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# REPRESENTATIONS OF THE LORENTZ GROUP AND PROJECTIVE GEOMETRY

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## SUMMARY

In the beginning of this century, with the origination of the theory of relativity, extensive geometrical investigations of the 4-dimensional space $R_4$ (space-time) with invariant pseudo-norm $x_0^2-x_1^2-x_2^2-x_3^2$ were done by Klein, Minkowski and others.
Later on this geometrical research developed in two directions, a differential-geometrical approach (related to the general theory of relativity) and an algebraical-geometrical one (related to quantum mechanics).
In this tract we follow the latter direction. After 1925 more abstract techniques, such as representation theory and spinor calculus were introduced in this research by Cartan, Weyl, Veblen, Schouten and others. Especially the last decade with the use of group theoretical methods in physics there is a growing interest in these algebraical techniques in the geometrical study of $R_4$ and other spaces in which these groups act. More recently also interesting topological investigations of $R_4$ have been carried out, e.g. by Zeeman.

The subject of this tract may be seen as being situated in this border land between geometry on the one side and representation theory and physics on the other side.
In *chapter I* we give an introduction to the representation theory of the Lorentz group. Besides the standard theory of the finite-dimensional representations which are used in relativistic quantum mechanics, we give in section 7 of this chapter a brief sketch of the theory of infinite-dimensional representations of the Lorentz group of Gel'fand and Neumark. In *chapter II* we start with the geometry of the Lorentz group. Especially the fact that the Lorentz group may be studied as a three-dimensional transformation group gives much insight into the geometrical structure of the Lorentz group. In this way, for instance, a brief description of the so-called spin-space of Veblen can be given.

In *chapter III*, which is the principal part of this tract, we relate the foregoing geometrical investigations with physics. In fact, it is an investigation of the projective-geometrical background of some equations which are well-known in physics as the *Proca, Maxwell, Weyl* and *generalized Weyl equation*. These equations are all linear first order equations in an n-component function $\psi(x)$, i.e.

$$L(\partial_\mu, \psi(x)) = 0, \text{ where } x \in R_4 \text{ and } \partial_\mu = \frac{\partial}{\partial x^\mu} .$$

The idea is that by developing $\psi(x)$ in plane waves, i.e. $\psi(x) = \psi(p)e^{ip.x}$, one obtains equations $L(ip^\mu, \psi(p)) = 0$, in the so-called *momentum space* which may be studied with the aid of projective geometry. We pay particular attention to the zero-mass equations. There we can make the following observation: the fact that photons are only transversaly polarized and the fact that there exist only right-hand neutrinos and left-hand anti-neutrinos is closely related to the fact that the complex light cone is covered by two systems of isotropic planes. Also, the fact that the Maxwell equations can be brought into neutrino form will be given a clear geometrical meaning.

*This chapter is already published as "zero mass equations and projective geometry" (thesis) by the author.*

In *chapter IV* which is again pure geometrical, the 3-dimensional transformation group corresponding to the Lorentz group is studied in more detail. It follows that every Lorentz transformation may be described as a screw in 3-dimensional hyperbolic geometry. For a compact description of this geometry, the method of Cartan is indispensable (this method is a description of orthogonal transformations in $R_n$ by a generalisation of the quaternion concept). In section 2 of chapter IV, this method is also applied to the 6-dimensional space of anti-symmetric tensors $p^{\mu\nu}$, and especially, the group which leaves the so-called *configuration of Kummer* invariant is studied.

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# REPRESENTATIONS OF THE LORENTZ GROUP AND PROJECTIVE GEOMETRY

## CONTENTS

# Chapter I

## REPRESENTATIONS OF THE LORENTZ GROUP

## A. THE ALGEBRAIC METHOD

### 1. Introduction

The actual study of the Lorentz group found its origin in the theory of relativity. In this theory one of the principal postulates is the *invariance of the velocity of light*, that is to say: the measured velocity c of light is independent of the motion of the observer and the time-space coordinate system (t, x, y, z) connected with him.
If the progress of an electromagnetic spherical wave is described by the formula

$$c^2t^2 = x^2 + y^2 + z^2,$$

then the principal postulate requires that we study only coordinate transformations into systems in which this formula preserves its form. Such a transformation is called a *Lorentz transformation*. In the usual four-dimensional notation one describes the components of a vector x by $x^\mu$ ($\mu$=0,1,2,3) i.e. $(ct, x, y, z) \equiv (x^0, x^1, x^2, x^3)$.
With the development of quantum mechanics around 1930 one became interested in the study of vector-functions $\psi(x) \equiv (\psi^1(x), \psi^2(x), \ldots \psi^n(x))$ defined on the four-dimensional vector space $R_4$, where the *function values* belong to an $R_n$.
A Lorentz transformation, x' = Lx, induces a linear transformation D(L) in this n-dimensional space of *function values* and by this a so-called n-dimensional representation of the Lorentz group.
The classification of all n-dimensional irreducible representations of the Lorentz group was given by Cartan and Weyl using spinor calculus, and was applied to quantum mechanics by van der Waerden.
In connection with the special interests of physicists in time and space reflections a systematic treatment of the representations of the Lorentz group including reflections was given (among others Watanabe 1951).

A closely related subject is the theory of linear partial differential equations, i.e. $L(\psi(x)) = 0$, which are invariant under Lorentz transformations. The equations of Maxwell and Klein-Gordon were already known, but the theory found its real start with the equations of Dirac in 1928. Since then the theory of Lorentz invariant equations has been uniformized by Pauli, Fierz and especially by Bhabha (1945-1949).
Dirac (1945) noted the existence of *infinite*-dimensional irreducible representations of the Lorentz group. In 1946-1954 Gel'fand and Neumark were able to give a classification of all infinite-dimensional representations of the Lorentz group.
They showed that in particular every infinite-dimensional representation may be realized in a suitable space of complex functions f(z). These results were applied to the study of the infinite-dimensional space of the *functions*. $\psi(x) = (\psi^1(x), \dots , \psi^n(x))$ , where $x \in R_4$. It is well-known that the spherical functions $\psi_{lm}$ form a basis of all single-valued representations of the rotation group. From 1955 onwards similar methods for the Lorentz group (the harmonical analysis of the Lorentz group) are developed.
Nowadays the study of infinite-dimensional representations of arbitrary semi-simple non-compact Lie groups is influenced by the representation theory of the Lorentz group.

Chapter I gives a brief introduction to the representation theory of the Lorentz group. Proofs and examples are given only if they contribute to the understanding of the subject.
In part A of chapter I the so-called *algebraic method* is treated. After the most important definitions in section 1: Section 2 serves as an example of the principal concepts of the representation theory. The fundamental theory may be found in section 3. The theory of Lorentz covariant equations is treated in section 4 ( and section 8), in which special attention is given to the equation of Dirac.
In part B of chapter I, the theory is developed starting with the theory of Lie groups, the so-called *infinitesimal method*. The more "technical"

information is given in the sections 6 and 8, while in sections 5 and 7 some theoretical background is given; however, this is not necessary for the understanding of the sections 6 and 8. In section 7 special attention is given to the work of Gel'fand and Neumark.
A short note on the use of the representation theory of the Lorentz group in elementary particle physics may be found in the sections 3.4 and 8.3.

## 1.1. The Lorentz group and its subgroups

In the *real* space $R_4$, with *four-vectors* $x \equiv x^{\mu}$, there is defined a pseudo norm:

$$(x)^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2.$$

Definition A *Lorentz transformation* L is a linear transformation $x' = Lx$, which leaves the norm of x invariant, i.e. $(x)^2 = (x')^2$.

All Lorentz transformations form by definition the *full or general Lorentz group L.*

Thus a Lorentz transformation leaves quadratic surfaces: $(x)^2$ = constant invariant, e.g. the light cone, $(x^2) = 0$, a hyperboloid of two sheets, $(x^2) = +1$ and a hyperboloid of one sheet, $(x^2) = -1$. This is illustrated in fig. 1.1 for the coordinates $x^0$, $x^2$, $x^3$, ($x^1 = 0$). The components of the matrix L are denoted by $L^{\mu'}{}_{\mu}$. We write $x^{\mu'} = L^{\mu'}{}_{\mu} x^{\mu}$, where the Einstein convention is used. In our notations Greek letters $\mu$, $\nu$, ... always take the values 0, 1, 2, 3 While the letters i, j are reserved for the space values 1, 2, 3, thus $\vec{x} \equiv x^i$. The symbol $x^4$ is used for ict. Sometimes it is more convenient to write $x = Lx'$ since each Lorentz transformation has an inverse. In an alternative way one may interprete L as a coordinate transformation as well as a point transformation. In the latter case one often writes: $x'^{\mu} = L^{\mu}{}_{\nu} x^{\nu}$. We may write the norm $(x)^2$, using the so called *metric tensor* $g \equiv g^{\mu\nu} \equiv g_{\mu\nu}$, as

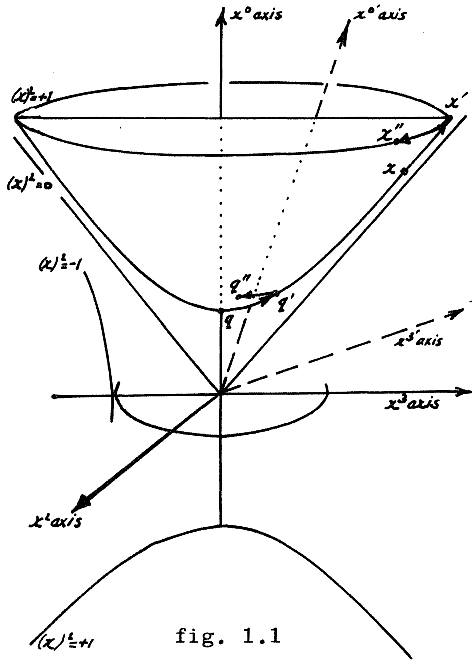

fig. 1.1

$$(x)^2 = g_{\mu\nu} x^{\mu} x^{\nu}, \text{ where } g = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix} \qquad (1\text{-}1)$$

or $(x)^2 = x^T g x$, where $x^T$ is the transpose of x.

After substitution of $x = Lx'$, we obtain for every L.

$$L^T g\, L = g, \text{ and also } LgL^T = g. \qquad (1\text{-}2)$$

From this it follows that $\det L = \pm 1$ and $(L^0{}_0)^2 - (L^0{}_i\, L^0{}_i) = 1$ so that $(L^0{}_0)^2 \geq 1$. In view of these properties we shall define two important subgroups of L.

1. <u>The reflection group.</u> This group consists of the following 4 reflections; the identity E, the space reflection P, the time reflection T and the total reflection J = PT

$$P = \begin{pmatrix} 1 & . & . & . \\ . & -1 & . & . \\ . & . & -1 & . \\ . & . & . & -1 \end{pmatrix} \quad T = \begin{pmatrix} -1 & . & . & . \\ . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \end{pmatrix} \quad J = \begin{pmatrix} -1 & . & . & . \\ . & -1 & . & . \\ . & . & -1 & . \\ . & . & . & -1 \end{pmatrix}. \qquad (1\text{-}3)$$

Sometimes we use the notation $S_\mu$ to indicate E, P, T or J.
During the last years this group has received much attention from physicists, see section 3.4.

2. <u>The restricted Lorentz group $L_+^\uparrow$ *)</u> This group consists of all Lorentz transformation $\Lambda$, such that

$$\det \Lambda = +1 \text{ and } \Lambda^0{}_0 \geq 1. \qquad (1\text{-}4)$$

The last condition implies that a vector directed upwards in the figure i.e. q(1, 0, 0, 0) remains upwards after Lorentz transformation, by which the index $\uparrow$ is explained. Every Lorentz transformation L can be written as the product $L = S_\mu \Lambda$ of a reflection $S_\mu$ and a restricted Lorentz transformation $\Lambda$.
Important subgroups of $L_+^\uparrow$ are

a. <u>The three-dimensional rotation group $O_{3+}$</u>. This group leaves invariant the time component $x^0$ and hence the euclidean norm $(x^1)^2 + (x^2)^2 + (x^3)^2$.

Thus a rotation r has the form: $r = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & & \\ 0 & & & \end{pmatrix}$

where the shaded matrix is a $3 \times 3$ orthogonal matrix with det = +1.
Every rotation is determined by a vector $\vec{\phi}\,(\phi^{23}, \phi^{31}, \phi^{12})$ where $\vec{\phi}$ is

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*) One also calls this group the *proper* Lorentz groups (Roman, Corson) but it seems better to reserve this term for the larger subgroup $L_+$ (det L = + 1) (Barut, Hilgevoord). Another larger group is the *orthochroneous* group $L\uparrow$ $(L^0{}_0 \geq 1)$.

directed along the rotation axis (as a right-handed screw) and $|\vec{\phi}| = \phi$ is the rotation angle ($0 \leq \phi \leq \pi$).
For instance a rotation in the $(x^1, x^2)$-plane is given by

$$r_{12}(0,\ 0,\ \phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (\phi = \phi^{12}). \tag{1-5}$$

Thus the rotation group is a three-parameter group (see section 5).

2b. Hyperbolic screws or pure Lorentz transformations along the $x^3$-axis.
One can easily verify that the transformations $h_{03}(\phi)$ in the $(x^0,\ x^3)$-plane

$$h_{03}(\phi) = \begin{pmatrix} \cosh\phi & 0 & 0 & \sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh\phi & 0 & 0 & \cosh\phi \end{pmatrix} \quad (\phi = \phi^{03}). \tag{1-6}$$

leave $(x^0)^2 - (x^3)^2$ invariant. In figure 1.1 is drawn the transformation $h_{03}(\phi)$: $x \to x'$ , $q \to q'$. All $h_{03}(\phi)$ form a one-parameter subgroup.
By rotating the $(x^0\ ,\ x^3)$-plane the hyperbolic screw $h_{03}(\phi)$ transforms into an arbitrary hyperbolic screw $h = r^{-1} h_{03}\, r$ along the $\vec{q}\,''$-axis, i.e. in the $(x^0, q'')$-plane. (it follows that h is hermitian). Every point q" on the upper branch of the hyperboloid $x^2 = +1$ determines one hyperbolic screw sothat $q'' = hq$, $q = (1,\ 0,\ 0,\ 0)$. Thus all hyperbolic screws can be mapped uniquely onto the upper branch of $x^2 = +1$. (Since this upper branch extends to infinity and therefore is not compact it follows that the Lorentz group is not compact, see section 5).
In analogy to the rotation group we also often prefer to determine the screw $h_{03}(\phi)$ by the vector $(0,\ 0,\ \phi)$ and an arbitrary hyperbolic screw by the vector $(\phi^{01},\ \phi^{02},\ \phi^{03}) = r(0,\ 0,\ \phi)$ which lies in the $(x^0\ ,\ q'')$-plane.
Further, every Lorentz transformation $\Lambda$ is the product of a rotation and a hyperbolic screw. For suppose that we have $\Lambda : q \to q''$, then there is one hyperbolic screw h so that $h : q \to q''$. The transformation $\Lambda h^{-1}$

leaves q and thus the $x^0$-axis invariant, from which follows $\Lambda h^{-1} = r$.

Theorem 1.1. *Every full Lorentz transformation L is the product of a reflection $S_\mu$ and a restricted Lorentz transformation $\Lambda$, i.e. $L = \Lambda S_\mu$ Every restricted Lorentz transformation $\Lambda$ is the product of a rotation r and a hyperbolic screw h, i.e. $\Lambda = rh$.*

Hence a restricted Lorentz transformation is determined by the parameters: $(\phi^{01}, \phi^{02}, \phi^{03}; \phi^{23}, \phi^{31}, \phi^{12})$, and $L_+^\uparrow$ is a six-parameter group.

The fact that the time and space components are interrelated in a hyperbolic screw has important physical significance, about which we have to say some words in order not to loose touch with physics completely.
If we substitute $\tanh\phi = \frac{\sinh\phi}{\cosh\phi} = \beta$ in (1-6) we get

$$\begin{pmatrix} ct \\ \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{1-\beta^2} & \frac{\beta}{1-\beta^2} \\ \\ \frac{\beta}{1-\beta^2} & \frac{1}{1-\beta^2} \end{pmatrix} \begin{pmatrix} ct' \\ \\ z' \end{pmatrix} .$$

This is a well-known formula in the theory of relativity. It expresses the transformation of the coordinates of the system Oxyzt into the system Ox'y'z't' which travels with a velocity v in positive direction along the z-axis ($\beta = \frac{v}{c}$). The physical interpretation of this leads to phenomenon of the Lorentz contraction of rigid bodies and the fact that clocks slow down.
Further if two hyperbolic screws are performed in succession we get the addition law of velocities in the theory of relativity. For (1-6) implies

$$h_{03}(\phi_1)\, h_{03}(\phi_2) = h_{03}(\phi_1 + \phi_2) .$$

By substituting $v = c \tanh(\phi_1 + \phi_2)$ and $v_i = c \tanh \phi_i$ $(i = 1, 2)$ it follows that $v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c}}$

See Bergman p. 43.

Other four-component quantities, with the same behaviour as time-space vectors $x^\mu$, which we shall meet later on are obtained if we introduce the "distance" s along curves by integrating the expression

$$\Delta s^2 = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$$

If we now describe the time-space behaviour of a particle by the curve $x^\mu(s)$ (world line) then the *four-velocity* $u^\mu = \frac{dx^\mu(s)}{ds}$ is also a vector, because "ds" is an invariant; s is called the *proper time* of the particle. Hence $u^\mu = (1, \frac{\vec{v}}{c}) \frac{dx^0}{ds}$ holds were $\frac{dx^0}{ds} = \frac{1}{1-\beta^2}$ $(\beta = \frac{v}{c})$ and $(u)^2 = 1$

One defines the *four-momentum* by

$$p^\mu = m_0 c u^\mu, \qquad (1-7)$$

$m_0$ is the restmass of the particle, and substituting the mass $m = \frac{m_0}{1-\beta^2}$, energy $E = mc^2$ and impuls $\vec{p} = m\vec{v}$ into $p^\mu$ one obtains $p^\mu = (\frac{E}{c}, \vec{p})$.

The *four-current density* is defined by

$$j^\mu = \rho_0 c u^\mu \qquad (1-8)$$

where $\rho_0$ is the rest-charge density, with the charge $\rho$ and current density $\vec{j} = \rho\vec{v}$ we obtain $j^\mu = (\rho c, \vec{j})$.

See Barut p. 48, 94.

## 1.2. Representations of the Lorentz group, definitions

Definitions. An *n-dimensional representation* D of the Lorentz group L is a homomorphic continuous mapping D of Lorentz transformations L onto n × n matrices which will be denoted by D(L).

In a *homomorphic* mapping products are preserved. Thus if

$$D : L_1 \to D(L_1)$$

and $$L_2 \to D(L_2)$$

then $$L_1\, L_2 \to D(L_1\, L_2) = D(L_1) \cdot D(L_2) .$$

The identity E corresponds with the $n \times n$ unit matrix D(E).
If the mapping is one to one, then the mapping is called *isomorphic* and the representation *faithfull*. Although the definition of a representation may be generalized to a mapping onto bounded operators D(L), which act in an infinite-dimensional linear space; we restrict ourselves for the present to finite-dimensional representations which act in $R_n$. See remark 6.1.
A representation D(L) is called *(completely) reducible* if there is exactly *one* coordinate transformation S in $R_n$ so that for *all* L the transformed matrices $D'(L) = S^{-1}\, D(L)\, S$ are in a diagonal-block form, for instance

$$D'(L) = \begin{pmatrix} D_1(L) & 0 \\ 0 & D_2(L) \end{pmatrix} ,$$

where $D_1(L)$ and $D_2(L)$ are $n_1 \times n_1$ and $n_2 \times n_2$ matrices respectively. Thus the n-dimensional representation space $R_n$ contains an invariant $n_1$-dimensional linear space $R_{n_1}$ and an invariant $n_2$-dimensional space $R_{n_2}$ $(n = n_1 + n_2)$.
With the symbol " $\dot{+}$ " (*tensorsum*) one describes

$$D'(L) = D_1(L) \dot{+} D_2(L) \text{ or } R_n = R_{n_1} \dot{+} R_{n_2} .$$

If there does not exist an invariant linear subspace (except the trivial ones: 0 and $R_n$) one calls the representation *irreducible*.
Representation D and D' which differ only by a coordinate transformation S, determine the same linear transformation in $R_n$ and are called *equivalent*.
In the representation theory often use is made of the following two lemma's of Schur.

Theorem 1.2. First lemma of Schur. *If D(A) is an irreducible representation of a group then a matrix S commutes with all matrices D(A) if and only if S is a multiple of the unit matrix.*

$$D(A) \text{ irred.}: D(A)S = SD(A) \text{ (for all A)} \Longleftrightarrow S = \lambda E . \quad (1\text{-}9a)$$

This lemma gives a criterion for irrreducibility.

Theorem 1.3. Second lemma of Schur. *If D(A) and E(A) are two non-equivalent representations then*

$$D(A)S = SE(A) \text{ (for all A)} \Longleftrightarrow S = 0 \quad (1\text{-}9b)$$

If there are given irreducible representations $D_1(L)$, $D_2(L)$, ... we can construct an infinite number of reducible representations by taking tensorsums: $D(L) = D_1(L) \dot{+} D_2(L) \dot{+} \ldots$ .
Hence only the (non-equivalent) irreducible representation of a group give essential information about the possible representations of a group and therefore the first task of the representation theory of a group is to classify all irreducible representations of this group.
One of the most important aids in constructing irreducible representations of a group is formed by the concept of *tensorproduct* which will be considered in the following section.
For literature concerning the general representation theory of groups see Ljubarski, Hamermesh, Lauwerier and the literature cited there in.

## 2. Tensor representations of the Lorentz group

This section serves as an introduction to the concepts of tensor product, reducibility with respect to a subgroup, pseudo tensors etc. Therefore special attention is devoted to the first-rank (vector) and second-rank tensor representations.

### The vector representation

This is the trivial representation $L \to L$, i.e. the mapping of the group element L onto the $4 \times 4$ matrix L. The vector representation is irreducible, since there is no invariant linear sub-space, for the only invariant surfaces are hyperboloids and one cone, each of which span the whole space $R_4$.

If we restrict ourselves to the subgroup of spatial rotations $O_{3+}$ then, because the matrices r are of the form

$$r = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \end{pmatrix}$$

the vector representation is reducible into a one and a three-dimensional representation. Instead of the dimension n of the irreducible subspaces, with respect to the rotation group, one often uses the *spin value* j defined by $j = \frac{1}{2}(n-1)$ *). Thus in the vector representation we have the spin values j = 0, 1.

Equivalent to the vector representation is the representation one obtains by performing a coordinate transformation, e.g. a reflection: gx. One denotes the components of gx with lower indices i.e. $x_\mu = g_{\mu\nu} x^\nu$ (note that $x_\mu x^\mu = x^2$).

The matrix L transforms into $g^{-1} L g = (L^{-1})^T$ (formula (1-2)) or in components one writes

$$L^{\mu'}{}_{\mu} \to L_{\mu'}{}^{\mu} \overset{d}{=} g_{\mu'\nu'} g^{\mu\nu} L^{\nu'}{}_{\nu} \tag{1-10}$$

Note that the index $\mu'$ is lowered by $g_{\mu'\nu'}$ and that the index $\mu$ is raised by $g^{\mu\nu}$. In this way we obtain a consistent notation of *contravariant* vectors $x^\mu$ which transform by $x^{\mu'} = L^{\mu'}{}_{\mu} x^\mu$ and *covariant* vectors $x_\mu$ which transform by $x_{\mu'} = L_{\mu'}{}^{\mu} x_\mu$.

There holds that

$$L^{\mu}{}_{\lambda} L_{\nu}{}^{\lambda} = \delta^{\mu}_{\nu} \tag{1-11}$$

Thus besides the relation $x^{\mu'} = L^{\mu'}{}_{\mu} x^\mu$ we have $x^\mu = L_{\mu'}{}^{\mu} x^{\mu'}$.

We obtain a non-trivial representation, equivalent to the vector representation, by defining now the four operators

$$\partial_\mu = \frac{\partial}{\partial x^\mu}, \tag{1-12}$$

we will show that $\partial_\mu$ is transformed as a covariant vector.

Therefore we consider the space of all functions f(x), ($x \in R_4$), which

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*) Physicist would prefer perhaps the expression "possible spin values" One also uses "the *weight* j", but "*highest weight j*" would be better, see p. 26 and section 7.2.

transforms as

$$f'(x') = f(x) \quad x' = Lx. \qquad (1\text{-}12a)$$

Or replacing the argument x' by x we get the transformation D(L) such that $D(L)f(x) = f'(x) = f(L^{-1}x)$

It follows that D(L) is a representation because D(L) is a linear operator

and $\qquad D(M)\ f'(x) = f(L^{-1}M^{-1}x)$,

thus $\qquad D(M)D(L) = D(ML)$ .

Differentiating f'(x') with respect to $x^{\mu'} = L^{\mu'}{}_{\mu}x^{\mu}$ we obtain

$$\partial_{\mu'}\ f'(x') = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\ \partial_{\mu}\ f(x)$$

or $$\partial_{\mu'}\ f'(x') = L_{\mu'}{}^{\mu}\ \partial_{\mu}\ f(x), \qquad (1\text{-}12b)$$

which indicates that $\partial_{\mu}$ indeed transforms covariant.

Through here the well-known comma notation is justified

$$f'_{,\mu'} = L_{\mu'}{}^{\mu} f_{,\mu}$$

The second-rank tensor representation

Starting from the vectors $x^{\mu}$ and $y^{\mu}$ one may form the set of 16 components

$$X = (x^{\mu} y^{\mu})$$

which may be interpreted as a vector in a 16-dimensional space. X is called the *tensor product* of x and y and all X span the 16-dimensional space $R_{16}$ of *second-rank tensors* $X(x^{\mu\nu})$, in general det $X \neq 0$. To a Lorentz transformation $x^{\mu'} = L^{\mu'}{}_{\mu}x^{\mu}$ there corresponds a transformation of X i.e.

$$x^{\mu'\nu'} = L^{\mu'}{}_{\mu}\ L^{\nu'}{}_{\nu}\ x^{\mu\nu}$$

or $$X' = LXL^{T}. \qquad (1\text{-}13)$$

Thus the vector X is transformed by the 16 × 16 matrix $L \times L = (L^{\mu'}{}_{\mu}L^{\nu'}{}_{\nu})$, where $\mu'\nu'$ indicate the rows and $\mu\nu$ the columns

$$L \times L = \begin{pmatrix} L^{0'}{}_0 L \; . \; . \; . \; . \; L^{0'}{}_3 L \\ \vdots \qquad\qquad \vdots \\ L^{3'}{}_0 L \; . \; . \; . \; . \; L^{3'}{}_3 L \end{pmatrix} \tag{1-14}$$

The mapping $L \rightarrow L \times L$ is called the *second-rank tensor representation* of the Lorentz group L and is reducible, as it is possible to write X as the sum of a symmetric and an anti-symmetric tensor

$$X = S + F, \text{ where } S = \tfrac{1}{2}(X + X^T) \text{ and } F = \tfrac{1}{2}(X - X^T).$$

The tensors S and F span a 10-dimensional *linear* space $R_{10}$ and a 6-dimensional *linear* space $R_6$ respectively.

$$S = \left(\begin{array}{c|c} S^{00} & S^{0i} \\ \hline S^{0i} & S^{ij} = S^{ji} \end{array}\right), \quad F = \begin{pmatrix} 0 & F^{01} & F^{02} & F^{03} \\ -F^{01} & 0 & F^{12} & -F^{31} \\ -F^{20} & -F^{12} & 0 & F^{23} \\ -F^{03} & F^{31} & -F^{23} & 0 \end{pmatrix} \tag{1-15}$$

A Lorentz transformation $X' = L\,X\,L^T$ leaves both spaces invariant, i.e. it transforms an (anti-) symmetric tensor into an (anti-) symmetric tensor. Thus the second-rank tensor representation is reducible. The space $R_{10}$ of symmetric tensors is still further reducible into the invariant space $R_1$, formed by the metric tensor i.e.

$$g' = LgL^T = g, \quad \text{see (1-2)},$$

and the 9-dimensional space $R_9$ of symmetric tensors with vanishing "trace". The "trace" X is defined by

$$\text{"trace" } X = X^{00} - X^{11} - X^{22} - X^{33} = X^{\mu\nu} g_{\nu\mu},$$

thus "trace" $X = X^{\mu}{}_{\mu}$ and with (1-11) it is easy to prove that $X^{\mu}{}_{\mu}$ is a *scalar* i.e. $X^{\mu'}{}_{\mu'} = X^{\mu}{}_{\mu}$. Thus $R_9$ is an invariant space. It will be proved later on that these representation spaces do not contain further invariant subspaces, thus that they are irreducible, (see formula (1-36)). Hence we have:

$$R_{16} = R_1 \dotplus R_6 \dotplus R_9$$

However, with respect to the *restricted group* $L_+^\uparrow$, the space $R_6$ of anti-symmetric tensors is further reducible. Therefore we write the matrix $F^{\mu\nu}$, from formula (1-15), in abbreviated form as

$$F^{\mu\nu} \equiv (\vec{E}, \vec{H}),$$

where $E^i = F^{0i}$ and $H^i = F^{jk}$ (i, j, k = 1, 2, 3 and cycl.) *)
Consider now the 3-dimensional spaces $R_3$ and $\dot{R}_3$ of anti-symmetric tensors of form

$$G = (\vec{G}, -i\vec{G}) \in R_3 \, , \qquad (1\text{-}16a)$$

$$\dot{G} = (\dot{\vec{G}}, +i\dot{\vec{G}}) \in \dot{R}_3 \, . \qquad (1\text{-}16b)$$

Every $F^{\mu\nu}$ may be written as the sum of anti-symmetric matrices of type $G$ and $\dot{G}$, i.e.

$$(\vec{E}, \vec{H}) = \tfrac{1}{2}(\vec{G}, -i\vec{G}) + \tfrac{1}{2}(\dot{\vec{G}}, + i\dot{\vec{G}}) \text{ where } \vec{G} = \vec{E} + i\vec{H} \text{ and } \dot{\vec{G}} = \vec{E} - i\vec{H} \qquad (1\text{-}16c)$$

We will prove that $R_3$ remains invariant, thus that $G' = \Lambda\, G\, \Lambda^T$ is also of the same form as $G$, i.e. $G' = (\vec{G}', -i\vec{G}')$. By which follows that $\vec{G}' = \vec{E}' + i\vec{H}'$ and that $G^{\mu\nu}$ and $\dot{G}^{\mu\nu}$ defined by (1-16c) indeed transform contravariant. Therefore we note that $G^T g\, G = g\,(-\vec{G}^2)$, thus all $G$ and $\dot{G}$ form two linear spaces of anti-symmetric tensors which are also Lorentz transformations if $\vec{G}^2 \neq 0$. (More exactly they are complex Lorentz transformations up to a numerical factor). After a restricted Lorentz transformation we obtain the matrix $G' = \Lambda G \Lambda^T$, which is again anti-symmetric and a Lorentz transformation, because $G'^T g\, G' = g\rho$. This implies that $G'$ has necessarily the form $G' = (\vec{G}', - i\vec{G}')$ or $G' = (\vec{G}', + i\vec{G}')(\vec{G}'^2 \neq 0)$. All restricted Lorentz transformations $\Lambda$ are continuously connected with the identity (section 5), thus starting from $G = (\vec{G}, - i\vec{G})$ it follows that $G' = (\vec{G}', - i\vec{G}')$. Thus $R_3$ and $\dot{R}_3$ are two invariant subspaces under the restricted group $L_+^\uparrow$. Only with a space of time reflection $G' = PGP^T$, or $G' = TGT^T$ we obtain

$$G' = (- \vec{G}, - i\vec{G})$$

and the spaces $R_3$ and $\dot{R}_3$ are interchanged.

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*) With this expression we always mean:

$$H^1 = F^{23},\ H^2 = F^{31},\ H^3 = F^{12},\ H^1 = -F^{32},\ \ldots \text{ etc.}$$

*Summarising, the space of second-rank tensors is reduced with respect to the restricted group into*

$$R_{16} = R_1 + R_3 + \dot{R}_3 + R_9.$$

The actual proof that these representation spaces do not contain further invariant subspaces, thus that they are irreducible will be given later on by using spinor calculus, see formula (1-36).

Finally we mention some notations which are often used. One writes

$$F_{\overline{\mu\nu}} = (\vec{H}, \vec{E})(-i) \quad \text{or} \quad {}^{*}F_{\mu\nu} \tag{1-16d}$$

and raising the indices $\overline{\mu\nu}$ we have

$$F^{\overline{\mu\nu}} = (\vec{H}, -\vec{E})(i) \text{ or } {}^{*}F^{\mu\nu}, \text{ which is called the } \textit{dual} \text{ of } F^{\mu\nu} \tag{1-16e}$$

It follows that $G^{\mu\nu} = F^{\mu\nu} + F^{\overline{\mu\nu}}$ , see (1-16c). Because $G^{\mu\nu}$ transforms contravariant it follows that $F^{\overline{\mu\nu}}$ transforms contravariant.

Using this, the following invariants in the space of anti symmetric tensors may be constructed

$$\tfrac{1}{4} F_{\overline{\mu\nu}} F^{\mu\nu} = \vec{E} \cdot \vec{H} \, (-i) \tag{1-16f}$$

and $$\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu} = \vec{H}^2 - \vec{E}^2.$$

The invariance of these expressions follow from the relation (1-11). For more detailled information, see the appendix of chapter II.

Once more we restrict ourselves to the subgroup $O_{3+}$. Then a further reduction of the spaces $R_6$ and $R_9$ is possible; where $R_6$ is the space of anti-symmetric tensors $F \equiv (\vec{E}, \vec{H})$ and $R_9$ the space of symmetric tensors T with trace zero. One may verify that after substitution of the rotation:

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & r & \\ 0 & & & \end{pmatrix} \quad \text{into} \quad F' = \Lambda\, F\, \Lambda^T \quad \text{and} \quad T' = \Lambda\, T\, \Lambda^T ,$$

$\vec{E}$ and $\vec{H}$ transform as 3-dimensional vectors $\vec{E}' = r\,\vec{E}$, $\vec{H}' = r\,\vec{H}$ and that the components $T^{oo}$, $T^{oi}$, $T^{ik}$ (i, k = 1, 2, 3) transform as a scalar, a 3-vector and a 3 x 3 tensor respectively.

We shall illustrate this with some examples from physics. By noting that as long we restrict ourselves to the rotation group $O_{3+}$ we obtain the *classical* behaviour of the tensor components and the irreducible subspaces corresponds with classical entities. In the theory of relativity the anti-symmetric tensor $F = (\vec{E}, \vec{H})$ describes the electromagnetic field, where $\vec{E}$ is the electric field strength and $\vec{H}$ the magnetic strength. (in a point x).

"Classically", that is to say with respect to the rotation group, $\vec{E}$ and $\vec{H}$ indeed transform as 3-dimensional vectors. Yet if one substitutes the hyperbolic screw $\Lambda = h_{03}(\phi)$ from formula (1-6) into $\Lambda F \Lambda^T$ one obtains the important transformation laws of the electromagnetic field in the theory of relativity in which the components of $\vec{E}$ and $\vec{H}$ are mixed. Thus with respect to the restricted Lorentz group the spaces $(\vec{E}, 0)$ and $(0, \vec{H})$ do not form invariant subspaces, which is in accordance with (1-16c). The vector $\vec{G}$ is introduced by Laporte and Uhlenbeck in 1931.

The symmetric tensor T appears as the symmetric energy-impuls tensor, where $T^{oo}$ determines the energy-density of the electromagnetic field, $T^{oi}$ the vector $\vec{S} = \vec{E} \times \vec{H}$ of Pointing and $T^{ik}$ the stress tensor of Maxwell. It is known that "classically" they behave as a scalar,a 3-vector and a (3 x 3)-tensor respectively, but in the theory of relativity they are obviously regarded as one system that is to say they form *one* irreducible space).

## The rth-rank tensor representation, pseudotensors

We may now construct the *rth-rank tensor representation*

$$L \to \underbrace{L \times L \times \ldots L}_{r} \equiv (L^{\mu_1'}{}_{\mu_1} \; L^{\mu_2'}{}_{\mu_2} \ldots L^{\mu_r'}{}_{\mu_r}) \equiv [L]^r, \qquad (1\text{-}17)$$

which acts on tensors $(x^{\mu_1 \ldots \mu_r})$. We recall that in the representation theory, lower and upper indices are equivalent, see p. 11.
The rth-rank tensor representation is reducible for $r > 1$ into spaces of tensors with a certain symmetry in the indices and with "traces" equal to zero. See Hamermesh.
The irreducible representations of the rotation group induced by the tensor representation are always odd-dimensional, thus the spin values are always integral: $j = 0, 1, 2, \ldots$ . (see the 2nd-rank tensor representation).
Apart from the tensor representations there also exist the so-called pseudo-tensor representations i.e.

$$L \to \rho [L]^r$$

If one chooses $\rho = 1$ for all Lorentz transformations one obtains the ordinary *proper tensors* which we have already considered.
We will denote this by $\rho = \rho\,(L)$ so that

$$L = \quad L_+^\uparrow \quad PL_+^\uparrow \quad TL_+^\uparrow \quad JL_+^\uparrow \qquad (J = PT)$$

$$\rho = \quad 1 \quad 1 \quad 1 \quad 1$$

If one substitutes $\rho = \dfrac{L^0{}_0}{|L^0{}_0|}$ then $\rho$ takes on the values

$$\rho = \quad 1 \quad 1 \quad -1 \quad -1$$

and we obtain tensors which have a "pseudo" behaviour with respect to time reflection which are called *time-pseudo tensors*. Continuing in this way we obtain the following table.

| kind | | | | P | T |
|---|---|---|---|---|---|
| 0. | $L \to [L]^r$ | acting on *proper tensors* | | + | + |
| 1. | $L \to \dfrac{L^0{}_0}{\lvert L^0{}_0 \rvert} [L]^r$ | " | " *time-pseudotensors* | + | - |
| 2. | $L \to \dfrac{L^0{}_0}{\lvert L^0{}_0 \rvert} \det L \; [L]^r$ | " | " *space-pseudotensors* | - | + |
| 3. | $L \to \det L \; [L]^r$ | " | " *pseudotensors* | - | - |

(1-18)

(Watanabe 1951)

E.g.: scalar $P, T : c \to c.$

pseudo scalar $P, T : c \to -c.$

a *scalar* remains invariant with space or time reflection, while a *pseudo scalar* changes sign.

The question of equivalence of the different pseudotensor representations will be answered in the next section.

## 3. Spinor representations of the Lorentz group

Until 1930 in physics one was only acquainted with tensors. About that time it became necessary, in connection with the development of quantum mechanics, to give a classification of *all* possible representations of the Lorentz group.

In quantum mechanics the space-time (and spin) behaviour of a particle is described by a state function $\psi(x)$ (field), with a certain number n of components $\psi(x) \equiv (\psi^1(x), \ldots , \psi^n(x))$, where $\psi(x)$ may be a scalar-function, a tensor-function, generally any function $\psi$ with values $\psi^i(x)$, which transform by a representation D(L) of the Lorentz group

$$\psi'(x') = D(L)\,\psi(x) \qquad\qquad x' = Lx$$

Hence a classification of all irreducible representations of the Lorentz group gives a classification of all functions $\psi(x)$, and thus gives a classification of all possible elementary particles in their space-time symmetries, see section 3.4.

It appears that next to the tensor representation $L \to [L]^r$ with integral spinvalues, there exists the so-called spinor representations with half-integral spinvalues $j=0, \frac{1}{2}, 1, \ldots$

Although the formalism of spinor calculus of m-dimensional orthogonal groups was already developed in 1913 by Cartan (and in a different way by Weyl in 1935), the application of the spinor calculus of the Lorentz group in quantum mechanics was made by van der Waerden in 1929, in connection with the equation of Dirac.

## 3.1. Spinor representations of the restricted group $L_+^\uparrow$

Mathematically spinors arise from the inverse problem of taking 'tensor products', roughly speaking from the problem of taking 'tensor roots'. We consider the problem of decomposing an Lorentz transformation $\Lambda$ as the tensor product of two matrices A and B such that

$$\Lambda = A \times B$$

where A and B are two-dimensional representations of the Lorentz group. Assuming the existence of such a decomposition we may give the following definition.

Definition *A two-dimensional representation $A(\Lambda)$ of the restricted Lorentz group $L_+^\uparrow$ is called a spinor representation and vectors $\psi$ which are transformed by $A(\Lambda)$ are called spinors.*

The existence of spinors representations is proved in the following two lemma's.

Lemma 3.1. *There exists a spinor representation $A(\Lambda)$ such that the 4-dimensional vector representation $\Lambda$ is equivalent to the tensor product $A \times \bar{A}$* *)

PROOF. We map every vector x in $R_4$ on a $2 \times 2$ matrix X with components $X^{ac}$ (a, c = 0,1)

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \qquad (1\text{-}19)$$

and note that $\det X = (x)^2$ holds and that X is hermitian, i.e. $X^+ = X$. From this it follows that the transformation

$$X' = AXA^+ \qquad (\det A = +1), \qquad (1\text{-}20a)$$

is a Lorentz transformation. For, X' is also hermitian, thus X' can be written in the form (1-19) with $x^{\mu'}$ instead of $x^\mu$. It follows that (1-20a) induces a linear transformation in $R_4$.
Moreover $\det X' = \det X$ or $(x')^2 = (x)^2$ and thus (1-20a) determines a Lorentz transformation $\Lambda$ in $R_4$.

*) The matrices $\bar{A}$ and $A^+ = \bar{A}^T$ denote the complex conjugate and the hermitian conjugate of the matrix A respectively.

We observe that the vector $X(X^{00}, X^{01}, X^{10}, X^{11})$ is obtained from $x^\mu$ by a coordinate transformation $X = Tx$. If we now write (1-20a) in components

$$X^{a'c'} = A^{a'}{}_a \bar{A}^{c'}{}_c X^{ac} \qquad (1\text{-}20b)$$

we obtain the transformation

$$X' = (A \times \bar{A})\, X$$

and it follows that

$$\Lambda = T(A \times \bar{A})T^{-1} \qquad (1\text{-}21)$$

Using the conditions (1-4) one may verify that $\Lambda$ is a restricted Lorentz transformation. Conversely one may show that it is possible to describe every restricted Lorentz transformation $\Lambda$ by a $2 \times 2$ matrix A, see section 5.

In this way we have constructed a two-dimensional irreducible representation of the Lorentz group A, i.e.

$$\Lambda \longleftrightarrow \pm A \;(\det A = +1)$$

and it follows that the 4-dimensional vector representation is equivalent to the tensor product $A \times \bar{A}$.

The matrices A are determined within sign, see formula (1-21), and the representation is essentially two-valued. If one denotes the group of $2 \times 2$ complex unimodular matrices by $SL_2$, the unit matrix by E and the group consisting of E and -E by $Z_2$ one may write

$$SL_2/Z_2 \simeq L_+^\uparrow ,$$

where $\simeq$ is the symbol for isomorphism.

<u>Remark 3.1.</u> In addition to the above proof we mention that if A is *unitary*, i.e. $A^+ = A^{-1}$, then

$$\text{tr } X' = \text{tr } AXA^+ = \text{tr } AXA^{-1} = \text{tr } X , \text{ *)}$$

thus $\qquad x^{0'} = x^0$

It follows that the unitary unimodular matrices A, forming the group $SU_2$,

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*) The trace of an $n \times n$ matrix X is defined by $\text{tr } X = \sum_i X_{ii}$ and there holds tr (AB) = tr (BA).

correspond with rotations. One may write

$$SU_{2/Z_2} \simeq O_{3+} \ .$$

We now consider two special matrices A and B.

$$A = \begin{pmatrix} e^{-i\frac{1}{2}\theta} & 0 \\ 0 & e^{+i\frac{1}{2}\theta} \end{pmatrix}, \quad B = \begin{pmatrix} e^{+\frac{1}{2}\theta} & 0 \\ 0 & e^{\frac{1}{2}\theta} \end{pmatrix} \qquad (1\text{-}22)$$

After substitution of A and B in (1-20a) it follows that A describes the a rotation $\theta$ about the z-axis, see (1-5), and that B describes a hyperbolic screw $\theta$ in the $(x^0, x^3)$-plane. The matrix B is hermitian and after applying a rotation R, R is unitary, we obtain an arbitrary hyperbolic screw given by the hermitian matrix $R^{-1}BR$, see (1-6).

<u>Notations</u>

One often writes $X = x^\mu \sigma_\mu$

The four matrices $\sigma_\mu$ are

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (1\text{-}23)$$

and in particular the three matrices $\sigma_i$ are called the *Pauli matrices*

One may verify that

$$\sigma_1\sigma_2 - \sigma_2\sigma_1 = 2i\sigma_3 \text{ and cycl.} \qquad (1\text{-}24)$$

$$\text{or} \quad [\sigma_1, \sigma_2] = 2i\sigma_3 \ ,$$

holds.

One denotes the components of a spinor by $\psi^a$ (a=0,1) and the letters a, b, c, ... are used for spin-indices.

The components of the spinor $\overline{\psi^a}$ which transforms by $\bar{A}$, one denotes by $\psi^{\dot{a}}$ (a with a dot). Equation (1-20b) then implies the components of the vector X must be rewritten as $X^{a\dot{c}}$. This notation expresses also that the four-dimensional space $R_4$ is the tensor product of the 2-dimensional spaces of spinors $(\psi^a)$ and $(\psi^{\dot{c}})$, thus spanned by

$$(X^{a\dot{c}}) = (\psi^a \psi^{\dot{c}}) \qquad (1\text{-}25)$$

In components one writes the relation $X = x^\mu \sigma_\mu$ as $X^{a\dot{c}} = x^\mu \sigma_\mu{}^{a\dot{c}}$. The transformation of two spinors may be written as

$$\begin{pmatrix} \psi^{0'} & \phi^{0'} \\ \psi^{1'} & \phi^{1'} \end{pmatrix} = A \begin{pmatrix} \psi^{0} & \phi^{0} \\ \psi^{1} & \phi^{1} \end{pmatrix} \quad \text{thus det} \begin{pmatrix} \psi^{0} & \phi^{0} \\ \psi^{1} & \phi^{1} \end{pmatrix} = \psi^{0}\phi^{1} - \psi^{1}\phi^{0} \qquad (1\text{-}26)$$

is an invariant. One notes $\phi_a = C_{ac}\, \phi^c$, where $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and with (1-26) we obtain the invariant indefinite "inner product"

$$\langle \phi, \psi \rangle = \psi^a\, \phi_a = C_{ac}\, \psi^a\, \phi^c \; .$$

Substituting $\phi = A\phi'$ and $\psi = A\psi'$ in the scalar $\psi^T C\phi$ we have

$$A^T CA = C \text{ (det } A), \qquad (1\text{-}27)$$

which may be also verified by direct calculation or by the remark that $C_{ac}$ is the two-dimensional *Levi-Cevità* symbol, which satisfies the relation det $A = C_{ac}\, A^a{}_1\, A^c{}_2$

The spinors $\psi_a$ and $\psi^a$ transform equivalently, for they differ only by a coordinate transformation C. This is not true for the spinors $\psi^a$ and $\psi^{\dot{a}}$.

<u>Lemma 3.2.</u> *The representations $\Lambda \to \pm A(\Lambda)$ and $\Lambda \to \pm \overline{A(\Lambda)}$ of the restricted group are not equivalent.*

PROOF. From (1-27) we obtain, that for every unimodular A

$$C\bar{A}C^{-1} = (A^+)^{-1} \qquad (1\text{-}28)$$

holds. The representation $\bar{A}$ is equivalent to $(A^+)^{-1}$. For the rotation group, $(A^+)^{-1} = A$ holds and therefore $\bar{A}$ is equivalent to A and there is only *one* type of spinors. Suppose now that the representations $A(\Lambda)$ and $\overline{A(\Lambda)}$ are equivalent for arbitrary Lorentz transformations $\Lambda$.

Then there is one and only one matrix T such that

$$T(A^{+})^{-1}T = A. \qquad (1\text{-}29)$$

for all unimodular A.
Because T is commuting with the irreducible subgroup of unitary matrices, the lemma of Schur implies that necessarily $T = \lambda E$ holds, see (1-9a)
Substituting this in (1-29) it follows that $(A^{+})^{-1} = A$, which is not true for every unimodular matrix. Hence the representations $A(\Lambda)$ and $\overline{A(\Lambda)}$ of the restricted group are not equivalent.

With the spinors $\psi^{a}$ and $\psi^{\dot{a}}$, one may construct tensor (spinor) product. In order to do this we first consider spinors with respect to the subgroup $O_{3+}$ of spatial rotations.
The spinors $\psi^{a}$ and $\psi^{\dot{a}}$ are equivalent in this case and we only need to consider the spinor products

$$\psi^{a_1}\phi^{a_2}\ldots\chi^{a_r} \qquad (a_i = 0, 1)$$

of spinors $\psi^{a}$ without dot. In particular, the space of r th-rank *symmetric* spinors is spanned by spinors with components

$$\psi^{a_1}\psi^{a_2}\ldots\psi^{a_r} \qquad (a_i = 0, 1).$$

One often writes $r = 2j$ and with $\psi^{0} = u$ and $\psi^{1} = v$ these components become

$$u^{2j-k}v^{k} \qquad (k = 0, \ldots, 2j), \text{ or } u^{h}v^{k} \quad (h+k = 2j)$$

All spinors $(u^{2j-k}v^{k})$ span a $(2j+1)$-dimensional space $R^{j}$. The representation which acts in $R^{j}$ is called $D^{j}$.

<u>Theorem 3.3.</u> *Every irreducible representation of the rotation group* $O_{3+}$ *is equivalent to the spinor representation* $D^{j}$, *which acts in the* $(2j+1)$*-dimensional space* $R^{j}$ *of symmetric spinors of* $(2j)$*th-rank*

$$(\psi^{a_1\ldots a_{2j}}) \qquad (j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots)$$

PROOF. See theorem 6.2. where the theory of the infinitesimal operators is used.

Remark 3.2.

The representation acting on $(u^h v^k)$ is not unitary. In order to bring this representation in unitary form, we first observe that the representation $[A]^r = A \times A \times \ldots A$, which acts in space of spinors $(\psi^{a_1} \phi^{a_2} \ldots \chi^{a_r})$ of (2j)th-rank, is unitary because it is the tensor product of unitary matrices A. We now consider the set (h, k) of all components $(\psi^{a_1} \phi^{a_2} \ldots \chi^{a_r})$ with h indices 0 and k indices 1 (h + k = r). The transformation to the subspace of *symmetric* spinors $(\psi^{0\ldots0\,1\ldots1})$ is given by the 'symmetric' sum:

$$\binom{r}{k} \psi^{\overbrace{0\ldots0}^{h}\,\overbrace{1\ldots1}^{k}} = \sum_{\psi^{a_1}\ldots\chi^{a_r} \in (h,k)} \psi^{a_1} \ldots \chi^{a_r} \qquad (*)$$

If we put $\psi^0 = \ldots = \chi^0 = u$ and $\psi^1 = \ldots = \chi^1 = v$ we obtain

$$\psi^{\overbrace{0\ldots0}^{h}\overbrace{1\ldots1}^{k}} = u^h v^k ,$$

due to the factor $\binom{r}{k}$ which is the number of terms in the sum. Since these terms are orthonormal coordinates the sum is normalized with aid of the theorem of Pythagoras by multiplying it with the factor $\binom{r}{k}^{-\frac{1}{2}}$. Hence we obtain the (2j+1) orthonormal components

$$\sqrt{\binom{r}{k}}\ \psi^{0\ldots0\,1\ldots1} = \sqrt{(2j)!}\ \frac{u^h v^k}{\sqrt{h!k!}} \qquad (1\text{-}30a)$$

The components $u^h v^k$ are sometimes replaces by $u^{j+m} v^{j-m}$ and the orthonormal components (1-30a) by $\psi^m$ $(-j \leq m \leq +j)$ (one may again verify that the $\psi^m$ are orthonormal coordinates by the relation $\sum \psi^m \bar{\psi}^m = (u\bar{u} + v\bar{v})^{2j}$ = invariant, see Weyl p. 137.).

For a moment we restrict ourselves to the subgroup $O_{2+}$ of rotations about the z-axis ($x^3$-axis).
Formula (1-22) shows that

$$u' = e^{-i\frac{\theta}{2}} u \text{ and } v' = e^{+i\frac{\theta}{2}} v$$

and thus the representation $D^{\frac{1}{2}}$ is reducible with respect to $O_{2+}$.
In general we have the (2j+1)- coordinates

$$v^{2j}, \quad u\,v^{2j-1}, \ldots , u^{j+m}\,v^{j-m}, \ldots , u^{2j} \quad (-j \leq m \leq +j)$$

with respect to the basic vectors

$$e_{-j},\ e_{-j+1}, \ldots , e_{m}, \ldots , e_{+j} \qquad (1\text{-}30b)$$

and having the eigenvalues

$$e^{-i(-j)\theta},\ e^{-i(-j+1)\theta}, \ldots , e^{-im\theta}, \ldots , e^{-ij\theta}$$

with respect to rotations $\theta$ about the z-axis. Introducing the infinitesimal rotation $J_3 = i\left(\frac{\partial D(\theta)}{\partial\theta}\right)_{\theta=0}$ about the z-axis, see section 6, we have the following eigenvalues with respect to $J_3$

$$-j,\ -j+1, \ldots , m, \ldots +j\ . \qquad (1\text{-}30c)$$

<u>Remark 3.3.</u>

Finally we note that sometimes one considers in an equivalent way the *dual space* of linear forms $\sum a_m u^{j+m} v^{j-m}$. In this case the expressions $\psi^m = u^{j+m} v^{j-m}$ are not *components* of a vector but *functions* which span the representation space of $D^j$.

To indicate that the vectors $e_m$ span the representation space $D^j$ one often denotes these vectors with two indices $e_{jm}$, and if the vectors $e_{jm}$ $(-j \leq m \leq +j)$ are normalized then the system $e_{jm}$ is called a *canonical basis* of $D^j$. In the Dirac notation one writes $e_{jm} \equiv |jm\rangle$. The index m is called the *weight* of $e_{jm}$ and thus j is the *highest weight* which characterizes the irreducible representation $D^j$. We will now obtain such a theorem for the Lorentz group.
In section 7.2. we mention "the unitary trick of Weyl", which states that every irreducible finite-dimensional representation of the Lorentz group $L_+^\uparrow$ corresponds with an irreducible representation of the real orthogonal group $O_{4+}$ which leaves invariant the form

$$||y||^2 = y_0^{\,2} + y_1^{\,2} + y_2^{\,2} + y_3^{\,2}\ .$$

The irreducible representations of $O_{4+}$ are easy to determine. To do so, we transform the real vectors $(x_0, \vec{x})$ into

$$(y_0, \vec{y}) = (x_0, i\vec{x}).$$

The $2 \times 2$ *hermitian* matrices X from (1-19) transform into $2 \times 2$ *unitary* matrices Y. (unitary within a factor, i.e. $U^+ = \rho U^{-1}$), so that $\det Y = y_0^2 + y_1^2 + y_2^2 + y_3^2$.

An arbitrary orthogonal transformation in the real $R_4$ is given by

$$Y' = UYV ,$$

(where U and V are arbitrary unitary, unimodular matrices), for Y' is unitary too and det Y' = det Y.

So we have, cf. (1-20a, b) and (1-21)

Lemma 3.4 *The real proper orthogonal group $O_{4+}$ in 4 dimensions is (1 <—> 2) isomorphic with the tensor product $SU_2 \times SU_2'$.*

The accent denotes that the matrices U and V which appear in the tensor product are independent.

For compact groups A and B there holds that all irreducible representations of the tensor product $A \times B$ are given by all tensor products D(A) of A and D'(B) of B.

Consequently all irreducible representation $O_{4+}$ are of the form $D^{j0} \times D^{0j'}$, where $D^{j0}$ and $D^{0j'}$ are irreducible representations of $SU_2$ and $SU_2'$ respectively.

One notes $D^{jj'} = D^{j0} \times D^{0j'}$.

In a similar way one may start with spinors.

The spinors $\psi^a$ and $\psi^{\dot{a}}$ are not equivalent with respect to the Lorentz group. Starting with these spinors, we now obtain spaces of (2j, 2j')th-rank *symmetric* spinors with components

$$\underbrace{\psi^{a_1}\psi^{a_2}\ldots}_{2j} \qquad \underbrace{\psi^{\dot{c}_1}\psi^{\dot{c}_2}\ldots}_{2j'} \qquad (a_i = 0, 1 \text{ and } \dot{c}_i = \dot{0}, \dot{1}) \qquad (1\text{-}31)$$

By *symmetric* is meant symmetric in the indices "a" without dot and symmetric in the indices "$\dot{c}$" with dot.

With $\psi^0 = u$ and $\psi^1 = v$ we may also write $u^h v^k \bar{u}^{h'} \bar{v}^{k'}$ or

$$\psi^{m}\psi^{-\dot{m}'} = \frac{u^{j+m}\, v^{j-m}\, \bar{u}^{j'-m'}\, \bar{v}^{j'+m'}}{\sqrt{(j+m)!\,(j-m)!\,(j'-m')!\,(j'+m')!}}\sqrt{(2j)!\,(2j')!}\,, \qquad (1\text{-}31a)$$

... etc.

One denotes this representation by $D^{jj'}$ and the representation $D^{jj'}$ is $(2j+1)\cdot(2j'+1)$-dimensional.

Theorem 3.5. *Every finite irreducible representation of the restricted group $L_+^\uparrow$ is equivalent to the spinor representation $D^{jj'}$ ($j,j'=0,\frac{1}{2},1,\ldots$) which acts in the space of $(2j,2j')$th-rank symmetric spinors ($\psi^{a_1\ldots a_{2j}\,\dot{c}_1\ldots\dot{c}_{2j'}}$).*

Thus we obtain the following list of irreducible representations of $L_+^\uparrow$.

| Representation | acting on spaces spanned by the spinors | dim | spin values | equivalent to |
|---|---|---|---|---|
| $D^{00}$ | $c$ | 1 | 0 | scalar |
| $D^{\frac{1}{2}0}$ | $\psi^{a}_{\cdot}$ | 2 | $\frac{1}{2}$ | spinor (lemma 3.1) |
| $D^{0\frac{1}{2}}$ | $\psi^{a}$ | $\dot{2}$ | $\frac{1}{2}$ | spinor (lemma 3.2) |
| $D^{10}$ | $\psi^{a}_{\cdot}\psi^{c}_{\cdot}$ | 3 | 1 | tensor sum equivalent to anti symmetric tensors (see (1-36) |
| $D^{01}$ | $\psi^{a}\psi^{c}$ | $\dot{3}$ | 1 | |
| $D^{\frac{1}{2}\frac{1}{2}}$ | $\psi^{a}\psi^{\dot{c}}$ | 4 | 0,1 | -vectors |
| . | . | . | . | . |
| . | . | . | . | . |
| . | . | . | . | . |
| $D^{11}$ | $\psi^{a_1}\psi^{a_2}\psi^{\dot{c}_1}\psi^{\dot{c}_2}$ | 9 | 0,1,2 | symmetric tensors with trace zero |
| . | . | . | . | . |
| . | . | . | . | . |
| . | . | . | . | . |
| $D^{jj'}$ | $\psi^{a_1}\ldots\psi^{a_{2j}}\psi^{\dot{c}_1}\ldots\psi^{\dot{c}_{2j}}$ | $(2j+1)(2j'+1)$ | $\lvert j-j'\rvert,\ldots j+j'$ | |
| . | . | . | . | . |
| . | . | . | . | . |

(1-32)

The representation $D^{\frac{1}{2}\frac{1}{2}}$ is the vector representation, see formula (1-25), and by forming tensor products $D^{\frac{1}{2}\frac{1}{2}} \times D^{\frac{1}{2}\frac{1}{2}}$ (section 3.2) one may prove that the tensor representations are characterized by the fact that both j and j' are integral or half-integral.

## 3.2. Products of representations

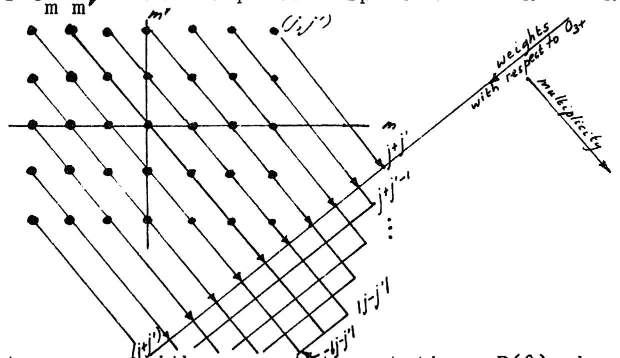
### Products of representations of $O_{3+}$

In the fourth column of (1-32) we have considered the behaviour of spinors under the rotation group $O_{3+}$. Since the spinors $\psi^{a}$ and $\psi^{\dot{a}}$ are equivalent with respect to this group (we will denote equivalence with respect to $O_{3+}$ by the symbol "~"), we may write

$$D^{j0} \sim D^{0j} \sim D^{j}$$

$$\text{and} \quad D^{jj'} = D^{j0} \times D^{0j'} \sim D^{j} \times D^{j'}.$$

The representation $D^{j} \times D^{j'}$ has the property of being reducible with respect to $O_{3+}$. To show this, we draw a "weight diagram". Every point (m,m') determines a vector $e_m e_{m'}$ from the product space $R^{jj'} = R^{j0} \times R^{0j'}$.




The eigenvalue of the vector $e_m e_{m'}$ with respect to rotations $D(\theta)$ about the z-axis is given by (see formula (1-13))

$$D(\theta)e_m e_{m'} = D(\theta)e_m \cdot D(\theta)e_{m'} = e^{-i(m+m')} e_m e_{m'} .$$

Thus the *weight* of $e_m e_{m'}$ is (m+m'). We shall write

$$e_m e_{m'} = e_n \quad (n = m+m'). \tag{1-33}$$

So the line $m + m' = n$ in the figure *(co-diagonal)* contains all vectors $e_m e_{m'}$ with weight n. Clearly there is *one* vector with weight $(j+j')$, a *two-dimensional* space of vectors with weight $(j+j')-1$, ... etc. The existence of a row of vectors with weights

$$j + j', j + j' - 1, \dots, -j - j'$$

expresses that the representation $D^j \times D^{j'}$ contains the irreducible representation $D^{j+j'}$. More precisely, we may say that by applying all rotations D(r) to the uniquely determined vector $e_j e_{j'} = e_{j+j'}$ with highest weight, an irreducible representation space of $O_{3+}$ is obtained by all $D(r)(e_{j+j'})$. The vector $e_{j+j'-1}$ is a certain linear combination of $e_{j-1}e_{j'}$ and $e_j e_{j'-1}$, ... etc. We now consider the remaining vectors. There is a second row of vectors with weights

$$-(j+j'-1), \dots, +(j+j'-1)$$

which expresses that $D^j \times D^{j'}$ also contains the irreducible representations $D^{j+j'-1}$. In this manner one obtains the famous *Clebsch-Gordan series.*

Theorem 3.6. *For irreducible representations of the rotation group hold the following product rule.*

$$D^j \times D^{j'} = \sum_{l=|j-j'|}^{j+j'} D^l. \qquad (1\text{-}34a)$$

Thus the irreducible representation $D^{jj'}$ of the Lorentz group contains the spin values $|j-j'|, \dots j+j'$, see fourth column of (1-32).

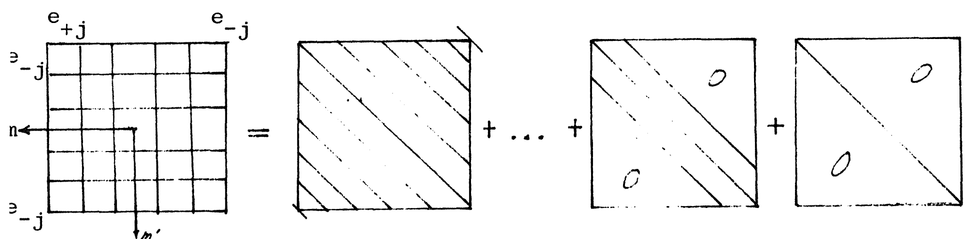
The canonical basis of the representation $D^\ell$, i.e. $e_{\ell n}$ $(-\ell \leq n \leq \ell)$ is a certain linear combination of the vectors $e_{jm} e_{j'm'}$ $(m+m' = n)$. Hence if $\ell$, j, j' are given, then there is an uniquely determined matrix $B(j,j',\ell)$ with "input" index mm' and "output" index n so that

$$e_{\ell n} = \sum_{\substack{m,m' \\ m+m'=n}} B_{\ell n\ jm\ jm'}\ e_{jm}\ e_{j'm'}\ . \qquad (1\text{-}34b)$$

The *Clebsch-Gordan coefficients* $B_{\ell n\ jm\ j'm'}$ or $(jm\ j'm'|\ell n)$ have been tabulated. Just like the 2nd order tensors $x^{\mu\nu}$ are given by a matrix, we observe that the vectors $e_{jm}e_{j'm'}$ span a space of second-order tensors ($c^{mm'}$) from which the matrix is given by the preceding figure, and are transformed by the representation $D^j \times D^{j'}$. The vectors $e_{\ell n}$ form a subspace of these second-order tensors, which we may visualize by putting the contribution $B_{\ell n\ jm\ j'm'}$ of each vector $e_{jm}e_{j'm'}$ to $e_{\ell m}$ $(m + m' = n)$ in the corresponding place $(m,m')$ in the figure. The vector $e_{\ell n}$ is thus represented by the row numbers on the line $m+m' = n$, and the space $R^\ell$ is spanned by the "lines" $n = -\ell, \ldots, +\ell$. For $\ell = j+j'$ we obtain a tensor entirely filled with non-zero numbers, and we shall call it a $(2\ell+1)$-codiagonal matrix $(\ell = j+j')$. For $\ell = j+j'-1$ we obtain a band matrix with zero's in the corners: $(j,j')$ and $-(j,j')$, which is a $(2\ell+1)$- codiagonal matrix $(\ell = j+j'-1)$, ... etc. until $\ell = |j-j'|$. Consequently, in the product basis $e_{jm}\ e_{j'm'}$ the tensor $c_{jm}\ c_{j'm'}\ e_{jm}\ e_{j'm'}$ reduces into the $(2\ell+1)$-codiagonal matrices:

$$\ell = j+j', \ldots, |j-j'|.$$

Example, reduction of $D^j \times D^j$:




(1-34c)

2nd-order tensor (2j+1)-vector 3-vector scalar

We shall use this result in section 7.3.

<u>Product of representations of the restricted Lorentz group $L_+^\uparrow$</u>

From formula (1-34a) we may obtain the following product rules for the restricted group:

$$D^{j0} \times D^{k0} = \sum_{\ell=|j-k|}^{j+k} D^{\ell 0}$$

and
$$D^{0j'} \times D^{0k'} = \sum_{\ell'=|j'-k'|}^{j'+k'} D^{0\ell'} .$$

In general one obtains

$$\begin{aligned} D^{jj'} \times D^{kk'} &= (D^{j0} \times D^{0j'}) \times (D^{k0} \times D^{0k'}) \\ &= (D^{j0} \times D^{k0}) \times (D^{0j'} \times D^{0k'}) \\ &= \sum_{\ell=|j-k|}^{j+k} D^{\ell 0} \times \sum_{\ell'=|j'-k'|}^{j'+k'} D^{0\ell'} \end{aligned}$$

so we have:

Theorem 3.7. *For irreducible representations of the restricted Lorentz group holds the following relation, i.e.*

$$D^{jj'} \times D^{kk'} = \sum_{l=|j-k|}^{j+k} \sum_{l'=|j'-k'|}^{j'+k'} D^{ll'} . \qquad (1\text{-}35)$$

In particular, the second-order tensor representation $D^{\frac{1}{2}\frac{1}{2}} \times D^{\frac{1}{2}\frac{1}{2}}$ decomposes as

$$D^{\frac{1}{2}\frac{1}{2}} \times D^{\frac{1}{2}\frac{1}{2}} = D^{11} \dot{+} (D^{10} \dot{+} D^{01}) + D^{00}. \qquad (1\text{-}36)$$

Thus the second-rank tensor representation decomposes into a 9-, 3-, $\dot{3}$- and 1-dimensional irreducible representation of the restricted group. They are the three irreducible representations, which work in the spaces of symmetric tensors with "trace" zero, anti-symmetric tensors (of type G and $\dot{G}$) and the metric tensor respectively (see p. 15)

## 3.3. Spinor representations of the full group L.

Introduction, bispinors.

We now consider the effect of reflections in the space of spinors. Before developing the general theory, some introductory remarks will be made first: A space reflection

$$P : x(x^0, \vec{x}) \to x'(x^0, -\vec{x})$$

may be written with the aid of the 2 × 2 matrices X as,

$$P : X \to X' = C\bar{X}C^{-1}, \text{ where } C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (1\text{-}37a)$$

or in components $P : X^{a\dot{c}} \to X^{a'\dot{c}'} = X_{\dot{a}c}$

Therefore the space of spinors, is transformed by

$$P: \begin{cases} \psi^a \to \psi^{a'} = \alpha\, \overline{C_{ac}\psi^c} = \alpha\, \psi_{\dot{a}}, \text{ where } \alpha \text{ is arbitrary complex} & (1\text{-}37b) \\ \psi^{\dot{c}} \to \psi^{\dot{c}'} = \alpha^{-1}\, \overline{C_{\dot{a}\dot{c}}\, \psi^{\dot{c}}} = \alpha^{-1}\, \psi_c & (1\text{-}37c) \end{cases}$$

Lowering the index $\dot{c}$, and using the fact that $C^2 = -1$, we have for

$$\psi_{\dot{c}} \to \psi_{\dot{c}'} = -\alpha^{-1}\, \psi^c \qquad (1\text{-}37d)$$

Accordingly to (1-37) the 2 × 2 transformation matrices A are transformed by

$$P : A \to C\bar{A}C^{-1} = (A^+)^{-1} ,$$

see formula (1-28).

Thus in the 2-dimensional space of spinors a space reflection is represented by the matrix C and complex conjugation. Yet the last operation is not a linear operation, *so that the space of 2-component spinors ($\psi^a$) is not suitable for a linear representation of the full group L.* However, by introducing 4-component spinors ($\psi^a$, $\psi_{\dot{a}}$) a 4-dimensional representation space of the full group is obtained. The representation matrices which act on ($\psi^a$, $\phi_{\dot{a}}$) are of the form

$$D(\Lambda) = \begin{bmatrix} A & 0 \\ 0 & C\bar{A}C^{-1} \end{bmatrix}, \quad D(P) = \begin{bmatrix} 0 & iE \\ iE & 0 \end{bmatrix}, \quad D(T) = \begin{bmatrix} 0 & iE \\ -iE & 0 \end{bmatrix} \qquad (1\text{-}38)$$

We have taken $\alpha = i$ in D(P), then the Dirac equation which is invariant with respect to D(P) can be written in the simple form (1-52)
If one takes $\alpha = 1$ then an extra negative sign must be placed in the lowest row of (1-52).[*)] With respect to the restricted group $L_+^\uparrow$ this representatic $D(\Lambda)$ is reducible into $D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}$, and with respect to the orthochroneous group $L^\uparrow$ this representation is irreducible. The proof is based on the lemma of Schur, for one may show with formulae (1-9a, b) that a matrix $S = \begin{pmatrix} Q & T \\ R & S \end{pmatrix}$ which commutes with all $D(\Lambda)$ and D(P) is necessarily a multiple of the unit matrix. By which follows that the representation $D(L^\uparrow)$ is irreducible.
Considering the full group, we note that D(P) D(T) = -D(T) D(P). Consequently in order that $D(S_\mu)$ is a representation of the *commutative* reflection group, it follows that $D(S_\mu)$ is essentially two-valued, i.e.

$$E,\ P,\ T,\ J \to \pm D(E),\ \pm D(P),\ \pm D(T),\ \pm D(J).$$

One obtains also a representation of the full group if one takes $D(T) = \pm D(P)$, however it follows from (1-37) that this representation belongs to the *time-pseudo* vectors $X^{a\dot{c}}$. In general there holds that

$$P : D^{jj'} \to D^{j'j} \text{ or } P : D(\Lambda) \to D(P\Lambda P^{-1}) = D((\Lambda^{-1})^T), \qquad (1\text{-}38a)$$

$D(\Lambda^{-1})^T$ is called the *conjugate representation,* of $D(\Lambda)$, and one may introduce *bispinors* i.e.
$(\psi^{a_1 \ldots a_{2j}}{}_{\dot{c}_1 \ldots \dot{c}_{2j'}}, \psi_{\dot{a}_1 \ldots \dot{a}_{2j}}{}^{c_1 \ldots c_{2j'}})$ transforming by the representation $D^{jj'} \dotplus D^{j'j}$ and which is irreducible with respect to the orthochroneous group $L^\uparrow$ if $j \neq j'$. Only in the special case $(j = j')$ a reflection has the form

$$D(P):\ \psi^{a_1 \ldots a_{2j}}_{\dot{c}_1 \ldots \dot{c}_{2j}} \to \psi^{c_1 \ldots c_{2j}}_{\dot{a}_1 \ldots \dot{a}_{2j}}(\alpha)$$

---

*) Sometimes one defines the operation of lowering indices in the space of dotted spinors by $(\psi_{\dot{a}}) = -C(\psi^a)$. Then (1-37d) becomes $\psi_{\dot{c}} \to \alpha^{-1}\,\psi^c$ and with $\alpha = 1$, one has $D(P) = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$

General theory

For a general treatment of the irreducible representations of the full group we consider a representation $D^{jj'}$ of the restricted group $L_+^\uparrow$ and put the question in which way this representation may be extended to an irreducible representation D of the full group L. There are two ways of doing this:

1. The mapping of the reflections $S_\mu$ = E, P, T, J onto D(E), D(P), D(T), D(J) is *single valued* or can be made single valued.

By virtue of the homomorphism, the operators $D(S_\mu)$ form a *commutative group* group i.e.

$$D(P)D(T) = D(T)D(P). \tag{1-39}$$

In particular $\quad D(L)D(J) = D(J)D(L)$

holds for every full Lorentz transformation L and because we require that D(L) is irreducible it follows from the lemma of Schur that

$$D(J) = \pm E$$

$(D^2(J) = E)$ and with (1-39), that

$$D(T) = \pm D(P). \tag{1-40}$$

Hence, of the 4 reflections $S_\mu$ only the operator D(P) is essential and the representations D(L) of the full group L is simultaneously irreducible with the induced representation $D(L\uparrow)$ of the orthochroneous group. Now we consider the induced representation $D(L_+^\uparrow)$ of the restricted group. The last one needs not be irreducible but may have the form $D(L_+^\uparrow) = D^{jj'} \dotplus \ldots$ There exist two possibilities.

1a. The representation $D(L_+^\uparrow)$ is also irreducible, so that it has the form $D(L_+^\uparrow) = D^{jj'}$. In other words the irreducible representation spaces of $L\uparrow$ and $L_+^\uparrow$ coïncide. It follows from (1-38a) that the relation $D(L_+^\uparrow) = D^{jj'}$ if and only if j = j' holds. Further one may prove that the reflection D(P) is necessarily equal to a certain matrix S, determined within sign :

$$D(P) = \pm \tilde{S}. \tag{1-41}$$

The different signs in (1-41) determine two non-equivalent representations $D^{jj+}$, $D^{jj-}$ of the group $L\uparrow$.

For, suppose that they are equivalent. Then there is exactly one coordinate transformation T with

$$T(D^{jj+})\,T^{-1} = D^{jj-}$$

$$\text{and} \qquad T(\tilde{S})\,T^{-1} = -\tilde{S} \qquad (1\text{-}42)$$

Because T commutes with all irreducible operators $D^{jj}(\Lambda)$ of the group, it follows that $T = \lambda E$. The condition (1-42) with $T = \lambda E$ however is not true. Hence $D^{jj+}$ and $D^{jj-}$ are not equivalent. Thus by choice of the different signs in (1-40) and (1-41) one obtains four non-equivalent irreducible representations of the full group L. E.g., beside the vector representation $D^{\frac{1}{2}\frac{1}{2}}$ there exist the pseudo vector, time-pseudo vector and space-pseudo vector representations, see (1-18). In this case the operator $\tilde{S}$ is given by P = g .

1b. In the second case the representation $D(L_+^\uparrow)$, is reducible thus $j \neq j'$, and has the form $D(L_+) = D^{jj'} \dotplus D^{j'j}$ $(j \neq j')$.

In this case the representation matrices have the following form:

$$D(\Lambda) = \begin{pmatrix} D^{jj'}(\Lambda) & 0 \\ 0 & D^{j'j}(\Lambda) \end{pmatrix} \qquad D(P) = \begin{pmatrix} 0 & \tilde{S} \\ \tilde{S} & 0 \end{pmatrix}$$

(see formula (1-38)).

Because there are two possible signs in (1-40) one now obtains two non-equivalent representations of the full group L.

2. It is possible that the representation matrices are anti-commuting, i.e.

$$D(P)D(T) = -\,D(T)D(P).$$

Because the reflection group is commutative, this case is only possible if the representation of the reflection group is essentially two valued i.e.

$$E,\ P,\ T,\ J \rightarrow \pm\, D(E),\ \pm\, D(P),\ \pm\, D(T),\ \pm\, D(J).$$

The operators D(S ) may be brought in the following form

$$D(P) = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \quad D(T) = \begin{pmatrix} 0 & iE \\ -iE & 0 \end{pmatrix} \quad D(J) = \begin{pmatrix} -iE & 0 \\ 0 & iE \end{pmatrix}.$$

For detailed information see Gel'fand p. 18? and p. 207. Especially the theory of reflections in the space of 4-component spinors see Rzewuski p. 71-90.

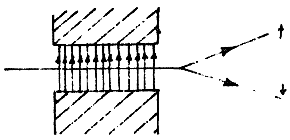
## 3.4. Applications of the representation theory in elementary particle physics.

We conclude this section with a short and simple note on the use of the representation theory of the Lorentz group into elementary particle physics, anticipating section 4 and 8.

In the beginning of this section, we have remarked that by classifying all irreducible representations of the Lorentz group one obtains a classification of all elementary particles with respect to their space-time behaviour.

We give some examples.

1.



If a beam of electrons is sent through an inhomogenous magnetic field, the beam splits into two separate parts.

(A simplified Stern-Gerlach experiment, see Feynman[III] p. 6-1 ff.)

This is explained by the assumption that electrons can only have an angular momentum: spin-up or spin-down, thus angular momentum is quantized, and electrons have an internal freedom of dimension two with respect to spatial rotations (spin $j = \frac{1}{2}$). From this it follows that $\psi(x)$, which describes the behaviour of electrons, transforms by the representation $D^{\frac{1}{2}0}$ or $D^{0\frac{1}{2}}$, which contains the spin value $j = \frac{1}{2}$, see (1-32). It appears that the properties of electrons are invariant with respect to spatial reflection, and thus in *relativistic* quantummechanics one describes electrons (and protons, neutrons) by the representation

$$D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}.$$

The function $\psi(x)$ appears in the well-known *equations of Dirac*. The fact that $\psi$ is four-dimensional is related to the fact that $\psi$ describes negatively charged *electrons* as well as the positively charged *positrons*.

2. On the contrary, spinors $\phi$ which transform by the representation $D^{\frac{1}{2}0}$ only, are used for describing *neutrino's*. This is a particle of spin $\frac{1}{2}$ and zero-mass, which travels with the velocity of light. The function appears in the so-called *equation of Weyl*, which is not invariant under

space reflection. This equation will be discussed in the next section. The discovery that not all laws of nature are invariant under space reflection was made in 1956 and is called the violation of parity.

3. In the same way a particle, with *non-zero* mass and with spin 1 has an internal freedom of dimension 3, e.g. a *H-atom in the ground-state* and so-called *vector-mesons.*

By a Stern-Gerlach experiment a beam of such particles will split into three separated beams. In the case of an H-atom the magnetic field must be infinitly small. See Feynman, p. 5-1 ff. and p. 12-13.

One may describe such particles by representations which contain the representation $D^{10} \dot{+} D^{01}$. Yet, in order to construct a Lorentz invariant equation for $\psi$ one needs an additional field $\phi^{\mu}(x)$, which satisfies the so-called *supplementary conditions* and which transforms under the representation $D^{\frac{1}{2}\frac{1}{2}}$ (see §8). Hence, particles with spin 1 and non-zero mass may be described by the representation

$$D^{\frac{1}{2}\frac{1}{2}} \dot{+} D^{10} \dot{+} D^{01}.$$

The corresponding equation is *the equation of Proca,* which will be considered in the next section.

4. Similarly particles with spin $\frac{3}{2}$ may be described by the representation

$$D^{\frac{1}{2}0} \dot{+} D^{0\frac{1}{2}} \dot{+} D^{1\frac{1}{2}} \dot{+} D^{\frac{1}{2}1}.$$

Here the function $\psi$ satisfies the so-called *Pauli-Fierz equation.* The spin value $j = \frac{3}{2}$ is contained in the last two terms and the supplementary conditions are responsible for the terms $D^{\frac{1}{2}0} \dot{+} D^{0\frac{1}{2}}$ (see section 8)

5. There exist also particles with spin 1 and *zero* mass viz. light particles or *photons,* which bear the energy of the electromagnetic field. Their behaviour is described by the *equation of Maxwell* in which the six-dimensional representation

$$D^{10} \dot{+} D^{01}$$

appears (see section 2).

6. Finally we mention that nowadays the pseudo representations play

an important role too. For instance the *pi-meson* is a spinless particle, thus it may be described by the representation $D^{00}$ which works on scalar functions $\psi(x^0,\vec{x})$. Experiments show that $\psi(x^0,-\vec{x}) = -\psi(x^0,\vec{x})$ must hold so that the pi-meson is a so-called pseudo-scalar particle.
Scalar particles may be described by the *Klein-Gordon equation.*
For further details see Roman p. 256.
In the next section we shall give a detailed discussion of the equations of Weyl, Dirac, Maxwell, Proca mentioned above.
After the classification of all representations of a group, the next task of the representation theory as far as "physical" groups are concerned, is to describe all partial linear differential equations, which are invariant with respect to that group.

## 4. Lorentz covariant equations

First we introduce the concept of a *field*, by considering in particular the tensor field $f^{\mu\nu}(x)$. This field is a 16-component function defined on $R_4$, which is transformed by a Lorentz transformation $x' = Lx$ as

$$f^{\mu'\nu'}(x') = (L \times L) f^{\mu\nu}(x). \qquad *)$$

Starting with tensor fields and remarking that $\partial_\mu$ is also a vector (section 2.) one may construct Lorentz covariant equations simply by coupling lower and upper indices. For instance the formula

$$\partial_\mu f^{\mu\nu}(x) = g^\nu(x)$$

transforms after a Lorentz transformation into

$$\partial_{\mu'} f^{\mu'\nu'}(x') = g^{\nu'}(x')$$

the proof of which depends on formula (1-11).

The fact that the components on the left- and the right-hand side of an equation transform in the same manner is expressed by saying that the equation is *Lorentz covariant*, and so Lorentz covariance implies that the *form* of these equations remains *invariant*. In general, one requires that all physical laws are independent of the choice of our coordinate system and therefore one requires that they are expressible as Lorentz covariant equations.

Below we shall give some examples of Lorentz covariant equations, constructed in this way. The more general theory will be given in section 8.

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*) It is important to remark, that the name "tensor field" denotes the behaviour of the space of *function-values*, thus the field $f^{\mu\nu}(x)$ is a second-rank tensor field. However, the space H of *functions* $f^{\mu\nu}$ is in general infinite-dimensional and transforms by

$$f^{\mu\nu}(x) \rightarrow f^{\mu'\nu'}(x) = L \times L\, f\ (L^{-1}x).$$

This representation is *not* equivalent with $L \times L$. For instance with respect to spatial rotations, the spherical functions $Y(\vec{x})$ are *scalar*-functions, because $Y(\vec{x})$ has only one component. Yet the space H of all functions $Y(\vec{x})$ is infinite-dimensional, spanned by the spherical functions $Y_{lm}(\vec{x})$, which supply all representations $D_l$ of the rotation group with integer $l = 0, 1, 2, \ldots$ .

4.1. Tensor equations

1. We consider the equation of *continuity of electric charge*

$$\frac{1}{c}\dot{\rho} + \nabla\vec{j} = 0. \qquad (1\text{-}43)$$

After substitution of $\partial_\mu = (\frac{\partial}{c\partial t},\nabla)$ and $j^\mu = (\rho,\vec{j})$, which is a vector see formula (1-8), there follows

$$\partial_\mu j^\mu = 0. \qquad (1\text{-}44)$$

Consequently the equation of continuity of electric charge is Lorentz covariant.

2. The *equations of Maxwell* in vacuum are:

$$\left.\begin{aligned} \nabla \cdot \vec{E} &= \rho \\ \nabla \times \vec{H} - \frac{1}{c}\dot{\vec{E}} &= \vec{j} \end{aligned}\right\} \qquad (1\text{-}45a)$$

$$\left.\begin{aligned} \nabla \cdot \vec{H} &= 0 \\ \nabla \times \vec{E} + \frac{1}{c}\dot{\vec{H}} &= 0 \end{aligned}\right\}. \qquad (1\text{-}45b)$$

We substitute $F^{\mu\nu} \equiv (\vec{E},\vec{H})$ in equation (1-45a) and $\overline{F^{\mu\nu}}(\vec{H},-\vec{E})$ in equation (1-45b) and obtain

$$\partial_\nu F^{\mu\nu} = j^\mu \qquad (1\text{-}46a)$$

$$\partial_\nu \overline{F^{\mu\nu}} = 0 \qquad (1\text{-}46b)$$

from which follows the covariance of Maxwell equations. (The proof that $\overline{F^{\mu\nu}}$ is also a covariant tensor is given in chapter II, appendix theorem B-1.) Inversely if one requires a priori that the equations of Maxwell are covariant, it follows that $F^{\mu\nu}(\vec{E},\vec{H})$ is an antisymmetric *tensor* which transforms by the representation $D^{10} \dot{+} D^{01}$.

In absence of charge there holds $j^\mu = 0$ and if we introduce the antisymmetric tensors

$$G^{\mu\nu} = F^{\mu\nu} + i\bar{\bar{F}}^{\mu\nu} \text{ thus } G^{\mu\nu} = (\vec{G}, -i\vec{G}) \text{ where } \vec{G} = \vec{E} + i\vec{H}$$

$$\dot{G}^{\mu\nu} = F^{\mu\nu} - i\bar{\bar{F}}^{\mu\nu} \text{ thus } \dot{G}^{\mu\nu} = (\dot{\vec{G}}, -i\dot{\vec{G}}) \text{ where } \dot{\vec{G}} = \vec{E} - i\vec{H}$$

see formula (1-16a,b,c), one may write equation (1-46a) and (1-46b) as

$$\partial_\nu G^{\mu\nu} = 0,$$

$$\partial_\nu \dot{G}^{\mu\nu} = 0.$$

The tensors $G^{\mu\nu}$ and $\dot{G}^{\mu\nu}$ are transformed by the representations $D^{10}$ and $D^{01}$ respectively. If we restrict ourselves to *real* $x^\mu$, $E^i$, $H^i$ then (1-46a) and (1-46b) may be written as one equation, i.e.

$$\partial_\nu G^{\mu\nu} = 0$$

in components:

$$(\partial_0, \partial_1, \partial_2, \partial_3) \begin{pmatrix} 0 & -G^1 & -G^2 & -G^3 \\ G^1 & 0 & iG^3 & -iG^2 \\ G^2 & -iG^3 & 0 & iG^1 \\ G^3 & iG^2 & -iG^1 & 0 \end{pmatrix} = 0 \qquad \text{(1-47a)}$$

and interchanging the role of $\partial_\mu$ and $G^{\mu\nu}$ one obtains

$$\begin{pmatrix} -\partial_0 & \partial_1 & \partial_2 & \partial_3 \\ -\partial_1 & -\partial_0 & i\partial_3 & -i\partial_2 \\ -\partial_2 & -i\partial_3 & -\partial_0 & i\partial_1 \\ -\partial_3 & +i\partial_2 & -i\partial_1 & -\partial_0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ G^1 \\ G^2 \\ G^3 \end{pmatrix} = 0. \qquad \text{(1-47b)}$$

This equation is written in abbreviated form as

$$(\beta^\mu \partial_\mu)\psi = 0.$$

The $\beta^\mu$ are 4 × 4 matrices and follow from (1-47b).
In general, equations of the form

$$(\beta^\mu \partial_\mu + i\kappa)\psi = 0,$$

where $\beta^\mu$ are $n \times n$ matrices and $\kappa$ a scalar, are called Lorentz covariant equations in *Bhabha form* (see section 8).

3. *The Proca-equation.* In this equation the 10-component quantity $\psi \equiv (\phi^\mu, F^{\mu\nu})$ appears which is transformed by the representation $D^{\frac{1}{2}\frac{1}{2}} \dot{+} D^{10} \dot{+} D^{01}$ i.e.

$$\partial^\nu F_{\mu\nu} = -\kappa^2 \phi_\mu \tag{1-48a}$$

$$\partial_\nu \phi_\mu - \partial_\mu \phi_\nu = F_{\mu\nu}$$

If $\kappa \neq 0$, this equation may be written in the following abbreviated form

$$\begin{array}{c} \\ 4 \\ 6 \end{array} \begin{pmatrix} \overset{\longleftarrow 4 \longrightarrow}{0} & \overset{\longleftarrow 6 \longrightarrow}{D^T} \\ D & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \frac{1}{-i\kappa} F \end{pmatrix} = -i\kappa \begin{pmatrix} \phi \\ \frac{1}{-i\kappa} F \end{pmatrix} . \tag{1-48b}$$

We may consider for instance the lowest row. The $6 \times 4$ matrix D contains the symbols $\partial_\nu$ in such a way that the vector $D\phi$ has components $\partial_1\phi_0 - \partial_0\phi_1 = F_{01}$, ... etc. It follows that the vector $D^T F$ is equal to $\partial^\nu F_{\mu\nu}$ (after the subsitution $(\partial_0, \partial_i) \rightarrow (\partial^0, -\partial^i)$). In components we obtain:

$$\left(\begin{array}{cccc|ccc|ccc}
 & & & & \partial^1 & \partial^2 & \partial^3 & \cdot & \cdot & \cdot \\
 & & & & -\partial^0 & \cdot & \cdot & \cdot & -\partial^3 & \partial^2 \\
 & & & & \cdot & -\partial^0 & \cdot & \partial^3 & \cdot & -\partial^1 \\
 & & & & \cdot & \cdot & -\partial^0 & -\partial^2 & \partial^1 & \cdot \\
\hline
\partial^1 & -\partial^0 & \cdot & \cdot & & & & & & \\
\partial^2 & \cdot & -\partial^0 & \cdot & & & & & & \\
\partial^3 & \cdot & \cdot & -\partial^0 & & & & & & \\
\hline
\cdot & \cdot & \partial^3 & -\partial^2 & & & & & & \\
\cdot & -\partial^3 & \cdot & \partial^1 & & & & & & \\
\cdot & \partial^2 & -\partial^1 & \cdot & & & & & &
\end{array}\right)
\begin{pmatrix} \phi^0 \\ \phi^1 \\ \phi^2 \\ \phi^3 \\ \frac{1}{-i\kappa} F^{01} \\ \frac{1}{-i\kappa} F^{02} \\ \frac{1}{-i\kappa} F^{03} \\ \frac{1}{-i\kappa} F^{23} \\ \frac{1}{-i\kappa} F^{31} \\ \frac{1}{-i\kappa} F^{12} \end{pmatrix}
= -i\kappa
\begin{pmatrix} \phi^0 \\ \phi^1 \\ \phi^2 \\ \phi^3 \\ \frac{1}{-i\kappa} F^{01} \\ \frac{1}{-i\kappa} F^{02} \\ \frac{1}{-i\kappa} F^{03} \\ \frac{1}{-i\kappa} F^{23} \\ \frac{1}{-i\kappa} F^{31} \\ \frac{1}{-i\kappa} F^{12} \end{pmatrix} \tag{1-48c}$$

Thus in Bhabha form the equations of Proca are written as

$$(\beta^{\mu}\partial_{\mu} + i\kappa)\psi = 0 ,$$

where the 10 x 10 matrices $\beta^{\mu}$ are the so-called *Kemmer matrices* which generate a group of 126 elements (Roman p. 147).

4. For the sake of completeness we mention the second order *Klein-Gordon* equation

$$(\partial^{\mu}\partial_{\mu} + \kappa^{2})\psi = 0 \quad \text{or} \quad (\Box + \kappa^{2})\psi = 0 \tag{1-49}$$

which is a relativistic generalization of the Schrödinger equation.

4.2. Spinor equations. The equations of Dirac.

The operator $\partial_\mu$ transforms as a vector $x_\mu$. Because $x_\mu$ corresponds with the 2x2 matrix $X \equiv X_{\dot{a}c}$ (formula (1-37a)) the matrix

$$D = \begin{pmatrix} \partial_0 + \partial_3 & \partial_1 - i\partial_2 \\ \partial_1 + i\partial_2 & \partial_0 - \partial_3 \end{pmatrix}$$

transforms as the matrix X and so one denotes the components of D by $\partial_{\dot{a}c}$. Similarly as done in the previous section we may construct covariant spinor equations by contractions with respect to lower with upper indices.

1. Equation of Weyl

This is the simplestequation in spinor calculus viz. $D\psi = 0$. Written out in components

$$\partial_{\dot{a}c}\psi^c = 0. \tag{1-50}$$

Using the Pauli matrices $\sigma^\mu$, one obtains $D = \partial_\mu\sigma^\mu$ and

$$(\sigma^\mu\partial_\mu)\psi = 0, \tag{1-51}$$

which is the equation of Weyl in Bhabha-form *).

2. Equation of Dirac

Now we consider an inhomogenous equation in which the four-component spinor $(\phi^a, \chi_{\dot{c}})$ appears, i.e.

$$\begin{aligned} \partial^{a\dot{c}}\chi_{\dot{c}} &= -i\kappa\,\phi^a \\ \partial_{\dot{c}a}\phi^a &= -i\kappa\,\chi_{\dot{c}} \end{aligned} \quad \text{or} \quad \begin{pmatrix} 0 & \partial^{a\dot{c}} \\ \partial_{\dot{c}a} & 0 \end{pmatrix} \begin{pmatrix} \phi^a \\ \chi_{\dot{c}} \end{pmatrix} = -i\kappa \begin{pmatrix} \phi^a \\ \chi_{\dot{c}} \end{pmatrix} \tag{1-52}$$

This equation is known as the *equation of Dirac* and due to the notation of upper and lower indices the covariance of the Dirac equations is clear. We remark that out of $\partial_{\dot{c}a}$ we may construct $\partial^{a\dot{c}}$ by the transformation $(\partial_0, \partial_i) \to (\partial_0, -\partial_i)$. Thus in components,

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*) The components $\sigma^\mu{}_{\dot{a}c}$ of $\sigma^\mu$ are obtained from the matrices $\sigma_\mu$ with components $\sigma_\mu{}^{a\dot{c}}$ (expression (1-23)) by raising $\mu$, lowering of $a\dot{c}$, and transposing the dot, which is the inverse operation. Thus for the matrices $\sigma^\mu$ from equation (1-51) there holds $\sigma^\mu = \sigma_\mu$.

$$\begin{pmatrix} 0 & 0 & \partial_0-\partial_3 & -\partial_1+i\partial_2 \\ 0 & 0 & -\partial_1-i\partial_2 & \partial_0+\partial_3 \\ \partial_0+\partial_3 & \partial_1-i\partial_2 & 0 & 0 \\ \partial_1+i\partial_2 & \partial_0-\partial_3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \phi^0 \\ \phi^1 \\ \chi_{\dot{0}} \\ \chi_{\dot{1}} \end{pmatrix} + i\kappa \begin{pmatrix} \phi^0 \\ \phi^1 \\ \chi_{\dot{0}} \\ \chi_{\dot{1}} \end{pmatrix} = 0 ,$$

or abbreviated in Bhabhaform

$$(\gamma^\mu \partial_\mu + ik) = 0 \tag{1-53}$$

The matrices $\gamma^\mu$, where

$$\gamma^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} \text{ and } \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix} , \tag{1-54}$$

are known as the Dirac matrices. They satisfy the following relations,

$$(\gamma^o)^2 = 1, \; (\gamma^1)^2 = -1$$

$$\gamma^0\gamma^1 = -\gamma^1\gamma^0, \; \ldots \text{ etc.}$$

(the $\gamma$-matrices are anti-commuting).
Combining these properties one may write

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}. \tag{1-55}$$

From this one may verify that the $\gamma$-matrices have the fundamental property

$$(\gamma^\mu x_\mu)^2 = x^\mu x_\mu .$$

It is understood that on the right side multiplication with the unit-matrix is carried out. Applying the operator $(\gamma^\mu \partial_\mu - i\kappa)$ on the expression (1-53) one obtains with (1-55)

$$(\partial^\mu \partial_\mu + \kappa^2)\psi = 0 \text{ or } (\Box + \kappa^2)\psi = 0.$$

Thus every component of $\psi$ satisfies the Klein-Gordon equation.
Historically the aim of Dirac was to decompose the Klein-Gordon equation into two first order equations. Therefore he introduced four symbolical quantities $\gamma^\mu$ which apparently had to satisfy (1-55) and by this it

followed that the $\gamma^\mu$ should be $4 \times 4$ matrices.
The proof is an elegant application of the representation theory of finite groups and will also be used for proving the covariance of the Dirac equations independently of spinor calculus. The following is a sketch of the proof.
We have seen the $\gamma$-matrices generate, with respect to addition and multiplication, a *ring*, the so-called *Dirac-ring*.
Inversely, we shall show here that the conditions (1-55) are sufficient to determine the $\gamma$-matrices uniquely.
We suppose that the $\gamma$-matrices are unknown. We have only four elements of an *abstract* Dirac-ring. Thus multiplication and addition are defined and the symbols $\gamma^\mu$ satisfy the relation (1-55).
With respect to multiplication, these $\gamma^\mu$ generate a group G of 32 elements $\gamma$, i.e.

$$\underset{2}{\pm 1}, \ \underset{8}{\pm \gamma^\mu}, \ \underset{12}{(\pm \gamma^0\gamma^1, \ldots, \pm \gamma^2\gamma^3)}, \ \underset{8}{(\pm \gamma^0\gamma^1\gamma^2, \ldots, \pm \gamma^1\gamma^2\gamma^3)}, \ \underset{2}{\pm \gamma^0\gamma^1\gamma^2\gamma^3} \qquad (1\text{-}56)$$

We shall denote an arbitrary element of G by $\gamma$.
The problem is now which matrix representations $M(\gamma)$ of the Dirac ring, and in particular of the group G, are possible.
<u>Lemma 4.1</u> *trace $M(\gamma) = 0$ for every $\gamma \neq \pm 1$.*
PROOF. First we remark that for every $\gamma \neq \pm 1$ there is a $\delta \in G$ such that $\delta\gamma\delta^{-1} = -\gamma$.
For instance if one takes $\gamma = \gamma^0$ then one has $\delta = \gamma^0\gamma^1$ and if one takes $\gamma = \gamma^0\gamma^1$ then $\delta = \gamma^1\gamma^2$, ... etc.
It follows that $\quad M(\delta)\, M(\gamma)\, M^{-1}(\delta) = -M(\gamma)$
and thus $\quad \text{trace } M(\gamma) = - \text{trace } M(\gamma)$,
hence trace $M(\gamma) = 0$ if $\gamma \neq \pm 1$.
<u>Theorem 4.2, (theorem of Pauli or van der Waerden)</u> *There is only one irreducible representation of the Dirac ring and this representation is four-dimensional.*
PROOF. Suppose one has two irreducible presentations $M^i$ and $M^j$ which are respectively n- and n'-dimensional. On the group G one defines the

character $\chi(\gamma)$ as trace $M(\gamma)$. For finite groups, the orthonormality relations

$$\sum_{\gamma} \chi^{i}(\gamma)\,\overline{\chi^{j}(\gamma)} = \delta^{ij} g \text{ (g = order of the group)} \qquad (1\text{-}57)$$

hold.
Equations (1-57) form a criterion for equivalence. If $M^i$ and $M^j$ are non-equivalent ($i \neq j$) then this sum is equal to zero and if $M^i$ and $M^j$ are equivalent ($i = j$) this sum is equal to the order g of the group. Now in the case of the group G (formula (1-56)) it holds that trace $M(\gamma) = 0$ for $\gamma \neq \pm 1$. Thus all terms, except those with $\gamma = \pm 1$, vanish. It follows that the sum (1-57) is equal to 2nn'.
Since $2nn' \neq 0$, the representations $M^i$ and $M^j$ are necessarily equivalent and $n = n'$.
The order g of the group is 32, hence from (1-57) we have

$$2n^2 = 32$$

and it follows that $n = 4$.
In this way we see that the only irreducible representation of the $\gamma$-matrices is four-dimensional. (Dewitt p. 117).

The theorem mentioned above implies that if there are four matrices $\gamma^{\mu'}$ which satisfy the relations (1-55), then there is necessarily a coordinate transformation S such that

$$\gamma^{\mu'} = S^{-1}\,\gamma^{\mu}\,S$$

where the $\gamma^{\mu}$ are given by (1-54).
<u>Theorem 4.3.</u> *The matrices $\gamma^{\mu'} = L^{\mu'}{}_{\mu}\gamma^{\mu}$, where $L^{\mu'}{}_{\mu}$ is a Lorentz transformation, are again Dirac matrices, and $L^{\mu'}{}_{\mu}\gamma^{\mu} = S^{-1}(L)\gamma^{\mu}S(L)$ holds*
PROOF. We have to prove that the $\gamma^{\mu'}$ satisfy the relations (1-55)
i.e. $\gamma^{\mu'}\gamma^{\nu'} + \gamma^{\nu'}\gamma^{\mu'} = L^{\mu'}{}_{\mu}L^{\nu'}{}_{\nu}\,(\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu})$

$$= 2L^{\mu'}{}_{\mu}\,L^{\nu'}{}_{\nu}g^{\mu\nu} \qquad \text{(see formula (1-55))}$$

$$= 2g^{\mu'\nu'} \qquad \text{(see formula (1-2))}.$$

With the theorem mentioned above we may show the covariance of the Dirac

equations without using spinor calculus.
To do this we substitute in the Dirac equation

$$(\gamma^{\mu}\partial_{\mu} + i\kappa)\ \psi(x) = 0 \tag{1-58}$$

$x = L^{-1}x'$ and $\partial = L^{T}\partial'$, and we obtain

$$(L^{\mu'}{}_{\mu}\ \gamma^{\mu}\partial_{\mu'} + i\kappa)\ \psi(L^{-1}x') = 0.$$

The matrices $\gamma^{\mu'} = L^{\mu'}{}_{\mu}\gamma^{\mu}$ are Dirac matrices and so there is a coordinate transformation $S(L)$ such that

$$\gamma^{\mu'} = S^{-1}(L)\gamma^{\mu}\ S(L).$$

The matrices $S(L)$ give a 4-dimensional representation of the Lorentz group and after putting $\psi'(x') = S(L)\ \psi(x)$, the Dirac equation transforms into

$$(\gamma^{\mu}\partial_{\mu'} + i\kappa)\ \psi'(x') = 0 \tag{1-59}$$

Comparing this equation with (1-58), if follows that the Dirac equations are covariant.

## B. THE INFINITESIMAL METHOD

The fact that all representations of the Lorentz group can not be described by tensors was the motive to introduce spinors and by this it was possible to classify all n-dimensional representations of the Lorentz group. In spite of this important result it appears that the spinor calculus is in its turn insufficient.

Between 1946 and 1953 it was shown that besides the spinor representations there also exists infinite-dimensional irreducible representations of the Lorentz group. In 1946 this was shown for unitary representations (Gel'fand), and in 1953 it was proved by Neumark that all irreducible infinite representations are obtained. Fortunately apart from these representations there are no other representations of the Lorentz group with which we have to occupy ourselves in the future.

Gel'fand and Neumark have developed the representation theory of the Lorentz group by starting from the general theory of Lie groups (the infinitesimal method).

By this it is possible to describe *all* irreducible representations of the Lorentz group.

In particular it was shown that all finite-dimensional representations are formed by the spinor representations.

Moreover the infinitesimal method is a powerful method to classify all linear partial differential equations $(\beta^{\mu}\partial_{\mu} + i\kappa)\ \psi = 0$, which are Lorentz covariant. Sections 6 and 8 give the most important techniques, while sections 5 and 7, which may be read independently from the sections 6 and 8, contain some theoretical background of the infinitesimal method.

## 5. Topological properties of the Lorentz group.

Every Lorentz transformation L is the product of a reflection $S_\mu$ and a restricted Lorentz transformation $\Lambda(\phi_\sigma)$ i.e. $L = S_\mu \Lambda(\phi_\sigma)$, see section 1.1. Therefore every Lorentz transformation L is determined by 7 parameters, the "discontinuous" parameter $\mu$ for reflections and the 6 "continuous" parameters $\phi_\sigma$ which determine a restricted Lorentz transformation $\Lambda$. In the future we shall use the following six parameters.

$$\Lambda(\phi_{01}, \phi_{02}, \phi_{03}; 0, 0, 0) \text{ where } -\infty < \phi_{oi} < +\infty, \tag{1-60}$$

are hyperbolic screws and the transformations

$$\Lambda(0, 0, 0; \phi_{23}, \phi_{31}, \phi_{12}), \text{ where } 0 \leq \phi_{23}^2 + \phi_{31}^2 + \phi_{12}^2 \leq \pi \tag{1-61}$$

are spatial rotations.
Thus the restricted Lorentz group $L_+^\uparrow$ may be mapped onto a point-set V in the 6-dimensional space, the so-called *parameter space V of* $L_+^\uparrow$ or *group space V*. The multiplication of two Lorentz transformations $\Lambda(\phi_\sigma)$ and $\Lambda(\psi_\sigma)$ i.e.

$$\Lambda(\chi_\sigma) = \Lambda(\phi_\sigma)\ \Lambda(\psi_\sigma)$$

corresponds in the parameter space with continuous-differentiable functions

$$\chi_\sigma = f_\sigma(\phi_\sigma, \psi_\sigma).$$

A group G in which the elements g can be described by points $\phi_\sigma(\sigma = 1, \ldots r)$ in such a way that the group-multiplication corresponds with continuous differentiable functions is called a *Lie group*. Thus $L_+^\uparrow$ is a Lie group and with a suitable definition for "continuous differentiable" for the parameter $\mu$, the full Lorentz group L is also a Lie group.
If we take into account that every parameter space with metric $\phi^2 = \sum \phi_\sigma^2$ is a topological space then we may also say that the Lorentz group (and every Lie group) is a *topological group*.

First we consider some topological properties of the group space of L and their consequences in the representation theory.

Connectedness

The restricted Lorentz group $L_+^\uparrow$ is *connected.* This means that every two elements $\phi_\sigma$ and $\psi_\sigma$ of $L_+^\uparrow$ may be connected by a continuous curve *) which lies totally in the group space V of $L_+^\uparrow$, defined by the formulae (1-60) and (1-61).

The full Lorentz group L however is not connected but decomposes into four connected components:

$$L_+^\uparrow,\ PL_+^\uparrow,\ TL_+^\uparrow,\ JL_+^\uparrow$$

where P, T, J are the reflections defined in section 1.1.

*In view of the property of connectedness, only the restricted group* $L_+^\uparrow$ *may be described by the group* $SL_2$ *of unimodular* $2 \times 2$ *matrices A.*

In order to prove this, we first mention that in section 3.1. we have shown that every matrix $\pm$ A represents a restricted Lorentz transformation $\Lambda$, thus the group $SL_2$ is $(2 \to 1)$ mapped onto a subgroup $(SL_2)'$ of $L_+^\uparrow$. We still have to prove that inversely every restricted Lorentz transformation can be represented by a $2 \times 2$ unimodular matrix A, i.e. $(SL_2)' \equiv L_+^\uparrow$. Now the following theorem holds.

Theorem 5.1. *The group* $SL_2$ *is a connected 6-parameter group.*

PROOF. We consider the 8-dimensional space of all $2 \times 2$ complex matrices. All matrices with det A = 0 form a 6-dimensional quadratic surface, which does not separate this 8-dimensional space. Thus with every two arbitrary unimodular matrices $A_0$ and $A_1$ there exists a curve A(t) of nonsingular matrices such that

$$A_0 = A(0) \text{ and } A_1 = A(1).$$

After a continuous deformation i.e.

$$A'(t) = \frac{A(t)}{\sqrt{\det(A(t))}} \quad ,$$

we obtain a curve of *unimodular* matrices so that

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*) More precisely, a continuous unique map of the interval $[0,1]$.

$$A_0 = A'(0) \text{ and } A_1 = +A'(1) \text{ or } A_1 = -A'(1).$$

The sign + or - depends on the number of times (even or odd) that the path det (A(t)) winds around the origin O in the complex plane.
Thus all matrices A *in* $SL_2$ are connected in $SL_2$ with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or with $-E = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and because E and -E are connected by the curve

$$A(\theta) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{pmatrix} \qquad (0 \leq \theta \leq 2\pi)$$

It follows that the group $SL_2$ is connected and 6-dimensional just like $L_+^\uparrow$. Because in the general theory of topological groups the theorem that the connected component of the identity of a group is uniquely determined holds, it follows that $(SL_2)' \equiv L_+^\uparrow$.
Thus conversely every restricted Lorentz transformation may be represented by a 2 × 2 unimodular matrix $\pm$ A.

## The Universal covering group

The group $SL_2$ is $(2 \to 1)$ mapped onto the Lorentz group which one may write as

$$SL_2/Z_2 \simeq L_+^\uparrow,$$

where $Z_2$ is the subgroup of $SL_2$ consisting of the identity E and -E.
In general if a connected Liegroup G' is $(m \to 1)$ mapped by a continuous homomorphism onto a group G : $G'/Z_m \simeq G$, then the group G' is called a *covering group* of G.
Among all covering groups G' of G, there is one uniquely determined group $G^*$ which is *simply connected.* That is to say every closed curve in $G^*$ can be continuously deformed in the group space into a point (all curves in G are *homotopic to zero).*
$G^*$ is called the *universal covering group* of G. The importance of the universal covering group $G^*$ of a group G lies in the fact that every

multiple-valued representation of G may be replaced by a representation of $G^*$, which is single-valued, because $G^*$ is simple connected. In further investigation of all representations of $G^*$ we then use the theory of single-valued representations, as orthonormality relations, ... etc. We now wish to proof that $SL_2$ is not only connected, but also simply connected. This is not true for the Lorentz group. We have

Theorem 5.2. *The group $SL_2$ of unimodular 2 × 2 matrices, is the universal covering group of $L_+^\uparrow$.*

PROOF. We write every unimodular 2 × 2 matrix A with the Pauli matrices i.e. $A = a_\mu \sigma^\mu$, the $a_\mu$ are arbitrary complex numbers, thus A need not be hermitian *). The matrix A is unimodular and because $\det A = a^2$,

$$a_0^2 - a_1^2 - a_2^2 - a_3^2 = 1 \qquad (1\text{-}62)$$

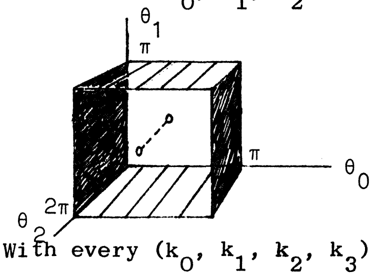
holds.

Thus the group space of $SL_2$ is formed by the complex 4-dimensional sphere $C_4$, given by (1-62). For the sub-group $SU_2$ of unitary unimodular matrices $(a_0, \vec{a}) = (k_0, i\vec{k})$ holds, where $\vec{k}$ is a real vector. Consequently the group space of $SU_2$ is formed by the real sphere $S_4 \subset C_4$ given by

$$k_0^2 + k_1^2 + k_2^2 + k_3^2 = 1. \qquad (1\text{-}63)$$

The proof of the simply connectedness of the real sphere $S_4$ (thus the fact that every closed curve on $S_4$ may be continuously deformed on $S_4$ into a point) may be found in Pontriagin II p. 30. Another proof one obtains by introducing spherical coordinates.

The real surface $S_4$ may be described by 4-dimensional real spherical coordinates $\theta_0$, $\theta_1$, $\theta_2$ such that




$$k_0 = \cos\theta_0 \qquad \theta_0 \in [0, \pi], \quad (1\text{-}64)$$

$$k_1 = \sin\theta_0 \cos\theta_1 \qquad \theta_1 \in [0, \pi], \quad (1\text{-}65)$$

$$k_2 = \sin\theta_0 \sin\theta_1 \cos\theta_2 \qquad \theta_2 \in [0, 2\pi], \quad (1\text{-}66)$$

$$k_3 = \sin\theta_0 \sin\theta_1 \sin\theta_2. \qquad (1\text{-}67)$$

With every $(k_0, k_1, k_2, k_3)$ on $S_4$ there exists *one* $\theta_0$, which satisfies (1-64). If $\theta_0 \neq 0, \pi$ then there exists *one* $\theta_1$ which satisfies (1-65) and

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*) Here μ is only an index having values from 0 to 3. We leave undecided in this chapter whether $a_\mu$ transforms as a vector or not.

if $\theta_1 \neq 0, \pi$ then there exists *one* $\theta_2$ which satisfies (1-66) and (1-67). If $\theta_0 = 0$ then all $\theta_1 \in [0, \pi]$, $\theta_2 \in [0, 2\pi]$ determine one point i.e. (1, 0, 0, 0), in the figure the corresponding plane $\theta_0 = 0$ ( and $\theta_0 = \pi$) is shaded.
Analogously if $\theta_1 = 0, \pi$ then all $\theta_2 \in [0, 2\pi]$ determine the point $(\cos\theta_0, \sin\theta_0, 0, 0)$ $(\cos\theta_0, -\sin\theta_0, 0, 0)$. The corresponding lines in the figure are drawing boldly. Finally, the points $(\theta_0, \theta_1, 0)$ and $(\theta_0, \theta_1, 2\pi)$ have to be identified. It is clear that every closed curve $\theta(t)$ in the group space of $SU_2$ may be continuously contracted to a point in the group-space by the transformation

$$\phi_\lambda(\theta_0, \theta_1, \theta_2)(t) = (\lambda\theta_0, \theta_1, \theta_2)(t). \quad (1 \geq \lambda \geq 0)$$

Because $\phi_1(\theta_0, \theta_1, \theta_2)(t) = (\theta_0, \theta_1, \theta_2)(t)$

and $\phi_0(\theta_0, \theta_1, \theta_2)(t) = (0, \theta_1, \theta_2)(t) \equiv k^\mu(1, 0, 0, 0)$

by which follows that $SU_2$ is simply connected.

We remark that there are closed curves, e.g. $\theta(t) \equiv (\theta_0, \theta_1, t\theta_2)$, which cannot be contracted to a point as long as we transform only the inner points of the cube, for $(\theta_0, \theta_1, 0) \equiv (\theta_0, \theta_1, 2\pi)$.
With respect to the group $SL_2$ the $k_\mu$ are arbitrary complex and the proofs mentioned above fail, for there appear singularities, (there are no spherical coordinates to describe the points (1, 1, i, 0), ... etc.), which are difficult to remove. However, one may prove that every closed continuous curve $k(t)$ on the complex sphere $C_4$ may be continuously transformed into a real closed continuous curve on $S_4$ and by this the simple connectedness of $C_4$ follows.
In order to do this we separate the real and imaginary part of $k(t)$

$$k(t) = u(t) + iv(t),$$

the fact that $k^2 = 1$ implies

$$u^2 - v^2 = 1 \qquad u \cdot v = 0, \tag{1-67a}$$

Thus

$$u^2 \geq 1 \tag{1-67b}$$

and we write

$$k = \frac{u}{\sqrt{u^2}} + (u - \frac{u^2}{\sqrt{u^2}}) + i\ v. \qquad (1\text{-}67c)$$

The continuous deformation of k(t) is now defined by

$$k_\lambda = \frac{u}{\sqrt{u^2}} + \lambda\ (u - \frac{u^2}{\sqrt{u^2}}) + i\ \rho\ v,\ \lambda \in [0,1]. \qquad (1\text{-}67d)$$

where the parameter $\rho$ is determined by the condition that for all $\lambda$ the curves $k_\lambda(t)$ ly on the complex sphere, i.e.

$$\left[\frac{u}{\sqrt{u^2}}\ (1 + \lambda\sqrt{u^2} - \lambda)\right]^2 + \rho^2\ v^2 = 1. \qquad (1\text{-}67e)$$

First we consider the set $T \subset [0,1]$ such that

$$v^2(t) = 0 \quad \text{for all } t \in T$$

then $\rho$ is not defined and we have the "deformation"

$$k_\lambda(t) = u(t) \quad \text{for all } \lambda \in [0,1].$$

Next we consider the set $S = [0,1] \setminus T$, thus $v^2(t) \neq 0$ if $t \in S$.
Then we have for $\rho$

$$\rho\ v = + \frac{v}{\sqrt{v^2}} \sqrt{(1-\lambda\ + \lambda\sqrt{u^2})^2 - 1} \qquad (1\text{-}67f)$$

By which follows that $\rho$ is a continuous function of $\lambda \in [0,1]$ and $t \in S$. This statement remains true if we consider the limits points $t_0$ of S such that $v^2(t_0) = 0$ and $u^2(t_0) = 1$.
Because $\frac{v(t)}{\sqrt{v^2(t)}}$ is a unit vector for all $t \in S$ and the expression under the $\sqrt{\ }$-sign tends to zero if $t \to t_0$, we may define $\rho(t_0)\ v(t_0) = 0$. Consequently the solutions $k_\lambda(t) = u(t)$ on T are continuously connected with the solutions k (t) on S. The curves u(t) and v(t) are closed and because for the expression under the $\sqrt{\ }$-sign

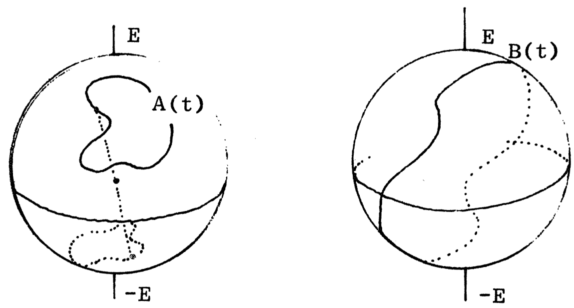
$$(1-\lambda+\lambda\sqrt{u^2})^2 - 1 \geq 0 \text{ for all } \lambda \in [0,1],\ t \in [0,1].$$

it follows that $\rho(t)v(t)$ and thus also $k_\lambda(t)$ is a closed curve. In this way we have constructed a continuous deformation of the closed curve

$$k_1(t) = k(t)$$

into the curve $$k_0(t) = \frac{u(t)}{\sqrt{u^2(t)}}$$

and by this the simple connectedness of $C_4$ follows.
The proof mentioned above may be easily extended to the complex sphere $C_n$ in n-dimensions.
The proof fails for the restricted Lorentz group $L_+^\uparrow$, because opposite points A and -A on the complex sphere $C_4$ must be identified. Thus $L_+^\uparrow$ is described by the space of rays through the origin




and the 2-valued function $\Lambda \to \pm A$, of rays $\to$ 2 points, consists of two single-valued branches i.e.

$$\Lambda \to E, A, \ldots$$
$$\Lambda \to -E, -A, \ldots$$

which are separated by the $(x_0 = 0)$ plane, e.g.

It is intuitively clear that a closed curve $\pm A(t)$ in the group space of $L_+^\uparrow$ which does not intersect the $(x_0 = 0)$ plane may be contracted to one point $\pm A$.
A curve B(t) however, which intersects the $(x_0 = 0)$ plane one time in the point $(0, \pm \vec{b})$ may not be contracted into a point.
Yet two curves B(t) and B'(t) (which intersects the $x^0 = 0$ plane 1 or 3, 5, ... times) may be continuously transformed into each other.
The fact that there exist in $L_+^\uparrow$ two classes of curves A(t) and B(t), two *homotopy* classes, one expresses by saying that $L_+^\uparrow$ is *double connected* and it follows that $SL_2$ is the universal covering group of $L_+^\uparrow$.

<u>Compactness</u>

The group space of the rotation group $O_{3+}$ is formed by the solid sphere $\Gamma$, see formulae (1-61). Because $\Gamma$ is a bounded and closed point set (which is equivalent to the statement that every infinite sequence of points of $\Gamma$ has a limit point in $\Gamma$) $\Gamma$ is a *compact* pointset and ac-

cordingly the rotation group $O_{3+}$ is called a *compact group*.
However, the group space of the restricted group $L_+^\uparrow$, which is defined by (1-60) and (1-61) is not compact.
The compactness of $O_{3+}$ implies that all irreducible representations of $O_{3+}$ are equivalent to an *unitary* representation in a *finite*-dimensional space, i.e.

$$D^j : O_{3+} \to \text{subgroup of } SU_{2j+1}.$$

For a non-compact group the irreducible representation need not be unitary and finite. The non-compactness of the Lorentz group implies that

Theorem 5.3.

1. *All finite-dimensional representation of the Lorentz group are not unitary (except the trivial one: $\Lambda \to 1$).*
2. *There exists infinite-dimensional irreducible representations on operators $D(\Lambda)$.*
3. *Every irreducible representation is determined by the number pair $(j^0, j^1)$ where $j^0 = 0, \frac{1}{2}, 1, \ldots$ and $j^1$ is arbitrary complex. Thus the Lorentz group has a non-countable family of irreducible representations.*

   A sketch of the proof is given in section 7.3.

Local compactness

Although the Lorentz group is not compact, the Lorentz group $L_+^\uparrow$ is *locally compact*. This means that every point of the group space of $L_+^\uparrow$ has a compact neighbourhood.
Gel'fand and Raikov have proved that every local compact group has a sufficient number of non-trivial irreducible unitary representations $D(\Lambda)$.

Theorem 5.3.

4. *The Lorentz group has unitary irreducible representations, and except the trivial one they are infinite-dimensional.*

## 6. The Lorentz group as a Lie group.

In section 5 we have shown that the Lorentz group is a Lie group, where every element $\Lambda$ is described by 6 parameters: $\Lambda \Longleftrightarrow \phi_\sigma$.
The matrix components of a representation $D(\Lambda)$ are continuous functions of $\phi_\sigma$. Using a theorem from the theory of Lie groups it follows that *the matrix components of $D(\Lambda(\phi_\sigma))$ are not only continuous functions, but analytic functions of $\phi_\sigma$.* (cf. Cohn p. 135, p.116.) Thus Taylor expansion of $D(\phi)$ gives

$$D(\phi) = E + \phi_\sigma \left(\frac{\partial D(\phi)}{\partial \phi_\sigma}\right)_{\phi=0} + \ldots . \quad ^{*)} \qquad (1\text{-}68)$$

Definition 6.1. The *infinitesimal operators* $D_\sigma$ of a representation $D(\phi)$ of the Lorentz group $L_+^\uparrow$ are defined by

$$D_\sigma = \left(\frac{\partial D(\phi)}{\partial \phi_\sigma}\right)_{\phi=0} . \qquad (1\text{-}69)$$

The importance of the theory of infinitesimal operators is due to the fact that $\phi_\sigma D_\sigma$ does not only determine $D(\phi)$ in first-order terms, but determines $D(\phi)$ completely. Because after a suitable parameter transformation, such that $\Lambda(s\phi_\sigma)\Lambda(t\phi_\sigma) = \Lambda((s+t)\phi_\sigma)$, one may prove that

$$D(\phi) = E + \phi_\sigma D_\sigma + \frac{1}{2!}(\phi_\sigma D_\sigma)^2 + \ldots \qquad (1\text{-}70)$$

holds, thus

$$D(\phi) = e^{\phi_\sigma D_\sigma}$$

This last expression also implies that the matrices $D(\phi)$ are reducible if and only if the six matrices $D_\sigma$ are reducible. Further, if an operator S commutes with all $D_\sigma$ then S commutes with all $D(\phi)$, and so on. In this way the problem of the determination of all representations $D(\Lambda)$ is reduced to the determination of all six-tuples $D_\sigma$.
To determine $D_\sigma$ we first start with the vector representation and we use the $4 \times 4$ matrices as given by formula (1-5), (1-6). After differentiating with respect to $\phi_\sigma$ and putting $\phi \equiv 0$ one obtaines the matrices

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*) For finite-dimensional operators D the components of $\frac{\partial D}{\partial \phi}$ are given by $\left(\frac{\partial D}{\partial \phi}\right)_{ij} = \left(\frac{\partial D_{ij}}{\partial \phi}\right)$, see also remark 6.1.

$$D_{23}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix},\quad D_{01}=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\ \ldots \text{ etc., so that } \phi_\sigma D_\sigma=\begin{pmatrix}0&\phi_{01}&\phi_{02}&\phi_{03}\\\phi_{01}&0&-\phi_{12}&\phi_{31}\\\phi_{02}&\phi_{12}&0&-\phi_{23}\\\phi_{03}&-\phi_{31}&\phi_{23}&0\end{pmatrix}$$

We use the convenient notation $A_1 = D_{23}$, .. etc. for the infinitesimal operators belonging to spatial rotations, and $B_1 = D_{01}$, .. etc. for the infinitesimal operators belonging to hyperbolic screws.
One may easily verify that

$$A_1A_2 - A_2A_1 = A_3 \text{ and cycl.}^{*)}$$

holds. Using the *Lie-product* $[A_1, A_2]= A_1A_2 - A_2A_1$ this *commutation rule* may be written as $[A_1, A_2] = A_3$.
Similar formulae hold for the other Lie products.
Now a theorem from the theory of (connected) Lie groups states that the commutation rules of a group are not only valid for the infinitesimal operators D of a special representation, but that they are valid for *all* representations of this group. Thus in general the following theorem holds:

Theorem 6.1. *The commutation relations of the Lorentz group are:*

$$[A_1\ ,\ A_2] = A_3 \text{ and cycl.} \tag{1-72}$$

$$[A_1\ ,\ B_2] = B_3 \text{ and cycl.} \tag{1-73}$$

$$[B_1\ ,\ B_2] = -A_3 \text{ and cycl.} \tag{1-74}$$

*The remaining combinations either vanish i.e.* $[A_1, B_1] = 0$, *... etc. or we obtain the anti-cyclic combinations i.e.* $[A_2, B_1] = -B_3$.
(A proof of these commutation relations will also be given in section 7.).
Because the infinitesimal operators $D_\sigma$ of a representation form a closed 'algebraic' system with respect to the Lie product "[,]", one says that the infinitesimal operators $D_\sigma$ form a *Lie algebra* $L_+^{\uparrow}$ given by the formulae (1-72, 73, 74).

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*) Hereby is meant $[A_1, A_2] = A_3$, $[A_2, A_3] = A_1$, $[A_3, A_1] = A_2$ and $[A_2, A_1] = -A_3$, ... etc.

If one interprets the Lie algebra as a vectorspace spanned by 6 *abstract* entities $D_\sigma$ and with a multiplication given by (1-72, 73, 74) one also calls $L_+^{\uparrow}{}'$ an *abstract* Lie algebra. The infinitesimal operators of an arbitrary representation of the group $L_+^{\uparrow}$ form a matrix representation of the abstract algebra $L_+^{\uparrow}{}'$.
Inversely there is a second theorem of the theory of Lie groups which states that if we have 6 finite-matrices $D_\sigma$, which satisfy the commutation rules (1-72, 73, 74) then the $D_\sigma$ are necessarily the infinitesimal operators of a uniquely determined representation $D(\phi)$. In other words, this theorem states that every matrix representation $D_\sigma$ of the Lie algebra $L_+^{\uparrow}{}'$ uniquely determines a representation D of $L_+^{\uparrow}$.
(In order to prove this theorem one must show conversely that the matrices $D(\phi) = e^{\phi_\sigma D_\sigma}$ form a representation of $L_+^{\uparrow}$).

The importance of this last theorem lies in the fact that by classifying all representations of the algebra $L_+^{\uparrow}{}'$ one classifies all representations of the group. This fact implies the reduction of a problem concerning a system of $\infty^6$ elements (the Lie group), into a problem concerning a well defined system of only 6 base elements (the Lie algebra). see Ljubarski p. 158 *)
Formerly the general solutions of the commutation rules (1-72) of the rotation group $O_{3+}$ were found by Cartan in 1913.
In order to describe these solutions we observe that $O_{3+}$ is a compact group. In fact the compactness of $O_{3+}$ also follows from the commutation rules (1-72) and the condition that the $A_k$ are the infinitesimal operators of a group (with real parameters $\phi_1$, $\phi_2$, $\phi_3$).
Hence we may suppose that the representation D is unitary: $(D^+)^{-1} = D$.
Differentiating with respect to $\phi_\sigma$, observing that $\partial_\sigma(DD^{-1}) = 0$ which implies that $\partial_\sigma(D^{-1}) = - D^{-2}\ \partial_\sigma(D)$, and putting all $\phi_\sigma = 0$ we obtain

$$D_\sigma^+ = -D_\sigma \quad \text{or} \quad (i\, D_\sigma)^+ = (iD_\sigma) \tag{1-75a}$$

Therefore one introduces the *hermitian* infinitesimal operators $J_k = iA_k$ with $[J_1, J_2] = i\, J_3$ and cycl. and further the operators $J_3$, $J_+ = J_1 + iJ_2$ and $J_- = J_1 - iJ_2$ . (or $J_\pm = iA_1 \mp A_2$).

*) See also remark 6.1. on page 63

So we have the following relations

$$[J_3, J_+] = +J_+ \qquad [J_3, J_-] = -J_- \tag{1-75b}$$

$$[J_+, J_-] = 2J_3 \tag{1-75c}$$

$$\text{and} \quad J_3^+ = J_3 \qquad J_+^+ = J_- \tag{1-75d}$$

Because $J_3$ is hermitian, there exists an orthogonal basis of n vectors $e_m$ which span the representation space and with eigenvalues m, i.e. $J_3 e_m = m e_m$. The number m is called the *weight* of $e_m$. We assume that $e_m$ are unit vectors. Thus a rotation θ about the z-axis is described by $D(\theta)e_m = e^{-im\theta} e_m$ and because D(θ) is unimodular, det D(θ) = + 1, it follows that the sum of all eigenvalues vanishes.

Further $J_+$ and $J_-$ are the so-called *step-operators*, such that $J_+ e_m$ and $J_- e_m$ have the eigenvalues m + 1 and m - 1 (or 0 and 0) respectively. Because (1-75b) implies that

$$J_3 (J_+ e_m) = J_+ J_3 e_m + J_+ e_m = (m+1)(J_+ e_m),$$

and thus it follows that

$$J_+ e_m = e_{m+1} \qquad (\times\ \alpha_{m+1}) \quad (\text{or} = 0) \tag{1-75e}$$

$$J_- e_{m+1} = e_m \qquad (\times\ \beta_{m+1}) \quad (\text{or} = 0).$$

Consequently the spectrum of $J_3$ consists of ladders of unit-spaced eigenvalues, i.e. .. , m-1, m, m + 1, ...

If one requires that D is irreducible it follows that there is only *one* such a ladder, from which we will call the highest weight j. From the fact that the sum of the eigen values m vanishes it follows that this ladder lies symmetrical with respect to the origin and thus $-j \leq m \leq +j$ and $j = 0, \frac{1}{2}, 1, \ldots$ . Using the relation $J_+^+ = J_-$ or $(J_+ e_{m-1}, e_m) = (e_{m-1}, J_- e_m)$ it follows that $\alpha_m = \beta_m$ and applying the operator $2J_3$ , see (1-75c), to a vector $e_m$ it follows that

$$\alpha_m^2 - \alpha_{m+1}^2 = 2m$$

Adding all terms from m till m = j, all terms on the left-hand side vanish, except $\alpha_m$ and we get

$$\alpha_m = \pm \sqrt{(j+m)(j-m+1)}$$

In general one normalizes the unit vectors $e_m$ in such a way that the formulae for $J_\pm$ holds with the + sign for $\alpha_m$

So we have: (see also Gel'fand p. 25, 73, Ljubarski p. 167,182)

<u>Theorem 6.2.</u> *With every irreducible representation of the rotation group there exist a uniquely determined $j = 0, \frac{1}{2}, 1, \ldots$ and a uniquely determined basis of $(2j+1)$ orthonormal vectors $e_m$ $(-j \leq m \leq +j)$, which span the representation space such that*

$$\begin{aligned} J_3\, e_m &= m\, e_m \\ J_-\, e_m &= \sqrt{(j+m)(j-m+1)}\,e_{m-1} \\ J_+\, e_m &= \sqrt{(j+m+1)(j-m)}\,e_{m+1} \end{aligned} \qquad (1\text{-}75f)$$

This irreducible representation is denoted by $D^j$ and we are back to the spinor representations if we substitute $e_m = u^{j+m}v^{j-m}$, see theorem 3.3 and formula (1-30b). For the infinitesimal operators $J_k$ of the spinor representation, see formula (1-79a).

We now consider representations of the Lorentz group. In general the induced representation of the rotation group is reducible. The solution of the system (1-72), (1-73), (1-74) is more difficult, because there exist also infinite-dimensional solutions. We will enter into details in the next sections.

<u>Remark 6.1.</u>

In the case of infinite-dimensional representations the theory is considerably more complicated. An infinite-dimensional representation is a continuous homomorphism onto a group of bounded operators in a Banach space. The definitions of an irreducible representation and of an infinitesimal operator need a supplement.

(Gel'fand etc. p. 176, Neumark p. 70-71, 94-95).

It remains true that by classifying all infinite representations of the Lie algebra one classifies all *possible* infinite representations of the

Lie group, however, it is not clear that given an "infinite" representation of the algebra, there actually *exists* a corresponding representation of the group.
The infinite-dimensional solutions of the Lorentz *algebra* $L_+^{\uparrow\,\prime}$, in view of unitary representations, were given by Gel'fand in 1944-6. Not before 1954 Neumark succeeded in proving that conversely to every representation of the Lorentz algebra there actually correponds a representation of the Lorentz group.
Hence, now all the irreducible representations of the Lorentz group are known.

The infinitesimal operators of the spinor representation

We end this section, by determining the infinitesimal operators $A_k$ and $B_k$ of the 2-dimensional spinor representation $\Lambda \to \pm A$, which will be used in section 8.2.
In order to calculate these operators we use the parameters $\phi_\sigma$ already introduced in section 5, and thus we first have to specify the dependence of the matrix A on $\phi_\sigma$. For that purpose we write every unimodular matrix as $A = a^\mu\sigma_\mu$, where $\sigma_\mu$ are the Pauli-matrices i.e.

$$A = \begin{pmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{pmatrix}$$

with det $A = a^2$ it follows that $(a^0)^2 - (a^1)^2 - (a^2)^2 - (a^3)^2 = 1$ must hold.
Analogously to page 54 we write

$$a^0 = \cos\frac{\theta}{2}, \vec{a} = -i \sin\frac{\theta}{2}\vec{k}. \qquad (1\text{-}76)$$

The numbers $\theta$ and $k^i$ are complex. If $\theta \neq 0$ then $\vec{k}^2 = 1$ and if $\theta = 0$ then $\vec{k}^2 = 0$.
We have already shown that if we take $\theta$ real and $\vec{k} = (0, 0, 1)$ we obtain the matrix

$$A = \cos\frac{\theta}{2}\sigma^0 - i\sin\frac{\theta}{2}\sigma^3 = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{pmatrix}, \qquad (1\text{-}77)$$

which corresponds to a spatial rotation $\theta$ about the $x^3$-axis, so that $\theta = \phi_{12}$.

In general there holds that an arbitrary real $\vec{k}$ ($\vec{k}^2 = 1$) substituted into (1-76) gives a spatial rotation about the axis $\vec{k}$ and with rotation angle $\theta$. (chapter II, formula (2-15)). Now we take $\theta$ purely imaginary $\theta = i\phi$ and $\vec{k} = (0, 0, 1)$. Then we obtain the matrix

$$A = \cosh\frac{\phi}{2}\,\sigma^0 + \sinh\frac{\phi}{2}\,\sigma^3 = \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}, \tag{1-78}$$

which corresponds to a hyperbolic screw in the $(x^0, x^3)$-plane, so that $\phi = \phi_{03}$. Taking $\vec{k}$ arbitrary real we obtain a hyperbolic screw in the $(x^0, \vec{k})$-plane, see also formula (2-17).

After differentiating the expressions (1-77), (1-78), ... etc. with respect to $\phi_{12}$, $\phi_{03}$, ... respectively, we obtain the infinitesimal operators

$$A_k = -\frac{i}{2}\,\sigma_k \text{ or } J_k = \tfrac{1}{2}\,\sigma_k \text{ and } B_k = \tfrac{1}{2}\,\sigma_k \text{ acting on } \psi^a. \tag{1-79a}$$

Using the commutation rules for the Pauli matrices i.e.

$$[\sigma_1, \sigma_2] = i\,\sigma_3 \text{ and cycl.,}$$

we again obtain the original commutation relations for $A_k$ and $B_k$ from formulae (1-72), ... (1-74).

In physical literature one often denotes the $2 \times 2$ spin matrices $J_k = iA_k$, thus $J_k = \frac{1}{2}\sigma_k$ by the letter $s_k$.

Next to the representation $D^{\frac{1}{2}0}$: $\Lambda \to \pm A$, there exists the representation $D^{0\frac{1}{2}}$ : $\Lambda \to \pm (A^+)^{-1}$ acting on spinors $\psi^a$ and $\psi_{\dot{a}}$ respectively,(lemma 3.2) and formula (1-37b) .

For rotations there holds that $(A^+)^{-1} = A$ and for hyperbolic screws that $(A^+)^{-1} = A^{-1}$ (remark 3.1). Hence by differentiating and using formula (1-75a) we obtain the infinitesimal operators of $D^{0\frac{1}{2}}$

$$A_k = -\frac{i}{2}\,\sigma_k \text{ or } J_k = \tfrac{1}{2}\,\sigma_k \text{ and } B_k = -\tfrac{1}{2}\,\sigma_k \text{ acting on } \psi_{\dot{c}} \tag{1-79b}$$

and also $A_k = +\frac{i}{2}\sigma_k$ or $J_k = -\frac{1}{2}\sigma_k$ and $B_k = \frac{1}{2}\sigma_k$ acting on $\psi^{\dot{c}}$ .

Consequently on the Dirac spinor $(\psi^{a}, \phi_{\dot{c}})$ acts the rotation matrices

$$-\frac{i}{2}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix},$$

which are sometimes called the *Dirac spin matrices.*

<u>Remark 6.3</u>

The fact that complex $\theta$ and $\vec{k}$ appear in (1-76) may be expressed by saying that the Lorentz group is isomorphic with the *complex* 3-dimensional rotation group.

Finally we note that if $\psi$ is an higher order spinor, e.g. $\psi = (\psi^{a_1}\psi^{a_2}\ldots)$, thus the $2 \times 2$ representation matrices $A = D(r)$ act on $\psi$ by

$$D(\psi^{a_1}\psi^{a_2}\ldots) = D(\psi^{a_1})\, D(\psi^{a_2})\ldots.$$

And infinitesimal it follows that the operators $J_k$ act on the components $\psi^{a_1}\psi^{a_2}\ldots$ in an *additive* way

$$J_k(\psi^{a_1}\psi^{a_2}\ldots) = J_k\,(\psi^{a_1})\,\psi^{a_2}\ldots + \psi^{a_1}(J_k\psi^{a_2})\ldots + \ldots$$

thus $$J_k(u^{j+m}v^{j-m}) = (j+m)\,(J_k u)v^{j-m}\ldots + (j-m)\,u^{j+m}\,(J_k v)^{j-m}.$$

We will use these formulae in section 7.2.

## 7. The adjoint representation of the Lorentz group

One can derive important properties of the infinitesimal operators by starting from the so-called *adjoint representation* or *infinitesimal representation*, which exists for every Lie group.
One arrives at this representation by studying the transformation properties of the infinitesimal operators themselves.
Consider for instance a scalar field $\psi(x)$ and the representation $\psi \to \psi'$ defined by $\psi'(x) = \psi(L^{-1}x)$.
Let L be a rotation in the $(x^2, x^3)$-plane. Henceforth we will denote the corresponding parameters $\psi_\sigma$ by $\psi^2{}_3$, thus

$$\psi'(x) = \psi(x^0, x^1, x^2\cos\psi^2{}_3 + x^3 \sin\psi^2{}_3, -x^2 \sin\psi^2{}_3 + x^3\cos\psi^2{}_3)$$

Differentiating with respect to $\psi^2{}_3$ and putting $\psi^2{}_3 = 0$ we obtain the infinitesimal operator

$$D^2{}_3\, \psi(x) = -(x^2\partial_3 - x^3\partial_2)\, \psi(x)$$

It follows that the infinitesimal operators transforms as tensors $F^\mu{}_\nu$ or equivalently, transform as *anti-symmetric* tensors $F^{\mu\nu}$. Hence in the case of scalar functions the infinitesimal operators transform under the representation $D^{10} \dot{+} D^{01}$, see table (1-32).
However the space of scalar functions $\psi(x)$ does not contain all irreducible representation spaces of the Lorentz group *), therefore we wish to prove this theorem in greater accordance with the theory of Lie groups.

Besides every representation D(L) of the full Lorentz group we define a representation which works in the space S of operators S by the formula

$$S' = D(L)\, S\, D^{-1}(L) \qquad (1\text{-}80a)$$

If D(L) is a finite-dimensional representation then one may write (1-80a) in components

$$S_{i'j'} = D_{i'i}\, D'_{j'j}\, S_{ij}\ ,\ \text{where } D' = (D^{-1})^T$$

In general there holds

$$S' = \overset{\times}{D}(L)\, S,\ \text{where } \overset{\times}{D}(L) = D(L) \times D'(L) \qquad (1\text{-}80b)$$

---

*) Cf. with the spherical functions $Y_{l,m}$ which form only irreducible representation spaces of the rotation group with *integer* spin value l.

We note that if $S = D(\Lambda)$, where $\Lambda$ is a restricted Lorentz transformation, then (1-80a) corresponds to the coordinate transformation $\Lambda' = L\Lambda L^{-1}$ in $R_4$, due to the homomorphism, i.e. $D(\Lambda)' = D(\Lambda')$.
We have written $D(\Lambda)'$ and not $D'(\Lambda)$ in order to indicate that the representation (1-80a) acts *here* in the $n^2$-dimensional space of *operators* $D(\Lambda)$ and not in the *function* space of all representations D.
We now consider the subspace D spanned by the infinitesimal operators. In the vector representation these operators are $4 \times 4$ matrices and because they are obtained from the matrices $\Lambda^\rho{}_\sigma$ it follows that they have components $\phi^\rho{}_\sigma$. Thus we have to rewrite formula (1-71) with $\phi^\rho{}_\sigma$ instead of $\phi_{\rho\sigma}$, if we take into account the transformation properties of the infinitesimal operators with respect to the representation (1-80a). Raising the index $\sigma$, we have that the components $\phi^{\rho\sigma} = (-\phi^0{}_i, \phi^j{}_k)$ form an antisymmetric tensor transforming by

$$\phi^{\rho'\sigma'} = L^{\rho'}{}_\rho L^{\sigma'}{}_\sigma \phi^{\rho\sigma} \qquad (1\text{-}80c)$$

or

$$\phi^{\rho\sigma} = L_{\rho'}{}^\rho L_{\sigma'}{}^\sigma \phi^{\rho'\sigma'}$$

The components $\phi^{\rho\sigma}$ are given with respect to the 6 infinitesimal operators $D_{\rho\sigma} = (-D^0{}_i, D^j{}_k) = (-B_i, A_i)$. Because $\phi^{\rho\sigma}$ are the parameters of the Lorentz group, it also follows that the 6 infinitesimal operators $D_{\rho\sigma} = (-B_i, A_i)$ of an *arbitrary* representation transform equivalent to the 6 basic vectors $F_{\mu\nu} = (\vec{E}_i, \vec{H}_i)$, where

$$\vec{E}_1 = (1, 0, 0 \mid 0, 0, 0) , \ldots \text{ etc.}$$
$$\vec{H}_1 = (0, 0, 0 \mid 1, 0, 0) , \ldots \text{ etc.}$$

which span the space of antisymmetric tensors $F^{\mu\nu} = (\vec{E}, \vec{H}) = (E^i\vec{E}_i, H^i\vec{H}_i)$.
So we have

Definition. The *adjoint representation* (or *infinitesimal representation*) is the representation which acts in the space spanned by the infinitesimal operators according to the formula

$$\overset{\times}{D}(L)\, D_{\rho\sigma} = D(L)\, D_{\rho\sigma}\, D^{-1}(L).$$

We will also use the matrix $D \equiv (D_{\rho\sigma})$ and write $\overset{\times}{D}(L)D = (\overset{\times}{D}(L)D_{\rho\sigma})$

If we exclude the trivial representation $\Lambda \to 1$ and require that the space D is 6-dimensional, we may state:

Theorem 7.1. *The space D spanned by the infinitesimal operators transforms under the adjoint representation equivalent with the 6-dimensional space of anti-symmetric tensors, i.e. if* $\overset{\times}{D}(L)D_{\rho\sigma} = \overset{\times}{D}_{\rho\sigma}$ *then*

$$\overset{\times}{D}_{\rho\sigma} = D_{\mu\nu}\, L^{\mu}{}_{\rho}\, L^{\nu}{}_{\sigma} \qquad (1\text{-}80d)$$

This formula may be also obtained by differentiating the formula

$$D(\Lambda)^{\times} = D(L)\; D(\Lambda)\; D(L^{-1}) \qquad (1\text{-}80e)$$

with respect to $\phi^{\rho\sigma}$ and using formula (1-80c)

Remark 7.1.

Theorem 7.1 has a number of important consequences. Especially the fact that the operators $B_k \sim \vec{E}_k$ transform like 3-dimensional vectors under rotations was the general starting point of Gel'fand a.o. to construct the infinite-dimensional representations of the Lorentz group, see section 7.3.

In the sections 7.1, 7.2, 7.3 we will apply the properties of the space of anti-symmetric tensors to the space D of infinitesimal operators. Some applications which are used in more physical literature are mentioned below.

The infinitesimal operators of little groups.

Finally we make the following important observations. For that purpose it is necessary to write the vector representation in the form $\Lambda = D(\Lambda^{\mu}{}_{\nu})$, where $\Lambda^{\mu}{}_{\nu}$ is the group element and $\Lambda$ the corresponding linear transformation in $R_4$. However the considerations below, also holds for arbitrary representations.

So far we have considered the transformation

$$\Lambda' = \mathbf{L}\,\Lambda\,L^{-1}$$

$$\text{or} \quad \Lambda^{\rho'}{}_{\sigma'} = L^{\rho'}{}_{\rho}\, L_{\sigma'}{}^{\sigma}\, \Lambda^{\rho}{}_{\sigma} \qquad (1\text{-}80f)$$

as a *point transformation* which maps the point $\Lambda^{\rho}{}_{\sigma}$ onto the point $\Lambda^{\rho'}{}_{\sigma'}$, and thus the linear transformation $\Lambda = D(\Lambda^{\rho}{}_{\sigma})$ onto $\Lambda' = D(\Lambda^{\rho'}{}_{\sigma'}) = D(\Lambda^{\rho}{}_{\sigma})'$. However if L is a *coordinate transformation* then the components $\Lambda^{\rho}{}_{\sigma}$ in the coordinate system $x^{\nu}$, determine the same linear transformation as the components $\Lambda^{\rho'}{}_{\sigma'}$ in the coordinate system $x^{\mu'}$. One may express this by introducing a new representation D' such that

$$D'(\Lambda^{\rho'}{}_{\sigma'}) = D(\Lambda^{\rho}{}_{\sigma}),(\Lambda^{\rho}{}_{\sigma}) = L^{-1}(\Lambda^{\rho'}{}_{\sigma'})L \qquad (1\text{-}80g)$$

or replacing the argument $\Lambda^{\rho'}{}_{\sigma'}$ by $\Lambda^{\rho}{}_{\sigma}$ we get

$$D'(\Lambda^{\rho}{}_{\sigma}) = D(L^{-1})\, D(\Lambda^{\rho}{}_{\sigma})\, D(L),$$

cf. (1-80e). Raising the index $\sigma$, differentiating (1-80g) with respect to the parameter $\phi_{\rho\sigma}$ and putting all $\phi = 0$ we obtain

$$D'_{\rho'\sigma'} = L_{\rho'}{}^{\rho}\, L_{\sigma'}{}^{\sigma}\, D_{\rho\sigma} \qquad (1\text{-}80h)$$

It follows that the operators $D_{\rho\sigma}$ transform as the covariant *components* of a vector, as contrasted with (1-80d).
In fact one may verify that if one considers the space of functions f(x) defined on $R_4$ ($x \in R_4$), transforming by

$$D(\Lambda)\, f(x) = f(\Lambda^{-1} x),$$

then under a coordinate transformation L, the operators $D(\Lambda)$ transform according to (1-80g).

As an application we consider the four operators $W_{\mu}$, i.e.

$$W_{\mu} = D_{\overline{\mu\nu}}\, P^{\nu} \text{ where } P^{\nu} = -\,i\,\partial^{\nu} = -i\,\frac{\partial}{\partial x_{\nu}} \text{ and } D_{\overline{\mu\nu}} = (A_i, B_i)$$

(see 1-16d), or theorem B-1 on p. 134, (1-80k)

which acts on plane waves $\psi(x,p) = \psi(p)e^{ip.x}$ $(p\cdot x = p_{\mu}\cdot x^{\mu})$., where $(p^2, p_0 > 0)$. It follows that $D_{\overline{\mu\nu}}\, P^{\nu} = D_{\overline{\mu\nu}}p^{\nu}$ .
Because $D_{\overline{\mu\nu}}p^{\nu}$ is an invariant expression, under the adjoint representation as defined above, we choose the coordinate systems such that $p^{\nu} = (1, 0, 0, 0)$. It follows that

$$W_{\mu} = (0, A_1, A_2, A_3)$$

which are the three rotation operators which leave the vector $p^\nu$ invariant. A group G(p) which leave a vector p invariant, is called the *little group* of p and if $p^2 > 0$ then G(p) is obviously isomorphic with the rotation group $O_3$. Consequently in an arbitrary coordinate system the expressions

$$W_\mu = D_{\overline{\mu\nu}} P^\nu \qquad (1\text{-}80l)$$

which act on plane waves $\psi(x, p)$ are the infinitesimal operators of the little group of p.

The Casimir operators of the Lorentz group

It is further known that the space of anti-symmetric tensors $F^{\mu\nu}(\vec{E}, \vec{H})$ contains the complex number $\vec{G}^2 = (\vec{E} + i\vec{H})^2$ as an invariant, see (1-16f) or chapter II, section 3.2. Analogously the space of infinitesimal operators contains the invariant operator

$$(\vec{A} + i\vec{B})^2 = \sum_k (A_k + iB_k)^2,$$

that is to say an operator which is invariant with respect to the adjoint representation i.e.

$$S = D(L)\, S\, D^{-1}(L).$$

If we separate the real and imaginary part we obtain the two operators

$$F = \vec{A}^2 - \vec{B}^2$$

$$\text{and} \qquad G = \vec{A}.\vec{B} \qquad (1\text{-}80m)$$

which commute with all the infinitesimal operators of the Lorentz group. Such operators, F, G are known as the *Casimir operators* of the Lorentz group. If D(L) is an irreducible representation then the operators F and G are multiples of the unit matrix fE, gE. (lemma of Schur) and irreducible representations of the Lorentz group may be characterized by f and g. See Joos p. 65.

## 7.1. The commutation rules of the Lorentz group.

For the study of invariant subspaces in S under the representation $D^{\times}(L)$, formula (1-80b), it is sufficient to study the behaviour of S under the infinitesimal operators $D^{\times}_{\sigma}$. These are defined by

$$\frac{\partial}{\partial\phi_\sigma} D^{\times}(L)S\Big|_{\phi=0} = \frac{\partial}{\partial\phi_\sigma} D(L)\, S\, D^{-1}(L)\Big|_{\phi=0}$$

$$= D_\sigma S - SD_\sigma$$

Thus $$D^{\times}_{\sigma} S = \left[D_\sigma, S\right] \qquad (1\text{-}81)$$

In order to obtain the commutation rules of the Lorentz group without referring to the general theory of Lie groups, we first consider the antisymmetric tensor $F^{\mu\nu} = (\vec{E}, \vec{H})$ transforming by

$$L^{\times} F = L F L^{T}$$

and infinitesimally by

$$L^{\times}_{\sigma} F = L_\sigma F + F L^{T}_{\sigma}$$

It follows that under the rotation $A_1$ and the hyperbolic screw $B_1$ the vectors $\vec{E}_2$ and $\vec{H}_2$ transform as follows.

$$A^{\times}_{1} (\vec{E}_2, \vec{H}_2) = (\vec{E}_3, \vec{H}_3) \text{ and cycl.}$$

$$B^{\times}_{1} (\vec{E}_2, \vec{H}_2) = (\vec{H}_3, -\vec{E}_3) \text{ and cycl.}$$

Thus the vectors $\vec{E}$ and $\vec{H}$ transform under rotations as 3-vectors and they become mixed under hyperbolic screws. Because we may identify $(\vec{E}_2, \vec{H}_2) \sim (-B_2, A_2)$ the above equations $A^{\times}_{1} \vec{E}_2 = \vec{E}_3$, ... become $A^{\times}_{1} B_2 = B_3$, ... thus

$$\left[A_1, (B_2, A_2)\right] = (B_3, A_3) \text{ and cycl.} \qquad (1\text{-}82a)$$

and $$\left[B_1, (B_2, A_2)\right] = (-A_3, B_3) \text{ and cycl.}$$

Hence the commutation rules of the Lorentz group express the behaviour of the spaces A and B of infinitesimal operators under the adjoint representation. (A is spanned by $A_i$ and B is spanned by $B_i$).

These commutation rules can be easily formulated in a way which holds for arbitrary (pseudo)orthogonal groups $O^p_n$, which leave invariant the norm:

$$g_{\mu\nu}\, x^\mu x^\nu = x_\mu x^\mu = x_0^2 + x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 \ldots - x_n^2 .$$

Therefore we note that also for these groups, the following formulae hold

$$D^{\times\mu\nu}\, D = [D^{\mu\nu}, D] \qquad D \equiv (D^{\rho\sigma}) \qquad \text{cf.(1-81)}$$

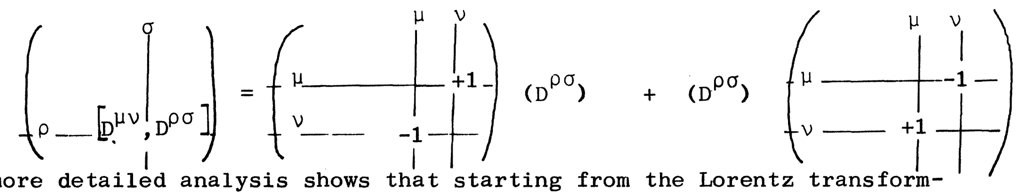
and $\quad D^\times(L)\, D = L\, D\, L^T$

Differentiating the last expression with respect to $\phi_{\mu\nu}$ and putting all $\phi = 0$ we obtain,

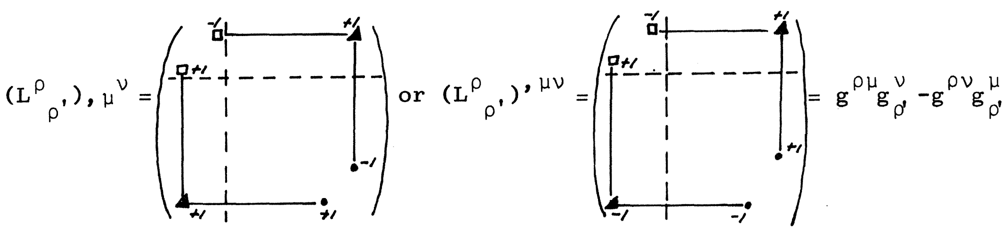
$$D^{\times\mu\nu} D = L'^{\mu\nu} D + D L'^{\mu\nu T} , \quad \text{where } L'^{\mu\nu} = \left(\frac{\partial L}{\partial \phi_{\mu\nu}}\right)_{\phi=0}$$

Thus $\quad [D^{\mu\nu}, D] = L'^{\mu\nu}\, D + D L'^{\mu\nu T}$

or in the components $D^{\rho\sigma}$ of the tensor D:




A more detailed analysis shows that starting from the Lorentz transformations $L^\rho{}_{\rho'}$, the infinitesimal operators $D_\mu{}^\nu = (L^\rho{}_{\rho'})'{}_\mu{}^\nu$ have only 2 non-zero components $(\mu\nu)$ and $(\nu\mu)$. In the figure they are marked with • or ▲ or □ corresponding to the cases that $(\mu\nu)$, $(\nu\mu)$ are space-like, space- and time-like, time-like, i.e.




Hence we obtain the commutation rule:

$$[D^{\mu\nu}, D^{\rho\sigma}] = g^{\mu\rho}\, D^{\nu\sigma} - g^{\nu\rho}\, D^{\mu\sigma} + g^{\mu\sigma}\, D^{\rho\nu} - g^{\nu\sigma} D^{\rho\mu} . \qquad \text{(1-82b)}$$

7.2. Finite-dimensional representations.

The space of antisymmetric tensors $F^{\mu\nu} = (\vec{E}, \vec{H})$ can be decomposed by the following formula

$$(\vec{E}, \vec{H}) = \tfrac{1}{2}(\vec{G}, -i\vec{G}) + \tfrac{1}{2}(\dot{\vec{G}}, +i\dot{\vec{G}}), \text{ where } \vec{G} = \vec{E} + i\vec{H} \text{ and } \dot{\vec{G}} = \vec{E} - i\vec{H}$$

for the basic vectors we have

$$\vec{G}_k = \vec{E}_k - i\,\vec{H}_k \text{ and } \dot{\vec{G}}_k = \vec{E}_k + i\,\vec{H}_k$$

see formula (1-16c). There holds

(a) $\vec{G}$ transforms like a 3-dimensional vector, for $\vec{G}^2$ is invariant, see (2-30).

(b) $\dot{\vec{G}}$ transforms like a 3-dimensional vector, for $\dot{\vec{G}}^2$ is invariant, see (2-30).

(c) $\vec{G}$ and $\dot{\vec{G}}$ determine two invariant spaces which have only the zero vector in common.

Because the space D of infinitesimal operators $(D_{oi}, D_{jk}) = (-B_i, A_i) \sim (E_k, H_k)$ transforms equivalent with the space of antisymmetric tensors, we reduce D by forming the vectors

$$G_k = \tfrac{1}{2}(B_k + iA_k) \text{ and } \dot{G}_k = \tfrac{1}{4}(-B_k + iA_k) \tag{1-83}$$

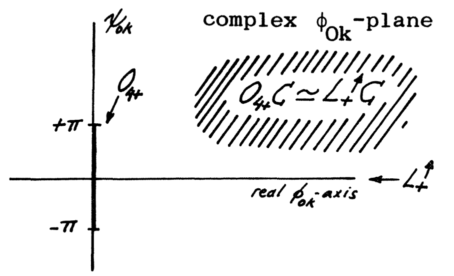
which span the spaces C and D respectively. The properties (a), (b), (c) correspond to

$$\begin{aligned} &(a^1) \quad [G_1, G_2] = iG_3 \text{ and cycl.} \\ &(b^1) \quad [\dot{G}_1, \dot{G}_2] = i\dot{G}_3 \text{ and cycl.} \\ &(c^1) \quad [G_i, G_j] = 0 \quad \text{for all } i, j = 1, 2, 3. \end{aligned} \tag{1-84}$$

Proving $(c^1)$, we observe that $[G_i, G_j]$ is necessarily a vector from the space G as well as a vector from the space $\dot{G}$. Because G and $\dot{G}$ have only the zero-vector in common, we obtain the relations $(c^1)$.

In order to solve the relations (1-84), we note that by the "unitary trick" it is possible to consider the operators from (1-84) as hermitian.

<u>The unitary trick (Weyl).</u> Let us be given a finite-dimensional




representation $D(\phi_\sigma)$ of the Lorentz group $L_+^\uparrow$ ; the numbers $\phi_\sigma$ are real, see the $\phi_{Ok}$-axis in in the figure. $\phi_{Ok}$ belongs to hyp. screws. Because D is an analytic function of $\phi_\sigma$, it follows by analytic continuation that $D(\phi_\sigma)$ is also a representation of the *complex* Lorentz group $L_+^\uparrow C$ which is analytic in the parameters $\phi_\sigma$.

The fact that all $\phi_\sigma$ are complex implies that all matrices $\Lambda^\mu{}_\nu$ which leave invariant the norm $x_0^2 - x_1^2 - x_2^2 - x_3^2$ are complex. Now the complex Lorentz group $L_+^\uparrow C$ is isomorphic with the complex 4-dimensional orthogonal group $O_{4+} C$, this follows after applying the transformation $(y_0, \vec{y}\,) = (x_0, i\vec{x})$ to $R_4$, which transforms the invariant expression $x_0^2 - \vec{x}^2$ into $y_0^2 + \vec{y}^2$ .

In particular the matrices

$$\begin{pmatrix} \cosh\phi_{ok} & \sinh\phi_{ok} \\ \sinh\phi_{ok} & \cosh\phi_{ok} \end{pmatrix} \quad \text{are transformed into} \quad \begin{pmatrix} \cos i\phi_{ok} & -\sin i\phi_{ok} \\ \sin i\phi_{ok} & \cos i\phi_{ok} \end{pmatrix},$$

thus for the group $O_{4+} C$ the complex parameters $(\psi_{oj}, \psi_{kl}) = (i\phi_{oj}, \phi_{kl})$ are more suitable. If we restrict ourselves to *real* $\psi$ we obtain a representation of the real orthogonal group $O_{4+}$, see figure. Hence it follows that to every representation $D(\phi)$ of the **real** Lorentz group $L_+^\uparrow$ there corresponds a representation $D(\psi)$ of the real orthogonal group $O_{4+}$, which is a compact group.

It may be proved that the properties equivalence and irreducibility are preserved in this correspondence (see for instance Hamermesh p. 388).

Thus the classification problem of all finite representations of the Lorentz group reduces to the problem of the classification of all (unitary) representations of the *compact* group $O_{4+}$. This procedure is called by Weyl: *the unitary trick* and is only valid for *finite*-dimensional representations of (pseudo)orthogonal groups. (However the Lorentz group is not compact, so there exists also infinite-dimensional representations, see remark 6.1.).

After differentiating with respect to $\psi_{0j}$ and $\psi_{kl}$ we obtain the infinitesimal operators $-i\ B_k$, $A_k$ of $O_{4+}$. Because $O_{4+}$ is a compact group it follows that these infinitesimal operators are anti-hermitian. Thus the operators $B_k$, $iA_k = J_k$, or $G_k$ and $\dot{G}_k$ of the Lorentz group are hermitian and with formula (1-84) it follows that the operators $G_k$ and $\dot{G}_k$ are the infinitesimal operators of the *rotation group*.

Applying theorem 6.2 we obtain that all eigenvectors $e_m$ of $G_3$ i.e. $G_3\ e_m = me_m$ $-j \leq m \leq +j$ (see formula (1-75)) form a space $E_m$. In view of property (c), $E_m$ is an invariant subspace for the operators $\dot{G}_k$, i.e.

$$G_3\ (\dot{G}_k e_m) = m(\dot{G}_k e_m),\ \text{hence}\ \dot{G}_k e_m \in E_m$$

Thus in the space $E_m$ there are eigenvectors $e_{mm'}$ of $\dot{G}_3$ i.e. $\dot{G}_3\ e_{mm'} = m'e_{mm'}$ $(-j' \leq m' \leq +j')$. If the representation $D(\Lambda)$ of the Lorentz group is irreducible and finite, all the $e_{mm'}$ span the representation space of $D(\Lambda)$. In the theory of Lie groups $(m,m')$ is called the *weight* of $e_{mm'}$ and analogously to the rotation group the irreducible representation $D(\Lambda)$ is characterized by the *highest weight* $(j, j')$. Thus by the infinitesimal method we re-obtain the spinor representations $D^{jj'}$. Ljubarski p. 248. (For the infinite-dimensional case see section 7.3.)

In order to relate the infinitesimal method with the algebraical method, we consider the spinors

$$e_{mm'} = \underbrace{u^{j+m}\ v^{j-m}}_{e_{m0}}\ \underbrace{\bar{u}^{j'-m'}\ \bar{v}^{j'+m'}}_{e_{0m'}} . \tag{1-85a}$$

Considering rotations about the z-axis we have

$$D(\theta)\ e_{mm'} = (e^{-im\ \theta})\ u^{j+m} v^{j-m}\ (e^{-im'\theta})\ \bar{u}^{j'-m'} \bar{v}^{j'+m'} \tag{1-85b}$$

The corresponding infinitesimal operator , $J_3 = i \left(\frac{\partial D(\theta)}{\partial \theta}\right)_{\theta=0}$, is given by

$$\begin{aligned} J_3\ e_{mm'} &= G_3(e_{m0})\ e_{0m'} + e_{m0}\ (\dot{G}_3\ e_{0m'}) \\ &= m\ e_{mm'} + m'\ e_{mm'} \\ &= (m+m')\ e_{mm'} \end{aligned} \tag{1-85c}$$

Consequently the operator $J_3$ consists of an operator $G_3$ which acts only in the space $R^{j0}$ of undotted spinors and an operator $\dot{G}_3$ which acts only the space $R^{0j'}$ of dotted spinors.

In fact using the action of the infinitesimal operators $A_k$ and $B_k$ on the spinors $(\psi^a)$, $(\psi^{\dot{a}})$, see (1-79a,b), and using the definition (1-83) of the infinitesimal operators $G_k$ and $\dot{G}_k$ it follows that the operators $G_k$ and $\dot{G}_k$ acts on each index a or $\dot{a}$ by the table.

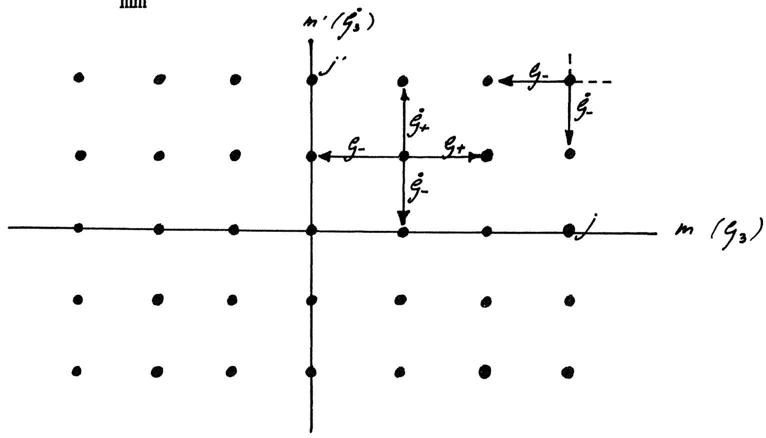
| | $R^{j0}$ | $R^{0j'}$ |
|---|---|---|
| $G_k$ | $\frac{1}{2}\sigma_k$ | 0 |
| $\dot{G}_k$ | 0 | $\frac{1}{2}\sigma_k$ |

(1-85d)

Consequently the indices m, and m' which are defined in (1-85a) are the eigenvalues of $e_{mm'}$, with respect to the operators $G_3$ and $\dot{G}_3$, see (1-85).

We draw a so-called *weight-diagram* or a *Cartan-Stiefel diagram*.

Every vector $e_{mm'}$, is represented in this diagram by its weight (m, m')




(1-85e)

Besides the operators $G_k$ and $\dot{G}_k$, which belong to the representation space of $D^{10}$ and $D^{01}$ respectively, one introduces a canonical basis by defining

$$G_- = G_1 - iG_2 \; , \; G_0 = G_3 \; , \; G_+ = G_1 + iG_2 \qquad (1\text{-}85f)$$

$$\dot{G}_- = \dot{G}_1 - i\dot{G}_2 \; , \; \dot{G}_0 = \dot{G}_3 \; , \; \dot{G}_+ = \dot{G}_1 + i\dot{G}_2 \qquad (1\text{-}85g)$$

The transformation property $J_3\, x_{\pm} = \pm\, x_{\pm}$, which holds for a canonical basis $x_-$, $x_0$, $x_+$, takes here the form $[J_3, G_{\pm}] = \pm\, G_{\pm}$ (see (1-81)). Observing that in the space G of undotted spinors $J_3 = G_3$ and in the space $\dot{G}$ of dotted spinors $J_3 = \dot{G}_3$, we have the commutation rules

$$[G_3, G_-] = -G_- \, , \; [G_3, G_+] = + G_+ \qquad (1\text{-}85h)$$
$$[\dot{G}_3, \dot{G}_-] = -G_- \, , \; [\dot{G}_3, \dot{G}_+] = + \dot{G}_+$$

It follows that the operators $G_{\pm}$ , $\dot{G}_{\pm}$ are the so-called *step-operators* which act on the vectors $e_{mm'}$, defined in (1-85a), by

$$G_+ e_{mm'} = e_{m+1,\, m'} \qquad G_- e_{mm'} = e_{m-1,\, m'}$$
$$\dot{G}_+ e_{mm'} = e_{m,\, m'+1} \qquad \dot{G}_- e_{mm'} = e_{m,\, m'-1} \qquad (1\text{-}85k)$$

or $G_+ e_{mm'} = 0$, ... etc. (see 1-85e).

This can easily be proven by using (1-85h), i.e.

$$G_3(G_+ e_{mm'}) = G_+ G_3 e_{mm'} + G_+ e_{mm'}$$
$$= m\, G_+ e_{mm'} + G_+ e_{mm'} = (m+1)\, G_+ e_{mm'}$$

thus $G_+ e_{mm'} = \lambda\, e_{m+1\, m'}$

Or using the definitions (1-85a, f, g) and we have that $G_+$ acts on the spinor (u, v) by $G_+ : v \to u$ . By which follows the relation (1-85b) with $\lambda = 1$.

## 7.3. Infinite-dimensional representations of the Lorentz group

The third important consequence of theorem 7.1 is that it is possible to determine the matrices $B_k$ also in the infinite-dimensional case and thus to classify *all* the irreducible representations of the Lorentz group. (Gel'fand etc. p. 133 and p. 188.) The operator $B_k$ transforms under the adjoint representation according to $\overset{\times}{B}_k = D(L) \times D'(L)\ B_k$, cf.(1-80b). In remark 7.1 we have noted that under spatial rotations i.e. $D(r) \times D'(r)$, every operator $B_k$ transforms like a 3-dimensional vector, see also the first line of (1-82a).

Thus $B_k$ belongs to the 3-dimensional irreducible component $D^j$ ($j = 1$), which is contained in the tensor product $D(r) \times D'(r)$, or irreducible components $D^1$ if there are more.

We shall select a suitable basis $\xi^i$ and $\xi_i$, for $D(r)$ and $D'(r)$ respectively, we note that $D'(r) = D^{-1T}(r)$ is a "covariant" representation. If the components $x^i{}_{i'}$ of the *vector* $B_k$ with respect to the tensorbasis $\xi^i\xi_{i'}$ are determined, then the *operator* $B_k$ is known. Now the representation $D(r)$ of the rotation group acting in the space R is in general reducible into irreducible representations $D^j$, i.e.

$$D = \sum D^j \quad \text{or} \quad R = \sum R^j. \tag{1-86}$$

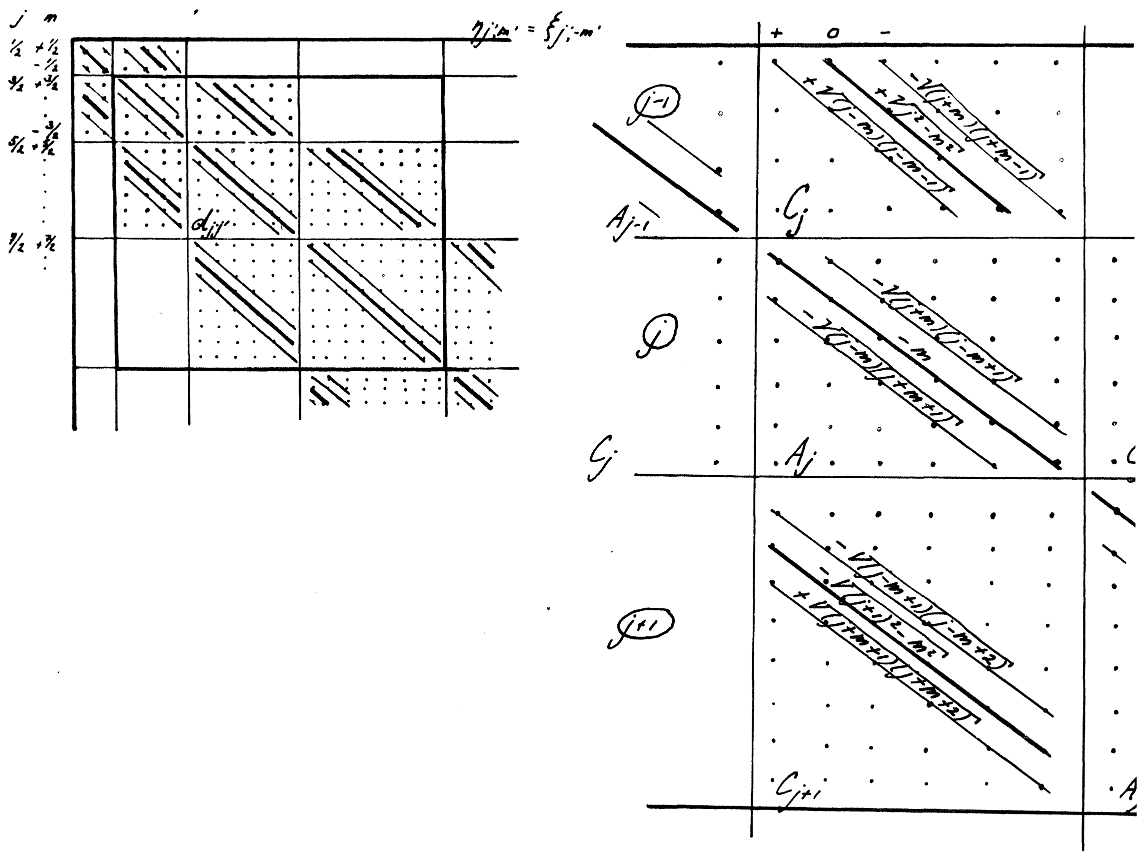
For every $D^j$ one may select the canonical basis $\xi_{jm}$ of $R^j(-j \leq m \leq +j)$ and for every $D^{j'}$ one may select the canonical basis $\eta_{j'm'}$ of $R^{j'}$ $(-j' \leq m' \leq j')$, using $D^{-1T}(r) = D^T(r^{-1})$ one may proof that $\eta_{j'-m'} = \xi_{j'm'}$.

If we require moreover that the representation $D(\Lambda)$ of the restricted Lorentz group acting in R is irreducible, then from observation (1) on page 81 it follows that each irreducible component $D^j$ of the rotation group appears only once.

All vectors $\xi_{jm}\xi_{j'm'}$ span the product space $R \times R'$.

In order to visualize the zero's and non-zero's of the matrix components $c_{jmj'm'}$ of $B_k$, we draw the following figure.

In the figure all possible vectors $\xi_{jm}$ (where $j = 0, \frac{1}{2}, 1, \ldots$ and $-j \leq m \leq +j$) are indicated by the numbers jm on the vertical line, in the same way the vectors $\eta_{j'm'} = \xi_{j'-m'}$ are given by the number j'm' on the horizontal line. The fact that only j is half-integer $j = \frac{1}{2}, \frac{3}{2}, \ldots$ and that



every j appears only once will be explained below.

The vectors $\xi_{jm}\xi_{j'-m'}$ from the tensor product are given by the $(2j + 1)(2j' + 1)$ lattice points in the square $(j, j')$.

Each product $R^{j} \times R^{j'}$ is reduced by

$$R^{j} \times R^{j'} = \sum_{\ell=|j-j'|}^{j+j'} R^{\ell}(j, j')$$

and consequently only tensor products with $j' = j-1, j, j+1$ contain a 3-dimensional representation space $R^{1}(j, j')$. Therefore only the spaces $R^{j} \times R^{j'}$ with $j' = j-1, j, j+1$ may contribute to the operator $B_k \in R^{1}$, where

$$R^{1} = \sum d(j, j')\, R^{1}(j, j'). \qquad (1\text{-}87)$$

The corresponding squares in the figure are filled with dots and we obtain two infinite chains of blocks, one with weights $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ (see figure), the two-valued spinor representations, and one with weights $j = 0, 1, 2, \ldots$, the tensor representations.

Inspection of this diagram leads to the following observations.

(1) It is clear that the operators $B_k$ transform the spaces $R^j$ into the spaces $R^{j-1}$, $R^j$, $R^{j+1}$ and thus *every irreducible* representation of the Lorentz group is given by a large square along the main diagonal, containing a system of *connected* dotted blocks with weight factors $d(jj')$, see figure. If the lowest weight of the rotation group appearing is $j_0$, then the weights $j_0$, $j_0+1$, $j_0+2$, ... also appear ($j_0 = 0, \frac{1}{2}, 1, \ldots$). Thus if the representation $D(\Lambda)$ is irreducible, then each irreducible component $D^j$ of the rotation group appears only once.

(2) The three-dimensional subspace $R^1(j, j')$ of $R^j \times R^{j'}$ is spanned by the three diagonals of each block (see §3.2 and formula (1-34c). The upper, main, and lower diagonal are the three vectors: $\zeta_{1k} = \zeta_{1,-1}, \zeta_{1,0}, \zeta_{1,+1}$, where $\zeta_{1,-1} = \dfrac{x_1 - ix_2}{\sqrt{2}}$, $\zeta_{1,0} = x_3$, $\zeta_{1,+1} = (-)\dfrac{x_1+ix_2}{\sqrt{2}}$.

Thus the operator $B_0 = iB_3$ is given by all main diagonals $\zeta_{1,0}(j,j')$ each labelled with a number $d(j, j')$, which give the contribution of each space $R^1(j, j')$ to $R^1$, see (1-87). Similarly the operators $B_- = iB_1 + B_2$ and $B_+ = iB_1 - B_2$ are given by the upper and lower diagonals respectively. It follows that the operators $B_-$, $B_0$, $B_+$ transform the vectors $\xi_{jm}$ with eigenvalue m into vectors $\xi_{\ell,m-1}$, $\xi_{\ell m}$, $\xi_{\ell,m+1}$ respectively ($\ell = j-1, j, j+1$).

Compare the formulae of Gel'fand on p. 188.

In the actual calculation of the components of the vector $B_k$, we first express the vectors $\zeta_{\ell k}(j, j')$ which span the space $R^1(j, j')$, as a linear combination of the vectors $\zeta_{jm}\zeta_{j'-m'}$.

Using the Clebsch-Gordan coefficients we have

$$\zeta_{\ell k}(j, j') = \sum_{m+m'=k} B_{\ell k\ jm\ j'-m'}\ \zeta_{jm}\zeta_{j'-m'}$$

(see section 3.2 formula (1-34b) .The C.G. coefficients may be found for instance in Gel'fand p. 152).
In the second figure the coefficients $B_{\ell k\ jm\ j'm}$ (k = -1, 0, +1) are written on the corresponding lines m+m' = k, where we have omitted the factors which are only dependent of j; these are incorporated into the coefficients d(j, j').
Thus the operators $B_k$ in turn may be written as a linear combination of all $\zeta_{\ell k}(j, j')$

$$B_k = d(j, j')\ \zeta_{\ell k}(j, j') \text{ (summation over } j = j_0, j_0+1, \dots \text{ and } j' = j-1, j, j+1).$$

Now it may be easily verified, that by a coordinate transformation

$$\xi'_{jm} = h(j)\ \xi_{jm},$$

which does not alter the formulae for the infinitesimal rotation operators $A_i$, the factors d(j, j') may be transformed in such a way that

$$d(j-1, j) = d(j, j-1) = C_j$$

$$\text{and } d(j, j) = A_j.$$

The "weights" $A_j$, $C_j$ are written in the corresponding blocks and thus the formulae for the infinitesimal operators belonging to hyperbolic screws are

$$B_+\ \xi_{jm} = +\ C_j\sqrt{(j-m)(j-m-1)}\ \xi_{j-1\ m+1} - A_j\sqrt{(j-m)(j+m+1)}\ \xi_{j\ m+1}$$

$$\dots \text{ etc.} \qquad + C_{j+1}\sqrt{(j+m+1)(j+m+2)}\ \xi_{j+1\ m+1}.$$

The coefficients $A_j$, $C_j$ still have to be determined. For that purpose we use the second line of (1-82) and obtain the commutation rules

$$[B_+, B_0] = A_+, \text{ where } B_+ = iB_1 - B_2,\ B_0 = iB_3,\ A_+ = iA_1 - A_2.$$

Substituting the expressions for $B_+$, $B_0$, which are now known, and substituting $A_+ = J_+$ (formula 1-75b) in this commutation rule one obtains two simple recurrence relations for the $A_j$ and $C_j$.
It appears that there are only two independent coefficients: the number

$j_0$ indicating the lowest spin value, and the weight $A_{j_0}$ which may be arbitrary complex.

Putting $A_{j_0} = \frac{ij_1}{j_0+1}$, where $j_1$ is arbitrary complex there follows

$$A_j = \frac{ij_0j_1}{j(j+1)} \quad \text{and} \quad C_j = \frac{i}{j}\sqrt{\frac{(j^2-j_0^2)(j^2-j_1^2)}{4j^2-1}} \; .$$

Thus all irreducible representations of the Lorentz group are classified by the pair of numbers $(j_0, j_1)$ or equivalently by $-(j_0, j_1)$. We note that Gel'fand uses the notation $\tau(\ell_0, \ell_1)$.

It is clear that the irreducible representation is finite if there is a coefficient $C_j = 0$, thus if $j_1$ is real and $j = |j_1|$.

The irreducible representations of the rotation group which appear in $(j_0, j_1)$ have weights

$$j_0, \; j_0+1, \; j_0+2 \; , \; \dots \; .$$

In the finite case the sequence ends with $|j_1|-1$. The coefficients $j_0$ and $j_1$ are related to the coefficients $j$, $j'$ of $D^{jj'}$ by

$$j_0 = |j-j'| \qquad\qquad j_0 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2} \; ,$$

$$j_1 = (j+j'+1) \times \text{sign}\,(j-j') \quad j_1 = \pm 1, \pm \tfrac{3}{2}, \; \dots$$

(see formula (1-34a)).

If $j_1$ is an arbitrary complex number $j_1 \neq \pm 1, \pm \frac{3}{2}, \dots$ then all $C_j \neq 0$, and the chain of blocks in the figure is infinite. In this case we have obtained an infinite-dimensional irreducible representation of the Lorentz group, see remark 6.1.

<u>Unitary representations of the Lorentz group</u>

For an unitary representation the infinitesimal operators $D_\sigma$ must be anti-hermitian, because $D^+ = D^{-1}$ implies $D_\sigma^+ = -D_\sigma$ and thus the operators $iD_\sigma = iA_3, iB_3, \dots$ must be hermitian. For the rotation operators $iA_k$ this property is already satisfied and it is clear from the foregoing figure that the $iB_k$ are hermitian if the coefficients $A_j$ are real and

$C_j$ are purely imaginary. Using the expressions for $A_j$ and $C_j$ mentioned above we obtain that $A_j$ is real implies that (1) $j_1$ is purely imaginary or (2) $j_0 = 0$, and that $C_j$ is purely imaginary implies that the term under the $\sqrt{\ }$-sign is positive:
Because $(j^2 - j_0^2)$ and $(4j^2 - 1)$ are positive there follows $j^2 - j_1^2 \geq 0$.
In case (1) this equality is satisfied. In case (2) it follows, because j=1 is the lowest weight, that $j_1$ is real and $|j_1| < 1$. Summarizing we obtain unitary irreducible (infinite-dimensional) representations in two cases, i.e.

(1) $j_0 = 0, \frac{1}{2}, 1, \ldots$ and $j_1$ = pure imaginary (*the main series* of representations)

(2) $j_0 = 0$ and $j_1$ = real and $|j_1| \leq 1$ (*the supplementary series* of representations).

## 8. Lorentz covariant equations in Bhabha form

In this section we study, in a manner resembling the methods used in the foregoing section, the form of linear partial differential equations which are Lorentz covariant.
It appears that the use of infinitesimal operators facilates considerably the solution of this problem.

## 8.1. General theory

We shall use the following notations, $\psi(x)$ denotes a vector function, $\psi(x) \equiv (\psi^1(x), \ldots, \psi^n(x))$, which transforms by a representations D(L) of the Lorentz group i.e.
if $x' = Lx$ then $\psi'(x') = D(L)\,\psi(x)$ and the operator $\partial \equiv (\partial_0, \partial_1, \partial_2, \partial_3)$ transforms by $\partial' = (L^{-1})^T\partial$.
The symbol $L^\mu$ ($\mu = 0, 1, 2, 3$) denotes an $n \times n$ matrix which appears in a Lorentz covariant equation (in general $L^\mu$ is an operator). In physics the letter $\beta^\mu \equiv L^\mu$ is also used.

Definition The differential equation

$$(L^\mu\partial_\mu + i\kappa)\cdot\psi(x) = 0 \qquad (1\text{-}87)$$

is called Lorentz covariant if its form remains invariant under a Lorentz transformation, i.e. after a Lorentz transformation L there must hold

$$(L^\mu\partial_{\mu'} + i\kappa)\cdot\psi'(x') = 0.$$

One may also define equations which are only invariant with respect to the restricted group $L_+^\uparrow$. The problem is to determine all possible Lorentz covariant equations, thus to determine all quartets $L^\mu$ *).

Theorem 8.1

*The equation $(L^\mu\partial_\mu + i\kappa)\psi = 0$ with $\kappa \neq 0$ is Lorentz covariant if for all Lorentz transformations $L \equiv L^\mu{}_\nu$ there holds*

$$L^\mu{}_\nu D(L)\, L^\nu D^{-1}(L) = L^\mu \qquad (1\text{-}88)$$

PROOF. We substitute $\psi(x) = D^{-1}(L)\psi'(x')$ and $\partial = L^T\partial'$ into (1-87) and obtain

$$(L^\nu L^\mu{}_\nu\partial_{\mu'} + i\kappa)\cdot D^{-1}(L)\psi'(x') = 0. \qquad (1\text{-}89)$$

Premultiplying this equation by D(L), observing that D(L) commutes with the numbers $L^\mu{}_\nu$, but not with $L^\nu$, we find

$$(L^\mu{}_\nu D(L)\, L^\nu D^{-1}(L)\partial_{\mu'} + i\kappa)\cdot\psi'(x') = 0.$$

After comparing this equation with (1-87) we obtain

*) The first proposal to formulate Lorentz covariant equations in the form (1-87) was done by Bhabha in 1946.

$$L^{\mu} = L^{\mu}{}_{\nu} D(L)\, L^{\nu} D^{-1}(L).$$

Remark 8.1

We observe that if $\kappa = 0$ (zero-mass case) then it is already sufficient for the equation to be Lorentz covariant, that there is an additional m-dimensional representation E(L) so that after multiplying (1-89) from the left by E(L) there holds

$$L^{\mu} = L^{\mu}{}_{\nu}\, E(L)\, L^{\nu} D^{-1}(L).$$

In this case the matrices $L^{\mu}$ are $m \times n$.

Example 8.1

We substitute the rotations $L^{\mu}{}_{\nu} \equiv r^{\mu}{}_{\nu}$ into (1-88), after which in the first row ($\mu = 0$) only the term with $r^{0}{}_{0} = 1$ remains

$$L^{0} = D(r)\, L^{0}\, D^{-1}(r).$$

Thus $L^{0}$ commutes with all rotations $D(r)$ : $[L^{0}, D(r)] = 0$.

Example 8.2

Now we substitute a space reflection $L = P$. In the same way as in example 8.1 we obtain that $L^{0}$ commutes with $D(P)$,

i.e. $[L^{0}, D(P)] = 0$.

The matrix $L = \Lambda$ belongs to the restricted Lorentz group $L_{+}^{\uparrow}$ and formula (1-88) describes in fact an infinite number of equations. We restrict this number by using the infinitesimal operators.

First we write equation (1-88) in *abbreviated* form:

$$\Lambda\, D(\Lambda)\, \vec{L}\, D^{-1}(\Lambda) = \vec{L} \qquad (1\text{-}90)$$

Further we use the notation $D_{\sigma}$ and $\Lambda_{\sigma}$ for the infinitesimal operators of the representation $D(\Lambda)$ and the vector representation $\Lambda$ respectively.

Theorem 8.2. *In order that the equation (1-87) is Lorentz covariant there must hold* $[D_{\sigma}, \vec{L}] = -\Lambda_{\sigma}\vec{L}$ (1-91)

PROOF. We differentiate (1-90) with respect to the parameter $\phi_{\sigma}$ and put $\phi = 0$.

Using $\left(\frac{\partial D^{-1}(\Lambda)}{\partial \phi_{\sigma}}\right)_{\phi=0} = -D_{\sigma}$ we obtain

$$\Lambda_\sigma \vec{L} + \Lambda(0)\,(D_\sigma \vec{L} - \vec{L} D_\sigma) = 0$$

from which theorem 8.2 follows, cf. (1-81).

Example 8.3

We substitute rotations in (1-91), e.g. $D_\sigma = A_1$, in order to obtain the formulae

$$\begin{pmatrix} [A_1, L^0] \\ [A_1, L^1] \\ [A_1, L^2] \\ [A_1, L^3] \end{pmatrix} = - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} L^0 \\ L^1 \\ L^2 \\ L^3 \end{pmatrix}$$

The commutation rules $[A_1, L^0] = [A_1, L^1] = 0$ and $[A_1, L^2] = L^3$, $[A_1, L^3] = -L^2$ follow. The relation $[A_i, L^0] = 0$, holds for all rotations which is equivalent to the statement made in example 8.1.

Example 8.4

Now we substitute hyperbolic screws, e.g. $D_\sigma = B_1$, in order to obtain

$$\begin{pmatrix} [B_1, L^0] \\ [B_1, L^1] \\ [B_1, L^2] \\ [B_1, L^3] \end{pmatrix} = - \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} L^0 \\ L^1 \\ L^2 \\ L^3 \end{pmatrix}$$

From which the formulae $[B_i, L^0] = -L^i$ (1-92)

and $[B_i, L^i] = -L^0$ (1-93)

follow.

The following conclusion may be made: If a representation $D(\Lambda)$ in the space of $\psi(x)$ with infinitesimal operators $D_\sigma$ is given, then in order to obtain the corresponding Lorentz covariant equations, it is sufficient to determine only the matrix $L^0$.

With the aid of the relation (1-92) the remaining matrices $L^i$ may be found. The matrix $L^0$ is determined by the relations (1-92) and (1-93). By eliminating $L^i$ in (1-93) we obtain

$$[B_i, [B_i, L^0]] = L^0. \tag{1-94}$$

Also in the subsequent theory, e.g. determination of the rest mass and spin of particles, the matrix $L^0$ will play a fundamental role. Summarizing the matrix $L^0$ must satisfy the relations

(1) $[D(r), L^0] = 0 \quad (r \in O_{3+})$

(2) $[B_i, [B_i, L^0]] = L^0$

(3) $[D(P), L^0] = 0.$

From these three equations one may obtain the general solution of the matrices $L^0$ and thus of the matrices $L^i$.

## 8.2. Construction of covariant equations

Instead of starting from the commutation relations (1) (2) (3) mentioned in the end of the preceding section, it is in individual cases easier to determine $L^0$ by using the more pictural method described in section 7. For this purpose we remark that the left- and right-hand side of equation (1-88) implies that $L^{\mu}$ transforms under the representation $D(L) \times D^{-1T}(L)$ as a 4-vector *). Thus

Theorem 8.3. *There only exist Lorentz covariant equations in $\psi(x)$ if the representation $D(L) \times D^{-1T}(L)$ contains the vector representation $D^{\frac{1}{2}\frac{1}{2}}$.*

We suppose that D is a representation of the full group, i.e. $D = \sum (D^{jj'} + D^{j'j})$, if $j \neq j'$. Then there holds that $D^{-1T}$ is equivalent to D. The proof of this depends on formula (1-28) on page 23 which shows that $A^{-1T}$ is equivalent with $\bar{A}$, thus $(D^{\frac{1}{2}0})^{-1T} \sim D^{0\frac{1}{2}}$ and by taking tensor-products we have

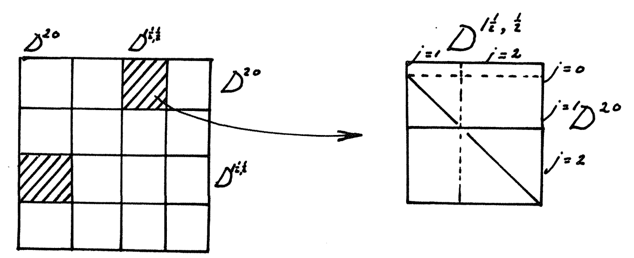
$$(D^{jj'})^{-1T} \sim D^{j'j} \quad \text{and} \quad D^{-1T} \sim D.$$

Now with the product formula for representations i.e.

$$D(L) \times D^{-1T}(L) = \sum D^{jj'} \times D^{kk'} = \sum_{\ell = j-k\,,\ \ell' = j'-k'}^{\ell = j+k,\ \ell' = j'+k'} D^{\ell\ell'}$$

it follows that whenever the representation $D^{jj'}$ appears in D(L) there necessarily exists a representation $D^{kk'}$ such that $k = j \pm \frac{1}{2}$, $k' = j' \pm \frac{1}{2}$ in order that there exists representations $D^{\ell\ell'} = D^{\frac{1}{2}\frac{1}{2}}$.

$L^0 =$



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*) Observe that $L^{\mu}$ does not transform as the component $x^{\mu}$ of a vector, but that *each* $L^{\mu}$ is a basis vector in the space of covariant vectors $(x_{\mu})$, because the condition of Lorentz covariance implies that after substitution of $L'^{\mu} = D(L)L^{\mu} D^{-1}(L)$ in (1-87) there holds $D' = L'^{\mu}\partial_{\mu} = L^{\mu}\partial'_{\mu}$.

In order to determine the operator $L^0$ we give similar arguments as in section 7. If $\xi_i$ is a basis of D(L) and $\xi_{i'}$ a basis of $D^{-1T}(L)$ then $\xi_i \xi_{i'}$ is a basis of $D(L) \times D^{-1T}(L)$ and the operator-vector $L^0$ is given by the components $x^{ii'}$ with respect to $\xi_i \xi_{i'}$ (see figure). On grounds of the arguments mentioned above only the blocks $D^{jj'} \times D^{kk'}$, with $k = j \pm \frac{1}{2}$, $k' = j' \pm \frac{1}{2}$, in the figure contribute to $L^0$. With respect to rotations $D^{kk'}$ reduces into $D^h$ ($h = |k-k'|, \dots k+k'$) and $D^{jj'}$ into $D^p$ ($p = |j-j'|, \dots j+j'$) respectively. Because $L^0$ is, with respect to rotations, a scalar it follows that $L^0$ has only components in the representation spaces $D^p \times D^p$ and $L^0$ is given by the $(2p+1) \times (2p+1)$ scalar matrices $c_p E$.
See figure on the right and formula (1-34c).

Example 8.5. We require that $\psi$ transforms by the representation $D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}$. There holds

$$(D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}) \times (D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}) = \dots \dotplus 2D^{\frac{1}{2}\frac{1}{2}} \dotplus \dots$$

Thus only the combinations $D^{\frac{1}{2}0} \times D^{0\frac{1}{2}}$ contain the vector representation and it follows that a Lorentz covariant equation in $\psi$ with $\kappa \neq 0$ may be constructed.

$$L^0 = \begin{array}{c|c|c|l} & D^{\frac{1}{2}0} & D^{0\frac{1}{2}} & \\ \hline & 0 & B_{12} & D^{\frac{1}{2}0} \\ \hline & B_{21} & 0 & D^{0\frac{1}{2}} \end{array}$$

For this purpose we consider the $4 \times 4$ matrix $L^0$.
It follows that only the blocks $B_{12} = (D^{\frac{1}{2}0} \times D^{0\frac{1}{2}})$ and $B_{21} = (D^{0\frac{1}{2}} \times D^{\frac{1}{2}0})$ can contribute to $L^0$. With respect to rotations $L^0$ is a scalar,
hence the blocks $B_{12}$ and $B_{21}$ are scalar matrices $c_{12}E$ and $c_{21}E$.
By a space reflection P the spinors belonging to $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$ are interchanged and thus there must hold $c_{12} = c_{21}$.
We have obtained

$$L^0 = c \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad (1\text{-}95)$$

Using the infinitesimal operators $B_i = \frac{1}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}$ (formula (1-79a,b)) and the relation $[B_i, L^0] = -L^i$ we obtain the matrices

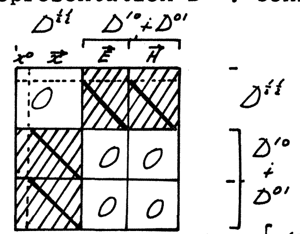
$$L^0 = \begin{pmatrix} 0 & \sigma^0 \\ \sigma^0 & 0 \end{pmatrix} \qquad L^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \tag{1-96}$$

which are the Dirac matrices $\gamma^\mu$ from section 4.2. and we have re-obtained the *Dirac-equation*.

<u>Example 8.6.</u>

On the contrary there is no equation with $\kappa \neq 0$ such that $\psi$ transforms by $D^{10} \dot{+} D^{01}$. However, if we require that $\psi$ transforms by $D^{10} \dot{+} D^{\frac{1}{2}\frac{1}{2}} \dot{+} D^{01}$ we obtain the combinations $D^{10} \times D^{\frac{1}{2}\frac{1}{2}}$ and $D^{\frac{1}{2}\frac{1}{2}} \times D^{01}$ which contain the vector representation $D^{\frac{1}{2}\frac{1}{2}}$. Consequently, in the matrix $L^0$ we only have to consider the shaded blocks and because $L^0$ is a scalar with respect to rotations it follows that $L^0$ consists only of scalar matrices. By a reflection in $D(P)$ : $(x, \vec{E}, \vec{H}) \to (Px, -\vec{E}, \vec{F})$, it

$L^0$ =




is shown that the relation $[D(P), L^0] = 0$ implies that $L^0$ is of the form

$$L^0 = \begin{pmatrix} 0 & a & 0 \\ b & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover if one requires that $L^0$ must be hermitian (see section 8.3) there follows

$$L^0 = \begin{pmatrix} 0 & E & 0 \\ E & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1-97}$$

Together with the remaining $10 \times 10$ matrices $L^i$ we have re-obtained the Kemmer matrices, which appear in the *Proca equations* (see section 4.1, formula (1-48c))

<u>Example 8.7</u>

There does exist an equation with $\kappa = 0$, in which $\psi$ transforms by $D^{10} \dot{+} D^{01}$. For that purpose we choose the additional representation E(L) as

$$E(L) = D^{\frac{1}{2}\frac{1}{2}} \qquad \text{(see remark 8.1)}$$

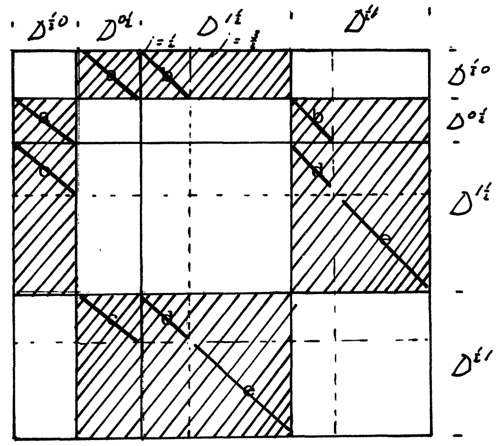
The combinations $D^{10} \times D^{\frac{1}{2}\frac{1}{2}}$ and $D^{01} \times D^{\frac{1}{2}\frac{1}{2}}$ contain, as in example 8.6, the vector representation. If one chooses

$$E(L) = D^{\frac{1}{2}\frac{1}{2}+} \dot{+} D^{\frac{1}{2}\frac{1}{2}-}$$

(the tensorsum of the vector and pseudo-vector representation) one obtains the *equation of Maxwell*.

<u>Example 8.8</u>

The *Pauli-Fierz equation*. We require that the vector $\psi$ transforms by the representation $D^{\frac{1}{2}0} \dot{+} D^{0\frac{1}{2}} \dot{+} D^{1\frac{1}{2}} \dot{+} D^{\frac{1}{2}1}$

$L^0 =$ 

The vector representation is contained in the 4 combinations

$$\begin{matrix} D^{\frac{1}{2}0} & \times & D^{0\frac{1}{2}} \\ \times & & \times \\ D^{1\frac{1}{2}} & \times & D^{\frac{1}{2}1} \end{matrix}$$

(see the shaded blocks)

We consider in particular the tensor product $D^{1\frac{1}{2}} \times D^{\frac{1}{2}1}$. With respect to rotations $D^{1\frac{1}{2}}$ decomposes into $D^{1} \times D^{\frac{1}{2}} = D^{\frac{3}{2}} \dot{+} D^{\frac{1}{2}}$, a four- and two-dimensional representation, see figure. The same holds for $D^{\frac{1}{2}1}$.

Thus the scalar $L^0$, with respect to rotations, is contained in the combinations $D^{\frac{1}{2}} \times D^{\frac{1}{2}}$ and $D^{3/2} \times D^{3/2}$, see the letters d and e in the figure. The remaining coefficients a, b, c follows by the same argument as in example 8.6.

Now we consider all submatrices with spin value $\frac{1}{2}$ and spin value $\frac{3}{2}$.

$$L^0_{\frac{1}{2}} = \begin{pmatrix} 0 & a & b & 0 \\ a & 0 & 0 & b \\ c & 0 & 0 & d \\ 0 & c & d & 0 \end{pmatrix} \qquad L^0_{\frac{3}{2}} = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}, \tag{1-98}$$

and choose $a = b = -d = \frac{1}{2}$ and $e = 1$.

In this case we obtain the equation of Pauli -Fierz. The choice of a,..e ensures us that $L^0$ is hermitian with respect to some inner product, see section 8.3.

8.3 Spin and rest mass of particles

In physics the number of covariant equations is limited by the condition that $L^0$ must be hermitian.

This depends on the fact, that one requires that for $\psi(x)$ there must be an invariant functional i.e. $\mathcal{L}[\psi(x)] = \mathcal{L}[\psi'(x')]$, the so-called invariant Lagrangian. One may prove that invariant Lagrangians exist only if the following two requirements are fullfilled:

(1) There exists some invariant inner-product $(\psi_1,\psi_2) = \psi_1^T H \psi_2$ i.e.

$$(\psi_1, \psi_2) = (D(L)\psi_1, D(L)\psi_2).$$

This inner-product does not have to be positive definite.

(2) If the matrices $L^\mu$ are hermitian with respect to this inner-product, i.e. $L^+H = HL$. It is sufficient that $L^0$ is hermitian in order that all $L^\mu$ are hermitian.

Gel'fand p. 284.

In the finite-dimensional case there is alsways an invariant inner-product. In section 3 we have shown that in the space of spinors the expression $\phi^a\chi_a$ is an invariant and in the space of 4-component spinors, which transform by $D^{\frac{1}{2}0} \dotplus D^{0\frac{1}{2}}$, we define

$$(\psi, \psi) = \overline{\phi^a\chi_a} + \phi^a\overline{\chi_a} = \overline{\phi}^a\chi_{\dot a} + \chi_{\dot a}\overline{\phi}^a$$

$$= \overline{(\phi^1, \phi^2, \chi_{\dot 1}, \chi_{\dot 2})} \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \\ \phi_{\dot 1} \\ \phi_{\dot 2} \end{pmatrix} = \psi^+H\psi.$$

Noting that $H = \gamma^0$, one often uses in physics the notation $\bar\psi = \psi^+\gamma^0$, by which one can form scalars $\bar\psi\,\psi$, vectors $\bar\psi\gamma^\mu\psi$, ... etc.

By generalizing this expression to spinors belonging to the representation $D^{jj'} + D^{j'j}$ we obtain

$$(\phi, \chi) = \overline{\phi^{a_1\ldots}{}_{\dot c_1\ldots}}\, \chi_{\dot a_1\ldots}{}^{c_1\ldots} + \overline{\chi_{\dot a_1\ldots}{}^{c_1\ldots}}\, \phi^{a_1\ldots}{}_{\dot c_1\ldots}$$

In the infinite-dimensional case there exists only an invariant hermitian form for unitary representations.

<u>Example 8.9</u>

One may verify, for instance, that the matrix $L^0$ in example 8.8 is hermitian, $(L^0)^+H = HL^0$, where

$$H = \begin{pmatrix} & E & & \\ E & & & \\ & & & -E \\ & & -E & \end{pmatrix}$$

and the same holds for the matrix $L^0$ from example 8.6 where

$$H = \begin{pmatrix} g & & \\ & -E & \\ & & E \end{pmatrix}$$

Besides the condition that $L^0$ must be hermitian one requires further that the $\psi(x)$ span a space, which is invariant under translations in $R_4$: $x' = x + a$. The group, which contains Lorentz transformations as well as translations is called the *Poincaré-group*. The importance of the Poincaré-group lies in the fact that a physical law must be independent of place and time. For that purpose one requires that the covariant equation has solutions in the form of plane waves i.e.

$$\psi(x) = \psi(p)\, e^{ipx} \quad (p.x = p_\mu x^\mu)\text{, in a homogeneous field,}$$

where p is the energy-momentum vector $(\frac{E}{c}, m\vec{v})$ from formula (1-7). Substitution in

$$(L^\mu \partial_\mu + i\kappa)\psi(\kappa) = 0 \qquad (1\text{-}99)$$

yields

$$(L^\mu p_\mu - \kappa)\psi(\mu) = 0 \qquad (1\text{-}100)$$

In order that this equation has a solution $\psi(x) \neq 0$, there must hold

$$\det(L^\mu p_\mu - \kappa) = 0.$$

The investigation of equation (1-100) demands the fact that there holds

$$D(\Lambda)L^\nu D^{-1}(\Lambda) = \Lambda_\mu{}^\nu L^\mu,$$

see formula (1-88). Left- and right-multiplying equation (1-100) by $D(\Lambda)$ and $D^{-1}(\Lambda)$ respectively we obtain

$$\det(L^\mu p_\mu - \kappa I) = \det(L^\mu p'_\mu - \kappa I) \text{ where } p'_\mu = L_\mu{}^\nu p_\nu.$$

There are two cases which are important in physics.

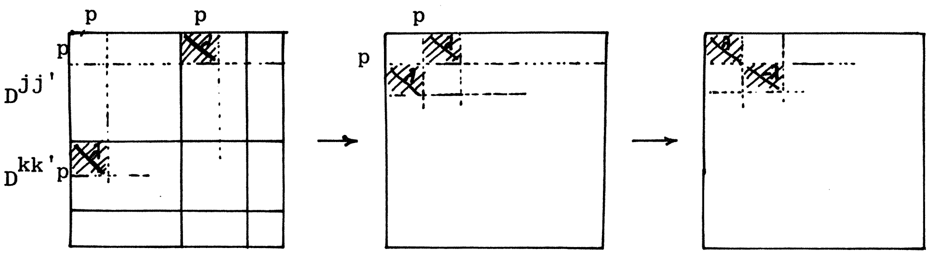
I) $p^2 > 0$. Then there is a Lorentz transformation $L_\mu{}^\nu$ such that $p'_\mu = (p_0, 0, 0, 0)$. The number $\sqrt{p^2} = p_0$ is known as the *rest mass* of the particle and equation (1-99) becomes

$$(L^0 p_0)\ \psi(p) = \kappa\ \psi(p).$$

If $\lambda$ is an eigenvalue of $L^0$ then $\kappa$ is related to the rest mass $p_0$ by $\lambda p_0 = \kappa$ or, in other words, the relation $p_0 = \frac{\kappa}{\lambda}$ is a condition in order that $\psi(x) = \psi(p)e^{ip.x}$ is a solution of equation (1-98).

We consider only real $\lambda \neq 0$, because if $\lambda = 0$ then $\kappa = 0$ and $p_\mu$ is entirely indetermined.

The eigenvalues $\psi_\lambda$ are easily determined. It is known from the general theory that the matrix $L^0$ consists of scalar matrices $\lambda E$, subblocks $D^p \times D^p$, which are contained in the blocks $D^{jj'} \times D^{kk'}$ $(k,k')=(j\pm\frac{1}{2}, j'\pm\frac{1}{2})$, see theorem 8.3 and figure (1-101a). After re-ordening rows and columns we obtain the matrix (1-101b) and using transformations $\begin{pmatrix} E & E \\ E & -E \end{pmatrix}$ the matrix (1-101b) may be brought in the form (1-101c).




(a) (b) (c) (1-101)

From which follows:

If $\lambda$ is an eigenvalue then $-\lambda$ is also an eigenvalue. Thus if there is a particle p with rest mass $p_0 > 0$ then there is also a particle $\tilde{p}$ with $\tilde{p}_0 = -p_0$, the so-called anti-particle with mass $|\tilde{p}_0|$.

If the eigenvector $\psi_\lambda$ $(\lambda \neq 0)$ belongs to the irreducible representation space $D^j$ of the rotation group $O_{3+}$ then the number j is called the *spin value of the particle.* In this case the spin value is uniquely determined.

From formula (1-95) and the considerations given above it follows that the Dirac equation describes particles with spin value $\frac{1}{2}$ and from formula (1-97) it follows that the Proca equation describes particles with uniquely determined spin 1.
The condition $\lambda \neq 0$ plays an essential role here.
From formula (1-98) it follows that the eigenvalue $\lambda$ of $L^0_{\frac{1}{2}}$ is equal to zero, thus $L^0_{\frac{3}{2}}$ is the only matrix, which has non-zero eigenvalues and so the Pauli-Fierz equation describes particles with uniquely determined spin value $\frac{3}{2}$ .
So far we have considered momentum vectors $(p^0, \vec{0})$, the corresponding $x^\mu$-system is called the *rest system* of the particle. For an arbitrary $p^{\mu'}$ there is a coordinate transformation such that $(p^0, 0) \to p^{\mu'}$ .
The rotation group $O_{3+}$ transforms into the group G(p), which leaves p invariant. G(p) is called the *stationary subgroup* or *little group* of p, introduced by Wigner in 1939. From the foregoing considerations it is seen that G(p) is isomorphic with $O_{3+}$ (if $p^2 \neq 0$ !) and that it is also possible to derive the spin values j by starting from the irreducible representations of the little group G(p).
II) In the second case there holds $p^2 = 0$, thus p lies on the light cone. This case corresponds with particles which travels with the velocity of light. It is possible to transform $p^\mu$ arbitrarily near to (0, 0, 0, 0) without changing the value of $\det(L^\mu p_\mu - \kappa I) = 0$. This expression is a continuous function of p and there follows

$$\det (L^\mu p_\mu - \kappa I) = \det (-\kappa I) = 0$$

and thus $\kappa = 0$.

Hence equations in which $\kappa = 0$ describe particles with zero mass which travel with the velocity of light. Because in this case there is no transformation $p^{\mu} \to (p^{0}, \vec{0})$ $(p^{0} \neq 0)$ it follows that such particles have no rest system. The little group G(p) is not isomorphic with $O_{3+}$ but with the group of rotations and translations in a plane. See chapter II, formula (2-18).

Chapter II

## GEOMETRY WITH SPINORS

In this chapter we consider first elementary projective geometry of three dimensions and show that the Lorentz group $L_+^\uparrow$ may be described as a special 3-dimensional transformation group. (section 1.)
Next we introduce in this 3-dimensional space the concept of a spinor. By Cartans method for n-dimensional orthogonal groups $O_n$ spinors are defined as *derived* quantities. That is to say, every isotropic plane is labelled by a row of numbers (spinor), which belongs to a *second* linear space and is transformed by a representation of $O_n$. However in the special case of the Lorentz group, spinors may be defined as concrete points in the 3-dimensional space itself and by this one may do the corresponding 3-dimensional projective geometry with spinors rather than with vectors.
Using this, we study the transformation properties of the real and complex lightcone (sections 2, 3). As an application of this we consider the vector $\vec{G} = \vec{E} + i\vec{H}$, which is lying on the complex light cone and which is introduced in electrodynamics by Laporte and Uhlenbeck. In section 4 we obtain, with this method an easy description of the so-called spin-value of Veblen. Also a condition for Clifford parallelism in 3 and (4k-1) dimensions is derived by using the representation theory.

## 1. The Lorentz group as a three-dimensional transformation group

For every vector $x^\mu$ in $R_4$, we denote the corresponding ray through the origin O and $x^\mu$ by $\rho x^\mu$ $(-\infty < \rho < +\infty)$. A linear transformation in $R_4$ leaves the origin O fixed and thus the group of linear transformations in $R_4$ corresponds to a group of transformations in the space of rays $\rho x^\mu$. Every ray $\rho x^\mu$ has one intersection point with the hyperplane $x^0=1$ (we include the points at infinity in this hyperplane) and every plane through O has one intersection line with the hyperplane $R_3(x_0=1)$. Thus the linear group in $R_4$ corresponds to a transformation group in $R_3$ which transforms lines into lines, the so-called *projective group* in three dimensions. After dividing the formula $x^{\mu'} = A^{\mu'}{}_{\mu} x^\mu$ by the factor $x^{0'}$ and substituting $(1, \frac{x^{1'}}{x^{0'}}, \frac{x^{2'}}{x^{0'}}, \frac{x^{3'}}{x^{0'}}) = (1, x', y', z')$ and after dividing the same formula by $x^0$ and substituting x, y, z, one sees that these transformations correspond to the group of *broken* linear transformations in $R_3$ in which the linear group in $R_3$ form a subgroup. In particular, the light cone in $R_4$ intersects the space $R_3$ in the unit sphere (see fig. 2.3) and thus the Lorentz group corresponds to the projective group in three dimensions, which leaves the unit sphere invariant. In order to formulate and extend the correspondence $\rho R_4 \to R_3$ in a more precise way we introduce homogenous coordinates for points, planes and lines in $R_3$. The definitions given in section 1.1 are taken from elementary projective geometry.

## 1.1. Homogenous coordinates in $R_3$, definitions.

Point coordinates.

We consider an arbitrary three-dimensional vectorspace $R_3$. Points of $R_3$ will be described either by *three affine coordinates* $x^i \equiv \vec{x}(x,y,z)$ or by *four homogeneous coordinates* $x^\mu$ ($\mu=0,1,2,3$)*)

so that

$$\vec{x}(x,y,z) = \left(\frac{x^1}{x^0}, \frac{x^2}{x^0}, \frac{x^3}{x^0}\right) \quad (x^0 \neq 0) \qquad (2\text{-}1)$$

Thus two rows $x^\mu$ and $\rho x^\mu$ determine the same point in $R_3$. To express the fact that a point is only determined by the ratio of the components $x^\mu$, one also writes $(x^0:x^1:x^2:x^3)$. In particular, the origin has the coordinates (1,0,0,0) and the point (0,1,0,0) with $x^0=0$ is the point X at infinity as "implied" by (2-1). All points with $x^0=0$ form the plane XYZ at infinity. The $R_3$ extended in this way, is called the three-dimensional *projective space* $P_3$. There is no point with coordinates (0, 0, 0, 0).

It follows that the space $P_3$ is equivalent to the three-dimensional space of rays $\rho x^\mu$ in $R_4$ through the origin and that $R_3$ is the subspace $x^0=1$ in $R_4$.

The rays with $x^0 \neq 0$ are represented by their intersection with $R_3$ at the point $\left(1, \frac{x^1}{x^0}, \frac{x^2}{x^0}, \frac{x^3}{x^0}\right)$ and the rays with $x^0=0$ which are parallel with $R_3$ correspond to points at infinity of $R_3$. (cf. Heyting p. 5, 6).

In particular, the light cone in $R_4$ given by

$$(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = 0,$$

corresponds to the unit sphere $x^2 + y^2 + z^2 = 1$ in $R_3$. However, we prefer the description of the unit sphere in homogenous coordinates,

$$x_\mu x^\mu = g_{\mu\nu} x^\mu x^\nu = 0; \qquad (2\text{-}2)$$

see formula (1-1).

---

*) See notation-convention in chapter I sections 1.1 and 3.1

In the same way, a plane Ors in $R_4$ which passes through the origin O and the vectors r and s corresponds to its intersection line rs in the space $R_3$ ($x^0=1$).

In homogeneous coordinates, the equation of the line rs (plane Ors) is

$$x^\mu = \alpha r^\mu + \beta s^\mu \quad .$$

If we substitute the values $x^0=r^0=s^0=1$, we obtain

$$\vec{x} = \alpha\vec{r} + \beta\vec{s} \;\; (\alpha + \beta = 1) \text{ or } \vec{x} = \vec{r} + \rho(\vec{s}-\vec{r}),$$

the well-known formula for a straight line in affine geometry. Similar formulae hold for a space Orst in $R_4$ which corresponds to a plane rst in $R_3$.

In the following, we will also introduce coordinates for planes rst and lines rs.However we will not use these definitions before section 4.

Plane coordinates.

To every point $p^\mu$ in $R_4$, is associated a 3-plane $p^\mu x_\mu =0$ which is orthogonal to $p^\mu$; the numbers $p_\mu$ are called the "space coordinates" of $p_\mu x^\mu=0$. In $R_3$ the equation $p^\mu x_\mu=0$ describes a plane which is called the *polar plane* of the point $p^\mu$ and the numbers $p_\mu$ are its *plane coordinates*. (The construction of $p_\mu$ in the figure 2.1 follows from projective geometry).

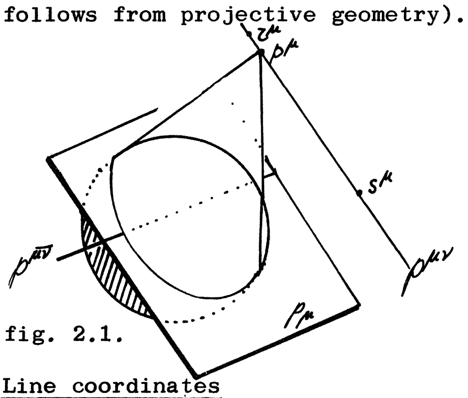



fig. 2.1.

Thus the polar plane $p_\mu$ represents the 3-dimensional space in $R_4$ which is orthogonal to the vector $p^\mu$. In particular, if $p^\mu$ lies on the unit sphere, i.e. $p_\mu p^\mu=0$, then $p_\mu$ is the tangent plane to the point $p^\mu$.

Line coordinates

One may also define line coordinates in $R_3$ (or 2-plane coordinates in $R_4$).

Definition The *line coordinates* of a line p through the points $r^\mu$ and $s^\mu$ in the 3-dimensional projective space are

$$p^{\mu\nu} = r^\mu s^\nu - r^\nu s^\mu \qquad (\mu,\nu = 0,1,2,3) \qquad (2\text{-}3)$$

The $p^{\mu\nu}$ form an anti-symmetric matrix and one may easily prove that the $p^{\mu\nu}$ are determined within a factor $\rho$ and that they are independent of the choice of the points r and s on p (for all properties concerning line coordinates, see the appendix of this chapter). We use the notation p = [rs], where [ ] is a kind of exterior product, and we shall write.

$$p^{\mu\nu} = \begin{bmatrix} r^0 & r^1 & r^2 & r^3 \\ s^0 & s^1 & s^2 & s^3 \end{bmatrix}^{\mu\nu}$$

$$= (p^{01}, p^{02}, p^{03} \mid p^{23}, p^{31}. p^{12}). \quad (2\text{-}4)$$

Thus we take the six sub-determinants formed by the $\mu$ th and $\nu$ th column and write them as the six-vector $(p^{0i}, p^{jk}) = (\vec{p}', \vec{p}'')$ (i,j,k = 1,2,3 and cycl.) We have

$$\vec{p}' = r^0\vec{s} - s^0\vec{r} \quad (2\text{-}5)$$

and $$p'' = \vec{r} \times \vec{s} \quad (2\text{-}6)$$

by which follows $\vec{p}' \cdot \vec{p}'' = 0$ . (2-7)

Conversely, the condition (2-7) implies that the anti-symmetric tensor $p^{\mu\nu}$ gives the coordinates of a line p.

The dual line

If the point $p^\mu$ moves along the line $p^{\mu\nu}$,

$$p^\mu = \alpha r^\mu + \beta s^\mu \quad ,$$

Then the corresponding polar planes, i.e.

$$p_\mu = \alpha r_\mu + \beta s_\mu \quad ,$$

are moving around a line $p^{\overline{\mu\nu}}$, the so-called *dual line* or *polar line* of $p^{\mu\nu}$ (see figure 2.1). One may prove, see appendix, that

$$p^{\mu\nu} = (\vec{p}', \vec{p}'') \text{ implies that } p^{\overline{\mu\nu}} = (\vec{p}'', -\vec{p}')\,\rho \ .$$

The number $\rho$ is arbitrary.

In order to make the map $p^{\mu\nu} \rightarrow p^{\overline{\mu\nu}}$ involutoric, i.e., $p^{\overline{\overline{\mu\nu}}} = p^{\mu\nu}$, one may choose $\rho = i$, see also formula (2-50).

Returning to four-dimensional considerations, we remark that $p^{\mu\nu}$ and

$p^{\overline{\mu\nu}}$ represent two planes Ors and $\overline{\text{Ors}}$ which are entirely orthogonal to each other, i.e., every vector in Ors is orthogonal to every vector in $\overline{\text{Ors}}$.

We note that in $R_4$ the $p^{\mu}$ are non-homogenous (vectors) and one has also non-homogeneous $p^{\mu\nu}$, the so-called *bivectors* ("plane with a screw sense"). However, for the transformation properties of $p^{\mu\nu}$, we may restrict ourselves to homogeneous $p^{\mu\nu}$. Yet, if one introduces a (hyperbolic) metric in $R_3$ it is also possible to construct a correspondence between bivectors and non-homogeneous entities in $R_3$, see chapter IV.

## 1.2. The Lorentz group and its little groups

In the introduction of section 1 we observed that the linear group in $R_4$ corresponds to the projective group in $P_3$, because every linear transformation A corresponds with a projective transformation A. However all dilations in $R_4$ (multiplication by a scalar) are mapped onto the identity in $P_3$. In order to construct a (1-1) map we exclude dilations by restricting ourselves to the enlarged unimodular group in $R_4$ ($\det A = \pm 1$). It follows that if a projective transformation $\rho A$ is given, then $\det \rho A > 0$ or $\det \rho A < 0$ and there exists a *real* $\rho$ such that $\det \rho A = \rho^4 \det A = +1$ or $= -1$ respectively. The number $\rho$ is determined within sign. Hence with every projective transformation in $P_3$ there exists the two transformations $\pm \rho A$ in $R_4$. In particular the orthochroneous group $L\uparrow$ which leaves the light cone invariant corresponds to the projective group in $P_3$ which leaves the "unit sphere" invariant, the so-called *quadratic group*. The number $\rho$ is now determined by the condition $\rho L^{0'}{}_{0} \geq 1$. So we have

Theorem 2.1. *The orthochroneous Lorentz group $L\uparrow$ is isomorphic with the quadratic group in $P_3$ and the restricted Lorentz group $L_+^\uparrow$ is isomorphic with the restricted quadratic group ($\det \rho A > 0$).*

See also Veblen and Neumann p. 1-10.

It follows by the foregoing that this theorem may be generalized to the complex n-dimensional unimodular group $SL_n$, which is $(n \to 1)$ isomorphic with the complex projective group $P_{n-1}$, Because with every $\rho A \in P_{n-1}$ there are n numbers $\rho = \frac{1}{\sqrt[n]{\det A}}$ such that $\det \rho A = +1$.

We will use this in section 2, where we observe that the 2-dimensional unimodular group $SL_2$ is $(2 \to 1)$ isomorphic with the projective group on a line.

It is possible to extend theorem 2.1 to the full Lorentz group (including time reflection) by covering the projective space $P_3$ with two sheets, see ch. IV, section 1.

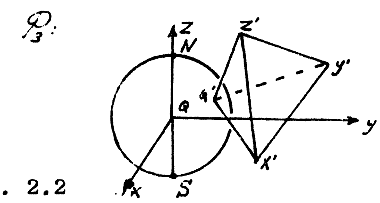



fig. 2.2

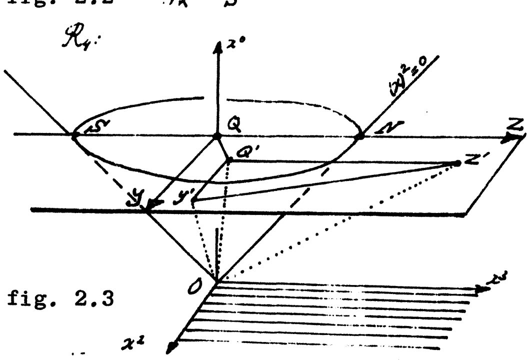
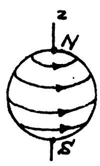
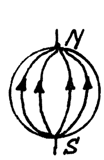



fig. 2.3

Finally we make a few remarks about the behaviour of the plane XYZ at infinity in $P_3$. The point Q and the points X, Y, Z at infinity are transformed by $\Lambda^{\mu'}{}_{\mu}$ into the points Q', X', Y', Z', which are in general affine points ($x^{0'} \neq 0$), $Q(1,0,0,0) \rightarrow Q'(\Lambda^{0'}_{0}, \Lambda^{1'}_{0}, \Lambda^{2'}_{0}, \Lambda^{3'}_{0})$, see fig. 2.2. If we return to 4-dimensional considerations we see that the simplex Q'X'Y'Z' corresponds to the $x^{0'}, x^{1'}, x^{2'}, x^{3'}$-axis, see fig. 2.3. Hence it follows that the simplex Q'X'Y'Z' is totally equivalent to the simplex QXYZ with respect to the projective group. After coordinate transformation to the simplex Q'X'Y'Z' these basic points are designated by the coordinates Q'=(1,0,0,0), ... etc.

The rotation group $G_+(Q) = O_{3+}$, which leaves the origin Q fixed is transformed into another subgroup $G_+(Q')$, which leaves Q' invariant. The group $G_+(Q)$ is isomorphic with $G_+(Q')$. These groups are known in physics as *stationary subgroups* or *little groups* of particular interest is the little group $G_+(p)$ of a point which lies on the unit sphere, $p^2 = 0$. If p is taken on the North Pole N(1,0,0,0,1) then the corresponding transformations which generate $G_+(P)$ are given as 2 × 2 matrices $A(\Lambda)$ in formulae (2-7a, b, c).

$P_3$:  $\begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{+i\frac{\phi}{2}} \end{pmatrix}$,  $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$,  $\begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}$.

(a) (b) (c) (2-7)

These transformations correspond with (a) rotations $\phi$ about the z-axis and (b) transformations $\alpha$ ($\alpha$ complex) which leave the only ray $\rho(1,0,0,1)$ point wise invariant.

We mention also (2-7 c), viz. hyperbolic screws $h_{03}(\phi)$ along the z-axis, formula (1-6). However, there holds $h_{03}(\phi)$: $p \rightarrow e^{\phi}p$ and $h_{03}(\phi)$ belongs to a larger group $G^{*}_{+}(p)$: $p \rightarrow \rho p$ which leaves the ray $\rho p^{\mu}$ invariant, but which we will exclude here.

Finally we note that the choice of the "affine" space $R_3(x^0=1)$, which is taken to represent the rays $\rho x^\mu$, is easy but not necessary. One may equivalently choose instead of $R_3$ = QXYZ the plane TXYZ at infinity in $R_4$ with points

$$(0, x^0, x^1, x^2, x^3) = (\frac{x^0}{0}, \frac{x^1}{0}, \frac{x^2}{0}, \frac{x^3}{0}),$$ cf. formula (2-1).

See fig. 2.4 which is "obtained" from figure 2.3. by projective transformation in $R_4$. Especially if one studies the larger *Poincaré group*, which consists of all transformations supplied with all translations $x' = x+a$, one is in general interested in properties which are invariant with respect to translations.

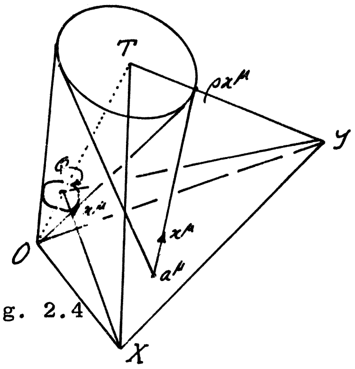


fig. 2.4

Hence one studies classes of parallel rays $\rho x^\mu$, $a^\mu+\rho x^\mu$ which are represented by one point $\rho x^\mu$ in the 3-plane TXYZ.

## 2. The real light cone

In chapter I, section 3.1 we mapped every vector x in $R_4$ onto a 2 × 2 hermitian matrix $X \equiv (X^{ac})$ (a, c = 0, 1)

$$X = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} . \tag{2-8}$$

Because det $X = x^2$, the "norm" of x, it follows that the transformation

$$X' = AXA^+ \quad (\det A=+1) \tag{2-9}$$

represents a restricted Lorentz transformation.
We proceed in the same way with the points $\rho x^\mu$ in the 3-dimensional space $P_3$, but observe that the $\rho x^\mu$ are homogeneous coordinates and thus the restriction to unimodular matrices A is only a matter of normalization. In the following we shall write $X^\mu$ for the components of the matrix X, i.e.,

$$X = \begin{pmatrix} X^0 & X^2 \\ X^1 & X^3 \end{pmatrix} . \tag{2-10}$$

We note that the transformation $T = x^\mu \to X^\mu$ is a coordinate transformation ($\frac{1}{\sqrt{2}}$ T is unitary) we will denote the base points by $e_\mu$ and $E_\mu$, respectively. For the description of spinors, the coordinate system $X^\mu$ is more appropriate and in general we follow the convention of writing the vector components $X^\mu$ in a square.

We consider now vectors $\rho x^\mu$ lying on the light cone with $x^0 \geq 0$, (or points x lying on the unit sphere $x^2 = 0$ in the three-dimensional language). Because det X=0, rows and columns of X are dependent and the matrix X is of the form

$$X = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} (\phi^0, \phi^1) = \psi\phi^T . \tag{2-11}$$

Moreover, if we require that all $x^\mu$ are real, then X is hermitian, $X^+ = X$ and it follows that

$$X = \psi\psi^{+} \tag{2-12}$$

It follows that $x^{0} = \psi^{0}\psi^{0} + \psi^{1}\psi^{1} > 0$ or in components $X = \psi^{a}\,\psi^{\dot{c}}$ .
The 2-dimensional vector $\psi$ may be expresses in the coordinates $x^{\mu}$
and is determined within a factor $e^{\frac{i\theta}{2}}$ by (2-12). After substitution of
$X = \psi\psi^{+}$ and $X' = \psi'\psi'^{+}$ in formula (2-9), we get

$$\psi' = A\psi \ .$$

Hence $\psi$ is transformed by a two-dimensional representation of the Lorentz group, i.e., $D^{\frac{1}{2}0}$: $\Lambda \to \pm A(\Lambda)$, and thus $\psi$ is a spinor.
*Thus the points p of the upper light cone are (1-1)-mapped on the rays* $e^{i\frac{\theta}{2}}\psi$ *in the space* $R_2$ *of spinors.*
See Veblen and Neumann p. 1-2.
We also have that every light ray $\rho p^{\mu}$ is (1-1) mapped on the ray $re^{i\frac{\theta}{2}}\psi$. Now by going from rays in $R_n$ to points in the projective space $P_{n-1}$, we observe that $\rho p^{\mu}$ corresponds with a point on the unit sphere and that $re^{i\theta}(\psi^{0}, \psi^{1})$ corresponds with the complex number $\psi = \frac{\psi^{1}}{\psi^{0}}$ . Thus the unit sphere is (1-1) covered with complex numbers $\psi(p)$. It is easy to prove that this map $\psi(p)$ can also obtained by performing stereografic projection $x \Longleftrightarrow x'$ of the unit sphere upon the (x+iy)-plane. The South Pole S(1,0,0,-1) being the centre of projection.
Thus we will prove that

$$\psi(X) = x' + iy' \ , \text{ see fig. 2.5.}$$

The next sub-section is an application of the above "decomposition" of the vector X , but is not necessary for the understanding of the following sections.

Stereographic projection

We observe that the $2 \times 2$ matrix X' is a linear combination of the matrices

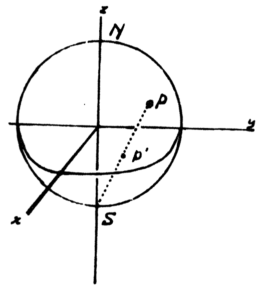



fig. 2.5

$$X = \begin{pmatrix} \psi^0\psi^{\dot 0} & \psi^0\psi^{\dot 1} \\ \psi^1\psi^{\dot 0} & \psi^1\psi^{\dot 1} \end{pmatrix} \text{ and } S = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

in such a way that the diagonal elements of X' are equal (which implies that z' = 0). We get

$$X' = \begin{pmatrix} \psi^0\psi^{\dot 0} & \psi^0\psi^{\dot 1} \\ \psi^1\psi^{\dot 0} & \psi^0\psi^{0} \end{pmatrix}$$

and after dividing by the homogeneous factor $\psi^0\psi^{\dot 0}$, we obtain

$$X' = \begin{pmatrix} 1 & \psi^{\dot 1}/\psi^{0} \\ \psi^1/\psi^0 & 1 \end{pmatrix} .$$

Hence $x' + iy' = \frac{\psi^1}{\psi^0}$. Thus if we identify X' with the complex number $x'+iy' = \psi(x')$, we see that $\psi$ is transformed into

$$\psi' = \frac{A^1_{\ 0} + A^1_{\ 1}\,\psi}{A^0_{\ 0} + A^0_{\ 1}\,\psi}$$

and that $\psi$ in its homogeneous form $(\psi^0, \psi^1)$ is transformed as a spinor. Consequently by describing the points on the unit sphere by complex numbers $\xi$ (the *Gaussian number sphere*), which are written in homogeneous form, one obtains the 2-dimensional space of spinors.

It is important to note that projection from the South Pole S (North Pole N) gives a ***right-handed (left-handed) Gaussian sphere*** with points $\psi$ (or $\dot\psi$). We note that the right-handed Gaussian sphere is a one-dimensional complex space, with N = 0 and S = $\infty$ and with homogeneous coordinates the Gaussian sphere is a one-dimensional complex projective space which is spanned by N = (1,0) and S = (0,1). So we have

Theorem 2.1. *If $S$ is the Gaussian sphere consisting of all light rays $\rho x^{\mu}$ in the ray space $P_3$, then the 2-dimensional spinor representation $D^{\frac{1}{2}0}$ (or $D^{0\frac{1}{2}}$) is given by the projective group of the right-handed (or left-handed) Gaussian sphere $S$ into itself.*

Applying total reflection J in $R_4$ : $p^{\mu} \to q^{\mu} = - p^{\mu}$ or with the 2 × 2 matrices $P \to Q = - P$ it follows that considering light vectors q with $q^0 < 0$, formula (2-12) has to be replaced by

$$Q = - \psi(q)\psi^{+}(q). \qquad (2\text{-}13)$$

Thus the light vectors $p^{\mu}$ and $- p^{\mu}$ are mapped onto the same spinor, $\psi(p) = \psi(-p)$.

We wish now to determine, by using stereographic projection, the form of the 2 × 2 matrices $A(\Lambda)$, where $\Lambda$ is (1) an arbitrary rotation, (2) an arbitrary hyperbolic screw, and (3) a so-called $\alpha$-transformation.

(1) Rotations We consider first a rotation $\theta$ around the vector $\vec{n}$ along the z-axis, $\psi' = e^{i\theta}\psi$, and thus by writing $\psi = \frac{\psi^1}{\psi^0}$ or $(\psi^0, \psi^1)$, we obtain the matrix

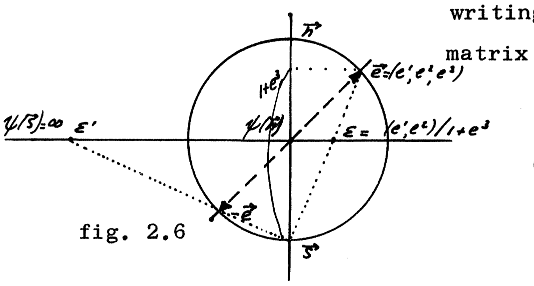


fig. 2.6

$$\Theta = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{pmatrix} . \qquad (2\text{-}14)$$

By a rotation S we now transform the vector $\vec{n}$ into an arbitrary vector $\vec{e}$. The projection of $\vec{e}$ in the (x-y)-plane is $\frac{(e^1, e^2)}{1 + e^3}$ and by this we obtain in the complex plane the transformation

$$\psi(N) = 0 \to \varepsilon' = \frac{e^1+ie^2}{1+e^3} \text{ or } (1,0) \to (1+e^3, e^1+ie^2)$$

$$\psi(S) = \infty \to \varepsilon' = -\frac{e^1+ie^2}{1-e^3} \text{ or } (0,1) \to (1-e^3, -e^1-ie^2) .$$

The columns of the transformation matrix S are thus given by the vectors ε and ε' ,

$$S = \begin{pmatrix} 1 + e^3 & 1 - e^3 \\ e^1 + ie^2 & -e^1 - ie^2 \end{pmatrix} .$$

A rotation θ around the unit vector $\vec{e}$ is thus given by

$$S\,\theta S^{-1} = \begin{pmatrix} 1+e^3 & 1-e^3 \\ e^1+ie^2 & -e^1-ie^2 \end{pmatrix} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{pmatrix} \begin{pmatrix} -e^1-ie^2 & -1+e^3 \\ -e^1-ie^2 & 1+e^3 \end{pmatrix} \frac{1}{-2(e^1+ie^2)} .$$

Using $(e^1)^2+(e^2)^2 = 1 - (e^3)^2$, we obtain:

$$A = \begin{pmatrix} \cos\frac{\theta}{2} - i \sin\frac{\theta}{2} e^3 & -i \sin\frac{\theta}{2} (e^1-ie^2) \\ -i \sin\frac{\theta}{2} (e^1+ie^2) & \cos\frac{\theta}{2} + i \sin\frac{\theta}{2} e^3 \end{pmatrix} ,$$

or with the Pauli matrices $\sigma_\mu$, see formula (1-23),

$$A = \cos\frac{\theta}{2}\sigma_0 - i \sin\frac{\theta}{2} e^k\sigma_k . \qquad (2\text{-}15)$$

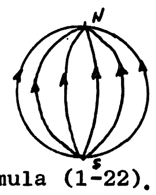
We shall write $A = a^\mu\sigma_\mu$ and so, with respect to the basis $\sigma_\mu$, every rotation θ around a vector $\vec{e}$ is determined by the vector

$$a^\mu = (\cos\frac{\theta}{2} , - i\sin\frac{\theta}{2}\vec{e}) \; (\vec{e}^{\,2} = 1) . \qquad (2\text{-}16)$$

In view of the vector i in (2-15), it follows that the matrix Λ is unitary for rotations.

<u>(2) Hyperbolic screws</u>

By substituting $\theta = i\psi$ in (2-6), we obtain the hyperbolic screw

fig. 2.7 


$$\begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix}$$

See formula (1-22).

It corresponds to the multiplication transformation $\psi' = e^{-\phi}\psi$

in the complex plane and on the sphere to a transformation with N and S as fixed points. A hyperbolic screw $\phi$ along the vector $\vec{e}$ is determined by

$$a^\mu = (\cos \frac{i\phi}{2} , -i \sin \frac{i\phi}{2} \vec{e})$$

$$= (\mathrm{ch} \frac{\phi}{2} , \mathrm{sh} \frac{\phi}{2} \vec{e}) . \qquad (2\text{-}17)$$

In view of the fact that all $a^\mu$ are real, it follows that for hyperbolic screws A is hermitian (cf. remark 3.1 in chapter I).

<u>(3) $\alpha$-transformations</u> We wish to consider now especially those transformations which have S as the *only* invariant point on the sphere. With a translation $\psi' = \psi + \alpha$ in the complex plane, it is clear that there is only *one* invariant point, the point at infinity. Thus by the stereographic projection we obtain a transformation of the sphere which has S as the only fixed-point. In homogeneous form we may write

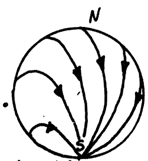


fig. 2.8

$$\begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

With formula (2-16) and (2-17), one may calculate now the infinitesimal operators of the spinor representation (section 6, chapter I). If the vector $\vec{e}$ is arbitrary complex with $\vec{e}^{\,2} = 1$, one may prove that the matrix A determines an arbitrary screw around an axis not necessarily through O. (see chapter IV).

Finally we consider the little group $G_+(p)$ of all restricted Lorentz transformations which leave p invariant $G_+(p)$: $p \to p$.

If one takes for the North Pole N = (1,0,0,1) then the transformations which generate $G_+(N)$ are given in formula (2-18a) and (2-18b)

$$\begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{+i\frac{\phi}{2}} \end{pmatrix} \qquad \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} e^{\frac{\phi}{2}} & 0 \\ 0 & e^{-\frac{\phi}{2}} \end{pmatrix} \qquad (2\text{-}18)$$

(a) (b) (c)

We also mention (2-18c) which is a hyperbolic screw $h_{03}(\phi)$ along the z-axis. However there holds $h_{03}(\phi)$ : $p \to e^{\phi}p$ and $h_{03}(\phi)$ belongs to the larger group $G_{+}^{*}(p)$: $p \to \rho p$ which leave the ray $\rho p^{\mu}$ invariant.

Remark 2.1. Connection with complex quaternions

If we go from the basis $\sigma_{\mu}$ to $j_0 = \sigma_0 = 1$ and $j_k = -i\sigma_k$, then the matrices $j_{\mu}$ are the quaternions of Hamilton, since

$$j_1^2 = j_2^2 = j_3^2 = -1,\ j_1 j_2 = j_3,\ j_2 j_1 = -j_3 \text{ and cycl.} \qquad (2\text{-}19a)$$

With respect to the basis $j_{\mu}$, a rotation is denoted by the real 4-vector

$$j = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\vec{e}\ .$$

For an arbitrary unimodular matrix, the number $\theta$ and the vector $\vec{e}$ are complex and one may state that the restricted Lorentz group $L_{+}^{\uparrow}$ is isomorphic with the complex rotation group in three dimensions (cf. Cartan II p. 73) or equivalently, $L_{+}^{\uparrow}$ is isomorphic with the group of complex quaternions with norm $j_0^2 + j_1^2 + j_2^2 + j_3^2 = 1$.
Many calculations with 2 × 2 matrices are most easily performed by using the multiplication rule of quaternions

$$(a_0+\vec{a})\ (b_0+\vec{b}) = (a_0 b_0 - \vec{a}.\vec{b}) + a_0\vec{b} + b_0\vec{a} + \vec{a} \times \vec{b}. \qquad (2\text{-}19b)$$

3. The complex light cone

In this section we treat the complex light cone $x^2=0$. The representations $D^{\frac{1}{2}0}$, $D^{10}$ (and $D^{0\frac{1}{2}}$, $D^{01}$) will be realized by concrete points on $x^2 = 0$ and using this we will treat the properties of complex Lorentz transformations. Here, the geometrical theory, which in fact developed by F. Klein [Nicht Eukl. Geom. p. 112 p. 238-240], will be more compactly formulated by using the matrix method of E. Cartan. These results will be applied in section 4.

## 3.1. The spinor representations $D^{\frac{1}{2}0}$ and $D^{0\frac{1}{2}}$

We shall treat in greater detail the transformation $(X^\mu) = T(x^\mu)$, formula (2-10), where a factor $\frac{1}{\sqrt{2}}$ has been added in order to make T unitary.

$$\begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \qquad (2\text{-}20)$$

If we interpret this transformation as a coordinate transformation, then the matrix $T^{-1} = T^{+}$ contain in its columns the new basic points. $E_0$, $E_1$, $E_2$, $E_3$. By Felix Klein $E_0$, $E_1$, $E_2$, $E_3$ is called the *invariant tetraeder* (invariant with respect to rotations about the z-axis) in physics one calls the new vectors in $R_4$ : $E_0$, $E_1$, $E_2$, $E_3$ = $\rho l^\mu$, $\rho m^\mu$, $\rho \bar{m}^\mu$, $\rho n^\mu$ a *null tetrad*

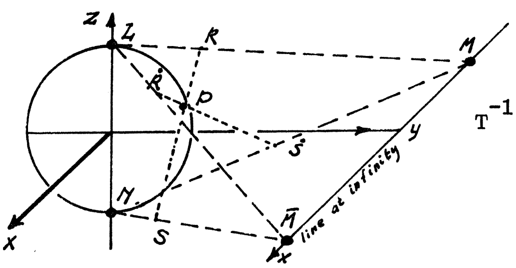



fig. 2.9

$$T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \qquad (2\text{-}21)$$

$$\quad E_0 \quad E_1 \quad E_2 \quad E_3$$

In particular, we have the isotropic points $E_1(0,1,-i,0)$ and $E_2(0,1,i,0)$ lying in the (x,y)-plane at infinity for $E_{10} = E_{20} = 0$ (see fig. 2.9.). Later on we will write again the vector X as a 2 × 2 matrix; thus an arbitrary point X on the *complex* unit sphere is of the form $X = \psi\phi^T$, see (2-11), or

$$X = \phi^0 \begin{pmatrix} \psi^0 & 0 \\ \psi^1 & 0 \end{pmatrix} + \phi^1 \begin{pmatrix} 0 & \psi^0 \\ 0 & \psi^1 \end{pmatrix} \qquad (2\text{-}22)$$

Thus X is a linear combination of the points

$$R = \begin{pmatrix} \psi^0 & 0 \\ \psi^1 & 0 \end{pmatrix} \quad \text{and } S = \begin{pmatrix} 0 & \psi^0 \\ 0 & \psi^1 \end{pmatrix}$$

from which follows that X lies on the line RS and is determined by the ratio $(\phi^0 : \phi^1)$. The line RS itself is determined by the spinor $\psi$. Every point of RS($\psi$) is of the form $X = \psi\phi^T$ and because det X = 0 for all $\phi$, it follows that the line RS lies wholely on the complex unit sphere and RS is called an *isotropic* line. Thus the spinor $\psi$ determines the isotropic line RS as well as the real point $X = \psi\psi^+$ on it. More important is that the spinor $\psi$ may be identified with the points R or S,

$$R = (\psi^0, \psi^1, 0, 0) \qquad (2\text{-}23)$$
$$S = (0, 0, \psi^0, \psi^1) \ .$$

Hence the Lorentz group induces the spinor representation $D^{\frac{1}{2}0}$ on the *fixed* $E_0E_1$ and $E_2E_3$-axis.

Similarly we may write

$$X = \psi^0 \begin{pmatrix} \phi^0 & \phi^1 \\ 0 & 0 \end{pmatrix} + \psi^1 \begin{pmatrix} 0 & 0 \\ \phi^0 & \phi^1 \end{pmatrix} \ . \qquad (2\text{-}24)$$

Thus the real point $X = \psi\psi^+$ also determined another isotropic line $\dot{R}\,\dot{S}$ with $X = \phi\psi^+$, and where $\dot{R}$ and $\dot{S}$ lie on the $E_0\ E_2$ and $E_1\ E_3$ axis, respectively, i.e.,

$$\dot{R} = (\psi^{\dot{0}}, 0, \psi^{\dot{1}}, 0) \qquad (2\text{-}25)$$
$$\dot{S} = (0, \psi^{\dot{0}}, 0, \psi^{\dot{1}}) \ .$$

## 3.2. The representations $D^{10}$ and $D^{01}$

Of particular interest is the intersection $\Omega$ of the complex unit sphere with the plane ($x^0$=0) at infinity.

The intersection point $G(0,\vec{G})$ on $\Omega$ lies on an isotropic line RS and thus the corresponding 2 × 2 matrix G is a linear combination of R and S. Because $G^0$=0, the diagonal elements of G are

$$G^3 \text{ and } -G^3$$

and thus we must take a linear combination of R and S in such a way, that the diagonal elements have opposite signs. It follows that

$$G = \psi^1 \begin{pmatrix} \psi^0 & 0 \\ \psi^1 & 0 \end{pmatrix} - \psi^0 \begin{pmatrix} 0 & \psi^0 \\ 0 & \psi^1 \end{pmatrix}$$

$$= \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} (\psi^1, -\psi^0) = \psi(C\psi)^T, \text{ where } C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (2\text{-}26)$$

One notes that $\psi_a = C_{ac}\ \psi^c$ (see formula (1-26)) and thus $G = (\psi^a \psi_c)$.

The representation acting on $(\psi^a \psi_c)$ is equivalent with the representation acting on the space of spinors $(\psi^a \psi^c)$ and thus the *intersection*

$$G = (\psi^0_0,\ \psi^1_0,\ \psi^0_1,\ \psi^1_1) \quad (\psi^a_c = \psi^a \psi_c) \qquad (2\text{-}27)$$

$$\dot{G} = (\psi^{\dot{0}}_{\dot{0}},\ \psi^{\dot{1}}_{\dot{0}},\ \psi^{\dot{0}}_{\dot{1}},\ \psi^{\dot{1}}_{\dot{1}}) \qquad (2\text{-}28)$$

point G of RS with the *fixed reference* plane ($x^0$=0) is transformed under the representation $D^{10}$ (see table (1-32)).

Thus the point G is not transformed as a vector, i.e., $X' = AXA^T$. Using the fundamental property of the matrix C ,

$$CA = (A^{-1})^T\ C\ ,$$

it follows from (2-26) that G is transformed by

$$G' = AGA^{-1}. \qquad (2\text{-}29)$$

Hence trace G' = trace G $= 2G^0 = 0$ and we see again that the point G remains in the plane at infinity as an intersection point of the line RS with the fixed reference plane at infinity.
In the 4-dimensional case, $\rho G$ is the intersection line of the complex light cone and the $(x^1x^2x^3)$-space. In this non-homogeneous case, it is also true that

$$\det G' = \det G \quad \text{or} \quad \vec{G}'^2 = \vec{G}^2. \tag{2-30}$$

Thus $\vec{G}$ is transformed as a vector under the complex 3-dimensional rotation group. We separate the real and imaginary part of G, $\vec{G} = \vec{E} + i\vec{H}$, and obtain the invariants

$$\vec{E}^2 - \vec{H}^2 \quad \text{and} \quad \vec{E} \cdot \vec{H} \, . \tag{2-31}$$

These are important invariants of the electromagnetic field in the theory of relativity. In formula (1-80m) on page 71 we have used these invariants to obtain the Casimir operators of the Lorentz group.

## 3.3. Complex Lorentz transformations

One may extend the Lorentz group L to the complex Lorentz group L(C) which leaves invariant the "norm", $x^2 = (x^0)^2-(x^1)^2-(x^2)^2-(x^3)^2$. However, it is not necessary that real $x^\mu$ are mapped onto real $x^\mu$. The real Lorentz group L is a subgroup of L(C). Besides L(C), there is another extension of L which leaves invariant the norm, $x_0\bar{x}_0 - x_1\bar{x}_1 - x_2\bar{x}_2 - x_3\bar{x}_3$, see Barut p. 34. We note further that L(C) is isomorphic with the 4-dimensional complex orthogonal group $O_4(C)$. Using the 2 × 2 matrices X from formula (2-2), it follows that the transformation

$$X'' = AXB^T \quad (\det A = \det B = 1) \tag{2-32}$$

is a complex Lorentz transformation. Because det X'' = det X, then $x''^2 = x^2$. Thus the transformation

$$T = A \times B = (A \times E).\ (E \times B) \tag{2-33}$$

acts on the vector $X^\mu$ and det T = +1. Conversely, every complex Lorentz transformation T with det T = +1, $T \in L_+(C)$, is of the form (2-33). Thus in group theoretical terms one may write

**Theorem 3.1.** $O_{4+}(C) \sim SL_2 \times SL_2'$ *(the homomorphism is: 1 ← 2)*

Now we write the point $x_0$ on the complex unit sphere as

$$X_0 = \psi_0\phi_0^T, \text{ where } \psi_0 \text{ and } \phi_0 \text{ are 2-component spinors.}$$

Then $\psi_0$ determines the isotropic line $X = \psi_0\phi^T$ through $X_0$, with $\phi$ as parameter and $\phi_0$ determines an isotropic line $X = \psi\phi_0^T$ through $X_0$ with $\psi$ as parameter. The above and (2-33) imply

**Theorem 3.2.** *Every complex Lorentz transformation consists of a transformation*

$$X' = AX$$

*which transforms the isotropic lines* $\psi_0$ *but leaves invariant the system* $\phi_0$, *followed by a transformation*

$$X'' = X'B^T$$

*which transforms the isotropic lines* $\phi_0$ *but leaves invariant the system* $\psi_0$.

Obviously, for the real Lorentz group $L_{+}^{\uparrow}$, the matrix B is chosen in such a way, i.e., $B = \bar{A}$, that the reality conditions in $R_4$ are restored; if X is hermitian ($x^{\mu}$ real), then X" is hermitian ($x^{\mu''}$ real).

# 4. Isotropic planes

## 4.1. The representation $D^{10} + D^{01}$

We wish to determine the coordinates $g^{\mu\nu}$ of the isotropic lines (planes) of the unit sphere (light cone). In the first part of this section, we work in the coordinate system $x^{\mu}$, whereas later on we work in the coordinate system $X^{\mu}$.

In section 1 we have mentioned that along with every point $p^{\mu}$, its polar plane $p_{\mu}$ may be introduced and along with the line $p^{\mu\nu}$ with points $p^{\mu}$, its polar (dual) line $p^{\overline{\mu\nu}}$ may be introduced which is the carrier of all polar planes $p_{\mu}$. Algebraically we have

$$p^{\mu\nu} = (\vec{E}, \vec{H}) \Rightarrow p^{\overline{\mu\nu}} = (\vec{H}, -\vec{E})i .$$

Now an isotropic line g lies entirely on the complex sphere. Thus if a point $p^{\mu}$ moves along g, then the polar (tangent) plane moves around g. One says that the isotropic lines are *selfdual*, i.e.,

$$g^{\mu\nu} = \pm g^{\overline{\mu\nu}} .$$

This implies that $\vec{H} = \mp i \vec{E}$ and we obtain the two systems of isotropic lines

$$g^{\mu\nu} = (\vec{G}, -i\vec{G}) \tag{2-34}$$

$$\dot{g}^{\mu\nu} = (\vec{\dot{G}}, +i\vec{\dot{G}}) . \tag{2-35}$$

As a consequence of (2-7) it follows that

$$\vec{G}^2 = \vec{\dot{G}}^2 = 0 . \tag{2-36}$$

Thus $\vec{G}$ and $\vec{\dot{G}}$ are necessarily complex vectors.

By a Lorentz transformation, the unit sphere is transformed into itself and thus the two systems of isotropic lines form two invariant spaces for $L_+^{\uparrow}$, but are transformed into each other by a space reflection $P \equiv (g_{\mu\nu})$ ,

$$P(\vec{G}, -i\vec{G})P^T = (-\vec{G}, -i\vec{G}) . \tag{2-37}$$

Because $\vec{G}$ is transformed as an ordinary 3-vector under rotations $\vec{G}' = r\vec{G}$, it is easy to prove that (2-34) and (2-35) determine two *irreducible linear* spaces under $L_+^\uparrow$. This means we have obtained a geometrical interpretation of the reduction of the space of anti-symmetric tensors (see section 2 chapter I).

Theorem 4.1. *The spaces $R_3$ and $\dot{R}_3$, into which the space of anti-symmetric tensors may be reduced,*

$$2(\vec{E}, \vec{H}) = (\vec{G}, i\vec{G}) + (\dot{\vec{G}}, -i\dot{\vec{G}}), \vec{G} = \vec{E} + i\vec{H}\ ,\ \dot{\vec{G}} = \vec{E} - i\vec{H}\ , \quad (2\text{-}38)$$

*are spanned by the two systems of isotropic lines (2-34) and (2-35), respectively ($\vec{G}^2 = \dot{\vec{G}}^2 = 0$).*

By the general representation theory of the Lorentz group, see table (1-32), it follows that the only 3-dimensional representations of $L_+^\uparrow$ are $D^{10}$ and $D^{01}$. Thus the space of anti-symmetric tensors is transformed by the representation $D^{10} \dot{+} D^{01}$.
Without using the general theory, one may also prove that the space of anti-symmetric tensors is transformed by $D^{10} + D^{01}$. Therefore, we observe that the first row $g^{0\nu} \equiv (0,\vec{G})$ of $g^{\mu\nu}$, which is defined by

$$g^{0\nu} = r^0 s^\nu - s^0 r^\nu \quad ,$$

is a linear combination of two points r and s so that $g^{00} = 0$. From (2-34) it follows that $(0,\vec{G})$ is the intersection point of the isotropic line $rs \equiv g^{\mu\nu}$ with the plane at infinity.
Using now section 3.1., it follows that $\vec{G}$ is transformed by $D^{10}$ and, analogously, we obtain that $\dot{\vec{G}}$ is transformed by the representation $D^{01}$.

We obtain the same result if we calculate the coordinates $G^{\mu\nu}$ of the isotropic lines g in the coordinate system $X^\mu$. From the points R and S on the line g, which we have already obtained in formula (2-17), we get

$$G^{\mu\nu} = \det \begin{bmatrix} \psi^0\psi^1 & 0 & 0 \\ 0 & 0 & \psi^0\psi^1 \end{bmatrix}^{\mu\nu} = (0,\ \psi^0\psi^0,\ \psi^0\psi^1 \mid 0,\ -\psi^1\psi^1,\ \psi^1\psi^0). \quad (2\text{-}39)$$

If we restrict ourselves to the components of $G^{\mu\nu}$ which are unequal to zero, we get

$$\begin{pmatrix} G^{02} & G^{03} \\ G^{12} & G^{13} \end{pmatrix} = \begin{pmatrix} \psi^0\psi^0 & \psi^0\psi^1 \\ \psi^1\psi^0 & \psi^1\psi^1 \end{pmatrix} = \psi\psi^T. \tag{2-40}$$

Thus by definition, the coordinates $(\psi^a \psi^c)$ of the isotropic lines are transformed by the representation $D^{10}$, i.e.,

$$\psi^{a'}\psi^{c'} = A^{a'}{}_a A^{c'}{}_c \psi^a\psi^c.$$

Similarly, we obtain for the other system of isotropic lines $\dot{G}^{\mu\nu}$ the coordinates $(\psi^{\dot{a}}\psi^{\dot{c}})$ transformed by $D^{01}$. If we write all components $G^{\mu\nu}$, we obtain the matrix $G^{\mu\nu} = \begin{pmatrix} 0 & \psi\psi^T \\ -\psi\psi^T & 0 \end{pmatrix} = C \times (\psi\psi^T)$, where $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and for $\dot{G}^{\mu\nu}$ we obtain $\dot{G}^{\mu\nu} = (\overline{\psi\psi^T}) \times C$. Summarizing:

<u>**Theorem 4.2.**</u> *After the coordinate transformation $x^\mu \to X^\mu$, the coordinates $g^{\mu\nu} = (\vec{G}, -i\vec{G})$ of the isotropic lines g may be decomposed by*

$$\begin{pmatrix} G^3 & G^1 - iG^2 \\ G^1 + iG^2 & -G^3 \end{pmatrix} = (\psi^a\psi_c) \text{ or } G^{\mu\nu} = \begin{pmatrix} 0 & \psi^a\psi^c \\ -\psi^a\psi^c & 0 \end{pmatrix} = C \times (\psi^a\psi^c),$$

A similar theorem holds for the isotropic lines $\dot{g}$.

## 4.2. The representation $D^{\frac{1}{2}0} + D^{0\frac{1}{2}}$ (the spin-space of Veblen)

By the foregoing method we also get an easy description of the (geometrical) *spin-space* $\tilde{P}_3$ introduced by Veblen.
Therefore, we consider the complex Lorentz group

$$X'' = AXB^T \quad (\det A = \det B = 1)$$

and restrict ourselves to the six parameter subgroup I,

$$X' = AX \quad (\det A = 1),$$

which is homomorphic with the Lorentz group. With respect to the group I, all points $R(\psi^0, \psi^1, 0, 0)$ and all points $S(0,0,\psi^0,\psi^1)$ form two invariant subspaces, both transformed by the representation $D^{\frac{1}{2}0}$. Every point X in $P_3(I)$ may be covariantly written as $X = \lambda R + \mu S$, where $\lambda$ and $\mu$ are scalars with respect to I. It follows that $P_3(I)$ is the representation space of $D^{\frac{1}{2}0} + D^{\frac{1}{2}0}$.
Now we transform $P_3$ into $\tilde{P}_3$ by the transformation

$$X^2,\ X^3 \rightarrow \bar{X}^2,\ \bar{X}^3 \text{ (complex conjugation)} .$$

The unit-sphere $X^0X^3 - X^1X^2 = 0$ is transformed into a so-called *anti-quadric* $X^0\bar{X}^3 - X^1\bar{X}^2 = 0$. In particular the spinors $R(\psi^0, \psi^1, 0, 0)$ remain invariant but $S(0,0,\psi^0,\psi^1)$ is transformed into $\dot{S}(0,0,\psi^{\dot{0}},\psi^{\dot{1}})$. Because every point of $P_3$ may be written in the form $X = \lambda R + \lambda\dot{S}$, it follows that $\tilde{P}_3$ is the representation space of

$$D^{\frac{1}{2}0} + D^{0\frac{1}{2}}.$$

Analogously to (2-39) and (2-40), the line coordinates of the "isotropic" lines RS are now given by $(\psi^a\psi^{\dot{c}}) = \psi\psi^+$.
It follows that the line coordinates $X = \psi\psi^+$ span the original real space $X = \psi\psi^+$. Thus by the method of Veblen, the original 3-dimensional space $P_3$ acts as the image space of the "isotropic" lines in the spin-spare $\tilde{P}_3$. See Veblen, geometry of four-component spinors.

## 4.3. Clifford parallels in 3-, 7-, ... (4k-1)-dimensions

### Clifford parallels in 3-dimensions.

We consider the complex Lorentz group.

Definition Two lines p and q are called *Clifford parallel* if there exists a Lorentz transformation (not the identity) which leaves p *and q* invariant.

We seek the condition which the line coordinates $p^{\mu\nu}$ and $q^{\mu\nu}$ must obey in order that p and q are Clifford parallel.

In section 4.1 chapter II, we have proved that every line $p_0^{\mu\nu}$ may be decomposed in the following way:

$$p_0 = G_0 + \dot{G}_0 \quad ,$$

where $G_0$ and $\dot{G}_0$ belong to the invariant subspaces $R_3$ and $\dot{R}_3$, respectively, formula (2-38). Now every Lorentz transformation $T = A \times B$ is theproduct of a Lorentz transformation $(A \times E)$ which works in the space $R_3$ but not in the space $\dot{R}_3$, and a transformation $(E \times B)$ which works in the space $\dot{R}_3$ but not in the space $R_3$, formula (2-33). Suppose now that $T = A \times B$ leaves the line $p_0$ invariant. Then the transformation $(A \times E)$ has the eigenvector $G_0$ in $R_3$ and the transformation $(E \times B)$ has the eigenvector $\dot{G}_0$ in $\dot{R}_3$. Thus $(A \times E)$ leaves invariant, along with $p_0$, all lines

$$p_r = G_0 + \dot{G}_r \quad ,$$

($\dot{G}_r \in \dot{R}_3$, $\dot{G}_r$ arbitrary so that $p_r$ is a line) and $(E \times B)$ leaves invariant, along with $p_0$, all lines

$$p_\ell = G_\ell + \dot{G}_0 \; .$$

In this way we obtain a system of lines $p_r$ which are called *right* Clifford parallel with $p_0$, and a system of lines $p_\ell$ which are called *left* Clifford parallel with $p_0$. Thus:

Theorem 4.3. *Suppose that there is given a line $p_0 = G_0 + \dot{G}_0$. All lines $p_r$ for which $G_r = G_0$ $(\vec{E}_r + i\vec{H}_r = \vec{E}_0 + i\vec{H}_0)$ are right Clifford parallel with $p_0$ and all lines $p_\ell$ for which $G = G_0$ $(\vec{E}_\ell - i\vec{H}_\ell = \vec{E}_0 - i\vec{H}_0)$ are left Clifford parallel with $p_0$.*

This proof holds in the complex case and may be generalized to n-dimensions. For another proof in 3-dimensional elliptic geometry based on the introduction of a metric, see Godeaux.

## Clifford parallel planes in(4k-1) dimensions

We wish to generalize the foregoing theorem. Therefore, we consider the linear space $R_{2\nu}$ or, equivalently, the projective space $P_{2\nu-1}$. In order to avoid lowering and raising of indices, we take the norm

$$x^2 = x_0^2 + x_1^2 + \dots x_{2\nu-1}^2$$

and we consider the complex orthogonal group which leaves invariant this norm. Similar to our introduction of *bi*vectors $p_{\mu\nu}$ in $R_4$, we consider now $\nu$-vectors in $R_{2\nu}$. These are anti-symmetric tensors p whose components have $\nu$ indices $p_{\mu_1\dots\mu_\nu}$. In order that $p_{\mu_1\dots\mu_\nu}$ are the coordinates of a $\nu$-plane, the components $p_{\mu_1\dots\mu_\nu}$ obey some quadratic relations; compare with formula (2-7) and see Veblen and Neumann p. (4.8). We call two $\nu$-vectors p and q *Clifford parallel* if there exists an orthogonal transformation (not the identity) which leaves p *and* q invariant. We will derive the conditions which $p_{\mu_1\dots\mu_\nu}$ and $q_{\mu_1\dots\mu_\nu}$ must obey in order that p and q are Clifford parallel.
One may define, analogously to $R_4$, a duality relation in $R_{2\nu}$

$$p_{\overline{\mu_1\dots\mu_\nu}} = i^\nu p_{\mu_{\nu+1}\dots\mu_{2\nu}} \quad (\mu_1,\dots\mu_2 = \text{even permutation of } 0,1,\dots 2\nu-1).$$

In this way one obtains two families of *selfdual* $\nu$-vectors,

$$G^+ : p_{\overline{\mu_1\dots\mu_\nu}} = p_{\mu_1\dots\mu_\nu},$$

and anti-selfdual $\nu$-vectors,

$$G^- : p_{\overline{\mu_1\dots\mu_\nu}} = -p_{\mu_1\dots\mu_\nu},$$

see Weyl and Brauwer p. 427. These two families correspond to the fact that in an even-dimensional space $R_{2\nu}$, the cone $x^2 = 0$ bears two families

of so called isotropic planes.
If we separate the components of a $\nu$-vector into two groups,

$$(p_{\mu_1 \ldots \mu_\nu}) \equiv (p_{0i_2 \ldots i_\nu}, p_{i_{\nu+1} \ldots i_{2\nu}}) (i_2, i_3, \ldots i_{2\nu})$$ forms an even perm of 1, ... $2\nu$-1,) then the duality relation in $R_{2\nu}$ takes the form

$$(p_{0i_2 \ldots i_\nu}, p_{i_{\nu+1} \ldots i_{2\nu}}) \to i^\nu (p_{i_{\nu+1} \ldots i_{2\nu}}, (-)^\nu p_{0i_2 \ldots i_\nu}).$$

The selfdual and anti-selfdual tensors are characterized by

$$G^+ = (p_{0i_2 \ldots i_\nu}, +(-i)^\nu p_{0i_2 \ldots i_\nu}) ,$$

and

$$G^- = (p_{0i_2 \ldots i_\nu}, -(-i^\nu) p_{0i_2 \ldots i_\nu})$$

respectively, and belong to a space $G_\nu^+$ and $G_\nu^-$, respectively. Now every $\nu$-vector p may be decomposed into the form

$$P = G^- + G^+$$

We now restrict ourselves to the case that $\nu$ is even: $\nu = 2k$
By a generalization of the 4-dimensional spinor theory, one may prove that every orthogonal transformation T in $R_{2\nu} = R_{4k}$ is the product of a transformation $T^+$ works in the space $G^+$ of the first system of isotropic planes $G^+$ but leaves the system $G^-$ invariant, and a transformation $T^-$ which works in the space $G^-$ of the second system of isotropic planes $G^+$ but leaves the system $G^-$ invariant, see Cartan II p. 49.
Suppose now that the orthogonal transformation $T = T^+T^-$ leaves the $\nu$-vector $p_0$ invariant,

$$p_0 = G_0^- + G_0^+ .$$

It follows that $T^+$ has the eigenvector $G_0^+$ in $G^+$ and $T^-$ has the eigenvector $G_0^-$ in $G^-$. Thus we obtain the following system r of $\nu$-vectors.

$$p_r = G_0^+ + G_r^-$$

which remains invariant with $p_0$ under $T^+$, and a system $\ell$ of $\nu$-vectors

$$p_\ell = G_\ell^+ + G_0^- ,$$

which remains invariant under $T^-$. Summarizing, we obtain the following theorem:

Theorem 4.4. *Suppose that there is a 2k-vector $p_0 = G_0^+ + G_0^-$ in the space $R_{4k}$, then there is a set r of 2k-vectors $p_r$, so that $G_r^+ = G_0^+$, which are right Clifford parallel with $p_0$, and there is a set $\ell$ of 2 k-vectors so that $G_\ell^- = G_0^-$, which are left Clifford parallel with $p_0$.*

# APPENDIX

## General properties of line coordinates and bivectors

In the following sections A, B, C, we mention some well-known properties of line coordinates $p^{\mu\nu}$ and the associated dual $p^{\overline{\mu\nu}}$.

A. Definition. The coordinates of the line $x^{\mu} = \lambda r^{\mu} + \kappa s^{\mu}$ (2-41)

are
$$p^{\mu\nu} = r^{\mu}s^{\nu} - r^{\nu}s^{\mu} \quad (\mu,\nu = 0,1,2,3). \qquad (2\text{-}42)$$

Theorem A-1. *All straight lines p in $P_3$ can be mapped 1-1 onto the points of a quadratic surface in the five dimensional projective space $R_5$.*

PROOF. (1). The line coordinates $p^{\mu\nu}$ are independent of the choice of the points r and s on p. For, if we choose r' and s' on p, i.e.,

$$\begin{pmatrix} r' \\ s' \end{pmatrix} = \begin{pmatrix} \lambda & \mu \\ \sigma & \tau \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix}, \text{ then } p^{\mu'\nu'} = \det \begin{pmatrix} \lambda & \mu \\ \sigma & \tau \end{pmatrix} p^{\mu\nu}.$$

Thus the coordinates $p^{\mu\nu}$ of a line p are determined within a factor $\rho$ and because $p^{\mu\nu}$ has six independent components $(p^{0i}, p^{jk})(i,j,k=1,2,3)$, every line p is mapped onto a ray $\rho p^{\mu\nu}$ in $R_6$ or, equivalently, a point $\rho p^{\mu\nu}$ in the projective space $P_5$.

(2) Conversely, suppose that the numbers $(p^{0i}, p^{jk}) \equiv (\vec{p}', \vec{p}'')$ are the coordinates of a line. Then (2-41) and (2-42) imply that the rows of $p^{\mu\nu}$, e.g., $p^{0\nu} = r^{0}s^{\nu}-s^{0}r^{\nu}$, are linear combinations of r and s and thus that the row $p^{0\nu}$ is the intersection point of the line p with the plane $x^{0}=0$. In general, the rows $p^{\mu\nu}$ $(\mu=0,1,2,3)$ are the intersection points of the line p with the planes $x^{\mu} = 0$. Hence it follows that the six line coordinates $p^{\mu\nu}$ determine the line p uniquely.

(3) The definition of $p^{\mu\nu}$ implies that for $\vec{p}' \equiv p^{0i}$ and $\vec{p}'' \equiv p^{jk}$ (i,j,k=1,2,3 and cycl.),

$$\vec{p}' = r^{0}\vec{s} - s^{0}\vec{r}, \qquad \vec{p}'' = \vec{r} \times \vec{s},$$
$$\text{and } \vec{p}'.\vec{p}'' = 0 \qquad (2\text{-}43)$$

Thus in the space $P_5$, a quadratic surface D (matrix D) is given so that the following hold for all line coordinates:

$$p^T Dp = 0 \ , \quad \text{where } D = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \text{ and } p=(\vec{p}', \vec{p}'').$$

(4) Suppose now that we have a point $p(\vec{p}', \vec{p}'')$ in $P_5$ with

$$p^{1'}p^{1''} + p^{2'}p^{2''} + p^{3'}p^{3''} = 0$$

$$\text{or} \quad p^{01}p^{23} + p^{02}p^{31} + p^{03}p^{12} = 0. \qquad (2\text{-}44)$$

We must prove that p corresponds to a line. Because there is in $P_5$ no point $(\vec{0},\vec{0})$, we suppose that $p^{1'} \neq 0$ and take the first two rows of $p^{\mu\nu}$, i.e., $p^{0\nu}(0, p^{1'}, p^{2'}, p^{3'})$ and $p^{1\nu}(-p^{1'}, 0, p^{3''}, -p^{2''})$.
It is easy to prove that condition (2-44) is sufficient for the following relation:

$$(\vec{p}', \vec{p}'') = p^{1'} \begin{bmatrix} 0, & p^{1'}, & p^{2'}, & p^{3'} \\ -p^{1'}, & 0, & p^{3''}, & -p^{2''} \end{bmatrix}^{\mu\nu} .$$

Thus the coordinates $p^{\mu\nu}$ determine a straight line through the points $p^{0\nu}$ and $p^{1\nu}$. (q.e.d.)

One writes the form (2-44) as follows: $\frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta}p^{\alpha\beta}p^{\gamma\delta} = 0$.
Here $\varepsilon_{\alpha\beta\gamma\delta}$ is the Levi-Cività symbol and is defined by:

$$\varepsilon_{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha,\ \beta,\ \gamma,\ \delta \text{ form an even permutation of } 0,1,2,3, \\ -1 & \text{if } \alpha,\ \beta,\ \gamma,\ \delta \text{ form an odd permutation of } 0,1,2,3, \\ 0 & \text{if two indices are equal.} \end{cases}$$

The fundamental property of the Levi-Cività symbol $\varepsilon$ is that

$$\varepsilon_{\alpha\beta\gamma\delta}\, a_{\alpha\alpha'} a_{\beta\beta'} a_{\gamma\gamma'} a_{\delta\delta'} = \det(a)\, \varepsilon_{\alpha'\beta'\gamma'\delta'}$$

We can verify this by taking $\alpha', \beta', \lambda', \delta' = 0,1,2,3$.

## B. The covariant coordinates $p_{\mu\nu}$ and $p_{\overline{\mu\nu}}$

A Lorentz transformation $L \equiv L^{\mu}{}_{\mu'}$ induces, in the space spanned by all $p^{\mu\nu} = r^{\mu}s^{\nu} - r^{\nu}s^{\mu}$, a transformation $L \times L \equiv L^{\mu}{}_{\mu'}L^{\nu}{}_{\nu'}$, i.e.,

$$x^{\mu} = L^{\mu}{}_{\mu'}x^{\mu'} \rightarrow p^{\mu\nu} = L^{\mu}{}_{\mu'}\, L^{\nu}{}_{\nu'}p^{\mu'\nu'} \quad . \tag{2-45}$$

One says that $p^{\mu\nu}$ is a *contravariant* entity . We remark that $p^{\mu\nu}$ is only transformed under the irreducible part of $L \times L$ which works in the space of antisymmetric tensors. A tensor transformed by the representation $(L^{-1})^T \times (L^{-1})^T$ is called a *covariant* entity. In order to construct covariant entities from $p^{\mu\nu}$ we first consider the index $\mu$ in (2-45),

$$p^{\mu'\cdot\cdot} = L^{\mu}{}_{\mu'}\ \ldots p^{\mu'\cdot\cdot} \quad .$$

By lowering this index with $g^{\mu\nu}$ and using $gL=(L^{-1})^T g$ , so that $g_{\gamma\mu}L^{\mu}{}_{\mu'} = L_{\gamma}{}^{\gamma'}g_{\gamma'\mu'}$ we obtain

$$p_{\mu}{}^{\cdot\cdot} = L_{\mu}{}^{\mu'}\ldots p_{\mu'}{}^{\cdot\cdot} \quad .$$

If we also lower $\nu$, the quantity $p_{\mu\nu}$ is transformed under the representation $(L^{-1})^T \times (L^{-1})^T$.

There is also another way to construct covariant quantities.
We lower indices in (2-45) with the operator $\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}$ ,

$$\text{i.e.,} \quad p^{\mu\nu} \rightarrow p_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2}\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}p^{\mu\nu},$$

into components $(\vec{p}', \vec{p}'') \rightarrow (\vec{p}'', \vec{p}')$ .

From the fundamental property of the Levi-Civita symbol $\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}L^{\bar{\mu}}{}_{\bar{\mu}'}L^{\bar{\nu}}{}_{\bar{\nu}'}L^{\mu}{}_{\mu'}L^{\nu}{}_{\nu'} = (\det L)\,\varepsilon_{\bar{\mu}'\bar{\nu}'\mu'\nu'}$, it follows that $\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}L^{\mu}{}_{\mu'}L^{\nu}{}_{\nu'} = (\det L)L_{\bar{\mu}}{}^{\bar{\mu}'}L_{\bar{\nu}}{}^{\bar{\nu}'}\varepsilon_{\bar{\mu}'\bar{\nu}'\mu'\nu'}$ and thus after multiplying (2-45) with $\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}$, we obtain

$$p_{\bar{\mu}\bar{\nu}} = L_{\bar{\mu}}{}^{\bar{\mu}'}L_{\bar{\nu}}{}^{\bar{\nu}'}\ p_{\bar{\mu}'\bar{\nu}'}\ (\det L)$$

This implies that $p_{\bar{\mu}\bar{\nu}}$ is a covariant entity. The fact that (det L) appears implies that $p_{\bar{\mu}\bar{\nu}}$ is a *pseudo tensor*, that is to say in the non-homogeneous case $p_{\bar{\mu}\bar{\nu}}$ gets an extra negative sign under (space and time) reflections.
Summarizing:

Theorem B-1 *Besides the contravariant* $p^{\mu\nu} \equiv (\vec{p}\,', \vec{p}\,'')$, *we may construct two types of covariants, namely,* $p_{\mu\nu}$ *and* $p_{\bar{\mu}\bar{\nu}}$ *defined by:*

$$p_{\mu\nu} = g_{\mu\mu'}g_{\nu\nu'}p^{\mu'\nu'} \quad \text{into components } p_{\mu\nu} \equiv (-\vec{p}\,', \vec{p}\,'')$$

*and*
$$p_{\bar{\mu}\bar{\nu}} = \tfrac{1}{2}\varepsilon_{\bar{\mu}\bar{\nu}\mu\nu}\, p^{\mu\nu} \quad \text{into components } p_{\bar{\mu}\bar{\nu}} \equiv (\vec{p}\,'', \vec{p}\,').$$

*Moreover, the tensor* $p_{\bar{\mu}\bar{\nu}}$ *is a pseudo tensor.*

Geometrical meaning of $p_{\mu\nu}$ and $p_{\bar{\mu}\bar{\nu}}$

The equation of an arbitrary plane V with points $x^\mu$ is of the form $r_\mu x^\mu = 0$. The covariant components $r_\mu$ are the so-called *plane coordinates* of V and in projective geometry one may prove that the relation , $r_\mu = g_{\mu\nu}r^\nu$, implies that the plane $r_\mu$, $x^\mu g_{\mu\nu} r^\nu = 0$, is the polar-plane of the point $r^\mu$. (See construction of $r_\mu$ in the figure).

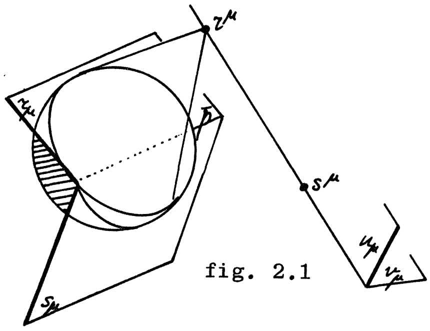


fig. 2.1

The polar-plane $r_\mu$ represents the 3-dimensional space in $R_4$ which is orthogonal to $r^\mu$.

The line p, which joins *the points* $r^\mu$ and $s^\mu$ (the *ray* p), has line coordinates $p^{\mu\nu}$ and one also says that the $p^{\mu\nu}$ are *ray coordinates*.

Similarly, the *planes* $r_\mu$ and $s_\mu$ have an intersection line $\bar{p}$ (the *axis* $\bar{p}$) with equation $t_\mu = \lambda r_\mu + k s_\mu$, and in an analogous way, one may show that the components

$$p_{\mu\nu} = r_\mu s_\nu - r_\nu s_\mu$$

are independent of the choice of the plane $t_\mu$ through $\bar{p}$. Therefore, one says that the $p_{\mu\nu}$ are *axis-coordinates* of the axis $\bar{p}$. One calls $\bar{p}$ the *dual (polar) line* of p and p is unequal to $\bar{p}$ in general. The lines p and $\bar{p}$ thus represent two orthogonal planes Op and O$\bar{p}$ in $R_4$. Now we consider the line p as an axis, that is to say, p is given as the intersection line of the planes u and v. To avoid misunderstanding, we denote the axis coordinates of p by $q_{\mu\nu}$, i.e., $q_{\mu\nu} = u_\nu v_\mu - u_\nu v_\mu$.

By comparing the covariant $q_{\mu\nu}$ with the covariant $p_{\bar{\mu}\bar{\nu}}$, which we have introduced in the foregoing section, in a suitable choosen coordinate system

$$p^{\mu\nu} = \begin{bmatrix} 0,1,0,0 \\ 1,0,0,0 \end{bmatrix}^{\mu\nu} = (-1,0,0 \mid 0,0,0) \rightarrow p_{\bar{\mu}\bar{\nu}} = (0,0,0 \mid 1,0,0)$$

$$q_{\mu\nu} = \begin{bmatrix} 0,0,0,1 \\ 0,0,1,0 \end{bmatrix}_{\mu\nu} = (0,0,0 \mid -1,0,0).$$

It follows that $p_{\bar{\mu}\bar{\nu}} = \rho q_{\mu\nu}$ (2-46)

and thus the $p_{\bar{\mu}\bar{\nu}}$ are the axis coordinates of the line $p^{\mu\nu}$ itself. Similarly, we may introduce $p^{\bar{\mu}\bar{\nu}} = \varepsilon^{\bar{\mu}\bar{\nu}\mu\nu} p_{\mu\nu}$, which are the ray coordinates of the dual line $\bar{p}$.

An arbitrary plane $w_\nu$ through the line p contains $r^\mu$ and $s^\mu$ and thus $p^{\mu\nu} w_\nu = 0$. An arbitrary point $x^\mu$ on p lies in the planes $v_\mu$ and $w_\mu$ and thus $q_{\mu\nu} x^\nu = 0$. Using (2-46) we obtain:

Theorem B-2 *The equation of a line p is $p^{\mu\nu} w_\nu = 0$ for planes $w_\nu$, and $p_{\bar{\mu}\bar{\nu}} x^\nu = 0$ for points $x^\nu$.*
*The equation of the dual line p is for planes $p^{\bar{\mu}\bar{\nu}} w_\nu = 0$* *(2-47)*
*and for points $p_{\mu\nu} x^\nu = 0$.* *(2-48)*

## C. Invariants in the six-dimensional space of anti-symmetric tensors

If we restrict ourselves to inhomogeneous $p^{\mu\nu}$, i.e., bivectors, we find that there are two important invariants.

Theorem C-1 *The six dimensional space of anti-symmetric tensors posesses with respect to the restricted Lorentz group, two invariants, namely,*

$$F = \tfrac{1}{2} p_{\mu\nu} p^{\mu\nu} = \vec{p}''^2 - \vec{p}'^2 \text{ and } G = \tfrac{1}{2} p_{\bar{\mu}\bar{\nu}} p^{\mu\nu} = \vec{p}' . \vec{p}'' . \qquad (2\text{-}49)$$

*G is a pseudoscalar, which means that under a reflection, G changes sign.*

The proof of this theorem follows at once from the covariance of $p_{\mu\nu}$ and $p_{\bar{\mu}\bar{\nu}}$.

Definition Two bivectors are called *dual* if they determine two dual lines $p^{\mu\nu}$ and $p^{\bar{\mu}\bar{\nu}}$ and if the "norms" F of $p^{\mu\nu}$ and $p^{\bar{\mu}\bar{\nu}}$ are equal. Starting from $p^{\mu\nu}$, we obtain $p^{\bar{\mu}\bar{\nu}}$ by

$$p^{\mu\nu} \rightarrow p_{\mu\nu} \rightarrow p^{\bar{\mu}\bar{\nu}}$$

$$\text{or} \quad (\vec{p}\,',\vec{p}\,'') \rightarrow (-\vec{p}\,', \vec{p}\,'') \rightarrow (\vec{p}\,'',-\vec{p}\,') \quad .$$

The requirement that $p^{\mu\nu}$ and $p^{\bar{\mu}\bar{\nu}}$ are dual is

$$p_{\mu\nu}p^{\mu\nu} = p_{\bar{\mu}\bar{\nu}}p^{\bar{\mu}\bar{\nu}} \text{ or } \vec{p}\,''^2-\vec{p}\,'^2 = \rho^2(\vec{p}\,'^2-\vec{p}\,''^2),$$

so that $\rho = \pm i$.

Hence the dual bivector of $(\vec{p}\,', \vec{p}\,'')$ is $\pm i(\vec{p}\,'',-\vec{p}\,')$. (2-50)

The reduction of the space of anti-symmetric tensors into two invariant subspaces is discussed in detail in chapter I section 2, and in chapter II section 4.

chapter III

## GEOMETRY OF ZERO-MASS EQUATIONS

In this chapter, we study the projective geometrical background of some equations which are known in physics as the Proca equation, the Maxwell equations, the Weyl equation and the generalized Weyl equation.
These equations are linear first order equations in an n-component function $\psi(x)$, i.e. $L(\partial_\mu, \psi(x)) \equiv 0$.
The idea is that by developing $\psi(x)$ in plane waves i.e. $\psi(x) = \psi(p)e^{ip.x}$ one obtains equations in "momentum space" $L(ip_\mu, \psi(p)) \equiv 0$ which may be studied with aid of projective geometry.
According to this method, the generalized Weyl equation describes in fact the system of isotropic planes on the light cone, and the one-dimensionality of the representation space to which $\psi(p)$ belongs (for a fixed p) can be very clearly shown. Therefore this treatment gives the geometrical back ground of the fact that photons are only transversely polarized, that there exist only right-handed neutrino's (and only left-handed anti-neutrino's) and that the equations of Maxwell may be brought in neutrino form.
In the sections 1, 2 and 3 of this chapter we treat the geometrical relations which appear in some special cases (the equations of Proca, Maxwell and Weyl); this serves at the same time as an introduction to section 4 (the generalized Weyl equation). Solutions of these equations may be written in terms of generalized spherical functions.
Finally we note that in each section the study of the properties of $\psi(p)$ is given first, followed by the properties of the covariant equations which $\psi(p)$ obeys.

## 1. Geometry of the Proca equation

### 1.1. Description of all lines in $R_4$ by the representation $D^{\frac{1}{2}\frac{1}{2}} + D^{10} \dotplus D^{01}$

Analogously to $R_3$, we now introduce in $R_4$ along with the affine coordinates (ct, x, y, z) the five homogeneous coordinates $x^{\sigma}(\sigma = -1, 0, \ldots, 3)$*), such that

$$(ct, x, y, z) = (\frac{x^0}{x^{-1}}, \frac{x^1}{x^{-1}}, \frac{x^2}{x^{-1}}, \frac{x^3}{x^{-1}}) \quad (x^{-1} \neq 0).$$

The rows $x^{\sigma}$ and $\rho x^{\sigma}$ determine the same point in $R_4$.
The equation of a line P through the points $r^{\mu}$ and $s^{\mu}$, i.e.

$$x^{\mu} = r^{\mu} + \rho\ (s^{\mu} - r^{\mu}),$$

becomes, in homogeneous coordinates, the equation

$$x^{\sigma} = \lambda r^{\sigma} + \kappa s^{\sigma}.$$

This may readily be seen after dividing both terms by $x^{-1}$; see also chapter II section 1.1.
The 4-dimensional *line coordinates* $p^{\sigma\tau}$ of P are defined by

$$p^{\sigma\tau} = r^{\sigma}s^{\tau} - r^{\tau}s^{\sigma}. \qquad (3\text{-}1)$$

The following are properties of the 4-dimensional line coordinates.
1) Non-homogeneous line coordinates or line vectors. If we choose two other points r' and s' on P we get $p^{\sigma'\tau'} = \rho p^{\sigma\tau}$ (as in $R_3$).
The result is that the coordinates of a line are homogeneous, thus independent of the choice of r and s on P.
The Lorentz group is an affine group, for the 3-plane at infinity remains invariant. Hence we take affine coordinates, i.e. $r^{-1} = s^{-1} = 1$ and thus $p^{-1\mu} = s^{\mu} - r^{\mu}$. The point and line coordinates are no longer homogeneous and it follows that $p^{\sigma'\tau'} = p^{\sigma\tau}$ ($\rho = 1$) if and only if $s^{\mu'} - r^{\mu'} = s^{\mu} - r^{\mu}$.

---

*) $\sigma, \tau$ take on the values -1, 0, 1, 2, 3, and, in accordance with the notation convention of chapter I section 1.1 and 3.1, the indices

| | | |
|---|---|---|
| $\mu, \nu, \ldots$ | take on the values | 0, 1, 2, 3 |
| i, j, k, ... | " " " " | 1, 2, 3 |
| a, b, c, ... | " " " " | 0, 1 |

If we call the class of all vectors rs which may be obtained from each other by translations along the line P, a *line vector*, then it follows that the non-homogeneous $p^{\sigma\tau}$ determines a line vector in $R_4$.

<u>Notations:</u> We use the notation

$$p^{\sigma\tau} = \begin{bmatrix} r^{-1} & r^{0} & r^{1} & r^{2} & r^{3} \\ s^{-1} & s^{0} & s^{1} & s^{2} & s^{3} \end{bmatrix}^{\sigma\tau} = \begin{bmatrix} 1 & r \\ 1 & s \end{bmatrix}^{\sigma\tau}$$

One may consider the components $p^{\sigma\tau}$ as an anti-symmetric (5 × 5) tensor as well as a ten-dimensional vector. For that we first take the four components $p^{-1\mu} = s^{\mu} - r^{\mu}$, which we shall call $p^{\mu}$; there remains the anti-symmetric tensor

$$p^{\mu\nu} = [rs]^{\mu\nu}. \text{ Thus } p^{\sigma\tau} = (p^{\mu}, p^{\mu\nu}) = (s-r, [rs]).$$

Analogously we have written the anti-symmetric tensor $p^{\mu\nu}$ in chapter II, formulae (2-4), ... (2-7) as a six-vector $(\vec{p}', \vec{p}'')$ by first taking the components $p^{oi} = r^{o}s^{i} - s^{o}r^{i}$ of $\vec{p}'$ and afterwards the components $p^{jk} = r^{j}s^{k} - s^{j}r^{k}$ of $\vec{p}''$ (i, j, k = 1, 2, 3 and cycl.).
Thus the anti-symmetric 5 × 5 matrix $p^{\sigma\tau}$ looks as follows

$$(p^{\sigma\tau}) = \begin{pmatrix} 0 & p^{0} & p^{1} & p^{2} & p^{3} \\ -p^{0} & 0 & p^{1'} & p^{2'} & p^{3'} \\ -p^{1} & -p^{1'} & 0 & p^{3''} & -p^{2''} \\ -p^{2} & -p^{2'} & -p^{3''} & 0 & p^{1''} \\ -p^{3} & -p^{3'} & p^{2''} & -p^{1''} & 0 \end{pmatrix} \qquad \begin{matrix} p = s-r \\ \vec{p}' = r^{0}\vec{s} - s^{0}\vec{r} \\ \vec{p}'' = \vec{r} \times \vec{s} \end{matrix}$$

The contravariant tensor $p_{\overline{\mu\nu}}$ was defined by $(\vec{p}'', -\vec{p}')i$ (see also appendix to chapter II).

2) <u>Relations between the components $(p^{\mu}, p^{\mu\nu})$</u> There is dependence between the 10 line coordinates $(p^{\mu}, p^{\mu\nu})$. To show this, we note that

$$\begin{bmatrix} 1 & r^{\mu} \\ 1 & s^{\mu} \end{bmatrix} = \begin{bmatrix} 1 & r^{\mu} \\ 0 & s^{\mu} - r^{\mu} \end{bmatrix}$$

$$\text{or } (p^{\mu}, p^{\mu\nu}) = (p^{\mu}, r^{\mu}p^{\nu} - r^{\nu}p^{\mu})$$

which implies that the vector $p^{\mu}$ lies in the plane $p^{\mu\nu}$. Using formula

(2-47) we may equivalently write

$$p_{\bar{\mu}\bar{\nu}}p^{\nu} = 0. \qquad (3\text{-}2)$$

Further, for the anti-symmetric tensor $p^{\mu\nu} = (\vec{p}', \vec{p}'')$,

$$\vec{p}'.\vec{p}'' = 0 \ . \qquad (3\text{-}3)$$

Conversely (3-2) and (3-3) imply that $(p^{\mu}, p^{\mu\nu})$ are the coordinates of a line. Because from (3-2, 3) follows that the 5 cofactors of the diagonal elements $p^{\sigma\tau}$ are zero.

This is sufficient for an anti-symmetric matrix to make the cofactors of *all* elements be zero. All 3 × 3 sub-determinants of the 5 × 5 anti-symmetric matrix also vanish. Hence it follows that the rank of $p^{\sigma\tau}$ is two.

Suppose that $p^{-1\tau}$ and $p^{0\tau}$ are two independent rows. Then the rows of $p^{\sigma\tau}$ are linear combinations of $p^{-1\tau}$ and $p^{0\tau}$ such that $p^{\sigma\sigma} = 0$.

Consequently,

$$(p^{\sigma\tau}) = p^{0\sigma}(p^{-1\tau}) - p^{-1\sigma}(p^{0\tau}) \qquad \times \rho_{\sigma}$$

The anti-symmetry of $p^{\sigma\tau}$ implies $\rho_{\sigma} = 1$. Hence $(p^{\sigma\tau})$ are the coordinates of the line which joins the points $(p^{-1\tau})$ and $(p^{0\tau})$, and it follows that the relations (3-2, 3) are necessary and sufficient for $(p^{\mu}, p^{\mu\nu})$ to be the coordinates of a line.

3) Transformation properties of $p^{\sigma\tau}$. The components $p^{\sigma\tau}$ are written as $p^{\sigma\tau} = (p^{\mu}, p^{\mu\nu})$. The vector $p^{\mu}$ is transformed under the vector representation $D^{\frac{1}{2}\frac{1}{2}}$ and the anti-symmetric tensor $p^{\mu\nu}$ is transformed under the representation $D^{10} + D^{01}$. (Chapter II, §4.1) From this follows:

Theorem 1.1 *With respect to the restricted Lorentz group, all lines $p^{\sigma\tau}$ in $R_4$ transform under the representation $D^{\frac{1}{2}\frac{1}{2}} \dot{+} D^{10} \dot{+} D^{01}$.*

We now consider the behaviour of $p^{\sigma\tau}$ under the larger *Poincaré group*, which consists of all Lorentz transformations supplied with all translations in $R_4$.

With a translation, $x' = x + a$, the components

$$p^{\sigma\tau} = \begin{bmatrix} 1 & r \\ 1 & s \end{bmatrix}^{\sigma\tau}$$ transform into

$$p^{\sigma'\tau'} = \begin{bmatrix} 1 & r + a \\ 1 & s + a \end{bmatrix}^{\sigma'\tau'}$$

$$\text{or } (p^{\mu'}, p^{\mu'\nu'}) = (s-r, \begin{bmatrix} r + a \\ s + a \end{bmatrix}) = (s-r, [r, s] + [a, (s-r)]) \quad .$$

It follows that the component $p^{\mu}$ is translation invariant and $p^{\mu}$ and $p^{\mu\nu}$ transform according to the (10 × 10) linear transformation

$$T : \begin{array}{l} p^{\mu} \to p^{\mu} \\ p^{\mu\nu} \to p^{\mu\nu} + (a^{\mu}p^{\nu} - a^{\nu}p^{\mu}) \end{array} \tag{3-4}$$

The consequence of this is that the irreducible space of tensors $G^{\mu\nu} = (\vec{G}, -i\vec{G})$ transforming under $D^{10}$ and the irreducible space of tensors $\dot{G}^{\mu\nu} = (\vec{\dot{G}}, + i\vec{\dot{G}})$ transforming under $D^{01}$ are transformed into each other, and the 6-dimensional space of anti-symmetric tensors for *one* irreducible space with respect to the Poincaré group.

## 1.2. The Proca equation

A point $x^\mu$ lies on a line $P(p^\mu,\ p^{\mu\nu})$ if

$$p^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu. \qquad (3\text{-}5)$$

This may be seen noting that x lies on P if and only if the coordinates of the line vector which joins x and x+p, i.e.

$$\begin{bmatrix} 1, & x \\ 1, & x+p \end{bmatrix} = (p^\mu,\ x^\mu p^\nu - x^\nu p^\mu)$$

are exactly the same as $P(p^\mu,\ p^{\mu\nu})$. The above statement follows immediately from this. Moreover, we now suppose the point $x^\mu$ lies in a space orthogonal to the vector $p^\nu$, i.e.

$$x^\mu p_\mu = 0. \qquad (3\text{-}6)$$

In the four-dimensional space, this means that $p^\nu$ is a tangent vector of the hyperboloïd on which $x^\mu$ lies, see fig. 3.1.

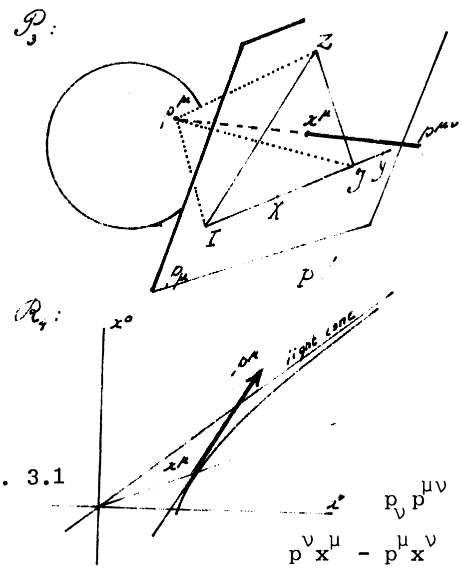



fig. 3.1

Combining (3-5) and (3-6) we obtain $p^{\mu\nu} p_\nu = (p^\nu p_\nu) x^\mu$ and we can replace (3-5) and (3-6) by the system

$$p_\nu p^{\mu\nu} = \kappa^2 x^\mu \qquad (3\text{-}7)$$

$$p^\nu x^\mu - p^\mu x^\nu = p^{\mu\nu}. \qquad (3\text{-}8)$$

Because only the case $p^\nu p_\nu \geq 0$ is important in physics, we have set $p^\nu p_\nu = \kappa^2$, where $\kappa$ is real.

Assume now that $p^{\mu\nu}$ and $x^\mu$ are functions of the vector $y^\mu$ and substitute $p_\nu \to -i\partial_\nu = -i \dfrac{\partial}{\partial y^\nu}$, then we obtain the following system

$$\partial_\nu (ip^{\mu\nu}) = -\kappa^2 x^\mu \qquad (3\text{-}7a)$$

$$\partial^\nu x^\mu - \partial^\mu x^\nu = (ip^{\mu\nu}) \qquad (3\text{-}8a)$$

which is transforming equivalent to (3-7, 8). Conversely, if the ten-dimensional vector $\psi$, where $\psi(y) \equiv (x^{\mu}(y), p^{\mu\nu}(y))$ is developed in plane waves i.e $\psi(y) = \psi(p)e^{ip\cdot y}$ $(p\cdot y = p^{\mu}y_{\mu})$. We return to equations (3-7, 8).
Equations (3-7a, 8a), known in elementary-particle physics as the *Proca equation*, are used for describing particles with spin = 1; see formula (1-48a). The vector $p = (\frac{E}{c}, \vec{p})$ is called the *four-momentum*; see formula (1-7) and $\kappa$ is equal to the *rest mass* of the particle, within a constant factor. We shall call in future the equations (3-7, 8) also the Proca equation. If one chooses $(p^{\mu})$ as time-axis (new $x^{0}$-axis) and three vectors $x^{\mu}$, with $x^{\mu}p_{\mu} = 0$, one obtains the *rest system* of the vector $p^{\mu}$ (or particle). We call the vector $x^{\mu}$ from the equation (3-7, 8) the point of *application* of the line vector $P(p^{\mu}, p^{\mu\nu})$ which belongs to the spatial part of the rest system of P. So we have

<u>Theorem 1.2.</u> *The Proca equation in momentum space gives the impliciet relation which exists between the line vectors P* $(p^{\mu}, p^{\mu\nu})$ *and its spatial points of applications* $x^{\mu}$ *in the rest system of P.*
Considering now translations in the plane $p_{\nu}$ orthogonal to $p^{\nu}$, we have

<u>Theorem 1.3.</u> *The Proca equation (3-7, 8) is invariant under translation in the 3-plane* $p_{\nu}$.

PROOF. Using formula (3-4), a translation $a^{\mu}$ in the plane $p_{\nu}$ has the form

$$x^{\mu} = x'^{\mu}+a^{\mu}$$

$$\text{and } (p^{\mu}, p^{\mu\nu}) = (p'^{\mu}, p'^{\mu\nu}+a^{\mu}p'^{\nu} - a^{\nu}p'^{\mu}) \quad (a^{\mu}p_{\mu} = 0),$$

where the accents are here set on the right side.
Substituting these expressions into (3-7, 8) one may verify that these equations are invariant under translation in the plane $p_{\nu}$.
We obtain another proof by observing that the equations (3-7, 8) are homogeneous in the components $(x^{\mu}, p^{\mu\nu})$. Thus every linear combination of two solutions is again a solution of (2-7, 8). Because $(a^{\mu}, a^{\mu}p'^{\nu}-a^{\nu}p'^{\mu})$ is a solution, it follows that $(x'^{\mu}, p'^{\mu\nu})$ is a solution of these equations.
Finally we make some remarks which will be used in the following sections.

<u>Remark 1.1.</u>

In order to study properties of the solutions $(a^{\mu}(y), p^{\mu\nu}(y))$ for a fixed $p^{\mu}$ one introduces the *little group* $G_{+}(p)$ which consists of all *restricted Lorentz transformations* which leave $p^{\mu}$ invariant.
$G_{+}(p)$ is isomorphic to the 3-dimensional rotation group $O_{3+}$, see section 1.2. of chapter II.
The rest system of $p^{\mu}$ is called PXYZ, then we may consider the sub-group $O_{2+}$ of $G_{+}(p)$ of rotations about the z-axis, see fig. 3.1.
It follows that, with respect to $O_{2+}$, we obtain 3 eigenvectors $(x^{\mu}, p^{\mu\nu})$, viz. the z-axis $(Z, [P,Z])$ with eigenvalue $\lambda = 1$ and the isotropic lines $(I, [P, I])$ and $(J, [P,J])$ in the (x, y)-plane with eigenvalues $\lambda = e^{-i\theta}, e^{+i\theta}$. Infinitesimally one obtains

$$m = i \frac{\partial}{\partial\theta} (\lambda)_{\theta=0} \text{ thus } m = -1, 0, +1.$$

# 2. Geometry of the Weyl equation.

## 2.1. Introduction

We require that the line $P(p^{\mu}, p^{\mu\nu})$ is a *light line* - thus that the direction vector $p^{\mu}$ of P is a *light vector*, i.e.

$$p^2 \equiv p_0^2 - p_1^2 - p_2^2 - p_3^2 = 0.$$

In order to study translation-invariant properties of P, it is sufficient to restrict ourselves to lines $P(p^{\mu}, 0)$ through the origin. Thus in the future every line $P(p^{\mu}, 0)$ represents the whole class of lines parallel with P.

We wish to study the existence and properties of the space of functions $\psi(p) \equiv (\psi^1(p), \ldots, \psi^n(p))$ which may be defined on the light cone by using methods from projective geometry. We require that the function value $\psi(p)$ belongs to an n-dimensional irreducible representation space $R^{jj'}$ of the restricted Lorentz group $L_+^{\uparrow}$, i.e.

$$\psi'(p') = D(\Lambda)\psi(p) \quad (p' = \Lambda p).$$

Considering now the subgroup $G_+(p)$ of $L_+^{\uparrow}$, the *little group* of p which contains all Lorentz transformations which leave p invariant, it follows that all $\psi'(p) = D(G_+)\psi(p)$ span a subspace $R_p$ of $R^{jj'}$ which is invariant with respect to $G_+$. We will require that $R_p$ is an irreducible representation space of $G_+$. Summarizing we require:

I. *The function value $\psi(p)$ is transformed by an irreducible representation $D(G_+(p))$ which is contained in the irreducible representation $D(L_+^{\uparrow})$ of the restricted Lorentz group* *)

The research of properties of the function $\psi(p)$ is facilitated by the fact that if $p^2 = 0$ then every irreducible representation of $G_+(p)$ is *one-dimensional*, i.e.

$$e^{i\theta}\psi(p) = D(G_+)\psi(p) , \qquad \text{see section 4.}$$

*) The physical meaning of this condition is that the spin value j and the possible projections m on the z-axis of a particle are uniquely determined.

Therefore as long as we consider representations of the restricted group we replace I by requirement II:

II. *The function value $\psi(p)$ is transformed by an irreducible representation of the restricted group and is invariant with respect to $G_+(p)$. Considering the full Lorentz group, it is necessary in order to avoid confusion to distinguish the proper little group $G_+(p)$ which is contained in $L_+^\uparrow$ from the full little group $G(p)$. Because irreducible representations of $G(p)$ are in general 2-dimensional.*

For instance if P is a space reflection, $P(p^0, \vec{p}) = (p^0, -\vec{p})$, and r is a rotation $r(p^0, -\vec{p}) = (p^0, \vec{p})$, then rP belongs to the little group G(p) but not to $G_+(p)$.
With the representation space $R^{jj'}$ and the representation space $R^{j'j}$ ($j \neq j'$) of the restricted group we may form an irreducible representation space of the *full* group, i.e.

$$R^{jj'} \dot{+} \dot{R}^{j'j}, \tag{3-9}$$

where both spaces are transformed into each other with space reflection. See chapter I section 3.3.
Hence, if two vectors $\psi(p)$ and $\dot{\psi}(p)$ are given, transforming by ***irreducible*** *one*-dimensional representations of the *restricted* little group $G_+^\uparrow$ and which are contained in the space $R^{jj'}$ and $\dot{R}^{j'j}$ respectively, we may construct the *irreducible two*-dimensional representation space of the *full* little group G(p) by taking the linear combinations

$$\lambda\psi(p) + \mu\dot{\psi}(p). \tag{3-10}$$

Because the representation space $R^{jj}$ ($j' = j$) is at the same time a representation space of the full group, the irreducible representation of the full little group remains in this case one-dimensional.

In section 2 and 3 we study the properties of $\psi(p)$ with the aid of projective geometry. In section 2 we require that $\psi(p)$ be transformed by the representation $D^{\frac{1}{2}0} \dot{+} D^{0\frac{1}{2}}$ in section 3 that $\psi(p)$ be transformed by the representation $D^{10} \dot{+} D^{01}$. Finally in section 4, where we treat the general case, we require that $\psi(p)$ be transformed by the representation $D^{jj'} + D^{j'j}$.

## 2.2. Description of the light vectors by the representation $D^{\frac{1}{2}0} \dot{+} D^{0\frac{1}{2}}$

In chapter II formula (2-12), every light vector p was mapped onto a $2 \times 2$ matrix P i.e.,

$$P = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}$$

with det P = 0. By this the matrix P was written in the form

$$P = \psi\psi^+, \qquad (3\text{-}11)$$

where $\psi \equiv \psi^a$ (a = 0, 1) is a spinor transforming by a two-dimensional representation of the restricted Lorentz group. Hence, every light vector p is mapped onto a spinor $(\psi^0, \psi^1)$, or, more precisely, every light vector $p(p^0 > 0)$ is (1-1) mapped onto the ray of spinors $e^{i\phi}(\psi^0, \psi^1)$. It follows, by this, that $\psi(p)$ is the only invariant (spinor) on the sphere under the little group $G_+(p)$, which leaves p invariant.
Hence we have obtained the one-dimensional irreducible representation space of the little group $G_+(p)$.
We further refer to the fact that this map can also be obtained by using the fact that every $(p^0, \vec{p})$, on the light cone, lies in the space $R_{p^0}$ of all x such that $(x^0 = p^0)$ and on the sphere $\vec{p}^2 = p_0^2$ in $R_{p_0}$ (see fig. 3.2.)
By describing the points on the sphere in $R_{p_0}$ by the complex number $\frac{\psi^0}{\psi 1}$ (the Gaussian sphere), or in homogeneous form by $\psi \equiv (\psi^0, \psi^1)$ or normalized such that $\psi \cdot \bar{\psi} = 2p_0$, one obtains the spinor representation $D^{\frac{1}{2}0}$ mentioned above. By complex conjugation, one obtains spinors $\bar{\psi} \equiv \psi^{\dot{a}}$ transforming by the representation $D^{0\frac{1}{2}}$. If we take the z-axis along the vector $\vec{p}$, then $\psi(p)$ corresponds with the point N, the North Pole on the Gaussian sphere and $\bar{\psi}(p)$ with $\bar{N}$, the North Pole on the complex conjugated Gaussian sphere. Further we have that the light vectors $(p^0, \vec{p})$ and $(-p^0, -\vec{p})$ are mapped onto the same spinor $\psi(p) \equiv \psi(q)$.

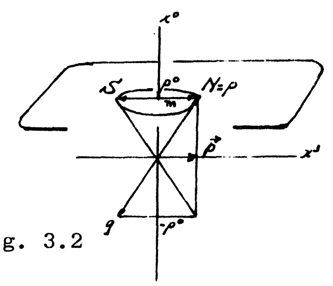



fig. 3.2

Considering now along with the spinor $\psi(p_0, \vec{p})$ $(p_0 > 0)$ also the spinor $\psi(-p_0, \vec{p})$ we may compare these by taking the z-axis along $\vec{p}$. It follows that the spinors $\psi(-p_0, \vec{p}) = \psi(p_0, -\vec{p})$ is equal to S, the South Pole of the Gaussian sphere. So we have

<u>Theorem 2.1.</u> *If p is a point on the light cone $p^2 = 0$ then every $\psi(p^0, \vec{p})$ $(p^0 > 0)$ transforming by an irreducible representation of the little group and which is contained in the representations $D^{\frac{1}{2}0}$ or $D^{0\frac{1}{2}}$ is (after suitable coordinate transformation) given by the North Poles N or $\bar{N}$ respectively and if $(p^0 < 0)$ by the South Poles S or $\bar{S}$ respectively.*

If we take $\psi(p) = N$ then, with respect to rotations $\theta$ around the z-axis, $\psi(p)$ is multiplied with $e^{-i\frac{\theta}{2}}$ thus $\psi(p)$ is the vector $e_{\frac{1}{2}}$ from the representation space of $D^{\frac{1}{2}0}$. Substituting this in the above theorem we obtain the following table

| $\psi(p) \approx$ | $D^{\frac{1}{2}0}$ | | $\dot{D}^{0\frac{1}{2}}$ | |
|---|---|---|---|---|
| $(p^0 > 0)$ | $N = e_{\frac{1}{2}}$ | | | $\dot{N} = \dot{e}_{-\frac{1}{2}}$ |
| $(p^0 < 0)$ | | $S = e_{-\frac{1}{2}}$ | $\dot{S} = \dot{e}_{+\frac{1}{2}}$ | |

(3-12)

Cf Gel'fand, p. 338.

*We observe that we have obtained this result without referring to the invariant equation which $\psi(p)$ satisfies and that the properties of $\psi(p)$ simply follow from the geometrical structure of the light cone.*

We further note that every other map $p \to \tilde{\psi}(p)$ which may be constructed such that $\tilde{\psi}(p)$ is transformed by $D^{\frac{1}{2}0}$ can only differ a coordinate transformation from $\psi(p)$, constructed in (3-11), and thus has the same properties.

## 2.3. The Weyl equation.

Here we study the implicit relations which must exist between p and $\psi(p)$. The fact that the matrix P may be decomposed by (3-11) such that the elements of one row are in the ratio $(\psi^0 : \psi^1)$ is expressed by the equation

$$(\psi^1, -\psi^0)P = 0 \tag{3-13}$$

With the matrix $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\psi_a = C_{ab}\psi^b$, thus $(\psi_0, \psi_1) = (\psi^1, -\psi^0)$ (see formula (1-26)), we obtain the equation

$$p^{a\dot{c}}\psi_a = 0. \tag{3-14}$$

Using the fact that P is Hermitian, $p^{a\dot{c}} = p^{\dot{c}a}$, we get by raising and lowering indices

$$p_{\dot{c}a}\,\psi^a = 0 \tag{3-15}$$

Finally, using the Pauli-matrices $\sigma^\mu$, i.e. $p_{\dot{c}a} = \sigma^\mu{}_{\dot{c}a}\,p_\mu$, we may write

$$(\sigma^0 p_0 + \sigma^1 p_1 + \sigma^2 p_2 + \sigma^3 p_3)\ (\psi^a) = 0\ . \tag{3-16}$$

If we substitute $p_\mu \to -i\partial_\mu$ and $\psi(p) \to \psi(p, x)$, just as we did in section 1.2, chapter III, we obtain the *equation of Weyl*. Conversely, if we develop $\psi(p, x)$ in plane waves, we reobtain equation (3-16)
We observe that $\psi$, which appears in the decomposition (3-11), determines not only the point P but also the isotropic bivector of the system g through it. (chapter II, section 4.1) Hence we have

Theorem 2.2. *The isotropic bivectors $\psi(p)$ from the system g through p are described by the equation of Weyl.*

In the same way we may express the decomposition (3-11) by writing.

$$P\begin{pmatrix} \psi^{\dot{1}} \\ -\psi^{\dot{0}} \end{pmatrix} = 0 \quad \text{or } P_{a\dot{c}}\,\psi^{\dot{c}} = 0 \tag{3-17}$$

which is equivalent with the equations

$$(\sigma^0 p_0 - \sigma^1 p_2 - \sigma^2 p_2 - \sigma^3 p_3)\ (\psi^{\dot{a}}) = 0 \tag{3-18}$$

Equation (3-16) and (3-18) are related to each other by spatial reflection, showing that the equation of Weyl is obviously not invariant for spatial reflection. It follows that *equation (3-18) describes the isotropic bivectors of the second system* $\dot{g}$.

Remark 2.1.

The equation of Weyl is used in physics for describing the behaviour of neutrinos (chapter I section 3.4). The four-momentum $(p^0, \vec{p})$ $(p^0 > 0)$, of the particle lies on the light cone. Hence, a neutrino is a particle which travels with the velocity of light and has zero mass.

We have obtained that $\psi(p)$ is the vector $e_{\frac{1}{2}}$ (the representation space of the little group $G_+(p)$ is one-dimensional). The physical meaning of this is that there is only one state of neutrinos in which the spin is *right-handed*.

The spin projection $m = +\frac{1}{2}$ is *parallel* with p (see fig. 3.2).

Considering now anti-neutrinos with four-momentum $(-p^0, \vec{p})$ it follows that $\psi(-p^0, \vec{p})$ is the vector $e_{-\frac{1}{2}}$. Hence there is only one state of anti-neutrinos in which the spin is *left-handed*.

The spin projection $m = -\frac{1}{2}$ is *anti-parallel* with $\vec{p}$ *).

Thus a consequence of the restriction to the representation $D^{\frac{1}{2}0}$ is that not all laws of nature (the equation of Weyl) are invariant with respect to spatial reflection.

The consequence of the fact that the irreducible representation space of the little group $G_+(p)$ is one-dimensional is that there is only one type of neutrino.

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*) We note that it is merely a convention which type of neutrino is called a neutrino and which an anti-neutrino. In the terminology used in physics, anti-neutrinos are right-handed and appear in $\beta$-decay:

$$n \rightarrow p + e^- + \bar{\nu}.$$

# 3. Geometry of the Maxwell equations

## 3.1. Introduction

In section 2 we have obtained the spinors $\psi^{a}(p)$, $\psi^{\dot{a}}(p)$ and now we may form tensor (spinor) products. In this section we take the spinor products $\psi^{a}\psi^{c}$ and $\psi^{\dot{a}}\psi^{\dot{c}}$ which are transformed by the representations $D^{10}$ and $D^{01}$ respectively, see table (1-32). Hence we require that the invariant function $\psi(p)$ is transformed by the representation $D^{10} \dot{+} D^{01}$ and using table (1-32) it follows that we may equivalently state that $\psi(p)$ is transformed as an anti-symmetric tensor $p^{\mu\nu}(p)$.
Therefore the following will be formulated with tensors, without using the theory of spinors, and we will return to spinor calculus in section 4.4.

Remark 3.1.
We start here with *geometry* by interpreting the $p^{\mu\nu}(p)$ as planes in $R_4$. Afterwards, in remark 4.1., we shall show that the geometrical properties of certain families of planes are closely related to properties which are obtained in the literature (De Vos). This depends on the fact that every other map $p \to \Omega^{\mu\nu}(p)$, where $\Omega^{\mu\nu}$ is an element of an "abstract" vector space which is transformed under $D^{10} \dot{+} D^{01}$ has the same transformation properties as the anti-symmetric tensor $p^{\mu\nu}$, since the representation matrices acting on $p^{\mu\nu}$ and $\Omega^{\mu\nu}$ are the same or can only differ by a coordinate transformation.

## 3.2. Description of all light vectors by the representation $D^{10} \dot{+} D^{01}$

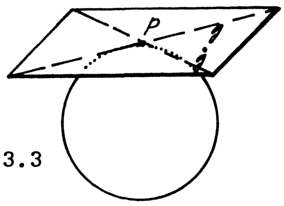


fig. 3.3

In the three-dimensional picture of the Lorentz group, light vectors $p^\mu$ correspond with points p on the unit sphere.
We will construct a map of the point p onto a line $p^{\mu\nu}(p)$, or linear sum of lines, which is invariant with respect to the little group $G_+(p)$. *We shall prove that $p^{\mu\nu}(p)$ is given by the isotropic lines $g^{\mu\nu}$ and $\dot{g}^{\mu\nu}$ through p.*

1. The isotropic lines g and $\dot{g}$ through $p^\mu$ are defined as the intersection of the complex sphere C with the tangent plane $p_\mu$ at p. Because the tangent plane $p_\mu$ and C are invariant under $G_+(p)$, it follows that we have the invariant lines $g^{\mu\nu}(p)$ and $\dot{g}^{\mu\nu}(p)$.

2. If m is a line which is also invariant under $G_+(p)$, then the two or one complex intersection points of m with C are also invariant. Because p is the only invariant and real point on C it follows that m necessarily coincides with g or $\dot{g}$.
All lines $p^{\mu\nu}(\vec{p}', \vec{p}'')$ are (1-1) mapped onto points P on the quadratic surface $\Gamma$: $\vec{p}' \cdot \vec{p}'' = 0$ in the 5-dimensional space $P_5$ of rays $\rho p^{\mu\nu}$ (see p. 131). Consequently $g = g^{\mu\nu}$ and $\dot{g} = \dot{g}^{\mu\nu}$ are the only invariant points on $\Gamma$.

3. Finally, if $a^{\mu\nu} = (\vec{a}', \vec{a}'')$ is an arbitrary anti-symmetric tensor ($\vec{a}'.\vec{a}''$ need not to be zero) which is invariant under $G_+(p)$ then the lines ga and $\dot{g}$a in $P_5$ are invariant and with that their intersection point with $\Gamma$. Because g and $\dot{g}$ are the only invariant points on $\Gamma$ it follows that ga and $\dot{g}$a coincides with $g\dot{g}$ and a with g or $\dot{g}$, by which the statement made in the beginning of this intersection is proved.
Returning to four-dimensional considerations we have to replace "isotropic lines" by "isotropic bivectors".
Below in formula (3-21) we have for the coordinates $g^{\mu\nu}$ and $\dot{g}^{\mu\nu}$ that they are of the form

$$g^{\mu\nu} = (\vec{G}, -i\vec{G})$$
$$\dot{g}^{\mu\nu} = (\dot{\vec{G}}, +i\dot{\vec{G}}).$$

This implies that $g^{\mu\nu}(p)$ and $\dot{g}^{\mu\nu}(p)$ transform by the representations $D^{10}$ and $D^{01}$ respectively and that for a fixed p they form the one-dimensional irreducible representation spaces of the little group $G_+(p)$. Considering now the *full* little group G(p), the bivectors $g^{\mu\nu}$ and $\dot{g}^{\mu\nu}$, span the 2-simensional irreducible space $R_p$ of G(p), which consists of all transport bivectors through p, i.e.

$$\lambda g^{\mu\nu} + \mu \dot{g}^{\mu\nu}, \tag{3-18a}$$

see (3-10).

So we have the following theorem.

<u>Theorem 3.1.</u> *If p is a point on the light cone $p^2 = 0$ and $R_p$ is the 2-dimensional representation space of the full little group G(p), contained in the representation of $D^{10} + D^{01}$, then $R_p$ is formed by all tangent bivectors in p and spanned by the two isotropic bivectors $g = (g^{\mu\nu})$ and $\dot{g} = (\dot{g}^{\mu\nu})$ through p.*

If the spatial component $\vec{p}$ is taken as z-axis then with respect to rotations $r(\theta)$ about the z-axis $g^{\mu\nu}$ and $\dot{g}^{\mu\nu}$ are transformed into $e^{-i\theta}g^{\mu\nu}$ and $e^{+i\theta}\dot{g}^{\mu\nu}$ see formula (3-23).

Using the infinitesimal operator $J_3 = i\left(\frac{\partial\, r(\theta)}{\partial\theta}\right)_{\theta=0}$ we have that $g^{\mu\nu}$ and $\dot{g}^{\mu\nu}$ have eigenvalues m = + 1, - 1 respectively and therefore we shall write $G_+ = g^{\mu\nu}$ and $\dot{G}_- = \dot{g}^{\mu\nu}$.

The effect of a total reflection $p^\mu \to -p^\mu$ is $g^{\mu\nu} \to g^{\mu\nu}$ . Thus all vectors $(p^0, \vec{p})$ are mapped on the same value $g^{\mu\nu}$ in $R_6$ as the vectors $-(p^0, \vec{p})$.

In analogy to table (3-12) we obtain the table

| $\psi(p)$ | $D^{10}$ | | | $D^{01}$ | | |
|---|---|---|---|---|---|---|
| $(p^0 > 0)$ | $G_{+1}$ | X | X | X | X | $\dot{G}_{-1}$ |
| $(p^0 < 0)$ | X | X | $G_{-1}$ | $\dot{G}_{+1}$ | X | X |

(3-18b)

Generalized spherical functions

In order to determine the coordinates $g^{\mu\nu}$, $\dot{g}^{\mu\nu}$ of the isotropic bivectors explicitely as functions of p. We consider the plane $x = \lambda p + ug$ through the vectors $p = p^0(1, \vec{p})$ and $g = (0, \vec{G})$ and require that this plane be isotropic, i.e. lies on the complex cone $x^2 = 0$. It follows that $p^2 = 0$, $p.g. = 0$, $g^2 = 0$, or, what is the same,

$$\vec{p}^2 = p_0^{\ 2};\ \vec{p}.\vec{G} = 0;\ \vec{G}^2 = 0. \qquad (3\text{-}19)$$

We substitute $\vec{G} = \frac{\vec{G}' + i\vec{G}''}{\sqrt{2}}$ into (3-19), where $\vec{G}'$ and $\vec{G}''$ are real vectors, we shall also write $\vec{G}' = \vec{E}$ and $\vec{G}'' = \vec{H}$ and obtain

$$\vec{p}.\vec{E} = \vec{p}.\vec{H} = 0;\ \vec{E}^{\ 2} = \vec{H}^{\ 2} \text{ and } \vec{E}.\vec{H} = 0\ . \qquad (3\text{-}20)$$

Hence it follows that $\vec{p}$, $\vec{E}$, $\vec{H}$ form an orthogonal triad, and supposing that in this order they are oriented as a right-handed screw, then $\dot{\vec{G}} = \frac{\vec{E} - i\vec{H}}{\sqrt{2}}$ gives the left-handed screw.

Using (3-20) we have that the coordinates of the isotropic bivectors are

$$g^{\mu\nu} = p^0 \begin{bmatrix} 1, & \vec{p} \\ 0, & \vec{G} \end{bmatrix}^{\mu\nu} = p^0(\vec{G},\ \vec{p} \times \vec{G}) \text{ and thus } g^{\mu\nu} = p^0(\vec{G},\ -i\vec{G}) \qquad (3\text{-}21)$$

$$\dot{g}^{\mu\nu} = p^0(\dot{\vec{G}},\ +i\dot{\vec{G}})$$

and hence we obtain the map $p \to p^0(\vec{G}, -i\vec{G})$, or

$p \to p^0\vec{G}$ with respect to the basis $(\vec{e}^{\,k}, -i\vec{e}^{\,k})$, *) (3-22)

and analogously the isotropic lines $\dot{g}^{\mu\nu}$ give the map

$p \to p^0\dot{\vec{G}}$ with respect to the basis $(\vec{e}^{\,k}, +i\vec{e}^{\,k})$ (3-22a)

This map is determined within a factor $re^{-i\theta}$, for the transformation

$$\vec{s} = re^{-i\theta}\vec{G} \text{ corresponds with: } \begin{pmatrix} \vec{s}\,' \\ \vec{s}\,'' \end{pmatrix} = r \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \qquad (3\text{-}23)$$

and it follows that also the vector $\vec{s}$ satisfies the conditions (3-20). However, with respect to the little group $G_+(p)$ the number r is invariant (r is a scalar). In order to show this, we take $p = (1, 0, 0, 1)$ and $g = (0, 1, -i, 0)$. Using formula (2-23), it follows that p and g

---

*) Although the 4-vector $(0, \vec{G})$ is transformed by the representation $D^{\frac{1}{2}\frac{1}{2}}$, we note that vector $\vec{G}$, which appears in $(\vec{G}, -i\vec{G})$ is a 3-vector and transforming by the representation $D^{10}$ (see also chapter II, section 3.2).

correspond with the spinors (1, 0) and (0, 1) respectively. Now applying the transformation $A = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ of the little group to $g \equiv (0, 1)$, it follows that g is transformed as:

$$A : g \rightarrow g' = \alpha p + g.$$

Thus the coordinates of the bivector $g^{\mu\nu}$ are transformed

$$\text{from } g^{\mu\nu} = \begin{bmatrix} p \\ g \end{bmatrix}^{\mu\nu} \text{ into } g'^{\mu\nu} = \begin{bmatrix} p \\ g' \end{bmatrix}^{\mu\nu} = \begin{bmatrix} p \\ \alpha p+g \end{bmatrix}^{\mu\nu} = \begin{bmatrix} p \\ g \end{bmatrix}^{\mu\nu} .$$

Hence the factor r and thus $|\vec{E}|$ and $|\vec{H}|$ are invariant with respect to $G_+(p)$.

In order to fix the map $p \rightarrow p^0\vec{G}$, we take for $\vec{E}$ and $\vec{H}$ unit-vectors attached in the tangent plane of p (see fig. 3.4) and determine the ray $p^0(1, \vec{p})$ by the spherical coordinates $\theta$, $\phi$. Now there is exactly one rotation D, such that the unit vectors $\vec{e}^x$ and $\vec{e}^y$ attached to the North Pole transform into $\vec{E}$ and $\vec{H}$. If $\phi_1$, $\theta$, $\phi_2$ are the Euler angles, then by a rotation $\theta$ around the x-axis, followed by a rotation $\phi_1 = \phi + \frac{\pi}{2}$ around the z-axis the North Pole N is transformed into P, the vector $\vec{e}^x$ into $\vec{e}^{x'} = \vec{e}^{\phi}$, and the vector $\vec{e}^y$ into $\vec{e}^{y'} = -\vec{e}^{\theta}$.

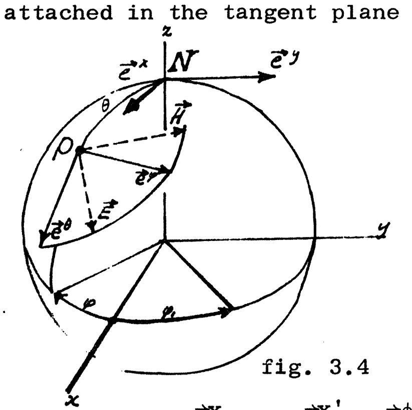

fig. 3.4

($\vec{e}^{\theta}$ and $\vec{e}^{\phi}$ are unit vectors attached in p and positive-directed along the $\theta$- and $\phi$-lines on the sphere).

Finally, after a rotation $\phi_2$ around p (or starting with a rotation $\phi_2$ around the z-axis) the vectors $\vec{e}^{x'}$ and $\vec{e}^{y'}$ are transformed into $\vec{E}$ and $\vec{H}$.

Hence in the first and second columns of the matrix $D(\phi_1, \theta, \phi_2)$, stand the vectors $\vec{E}$ and $\vec{H}$ respectively. We bring the matrix $D(\phi_1, \theta, \phi_2)$ in canonical form; that is to say, we transform to a basis consisting of eigenvectors with respect to rotations around the z-axis; i.e.

$$D^1 = T^{-1}DT \text{ where } T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ -i & 0 & -i \\ 0 & \sqrt{2} & 0 \end{pmatrix} .$$

We note that by multiplying D from the right with T, we obtain the vector $\vec{G}$ in the third row or (+)-row of $D^1$ and we use the index 1 in $D^1$ in order to indicate that $D^1$ is the representation matrix in canonical form, of the representation $D^j (j = 1)$.
$D^1$ is the vector representation of the rotation group.
Hence it follows that $\vec{G}$ transforming by the representation $D^{10}$ is given by the components $D^1_{n,+1} (\phi_1, \theta, \phi_2)$ $(n = -1, 0, +1)$, and, by the same arguments, one can prove that the vector $\vec{G}$ transforming by the representation $D^{01}$ is given by the components $D^1_{-n,-1}$ $(n = -1, 0, +1)$.
If there is given a unitary representation $D^j(\phi_1, \theta, \phi_2)$ of the rotation group, then the matrix components $D^j_{mn} (\phi_1, \theta, \phi_2)$ are known as *generalized spherical functions*, see Gel'fand p. 78-106.
Using formulae (3-22, 23 and 3-10), we have the following map:
$p \to \psi \equiv (\vec{E}, \vec{H}) = \lambda p_0 (\vec{G}, -i\vec{G}) + \mu p_0 (\vec{\bar{G}}, +i\vec{\bar{G}})$. Substituting the generalized spherical functions $D^1$ we obtain:

<u>Theorem 3.2.</u> *If $p = p^0 (1, \sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$ is a point on the light come, then the function $\psi(p) \equiv (\vec{E}, \vec{H})$ transformed by an irreducible representation of the full little group, contained in the representation $D^{10} \dotplus D^{01}$ is given with aid of the generalized spherical functions $D^j_{mn} (\phi_1, \theta, \phi_2)$, i.e.*

$$E_n = p_0(\lambda D^1_{n,1} + \mu D^1_{-n,-1}) \qquad \text{where } n = -1, 0, +1.$$

$$H_n = -ip_0(\lambda D^1_{n,1} - \mu D^1_{-n,-1}) \qquad \phi_1 = \phi + \frac{\pi}{2} \text{ and } \phi_2 \text{ is arbitrary.}$$

## 3.3. The Maxwell equations

In the foregoing subsection, the mapping $p^{\mu} \to g^{\mu\nu}$ of light vectors onto isotropic bivectors is given explicity. Now we study the implicit relations which exist between the vectors p and $g^{\mu\nu}$.

Theorem 3.3. *All tangent bivectors are described by the equations:*

$$p^{\bar{\mu}\bar{\nu}}p_{\nu} = 0 \quad \text{and} \tag{3-24a}$$

$$p^{\mu\nu}p_{\nu} = 0, \tag{3-24b}$$

*"the Maxwell equations".*

PROOF. In the three-dimensional terminology, it follows that equation (3-24a) expresses that the point $p^{\nu}$ lies on the line $p^{\mu\nu}$ and equation (3-24b) expresses that this line lies in the plane $p_{\nu}$, see chapter II, appendix, theorem B-2.

Thus $p^{\nu}$ lies in its correlated plane $p_{\nu}$, $p_{\nu}p^{\nu} = 0$, hence $p^{\nu}$ lies on the unit sphere and $p^{\mu\nu}$ is a tangent line in $p^{\mu}$.

For the fixed four-vector $p^{\mu}$, the tangent bivectors span a two-dimensional space.

Analogously to § 1.2 we observe that if we substitute $p_{\nu} \to -i\partial_{\nu}$ and $p^{\mu\nu} \to F^{\mu\nu}(x)$ we obtain the equations

$$\partial_{\nu} F^{\bar{\mu}\bar{\nu}} = 0 \text{ and} \tag{3-25a}$$

$$\partial_{\nu} F^{\mu\nu} = 0 , \tag{3-25b}$$

which are known in physics as the Maxwell equations. (See chapter I).

Conversely, if we develop the 6-dimensional vector $\psi(x) \equiv F^{\mu\nu}(x)$ in plane-waves, i.e. $\psi(x) = \psi(p)e^{ip.x}$ $(p.x = p^{\mu}x_{\mu})$, we re-obtain the equations (3-24a, b). The vector $p = (\frac{E}{c}, \vec{p})$ is called the four-moment, and hence, in § 3.1, we have treated properties of the solution of the Maxwell equations in to so-called momentum space.

Remark 3.1. In quantum mechanics, the Maxwell equations (3-25a, b) determine the state-function $\psi(p) \equiv F^{\mu\nu}(p)$ of particles which are carriers of electromagnetic interaction, i.e. photons.

Considering rotations around the $\vec{p}$-axis the representation space of $\psi(p)$ is spanned by the eigenvectors $g^{\mu\nu}(p)$ and $\bar{g}^{\mu\nu}(p)$ with eigenvalues

$$e^{-im\theta} \qquad (m = \pm 1).$$

One expresses this by saying that a photon is a spin-one particle with projection $m = +1$ and $m = -1$ on the $\vec{p}$-axis. These two states correspond with right-handed circular-polarized and left-handed circular-polarized light. See Feynman p. 11-9. As contrasted with photons, a spin-one particle $\psi(p)$ with given p and *non*-zero mass has *three* independent states $m = -1, 0, +1$. Hence the physical consequence of the fact that for fixed p the space of all $\psi(p) \equiv p^{\mu\nu}(p)$ is *two*-dimensional is that *photons are only transversely polarized,* and the fact that the eigenvalue $m = 0$ does not appear follows from the considerations made in the beginning of section 3.1.

<u>An alternative form of Maxwell's equations (I).</u>

We can write equations (3-24a, b) in the form (3-26) or (3-27).

$$\left[\begin{array}{cccc|cccc} & & & & \cdot & -H^1 & -H^2 & -H^3 \\ & & & & H^1 & \cdot & E^3 & -E^2 \\ & & & & H^2 & -E^3 & \cdot & E^1 \\ & & & & H^3 & E^2 & -E^1 & \cdot \\ \hline \cdot & E^1 & E^2 & E^3 & & & & \\ -E^1 & \cdot & H^3 & -H^2 & & & & \\ -E^2 & -H^3 & \cdot & H^1 & & & & \\ -E^3 & H^2 & -H^1 & \cdot & & & & \end{array}\right] \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = 0 \qquad (3\text{-}26)$$

where $p^{\mu\nu} \equiv (\vec{E}, \vec{H})$ thus $p^{oi} = E^i$ and $p^{jk} = H^i$ where $i, j, k = 1, 2, 3$ and cycl.

Interchanging the role of $p_\nu$ and $p^{\mu\nu}$ we get

$$
\left(\begin{array}{ccc|ccc}
\cdot & \cdot & \cdot & -p_1 & -p_2 & -p_3 \\
\cdot & -p_3 & p_2 & p_0 & \cdot & \cdot \\
p_3 & \cdot & -p_1 & \cdot & p_0 & \cdot \\
-p_2 & p_1 & \cdot & \cdot & \cdot & p_0 \\
\hline
p_1 & p_2 & p_3 & \cdot & \cdot & \cdot \\
-p_0 & \cdot & \cdot & \cdot & -p_3 & p_2 \\
\cdot & -p_0 & \cdot & p_3 & \cdot & -p_1 \\
\cdot & \cdot & -p_0 & -p_2 & p_1 & \cdot
\end{array}\right)
\begin{pmatrix} E^1 \\ E^2 \\ E_3 \\ \hline H^1 \\ H^2 \\ H^3 \end{pmatrix} = 0 \qquad (3\text{-}27)
$$

Equation (3-26) is of the form

$$
\begin{pmatrix} 0 & p^{\bar{\mu}\bar{\nu}} \\ p^{\mu\nu} & 0 \end{pmatrix} \begin{pmatrix} p_\nu \\ p_\nu \end{pmatrix} = \begin{pmatrix} s^{\bar{\mu}} \\ t^{\mu} \end{pmatrix} = 0 \ .
$$

The vector $t^\mu$ is an ordinary vector, while the vector $s^{\bar{\mu}}$ is a *pseudo vector*. This means that $s^{\bar{\mu}}$ obtains an extra negative sign with space and time reflection because $p^{\bar{\mu}\bar{\nu}}$ is a pseudo tensor (see chapter II, appendix , theorem B-1).

If required, we can write system (3-27) in canonical form. That is to say, first decompose $p^{\mu\nu}(\vec{E}, \vec{H})$ into its irreducible components belonging to $D^{10}$ and $D^{01}$, i.e.,

$$\vec{G} = \vec{E} + i\vec{H} \ , \ \dot{\vec{G}} = \vec{E} - i\vec{H}.$$

Applying this transformation to (3-27), we obtain an equation, say (3-27'). Then transform to eigenvectors with respect to rotations around the $x^3$-axis. Thus for the column of (3-27') we have the transformation:

$$T : (G^k, \dot{G}^k) \to (G^-, G^3, G^+; \dot{G}^-, \dot{G}^3, \dot{G}^+),$$

and for the rows we have that

$$S : (s^{\bar{\mu}}, t^\mu) \to (s^0, s^-, s^3, s^+; t^0, t^-, t^3, t^+).$$

In this way, the $8 \times 6$ matrix B in (3-27) transforms into $SBT^{-1}$ which may be written as the sum of four matrices $\beta_\mu$: $\beta_\mu p^\mu = SBT^{-1}$, and equation (3-27) takes the *canonical* Bhabha form: $(\beta_\mu p^\mu)\psi = 0$. The four matrices $\beta_\mu$ (in canonical form) can be found in Gel'fand p. 313.

Remark 3.2.

If we restrict ourselves to the condition that $\vec{E}$, $\vec{H}$, $\vec{p}$ are real then we can bring the equations (3-24a) and (3-24b) into the form (1-47a, b) chapter I, in our notation $g^{\mu\nu}p_\nu = 0$ (or $g^{\mu\nu}p_\nu = 0$), which expresses the incidence of the real plane $p_\nu$ (point $p^\nu$) with the isotropic bivector $g^{\mu\nu}$.

An alternative form of Maxwell equations (II)

From the Proca equations

$$p_\nu F^{\mu\nu} = \kappa^2 \phi^\mu \qquad (3\text{-}28)$$

$$p^\nu\phi^\mu - p^\mu\phi^\nu = F^{\mu\nu} \qquad (3\text{-}29)$$

(see 3-7, 8) one obtains Maxwell equations by putting $\kappa = 0$ (the mass of a photon is zero).

The same procedure is not possible with the Kemmer equation (1-48b, c) in view of the appearance of $\kappa$ in the denominator. Nevertheless, it is possible to write the equations (3-28, 29) with $\kappa = 0$ in the form

$$\left[\begin{pmatrix} 0 & D^T \\ D & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix}\right] \begin{pmatrix} \phi^\mu \\ F^{\mu\nu} \end{pmatrix} = 0 \qquad (3\text{-}30)$$

where D is the matrix, containing $p^\nu$ in such a way that (3-30) is the same equation as (3-28, 29). See also the text after formula (1-48b).

Thus equation (3-30) may be written in the Bhabha-like form

$$(\beta^\mu p_\mu + \varepsilon)\psi = 0 \quad \text{where } \varepsilon = \begin{pmatrix} 0 & 0 \\ 0 & E \end{pmatrix} \; ^{*)} \qquad (3\text{-}31)$$

and where $\beta^\mu$ are the $10 \times 10$ Kemmer-matrices.

We will show that this equation is relativistic invariant.

---

*) The possibility of writing the Maxwell equations into this form was suggested by a remark in Roman p. 155.

Consider first the Kemmer equation

$$(\beta^{\mu}p_{\mu} + i\kappa)\psi = 0 \qquad (3\text{-}32)$$

where $\psi$ is transforming by the $10 \times 10$ representation matrices $D(L)$. We substitute $\psi(x) = D^{-1}(L)\,\psi'(x')$, $x' = Lx$ and $\partial = L^{T}\partial'$, into (3-32) and left multiply by $D(L)$, to get

$$L^{\mu}{}_{\nu}D(L)\beta^{\nu}D^{-1}(L)\partial'_{\mu}\psi' + i\kappa\psi' = 0$$

(cf. theorem 8.1 chapter I). The Kemmer equation is relativistic invariant; hence, it follows that

$$L^{\mu}{}_{\nu}D(L)\beta^{\nu}D^{-1}(L) = \beta^{\mu} \text{ for all } L.$$

Now equation (3-31) is relativistic invariant if

$$D(L)\,\varepsilon D^{-1}(L) = \varepsilon$$

and this can easily be proven by noting that $D(L)$ is of the form

$$D(L) = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \cdot \begin{matrix} 4 \\ 6 \end{matrix}$$
$$\quad\;\; 4 \quad 6$$

## 3.4. The Maxwell equations in neutrino and spinor form

We now return to spinor calculus. The spinor $\psi$ determines the point P on the light cone as well as the isotropic bivector through it (section 3.1. chapter II).

By the foregoing geometrical treatment of the Maxwell and Weyl equations it is easy to see that both equations express, in a certain sense, the same thing: the incidence of the point P with its corresponding isotropic bivector.

In the Maxwell equations we clearly use the coordinates $g^{\mu\nu}(p)$ or $\dot{g}^{\mu\nu}(p)$ for the isotropic bivectors, whereas, in the equation of Weyl, we use the spin coordinates $\overset{a}{\psi}(p)$ for the same bivectors.

Using formula (2-26) of chapter II, it follows that if the isotropic bivector is given by

$$g^{\mu\nu}(\vec{G}, -i\vec{G}), \; \vec{G} = \vec{E} + i\vec{H},$$

then the $2 \times 2$ matrix G corresponding to $\vec{G}$ may be decomposed by

$$G = \begin{pmatrix} G_3 & G_1 - iG_2 \\ G_1 + iG_2 & -G_3 \end{pmatrix} = \begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} (\psi^1, \; - \psi^0) \; . \qquad (3\text{-}33)$$

We obtain $\begin{pmatrix} \psi^0 \\ \psi^1 \end{pmatrix} = \sigma \begin{pmatrix} G_3 \\ G_1 + iG_2 \end{pmatrix}$, where $\sigma = \dfrac{\pm 1}{\sqrt{G_1 + iG_2}}$ .

$$\text{Thus} \quad \begin{pmatrix} \psi^0 \\ \psi^1_1 \end{pmatrix} = \pm \begin{pmatrix} \dfrac{G_3}{\sqrt{G_1 + iG_2}} \\ \sqrt{G_1 + iG_2} \end{pmatrix} = \pm \begin{pmatrix} \dfrac{E^3 + iH^3}{\sqrt{(E^1 + iH^1) + i(E^2 + iH^2)}} \\ \sqrt{E^1 + iH^1) + i(E^2 + iH^2)} \end{pmatrix} \qquad (3\text{-}34)$$

Substituting equation (3-34) in the Weyl equation (3-15) and in equation (3-17), we obtain the *Maxwell equations in neutrino form*.

See also Barut p. 98, Laporte and Uhlenbeck, Whittaker.

<u>Remark 3.3.</u>

The same result can be obtained by writing the Maxwell equations (3-24 a, b), i.e.

$$p_\nu p^{\mu\nu} = 0$$

$$p_\nu p^{\overline{\mu\nu}} = 0$$

in the form
$$\begin{cases} p_\nu g^{\mu\nu} = 0, \text{ where } g^{\mu\nu} = p^{\mu\nu} + p^{\bar{\mu}\bar{\nu}} & \text{(3-35a)} \\ p_\nu \dot{g}^{\mu\nu} = 0 \text{ and } \dot{g}^{\mu\nu} = p^{\mu\nu} - p^{\bar{\mu}\bar{\nu}} & \text{(3-35b)} \end{cases}$$

If we restrict ourselves to *real* $p^\mu$ and $p^{\mu\nu}$ then the equation (3-35a) is sufficient for obtaining (3-24a, b) since it can be separated into a real and imaginary part. However we shall consider the complex case and use the system (3-35a, b).

After the coordinate transformation

$$p\ (p^0,\ p^1,\ p^2,\ p^3) \rightarrow P\ (P^{0\dot{0}},\ P^{1\dot{0}},\ P^{0\dot{1}},\ P^{1\dot{1}})$$

the equations (3-35a, b) are transformed into

$$P_\nu G^{\mu\nu} = 0$$

$$P_\nu \dot{G}^{\mu\nu} = 0$$

we have that $G^{\mu\nu} = \psi^{ac}$ and $\dot{G}^{\mu\nu} = \psi^{\dot{a}\dot{c}}$, see (2-40), thus

$$P_{\dot{c}a_1}\ \psi^{a_1 a_2} = 0 \qquad \text{(3-36a)}$$

$$P_{\dot{c}_1 a}\ \psi^{\dot{c}_1 \dot{c}_2} = 0 \qquad \text{(3-36b)}$$

These are the *Maxwell equations in spinor form* and they form a generalization of the Weyl equation.

Using the fact that $\psi^{ac} = \psi^a \psi^c$ ,formula (2-40), or using the fact that $p^2 = 0$, thus that $P_{\dot{c}a_1} = \psi_{\dot{c}}\ \psi_{a_1}$ , formula (3-11), it follows from (3-36a) that the components of a column of $\psi^{a_1 a_2}$ are in the ratio $(\psi^0 : \psi^1)$. Hence $\psi^{a_1 a_2} = \psi^{a_1} \psi^{a_2}$ and one obtains from (3-36a) the equation of Weyl. The system (3-36a, b) reduces into

$$\begin{pmatrix} 0 & P_{\dot{c}a} \\ P^{c\dot{a}} & 0 \end{pmatrix} \begin{pmatrix} \psi^{a_1} \\ \psi_{\dot{a}} \end{pmatrix} = 0 \qquad \begin{matrix}\text{(3-36c)} \\ \text{(3-36d)}\end{matrix}$$

which is in fact a special case of the Cartan form $X\xi = 0$, see Cartan II p. 21.

## 4. Geometry of the generalized Weyl equation

### 4.1. Description of all light vectors by the representation $D^{jj'} \dot{+} D^{j'j}$.

We take $\psi(p) \equiv (\psi^1(p), \ldots, \psi^n(p))$ which is an n-component function defined on the light cone $p^2 = 0$. We refer to section 2.1. of this chapter, where we have introduced the little group $G_+(p)$, a subgroup of the restricted Lorentz group $L_+^\uparrow$ which leaves the vector p invariant. Thus every representation $D(L_+^\uparrow)$ of the Lorentz group induces a representation of the little group which in general is not irreducible. After having studied some special cases in section 2 and 3, we consider here the general case and we require that $\psi$ be *transformed by an irreducible representation of the little group $G_+(p)$, which is contained in the representation $D^{jj'}$ of the restricted group.* *)
Using the fact that the representation space $R^{jj'}$, where the representation $D^{jj'}$ acts is spanned by the vectors $e_{m,m'}$, $(-j \leq m \leq +j, -j' \leq m' \leq +j')$, we have

Theorem 4.1. *If p is a light vector, $p^2 = 0$ and $p^0 > 0$ then every irreducible representation of the little group $G_+(p)$ contained in the irreducible representation $D^{jj'}$ of the restricted Lorentz group is one-dimensional and if the spatial component $\vec{p}$ of p is taken as z-axis then $\psi(p)$ is given by the vector $e_{j,-j'}$. The vector $e_{j,-j'}$ has in this case the eigenvalue $e^{-i(j-j')\theta}$ with respect to rotations $\theta$ around the $\vec{p}$-axis.*

PROOF. In chapter II, formula (2-12), every light vector p is mapped onto a spinor $\psi(p) = (u, v)$. More precisely, p is (1-1) mapped onto the ray of spinors $e^{i\theta}(u, v)$. Considering the little group $G_+(p_0)$ of a fixed vector $p_0$, it follows that the fact that $p_0$ is the only invariant vector on the light cone implies that the spinor $\psi(p_0) = (u_0, v_0)$ is the only invariant spinor with respect to $G_+(p_0)$. Hence the 2-dimensional space $R_2$ of spinors contains *only* a one-dimensional irreducible subspace given by $\psi(p_0)$.

---

*) Although the transformation properties of the function $\psi(p)$ defined on the light cone were already studied by Dirac and Majorana, the first systemic treatment was given by Wigner. Wigner also studies representations of the form $D^{\xi}: A\begin{pmatrix} 1 & z_1+iz_2 \\ 0 & 1 \end{pmatrix} \rightarrow e^{i(\xi_1 z_1 + \xi_2 z_2)}$ which are contained in irreducible representations of the *Poincaré group* (Wigner p. 197, Hamermesh p. 486). (cont. see next page)

Because the space $R_2$ is not completely reducible ($R_2$ is not the tensor sum of two invariant subspaces) and $R_2$ is not irreducible, one says that $R_2$ is not *completely reducible* with respect to $G_+(p_0)$.
Now by taking tensor products, one obtains spinors with components

$$p \rightarrow u^h v^k \bar{u}^{h'} \bar{v}^{k'}, \text{ where } h+k = 2j \text{ and } h'+k' = 2j', \tag{3-37}$$

or $\quad p \rightarrow \psi_{mm'} = u^{j+m} v^{j-m} \bar{u}^{j-m'} \bar{v}^{j'+m'}$

which span the representation space of $D^{jj'}$ if p takes on all values on $p^2 = 0$ (Chapter I, formula (1-31a)).
It follows by the foregoing considerations that from all spinors (3-37), only the spinor with components

$$u_0^h v_0^k \bar{u}_0^{h'} \bar{v}_0^{k'} \quad . \tag{3-38}$$

remains invariant under the little group $G_+(p_0)$.
We take the spatial component $\vec{p}_0$ as z-axis thus $p_0 = (a, 0, 0, a)$ by which follows $\psi(p_0) = (u_0, 0)$ where $u_0 = \pm\sqrt{2a}$ and the corresponding spinor (3-37) transforming by $D^{jj'}$ has only one non-zero component $\psi_{jj'} = u_0^{2j}\ \bar{u}^{2j'}$ .
In this coordinate system the group $G_+(p_0)$ is generated by the transformations

$$A = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{+i\frac{\theta}{2}} \end{pmatrix} \tag{3-39}$$

(formula (2-18a, b) chapter II).
It is clear that the spinor $u^{2j} \cdot \bar{u}_0^{2j'}$ has the eigenvalue $e^{-i(j-j')}$ with respect to rotations B.
It remain to show that $e_{j,-j'}$ is the *only* irreducible subspace under $G_+(p)$ in the representation space of $D^{jj'}$, or what is the same thing, we have to prove that every subspace V which is invariant under $G_+(p)$ necessarily contains the vector $e_{j,-j'}$, by which follows that V is an

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*) However, by the proof given here, which requires only simple algebraic methods, it is also possible to expand the function $\psi(p)$ into generalized spherical functions and to develop in the foregoing sections, the projective geometrical background of this theorem.
Prof.dr. J. Hilgevoord drew my attention to the fact that this theorem may be formulated for the representation $D^{jj'}$ instead of $D^{j0}$, $D^{0j'}$

irreducible space if and only if V coïncides with the space $\lambda e_{j,-j'}$.
For that purpose we take an arbitrary vector $x \in V$, the components of x are a linear combination of

$$x^{kk'} = u^h v^k \bar{u}^{h'} \bar{v}^{k'} ,$$

for different p (we omit the $\sum_p$-sign)
The transformation

$$A: u \to u + vz, \ v \to v$$

is represented in V by

$$x^{kk'} = u^h v^k \bar{u}^{h'} \bar{v}^{k'} \to \bar{x}^{kk'} = (u+vz)^h v^k (\bar{u}+\bar{v}\bar{z})^{h'} \bar{v}^{k'}$$

$$= u^h v^k \bar{u}^{h'} \bar{v}^{k'} + \square v^{k+1} \square \bar{v}^{k'} + \square v^k \square \bar{v}^{k'+1} + \dots + v^{2j} \bar{v}^{2j'} \quad {*})$$

If we require that V be an invariant space, then V must also contain the vector $y = x - \bar{x}$.

$$y^{kk'} = \square v^{k+1} \square \bar{v}^{k'} + \square v^k \square \bar{v}^{k'+1} + \dots + v^{2j} \bar{v}^{2j'} .$$

Hence, if the space V contains the vector x with components,

$$x(\square v^k \square \bar{v}^{k'} , \dots , v^{2j} \bar{v}^{2j'} ),$$

then it also contains a vector. $y(\square v^{k+1} \square \bar{v}^{k'} + \square v^k \square \bar{v}^{k'+1} + \dots , \dots, 0)$
Continuing this procedure $(2j+1)(2j'+1)-1$ times, it follows that the space V contains the vector

$$e_{j,-j'} = (u^{2j} \bar{u}^{2j'}, 0, \dots , 0) \qquad (3\text{-}40)$$

This proves that the only irreducible representation space of $G_+(p)$ is given by the vector $e_{j,-j'}$.
Finally, we note that every other map $p \to \tilde{\psi}(p)$ which can be constructed such that $\tilde{\psi}(p)$ transforms by the spinor representation $D^{\frac{1}{2}0}$ can differ only by a coordinate transformation from the foregoing and thus has the same properties.

<u>Remark 4.1.</u> The representation $D^{jj}$ $(j = j')$ is also an irreducible

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*) $\square$ stands for the components u .

representation of the full group. So the above theorem holds for irreducible representations $D^{jj}$ of the full group.
If $j \neq j'$ then $D^{jj'}$ is contained in the irreducible representation $D^{jj'} + D^{j'j}$ of the full group, where the dot denotes that $\dot{D}^{jj'}$ is the conjugate representation of $D^{jj'}$. In this case, we have the full little group G(p) which contain reflection like transformation. Thus theorem 4.1. becomes

<u>Theorem 4.2.</u> *Every irreducible representation of the full little group $G(p)$ $(p^2 = 0)$ which is contained in the irreducible representation $D^{jj'} + D^{j'j}$ $(j \neq j')$ of the full Lorentz group is two-dimensional and spanned by the vectors*

$$e_{j,-j'} \quad \text{and} \quad e_{j',-j}$$

<u>Remark 4.2.</u> In order to relate theorem 4.1. to the literature we note that De Vos has proved that every field $\psi(p)$ defined on the light cone $p^2 = 0$, satisfies the equation

$$\Omega_{\overline{\mu\nu}}\, \psi(p)=0 \text{ ,cf. De Vos (1.20),} \qquad (*)$$

where $\Omega_{\overline{\mu\nu}}$ $(\Omega_{\overline{\mu\nu}} = -\Omega_{\overline{\mu\nu}})$ is a set of operators which belong to the 6-dimensional space I spanned by the infinitesimal operators $I_{\mu\nu}$ of the Lorentz group. The operators $\Omega_{\mu\nu}$ are defined in D.V. (1.10) and satisfies the equation

$$\Omega^{\overline{\mu\nu}} p_{\nu} = 0 \text{ ,} \qquad \text{cf. D.V. (1.13)}$$

$$\Omega^{\mu\nu} p_{\nu} = 0 \text{ ,} \qquad \text{cf. D.V. (1.15).}$$

In order to obtain solutions of these equations we note that these equations are in fact the Maxwell equations (3-24a, b) with operators $\Omega^{\mu\nu}$ instead of coordinates $p^{\mu\nu}$. In the table on page 153 it was shown that if p is taken on the North Pole N(1, 0, 0, 1) then the solutions of the equations (3-24a, b) are given by the isotropic $G_+$ and $\dot{G}_-$ through P. These planes belong to the representation space of $D^{10}$ and $D^{01}$ respectively and are eigenvectors of the infinitesimal rotation $J_3$, i.e.

$$J_3\, G_+ = G_+ \,, \quad J_3\, \dot{G}_- = -\dot{G}_-$$

We note that $J_3$ consists of a component $G_3$ which acts only in the space $D^{10}$ and a component $\dot{G}_3$ which acts only in the space $D^{01}$, see formula (1-85d), so we may also write

$$G_3\, G_+ = G_+ \,, \quad \dot{G}_3\, \dot{G}_- = -\dot{G}_- \qquad (**)$$

Considering now the space I of infinitesimal operators, we first refer to section 3.1., remark 3.1. There we have observed that with respect to the representation theory, this space I has the same properties of the space of antisymmetric tensors. An infinitesimal rotation $J_3$ acts in this space of operators $I_{\mu\nu}$ by the commutation role

$$[J_3, I_{\mu\nu}] = I'_{\mu\nu} \,, \text{ see (1-81).}$$

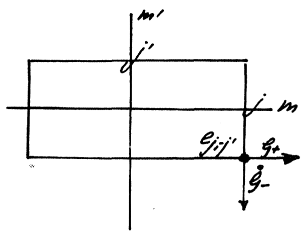
Consequently the solutions of D.V. (1.13) and D.V. (1.15) are two operators $G_+$, $\dot{G}_-$ which belong to the representation space of $D^{10}$ and $D^{01}$ respectively, such that according to (**)

$$[G_3, G_+] = G_+ \text{ and } [\dot{G}_3, \dot{G}_-] = -\dot{G}_- \,.$$

Using (1-85h) it follows that $G_+$ and $\dot{G}_-$ are the so-called *step-operators* acting on the first and second indices of the vectors $e_{mm'}$, respectively. Consequently the solution of (*) is thus a vector $\psi$ such that

$$G_+\, \psi = 0 \,, \quad \dot{G}_-\, \psi = 0$$

cf. D.V. (1-35), and it follows that the only solution of this equation is the vector $e_{j,-j'}$, see the weight diagram (1-85e).



## 4.2. Generalized spherical functions

We now wish to give the map $p \to u^{j+m}v^{j-m}\,\bar{u}^{j'-m'}\bar{v}^{j'+m'}$ in an explicit form. To that purpose we determine the light vector p by the component $p^0$ and the spherical coordinates of the ray $\rho p$.

$$\begin{aligned} p &= p^0(1,\ p_x,\ p_y,\ p_z) \qquad (p^0 > 0) \\ &= p^0(1,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta) \qquad (3\text{-}41) \end{aligned}$$

Substituting this in formula (2-12) we obtain

$$P = p^0\begin{pmatrix} 1+\cos\theta & e^{-i\phi}\sin\theta \\ e^{+i\phi}\sin\theta & 1-\cos\theta \end{pmatrix} = \psi\psi^+ \text{ where } \psi = \binom{u}{v}.$$

Hence

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{p_0}\begin{pmatrix} \sqrt{1+\cos\theta} \\ \sqrt{1-\cos\theta}\; e^{i\phi} \end{pmatrix} . \qquad (3\text{-}42)$$

We observe that (u, v) is determined within a factor $e^{-i\frac{1}{2}\phi}$. The meaning of this is the following. The spinor (u, v) is defined on the unit sphere $P(\theta,\ \phi)$. Now there is a rotation $r(\phi_1,\ \theta',\ \phi_2)$ such that the North Pole N is transformed into P. The angles $\phi_1,\ \theta',\ \phi_2$ are the *Euler angles* such that $r(\phi_1,\ \theta,\ \phi_2)$ is the product of a rotation $\phi_2$ around the z-axis, a rotation $\theta$ about the x-axis and a rotation $\phi_1$ around the z-axis (see figure 3.5). It is clear from

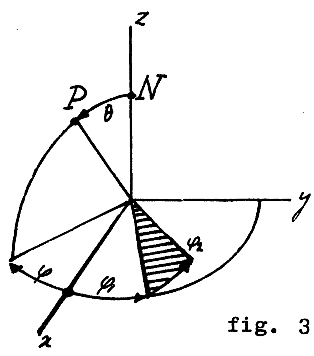



fig. 3.5

the figure that $\phi = \phi_1 - \frac{\pi}{2}$, $\theta' = \theta$. We substitute these expressions into (3-42) and observe that we may choose the rotation $\phi_2$ arbitrary, this corresponds to a multiplication of (u, v) by a factor $e^{-i\frac{1}{2}\phi_2}$. We substitute $\phi_2 \to \phi_2 + \phi_1$ and obtain the spinor

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sqrt{p_0} \begin{pmatrix} \frac{1}{\sqrt{2}} \sqrt{1+\cos\theta}\; e^{-i\frac{\phi_1}{2}} \\ \frac{-i}{\sqrt{2}} \sqrt{1-\cos\theta}\; e^{+i\frac{\phi_1}{2}} \end{pmatrix} e^{-i\frac{\phi_2}{2}} \tag{3-43}$$

The factor $\frac{1}{\sqrt{2}}$ is added to normalize (u, v) : $u\bar{u} + v\bar{v} = p_0$.
By this procedure, we have obtained the result that the spinor $\psi(u, v)$ is defined not only on the unit sphere but also on the rotation group.
By taking tensor products

$$\phi_m = u^{j+m} v^{j-m} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \quad ,$$

(see formula (1-30a)) one obtains spinors with components

$$\psi_m = p_0^{\,j} (1+\cos\theta)^{\frac{j+m}{2}} (1-\cos\theta)^{\frac{j-m}{2}}\; e^{-im\phi_1}\, e^{-ij\phi_2}\, C, \tag{3-44}$$

where $$C = \frac{(-i)^{j-m}}{2^j} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}.$$

It follows that

$$\psi_m = p_0^{\,j}\, D^j_{m,j}\, (\phi_1, \theta, \phi_2) \; , \tag{3-45}$$

where $D^j_{m,j}\,(\phi_1, \theta, \phi_2)$ are the so-called generalized spherical functions (see the formulae of $D^j_{m,j}$ in Gel'fand p. 85).
Finally, we construct the tensor products:

$$e_{jm}\, e_{j'm'} = u^{j+m}\, v^{j-m}\, \bar{u}^{j'-m'}\, \bar{v}^{j'+m'} \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \sqrt{\frac{(2j')!}{(j'+m')!(j'-m')!}}$$

and obtain the map:

$$p(p^0, \theta, \phi) \to e_{jm} e_{j'm'} = p_0^{\,j} p_0^{\,j'}\; D^j_{m,j}\, (\phi_1, \theta, \phi_2)\; D^{j'}_{-m',-j'}\, (\phi_1, \theta, \phi_2) \; ,$$

where $\phi_1 = \phi + \frac{\pi}{2}$ .

We observe that if $\psi(p)$ belongs to a *reducible* representation $D = \sum_{j,j'} D^{jj'}$ of the Lorentz group and $\psi(p) = \sum_{j,j'} c_{jj'} e_{jmj'm'}(p)$ then it remains true that $\psi(p)$ is invariant with respect to the little group $G_+(p)$; i.e.

$$D(G_+)\ \psi(p) = \lambda\psi(p) \tag{3-46}$$

Conversely, the function $\psi(p)$ is invariant only if $\psi(p)$ is the tensor-sum of one-dimensional irreducible representation of the little group $G_+(p)$. Hence we have:

Theorem 4.3. *Every function $\psi(p)$ defined on the light cone $p = p^0$ (1, sin $\theta$ cos $\phi$, sin $\theta$ sin $\phi$, cos $\theta$) such that the function values belong to a finite-dimensional representation space of the Lorentz group and such that $\psi(p)$ is invariant with respect to the little group $G_+(p)$, may be written after a suitable coordinate transformation, in the form*

$$\psi(p) = \sum c_{jj'} p_0^{\,j} p_0^{\,j'}\ D^j_{m,j}\ (\phi_1, \theta, \phi_2)\ D^{j'}_{-m', -j'}(\phi_1, \theta, \phi_2), \text{ where}$$

$$\phi_1 = \phi + \frac{\pi}{2}\ . \tag{3-47}$$

(In section 3.1., theorem 4.3 is illustrated by a geometrical example). The occurence of the generalized spherical functions follows also by more qualitative arguments.

If a rotation $r(\phi_1, \theta, \phi_2)$ is given, then the matrices of a unitary representation $D^j(\phi_1, \theta, \phi_2)$ are functions of $\phi_1$, $\theta$ and $\phi_2$. The matrix components $D^j_{mn}\ (\phi_1, \theta, \phi_2)$ are by definition the *generalized spherical functions.*

Now the map $\psi : p \to \psi(p)$, which is given in formula (2-12) and (3-37), is defined in such a way that the spinor $\psi(rP)$ belonging to the point rP may be obtained by transforming $\psi(p)$ by the corresponding representation matrix $D^j(r)$; i.e.,

$$\psi(rP) = D^j(r)\ \ \psi(P). \tag{3-48}$$

If the North Pole N and the point P are connected by the rotation r, i.e.,

$$P = rN \quad r = r(\phi_1, \theta, \phi_2) ,$$

it follows that

$$\psi(P) = \psi(rN) = D^j(r)\, \psi(N),$$

or, in components

$$\psi_m(P) = D^j_{mi}(r)\, \psi_i(N) \quad (m, i = -j, -j+1, \dots , +j). \tag{3-49}$$

Because $\psi_i(N)$ has the components $(v^{2j}, v^{2j-1}, \dots , u^{2j})$ where $(u, v) = (1,0)$, i.e., $\psi_i(N) \equiv (0, 0, 0, \dots , 1)$, it follows that $\psi(p)$ is equal to the last column $(i = j)$ of $D^j_{mi}(r)$.

$$\psi_m(P) = D^j_{mj}\, (\phi_1, \theta, \phi_2).$$

<u>Remark 4.2.</u> We add the following note. Consider the $n^2$-dimensional space R which is spanned by the *functions* $D^j_{mn}(r)$.
The *regular representation* T acting in R is defined by

$$T(r_0)\, D^j_{mn}(r) = D^j_{mn}(rr_0) \tag{3-50}$$

$$= D^j_{mi}(r)\, D^j_{in}(r_0).$$

It follows that the *functions* in the m-th row (which are now *vectors* and not *components* as above) are transformed into each other by the matrices $D^j_{in}\,(r_0)$.
Hence, in the space $R^m$ of the functions $D^j_{mi}\,(r)$ $(-j \leq i \leq +j)$ of the m-th row, the representation $D^j$ acts with the representation matrices $D^j_{in}\,(r_0)$.
For instance, a rotation $\theta$ around the z-axis is represented by

$$D(r_0) = \begin{pmatrix} e^{-ij\theta} & & 0 \\ & e^{-i(j-1)\theta} & \\ 0 & & \ddots \; e^{+ji\theta} \end{pmatrix} . \tag{3-51}$$

From this, it follows that the vectors $e_i = D^j_{mi}\,(r)$ form the canonical basis in $R^m$. Now the vectors $e_j = D^j_{mj}\,(r)$ are given for every m in formula (3-44). Hence, applying the operator $J_- e = ce_{j-1}$, one can construct the remaining generalized spherical functions.

## 4.3. The generalized Weyl equation

Supposing we are given the representation $D^{jj'}$, then, just as in 2.3, section 3.4 by constructing the equation of Weyl, we note that

$$\psi^{a_1a_2\cdots}{}_{\dot{c}_1\dot{c}_2} = \psi^{a_1}\psi^{a_2}\cdots\psi_{\dot{c}_1}\psi_{\dot{c}_2}\cdots$$

if and only if

$$p_{\dot{c}a_1}\psi^{a_1a_2\ldots}{}_{\dot{c}_1\dot{c}_2\ldots} = 0 \qquad (3\text{-}52)$$

and $$p^{a\dot{c}_1}\psi^{a_1a_2\ldots}{}_{\dot{c}_1\dot{c}_2\ldots} = 0\ . \qquad (3\text{-}53)$$

Hence the spinor products constructed in formula (3-37) obeys the so-called generalized Weyl equation and these equations can be brought into the Weyl form (3-36c, d). Without using the results of §4.1., one can show that the solutions of this equation form a one-dimensional space and have eigenvalues $e^{-i(j-j')\theta}$ with respect to spatial rotations $\theta$ around the $\vec{p}$-axis.

Therefore, we take $p = (p^0, 0, 0, p^0)$ $(p^0 > 0)$, and thus the corresponding $2 \times 2$ matrix P is given by

$$p_{\dot{c}a_1} = \begin{pmatrix} 2p_0 & 0 \\ 0 & 0 \end{pmatrix},$$

and the generalized Weyl equation becomes

$$(2p_0)\,\psi^{0a_2\ldots}{}_{\dot{c}_1\dot{c}_2\ldots} = 0$$

$$(2p_0)\,\psi^{a_1a_2\ldots}{}_{0\dot{c}_2\ldots} = 0\ .$$

Hence the only non-vanishing component is $\psi^{11\ldots}{}_{11\ldots}$ with eigenvalue $e^{-i(j-j')}$.

Summarizing the geometrical content of the generalized Weyl equation, it follows from the results in section 2 and 3 that the generalized Weyl equation describes in fact the isotropic bivectors on the light cone. The fact that only the eigenvalues $e^{\pm i(j-j')}$ appear with respect to rotations $\theta$ around the $\vec{p}$-axis depends on the non-invariance of the $\vec{p}$-axis under the little group $G_+(p)$ (section 3.1 first page) and the possibility that the solutions of the generalized Weyl equation can be written with aid of the generalized spherical function follows from the constructions given in formula (3-22) or formula (3-42).

## Chapter IV

## THE LORENTZ GROUP AS A THREE-DIMENSIONAL TRANSFORMATION GROUP

In chapter II, we remarked that the Lorentz group can be studied as the three-dimensional projective group which leaves the unit sphere invariant. In this chapter, we study this group in more detail.

In section 1., we shall show that every Lorentz transformation may be described as a *screw* in hyperbolic geometry. For that purpose it is necessary to introduce the concepts of (hyperbolic) distance and angle. For a certain family of planes it is also possible to introduce a euclidean distance. In this study, we take advantage of what we know already about the four-dimensional geometry of the Lorentz group.

For a depper analysis we need the method of Cartan. Cartan developed a method to describe (pseudo-) orthogonal transformations in $R_n$ by introducing a certain multiplication between vectors. In this way we obtained a so-called Clifford algebra, which is a generalization of the quaternion concept.
In section 1.4, we show that this method is indispensable in hyperbolic geometry for the construction of screws with the aid of reflections.
In section 2., we give an application of this method to the 5-dimensional line space. In particular, it is shown that the anti-symmetric matrices which leave invariant the so-called configuration of Kummer are obtained from a 64-dimensional Clifford algebra.

# 1. Three-dimensional hyperbolic geometry

## 1.1. Geometrical introduction

In this section we still describe a Lorentz transformation as a screw in hyperbolic geometry. Before going on with the analytic treatment, we first give a more geometrical classification of Lorentz transformations. As in chapter II section 2, every point $x(x^0, x^1, x^2, x^3)$ is mapped onto the $2 \times 2$ matrix x (we use small letters for $2 \times 2$ matrices in this chapter)

$$x = \begin{pmatrix} x^0+x^3 & x^1-ix^2 \\ x^1+ix^2 & x^0-x^2 \end{pmatrix},$$

which is transformed by $x' = axa^+$ (det a = 1). (4-1)

In particular, points on the complex sphere det $x = (x)^2 \equiv 0$ may be written as

$$x = \psi\phi^+ ,$$

where $\psi$ and $\phi$ are two-dimensional vectors transformed by the two-dimensional representation a of the Lorentz group, i.e.

$$\psi' = a\psi \text{ and } \phi' = a\phi; \qquad (4\text{-}2)$$

see formula (2-11).

We shall determine the eigenvectors x of the transformation (4-1) which lie on the complex sphere. To that purpose we determine the two eigenvectors $\psi$ and $\phi$ of the transformation (4-2).

There are two cases:

(I) $\psi = \phi$. Hence $x = \psi\psi^+$, and it follows that x is hermitian. Thus, there is only *one* invariant and *real* point on the sphere. It follows that, if x is taken on the positive z-axis, then the transformation a has the form $a = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$. We shall call this transformation a *horiscrew* (see fig. 4.1 and p. 114).

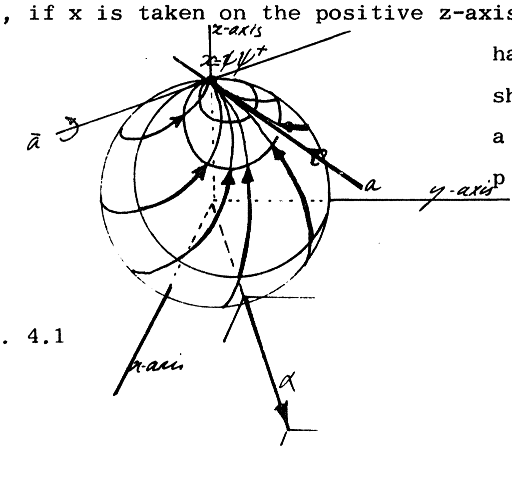



fig. 4.1

(II) $\psi \neq \phi$. In this case, we have the four invariant points $X = \psi\psi^+$, $\phi\phi^+$, $\psi\phi^+$ and $\phi\psi^+$, which form an *invariant tetraeder*.
The real points $\psi\psi^+$ and $\phi\phi^+$ determine an invariant line a which intersects the real sphere, and, since the polar line $\bar{a}$ remains invariant too, it follows that the complex-conjugate points $\psi\phi^+$ and $\phi\psi^+$ of the complex sphere are on $\bar{a}$.
If there exists a fifth invariant point P such that P lies not in a plane through 3 invariant points, then the transformation a is the identity.
If the transformation a is unequal to the identity, it follows that P lies in one of the 4 planes of the invariant tetraeder.

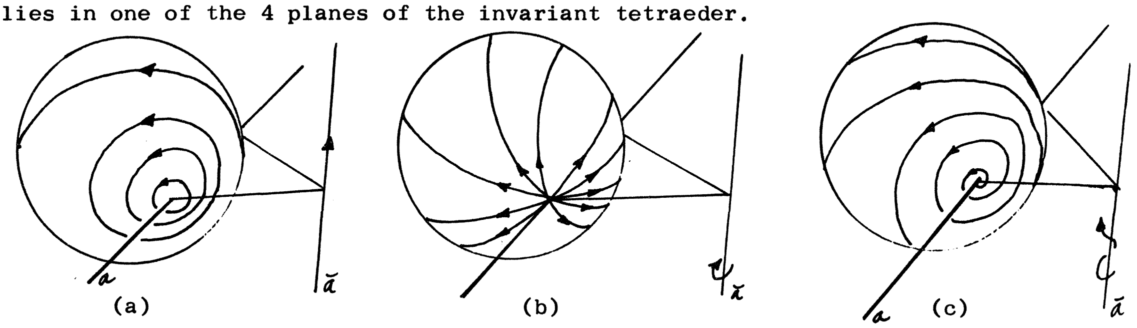



fig. 4.2.

(a) If P lies in a plane of the tetraeder through a, then a is also point-wise invariant. The line a is called a *rotation-axis*.
(b) If P lies in a plane of tetraeder through a the polar line of a, then $\bar{a}$ is point-wise invariant and a rotation about $\bar{a}$ corresponds to a *shift* along a .
(c) If there is no invariant fifth point, then the transformation which leaves a and $\bar{a}$ invariant is a product of a rotation and a shift, and we have a *screw*
We will call the transformation I a horiscrew, the cases IIa and IIb being degenerated screws. Thus we have that every Lorentz transformation determines a screw in the three-dimensional projective space $P_3$.

## 1.2. Introduction of a metric

For the analytic treatment of screws, it is necessary to introduce the concepts of *distance* and *angle*, and thus to introduce a metric in the three-dimensional projective space $P_3$. We shall use the invariance of $x \cdot y = x^{\mu} y_{\mu}$.

First, we normalize every point in $P_3$ in such a way that, for points x interior to the unit sphere, $x^2 = +1$, and for points x exterior to the unit sphere, $x^2 = -1$.

In this way, the coordinates of a point x are determined within sign $x \Longleftrightarrow \pm x^{\mu}$. Now there are two possibilities:

(A) One may consider the points $x^{\mu}$ and $-x^{\mu}$ as the same. see Godeaux p. 15 , but we prefer:

(B). One may say that the projective space $P_3$ is twice covered, once with points such that $x^0 > 0$ and once with points such that $x^0 < 0$.

The advantage of method B is explained in remark 4.1.

However, in order to introduce a distance (pq) between two points p and q on a line a, it is necessary that p and q be *connected* by a Lorentz transformation. That is to say, there must exist a restricted Lorentz transformation which leaves the line a invariant and transforms p into q. Starting first from the point $p^{\mu}$ and afterwards from the point $-p^{\mu}$, and considering the points $p^+$ and $p^-$ which are connected with $p^{\mu}$ and $-p^{\mu}$ respectively, it follows that a is covered by at least two components $P^+$ and $P^-$, each of which is connected. Below, we study in which cases the component $P^+$ is connected with $P^-$.

The best tool for this study is first to consider, instead of the 3-dimensional projective group, the four-dimensional Lorentz group.

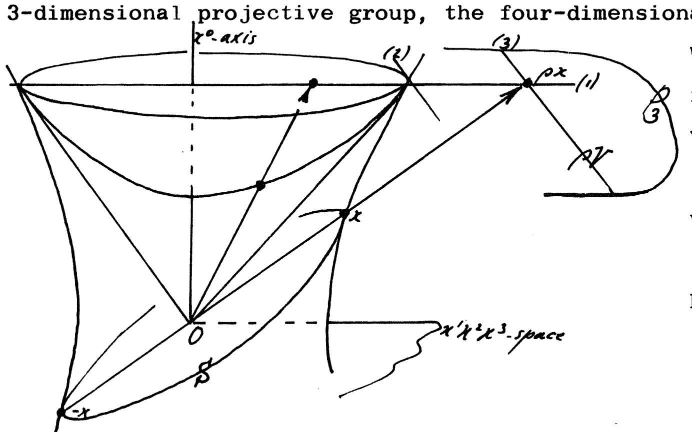



fig. 4.3

We identify the projective space $P_3$ of rays $\rho x^{\mu}$ with the 3-dimensional space, at infinity, in $R_4$ with coordinates $(0, x^0, x^1, x^2, x^3)$,see p.108. By normalizing the points

$\rho x^\mu$ such that $x^2 = +1$ or $x^2 = -1$, we have in fact performed central projection from the origin O, whereby the surfaces $x^2 = +1$ and $x^2 = -1$ in $R_4$ are mapped onto the interior and exterior of the unit sphere in $P_3$ respectively. Considering now the interior points of the unit sphere, it follows that the two branches $P^+(p^0 > 0)$ and $P^-(p^0 < 0)$ lie on the two branches of the hyperboloïd $x^2 = +1$. Clearly, $P^+$ and $P^-$ are not connected and are separated by points of the unit sphere. Hence, we have:
(1) Every line a which has two points in common with the surface of the unit sphere is separated into two segments (interior and exterior points) and every segment is covered by two *separated* branches $P^+$ and $P^-$ ($P^+$ is obtained starting from $p^\mu$ and $P^-$ is obtained starting from $-p^\mu$

We will choose the coordinates $x^\mu$ of interior points such that $x^0 > 0$. As long as we consider the orthochroneous Lorentz group L↑, the branch ($x^0 > 0$) is transformed into itself and we need not consider the branch ($x^0 < 0$) for interior points.
(2). A line a which has only one point in common with the surface of the unit sphere consists of one segment which is covered by two *separated* branches $P^+$ and $P^-$ which correspond to two straight and parallel lines on the hyperboloïd $x^2 = -1$ in $R_4$. One can only introduce a distance (pq) if p and q be on the same branch of a.
(3) Only in the case that a line has no point in common with the surface of the unit sphere does the line consist of one segment and is it covered by two *connected* components $P^+$ and $P^-$. In particular, a 2-plane $\rho V$ in the projective space which does not intersect the unit sphere is covered by points of the sphere S ($x^2 = -1$) which lies in the 3-space V through the origin O. See fig. 4.3. It is significant in this case to introduce a distance $|p^+ q^-|$ between points which lie on different branches. Corresponding to these three cases, we shall introduce a hyperbolic, parabolic (euclidean) and elliptic metric on the line a.

Remark 4.1. Considering the sphere S, we note that in the case (A), mentioned a page before, we have identified the points $+x^\mu$ and $-x^\mu$. One may prove that the surface of a sphere where opposite points are

identified is doubly connected (see p. 44). Hence, the projective plane $\rho V$ is doubly connected.
However, by distinquishing the points + x and -x (case B), we have covered the projective plane $\rho V$ by the simply connected sphere S (the universal covering space of $\rho V$). In this way, the two-valuedness of the angle between two planes in $P_3$ : $\phi$ or $\pi - \phi$ is replaced by *one* angle between two half-planes. By this procedure we have also covered the non orientable projective plane $\rho V$ by the orientable sphere S. In Klein p. 16, 152, this is expressed by saying that the one-sided projective plane is covered by the two-sided sphere.
Also, for the introduction of a parabolic measure, it is simpler to distinguish points x and -x.

(1) Hyperbolic measure

Definition. The *(hyperbolic) distance* $|pq|$ of two points p and q inside the unit sphere is defined by

$$\cos i\ |pq| = p^\mu q_\mu \quad \text{or} \quad \cosh |pq| = p^\mu q_\mu \ , \qquad (4\text{-}2)$$

$$\text{where } p^2 = q^2 = +1 \text{ and } p^0 q^0 > 0 \ . \ |pq| \geq 0$$

We observe that, if we take p = (1, 0, 0, 0), then, by the hyperbolic screw $h_{01}$ , i.e.

$$\begin{pmatrix} y^0 \\ y^1 \end{pmatrix} = \begin{pmatrix} \cosh \psi'' & \sinh \psi'' \\ \sinh \psi'' & \cosh \psi'' \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix} , \qquad (4\text{-}3)$$

p (1, 0, 0, 0) is transformed into q(cosh $\psi''$, sinh $\psi''$, 0, 0) = (cosh $\psi''$) p + sinh $\psi''$ X; hence $p \cdot q = \cosh \psi''$ and $|p \cdot q| = \psi''$, and the transformation (4-3) is called a *shift* along the line p q over a distance $\psi''$ (see fig. 4.2 (b)).
Because $p \cdot q$ is an invariant, it further follows from this example that $p \cdot q \geq 1$ always holds, and thus the distance $|pq|$ is always real. In the future we will take $|pq|$ positive.
Because there are two real intersection points m and n of the line pq with the unit sphere which lie at infinity ($|pm| = |pn| = \infty$), the metric defined above is called *hyperbolic* and the interior of the sphere metrized in this way is called the three-dimensional *hyperbolic space* $H_3$.

*From the invariance of $|pq|$ it follows that the Lorentz group is isomorphic with the group of displacements in hyperbolic geometry.*

Considering now planes $p_\mu$, $q_\mu$ which belong to the hyperbolic space $H_3$, i.e. have points in common with $H_3$; the intersection line a of $p_\mu$ and $q_\mu$ may have 0, 1 or 2 real points in common with the unit sphere. In this case with 0 points, the *distance* between the planes $p_\mu$, $q_\mu$ may be defined by using the distance on the dual line $\bar{a}$. If the intersection a belongs to $H_3$, we shall introduce the concept of *angle* between $p_\mu$, $q_\mu$, or, what is the same, we shall introduce the distance between points $p^\mu$, $q^\mu$, which lie on the dual line $\bar{a}$ which has no points in common with the unit sphere.

(2) Elliptic measure.

Definition. The *(elliptic) distance* $|pq|$ between two points $p^\mu$ $q^\mu$ on a line $\bar{a}$ which has no real points in common with the unit sphere is defined by

$$\cos |pq| = -p_\mu q^\mu, \text{ where } p^2 = q^2 = -1$$

The definition given above is at the same time a definition of the *angle* between the planes $p_\mu$, $q_\mu$ through the line a which has two real points in common with the unit sphere.

The minus sign is chosen in order that $|pp| = 0$. We observe that if we take the point $p^\mu(0, 1, 0, 0)$ ($p_\mu$ is the yz-plane) and apply a rotation

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} \cos \psi' & -\sin \psi' \\ \sin \psi' & \cos \psi' \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}, \qquad (4\text{-}4)$$

then we obtain the point $q^\mu$ $(0, \cos \psi', \sin \psi', 0)$, and thus $|pq| = \psi'$. Hence, the transformation (4-4) is a *rotation* through the *angle* $\psi'$ about the line a, or a *shift* over the *distance* $\psi'$ along the line $\bar{a}$.

We note that the point $p^+ \equiv p^\mu$ lying on the branch $P^+$ is connected to the point $p^- \equiv -p^\mu$ lying on the branch $P^-$. It follows that $|p^+ p^-| = \pi$, the largest distance between points in elliptic geometry, and the longest arc in elliptic geometry $(p^+ \to p^- \to p^+)$ has length $2\pi$. Because there are no real points which lie at infinity, this metric is called *elliptic*.

(3) Parabolic measure.

Finally, we suppose that the intersection line of the planes $p_\mu$ and $q_\mu$ which belong to $H_3$ is a tangentline a of the unit sphere. It follows that $p^\mu q_\mu = -1$, and hence the angle between $p_\mu$ and $q_\mu$ is zero. However it is possible to introduce a *parabolic (euclidean) measure* for the bundle of planes through a, or, what is the same, a measure on the dual line $\bar{a}$. In this case, there is only one point which lies at infinity, the intersection point of a and $\bar{a}$, and the metric is therefore called parabolic. See fig. 4.1 and fig. 4.4.

Definition The *distance* $|pq|$ between the planes $p_\mu$, $q_\mu$ from which the intersection line a is a tangent line of the unit sphere is defined by

$$|pq| = |p^0 - q^0|,$$

where $p^2 = q^2 = -1$ and $p^\mu$ and $q^\mu$ belong to the same connected component.

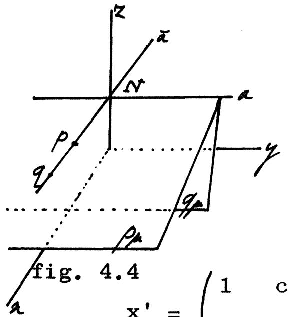



fig. 4.4

Consider e.g. tangent lines a and $\bar{a}$ through the North Pole and the point $p^\mu(p^0, 1, 0, p^0)$ on $\bar{a}$ (observe that $p^2 = -1$), see fig. 4.4. With a horiscrew along the line $\bar{a}$, i.e.

$$x' = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^0 + x^3 & x^1 - cx^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \quad \text{(c is real)} \qquad (4\text{-}5)$$

the point $p(p^0, 1, 0, p^0)$ is transformed into $q(p^0 + c, 1, 0, p^0 + c)$. Hence $|pq| = c$ and (4-5) is a "translation" along the line $\bar{a}$ of a distance c.

Applying an arbitrary horiscrew $\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ (y complex), one obtains $p(p^0, 1, 0, p^0) \rightarrow p'(p^0 + \mathrm{Re}\gamma, 1, 0, p^0 + \mathrm{Re}\gamma)$. It follows that the measure $|p^0 - q^0|$ of the points p and q on the tangent line $\bar{a}$ is invariant with respect to rotations and horiscrews (translations).

Considering now all Lorentz transformations which leave the North Pole invariant, we earlier observed in that they form the group $G^*(N)$ isomorphic with the 2-dimensional similarity group (rotations, translations, multiplication with a factor). See section 1, chapter II p. 107.

Hence, only the last transformation, i.e. a shift along the z-axis $(p^0, 1, 0, p^0) \rightarrow (p^0 e^\phi, 1, 0, p^0 e^\phi)$ , causes the distance between the planes of the bundle a to be multiplied by a factor $e^\phi$.
Suppose now that $\Lambda$ is an arbitrary Lorentz transformation $\Lambda : p \rightarrow p'$; then there exists a rotation r such that $r : p \rightarrow p'$. It follows that $\Lambda r^{-1} : p \rightarrow p$, and thus $\Lambda r^{-1} \in G^*(p)$ and every Lorentz transformation is the product of a rotation and a similarity transformation from $G^*(p)$. Thus, under the Lorentz group, the distance $|pq|$ between the planes of a horibundle a is only multiplied by a factor.

## 1.3. Screws in hyperbolic geometry

Suppose that a restricted Lorentz transformation is given by

$$x' = a\,x\,a^{+}.$$

We write the unimodular matrix a in the same way as x, i.e.

$$a = \begin{pmatrix} a^0 + a^3 & a^1 - ia^2 \\ a^1 + ia^2 & a^0 - a^3 \end{pmatrix}, \quad \det A = (a)^2 = 1.$$

Every Lorentz transformation a is thus described by the complex 4-vector $(a^0, \vec{a})$. Since $(a^0)^2 - \vec{a}^2 = 1$, the components $a^\mu$ can be parametrized as follows, if $\vec{a}^2 \neq 0$ or if $\vec{a}^2 = 0$ *and* $\vec{a} = \vec{0}$

$$(a^0, \vec{a}) = (\cos \frac{\psi}{2}, - i \sin \frac{\psi}{2} \vec{k}), \tag{4-6}$$

where $\psi$ and $\vec{k}$ are complex. The pair $\psi$, $\vec{k}$ have determined within sign. If $\psi \neq 0$, then $\vec{k}^2 = 1$. We shall write

$\psi = \psi' + i\psi''$ and $\vec{k} = \vec{k}' + i\vec{k}''$, where $\psi'$, $\psi''$, $\vec{k}'$ and $\vec{k}''$ are real.

Theorem 1.1. *If a restricted Lorentz transformation is given by the $2 \times 2$ unimodular matrix $a = (\cos \frac{\psi}{2}, - i \sin \frac{\psi}{2} \vec{k})$, then the complex vector $\vec{k} = \vec{k}' + i\vec{k}''$ gives the screw axis $k$ $(\vec{k}'', \vec{k}'')$,* *) *and the complex number $\psi = \psi' + i\psi''$ gives the pitch of the screw; i.e., $\psi'$ is the rotation angle about $k$ and $\psi''$ is the distance of the shift along $k$. If $a = (a^0, \vec{a})$ with $\vec{a}^2 = 0$ and $\vec{a} \neq \vec{0}$ then a determines a horiscrew along $(\vec{a}', \vec{a}'')$ over a distance $2|\vec{a}'|$.*

PROOF. With a Lorentz transformation b, the matrix a transforms as follows

$$a \to a' = bab^{-1}. \tag{4-7}$$

Thus, a consists of the scalar tra = $2a_0$ and a bivector $(0, \vec{a})$ which transforms by the representation $D^{10}$; see formula (2-29). Hence, a is transformed under the representation $D^{00} + D^{10}$. Supposing b = a, then a and the bivector $(0, \vec{a})$ belonging to it remain invariant under (4-7).

---

*) $(\vec{k}', \vec{k}'')$ are the six line coordinates $(k^{0i}, k^{jk})$ i, j, k = 1, 2, 3 and cycl.

Now the only invariant bivector belonging to the transformation a is the screw axis k with line coordinates $(\vec{k}', \vec{k}'')$. The line k has the component $\vec{k} = \vec{k}' + i\vec{k}''$ in the representation space of $D^{10}$. Thus, it follows that $\vec{a} = (\vec{k}' + i\vec{k}'')\rho$ . We normalize $\vec{k}$ such that $\vec{k}^2 = +1$, $\vec{k}$ is determined within sign. Because $a_0^2 - \vec{a}^2 = 1$, it follows that $\rho = \pm i \sin \frac{\psi}{2}$.

We choose the sign of $\psi$ and $\vec{k}$ such that $\vec{a} = - i \sin \frac{\psi}{2} \vec{k}$ . We now consider $\psi$. Since $a_0 = \cos \frac{\psi}{2}$ is a scalar, we may transform a by (4-7) in order to obtain the meaning of $\psi$.

We distinguish two cases:

$\underline{\vec{a}^2 \neq 0}$ In this case we transform in such a way that the axis $(\vec{a}', \vec{a}'')$ goes over into the z-axis, thus

$$a \to a^*(\cos \frac{\psi}{2}, - i \sin \frac{\psi}{2} \vec{a}^*), \text{ where } \vec{a}^* = (0, 0, 1), \text{ or}$$

$$a^* = \begin{pmatrix} e^{-i\frac{\psi}{2}} & 0 \\ 0 & e^{+i\frac{\psi}{2}} \end{pmatrix} = \begin{pmatrix} e^{-i\frac{\psi'}{2}} & 0 \\ 0 & e^{+i\frac{\psi'}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{\psi''}{2}} & 0 \\ 0 & e^{-\frac{\psi''}{2}} \end{pmatrix} . \qquad (4\text{-}8)$$

This transformation is the product of a shift of a distance $\psi''$ along the z-axis in positive direction followed by a rotation $\psi'$ about the z-axis.

Hence, a is a right-handed screw about the $(\vec{a}', \vec{a}'')$ axis if $\psi' \psi'' > 0$.

$\underline{\vec{a}^2 = 0}$ The relation $\vec{a}'^2 - \vec{a}''^2 = 0$ holds; it follows that the screw axis $(\vec{a}', \vec{a}'')$ is a tangent line to the sphere. *)

We rotate the matrix a by (4-7) such that $(\vec{a}', \vec{a}'')$ becomes the line $(1, 0, 0 \mid 0, 1, 0) \, |\vec{a}'|$ through the North Pole (1, 0, 0, 1) and the point X(0, 1, 0, 0). The lenghts $|\vec{a}'|$ and $|\vec{a}''|$ are invariant with respect to rotations, and it follows that the matrix $a^* = (1, \vec{a}^*)$ where $\vec{a}^* = (1, i, 0) \, |\vec{a}'|$. Thus,

$$a^* = \begin{pmatrix} 1 & 2|\vec{a}'| \\ 0 & 1 \end{pmatrix}$$

This is a horiscrew along the x-axis over a distance $2|\vec{a}'|$.

---

*) If r and s are two points on a, then the condition that the line $x = \lambda r + \mu s$ has *one* intersection point with the sphere $x^2 = 0$, is that $(r.s)^2 - r^2 s^2 = 0$. One may also verify that $\vec{a}'^2 - \vec{a}''^2$ is equal to $(r.s)^2 - r^2 s^2$ by substituting $\vec{a}' = r^0 \vec{s} - s^0 \vec{r}$ and $\vec{a}'' = \vec{r} \times \vec{s}$ and adding the term $(r^0 s^0)^2 - (r^0 s^0)^2$. Hence, we obtain the condition $\vec{a}'^2 - \vec{a}''^2 = 0$ for a to be a tangent line.

Remark 4.2. Concerning reflections.

Because $\psi$ is a scalar, it follows that the matrices a fall into two disjunct groups, viz. $\psi'\cdot\psi'' > 0$ (right-handed screws) and $\psi'\cdot\psi'' < 0$ (left-hand screws).

With a reflection, the matrix a which is transformed by the representation $D^{00} \dot{+} D^{10}$ goes over into the matrix $\bar{c}\bar{a}c^{-1}$, $c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ see (1-37b), which is transformed by $D^{00} \dot{+} D^{01}$.

The number $\psi$ is transformed into $\bar{\psi}$, and one sees that with reflection the families of left-handed and right-handed screws are interchanged.

Bivectors. A plane $x^\mu = \alpha r^\mu + \beta m^\mu$ in $R_4$ corresponds with a line in the projective space $P_3$. After normalisation ($x^2 = \pm 1$), the vectors $x^\mu$ in the hyperbolic space $H_3$ are no longer homogeneous and neither are the line coordinates $a^{\mu\nu}$. In $R_4$ the non-homogeneous $a^{\mu\nu}$ is called a *bivector*, and in this sub-section we shall consider the corresponding object in the hyperbolic space $H_3$.

We consider all Lorentz transformations $\Lambda$ in $R_4$ which leave the plane $a^{\mu\nu}$ invariant. Then the vectors r and m, in a, are transformed into r' and m', in a. Hence the bivector $a^{\mu\nu}$ is transformed into $a^{\mu'\nu'} = \rho a^{\mu\nu}$. Because $\frac{1}{2} a_{\mu\nu} a^{\mu\nu} = \vec{a}''^2 - \vec{a}'^2$ is an invariant (see formula (2-49)), we have that $\rho = \pm 1$ and we may take $\rho = +1$ as long as we consider transformations which do not invert the screw sense in a (the order of the rays).

Hence, it follows that we can consider the bivector $a^{\mu\nu}$ as the class of all (r', m') in the plane a which are generated from the pair (r, m) by a Lorentz transformation in the plane a. In the hyperbolic space $H_3$ we have accordingly

Theorem 1.2. *One may consider the bivector $a^{\mu\nu}$ as the line vector (r,m) which is the class of all (r', m') generated from the pair (r, m) by a shift along the line a.*

Instead of using the coordinates $(\vec{a}', \vec{a}'')$ ($\vec{a}' \equiv a^{0i}$, $\vec{a}'' \equiv a^{jk}$ where i, j, k = 1, 2, 3) of the bivector $[r, m]$, we perform a coordinate

transformation in the 6-dimensional space of bivectors. Then $[r, m]$ is given by $(\vec{a}, \overset{\ast}{a})$ where $\vec{a} = \vec{a}' + i\vec{a}''$ and $\overset{\ast}{a} = \vec{a}' - i\vec{a}''$.
Sometimes we shall denote the component $\vec{a}$ of $[r, m]$ which belongs to the representation space of $D^{10}$ by $[r, m]^{+}$ and the component $\overset{\ast}{a}$ of $[r, m]$ which belongs to the representation space of $D^{01}$ by $[r, m]^{-}$.

Suppose two interior points r and s on the line a are given. We wish to construct the Lorentz transformation $= (a_0, \vec{a}) = (\cosh \frac{\psi''}{2}, \sinh \frac{\psi''}{2} \vec{k})$ which transforms r into s by a shift along a. It follows from the foregoing theorem that

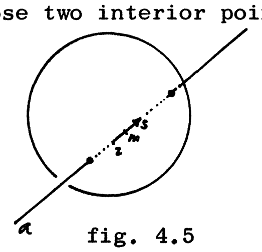


fig. 4.5

$$|rs| = \psi'',$$

the distance between r and s. We determine a point in the *middle* of r and s, thus

$$\cosh \frac{\psi''}{2} = r \cdot m = a_0$$

Further, if $a^{\mu\nu} = r^{\mu}m^{\nu} - r^{\nu}m^{\mu}$ are the line coordinates of the line a, it follows that the component $\vec{a}$ of the transformation is equal to $(\vec{a}'+i\vec{a}'')\rho$; after substituting $\vec{a}' = r^0 \vec{m} - m^0 \vec{r}$ and $\vec{a}'' = \vec{r} \times \vec{m}$ and adding the term $+(r^0m^0)^2-(r^0m^0)^2$, we have that

$$\vec{a}^2 = \left[(r\cdot m)^2 - r^2m^2\right] \rho^2 = (a_0^2 - 1)\rho^2,$$

and thus $\rho = +1$. We shall write $\vec{a} = [r, m]^{+}$ and we have the following theorem.

<u>Theorem 1.3.</u> *If r and s are two interior points on the line a, and m is the middle of r and s, then*

$$a(a^0, \vec{a}) = (r\cdot m, [r, m]^{+}) \qquad (4\text{-}9)$$

*is a shift along a which transforms r into s.*

Analogous formulae may be obtained if r and s are exterior points. Hence we may associate every bivector $[r, m]$ with a shift $r \to s$ along the line r m .

1.4. Reflections in hyperbolic geometry

Every linear transformation in $R_n$ which leaves a quadratic norm invariant can be represented by a product of reflections. Cartan has developed a method for describing such transformations. An application of this method to the line space $P_5$ is given in section 2 of this chapter. Next we give an application tp $P_3$.

Let us map every vector $x^\mu$ in $R_4$ ($\rho x^\mu$ is a point in $P_3$) onto a $4 \times 4$ matrix X (we use small letters x to denote $2 \times 2$ matrices)

$$\text{i.e., } x^\mu \rightarrow X = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}, \text{ where } x = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix}, \tag{4-10}$$

and the $2 \times 2$ matrix $\bar{x}$ is constructed from x by space reflection, i.e. $(x^0, x) \rightarrow (x^0, -x)$.
The fundamental property of X is

$$X^2 = x^\mu x_\mu = (x^0)^2 - (x^2)^2 - (x^2)^2 - (x^3)^2. \tag{4-11}$$

(It is understood that multiplication by the unit matrix is carried out on the right side.).This property has the consequence that for the inner product of two vectors one may write

$$x^\mu y_\mu = \tfrac{1}{2}(XY + YX). \tag{4-12}$$

In order to prove this, we observe that the map (4-10) implies

$$\lambda x^\mu + \kappa y^\mu \rightarrow \lambda X + \kappa Y .$$

Hence, $$(\lambda x^\mu + \kappa y^\mu)^2 = (\lambda X + \kappa Y)^2 .$$

By comparing the terms on the left- and right-hand sides we obtain (4-12).
The images of the "orthonormal" vector $e_\mu$ ,

$$e_0 = (1, 0, 0, 0) \quad e_0^2 = +1$$
$$e_1 = (0, 1, 0, 0) \quad e_1^2 = -1 , \ldots \text{etc.},$$

are the Dirac matrices $\gamma_\mu$. From (4-12), we obtain the well-known orthonormality rules *)

$$\tfrac{1}{2}(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = g_{\mu\nu}, \text{ where } g = \begin{pmatrix} 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & -1 \end{pmatrix}. \qquad (4\text{-}13)$$

The $\gamma$-matrices are called *Clifford-numbers* and form a *Clifford algebra.* In this special case, where $\mu, \nu = 0, \ldots, 3$ one also speaks of a *Dirac algebra.* Consider the 3-plane $a_\mu$, where $a^\mu$ is a space-like unit vector, $a_\mu a^\mu = -1$. A reflection in the 3-plane $a_\mu$ has the form

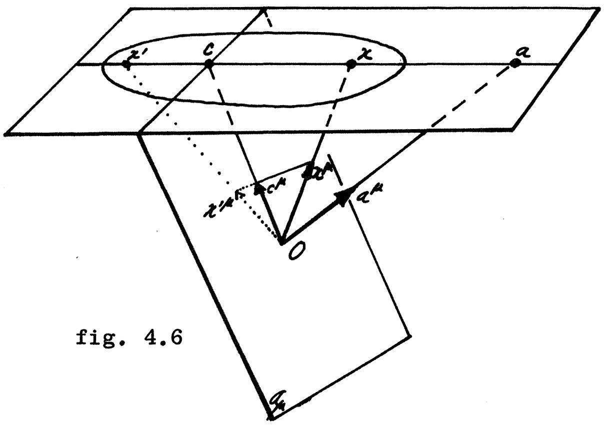



fig. 4.6

$$x'^\mu = x^\mu - 2(x^\mu a_\mu)a^\mu \quad (a^2 = -1)$$

(see fig. 4.6). Translating this equation by using the 4 × 4 matrices, we obtain

$$X' = -AXA, \text{ where } A = i\begin{pmatrix} 0 & a \\ a^{-1} & 0 \end{pmatrix} \; (A^2 = -1). \qquad (4\text{-}14)$$

A reflection in $R_4$ corresponds to a point-plane reflection in $P_3$. The point $a = \rho a^\mu$ and all points $c = \rho c^\mu$ in the 2-plane $\rho a_\mu$ remain invariant. Points $x = \rho x^\mu$ on the line (a c) are transformed into x' on (a c) such that the points a, c and x, x' form a harmonic sequence. The product of an even number of reflections is a restricted Lorentz transformation, i.e.

$$X' = TXT^{-1}, \text{ where } T = BA = (-)\begin{pmatrix} ba^{-1} & 0 \\ 0 & b^{-1}a \end{pmatrix}.$$

The 2 × 2 matrices x transform by

$$x' = (ba^{-1})\, x\, (a^{-1}b).$$

If $a^\mu$ and $b^\mu$ are real, than the 2 × 2 matrices a and b are hermitian and one may write

$$x' = t\, x\, t^* \qquad (4\text{-}15)$$

---

*) Analogously, in $R_3$ we map every vector $\vec{x}$ onto the matrix $x = \begin{pmatrix} x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -x^3 \end{pmatrix}$. The base vectors (1, 0, 0), ... etc. are mapped onto the Pauli-matrices $\sigma_1, \sigma_2, \sigma_3$ with orthonormality rules

$$\tfrac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i) = \delta_{ij}.$$

For the inner product $r^{\mu}s_{\mu}$, we have obtained formula (4-12). Next we consider the exterior product

$$p^{\mu\nu} = r^{\mu}s^{\nu} - r^{\nu}s^{\mu} \qquad (\mu,\ \nu = 0,\ \ldots\ 3)$$

or, in reduced form,

$$[rs] \equiv (\vec{p},\ \dot{\vec{p}}) = ([rs]^{+},\ [rs]^{-}),$$

see the notation convention on p. 187.
One may verify that the matrix

$$P = \tfrac{1}{2}(RS - SR) \tag{4-16}$$

has the form

$$P = \begin{pmatrix} -p & 0 \\ 0 & p \end{pmatrix}, \text{ where } p = \begin{pmatrix} p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^3 \end{pmatrix}, \tag{4-17}$$

$$\text{or} \quad P = (-)\begin{pmatrix} p^j\sigma_j & 0 \\ 0 & 0 \end{pmatrix} + (-)\begin{pmatrix} 0 & 0 \\ 0 & \dot{p}^j\sigma_j \end{pmatrix}.$$

Hence, we write

$$\tfrac{1}{2}(RS - SR) = -[rs] \tag{4-18}$$

(it is understood that the components $(p^j,\ \dot{p}^k)$ on the right side are given with respect to the basis $I_j = \begin{pmatrix} \sigma_j & 0 \\ 0 & 0 \end{pmatrix}$, $\dot{I}_k = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_k \end{pmatrix}$ .

Applying a Lorentz transformation, it follows that

$$P' = TPT^{-1}$$

and one notes again that $p^j I_j$ and $\dot{p}^k \dot{I}_k$ form two invariant subspaces.

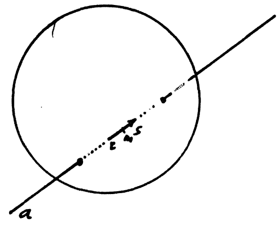


fig. 4.7

A Lorentz transformation is the product of 4 reflections. This fact can be most easily analytically treated by the method of Cartan We shall show this as an application of the foregoing.

A shift A along the line a which transforms r into s can be constructed by a reflection in the point m, i.e.

$$m: r \to s \;;$$

However, this reflection reverses the order of points on a. Hence, by performing a reflection in r first and the reflection in m afterwards we obtain the shift A along a.
Applying the reflection formula (4-14), we get

$$X' = MRXRM.$$

Thus, the 4 × 4 transformation matrix A can be written as,

$$A = MR = \tfrac{1}{2}\,(MR + RM) + \tfrac{1}{2}(MR - RM)$$

Using (4-12) and (4-18) for the inner and outer product, we obtain

$$A = (r\cdot m) + [r\ m].$$

Thus we have the 2 × 2 matrices x such that

$$x' = axa^{+},\ \text{where } a = (r\cdot m) + [r,\ m]^{+}$$
$$a = \cosh \frac{\psi''}{2} + \sinh \frac{\psi''}{2}\,\vec{k}\ .$$

cf. formula (4-9).
In an analogous way, a shift $\bar{a}$ along the dual line $\bar{a}$ may be calculated, and we obtain the two transformations

$$a = (\cos \frac{i\psi''}{2},\ - i \sin \frac{i\psi''}{2}\,\vec{k})$$

$$\text{and} \quad \bar{a} = (\cos \frac{\psi'}{2}\ ,\ - i \sin \frac{\psi'}{2}\,\vec{k}\ )$$

An arbitrary Lorentz transformation is obtained by the product $a\bar{a}$ , thus by performing 4 reflections

$$a\bar{a} = (\cos \frac{\psi}{2}\ ,\ - i \sin \frac{\psi}{2}\,\vec{k}),\ \text{where } \psi = \psi' + i\psi''\ .$$

The multiplication $a\bar{a}$ can be most easily performed by using the relation (2-19b).

## 2. Null correlations

The space of anti symmetric tensors $a^{\mu\nu}$ is spanned by the 6 infinitesimal operators of the Lorentz group.

In chapter I, section 7.1. we have observed that under the Lie product $[A, B]$ these infinitesimal operators form a 6-dimensional algebra. In this section we take the ordinary matrix product AB and study the properties of the algebra generated by the infinitesimal operators under this product. In order to do this we note that the antisymmetric tensor $a^{\mu\nu}$ appears in projective geometry as a null correlation, i.e. a (1-1) map of points onto planes.

By an application of the method of Cartan to the 6-dimensional space $R_6$ of anti-symmetric tensors, we shall show that this method is indispensable for a uniform and compact description of properties of this space.

In particular, we apply this method to the study of the so-called configuration of Kummer. It follows that the 6 "orthogonal" anti-symmetric matrices (infinitesimal operators of the Lorentz group), which span the $R_6$ leave this configuration invariant and form a 32-dimensional Clifford algebra.

## 2.1. Null correlations, introduction.

All straight lines in the three-dimensional space $P_3$ can be described by the 6 line coordinates $p^{\mu\nu}$, or

$(p^{i'}, p^{i''})$, where $p^{i'} = p^{0i}$ and $p^{i''} = p^{jk}$ with i, j, k = 1, 2, 3,

which satisfy to the relation

$$\Gamma : (p)^2 = p^{1'}p^{1''} + p^{2'}p^{2''} + p^{3'}p^{3''} = 0. \qquad (4\text{-}19)$$

Hence, the study of all lines in $P_3$ (or all bivectors in $R_4$) may be reduced to the study of the quadratic surface $\Gamma$ in $P_5$ (in $R_6$). (See formula (2-44) and the appendix to chapter II).

With the aid of formula (4-19) we can introduce an invariant inner product (p, q) of two vectors $p(\vec{p}', \vec{p}'')$ and $q(\vec{q}', \vec{q}'')$ in $R_6$:

$$(p, q) = \tfrac{1}{2}\, p_{\overline{\mu\nu}}\, q^{\mu\nu} = \vec{p}'\,\vec{q}'' + \vec{p}''\,\vec{q}' \ . \qquad (4\text{-}20)$$

It follows that $p^{\mu\nu}$ are *point coordinates* and $p_{\overline{\mu\nu}}$ are *plane coordinates* of the plane orthogonal to the point $p^{\mu\nu}$. Analogously to formula (4-10), we map every vector $p^{\mu\nu}$ onto an 8 × 8 matrix P; i.e.

$$p^{\mu\nu} \rightarrow P = \begin{pmatrix} 0 & p^{\mu\nu} \\ -p_{\overline{\mu\nu}} & 0 \end{pmatrix} \qquad (4\text{-}21)$$

Besides the coordinates $p^{\nu}$ and $q_{\nu}$ for points and planes in $P_3$, we introduce also the 8-dimensional vector $r(p^{\nu}, q_{\nu})$ . The incidence of the point $p^{\nu}$ or plane $q_{\nu}$ with the line $p^{\mu\nu}$ is given by $p_{\overline{\mu\nu}}p^{\nu} = 0$ or $p^{\mu\nu}q_{\nu} = 0$ respectively (formula 2-47, 48). Hence, we obtain the formula

$$Pr = \begin{pmatrix} 0 & p^{\mu\nu} \\ -p_{\overline{\mu\nu}} & 0 \end{pmatrix} \begin{pmatrix} p^{\nu} \\ q_{\nu} \end{pmatrix} = 0 \ . \qquad (4\text{-}22)$$

In components we have

$$Pr = \left(\begin{array}{cccc|cccc} & & & & \cdot & p^{1'} & p^{2'} & p^{3'} \\ & & & & -p^{1'} & \cdot & p^{3''} & -p^{2''} \\ & & & & -p^{2'} & -p^{3''} & \cdot & p^{1''} \\ & & & & -p^{3'} & p^{2''} & -p^{1''} & \cdot \\ \hline \cdot & p^{1''} & -p^{2''} & -p^{3''} & & & & \\ p^{1''} & \cdot & -p^{3'} & p^{2'} & & & & \\ p^{2''} & p^{3'} & \cdot & -p^{1'} & & & & \\ p^{3''} & -p^{2'} & p^{1'} & \cdot & & & & \end{array}\right) \left(\begin{array}{c} p^0 \\ p^1 \\ p^2 \\ p^3 \\ \hline q_0 \\ q_1 \\ q_2 \\ q_3 \end{array}\right) = 0 \qquad (4\text{-}23)$$

$$\text{or } Rp = \left(\begin{array}{ccc|ccc} q_1 & q_2 & q_3 & \cdot & \cdot & \cdot \\ -q_0 & \cdot & \cdot & \cdot & -q_3 & q_2 \\ \cdot & -q_0 & \cdot & q_3 & \cdot & -q_1 \\ \cdot & \cdot & -q_0 & -q_2 & q_1 & \cdot \\ \hline \cdot & \cdot & \cdot & -p' & -p^2 & -p^3 \\ \cdot & p^3 & -p^2 & p^0 & \cdot & \cdot \\ -p^3 & \cdot & p^1 & \cdot & p^0 & \cdot \\ p^2 & -p^1 & \cdot & \cdot & \cdot & p^0 \end{array}\right) \left(\begin{array}{c} p^{1'} \\ p^{2'} \\ p^{3'} \\ \hline p^{1''} \\ p^{2''} \\ p^{3''} \end{array}\right) = 0 \, . \qquad (4\text{-}24)$$

The fundamental property of the matrix P is that

$$P^2 = (p)^2 \, .$$

(It is understood that on the right hand side multiplication with the unit matrix is carried out).

Analogously to (4-12), we may write the inner product (p, q) using the $8 \times 8$ matrices in the form

$$(p, q) = \tfrac{1}{2} (PQ + QP). \qquad (4\text{-}25)$$

Remark 4.3. If $r = (p^{\nu}, q_{\nu})$ is an arbitrary vector not equal to the zero vector, then the rank of the matrix R is three; thus there are three independent solutions $p^{\mu\nu}$ for which $Pr = 0$ and which span a two-dimensional plane (2-plane) in $P_5$, or, what is the same thing, it follows that the vector $r(p^{\nu}, q_{\nu})$ describes a 3-plane in $R_6$. Furthermore $P^2 r = 0$; thus $(p)^2 = 0$, which implies that the system (4-24) describes a family of planes which lie on $\Gamma$.
Hence, we see that the quadratic *cone* $\Gamma$ in the *even* dimensional space $R_6$ is covered by two systems of 3-planes viz. $(p^{\nu}, 0)$ and $(0, q^{\nu})$. This is well-known in projective geometry and is in analogy with four-dimensional spinor calculus.
In $R_4$, the four-simensional spinors $(\psi^{a}, \phi_{\dot{a}})$ determine two families of isotropic planes on the light cone.
If we restrict ourselves to $p^{\nu} = q^{\nu}$, then the equation (4-24) is equivalent to the equations of Maxwell, formula (3-24a, b).

By considering reflections in $R_6$ and applying formula (4-14) we obtain the result that a reflection with respect to the 5-plane $a_{\mu\nu}$ has the form

$$P' = -APA \text{ , where } A^2 = 1 \qquad (4\text{-}26)$$

In order that the relation $Pr = 0$ remains invariant, it follows that r is transformed as

$$r' = Ar$$

In components, we obtain

$$p'^{\mu} = a^{\mu\nu}q \quad \text{and } q'_{\mu} = -a_{\overline{\mu\nu}}p^{\nu}.$$

This transformation maps points $p^{\nu}$ in $P_3$ onto planes $q'_{\mu}$ and is there fore called a *correlation*. We have

$$a_{\overline{\mu\nu}}\rho^{\nu}p^{\mu} = 0$$

for every p , because $a_{\overline{\mu\nu}}$ is anti-symmetric. Thus, every point lies in

its correlated plane, and $a_{\overline{\mu\nu}}$ is called a *null correlation*.
(Supposing that $a_{\mu\nu}$ is symmetric, then $a_{\mu\nu}p^{\mu}p^{\nu} = 0$ determines a quadratic surface in $P_3$ and one says that $a_{\mu\nu}$ determines a *pole correlation* or *polarity)*.
A ray in $R_6$ corresponds with a point in $P_5$ and a reflection in $R_6$ corresponds with a point-plane reflection (or central-involutoric collineation) in $P_5$.
Hence we have.

Theorem 2.1. *A null correlation* $\rho a_{\overline{\mu\nu}}$ *in* $P_3$ *corresponds, in the 5-dimensional line space, to a central-involutoric collineation which leave the point* $a^{\mu\nu}$ *and the plane* $a_{\overline{\mu\nu}}$ *invariant.*

For a more synthetic proof, see Barran p. 391.

2.2. The Configuration of Kummer.

We now consider the fundamental form

$$(x)^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 ,$$

where the $x_\mu$ are arbitrary complex numbers. The corresponding group is the 4-dimensional complex orthogonal group, which is isomorphic to the complex Lorentz group $L\mathbb{C}$.

The 6-dimensional space of anti-symmetric tensors $p^{\mu\nu}(\vec{p}', \vec{p}'')$ reduces now with respect to the group $L\mathbb{C}$ into two subspaces of selfdual tensors $(\vec{p}, \vec{p})$ and anti-selfdual tensors $(\vec{q}, -\vec{q})$ we shall write

$$(\vec{p}, \vec{p}) = p^i \beta_i \text{ and } (\vec{q}, -\vec{q}) = q^i \beta_{i'} (-i), \tag{4-27}$$

where $\beta_i$, $\beta_{i'}$ (i = 1,2,3) are six anti-symmetric matrices

$$\beta_1 = (1,0,0|\ 1,0,0) \quad \text{with norm } \beta_1^2 = +1$$

$$\beta_{1'} = (0,0,0|\ -1, 0, 0)\ i \quad \text{with norm } \beta_{1'}^2 = +1$$

$$\ldots\ldots \text{ etc.}$$

Because the matrices $\beta$ are (anti-)selfdual, it follows that the 6 vectors $\beta_i$, $\beta_{i'}$ are mapped onto the 8 × 8 matrices

$$B_1 = \begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} , \quad B_{1'} = \begin{pmatrix} 0 & \beta_{1'} \\ +\beta_1 & 0 \end{pmatrix} , \ \ldots \text{ etc.} \tag{4-28}$$

The orthonormality relations take the form

$$B_\xi B_\eta + B_\eta B_\xi = 2\,\delta_{\xi\eta}, \quad (\xi, \eta = 1,2,3,1',2',3'), \tag{4-29}$$

from which we derive

$$\beta_i\beta_j + \beta_j\beta_i = -2\delta_{ij},$$

$$\beta_{i'}\beta_{j'} + \beta_{j'}\beta_{i'} = 2\delta_{i'j'},$$

$$\beta_i\beta_{j'} + \beta_j\beta_{i'} = 0, \tag{4-30}$$

moreover, $\beta_1\ \beta_2\ \beta_3 = e$ and $\beta_{1'}\beta_{2'}\ \beta_{3'} = -ie$.

In condensed form, we get

$$\beta_i\beta_j = \delta_{ij} + \varepsilon_{ijk}\,\beta_k \qquad (\varepsilon_{123} = 1)$$

and $$\beta_{i'}\beta_{j'} = \delta_{i'j'} - i\varepsilon_{i'j'k'}\,\beta_{k'} \quad (\varepsilon_{1'2'3'} = 1).$$

(cf. Barut p. 98).

The $\beta_i$ and $\beta_{i'}$ generate a group of 64 elements, namely

| | |
|---|---|
| $e$ | 1 collin |
| $\beta_1, \beta_2, \beta_3, \beta_{1'}, \beta_{2'}, \beta_{3'}$ | 6 correl |
| $\beta_1\beta_2, \ldots \qquad ,\beta_{2'}\beta_{3'}$ | 15 collin |
| $\beta_1\beta_2\beta_3, \ldots \qquad ,\beta_{1'}\beta_{2'}\beta_{3'}$ | 20 correlt |
| $\beta_1\beta_2\beta_3\beta_{1'}, \ldots \quad ,\beta_3\beta_{1'}\beta_{2'}\beta_{3'}$ | 15 collin |
| $\beta_1\beta_2\beta_3\beta_{1'}\beta_{2'}, \ldots ,\beta_2\beta_3\beta_{1'}\beta_{2'}\beta_{3'}$ | 6 correl |
| $\beta_1\beta_2\beta_3\beta_{1'}\beta_{2'}\beta_{3'} = -1$ | 1 collin |
| | 64 |

In the projective space $P_3$, two matrices $\beta$ and $\rho\beta$ determine the same transformation. Thus, we can identify the elements from the first row (15) with the second row (15). For instance,

$$\begin{cases} \beta_{2'}\beta_{3'} = \beta_{1'}\beta_{1'}\beta_{2'}\,\beta_{3'} = -i\beta_{1'} \\ \beta_1\beta_2\beta_3\beta_{1'} = \beta_{1'} \quad , \end{cases}$$

where we have used (4-30).

There remains a *projective group* of 32 elements, namely the first 32 elements of the above scheme *).

If we subject a point P in the projective space $P_3$ to this group of 32 elements, we get 1 + 15 = 16 points and 6 + 10 = 16 planes. Every point lies in the planes to P correlated, and 16 points lie in every plane. This configurates $16_6$ is known as the *configuration of Kummer* and we have

---

*) One may also identify the correlation $\beta_{3'}$ (point → plane) with the collineation (point → point) $\beta_{1'}\beta_{2'} = -\beta_{3'}$, both are described by the same matrix (within a numerical factor). There remains a group of 16 elements generated by the Dirac matrices, i.e.

$$\gamma_0 = i\beta_1\beta_{3'} \quad , \gamma_1 = \beta_3 \; , \gamma_2 = +\,\beta_1\beta_{2'} \quad , \gamma_3 = \beta_2.$$

**Theorem 2.2.** *The six orthonormal vectors* $\beta_i$, $\beta_i$, *which span the irreducible spaces of anti-symmetric tensors* $(\vec{p}, \vec{p})$ *and* $(\vec{q}, -\vec{q})$ *respectively form a projective group of 32 elements with relations (4-30), which leaves invariant the configuration of Kummer.*

## 2.3. Geometrical analysis of the Kummer group.

The 6 elements $\beta_\xi$ ($\xi = 1,2, \ldots 3'$)

A null correlation transforms a point x into a plane x', and the line which joins two points x and y (the *ray* xy) is transformed by a null correlation into the intersection line of the planes x' and y' (the *axis* x'y').

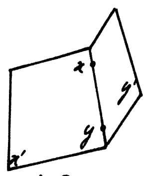


fig. 4.8

In particular, if an arbitrary y lies in the plane x', i.e.

$$a_{\mu\nu}\, x^\mu y^\nu = 0,$$

then it follows that the point x lies in the plane y', thus the ray xy coincides with the axis x'y' and the line xy remains invariant under the null correlation $a_{\bar{\mu}\bar{\nu}}$.

Definition. All lines in $P_3$ which are invariant under a given null correlation form a *linear complex.*

It follows from the above considerations that a linear complex is formed by all lines through the points x of $P_3$ which lie in the plane x' correlated to x.

In the line space $P_5$, the null correlations $a_{\bar{\mu}\bar{\nu}}$ corresponds to a point-plane reflection. Because the points of the plane $a_{\bar{\mu}\bar{\nu}}$ remain invariant, it follows that the intersection points of the plane $a_{\bar{\mu}\bar{\nu}}$ with the quadratic surface $\Gamma$ ($a_{\bar{\mu}\bar{\nu}}\, a^{\mu\nu} = 0$) are the image points of the lines of the linear complex a.

Hence, the equation of a linear complex is given by

$$a_{\bar{\mu}\bar{\nu}}\, p^{\mu\nu} = 0 \quad \text{and} \quad p_{\bar{\mu}\bar{\nu}}\, p^{\mu\nu} = 0 \; .$$

Remark 4.4. On the relation between linear complexes and Clifford parallels.

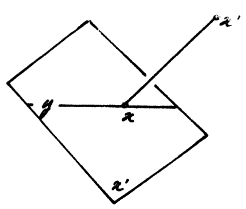


fig. 4.9

We make the collineation $a^\mu{}_\nu$ (point → point) from the null correlation $a = a_{\mu\nu}$ by raising with the metric tensor $g_{\mu\nu}$ one index.

Supposing that the null correlation $a_{\mu\nu}$ has the form $a \equiv (\vec{a}, \varepsilon\, \vec{a})$ ($\varepsilon = \pm 1$), then, because a is anti-symmetric and orthogonal, we have

$$(ga)^{-1} = -(ga).$$

Thus, $a^{\mu}_{\ \nu}$ is involutoric. It follows that if

$$\rho x'^{\mu} = a^{\mu}_{\ \nu} x^{\nu}$$

then $$\rho x^{\mu} = a^{\mu}_{\ \nu} x'^{\nu}$$

and all lines xx' are invariant under the collineation $a^{\mu}_{\ \nu}$ and thus they are Clifford parallel (see also Barrau p. 20).

In order to visualize the linear complex $a_{\mu\nu}$ and the corresponding Clifford parallels, we apply a coordinate transformation in such a way that the correlation $a_{\mu\nu}$ $(\vec{a}, \varepsilon\ a)$ transforms the point T(1, 0, 0, 0) into the plane TXY(0, 0, 0, 1), and the point Z(0, 0, 0, 1) into the plane ZXY(1, 0, 0, 0). TZ is called the *axis* of the complex (Klein, Vorl. Höhere G., p. 63).

The tensor $a(\vec{a}, \varepsilon\ \vec{a})$ becomes

$$a_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \varepsilon & 0 \\ 0 & -\varepsilon & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad (\varepsilon = \pm 1).$$

In particular, the points x(1, x, 0, 0) of the x-axis are transformed by the collineation $a^{\mu}_{\ \nu}$ into points x'(0, 0, $-\varepsilon$x, 1) on the ZY-axis.

In figure 4.10., the Clifford parallels xx' are drawn for $\varepsilon$ = +1, -1.

The planes $(x_{\mu})$, through the point x and orthogonal to the line xx', which contain all lines of the linear complex are omitted.

Thus, the reduction of the space of anti-symmetric tensors into matrices of type $(\vec{a}, \varepsilon\ \vec{a})$, $\varepsilon = \pm 1$, can also be interpreted as the reduction of linear complexes into right- and left-handed systems of planes.

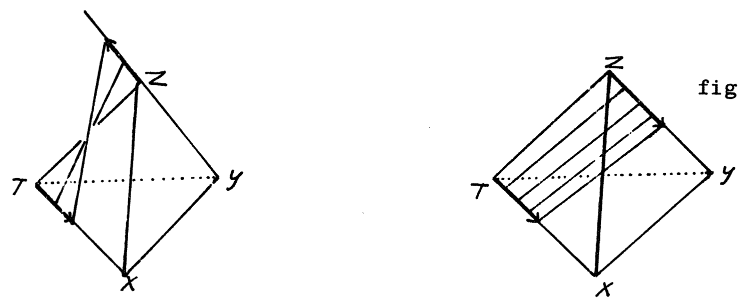



fig. 4.10

$\varepsilon$ = 1 right-handed $\qquad$ $\varepsilon$ = -1 left handed

The 15 elements $\beta_\xi\beta_\eta$ $(\xi, \eta = 1,2, \dots 3')$

The product ab of two null correlations a and b leaves all lines which belong to the intersection of the linear complexes a and b invariant.

Definition. *The intersection of two linear complexes is called a linear congruence.*

Theorem 2.7. *A linear congruence consists of all lines which intersect two fixed lines p and q* (Klein p. 87).

PROOF. The null correlations $a_{\overline{\mu\nu}}$ and $b_{\overline{\mu\nu}}$ determine two point-plane reflections in $P_5$ with center $a = a^{\mu\nu}$ and $b = b^{\mu\nu}$. The product ab of the two point-plane reflections leaves the line at invariant and in particular the intersection points p and q of ab with the fundamental form Γ. Thus in the 3-dimensional space $P_3$, there are two lines $p^{\mu\nu}$ and $q^{\mu\nu}$ which remain invariant under the transformation ab. Considering now all lines m which remain invariant under ab and thus belong to the linear complexes $a_{\overline{\mu\nu}}$ *and* $b_{\overline{\mu\nu}}$, i.e.

$$(a, m) = 0 \qquad a = \lambda p + \mu q$$

$$\text{and} \quad (b, m) = 0 \qquad b = \rho p + \sigma q ,$$

it follows that $(p, m) = (q, m) = 0$ *). Hence, every line m invariant under ab cuts p and q, and we have obtained the intersection of the two linear complexes $a_{\mu\nu}$ and $b_{\mu\nu}$

We note that the intersection points P and Q of the line m with p and q respectively remain invariant and thus the lines p and q remain pointwise invariant.

Further, we notice that through every point A not on p and q, there exists exactly one line which belongs to the congruence ab.

Supposing now that a and b are orthogonal, i.e. $(a, b) = 0$, we have:

Theorem 2.4. *The product of two orthogonal null correlations is a reflection in two lines.*

---

*) $(p, m) = 0$ is the condition that the lines p, and m lie in one plane. Therefore, we note that if u and v are two points which lie on p, and w and z are two points on m, then $\det(u, v, w, z) = 0$. If we write this determinant as the product of $2 \times 2$ sub determinants $p^{\mu\nu} = u^\mu v^\nu - u^\nu v^\mu$ and $m^{\mu\nu} = w^\mu z^\nu - w^\nu z^\mu$, we obtain the condition $(p, m) = p_{\overline{\mu\nu}}\, m^{\mu\nu} = 0$.

PROOF.

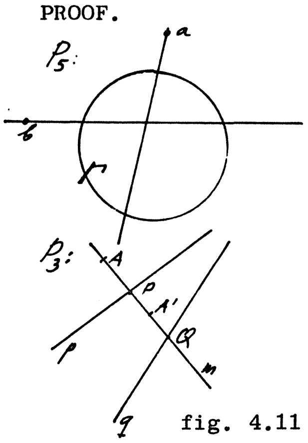



fig. 4.11

In the 5-dimensional space $P_5$, the relation $(a, b) = 0$ implies that b lies in the polar plane of a and conversely that a lies in the polar plane of b. The plane-point reflection b leaves the point $a = a^{\mu\nu}$ and the plane $a_{\overline{\mu\nu}}$ invariant; hence, $a = bab^{-1}$ and $(ab)^2 = a^2b^2 = 1$.
Because (ab) is involutoric, it follows that if a point A is transformed into A' and P and Q are the invariant points on m = AA', then A, A' and P, Q form a harmonic sequence and the transformation $A \to A'$ is a reflection on m.

Hence, we see that the row of the 15 elements $\beta_1\beta_2, \ldots, \beta_{2'}\beta_{3'}$ consists of reflection in two lines.

<u>The 10 elements $\beta_\xi\beta_\eta\beta_\zeta$ $(\xi, \eta, \zeta = 1,2,\ldots 3')$.</u>

The product of three correlations is again a correlation. Using the anti-symmetry of the matrices $\beta_i$ , $\beta_{i'}$ , we obtain, for instance,

$$(\beta_1\beta_2\beta_{1'})^T = -\beta_{1'}\beta_2\beta_1 ,$$

and, applying the relations (4-30), we have

$$(\beta_1\beta_2\beta_{1'})^T = \beta_1\beta_2\beta_{1'} .$$

It follows that the row (20) consists of 10 symmetric correlations (pole correlations) which determine 10 quadratic surfaces. From the foregoing, it follows that the product of three null correlations a, b, c leaves the intersection of the linear complexes a, b, c invariant. One can show that this intersection is given by *one* system of describing lines of a quadratic surface. (F. Klein, Vorl. höh. Geom. p. 88).
To do so, we observe that the product abc of the 3 points-plane reflections leaves the plane $\lambda a + \mu b + \nu c$ invariant and, in particular, the intersection $\gamma$ of this plane with $\Gamma$.
Taking three solutions $p_1$, $p_2$ and $p_3$ on $\gamma$, it follows that there are three lines $p_1$, $p_2$, $p_3$ in the projective space which remain invariant under abc (and every line $\lambda_i p_i \in \gamma$).

From the foregoing theory, it follows that every line on which belongs to the linear complexes a, b and c intersects $p_1$, $p_2$ and $p_3$. If A is a point on $p_3$, then m is the intersection line of the planes $Ap_1$ and $Ap_2$. Since the two bundles $Ap_1$ and $Ap_2$ are projectively correlated, it follows that their intersection lines are the describing lines of a quadratic surface.

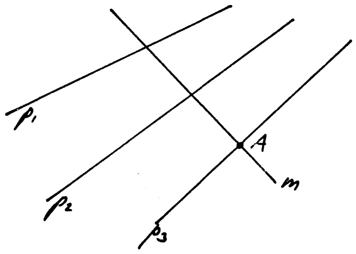



fig. 4.12

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## NAME AND SUBJECT INDEX

Underlined numbers refer to pages where the concepts are defined.

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A star (*) before the number means "to appear".