \frac{U(\frac{1}{(n_t+1)a})-U(\frac{1}{n_t})}{U(\frac{1}{n_t a})-U(\frac{1}{n_t})} \le \frac{U(\frac{1}{n_t(a+\varepsilon)})-U(\frac{1}{n_t})}{U(\frac{1}{n_t a})-U(\frac{1}{n_t})} .$$

As by (2.4.4) the lefthand side of (2.4.5) tends to 1 and by (2.4.3) the righthand side tends to $(\log a)^{-1} \log(a+\varepsilon)$ for $t \downarrow 0$, we have proved

$$\lim_{t\downarrow 0} \frac{U(tx)-U(t)}{U(n_t^{-1}x)-U(n_t^{-1})} = 1 \qquad \text{for all } 0 < x < 1.$$

Again with (2.4.3) we now have (2.4.1) for all x and y in (0,1). Now take $0 < x < 1$, $y > 1$ and $y^{-1}x < 1$. Then with $s = ty$ we have

$$\frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{U(s\frac{x}{y})-U(\frac{s}{y})}{U(s)\ -U(\frac{s}{y})} = 1 - \frac{U(s)-U(s.\frac{x}{y})}{U(s)-U(s.\frac{1}{y})} .$$

As the righthand side tends to $(\log y)^{-1} \log x$ for $s \to \infty$ we have proved (2.4.1) for $0 < x < 1$ and $y > 1$. Hence (2.4.1) is true for all positive x and y ($y \neq 1$).

b) Next suppose (2.4.1) is true for all positive x and y ($y \neq 1$). Take

$$(2.4.6) \qquad \begin{cases} b_n = U(\frac{1}{n}) \\ a_n = U(\frac{1}{ne}) - U(\frac{1}{n}) . \end{cases}$$

We want to prove

$$F^n(a_n x+b_n) \overset{w}{\to} \exp(-e^{-x}) \qquad \text{for } n \to \infty.$$

Instead we shall prove the clearly equivalent statement: for each sequence of integers $n(i) \to \infty$ and (possibly defective) distribution function Φ for which

$$(2.4.7) \qquad F^{n(i)}(a_{n(i)}x+b_{n(i)}) \overset{w}{\to} \Phi(x) \quad \text{for } i \to \infty,$$

we have necessarily

$$(2.4.8) \qquad \Phi(x) = \exp(-e^{-x}) \qquad \text{for all } x \in \mathbb{R}.$$

Suppose (2.4.7) holds for some sequence $\{n(i)\}$ and some Φ. Take a fixed continuity point $x \neq 0$ of Φ. Then by (2.4.1) and theorem 2.1.1 (this side of the theorem goes through even for degenerate and defective distribution functions) relation (2.4.7) is also true with

$$a_{n(i)} = \frac{U(\frac{1}{n(i).e^x})-U(\frac{1}{n(i)})}{x} .$$

Then (2.4.7) reduces to

$$\lim_{i\to\infty} F^{n(i)}(U(\frac{1}{n(i).e^x})) = \Phi(x)$$

and by lemma 2.2.2 we have

$$(2.4.9) \qquad \lim_{i\to\infty} n(i)\ \{1 - F(U(\frac{1}{n(i).e^x}))\} = -\log \Phi(x).$$

As for all $\varepsilon > 0$ and sufficiently large i

$$U(\frac{1}{n(i).e^x}) < U(\frac{1}{n(i).e^{x+\varepsilon}}),$$

we have (cf. (1.1.15))

$$\frac{n(i)}{n(i).e^{x+\varepsilon}} \le n(i)\ \{1 - F(U(\frac{1}{n(i)e^{x+\varepsilon}})-0)\} \le$$

$$\le n(i)\ \{1 - F(U(\frac{1}{n(i)e^x}))\} \le \frac{n(i)}{n(i).e^x}$$

and thus from (2.4.9) it follows that

$$-\log \Phi(x) = e^x$$

for all continuity points $x \neq 0$ of Φ. Hence (2.4.8) is true for all x. □

In the sequel we need the following obvious extension of theorem 2.1.1.

Lemma 2.4.1 Suppose that for a family $\{F_t\}$ of distribution functions (with $t \in \mathbb{R}$ and $-\infty < t < t_0$ for some t_0 with $-\infty < t_0 \le \infty$) there exist functions $a : \mathbb{R} \to \mathbb{R}^+$ and $b : \mathbb{R} \to \mathbb{R}$ and a non-degenerate distribution function G such that

$$F_t(a(t)x+b(t)) \xrightarrow{w} G(x) \qquad \text{for } t \uparrow t_0.$$

For functions $\alpha : \mathbb{R} \to \mathbb{R}^+$ and $\beta : \mathbb{R} \to \mathbb{R}$ and a non-degenerate distribution function G^* we have

$$F_t(\alpha(t)x+\beta(t)) \overset{w}{\to} G^*(x) \qquad \text{for } t \uparrow t_0$$

if and only if there exist real constants A and B (A>0) with

$$\frac{\alpha(t)}{a(t)} \to A \quad \text{and} \quad \frac{\beta(t)-b(t)}{a(t)} \to B \quad \text{for } t \uparrow t_0$$

and

$$G^*(x) = G(Ax+B) \qquad \text{for all } x \in \mathbb{R}.$$

Corollary 2.4.1 A distribution function F is in the domain of attraction of Λ if and only if there exist functions $a : \mathbb{R} \to \mathbb{R}^+$ and $b : \mathbb{R} \to \mathbb{R}$, such that

(2.4.10) $$\lim_{s\to\infty} s\{1 - F(a(s).x+b(s))\} = e^{-x} \quad \text{for all } x \in \mathbb{R}.$$

Moreover then (2.4.10) holds with (for the function U see (2.4.2))

(2.4.11) $$\begin{cases} b(s) = U(\frac{1}{s}) \\ a(s) = U(\frac{1}{se}) - U(\frac{1}{s}) \qquad \text{for all } s \in \mathbb{R}^+. \end{cases}$$

Proof Suppose $F \in \mathcal{D}(\Lambda)$. Take a and b as in (2.4.11). By corollary 2.1.1

$$F^n(a(n).x+b(n)) \overset{w}{\to} \exp(-e^{-x}).$$

By lemma 2.2.2 then it follows that

$$n . \{1 - F(a(n).x+b(n))\} \to e^{-x} \quad \text{for } n \to \infty \text{ and all } x \in \mathbb{R},$$

and hence

$$s . \{1 - F(a([s])x+b([s]))\} \to e^{-x} \quad \text{for } s \to \infty \text{ and all } x \in \mathbb{R}.$$

As in part a) of the proof of theorem 2.4.1 we can prove

$$\frac{a(s)}{a([s])} \to 1 \quad \text{and} \quad \frac{b(s)-b([s])}{a(s)} \to 0 \quad \text{for } s \to \infty.$$

Application of lemma 2.4.1 gives (2.4.10).

The converse is simple. □

The next theorem is due to Gnedenko.

<u>Theorem 2.4.2</u> A distribution function F belongs to $\mathcal{D}(\Lambda)$ if and only if there exists a function $f : \mathbb{R} \to \mathbb{R}^+$ such that

$$\lim_{t \uparrow x_0} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all real } x. \tag{2.4.12}$$

Here x_0 is the endpoint of F.

<u>Proof</u> a) Suppose $F \in \mathcal{D}(\Lambda)$. Then (2.4.10) holds with (2.4.11). If we substitute s(t) for s in (2.4.10) where

$$s(t) = \frac{1}{1-F(t)} \qquad \text{for all } t \in \mathbb{R},$$

we obtain

$$\lim_{t \uparrow x_0} \frac{1-F(b(s(t))+xa(s(t)))}{1-F(t)} = e^{-x} \quad \text{for all } x \in \mathbb{R}.$$

By the definition of b and s we have for all $\varepsilon > 0$ and all t sufficiently close to x_0

$$0 \le \frac{t-b(s(t))}{a(s(t))} \le \frac{b(s(t)(1+\varepsilon))-b(s(t))}{a(s(t))}.$$

By theorem 2.4.1 the righthand side tends to $\log(1+\varepsilon)$ for $t \uparrow x_0$, hence

$$\lim_{t \uparrow x_0} \frac{t-b(s(t))}{a(s(t))} = 0.$$

Application of lemma 2.4.1 with $F_t(x) = \exp\{-(1-F(t))^{-1}(1-F(x))\}$ and $G(x) = \exp(-e^{-x})$ gives (2.4.12) (the fact that $F_t(-\infty) > 0$ is immaterial).

b) Suppose (2.4.12) holds for some positive function f. Substitution of

$$t(s) = U(\frac{1}{s})$$

for t in (2.4.12) with U defined by (2.4.2), gives

$$\lim_{s\to\infty} \frac{1-F(f(t(s)).x+t(s))}{1-F(t(s))} = e^{-x} \quad \text{for all real } x.$$

By the definition of t we have for all $\varepsilon > 0$ (cf. (1.1.15))

$$1 - F(t(s)) \leq \frac{1}{s} \leq 1 - F(t(s)-0) \leq 1 - F(t(s)-\varepsilon\, f(t(s)))$$

or

$$\frac{1-F(t(s))}{1-F(t(s)-\varepsilon\, f(t(s)))} \leq s \,.\, \{1 - F(t(s))\} \leq 1.$$

Hence by (2.4.12)

$$s \,.\, \{1 - F(t(s))\} \to 1 \qquad \text{for } s \to \infty$$

and (2.4.10) is true with $b(s) = t(s)$ and $a(s) = f(t(s))$ for all $s \in R^+$. By corollary 2.4.1 it now follows that $F \in \mathcal{D}(\Lambda)$. □

Corollary 2.4.2 If (2.4.12) holds for a distribution function with infinite endpoint, then

$$(2.4.13) \qquad \frac{f(t)}{t} \to 0 \qquad \text{for } t \to \infty.$$

If (2.4.12) holds for a distribution function with finite endpoint x_0, then

$$(2.4.14) \qquad \frac{f(t)}{x_0-t} \to 0 \qquad \text{for } t \uparrow x_0.$$

Proof By (2.4.12) for each $\varepsilon > 0$ and each real x there exists a $t_0(x,\varepsilon) < x_0$ such that for all t with $t_0 < t < x_0$

$$(2.4.15) \qquad 1 - \varepsilon < F(t+xf(t)) < 1.$$

If $x_0 = \infty$, this yields

$$t + x.f(t) > 0,$$

hence for each negative x

$$0 < \frac{f(t)}{t} < -\frac{1}{x} \qquad \text{for all } t > t_0.$$

This proves (2.4.13).

If $x_0 < \infty$, relation (2.4.15) yields

$$t + x\, f(t) < x_0,$$

hence for each positive x

$$0 < \frac{f(t)}{x_0-t} < \frac{1}{x} \qquad \text{for all } t \text{ with } t_0(x) < t < x_0.$$

This proves (2.4.14). □

Corollary 2.4.3 a) Suppose that F_1 and F_2 are distribution functions with infinite endpoint and the function $U : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$U(x) = \frac{1-F_1(x)}{1-F_2(x)}$$

is regularly varying at infinity. Then $F_1 \in \mathcal{D}(\Lambda)$ if and only if $F_2 \in \mathcal{D}(\Lambda)$.

b) Suppose F_1 and F_2 are distribution functions with a common finite endpoint x_0 and the function $V : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$V(x) = \frac{1-F_1(x_0-1/x)}{1-F_2(x_0-1/x)}$$

is regularly varying at infinity. Then $F_1 \in \mathcal{D}(\Lambda)$ if and only if $F_2 \in \mathcal{D}(\Lambda)$.

Proof a) Suppose U is ρ-varying ($\rho \in \mathbb{R}$) and $F_1 \in \mathcal{D}(\Lambda)$. Then (2.4.12) holds for $F = F_1$ and some positive function f. By corollary 2.4.2 we have for all real x

$$\frac{t+x.f(t)}{t} \to 1 \qquad \text{for } t \to \infty.$$

By corollary 1.2.1 part 2) it follows that

$$(2.4.16) \qquad \frac{U(t+x.f(t))}{U(t)} \to 1 \qquad \text{for } t \to \infty$$

for all real x. Combining (2.4.12) with $F = F_1$ and (2.4.16) we get

$$\lim_{t\to\infty} \frac{1-F_2(t+x.f(t))}{1-F_2(t)} = e^{-x} \qquad \text{for all real } x.$$

Hence by theorem 2.4.2 we have $F_2 \in \mathcal{D}(\Lambda)$.

b) Suppose V is ρ-varying ($\rho\in\mathbb{R}$) and $F_1 \in \mathcal{D}(\Lambda)$. Then (2.4.12) holds for $F = F_1$ and some positive function f. By corollary 2.4.2 we have for all real x

$$\frac{x_0-t-xf(t)}{x_0-t} \to 1 \qquad \text{for } t \uparrow x_0.$$

By corollary 1.2.1 property 2 it follows that

$$(2.4.17) \qquad \frac{V(\frac{1}{x_0-t-xf(t)})}{V(\frac{1}{x_0-t})} \to 1 \qquad \text{for } t \uparrow x_0$$

for all real x. Combining (2.4.12) with $F = F_1$ and (2.4.17) we get

$$\lim_{t\uparrow x_0} \frac{1-F_2(t+x.f(t))}{1-F_2(t)} = e^{-x} \qquad \text{for all real } x.$$

Hence by theorem 2.4.2 we have $F_2 \in \mathcal{D}(\Lambda)$. □

<u>Lemma 2.4.2</u> Suppose $F \in \mathcal{D}(\Lambda)$, i.e. (2.4.12) holds for some positive function f. Then if b is a real constant, we have

$$\lim_{t\uparrow x_0} \frac{1-F(\beta(t)+x.\alpha(t))}{1-F(t)} = e^{-x-b} \quad \text{for all } x \in \mathbb{R}$$

if and only if

$$\frac{\alpha(t)}{f(t)} \to 1 \quad \text{and} \quad \frac{\beta(t)-t}{f(t)} \to b \qquad \text{for } t \uparrow x_0.$$

<u>Proof</u> Application of lemma 2.4.1 with $F_t(x) = \exp\{-(1-F(t))^{-1}(1-F(x))\}$ and $G(x) = \exp(-e^{-x})$, $a(t) = f(t)$ and $b(t) = t$. The fact that $F_t(-\infty) > 0$ is again immaterial. □

<u>Lemma 2.4.3</u> If $F \in \mathcal{D}(\Lambda)$ with endpoint x_0, there exists a continuous and on $(-\infty,x_0)$ strictly increasing distribution function G (also with endpoint x_0), such that

$$\frac{1-F(x)}{1-G(x)} \to 1 \qquad \text{for } x \uparrow x_0. \tag{2.4.18}$$

<u>Proof</u> a) Suppose first $x_0 = \infty$. We use theorem 2.4.2. As both sides of (2.4.12) are monotone functions of x and as e^{-x} is a continuous function, (2.4.12) holds uniformly on finite intervals. This means that for each bounded real function x(t)

$$\lim_{t\to\infty} \left\{\frac{1-F(t+x(t).f(t))}{1-F(t)} - e^{-x(t)}\right\} = 0.$$

Taking $x(t) = -t^{-1}$ we obtain

$$\lim_{t\to\infty} \frac{1-F(t-t^{-1}f(t))}{1-F(t)} = 1. \tag{2.4.19}$$

We now define a sequence $\{F_n\}_{n=0}^{\infty}$ of distribution functions. Take a constant $c_0 > 0$ such that $F(c_0) > 0$. Take for F_0 a distribution function with

$$F_0(t) = \begin{cases} \text{continuous and strictly increasing} & \text{for } t \le c_0 \\ F(t) & \text{for } t > c_0. \end{cases}$$

Let $\{t_n\}_{n=1}^{\infty}$ be an enumeration of the discontinuity points of F which exceed c_0. Define F_1 by

$$F_1(t) = \begin{cases} F_0(t) & \text{for } t \notin (t_1',t_1) \\ \text{linear} & \text{for } t \in [t_1',t_1], \end{cases}$$

where $t_1' = t_1 - t_1^{-1} f(t_1)$. Then F_1 has no discontinuity point at t_1. For general $n \geq 2$ we define F_n by induction. Take $t_n' = t_n$ if $t_n \in \bigcup_{k=1}^{n-1} (t_k',t_k)$. Otherwise we choose t_n' in such a way that $t_n - t_n^{-1} f(t_n) \leq t_n' < t_n$ and

$$(t_n',t_n) \cap \bigcup_{k=1}^{n-1} (t_k',t_k) = \emptyset.$$

Then F_n is defined by

$$F_n(t) = \begin{cases} F_{n-1}(t) & \text{for } t \notin (t_n',t_n) \\ \text{linear} & \text{for } t \in [t_n',t_n]. \end{cases}$$

Clearly the intervals (t_1',t_1), (t_2',t_2),... are disjoint, hence for all t the sequence $F_n(t)$ has a limit $H(t)$ for $n \to \infty$. Clearly H is a continuous distribution function but not necessarily strictly increasing. We now prove

$$\text{(2.4.20)} \qquad \frac{1-F(t)}{1-H(t)} \to 1 \qquad \text{for } t \to \infty.$$

e need only prove (2.4.20) for those t for which $H(t) \neq F(t)$. For such a we have $t \in (t_n',t_n)$ for some n. From the construction of F_n it follows hat

$$\frac{1-F(t_n)}{1-F(t_n-t_n^{-1}f(t_n))} \leq \frac{1-H(t)}{1-F(t)} \leq \frac{1-F(t_n-t_n^{-1}f(t_n))}{1-F(t_n)} .$$

ience by (2.4.19) we have proved (2.4.20).

At this point we have proved lemma 2.4.3 but for the "strictly in- :reasing" if we take $G = H$. To construct the required strictly increasing ; we use H in the following manner. Suppose $\{u_n\}$ is an enumeration of the initial points of the intervals where H is constant and let $\{v_n\}$ be the corresponding endpoints. We define a sequence $\{H_n\}_{n=1}^{\infty}$ of distribution functions. Take $w_1 \in [u_1 - u_1^{-1} f(u_1), u_1)$ in such a way that $w_1 \notin \bigcup_{k=2}^{\infty} (u_k,v_k)$. Define H_1 by

$$H_1(t) = \begin{cases} H(t) & \text{for } t \notin (w_1,v_1) \\ \text{linear} & \text{for } t \in [w_1,v_1]. \end{cases}$$

Then H_1 is strictly increasing on (w_1,v_1). For general $n \geq 2$ we define H_n by induction. Take $w_n = v_n$ if $(u_n,v_n) \subset \bigcup_{k=1}^{n-1} (w_k,v_k)$. Otherwise we choose $w_n \in [u_n - u_n^{-1} f(u_n), u_n)$ in such a way that

$$\begin{cases} (w_n,u_n) \cap \bigcup_{k=1}^{n-1} (w_k,v_k) = \emptyset, \\ w_n \notin \bigcup_{k=n+1}^{\infty} (u_k,v_k). \end{cases}$$

These H_n tend weakly to some distribution function G which clearly is continuous and strictly increasing. In the same way as we proved (2.4.20) it can be shown that

$$(2.4.21) \qquad \frac{1-H(t)}{1-G(t)} \to 1 \qquad \text{for } t \to \infty.$$

From (2.4.20) and (2.4.21) we have (2.4.18).

b) Suppose now $x_0 < \infty$. We take $x(t) = t-x_0$ in (2.4.12) and obtain

$$\lim_{t \uparrow x_0} \frac{1-F(t-(x_0-t).f(t))}{1-F(t)} = 1.$$

The rest of the proof is completely analogous to part a). □

2.5 THE DOMAIN OF ATTRACTION OF Λ

In this section the results of section 1.4 are used to obtain necessary and sufficient conditions for a distribution function to belong to $\mathcal{D}(\Lambda)$. First we give a specification of the auxiliary function in Gnedenko's characterization (theorem 2.4.2).

Theorem 2.5.1 A distribution function F belongs to the domain of attraction of Λ if and only if

$$(2.5.1) \qquad \lim_{t \uparrow x_0} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}$$

with

$$(2.5.2) \qquad f(t) = \frac{\int_t^{x_0} \{1-F(s)\}ds}{1-F(t)} \qquad \text{for all real } t < x_0.$$

Here x_0 is the endpoint of F defined by

$$x_0 = \sup\{x \mid F(x) < 1\}.$$

Proof a) Suppose (2.5.1) holds with (2.5.2). Then by theorem 2.4.2 we have $F \in \mathcal{D}(\Lambda)$.

b) Suppose $F \in \mathcal{D}(\Lambda)$. By lemma 2.4.3 there exists a continuous and on $(-\infty,x_0)$ strictly increasing distribution function G with

$$(2.5.3) \qquad \frac{1-F(x)}{1-G(x)} \to 1 \qquad \text{for } x \uparrow x_0.$$

From theorem 2.4.2 it is clear that $G \in \mathcal{D}(\Lambda)$. We first show that

$$(2.5.4) \qquad \lim_{t\uparrow x_0} \frac{1-G(t+x.g(t))}{1-G(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}$$

with

$$(2.5.5) \qquad g(t) = \frac{\int_t^{x_0} \{1-G(s)\}ds}{1-G(t)} \qquad \text{for all real } t < x_0.$$

By theorem 2.4.1 we have for all positive x and y $(y\neq 1)$

$$\lim_{t\downarrow 0} \frac{U(tx)-U(t)}{U(ty)-U(t)} = \frac{\log x}{\log y},$$

where

$$U(x) = \inf\{y \mid 1 - G(y) \leq x\} \text{ for all } x \in (0,1).$$

As U is continuous and strictly increasing, theorem 1.4.3 gives

$$(2.5.6) \qquad \lim_{t\downarrow 0} \frac{U(\frac{t}{e})-U(t)}{\frac{1}{t}\int_0^t U(s)ds-U(t)} = 1.$$

As $G \in \mathcal{D}(\Lambda)$, by corollary 2.4.1 we have equation (2.4.10) with the functions a and b defined by (2.4.11). By (2.5.6) and lemma 2.4.1 (for $G_s(x) = 1 - s\{1 - G(x)\}$) we obtain

$$\lim_{s\to\infty} s\{1 - G(a(s).x+b(s))\} = e^{-x} \quad \text{for all } x \in \mathbb{R}$$

with for all $s \in \mathbb{R}^+$

$$\begin{cases} b(s) = U(\frac{1}{s}) \\ a(s) = s \, . \displaystyle\int_0^{1/s} U(p)dp - U(\frac{1}{s}). \end{cases}$$

If we regard s in this equation as a function of t given by

$$s(t) = \frac{1}{1-G(t)} \qquad \text{for all real } t < x_0,$$

we obtain (2.5.4) with (2.5.5) (it is not hard to see that a(s(t)) reduces to g(t) as given in (2.5.5)). To prove the theorem for F we observe that by lemma 1.2.1 b)

$$\frac{\displaystyle\int_x^{x_0} \{1-F(t)\}dt}{\displaystyle\int_x^{x_0} \{1-G(t)\}dt} \to 1 \qquad \text{for } x \uparrow x_0$$

and hence

$$\frac{f(t)}{g(t)} \to 1 \qquad \text{for } x \uparrow x_0.$$

With (2.5.3) and (2.5.4) we have

$$\lim_{t\uparrow x_0} \frac{1-F(t+x.g(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}.$$

Application of lemma 2.4.2 gives (2.5.1). □

<u>Corollary 2.5.1</u> If $F \in \mathcal{D}(\Lambda)$, then

$$F^n(a_n x+b_n) \overset{w}{\to} \exp(-e^{-x})$$

with for n = 1, 2, 3, ...

$$\begin{cases} b_n = \inf\{x \mid 1 - F(x) \le 1/n\} \\ a_n = \dfrac{\int_{b_n}^{x_0} \{1-F(t)\}dt}{1-F(b_n)} \end{cases} .$$

<u>Proof</u> By corollary 2.4.1 we have (use (2.4.10) with $x = 0$)

$$\lim_{n\to\infty} n\{1 - F(b_n)\} = 1.$$

Hence replacing t in (2.5.1) by b_n we obtain

$$\lim_{n\to\infty} n\{1 - F(a_n x+b_n)\} = \lim_{n\to\infty} \frac{1-F(a_n x+b_n)}{1-F(b_n)} = e^{-x} \quad \text{for all } x \in \mathbb{R}.$$

Application of lemma 2.2.2 gives the statement of the corollary. □

<u>Lemma 2.5.1</u> Suppose $F \in \mathcal{D}(\Lambda)$, i.e. (2.5.1) holds with the function f defined by (2.5.2). Then with the same function f we have

$$(2.5.7) \qquad \lim_{t\uparrow x_0} \frac{1-F_1(t+x.f(t))}{1-F_1(t)} = e^{-x} \quad \text{for all } x \in \mathbb{R},$$

where

$$(2.5.8) \qquad F_1(x) = 1 - \int_x^{x_0} \{1 - F(t)\}dt \quad \text{for all } x < x_0.$$

<u>Proof</u> By (2.5.1) and corollary 2.4.2 we have $t + x\, f(t) \uparrow x_0$ for $t \uparrow x_0$ and all $x \in R$. Hence in (2.5.1) we may replace t by $t(u) = u + y\, f(u)$, where y is an arbitrary real number and $u < x_0$, to obtain

$$\lim_{u\uparrow x_0} \frac{1-F(u+yf(u)+x.f(u+y.f(u)))}{1-F(u+y.f(u))} = e^{-x} \quad \text{for all } x \in \mathbb{R}.$$

In this relation we may replace $1 - F(u+y.f(u))$ by $e^{-y}.\{1 - F(u)\}$ in virtue of theorem 2.5.1. This leads to

$$\lim_{t\uparrow x_0} \frac{1-F(t+y.f(t)+x.f(t+y.f(t)))}{1-F(t)} = e^{-x-y} \quad \text{for all } x \in \mathbb{R}.$$

Combining this equation with (2.5.1) we get by lemma 2.4.2

$$(2.5.9) \qquad \lim_{t\uparrow x_0} \frac{f(t+y.f(t))}{f(t)} = 1 \qquad \text{for all } y \in \mathbb{R}.$$

As

$$f(t) = \frac{1-F_1(t)}{1-F(t)} \qquad \text{for all } t < x_0,$$

we have by (2.5.9)

$$\lim_{t\uparrow x_0} \frac{1-F_1(t+y.f(t))}{1-F_1(t)} \cdot \frac{1-F(t)}{1-F(t+y.f(t))} = 1$$

and hence by (2.5.1) the proof is complete. □

The next theorems provide necessary and sufficient conditions for $F \in \mathcal{D}(\Lambda)$. It is convenient to use the notation

$$x_1 = \begin{cases} 1 & \text{if } x_0 = \infty \\ x_0 - 1 & \text{if } x_0 < \infty, \end{cases}$$

where x_0 is the endpoint of F.

Theorem 2.5.2 For a distribution function F the following assertions are equivalent.

a) F belongs to $\mathcal{D}(\Lambda)$.

b) The integral $\int_x^{x_0}\int_y^{x_0} \{1 - F(t)\}dtdy$ is finite and

$$(2.5.10) \qquad \lim_{x\uparrow x_0} \frac{\{1-F(x)\}\left\{\int_x^{x_0}\int_y^{x_0} \{1-F(t)\}dtdy\right\}}{\left\{\int_x^{x_0} \{1-F(t)\}dt\right\}^2} = 1.$$

c) There exist real constants c_1 and c_2 and real-valued functions c, a and b defined on $(-\infty,x_0)$ *) with

$$(2.5.11)\qquad \begin{cases} c(x) > 0 \text{ for all } x < x_0,\ \lim\limits_{x\uparrow x_0} c(x) = c_1 > 0, \\ \lim\limits_{x\uparrow x_0} a(x) = 1, \\ \begin{cases} c_2 + \int_1^x b(t)dt > 0 \text{ for all } x \in \mathbb{R} & \text{if } x_0 = \infty \\ \int_x^{x_0} b(t)dt > 0 \quad \text{for all } x < x_0 & \text{if } x_0 < \infty \end{cases} \\ \text{and } \lim\limits_{x\uparrow x_0} b(t) = 0, \end{cases}$$

such that for $x < x_0$

$$(2.5.12)\qquad 1 - F(x) = c(x) \, . \, \exp\left\{- \int_{x_1}^{x} \frac{a(t)}{f(t)}\, dt\right\},$$

where for all $x < x_0$

$$(2.5.13)\qquad f(x) = \begin{cases} c_2 + \int_1^x b(t)dt & \text{if } x_0 = \infty \\ \int_x^{x_0} b(t)dt & \text{if } x_0 < \infty. \end{cases}$$

Proof a) $\Rightarrow$ b): Suppose $F \in \mathcal{D}(\Lambda)$. By theorem 2.5.1

$$(2.5.14)\qquad \lim_{t\uparrow x_0} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}$$

with

$$(2.5.15)\qquad f(t) = \frac{\int_t^{x_0} \{1-F(s)\}ds}{1-F(t)} \qquad \text{for all } t < x_0.$$

*) We suppose that a and b are such that all integrals from (2.5.11) and (2.5.12) exist.

By lemma 2.5.1 we have (with f as defined in (2.5.15))

$$\lim_{t \uparrow x_0} \frac{1-F_1(t+x.f(t))}{1-F_1(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R} \tag{2.5.16}$$

with

$$F_1(x) = \max\{0,\ 1 - \int_x^{x_0} \{1 - F(t)\}dt\} \quad \text{for all } x < x_0.$$

Hence by theorem 2.4.2 we have $F_1 \in \mathcal{D}(\Lambda)$. But then by theorem 2.5.1 equation (2.5.16) also holds with f replaced by f^*, where

$$f^*(t) = \frac{\int_t^{x_0} \int_u^{x_0} \{1-F(s)\}ds du}{\int_t^{x_0} \{1-F(s)\}ds} \qquad \text{for all } t < x_0.$$

By lemma 2.4.2 then

$$\frac{f^*(t)}{f(t)} \to 1 \qquad \text{for } t \uparrow x_0.$$

This gives (2.5.10).

b) $\Longrightarrow$ c): Suppose (2.5.10) holds. Define the function b by

$$b(x) = -1 + \frac{\{1-F(x)\}\left\{\int_x^{x_0} \int_y^{x_0} \{1-F(t)\}dt dy\right\}}{\left\{\int_x^{x_0} \{1-F(t)\}dt\right\}^2} \quad \text{for all } x < x_0. \tag{2.5.17}$$

By assumption we have $b(x) \to 0$ for $x \uparrow x_0$. Obviously b is summable on finite intervals and for some constant c_2 we have (as the two sides are absolutely continuous with the same derivative a.e.)

$$c_2 + \int_{x_1}^{x} b(t)dt = \frac{\int_x^{x_0} \int_y^{x_0} \{1-F(t)\}dt dy}{\int_x^{x_0} \{1-F(t)\}dt} \quad \text{for all } x < x_0. \tag{2.5.18}$$

We denote the (clearly positive and continuous) righthand side by $f(x)$. For $x_0 < \infty$ we compute c_2. As

$$\int_x^{x_0} \int_y^{x_0} \{1 - F(t)\}dtdy \leq \int_x^{x_0} \int_x^{x_0} \{1 - F(t)\}dtdy =$$

$$= (x_0-x) \cdot \int_x^{x_0} \{1 - F(t)\}dt,$$

we have

$$\frac{f(x)}{x_0-x} \leq 1$$

and hence $f(x) \to 0$ for $x \uparrow x_0$. As b is bounded near x_0, we may conclude $c_2 = - \int_{x_1}^{x_0} b(t)dt$. So we find for all $x < x_0$

$$f(x) = \begin{cases} c_2 + \int_1^x b(t)dt & \text{if } x_0 = \infty \\ - \int_x^{x_0} b(t)dt & \text{if } x_0 < \infty. \end{cases}$$

Integrating $\{f(t)\}^{-1}$ we get (by ordinary Riemann integration)

$$\int_{x_1}^{x} \frac{dt}{f(t)} = c_3 - \log \left\{ \int_x^{x_0} \int_y^{x_0} \{1 - F(t)\}dtdy \right\} \qquad (2.5.19)$$

for all $x < x_0$.

Taking exponentials on both sides in (2.5.19) and using the definitions of f and b we obtain

$$1 - F(x) = e^{c_3} \cdot \{1 + b(x)\} \cdot \{f(x)\}^{-2} \cdot \exp\left\{ - \int_{x_1}^{x} \frac{dt}{f(t)} \right\} \qquad (2.5.20)$$

for all $x < x_0$.

As

$$\log f(x) = \log f(x_1) + \int_{x_1}^{x} \frac{b(t)}{f(t)}\, dt \quad \text{for all } x < x_0,$$

we may replace (2.5.20) by

$$1 - F(x) = c(x) \, . \, \exp\left\{ - \int_{x_1}^{x} \frac{a(t)}{f(t)} \, dt \right\} \quad \text{for all } x < x_0 \tag{2.5.21}$$

with for those x

$$\begin{cases} c(x) = e^{c_3} \{f(x_1)\}^{-2} \, . \, \{1 + b(x)\} \\ a(x) = 1 + 2\, b(x). \end{cases} \tag{2.5.22}$$

This provides the representation as given in part c) of the theorem.

c) $\Rightarrow$ a): By theorem 2.4.2 it is sufficient to prove that F satisfies

$$\lim_{t \uparrow x_0} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R} \tag{2.5.23}$$

with f defined by (2.5.13).

First we remark that by (2.5.13)

$$\begin{cases} \lim_{t \to \infty} t^{-1} \, . \, f(t) = 0 & \text{if } x_0 = \infty \\ \lim_{t \uparrow x_0} (x_0 - t)^{-1} f(t) = 0 & \text{if } x_0 < \infty. \end{cases} \tag{2.5.24}$$

From this it follows that for each fixed real x

$$t + x \, . \, f(t) < x_0$$

for all sufficiently large $t < x_0$ and

$$t + x \, . \, f(t) \to x_0 \qquad \text{for } t \uparrow x_0.$$

Suppose first x > 0. We start by proving

$$\lim_{t \uparrow x_0} \frac{f(t)}{f(t+yf(t))} = 1 \tag{2.5.25}$$

uniformly for $0 \leq y \leq x$. Choose $\varepsilon > 0$ and $t_0(\varepsilon)$ such that for $t_0(\varepsilon) \leq t < x_0$

$$-\varepsilon < b(t) < \varepsilon,$$

then for all $y \in [0,x]$

$$-\varepsilon.y.f(t) < \int_t^{t+y.f(t)} b(s)ds = f(t+y.f(t)) - f(t) < \varepsilon.y.f(t).$$

Hence we have

$$1 - y.\varepsilon < \frac{f(t+y.f(t))}{f(t)} < 1 + y.\varepsilon$$

and (provided $\varepsilon < x^{-1}$)

$$\frac{1}{1+x.\varepsilon} \leq \frac{1}{1+y.\varepsilon} < \frac{f(t)}{f(t+y.f(t))} < \frac{1}{1-y.\varepsilon} \leq \frac{1}{1-x.\varepsilon} .$$

This yields (2.5.25). Next we prove (2.5.23). Using (2.5.12) we have

(2.5.26)
$$\frac{1-F(t+x.f(t))}{1-F(t)} =$$

$$= \frac{c(t+x.f(t))}{c(t)} \exp\left\{ - \int_0^x a(t+yf(t)) \frac{f(t)}{f(t+yf(t))} dy \right\} .$$

As for $t \uparrow x_0$ the integrand in (2.5.26) tends to 1 uniformly for $0 \leq y \leq x$, relation (2.5.23) is proved.

For $x < 0$ and $x_0 = \infty$ we take $t_0(\varepsilon)$ as before and $t_1(\varepsilon)$ such that for $t \geq t_1(\varepsilon)$

$$t + x\ f(t) = t\{1 + x \frac{f(t)}{t}\} \geq t_0(\varepsilon).$$

For $x < 0$ and $x_0 < \infty$ we take $t_0(\varepsilon)$ as before and $t_1(\varepsilon)$ such that for $t \geq t_1(\varepsilon)$

$$t + x\ f(t) = x_0 + t - x_0 + x\ f(t) =$$

$$= x_0 - (x_0-t)\ \{1 - x \frac{f(t)}{x_0-t}\} \geq t_0(\varepsilon).$$

These choices are possible by (2.5.24). In the same way as before we now obtain (2.5.25) uniformly for $x \le y \le 0$. Using (2.5.12) as before we obtain (2.5.23) for $x < 0$. □

Remark 2.5.1 From (2.5.21) we see that F can be expressed (apart from one or two constants) in terms of the single auxiliary function b defined by (2.5.17).

The next theorem gives an alternative formulation for part c) of theorem 2.5.2.

Theorem 2.5.3 A distribution function F belongs to $\mathcal{D}(\Lambda)$ if and only if there exist a real constant c_1 and real valued functions c, a and f defined on $(-\infty, x_0)$ with

$$(2.5.27) \quad \begin{cases} c(x) > 0 \text{ for all } x < x_0, \lim_{x \uparrow x_0} c(x) = c_1 > 0, \\ \lim_{x \uparrow x_0} a(x) = 1, \\ f(x) \text{ positive and differentiable for all } x < x_0 \\ \text{and } \lim_{x \uparrow x_0} f'(x) = 0, \\ \text{moreover } \lim_{x \uparrow x_0} f(x) = 0 \text{ if } x_0 < \infty, \end{cases}$$

such that (2.5.12) holds for all $x < x_0$.

Proof a) Suppose (2.5.12) holds for all $x < x_0$ together with (2.5.27). By (2.5.27) there exists an $x_2 < x_0$ such that f' is bounded on (x_2, x_0). By a well-known theorem on Lebesgue-integration (see [15] p. 368) we have for $x_2 < x < x_0$

$$f(x) = \begin{cases} f(x_2) + \int_{x_2}^{x} f'(t)dt & \text{if } x_0 = \infty \\ \int_{x}^{x_0} f'(t)dt & \text{if } x_0 < \infty, \end{cases}$$

so that by theorem 2.5.2 it follows that $F \in \mathcal{D}(\Lambda)$.

b) Suppose $F \in \mathcal{D}(\Lambda)$. If we define f on $(-\infty, x_0)$ by

$$f(x) = \frac{\int_x^{x_0}\int_y^{x_0}\int_z^{x_0} \{1-F(t)\}dtdzdy}{\int_x^{x_0}\int_y^{x_0} \{1-F(t)\}dtdy},$$

then for all $x < x_0$ f has a derivative f' given by

$$f'(x) = -1 + \frac{\left\{\int_x^{x_0} \{1-F(t)\}dt\right\}\left\{\int_x^{x_0}\int_y^{x_0}\int_z^{x_0} \{1-F(t)\}dtdzdy\right\}}{\left\{\int_x^{x_0}\int_y^{x_0} \{1-F(t)\}dtdy\right\}^2}.$$

By lemma 2.5.1 and theorem 2.5.2 b) we have

$$\lim_{x \uparrow x_0} f'(x) = 0.$$

Now a procedure analogous to that in the proof of b) $\Longrightarrow$ c) of theorem 2.5.2 gives the representation for F. □

The next theorem gives an alternative formulation for part b) of theorem 2.5.2. We omit the proof which uses partial integration.

<u>Theorem 2.5.4</u> The distribution function F belongs to $\mathcal{D}(\Lambda)$ if and only if $\int_{x_1}^{x_0} x^2 dF(x) < \infty$ and

$$\text{(2.5.28)} \qquad \lim_{x \uparrow x_0} \frac{\{1-F(x)\}.\int_x^{x_0} t^2 dF(t) - \left\{\int_x^{x_0} tdF(t)\right\}^2}{\left(\int_x^{x_0} tdF(t) - x\{1-F(x)\}\right)^2} = 1.$$

Condition c) of theorem 2.5.2 can be formulated in a somewhat sharper form as is shown in the next lemma.

<u>Lemma 2.5.2</u> For all x_0 with $-\infty < x_0 \leq \infty$ and all functions a and b defined on $(-\infty, x_0)$ and satisfying the appropriate conditions of (2.5.11),

there exists a function c satisfying the appropriate conditions of (2.5.11) such that the function F defined by (2.5.12) (via (2.5.13)) is a distribution function with endpoint x_0 (which then by theorem 2.5.2 c) belongs to $\mathcal{D}(\Lambda)$).

Proof By (2.5.11) there exists an x_2 with $x_1 \leq x_2 < x_0$ such that $a(x) > 0$ for $x_2 < x < x_0$. Define c by

$$c(x) = \exp\left\{\int_{x_1}^{\min(x,x_2)} \frac{a(t)}{f(t)}\,dt\right\} \qquad \text{for all } x < x_0$$

and F by (2.5.12). Then F is continuous and non-decreasing with $\lim_{t\downarrow-\infty} F(x) = 0$. To show $F(x) \to 1$ for $x \uparrow x_0$ we observe that (2.5.11) and (2.5.13) imply

$$\begin{cases} \lim_{x\to\infty} \dfrac{f(x)}{x} = 0 & \text{if } x_0 = \infty \\ \lim_{x\uparrow x_0} \dfrac{f(x)}{x_0-x} = 0 & \text{if } x_0 < \infty. \end{cases}$$

As for $x < x_0$

$$-\log\{1-F(x)\} = \begin{cases} -\log c(x) + \displaystyle\int_1^x \frac{a(t)/t}{f(t)/t}\,dt & \text{if } x_0 = \infty \\ -\log c(x) + \displaystyle\int_{x_1}^x \frac{a(t)/(x_0-t)}{f(t)/(x_0-t)}\,dt & \text{if } x_0 < \infty, \end{cases}$$

we have $-\log\{1-F(x)\} \to \infty$ for $x \uparrow x_0$. Hence F is a distribution function. □

We conclude this section with two corollaries of theorem 2.5.2 concerning the behaviour of $1 - F$ near x_0.

Corollary 2.5.2 If $F \in \mathcal{D}(\Lambda)$, then

$$\lim_{x\uparrow x_0} \frac{\log \int_x^{x_0} \{1-F(t)\}dt}{\log\{1-F(x)\}} = 1. \tag{2.5.29}$$

<u>Proof</u> Take b and f as in the proof of b) $\Longrightarrow$ c) of theorem 2.5.2. By (2.5.19) we have for all $x < x_0$

$$(2.5.30)\qquad \int_x^{x_0} \{1 - F(t)\}dt = \{f(x)\}^{-1} \cdot \int_x^{x_0}\int_y^{x_0} \{1 - F(t)\}dtdy =$$

$$= e^{c_3} \cdot \{f(x)\}^{-1} \cdot \exp\left\{ - \int_{x_1}^{x} \frac{1}{f(t)}\, dt \right\} =$$

$$= e^{c_3} \cdot \{f(x_1)\}^{-1} \cdot \exp\left\{ - \int_{x_1}^{x} \frac{1+b(t)}{f(t)}\, dt \right\} .$$

Now (2.5.29) is an easy consequence of (2.5.12), (2.5.30) and lemma 1.2.1 a). □

<u>Corollary 2.5.3</u> If $F \in \mathcal{D}(\Lambda)$, then

$$\begin{cases} \lim_{x\to\infty} \dfrac{\log\{1-F(x)\}}{\log x} = -\infty & \text{if } x_0 = \infty, \\ \lim_{x\uparrow x_0} \dfrac{\log\{1-F(x)\}}{\log(x_0-x)} = \infty & \text{if } x_0 < \infty. \end{cases}$$

(Hence if $x_0 = \infty$, $\lim_{x\to\infty} x^\alpha\{1 - F(x)\} = 0$ for all $\alpha > 0$ and thus $\int_0^\infty x^\alpha dF(x) < \infty$ for all $\alpha > 0$.)

<u>Proof</u> The result follows easily from the representation in part c) of theorem 2.5.2. □

2.6. A UNIFYING APPROACH

For distribution functions with finite endpoint we now combine the results with respect to the domains of attraction of the possible limit distributions Ψ_α and Λ in the following theorem.

<u>Theorem 2.6.1</u> Suppose F is a distribution function with finite endpoint x_0. The distribution functions

$$F^n(a_n x+b_n)$$

converge weakly to a non-degenerate distribution function for a proper choice of the constants $a_n > 0$ and b_n if and only if for some value c from $(\frac{1}{2},1]$

$$\lim_{x\uparrow x_0} g(x) = c \tag{2.6.1}$$

where

$$g(x) = \frac{\{1-F(x)\}.\left\{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}dt\,dy\right\}}{\left\{\int_x^{x_0}\{1-F(t)\}dt\right\}^2} \qquad \text{for all } x < x_0. \tag{2.6.2}$$

If $c = 1$, then $F \in \mathcal{D}(\Lambda)$. If $c < 1$, then $F \in \mathcal{D}(\Psi_\alpha)$ with $\alpha = (1-c)^{-1} - 2$.

Proof a) Suppose (2.6.1) holds with $\frac{1}{2} < c \leq 1$. Suppose first $c = 1$. Then by theorem 2.5.2 we have $F \in \mathcal{D}(\Lambda)$. Next suppose $\frac{1}{2} < c < 1$. As in the part b) $\Longrightarrow$ c) of the proof of theorem 2.5.2 we have

$$\int_x^{x_0}\{1-g(t)\}dt = \frac{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}dt\,dy}{\int_x^{x_0}\{1-F(t)\}dt} \qquad \text{for all } x < x_0$$

and hence by (2.6.1)

$$\frac{\int_x^{x_0}\int_y^{x_0}\{1-F(t)\}dt\,dy}{(x_0-x)\int_x^{x_0}\{1-F(t)\}dt} \to 1-c \qquad \text{for } x \uparrow x_0. \tag{2.6.3}$$

Using the definition of g and (2.6.1) we obtain

$$\frac{\int_x^{x_0}\{1-F(t)\}dt}{(x_0-x)\{1-F(x)\}} \to c^{-1}(1-c) \qquad \text{for } x \uparrow x_0 \tag{2.6.4}$$

or

$$(2.6.5)\qquad \lim_{y\to\infty} \frac{y.\int_y^{\infty} \{1-F(x_0-1/t)\}\frac{dt}{t^2}}{1-F(x_0-1/y)} = c^{-1}(1-c).$$

From theorem 1.2.1 b) it follows that $1 - F(x_0-1/x)$ is regularly varying at $x = \infty$ with exponent $2 - (1-c)^{-1}$. Then by theorem 2.3.2 we have $F \in \mathcal{D}(\Psi_\alpha)$ with $\alpha = (1-c)^{-1} - 2$.

b) Suppose first $F \in \mathcal{D}(\Lambda)$. By theorem 2.5.2 we have (2.6.1) with $c = 1$.

Next suppose $F \in \mathcal{D}(\Psi_\alpha)$ for some $\alpha > 0$. Then by theorem 2.3.2 the function $1 - F(x_0-1/x)$ is $-\alpha$-varying at $x = \infty$ and by theorem 1.2.1 a) the equations (2.6.5) and (2.6.4) hold with $c = 1 - (2+\alpha)^{-1}$. By (2.6.5) the function $\int_y^{\infty} \{1-F(x_0-1/t)\}\frac{dt}{t^2}$ is $(-\alpha-1)$-varying at infinity and hence by theorem 1.2.1 a) equation (2.6.3) holds. Combining (2.6.3) and (2.6.4) we get (2.6.1). □

<u>Remark 2.6.1</u> If for a distribution function F with $x_0 < \infty$ the limit of the function g defined in (2.6.2) exists for $x \uparrow x_0$, then necessarily $\frac{1}{2} \le c \le 1$. To prove this we note that the derivation of (2.6.4) holds for arbitrary c and that the left-hand side of (2.6.4) is between 0 and 1. In theorem 2.6.1 we have excluded $c = \frac{1}{2}$. This case is equivalent with slow variation of $1 - F(x_0-1/x)$ at $x = \infty$ (the method of part a) of the proof of theorem 2.6.1 actually covers this case as well). Using theorem 1.1.3 a), remark 1.1.1 and lemma 2.2.2 we can show that (2.6.1) with $c = \frac{1}{2}$ is equivalent with the possibility of choosing positive numbers a_n such that

$$(2.6.6)\qquad \lim_{n\to\infty} F^n(a_n x+x_0) = \begin{cases} e^{-1} & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

It is now clear why $c = \frac{1}{2}$ had to be excluded in theorem 2.6.1.

Next we derive a similar theorem for distribution functions with endpoint at infinity. In this case the criterion of theorem 2.6.1 would not work: if for example $F \in \mathcal{D}(\Phi_{\frac{1}{2}})$, then g is not defined because here $\int_0^{\infty} \{1-F(t)\}dt = \infty$.

Theorem 2.6.2 Suppose F is a distribution function with endpoint at infinity. The distribution functions

$$F^n(a_n x+b_n)$$

converge weakly to a non-degenerate distribution function for a proper choice of the constants $a_n > 0$ and b_n if and only if for some value c from [1,2)

$$\lim_{x\to\infty} h(x) = c \tag{2.6.7}$$

where

$$h(x) = \frac{\{1-F(x)\}.\left\{\int_x^\infty \int_y^\infty \{1-F(t)\}\frac{dt}{t^3}\,dy\right\}}{x^3.\left\{\int_x^\infty \{1-F(t)\}\frac{dt}{t^3}\right\}^2} \quad \text{for all real } x. \tag{2.6.8}$$

If c = 1, then $F \in \mathcal{D}(\Lambda)$. If c > 1, then $F \in \mathcal{D}(\Phi_\alpha)$ with $\alpha = (c-1)^{-1} - 1$.

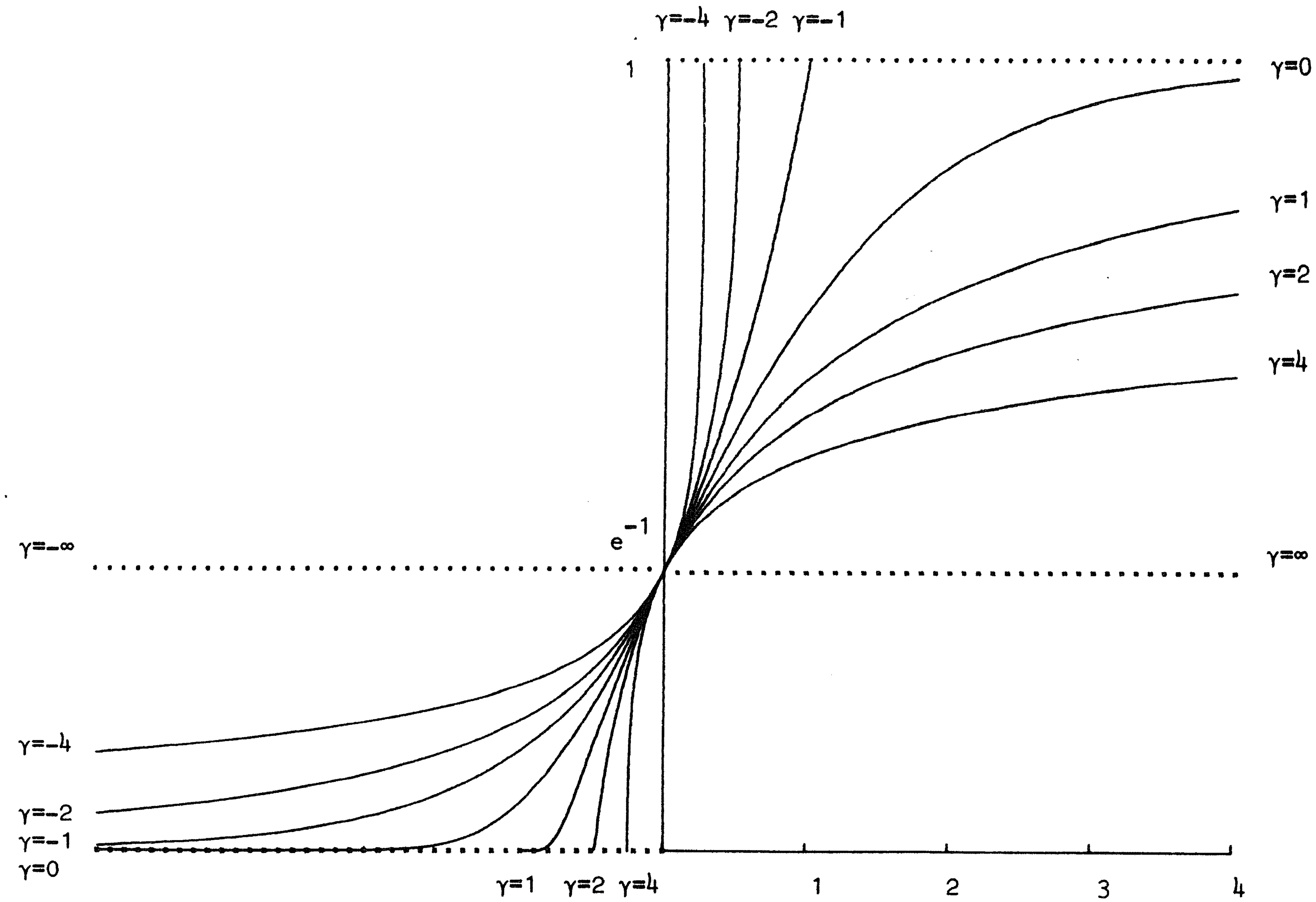
Remark The repeated use of $x^{-3}\{1-F(x)\}$ in (2.6.8) (3 times) is not necessary. Actually we may formulate a similar theorem using each time $x^{-p}\{1-F(x)\}$ for an arbitrary real $p \geq 2$.

Proof a) Suppose (2.6.7) holds with $1 < c \leq 2$. Suppose first c = 1. Then by theorem 2.5.2 and corollary 2.4.3 we have $F \in \mathcal{D}(\Lambda)$. Next suppose $1 < c < 2$. As in the part b) $\Rightarrow$ c) of the proof of theorem 2.5.2 we obtain for some constant c_1

$$\frac{\int_x^\infty \int_y^\infty \{1-F(t)\}\frac{dt}{t^3}\,dy}{\int_x^\infty \{1-F(t)\}\frac{dt}{t^3}} = c_1 + \int_1^x \{g(t)-1\}dt \quad \text{for all real } x$$

and hence by (2.6.7)

$$\frac{\int_x^\infty \int_y^\infty \{1-F(t)\}\frac{dt}{t^3}\,dy}{x.\int_x^\infty \{1-F(t)\}\frac{dt}{t^3}} \to c - 1 \qquad \text{for } x \to \infty. \tag{2.6.9}$$



The stable distribution G_γ.

Using the definition of h and (2.6.7) we obtain

$$(2.6.10) \qquad \frac{x^2 . \int_x^\infty \{1-F(t)\}\frac{dt}{t^3}}{1-F(x)} \to c^{-1}(c-1) \qquad \text{for } x \to \infty.$$

From theorem 1.2.1 b) it follows that 1 - F is regularly varying at infinity with exponent $1 - (c-1)^{-1}$. Then by theorem 2.3.1 we have $F \in \mathcal{D}(\Phi_\alpha)$ with $\alpha = (c-1)^{-1} - 1$.

b) Suppose first $F \in \mathcal{D}(\Lambda)$. By corollary 2.4.3 and theorem 2.5.2 we have (2.6.7) with c = 1.

Next suppose $F \in \mathcal{D}(\Phi_\alpha)$ for some $\alpha > 0$. Then by theorem 2.3.1 the function 1 - F is $-\alpha$-varying at infinity and by theorem 1.2.1 a) equation (2.6.10) holds with $c = (1+\alpha)^{-1} + 1$. By (2.6.10) the function $\int_x^\infty \{1-F(t)\}\frac{dt}{t^3}$ is $(-\alpha-2)$-varying at infinity and hence by theorem 1.2.1 a) equation (2.6.9) holds. Combining (2.6.9) and (2.6.10) we get (2.6.7). □

Remark 2.6.2 Here we can make a similar remark as after theorem 2.6.1. If h tends to some value c, then we now necessarily have $1 \leq c \leq 2$. In theorem 2.6.2 the value c = 2 is excluded. This case is equivalent with slow variation of 1 - F at infinity. This again is equivalent with the possibility of choosing positive numbers a_n such that

$$(2.6.11) \qquad \lim_{n\to\infty} F^n(a_n x) = \begin{cases} 0 & \text{for } x < 0 \\ e^{-1} & \text{for } x > 0. \end{cases}$$

Our figure shows the possible limit distributions G normalized in such a way that

$$(2.6.12) \qquad \begin{cases} G(0) = e^{-1} \\ G'(0) = e^{-1}. \end{cases}$$

For these limit distributions we find it convenient to use the following extension of von Mises' parametrization (see [15]) depending on γ.

$$\text{(2.6.13)}\quad \text{If } -\infty < \gamma < 0, \text{ then } G_\gamma(x) = \begin{cases} \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{for } x < -1/\gamma \\ 1 & \text{for } x \geq -1/\gamma. \end{cases}$$

$$\text{(2.6.14)}\quad \text{If } \gamma = 0, \quad \text{then } G_\gamma(x) = \exp(-e^{-x}) \quad \text{for all } x.$$

$$\text{(2.6.15)}\quad \text{If } 0 < \gamma < \infty, \text{ then } G_\gamma(x) = \begin{cases} 0 & \text{for } x \leq -1/\gamma \\ \exp\{-(1+\gamma x)^{-1/\gamma}\} & \text{for } x > -1/\gamma. \end{cases}$$

Note that the distribution function G_γ is of type $\Phi_{-1/\gamma}$ for $\gamma < 0$, of type Λ for $\gamma = 0$ and of type $\Phi_{1/\gamma}$ for $\gamma > 0$. Clearly $G_\gamma(x)$ is for fixed real x a continuous and non-increasing function of γ. The defective distributions obtained in (2.6.6) and (2.6.11) now appear as $G_{-\infty}$ and G_∞ respectively.

It can be seen that for $F \in \mathcal{D}(G_\gamma)$ with $-\infty < \gamma < \infty$ a possible choice for the stabilizing constants is

$$\text{(2.6.16)}\quad \begin{cases} b_n = \inf\{x \mid 1 - F(x) \leq 1/n\} \\ a_n = \begin{cases} \dfrac{b_n}{1-F(b_n)} \displaystyle\int_{b_n}^{\infty} \frac{1-F(t)}{t}\,dt & \text{if } x_0 = \infty \\ \dfrac{x_0-b_n}{1-F(b_n)} \displaystyle\int_{b_n}^{x_0} \frac{1-F(t)}{x_0-t}\,dt & \text{if } x_0 < \infty. \end{cases} \end{cases}$$

With this choice the limit distribution satisfies (2.6.12) i.e. we get a distribution function from the family $\{G_\gamma\}$.

It is possible to specify a one-to-one correspondence between the distribution functions with an arbitrary but fixed finite endpoint and those with an infinite endpoint and all mass concentrated on $(0,\infty)$ in such a way that either each of two corresponding distribution functions belongs to the domain of attraction of some (not necessarily the same) stable law or both do not. This is the content of the next theorem.

<u>Theorem 2.6.3</u> Let x_0 be a real number.

a) Let F be a distribution function with finite endpoint x_0 and γ a non-positive real number. If $F \in \mathcal{D}(G_\gamma)$, the distribution function F^* defined by

$$F^*(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ F(x_0 - 1/x) & \text{for } x > 0 \end{cases}$$

is in $\mathcal{D}(G_{-\gamma})$.

b) Let F be a distribution function with endpoint at infinity and all mass concentrated on $(0,\infty)$ and γ a non-negative real number. If $F \in \mathcal{D}(G_\gamma)$, then the distribution function F^* with endpoint x_0 defined by

$$F^*(x) = \begin{cases} F(\frac{1}{x_0 - x}) & \text{for } x < x_0 \\ 1 & \text{for } x \geq x_0 \end{cases}$$

is in $\mathcal{D}(G_{-\gamma})$.

<u>Proof</u> For $\gamma \neq 0$ the theorem is a simple consequence of theorem 2.3.1 and theorem 2.3.2. Hence suppose $\gamma = 0$.

a) Suppose $F \in \mathcal{D}(\Lambda)$ and $x_0(F) < \infty$. We use the representation of theorem 2.5.3 and find for F^*

$$1 - F^*(x) = c(x_0 - 1/x).\exp\left\{ - \int_{x_1}^{x} \frac{a(x_0 - 1/t)}{f(x_0 - 1/t)} \cdot \frac{dt}{t^2} \right\}$$

for all $x > 0$.

Defining for all $x > 0$

$$\text{(2.6.17)} \qquad \begin{cases} c^*(x) = c(x_0 - 1/x) \\ a^*(x) = a(x_0 - 1/x) \\ f^*(x) = x^2\, f(x_0 - 1/x) \end{cases}$$

we have

$$1 - F^*(x) = c^*(x).\exp\left\{ - \int_{x_1}^{x} \frac{a^*(t)}{f^*(t)}\, dt \right\} \text{ for all } x \in \mathbb{R}^+.$$

By theorem 2.5.3 it is now sufficient to prove

$$\frac{d}{dx} f^*(x) \to 0 \qquad \text{for } x \to \infty. \tag{2.6.18}$$

By (2.6.17) we have

$$\frac{d}{dx} f^*(x) = 2x\, f(x_0-1/x) + f'(x_0-1/x).$$

By (2.5.27) the latter term tends to zero for $x \to \infty$ and by de l'Hôpital's rule

$$2 \cdot \frac{f(x_0-1/x)}{1/x} \sim 2.f'(x_0-1/x) \to 0 \qquad \text{for } x \to \infty,$$

hence (2.6.18) is true and $F^* \in \mathcal{D}(\Lambda)$.

b) The proof is analogous to that of part a). □

<u>Remark</u> An alternative way of stating the result of theorem 2.6.3 is the following. Suppose that $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$, ... are independent random variables and have a common distribution with finite endpoint x_0. The sequence of partial maxima of $\{\underline{x}_n\}$ (with a proper normalization) converges weakly if and only if the sequence of partial maxima of

$$\frac{1}{x_0-\underline{x}_1}, \frac{1}{x_0-\underline{x}_2}, \frac{1}{x_0-\underline{x}_3}, \ldots$$

(with a proper normalization) converges weakly.

2.7. A SPECIAL CASE: CONNECTION WITH VON MISES' WORK

In the sections 2.4 and 2.5 we started from Gnedenko's results. In this section we give a connection with von Mises' work [14]. We show that under rather mild conditions on the derivatives of F von Mises' sufficient conditions for the domains of attraction are also necessary conditions.

<u>Theorem 2.7.1</u> Suppose the distribution function F has a positive derivative F' for all x larger than some value x_2 (hence $x_0(F) = \infty$).

a) (R. von Mises) If for some positive α

$$\lim_{x\to\infty} \frac{x.F'(x)}{1-F(x)} = \alpha, \tag{2.7.1}$$

then $F \in \mathcal{D}(\Phi_\alpha)$.

b) If F' is non-increasing and $F \in \mathcal{D}(\Phi_\alpha)$, then (2.7.1) holds.

<u>Proof</u> a) Suppose (2.7.1) holds. By a well-known theorem on Lebesgue integration (cf. [15] p. 368) we have

$$1 - F(x) = \int_x^\infty F'(t)dt \qquad \text{for all real } x.$$

By theorem 1.2.1 b) it follows from (2.7.1) that F' is $(-\alpha-1)$-varying at infinity. Hence $1 - F$ is $-\alpha$-varying by lemma 1.2.2 b). So finally we have $F \in \mathcal{D}(\Phi_\alpha)$ by theorem 2.3.1.

b) Suppose F' is non-increasing and $F \in \mathcal{D}(\Phi_\alpha)$. For all x and y with $x < y$ and all t with $tx > x_2$ we have

$$\frac{\{1-F(tx)\}-\{1-F(ty)\}}{1-F(t)} = \int_x^y \frac{t.F'(st)}{1-F(t)}\,ds = \frac{t.F'(t)}{1-F(t)} \int_x^y \frac{F'(st)}{F'(t)}\,ds.$$

As $1 - F$ is $-\alpha$-varying by theorem 2.3.1, the lefthand side tends to $x^{-\alpha} - y^{-\alpha}$ as t goes to infinity. The last integrand is at most 1 for $1 < x < y$ and so

$$x^{-\alpha} - y^{-\alpha} \le \liminf_{t\to\infty} \frac{t.F'(t)}{1-F(t)} \cdot (y-x) \quad \text{for all } x,y \text{ with } 1 < x < y.$$

This implies

$$\liminf_{t\to\infty} \frac{t.F'(t)}{1-F(t)} \ge \alpha.$$

On the other hand, starting with $x < y < 1$ we obtain

$$\limsup_{t\to\infty} \frac{t.F'(t)}{1-F(t)} \le \alpha. \;\Box$$

<u>Remark 2.7.1</u> Comparison of part b) of theorem 2.7.1 and the lemma on page 422 of [2] shows that this lemma could have been used to prove part b) and that (surprisingly) the present proof of part b) contains a far less complicated proof of this lemma.

Next we state the corresponding theorem for distribution functions in $\mathcal{D}(\Psi_\alpha)$.

<u>Theorem 2.7.2</u> Suppose the distribution function F has a finite endpoint x_0 and a positive measurable derivative F' for all x in some interval (x_2,x_0).

a) If for some positive α

$$(2.7.2) \qquad \lim_{x\uparrow x_0} \frac{(x_0-x).F'(x)}{1-F(x)} = \alpha,$$

then $F \in \mathcal{D}(\Psi_\alpha)$.

b) If F' is non-increasing and $F \in \mathcal{D}(\Psi_\alpha)$, then (2.7.2) holds.

<u>Proof</u> Using theorem 2.6.3 we can reduce the present theorem to theorem 2.6.1. □

For distribution functions in $\mathcal{D}(\Lambda)$ we have an analogous result.

<u>Theorem 2.7.3</u> Suppose the distribution function F has a positive measurable derivative F' for all x in some interval (x_2,x_0), where x_0 is the endpoint of F.

a) If

$$(2.7.3) \qquad \lim_{x\uparrow x_0} g(x) = 1,$$

where

$$(2.7.4) \qquad g(x) = \frac{F'(x).\left\{\int_x^{x_0} \{1-F(t)\}dt\right\}}{\{1-F(x)\}^2} \qquad \text{for all } x < x_0,$$

then $F \in \mathcal{D}(\Lambda)$.

b) If F' is non-increasing and $F \in \mathcal{D}(\Lambda)$, then (2.7.3) holds.

<u>Proof</u> a) Define the function f by

$$(2.7.5) \qquad f(x) = \frac{\int_x^{x_0} \{1-F(t)\}dt}{1-F(x)} \qquad \text{for all } x < x_0.$$

Then we have for $x_2 < x < x_0$

$$f'(x) = -1 + g(x)$$

and by (2.7.3)

$$f'(x) \to 0 \qquad \text{for } x \uparrow x_0.$$

As in the part b) $\Rightarrow$ c) of the proof of theorem 2.5.2 we get

$$1 - F(x) = e^{c_3} . \{f(x_2)\}^{-1} . \exp\left\{ - \int_{x_2}^{x} \frac{g(t)}{f(t)} dt \right\}$$

$$\text{for } x_2 < x < x_0$$

and hence $F \in \mathcal{D}(\Lambda)$ by theorem 2.5.3.

b) Suppose F' is non-increasing and $F \in \mathcal{D}(\Lambda)$. We proceed as in the proof of part b) of theorem 2.7.1. Take f as in (2.7.5). Then we have for all x and y with $x < y$ and all t with $t + x.f(t) > x_2$

$$\frac{\{1-F(t+xf(t))\}-\{1-F(t+yf(t))\}}{1-F(t)} = g(t) . \int_x^y \frac{F'(t+sf(t))}{F'(t)} ds.$$

By theorem 2.5.1 the lefthand side tends to $e^{-x} - e^{-y}$ for $t \uparrow x_0$. The integrand in the righthand side is at most 1 for $0 < x < y$ and so

$$e^{-x} - e^{-y} \leq \liminf_{t \uparrow x_0} g(t) . (y-x) \qquad \text{for } 0 < x < y.$$

This implies

$$\liminf_{t \uparrow x_0} g(t) \geq 1.$$

On the other hand, starting with $x < y < 0$ we obtain

$$\limsup_{t \uparrow x_0} g(t) \leq 1. \quad \square$$

Now we give the connection with von Mises' result for $\mathcal{D}(\Lambda)$.

<u>Theorem 2.7.4</u> Suppose the distribution function F has a negative second derivative F'' for all x in some interval (x_2, x_0), where x_0 is the endpoint of F.

a) (R. von Mises) If

$$\lim_{x \uparrow x_0} \frac{F''(x).\{1-F(x)\}}{\{F'(x)\}^2} = -1, \tag{2.7.6}$$

then $F \in \mathcal{D}(\Lambda)$.

b) If F'' is non-decreasing and $F \in \mathcal{D}(\Lambda)$, then (2.7.6) holds.

<u>Proof</u> As F'' is negative, there exists an x_3 with $x_2 \leq x_3 < x_0$ such that $F'(x)$ is continuous and strictly decreasing with

$$0 < F'(x) < 1$$

for all x with $x_3 \leq x < x_0$. Hence the function F_0 defined by

$$F_0(x) = \begin{cases} 0 & \text{for } x < x_3 \\ 1 - F'(x) & \text{for } x \geq x_3 \end{cases} \tag{2.7.7}$$

is a distribution function.

a) By theorem 2.7.3 a) we have $F_0 \in \mathcal{D}(\Lambda)$. By lemma 2.5.1 it follows that $F \in \mathcal{D}(\Lambda)$.

b) By theorem 2.7.3 b) we have (2.7.3). Hence by theorem 2.5.2 it follows that $F_0 \in \mathcal{D}(\Lambda)$. Again applying theorem 2.7.3 b) we get (2.7.6). $\square$

Remark 2.7.2 Throughout section 2.7 we may take Radon-Nikodym derivatives with respect to Lebesgue-measure instead of ordinary derivatives.

2.8. ANOTHER CHARACTERIZATION OF $\mathcal{D}(\Lambda)$

In section 2.5 we have derived several (closely related) characterizations of the domain of attraction of Λ. In this section we give a characterization of a somewhat different character.

Theorem 2.8.1 a) If $F \in \mathcal{D}(\Lambda)$, then for all positive α

$$(2.8.1) \qquad \lim_{x \uparrow x_0} r_\alpha(x) = \frac{1}{\alpha},$$

where

$$(2.8.2) \qquad r_\alpha(x) = \frac{\int_x^{x_0} \{1-F(t)\}^\alpha \, dt}{\{1-F(x)\}^{\alpha-1} \cdot \int_x^{x_0} \{1-F(t)\} dt} \qquad \text{for all } x < x_0.$$

b) If for some positive $\alpha \neq 1$ equation (2.8.1) holds, then $F \in \mathcal{D}(\Lambda)$.

Proof a) Suppose $F \in \mathcal{D}(\Lambda)$. By theorem 2.5.1

$$(2.8.3) \qquad \lim_{t \uparrow x_0} \frac{1-F(t+xf(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}$$

with

$$f(t) = \frac{\int_t^{x_0} \{1-F(s)\} ds}{1-F(t)} \qquad \text{for all } t < x_0.$$

As by (2.8.3) for all positive α

$$\lim_{t \uparrow x_0} \frac{\{1-F(t+x.\frac{f(t)}{\alpha})\}^\alpha}{\{1-F(t)\}^\alpha} = e^{-x} \qquad \text{for all } x \in \mathbb{R},$$

by theorem 2.4.2 the distribution function F_α defined by

$$F_\alpha(x) = 1 - \{1-F(x)\}^\alpha \qquad \text{for all } x < x_0,$$

belongs to $\mathcal{D}(\Lambda)$. But then by theorem 2.5.1

$$\lim_{t\uparrow x_0} \frac{1-F_\alpha(t+xf^*(t))}{1-F_\alpha(t)} = e^{-x} \qquad \text{for all } x \in R$$

with

$$f^*(t) = \frac{\int_t^{x_0} \{1-F(s)\}^\alpha ds}{\{1-F(t)\}^\alpha} \qquad \text{for all } t < x_0.$$

Hence by lemma 2.4.2

$$f^*(t) \sim \frac{f(t)}{\alpha} \qquad \text{for } t \uparrow x_0.$$

This proves (2.8.1)

b) Suppose (2.8.1) holds for some positive $\alpha \neq 1$. Then

$$(2.8.4) \qquad \frac{1-F(x)}{1-G(x)} \to 1 \qquad \text{for } x \uparrow x_0$$

with

$$(2.8.5) \qquad G(x) = 1 - \alpha^{\frac{1}{\alpha-1}} \cdot \left\{ \frac{\int_x^{x_0} \{1-F(t)\}^\alpha dt}{\int_x^{x_0} \{1-F(t)\}dt} \right\}^{\frac{1}{\alpha-1}} \qquad \text{for all } x < x_0.$$

By (2.8.5) $\log(\frac{1}{1-G(x)})$ is differentiable for almost all $x < x_0$ with for those x

$$(2.8.6) \qquad \frac{d}{dx} \log(\frac{1}{1-G(x)}) = -\frac{1}{\alpha-1} \frac{\int_x^{x_0} \{1-F(t)\}dt}{\int_x^{x_0} \{1-F(t)\}^\alpha dt} \left\{ -\frac{\{1-F(x)\}^\alpha}{\int_x^{x_0} \{1-F(t)\}dt} + \right.$$

$$\left. + \frac{\{1-F(x)\}\left\{\int_x^{x_0} \{1-F(t)\}^\alpha dt\right\}}{\left\{\int_x^{x_0} \{1-F(t)\}dt\right\}^2} \right\} =$$

$$= -\frac{1}{\alpha-1}\frac{\{1-F(x)\}^{\alpha}}{\int_x^{x_0}\{1-F(t)\}^{\alpha}dt}\{-1+r_{\alpha}(x)\}.$$

Define the function g by

$$(2.8.7)\qquad g(x) = \frac{\{1-G(x)\}\{1-F(x)\}^{\alpha}\{1-r_{\alpha}(x)\}}{(\alpha-1).\int_x^{x_0}\{1-F(t)\}^{\alpha}dt} \qquad \text{for all } x < x_0.$$

From (2.8.6) we see that for almost all $x < x_0$

$$g(x) = \frac{d}{dx}G(x).$$

Hence, as by (2.8.5) G is absolutely continuous, we must have (cf. [16] p. 362)

$$(2.8.8)\qquad 1 - G(x) = \int_x^{x_0} g(t)dt.$$

From the definition of g and (2.8.1) it follows that

$$(2.8.9)\qquad \frac{g(x)}{1-G(x)} \sim \frac{1}{\alpha}\cdot\frac{\{1-F(x)\}^{\alpha}}{\int_x^{x_0}\{1-F(t)\}^{\alpha}dt} \qquad \text{for } x \uparrow x_0.$$

By (2.8.5) we have $g(x) > 0$ for all x in some interval (x_2,x_0). Hence for $x > x_2$ the function G coincides with a distribution function. By corollary 2.4.3, theorem 2.7.3 and remark 2.7.2 it is then sufficient to show

$$(2.8.10)\qquad \frac{g(x).\left\{\int_x^{x_0}\{1-G(t)\}dt\right\}}{\{1-G(x)\}^2} \to 1 \qquad \text{for } x \uparrow x_0.$$

By lemma 1.2.1 b) we get from (2.8.3)

$$(2.8.11)\qquad \frac{\int_x^{x_0}\{1-F(t)\}dt}{\int_x^{x_0}\{1-G(t)\}dt} \to 1 \qquad \text{for } x \uparrow x_0.$$

Now (2.8.10) follows from (2.8.9), (2.8.4), (2.8.11), (2.8.2) and (2.8.1). □

2.9. WEAK LAW OF LARGE NUMBERS; RELATIVE STABILITY

Up to now we only considered non-degenerate limit laws for sequences of partial maxima. In applications degenerate limit laws are important too. We begin with the definition of relative stability for a sequence of maxima.

Definition 2.9.1 A distribution function F is called relatively stable (notation $F \in RS$) if there exists a sequence $\{a_n\}$ of positive constants such that

$$(2.9.1) \qquad \lim_{n\to\infty} F^n(a_n x) = \begin{cases} 0 & \text{for } x < 1 \\ 1 & \text{for } x > 1. \end{cases}$$

In the following only distribution functions with endpoint at infinity are considered: It is not difficult to see that (2.9.1) cannot hold if $x_0(F) \le 0$ and that (2.9.1) holds with $a_n = x_0$ for $n = 1, 2, \ldots$ if $0 < x_0(F) < \infty$. So both cases are uninteresting.

Theorem 2.9.1 For a distribution function F with endpoint at infinity the following assertions are equivalent.

a) $F \in RS$.

b) (Gnedenko) $1 - F$ is $-\infty$-varying at infinity.

c) The integral $\int_0^\infty \{1-F(t)\}dt$ is finite and

$$(2.9.2) \qquad \lim_{x\to\infty} \frac{x.\{1-F(x)\}}{\int_x^\infty \{1-F(t)\}dt} = \infty.$$

Proof a) $\Leftrightarrow$ b): by lemma 2.2.2 relation (2.9.1) is equivalent with

$$(2.9.3) \qquad \lim_{n\to\infty} n\,.\,\{1-F(a_n x)\} = x^{-\infty} \qquad \text{for } x \neq 1.$$

As F has its endpoint at infinity, (2.9.3) for x = 2 implies

$$\lim_{n\to\infty} a_n = \infty. \tag{2.9.4}$$

Now by theorem 1.1.3 b) and remark 1.1.1 relation (2.9.3) is equivalent to $-\infty$-variation of $1 - F$ at infinity.

b) $\Longleftrightarrow$ c): This equivalence is a trivial consequence of theorem 1.3.2 for the particular case $\alpha = 0$. □

Remark 2.9.1 According ro remark 1.1.1 under the conditions of theorem 2.9.1 relation (2.9.1) holds if we take

$$a_n = \inf\{x \mid 1 - F(x) \leq 1/n\} \quad \text{for } n = 1, 2, \ldots .$$

Remark 2.9.2 By partial integration it can be shown that (2.9.1) is equivalent to

$$\lim_{x\to\infty} \frac{x.\{1-F(x)\}}{\int_x^\infty t dF(t)} = 1.$$

Corollary 2.9.1 (Gnedenko) If $F \in \mathcal{D}(\Lambda)$ and F has its endpoint at infinity, then $F \in RS$.

Proof By theorem 2.5.1 we have

$$\lim_{t\to\infty} \frac{1-F(t+x.f(t))}{1-F(t)} = e^{-x} \qquad \text{for all } x \in \mathbb{R}$$

with

$$f(t) = \frac{\int_t^\infty \{1-F(s)\}ds}{1-F(t)} \qquad \text{for all } t \in \mathbb{R}.$$

By corollary 2.4.2 then

$$\frac{f(t)}{t} \to 0 \qquad \text{for } t \to \infty.$$

By theorem 2.9.1 the proof is complete. □

Similar to what we did in section 2.7 for non-degenerate limit distributions, we now give a condition for relative stability in terms of the derivative of F.

<u>Theorem 2.9.2</u> Suppose the distribution function F has its endpoint at infinity and a positive derivative F' for all x larger than some value x_2 with F' summable on (x_2,∞).

a) (Geffroy [5]) If

$$\lim_{x\to\infty} \frac{x.F'(x)}{1-F(x)} = \infty, \tag{2.9.5}$$

then $F \in RS$.

b) If F' is non-increasing and $F \in RS$, then (2.9.5) holds.

<u>Proof</u> a) Define the function b by

$$b(x) = \frac{x.F'(x)}{1-F(x)} \qquad \text{for all } x > x_2.$$

As in the beginning of the proof of theorem 1.2.1 we get for some positive constant c_0 and all $x > x_2$

$$1 - F(x) = c_0 \cdot \exp\left\{ - \int_{x_2}^{x} \frac{b(t)}{t}\, dt \right\}.$$

Hence for all $x > 0$ and all t such that $\min(t,tx) > x_2$

$$\frac{1-F(tx)}{1-F(t)} = \exp\left\{ - \int_{1}^{x} \frac{b(ts)}{s}\, ds \right\}. \tag{2.9.6}$$

As $b(t) \to \infty$ for $t \to \infty$, the righthand side of (2.9.6) tends to $x^{-\infty}$ for all positive x and hence by theorem 2.9.1 we have $F \in RS$.

b) Suppose F' is non-increasing and $F \in RS$. For all $x > 1$ and all $t > x_2$ we have

$$\frac{\{1-F(t)\}-\{1-F(tx)\}}{1-F(t)} = \int_1^x \frac{tF'(st)}{1-F(t)}\,ds = \frac{t.F'(t)}{1-F(t)} \cdot \int_1^x \frac{F'(st)}{F'(t)}\,ds.$$

By theorem 2.9.1 the function 1 - F is $-\infty$-varying, hence the lefthand side tends to 1 as t goes to infinity. The last integrand is at most 1 and so for all $x > 1$

$$\liminf_{t\to\infty} \frac{t.F'(t)}{1-F(t)} \geq \frac{1}{x-1} .$$

This proves (2.9.5). □

Now we turn to the weak law of large numbers. First we give the definition of this law for sequences of partial maxima.

Definition 2.9.2 A distribution function F is said to satisfy the weak law of large numbers (notation $F \in$ WLLN) if there exists a sequence $\{b_n\}_{n=1}^{\infty}$ of real numbers such that

$$\text{(2.9.7)} \qquad \lim_{n\to\infty} F^n(x+b_n) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases}$$

In the following we only consider distribution functions with endpoint at infinity: For distribution functions with finite endpoint x_0 equation (2.9.7) holds with $b_n = x_0$ for $n = 1, 2, \ldots$.

Theorem 2.9.3 For a distribution function F with endpoint at infinity the following assertions are equivalent.

a) $F \in$ WLLN.

b) (Gnedenko) For all positive x

$$\text{(2.9.8)} \qquad \lim_{t\to\infty} \frac{1-F(t+x)}{1-F(t)} = 0.$$

c) The integral $\int_0^\infty \{1-F(t)\}dt$ is finite and

$$\text{(2.9.9)} \qquad \lim_{x\to\infty} \frac{1-F(x)}{\int_x^\infty \{1-F(t)\}dt} = \infty.$$

Proof We use the fact that $F \in$ WLLN if and only if the distribution function G defined by

$$(2.9.10) \qquad G(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ F(\log x) & \text{for } x > 0 \end{cases}$$

is relatively stable. Then the equivalence of the parts a) and b) of the present theorem is established by noting that F satisfies (2.9.8) if and only if 1 - G is $-\infty$-varying. To prove the equivalence of b) and c) note that by theorem 1.3.2 (with $\alpha = -1$) 1 - G is $-\infty$-varying if and only if

$$\lim_{x\to\infty} \frac{1-G(x)}{\int_x^\infty \{1-G(t)\}\frac{dt}{t}} = \infty.$$

This relation is equivalent with (2.9.9). □

Remark 2.9.3 According to remark 2.9.1 under the conditions of theorem 2.9.3 relation (2.9.7) holds if we take

$$b_n = \inf\{x \mid 1 - F(x) \leq 1/n\} \text{ for } n = 1, 2, \ldots .$$

Theorem 2.9.4 Suppose the distribution function F has its endpoint at infinity and a positive derivative F' for all x larger than some value x_2 with F' summable on (x_2,∞).

a) (Geffroy [5]) If

$$(2.9.11) \qquad \lim_{x\to\infty} \frac{F'(x)}{1-F(x)} = \infty,$$

then $F \in$ WLLN.

b) If F' is non-increasing and $F \in$ WLLN, then (2.9.11) holds.

Proof Define G by (2.9.10). Then (2.9.11) holds for F if and only if (2.9.5) holds for G. Using this fact we can reduce the present theorem to theorem 2.9.2. □

2.10. TWO OPEN PROBLEMS

The results of this work give rise to two problems, one of them being of interest from an analytic and the other one from a probabilistic point of view. First we mention the problem suggested by the results of chapter I.

a. If we look at the three relations which can serve as definitions of regular variation (definition 1.1.1, theorem 1.2.1 and theorem 1.2.2), we see that for functions U which are summable on finite intervals the three statements are equivalent.

Even - as is shown in [1] - if we only suppose the functions U to be measurable, we have three equivalent statements (cf. remark 1.2.2). If U is ρ-varying with $0 < \rho < \infty$, there exists by corollary 1.2.1 part 7 a non-decreasing U^* such that

$$U(x) \sim U^*(x) \qquad \text{for } x \to \infty. \tag{2.10.1}$$

The extension of regular variation described in section 1.5 also contains three relations (definition 1.5.1, theorem 1.5.3 b) and theorem 1.5.3 c)) which are parallel to the three relations for regularly varying functions. However here the perfect symmetry of the previous statements is lacking. To deduce (1.5.3) and (1.5.5) of theorem 1.5.3 from definition 1.5.1 we used the monotonicity of U. On the other hand the relations (1.5.3) and (1.5.5) may be fulfilled for functions U which are not monotone; for such functions (1.5.1) also holds and there is a non-decreasing function U^* such that (2.10.1) holds. This leads to the question whether theorem 1.5.3 also holds for every function U which is summable on finite intervals (and eventually not monotone). Maybe some extra conditions have to be imposed on the function f. It might even be possible to replace the monotonicity condition on U by the condition that U is measurable.

The application in chapter II only concerns monotone functions, so this question is only of interest from an analytic point of view.

An analogous question can be put concerning the auxiliary function f. In section 2.5 it is proved that (relation (2.5.25))

(2.10.2) $$\lim_{t\to\infty} \frac{f(t+x.f(t))}{f(t)} = 1 \qquad \text{for all } x \in \mathbb{R}$$

and (relation (2.5.13) and lemma 2.4.2)

(2.10.3) $$f(x) \sim \int_0^x b(t)dt \qquad \text{for } x \to \infty$$

where

(2.10.4) $$\lim_{x\to\infty} b(x) = 0.$$

The question is whether for functions f which are such that $\frac{1}{f}$ is summable on finite intervals (or simply measurable), (2.10.2) and (2.10.3) (together with (2.10.4)) are equivalent. It was shown in section 2.5 that (2.10.3) (with (2.10.4)) implies (2.10.2).

b. For distribution functions in the domain of attraction of one of the stable distributions Φ_α, Ψ_α or Λ, there is a representation in terms of auxiliary functions tending to certain limits for $x \uparrow x_0$. It is even possible to construct a representation of F with the aid of a single auxiliary function (remark 2.3.2, remark 2.3.4 and remark 2.5.1). It seems plausible that the speed of convergence if $F^n(a_n x+b_n)$ to the stable law G(x) is connected with the speed of convergence of this auxiliary function to its limit for $x \uparrow x_0$. It is not even clear how one should start investigating this matter.

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