Printed at the Mathematical Centre, 49, 2e Boerhaavestraat 49, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

MATHEMATICAL CENTRE TRACTS 26

SELECTED STATISTICAL PAPERS 1

EUROPEAN MEETING 1968

MATHEMATISCH CENTRUM AMSTERDAM 1968

AMS(MOS) subject classification scheme (1970): 60J10, 60K05, 62E10, 62F10, 62C05, 62G10, 62J05.

PREFACE

This volume of the series "Mathematical Centre Tracts" is published on the occasion of the European Meeting 1968 on Statistics, Econometrics and Management Science in Amsterdam. With permission of the Organizing Committee of this Meeting, the Statistical Department of the Mathematical Centre has invited some authors of papers on Statistics and Probability Theory to publish their work in the form of this Tract. This first volume contains eight papers. The authors come from seven countries and their subjects vary from renewal processes to slippage tests. The papers appear in an almost random order, determined mainly by the date of arrival of the manuscript. It is hoped that a second volume of the same kind will be published shortly after the Meeting.

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POISSON PROCESSES AS RENEWAL PROCESSES INVARIANT UNDER TRANSLATIONS

by Murali Rao and Hans Wedel (Sweden)
University of Göteborg

INTRODUCTION: Thedéen has proved [3] that a renewal process whose renewals are "stationary under translation" is necessarily Poisson. In search for some sort of generalization of this interesting result we arrived at a very simple proof which we present.

Let $\{X_n : n = \pm 1, \pm 2, \ldots\}$ be a sequence of random variables such that a.s.

$$\ldots X_{-2} < X_{-1} < 0 < X_1 < X_2 \ldots .$$

Put $Y_0 = X_{-1}$, $Y_1 = X_1$, $Y_n = X_n - X_{n-1}$ for $n \neq 0, 1$. Assume that:

i) $\{(Y_0,Y_1), Y_n \; n \neq 0, 1\}$ is a set of independent random variables.

ii) $\{Y_n : n \neq 0, 1\}$ are independent, identically distributed positive random variables with $P[Y_n \leq y] = F(y)$, $F(0) = 0$ and $E[Y_n] = \frac{1}{m} < \infty$.

iii) $E[N(I)] = m|I|$ where $|I|$ denotes the Lebesgue measure of I, and $N(I)$ = number of $X_n \in I$. Then iii) is equivalent to

$$P(Y_i > u) = \int_u^\infty (1 - F(t))dt \text{ for } i = 0, 1; \text{ see } [1, \text{p. } 354].$$

Let $\{\xi_n : n = \pm 1, \pm 2, \ldots\}$ be a sequence of random variables which is independent of the sequence $\{X_n\}$. We shall assume that for all n,m $n \neq m$ (ξ_n, ξ_m) have the same joint distribution G and that the support group of G, i.e. the group generated by the support of G, has an element of the form (0,d) with $d > 0$; if ξ_n and ξ_m are independent and have a nondegenerate distribution then this is certainly true.

Put $Z_n = X_n + \xi_n$, $n = \pm 1, \pm 2, \ldots$ and $\tilde{N}(I)$ = number of $Z_n \in I$.

<u>Theorem 1.</u>

Let X_n, ξ_n, Z_n be as above. If $E[\tilde{N}(I)\,\tilde{N}(J)] = E[N(I)\,N(J)]$ for all I, J then $\{X_n\}$ is Poisson, i.e. $F(y) = 1 - e^{-my}$.

<u>Proof</u>. Put $\Phi(I,J) = E[N(I)\ N(J)] - E[N(I \cap J)] = \sum_{n \neq m} P[X_n \in I,\ X_m \in J]$.
Using independence of $\{\xi_n\}$ and $\{X_n\}$ we get
$E[\tilde{N}(I)\ \tilde{N}(J)] - E[\tilde{N}(I \cap J)] = \iint \Phi(I - u,\ J - v)dG(u,v)$.
The condition $E[N(I)] = m|I|$ implies $E[\tilde{N}(I)] = m|I|$. Thus $E[\tilde{N}(I)\ \tilde{N}(J)] - E[\tilde{N}(I \cap J)] = E[N(I)\ N(J)] - E[N(I \cap J)]$: $\Phi = \Phi * G$.

A simple consequence of the renewal theorem is that for any finite intervals I, J, $E[N(I + h)\ N(J + k)]$ is a bounded function of (h,k). The Choquet-Deny theorem [2, p. 152] applies and we deduce that every point of support of G is a period for Φ. The set of periods for Φ is a group and this group contains the element (0,d) and hence (0,kd) where k is any positive integer (indeed any integer). Thus for all I, J and all positive integers k, $\Phi(I,J) = \Phi(I,\ J + kd)$. Take $I = (0,x]$, $J = (0,x]$ with $x < kd$. Then $I \cap (I + kd) = \emptyset$.

Also $\Phi(I,\ I + kd) = \sum_{n \neq m} P[X_n \in I,\ X_m \in I + kd] = \sum_{m,n \geq 1} P[X_n \in I,\ X_m \in I + kd] =$

$$= \sum_{n=1}^{\infty} \sum_{m>n} P[X_n \in I,\ X_m \in I + kd] = \sum_{n=1}^{\infty} \int_0^x H(I + kd - u)\ d(F_0 * F^{(n-1)*})(u) =$$

$$= m \int_0^x H(I + kd - u)du \text{ where } H(x) = \sum_{k=1}^{\infty} F^{k*}(x) \text{ and iii) implies}$$

$$mx = \sum_{k=0}^{\infty} F_0 * F^{k*}(x).\ x > 0.$$

Similar calculations give $\Phi(I,I) = 2m \int_0^x H(x - u)du = 2m \int_0^x H(u)du$.

$$\text{Thus } 2 \int_0^x H(u)du = \int_0^x H(I + kd - u)du = \int_0^x [H(x + kd - u) -$$

$$- H(kd - u)]du = \int_0^x [H(kd + u) - H(kd - u)]du.$$

This equality for all $x < kd$ implies $2H(u) = H(kd + u) - H(kd - u)$; $u < kd$.

It is possible to show that the only solution of this functional equation is $H(x) = \lambda x$. However we take a short cut.

Suppose d_0 is any positive number such that $F(d_0) > 0$ and $F(d_0-) = 0$. Then F^{n*} has an atom at nd_0 and thus H has a mass at every positive integral multiple of d_0, but $H(u) = 0$ for $u < d_0$. Choose k so that $kd > d_0$. For $u < d_0$, $H(u) = 0$ and the functional equation for H shows that $H(kd - u) = H(kd + u)$; $u < d_0$. This is absurd since every interval of length larger than d_0, has a multiple of d_0 and H has a mass at such a point. Thus F certainly cannot be arithmetic. As $k \to \infty$ the renewal theorem shows that $2H(u) = 2um$. This is equivalent to F being exponential. Q.E.D.

Theorem 2.

If the support group of G is dense in the plane, condition iii) can be removed and a sufficient condition for the preceding theorem is

$$E\left[N(I)\ N(J) - N(I \cap J)\right] = E\left[\tilde{N}(I)\ \tilde{N}(J) - \tilde{N}(I \cap J)\right].$$

Proof. With the same notation as above the Choquet-Deny theorem shows that $\Phi(I,J) = \lambda |I|\ |J|$, thus

$$\lambda hx = E\left[N[-h,0)\ N(0,x]\right] = \sum_{n \neq m} P\left[X_n \in [-h,0), X_m \in (0,x]\right] =$$

$$= \sum_{\substack{n<1 \\ m \geq 1}} \int_0^x \int_0^h P\left[X_n \in [-h,0),\ X_m \in (0,x] \mid y_0 = u,\ y_1 = v\right] dK(u,v) =$$

$$= \int_0^x \int_0^h \sum_{n \leq -1} P\left[X_n \in [-h,0) \mid y_1 = u\right] \sum_{m \geq 1} P\left[X_m \in (0,x] \mid y_1 = v\right] dK(u,v)$$

$$= \int_0^h \int_0^x U(x - v)\ U(h - u)\ dK(u,v)$$

where $U(I) = \sum_{h=0}^{\infty} F^{n*}(I)$ and K is the joint distribution of y_0 and y_1.

Since the left side is a product measure this implies that dK is a product measure:

Say $dK(u) \cdot dK(v) = dK(u,v)$.

Further $\lambda x = \int_0^x U(x - u)\, dK(v) \cdot \frac{1}{h} \int_0^h U(h - v)\, dK(v)$.

Put $x = 1$, we see $\frac{1}{h} \int_0^h U(h - v)\, dK(v) = \mu$ where μ is a constant. This implies $(K * U)(x) = \mu x$, i.e. $E[N(I)] = \mu |I|$.

Put $K * U = V$ and $I = J = (0,x]$ then $\Phi(I,I) = \lambda x^2$, $\Phi(I,I) =$

$= \sum_{n \neq m} P[x_n \in I,\ x_m \in I] = 2V * H = \mu\, 2 \int_0^x H(t)dt$, thus we have $H(x) = \frac{\lambda}{\mu} x$.

I.e. F is exponential. Q.E.D.

Acknowledgement: It was H. Bergström who pointed out that our conditions are sufficient. We are grateful to him and P. Jagers for valuable discussions.

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SOME ANALYTICAL ASPECTS OF ESTIMATION THEORY

by A.M. Kagan (USSR)
Mathematical Institute of the Academy of Sciences,
Leningrad

In the paper the close connection between certain problems of the statistical theory of estimation and analytical problems of the characterization of distributions is demonstrated.

More precisely let $(x_1, \ldots, x_n)$ be a repeated sample from the population with distribution function (d.f.) $F(x;\theta)$, where θ is a parameter. Suppose that $g(x_1, \ldots, x_n)$ is an admissible or optimal (in a certain sense) estimator of the parametric function $\gamma(\theta)$. What are the conditions imposed on $F(x;\theta)$ by admissibility or optimality of more or less simple estimators $g(x_1, \ldots, x_n)$ - that is the question we discuss in the paper. We mention also a number of related results concerning sufficiency of statistics and Fisher's information.

Recently certain results have been obtained for the exponential families and for the families with group parameters - location and scale. We shall restrict ourselves with the families depending on the scale parameter because the principal results for the exponential families and for the families with the location parameter were reported in Linnik's paper [1] at the previous European Meeting of Statisticians (London, 1966).

Everywhere the quadratic loss function will be used; it means that the quality of an estimator $g(x_1, \ldots, x_n)$ of $\gamma(\theta)$ is measured by $E_\theta(g - \gamma(\theta))^2$. The agreement automatically defines the conceptions of admissibility and optimality.

1. ESTIMATION OF POLYNOMIALS OF SCALE PARAMETER.

Let $(x_1, \ldots, x_n)$ be a repeated sample from the population with d.f. $F(\frac{x}{\sigma})$ depending on the scale parameter $\sigma \in (0,\infty)$. Everywhere in the paper $F(x)$ is supposed to be concentrated on $(0,\infty)$. Assume that

$$\int_0^\infty x\,dF(x) = \alpha_1 < \infty, \quad \int_0^\infty x^2\,dF(x) = \alpha_2 < \infty;$$

then
$$\alpha_1^{-1}\bar{x} = \frac{x_1 + \dots + x_n}{\alpha_1 n}$$

will be an unbiased estimator of σ with finite variance. It is easily to see that apart from the trivial case of degenerate $F(x)$ the best estimator of σ of the form $c\bar{x}$ (we shall denote it $c_n^0\bar{x}$) which has a bias, is better than $\alpha_1^{-1}\bar{x}$, i.e. it satisfies the condition

$$E_\sigma(c_n^0\bar{x} - \sigma)^2 < E_\sigma(\alpha_1^{-1}\bar{x} - \sigma)^2 \text{ for all } \sigma \in (0,+\infty).$$

That is why it is natural to clear up the conditions of admissibility of $\alpha_1^{-1}\bar{x}$ among all unbiased estimators of σ and the conditions of admissibility of $c_n^0\bar{x}$ among all estimators of σ.

The next theorems were proved in [2]; there $F(x)$ was a priori supposed to satisfy the condition

$$\int_0^\infty x^k\,dF(x) < \infty, \quad k = 1, 2, \dots . \tag{1}$$

Theorem 1.1. Let $F(x)$ satisfy the condition (1). Then the necessary and sufficient condition for $\alpha_1^{-1}\bar{x}$ to be admissible among unbiased estimators of $\sigma \in (0,\infty)$ for two sample sizes $n = n_1$, $n = n_2$, $n_2 > n_1 \geq 3$, is that $F(x)$ is either a degenerate d.f. or a d.f. of the gamma-distribution.

Theorem 1.2. Let $F(x)$ satisfy the condition (1). The necessary and sufficient condition for $c_n^0\bar{x}$ to be admissible among all estimators of $\sigma \in (0,\infty)$, for two sample sizes $n = n_1$, $n = n_2$, $n_2 > n_1 \geq 3$, is that $F(x)$ is either a degenerate d.f. or a d.f. of the gamma-distribution.

It should be noticed that using analytical results obtained recently by C.G. Khatri and C.R. Rao [3] one can avoid the condition (1).

We shall now outline briefly the scheme of the proof of Theorems 1.1 and 1.2. Sufficiency is proved in the following manner. The case of degenerate $F(x)$ is trivial. If $F(x)$ is a function of the gamma-distribution then $\bar{x}$ will be a complete sufficient statistic for the family $F(\frac{x_1}{\sigma}) \ldots F(\frac{x_n}{\sigma})$. Hence according to the Rao-Blackwell-Kolmogorov theorem it follows that in this case $\alpha_1^{-1}\bar{x}$ is for all n not only admissible but the best unbiased estimator of $\sigma \in (0,\infty)$. The proof of the admissibility of $c_n^0\bar{x}$ in this case is also based on sufficiency of $\bar{x}$ and on the Cramér-Rao inequality (cf. [4]). Necessity of the conditions of Theorems 1.1 and 1.2 is proved almost in the same manner. We shall restrict ourselves to Theorem 1.2.

Let us consider the estimator $S_n(x_1, \ldots, x_n) = S_n$,

$$S_n = c_n^0\bar{x} \frac{E_1(c_n^0\bar{x}|y)}{E_1((c_n^0\bar{x})^2|y)}, \tag{2}$$

where $y = (\frac{x_2}{x_1}, \ldots, \frac{x_n}{x_1})$. It can be proved that

$$E_\sigma(S_n - \sigma)^2 \leq E_\sigma(c_n^0\bar{x} - \sigma)^2 \tag{3}$$

and the equality sign in (3) holds - simultaneously for all $\sigma \in (0,\infty)$ - if and only if

$$E_1(c_n^0\bar{x}|y) = E_1((c_n^0\bar{x})^2|y) \tag{4}$$

with probability 1.

Analytically the condition (4) is convenient enough. Denoting

$$x_i = e^{\xi_i},$$

$$G(u) = P\{\xi_i < u; \sigma = 1\}$$

we can rewrite the condition (4) as

$$E\{\frac{1}{n}\sum_1^n e^{\xi_i}|\xi_2 - \xi_1, \ldots, \xi_n - \xi_1\} =$$
$$= c_n^0 E\{(\frac{1}{n}\sum_1^n e^{\xi_i})^2|\xi_2 - \xi_1, \ldots, \xi_n - \xi_1\}. \tag{5}$$

Multiplying both sides of (5) by $\exp(t_2(\xi_2 - \xi_1) + \ldots + t_n(\xi_n - \xi_1))$ and taking the expectations of both sides we obtain the following functional equation for the Laplace transform

$$P(z) = \int_{-\infty}^{+\infty} e^{zu} dG(u):$$

$$B_n\{P(1 - \sum_{i=2}^{n} t_i) \prod_{i=2}^{n} P(t_i) + P(- \sum_{i=2}^{n} t_i) \sum_{k=2}^{n} [P(1+t_k) \prod_{\substack{i=2 \\ i\neq k}}^{n} P(t_i)]\} =$$

$$= P(2 - \sum_{i=2}^{n} t_i) \prod_{i=2}^{n} P(t_i) + P(- \sum_{i=2}^{n} t_i) \sum_{k=2}^{n} [P(2+t_k) \prod_{\substack{i=2 \\ i\neq k}}^{n} P(t_i)] +$$

$$+ 2\{P(1 - \sum_{i=2}^{n} t_i) \sum_{k=2}^{n} [P(1 + t_k) \prod_{\substack{i=2 \\ i\neq k}}^{n} P(t_i) + \tag{6}$$

$$+ P(- \sum_{i=2}^{n} t_i) \sum_{j>k\geq 2}^{n} [P(1 + t_i) P(1 + t_k) \prod_{\substack{i=2 \\ i\neq j \\ i\neq k}}^{n} P(t_i)]\},$$

where B_n = constant and Re $t_i = 0$.

From (6) the desired result follows after certain analytic transformations.

Note that if $F'(x) = f(x)$ exists then the estimator (2) takes the form obtained originally by E. Pitman [5]:

$$S_n = \frac{\int_0^\infty u^n \prod_1^n f(ux_i)du}{\int_0^\infty u^{n+1} \prod_1^n f(ux_i)du} .$$

A development of the method used in [2] allowed to obtain the following theorem (see [6]):

Theorem 1.3. Suppose that $F(x)$ satisfies the condition (1). The necessary and sufficient condition for a polynomial $q(\bar{x}) = a_0\bar{x}^k + \ldots + a_k$, $a_0 \neq 0$, of degree $k \geq 1$ to be optimal for all $\sigma \in (0,\infty)$ among unbiased estimators of $\pi_k(\sigma) = E_\sigma q(\bar{x})$ for k sample sizes $n = m, m+1, \ldots, m+k-1$, $m \geq 3$, is that $F(x)$ is either a degenerate d.f. or a d.f. of the gamma-distribution.

The proof of Theorem 1.3 appears to be equivalent to solving the following equation for P(z):

$$\sum_{j=0}^{k} \Big[P(j - \sum_{2}^{n} t_i) \sum_{\substack{m_2+\ldots+m_n=k-j \\ m_i \geq 0}} \prod_{2}^{n} P(m_k + t_k)\Big] = \tag{7}$$

$$= b_n P(-\sum_{2}^{n} t_i) \prod_{2}^{n} P(t_i), \quad n = m, m+1, \ldots, m+k-1,$$

b_n = const., Re $t_i = 0$.

2. SUFFICIENCY AND PARTIAL SUFFICIENCY

Under different conditions on $F(x)$ there was proved in the papers [7,8,9] that sufficiency of the statistic $\bar{x}$ for the family

$$F(\frac{x_1}{\sigma}) \ldots F(\frac{x_n}{\sigma}), \quad \sigma \in (0,\infty) \tag{8}$$

is equivalent to the fact that $F(x)$ is a d.f. of the gamma-distribution. It appears [9] that the independence of σ of the conditional expectation $E_\sigma(Q\,|\bar{x})$ for a separate polynomial Q of general form imposes strong conditions upon $F(x)$. In particular the following theorem holds.

Theorem 2.1. If 1^o. $\int_0^\infty x^2 dF(x) < \infty$, 2^o. the n-th convolution $F^{*n}(x)$ is absolutely continuous, 3^o. $E_\sigma(x_1^2\,|\bar{x})$ is independent of σ, then $F(x)$ is a d.f. of the gamma-distribution.

Now we are going to generalize the conception of sufficiency (cf. [10]). Assume that for some integer $k \geq 1$

$$\int_0^\infty x^{2k}\, dF(x) < \infty. \tag{9}$$

Under this condition the set of all polynomials $Q(x_1, \dots, x_n)$ of degree $\leq$ k forms a Hilbert space if one defines the scalar product of elements Q_1 and Q_2 as

$$(Q_1, Q_2)_\sigma = E_\sigma(Q_1\, Q_2). \tag{10}$$

We shall denote this space by $L_k^{(2)}$ and its subspace formed by all polynomials $q(\bar{x}) = a_0\bar{x}^k + \dots + a_k$ of the sample mean $\bar{x}$ will be denoted by $\mathcal{T}_k$.

The subspace $\mathcal{T}_k$ is said to be $L_k^{(2)}$-sufficient for the family (8) if for any $Q \in L_k^{(2)}$ there exists an element $q \in \mathcal{T}_k$ independent of $\sigma \in (0,\infty)$ such that

$$\hat{E}_\sigma(Q|\mathcal{T}_k) = q$$

where $\hat{E}_\sigma(\cdot|\mathcal{T}_k)$ denotes the projection into $\mathcal{T}_k$ when the scalar product in $L_k^{(2)}$ is defined by the formula (10) with given σ.

Theorem 2.2 (cf. [9]). The necessary and sufficient condition for $\mathcal{T}_k$ to be $L_k^{(2)}$-sufficient for the family (8) is that either $F(x)$ is a degenerate d.f. or the first 2k moments of $F(x)$ coincide with the corresponding moments of the gamma-distribution.

From Theorem 2.2 it follows that if the first 2k moments of $F(x)$ coincide with the corresponding moments of the gamma-distribution then any polynomial $Q \in L_k^{(2)} \setminus \mathcal{T}_k$ will be inadmissible among unbiased estimators of $E_\sigma Q$.

3. EXTREMAL ROLE OF THE GAMMA-DISTRIBUTION IN INFORMATIONAL SENSE

Suppose that $F(x)$ has the density $f(x)$. The integral

$$\mathcal{J}_f(\sigma) = \int_{f(\frac{x}{\sigma})>0} \left[\frac{\partial \log \frac{1}{\sigma} f(\frac{x}{\sigma})}{\partial\sigma}\right]^2 \frac{1}{\sigma} f(\frac{x}{\sigma})dx \tag{11}$$

is well known as Fisher information for the family of densities $\frac{1}{\sigma} f(\frac{x}{\sigma})$.

Suppose that the following conditions are satsfied.

1. $f(x)$ is continuously differentiable,

2. $\int_0^\infty x^2 f(x)dx < \infty$,

3. $\lim_{x\to 0} xf(x) = 0$, $\lim_{x\to\infty} x^2 f(x) = 0$.

Theorem 3.1 (see [11]). Within the class of densities with given moments α_1, α_2 satisfying the conditions 1-3, $\min_f \mathcal{I}_f(\sigma)$ is attained - simultaneously for all $\sigma \in (0,\infty)$ - by the gamma-distribution.

The comparison of the results of this paper with the results of [10, 12,13] shows that for problems concerning the scale parameter the gamma-distribution plays the same role as the normal law does for problems concerning the location parameter.

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THE BEHAVIOR OF SOME TESTS FOR ORDERED ALTERNATIVES UNDER INTERIOR SLIPPAGE

by Hans K. Ury (U.S.A)
San Francisco Medical Center, University of California
and California State Department of Public Health

0. INTRODUCTION AND SUMMARY

For testing the standard null hypothesis against ordered alternatives in a one-way analysis of variance with k samples, Bartholomew [1], [3] proposed some test statistics for the case of underlying normal distributions, and Chacko gave the corresponding nonparametric test in [8]. In this note it is shown that under "interior slippage" (i.e., when one population other than the first or k^{th} is larger or smaller than the others), the probability that these tests will reject the null hypothesis simultaneously in favor of *both* the alternatives of upward and downward ordering goes to 1 in the limit, as the sample sizes grow sufficiently large, regardless of the significance level.

Since a nonparametric test against trend by Terpstra [13] and Jonckheere [10] is shown to behave somewhat more "normally" under interior slippage, it is suggested that this test might well be preferable at least for small k, particularly for k = 3.

1. NOTATION AND THE TESTS

Since the underlying models for the tests of (a) Bartholomew, (b) Chacko, and (c) Terpstra are different, they will be given separately.

(a) Here we have k independent normal random variables, $X_1, \ldots, X_k$ with unknown means $\mu_1, \ldots, \mu_k$ and a common but unknown variance σ^2. Let x_{ij} ($i = 1, \ldots, k$; $j = 1, \ldots, n_i$) be independent observations on the k variables, with x_{ij} the j^{th} observation from the i^{th} variable. Let $\bar{x}_i = \sum_{j=1}^{n_i} x_{ij}/n_i$ and $s_i^2 = \sum_{j=1}^{n_i} (x_{ij} - \bar{x}_i)^2/n_i$ denote the sample mean and variance for the i^{th} variable. The null hypothesis $H_0: \mu_1 = \ldots = \mu_k$ is tested against either the alternative of upward ordering, $H_1: \mu_1 \leq \ldots \leq \mu_k$ or against that of downward ordering, $H_2: \mu_1 \geq \ldots \geq \mu_k$, with at least one inequality strong in each case and with σ^2 unspecified for all three hypotheses. Denote by

(1) $$\bar{x}_{[t,s]} = (n_t\bar{x}_t + n_{t+1}\bar{x}_{t+1} + \ldots + n_s\bar{x}_s)/(n_t + n_{t+1} + \ldots + n_s)$$

the pooled sample mean of $\bar{x}_t, \bar{x}_{t+1}, \ldots, \bar{x}_s$, where s and t are positive integers with $1 \leq t < s \leq k$.

The MLE's (maximum likelihood estimates) of the μ's under H_0 are $\hat{\mu}_1 = \ldots = \hat{\mu}_k = \bar{x}_{[1,k]}$. Under H_1 or H_2 the MLE's are obtained by pooling successive sample means which violate the restriction specified by the alternative, continuing this procedure until no violations exist among the remaining pooled or unpooled means. If there are m distinct estimates obtained by pooling, respectively, the first t_1 means, the next t_2 means, ..., and the last t_m means, $t_j > 0$, $\sum_{j=1}^{m} t_j = k$, and if we set $\tau_0 = 0$,

(2) $$\tau_i = t_1 + t_2 + \ldots + t_i \quad (i = 1, 2, \ldots, m)$$

$$\tau_m = k,$$

then

(3) $$\hat{\mu}_{\tau_i+1} = \hat{\mu}_{\tau_i+2} = \ldots = \hat{\mu}_{\tau_{i+1}} = \bar{x}_{[\tau_i+1, \tau_{i+1}]} \quad (i = 0, 1, \ldots, m-1).$$

Denote the m distinct estimates by $\bar{x}_{t_j}$ $(j = 1, \ldots, m)$, where $\bar{x}_{t_j} = \bar{x}_{[\tau_{j-1}+1, \tau_j]}$. Let the sum of the sample sizes pooled into $\bar{x}_{t_j}$ be denoted by N_{t_j}. [It follows from Brunk [5], [6] and van Eeden [9] that these MLE's are unique and can be formally represented as $\hat{\mu}_i = \max_{1 \leq r \leq s} \min_{i \leq s \leq k} \bar{x}_{[r,s]}$ for the case of H_1, and as $\hat{\mu}_i = \min_{1 \leq r \leq i} \max_{i \leq s \leq k} \bar{x}_{[r,s]}$ for H_2.]

Finally, let $p_{m,k}$ stand for the probability of obtaining exactly m distinct estimates out of a possible k. Under H_0, and for equal sample sizes, Chacko [7] and Miles [11] showed that

(4) $$p_{m,k} = |S_k^m| / k!$$

where $|S_k^m|$ is the coefficient of z^m in $z(z+1) \ldots (z+k-1)$ (i.e., is the modulus of a Stirling's number of the First Kind). They also showed that (4) holds not only for $X_1, \ldots, X_k$ normally distributed, but whenever their joint distribution is a symmetric function of them.

Bartholomew's likelihood ratio test at significance level α [3] calls for rejection of H_0 when $T_k^1 \geq C_1$, where the test statistic

(5) $$T_k^1 = \sum_{i=1}^{k} n_i [\hat{\mu}_i - \bar{x}_{[1,k]}]^2 / s_0^2 = \sum_{j=1}^{m} N_{t_j} [\bar{x}_{t_j} - \bar{x}_{[1,k]}]^2 / s_0^2,$$

with $s_0^2 = \sum_{i=1}^{k} n_i [\bar{x}_i - \bar{x}_{[1,k]}]^2 + \sum_{i=1}^{k} n_i s_i^2$, and where C_1 is determined by

(6) $$\alpha = \sum_{m=2}^{k} p_{m,k} P[\beta_{[(m-1)/2,(N-m)/2]} \geq C_1].$$

Here $\beta_{[(m-1)/2,(N-m)/2]}$ is a random variable having the Beta distribution with parameters $(m-1)/2$ and $(N-m)/2$, and $N = \sum_{i=1}^{k} n_i = \sum_{j=1}^{m} N_{t_j}$.

(a') When σ^2 is known, the test statistic is

(7) $$T_k^2 = \sum_{i=1}^{k} n_i [\hat{\mu}_i - \bar{x}_{[1,k]}]^2 / \sigma^2 = \sum_{j=1}^{m} N_{t_j} [\bar{x}_{t_j} - \bar{x}_{[1,k]}]^2 / \sigma^2,$$

and H_0 is rejected at level α if $T_k^2 \geq C_2$, with C_2 determined by

(8) $$\alpha = \sum_{m=2}^{k} p_{m,k} P[\chi_{m-1}^2 \geq C_2],$$

where χ_{m-1}^2 is a random variable having the Chi square distribution with m-1 degrees of freedom.

(a") The computation of the $p_{m,k}$ imposes limitations on the use of T_k^1 and T_k^2. For unequal sample sizes, these probabilities have been determined only for k = 3, 4, and 5 [1]. For equal n_i, they were tabulated for $k \leq 12$ in [11], and tables of the Stirling Numbers of the First Kind can of course be used for $k \leq 50$. Barton and Mallows [4] give some approximations which should prove useful for k quite large. But in the general case of unequal samples (or for moderately large k) one could use a conditional test, suggested by Tukey and mentioned in [2], which does not require a knowledge of the $p_{m,k}$. To apply this test, one simply computes the test statistic as in (5) or (7) and determines significance by referring to the percentage points of $\beta[(m_0-1)/2, (N-m_0)/2]$ or $\chi^2_{m_0-1}$, respectively, where m_0 is the observed value of m. The corresponding test statistics will be denoted T_k^{1c} and T_k^{2c}.

(b) Since (4) holds only for symmetrically dependent random variables, Chacko's rank test [8] calls for equal sample sizes. Thus, let k independent random samples be drawn from populations with unknown continuous (to avoid ties) cumulative distributions F_i $(i = 1, \ldots, k)$ respectively. We now have H_0: $F_1 = \ldots = F_k$; H_1: $F_1 \geq \ldots \geq F_k$; H_2: $F_1 \leq \ldots \leq F_k$ with at least one inequality strong in each case. Chacko's test procedure consists of replacing each x_{ij} by its rank R_{ij} in the overall sample and, letting $\bar{R}_i = (1/n) \sum_{j=1}^{n} R_{ij}$, formally operating on the $\bar{R}_i$ as one previously did on the $\bar{x}_i$, pooling when necessary, to obtain a final distinct set of m quantities $\bar{R}_{t_1}, \ldots, \bar{R}_{t_m}$. Let $N = nk$. Then the test statistic is

$$(9) \qquad T_k^3 = \frac{12n}{N(N+1)} \sum_{j=1}^{m} t_j \left[\bar{R}_{t_j} - \frac{N+1}{2} \right]^2 .$$

As shown in [8], H_0 is for large n rejected at level approximately α if $T_k^3 \geq C_2$, where C_2 is determined by (8).

(b') For unequal sample sizes one could again use a conditional test analogous to those mentioned in (a"). This test will be denoted T_k^{3c}.

(c) The underlying model for Terpstra's [13] or Jonckheere's [10] test is

$$x_{ij} = \alpha + \beta i + \varepsilon_{ij},\ i = 1,\dots,k;\ j = 1,\dots,n_i,$$

where α and β are unknown constants and where the ε_{ij} have a common continuous cdf F. H_0, H_1 and H_2 now correspond to $\beta = 0$, $\beta \geq 0$ and $\beta \leq 0$, respectively. The test statistic will be stated as follows:

$$(10) \qquad T_k^4 = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} h_{ij},$$

where $h_{ij} = \sum_{r=1}^{n_i} \sum_{s=1}^{n_j} h_{ir,js}$

and $h_{ir,js} = \begin{cases} 1 \text{ if } x_{ir} < x_{js} \\ -1 \text{ if } x_{ir} > x_{js} \quad (i < j). \end{cases}$

It follows from [13] or [10] that T_k^4 is under H_0 asymptotically normally distributed with zero mean and variance

$$D^2 = \frac{1}{18} \left\{N(N+1)(2N+1) - \sum_{i=1}^{k} n_i(n_i+1)(2n_i+1)\right\},$$

where $N = \sum_{i=1}^{k} n_i$. Therefore if all n_i are large, H_0 is rejected at level approximately α in favor of H_1 if $T_k^4 \geq t_\alpha D$, and in favor of H_2 if $T_k^4 \leq -t_\alpha D$, where t_α is the $100(1-\alpha)\%$-point of a standard normal distribution.

2. BEHAVIOR OF THE TESTS UNDER INTERIOR SLIPPAGE

We define interior slippage as follows for model (a): $\mu_1 = \dots = \mu_{m-1} = \mu_{m+1} = \dots = \mu_k = \mu$; $\mu_m = \mu + \Delta$, with $\Delta \neq 0$. The result for Bartholomew's test will be shown in detail for the case of known σ^2.

THEOREM 1. Under interior slippage, when testing H_0 against either H_1 or H_2, $\lim P[T_k^2 \geq C_2] = 1$ as $N \to \infty$ with $n_i/N \geq a > 0$ for $i = 1, \dots, k$.

PROOF. It suffices to show that $\lim P[T_k^2 < \infty] = 0$ as $N \to \infty$ with $n_i/N \geq a > 0$. Let us assume the slippage is upward, i.e. $\Delta > 0$.

(The case $\Delta < 0$ is similar.)

(i) Testing H_0 against H_1: To begin with, the probability of complete amalgamation, $m = 1$, is zero in the limit. This follows immediately from Theorem 1 of [8], which states that a necessary (and sufficient) condition for complete pooling is

(11) $$\bar{x}[1,j] > \bar{x}[1,k] \quad \text{for} \quad j = 1, \ldots, k-1.$$

By the consistency of $\bar{x}_i$ as estimator of μ_i we have $\lim P[\bar{x}[1,m-1] < < \bar{x}[1,k]] = 1$. Therefore there will be, with limiting probability 1, a contribution to T_k^2 from at least the first sample mean. (We can ignore the contribution from $\bar{x}[m,k]$.)

Again from the consistency of $\bar{x}_i$ and of $\bar{x}[1,k]$ we know that for any $\varepsilon_1 > 0$ there exists an N_1 such that for $N \geq N_1$,

(12) $$P[|\bar{x}_1 - \mu| > \varepsilon_1] \leq \varepsilon_1,$$

and for any $\varepsilon_2 > 0$ there exists an N_2 such that for $N \geq N_2$,

(13) $$P[|\bar{x}[1,k] - \mu| < \Delta a - \varepsilon_2] \leq \varepsilon_2.$$

Hence for $N \geq \max(N_1, N_2)$,

(14) $$P[|\bar{x}_1 - \bar{x}[1,k]| < \Delta a - \varepsilon_1 - \varepsilon_2] \leq \varepsilon_1 + \varepsilon_2,$$

and therefore $\lim P[T_k^2 > n_1(\Delta a - \varepsilon_1 - \varepsilon_2)^2/\sigma^2] = 1$. Since Δ, a, ε_1, ε_2, and σ^2 are constant, this proves the theorem for (i).

(ii) Testing H_0 against H_2: An analogous proof holds in this case. The necessary (and sufficient) condition for complete amalgamation is now

(15) $$\bar{x}[1,j] < \bar{x}[1,k] \text{ for } j = 1,\ldots,k-1,$$

and we now have $\lim P[\bar{x}[1,m] > \bar{x}[1,k]] = 1$. We eventually obtain $\lim P[T_k^2 > n_k(\Delta a - \varepsilon_1 - \varepsilon_2)^2/\sigma^2] = 1$, which proves it for (ii).

COROLLARY 1. Under interior slippage, when testing H_0 against either H_1 or H_2, $\lim P[T_k^1 \geq C_1] = 1$ as $N \to \infty$ with $n_i/N \geq a > 0$ for $i = 1,\ldots,k$.

The proof is similar to that of Theorem 1. One shows that $\lim P[T_k^1 < a/g] = 0$, where $g = \Delta^2/a^2$.

COROLLARY 2. Under interior slippage, when testing H_0 against either H_1 or H_2, $\lim P[T_k^{jc} \geq C_j] = 1$ for $j = 1, 2$ as $N \to \infty$ with $n_i/N \geq a > 0$, $i = 1, \ldots, k$.

For model (b), interior slippage means $F_1(x) = \ldots = F_{m-1}(x) = F_{m+1}(x) = \ldots = F_k(x) = F(x)$, $F_m(x) = F(x+\theta)$, $\theta > 0$ (for upward slippage). The equivalent result for Chacko's test follows.

THEOREM 2. Under interior slippage, when testing H_0 against H_1 or H_2, $\lim P[T_k^3 \geq C_2] = 1$ as $N \to \infty$.

PROOF. Because of the equal sample sizes, the m^{th} population will be entitled to a mean rank of $[(N+1)/2]\ (1+\Delta)$ and the others, to mean ranks of $[(N+1)/2][1 - \Delta/(k-1)]$ in an overall sample of N, for some $0 < \Delta < 1$. For testing H_0 against H_1, a proof similar to that of Theorem 1 establishes that for any $\varepsilon > 0$ there exists an N_0 such that for $N \geq N_0$,

(16) $$P\left[\left|\bar{R}_1 - \frac{N+1}{2}\right| < \frac{N+1}{2}\,\frac{\Delta}{k-1} - \varepsilon\right] \leq \varepsilon,$$

and hence

(17) $$\lim P\left\{T_k^3 > \frac{12n}{N(N+1)}\left[\frac{N+1}{2}\,\frac{\Delta}{k-1} - \varepsilon\right]^2\right\} = 1,$$

or, in effect, $\lim P[T_k^3 > dn] = 1$ for some constant $d > 0$. This proves Theorem 2 for the case of H_0 against H_1. The proof for the case H_0 against H_2 is similar.

COROLLARY 1. Under interior slippage, when testing H_0 against H_1 or H_2, $\lim P[T_k^{3c} \geq C_2] = 1$ as $N \to \infty$.

For model (c), interior slippage can be defined by $x_{ij} = \alpha + \varepsilon_{ij}$, $i = 1, \ldots, m-1, m+1, \ldots, k$, and $x_{mj} = \alpha + \Delta + \varepsilon_{ij}$, with $\Delta > 0$ for upward slippage.

It is easiest here to consider the expected value of the test statistic,

(18) $$ET_k^4 = n_m \left(\sum_{i=1}^{m-1} n_i - \sum_{i=m+1}^{k} n_i \right).$$

In particular, for the case of equal sample sizes,

(19) $$ET_k^4 = n(2m - k - 1).$$

We therefore see at once that $ET_k^4 = 0$ if $m = (k+1)/2$ for k odd. For equal sample sizes it can be shown, analogously to the earlier proofs, that if $m < (k+1)/2$, the limit of the probability of rejecting H_0 in favor of H_2 is 1, and hence the limit of the probability of rejecting H_0 in favor of H_1 is zero. The opposite holds for $m > (k+1)/2$. These situations are, in turn, reversed for $\Delta < 0$.

For unequal sample sizes, no such statement can be made. The probabilities depend on whether $\sum_{i=1}^{m-1} n_i < \sum_{i=m+1}^{k} n_i$ or vice versa.

3. COMMENTS

In Section 2 we dealt with what Mosteller[12] calls "the error of the third kind": rejecting H_0 correctly, but for the wrong reason. It is of course a matter of opinion whether it is worse in general to accept or reject H_0 under such circumstances. However, in particular cases it does seem reasonable "to prefer one wrong decision over the other".

For example, if a population has slipped upward for $m > (k+1)/2$, particularly for $m = k-1$, it seems worse to reject H_0 in favor of H_2 than to reject it in favor of H_1; the opposite situation would hold for $m < (k+1)/2$, especially $m = 2$. The Terpstra test will only make the less poor decision in these cases, at least for reasonably equal sample sizes. The other tests can make both.

If the central population has slipped, $m = (k+1)/2$, it seems preferable to accept H_0, since no "trend" of any sort can possibly be claimed. Terpstra's test will accept H_0 here, not only for equal sample sizes but whenever $\sum_{i=1}^{m-1} n_i = \sum_{i=m+1}^{k} n_i$.

T_k^4 is of course a test designed for a more fully specified model than T_k^j, j = 1, 2, 3. As shown in [3], it should have better power than the tests against ordering when there is a (reasonably) linear trend, and poorer power when there is considerable variation in the differences between successive means. (Asymptotic efficiency comparisons have not been possible because the tests have different limiting distributions.) However, for k = 3 or 4, the difference in power between T_k^4 and the other tests appears to be quite small.

T_k^4 cannot be used unless a complete *a priori* ranking of the μ's is feasible. But if this can be done, then in view of its comparable power and the protection which it offers against really bad slippage decisions, T_k^4 would seem to be preferable for k = 3 and 4, and possibly 5. This is particularly so for k = 3, since here any interior slippage must be that of the central population.

It may also be noted that T_k^4, unlike T_k^1 and T_k^2, does not presuppose underlying normal distributions and, unlike T_k^3, can be used with unequal n_i's. (The use of the three conditional tests mentioned may be hard to justify, since the value of m obtained is clearly not irrelevant for deciding between H_0 and H_1 or H_2; also, some power studies in [2] indicate that these tests have distinctly lower power than their unconditional counterparts.)

Incidentally, the two tests given by Whitney [14] for k = 3 show the same behavior under interior slippage as T_k^4.

Finally, when several interior populations have slipped in the same direction but by arbitrary amounts, the preceding results will of course hold true. When some have slipped upwards and some downwards, it is possible to have a "weighted mean slippage" of zero; in this case only one of the alternative hypotheses would be rejected. In general, however, both H_1 and H_2 would again be rejected with probability 1 in the limit.

ACKNOWLEDGEMENTS. I wish to express my thanks to Dr. Alvin Wiggins and to Dr. Robert Elashoff for their helpful comments.

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THE DEVELOPMENT OF THE CONCEPT OF STATISTICAL DECISION THEORY

By D. Bierlein, Germany
Technische Hochschule, Karlsruhe

1. A. WALD'S MODEL AND ITS PRACTICAL APPLICATION

What is a "good" statistical procedure? - Up to the middle of the 20^{th} century the answer to this question generally was given by an ad-hoc definition of an "optimum" property for a special case. Dissatisfaction with that divergent tendency has led A. Wald (about 1939) to the outline of statistical decision theory. Its concern is a uniform model for quite different statistical decision problems, with an accent on the consideration of the consequences of every decision. The model for a general statistical decision situation is - as is well known - characterized by the following data:

1) sample space M (= set of potential results of the planned observation of a random vector $\mathfrak{m}$)
2) set $\mathcal{F}$ of distribution functions which have to be taken into consideration for $\mathfrak{m}$
3) set D of possible decisions
4) loss function W by which for every element of $\mathcal{F}$ the consequence of a decision is evaluated.

A statistical procedure δ for a decision problem formalized in this way is a map from M into D and may be interpreted as a strategy which attaches a decision to every potential information. If, concerning δ, you make the additional assumption that for every $F \in \mathcal{F}$ the expectation of the loss function exists - and therefore the risk function $r_\delta | \mathcal{F}$ - you get to the definition of the *statistical decision function*. Let Δ be the set of all decision functions of the decision problem $(M, \mathcal{F}, D, W)$. The question which statistical procedure from Δ should be used is reduced to the choice of a suitable principle of optimality. The main principles in statistical decision theory are the dominance principle, related to classes of unbiased or invariant procedures, and the

minimax principle.
The task of *mathematical statistics* is to find explicitly, for as extensive classes of statistical decision problems as possible, the optimal procedures based on the most important principles of optimality or at least to give a practicable algorithm for finding them. For the *practical statistician* or for the consumer of statistics in economics, engineering and empirical sciences there remains the task to give, in an adequate manner, a concrete value to the parameters M, $\mathcal{J}$, D and W of the statistical decision problem. This task meets some difficulties: for example it strikes against the old custom of many experimental scientists to use rigorously α-level tests with a value α which depends only on the scientist's special branch. But, there are even men in statistical practice who increasingly regret that an α-level test does not give any concrete recommendation how to act. This is just recently pointed out again by J. Wolfowitz [1]). My own experience as statistical consultant mainly to engineers encourages me to hope that in a large area of application the consumers of statistics will rather soon accept the framework of decision theory. As soon as mathematical statistics will present a way - with or without the help of a computer - to calculate the optimal procedure, in practice most of the classical standard methods will be replaced by more adequate procedures of statistical decision theory.
However, many of the classical procedures, which do not fit (or fit only in an insatisfactory manner) into the scheme of statistical decision theory, will remain practically important. In the first place that seems to be true for the methods of correlation and regression theory and, generally, for descriptive statistics. But in the model of a statistical decision problem they have their legitimate place too: In addition to intuition and special scientific investigations, they are useful for making precise the datum $\mathcal{J}$ (in several cases also D) of the statistical decision problem. That takes place in a preceding analysis of a separated sample with the help of descriptive statistics. In general the preceding inspections and considerations supply *more* information than may be inferred for the formal decision problem by a suitable choice of $\mathcal{J}$, but not enough information to be able to determine a precise a-priori

[1]) J. Wolfowitz: Remarks on the theory of testing hypotheses. The New York Statistician 18 (1967).

probability à la Bayes. This situation gave rise to several modifications of statistical decision theory: Hodges and Lehmann, for example, offer the "Restricted Bayes Solution" as an alternative to the minimax solution [2]). In the following we will discuss a generalization of Wald's decision model where $\mathcal{J}$ is replaced by the set Φ of all such probability measures on $\mathcal{J}$ which are compatible with the "pre-statistical information" of the decision maker (R. Bartoszýnski, D. Bierlein, O. Bunke, H. Richter and others).

2. EXAMPLES

How important a pre-statistical information may be in practice, will be demonstrated by means of some examples of point estimation of a probability:

Consider the situation that for a river there are constructed new installations regulating the stream, for instance a new dam. Let A be the event that a *flood* of well-defined power will occur at least once during a time-interval of ten years and put $x = Pr(A)$. It is required to estimate x as soon as possible, say after 10, 20 or 30 years; i.e. it is not possible to make a sample of size greater than 1, 2 or 3. But beyond that there may be given a number of flood-observations which were made during many decades before the construction of the new installations. Add to this the hydrologically established knowledge, that Pr(A) is not increased by the new construction, perhaps is even decreased. Both these facts - the knowledge of results of former observations made under similar conditions and the use of theoretical considerations - allow to estimate a subjective probability ϕ for that x does not exceed a certain bound x_o:

$$\phi(x \leq x_o) = \phi_o \text{ or } \phi(x \leq x_o) \geq \phi_o .$$

Already these crude specifications of ϕ can represent a pre-statistical information diminishing the minimax risk.

A quite similar situation occurs in medium range or long-term *weather-*

[2]) J.L. Hodges and E.L. Lehmann: The use of previous experience in reaching statistical decisions. Ann. Math. Stat. 23 (1952).

forecast. To be able to make observations in order to test a forecasting rule one has to wait for the - in general rare - years where the conditional meteorological situation, to which the rule attaches a forecast, is obtained. For that reason the pre-statistical information which results from theoretical meteorology and from observations made under similar situations should not be disregarded.

Further actual examples can be found in *techniques of astronautics.* Here the sample size is bounded drastically by high costs of experiments and by time limitation. The information resulting from former experiments with older construction units and a knowledge about the tendency of the effects of technical innovations should - compared with the information resulting from sampling - be of some importance for the decision. In this example too, it should not be difficult to express the pre-statistical information as a system of subjective probabilities for a certain number of subsets A_i of the parameter interval $[0, 1]$.

3. Φ-OPTIMAL PROCEDURES

To develop a general model for these examples, let us assume that for certain subsets A_i, B_k of $\mathcal{F}$ the subjective probabilities themselves or bounds for them are known:

$$\phi(A_i) = \phi_i \text{ for } i \in J_1, \quad \phi(B_k) \leq \psi_k \text{ for } k \in J_2.$$

By these data ϕ is, in general, not made suffciently precise in order to form the risk expectation $\int_{\mathcal{F}} r_\delta(F)d\phi$. The set of all precise a-priori probabilities is the set P of all probability measures p, for which

$$r_\delta(p) := \int_{\mathcal{F}} r_\delta(F)\, dp$$

exists for all $\delta \in \Delta$. Under these measures p the elements of the subset

$$\Phi := \{p \in P: p(A_i) = \phi_i \text{ for } i \in J_1,\ p(B_k) \leq \psi_k \text{ for } k \in J_2\}$$

are compatible with ϕ. For that Φ is not empty, the data for ϕ must, of course, not be self-contradictory.

In the generalized decision theory other non-empty subsets of P are also admitted as pre-statistical informations.

For a generalized statistical decision situation, characterized by the data M, Φ, D and W, a statistical decisions function δ^* is called *Φ-optimal*, if

$$\sup_{p\in\Phi} r_{\delta^*}(p) = \min_{\delta\in\Delta} \sup_{p\in\Phi} r_{\delta}(p),$$

i.e. δ^* is minimax strategy of player 2 in the zero-sum two-person game $(\Phi, \Delta, r_\delta(p))$. The term

$$v^*(\Phi,\Delta) = \inf_{\delta\in\Delta} \sup_{p\in\Phi} r_\delta(p),$$

i.e. the upper value of the game $(\Phi, \Delta, r_\delta(p))$, is called *$\Phi$-minimax risk.* A Φ-optimal procedure guarantees that the expected loss $r_\delta(p)$ does not exceed the bound $v^*(\Phi,\Delta)$.

4. EFFECTIVITY OF AN INFORMATION Φ

At first there is the question: Under what conditions is it possible to take advantage of a pre-statistical information, i.e. under what conditions is the minimax risk $v^*(\Phi,\Delta)$ less than $v^*(P,\Delta)$, which is equal to the minimax risk of the decision problem without using a pre-statistical information (or - more precisely - with respect to the trivial information $\phi(\xi) = 1$)?

We say Φ is *effective*, if

$$v^*(\Phi,\Delta) < v^*(P,\Delta),$$

and define as *effectivity* of Φ the term

$$\mathrm{Eff}\ (\Phi) := \frac{v^*(P,\Delta) - v^*(\Phi,\Delta)}{v^*(P,\Delta) - r_*},$$

where r_* is the infimum of the risk function $r_\delta(p)$ on $\Delta \times P$. [3]. It is easy to see that

$$0 = \mathrm{Eff}(P) \leq \mathrm{Eff}(\Phi) \leq \mathrm{Eff}(\{F_o\}) = 1$$

for all $F_o \in \mathcal{F}$ with

$$\inf_{d \in D} W(F_o, d) = r_*.$$

Necessary and sufficient conditions for Φ to be effective are formulated [4] making use of Wald's [5] intrinsic metric on the space P and of the condition that the games (P,Δ,r) and (Φ,Δ,r), respectively, are strictly determined.

Because of this aspect - and in another connection - criteria for that (Φ,Δ,r) is strictly determined are interesting. With the help of a minimax theorem of Ky Fan [6] such criteria are offered in a form which allowes applications to important special cases of point estimation [4].

One may ask about the relation between effective pre-statistical information and Bayesian a-priori probabilities. The guess, that every precise a-priori probability is an effective pre-statistical information, is wrong.

Indeed, if (P,Δ,r) is strictly determined, then for every minimax strategy p^* of "player 1"

$$v^*(\{p^*\}, \Delta) = v^*(P, \Delta)$$

and consequently

$$\mathrm{Eff}(\{p^*\}) = 0;$$

thus the information p^* is *not* effective.

---

[3]) r_* is zero, if the loss function W is reduced, that is, if $\inf_{d \in D} W(F,d)=0$ for all $F \in \mathcal{F}$.

On the other side, just that a-priori probability which is not seldom taken to be equivalent with absolute ignorance, namely the rectangular distribution L, is effective by all means: In the example of point estimation of a probability with quadratic loss function and sample size 1 you have

$$\text{Eff}\ (\{L\}) = \frac{1}{9}.$$

5. HOW TO GAIN A Φ-OPTIMAL PROCEDURE

The statistical decision theory often meets the reproof that the gap between theory and application is rather large. Is not this gap increased further by including an additional parameter into the model of a decision situation? - Just in the practically important case of a quite vague pre-statistical information a Φ-optimal procedure can be found in a relatively convenient manner. That may be demonstrated on the example of point estimation of a probability **x**: Let W be a quadratic loss function, and a pre-statistical information be given in the form

$$\phi(\mathbf{x} \leq \tau) = \lambda.$$

Then in the first instance you can use tables or diagrams of those pairs (τ, λ) for which the information

$$\phi(\mathbf{x} \leq \tau) = \lambda$$

is effective. If the pre-statistical information is effective, you can find the Φ-optimal estimator in a table, which till now has been computed for sample sizes 1 and 2 and is being prepared for sample sizes 3 and 4.

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The effectivity of Φ *and the* Φ*-optimal estimator* $\delta^{*}|M$ *as a function of* τ *and* λ.

<u>Sample size 1</u>

Eff(Φ)
$\delta^{*}(0)$ (in $^{o}/oo$)
$\delta^{*}(1)$

τ \ λ	0,01	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,99
0,01					11 215 738	80 164 720	192 123 705	341 88 691	517 57 676	717 31 656	922 10 507
0,1						8 222 737	51 182 717	128 148 697	232 118 673	386 100 574	853 57 230
0,2							0,5 243 746	19 209 726	90 199 645	307 153 528	765 97 268
0,3	14 293 752							2 244 737	61 212 663	243 174 551	630 138 341
0,4	118 389 767	26 308 761							32 226 687	170 197 586	462 175 426
0,5	282 483 793	91 367 780	8 282 760			(0 250 750)			8 240 718	91 220 633	282 207 517
0,6	462 574 825	170 414 803	32 313 774							26 239 692	118 233 611
0,7	630 659 862	243 449 826	61 337 788	2 263 756							14 248 707
0,8	765 732 903	307 472 847	90 355 801	19 274 791	0,5 254 757						
0,9	853 770 943	386 426 900	232 327 882	128 303 852	51 283 818	8 263 778					
0,99	922 493 990	717 344 969	517 324 943	341 309 912	192 295 877	80 280 836	11 262 785				

DB 8

<u>Sample size 2</u>

Eff(Φ)
$\delta^*(0)$ (in $^o/oo$)
$\delta^*(1)$
$\delta^*(2)$

τ \ λ	0,01	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	0,99
0,01				22 162 493 780	91 122 484 768	190 92 475 758	310 69 467 750	445 49 458 742	593 33 446 736	752 19 424 727	950 11 131 711
0,1					5 186 493 787	33 155 480 776	83 131 461 765	164 117 406 756	317 110 319 738	550 85 237 704	870 58 113 541
0,2	2 221 497 796						21 185 454 778	92 162 401 757	219 141 343 730	414 122 274 680	714 96 200 440
0,3	63 307 497 808	2 225 501 797						36 185 438 765	117 169 387 735	259 155 333 684	552 126 300 411
0,4	168 390 516 819	25 266 511 805							35 192 441 756	119 179 400 701	404 151 400 455
0,5	285 470 546 831	58 298 529 813	0,2 212 502 794			(0 207 500 793)			0,2 206 498 788	58 187 471 702	285 169 454 530
0,6	404 545 600 849	119 299 600 821	35 244 559 808							25 195 489 734	168 181 484 610
0,7	552 589 700 874	259 316 667 845	117 265 613 831	36 235 562 815						2 203 499 775	63 192 503 693
0,8	714 560 800 904	414 320 726 878	219 270 657 859	92 243 599 838	21 222 546 815						2 204 503 779
0,9	870 459 887 942	550 296 763 915	317 262 681 890	164 244 594 883	83 235 539 869	33 224 520 845	5 213 507 814				
0,99	950 289 869 989	752 273 576 981	593 264 554 967	445 258 542 951	310 250 533 931	190 242 525 908	91 232 516 878	22 220 507 838			

METHODS OF INVESTIGATING WHETHER A REGRESSION RELATIONSHIP IS CONSTANT OVER TIME

by R.L. Brown and J. Durbin (England)
Central Statistical Office, London, and
London School of Economics and Political Science

1. INTRODUCTION

Regression analysis of time-series data is usually based on the assumption that the regression relationship is constant over time. In some applications, particularly in the social and economic field, the validity of this assumption is open to question and it is important that methods of detecting and allowing for changes should be included in the analysis.

In this paper we consider a number of techniques for detecting departures from constancy. Although we shall present several formal tests of significance our approach is essentially that of Tukey's data analysis (Tukey, 1962), that is we try to develop techniques which bring out in a graphic way whatever departures from constancy are present in the data rather than parametrise in advance particular types of departure and develop formal significance tests which have high power against these particular alternatives.

The present paper should be regarded as a preliminary report on and summary of our work on this subject, a full acount of which will be published later (Brown and Durbin, 1969). The later paper will contain proofs of theoretical results and applications to real and artificial examples.

The basic regression model we are concerned with is

$$y_t = x_t'\beta_t + u_t \ , \quad t = 1, \ldots, T \qquad (1)$$

where x_t is the column vector of observations at time t on each of k regressors, β_t is the vector of regression coefficients, where we have attached the suffix t to indicate that β_t may not be constant, and $u_1, \ldots, u_T$ are independent normal variables with zero means and variances

$\sigma_1^2, \ldots, \sigma_T^2$. The first element in each x_t is unity, representing the constant term in the model, and the remaining elements are assumed to be non-stochastic. Thus autoregressive and other models containing lagged y's are excluded from consideration. The hypothesis H_0 we wish to investigate is $\beta_1 = \ldots = \beta_T = \beta$ and $\sigma_1^2 = \ldots = \sigma_T^2 = \sigma^2$; we are, however, more concerned about departures from equality among the β's than among the σ's.

2. METHODS BASED ON LEAST-SQUARES RESIDUALS

Assuming H_0 to be true, let b denote the least-squares estimate of β, i.e. $b = \left[\sum_1^T x_t x_t'\right]^{-1} \sum_1^T x_t y_t$, and let $z_1, \ldots, z_T$ denote the least-squares residuals, i.e. $z_t = y_t - x_t'b$, $t = 1, \ldots, T$. The desirability of examining the residuals as a means of detecting departures from model specification is now generally accepted; see Anscombe (1961) and Anscombe and Tukey (1963) for details of a variety of procedures. For the present problem a natural first step is to plot z_t as a function of t. If there were an abrupt and substantial change in β_t at some point one would expect this to be indicated fairly clearly on the diagram. Experience shows, however, that this is not a very effective method of detecting changes which are small or gradual.

In this respect the problem resembles that of detecting changes in the mean in industrial quality control for which the cumulative sum or cusum technique, introduced by Page (1954) and discussed further by Barnard (1959) and by Woodward and Goldsmith (1964), has been found to be a more effective tool for detecting small changes than the ordinary control chart. This suggests that instead of plotting out the individual z_t the cusums $Z_r = \frac{1}{s} \sum_1^r z_t$, $r = 1, \ldots, T$ should be plotted, where we have divided by the estimated standard deviation s to eliminate the irrelevant scale factor. The difficulty about this suggestion is that there seems to be no way of assessing the significance of the departure of the observed graph of Z_r against r from the mean-value line $E(Z_r) = 0$. The intractability of the problem arises from the fact that in general the covariance function $E(Z_r Z_s)$ does not reduce to a form that is manageable by standard Gaussian-process techniques (c.f. Mehr and McFadden, 1965). For instance, for the simple case of regression through the origin on a linear time trend the covariance function is asymptotically $r - 3r^2s^2/4T^3$ $(r < s)$ which is an unmanageable form.

An alternative is to consider the cusum of squares $\frac{1}{s^2}\sum_1^r z_t^2$. Although more tractable, this is still fairly difficult to deal with and is hard to interpret. Instead of considering this we prefer to make the transformation given in the following section which enables us to treat the problem in terms of cusums and cusums of squares of independent $N(0,\sigma^2)$ variables.

3. METHODS BASED ON RECURSIVE RESIDUALS

Let b_r be the least-squares estimate of β based on the first r observations and let

$$w_r = \frac{y_r - x_r'b_{r-1}}{\sqrt{1 + x_r'(X_{r-1}'X_{r-1})^{-1}x_r}}, \quad r = k+1, \ldots, T \tag{2}$$

where $X_{r-1}' = [x_1, \ldots, x_{r-1}]$. It can be shown that $w_{k+1}, \ldots, w_T$ are independent $N(0,\sigma^2)$. These quantities are easy to obtain recursively on a modern computer without the necessity of repeated matrix inversions in virtue of the relations

$$b_r = b_{r-1} + (X_r'X_r)^{-1}x_r'(y_r - x_r'b_{r-1}) \tag{3}$$

and

$$(X_r'X_r)^{-1} = (X_{r-1}'X_{r-1})^{-1} - \frac{(X'_{r-1}X_{r-1})^{-1}\, x_r x_r'(X_{r-1}'X_{r-1})^{-1}}{1 + x_r'(X_{r-1}'X_{r-1})^{-1}x_r}. \tag{4}$$

Denoting by S_r the residual sum of squares after fitting the model from the first r observations, we have the further relation

$$S_r = S_{r-1} + w_r^2. \tag{5}$$

To avoid difficulties due to ill-conditioning of the matrices $X_r'X_r$, it is recommended that all elements of x_t except the value unity in the leading position should be replaced by their differences from the overall sample mean.

(2) is a generalisation of the regression model of the Helmert transformation. Interesting applications of (3) and (4) to the fitting of regression models in the frequency domain and to the fitting of non-linear models are given by Duncan and Jones (1966) and by Walker and Duncan (1967). The basic relation (4) which enables repeated matrix inversions to be avoided is due to Bartlett (1951).

If β_t is constant up to time $t = t_0$ and differs from this constant value from then on, the w_r's will have mean zero up to $r = t_0$ but in general will have non-zero means subsequently. This suggests that plotting the cusum quantity

$$W_r = \frac{1}{s} \sum_{k+1}^{r} w_j, \quad r = k+1, \ldots, T \qquad (6)$$

against r should be a useful technique for detecting changes in β_t. As previously, s denotes the estimated standard deviation determined by $s^2 = S_T/(T - k)$.

We require a method of testing the significance of the departure of the sample path of W_r from its mean-value line $W_r = 0$. A suitable procedure is to find a pair of lines lying symmetrically above and below the line $W_r = 0$ such that the probability of crossing one or both lines is α, the required significance level. Since the variance of W_r increases with r, it is clearly desirable to choose the slopes of the lines so that they diverge with increasing r. We suggest taking slopes such that the probability that the point (r, W_r) lies outside the lines is a maximum for r half-way between $r = k$ and $r = T$. This leads to the lines joining the points $(k, \pm a\sqrt{T - k})$ and $(T, \pm 3a\sqrt{T - k})$ where a is determined by the equation

$$\Phi(3a) + e^{-4a^2} \{1 - \Phi(a)\} = \frac{1}{2}\alpha \qquad (7)$$

in which $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\frac{1}{2}u^2} du$.

These results are obtained by approximating W_r by the continuous Gaussian process having the same mean and covariance functions. We have assumed that the probability that a particular sample path of W_r crosses both lines is negligible which will be justifiable for values of α normally used for significance testing, say 0.1 or less. Useful values of a are: $\alpha = 0.05$, $a = 0.948$; $\alpha = 0.01$, $a = 1.143$.

We believe that the proper function of these lines is to provide a yardstick against which to assess the observed pattern of the sample path, though of course they can be used to provide a formal test of significance by rejecting if the sample path travels outside the region between the lines.

Another useful plot is that of the cusum of squares

$$s_r = \frac{\sum_{k+1}^{r} w_t^2}{\sum_{k+1}^{T} w_t^2} = \frac{S_r}{S_T} \quad , \quad r = k+1, \ldots, T \tag{8}$$

where we have standardised by dividing by the overall residual sum of squares S_T. On H_0, $E(s_r) = \frac{r - k}{T - k}$. This suggests drawing a pair of lines $s_r = \pm c_0 + \frac{r - k}{T - k}$ on the diagram parallel to the mean-value line such that the probability that the sample path crosses one or both lines is α, the significance level.

We are able to obtain the required significance values c_0 from the theory of Kolmogorov-Smirnov statistics in the study of the sample distribution function. This possibility arises because when T - k is even the joint distribution of $s_{k+2}, s_{k+4}, \ldots, s_{T+k-2}$ is the same as that of an ordered sample of $\frac{1}{2}(T - k) - 1$ independent observations from the uniform (0,1) distribution. Let

$$c^+ = \max_{j=1,\ldots,m-1} \left(s_{k+2j} - \frac{j}{m}\right)$$

$$c^- = \max_{j=1,\ldots,m-1} \left(\frac{j}{m} - s_{k+2j}\right)$$

where $m = \frac{1}{2}(T - k)$. Then c^+ and c^- have the same distribution and are distributed as Pyke's modified Kolmogorov-Smirnov statistic C_n^+ with $n = m - 1$; significance values have been computed by C.E. Rogers and are tabulated in Table 1 of Durbin (1969). We suggest that these values are used as approximations to the significance values of

$$\max_{i=1,\dots,T-k-1} \left(s_{k+i} - \frac{i}{T-k}\right) \quad \text{and}$$

$$\max_{i=1,\dots,T-k-1} \left(\frac{i}{T-k} - s_{k+i}\right),$$

entering the table at $m = \frac{1}{2}(T - k)$ when $T - k$ is even and interpolating linearly between the values for $m = \frac{1}{2}(T - k) - \frac{1}{2}$ and $m = \frac{1}{2}(T - k) + \frac{1}{2}$ when $T - k$ is odd. Our expectation is that the approximation should be good unless $T - k$ is small.

Let c_0 be the significance value obtained in this way corresponding to significance level $\frac{1}{2}\alpha$. The pair of lines $s_r = \pm c_0 + \frac{r-k}{T-k}$ are then drawn on the diagram plotting s_r against r. Since for the values of α normally used, say 0.1 or less, the probability of crossing both lines is negligible, we may take α as the probability of crossing either line.

It is sometimes appropriate to consider a one-sided test. For example, if it is assumed that $\beta_t = \beta^*$ for $t \leq r$ and $\beta_t = \beta^{**} \neq \beta^*$ for $t > r$ while $\sigma_t^2 = \sigma^2$ for all t, then $E(w_t^2) = \sigma^2$ for $t \leq r$ and $E(w_t^2) > \sigma^2$ for $t > r$. One would therefore expect the departure from the null hypothesis to be indicated by a tendency for the sample path of s_r to lie below the mean-value line, and would therefore use a one-sided test. For this purpose, one would take the significance value of c_0 corresponding to significance level α, not $\frac{1}{2}\alpha$.

But whether the two-sided or one-sided situations are envisaged we ourselves prefer to regard the lines constructed in this way as yardsticks against which to assess the observed sample path rather than as providing formal tests of significance.

These procedures based on the plot of s_r represent a development of a test of constancy proposed by Durbin (1960).

Further useful plots are obtained by graphing the components of b_r against r. If the regression relationship does indeed vary over time, these plots may serve to identify the source of the variation.

Finally, we remark that an alternative set of plots can be obtained by running the analysis backwards through time instead of forwards through time. The pictures provided by the two plots will differ according to where along the time scale any variation in the regression relationship takes place. Since both analyses are informative, we suggest that both should be carried out.

4. MOVING REGRESSIONS

Another useful way of investigating the time-variation of β_t and σ_t^2 is to plot out the estimated regression coefficients and residual variance obtained from a segment of ℓ successive observations, this segment being moved along the time scale. The significance of differences over time can then be assessed by a variant of the ordinary analysis-of-variance test for non-overlapping groups. This technique will be considered in Brown and Durbin (1969).

5. ILLUSTRATIONS

Some examples will be presented at the Conference to illustrate the procedures desfribed in section 3.

This work was done in the Research and Special Studies Division of the Central Statistical Office. Durbin's part of the work was done in the capacity of consultant to the Central Statistical Office.

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SOME EFFECTS OF ERRORS OF MEASUREMENT IN MULTIPLE REGRESSION [1)]

by W.G. Cochran (USA)
Department of Statistics, Harvard University

1. INTRODUCTION

In recent years there has been an increase in multiple regression studies on problems in which some of the independent variables represent quantities that are obviously difficult to measure, and are presumably measured with substantial errors. In the social sciences, for example, these variables may include measures of a person's skills at certain tasks or his attitudes and psychological characteristics, the data being obtained from a questionnaire, plus perhaps some kind of examination. Such studies raise the question: to what extent do these errors of measurement vitiate the uses to which the multiple regression is put? In examining this question, my results are less general than is desirable. The only tractable model is simpler than is needed for many applications. Even with this model, the effects of the errors are complex. I have, however, tried to indicate approximately what happens in situations representative of at least a substantial number of applications.

Frequent uses of multiple regression are: (1) to predict a variable y. The relevant quantity here is the residual variance $\sigma_y^2(1-R^2)$, where R is the population multiple correlation coefficient between y and the x's. (2) To study and try to interpret the values of the individual regression coefficients. The nature of the effects of errors of measurement on the values of the β_i has been indicated in a previous paper, Cochran, 1968. Consequently, this paper deals mainly with R^2, although the effects on the β_i will be discussed briefly in section 8.

2. MATHEMATICAL MODEL

Using capital letters to denote correctly measured values, we suppose that in the population the variate Y_u has a linear regression $\alpha + \sum \beta_i X_{iu} + d_u$ on the $\underline{k}$ X's, where $\underline{d}$ is the random residual from the

1) This work was assisted by a Contract between the Office of Naval Research and the Department of Statistics, Harvard University.

regression. Owing to errors of measurement, the variates actually recorded for Y_u and for the $\underline{i}$ X-variate are

$$(2.1) \qquad y_u = Y_u + a + e_u, \qquad x_{iu} = X_{iu} + a_i + e_{iu},$$

where $\underline{a}$ and the a_i represent overall constant biases of measurement, while e_u and the e_{iu} are fluctuating components which follow frequency distributions with means zero.

For this type of model, Lindley, 1947, gave the necessary and sufficient relations that must hold between the joint frequency function of the X_{iu} and that of the e_{iu} in order that the regression of y_u on the x_{iu} remain linear. In particular, if y_u and the X_{iu} follow a multivariate normal distribution, the e_{iu} must also follow a multivariate normal. This case is assumed here. Clearly, if Y_u, e_u, X_{iu} and the e_{iu} jointly follow a multivariate normal, it follows from relation (2.1) that y_u and the x_{iu} also follow a multivariate normal and hence that the regression of y_u on the x_{iu} is linear.

For the present it is assumed further that e_u is independent of Y_u and that any e_{iu} is independent of X_{iu} or any X_{ju} $(j \neq i)$ and of any other e_{ju}. These last assumptions are not essential to ensure linearity of the regression of y_u on the x_{iu}, and some remarks about the non-independent case will be made in section 7. For many applications in which both the X_{iu} and the e_{iu} appear non-normal, it would be desirable to bypass the normality assumptions, but I have no results for this situation.

The bias terms $\underline{a}$ and a_i in (2.1) affect the constant term in the regression of y on the x_i, but do not affect the multiple correlation coefficient between y and the x_i, and hence do not enter into the following sections on R^2.

3. EFFECT ON R^2 WHEN X's ARE INDEPENDENT

With $\underline{k}$ X-variates, the following notation will be used for the relevant population parameters.

σ_i^2 = variance of the correct X_{iu};

ε^2, ε_i^2 = variance of e_u, e_{iu};

ρ_{ij} = correlation coefficient between X_{iu} and X_{ju};

δ_i = correlation coefficient between X_{iu} and Y_u.

The symbol δ_i is used instead of the more natural ρ_{iy} because this helps to avoid confusion between different kinds of correlation in later discussion. The sign attached to each X_{iu} is assumed chosen so that $\delta_i \geq 0$.

The value of R^2, the population squared multiple correlation between Y and the X_i, is completely determined by the ρ_{ij} and the δ_i. Primes will be used to denote the corresponding correlations R'^2, ρ'_{ij}, δ'_i between the observed y and the x_i. From the assumptions we have

$$(3.1) \qquad \rho'_{ij} = \frac{\mathrm{Cov}(X_i+e_i)(X_j+e_j)}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_j^2+\varepsilon_j^2)}} = \frac{\rho_{ij}\sigma_i\sigma_j}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_j^2+\varepsilon_j^2)}} ,$$

$$(3.2) \qquad \delta'_i = \frac{\mathrm{Cov}(X_i+e_i)(y+e)}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_Y^2+\varepsilon^2)}} = \frac{\delta_i\sigma_i\sigma_Y}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_Y^2+\varepsilon^2)}} .$$

In psychometric writings the quantity $\sigma_i^2/(\sigma_i^2+\varepsilon_i^2)$ is often called the *coefficient of reliability* of x_i. We shall follow this terminology and define

$$g_i = \sigma_i^2/(\sigma_i^2+\varepsilon_i^2) = \text{coefficient of reliability of } x_i.$$

Similarly,

$$g_y = \sigma_Y^2/(\sigma_Y^2+\varepsilon^2) = \text{coefficient of reliability of } y.$$

Hence, from (3.1) and (3.2),

(3.3) $\rho'_{ij} = \rho_{ij}\sqrt{g_i g_j}, \qquad \delta'_i = \delta_i\sqrt{g_i g_y}$.

If the X's are mutually independent, it is well known that

(3.4) $$R^2 = \sum_{i=1}^{k} \delta_i^2 .$$

Since our assumptions guarantee that the x's are also independent in this case,

(3.5) $$R'^2 = \sum_{i=1}^{k} \delta_i'^2 = g_y \sum_{i=1}^{k} \delta_i^2 g_i .$$

Hence,

(3.6) $$R'^2 = R^2 g_y \sum_{i=1}^{k} \delta_i^2 g_i / \sum_{i=1}^{k} \delta_i^2 = R^2 g_y \bar{g}_w$$

where $\bar{g}_w$ is a weighted mean of the coefficients of reliability of the x_i.

Consider now the residual variance from the regression. With the correct measurements this is $\sigma_Y^2(1-R^2)$. With the fallible measurements it becomes

(3.7) $$\sigma_y^2(1-R'^2) = \sigma_Y^2+\varepsilon^2 - \sigma_y^2 g_y \bar{g}_w R^2 = \sigma_Y^2(1-R^2\bar{g}_w) + \varepsilon^2$$

since $\sigma_y^2 g_y = \sigma_Y^2$. Equation (3.7) contains the well-known result that under this model the effect of errors of measurement of y with variance ε^2 is simply to increase the residual variance by ε^2, the variance of these errors.

As regards errors of measurement of the x_i, two points are worth noting in relation to applications. For a given reliability of measurement, i.e. a given value of $\bar{g}_w$, the deleterious effect on the residual variance increases as R^2 increases, being greater when the prediction formula is very good than when it is mediocre. For example, suppose that $\bar{g}_w = 0.5$, representing a poor reliability in measurement of the x_i. If $R^2 = 0.9$, the residual variance is increased from $0.1\sigma_Y^2$ to $0.55\sigma_Y^2$, over a five-fold increase. With $R^2 = 0.4$, the increase is only from $0.6\sigma_Y^2$ to $0.8\sigma_Y^2$, a 33% jump.

Secondly, as would be expected, the quality of measurement of those X_i that are individually good predictors is much more important than that of poorer predictors. With $k = 2$, $\delta_1 = 0.9$, $\delta_2 = 0.3$, we have $R^2 = 0.90$, $(1-R^2) = 0.1$. If $g_1 = 0.5$, $g_2 = 1$, this gives $(1-R'^2) = 0.505$, but with $g_1 = 1$, $g_2 = 0.5$, $(1-R'^2) = 0.145$, a much smaller increase.

4. EFFECT OF CORRELATION BETWEEN X's: TWO VARIATES

After working several numerical examples, my approach was to try to construct an approximation of the form $R'^2 = R^2 g_y \bar{g}_w f$, where $\underline{f}$ is a correction factor to allow for the effect of correlations among the X's, being equal to 1 when the X's are independent. But with numerous X variables, all intercorrelated, it soon appeared that no simple correction factor was likely to be generally applicable. However, we will continue to study the relation of R'^2 to $R^2 g_y \bar{g}_w$. Further, since the effect of errors in y under this present model is always just to introduce the factor g_y, this factor will be omitted in what follows so as to concentrate attention on correlations among the X's.

With 2 X-variates having a correlation ρ, the values of R^2 and R'^2 work out as follows:

$$(4.1) \qquad R^2 = (\delta_1^2+\delta_2^2 - 2\rho\delta_1\delta_2)/(1-\rho^2) \;,$$

$$(4.2) \qquad R'^2 = (g_1\delta_1^2+g_2\delta_2^2 - 2g_1g_2\rho\delta_1\delta_2)/(1-g_1g_2\rho^2) \;.$$

For given δ_1, δ_2, the correlation ρ lies within the limits $\delta_1\delta_2 \pm \sqrt{(1-\delta_1^2)(1-\delta_2^2)}$, otherwise R^2 would exceed 1. Within these limits,

$$(4.3) \qquad R'^2 = R^2 \frac{(g_1\delta_1^2+g_2\delta_2^2 - 2g_1g_2\rho\delta_1\delta_2)}{(\delta_1^2+\delta_2^2 - 2\rho\delta_1\delta_2)} \cdot \frac{(1-\rho^2)}{(1-g_1g_2\rho^2)} \;.$$

Taking out $\bar{g}_w = (g_1\delta_1^2+g_2\delta_2^2)/(\delta_1^2+\delta_2^2)$ as a factor, we may write

$$R'^2 = R^2\bar{g}_w(A)(B) \tag{4.4}$$

where B is the term

$$B = (1-\rho^2)/(1-\rho^2 g_1 g_2) \ . \tag{4.5}$$

For $g_1 g_2 < 1$, $\rho \neq 0$, this term is always < 1. For fixed $g_1 g_2$ it decreases monotonically towards 0 as ρ moves from 0 towards either +1 or -1.

Factor A takes the form

$$A = \left(1 - \frac{2\rho\delta_1\delta_2}{\dfrac{\delta_1^2}{g_2} + \dfrac{\delta_2^2}{g_1}}\right) \Bigg/ \left(1 - \frac{2\rho\delta_1\delta_2}{\delta_1^2+\delta_2^2}\right) \tag{4.6}$$

For $0 < g_1 g_2 < 1$, it follows that $A > 1$ if ρ is positive while $A < 1$ if ρ is negative, provided that δ_1, δ_2 are both > 0.

Hence, if ρ is negative the factor $f = AB$ is always < 1, decreasing towards zero as ρ approaches -1. If ρ is positive the situation is not so clear, since $A > 1$ and $B < 1$. However, when ρ is small the factor A, which contains only linear terms in ρ, tends to dominate B which is quadratic in ρ. Thus when ρ is positive, $f = AB$ increases and is greater than 1 for a time, but then decreases as ρ increases further, becoming less than 1 if ρ is high enough. The only exception is the case $\delta_1 = \delta_2$, $g_1 = g_2 = g$: f then reduces to $(1+\rho)/(1+g\rho)$, which increases from $f = 1$ at $\rho = 0$ to $f = 2/(1+g)$ at $\rho = 1$. Incidentally, when $\delta_1 = \delta_2$, the range of ρ is from $(-1 + 2\delta_1^2)$ to +1.

The size of the product $g_1 g_2$ is also relevant to f. For given ρ, both A and B tend to approach 1 as $g_1 g_2$ increases towards 1. Thus the formula $R'^2 = R^2 g_w$ is closer to the truth when g_1 and g_2 are high.

In a previous paper, Cochran, 1961, the effect of ρ on the value of R^2 was studied in connection with applications to the discriminant function. From (4.1) it is clear that negative values of ρ are helpful to prediction, since when ρ is negative, R^2 always exceeds the value $(\delta_1^2+\delta_2^2)$ that it would have if ρ were 0. With ρ positive, R^2 decreases at first but has a minimum at $\rho = \delta_2/\delta_1$, where $\delta_2 < \delta_1$, and thereafter increases. It does not reach $(\delta_1^2+\delta_2^2)$ until $\rho = 2\delta_1\delta_2/(\delta_1^2+\delta_2^2)$, which is high if δ_1 and δ_2 are not too different. Thus, positive correlations are harmful to prediction unless ρ is high enough.

As an illustration, table 4.1 shows the values of R^2 and f for $\rho = -.0.5(0.1) + 0.9$, for six examples. In the first three, $\delta_1 = 0.6$, $\delta_2 = 0.4$, and in the second, $\delta_1 = 0.7$, $\delta_2 = 0.2$. The three pairs $g_1, g_2 = (0.9,0.7)$, $(0.8,0.6)$, and $(0.7,0.5)$ are given. The behavior of R^2 and f as described above may be noted, as well as the increasing departure of f from 1 as the product $g_1 g_2$ decreases. The principal difference between the cases $\delta_1 = 0.6$, $\delta_2 = 0.4$ and $\delta_1 = 0.7$, $\delta_2 = 0.2$ is as follows. When δ_1 and δ_2 differ greatly and ρ is positive, R^2 begins to increase and f to decrease for quite moderate values of ρ (around 0.3 for $\delta_1 = .7$, $\delta_2 = .2$), while when δ_1 and δ_2 are more nearly equal, R^2 decreases and f increases until ρ is closer to 1. The turning value of f is a complicated expression, but is usually close to that of R^2.

The complementary sets $g_1, g_2 = (0.7,0.9)$, $(0.6,0.8)$, $(0.5,0.7)$, not shown here, exhibit the same behavior with f lying a little nearer 1, except for high, positive ρ when f becomes less than 1.

Table 4.1

Values of $f = R'^2/R^2\bar{g}_w$ for six examples.

	$\delta_1 = .6,\ \delta_2 = .4$				$\delta_1 = .7,\ \delta_2 = .2$			
	g_i =	.9,.7	.8,.6	.7,.5	g_i =	.9,.7	.8,.6	.7,.5
ρ	R^2	f	f	f	R^2	f	f	f
−.5	–[x]	–	–	–	.893	0.84	0.78	0.74
−.4	.848	0.87	0.82	0.78	.764	0.89	0.85	0.81
−.3	.730	0.91	0.88	0.85	.675	0.93	0.90	0.88
−.2	.642	0.95	0.92	0.90	.610	0.96	0.94	0.93
−.1	.574	0.98	0.97	0.96	.564	0.98	0.98	0.97
0	.520	1.00	1.00	1.00	.530	1.00	1.00	1.00
.1	.477	1.02	1.03	1.04	.507	1.01	1.02	1.02
.2	.442	1.04	1.06	1.07	.494	1.02	1.02	1.03
.3	.413	1.06	1.08	1.10	.490	1.02	1.02	1.03
.4	.390	1.07	1.10	1.12	.498	1.00	1.00	1.01
.5	.373	1.08	1.11	1.14	.520	0.98	0.97	0.97
.6	.362	1.08	1.11	1.14	.566	0.94	0.91	0.90
.7	.361	1.07	1.09	1.12	.655	0.86	0.82	0.79
.8	.378	1.03	1.03	1.06	.850	0.73	0.67	0.63
.9	.463	0.86	0.85	0.85	–[x]	–	–	–
	$\bar{g}_w$ =	.838	.738	.638	$\bar{g}_w$ =	.885	.785	.685

[x] Impossible because $R^2 > 1$.

In these examples $\bar{g}_w$ lies between 0.638 and 0.885. As regards the crude approximation $R'^2 \doteq R^2\bar{g}_w$, in these examples this is correct to within $\pm$ 15% for ρ lying between -0.3 and +0.5, being much closer than this throughout most of table 4.1.

To summarize for two independent variates: when ρ is negative, $f < 1$ because the negative correlation $\rho' = \rho\sqrt{g_1 g_2}$ is less helpful to R'^2 than the negative correlation ρ is to R^2. When ρ is positive and small or modest, f exceeds 1, because the harmful positive correlation is decreased by the errors of measurement. If ρ becomes high enough, however, positive correlation becomes helpful and f drops below 1. For given ρ, f departs further from 1 as the product $g_1 g_2$ decreases.

With 3 X-variables denoted by the subscripts i, j, and k, the value of R^2 may be expressed as

$$(4.7) \qquad R^2 = \frac{\sum_i \delta_i^2 (1-\rho_{jk}^2) - 2\sum_{j>i} (\rho_{ij} - \rho_{ik}\rho_{jk})\delta_i\delta_j}{1 - \sum_{j>i} \rho_{ij}^2 + 2\rho_{12}\rho_{13}\rho_{23}}$$

while R'^2 has the corresponding value found by substituting $\delta_i' = \delta_i\sqrt{g_i}$, $\rho_{ij}' = \rho_{ij}\sqrt{g_i g_j}$. These expressions are discouraging to the prospect of finding an approximation for f that would be valid over a wide range of values of the g_i and the ρ_{ij}. With regard to R^2 itself, (4.6) suggests that with all $\delta_i > 0$, negative values of ρ_{ij} are likely to be helpful, since the only linear term in the ρ_{ij} is $-2\rho_{ij}\delta_i\delta_j$ in the numerator.

Before proceeding further, we digress to consider the values of the ρ_{ij} and the g_i likely to occur in practice.

5. SOME VALUES OF r_{ij} IN PRACTICAL APPLICATIONS

When the sign attached to each x_i is chosen so that $\delta_i \geq 0$, these decisions determine the sign attached to every ρ_{ij}. In studying the estimates r_{ij} of the ρ_{ij} found in 12 well-known examples of the discriminant function, Cochran, 1961, I noted that most of the r_{ij}

are positive and modest in size, while those that are negative are usually small. The same situation appears to hold in many applications of multiple regression. Table 5.1 shows the distributions of the r_{ij} in (i) the discriminant function examples, (ii) the numerical examples of a multiple regression given in 12 standard statistical texts, (iii) a single large example--the prediction of verbal ability scores of 12th grade white students in the north of the U.S. from 20 variables representing data on the student, the quality of the school, and the student's home environment, Coleman et al., 1966.

Table 5.1

Distributions of estimated correlations between x's

	Number of Cases				Number of Cases		
r_{ij}	D.F.	Texts	Verbal	r_{ij}	D.F.	Texts	Verbal
< -.5	1	1	0	0 to .1	15	5	58
-.5 to -.4	2	0	0	.1 to .2	22	8	41
-.4 to -.3	1	0	1	.2 to .3	25	7	25
-.3 to -.2	4	2	2	.3 to .4	18	9	7
-.2 to -.1	4	2	10	.4 to .5	6	7	3
-.1 to 0	9	5	36	.5 to .6	10	6	6
				.6 to .7	4	5	1
				.7 to .8	1	4	0
				> .8	0	3	0
Total	21	10	49	Total	101	54	141
$\bar{r}$	-0.19	-0.17	-0.09	$\bar{r}$	+0.30	+0.41	+0.16

The percentages of r's that are positive are 83%, 84%, and 74% in the three sets. The negative r's average to between -0.2 and 0, the averages of the positive r's being a little higher. While some allowance is needed for the sampling errors of the r_{ij}, since our interest is in the unknown ρ_{ij}, my impression is that most of the degrees of freedom are large enough so that the effect of sampling errors on the average r's should be small. In the discriminant function and text examples there may have been some selection towards more interesting examples, but this would probably affect the sizes of δ_i rather than the ρ_{ij}.

In calculations for 3 or more x's, these results led me to concentrate on ρ_{ij} less than 0.5, and mainly on two cases: (1) all ρ_{ij} positive, (2) only a minority negative.

6. THE PROBLEM OF ESTIMATING RELIABILITY

With variables that are hard to measure, the problem of estimating the reliability of measurement is also formidable, and I have not come across any set of g values that might be regarded as representative. Direct estimation of g is possible only when it is feasible to measure both the correct value X and the fallible value x for a sample of items. This situation is likely to be confined mainly to applications in which (1) g is high and (2) the fallible measurement is enough cheaper or more convenient to make it preferable to the correct measurement. Occasionally, an opportunity to measure X may present itself even though X is not usually available. Thus, the reliability of appraiser's estimates of the values of homes may be estimated by finding the actual selling-prices for these homes that happen to have been sold recently: data of Kish and Lansing, 1954, indicate a g of around 0.83 in this situation.

When X cannot be measured, assume first that for the <u>u</u>th item the correct measurement X_u is constant (i.e. not varying with time). If two independent measurements x_{u1}, x_{u2} of each item by the fallible instrument can be made, the quantity $Cov(x_{u1}, x_{u2})$ estimates σ_X^2, so that $\hat{g} = Cov(x_{u1}, x_{u2})/s_x^2$ is an estimate of g. This method is widely used in appraising the reliability of examinations, the two measurements being either random halves or alternative forms of an examination. Naturally, values of g over 0.9 are sought, though values between 0.7 and 0.9 may be considered acceptable if the skill in question is difficult to measure by examination. The assumption of independence is crucial in this approach. With a positive correlation between the errors e_{u1} and e_{u2}, $Cov(x_{u1}, x_{u2})$ overestimates σ_X^2 so that g is overestimated.

Alternatively, the same measurement may be made on the specimens at two different times. Examples are examinations given a week apart, or questions repeated to a respondent on a later occasion. If these questions refer to memory of a definite past event, the errors of measurement may be smaller on the first than on the second occasion. Fortunately, still assuming independence and constant X_u for given $\underline{u}$, all three quantities σ_X^2, $\sigma_{x_1}^2$, and $\sigma_{x_2}^2$ can be estimated, as can $g = \sigma_X^2/\sigma_{x_1}^2$, the reliability of the answers on the first occasion. With questions involving memory, however, positive correlation between errors is a constant danger, since the respondent may recall the same wrong answer on both occasions.

When the correct measurement varies with time, interpretation becomes more complex. For the $\underline{u}$th item on the $\underline{j}$th occasion, the simplest model is to write the correct value as X_u+t_{uj}, where X_u now represents an average value over time for the $\underline{u}$th item and t_{uj} represents the fluctuation over time for the true value. The observed value on the $\underline{j}$th occasions is $x_{uj} = X_u + t_{uj} + e_{uj}$. For simplicity, assume t_{uj} and e_{uj} independent from item to item and from occasion to occasion. Then if the objective is to measure the correct value of X on a specific occasion, i.e. to measure X_u+t_{uj}, the reliability of our measuring process is

$$g = (\sigma_X^2 + \sigma_t^2)/\sigma_x^2 .$$

This quantity is estimated by $\hat{g}$ if our data are a sample of two independent measurements of each specimen on the $\underline{j}$th occasion. But if our sample consists of independent measurements on two different occasions, $\hat{g} = \mathrm{Cov}(x_{uj}, x_{uj})/s_x^2$ estimates σ_X^2/σ_x^2, which can be a serious underestimate if σ_t^2 is large. For instance, Guilford, 1959, from measurements one day apart on the same subjects, reports estimated g values of 0.22 for respiration period, 0.36 for white blood cell count, 0.65 for blood sugar content, and 0.74 for systolic blood pressure. Presumably, these relatively low values are in part caused by real day to day variation in the values of the items.

Further, when X varies through time, the relevant quantity for prediction of y may not be the value of X on the first occasion, but some function of its levels over time, as for instance in the prediction of death rate from cigarette smoking history. This point may be put more generally. In difficult problems of measurement we may be attempting, through ignorance, to measure the wrong quantity. At its simplest, suppose that the relevant true measurement for the uth item in the population is X_u, which does not vary with time. The "true" value which we attempt to measure is X_u+a_u, where a_u is a random variable representing the extent to which we are measuring the wrong quantity. Our observed values for two independent fallible measurements are $X_u+a_u+e_{u1}$, $X_u+a_u+e_{u2}$. Hence, $\hat{g}$ estimates $(\sigma_X^2+\sigma_a^2)/\sigma_x^2$, whereas the relevant g is σ_X^2/σ_x^2.

For these reasons I am unable to name any narrow range of values of g which can represent practical experience in difficult measurement problems. My calculations have been done for the range $g \geq 0.5$: they should perhaps have been extended to lower g's. Ignorance of the values of the g_i for a specific application of interest is, of course, a considerable detriment to the use of any results of this paper. There is, however, increased interest in studying errors of measurement, as evidenced by the work of the U.S. Census Bureau, e.g. Hansen, Hurwitz, and Bershad, 1961, and later papers, by Kish's study, 1962, of interviewer variance, and by Mandel's study, 1959, of errors of measurement by different laboratories.

It should not be forgotten that g is a measure of precision of measurement *relative to the true variation in the population.* High values of g may be found with what appears quite sloppy and imprecise measurements, because the population is highly variable. A low value, such as $\hat{g} = 0.41$ reported by Kinsey, 1948, for "Age at first knowledge of venereal disease", (by repeating the question on a later occasion) may in part reflect the fact that the correct ages have a small standard deviation.

7. EFFECT OF ERRORS WITH MORE THAN TWO X VARIATES

Returning to the relation between R'^2 and R^2 with k X-variates (k>2), the only case which will be discussed algebraically is that in which $\rho_{ij} = \rho > 0$, $g_i = g$. In this case R^2 and R'^2 have the simple formulas

$$(7.1) \qquad R^2 = \frac{\sum \delta_i^2}{1 + (k-1)\rho} \left[1 + \frac{k\rho \sum (\delta_i - \bar{\delta})^2}{(1-\rho)\sum \delta_i^2}\right] ,$$

$$(7.2) \qquad R'^2 = \frac{g\sum \delta_i^2}{1 + (k-1)g\rho} \left[1 + \frac{gk\rho \sum (\delta_i - \bar{\delta})^2}{(1-g\rho)\sum \delta_i^2}\right] .$$

If the δ_i are approximately equal, i.e. the X_i are individually about equally good, the first terms in (7.1) and (7.2) dominate. Then we have

$$(7.3) \qquad f = \frac{R'^2}{gR^2} \cong \frac{1 + (k-1)\rho}{1 + (k-1)g\rho}$$

For ρ positive, this f exceeds 1 and increase steadily as ρ goes from 0 to 1.

The case $\delta_i = \delta$, $\rho = 1$ is of some interest. In measuring a trait of a subject, such as aggressiveness, a common practice is to ask k questions, the answer to each being a measure of aggressiveness. If all the questions measured aggressiveness correctly, we would have $\rho = 1$, $\delta_i = \delta$, making $R^2 = \delta^2$, reflecting the fact that in this event any one question contains all the information. If the questions have reliability g and independent errors, $\delta_i' = \sqrt{g}\delta$, $\rho' = g\rho$ and $R'^2 = kg\delta^2/\{kg + (1-g)\}$ from (7.2). For example, with g = .6 and k = 5 questions, $R'^2 = 3\delta^2/3.4 = 0.88\delta^2$, considerably better than the value $R'^2 = 0.6\delta^2$ that we would get by asking only one question. The argument here is the same as that used in the well-known correction for attenuation. If the errors of measurement in the different questions are positively correlated with one another though still uncorrelated with X, we do not do quite so well. With a correlation r between these errors, R'^2 works out as $kg\delta^2/\{kg + (1-g)(1+kr-r)\}$. Thus with g = .6, k = 5, r = .5, $R'^2 = 0.71\delta^2$, as against $0.88\delta^2$.

When the δ_i vary in (7.1) and (7.2) the ratio of the second terms inside the brackets is $g(1-\rho)/(1-g\rho)$. This ratio is less than g, and therefore less than 1, and decreases as ρ increases. Thus this term slows down the increase in f. In applications the two terms in R^2 are often of the same order of magnitude, so that f shows only a small rise above 1 for positive and moderate values of ρ.

Table 7.1 shows the values of f for seven examples for 3, 5, and 10 x's, selected from those worked. In these examples, the reliability g is the same for all x's, f being given for g = .9(.1).5. For each example the number of x-variates k, and the values for the δ_i and of the ρ_{ij} are given. At the foot of the table are the values of R^2 and of $R^2_{ind} = \sum \delta_i^2$, the value that R^2 would have if the X's were independent.

Table 7.1

Values of f for seven selected examples

	All ρ_{ij} positive			Some ρ_{ij} negative			
k =	3	5	10	3	3	5	10
δ_i =	.6,.5,.4	.5,$(.4)^2$ .3,.2	.5,.4,$(.3)^2$ $(.2)^3$,$(.1)^3$	.6,.5,.4	.6,.5,.4	.5,.4,.3 .2,.4	.5,.4,$(.3)^2$, $(.2)^2$,$(.1)^3$, .2
ij =	.3	.3	.3	.2,−.2, −.2	.3,.3, −.2	.3(j≠5) −.3(j=5)	.3(j≠10) −.2(j=10)
g	f	f	f	f	f	f	f
9	1.03	1.04	1.01	0.98	0.99	0.97	1.00
8	1.07	1.08	1.02	0.97	1.00	0.94	1.00
7	1.11	1.13	1.04	0.95	1.01	0.92	1.00
6	1.16	1.19	1.08	0.94	1.04	0.90	1.01
5	1.21	1.26	1.12	0.93	1.04	0.89	1.03
2 =	.497	.369	.390	.875	.630	.805	.511
2 ind =	.770	.700	.740	.770	.770	.700	.740

In general, the values of f in table 7.1 behave as would be expected from the results for k = 2 in section 4. In the three cases with ρ_{ij} = +0.3, this correlation produces a marked decrease in R^2 as compared with R^2_{ind}. Consequently, f rises above 1 because errors of measurement reduce the detrimental correlation to +0.3g.

When some ρ_{ij} are positive and some negative, we may in general regard the positive ρ_{ij} as harmful to prediction and the negative ρ_{ij} as helpful, though this is an oversimplification of what can happen with more than 2 x's. Two examples with substantial proportions of negative correlations have been included in table 7.1. The first example with k = 3 has 2 of the 3 ρ_{ij} negative. In the example with k = 5, all correlations between x_5 and the other x's are negative, making 4 negative ρ_{ij} out of 10. In both examples the net effect of the correlations is distinctly helpful, R^2 exceeding R^2_{ind}. As would be anticipated, f < 1 in both examples. In the two remaining examples, with k = 3 and k = 10, one ρ_{ij} out of 3 and 9 out of 45 are negative. The net effect of the correlations is a decrease in R^2 versus R^2_{ind}, though less marked than when all ρ_{ij} are positive. In both examples f rises only very slightly above 1.

When some ρ_{ij} are negative and the g_i are very unequal, f is more erratic. Its behavior still follows the lines indicated above. As an illustration, table 7.2 gives f for some unequal g_i for the example in table 7.1 with k = 5 and ρ_{i5} negative.

Table 7.2

Values of f with unequal g_i k=5, ρ_{ij} = .3,($j \neq 5$), = -.3(j=5)

$g_1 \ldots g_4$ =	1.0	.8	.6	.5
g_5 =	.5	.6	.8	1.0
f =	0.76	0.85	0.99	1.09
$\bar{g}_w$ =	.886	.754	.646	.614

In the first case in table 7.2, $g_1 \ldots g_4$ are without error, while g_5 has reliability only 0.5. The harmful correlations ρ = +.3 are unaltered:

the helpful ones are reduced to $-0.3\sqrt{.5} \cong -0.21$. Consequently, f is well below 1 although $\bar{g}_w = .886$ is quite high. The opposite case, with $g_1 \ldots g_4 = 0.5$, $g_5 = 1$, gives $f = 1.09$, the harmful correlations being reduced more than the helpful one. The two middle examples illustrate less extreme situations of the same type.

Since with electronic computers, multiple regression calculations having as many as 50 independent variates are becoming commoner, a summary statement could be made with greater confidence if examples of this size, with all ρ_{ij}, δ_i and g_i different, had been worked, and if more were known about the values of the g_i likely to occur in such problems and about the cruciality of the assumption of a multivariate normal. As a rough guide to the effects of errors of measurement, the relation $R'^2 \cong R^2 g_y \bar{g}_w$ may serve, the value of R'^2 being perhaps 10-20% higher than this if most correlations among the X's are positive and harmful, and perhaps 10-20% lower if we are lucky enough to have mainly helpful correlations. This rule assumes that the errors of measurement are independent of one another and of the correct X's.

If the e_i and e_j for two different X's have a correlation c_{ij} but are still independent of the true X_i or X_j, we have, in the notation of section 3,

$$\rho'_{ij} = \frac{\rho_{ij}\sigma_i\sigma_j + c_{ij}\varepsilon_i\varepsilon_j}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_j^2+\varepsilon_j^2)}} : \quad \delta'_i = \frac{\delta_i\sigma_i\sigma_Y}{\sqrt{(\sigma_i^2+\varepsilon_i^2)(\sigma_Y^2+\varepsilon^2)}} .$$

Thus $\delta'_i = \delta_i\sqrt{g_i g_y}$ as before, but if $c_{ij} > 0$, ρ'_{ij} now exceeds $\rho_{ij}\sqrt{g_i g_j}$. With most ρ_{ij} positive and harmful, it looks as if the effect of a positive c_{ij} will be that f will lie closer to unity.

Suppose now that e_i and X_i are correlated, with $\mathrm{Cov}(e_i, X_i) = c_i$. To take the simplest case, e_i is assumed uncorrelated with any other X_j or e_j, though in applications it may happen that e_i is correlated with some other X_j's. The variance of x_i now becomes $(\sigma_i^2+\varepsilon_i^2+2c_i)$, so that g_i becomes $\sigma_i^2/(\sigma_i^2+\varepsilon_i^2+2c_i)$. With the above assumptions, the equation $\rho'_{ij} = \rho_{ij}\sqrt{g_i g_j}$ still holds. However, e_i becomes correlated with y through its correlation with X_i. We have

(7.4) $$\mathrm{Cov}(y,x_i) = \mathrm{Cov}(y,X_i) + \mathrm{Cov}\{(\alpha + \sum\beta_j X_j + d),(e_i)\}$$

$$= \delta_i\sigma_y\sigma_i + \beta_i c_i .$$

Hence,

(7.5) $$\delta_i' = \rho_{yx_i} = \delta_i\sqrt{g_i} + \frac{\beta_i c_i}{\sigma_y\sigma_{x_i}} = \delta_i\sqrt{g_i} + \frac{\beta_i c_i\sqrt{g_i}}{\sigma_y\sigma_i} .$$

Equation (7.5) suggests that if β_i has the same sign as δ_i, as should happen with the predominant regression coefficients, a positive correlation between e_i and X_i will increase δ_i' and hence tend to increase R'^2. This might be expected since the e_i, as it were, are doing some of the work of the X_i. The most interesting case of a *negative* correlation is that studied by Berkson, 1950, and Box, 1961, in controlled experiments in which the fallible x_i are set at predetermined values, the errors of measurement e_i being therefore uncorrelated with the fallible x_i. Hence, $\mathrm{Cov}(e_i,x_i) = \mathrm{Cov}(e_i,X_i) + \varepsilon_i^2 = 0$, making $c_i = -\varepsilon_i^2$. In this case the effect on R'^2 is almost easily seen by considering the values of the regression coefficients in the next section.

8. EFFECTS ON REGRESSION COEFFICIENTS

The assumptions and notation are the same as in section 2, except that for the moment we assume that e_i and X_i have a covariance c_i. The symbols σ_{ij}, σ_{ij}' denote $\mathrm{Cov}(X_i,X_j)$ and $\mathrm{Cov}(x_i,x_j)$, where $\sigma_{ii} = \sigma_i^2$, $\sigma_{ii}' = \sigma_i^2+\varepsilon_i^2+2c_i$. The assumption of multivariate normality guarantees that y has a linear regression both on the X_i and on the x_i. These relations provide two expressions for $\mathrm{Cov}(y,x_i)$.

(8.1) $$\mathrm{Cov}(y,x_i) = \mathrm{Cov}\{(\alpha' + \sum_j \beta_j' x_j + d'),(x_i)\} = \sum_j \beta_j'\sigma_{ij}' ,$$

$$(8.2) \qquad \mathrm{Cov}(y,x_i) = \mathrm{Cov}\{(\alpha + \sum_j \beta_j X_j + d),\ (X_i+e_i)\} = \sum_j \beta_j \sigma_{ij} + \beta_i c_i$$

since the e_i are assumed uncorrelated with $\underline{d}$. Hence the relations connecting the β_i' and the β_i are

$$(8.3) \qquad \sum_j \sigma_{ij}' \beta_i' = \sum_j \sigma_{ij} \beta_j + \beta_i c_i \ .$$

Since $\sigma_{ii} = \sigma_{ii}' - \varepsilon_i^2 - 2c_i$, these relations may be written

$$(8.4) \qquad \sum_j \sigma' ij(\beta_j' - \beta_j) = -\beta_j (c_i + \varepsilon_i^2) \ .$$

Assuming σ_{ij}' non-singular, let its inverse be $\sigma^{ij'}$. Then

$$(8.5) \qquad \beta_i' = \beta_i - \sum_j \sigma^{ij'} \beta_j (c_j + \varepsilon_j^2) \ .$$

Thus the effect of errors of measurement is that β_i' is a linear combination of β_i and of all the other β's. The only case in which $\beta_i' \equiv \beta_i$ occurs when the values of all the x_j have been set at predetermined levels, making $(c_j+\varepsilon_j^2) = 0$ for all j (the Berkson case).

With a single x-variate, $\sigma^{11'} = 1/(\sigma^2+\varepsilon^2+2c)$ so that

$$(8.6) \qquad \beta' = \beta(\sigma^2+c)/(\sigma^2+\varepsilon^2+2c) \ .$$

Since with one x-variate $R^2\sigma_y^2 = \beta^2\sigma^2$ and $R'^2\sigma_y^2 = \beta'^2(\sigma^2+\varepsilon^2+2c)$, this gives

$$(8.7) \qquad \frac{R'^2}{R^2} = \frac{(\sigma^2+c)^2}{\sigma^2(\sigma^2+\varepsilon^2+2c)} \ .$$

For c positive, this ratio increases steadily as suggested in section 7, reaching the value 1 if X and e have correlation 1, making $c = \sigma\varepsilon$. In the Berkson case, with $c = -\varepsilon^2$, (8.8) $$\frac{R'^2}{R^2} = 1 - \frac{\varepsilon^2}{\sigma^2} ,$$

this being a reminder that although β is unchanged, the residual variance is increased by the errors in x. In the Berkson case with $\underline{k}$ variates, it is easily shown that

(8.9) $$\frac{R'^2}{R^2} = 1 - \frac{\sum \beta_i^2 \varepsilon_i^2}{\sum\sum \beta_i \beta_j \sigma_{ij}} .$$

We now assume the e_i and X_i uncorrelated, and briefly consider the β_i'. From (8.5) it is evident that the effects on a specific β_i' are complicated, depending on the signs and sizes of the other β_j and on the terms in the inverse matrix. As an approximation to applications in which most correlations among the X's are positive and modest, the following is the expression for β_i' when $\rho_{ij} = \rho$, $g_i = g$:

(8.10) $$\beta_i' = \frac{g(1-\rho)\beta_i}{1-g\rho} + \frac{g(1-g)\rho(\sum \beta_i)}{(1-g\rho)[1+(k-1)g\rho]} .$$

The first term, which predominates when g is high, amounts to a reduction of β_i' to a value somewhat less than $g\beta_i$. The second term is a common contribution to all the β_i', and is positive if $(\sum\beta_i)$ is positive. In the examples that I have worked, the net effect is to make β_i'/β_i slightly greater than g for the larger β_i and substantially greater than g for the smaller β_i. The differences between the β_i' are smaller than these between the β_i so that it becomes more difficult to distinguish the important from the unimportant regression coefficients.

A further consequence of the general relations (8.4), (8.5) is that if only one x_i, say x_1, is subject to error,

(8.11) $$\beta_1' = \frac{\beta_1}{1+\varepsilon_1^2\sigma^{11}} : \beta_i' = \beta_i - \frac{\varepsilon_1^2\sigma^{i1}\beta_1}{(1+\varepsilon_1^2\sigma^{i1})} \quad (i > 1) ,$$

Since $\sigma^{11} < 1/\sigma_1^2$ unless X_1 is uncorrelated with any other X_i, it follows that $\beta_1' < g_1\beta_1$ in this case. Also, every other β_i that is correlated with β_1 is affected by errors in x_1.

In examples worked with unequal g's, the β_i' for those x_i having low g_i are very substantially reduced, while some β_j' with higher g_j may exceed β_j because of the contributions from ε_i^2 in (8.5). Consequently in this situation, in ignorance of the g_i, interpretations based on the relative sizes of the β_j' can become highly misleading.

In a more positive vein, equations (8.4) and (8.5) would also enable us to estimate the β_i from the β_i', if we had good estimates of the g_i and if the model could be assumed to apply.

SUMMARY

Multiple regression studies in which some or all of the variables are difficult to measure, and therefore presumably are measured with substantial errors, are increasingly common, particularly in the social sciences. This paper attempts to discuss the effects of such errors of measurement on the utility of multiple regression, both when the objective is prediction and when it is interpretation of the regression coefficients. Several different mathematical models are possible, since there may be correlations between the error of measurement of a variable and the true value of that variable and also between the errors for different variables.

A multivariate normal distribution of the correct Y's and the correct X_i is assumed. The errors of measurement are assumed normal with variances ε_i^2 and at first independent of the correct values and of each other. Formulas available in the simplest cases and worked numerical examples indicate that the formula $R'^2 = R^2 g_y \bar{g}_w$ approximates the relation between the squared multiple correlation coefficients R'^2 and R^2 in the presence and absence of errors. Here, $g_y = \sigma_Y^2/(\sigma_Y^2+\varepsilon^2)$ is the coefficient of reliability of y, while $\bar{g}_w = \sum\delta_i^2 g_i/\sum\delta_i^2$ is a weighted mean of the coefficients of reliability of the x_i. The formula $R'^2 = R^2 g_y \bar{g}_w$ slightly underestimates R'^2 when the correlations among the X_i are positive, and may slightly overestimate R'^2 when a minority of these correlations are negative, but appears correct to within $\pm$ 20% in most cases. The effects of correlation

between the e_i and e_j for different x's and between e_i and X_i are also indicated.

With errors of measurement in the x_i, any regression coefficient β_i' becomes in general a linear function of all the β's in the regression equation. Interpretation of the sizes of these β_i' may become highly misleading. The problem of estimating the g_i in practice is also discussed.

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SOME RECENT DEVELOPMENTS IN THE THEORY OF MARKOV CHAINS

by J.F.C. Kingman (England)
University of Sussex and Stanford University

For many years the analysis of stochastic processes involving some degree of Markovian behaviour has depended, implicitly or explicitly, on the discovery of "regeneration points" for the process. This concept was introduced by Palm, and a systematic account has been given by Smith [8]. Roughly speaking, a regeneration point for a process X_t is a random time τ such that the process $X_{\tau+t}$ $(t > 0)$ is independent of X_u $(u < \tau)$ and has the same stochastic structure as X_t $(t > 0)$. Thus at time τ the process is "regenerated", and its random evolution from τ follows the same laws as governed the process from $t = 0$.

If a regeneration point exists, it is not difficult to see that there is a whole sequence $\tau_1, \tau_2, \cdots$ of these points, forming a renewal sequence in the sense that the positive random variables $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots$ are independent and identically distributed. The process X_t can then be split up into independent segments X_t $(\tau_{n-1} \leq t < \tau_n)$ which can be examined separately. In particular, many problems can be reduced to questions about the renewal theory of the sequemce $\{\tau_n\}$, (cf. [8], [9]).

For some processes, however, regeneration points exist in much greater profusion than appears in the rather severe theory just mentioned, and to ignore all but a single sequence is to sacrifice a good deal of information. For example, in a queueing system fed by a Poisson arrival stream, every point of time at which the queue is empty is a regeneration point of the process, and the set of such instants is not a sequence, but a collection of intervals. More generally, if X_t is a Markov process with initial state $X_0 = x$, then any time with $X_t = x$ regenerates the process.

In recent years a theory has been developed which generalizes classical renewal theory by allowing the set of regeneration points to be more substantial than a discrete sequence. This theory has application to the theory of continuous-time Markov chains (as developed for example in [2]), and in particular to the very difficult problem of characterizing Markov transition probabilities. The details may be found in [4] and [5] (and in other references cited in the latter paper).

Let Z_t $(t > 0)$ be a stochastic process taking only the values 0 and 1. Suppose that, for any T for which the event $\{Z_T = 1\}$ has positive probability, the processes $Z_t (t < T)$ and Z_t $(t > T)$ are independent conditionally on $\{Z_T = 1\}$, and that moreover the conditional distributions of the latter process Z_{t+u} $(u > 0)$ are the same as the distributions of the original process Z_u $(u > 0)$. Then the process Z is said to define a *regenerative phenomenon*. The phenomenon is said to *occur* at time t if $Z_t = 1$. A typical example of a regenerative phenomenon is provided by a Markov process X_t with initial state x, by means of the formula

$$Z_t = f(X_t), \tag{1}$$

where

$$f(x) = 1,$$
$$f(\xi) = 0, \quad (\xi \neq x).$$

Denote the probability of occurrence at time t by

$$p(t) = P\{Z_t = 1\}. \tag{2}$$

The regenerative condition on Z implies that, whenever $0 < t_1 < t_2 < \ldots$ $\ldots < t_k$, then

$$P\{Z_{t_1} = Z_{t_2} = \ldots = Z_{t_k} = 1\} = p(t_1)p(t_2-t_1)\ldots p(t_k-t_{k-1}). \tag{3}$$

It follows that the function p, called the p-*function* of the phenomenon, determines the finite-dimensional distributions of the process Z. The first problem is therefore to determine which functions p can arise in this way. This accomplished, a second problem is to describe the behaviour of the process Z in terms of properties of the p-function.

The first fact to notice about p-functions is that they have to satisfy certain functional inequalities. For instance, the probabilities

$$P\{Z_s = 0, Z_{s+t} = 1\} = P\{Z_{s+t} = 1\} - P\{Z_s = 1, Z_{s+t} = 1\}$$
$$= p(s+t) - p(s)\, p(t)$$

and

$$P\{Z_s = 0,\ Z_{s+t} = 0\} = P\{Z_s = 0\} - P\{Z_s = 0,\ Z_{s+t} = 1\}$$

$$= 1 - p(s) - p(s+t) + p(s)\, p(t)$$

must be non-negative, so that p necessarily satisfies

$$p(s)\, p(t) \le p(s+t) \le 1 + p(s)\, p(t) - p(s). \tag{4}$$

More complicated inequalities may similarly be derived from the fact that

$$P\{Z_{t_1} = Z_{t_2} = \dots = Z_{t_k} = 0\} \ge 0$$

and

$$P\{Z_{t_1} = Z_{t_2} = \dots = Z_{t_{k-1}} = 0,\ Z_{t_k} = 1\} \ge 0$$

for every k. Conversely, the Daniell-Kolmogorov theorem can be used to prove that this infinite family of functional inequalities is sufficient, as well as necessary, for a function p to be a p-function. The study of p-functions is therefore the study of the consequence of these inequalities.

Of particular importance among p-functions are those which satisfy

$$\lim_{t \to 0} p(t) = 1; \tag{5}$$

these are called *standard*. Their significance has recently been stressed by the proof [7] that any measurable p-function is either

(i) of the form $ap(t)$, where $0 < a \le 1$ and p is a standard p-function, or

(ii) equal to zero almost everywhere.

It is an easy consequence of (4) that a standard p-function is (uniformly) continuous on $t \ge 0$, and the corresponding process Z is continuous in probability.

The fundamental theorem in the theory of regenerative phenomena is an integral representation formula for the Laplace transform

$$r(\theta) = \int_0^\infty p(t)\, e^{-\theta t}\, dt$$

of the standard p-function p. This can always be expressed in the form

$$r(\theta) = \{\theta + \int (1-e^{-\theta x})\, \mu(dx)\}^{-1}. \tag{6}$$

where μ is a positive measure on the interval $(0,\infty]$, uniquely determined by p. Conversely, if μ is any positive measure on this interval with

$$\int (1-e^{-x})\, \mu(dx) < \infty, \tag{7}$$

then there is a unique continuous function with Laplace transform given by (6), and this is a standard p-function. The formula (6) therefore sets up a one-to-one correspondence between the standard p-functions and the positive measures satisfying (7).

Because $(1-e^{-x})$ is small near $x = 0$, condition (7) does not necessarily imply that μ is a finite measure. But if it has finite total mass q, μ has a simple interpretation. In this case it can be written $\mu = q\pi$, where π is the probability measure of a positive, but possibly infinite random variable. Then it can be shown that Z_t is a step function, constant on intervals whose lengths are independent random variables. The lengths of intervals on which $Z_t = 1$ have the negative exponential distribution with density qe^{-qt}, while the lengths of intervals on which $Z_t = 0$ have distribution π. Following Bartlett [1], we say that μ is a multiple of the recurrence time distribution of the phenomenon.

When μ has infinite total mass, the structure of Z is much more complicated to describe. Although such phenomena are in a sense pathological, they do occur in models of practical importance, for instance in the theory of dams.

Using (6), a number of properties of standard p-functions can be established:

(a) p(t) is strictly positive,

(b) p(t) has finite right and left derivatives $D^+p(t)$ and $D^-p(t)$ at every positive value of t, and $D^+p(t) - D^-p(t)$ is equal to the atom (if any) of μ at t, (in particular p is continuously differentiable in

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$t > 0$ if and only if μ has no atoms in $(0,\infty)$).

(c) the derivative $D^+p(0)$ exists, but may be $-\infty$, and $-D^+p(0)$ is the total mass of μ,

(d) $p(t)$ tends to a limit $p(\infty)$ as $t \to \infty$, and

$$p(\infty) = \{1 + \int x\mu(dx)\}^{-1}. \tag{8}$$

The form of equation (6) may suggest to some readers a connection with the theory of processes with non-negative independent increments, and such a connection does indeed exist. Writing

$$\Phi(\theta) = \int (1 - e^{-\theta x})\, \mu(dx),$$

it is known that there exists such a process Y_t with

$$E(e^{-\theta Y_t}) = e^{-t\Phi(\theta)}. \tag{9}$$

In terms of Y, define a process Z taking values 0 and 1 by

$$Z_t = 1 \iff s + Y_s = t \quad \text{for some } s.$$

Then it has been shown by Kendall (in an as yet unpublished study of the sample functions of regenerative phenomena) that Z defines a regenerative phenomenon whose p-function satisfies

$$\int_0^\infty p(t)e^{-\theta t}\, dt = \{\theta + \Phi(\theta)\}^{-1}. \tag{10}$$

The most important example of a regenerative phenomenon arises as in (1), where X_t is a Markov chain in the sense of Chung [2], a Markov process taking values in a countable set S. For any $i \in S$, we may take $X_0 = i$, and the phenomenon then occurs at time t if and only if the chain is in its initial state i. If the transition probabilities are written

$$p_{ij}(t) = P\{X_t = j \mid X_0 = i\} \tag{11}$$

then the p-function of the phenomenon is just the diagonal transition probability $p_{ii}(t)$. It is usual to assume that

$$p_{ij}(t) \to \delta_{ij} \quad (t \to 0),$$

in which case $p_{ii}(t)$ is a standard p-function, to which all the general results about such functions may be at once applied.

This observation yields nearly all the known properties of the function $p_{ii}(t)$ (as given for example in [2]). But there is one surprising exception, for it is known that every function of the form $p_{ii}(t)$ is continuously differentiable in $t > 0$. In view of the remark made under (b) above, this is equivalent to the statement that the corresponding measure μ has no atoms (except perhaps at ∞). This can even be strengthened to show that μ has a density in $(0,\infty)$.

It therefore follows that not every standard p-function is of the form p_{ii} in some Markov chain. On the other hand, it is not difficult to show that every standard p-function is the limit of a sequence of such functions p_{ii}. It is therefore a delicate, and at present unsolved, problem to determine which standard p-functions can arise from Markov chains. All that can be presented here is an account of the few known partial results.

Because (6) sets up a one-to-one correspondence, the problem of characterising the functions p_{ii} is equivalent to that of characterising the corresponding measures μ. Call μ a Markov measure if the standard p-function corresponding to it in (6) can be expressed in the form p_{ii} for some Markov chain. Then the following facts are known [6]:

(A) the value of the atom at ∞ is irrelevant to deciding whether or not a measure is a Markov measure,

(B) any multiple of a Markov measure is a Markov measure,

(C) any finite or countable sum (subject to (7)) of Markov measures is a Markov measure,

(D) any Markov measure of infinite total mass admits a decomposition as a countable sum of totally finite Markov measures,

(E) the convolution of two totally finite Markov measures is a Markov measure,

(F) any Markov measure has a density on $(0,\infty)$ which is lower-semicontinuous and strictly positive (unless it is identically zero),

(G) any measure on $(0,\infty)$ satisfying (7) and having a density of the form

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_{mn} t^m e^{-nt}, \tag{12}$$

where $a_{mn} \geq 0$, is a Markov measure.

The difficulty of the characterization problem arises from the tension between the continuous time parameter and the essèntially discontinuous nature of the stochastic process. If more general Markov processes are considered, the difficulty disappears. Indeed, let Z be any standard regenerative phenomenon, and let

$$X_t = t - \sup\{u \leq t;\ Z_u = 1\} \tag{13}$$

be the time elapsed at t since the last occurrence of the phenomenon. Then X is a Markov process, and the phenomenon defined by (1) with $x = 0$ is a trivial modification of Z with the same p-function.

It should be remarked, however, that for this process the phenomenon defined by (1) with any non-zero value of x is not standard. If p arises from a Markov process in which *all* states determine standard phenomena, then [6, III] similar restrictions on p apply as in the Markov chain case.

The direct application of the theory of p-functions to Markov chains involves the functions $p_{ij}(t)$ only for $i = j$, but the theory can be modified to deal also with the non-diagonal case $i \neq j$. The appropriate concept is that of a *quasi-Markov chain*, which is a stochastic process taking values 0, 1, 2, ..., N such that each of the states 1, 2, ..., N, though not the anomalous state 0, has the Markov property of regenerating the process. An example of such a process can be obtained from a Markov process X by taking distinct states $x_1, x_2, \ldots, x_N$ and setting

$$Z_t = f(X_t) \tag{14}$$

where

$$f(x_r) = r \qquad (r = 1, 2, \ldots, N)$$

$$f(\xi) = 0 \qquad (\xi \notin \{x_1, x_2, \ldots, x_N\}).$$

The analogue of the p-function is a matrix-valued function, whose Laplace transform has a representation similar to (6). In particular, in the countable case, a typical p-matrix with N = 2 is

$$\begin{pmatrix} p_{ii}(t) & p_{ij}(t) \\ p_{ji}(t) & p_{jj}(t) \end{pmatrix} . \tag{15}$$

The detailed consequences of the theory of quasi Markov chains for the characterization problem will be found in [6]; it suffices here to quote the main result. A function f(t) can be expressed in the form $p_{ij}(t)$, where i and j are distinct states of some Markov chain, if and only if f is expressible as a convolution

$$f = p_1 * d\mu * p_2, \tag{16}$$

or more explicitly

$$f(t) = \int_0^t \int_0^{t-u} p_1(t-u-v)\ \mu(du)\ p_2(v)\ dv,$$

where

(i) p_1 and p_2 are diagonal Markov transition functions,

(ii) μ is a totally finite measure on $[0,\infty)$ which, apart from a possible atom at 0, is a Markov measure, and

(iii) $\mu[0,\infty) \int_0^\infty p_1(t)\ dt \leq 1.$

It follows therefore that, if the class of Markov measures can once be determined, the characterization problem for the transition functions of Markov chains, both diagonal and non-diagonal, will have been solved, and the known theorems about the functions p_{ij} will fall into their natural perspective. This will not, however, dispose of all the outstanding problems even in the analytical part of Markov chain theory (leaving aside, that is, problems about sample function behavior). For example, Kendall and Speakman [3] have studied the function

$$g(t) = \inf_i p_{ii}(t), \tag{17}$$

and a systematic theory of such g-functions is urgently needed. Again, recent work of Williams (as yet unpublished) has added new interest to the problem of giving necessary and sufficient conditions for a set of numbers q_{ij} $(i, j \in S)$ to be expressible as the derivatives at the origin

$$q_{ij} = p'_{ij}(0) \tag{18}$$

of the transition functions of some Markov chain. Indeed, it would seem that the deep problems of the analytical theory of Markov chains are to characterize the various functions and matrices arising, of which the problem described in this brief survey is the simplest, if not the least demanding.

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ON SEQUENTIAL SEARCH

by B. Eichhorn (Israel)
Dept. of Statistics, Tel-Aviv University

THE PROBLEM IN GENERAL.

We face an unknown function f, belonging to a family of real functions $\mathcal{F}$, say on the interval $I= [0,1]$, with some given properties. For instance, the family of all monotone nonincreasing functions with exactly one zero on I.

We want to estimate a point (or points) of I where f assumes values of particular interest (for instance, the value zero or its maximum) by using some specified estimate for which a loss function giving "the loss due to estimation" L (estimate, f) ≥ 0 is defined. To help us make this estimate we are allowed to observe the values of the function f at points of the domain I which we can choose sequentially.

The problem varies according to the specific assumptions about the class $\mathcal{F}$ and the way f is obtained from it, by the kind of estimate to be used (point or interval), the requirements imposed on it and the loss function attached, by the kind of estimating procedure allowed and kind of optimization sought, e.g. a minimax procedure.

We distinguish between two kinds of sequential estimating procedures:

(a) The fixed sample size sequential procedure, or n-observation sequential procedure T_n, where the fixed number of observations is n. The class of all T_n admissible with respect to the particular problem that is discussed will be denoted by $\mathcal{T}_n$. We shall call it briefly an n-seq. procedure. Here we are allowed exactly n observations whose places we can choose sequentially, making the place of the ith observation a function of the places and values of the first i-1 observations.

(b) The "true" sequential procedure $T \varepsilon \mathcal{T}$, the class of all such admissible procedures, which we shall call just sequential procedures.

Here we do not fix ahead of time the number of observations. Instead we use a stopping rule $\delta = \{\delta_1, \delta_2, \dots\}$ which at each stage i tells us wether to stop or to continue and take the (i+1)st observation, or in a somewhat more general way tells us to stop with a certain probability δ_i, each δ_i being a function of the first i observations.

In this case, where the number of observations is not fixed, we assume a cost of observation in addition to the loss due to estimation. Usually we shall assume a constant cost $c > 0$ per observation. In this case we are concerned with the "total" cost which is defined as the sum of the cost of observation and the loss due to estimation

$$R(T,f) = L(T,f) + cn(T,f) .$$

R,L and n are functions of the procedure T and the function f. If T and f do not determine n and L completely but determine their distributions we define $R(T,f) = E(L(T,f)) + cE(n(T,f))$. We call a procedure nonrandomized if the places of observation are completely determined by the procedure T and the function f, each observation being a function of the information obtained so far. This is in contrast to randomized procedures where our next observation could be chosen randomly according to some probability distribution which is determined by the previous observations.

Let us state all this precisely.

DEFINITION, A nonrandomized n-seq. estimating procedure T_n is given by

(1) $$T_n = [x_1, g_2, \dots, g_n, \ell]$$

where $x_1 \in I$ is the first place of observation, the other places of observation x_k are given by

$$g_k : I^{k-2} \times R^{k-1} \longrightarrow I, \qquad 2 \leqq k \leqq n,$$

which are functions of the former x_i's and $f(x_i)$'s; ℓ is the estimate, which is a function of $x_2, \dots, x_n$ and $f(x_1), \dots, f(x_n)$, and its range depends on the particular problem and the kind of estimate we use. For instance if we use an interval estimate, ℓ will take values $[s,t]$ with $s,t \in I$ and $s \leqq t$.

DEFINITION. A nonrandomized sequential procedure T is given by

(2) $$T = \{\delta_0, \ell_0, x_1, \delta_1, \ell_1, g_2, \delta_2, \ell_2, \ldots\}$$

where x_1 and g_k, $(k \geq 2)$ are as in (1). δ_k, $k = 0,1,\ldots$ is the probability of stopping with k observations given the values of the first k observations. So having reached stage k we stop with probability

$$P_k = \frac{\delta_k}{1 - \sum_{i=0}^{k-1} \delta_i},$$

$\delta_k : I^k \times R^k \longrightarrow [0,1]$ (usually δ_k will be 0 or 1).

For any $T \in \mathcal{T}$ we require that for each $f \in \mathcal{F}$ together leading to the sequence x_1, $f(x_1)$, x_2, $f(x_2), \ldots$ we shall have

(3) $$\sum_{k=0}^{\infty} \delta_k \text{ (T, sequence)} = 1$$

but each δ_k depends only on the first 2k elements of the sequence. As each sequence is completely determined by T and f (T being nonrandomized) we can also write $\delta_k(T,f)$ and (3) becomes

(3') $$\sum_{k=0}^{\infty} \delta_k(T,f) = 1,$$

for all $T \in \mathcal{T}$ and for all $f \in \mathcal{F}$. Condition (3) assures us also that T will stop with probability 1. Finally the ℓ_i's are the estimates we would make if we stop after i observations; they are also functions of $x_1,\ldots,x_i$ and $f(x_1),\ldots,f(x_i)$.

In a former paper [5] we found the following result concerning minimax procedures. The theorem is stated in slightly more general terminology than of a search problem.

THEOREM 1. Let $\mathcal{F} \equiv \{f\}$ be a set of "states of nature," X be a set of possible places of observation on f, and $d_i \equiv D_i(x_1,\ldots,x_i,f(x_1),\ldots,f(x_i))$, $i = 1,2,\ldots$ be a set of admissible decisions, given the first i observations. Let there be a bounded loss function L(d,f), giving the loss for taking decision d for state f,

(4) $$0 \leq L(d,f) \leq 1.$$

The n-seq. nonrandomized procedure T_n and the "true" sequential decision procedure T are defined in analogy with the estimating procedures (1) and (2) respectively, as d_n replaces ℓ and d_i replaces ℓ_i; a constant cost of observation $c > 0$ is assumed in the latter case. *If* for every integer $n \geqq 0$ (and $\varepsilon > 0$) there exists a procedure $T_n^* \in \mathcal{T}_n$ and a number L_n^* such that

$$(5) \qquad \sup_{f \in \mathcal{F}} L(d_n(T_n^*, f), f) + (-\varepsilon) \leqq L_n^* \leqq \sup_{f \in \mathcal{F}} L(d_n(T_n, f), f)$$

for all $T_n \in \mathcal{T}_n$ and also

(A) the sequence $L_n^* - L_{n-1}^* \downarrow 0$ is strictly decreasing, and

(B) for each "true" sequential procedure T and any given integer $k > 0$, there exists $f^* = f^*(k,T)$ such that

$$(6) \qquad L(d_i(T, f^*), f^*) \geqq L_i^* , \qquad \text{for } i = 0,1,2,\dots,k,$$

then if we define n_o such that

$$(7) \qquad L_{n_o}^* - L_{n_o-1}^* > c \geqq L_{n_o+1}^* - L_{n_o}^*$$

or $n_o = 0$ if (7) does not hold for any n, $T_{n_o}^*$ is (ε) minimax among all nonrandomized sequential decision procedures $T \in \mathcal{T}$. That means, if $R(T,f) = E(L(T,f)) + cE(n(T,f))$,

$$(8) \qquad \sup_{f \in \mathcal{F}} R(T_{n_o}^*, f) = \sup_{f \in \mathcal{F}} EL(d_{n_o}(T_{n_o}^*, f), f) + cEn_o$$

$$= \sup_{f \in \mathcal{F}} L(d_{n_o}(T_{n_o}^*, f), f) + n_o c$$

$$\leq \sup_{f \in \mathcal{F}} \sum_{i=0}^{\infty} \left[L(d_i(T,f),f) + ic\right] \delta_i(T,f) + (\varepsilon)$$

for all $T \in \mathcal{T}$.

This was applied to the following slach problem.

Let $\mathcal{F}$ be the class of all unimodal functions on I, that means, for each $f \in \mathcal{F}$ there exists $x^{(f)} \in I$ such that

(9) f is strictly increasing for $x \leqq x^{(f)}$ and strictly decreasing for $x > x^{(f)}$, or strictly increasing for $x < x^{(f)}$ and strictly decreasing for x $x^{(f)}$.

We want to estimate $x^{(f)}$ by means of an interval estimate $[s,t]$ which has to contain the true $x^{(f)}$. Our loss due to estimation will be the length of this interval $L([s,t]) = t - s$. J.Kiefer [1] found an ε-minimax solution for the n-seq. case. His procedure, called the Fibonacci method, which we shall denote $T_n^*(\varepsilon)$, is ε-minimax, namely, for any given $\varepsilon > 0$

$$(10) \qquad \sup_{f\in\mathcal{F}} L(T_n^*(\varepsilon),f) - \varepsilon \le L_n^* \le \sup_{f\in\mathcal{F}} L(T_n,f) \text{ , for all } T_n \in \mathcal{T}_n,$$

where $L(T_n,f)$ is the loss due to estimation resulting from using procedure T_n on the function f. L_n^* is known to be $1/U_{n+1}$, U_n being the nth Fibonacci number defined as follows:

$$U_o = 0,\ U_1 = 1,\ U_n = U_{n-2} + U_{n-1} \quad \text{for } n \ge 2.$$

By showing that conditions A and B of Th. 1 hold for this problem we showed that the ε-minimax sequential procedure here is of a fixed size.

Let us consider the simpler search problem of finding a root (zero) of a monotone function.

PROBLEM 2.

Let $\mathcal{F}$ be the set of all monotone nonincreasing functions on I with *one* zero at $x^{(f)} \in I$. We want to estimate $x^{(f)}$ again using interval estimates that have to include the true $x^{(f)}$ and where the loss due to estimation is the length of this interval.

(a) For each fixed n we have a minimax procedure T_n^* which is the Bolzano method of taking the next observation at the middle of the "interval of uncertainty". Thus T_n^* takes

$$x_o = \tfrac{1}{2}$$
$$x_1 = \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{sgn}\, f(x_o)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$x_k = x_{k-1} + (\tfrac{1}{2})^k\,\mathrm{sgn}\, f(x_{k-1})$$

(D.J. Wilde [4]). $L_n^* = 1/2^n$ which is the length of the last interval of uncertainty.

Conditions (A) and (B) of Theorem 1 are easily seen to hold here, and using the theorem we have for part (b) of this problem we get:

COROLLARY. For problem 2, the Bolzano procedure $T^*_{n_o}$ with n_o such that

$$\frac{1}{2^{n_o+1}} \leq c < \frac{1}{2^{n_o}}$$

is minimax among all "true" sequential procedures $T \in \mathcal{T}$.

PROBLEM 3. A PRIORI DISTRIBUTIONS ON $x^{(f)}$ AND OPTIMAL PROCEDURES.

Let $\mathcal{F}$ be the class of all monotone nonincreasing functions on I with one zero $x^{(f)} \in I$. This time we assume that f is picked in such a way that there is a continuous *a priori* distribution G of $x^{(f)}$ on I. Again there is a constant cost $c > 0$ per observation and for each kind of estimate and loss function we want to find an *optimal* procedure, that is, the one which minimizes the *expected* cost.

For the fixed size procedure we aim to minimize $E(L(T_n,f))$, for the sequential procedure to minimize $E(R(T,f))$. We consider a few particular cases.

(a) We use a point estimate $\hat{x}$ and assume loss due to estimation

$$L(\hat{x},x^{(f)}) = |\hat{x} - x^{(f)}| . \tag{11}$$

If we are given the a priori G with density g on I and have to estimate $x^{(f)}$ without taking any observations, the best estimate, namely the one which minimizes E(L), is M *the median of* G.

From here we would like to proceed to the n-seq. procedure. It may seem that if we take an observation the best place to look for information is where we were going to make our estimate, namely at M. However, this is not true in general as we can see from an example. Consider the a priori distribution G with density $g(x) = 2x$. Let us take one observation and then estimate $x^{(f)}$. The median of G is $M = \frac{1}{\sqrt{2}}$ and the expected loss due to estimation is

$$E(L(M)) \approx 0.096$$

It is easy to calculate the optimal place of observation $X_1 \approx 0.645$ with expected loss $E(L(X_1)) \approx 0.02$.

For this G and other "nice" a-priori distributions one could, in principle, find n-seq. optimal solutions by working backwards from stage n to 0. A particularly nice G is the uniform one.

(b) Assume a uniform apriori distribution G for the problem in (a).

THEOREM 2. The n-seq. procedure T_n^* defined by

(12) $$T_n^* = \{ \frac{1}{2}, x_1 + \operatorname{sgn} f(x)\frac{1}{2^2}, \ldots, x_{n-1} + \operatorname{sgn} f(x_{n-1}) \frac{1}{2^n},$$
$$\frac{1}{2}[\max(0, x_i \mid f(x_i) \geq 0) + \min(1, x_i \mid f(x_i) \leq 0)]\}$$

is optimal in $\mathcal{T}_n$, namely

(13) $$E(L(T_n^*)) \leq E(L(T_n)) \qquad \text{for all } T_n \in \mathcal{T}_n$$

PROOF. First we notice that after taking the n observations the best estimate is in the middle of the interval of uncertainty V_n (we shall use V_n ambiguously to denote also its length),

$$V_n = [\max(o, x_i \mid f(x_i) \geq 0), \min(1, x_i \mid f(x_i) \leq 0)]$$

giving expected loss

$$E(L \mid V_n) = \frac{1}{4} V_n \quad .$$

This is clear since the aposteriori distribution is uniform on V_n. Consequently

(14) $$E(L(T_n)) \geq \frac{1}{4} E(V_n(T_n)) \quad ,$$

and we need only prove that $E(V_n(T_n^*)) \leq E(V_n(T_n))$ for all $T_n \in \mathcal{T}_n$.

From here on we consider interval estimates which are the intervals of uncertainty. This will only change the scale of the loss function, nultiplying it by 4, and all results will hold as well for the point estimate.

Let T_n be any procedure in $\mathcal{T}_n$ and V_k be the interval of uncertainty after k observations. If T_n takes the next observation at x_{k+1} dividing V_k into two possible V_{k+1}'s of size $y_{k+1}V_k$ and $(1-y_{k+1})V_k$, $0 \leq y_k \leq 1$. At this stage we have a uniform apriori on V_k so we obtain

$$E(V_{k+1}(T_n)|V_k) = [\, y_{k+1}^2 + (1 - y_{k+1})^2]V_k \geq \frac{1}{2} V_k \quad .$$

This inequality holds for all y_{k+1} and any V_k . We can conclude that

$$E(V_{k+1}(T_n)) \geq \frac{1}{2} E(V_k(T_n))$$

and therefore

(15) $$E(V_n(T_n)) \geq \frac{1}{2^n}$$

for all T_n .

Since $E(V_n(T_n^*)) = 1/2^n$ we have established (13) and proved T_n^* to be optimal.

(c) We now introduce a cost $c > 0$ per observation, and try to find an optimal "true" sequential procedure, using the interval of uncertainty as estimate. We want to minimize $E(R) = E(V_n) + cE(n)$.

Let us consider the procedure $T^*(c) = T_{n_o}^*$ for $n_o = n_o(c)$ satisfying

(16) $$\frac{1}{2^{n_o + 1}} < c \leq \frac{1}{2^{n_o}} \quad .$$

This procedure takes observations in the middle of the interval of uncertainty as long as one further observation will reduce the interval of uncertainty by at least c.

This however is not an optimal procedure. The following example will show this.

Let $c = 31/120$, then $T^*(c) = T_1^*$, and we get $R = 1/2 + 31/120 = 91/120$. Now consider instead the procedure T' which takes its first observation at $x_1 < 1/2$. If $f(x) \leq 0$ it stops with $V_1 = [0,x_1]$, if $f(x_1) > 0$ it takes a second observation at $x_2 = x_1 + 1/2(1 - x_1)$ and stops with

$V_2 = 1/2(1 - x_1)$, then

$$
\begin{aligned}
E(R(T')) &= x_1(x_1 + c) + (1-x_1)[\tfrac{1}{2}(1-x_1) + 2c] \\
&= \tfrac{3}{2}x_1^2 - (1+c)x_1 + \tfrac{1}{2} + 2c , \\
\frac{dE(R(T'))}{dx_1} &= 3x_1 - (1+c) .
\end{aligned}
$$

This is minimized for $x_1 = 1/3 + 1/3\, c = 151/360$. Therefore, let T' take $x_1 = 151/360$. We obtain $E(R(T')) = 65039/86400 < 91/120$. It follows that T* is not optimal. This is even true when $c = 1/4$, $E(R(T^*)) = 3/4$, while taking $x_1 = 5/12$ in T' leads to

$$E(R(T')) = \frac{71}{96} < \frac{72}{96} = \frac{3}{4} .$$

We can see, therefore, that for a small enough $\varepsilon > 0$ we may put $c = 1.4-\varepsilon$ and have T' take fewer observations than T* and yet have a smaller expected total cost.

These examples show that T* is not optimal, but we shall show that T* is still valuable. While the optimal procedure is hard to calculate in each case, T* is very simple and moreover:

THEOREM 3. T* is c-optimal. That means the expected cost from T* is less than c over the optimal expected cost. The proof of this will follow from the following theorem by D.Blackwell.

We introduce in our problem the following notation:

n = expected sample size,

λ = expected length of final interval.

We are interested in the question: Which pairs (n,λ) are attainable by sequential procedures?

THEOREM * If (n,λ) is attainable, $\lambda \geq 2^{-n}$.

PROOF. First we shall notice that if we prove the theorem for procedures that at each stage decide whether to stop or to take another observation without randomization, then it will follow also for procedures that allow randomization. This is a result of the convexity of the function 2^{-x}. So if we decide to stop with probability $0 \le s \le 1$ and state (n_1, λ_1) and go on with probability $1 - s$ to state (n_2, λ_2),

$$\lambda_i \ge 2^{-n_i} .$$

We are going to attain $(sn_1 + (1 - s)n_2, s\lambda_1 + (1 - s)\lambda_2)$ and from the convexity it follows that

$$s\lambda_1 + (1-s)\lambda_2 \ge s2^{-n_1} + (1-s)2^{-n_2} \ge 2^{-sn_1-(1-s)n_2} .$$

So we have to prove the theorem only for the first kind of procedures.

We look at procedures truncated at k steps and use a proof by induction on k. Among procedures truncated at 0, the only point is (0,1) and the result holds. Suppose the result is true for procedures truncated at k. A procedure truncated at k+1 can either take no observations in which case we get (0,1) or it definitely takes one observation. This means it specifies an initial x, $0 < x < 1$ and two procedures π_1 and π_2 truncated at k:

Use π_1 when $[0,x]$ occurs, π_2 when $[x,1]$ occurs. The resulting (n,λ) is then

$$x(1+n_1,\ x\lambda_1) + (1-x)(1+n_2,\ (1-x)\lambda_2)$$

where π_1 yields (n_1,λ_1), π_2 yields (n_2, λ_2), therefore, π_1 applied on an interval $[0,x]$ yields $(n_1, x\lambda_1)$, etc..

So we must show that if $\lambda_1 \ge f(n_1)$ and $\lambda_2 \ge f(n_2)$, then

$$x^2\lambda_1 + (1-x)^2\lambda_2 \ge f(1 + xn_1 + (1-x)n_2)$$

where $f(t) = 2^{-t}$. For this it is enough to show that

$$[x^2f(n_1) + (1-x)^2f(n_2)] \ge f[xn_1 + (1-x)n_2]/2$$

for $n_1 \ge 0$, $n_2 \ge 0$, $0 \le x \le 1$.

Say $n_1 \ge n_2$ and write $n_1 - n_2 = t$. Multiply both sides by $f(-n_2)$:

$$2[x^2 f(t) + (1-x)^2] \geq f(xt),$$

put

$$f(t) = a: \qquad 2[x^2 a + (1-x)^2] \geq a^x \ , \quad 2^{-k} \leq a \leq 1, \ 0 \leq x \leq 1 \ .$$

Fix x and maximize $a^x - 2x^2 a = \phi(a)$ over a. Let us maximize it over a larger range: $0 < a$, which may, if anything, increase the maximum which we want to show to be less than or equal to $2(1-x)^2$.
$\phi'(a) = xa^{x-1} - 2x^2 \downarrow \phi' = 0 : a^{x-1} = 2x: a = (2x)^{1/x-1}$. The maximum of $\phi(a)$ occurs at $a = (2x)^{1/x-1}$ and is $(1-x)^{x/x-1} - 2x^2(2x)^{1/x-1} =$ $(1-x)(2x)^{x/x-1}$. Is for $0 \leq x \leq 1$,

$$(1-x)(2x)^{x/x-1} \leq 2(1-x)^2?$$

i.e., is $\quad (2x)^{x/x-1}(1-x)^{-1} \leq 2$?

i.e., is $\quad (2x)^x(1-x)^{1-x} \geq 2^{x-1}$?

i.e., is $\quad x^x(1-x)^{1-x} \geq \frac{1}{2}$?

Yes. Put $\Psi(x) = x^x(1-x)^{1-x}$; then

$$\alpha = \log \Psi = x\log x + (1-x)\log(1-x)$$

is convex and symmetric about 1/2, assuming its minimum at x = 1/2. Since $\Psi(1/2) = 1/2$ this completes the proof of Theorem *.

Now we come back to prove Theorem 3.

PROOF. Using T* we get an expected cost

$$E(R(T^*,c)) = nc + 2^{-n}$$

as $2^{-n-1} < c \leq 2^{-n}$.

For any other T we get an expected number of observations n(T) = x and $E(L(T)) = \lambda \geq 2^{-x}$ due to Theorem *. It follows that

$$E(R(T,c)) \geq \min_{0 \leq x < \infty} (2^{-x} + xc) \ .$$

Put $\rho(x) = 2^{-x} + xc$, then $E(R\ ^{n}*,c)) = \rho(n)$,

$$\rho'(x) = -2^{-x}\ln 2 + c \quad .$$

So the minimum occurs at $c = 2^{-x} \ln 2 \simeq 2^{-x} \cdot 0.6931$. It follows that

$$2^{n-1} < 0.6931 \times 2^{-x} = c \leq 2^{-n},$$

therefore,

$$2^{-n-1} < 2^{-x} < 2^{-n+1} \quad .$$

So $\rho(x) \geq (n-1)c + 2^{-n-1} \geq \rho(n) - 2c$. We can do better by considering separate cases: If $x = n$,

$$\rho(x) = \rho(n).$$

If $x > n$, $\quad \rho(x) \geq nc + 2^{-n-1} \geq \rho(n) - c.$

If $x < n$, $\quad \rho(x) \geq (n-1)c + 2^{-n} = \rho(n) - c.$

Therefore, T* is c-optimal.

Finally let us note that the last two results, T^*_n optimal in $\mathcal{T}_n$ and T* c-optimal, remain true when we allow also randomized procedures.

PROOF. Let V_i be the interval of uncertainty after i steps and let us take the next observation at y_{i+1} in such a way that V_i is divided into two intervals of lengths xV_i and $(1-x)V_i$, where y_{i+1} is a random variable and therefore x is a random variable on $[0,1]$. Then

$$E(V_{i+1} | V_i, x) = E(x^2 + (1-x)^2)V_i \geq \{[E(x)]^2 + [1-E(x)]^2\}V_i .$$

Therefore, we could have taken $x = E(x)$ constant and done at least as well.

(d) Let us consider again a point estimate $\hat{X}$ for $X^{(f)}$ but with a square error loss function

$$L(X^{(f)}, \hat{X}) = (X^{(f)} - \hat{X})^2 \ .$$

For this case we obtained results exactly analogous with the results for the loss function $L = |X^{(f)} - \hat{X}|$ including theorems 2 and 3.

RANDOMIZED PROCEDURES.

Let us return to Problem 2. In view of the results obtained for uniform *a priori* distributions we may try to see what we can say about randomizations and minimax procedures.

THEOREM 4. For Problem 2(a) T^*_n is minimax among all randomized n-seq. procedures.

PROOF. We know that T^*_n is the optimal procedure (Bayes solution) among all randomized procedures when a uniform a priori distribution is assumed. T^*_n has constant risk $R = 1/2^n$ for all $x^{(f)} \in I$, therefore following a known theorem and a lemma found in Lehmann [6] the uniform distribution is "least favorable" and T^*_n is minimax among all *randomized* procedures this time.

Let us go through the proof as a preparation for the next theorem.

PROOF. T^*_n is constant over $x^{(f)} \in I$, therefore

$$\sup_{x \in I} R(T^*_n, x) = \int_I R(T^*_n, x)dx \leq \int_I R(T_n, x)dx \quad \text{by optimality of } T^*_n$$

$$\leq \sup_{x \in I} R(T_n, x)$$

for all $T_n \in \mathcal{T}_n$, where $\mathcal{T}_n$ is the class of all *randomized* n-seq. procedures. Q.E.D.

Now let us look at the problem 2(b). We know that T*(c) is c-optimal among all randomized procedures, given a uniform a priori distribution. T* has also a constant cost.

THEOREM 5. T*(c) is c-minimax among all randomized procedures.

PROOF. $$\sup_{x \in I} R(T^*, x) = \int_I R(T^*, x)dx \leq \int_I R(T, x)dx - c \leq \sup_{x \in I} R(T, x) - c$$

for all $T \in \mathcal{T}$. Q.E.D.

EXTENSIONS.

There are many possible extensions of the search problem some of which were explored by Kiefer [2] and Wilde [4] and several others. We did some further work considering restricting the class $\mathcal{F}$ by putting a bound on the slope of f. This produced several new concepts and interesting examples.

J. Kiefer defines in [2] the *order* of the search problem as the minimum number of observations needed in order to be sure to obtain some information on the location of $x^{(f)}$.

Here we considered only first and second order problems. In [2] Kiefer solves the minimax n-seq. problem for the third order search, the search for an inflection point. It would be interesting to see the effect of randomization on the second and third order procedures, as well as the effect of various a priori distributions on these problems. It seems likely also that one would be able to find "almost optimal procedures" for nonuniform a priori distributions at least for the first order problem.

Another important extension would be the consideration of different loss functions; for instance, for the first order search problem with an a priori distribution given, define the loss function as the left end of the interval of uncertainty. Professor B. McGuire has some unpublished results on this problem which arose from a simplified practical problem.

The most important extension, in my opinion, is the extension to higher dimensions. This may also be the hardest extension. Kiefer reports on the difficulties involved already in the two dimensional search in [2]. We have considered the search for two zeros on an interval I, or equivalently, the search for an indicator function of a subinterval of I. This problem may be reduced to a search for a point of the subinterval. Under the assumption that we know a positive lower bound to the length of the sought interval, we could find a minimax search procedure, yet this procedure was not satisfactory, being inadmissible, actually dominated by many other procedures out of which we could not find a best procedure.

Another extension would be considering different families of functions. This may not bring too many new results. Kiefer also reports on this in [2] and deals in particular with the problem of search on a lattice, where the domain of f is just a finite number of points. This is done for both one and two dimensions.

A last extension that is also mentioned by Kiefer is considering problems in which errors are involved in the observation. A paper on this subject was written by Kiefer and Wolfowitz [7] .

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