MATHEMATICAL CENTRE TRACTS 167

NONNEGATIVE MATRICES IN DYNAMIC PROGRAMMING

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MATHEMATISCH CENTRUM AMSTERDAM 1983

1980 Mathematics subject classification: 90C39, 15A48

ISBN 90 6196 264 1

ACKNOWLEDGEMENTS

This monograph is a revised version of my doctoral thesis written at the Department of Mathematics of the Eindhoven University of Technology. I wish to express my gratitude to prof.dr. J. Wessels, my dissertation advisor, for his constant encouragement and advice during the period I worked in Eidnhoven. Without him, this monograph would not have been written.

I am indebted to prof.dr. G.W. Veltkamp, prof.dr.ir. M.L.J. Hautus and prof.dr. F.W. Steutel for many valuable comments. Furthermore, I would like to mention prof.dr. J. Wessels and my colleagues in Eindhoven for creating an excellent working atmosphere.

Thanks are also given to prof.dr. G. de Leve for giving me the opportunity to prepare the final version of my thesis when working already in his group at the University of Amsterdam.

It is a pleasure to mention Jolanda van Haarlem-Moolekamp for the excellent typing of the manuscript. And finally, I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts, and all those at the Mathematical Centre who have contributed to its technical realization.

Contents

CHAPTER 1

INTRODUCTION

In this monograph we study dynamic programming models in which the transition law is specified by a set of nonnegative matrices. These models include e.g. Markov decision processes with additive and multiplicative utility function , input-output systems with substitution, controlled multitype branching processes, etc. The main objective of this monograph is to show that all these models can be studied within one general matrix-theoretical framework. This framework will be built up by using dynamic programming methods and will be based on the theory of sets of general nonnegative matrices. This explains the title.

Methods which have been developed to determine an optimal control in the above mentioned models with respect to various types of criterion functions, will follow as special cases from such a general framework. As an example we may think of a policy iteration method for a Markov decision process with respect to some "sensitive optimality" criterion or of methods to determine equilibrium prices in a Leontief substitution system. This indicates the generality of our model, a model in which the theory of generalized eigenvectors and generalized (sub)invariant vectors for sets of nonnegative matrices plays a central role.

In this introductory chapter we first give a short historical review of the problem field and a summary of our objectives (section 1.1). After that a more formal description is given of the model to be studied in this volume (section 1.2).

Section 1.3 lists a number of examples of models, arising from various fields in mathematics and in mathematical economics, which can be written in, or easily be transformed into our problem formulation. The contents of the subsequent chapters are summarized in section 1.4 and a list of notations is given in section 1.5.

1.1. A short history; main objectives

Since the publication of Bellman's "Dynamic Programming" in 1957 (BELLMAN [5]), interest in dynamic programming has expanded rapidly. In his book Bellman formalized the technique of backward induction which appeared to be fundamental for the analysis of sequential decision processes. In the last chapter of that volume some attention is paid to Markov decision processes. A deeper investigation of the use of dynamic programming for the control of Markov decision processes appeared three years later (HOWARD [29]). Also Shapley's paper on stochastic games is now recognized as fundamental to this field (SHAPLEY [53]). But, as Denardo remarked, the modern era started with the work of Blackwell (compare Denardo's contribution to the panel discussion in PUTERMAN [47]; see also BLACKWELL [8], [9]).

Markov decision processes with additive reward function have been studied with respect to several criteria, the classical ones being: the expected total reward criterion and the expected average reward criterion. More sensitive optimality criteria have been investigated by VEINOTT [64], SLADKY [54], and DENARDO AND ROTHBLUM [16]. Often the transition probability matrices in these models are allowed to be substochastic, i.e., a positive probability for fading of the system is allowed (cf. VEINOTT [64], ROTHBLUM [50], [51], HORDIJK [27] and WESSELS [71]).

Multiplicative Markov decision processes have been studied by HOWARD AND MATHESON [30] and by ROTHBLUM [49]. Other models which are in fact closely related (as far as structure is concerned), can be found in e.g. MORISHIMA [42] or BURMEISTER AND DOBELL [12] (Leontief substitution systems) and in PLISKA [46] (controlled multitype branching processes).

One of the objectives of this monograph is to analyze these models by using nonnegative matrix theory instead of probabilistic arguments (note that several models, which have been mentioned above, have no probabilistic interpretation at all, and that the associated nonnegative matrices are not stochastic in general). This takes us to our second subject. Nonnegative matrices and more general nonnegative operators play an important role in various fields of applied mathematics, e.g. probability theory, demography, numerical analysis and mathematical economics. Since the publication of the basic work of PERRON [45] and FROBENIUS [24], [25] an overwhelming number of papers appeared in the literature. To mention only a few important ones: BIRKHOFF [7], KARLIN [33] and VERE-JONES [65], [66]. Excellent overviews may be found in SENETA [52] and in BERMAN AND PLEMMONS [6]. Finally, some results concerning sets of finite-dimensional nonnegative matrices, closely related to some of our own work in part I of this monograph, are given in SLADKY [56], [58].

We conclude this section with a sketch of problems we examine and objectives we pursue in this monograph. The book is divided into two parts, the first one dealing with finite-dimensional systems, the second one with models of countably infinite dimension. Our main objective will be to give a systematic treatment of the theory of sets of nonnegative matrices in dynamic programming problems and to give a fairly complete analysis of the asymptotic behaviour of dynamic programming recursions. In order to keep the exposition lucid and reasonably simple we shall first treat the finite-dimensional case. In this case it is possible to develop explicit policy-iteration methods, which end after a finite number of steps, in order to characterize and to determine matrices which maximize the growth of the system. Brief attention will be paid to the continuous-time analogue of the above sketched models.

The second part of this book is devoted to the development of a theory for sets of countably infinite nonnegative matrices. Questions concerning invariant vectors and optimal contraction factors then arise and we shall try to answer them. The reader familiar with CHUNG [13] will recognize that some of our results are extensive generalizations of results in that volume. Our results are also related to well-known facts in potential theory for Markov chains (cf. KEMENY, SNELL AND KNAPP [35], and HORDIJK [27]). At several places we shall indicate applications of the results, e.g. in the theory of Markov decision processes and strongly excessive functions (cf. VAN HEE AND WESSELS [70]), and in the investigation of sensitive optimality criteria in controlled Markov chains (cf. SLADKY [54]).

1.2. Description of the model

In this section a formal description is given of the dynamic systems to be studied in this monograph. For notations the reader is referred to section 1.5.

Central in the book is the concept of a set of matrices with the product property. Let us first give the formal definition.

DEFINITION 1.1. Let K be a set of $k \times m$ matrices ($k,m \in \bar{\mathbb{N}}$) and let P_i denote the i-th row of a matrix $P \in K$. Then K has the *product property* if for each subset V of $\{1,2,\dots,k\}$ and for each pair of matrices $P(1), P(2) \in K$ the following holds:

The matrix $P(3)$, defined by

$$P(3)_i := \begin{cases} P(1)_i & \text{for } i \in V \\ P(2)_i & \text{for } i \in \{1,2,\dots,k\}\backslash V, \end{cases}$$

is also an element of K. □

Roughly speaking this means that for $i = 1,2,\dots,k$ there exists a collection C_i of row vectors of length m. K is the set of all $k \times m$ matrices with the property that their i-th row is an element of C_i, for $i = 1,\dots,k$.

Next we describe the finite-dimensional models to be studied in part I. Let $\mathbb{R}^N$ denote the N-dimensional Euclidean space. The set $\{1,2,\dots,N\}$ will often be called the state space and is then denoted by S. A nonnegative matrix P is a matrix with all its entries real and nonnegative. Let K now denote a finite set of nonnegative $N \times N$ matrices with the product property. One of our objectives is to obtain information about the asymptotic behaviour of the utility vector $x(n)$ (an N-dimensional column vector), obeying the dynamic programming recursion

$$(1.2.1) \qquad x(n+1) = \max_{P \in K} P\,x(n) \qquad n = 0,1,2,\dots$$

where the maximum is taken component-wise and $x(0)$ denotes a fixed strictly positive vector. For interpretations of (1.2.1) we refer to section 1.3. Here we only remark that the fact that K has the product property implies for each n the existence of a matrix $P(n) \in K$ such that

$$x(n+1) = P(n)\ x(n) \qquad n = 0,1,2,\ldots$$

In chapter 6 we briefly treat the continuous-time analogue of the discrete dynamic programming recursion defined above. A central role is then played by a collection of so-called ML-matrices with the product property. An ML-matrix is a square matrix with all its nondiagonal entries nonnegative. Let M denote a finite set of ML-matrices with the product property. We are now interested in the asymptotic behaviour of the vector function z(t), defined by

$$\text{(1.2.2)} \qquad \frac{dz}{dt}(t) = \max_{Q \in M} Q\ z(t) \qquad t \in [0,\infty),$$

with z(0) fixed, strictly positive (again the maximum is taken componentwise). Note that, since M has the product property, there exist matrices $Q(t) \in M$ such that

$$\frac{dz}{dt}(t) = Q(t)\ z(t) \qquad t \in [0,\infty).$$

For an example we refer to section 1.3.

The analysis of these models requires a detailed study of sets of nonnegative matrices (resp. sets of ML-matrices) with the product property. In part I we shall develop a theory for sets of finite-dimensional matrices, in part II infinite-dimensional models are investigated. The results in the second part may be viewed as rather far-reaching extensions of the R-theory for nonnegative matrices, initiated by VERE-JONES [65], [66].

1.3. Examples

In this section, we list as examples a number of special cases of the general models, sketched in the preceding section.

1.3.1. Markov decision processes with additive reward function

a. The discrete time case

Markov decision processes have been studied initially by BELLMAN [4], [5] and HOWARD [29]. Suppose a system is observed at discrete points of time. At each time point the system may be in one of a finite number of states, labeled by $1,2,\dots,N$. If, at time t, the system is in state i, one may choose an action, a say, from a finite action space A; this action results in a probability p_{ij}^a of finding the system in state j at time t+1. Furthermore a reward r_{ij}^a is earned when in state i action a is taken and the system moves to state j. Suppose

$$r_{ij}^a \geq 0 \; ; \; \sum_{j=1}^{N} p_{ij}^a \leq 1 \qquad i,j = 1,\dots,N; \; a \in A,$$

i.e., a positive probability that the process terminates is allowed.

Let $v(0)_i$ denote the terminal reward in state i and let $v(n)_i$ be the maximal expected return for the n-period problem (i.e., with n periods to go), when starting in state i. For convenience define

$$r_i^a = \sum_{j=1}^{N} p_{ij}^a \, r_{ij}^a \qquad i = 1,\dots,N; \; a \in A.$$

Bellman's *optimality principle* implies that the following recursion holds for $v(n)_i$ (cf. BELLMAN [5]):

$$\text{(1.3.1)} \qquad v(n)_i = \max_{a \in A} \{ r_i^a + \sum_{j=1}^{N} p_{ij}^a \, v(n-1)_j \} \qquad i = 1,\dots,N.$$

Recursion (1.3.1) can be written in vector notation when policies are introduced. A *policy* f is a function from $\{1,\dots,N\}$ to A. The set of all possible policies is denoted by F. Let P(f) be the (substochastic) matrix with entries $p_{ij}^{f(i)}$ and r(f) the vector with components $r_i^{f(i)}$ for $i,j = 1,2,\dots,N$; $f \in F$. From these definitions, it immediately follows that the collection of Nx(N+1) matrices

$$\{ (P(f), \; r(f)) \mid f \in F \}$$

has the product property. Instead of (1.3.1) we may write

$$(1.3.2)\qquad v(n) = \max_{f\in F} \{r(f) + P(f)\, v(n-1)\} \qquad n \in \mathbb{N}$$

where $v(n)$ denotes the vector with components $v(n)_i$, $i = 1,\ldots,N$. By introducing a simple dummy variable we obtain

$$(1.3.3)\qquad \begin{pmatrix} v(n) \\ 1 \end{pmatrix} = \max_{f\in F} \begin{pmatrix} P(f) & r(f) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v(n-1) \\ 1 \end{pmatrix} \qquad n \in \mathbb{N}$$

which is an example of the recursion (1.2.1), to be studied in part I of this monograph.

b. The continuous-time case.

As in the previous example we consider a system with a finite state space, $\{1,2,\ldots,N\}$ say, and a finite action space A. Suppose now the system is observed continuously. At each time point $t \in [0,\infty)$ the system is allowed to make a transition from one state to another one. It will be clear that the significant parameters are transition rates rather than transition probabilities (cf. CHUNG [13]).

We assume that a controller is allowed to react at each time point $t \in [0,\infty)$. If at time t the system is in state i, and action $a \in A$ is taken the system is supposed to make a transition to state j in a short time interval Δt with probability $\tilde{q}^a_{ij}\,\Delta t + \sigma(\Delta t)$ $(i,j = 1,\ldots,N)$. The probability of two or more transitions is of order $\sigma(\Delta t)$ if Δt is sufficiently small (we say that a function $h(t)$ is of order $\sigma(t)$ for t small if $\lim_{t\to 0} t^{-1}h(t) = 0$). The probability of making no transition in a short time interval Δt is then equal to $1 - \sum_{j=1}^{N} \tilde{q}^a_{ij}\,\Delta t$

Suppose furthermore that, if the system is in state i at time t and action a is chosen, a reward of $\tilde{r}^a_{ii}$ per unit time is earned during the time that the system remains in state i. If the system moves from state i to state j a reward r^a_{ij} is received $(i,j = 1,\ldots,N)$. Now, if $v(t)_i$ denotes the maximal expected return in a time interval of length t when starting in state i and $v(0)_i$ denotes the terminal reward in state i, it follows from Bellman's optimality principle that for $i = 1,\ldots,N$ and $t \in [0,\infty)$:

$$v(t+\Delta t)_i = \max_{a\in A} \{ (1-\sum_{j=1}^{N} \tilde{q}^a_{ij}\,\Delta t)(\tilde{r}^a_{ii}\,\Delta t + v(t)_i) + \sum_{j=1}^{N} \tilde{q}^a_{ij}\,\Delta t\,(r^a_{ij} + v(t)_j)\} + \sigma(\Delta t)$$

Define for $i,j = 1,\dots,N$ and $a \in A$

$$q_{ii}^{a} = -\sum_{j\neq i} \tilde{q}_{ij}^{a} \; ; \; q_{ij}^{a} = \tilde{q}_{ij}^{a} \; (j\neq i) \; ; \; r_{i}^{a} = \tilde{r}_{ii}^{a} + \sum_{j=1}^{N} \tilde{q}_{ij}^{a} \, r_{ij}^{a} .$$

Then, for $i = 1,\dots,N$ and $t \in [0,\infty)$, we obtain

$$(1.3.4) \qquad \frac{v(t+\Delta t)_i - v(t)_i}{\Delta t} = \max_{a\in A} \{r_i^a + \sum_{j=1}^{N} q_{ij}^{a} \, v(t)_j\} + \frac{\sigma(\Delta t)}{\Delta t}$$

Again, a policy f is defined as a function from $\{1,\dots,N\}$ to A. Let F denote the set of all possible policies, $Q(f)$ the matrix with entries $q_{ij}^{f(i)}$ and $r(f)$ the vector with components $r_i^{f(i)}$. If we take the limit in (1.3.4) as $\Delta t \to 0$ we obtain, in vector-notation:

$$(1.3.5) \qquad \frac{dv}{dt}(t) = \max_{f\in F} \{r(f) + Q(f)\, v(t)\} \qquad\qquad t \in [0,\infty).$$

Define a scalar function $v_{N+1}(t) \equiv 1$ for $t \in [0,\infty)$. Then we may write

$$(1.3.6) \qquad \begin{pmatrix} \frac{dv}{dt}(t) \\ \frac{dv_{N+1}}{dt}(t) \end{pmatrix} = \max_{f\in F} \begin{pmatrix} Q(f) & r(f) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v(t) \\ v_{N+1}(t) \end{pmatrix} \qquad t \in [0,\infty),$$

which is an example of the model to be studied in chapter 6. Note that the collection of matrices

$$\left\{ \begin{pmatrix} Q(f) & r(f) \\ 0 & 0 \end{pmatrix} \;\middle|\; f \in F \right\}$$

is a collection of ML-matrices with the product property.

1.3.2. <u>Risk-sensitive Markov decision processes</u>

Consider once again the discrete-time Markov decision process which has been described in part a of example 1.3.1. Suppose now that a decision maker represents his risk preference by a utility function u that assigns a real number to each of a number of possible outcomes. Thus, if r_i^a is the expected reward when in state i action a is chosen, the value for the decision maker is equal to $u(r_i^a)$; if $v(n)_i$ is the maximal expected return for the n-period problem, then the utility for the decision maker equals $u(v(n)_i)$.

In example 1.3.1, part a, we treated the case in which $u(x) = x$ for each possible return x, which implies risk-indifference. HOWARD AND MATHESON [30] treated the case in which the utility function has the following form:

$$(1.3.7) \qquad u(x) = -(\operatorname{sgn} \gamma) \exp(-\gamma x)$$

where $\gamma \neq 0$ is called the *risk aversion coefficient* and $\operatorname{sgn} \gamma$ denotes the sign of γ. A positive value of γ indicates risk aversion, a negative value indicates risk preference. Note that the function u(.), defined in (1.3.7), is increasing.

It follows that a stream of rewards $r_{i_1}, r_{i_2}, \ldots, r_{i_n}$ has a utility

$$-(\operatorname{sgn} \gamma) \exp(-\gamma(r_{i_1}+r_{i_2}+\ldots+r_{i_n}))$$

Now, let $v(n)_i$ denote the utility of staying in the system for n periods when starting in state i. Using the concept of "certain equivalent", HOWARD AND MATHESON [30] showed:

$$(1.3.8) \qquad v(n)_i = \max_{a \in A} \sum_{j=1}^{N} p_{ij}^a \exp(-\gamma r_{ij}^a)\, v(n-1)_j \qquad i = 1,..,N;\ n \in \mathbb{N}.$$

Defining

$$\tilde{p}_{ij}^a = p_{ij}^a \exp(-\gamma r_{ij}^a)$$

we obtain

$$(1.3.9) \qquad v(n)_i = \max_{a \in A} \sum_{j=1}^{N} \tilde{p}_{ij}^a\, v(n-1)_j \qquad i = 1,..,N;\ n \in \mathbb{N},$$

or, defining f, F, $\tilde{P}(f)$ and v(n) as usual,

$$(1.3.10)\quad v(n) = \max_{f\in F} \tilde{P}(f)\, v(n-1) \qquad n \in \mathbb{N}.$$

1.3.3. Controlled multitype branching processes

Consider a population consisting of individuals of N types, labeled 1,2,...,N, which is observed at time points 0,1,2,... . Each individual lives from one such time point to the next, at which moment he produces a random number of offspring; all these numbers are supposed to be independent. At time t an action is chosen (from a finite set A) for each individual. Different actions may be chosen for different individuals (possibly of the same type).

At each time point the state of the system is described by a vector $(s_1,\ldots,s_N)$, where s_i denotes the number of individuals of type i. Let $p_i(t_1,\ldots,t_N \mid a)$ denote the probability that (as a result of action $a \in A$) one individual of type i produces exactly t_j individuals of type j, $j = 1,2,\ldots,N$. Suppose furthermore, that, if for an individual of type i action a is chosen, a reward r_i^a is earned. It is not hard to verify that this system may be described by a Markov decision process with a countable state space (cf. PLISKA [46]).

Note that, in general, different actions may be selected for different individuals of the same type. A decision rule that selects the same action for all individuals of the same type and such that this selection is independent of the state $(s_1,\ldots,s_N)$ is called *static*. PLISKA [46] showed that the multitype branching process, described above, can be controlled by considering only static decision rules, and a collection of nonnegative $N \times N$ matrices with the product property. Let u_{ij}^a denote the expected number of individuals of type j among the offspring of one individual of type i when action a is chosen. Assume

$$0 \leqq u_{ij}^a < \infty \qquad i,j = 1,\ldots,N;\ a \in A.$$

Let, furthermore, $x(n)_i$ denote the maximal expected return when we start with exactly one individual of type i, no individuals of other types, when only static decision rules are considered, and with n periods to go. Then obviously

$$(1.3.11)\quad x(n)_i = \max_{a \in A} \{r_i^a + \sum_{j=1}^{N} u_{ij}^a\, x(n-1)_j\} \qquad i = 1,..,N;\ n \in \mathbb{N},$$

where $x(0)_i$ is a terminal reward. If we define a static policy f as a function from $\{1,\ldots,N\}$ to A, and U(f) denotes the matrix with entries $u_{ij}^{f(i)}$, while r(f), resp. x(n), are the vectors with components $r_i^{f(i)}$, resp. $x(n)_i$, then we may write

$$(1.3.12)\quad x(n) = \max_{f \in F} \{r(f) + U(f)\, x(n-1)\} \qquad n \in \mathbb{N},$$

where F denotes the set of all static policies. As before, (1.3.12) can be transformed into a recursion of the form (1.2.1):

$$\begin{pmatrix} x(n) \\ 1 \end{pmatrix} = \max_{f \in F} \begin{pmatrix} U(f) & r(f) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(n-1) \\ 1 \end{pmatrix} \qquad n \in \mathbb{N}.$$

It is interesting to note that PLISKA [46] showed that, if both static and nonstatic decision rules are considered, the maximal expected return for an n-period controlled multitype branching process, when starting in state $(s_1,\ldots,s_N)$, and summed over the total number of individuals at the start, is equal to

$$\sum_{i=1}^{N} s_i\, x(n)_i\ .$$

Hence there exists a static decision rule which is optimal. It follows that these problems can be handled either as a Markov decision process with a countable state space or as a more general dynamic programming problem with a set of finite-dimensional nonnegative matrices with the product property.

1.3.4. An input-output system with substitution

An economic system, consisting of N industries (or resources), is controlled at discrete points of time. We assume presence of a sufficient amount of labour (of homogeneous type). Each industry i produces a single commodity, also indicated by i (no joint production is allowed). Furthermore, there exists a finite set A of alternative technologies for each industry i. If industry i chooses technology $a \in A$, we denote by p_{ij}^a the number of units of commodity j (produced in the previous period) which is necessary for the production of one unit of commodity i. Furthermore, ℓ_i^a denotes the amount of labour, necessary for the production of one unit of commodity i, when technology a is chosen.

Let w be the (constant) wage rate and let $c(n)_i$ denote the cost of the production of one unit of commodity i at time point n. We assume $c(0)_i > 0$ for $i = 1,\dots,N$. Since we may expect that each industry is interested in minimizing its costs, we find

$$(1.3.13)\quad c(n)_i = \min_{a\in A} \{w\ell_i^a + \sum_{j=1}^{N} p_{ij}^a\, c(n-1)_j\} \qquad i = 1,\dots,N;\ n \in \mathbb{N}$$

(here we assumed that the production costs of one unit of a commodity is equal to its price on the market).

A technology vector f is a function from $\{1,\dots,N\}$ to A, which specifies for each industry a particular technology. The set of all technology vectors is denoted by F, P(f) denotes the matrix with entries $p_{ij}^{f(i)}$ and $\ell(f)$ the vector with components $\ell_i^{f(i)}$, for all i,j,f. With these definitions, (1.3.13) can be written as

$$(1.3.14)\quad c(n) = \min_{f\in F} \{ w\,\ell(f) + P(f)\ c(n-1)\} \qquad n \in \mathbb{N},$$

where c(n) denotes the vector with components $c(n)_i$, $i = 1,\dots,N$. As before, we find a recursion of the form (1.2.1):

$$\begin{pmatrix} c(n) \\ 1 \end{pmatrix} = \min_{f\in F} \begin{pmatrix} P(f) & w\ell(f) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c(n-1) \\ 1 \end{pmatrix} \qquad n \in \mathbb{N}.$$

Here, we have an example with "max" replaced by "min". These models can be treated in essentially the same way as the one, introduced in section 1.2. The model, described above, is an example of a *Leontief substitution system* (cf. MORISHIMA [41], BURMEISTER AND DOBELL [12]).

1.3.5. A terminating decision process

In BELLMAN [3], a multistage decision process is considered where, at each stage, one has the choice of one of a finite number of actions, 1,2,...,K say. The choice of action $a \in \{1,\dots,K\}$ results in a probability distribution with the following properties:

a. There is a probability p_i^a that one receives i units and the process continues $(i = 1,2,\dots,N)$;

b. There is a probability p_0^a that one receives nothing and the process terminates.

Now let n be a fixed integer and suppose a decision maker wants to maximize the probability that he receives at least a total number of n units before the process terminates. Let u_j denote the maximal probability of obtaining at least j units before termination of the process, then

$$(1.3.15)\qquad u_j = \begin{cases} \max\limits_a \sum\limits_{i=1}^{N} p_i^a u_{j-i} & j > 0 \\ 1 & j \leq 0. \end{cases}$$

Applying a simple transformation, this problem can again be written in the formulation, introduced in section 1.2. For $j = 1,2,\dots,n$ we have

$$\begin{pmatrix} u_j \\ u_{j-1} \\ \vdots \\ u_{j-N+1} \end{pmatrix} = \max_a \begin{pmatrix} p_1^a & \cdots\cdots & p_N^a \\ 1 & 0 \quad\cdots & 0 \\ 0 & 1 \ddots & \vdots \\ \vdots & \ddots \ddots & \vdots \\ 0 & \cdots\; 0 \quad 1 & 0 \end{pmatrix} \begin{pmatrix} u_{j-1} \\ u_{j-2} \\ \vdots \\ u_{j-N} \end{pmatrix},$$

where we start with $(u_0,\dots,u_{1-N}))^T = (1,\dots,1)^T$.
It follows that the decision maker has to solve an n-step sequential decision problem of type (1.2.1).

1.4. Summary of the subsequent chapters

As mentioned already, one of the main objectives of this monograph is to analyze the asymptotic behaviour of dynamic programming recursions (or quasi-linear equations, cf. BELLMAN [3]) of type (1.2.1), based on a set K of nonnegative square matrices with the product property. It will be clear that some insight in the structure of such sets of matrices is fundamental. In chapter 2 we first briefly repeat some well-known results concerning structure and properties of a single nonnegative matrix. A relatively large part of this chapter is devoted to what we will call a generalized eigenvector theory for square nonnegative matrices (cf. ROTHBLUM [48]). Chapters 3,4 and 5 deal with sets of finite-dimensional nonnegative matrices. In chapter 3 it is shown that a particular block-triangular structure exists for sets of nonnegative matrices which is closely related to the behaviour of dynamic programming recursions of type (1.2.1). In chapter 4, convergence results for these recursions are proved under rather special conditions. Indispensable for the analysis in this chapter is a result, recently proved by SCHWEITZER AND FEDERGRUEN [61], concerning geometric convergence in undiscounted Markov decision processes. The original proof of this result is extremely complicated; in appendix 4.A we present a new, relatively simple proof, together with some extensions. This geometric convergence result plays a key role again in chapter 5, where both convergence results for recursions of type (1.2.1) in the most general case are proved, and a theory concerning generalized eigenvectors for sets of nonnegative matrices with the product property is completed. Key words in the analysis are *spectral radius, index* and *generalized eigenvectors*. Brief attention will be paid to estimation methods for these characteristics. Typical for the finite case is that all proofs can be given in a constructive way; in particular it is possible to develop policy iteration methods for the construction of matrices which maximize the "growth" of systems of type (1.2.1).

In chapter 6 we briefly treat the continuous-time analogue of the model, studied in chapters 3,4 and 5. There we deal with a set of ML-matrices with the product property. Special attention is paid to an exponential convergence result for undiscounted continuous-time Markov decision processes (appendix 6.A), which may be viewed as an analogue of the main result of appendix 4.A in the discrete-time case.

Although a theory for sets of nonnegative matrices with the product property has been developed mainly for its usefulness in the analysis of dynamic programming recursions, the results are interesting in themselves; they provide a considerable generalization of the classical Perron-Frobenius theory. In part II (starting with chapter 7) an attempt is made to extend this theory to sets of countably infinite nonnegative matrices. Such an extension is relevant in connection with the study of denumerable Markov decision processes, invariant vectors for sets of nonnegative matrices etc. Chapter 7 is an introductory one in which Markov chains with a countable state space are discussed. *Strong ergodicity* and the *Doeblin condition* are some of the key concepts in the analysis. Although interesting in itself, the results mainly serve to explain and motivate the conditions of the theorems, proved in chapter 8. In that chapter the structure of countably infinite nonnegative matrices is analyzed; it turns out that a beautiful extension of the generalized eigenvector theory, treated in chapter 2, exists. Vere-Jones' *R-theory* (which deals only with irreducible nonnegative matrices of countably infinite dimension) is used as a starting point (cf. VERE-JONES [65], [66]). The results obtained are related to results in potential theory for Markov chains (cf. KEMENY, SNELL AND KNAPP [35]). In chapter 9, finally, we return to sets of (countably infinite) nonnegative matrices and show how results, similar to those in chapter 3 can be obtained. As a by-product of our analysis we obtain a semi-probabilistic interpretation of (generalized) eigenvectors and (generalized) invariant vectors which seems to be new even in the finite case.

1.5. Notational conventions

We shall be concerned with sets of nonnegative matrices with the product property (cf. definition 1.1). Unless stated otherwise all matrices will be square and of a fixed dimension. Throughout part I, N denotes the dimension of these matrices. Motivated by the theory of Markov processes the set $\{1,2,\ldots,N\}$ is called the state space and denoted by S. Part II deals with matrices of countably infinite dimension; in this case $S := \{1,2,\ldots\}$.

Matrices will be denoted by capitals $P,Q,\ldots$, (column) vectors by lower case letters $x,y,u,w,\ldots$. The identity matrix (ones on the diagonal, zeros elsewhere) is denoted by I, the vector with all components equal to one by e. The null matrix is denoted by $\underline{\underline{0}}$, the null vector by $\underline{0}$.

The n-th power of a matrix P is denoted by P^n; $p_{ij}^{(n)}$ denotes the ij-th entry of P^n . Instead of $p_{ij}^{(1)}$ we usually write p_{ij}. P_i denotes the i-th row of P. The i-th component of a vector x is denoted by x_i. We define $P^0 := I$.

As usual $\mathbb{N}$ denotes the set of positive integers, $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\bar{\mathbb{N}}_0 := \mathbb{N}_0 \cup \{\infty\}$. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, $\bar{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$, $\mathbb{R}_0^+ := \mathbb{R}^+ \cup \{0\}$. $\mathbb{R}^k$ denotes the k-fold cartesian product $\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}$ $(k \in \bar{\mathbb{N}})$.

A nonnegative square matrix P is a function from $S \times S$ to $\mathbb{R}_0^+$. If $p_{ij} > 0$ for all $i,j \in S$ the matrix P is called positive. If P is nonnegative (positive) we write $P \geqq \underline{\underline{0}}$ $(P > \underline{\underline{0}})$. We say that P is semi-positive and write $P \geq \underline{0}$, if $P \geqq \underline{0}$ and $P \neq \underline{0}$. Furthermore we write $P \geqq Q$ ($\geq Q$, $> Q$) if $P-Q \geqq \underline{0}$ ($\geq \underline{0}$, $> \underline{0}$). Similar definitions apply to vectors. Instead of "positive vector" often the words "strictly positive vector" will be used.

The transpose of a matrix P is denoted by P^T; the transpose of a (column) vector x is written as x^T. Subsets of the state space S will be denoted by A,B,C,D,... . If $C \subset S$ then by P^C the restriction of the square matrix P to $C \times C$ is denoted. Similarly, x^C is the restriction of the (column) vector x to C. If $\{I(1), I(2),\ldots,I(r)\}$ denotes a partition of the state space S then we often write $P^{(k,\ell)}$ for the restriction of P to $I(k) \times I(\ell)$, $k,\ell = 1,\ldots,r$. Note that $P^{(k,k)} = P^{I(k)}$, $k = 1,\ldots,r$.

If P is a square matrix of finite dimension then the *spectral radius* of P is defined as the modulus of its largest eigenvalue. Throughout this monograph the spectral radius of P is denoted by $\sigma(P)$.

In chapter 6 ML-matrices of finite dimension are considered. An ML-matrix is a square matrix with all its nondiagonal entries nonnegative. The name is adopted from SENETA [52], who uses the word in connection with the work of Metzler and Leontief in mathematical economics.

Lexicographical order symbols are used in several chapters. Let $(x(1),\ldots,x(n))$ and $(y(1),\ldots,y(n))$ be two sequences of real-valued vectors. We say that $(x(1),\ldots,x(n)) \succ (y(1),\ldots,y(n))$ if $x(1) > y(1)$ or if for some $k \in \{1,\ldots,n-1\}$ holds that $x(\ell) = y(\ell)$ for $\ell = 1,2,\ldots,k$ and $x(k+1) > y(k+1)$. Similar definitions hold for $\succeq$, $\succeqq$, $\prec$, $\preceq$ and $\preceqq$.

Let f(t) and g(t) be real-valued (vector) functions such that $g(t) > \underline{0}$ for $t \in \mathbb{R}$. Then $f(t) = \sigma(g(t))$ for $t \to a$ ($a \in \bar{\mathbb{R}}$) if $\lim_{t \to a} (g(t)_i)^{-1} f(t)_i = 0$ for all i. Furthermore $f(t) = O(g(t))$ for $t \to a$ if there exists a constant c such that $|f(t)| \leqq c\,(g(t))$ for t close to a.

The symbol := is used to define concepts. The symbol ~ is used for asymptotic equality; for instance $x(n) \sim y(n)$ for $n \to \infty$ means that for each $\varepsilon > 0$ there exists an integer n_0 such that $(1-\varepsilon)\, y(n) \leqq x(n) \leqq (1+\varepsilon)\, y(n)$ for $n \geqq n_0$. The symbol □ denotes the end of a proof, or the end of the formulation of a proposition, lemma or theorem if no proof is given. Also the end of a definition is marked by □. The Kronecker delta δ_{ij} is defined by $\delta_{ij} := 1$ if $i = j$, $\delta_{ij} := 0$ if $i \neq j$. By $\|\,..\,\|$ the usual sup-norm is denoted.

PART I

FINITE-DIMENSIONAL SYSTEMS

CHAPTER 2

NONNEGATIVE MATRICES: A STRUCTURE ANALYSIS

Any investigation of dynamic programming recursions of the type

$$(1.2.1) \qquad x(n) = \max_{P \in K} P\, x(n-1) \qquad\qquad n = 1,2,\ldots;\ x(0) > \underline{0}$$

with K a set of nonnegative square matrices with the product property, entails the study of products of nonnegative matrices, or, in the case that K contains only one matrix, of powers of that matrix. Clearly, powers of a square nonnegative matrix can be studied by familiar matrix-theoretical methods such as Jordan decomposition. The disadvantage of these methods however is that the nonnegativity of the entries is completely ignored. A graph-theoretical, rather than a matrix-theoretical, approach appears to be the natural answer to this objection (cf. SENETA [52], p. 9-12 and ROTHBLUM [48]). The authors mentioned exploit the idea that a square nonnegative matrix P of dimension N can be represented by a directed graph with N nodes in which a transition from node i to node j is possible if and only if $p_{ij} > 0$ $(i,j = 1,\ldots,N)$.

In this chapter a rather detailed analysis of the structure of a single square nonnegative matrix is presented. We follow the (graph-theoretical) terminology of ROTHBLUM [48], which is strongly motivated by the theory of Markov chains. In section 2.1, a brief review of some well-known definitions and results will be given (most of them without proof) which can be found, for instance, in SENETA [52] or BERMAN AND PLEMMONS [6]. We also give some immediate corollaries which will be needed later. In section 2.2, a fundamental decomposition result for one square nonnegative matrix is presented which describes the hierarchical structure of the underlying graph; this decomposition proves to be extremely useful for the analysis of the behaviour of powers of that matrix (cf. SLADKY [58], ZIJM [76]). Section 2.3

is devoted to an analysis of the structure of so-called generalized eigenvectors, associated with the spectral radius of a square nonnegative matrix, whereas section 2.4 relates these results to more familiar concepts in matrix theory.

The results obtained in this chapter imply some immediate corollaries on the behaviour of the vector x(n), defined by

$$x(n) = P^n x(0) \qquad n \in \mathbb{N} ; \; x(0) > \underline{0},$$

where P denotes a square nonnegative matrix. However, the great advantage of the methods developed here is that they can be extended to sets of nonnegative matrices with the product property, where they yield analogous results for dynamic programming recursions of the type (1.2.1). In order to facilitate the proofs of these extensions, state classifications are introduced in section 2.5, and the results of chapter 2 are reformulated in terms of these state classifications. In fact, state classifications relate in a very precise way the hierarchical structure of the graph, associated with a nonnegative matrix, to the behaviour of its powers; they will prove to play a key role in the forthcoming analysis.

Throughout this chapter P denotes a nonnegative $N \times N$ matrix; the state space S is defined by $S := \{1,2,\dots,N\}$.

2.1. Basic tools and definitions

In this section we briefly review some (mostly well-known) definitions and results concerning the structure of nonnegative matrices.

We start with a definition.

DEFINITION 2.1. We say that state i has *access* to state j under P if $p_{ij}^{(n)} > 0$ for some $n \in \mathbb{N}_0$ $(i,j \in S)$. □

Note that, since $p_{ii}^{(0)} = 1$, state i has always access to state i. Definition 2.1. reflects the idea that the positive-zero configuration of P can be represented by a directed graph. Accordingly, we consider P as a function from $S \times S$ to $\mathbb{R}_0^+$ rather than as a linear operator from $\mathbb{R}^N$ to $\mathbb{R}^N$.

Powers of square matrices are usually studied in terms of their eigenvalue structure (Jordan decomposition). For nonnegative square matrices another approach exists, based on accessibility relations between the states (cf. SENETA [52]). It can be shown that an analysis of the behaviour of powers of a square nonnegative matrix becomes much easier if in the underlying graph any two states have access to each other.

DEFINITION 2.2. P is called *irreducible* if any two states have access to each other. In all other classes we call P *reducible*. □

This definition implies that a square reducible nonnegative matrix can be written in block-triangular form, possibly after a permutation of the states. In other words: using the accessibility relations a hierarchical structure of the state space can be shown.

Irreducible nonnegative matrices can be either periodic or aperiodic. We need the following definition:

DEFINITION 2.3. Let P be irreducible. The *period* d_i of a state i with respect to P is defined by

$$d_i := \text{g.c.d}\ \{n \mid p_{ii}^{(n)} > 0,\ n \in \mathbb{N}\} \qquad i \in S. \qquad \square$$

A proof of the following result can be found in SENETA [52]:

PROPOSITION 2.1. Let P be irreducible. Then all states have the same period, d say, with respect to P. There exists a unique partition $\{C(1),\ldots,C(d)\}$ of S such that $i \in C(k)$ and $p_{ij} > 0$ implies $j \in C(k+1)$ if $k < d$ and $j \in C(1)$ if $k = d$. □

P is said to be *aperiodic* if d = 1, otherwise it is *periodic* with period d. Some authors use the word (a)cyclic instead of (a)periodic.

Powers of square matrices are usually studied by eigenvalue methods. The eigenvalues on the spectral circle, i.e., the eigenvalues with largest absolute value, play a special role, in fact they characterize the first-order asymptotic behaviour of P^n for $n \to \infty$. If P is nonnegative, these eigenvalues and their associated eigenvectors possess very nice properties; these properties are summarized below in the famous *Perron-Frobenius theorem*.

PROPOSITION 2.2. Let P be a square nonnegative matrix and let $\sigma(P)$ denote its *spectral radius*, i.e., $\sigma(P) := \max\ \{\ |\lambda|\ \mid \lambda \text{ an eigenvalue of } P\}$. Then $\sigma(P)$ itself is an eigenvalue of P with which can be associated semi-positive left- and right-eigenvectors. If P is irreducible these eigenvectors are unique up to multiplicative constants; furthermore they can be chosen strictly positive in this case. If P is irreducible, $\sigma(P)$ is simple. If P is irreducible with period d then there exist precisely d eigenvalues $\lambda_1,\ldots,\lambda_d$ with $|\lambda_k| = 1$, namely $\lambda_k = \sigma(P)\exp(i.2\pi k/d)$ for $k = 1,\ldots,d$. These eigenvalues are all simple. □

For a proof we refer to GANTMACHER [26] or SENETA [52]. Note that $\sigma(P)>|\lambda|$ for any eigenvalue $\lambda \neq \sigma(P)$, if P is irreducible and aperiodic.

The existence of strictly positive eigenvectors associated with the spectral radius $\sigma(P)$ of a square nonnegative matrix P immediately provides us with bounds for the vector $x(n) = P^n x(0)$, $n = 1,2,\ldots$, where $x(0)$ is any positive vector. Let u be a strictly positive right-eigenvector, associated with $\sigma(P)$, and choose constants $c_1,c_2 > 0$ with $c_1 u \leqq x(0) \leqq c_2 u$. Then

$$c_1(\sigma(P))^n u \leqq x(n) \leqq c_2(\sigma(P))^n u \qquad n \in \mathbb{N}.$$

This result suggests the question: what nonnegative matrices possess strictly positive eigenvectors. Irreducibility is a sufficient (but certainly not necessary) condition. Before answering this question, we need a few definitions.

DEFINITION 2.4. Let D be a proper subset of S. The restriction P^D of P to D x D is called a *principal minor* of P. □

For principal minors the following result holds.

PROPOSITION 2.3. The spectral radius $\sigma(P')$ of any principal minor P' of P does not exceed the spectral radius $\sigma(P)$ of P. If P is irreducible, we have $\sigma(P') < \sigma(P)$; if P is reducible, then $\sigma(P') = \sigma(P)$ for at least one irreducible principal minor P'. □

For a proof see GANTMACHER [26].

Reducible nonnegative matrices can be written in block-triangular form (possibly after permutation of the states) in such a way that the blocks on the diagonal are irreducible. This defines a partially hierarchical structure in the underlying graph. The irreducible blocks correspond to *classes*. More formally:

DEFINITION 2.5. A *class* of P is a subset C of S such that P^C is irreducible and such that C cannot be enlarged without destroying the irreducibility. C is called *basic* if $\sigma(P^C) = \sigma(P)$, otherwise C is called *nonbasic* (in which case $\sigma(P^C) < \sigma(P)$, according to proposition 2.3). □

The reader may note that P partitions the state space S into classes, C(1),C(2),...,C(n) say. If $P^{(i,j)}$ denotes the restriction of P to C(i)x C(j), i,j = 1,...,n, then, possibly after permutation of the states, P can be written in the following form:

$$(2.1.1)\quad P = \begin{bmatrix} P^{(1,1)} & P^{(1,2)} & \cdots & P^{(1,n)} \\ & P^{(2,2)} & \cdots & P^{(2,n)} \\ & & \ddots & \vdots \\ & & & P^{(n,n)} \end{bmatrix}$$

with $P^{(i,j)} = \underline{\underline{O}}$ for i > j, i,j = 1,...,n. Hence classes can be partially ordered by accessibility relations. We may speak of *access to (from) a class* if there is access to (from) some (or equivalently : any) state in that class.

DEFINITION 2.6. A class C, associated with P, is called *final*, if C has no access to any other class. A class C is called *initial*, if no other class has access to C. □

The question which class has access to which class is fundamental for the investigation of powers of nonnegative matrices. The existence of strictly positive eigenvectors also depends completely on the accessibility structure. We have

PROPOSITION 2.4. P possesses a strictly positive right- (left-) eigenvector if and only if its basic classes are precisely its final (initial) classes. □

For a proof we refer to GANTMACHER [26] again.

Matrices with strictly positive eigenvectors (and especially their powers) have very nice properties as has already been indicated above. The following lemma is fundamental for the analysis in the forthcoming sections. We have:

LEMMA 2.5. Let P have spectral radius $\sigma > 0$, and let there exist a strictly positive right-eigenvector, u say, associated with σ. Then there exists a nonnegative matrix P^*, defined by

$$P^* := \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} \sigma^{-k} P^k.$$

We have $PP^* = P^*P = \sigma P^*$ and $(P^*)^2 = P^*$. Furthermore, $p^*_{ij} > 0$ if and only if j is contained in a basic class of P and i has access to j under P. If the restriction of P to each of its basic classes is aperiodic, then

$$P^* = \lim_{n\to\infty} \sigma^{-n} P^n$$

Finally the matrix $(\sigma I-P+P^*)$ is nonsingular.

PROOF. The matrix $\bar{P}$, defined by

$$(*) \qquad \bar{p}_{ij} = \sigma^{-1} u_i^{-1} p_{ij} u_j \qquad\qquad i,j \in S$$

is *stochastic* (i.e., $\bar{P} \geq \underline{\underline{0}}$, $\bar{P}e = e$). For stochastic matrices the results are well known (cf. KEMENY AND SNELL [34]). By the inverse transformation of (*) all results for $\bar{P}$ are translated into the corresponding results for P. □

The matrix P^* is the projector on the null-space of $(\sigma I-P)$, along the range of $\sigma I-P$. The matrix $(\sigma I-P+P^*)$ is often called the *fundamental matrix*, corresponding to P (KEMENY AND SNELL [34]). Note that the restriction of P^* to each basic class of P is strictly positive.

Even if a nonnegative reducible matrix P does not possess a strictly positive eigenvector, associated with its spectral radius σ, it is easy to understand the fundamental role of σ with respect to the behaviour of P^n. The following characterization is useful.

LEMMA 2.6. Let P have spectral radius σ. Then

a. $\sigma = \lim_{n\to\infty} \| P^n \|^{1/n} = \inf_n \| P^n \|^{1/n}$

b. $\sigma = \sup \{\lambda | \exists\, w > \underline{0} : Pw \geqq \lambda w\} = \inf \{\lambda | \exists\, w > \underline{0} : Pw \leqq \lambda w\}$

c. For each $\lambda > \sigma$ there exists a vector $w > \underline{0}$ such that $Pw \leqq \lambda w$.

PROOF. a. follows from DUNFORD AND SCHWARTZ [21], p.567, b. follows from a. To establish c., take $w = (\lambda I-P)^{-1}e$. □

If $\lambda > \sigma$, $w > \underline{0}$ such that $Pw \leqq \lambda w$ one immediately sees that $\lambda^n w$ dominates $P^n x(0)$ if $x(0) \leqq w$. A vector w, satisfying $w > \underline{0}$ and $Pw \leqq \lambda w$, is often called λ-*subinvariant* (cf. chapter 8 of this monograph) or *strongly excessive* (cf. VAN HEE AND WESSELS [70]); these vectors play an important role in stochastic analysis and in potential theory for Markov chains (cf. KEMENY, SNELL AND KNAPP [35] or HORDIJK [27]).

A more detailed analysis of the role of the spectral radius with respect to powers of a nonnegative matrix will be given in the next section. Accessibility relations between basic (and nonbasic) classes will play a fundamental role again. We now conclude this section with two lemmas which are needed in the sequel.

LEMMA 2.7. Let P be irreducible, let σ be its spectral radius and let $x > \underline{0}$. Then $Px \geqq \sigma x$ implies $Px = \sigma x$. Analogously, $Px \leqq \sigma x$ implies $Px = \sigma x$.

PROOF. Multiplying $Px \geq \sigma x$ by the strictly positive left-eigenvector of P, associated with σ, yields $\sigma > \sigma$, a contradiction. Hence $Px = \sigma x$. Similarly, if $Px \leq x$. □

LEMMA 2.8. Let P have spectral radius σ and suppose $Px \geqq \lambda x$ for some real λ and some real vector x with at least one positive component. Then $\sigma \geqq \lambda$.

PROOF. Let $y := (\lambda I-P)x$. Then $y \leqq \underline{0}$. If $\lambda > \sigma$ then $\lambda I-P$ is nonsingular and

$$x = (\lambda I-P)^{-1}y = \sum_{n=0}^{\infty} \lambda^{-(n+1)}P^n y \leqq \underline{0},$$

which gives a contradiction. Hence $\sigma \geqq \lambda$. □

In the next section, the structure analysis of reducible nonnegative matrices is continued. The concepts introduced there are less familiar; again they have a strongly graph-theoretical interpretation.

2.2. Block-triangular decompositions

In this section a rather specific decomposition result for nonnegative matrices is presented, which is based on the already mentioned hierarchical structure in the underlying graph. Recall that classes are partially ordered by accessibility relations, cf. (2.1.1). We shall show the existence of a particular hierarchical order of basic and nonbasic classes, which is strongly related to the behaviour of powers of the associated nonnegative matrix.

Let P be a square nonnegative matrix (of finite dimension) and let S be the state space. Obviously, there must be a strong connection between the familiar Jordan canonical form of P and the partitioning of S in basic and nonbasic classes. The number of basic classes for instance is precisely equal to the algebraic multiplicity of the eigenvalue $\sigma(P)$. Namely, if C is a basic class of P, then $\sigma(P)$ is a simple eigenvalue of P^C, and furthermore each eigenvalue of P^C is an eigenvalue of P (use (2.1.1)). We shall show that there also exists a relationship between certain chain-structures of basic and nonbasic classes and the size of the Jordan-block, associated with the spectral radius $\sigma(P)$, if $\sigma(P)$ is degenerated (cf. PEASE [44]).

Consider the following example:

Example 2.1.

$$P = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \qquad x(0) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and define $x(n) = P^n\, x(0)$, $n = 1,2,\ldots$. Then

$$x(n) = \begin{pmatrix} x(n)_1 \\ x(n)_2 \end{pmatrix} = 2^n \begin{pmatrix} x_1 + 2n.x_2 \\ x_2 \end{pmatrix}$$

Notice the difference in behaviour between $x(n)_1$ and $x(n)_2$, caused by the fact that state 1 has access to state 2 and p_{11} and p_{22} are both equal to the spectral radius. In terms of classes : the matrix P possesses two basic classes : {1} and {2}, the first one having access to the second one, which implies an asymptotic behaviour of the vector x(n) of the form

$$x(n)_1 \sim n2^n c_1 \qquad\qquad x(n)_2 \sim 2^n c_2$$

where c_1 and c_2 depend on the starting vector x(0).

The next example is an extension of the previous one.

Example 2.2.

$$P = \begin{bmatrix} 2 & 4 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix} \qquad\qquad x(0) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now it is easy to verify that for $n \to \infty$ $x(n) = P^n x(0)$ obeys

$$x(n)_1 \sim n2^n d_1 \qquad x(n)_2 \sim 2^n d_2 \qquad x(n)_3 \sim 2^n d_3$$

where again d_1, d_2 and d_3 are constants, depending on x(0).

Apparently, the presence of a nonbasic class in a "chain between two basic classes" (definition follows below) does not really influence the asymptotic behaviour of x(n) (note that still the first basic class has access to the second one, but now via a nonbasic class). It is this relationship, between the position of basic and nonbasic classes and the behaviour of powers of a nonnegative matrix, that will be studied in this section.

What we need first is a quantitative indication of the position of a class. We start with the definition of a *chain*.

DEFINITION 2.7. By a *chain of classes* of P we mean a collection of classes {C(1), C(2),...,C(n)}, such that $p_{i_k j_k} > 0$ for some pair of states (i_k, j_k) with $i_k \in C(k)$, $j_k \in C(k+1)$, k = 1,2,...,n-1. We say that the chain *starts* with C(1) and *ends* with C(n). The *length of a chain* is the number of basic classes it contains. □

The position of a class is now defined as follows.

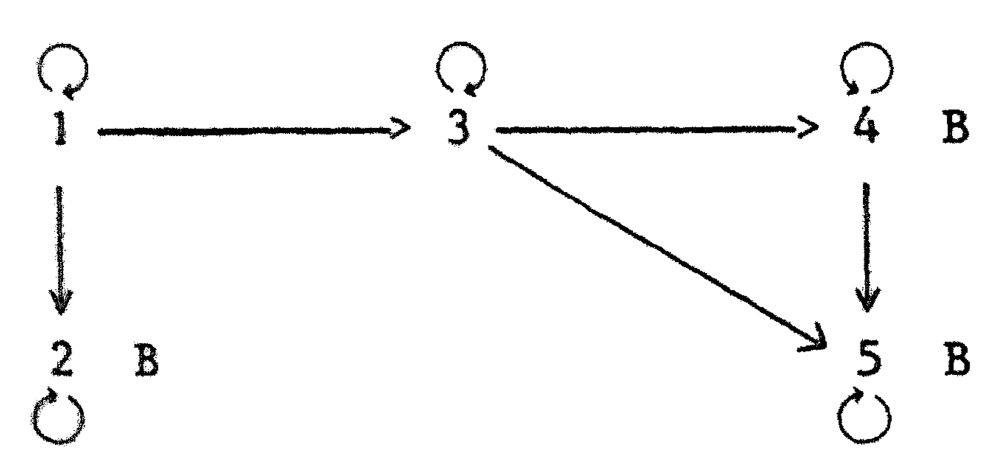
DEFINITION 2.8. The *height (depth)* of a class C of P is the length of the longest chain which ends (starts) with C. □

To illustrate these definitions consider the following example.

Eaxmple 2.3.

$$P = \begin{bmatrix} 1 & 4 & 1 & 0 & 0 \\ & 2 & 0 & 0 & 0 \\ & & 1 & 1 & 4 \\ & & & 2 & 4 \\ & & & & 2 \end{bmatrix}$$

Here the lower triangle of P contains only zeros. Each class of P contains exactly one state. The following graph shows the positions of these classes (B ≡ basic):



Classifying the classes according to their position, we obtain:

	Basic classes	{2}	{4}	{5}	Nonbasic classes	{1}	{3}
Height		1	1	2		0	0
Depth		1	2	1		2	2

Finally,we define the degree of a nonnegative matrix.

DEFINITION 2.9. The *degree* $\nu(P)$ of P is the length of its longest chain. □

In all the examples $\nu(P) = 2$.

The following decomposition result is now easily established.

LEMMA 2.9. Let P have spectral radius σ and degree ν. There exists a partition $\{D(\nu),\ldots,D(1),D(0)\}$ of the state space S such that D(k) is the union of all classes with depth k, for $k = 0,1,\ldots,\nu$. If $P^{(k,\ell)}$ denotes the restriction of P to $D(k) \times D(\ell)$, then $P^{(k,\ell)} = \underline{0}$ for $k < \ell$ $(k,\ell = 0,1,\ldots,\nu)$. After possibly permuting the states we may write

$$(2.2.1) \qquad P = \begin{pmatrix} P^{(\nu,\nu)} & P^{(\nu,\nu-1)} & \cdots & P^{(\nu,1)} & P^{(\nu,0)} \\ & P^{(\nu-1,\nu-1)} & \cdots & P^{(\nu-1,1)} & P^{(\nu-1,0)} \\ & & \ddots & \vdots & \vdots \\ & & & P^{(1,1)} & P^{(1,0)} \\ & & & & P^{(0,0)} \end{pmatrix}$$

We have $\sigma(P^{(k,k)}) = \sigma$ for $k = 1,\ldots,\nu$ and $\sigma(P^{(0,0)}) < \sigma$ (if D(0) is not empty). Furthermore, there exist vectors $u^{(k)} > \underline{0}$ such that

$$(2.2.2) \qquad P^{(k,k)}\, u^{(k)} = \sigma u^{(k)} \qquad\qquad k = 1,\ldots,\nu.$$

PROOF. Since the degree of P is ν, there exist classes with depth k, for $k = 1,\ldots,\nu$, and possibly classes with depth zero (nonbasic classes which do not have access to any basic class). Obviously, a class of depth k has no access to a class of depth $\ell > k$, hence $P^{(k,\ell)} = \underline{0}$ for $k < \ell$. Basic classes with depth k do not have access to any other class of depth k, whereas nonbasic classes with depth k must have access to at least one basic class of depth k. Furthermore, $\sigma(P^{(k,k)}) = \sigma$ for $k = 1,\ldots,\nu$ and $\sigma(P^{(0,0)}) < \sigma$ by proposition 2.3. and definition 2.5. Proposition 2.4. now implies the existence of vectors $u^{(k)} > \underline{0}$ such that (2.2.2) holds for $k = 1,\ldots,\nu$. □

Remark. Note that each state in D(k) has access to some state in D(k-1), for $k = \nu, \nu-1,\ldots,2$.

DEFINITION 2.10. The partition $\{D(\nu), D(\nu-1),\ldots,D(1), D(0)\}$, such that D(k) contains all classes with depth k $(k = \nu, \nu-1,\ldots,1,0)$, is called the *principal partition* of S with respect to P. □

Consider, once again, the matrix of example 2.3. We find $D(2)=\{1,3,4\}$, $D(1) = \{2,5\}$, $D(0) = \emptyset$. In other words, after permuting the states we have the following structure:

$$P = \begin{pmatrix} 1 & 1 & 0 & 4 & 0 \\ & 1 & 1 & 0 & 4 \\ & & 2 & 0 & 4 \\ & & & 2 & 0 \\ & & & & 2 \end{pmatrix} = \begin{pmatrix} P^{(2,2)} & P^{(2,1)} \\ & P^{(1,1)} \end{pmatrix}$$

Both $P^{(2,2)}$ and $P^{(1,1)}$ possess a strictly positive right-eigenvector, associated with eigenvalue 2. If we define an "aggregated" state space $S' = \{1',2'\}$ with $1' = D(1)$ and $2' = D(2)$, and an "aggregated" matrix P' on $S' \times S'$ by

$$P' = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix}$$

then the behaviour of powers of P' is in essence the same as the behaviour of powers of P. Note that P' has been investigated in example 1; there we saw that the position of the basic classes determined the asymptotic behaviour of $x(n) = P^n x(0)$ as $n \to \infty$. More generally we have

<u>LEMMA 2.10.</u> Let P have spectral radius σ and degree ν and let $\{D(\nu), D(\nu-1),\ldots,D(1),D(0)\}$ be the principal partition of S with respect to P. Choose $x(0) > \underline{0}$ and let $x(n) = P^n x(0)$, $n = 1,2,\ldots$. Then there exist constants c_1, c_2 and vectors $u^{(k)} > \underline{0}$, satisfying (2.2.2), such that for $n \in \mathbb{N}$

$$(*) \qquad c_1 u_i^{(k)} \leqq \binom{n}{k-1}^{-1} \sigma^{-n}\, x(n)_i \leqq c_2 u_i^{(k)} \qquad i \in D(k);\ k = 1,\ldots,\nu,$$

and

$$\lim_{n\to\infty} \sigma^{-n}\, x(n)_i = 0 \qquad i \in D(0). \qquad \square$$

The proof of lemma 2.10 is postponed until section 2.3, where it follows immediately from a general result concerning the structure of generalized eigenvectors, associated with the spectral radius of a square nonnegative matrix. For a direct proof of lemma 2.10., see ZIJM [76].

Lemma 2.10. shows the desired relationship between the behaviour of $x(n) = P^n x(0)$ and the position of the classes of P. The concept of "depth of a class C" appears to play a key role, which means that the maximal number of basic classes which can be found in a chain, starting with C, is relevant. A relationship, completely analogous to (*), exists between the height of a class and the behaviour of $x(0)^T P^n$, restricted to that class, as $n \to \infty$ (note that the depth of C with respect to P is equal to the height of C with respect to P^T).

We conclude this section with an extension of lemmas 2.9. and 2.10. which will be needed in the sequel. Note that the concepts "basic class", "depth" and "degree" are defined with respect to $\sigma(P)$. However, if $D(0) \neq \emptyset$ and $\sigma(P^{(0,0)}) \neq 0$, we may repeat the procedure, i.e. decompose $P^{(0,0)}$ in exactly the same way. Continuing in this way we finally obtain

<u>LEMMA 2.11.</u> Let P be a square nonnegative matrix. There exist an integer $r = r(P)$ and a partition $\{I(1), I(2), \ldots, I(r)\}$ of the state space S, such that the following properties hold:

a. Let $P^{(k,\ell)}$ denote the restriction of P to $I(k) \times I(\ell)$. Then $P^{(k,\ell)} = \underline{\underline{0}}$ if $k > \ell$; $k,\ell = 1,\ldots,r$.

b. For $k \leqq \ell$, we have $\sigma(P^{(k,k)}) \geqq \sigma(P^{(\ell,\ell)})$ with equality only if each state in $I(k)$ has access to some state in $I(\ell)$, $k,\ell = 1,\ldots,r$.

c. There exist strictly positive vectors $u^{(k)}$ such that

$$P^{(k,k)} u^{(k)} = \sigma(P^{(k,k)}) u^{(k)} \qquad k = 1,2,\ldots,r. \tag{2.2.3}$$

d. Choose $x(0) > \underline{0}$ and let $x(n) = P^n x(0)$ for $n = 1,2,\ldots$. For each $k \in \{1,2,\ldots,r\}$ define the integer t_k by

$$t_k := \min \{\ell \mid 0 < \ell \leqq r-k,\ \sigma(P^{(k+\ell,k+\ell)}) < \sigma(P^{(k,k)})\}$$

and $t_k := r-k+1$ if the minimum does not exist.

Then, if $\sigma(P^{(r,r)}) \neq 0$, there exist positive constants c_1, c_2, depending on $x(0)$ only, such that

$$c_1 u_i^{(k)} \leqq \binom{n}{t_k - 1}^{-1} \sigma(P^{(k,k)})^{-n} x(n)_i \leqq c_2 u_i^{(k)}$$

for $i \in I(k)$, $k = 1,\ldots,r$ and $n \in \mathbb{N}$. □

DEFINITION 2.11. The partition $\{I(1),I(2),\ldots,I(r)\}$, discussed in lemma 2.11 is called the *spectral partition* of S with respect to P.

In the next section we discuss generalized eigenvectors, associated with the spectral radius of a square nonnegative matrix. It will appear that a strong relationship exists with the decomposition result of lemma 2.9.

2.3 Generalized eigenvectors

In the preceding sections we have seen that nonnegative matrices with strictly positive eigenvectors have nice properties, in particular with respect to their powers (note that for these matrices the integer $r(P)$, defined in lemma 2.11, is equal to one). One of the most important cases where such a strictly positive eigenvector does not exist is the case with the degree of P larger than one. In this case the spectral radius $\sigma(P)$ is degenerated as an eigenvalue (i.e., the number of independent eigenvectors associated with $\sigma(P)$ is smaller than its algebraic multiplicity), which implies the existence of *generalized eigenvectors* (cf. PEASE [44]). In this section, the structure of these generalized eigenvectors is studied and related to accessibility relations between basic classes and so to the decomposition result of lemma 2.9.

Let us start with a formal definition.

DEFINITION 2.12. Let P have spectral radius σ and for $k \in \mathbb{N}$ let $N^k(P)$ be the null space of $(P-\sigma I)^k$. The *index* $\eta(P)$ of P is the smallest nonnegative integer k such that $N^k(P) = N^{k+1}(P)$. □

If P is an N x N matrix with spectral radius σ and index η, then $\eta \leqq N$ and

$$N^1(P) \subsetneqq N^2(P) \subsetneqq \ldots\ldots \subsetneqq N^\eta(P) = N^k(P) \qquad \text{for } k \geqq \eta$$

(compare e.g. DUNFORD AND SCHWARTZ [21], p.556). The elements of $N^\eta(P)$ are called *generalized eigenvectors*. If $x \in N^k(P) \setminus N^{k-1}(P)$, we call x a *generalized eigenvector of order k*.

ROTHBLUM [48] showed that generalized eigenvectors, associated with the spectral radius of a square nonnegative matrix, have nice properties. Before discussing his results we consider once again the matrix of example 2.1.

Example 2.1. (continued). Define P, w(1) and w(2) by

$$P = \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} , \; w(1) = \begin{pmatrix} 4 \\ 0 \end{pmatrix} , \; w(2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

Then, clearly,

$$P\,w(2) = 2w(2) + w(1) \; , \quad P\,w(1) = 2w(1) \; .$$

Hence w(2) is a generalized eigenvector of order 2. Note that w(2) is strictly positive.

One may wonder whether in general the generalized eigenvector of highest order can be chosen strictly positive. It is intuitively clear that the position of the zeros in any generalized eigenvector must be related to the block-triangular structure, presented in lemma 2.9. The following result can be established (ROTHBLUM [48]).

THEOREM 2.12. Let P have spectral radius σ and degree ν. Then for $k = 1,..,\nu$ there exist generalized eigenvectors w(k) of order k such that

$$(2.3.1) \qquad P\,w(\nu) = \sigma\,w(\nu)$$

$$(2.3.2) \qquad P\,w(k) = \sigma\,w(k) + w(k+1) \qquad\qquad k = 1,\ldots,\nu-1.$$

Let $\{D(\nu), D(\nu-1),\ldots,D(1), D(0)\}$ be the principal partition of S with respect to P. Then the vectors w(k) can be chosen in such a way that for $k = 1,\ldots,\nu$

$$w(k)_i > 0 \qquad\qquad \text{for } i \in \bigcup_{\ell=k}^{\nu} D(\ell),$$

and

$$w(k)_i = 0 \qquad\qquad \text{for } i \in \bigcup_{\ell=0}^{k-1} D(\ell).$$

PROOF. As before, let $P^{(k,\ell)}$ denote the restriction of P to $D(k) \times D(\ell)$, and for $k = 1,2,\ldots,\nu$ define

$$R(k) := \begin{bmatrix} P^{(k,k)} & P^{(k,k-1)} & \cdots & P^{(k,1)} & P^{(k,0)} \\ & P^{(k-1,k-1)} & \cdots & P^{(k-1,1)} & P^{(k-1,0)} \\ & & \ddots & \vdots & \vdots \\ & & & P^{(1,1)} & P^{(1,0)} \\ & & & & P^{(0,0)} \end{bmatrix}$$

Noth that $R(\nu) = P$. We shall prove, by induction with respect to k, that for $k = 1,2,\ldots,\nu$ there exists a sequence of generalized eigenvectors $y(1),\ldots,y(k)$ such that

$$(2.3.3)\qquad \begin{cases} R(k)\ y(\ell) = \sigma\ y(\ell) + y(\ell+1) & \ell = 1,\ldots,k-1 \\ R(k)\ y(k) = \sigma\ y(k)\,, \end{cases}$$

with $y(\ell)_i = 0$ for $i \in \bigcup_{n=0}^{\ell-1} D(n)$, $y(\ell)_i > 0$ for $i \in \bigcup_{n=\ell}^{k} D(n)$.

By lemma 2.9 there exists a vector $y(1)$, defined on $D(0) \cup D(1)$, such that

$$R(1)\ y(1) = \sigma\ y(1),$$

with $y(1)_i = 0$ for $i \in D(0)$, $y(1)_i > 0$ for $i \in D(1)$.

Suppose, there exist vectors $x(1), x(2),\ldots,x(k)$, defined on $\bigcup_{n=0}^{k-1} D(n)$, such that

$$(2.3.4)\qquad R(k-1)\ x(\ell) = \sigma\ x(\ell) + x(\ell+1) \qquad \ell = 1,\ldots,k-1,$$

with $x(\ell)_i = 0$ for $i \in \bigcup_{n=0}^{\ell-1} D(n)$ and $x(\ell)_i > 0$ for $i \in \bigcup_{n=\ell}^{k-1} D(n)$ $(\ell = 1,..,k)$

We want to find vectors $y(1), y(2),..,y(k)$, defined on $\bigcup_{n=0}^{k} D(n)$, such that (2.3.3) holds. It seems natural to take

$$y(\ell)_i := x(\ell)_i \qquad i \in \bigcup_{n=0}^{k-1} D(n),\ \ell = 1,\ldots,k.$$

Note that $x(k) = \underline{0}$. If $x(n)^{(\ell)}$ denotes the restriction of $x(n)$ to $D(\ell)$ for $n = 1,\dots,k$ and $\ell = 0,1,\dots,k-1$, and $y(n)^{(k)}$ the restriction of $y(n)$ to $D(k)$ for $n = 1,\dots,k$, then (2.3.3) reduces to

$$
(2.3.5)\quad \begin{cases}
P^{(k,k)}y(k)^{(k)} & = \sigma\, y(k)^{(k)} \\
P^{(k,k)}y(k-1)^{(k)} + P^{(k,k-1)}x(k-1)^{(k-1)} & = \sigma\, y(k-1)^{(k)} + y(k)^{(k)} \\
\vdots \qquad\qquad \vdots & \quad \vdots \qquad\qquad \vdots \\
P^{(k,k)}y(1)^{(k)} + \sum_{\ell=1}^{k-1} P^{(k,\ell)}x(1)^{(\ell)} & = \sigma\, y(1)^{(k)} + y(2)^{(k)}.
\end{cases}
$$

Note that $P^{(k,k)}$ possesses a strictly positive eigenvector, associated with σ. By lemma 2.a.1 in the appendix to this chapter, there exists a solution $\{y(k)^{(k)},\dots,y(1)^{(k)}\}$ of (2.3.5) with

$$y(k)^{(k)} = (P^{(k,k)})^{*}\, P^{(k,k-1)}\; x(k-1)^{(k-1)}.$$

Since the restriction of $(P^{(k,k)})^{*}$ to the basic classes of $P^{(k,k)}$ is strictly positive (lemma 2.5) and since each state in $D(k)$ has access to some state in $D(k-1)$, it follows from $x(k-1)^{(k-1)} > \underline{0}$ that

$$y(k)^{(k)} > \underline{0}\ .$$

We now have found a solution $\{y(1),\dots,y(k)\}$ of (2.3.3). Note that, if $\{y(1),\dots,y(k)\}$ satisfies (2.3.3), then this holds also for $\{w(1),\dots,w(k)\}$, defined by

$$
\begin{cases}
w(k) = y(k) & \\
w(\ell) = y(\ell) + \alpha\, w(\ell+1) & \ell = k-1,k-2,\dots,1;\ \alpha\in\mathbb{R}\,.
\end{cases}
$$

Since $y(k)^{(k)} > \underline{0}$, we can choose α so large, that $w(\ell)_i > 0$ for $i \in D(k)$, $\ell = 1,\dots,k$. This proves the desired results for $R(k)$. The proof can now be completed by induction. □

<u>COROLLARY</u>. Let P have spectral radius σ and index ν and suppose that each

nonbasic class has access to some basic class. Then there exists a generalized eigenvector $w(\nu)$ of order ν, associated with σ, such that $w(\nu) > \underline{0}$. □

It is easily verified that the generalized eigenvector of order 2, associated with the matrix P of example 3 in section 2.2, can be chosen strictly positive; take e.g. $w(2) = e$.

Theorem 2.12 enables us to show a relationship between the position of basic and nonbasic classes of P and the behaviour of the powers of P. The following result holds.

THEOREM 2.13. Let P have spectral radius σ and degree ν, let $\{D(\nu), D(\nu-1),\ldots, D(1), D(0)\}$ be the principal partition of S with respect to P, and let $\{w(\nu), w(\nu-1),\ldots,w(1)\}$ be defined as in theorem 2.12. Choose $x(0) > \underline{0}$ and let $x(n) := P^n x(0)$, $n = 1,2,\ldots$. Then there exist constants $c_1, c_2 > 0$, such that

$$c_1 \sum_{k=1}^{\nu} \binom{n}{k-1} \sigma^{n-k+1} w(k)_i \leqq x(n)_i \leqq c_2 \sum_{k=1}^{\nu} \binom{n}{k-1} \sigma^{n-k+1} w(k)_i \qquad i \in S \setminus D(0),\ n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \sigma^{-n} x(n)_i = \underline{0} \qquad i \in D(0).$$

PROOF. Let, as before, $P^{(k,\ell)}$ denote the restriction of P to $D(k) \times D(\ell)$ and $w(k)^{(\ell)}$ the restriction of $w(k)$ to $D(\ell)$, for all k,ℓ. Hence $P^{(k,\ell)} = \underline{\underline{0}}$ for $k < \ell$ and $w(k)^{(\ell)} = \underline{0}$ for $k > \ell$. Since $\sigma(P^{(0,0)}) < \sigma$, there exist (by lemma 2.6.c) a nonnegative real number $\lambda < \sigma$ and a vector $w(\lambda) > \underline{0}$, defined on $D(0)$, such that

$$P^{(0,0)} w(\lambda) \leqq \lambda w(\lambda).$$

Choose $w(\lambda)$ such that $P^{(k,0)} w(\lambda) \leqq w(k)^{(k)}$ for $k = 1,\ldots,\nu$ and let $c > 0$ be chosen such that $x(0)_i \leqq c\, w(1)_i$ for $i \in S\setminus D(0)$ and $x(0)_i \leqq c\, w(\lambda)_i$ for $i \in D(0)$. Then, by induction, it is easily shown that

$$x(n)_i \leqq c\lambda^n w(\lambda)_i \qquad\qquad i \in D(0)$$

$$x(n)_i \leqq c\{\sum_{k=1}^{\nu} \binom{n}{k-1} \sigma^{n-k+1} w(k)_i + \sigma^{n-1}(1-\lambda\sigma^{-1})^{-1} w(\ell)_i\}$$

$$i \in D(\ell),\ \ell = 1,\dots,\nu.$$

The other inequalities are proved similarly. By choosing approciate c_1 and c_2 the theorem follows. □

Lemma 2.10 in section 2.2 is an immediate corollary of theorem 2.13.

Theorems 2.12 and 2.13 can be extended in the same way as lemmas 2.9 and 2.10 (which were extended to lemma 2.11) by decomposition of the matrix $P^{(0,0)}$ if $D(0) \neq \emptyset$. Details are left to the reader.

The results in sections 2.2 and 2.3 and particularly theorem 2.12 can be viewed as extensions of the Perron-Frobenius theorem. Instead of using familiar matrix-theoretical arguments, we preferred another approach, leading to these results. The advantage of a characterization of nonnegative eigenvectors and generalized eigenvectors in terms of classes and accessibility relations between these classes is that it can be extended to sets of nonnegative matrices (cf. chapters 3 until 5). Once having proved these extensions, convergence results for dynamic programming recursions of the type (1.2.1) are easily established. These convergence results then also hold for the "one-matrix" case; we do not discuss them here, since they follow immediately from results in the forthcoming chapters.

2.4. Some further results

In this section, we relate some of the preceding results to concepts in more familiar matrix theory. We saw already that the number of basic classes of a square nonnegative matrix P is equal to the algebraic multiplicity of its eigenvalue $\sigma(P)$. In this section it is shown that the degree $\nu(P)$ of a nonnegative matrix P is equal to its index $\eta(P)$. Moreover, a basis, consisting of nonnegative vectors only, for the algebraic eigenspace $N^{\eta(P)}(P)$ is constructed. Some of these results have been proved in ROTHBLUM [48]; however, the proofs given here are completely different. The results are not used in the analysis in the forthcoming chapters; we merely state them for completeness.

Theorem 2.12 implies the existence of a generalized eigenvector of order $\nu = \nu(P)$ for a nonnegative matrix P with spectral radius $\sigma(P)$ and degree $\nu(P)$ (i.e. the vector $w(1)$, defined by (2.3.2)). It follows from definition 2.12 and theorem 2.12 that $\nu(P) \leqq \eta(P)$, the index of P. It is not hard to show that $\eta(P) = \nu(P)$. First we need

<u>LEMMA 2.14</u>. Let P have spectral radius σ and let $\{x(1),\ldots,x(m)\}$ be a set of generalized eigenvectors such that

$$(2.4.1)\quad \begin{cases} P\,x(m) = \sigma\,x(m) \\ P\,x(k) = \sigma\,x(k) + x(k+1) & k = 1,\ldots,m-1. \end{cases}$$

Let $D(0) \subset S$ be the union of all classes with depth zero. Suppose $D(0) \neq \emptyset$. Then

$$x(k)_i = 0 \qquad i \in D(0),\ k = 1,\ldots,m.$$

<u>PROOF</u>. We proceed by backward induction. Note that $D(0) \neq \emptyset$ implies $\sigma > 0$. If $P\,x(m) = \sigma\,x(m)$ then

$$\sum_{j\in D(0)} p_{ij}\,x(m)_j = \sigma\,x(m)_i \qquad i \in D(0),$$

which implies $x(m)_i = 0$ for $i \in D(0)$, since $\sigma(P^{(0,0)}) < \sigma$. Suppose $x(k)_i = 0$ for $i \in D(0)$ and $k = m, m-1,\ldots, n+1$ (with $n \geqq 1$). Then $P\,x(n) = \sigma\,x(n) + x(n+1)$ implies

$$\sum_{j\in D(0)} p_{ij}\,x(n)_j = \sigma\,x(n)_i \qquad i \in D(0).$$

Hence $x(n)_i = 0$ for $i \in D(0)$. □

Now, let P, σ and $\{x(1),\ldots,x(m)\}$ be defined as in lemma 2.14. From (2.4.1) it easily follows that

$$(2.4.2)\quad \lim_{n\to\infty} \binom{n}{m-1}^{-1} \sigma^{m-n-1}\, P^n\, x(1) = x(m).$$

Using this, we shall prove :

THEOREM 2.15. Let P have spectral radius σ, degree ν and index η. Then $\eta = \nu$.

PROOF. Suppose $\eta > \nu$ (recall that $\eta < \nu$ is impossible). Let $\{x(1), x(2),\ldots,x(\nu), x(\nu+1)\}$ be a set of vectors satisfying (2.4.1) for $m = \nu+1$. By lemma 2.14 we may choose a constant $c > 0$ such that

$$-cx(1) \leqq x(1) \leqq cx(1) \tag{2.4.3}$$

with w(1) defined as in theorem 2.12. Since w(1) is a generalized eigenvector of order ν, we have

$$\lim_{n\to\infty} \binom{n}{\nu}^{-1} \sigma^{-n+\nu} P^n w(1) = \underline{0}.$$

Hence, by (2.4.3) and (2.4.2),

$$\lim_{n\to\infty} \binom{n}{\nu}^{-1} \sigma^{-n+\nu} P^n x(1) = x(\nu+1) = \underline{0}.$$

The conclusion is : $\eta = \nu$. □

Finally, we show how a basis, consisting of semi-positive generalized eigenvectors, can be constructed for the algebraic eigenspace $N^{\eta(P)}(P)$. Recall that the number of basic classes of a square nonnegative matrix P is equal to the algebraic multiplicity α of its eigenvalue $\sigma(P)$. Choose one particular basic class C and let B be the set of states which have access to C. Note that $C \subset B$, $\sigma(P^B) = \sigma(P)$ and $\nu(P^B) = k$, where k denotes the height of C with respect to P. According to theorem 2.12, there exists a generalized eigenvector u^B of order k for P^B, associated with $\sigma(P^B) = \sigma(P)$, which is strictly positive on B (each class in B has access to C). The vector u, defined by

$$u_i = \begin{cases} u_i^B & i \in B \\ 0 & i \in S\backslash B \end{cases}$$

is then a semi-positive generalized eigenvector of order k for P, with respect to $\sigma(P)$. Repeating this procedure for all basic classes of P, we obtain a set of α semi-positive generalized eigenvectors. The proof of the fact that these generalized eigenvectors are independent (and hence form a basis) is trivial.

2.5. State classifications

Before turning to sets of nonnegative matrices with the product property we introduce the concept of state classifications, which enables us to simplify the proofs in the next chapter considerably. We start with two definitions.

DEFINITION 2.13. Let C be a class of states, associated with P. The *accessibility set A(C)* is then the set of all states to which C has access. □

DEFINITION 2.14. We say that a class C associated with P has *growth rate* ρ and *growth index* k, if $\sigma(P^A) = \rho$ and $\nu(P^A) = k$, where A is the accessibility set of C. □

Note that, if $\sigma(P^A) = \sigma(P)$, then the growth index k of C is precisely equal to its depth. In fact, in lemma 2.11 the classes of P are ordered according to their growth rate and growth index. The reader may verify that a class in I(k) has growth rate $\sigma(P^{(k,k)})$ and growth index t_k (compare the definitions of I(k), $P^{(k,k)}$ and t_k in lemma 2.11).

A state classification is now defined as follows.

DEFINITION 2.15. A state $i \in S$ has *growth rate* ρ and *growth index* k, with respect to P, if this is so for the class that contains i.
Notation: $(\sigma_i(P),\ \nu_i(P)) = (\rho,k)$. □

Hence, in lemma 2.11, we have $(\sigma_i(P),\ \nu_i(P)) \succ (\sigma_j(P),\ \nu_j(P))$ for $i \in I(m)$, $j \in I(n)$ with $m < n \leq r$ (here $\succ$ means lexicographically greater).

The following assertions are easily verified.

P possesses a strictly positive generalized eigenvector of highest order if and only if $\sigma_j(P) = \sigma(P)$ for all $j \in S$. There exists a strictly positive right-eigenvector, associated with $\sigma(P)$, if and only if $(\sigma_j(P),\ \nu_j(P)) = (\sigma(P),\ 1)$ for all $j \in S$.

The names "growth rate" and "growth index" are suggested by the following : let P be a square nonnegative matrix, x(0) be a strictly positive vector and let $x(n) := P^n x(0)$. Then

$$\sigma_j(P) = \sup \{\lambda \mid \lambda > 0 \; ; \; \limsup_{n\to\infty} \lambda^{-n} x(n)_j > 0\},$$

$$\nu_j(P) = \sup \{k \mid k \in \mathbb{N} \; ; \; \limsup_{n\to\infty} \binom{n}{k-1}^{-1} (\sigma_j(P))^{-n} x(n)_j > 0\}$$

(compare lemma 2.11).

In the next chapter, state classifications will be used in order to develop iteration procedures for a set K of nonnegative matrices with the product property. The objective then is to find a matrix $\hat{P}$ such that

$$(\sigma_j(\hat{P}), \nu_j(\hat{P})) \succeq (\sigma_j(P), \nu_j(P))$$

for all $P \in K$, i.e., a matrix that maximizes the growth of the dynamic system (1.2.1).

Appendix 2.A. A fundamental set of equations.

In this appendix we treat a technical detail, needed for the proof of lemma 2.12.

LEMMA 2.a.1. Let P have spectral radius σ and a strictly positive right-eigenvector associated with σ. Let, furthermore $(r(1),\dots,r(k-1))$ be a given sequence of vectors. Then there exists a solution $(y(1),\dots,y(k))$ of the set of equations

$$(2.a.1)\quad \begin{cases} Py(k) = \sigma\, y(k) \\ Py(k-1) + r(k-1) = \sigma\, y(k-1) + y(k) \\ \quad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ Py(1) + r(1) = \sigma\, y(1) + y(2) \end{cases}$$

PROOF. By lemma 2.5, P^* exists and $(\sigma I-P+P^*)$ is non-singular. Iteration of the first equation yields

$$\sigma^{-n} P^n y(k) = y(k) \qquad n \in \mathbb{N},$$

so, by definition of P^*,

$$(2.a.2)\quad P^* y(k) = y(k).$$

Multiplying both sides of the equalities (2.a.1) with P^*, we obtain

$$(2.a.3)\quad P^* r(\ell-1) = P^* y(\ell). \qquad \ell = 2,\dots,k.$$

Now, add the equation

$$(2.a.4)\quad P^* y(1) = \underline{0}.$$

Then, by combination of (2.a.1), (2.a.2), (2.a.3) and (2.a.4) it is easy to verify that a unique solution $(y(1),\dots,y(k))$ of (2.a.1) - (2.a.4) exists, namely

$$
\begin{aligned}
y(k) &= P^* \, r(k-1) \\
y(k-1) &= (\sigma I - P + P^*)^{-1} \, (r(k-1) + P^* \, r(k-2) - y(k)) \\
&\;\;\vdots \\
y(2) &= (\sigma I - P + P^*)^{-1} \, (r(2) + P^* \, r(1) - y(3)) \\
y(1) &= (\sigma I - P + P^*)^{-1} \, (r(1) - y(2)).
\end{aligned}
$$

Hence, certainly a (not necessarily unique) solution of (2.a.1) exists. □

CHAPTER 3

SETS OF NONNEGATIVE MATRICES: BLOCK-TRIANGULAR STRUCTURES

In this chapter we deal with a finite set K of nonnegative $N \times N$ matrices with the product property. In other words: for each $i \in \{1,2,..,N\}$ there exists a finite set, A_i say, of N-dimensional nonnegative row vectors and K is the set of all $N\times N$-matrices P with $P_i \in A_i$, $i = 1,\dots,N$ (cf. definition 1.1). The set $\{1,\dots,N\}$ is called the state space and denoted by S.

In particular, we are concerned with the properties of a nonlinear mapping which often appears in a dynamic programming context and which for each vector $x \in \mathbb{R}^N$ is defined by

$$(3.0.1) \qquad x \to \max_{P\in K} Px$$

From definition 1.1 it follows that, for each $x \in \mathbb{R}^N$, there exists a matrix $\tilde{P} = \tilde{P}(x) \in K$ such that

$$\tilde{P}x = \max_{P\in K} Px,$$

a property, which is usually referred to as the *optimal choice property* (cf. SENETA [52]). The main objective of this chapter is to show that a decomposition result, similar to the one presented in lemma 2.11, exists for the nonlinear mapping, defined by (3.0.1). This result will prove to be fundamental for the whole monograph. It will be exploited in chapter 4 in order to analyze the asymptotic behaviour of the dynamic programming recursion

$$(3.0.2) \qquad x(n) = \max_{P\in K} Px(n-1) \qquad n = 1,2,\dots;\ x(0) > \underline{0},$$

under some restrictions on the set K. In chapter 5 the most general case will be treated. Simultaneously, a generalized eigenvector theory, similar to the results of theorem 2.12, will be developed for the nonlinear mapping, defined by (3.0.1).

Intuitively, it will be clear that matrices with maximal spectral radius and maximal index will play a special role in recursions of the type (3.0.2). Recalling the definitions of growth rate and growth index (cf. section 2.5) and the results of lemma 2.11, one may even conjecture that matrices $\hat{P} \in K$, for which

$$(3.0.3) \qquad (\sigma_i(\hat{P}),\ \nu_i(\hat{P})) \succeq (\sigma_i(P),\ \nu_i(P)) \qquad\qquad i \in S,\ P \in K$$

maximize the asymptotic growth of the vector x(n), defined by (3.0.2), as n tends to infinity. In section 3.2, we show how such matrices can be found in a constructive way (by developing an iterative procedure). In order to keep the exposition transparent (and to demonstrate the techniques that will be used), we start with the simple case where all matrices are irreducible (section 3.1). The results of this chapter mainly stem from SLADKY [58] and ZIJM [75].

3.1. Sets of irreducible nonnegative matrices

Since, by the Perron-Frobenius theorem, a square irreducible nonnegative matrix P possesses a strictly positive right eigenvector u, associated with its spectral radius σ, it is possible to establish bounds for the vector x(n), defined by

$$x(n) = Px(n-1) \qquad\qquad n \in \mathbb{N},\ x(0) > \underline{0}.$$

This has been observed already in section 2.1; we saw that

$$c_1\sigma^n u \leqq x(n) \leqq c_2\sigma^n u \qquad\qquad n \in \mathbb{N},$$

if c_1, c_2 are chosen such that

$$c_1 u \leqq x(0) \leqq c_2 u.$$

The question we want to investigate in this section is whether similar results can be obtained for dynamic programming recursions of the type

$$(3.0.2) \qquad x(n) = \max_{P \in K} Px(n-1) \qquad\qquad n \in \mathbb{N}\ ;\ x(0) > \underline{0},$$

if we assume that K consists of irreducible nonnegative matrices only. In other words: one may wonder whether there exists a strictly positive vector u and a nonnegative real number σ such that

$$\max_{P \in K} Pu = \sigma u$$

If the answer is affirmative, then immediately the question arises whether a generalized eigenvector theory, analogous to the theory presented in chapter 2, can be developed for sets of reducible nonnegative matrices.

This section is meant to give an indication that such extensions are indeed possible. The following result has been proved by MANDL AND SENETA [39].

LEMMA 3.1. Let K contain only irreducible matrices and let $\hat{\sigma} := \max \{\sigma(P) \mid P \in K\}$. Then there exists a $\hat{P} \in K$, with spectral radius $\hat{\sigma}$ and associated right eigenvector $\hat{u} > \underline{0}$ such that

$$(3.1.1) \qquad \hat{P}\hat{u} = \max_{P \in K} P\hat{u} = \hat{\sigma}\hat{u}$$

PROOF. Consider the following iteration procedure:

a. Choose $P(0) \in K$ arbitrary.

b. For $m = 0,1,2,\ldots$, choose $P(m+1) \in K$ such that

$$P(m+1)u(m) = \max_{P \in K} Pu(m),$$

with

$$P(m+1)_i := P(m)_i \quad \text{if} \quad (P(m)u(m))_i = (\max_{P \in K} Pu(m))_i .$$

Here u(m) denotes the strictly positive right eigenvector, associated with the spectral radius σ_m of P(m).

c. Stop, if $P(m+1) = P(m)$. Define $\hat{\sigma} := \sigma_m, \hat{u} := u(m)$ and $\hat{P} = P(m)$.

Obviously $P(m+1)u(m) \geqq P(m)u(m) = \sigma_m u(m)$ hence $\sigma_{m+1} \geqq \sigma_m$ by lemma 2.6, whereas $\sigma_{m+1} = \sigma_m$ implies $P(m+1)u(m) = \sigma_m u(m) = P(m)u(m)$ by lemma 2.7. In the latter case we have $P(m+1) = P(m)$. In other words: $P(m+1) \neq P(m)$ implies $\sigma_{m+1} > \sigma_m$, which means that the procedure does not cycle and, hence, ends after a finite number of steps, since K is finite. □

It follows from lemma 3.1 that for

$$x(n) = \max_{P \in K} Px(n-1) \qquad n \in \mathbb{N};\ x(0) > \underline{0} \tag{3.0.2}$$

we have

$$c_1 \hat{\sigma}^n \hat{u} \leqq x(n) \leqq c_2 \hat{\sigma}^n \hat{u} \qquad n \in \mathbb{N},$$

if c_1, c_2 are chosen such that

$$c_1 \hat{u} \leqq x(0) \leqq c_2 \hat{u}.$$

Hence, upper and lower bounds for $x(n)$, defined by (3.0.2), exists. A more precise description of the asymptotic behaviour of $x(n)$, for $n \to \infty$, will be given in chapter 4.

Lemma 3.1 shows that it is possible to extend results for one square nonnegative matrix to the nonlinear mapping defined by (3.0.1), at least under special conditions. In the next section we shall attempt to generalize lemma 3.1 to the case where K is a finite set of (possibly) reducible nonnegative matrices with the product property; the procedure in the proof of lemma 3.1 then appears as a special case of a general iteration procedure.

3.2. Sets of reducible nonnegative matrices

In section 2.2, lemma 2.11, we presented a decomposition result for square nonnegative matrices which was strongly related to the behaviour of their powers. In this section, the extension of this result to sets of square nonnegative matrices with the product property is given. The existence of such an extension implies that it is possible to establish bounds, similar to those in lemma 2.11.d, for the vector $x(n)$, defined by (3.0.2).

The precise formulation of the main result of this section reads as follows:

THEOREM 3.2. Let K be a finite set of square nonnegative matrices with the product property. There exists an integer r and a partition $\{I(1),\dots,I(r)\}$ of the state space S, such that the following properties hold:

a. Let $P^{(k,\ell)}$ denote the restriction of P to $I(k) \times I(\ell)$. Then $P^{(k,\ell)} = \underline{\underline{0}}$ if $k > \ell$ $(k,\ell = 1,\dots,r)$ for each $P \in K$.

b. There exists a matrix $\hat{P} \in K$ and strictly positive vectors $\hat{u}^{(k)}$, defined on $I(k)$, such that

$$\hat{P}^{(k,k)}\hat{u}^{(k)} = \max_{P \in K} P^{(k,k)}\hat{u}^{(k)} = \hat{\sigma}_k \hat{u}^{(k)} \qquad k = 1,2,\dots,r, \tag{3.2.1}$$

where

$$\hat{\sigma}_k := \sigma(\hat{P}^{(k,k)}) \qquad k = 1,2,\dots,r.$$

For $k \leqq \ell$ we have $\hat{\sigma}_k \geqq \hat{\sigma}_\ell$ with equality only if each state in $I(k)$ has access to some state in $I(\ell)$ under $\hat{P}$.

c. Let $x(0) > \underline{0}$ and let $x(n)$ be defined by (3.0.2). For each $k \in \{1,2,\dots,r\}$ let the integer t_k be defined by

$$t_k := \begin{cases} \min \{j \mid j > 0,\ \hat{\sigma}_{k+j} < \hat{\sigma}_k\} & \text{if such a j exists} \\ r-k+1 & \text{otherwise.} \end{cases}$$

Then there exist positive constants $c_1, c_2 > 0$ such that

$$c_1 u_i^{(k)} \leqq \binom{n}{t_k - 1}^{-1} \hat{\sigma}_k^{-n}\ x(n)_i \leqq c_2 u_i^{(k)}$$

for $i \in I(k)$, $k = 1,2,\dots,r$ and $n \in \mathbb{N}$. □

The proof of theorem 3.2 will be split up into several lemmas. Recalling the definitions of growth rate and growth index (section 2.5) one easily verifies that the matrix $\hat{P}$, introduced in theorem 3.2.b, satisfies

$$(\sigma_i(\hat{P}),\ \nu_i(\hat{P})) \succeq (\sigma_i(P),\ \nu_i(P)) \qquad i \in I(k);\ k = 1,\dots,r.;\ P \in K, \tag{3.2.2}$$

since, by lemma 2.6,

$$\sigma(P^{(k,k)}) \leqq \hat{\sigma}_k \qquad k = 1,\ldots,r;\ P \in K,$$

and $P^{(k,\ell)} = \underline{\underline{0}}$ for $k > \ell$ and for all $P \in K$.

This section will be devoted to the development of an iterative procedure for finding such a matrix $\hat{P}$ in a constructive way. Note that we have to determine both a "simultaneous block-triangular structure" for all matrices in K and a set of vectors $\hat{u}^{(k)}$ $(k = 1,2,\ldots,r)$ such that (3.2.1) holds. Analogous to the irreducible case we can construct these vectors by an iterative procedure (although the matrices $P^{(k,k)}$ are not irreducible in general); however, difficulties arise, if, during an improvement step, the block-triangular structure changes. For this reason, we first apply a so-called completion procedure: starting with an arbitrary matrix we reach, after a finite number of steps, some kind of ultimate block-triangular structure. Once having obtained this ultimate structure, we may try to improve the eigenvectors. With the resulting matrix we again start a completion procedure, etc.

So theorem 3.2 will be established by constructing an iteration algorithm each step of which consists of a completion procedure and an improvement procedure. The proof of the fact that these procedures do not cycle is rather complicated, therefore it is divided into a number of steps. Let us start with a description of the completion procedure.

<u>Completion procedure</u>

<u>Start</u>: Let $P(0) \in K$ be given.

For $m = 0,1,2,\ldots$, apply the following iteration step until the stopping condition is satisfied.

<u>Iteration step</u>: By permuting the states, we may write $P(m)$ (the outcome of the m-th iteration step) in the block-triangular form of lemma 2.11. Let $\{I_1(m),\ldots,I_{r_m}(m)\}$ be the spectral partition of S with respect to $P(m)$ (cf. definition 2.11).

Now, there are two possibilities.

a. If, for $k = 1,2,\ldots,r_m-1$ and for all $P \in K$

$$p_{ij} = 0 \qquad \text{for all } i \in \bigcup_{\ell=k+1}^{r_m} I_\ell(m) \text{ and all } j \in I_k(m),$$

then we define $P(m+1) := P(m)$.

b. If, on the contrary, there exists an integer t, $1 \leqq t \leqq r_m$, such that, for $1 \leqq k < t$ and for all $P \in K$

$$p_{ij} = 0 \qquad \text{for all } i \in \bigcup_{\ell=k+1}^{r_m} I_\ell(m) \text{ and all } j \in I_k(m),$$

whereas, for $k = t$, there is a $P \in K$ such that

$$p_{ij} > 0 \qquad \text{for some } i \in \bigcup_{\ell=k+1}^{r_m} I_\ell(m) \text{ and some } j \in I_k(m),$$

then we proceed as follows.

For $P \in K$, let $D(m,P)$ denote the set of all states which have access to $I_k(m)$ under P and let

$$D(m) := \bigcup_{P \in K} D(m,P).$$

Since K has the product property, there exists a matrix, $\tilde{P}$ say, such that all states in $D(m) \backslash I_t(m)$ have access to $I_t(m)$ under $\tilde{P}$. Now, determine $P(m+1)$ such that

$$P(m+1)_i = \begin{cases} \tilde{P}_i & , \quad i \in D(m) \backslash I_t(m) \\ P(m)_i & \quad \text{otherwise}. \end{cases}$$

<u>Stop</u>: if $P(m+1) = P(m)$. Define $\bar{P}(0) := P(m)$. $\bar{P}(0)$ is called a *completion* of $P(0)$.

That the completion procedure does not cycle, and hence ends after a finite number of steps, is a consequence of the following result.

<u>LEMMA 3.3.</u> Let $P(m)$ be the matrix, resulting from the m-th iteration step in a completion procedure ($m = 0,1,2,\ldots$). Then

$$(3.2.3) \qquad (\sigma_i(P(m+1)),\ \nu_i(P(m+1))) \succeq (\sigma_i(P(m)),\ \nu_i(P(m))) \qquad \forall i \in S,$$

with equality for all states if and only if $P(m+1) = P(m)$.

PROOF. Suppose $P(m+1) \neq P(m)$. Let $(\sigma_i(P(m)), \nu_i(P(m))) =: (\rho,\eta)$ for $i \in I_t(m)$. In order to simplify notations we define

$$A := I_t(m) \qquad D := D(m) \qquad E := \left(\bigcup_{\ell=t}^{r_m} I_\ell(m)\right)\setminus D(m)$$

Now each state in D has access to some state in A under $P(m+1)$. Since $P(m+1)_i = P(m)_i$ for $i \in A$ it follows that each final class C of $P(m+1)^D$ contains a final (and hence basic) class B of $P(m)^A$. Now, there are two possibilities:

either: $B \subsetneq C$, in which case $\sigma(P(m+1)^C) > \sigma(P(m+1)^B) = \sigma(P(m)^B) = \rho$ by proposition 2.3 and the fact that $P(m+1)_i = P(m)_i$ for $i \in B$,

or : $B = C$, in which case $\sigma(P(m+1)^C) = \sigma(P(m)^B) = \rho$.

C has access only to classes in E (under $P(m+1)$). If $\sigma(P(m+1)^C) = \rho$, it follows that $C = B$, in which case $\nu_i(P(m+1)) = \nu_i(P(m))$ for $i \in C$. It follows that

$$(\sigma_i(P(m+1)), \nu_i(P(m+1))) \succeq (\rho,\eta)$$

at least for all states i in a final class of $P(m+1)^D$ and hence, immediately for all states in D. Since $P(m+1)_i = P(m)_i$ for $i \in E$, it follows that (3.2.3) holds for all states in E. Finally, one easily verifies that (3.2.3) now holds for all $i \in S$, again since $P(m+1)_i = P(m)_i$ for $i \in S\setminus D$.

Equality in (3.2.3) for all states holds if and only if $A\setminus D = I_t(m) \setminus D(m) = \emptyset$, in which case $P(m+1) = P(m)$. □

Once having obtained a completion $\bar{P}(0)$ of $P(0)$ (i.e. an ultimate block-triangular structure) we try to improve the eigenvectors. This improvement procedure reads as follows.

Improvement procedure

Start: Let $\bar{P}(0)$ be the result of a completion procedure and let $\{I_1(0),\ldots,I_{r_0}(0)\}$ denote the spectral partition of S with respect to $\bar{P}(0)$. Finally, let $\{u(0)^{(k)};\ k = 1,2,\ldots,r_0\}$ be the set of associated right eigenvectors, given by (2.2.3).

Set $P(r_0+1) = \bar{P}(0)$. For $m = r_0, r_0-1,\ldots,2,1$, apply the following improvement step:

Improvement step: Suppose we have obtained $P(m+1)$. Determine $\tilde{P} \in K$ such that

$$(3.2.4)\qquad \sum_{j\in I_m(0)} \tilde{p}_{ij}\, u(0)_j^{(m)} = \max_{P\in K} \sum_{j\in I_m(0)} p_{ij}\, u(0)_j^{(m)} \qquad i \in I_m(0),$$

with

$$\tilde{P}_i := \bar{P}(0)_i \quad \text{if} \quad \sum_{j\in I_m(0)} \bar{p}(0)_{ij}\, u(0)_j^{(m)} = \max_{P\in K} \sum_{j\in I_m(0)} p_{ij}\, u(0)_j^{(m)}.$$

Define P(m) by

$$P(m)_i = \begin{cases} \bar{P}(0)_i & i \in \bigcup_{k=1}^{m-1} I_k(0) \\ \tilde{P}_i & i \in I_m(0) \\ P(m+1)_i & i \in \bigcup_{k=m+1}^{r_m} I_k(0) \end{cases}$$

Stop: P(1) is called an *improvement* of $\bar{P}(0)$. □

That P(1) is really an improvement of $\bar{P}(0)$ follows from a combination of the following two results. First we have

LEMMA 3.4. In the improvement procedure we have for $m = r_0, r_0-1, \dots, 2, 1$:

$$(3.2.5)\qquad (\sigma_i(P(m)),\ \nu_i(P(m))) \succeq (\sigma_i(P(m+1)),\ \nu_i(P(m+1))) \qquad i \in S.$$

PROOF. Let $(\sigma_i(\bar{P}(0)),\ \nu_i(\bar{P}(0))) =: (\rho,\eta)$ for $i \in I_m(0)$. For notational convenience define

$$A := I_m(0) \qquad E := \bigcup_{k=m+1}^{r_0} I_k(0).$$

Now, let C be a final class of $P(m)^A$. Since, by (3.2.4),

$$\sum_{j\in A} p(m)_{ij}\, u(0)_j^{(m)} \geqq \sum_{j\in A} \bar{p}(0)_{ij}\, u(0)_j^{(m)} = \rho\, u(0)_i^{(m)} \qquad i \in C,$$

it follows that $\sigma(P(m)^C) \geqq \rho$ (cf. lemma 2.6). If $\sigma(P(m)^C) = \rho$, then by lemma 2.7

$$\sum_{j\in A} p(m)_{ij}\; u(0)_j^{(m)} = \sum_{j\in C} p(m)_{ij}\; u(0)_j^{(m)} = \rho u(0)_i^{(m)} = \sum_{j\in A} \bar{p}(0)_{ij}\; u(0)_j^{(m)}$$

for all $i \in C$, which implies that

$$P(m)_i = \bar{P}(0)_i = P(m+1)_i \qquad i \in C.$$

Furthermore, C has possible access only to classes in E under P(m). Since the same arguments hold for any final class of $P(m)^A$, it follows that (3.2.5) holds for all states in A. By arguments, similar to those in the proof of lemma 3.3, (3.2.5) now holds for all $i \in S$. □

The following lemma is useful in the case that equality holds in (3.2.5) for all states.

LEMMA 3.5. Let K be a finite set of nonnegative square matrices with the product property. Let P(0) be a matrix with spectral radius σ, having a strictly positive eigenvector u(0). Determine P(1) such that

$$P(1)\; u(0) = \max_{P\in K} P\; u(0),$$

with

$$P(1)_i := P(0)_i \quad \text{if } (P(0)u(0))_i = (\max_{P\in K} P\; u(0))_i.$$

Suppose

$$(3.2.6) \qquad (\sigma_i(P(1)),\; \nu_i(P(1))) = (\sigma,1) \qquad \forall\; i \in S.$$

Then there exists a strictly positive eigenvector u(1) for P(1), associated with σ, such that $u(1)_i = u(0)_i$ for i belonging to a basic class of P(1). Furthermore

$$u(1) \geqq u(0),$$

with equality if and only if P(1) = P(0).

PROOF. Since (3.2.6) holds, the basic classes of P(1) are precisely its final classes. Hence, a strictly positive eigenvector $u(1)$, associated with σ, exists for $P(1)$. As in the proof of lemma 3.4, one has $P(1)_i = P(0)_i$ for i in a final class of $P(1)$. Hence, a final, basic class of $P(1)$ is also a final, basic class of $P(0)$. It follows that $u(1)$ may be chosen such that $u(1)_i = u(0)_i$ for i belonging to a basic class of $P(1)$.

Now let A denote the set of states that do not belong to a basic class of $P(1)$. As

$$P(1)\ (u(0)-u(1)) \geqq \rho(u(0)-u(1)),$$

it follows from $u(1)_i = u(0)_i$ for $i \in S\backslash A$, that

$$P(1)^A\ (u(0)^A - u(1)^A) \geqq \rho\,(u(0)^A - u(1)^A)$$

If $u(1)_i < u(0)_i$ for some $i \in A$, then lemma 2.8 implies

$$(P(1)^A) \geqq \rho\ ,$$

contradicting the definition of A. Hence $u(1) \geqq u(0)$ if $u(1)_i = u(0)_i$ for $i \in S\backslash A$. Finally, $u(1) = u(0)$ implies

$$\text{(3.2.7)}\qquad \max_{P \in K} Pu(0) = P(1)u(0) = \sigma u(0) = P(0)u(0),$$

i.e., $P(1) = P(0)$. This completes the proof. □

Translating this result back to the situation of lemma 3.4, it follows that, if equality holds in (3.2.5) for some $m \in \{r_0, r_0-1, \ldots, 2, 1\}$ and for all $i \in S$, then there exists a vector $u(1)^{(m)}$, defined on $I_m(0)$, which is strictly positive and which obeys

$$\sum_{j \in I_m(0)} p(m)_{ij}\ u(1)_j^{(m)} = u(1)_i^{(m)} \qquad i \in I_m(0),$$

and

$$u(1)_i^{(m)} \geqq u(0)_i^{(m)} \qquad i \in I_m(0),$$

with equality for all states i which belong to any class $C \subset I_m(0)$ of $P(m)$ satisfying $\sigma(P(m)^C) = \rho$ (here ρ is defined as in the proof of lemma 3.4).

We have $P(m) = P(m+1)$ if and only if $u(1)^{(m)} = u(0)^{(m)}$, in which case

$$(3.2.8) \qquad \sum_{j\in I_m(0)} p(m)_{ij} u(0)_j^{(m)} = \max_{P\in K} \sum_{j\in I_m(0)} p_{ij}\, u(0)_j^{(m)} = \rho u(0)_i^{(m)} \quad i \in I_m(0).$$

The reader may notice that the proof of lemma 3.1 consists of an iterative procedure each step of which is in fact an improvement procedure. A completion procedure is not needed at all in the irreducible case where the state space cannot be decomposed.

After finishing an improvement procedure, it may be necessary that again a completion procedure is started, since the spectral partition of S with respect to the preceding completion does not correspond in general to the spectral partition of S with respect to the improvement of this preceding completion.

It is now easy to verify that parts a and b of theorem 3.2 follows by applying the algorithm, described below.

<u>Optimal growth iteration procedure</u>

<u>Start</u>: Choose $P(0) \in K$.

For $m = 0,1,2,\ldots$, find a matrix $P(m+1)$ as follows:

<u>Iteration step</u>: Calculate $\bar{P}(m)$, the completion of $P(m)$.
Find $P(m+1)$, an improvement of $\bar{P}(m)$.

<u>Stop</u>: if $P(m+1) = P(m)$. Let $\{I_1(m),\ldots,I_{r_m}(m)\}$ denote the spectral partition of S with respect to $P(m)$ and let $\{u^{(1)},\ldots,u^{(r_m)}\}$ be the set of associated strictly positive eigenvectors, described in lemma 2.11 (formula (2.2.3)). Define $\hat{P} := P(m)$, $r := r_m$, $\{I(1),\ldots,I(r)\} := \{I_1(m),\ldots,I_{r_m}(m)\}$ and $\{\hat{u}^{(1)},\ldots,\hat{u}^{(r)}\} := \{u^{(1)},\ldots,u^{(r_m)}\}$. □

It follows immediately from the lemmas 3.3, 3.4 and 3.5 that this iterative procedure does not cycle and, hence, ends after a finite number of steps. One easily verifies that the resulting matrix $\hat{P}$, the partition $\{I(1),\ldots,I(r)\}$ and the collection of vectors $\{u^{(1)},\ldots,u^{(r)}\}$ are precisely those described in theorem 3.2. It follows from (3.2.2) that $\hat{P}$ is indeed an "optimal growth" matrix. We now call $\{I(1),\ldots,I(r)\}$ the *spectral partition* of S with respect to the set K.

The proof of part c of theorem 3.2 is postponed until chapter 5, where it follows from a more general result. The reason for including part c in the formulation of theorem 3.2 is that it shows exactly the relationship between the spectral partition of S with respect to K and the first order asymptotic behaviour of the sequence $\{x(n);\ n = 0,1,\ldots\}$, defined by the dynamic programming recursion (3.0.2).

Theorem 3.2 is the generalization of lemma 2.11 to sets of square nonnegative matrices with the product property or, more precisely, to the nonlinear operator defined by (3.0.1). Lemma 2.11 in turn has been formulated as a slight extension of lemmas 2.9 and 2.10 where we only considered the principal decomposition of S with respect to P. Although it may seem superfluous, we also give the direct generalization of lemma 2.9 to sets of square nonnegative matrices with the product property. This formulation will be used as a starting point in the forthcoming analysis concerning generalized eigenvectors for sets of nonnegative matrices. We first presented theorem 3.2 because we preferred to give a constructive proof by means of an iterative method (which implies that the complete state space has to be considered). The following result is the analogue of lemma 2.9 for the set K; it follows, of course, immediately from theorem 3.2.

THEOREM 3.6. Let K be a finite set of square nonnegative matrices with the product property. Let $\hat{\sigma} := \max\{\sigma(P)\,|\,P \in K\}$ and let $\nu := \max\{\nu(P)\,|\,P \in K,\ \sigma(P) = \hat{\sigma}\}$. Then there exists a partition $\{D(\nu), D(\nu-1),\ldots,D(1),D(0)\}$ of the state space S such that the following properties hold:

a. Let $P^{(k,\ell)}$ denote the restriction of P to $D(k) \times D(\ell)$.
Then $P^{(k,\ell)} = \underline{\underline{0}}$ for $k < \ell$, $k,\ell = 0,1,\ldots,\nu$, $P \in K$.

b. There exist a matrix $\hat{P} \in K$, with $\sigma(\hat{P}) = \hat{\sigma}$ and $\nu(\hat{P}) = \nu$, and strictly positive vectors $\hat{u}^{(k)}$, defined on $D(k)$, such that

$$\hat{P}^{(k,k)}\,\hat{u}^{(k)} = \max_{P \in K} P^{(k,k)}\,\hat{u}^{(k)} = \hat{\sigma}\,\hat{u}^{(k)} \qquad k = 1,2,\ldots,\nu.$$

For $k = 0,1,\ldots,\nu$, the set $D(k)$ is the union of all classes with depth k, with respect to $\hat{P}$. Furthermore

$$\max_{P \in K} \sigma(P^{(0,0)}) < \hat{\sigma}.$$

c. Choose $x(0) > \underline{0}$ and let $x(n)$ be defined by

$$x(n) = \max_P P\, x(n-1) \qquad n \in \mathbb{N}.$$

Then there exist constants $c_1, c_2 > 0$ such that

$$c_1 \hat{u}_i^{(k)} \leqq \binom{n}{k-1}^{-1} \hat{\sigma}^{-n} x(n)_i \leqq c_2 \hat{u}_i^{(k)} \qquad i \in D(k);\ k = 1,\ldots,\nu,$$

whereas

$$\lim_{n\to\infty} \hat{\sigma}^{-n} x(n)_i = 0 \qquad i \in D(0). \qquad \square$$

DEFINITION 3.1. The partition $\{D(\nu), D(\nu-1), \ldots, D(1), D(0)\}$, introduced in theorem 3.6, is called the *principal partition* of S with respect to K. □

A direct proof of theorem 3.6 can be found in ZIJM [75]. The proof given there is less constructive than the one presented here; the author starts with the principal partition of S with respect to K and then applies the improvement procedure a number of times until no further improvement is possible. Theorem 3.2 is then obtained as an immediate extension of theorem 3.6.

Theorem 3.2 has also been proved by SLADKY [58], using different methods. The proof presented here may be viewed as a combination of ideas in SLADKY [58] and ZIJM [75]. The idea of using state classifications can be found in SLADKY [58]. The proofs of lemmas 3.3, 3.4 and 3.5 (i.e. the fact that the two subroutines do not cycle) in their present form are taken mainly from ZIJM [75].

The existence of the principal partition of S with respect to K and its connection with the asymptotic behaviour of certain dynamic programming recursions (expressed in part c of theorem 3.6) will appear to be fundamental throughout this monograph. In the next two chapters, we shall use this partition extensively in order to establish convergence results for dynamic programming recursions of the type (3.0.2).

CHAPTER 4

CONVERGENCE OF DYNAMIC PROGRAMMING RECURSIONS: THE CASE $\nu = 1$

In the preceding chapter we obtained a decomposition result for the finite state space S with respect to a set K of square nonnegative matrices with the product property. Its importance derives from the fact (not yet proved) that there exists a connection (cf. theorem 3.6.c) with the first order asymptotic behaviour of dynamic programming recursions of the type

$$(4.0.1) \qquad x(n) = \max_{P \in K} P\, x(n-1) \qquad\qquad n \in \mathbb{N}\,;\; x(0) > \underline{0}.$$

In particular, if $\{D(\nu), D(\nu-1), \ldots, D(1), D(0)\}$ denotes the principal partition of S with respect to K (cf. definition 3.1) and $\hat{\sigma}$ is defined by

$$\hat{\sigma} := \max_{P \in K} \sigma(P)$$

then we claim that positive upper and lower bounds exist for the sequence

$$\binom{n}{k-1}^{-1} \hat{\sigma}^{-n}\, x(n)_i \qquad\qquad i \in D(k);\; k = 1, \ldots, \nu.$$

Now, although boundedness gives at least a first idea concerning the asymptotic properties of $x(n)$, it is well known that in the "one matrix" case much stronger results can be proved. Consider - for instance - a square irreducible nonnegative matrix P, with spectral radius σ, that is aperiodic. Then

$$\lim_{n\to\infty} \sigma^{-n} P^n\, x(0)$$

exists and is strictly positive if $x(0) \geq \underline{0}$ (exploit the Jordan canonical form of P, or lemma 2.5).

Also, when dealing with reducible matrices, more detailed results concerning the asymptotic behaviour of $P^n x(0)$ for $n \to \infty$ are known (compare e.g. PEASE [44]). The question arises whether these results can be extended to dynamic programming recursions of the type (4.0.1).

To answer this question is one of the objectives of part I of this monograph. In this chapter we make a first attempt; we treat the case $\nu = 1$ or, in other words, the case where the principal partition of S with respect to K takes the form $\{D(1), D(0)\}$. This means that we assume

$$(4.0.2) \qquad \max \{ \nu(P) \mid P \in K, \sigma(P) = \hat{\sigma}\} = 1,$$

where $\hat{\sigma}$ is defined by

$$(4.0.3) \qquad \hat{\sigma} := \max \{\sigma(P) \mid P \in K\}.$$

As usual, we start with the case where all $P \in K$ are irreducible (section 4.1). In section 4.2, we first establish boundedness of dynamic programming recursions, based on a finite set K of general square nonnegative matrices with the product property, satisfying assumption (4.0.2). The boundedness is then used to prove convergence results for dynamic programming recursions, again under condition (4.0.2).

It turns out that in this second section we need some results from the theory of nonstationary Markov decision processes. In particular a geometric convergence result for undiscounted Markov decision processes, recently proved by SCHWEITZER AND FEDERGRUEN [61], appears to be useful. In the appendix new proofs for these results are presented (compare also ZIJM [78]).

4.1. Dynamic programming recursions with irreducible nonnegative matrices

As in the preceding chapter, it turns out that, in order to gain some insight in the methods to be used, a separate treatment of the irreducible case is most helpful. Moreover, the results of this section will prove to be fundamental in the forthcoming analysis.

In this section, it will be shown that $\lim_{n\to\infty} \hat{\sigma}^{-n} x(n)$ exists (where $x(n)$ and $\hat{\sigma}$ are defined by (4.0.1) and (4.0.3), respectively), when K is a set of aperiodic, irreducible nonnegative matrices. Next, brief attention is paid to the periodic case. These cases have been analyzed already by SLADKY [55]. The following result will be useful.

LEMMA 4.1. Let P be a square irreducible aperiodic nonnegative matrix with spectral radius σ and let $\{x(n);\ n = 0,1,2,\dots\}$ be a set of vectors, such that $\sigma^{-n}x(n)$ is bounded (uniformly for $n \in \mathbb{N}$). Suppose

$$(4.1.1) \qquad x(n+1) \geqq P\,x(n) \qquad\qquad n \in \mathbb{N}.$$

Then, there exists a vector x, which satisfies

$$(4.1.2) \qquad \lim_{n\to\infty} \sigma^{-n}\, x(n) = x = \sigma^{-1}\, Px.$$

Furthermore, $x(0) \geq \underline{0}$ implies $x > \underline{0}$.

PROOF. Since $\{\sigma^{-n}x(n);\ n = 0,1,2,\dots\}$ is bounded, we may assume the existence of finite limit-points for this sequence. Suppose two different limit-points a and b exist. Iterating (4.1.1) yields

$$\sigma^{-(n+m)}\, x(n+m) \geqq \sigma^{-m}\, P^{m}\, \sigma^{-n}x(n) \qquad\qquad n,m \in \mathbb{N}.$$

Choose n fixed and let $m_1, m_2, \dots$ be a sequence such that $\lim_{k\to\infty} \sigma^{-(n+m_k)}\, x(n+m_k) = a$. By lemma 2.5 we find

$$a \geqq P^{*}\, \sigma^{-n}x(n).$$

The same conclusion can be obtained for each $n \in \mathbb{N}$. Choosing $n_1, n_2, \dots$ such that $\lim_{k\to\infty} \sigma^{-n_k}\, x(n_k) = b$ we find

$$a \geqq P^{*}b.$$

Analogously it is proved that

$$b \geqq P^{*}a.$$

Hence, $a \geqq P^{*}P^{*}a = P^{*}a$. Since $P^{*} > \underline{0}$, $a \geq P^{*}a$ implies $\underline{0} < P^{*}(a-P^{*}a) = P^{*}a - P^{*}a = 0$ (compare lemma 2.5), a contradiction. Hence $a = P^{*}a \leqq b$. Analogously, we find $a \geqq b$. Hence $a = b$. Since $a \geqq P^{*}x(0)$, we find $a > \underline{0}$ if $x(0) \geq \underline{0}$. □

THEOREM 4.2. Let K be a finite set of square irreducible nonnegative matrices with the product property. Let $\hat{\sigma} := \max \{\sigma(P) \mid P \in K\}$ and assume that at least one matrix $\bar{P} \in K$ exists, with $\sigma(\bar{P}) = \hat{\sigma}$, that is aperiodic. Choose $x(0) \geq \underline{0}$ and let $x(n)$ be defined by (4.0.1) for $n = 1,2,\ldots$. Then

$$\lim_{n\to\infty} \hat{\sigma}^{-n} x(n)$$

exists. Let x denote this limit, then $x > \underline{0}$ and

$$\max_{P\in K} Px = \hat{\sigma}x. \tag{4.1.3}$$

PROOF. By lemma 3.1, there exists a vector $\hat{u} > \underline{0}$ such that

$$\max_{P\in K} P\hat{u} = \hat{\sigma}\hat{u}.$$

Without restriction we may take $\hat{u} \geqq x(0)$. It follows that

$$\underline{0} \leqq \hat{\sigma}^{-n} x(n) \leqq \hat{u} \qquad n \in \mathbb{N}.$$

Since $x(n+1) \geqq \bar{P}x(n)$ for all n, lemma 4.1 implies the existence of $x := \lim_{n\to\infty} \hat{\sigma}^{-n} x(n)$. Formula (4.1.3) now follows from

$$\hat{\sigma} \,.\, \hat{\sigma}^{-(n+1)} x(n+1) = \max_{P\in K} P \,\hat{\sigma}^{-n} x(n) \qquad n \in \mathbb{N}_0,$$

by letting $n\to\infty$ in both sides of this equality. It has already been established in lemma 4.1 that $x > \underline{0}$. □

The reader can easily verify that $x = c\,\hat{u}$ for some constant $c > 0$, where $x = \lim_{n\to\infty} \hat{\sigma}^{-n} x(n)$ and $\hat{u}$ is defined in lemma 3.1.

The periodic case will not be treated extensively throughout this monograph. The proofs follow essentially the same lines, the only difficulties arising are of technical or notational type. The proof of the following, periodic, analogue of theorem 4.1 is left to the reader.

THEOREM 4.3. Let K be a finite set of square irreducible nonnegative matrices with the product property. Let $\hat{\sigma} := \max \{\sigma(P) \mid P \in K\}$, let $d(P)$ denote the period of P and define d by

$$d := \text{g.c.d} \{d(P) \mid P \in K,\ \sigma(P) = \hat{\sigma}\}$$

Choose $x(0) > \underline{0}$ and let $x(n)$ be defined by (4.0.1) for $n = 1,2,\ldots$. Then, for $\ell = 0,1,\ldots,d-1$,

$$\lim_{k\to\infty} \hat{\sigma}^{-(\ell+kd)} x(\ell+kd)$$

exists and is strictly positive. □

Theorems 4.2 and 4.3 show that it is indeed possible to establish convergence results for dynamic programming recursions of type (4.0.1). If K consists of exactly one matrix these results are well known. The results of this section will prove to be useful in the treatment of more complicated, reducible cases.

4.2. Convergence of dynamic programming recursions: reducible matrices.

In this section we prove that dynamic programming recursions of type (4.0.1), based on a finite set K of possibly reducible nonnegative square matrices with the product property, are bounded in some sense if condition (4.0.2) holds. To be more precise, if

$$(4.0.2) \qquad \max_{P\in K} \{\nu(P) \mid P \in K,\ \sigma(P) = \hat{\sigma}\} = 1,$$

with $\hat{\sigma} = \max \{\sigma(P) \mid P \in K\}$, then for $x(n)$, defined by (4.0.1), we have

$$\sup_n \hat{\sigma}^{-n} x(n)_i < \infty \qquad\qquad i \in S.$$

Note that this assertion is in fact part c of theorem 3.6 for the case $\nu = 1$. Once having obtained boundedness, convergence results are proved for recursions of type (4.0.1), again under condition (4.0.2).

One lemma is needed, which may be viewed as the analogue of part c of lemma 2.6. We have

LEMMA 4.4. Let K be finite and let $\hat{\sigma} := \max\{\sigma(P) \mid P \in K\}$. Then

a. $\hat{\sigma} = \inf\{\lambda \mid \exists\, w > \underline{0} \text{ such that } \max_{P\in K} P\, w \leqq \lambda\, w\}$.

b. For each $\lambda > \hat{\sigma}$ there exists a vector $w(\lambda) > \underline{0}$ such that

$$\max_{P\in K} P\, w(\lambda) < \lambda\, w(\lambda).$$

PROOF. Part a follows immediately from lemma 2.6. To establish part b, take

$$w(\lambda) := \max_{P\in K} (\lambda I - P)^{-1} e \qquad \text{(component-wise).}$$ □

The following lemma establishes boundedness of the sequence $\hat{\sigma}^{-n} x(n)$. We have

LEMMA 4.5. Let K be a finite set of square nonnegative matrices with the product property, let $0 < \hat{\sigma} := \max\{\sigma(P) \mid P \in K\}$ and let the principal partition of S with respect to K be $\{D(1), D(0)\}$ (i.e., (4.0.2) is assumed to hold). Then for $x(n)$, defined by (4.0.1), we have

$$(4.2.1) \qquad \lim_{n\to\infty} \hat{\sigma}^{-n} x(n)_i = 0 \qquad\qquad i \in D(0),$$

$$(4.2.2) \qquad c_1 \hat{u}_i^{(1)} \leqq \hat{\sigma}^{-n} x(n)_i \leqq c_2 \hat{u}_i^{(1)} \qquad\qquad n \in \mathbb{N};\ i \in D(1),$$

for some positive constants c_1, c_2, and $\hat{u}^{(1)}$ defined as in theorem 3.6.

PROOF. As usual, for $P \in K$ let $P^{(k,\ell)}$ denote the restriction of P to $D(k) \times D(\ell)$, for $k,\ell = 0,1$. Then $P^{(0,1)} = \underline{\underline{0}}$ for all P. Since $\max\{\sigma(P^{(0,0)}) \mid P \in K\} < \hat{\sigma}$, there exists a $\lambda < \hat{\sigma}$ and a vector $w(\lambda) > \underline{0}$, defined on D(0), such that

$$\max_{P\in K} P^{(0,0)}\, w(\lambda) \leqq \lambda\, w(\lambda)\,.$$

Choosing $c > 0$ such that $x(0)_i \leqq c w(\lambda)_i$ for $i \in D(0)$, we find

$$\hat{\sigma}^{-n} x(n)_i \leqq c \lambda^n \hat{\sigma}^{-n} w(\lambda) \qquad\qquad n \in \mathbb{N};\ i \in D(0),$$

which establishes (4.2.1), since $x(n) > \underline{0}$ for all n.

Choosing $c_1 > 0$ such that

$$c_1 \hat{u}_i^{(1)} \leqq x(0)_i \qquad\qquad i \in D(1),$$

and recalling that

$$\max_{P \in K} P^{(1,1)} \hat{u}^{(1)} = \hat{\sigma}\, \hat{u}^{(1)},$$

the left inequality in (4.2.2) follows immediately from

$$x(n)^{(1)} \geqq \max_{P \in K} P^{(1,1)} x(n-1)^{(1)} \qquad\qquad n = 1,2,\ldots,$$

where $x(n)^{(1)}$ denotes the restriction of $x(n)$ to $D(1)$. Finally, if we choose $w(\lambda) > \underline{0}$ and α such that

$$\max_{P \in K} P^{(1,0)} w(\lambda) \leqq \hat{u}^{(1)},$$

$$x(0)_i \leqq \alpha w(\lambda)_i \qquad\qquad i \in D(0),$$

$$x(0)_i \leqq \alpha \hat{u}_i^{(1)} \qquad\qquad i \in D(1),$$

then, by induction, we obtain

$$x(n)_i \leqq \alpha\{\hat{\sigma}^n + \hat{\sigma}^{n-1}/(1-\lambda\hat{\sigma}^{-1})\}\, \hat{u}_i^{(1)} \leqq c_2 \hat{\sigma}^n \hat{u}_i^{(1)} \qquad i \in D(1),$$

for an appropriate choice of c_2. □

A natural way to prove convergence of a sequence of (finite-dimensional) vectors (or scalars) is to establish boundedness first, after which one has to show that no two different limit-points exist. This method was followed in the proof of theorem 4.2 and will also prove to be useful here. Before we can formulate the main result of this section it is necessary to extend the definition of aperiodicity to reducible nonnegative square matrices.

DEFINITION 4.1. A square nonnegative matrix P is called *aperiodic* if the restriction of P to each of its basic classes is aperiodic. □

The following result is now fundamental.

THEOREM 4.6. Let K be a finite set of square nonnegative matrices with the product property, let $\hat{\sigma} := \max \{\sigma(P) \mid P \in K\}$ and let $\{D(1), D(0)\}$ be the principal partition of S with respect to K. Finally, assume that each matrix $P \in K$, for which $\sigma(P) = \hat{\sigma}$, is aperiodic. Then there exists a vector $x \geq \underline{0}$, with $x_i = 0$ for $i \in D(0)$ and $x_i > 0$ for $i \in D(1)$, such that

$$(4.2.3) \qquad \lim_{n\to\infty} \hat{\sigma}^{-n} x(n) = x,$$

where $x(n)$ is defined by (4.0.1). Furthermore, the vector x obeys

$$(4.2.4) \qquad \max_{P \in K} Px = \hat{\sigma} x.$$

PROOF. For convenience we define $\bar{x}(n) := \hat{\sigma}^{-n} x(n)$. We know from lemma 4.5 that

$$\lim_{n\to\infty} \bar{x}(n)_i = 0 \qquad\qquad i \in D(0).$$

Hence we concentrate on the states of D(1). Since (4.2.2) holds, we can define finite-valued vectors a and b (component-wise) by

$$a := \limsup_{n\to\infty} \bar{x}(n),$$

$$b := \liminf_{n\to\infty} \bar{x}(n).$$

Suppose $a \geq b$ (note that $b \geqq \underline{0}$ and that $a_i = b_i = 0$ for $i \in D(0)$). Choose a sequence $\{n_k \,;\, k = 0,1,2,\ldots\}$ such that

$$\lim_{k\to\infty} \bar{x}(n_k) = b$$

and such that $\lim_{k\to\infty} \bar{x}(n_k - 1)$ exists. Define $\bar{\bar{x}} := \lim \bar{x}(n_k - 1)$.
Then $\bar{\bar{x}} \geqq b$ and $\bar{\bar{x}}_i = b_i = 0$ for $i \in D(0)$. Since K is finite, there exist a matrix $\bar{P}$ and a subsequence $\{n_{k(\ell)} \,;\, \ell = 0,1,2,\ldots\}$ of $\{n_k \,;\, k = 0,1,2\ldots\}$ such that for $\ell = 0,1,2,\ldots$

$$(4.2.5) \qquad \bar{x}(n_{k(\ell)}) = \hat{\sigma}^{-1} \bar{P}\, \bar{x}(n_{k(\ell)} - 1) = \hat{\sigma}^{-1} \max_{P \in K} P\, \bar{x}(n_{k(\ell)} - 1).$$

For $\ell \to \infty$, we obtain

$$b = \hat{\sigma}^{-1}\bar{P}\,\bar{x} \geqq \hat{\sigma}^{-1}\bar{P}\,b\,.$$

Analogously, we find that for some $\tilde{P} \in K$

$$a \leqq \hat{\sigma}^{-1}\tilde{P}\,a.$$

Since $\bar{P}$ was found as an optimal matrix in (4.2.5), we also have

$$b \geqq \hat{\sigma}^{-1}\tilde{P}\,\bar{x} \geqq \hat{\sigma}^{-1}\tilde{P}\,b.$$

Combining these results we conclude

$$(4.2.6) \qquad a-b \leqq \hat{\sigma}^{-1}\tilde{P}\,(a-b),$$

and, since $a_i = b_i = 0$ for $i \in D(0)$, (4.2.6) reduces to

$$a^{(1)}-b^{(1)} \leqq \hat{\sigma}^{-1}\tilde{P}^{(1,1)}(a^{(1)}-b^{(1)}),$$

where $a^{(1)}$ and $b^{(1)}$ denote the restriction of a and b to D(1). Since we assumed $b \le a$, it follows from lemma 2.8 that $\sigma(\tilde{P}^{(1,1)}) = \hat{\sigma}$. Furthermore, since there exists a vector $\hat{u}^{(1)} > \underline{0}$, defined on D(1), such that

$$\tilde{P}^{(1,1)}\hat{u}^{(1)} \leqq \max_{P \in K} P^{(1,1)}\hat{u}^{(1)} = \hat{\sigma}\,\hat{u}^{(1)}$$

(cf. theorem 3.6.b), it follows that each basic class of $\tilde{P}^{(1,1)}$ is final (compare the proof of lemma 2.7). Let C be any basic class of $\tilde{P}^{(1,1)}$. Since

$$x(n)^C \geqq \tilde{P}^C\,x(n-1)^C \qquad\qquad n \in \mathbb{N},$$

and since $\sigma(\tilde{P}^C) = \hat{\sigma}$, it follows from lemma 4.1 that $a_i = b_i$ for $i \in C$.

Let, finally $E \subset D(1)$ denote the set of states in D(1) which are not contained in a basic class of $\tilde{P}^{(1,1)}$. Then, by the results just obtained, $b \le a$ implies $b^E \le a^E$, and from (4.2.6):

$$a^E-b^E \leqq \hat{\sigma}^{-1}\tilde{P}^E(a^E-b^E)$$

which implies $\sigma(\tilde{P}^E) \geqq \hat{\sigma}$ by lemma 2.8. Since this contradicts the definition of E, we must conclude that a = b.

It follows immediately from (4.2.2) that $a_i > 0$ for $i \in D(1)$. Finally, (4.2.4) follows from

$$\hat{\sigma}\ \bar{x}(n+1) = \max_{P \in K} P\ \bar{x}(n)$$

by letting n tend to infinity. □

It is well known that in the "one-matrix" case the convergence of $\hat{\sigma}^{-n}x(n)$ to its limit vector x is geometric, i.e. there exist constants $\rho < 1$ and $c > 0$ such that for all n

$$\|\hat{\sigma}^{-n}x(n)-x\| \leqq c\ \rho^n$$

(a lower bound for $\hat{\sigma}\rho$ is given by the modulus of the subdominant eigenvalue of the matrix involved). It turns out that the same result holds for a set K of square nonnegative matrices with the product property, the proof however is essentially harder; it is related to a geometric convergence result for undiscounted Markov decision processes (cf. SCHWEITZER AND FEDERGRUEN [61]), which will be treated extensively in the appendix to this chapter. The next theorem can be proved by using the results of this appendix. We have

THEOREM 4.7. Under the conditions of theorem 4.6 there exist constants $\rho < 1$ and $c > 0$ such that

$$\|\hat{\sigma}^{-n}x(n)-x\| \leqq c\ \rho^n \qquad n \in \mathbb{N},$$

where $x := \lim_{n\to\infty} \hat{\sigma}^{-n}x(n)$.

PROOF. Let $\bar{x}(n) := \hat{\sigma}^{-n}x(n)$ and let $\bar{x}(n)^{(k)}$, $x^{(k)}$ denote the restriction of $\bar{x}(n)$, x to D(k), k = 0,1. Since, for some $\lambda < \hat{\sigma}$ and some $w(\lambda) > \underline{0}$, defined on D(0), we have (cf. lemma 4.4)

$$\max_{P \in K} P^{(0,0)}w(\lambda) \leqq \lambda\ w(\lambda),$$

it follows that $\bar{x}(n)^{(0)}$ tends to $\underline{0}$ geometrically (choose $c > 0$ such that $\bar{x}(0)_i \leqq cw(\lambda)_i$ for $i \in D(0)$, then $\bar{x}(n)_i \leqq \hat{c}\, \lambda^n \hat{\sigma}^{-n}\, w(\lambda)_i$ for $i \in D(0)$).

Furthermore, since $\lim_{n\to\infty} \bar{x}(n) = x$ and since (4.2.4) holds, it is easy to verify that for $n \geqq n_0$, say, we have

$$x(n+1) = \max_{P \in K_1} P\, x(n),$$

where K_1 is defined by

$$K_1 := \{P \in K \mid P\, x = \hat{\sigma}\, x\}.$$

For states $i \in D(1)$, and for $n \geqq n_0$, recursion (4.0.1) can be written as

$$(4.2.7) \qquad \bar{x}(n+1)_i = \max_{P \in K_1} \{ \sum_{j \in D(1)} \hat{\sigma}^{-1} p_{ij}\, \bar{x}(n)_j + \sum_{j \in D(0)} \hat{\sigma}^{-1} p_{ij}\, \bar{x}(n)_j \}.$$

By (4.2.4) and the fact that $x^{(1)} > \underline{0}$, the following transformation can be applied. Define

$$\tilde{p}_{ij} := x_i^{-1}\, \hat{\sigma}^{-1}\, p_{ij}\, x_j \qquad i,j \in D(1),\ P \in K,$$

$$\tilde{x}(n)_i := x_i^{-1}\, \bar{x}(n)_i \qquad i \in D(1),\ n \geqq n_0,$$

$$\tilde{r}(P,n)_i := \hat{\sigma}^{-1}\, x_i^{-1} \sum_{j \in D(0)} p_{ij}\, \bar{x}(n)_j \qquad i \in D(1),\ n \geqq n_0,\ P \in K.$$

Then (4.2.7) can be written as

$$\tilde{x}(n+1)_i = \max_{P \in K_1} \{ \sum_{j \in D(1)} \tilde{p}_{ij}\, \tilde{x}(n)_j + \tilde{r}(P,n)_i \} \qquad i \in D(1),\ n \geqq n_0.$$

Since $\tilde{r}(P,n)_i$ tends to zero geometrically, for $n \to \infty$, for each $P \in K_1$ and for each $i \in D(1)$, and since

$$\{\tilde{P}^{(1,1)} \mid P \in K_1\}$$

is a set of stochastic matrices, theorem 4.a.5 of the appendix can be applied now to establish geometric convergence of $\tilde{x}(n)_i$ to 1 for $n \to \infty$, $i \in D(1)$. Hence $\bar{x}(n)^{(1)}$ tends to $x^{(1)}$ geometrically. □

Periodic analogues of these theorems can be proved again. Let, as before,

$$\hat{\sigma} := \max \{\sigma(P) \mid P \in K\},$$

and define

$$\tilde{K} := \{P \in K \mid \sigma(P) = \hat{\sigma}\} .$$

Now, let some $P \in \tilde{K}$ have k basic classes, $B(1),\ldots,B(k)$, and let $d_\ell(P)$ denote the period of $P^{B(\ell)}$, for $\ell = 1,\ldots,k$. Define

(4.2.8) $\quad d(P) := \ell.c.m. \{d_1(P),\ldots,d_k(P)\}.$

This can be done for each $P \in \tilde{K}$. Finally, let

(4.2.9) $\quad d := g.c.d. \{d(P) \mid P \in \tilde{K}\}.$

Then the following theorem may be formulated (the proof is left to the reader).

<u>THEOREM 4.8.</u> Let $\tilde{K}$, $\hat{\sigma}$ and d be defined as above. Suppose condition (4.0.2) holds. Then there exist vectors $w(\ell) \geq \underline{0}$ ($\ell = 0,1,\ldots,d-1$) such that for x(n), defined by (4.0.1), the following holds:

(4.2.10) $\quad \lim_{k\to\infty} \hat{\sigma}^{-(\ell+kd)} x(\ell+kd) = w(\ell) \qquad \ell = 0,1,\ldots,d-1.$

Furthermore, the following relationship holds:

(4.2.11) $\quad \max_{P \in K} P\, w(\ell) = \hat{\sigma}\, w(\ell+1) \qquad \ell = 0,1,\ldots,d-1.$

where $w(d) := w(0)$. □

Again, it can be proved that the convergence in (4.2.10) is geometric. In chapter 5, more general results concerning the asymptotic behaviour of x(n) for $n \to \infty$ will be proved. The results of this section (in particular theorems 4.6 and 4.7) will serve as a first step in the analysis of the general case.

Appendix 4.A. Geometric convergence in undiscounted Markov decision processes.

In this appendix we deal with a finite set K of *stochastic* N x N matrices. As before the state space is denoted by S. With each $P \in K$ a sequence of vectors $\{r(n,P) \mid n = 0,1,2,\dots\}$ is associated such that the set of matrices

$$(4.a.1) \qquad \{(P,\ r(0,P),\ r(1,P),\dots) \mid P \in K\}$$

has the product property. Furthermore, it is assumed that for each P the sequence $r(n,P)$ has a limit $r(P)$, and that

$$(4.a.2) \qquad \| r(n,P) - r(P) \| \leq \alpha\rho^n \qquad\qquad P \in K,\ n \in \mathbb{N}_0,$$

for some $\alpha > 0$, and some ρ, $0 \leq \rho < 1$. It follows that the set of N x (N+1) matrices

$$(4.a.3) \qquad \{(P,\ r(P)) \mid P \in K\}$$

also has the product property.

Now consider the following dynamic programming recursion:

$$(4.a.4) \qquad v(n+1) = \max_{P \in K} \{r(n,P) + P\,v(n)\} \qquad\qquad n \in \mathbb{N}_0,$$

where $v(0)$ is arbitrary.

The problem, to be solved in this appendix, can be stated as follows: what is the asymptotic behaviour of $v(n)$ for $n \to \infty$?

In the case that $r(n,P) = r(P)$ for $n \in \mathbb{N}_0$ and for each $P \in K$, the answer to this question can be found in SCHEITZER AND FEDERGRUEN [60], [61]. By a slight extension of their results also the more general problem (concerning the asymptotic behaviour of $v(n)$, defined by (4.a.4)) can be solved. Nevertheless, we prefer to give a separate treatment of the problem in this appendix, especially because the proofs in the two references mentioned above are relatively complicated and can be simplified considerably. The proofs in this appendix are mainly based on ZIJM [78].

The following lemma is needed (cf. DERMAN [17]).

LEMMA 4.a.1. Let K and $\{r(P) \mid P \in K\}$ be defined as above. Define g^* by

$$(4.a.5) \qquad g_i^* := \max_{P \in K} (P^* r(P))_i \qquad i \in S,$$

where P^* is defined as in lemma 2.5. Then there exists a $\tilde{P} \in K$ such that $g^* = \tilde{P}^* r(P)$. Furthermore

$$(4.a.6) \qquad \max_{P \in K} Pg^* = \tilde{P}g^* = g^*$$ □

Now consider the dynamic programming recursion

$$(4.a.7) \qquad x(n+1) = \max_{P \in K} \{r(P) + P\, x(n)\} \qquad n \in \mathbb{N}_0,$$

where $x(0)$ is arbitrary. BROWN [11] proved

LEMMA 4.a.2. For $x(n)$, defined by (4.a.7), and g^*, defined by (4.a.5), there exists a constant $\beta > 0$ such that

$$\| x(n) - ng^* \| \leqq \beta \qquad n \in \mathbb{N}_0 .$$ □

Now, if we compare the recursions (4.a.4) and (4.a.7) and if we take $v(0) = x(0)$, then it is easy to see (cf. (4.a.2)) that

$$(4.a.8) \qquad \| v(n) - x(n) \| \leqq \alpha \sum_{k=0}^{n-1} \rho^k \leqq \alpha(1-\rho)^{-1},$$

which, together with lemma 4.a.2, implies that $\{v(n) - ng^* \mid n = 0,1,\dots\}$ is also bounded.

The following result will be proved.

LEMMA 4.a.3. Let each $P \in K$ be aperiodic. Then there exists a vector w^*, depending on $v(0)$, such that

$$(4.a.9) \qquad \lim_{n\to\infty} (v(n) - ng^*) = w^*.$$

Define $K_1 := \{P \in K \mid Pg^* = g^*\}$. Then w^* obeys

$$(4.a.10) \qquad \max_{P \in K_1} \{r(P) + Pw^*\} = w^* + g^*.$$

PROOF. We have seen that the vector w(n), defined by

$$w(n) := v(n) - ng^* \qquad n \in \mathbb{N}_0,$$

is bounded. Note that (4.a.4) can be written as

$$(n+1)g^* + w(n+1) = \max_{P\in K} \{r(n,P) + nPg^* + P\,w(n)\} \qquad n \in \mathbb{N}_0.$$

Obviously, for n sufficiently large, $n \geq n_0$ say, only matrices $P \in K_1$, can be optimal in the n-th step. By (4.a.6) and the definition of K_1, the equation above reduces to

$$\text{(4.a.11)} \quad g^* + w(n+1) = \max_{P\in K_1} \{r(n,P) + P\,w(n)\} \qquad n \geq n_0.$$

Define

$$b := \limsup_{n\to\infty} w(n),$$

$$a := \liminf_{n\to\infty} w(n).$$

Now, let $n_1, n_2, \ldots$ be a sequence such that

$$\lim_{k\to\infty} w(n_k+1) = b,$$

and such that $x := \lim_{k\to\infty} w(n_k)$ exists. Then $x \leq b$. Putting $n = n_k$ in (4.a.11) and letting $k \to \infty$ we obtain (using 4.a.2)

$$g^* + b \leq \max_{P\in K_1} \{r(P) + P\,b\}.$$

Determine $\hat{P} \in K_1$, such that

$$r(\hat{P}) + \hat{P}\,b = \max_{P\in K_1} \{r(P) + P\,b\}.$$

Then, by induction, it follows that

$$\text{(4.a.12)} \quad mg^* + b \leq \sum_{k=0}^{m-1} \hat{P}^k r(\hat{P}) + \hat{P}^m b \qquad m \in \mathbb{N}.$$

On the other hand, by iteration of (4.a.11), we find

$$(4.a.13)\quad m\,g^* + w(n+m) \geqq \sum_{k=0}^{m-1} \hat{P}^{\,k} r(n+m-k-1,\ \hat{P}) + \hat{P}^{\,m} w(n) \qquad n \geqq n_0,\ m \in \mathbb{N}_0.$$

Combining (4.a.2), (4.a.12) and (4.a.13), it is easily shown that

$$b - \hat{P}^{\,m} b \leqq w(n+m) - \hat{P}^{\,m} w(n) + \alpha\rho^n \sum_{k=0}^{m-1} \rho^{m-k-1}\, e \qquad n \geqq n_0,\ m \in \mathbb{N}_0.$$

Choose a sequence $(m_1, m_2, \ldots)$ such that $\lim_{k\to\infty} w_{n+m_k} = a$. Then, obviously,

$$b - \hat{P}^* b \leqq a - \hat{P}^* w(n) + \alpha\rho^n \sum_{k=0}^{\infty} \rho^k\, e \qquad n \geqq n_0.$$

Finally, taking $n = n_1+1,\ n_2+1, \ldots$ we obtain

$$b - \hat{P}^* b \leqq a - \hat{P}^* b.$$

hence $b \leqq a$. It follows that $a = b$. Define $w^* := a$. Then, (4.a.10) follows immediately from (4.a.11), if n tends to infinity. □

The main objective of this appendix is to show that, under the aperiodicity assumption, the convergence of $v(n) - ng^*$ to w^* is geometric. Since (4.a.10) holds, we may define

$$K_2 := \{P \in K_1 \mid r(P) + P\, w^* = w^* + g^*\}.$$

Furthermore, let

$$e(n) := w(n) - w^* = v(n) - ng^* - w^* \qquad n \in \mathbb{N}_0.$$

Since $\lim_{n\to\infty} e(n) = \underline{0}$ by lemma 4.a.3 and since (4.a.2) holds, it follows that for n sufficiently large, $n \geqq \bar{n}$ say, (4.a.11) reduces to

$$(4.a.14)\quad e(n+1) = \max_{P\in K_2} \{r(n,P) - r(P)) + P\, e(n)\} \qquad n \geqq \bar{n}.$$

We have to prove that the convergence of $\{e(n) \mid n = \bar{n}, \bar{n}+1, \ldots\}$ to zero is geometric. In order to simplify this proof, we first treat the case where $r(n,P) = r(P)$ for all $n \in \mathbb{N}_0$ and for all $P \in K$. We have

LEMMA 4.a.4. Let g^* be defined by (4.a.5). Let $r(n,P) = r(P)$ for $n \in \mathbb{N}_0$. Choose $x(0) \in \mathbb{R}^N$ and let $x(n)$ be defined by (4.a.7). If each $P \in K$ is aperiodic, then

$$x^* := \lim_{n\to\infty} (x(n) - ng^*)$$

exists, and the convergence of $(x(n) - ng^*)$ to x^* (for $n \to \infty$) is geometric.

PROOF. The existence of $\lim_{n\to\infty} (x(n) - ng^*)$ follows from lemma 4.a.3 with $x(n) = v(n)$, $r(n,P) = r(P)$ for all $n \in \mathbb{N}_0$ and for all $P \in K$. In this case (4.a.14) becomes

$$\text{(4.a.15)} \qquad e(n+1) = \max_{P\in K_2} P\, e(n) \qquad\qquad n \geqq \bar{n},$$

with $\lim_{n\to\infty} e(n) = \underline{0}$. We have to prove that this convergence is geometric.

Since all matrices are stochastic, it follows immediately from $\lim_{n\to\infty} e(n) = \underline{0}$ and (4.a.15) that $e(n)$ cannot be strictly positive or strictly negative for $n \geqq \bar{n}$. Now, define

$$C(n) = \{i \in S \mid e(n)_i > 0\} \qquad\qquad \bar{C}(n) = S\backslash C(n),$$

$$D(n) = \{i \in S \mid e(n)_i < 0\} \qquad\qquad \bar{D}(n) = S\backslash D(n).$$

We shall prove that there exists an integer t and a constant $\varepsilon \in \mathbb{R}_0^+$, with $0 < \varepsilon \leqq 1$, such that

1. $\max_{i\in S} e(n+t)_i \leqq (1-\varepsilon) \max_{i\in S} e(n)_i \qquad n \geqq \bar{n},$

2. $\min_{i\in S} e(n+t)_i \geqq (1-\varepsilon) \min_{i\in S} e(n)_i \qquad n \geqq \bar{n}.$

Proof of 1.

Suppose $C(\bar{n}) \neq \emptyset$ (otherwise the result holds trivially). Define $R(\bar{n}) = C(\bar{n})$ and for $n > \bar{n}$ recursively

$$R(n) := \{i \in S \mid \exists\, P \in K_2\colon \sum_{j\in R(n-1)} p_{ij} = 1\}.$$

Clearly $R(n) \subset C(n)$. If $R(n) \neq \emptyset$ for $n = \bar{n} + 2^N$, then $R(k) = R(\ell)$ for some $k,\ell \in \mathbb{N}$ with $\bar{n} \leq k < \ell \leq \bar{n} + 2^N$, since there exist at most 2^N-1 nonempty subsets of S. Define $R := R(k) = R(\ell)$. By definition of $R(n)$ there exists a finite sequence of matrices $P(k+1), P(k+2),\ldots,P(\ell)$ such that for Q, defined by $Q := P(\ell)\, P(\ell-1)\ldots P(k+1)$, we have

$$\sum_{j\in R} q_{ij} = 1 \qquad\qquad i \in R.$$

Let $\delta := \min_{i\in R} e(k)$. Then $\delta > 0$. From the definition of $e(n)$ it follows immediately that

$$e(k+m(\ell-k))_i \geq \delta > 0 \qquad\qquad i \in R, m \in \mathbb{N},$$

contradicting the fact that $\lim_{n\to\infty} e(n) = 0$. It follows that $R(n) = \emptyset$ for $n = \bar{n} + 2^N$. Now define $t := 2^N$, and let $\varepsilon(P(1)P(2)\ldots P(t))$ be defined as the smallest <u>positive</u> entry of the matrix

$$P(1)P(2)\ldots P(t).$$

Finally, define

$$\text{(4.a.16)} \qquad \varepsilon := \min\,\{\varepsilon(P(1)P(2)\ldots P(t)) \mid P(1),P(2),\ldots,P(t) \in K_2\}.$$

Then $0 < \varepsilon \leq 1$ and, since $R(n+t) = \emptyset$,

$$\max_{i\in S} e(\bar{n}+t)_i \leq (1-\varepsilon) \max_{i\in S} e(\bar{n})_i.$$

The same result can be obtained when starting with $C(\bar{n}+1)$, $C(\bar{n}+2),\ldots$ Since t and ε do not depend on $\bar{n}$, part 1 follows.

<u>Proof of 2.</u>

Suppose $D(\bar{n}) \neq \emptyset$ (otherwise the result holds trivially). Define $U(\bar{n}) := D(\bar{n})$ and for $n > \bar{n}$ recursively

$$U(n) = \{i \in S \mid \sum_{j\in U(n-1)} p_{ij} = 1 \text{ for all } P \in K_2\}.$$

Then $U(n) \subset D(n)$ and $U(n) = \emptyset$ for $n \geq 2^N$ by arguments similar to those used in the proof of 1 (however, notice the difference in the definitions of $R(n)$ and $U(n)$). It follows that, for $t := 2^N$, there exists a sequence $P(1), P(2),\dots,P(t)$ such that the matrix $\tilde{Q}$, defined by $\tilde{Q} := P(t)P(t-1)..P(1)$ obeys

$$\sum_{j \notin D(\bar{n})} \tilde{q}_{ij} > 0$$

for all $i \in S$. Defining ε as in (4.a.16), it follows that

$$\min_{i\in S} e(\bar{n}+t)_i \geqq (1-\varepsilon) \min_{i\in S} e(\bar{n})_i .$$

Since t and ε do not depend on $\bar{n}$, 2 follows.

Combining 1 and 2 we obtain

$$(\max_{i\in S} e(n+mt)_i - \min_{i\in S} e(n+mt)_i) \leqq (1-\varepsilon)^m (\max_{i\in S} e(n)_i - \min_{i\in S} e(n)_i)$$

for $n \geqq \bar{n}$, $m \in \mathbb{N}_0$. This completes the proof since $0 < \varepsilon \leqq 1$. □

Once having proved lemma 4.a.4, we are ready to handle the more general case where we deal with a sequence $\{r(n,P);\ n = 0,1,\dots\}$ which converges to $r(P)$ geometrically for each $P \in K$. Recall that we have to show that $e(n)$, defined by

$$e(n) := v(n) - ng^* - w^*,$$

tends to zero geometrically (where each $P \in K_2$ is assumed to be aperiodic). By lemma 4.a.3 it is known that $\lim_{n\to\infty} e(n) = \underline{0}$. Furthermore, it has been observed that

$$\text{(4.a.14)} \quad e(n+1) = \max_{P\in K_2} \{r(n,P) - r(P) + P\, e(n)\} \qquad n \geqq \bar{n}.$$

Choose $k \geqq \bar{n}$. Define $\tilde{e}(k,n)$ by

$$\tilde{e}(k,n) = \max_{P\in K_2} P\, \tilde{e}(k,n-1) \qquad k \geqq \bar{n},\ n \in \mathbb{N},$$

with $\tilde{e}(k,0) := e(k)$. It follows from lemma 4.a.3 and the fact that all matrices in K_2 are aperiodic that $\lim_{n\to\infty} \tilde{e}(k,n)$ exists (take $r(n,P) = 0$ in lemma 4.a.3 for all n and P). Denote this limit by $\tilde{e}(k)$, then lemma 4.a.4 implies the existence of constants $c > 0$ and $\delta < 1$ such that

$$(4.a.17) \quad \| \tilde{e}(k,n) - \tilde{e}(k) \| \leqq c\,\delta^n \qquad k \geqq \bar{n},\ n \in \mathbb{N},$$

with c and δ independent of k (compare the proof of lemma 4.a.4). Using (4.a.2), it follows that

$$(4.a.18) \quad \| e(k+n) - \tilde{e}(k,n) \| \leqq \alpha\rho^k \sum_{\ell=0}^{n-1} \rho^\ell \qquad k \geqq \bar{n},\ n \in \mathbb{N}.$$

Hence, for n tending to infinity, we find

$$(4.a.19) \quad \| \tilde{e}(k) \| \leqq \alpha\rho^k (1-\rho)^{-1} \qquad k \geqq \bar{n}.$$

For $k = n$ and $n \geqq \bar{n}$, combination of (4.a.17), (4.a.18) and (4.a.19) yields

$$\| e(2n) \| \leqq \| e(2n)-\tilde{e}(n,n) \| + \| \tilde{e}(n,n)-\tilde{e}(n) \| + \| \tilde{e}(n) \| \leqq$$

$$\leqq \alpha\rho^n(1-\rho)^{-1} + c\,\delta^n + \alpha\rho^n(1-\rho)^{-1} \qquad n \geqq \bar{n}.$$

With $\gamma := \{\max(\rho,\delta)\}$ and $c_1 := 2\alpha(1-\rho)^{-1} + c$ we obtain

$$\| e(2n) \| \leqq c_1\, \gamma^n \qquad n \geqq \bar{n},$$

which establishes the geometric convergence of $\{e(n);\ n = 0,1,2,\ldots\}$ to zero (note that $\| e(2n+1) \| \leqq \| e(2n) \|$). Formally, summing up we have the following theorem.

<u>THEOREM 4.a.5.</u> Let each $P \in K$ be aperiodic. Let $r(n,P)$, $r(P)$ be defined as in the beginning of this appendix and let (4.a.2) hold. Then there exist vectors g^* and w^* such that for $v(n)$, defined by (4.a.4),

$$\lim_{n\to\infty} (v(n) - ng^*) = w^*,$$

and the convergence is geometric. □

Theorem 4.a.5 has proved its usefulness already in the proof of theorem 4.7. It will play a key role in the analysis in chapter 5 where convergence results for general recursions of type (4.0.1) will be proved.

CHAPTER 5

SENSITIVE ANALYSIS OF GROWTH

In chapter 3, it has been shown that a fundamental partition $\{D(\nu), D(\nu-1),\ldots,D(1), D(0)\}$ of the state space S exists, with respect to a finite set K of square nonnegative matrices with the product property. Theorem 3.6.c expresses the relationship between this fundamental partition and the first order asymptotic behaviour of x(n), defined by

$$(5.0.1) \qquad x(n+1) = \max_{P\in K} P\, x(n) \qquad\qquad x(0) > \underline{0};\ n \in \mathbb{N}.$$

In chapter 4, the behaviour of x(n) for $n \to \infty$ has been analyzed in more detail; convergence results have been proved for the case $\nu=1$. The general case (with no restrictions on ν) will be treated in this chapter. It turns out that in this general case there exist vectors $y(1), y(2),\ldots,y(\nu)$ such that, under some aperiodicity assumptions,

$$(5.0.2) \qquad \| x(n) - \{ \tbinom{n}{\nu-1}\hat{\sigma}^{\,n-\nu+1} y(\nu)+\ldots+\tbinom{n}{1}\hat{\sigma}^{\,n-1} y(2) + \hat{\sigma}^{\,n} y(1)\} \| \leqq c\rho^n$$

for some constants $c > 0$ and $\rho < \hat{\sigma}$, with $\hat{\sigma} := \max\{\sigma(P) \mid P \in K\}$. In order to prove (5.0.2), we have to solve a set of "nested" functional equations, a technical detail which will be treated in the appendix of this chapter.

Section 5.1 is devoted to the proof of (5.0.2). In section 5.2, this result will be used to prove structural properties of generalized eigenvectors of the mapping, defined by

$$(5.0.3) \qquad x \to \max_{P\in K} Px \qquad\qquad x \in \mathbb{R}^N.$$

In particular, an analogue of theorem 2.12 will be proved for this mapping. Section 5.3 is devoted to some procedures for estimating $\hat{\sigma},\nu$, and the vectors $y(1),y(2),\dots,y(\nu)$ in (5.0.2).

5.1. Convergence results for dynamic programming recursions: the general case.

In the special cases, treated in the preceding chapter, the limit vector x, defined by $x := \lim_{n\to\infty} \hat{\sigma}^{-n}x(n)$, obeys

$$\max_{P\in K} Px = \hat{\sigma}x.$$

This result is completely in accordance with the "one-matrix" case with index equal to one. It is well known that in the general case (without restrictions on the index) generalized eigenvectors also play a role in the asymptotic expansion of $P^n x(0)$ for $n\to\infty$. If P is aperiodic, then

$$P^n x(0) = \binom{n}{\nu-1}\sigma^{n-\nu+1}y(\nu)+\dots+\binom{n}{1}\sigma^{n-1}y(2) + \sigma^n y(1) + \mathcal{O}(\rho^n) \qquad (n\to\infty)$$

with $\sigma := \sigma(P)$, $\nu = \nu(P)$, $\rho < \sigma$ and $y(k)$ a generalized eigenvector of order k, associated with σ (exploit the Jordan canonical form of P; compare e.g. PEASE [44]).

In this section we show that analogous results can be obtained for dynamic programming recursions of the form

$$(5.0.1) \qquad x(n+1) = \max_{P\in K} P\,x(n) \qquad\qquad x(0) > \underline{0};\ n \in \mathbb{N},$$

with K a finite set of square nonnegative matrices with the product property. The main result of this section can be formulated as follows (cf. ZIJM [76]):

<u>THEOREM 5.1.</u> Let $\hat{\sigma} := \max\,\{\sigma(P) \mid P \in K\}$ and let $\{D(\nu),\dots,D(1),D(0)\}$ be the principal partition of S with respect to K ($\nu \in \mathbb{N}$). Suppose that all matrices $P \in K$ with $\sigma(P) = \hat{\sigma}$ and $\nu(P) = \nu$ are aperiodic. Then there exist unique vectors $y(1),y(2),\dots,y(\nu)$, and constants $c > 0$, $\rho < \hat{\sigma}$, such that for the sequence $x(n)$, defined by (5.0.1), the following holds

$$(5.1.1) \qquad \| x(n)-\{\binom{n}{\nu-1}\hat{\sigma}^{n-\nu+1}y(\nu)+\dots+\binom{n}{1}\hat{\sigma}^{n-1}y(2)+\hat{\sigma}^n y(1)\}\| \leqq c\rho^n \qquad n \in \mathbb{N}.$$

The vectors $y(k)$ satisfy

$$y(k)_i > 0 \qquad i \in D(k),\ k = 1,\dots,\nu,$$

$$y(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = 1,\dots,\nu.$$

Furthermore, the following relationship holds :

$$(5.1.2.\nu)\quad \max_{P\in K} P\, y(\nu) = \hat{\sigma}\, y(\nu)$$

$$(5.1.2.\ell)\quad \max_{P\in K_{\ell+1}} P\, y(\ell) = \hat{\sigma}\, y(\ell) + y(\ell+1) \qquad \ell = \nu-1,\nu-2,\dots,1,$$

with

$$(*)\quad \begin{cases} K_\nu := \{P \in K \mid P\, y(\nu) = \hat{\sigma}\, y(\nu)\}, \\ K_\ell := \{P \in K_{\ell+1} \mid P\, y(\ell) = \hat{\sigma}\, y(\ell) + y(\ell+1)\} \qquad \ell = \nu-1,\nu-2,\dots,2,1. \end{cases}$$

PROOF. The proof will be given by induction with respect to ν. For $\nu=1$ the results follow from theorem 4.6. Now assume that theorem 5.1 holds for $\nu = 1,2,\dots,t-1$ and let the principal partition of S with respect to K be given by $\{D(t),\dots,D(1),D(0)\}$. As before, we define for each $P \in K$ and for $m = 1,2,\dots,t$

$$P^{(m)} = \begin{bmatrix} P^{(m,m)} & P^{(m,m-1)} \dots & P^{(m,1)} & P^{(m,0)} \\ & P^{(m-1,m-1)} \dots & P^{(m-1,1)} & P^{(m-1,0)} \\ & \ddots & \vdots & \vdots \\ & & P^{(1,1)} & P^{(1,0)} \\ & & & P^{(0,0)} \end{bmatrix}$$

where, as usual, $P^{(k,\ell)}$ denotes the restriction of P to $D(k) \times D(\ell)$, for $k,\ell = 0,1,\dots,t$. Note that $P^{(t)} = P$ for all $P \in K$.

By the induction hypothesis there exist vectors $w(1),w(2),\dots,w(t-1)$, with

$$w(k)_i > 0 \qquad i \in D(k),\ k = 1,\dots,t-1,$$

$$w(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = 1,\dots,t-1,$$

such that for some constants c_1, c_2 and ρ, with $\rho < \hat{\sigma}$, the following holds

$$(5.1.3) \qquad c_1\rho^n \leqq x(n)_i - \{\binom{n}{t-2}\hat{\sigma}^{n-t+2} w(t-1)_i + .. + \hat{\sigma}^n w(1)_i\} \leqq c_2\rho^n, \qquad i \in \bigcup_{\ell=0}^{t-1} D(\ell),\ n \in \mathbb{N}.$$

These vectors $w(1), w(2), \ldots, w(t-1)$ satisfy

$$(5.1.4.(t-1)) \qquad \max_{P \in K} P^{(t-1)} w(t-1) = \hat{\sigma}\, w(t-1)$$

$$(5.1.4.\ell) \qquad \max_{P \in H_{\ell+1}} P^{(t-1)} w(\ell) = \hat{\sigma}\, w(\ell) + w(\ell+1) \qquad \ell = t-2, \ldots, 2, 1,$$

where $H_\ell \subset K$ denotes the set of matrices that maximize the left-hand side of (5.1.4.ℓ) ($\ell = t-1, t-2, \ldots, 2, 1$).

We want to find vectors $y(1)$, $y(2), \ldots y(t)$ such that (5.1.1) and (5.1.2) hold for $\nu = t$. Since the asymptotic behaviour of $x(n)_i$, for $n \to \infty$, is already completely determined by (5.1.3), for $i \in S \backslash D(t)$, it follows that we must choose

$$(5.1.5) \qquad y(k)_i := w(k)_i \qquad i \in \bigcup_{\ell=0}^{t-1} D(\ell),\ k = 1, 2, .. t-1,$$

$$(5.1.6) \qquad y(t)_i := 0 \qquad i \in \bigcup_{\ell=0}^{t-1} D(\ell).$$

Then, obviously, $K_\ell \subset H_\ell$ for $\ell = 1, 2, \ldots, t-1$ (K_ℓ is defined by (*)). What remains, is the determination of $y(k)_i$ for $i \in D(t)$, $k = 1, 2, \ldots, t$. Using (5.1.5), (5.1.6) and

$$w(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = 1, .., t-1,$$

it follows that we must have

$$(5.1.7.t) \quad \max_{P \in K} \{P^{(t,t)} y(t)^{(t)}\} = \hat{\sigma}\, y(t)^{(t)},$$

$$(5.1.7.k) \quad \max_{P \in K_{k+1}} \{P^{(t,t)} y(k)^{(t)} + \sum_{\ell=k}^{t-1} P^{(t,\ell)} w(k)^{(\ell)}\} = \hat{\sigma}\, y(k)^{(t)} + y(k+1)^{(t)},$$

$$k = t-1, \ldots, 1,$$

where $y(k)^{(t)}$ denotes the restriction of $y(k)$ to $D(t)$, for $k = 1,2,\dots,t$, and $w(k)^{(\ell)}$ the restriction of $w(k)$ to $D(\ell)$, for $k,\ell = 1,\dots,t-1$.

By the induction hypothesis $w(t-1)^{(t-1)} > \underline{0}$. Furthermore, by theorem 3.6, there exists a matrix $\hat{P} \in K$ such that each state in $D(t)$ has access to some state in $D(t-1)$ under $\hat{P}$. Note that $\hat{P}^{(t,t)}$ possesses a strictly positive right-eigenvector, associated with $\hat{\sigma}$ (cf. theorem 3.6.b). Applying lemma 2.5 we obtain

$$(\hat{P}^{(t,t)})^* \hat{P}^{(t,t-1)} w(t-1)^{(t-1)} > \underline{0}.$$

It now follows from theorem 5.a.1, in the appendix of this chapter, that there exists a solution $\{y(t)^{(t)},\dots,y(2)^{(t)}, y(1)^{(t)}\}$ of (5.1.7), with $y(t)^{(t)} > \underline{0}$, $y(\ell)^{(t)}$ uniquely determined for $\ell = 2,\dots,t$, whereas there is some freedom in the choice of $y(1)^{(t)}$.

Recalling the definitions of $y(k)_i$ for $i \in \bigcup_{\ell=0}^{t-1} D(\ell)$ and $k = 1,\dots,t$ (cf. (5.1.5) and (5.1.6)), it follows that we have found a solution $\{y(t),y(t-1),\dots,y(2),y(1)\}$ of (5.1.2).

Next, we must show that this solution satisfies (5.1.1), where we allow a possible change in the value of $y(1)^{(t)}$.

Define

$$z(n) := \binom{n}{t-1}\hat{\sigma}^{n-t+1} y(t) + \dots + \binom{n}{1}\hat{\sigma}^{n-1} y(2) + \hat{\sigma}^{n} y(1) \qquad n \in \mathbb{N}_0.$$

First, the boundedness of the sequence $\{\hat{\sigma}^{-n}(x(n)-z(n));\ n = 0,1,2,\dots\}$ will be established. Note that by (5.1.2)

$$(5.1.8) \qquad \max_{P\in K_2} P\, z(n) = z(n+1).$$

Since $\lim_{n\to\infty} \binom{n}{k}/\binom{n}{k-1} = \infty$, there exists an integer n_0, such that

$$(5.1.9) \qquad \max_{P\in K} P\, z(n) = \max_{P\in K_2} P\, z(n) \qquad n \geqq n_0.$$

Furthermore, it is possible to choose a constant $\alpha > 0$ such that

$$(5.1.10) \qquad x(n_0)_i \leqq z(n_0)_i + \alpha\hat{\sigma}^{n_0} y(t)_i \qquad i \in D(t),$$

since $y(t)_i > 0$ for $i \in D(t)$. Obviously, we also have for $n \geqq n_0$

$$(5.1.11)\quad \max_{P\in K} P(z(n)+\alpha\hat{\sigma}^{n} y(t)) = \max_{P\in K_2} P(z(n)+\alpha\hat{\sigma}^{n} y(t)) =$$
$$= z(n+1)+\alpha\hat{\sigma}^{n+1} y(t)$$

(note that $y(t)_i = 0$ for $i \in S\backslash D(t)$).

Finally, we choose β such that

$$(5.1.12)\quad \max_{P\in K_2} \{P^{(t,t-1)}e^{(t-1)}+\ldots+P^{(t,0)}e^{(0)}\} \leqq \beta\hat{\sigma}\, y(t)^{(t)}$$

(where $e^{(\ell)}$ is the restriction of e to $D(\ell)$, $\ell = 0,1,\ldots,t-1$).

Combining (5.1.3), (5.1.10), (5.1.11) and (5.1.12), it is easy to show (by induction) that, for $i \in D(t)$ and $n > n_0$,

$$(5.1.13)\quad x(n)_i \leqq z(n)_i + \alpha\hat{\sigma}^{n} y(t)_i + \beta c_2 \rho^{n_0} \hat{\sigma}^{n-n_0} (1-\rho\hat{\sigma}^{-1})^{-1} y(t)_i .$$

On the other hand, we may choose a constant $\delta > 0$, such that

$$(5.1.14)\quad x(0)_i \geqq y(1)_i - \delta\, y(t)_i = z(0)_i - \delta\, y(t)_i \qquad i \in D(t),$$

and since for $n \geqq 0$

$$(5.1.15)\quad \max_{P\in K} P(z(n) - \delta\hat{\sigma}^{n} y(t)) \geqq \max_{P\in K_2} P(z(n)-\delta\hat{\sigma}^{n} y(t)) = z(n+1)-\delta\hat{\sigma}^{n+1} y(t),$$

it follows inductively, by combining (5.1.3), (5.1.8), (5.1.12), (5.1.14) and (5.1.15) that

$$(5.1.16)\quad x(n)_i \geqq z(n)_i - \delta\hat{\sigma}^{n} y(t)_i - \beta c_1 \hat{\sigma}^{n} (1-\rho\hat{\sigma}^{-1})^{-1} y(t)_i \qquad i \in D(t),\ n \geqq n_0.$$

Combination of (5.1.13) and (5.1.16) yields the boundedness of $\{\hat{\sigma}^{-n}(x(n)_i - z(n)_i)\ ;\ n = 0,1,2,\ldots\}$, for $i \in D(t)$.

If we define

$$(5.1.17)\quad \bar{z}(n) := z(n) - \hat{\sigma}^{n} y(1) \qquad n \in \mathbb{N}_0,$$

then it follows immediately that $\{\hat{\sigma}^{-n}(x(n)_i - \bar{z}(n)_i); n = 0,1,\ldots\}$ is also bounded for $i \in D(t)$.

Up to now we have determined unique vectors $y(t), y(t-1), \ldots, y(2)$, as part of a solution of (5.1.2) for $\nu = t$, such that for some $c > 0$

(5.1.18) $\quad \| x(n) - \{\binom{n}{t-1}\hat{\sigma}^{n-t+1} y(t)+\ldots+\binom{n}{1}\hat{\sigma}^{n-1} y(2)\}\| \leq c\hat{\sigma}^{n} \qquad n \in \mathbb{N}.$

Now define, for $n \in \mathbb{N}_0$,

(5.1.19) $\quad v(n) := x(n) - \{\binom{n}{t-1}\hat{\sigma}^{n-t+1} y(t)+\ldots+\binom{n}{1}\hat{\sigma}^{n-1} y(2)\} = x(n) - \bar{z}(n).$

From the induction hypothesis we know that

$$\lim_{n\to\infty} \hat{\sigma}^{-n} v(n)_i = y(1)_i \qquad i \in \bigcup_{\ell=0}^{t-1} D(\ell).$$

Furthermore, the sequence $\{\hat{\sigma}^{-n} v(n)_i;\ n = 0,1,2,\ldots\}$ is bounded for $i \in D(t)$. What remains is the proof that the sequences converge geometrically to some v_i, for $n\to\infty$.

Note that (5.1.18) implies that for some $n_1 \in \mathbb{N}$, and $n \geq n_1$, we have

$$\max_{P\in K} P(\bar{z}(n)+v(n)) = \max_{P\in K_2} P(\bar{z}(n)+v(n)) = \bar{z}(n+1)-\hat{\sigma}^{n} y(2) + \max_{P\in K_2} Pv(n).$$

Since for $n \geq 0$,

$$\bar{z}(n+1)+v(n+1) = x(n+1) = \max_{P\in K} Px(n) = \max_{P\in K} P(\bar{z}(n)+v(n)),$$

we find

$$v(n+1) = \max_{P\in K_2} \{Pv(n) - \hat{\sigma}^{n} y(2)\} \qquad n \geq n_1.$$

Let

$$\bar{v}(n) := \hat{\sigma}^{-n} v(n) \qquad n \in \mathbb{N}_0.$$

Then we obtain

$$\bar{v}(n+1) = \max_{P\in K_2} \{\hat{\sigma}^{-1} P\bar{v}(n) - \hat{\sigma}^{-1} y(2)\} \qquad n \geq n_1.$$

Let $\bar{v}(n)^{(k)}$ denote the restriction of $\bar{v}(n)$ to $D(k)$ $(k = 1,2,\ldots,t;\ n \in \mathbb{N})$. Then in particular

(5.1.20) $\quad \bar{v}(n+1)^{(t)} = \max_{P\in K_2} \{\hat{\sigma}^{-1} P^{(t,t)}\bar{v}(n)^{(t)} + r(n,P)\} \qquad n \geq n_1,$

where, for each $P \subset K_2$ and $n \in \mathbb{N}$, $r(n,P)$ is defined by

$$r(n,P) := \hat{\sigma}^{-1} P^{(t,t-1)} \bar{v}(n)^{(t-1)} + .. + \hat{\sigma}^{-1} P^{(t,1)} \bar{v}(n)^{(1)} - \hat{\sigma}^{-1} y(2)^{(t)}.$$

It follows from the definition of $\bar{v}(n)$ and the induction hypothesis that, for each $P \in K_2$, $r(n,P)$ converges geometrically to $r(P)$ for $n \to \infty$, where $r(P)$ is for each $P \in K_2$ defined by

$$r(P) := \hat{\sigma}^{-1} P^{(t,t-1)} y(1)^{(t-1)} + .. + \hat{\sigma}^{-1} P^{(t,1)} y(1)^{(1)} - \hat{\sigma}^{-1} y(2)^{(t)}$$

Since $K_2 \subset K_t$ it follows that, for all $P \in K_2$,

$$\hat{\sigma}^{-1} P^{(t,t)} y(t)^{(t)} = y(t)^{(t)}$$

(recall that $y(t)^{(t)} > \underline{0}$). From (5.1.5) and multiplication of (5.1.7.1) with $(P^{(t,t)})^*$, it follows that

$$(5.1.21) \quad \max_{P \in K_2} (P^{(t,t)})^* \, r(P) = \underline{0}.$$

By a transformation, similar to the one used in the proof of theorem 4.7, K_2 is transformed in a set of stochastic matrices. By combination of (5.1.20) and (5.1.21) with the geometric convergence result of appendix 4.A (theorem 4.a.5), it follows that there exists a vector $v^{(t)}$, defined on $D(t)$, such that $\{\bar{v}(n)^{(t)};\ n = 0,1,2,...\}$ converges to $v^{(t)}$ geometrically, for $n \to \infty$. Recalling the definitions of $\bar{v}(n)$ and $v(n)$ (cf. (5.1.19)), we find that for some constants $c_3 > 0$ and $\delta < \hat{\sigma}$

$$(5.1.22) \quad \| x(n)^{(t)} - \{ \tbinom{n}{t-1} \hat{\sigma}^{n-t+1} y(t)^{(t)} + ... + \tbinom{n}{1} \hat{\sigma}^{n-1} y(2)^{(t)} + \hat{\sigma}^{n} v^{(t)} \} \| \leqq c_3 \delta^n$$

for all $n \in \mathbb{N}$.

For $n \to \infty$, (5.1.20) becomes

$$v^{(t)} = \max_{P \in K_2} \{ \hat{\sigma}^{-1} P^{(t,t)} v^{(t)} + r(P) \},$$

or, recalling the definition of $r(P)$,

$$(5.1.23)\quad \max_{P\in K_2} \{P^{(t,t)}v^{(t)}+P^{(t,t-1)}y(1)^{(t-1)}+..+P^{(t,1)}y(1)^{(1)}\} = \hat{\sigma}v^{(t)}+y(2)^{(t)}.$$

Recall that the vectors $y(t)^{(t)}$, $y(t)^{(t-1)}, \ldots y(t)^{(2)}$ are uniquely determined already by (5.1.7). Choosing $y(t)^{(1)} := v^{(t)}$ and $y(k)^{(\ell)}$ according to (5.1.5) and (5.1.6) for $k,\ell = 1,\ldots,t-1$, it follows from (5.1.22) and (5.1.23), together with the induction hypothesis, that theorem 5.1 holds for $\nu = t$.

By induction the theorem holds for each ν. □

Theorem 5.1 yields rather strong results concerning the asymptotic behaviour of dynamic programming recursions of the type

$$(5.0.1)\quad x(n+1) := \max_{P\in K} Px(n) \qquad x(0) > \underline{0},\ n \in \mathbb{N},$$

at least under suitable aperiodicity assumptions. The reader may note that part c of theorem 3.6 follows immediately from theorem 5.1, since

$$\max_{P\in K} P^{(k,k)}y(k)^{(k)} = \hat{\sigma}y(k)^{(k)} > \underline{0} \qquad k = 1,2,\ldots,\nu.$$

A number of well-known results concerning the asymptotic behaviour of value functions of Markov decision processes follow as special cases from theorem 5.1. As an example, we consider a model, arising from the study of sensitive optimality criteria in Markov decision processes. This model has been studied in VAN DER WAL [68] (compare also VAN DER WAL AND ZIJM [69], and ZIJM [77]).

Consider (for fixed $k \in \mathbb{N}_0$) the following dynamic programming recursion

$$(5.1.24)\quad v(n+1) = \max_{P\in K} \{\tbinom{n}{k}\, r(P) + Pv(n)\} \qquad n \in \mathbb{N}_0,$$

where K denotes a finite set of stochastic $N \times N$ matrices, $r(P)$ and $v(n)$ are vectors in $\mathbb{R}^N$ ($n \in \mathbb{N}_0$). We assume $v(0) > \underline{0}$, $r(P) \geqq \underline{0}$. Furthermore, the set of $N \times (N+1)$ matrices

$$\{(P,\ r(P)) \mid P \in K\}$$

is supposed to have the product property.

VAN DER WAL [68] showed that there exist vectors $y(1), y(2), .., y(k+2)$, and a constant $\rho < 1$, such that

$$(5.1.25)\quad v(n) = \binom{n}{k+1}y(k+2)+\ldots+\binom{n}{1}y(2)+y(1)+O(\rho^n)\qquad (n\to\infty).$$

However, by a simple trick, (5.1.24) can be reformulated as

$$\begin{pmatrix} v(n+1) \\ \binom{n+1}{k} \\ \vdots \\ \binom{n+1}{1} \\ 1 \end{pmatrix} = \max_{P\in K} \begin{pmatrix} P & r(P) & & & \\ & 1 & 1 & & \\ & & \ddots & \ddots & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} v(n) \\ \binom{n}{k} \\ \vdots \\ \binom{n}{1} \\ 1 \end{pmatrix} \qquad n \in \mathbb{N}_0$$

Hence, (5.1.25) follows immediately from theorem 5.1 (with $\nu = k+2$).

Recursions of type (5.1.24) play an important role in the study of so-called k-average optimality criteria in Markov decision processes, a concept introduced by SLADKY [54], as an extension of Veinotts' overtaking optimality criterion (VEINOTT [63], compare also SLADKY [57]). Note that for $k = 0$, we obtain the geometric convergence result of SCHWEITZER AND FEDERGRUEN [61] again (cf. appendix 4.A). Furthermore, the reader may verify that, for this example, equations (5.1.2) turn into the well-known policy-iteration equations for k-average optimal policies (cf. also SLADKY [54]).

The reader may note that theorem 5.1 can be extended by decomposing D(0) again, etc. etc. We then obtain results concerning the asymptotic behaviour of $x(n)$ with respect to the spectral partition of S (compare theorem 3.2).

Also "periodic analogues" of theorem 5.1 may be formulated. Details will not be given here. However, the reader may note, that a set K of square nonnegative matrices with the product property can easily be transformed into an equivalent set $\tilde{K}$, in which the aperiodicity assumptions of theorem 5.1 are fulfilled for each matrix. Apply the following data transformation (cf. SCHWEITZER [59]):

$$\tilde{P} = \delta P + (1-\delta)\sigma(P)I \qquad (0 < \delta < 1).$$

Clearly, class-structures and chain-structures of P and $\tilde{P}$ are completely identical, but each $\tilde{P}$ is aperiodic. $\tilde{P}$ possesses the same (generalized) eigenvectors as P, and

$$\sigma(\tilde{P}) = \sigma(P)$$

Furthermore, if P^* exists, $\tilde{P}^*$ exists and $\tilde{P}^* = P^*$.

The results concerning the asymptotic behaviour of x(n), for $n \to \infty$, can be used to obtain estimates for $\hat{\sigma}$, as well as for the vectors y(1),y(2),...,y(ν), determined in theorem 5.1. This topic will be treated in section 5.3. First, some attention is paid to an analogue of theorem 2.12 for sets of nonnegative matrices with the product property.

5.2. The structure of generalized eigenvectors.

As part of theorem 5.1 a solution {y(1),y(2),...,y(ν)} of the system of equations (5.1.2) was obtained with

$$(5.2.1) \qquad y(k)_i > 0 \qquad i \in D(k),\ k = 1,2,..,\nu,$$

$$(5.2.2) \qquad y(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = 1,2,..,\nu.$$

However, note that, if {y(1),y(2),...,y(ν)} is a solution of (5.1.2), then so is {w(1),w(2),...,w(ν)}, defined by

$$(5.2.3) \quad \begin{cases} w(\nu) := y(\nu) & \\ w(k) := y(k) + \alpha w(k+1) & k = \nu-1,\nu-2,\ldots,1, \end{cases}$$

where α can be chosen arbitrarily. In view of (5.1.1) and (5.2.2) α can be chosen so large, that

$$(5.2.4) \qquad w(k)_i > 0 \qquad i \in \bigcup_{\ell=k}^{\nu} D(\ell),\ k = 1,2,\ldots,\nu,$$

$$(5.2.5) \qquad w(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = 1,2,\ldots,\nu.$$

Furthermore, the reader may verify that for α sufficiently large

$$\max_{P\in K_{\ell+1}} P\,w(\ell) = \max_{P\in K_{\ell+2}} P\,w(\ell) = \ldots = \max_{P\in K} P\,w(\ell) \qquad \ell = 1,\ldots,\nu. \tag{5.2.6}$$

Combination of (5.2.3), (5.2.4), (5.2.5) and (5.2.6) yields the following theorem:

THEOREM 5.2. Let $\hat{\sigma} := \max\{\sigma(P) \mid P \in K\}$ and let $\{D(\nu),\ldots,D(1),D(0)\}$ be the principal partition of S with respect to K. Then there exists a set of semi-positive vectors $\{w(\nu),\ldots,w(2),w(1)\}$, such that

$$\max_{P\in K} P\,w(\nu) = \hat{\sigma}w(\nu)$$

$$\max_{P\in K} P\,w(k) = \hat{\sigma}w(k) + w(k+1) \qquad k = \nu-1,\ldots,2,1.$$

For $k = \nu,\nu-1,\ldots,2,1$ we have

$$w(k)_i > 0 \qquad i \in \bigcup_{\ell=k}^{\nu} D(\ell),$$

$$w(k)_i = 0 \qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell). \qquad \square$$

Theorem 5.2 is the analogue of theorem 2.12, for the set K. A direct proof (without using theorem 5.1) has been given by ZIJM [75]. The proof given there is constructive; it uses an iteration method which is also the basis of the results in appendix 5.A. The aperiodicity assumption (cf. theorem 5.1) can be removed.

5.3. Estimation of growth characteristics.

We have seen that the consecutive generalized eigenvectors appearing in the asymptotic expansion of x(n) (defined by (5.0.1)), for $n\to\infty$, can be obtained from functional equations of the type (5.1.2). However, in large scale systems, it requires an enormous amount of work to solve these equations. One wonders whether cheap approximation methods exist for obtaining these generalized eigenvectors. By cheap we mean relatively cheap compared with exact methods.

The most important growth characteristic to be estimated is the spectral radius $\hat{\sigma} = \max\{\sigma(P) \mid P \in K\}$. Several methods have been developed.

MANDL [38], for instance, developed a bisection procedure, of which each step results into an upperbound β_n and a lowerbound α_n for $\hat{\sigma}$. Defining $\lambda_n := \frac{1}{2}(\alpha_n+\beta_n)$ he examines, whether there exists a strictly positive, finite solution $x = x(\lambda_n)$ of

$$(5.3.1) \qquad \lambda_n x = \max_{P\in K} \{e+Px\}.$$

If such a solution exists then clearly $\hat{\sigma} < \lambda_n$ (lemma 2.6), otherwise $\hat{\sigma} \geq \lambda_n$. In the first case we take $\beta_{n+1} := \lambda_n$, $\alpha_{n+1} := \alpha_n$, in the second case $\beta_{n+1} := \beta_n$, $\alpha_{n+1} := \lambda_n$.

ZIJM [74] also uses (5.3.1) to determine a sequence $\{\lambda_n;\ n = 1,2,\ldots\}$, which tends to $\hat{\sigma}$ from above. Using in each step the solution $x(\lambda_n)$, a better approximation λ_{n+1} is calculated. Instead of $x(\lambda_n)$, also approximations of $x(\lambda_n)$ can be used for updating λ_n (see also ZIJM [73]).

SLADKY [55] gives upper- and lowerbounds for $\hat{\sigma}$ when K contains only aperiodic, irreducible matrices. These bounds are based on theorem 4.2. To be precise we have:

<u>LEMMA 5.3</u>. Let all matrices in K be irreducible, let $\hat{\sigma} := \max\{\sigma(P) \mid P \in K\}$ and let a $P \in K$ exist, with $\sigma(P) = \hat{\sigma}$, which is aperiodic. Let $x(n)$ be defined by (5.0.1), with $x(0) > \underline{0}$. Then, for α_n and β_n, defined by

$$\alpha_n := \min_{i\in S} (x(n+1)_i/x(n)_i) \qquad n \in \mathbb{N},$$

$$\beta_n := \max_{i\in S} (x(n+1)_i/x(n)_i) \qquad n \in \mathbb{N},$$

we have

$$\alpha_{n+1} \geq \alpha_n \qquad \beta_{n+1} \leq \beta_n \qquad n \in \mathbb{N},$$

and

$$(5.3.2) \qquad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \hat{\sigma}.$$

<u>PROOF</u>. Since

$$x(n+2) = \max_{P\in K} Px(n+1) \geq \alpha_n \max_{P\in K} Px(n) = \alpha_n\, x(n+1)$$

it follows that $\alpha_{n+1} \geqq \alpha_n$. Similarly, one has $\beta_{n+1} \leqq \beta_n$.

Furthermore, by theorem 4.2,

$$(5.3.3) \qquad \lim_{n\to\infty} \hat{\sigma}^{-n} x(n) = x > \underline{0},$$

and this convergence is geometric. This implies (5.3.2). □

The reader may verify that the proof of lemma 5.3 only depends on the existence of a strictly positive vector x, such that (5.3.3) holds, rather than on the irreducibility assumption. As a consequence, (5.3.2) remains valid whenever (5.3.3) holds. An important case, in which (5.3.3) is fulfilled, is the case of a *communicating* set (cf. BATHER [1]).

DEFINITION 5.1. K is said to be *communicating* if for each $i,j \in S$ there exists a $P \in K$ (depending on i and j), such that i has access to j under P. □

LEMMA 5.4. If K is communicating, and if there exists an aperiodic matrix $P \in K$, with $\sigma(P) = \hat{\sigma}$, then $\lim_{n\to\infty} \hat{\sigma}^{-n} x(n) = x > \underline{0}$, where x(n) satisfies (5.0.1).

PROOF. Let $\{D(\nu),\dots,D(1),D(0)\}$ be the principal partition of S with respect to K. If $\nu > 1$ or if $D(0) \neq \emptyset$, K is not communicating. Hence $\nu = 1$ and $D(0) = \emptyset$. By theorem 4.6 the result follows. □

A communicating set can also be defined in terms of the *incidence matrix* of K.

DEFINITION 5.2. The *incidence matrix* T of K is defined by

$$t_{ij} = \begin{cases} 1 & \text{if } p_{ij} > 0, \text{ for some } P \in K \\ 0 & \text{otherwise} \end{cases} \qquad i,j \in S.$$

□

It follows immediately that K is communicating if and only if its incidence matrix T is irreducible.

Lemma 5.4 implies that also in the communicating case one can obtain upper- and lowerbounds for $\hat{\sigma}$. In general, we can determine all classes of the incidence matrix T, $C(1),C(2),\dots,C(m)$ say. With each $P \in K$ we may associate a matrix $\tilde{P}$, as follows

$$\tilde{p}_{ij} = \begin{cases} p_{ij} & \text{if } i,j \in C(k), \text{ for some } k \in \{1,2,\dots,n\} \\ 0 & \text{otherwise} \end{cases}$$

for all $i,j \in S$.

Note that P and $\tilde{P}$ possess the same eigenvalues ($P \in K$). Let $\tilde{K} := \{\tilde{P} \mid P \in K\}$. It is easily verified that $\max\,(\nu(\tilde{P}) \mid \tilde{P} \in \tilde{K},\ \sigma(\tilde{P}) = \hat{\sigma}) = 1$ (compare the proof of lemma 5.4)

Choose $\tilde{x}(0) > \underline{0}$ and define

$$\tilde{x}(n+1) := \max_{\tilde{P} \in \tilde{K}} \tilde{P}\tilde{x}(n) \qquad n \in \mathbb{N}_0.$$

Clearly, the restriction of all $P \in K$ to $C(k) \times C(k)$ gives a communicating set, for $k = 1,2,\dots,m$. Define

$$\alpha_n^{(k)} := \min_{i \in C(k)} (\tilde{x}(n+1)_i / \tilde{x}(n)_i) \qquad k = 1,\dots,m;\ n \in \mathbb{N}_0,$$

$$\beta_n^{(k)} := \max_{i \in C(k)} (\tilde{x}(n+1)_i / \tilde{x}(n)_i) \qquad k = 1,\dots,m;\ n \in \mathbb{N}_0,$$

and

$$\alpha_n := \max_{k=1,\dots,m} \alpha_n^{(k)} \qquad \beta_n := \max_{k=1,\dots,m} \beta_n^{(k)} \qquad n \in \mathbb{N}_0.$$

Then, clearly,

$$\text{(5.3.4)} \qquad \alpha_n \leqq \hat{\sigma} \leqq \beta_n \qquad n \in \mathbb{N}_0,$$

and

$$\text{(5.3.5)} \qquad \lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = \hat{\sigma}.$$

In this way, sharp estimates for $\hat{\sigma}$ are obtained. Note that the convergence in (5.3.5) is geometric (although the original matrices $P \in K$ may have index larger than one).

Let us return to the original set K of square nonnegative matrices with the product property. Choose $x(0) > \underline{0}$, and let

$$x(n) := \max_{P \in K} Px(n-1) \qquad n \in \mathbb{N}. \tag{5.0.1}$$

Define, for $n \in \mathbb{N}_0$, differences $\Delta^0 x(n)$, $\Delta^1 x(n)$, $\Delta^2 x(n), \ldots$ by

$$\Delta^0 x(n) := x(n)$$

$$\Delta^k x(n) := \Delta^{k-1} x(n+1) - \Delta^{k-1} x(n) \qquad k \in \mathbb{N}.$$

Under appropriate aperiodicity assumptions $x(n)$ obeys

$$x(n) = \binom{n}{\nu-1}\hat{\sigma}^{\,n-\nu+1}\, y(\nu) + \ldots + \binom{n}{1}\hat{\sigma}^{\,n-1}\, y(2) + \hat{\sigma}^{\,n}\, y(1) + O(\rho^n) \qquad (n \to \infty)$$

with $\rho < \hat{\sigma}$ and $y(1), y(2), \ldots, y(\nu)$ as in theorem 5.1. Furthermore, define

$$\bar{y}(k) := \hat{\sigma}^{\,-k+1}\, y(k) \qquad k = 1,2,\ldots,\nu,$$

$$\bar{x}(n) := \hat{\sigma}^{\,-n}\, x(n) \qquad n \in \mathbb{N}_0 .$$

Then, clearly

$$\bar{x}(n) = \binom{n}{\nu-1}\bar{y}(\nu) + \ldots + \binom{n}{1}\bar{y}(2) + y(1) + O((\rho\hat{\sigma}^{-1})^n) \qquad (n \to \infty).$$

It is readily verified that

$$\Delta^{\nu-1}\bar{x}(n) = \bar{y}(\nu) + O((\rho\hat{\sigma}^{-1})^n) \qquad (n \to \infty).$$

In the same way we find from

$$\Delta^{\nu-2}\bar{x}(n) = \binom{n}{1}\bar{y}(\nu) + \bar{y}(\nu-1) + O((\rho\hat{\sigma}^{-1})^n) \qquad (n \to \infty),$$

that

$$\Delta^{\nu-2}\bar{x}(n) - \binom{n}{1}\Delta^{\nu-1}\bar{x}(n-1) = \bar{y}(\nu-1) + O((\rho\hat{\sigma}^{-1})^{-n}) \qquad (n \to \infty).$$

In general, if we define for $n \in \mathbb{N}$

$$h^{(\nu-1)}(n) := \Delta^{\nu-1}\bar{x}(n)$$

and for $k > 1$

$$h^{(\nu-k)}(n) := \Delta^{\nu-k}\bar{x}(n) - \{\binom{n}{1} h^{(\nu-k+1)}(n-1)+\ldots+\binom{n}{k-1}h^{(\nu-1)}(n-k+1)\},$$

then a simple calculation shows that

$$h^{(\nu-k)}(n) = \bar{y}(\nu-k+1) + O((\rho\hat{\sigma}^{-1})^n) \qquad k = 1,\ldots,\nu \qquad (n\to\infty).$$

In order to determine how good a particular estimate is and to specify upper- and lowerbounds for the vectors $\bar{y}(k)$, $k = 1,\ldots,\nu$, one should have an upperbound for ρ. A rough upperbound can be obtained from the proof of lemma 4.a.4 in appendix 4.A. In general, however, the determination of sharp upperbounds for ρ still remains an open problem. Furthermore, estimation methods like the one described above are not very efficient in numerical sense if ν is large (caused by the loss of significant digits). It seems to us that large parts of an "estimation theory" for the asymptotic expansion of $x(n)$ still have to be developed; this section is only meant as a first attempt. The topic, however, appears to be of increasing importance, e.g., in studying sensitive optimality criteria in Markov decision processes (cf. SLADKY [57], VAN DER WAL [68]).

Appendix 5.A. Nested functional equations

The proof of theorem 5.1 requires the existence of a solution of a set of "nested" functional equations (cf. (5.1.7)). In this appendix it is proved that such a solution does indeed exist and that it can be found in a constructive way. The proof is based on a generalization of Howard's policy iteration algorithm for the average reward criterion in Markov decision processes (cf. HOWARD [29], see also MILLER AND VEINOTT [82]).

We now state the problem to be solved in this appendix. Let K denote a finite set of nonnegative $N \times N$ matrices with the product property, let $\hat{\sigma} := \max \{\sigma(P) \mid P \in K\}$ and suppose that there exists a vector $\hat{u} > \underline{0}$, such that

$$(5.a.1) \qquad \max_{P \in K} P\hat{u} = \hat{P}\hat{u} = \hat{\sigma}\hat{u}$$

for some $\hat{P} \in K$ with $\sigma(P) = \hat{\sigma}$.

Let $t \in \mathbb{N}$, $t \geqq 2$ and suppose that for each $P \in K$ there exists a set of vectors $\{r(1,P), r(2,P),\dots,r(t-1,P)\}$. It is assumed that the set of $N \times (N+t-1)$ matrices

$$(5.a.2) \qquad \{(P, r(1,P),\dots,r(t-1,P)) \mid P \in K\}$$

also possesses the product property.

Finally, suppose that

$$(5.a.3) \qquad \hat{P}^* r(t-1,\hat{P}) > \underline{0}$$

(note that $\hat{P}^*$ is well defined, by lemma 2.5).

The problem we investigate in this appendix is whether a solution of the following set of functional equations exists.

$$\begin{array}{llll}
(5.a.4.t) & \max_{P \in K} \{Px(t)\} & = \hat{\sigma}x(t) & \\
(5.a.4(t-1)) & \max_{P \in K_t} \{Px(t-1) + r(t-1,P)\} & = \hat{\sigma}x(t-1) & + x(t) \\
\vdots & \vdots & \vdots & \vdots \\
(5.a.4.1) & \max_{P \in K_2} Px(1) + r(1,P)\} & = \hat{\sigma}x(1) & + x(2)
\end{array}$$

where $K_\ell \subset K$ denotes the set of all matrices that maximize the left hand side of (5.a.4.ℓ) ($\ell = t, t-1, \dots, 2, 1$). Hence $K_t \supset K_{t-1} \supset \dots \supset K_1$.

The following result holds:

THEOREM 5.a.1. Let K, $\hat{P}$, $\hat{\sigma}$, $\hat{u}$ and t be given as above. Let for each $P \in K$ a set $\{r(1,P), \dots, r(t-1,P)\}$ be defined, again as above. Suppose (5.a.1) and (5.a.3) are fulfilled, and assume that the set of matrices, given in (5.a.2), has the product property.

Then there exists a solution $(\bar{x}(t), \bar{x}(t-1), \dots, \bar{x}(1))$ of (5.a.4), with $\bar{x}(t)$, $\bar{x}(t-1), \dots, \bar{x}(2)$ unique, whereas there is some freedom in the choice of $\bar{x}(1)$. Furthermore, the vector $\bar{x}(t)$ is strictly positive.

PROOF. The existence will be proved by means of an iterative procedure. Define $P(0) := \hat{P}$. The set of equations

$$\begin{cases} P(0)x(t) & = \hat{\sigma}x(t) \\ P(0)x(t-1) + r(t-1,P(0)) & = \hat{\sigma}x(t-1) + x(t) \\ \vdots & \vdots \\ P(0)x(1) \quad + r(1,P(0)) & = \hat{\sigma}x(1) \quad + x(2) \\ P(0)^*x(1) & = \underline{0} \end{cases}$$

possesses a unique solution $(x(t,0), x(t-1,0), \dots, x(1,0))$, with $x(t,0) = P(0)^* r(t-1,P(0)) > \underline{0}$ (compare the proof of lemma 2.a.1).

Determine $P(1) \in K$ such that

$$\begin{array}{lll} (5.a.5.t) & P(1)x(t,0) & = \max\limits_{P \in K} \{Px(t,0)\} \\ (5.a.5.t-1) & P(1)x(t-1,0) + r(t-1,P(1)) & = \max\limits_{P \in H_t} \{Px(t-1,0) + r(t-1,P)\} \\ \vdots & \vdots & \vdots \\ (5.a.5.1) & P(1)x(1,0) \quad + r(1,P(1)) & = \max\limits_{P \in H_2} \{Px(1,0) \quad + r(1,P)\}, \end{array}$$

where $H_\ell \subset K$ denotes the set of all matrices that maximize the right-hand side of (5.a.5.ℓ) ($\ell = t, t-1, \dots, 2, 1$). We choose $P(1) := P(0)$ if $P(0) \in H_1$.

Define vectors $\psi(t,0)$, $\psi(t-1,0), \dots, \psi(1,0)$, such that

$$(5.a.6.t)\qquad P(1)x(t,0) = \hat{\sigma}x(t,0) + \psi(t,0)$$

$$(5.a.6.(t-1))\qquad P(1)x(t-1,0) + r(t-1,P(1)) = \hat{\sigma}x(t-1,0) + x(t,0) + \psi(t-1,0)$$

$$\vdots$$

$$(5.a.6.1)\qquad P(1)x(1,0) + r(1,P(1)) = \hat{\sigma}x(1,0) + x(2,0) + \psi(1,0).$$

From $P(0)x(t,0) = \hat{\sigma}x(t,0)$ and (5.a.5.t) it follows that $\psi(t,0) \geqq \underline{0}$. If $\psi(\ell,0)_i = 0$ for $\ell = t,t-1,\ldots,k+1$, then $\psi(k,0)_i \geqq 0$ ($i \in S$, $1 \leqq k \leqq t$). Furthermore,

$$P(1)x(t,0) \geqq \hat{\sigma}x(t,0) > \underline{0},$$

which implies that $\sigma(P(1)) = \hat{\sigma}$, and, moreover, that each nonbasic class of $P(1)$ has access to some basic class. Since $\hat{u} > \underline{0}$ and

$$P(1)\hat{u} \leqq \hat{\sigma}\,\hat{u},$$

each basic class of $P(1)$ must be final. Hence, by proposition 2.4, $P(1)$ possesses a strictly positive right eigenvector associated with $\hat{\sigma}$. As in the proof of lemma 2.a.1, it follows that a unique solution $\{x(t,1), x(t-1,1),\ldots,x(1,1)\}$ of the following equations exists:

$$(5.a.7.t)\qquad P(1)x(t) = \hat{\sigma}x(t)$$

$$(5.a.7.(t-1))\qquad P(1)x(t-1) + r(t-1,P(1)) = \hat{\sigma}x(t-1) + x(t)$$

$$\vdots$$

$$(5.a.7.1)\qquad P(1)x(1) + r(1,P(1)) = \hat{\sigma}x(1) + x(2)$$

$$(5.a.7.0)\qquad P(1)^{*}x(1) = \underline{0}.$$

It will be proved that

$$(5.a.8)\qquad (x(t,1), x(t-1,1),..,x(1,1)) \succeq (x(t,0), x(t-1,0),..,x(1,0)),$$

with equality if and only if $P(1) = P(0)$.

The proof of (5.a.8) consists of three steps. First we prove

$$(*)\qquad x(t,1) \geqq x(t,0).$$

PROOF of (*): Let $D \subset S$ be the set of states, which belong to a basic class of $P(1)$. Since $x(t,0) > \underline{0}$, we have $\psi(t,0)_i = 0$ for $i \in D$ (by lemma 2.7). Hence $\psi(t-1,0)_i \geqq 0$ for $i \in D$. Multiplying (5.a.6.(t-1)) by $P(1)^*$ yields

$$(5.a.9) \quad P(1)^* r(t-1,P(1)) = P(1)^* x(t,0) + P(1)^* \psi(t-1,0) \geqq P(1)^* x(t,0)$$

(cf. lemma 2.5). From (5.a.7.t) and (5.a.7.(t-1)) we derive

$$(5.a.10) \quad x(t,1) = P(1)^* x(t,1) = P(1)^* r(t-1,P(1)).$$

Since $\psi(t,0) \geqq \underline{0}$, (5.a.6.t) yields

$$(5.a.11) \quad P(1)^* x(t,0) \geqq x(t,0).$$

A combination of (5.a.9), (5.a.10) and (5.a.11) shows (*).

The second assertion to be proved is the following:

$$(**) \qquad x(\ell,1) = x(\ell,0) \text{ for } k+1 \leqq \ell \leqq t \Rightarrow x(k,1) \geqq x(k,0) \quad k=t-1,..,2,1.$$

PROOF of (**): Define $y(\ell) := x(\ell,1) - x(\ell,0)$ for $\ell = t,t-1,...,1$. Combining (5.a.6.ℓ) and (5.a.7.ℓ) we find

$$P(1)y(\ell) = \hat{\sigma}y(\ell) + y(\ell+1) - \psi(\ell,0) \qquad \ell = t,t-1,...,1.$$

Hence, $y(\ell) = \underline{0}$ for $k+1 \leqq \ell \leqq t$ implies $\psi(\ell,0) = \underline{0}$ for $k+1 \leqq \ell \leqq t$ and hence $P(0) \in H_{k+1}$. It follows that $\psi(k,0) \geqq \underline{0}$, while furthermore

$$(5.a.12) \quad P(1)y(k) = \hat{\sigma}y(k) - \psi(k,0).$$

Multiplying both sides of (5.a.12) with $P(1)^*$ yields $P(1)^*\psi(k,0) = \underline{0}$. Since $\psi(k,0) \geqq \underline{0}$, we find (cf. lemma 2.5)

$$(5.a.13) \quad \psi(k,0)_i = 0 \qquad i \in D.$$

For $k \geqq 2$ we reason as follows: $\psi(k,0)_i = 0$ for $i \in D$ implies $\psi(k-1,0)_i \geqq 0$ for $i \in D$, hence $P(1)^* \psi(k-1,0) \geqq \underline{0}$. Multiplying (5.a.6.(k-1)) and (5.a.7.(k-1)) by $P(1)^*$ and subtracting the two resulting equations yields

$$P(1)^* y(k) - P(1)^* \psi(k-1,0) = \underline{0}.$$

On the other hand, (5.a.12) implies (recall that $\psi(k,0) \geqq \underline{0}$)

$$P(1)\, y(k) \leqq y(k).$$

Hence, $y(k) \geqq P(1)^* y(k) \geqq P(1)^* \psi(k-1,0) \geqq \underline{0}$. This proves (**) for $k \geqq 2$.

For $k = 1$, (5.a.13) implies that we may choose $P(1)_i := P(0)_i$ for $i \in D$. In that case $P(1)^*_i = P(0)^*_i$ for $i \in D$, hence $x(1,1)_i = x(1,0)_i$ for $i \in D$ (since the values of $x(1,1)_i$ for $i \in D$ are completely determined by $P(1)_i$, $P(1)^*_i$ and $r(1,P(1))_i$ for $i \in D$). Using lemma 2.5, it follows that

$$P(1)^* x(1,1) = P(1)^* x(1,0).$$

Combining (5.a.12), the fact that $\psi(1,0) \geqq \underline{0}$, and the definition of $y(1)$, we find that

$$y(1) \geqq P(1)^* y(1) = \underline{0}.$$

This proves (**) for $k = 1$.

Finally, we need

(***) $\quad x(\ell,1) = x(\ell,0)$ for $\ell = t,t-1,..,2,1$ if and only if $P(1) = P(0)$.

PROOF of (***): As above, $y(\ell) = \underline{0}$ for $\ell = t,t-1,\ldots,1$ implies $\psi(\ell,0) = \underline{0}$ for $\ell = t,t-1,\ldots,1$, in which case $P(0) \in H_1$, hence $P(1) := P(0)$. The inverse implication is trivial.

Combination of (*), (**) and (***) now shows that (5.a.8) holds, with equality if and only if $P(1) = P(0)$.

It is now easy to define an iterative procedure, based on a repeated application of equations of the kind (5.a.5) and (5.a.7). In fact we studied the first step of such a procedure. Since K is finite, this procedure stops after a finite number of steps, m say, if we use $P(m) = P(m-1)$ as the stopping condition. The final solution $\{x(t,m),\ldots,x(1,m)\}$, say, certainly satisfies (5.a.4), and $x(t,m) > \underline{0}$. □

CHAPTER 6

CONTINUOUS-TIME DYNAMIC PROGRAMMING MODELS

In the preceding chapters we studied discrete-time systems, specified by a finite set K of square nonnegative matrices with the product property. In particular, the following dynamic programming recursion was analyzed:

(6.0.1) $$x(n) = \max_{P \in K} Px(n-1) \qquad n \in \mathbb{N},\ x(0) > \underline{0}.$$

By subtracting x(n-1) on both sides of the equation, we obtain

(6.0.2) $$x(n)-x(n-1) = \max_{P \in K} (P-I)x(n-1) \qquad n \in \mathbb{N},\ x(0) > \underline{0}.$$

Note that the set of matrices

(6.0.3) $$\{(P-I) \mid P \in K\}$$

also has the product property. Each matrix P-I is an example of a so-called *ML-matrix* (note that $P \geq \underline{\underline{0}}$). An *ML-matrix* is a square matrix with all its nondiagonal entries nonnegative (cf. definition 6.1).

In this chapter we study the continuous-time analogue of the nonlinear difference equation (6.0.2). This continuous-time analogue reads as follows (cf. HOWARD [29], MILLER [40], [41]):

(6.0.4) $$\frac{dx}{dt}(t) = \max_{Q \in M} Qx(t) \qquad t \in [0,\infty),\ x(0) > \underline{0},$$

where M is a set of ML-matrices with the product property and $x(.) : [0,\infty) \to \mathbb{R}^N$. Note that we do not assume that $Q+I \geq \underline{\underline{0}}$ for all $Q \in M$. However, since M is finite and since each $Q \in M$ is a ML-matrix (the formal definition is given in section 6.1), there exists an $\alpha > 0$ such that $Q+\alpha I \geq \underline{\underline{0}}$ for all $Q \in M$.

As expected, we have to study nonlinear differential equations, instead of nonlinear difference equations. For an example the reader is referred to section 1.3 (example 1.3.1.b).

In section 6.1 we briefly review some basic theory of ML-matrices. Section 6.2 is devoted to the model in which all matrices are irreducible. Results for the more general case are given in section 6.3. Not all proofs are given in detail (since the techniques are more or less analogous to the discrete-time case), we only indicate the essential steps. One of these essential steps, exponential convergence in continuous-time, undiscounted Markov decision processes will be treated in more detail (appendix 6.A); the result given there can be viewed as the continuous-time analogue of the geometric convergence result of appendix 4.A.

6.1. ML-matrices

In this section we present some results concerning ML-matrices which will be needed later. First the formal definition.

DEFINITION 6.1. An N x N matrix Q is called an *ML-matrix* if

$$q_{ij} \geqq 0 \qquad \text{for } i \neq j;\ i,j \in S,$$

where, as usual, S denotes the state space ($S := \{1,2,\ldots,N\}$). □

If Q is an ML-matrix, it follows that there exists a (nonunique) $\alpha \in \mathbb{R}_0^+$ such that $Q+\alpha I \geqq \underline{0}$.

ML-matrices play a role in a wide variety of areas, for instance, input - output models in mathematical economics, Markov processes and especially queueing problems in probability theory. They are named after Metzler and Leontief in connection with their work in mathematical economics (cf. SENETA [52]).

Let Q be an ML-matrix. Consider the following differential equation

$$(6.1.1) \qquad \frac{dx}{dt}(t) = Qx(t) \qquad t \in [0,\infty);\ x(0) > \underline{0},$$

with x(t) a vector function from $[0,\infty)$ to $\mathbb{R}^N$. It is well known that the solution of (6.1.1) can be written as

(6.1.2) $\quad x(t) = \exp(tQ)x(0) \qquad t \in [0,\infty),$

with

(6.1.3) $\quad \exp(tQ) := \sum_{k=0}^{\infty} \frac{t^k}{k!} Q^k \qquad t \in [0,\infty).$

Since I and Q commute, it follows easily from (6.1.3) that

$$\exp(tQ) = \exp(-\alpha tI)\exp(t(Q+\alpha I)) = \exp(-\alpha t)\exp(t(Q+\alpha I)).$$

By choosing α sufficiently large, it follows that

$$\exp(tQ) \geq \underline{\underline{0}} \qquad t \in [0,\infty).$$

Moreover, if all row-sums of Q are equal to zero, then exp(tQ) is stochastic. Furthermore, note that

$$\exp((t+s)Q) = \exp(tQ)\exp(sQ) \qquad s,t \in [0,\infty).$$

In the discrete-time case we have seen that the spectral radius of a square nonnegative matrix P plays a dominant role in recursions of type $x(n) = Px(n-1)$, $\underline{x}(0) > \underline{0}$ and $n \in \mathbb{N}$. In connection with differential equations of type (6.1.1), a different eigenvalue becomes important. Let Q be an ML-matrix. Define

(6.1.4) $\quad \tau(Q) := \max\{\mathrm{Re}(\lambda) \mid \lambda \text{ an eigenvalue of } Q\},$

where $\mathrm{Re}(\lambda)$ denotes the real part of λ. Let $Q+\alpha I \geq \underline{\underline{0}}$ for some $\alpha > 0$. It follows immediately that

$$\tau(Q) = \sigma(Q+\alpha I) - \alpha.$$

This shows that $\tau(Q)$ is an eigenvalue of Q and, moreover, that $\mathrm{Re}(\lambda) < \tau(Q)$ for any eigenvalue λ of Q with $\lambda \neq \tau(Q)$.

<u>DEFINITION 6.2</u>. Let Q be an ML-matrix and let $\tau(Q)$ be defined by (6.1.4). The value $\tau(Q)$ is called the *dominant eigenvalue* of Q. □

In general, $\tau = \tau(Q)$ is not the spectral radius of Q, however, $\exp(t\tau)$ is the spectral radius of $\exp(tQ)$. Therefore, we may expect τ to play a dominant role in the asymptotic behaviour of $x(t)$, given in (6.1.1), for $t \to \infty$.

DEFINITION 6.3. An ML-matrix Q is called *irreducible* if for each pair $i,j \in S$ there exist states $i_1, i_2, \ldots, i_k \in S$, with $i \neq i_1 \neq \ldots \neq i_k \neq j$, such that

$$q_{ii_1} q_{i_1 i_2} \cdots q_{i_{k-1} i_k} q_{i_k j} > 0. \qquad \square$$

The following result is an immediate consequence of the Perron-Frobenius theorem and the relationship between ML-matrices and square nonnegative matrices. We have:

PROPOSITION 6.1. Let Q be an ML-matrix with dominant eigenvalue τ. With τ can be associated nonnegative left- and right-eigenvectors. If Q is irreducible, these eigenvectors are unique up to multiplicative constants and can be chosen strictly positive. $\square$

The index of an ML-matrix is defined in an analogous way as for a nonnegative matrix (cf. definition 2.12).

DEFINITION 6.4. Let Q be an ML-matrix with dominant eigenvalue τ and let $N^k(Q)$ be the null space of $(Q-\tau I)^k$ for $k \in \mathbb{N}_0$. The *index* $\nu(Q)$ of Q is the smallest $k \in \mathbb{N}_0$ such that $N^k(Q) = N^{k+1}(Q)$. $\square$

It will be clear that all results concerning block-triangular structure, eigenvectors and generalized eigenvectors of a square nonnegative matrix (with respect to its spectral radius) can be translated straightforwardly into corresponding results for ML-matrices (with respect to its dominant eigenvalue). The same holds for all results concerning sets of nonnegative matrices with the product property (cf. chapter 3 and section 5.2). The reformulation of these results will be left to the reader.

Now let Q be an irreducible ML-matrix and suppose $Q+\alpha I \geqq \underline{0}$. If Q is irreducible, then clearly

$$\sum_{k=1}^{n} (Q+\alpha I)^k > \underline{0}$$

for n sufficiently large. Since

$$\exp(tQ) = \exp(-\alpha t)\sum_{k=0}^{\infty}\frac{t^k}{k!}(Q+\alpha I)^k,$$

it follows that $\exp(tQ) > \underline{\underline{0}}$ if Q is irreducible and $t > 0$. Hence, the period of $\exp(tQ)$ is 1.

The following result is important (compare also lemma 2.5).

PROPOSITION 6.2. If Q is an ML-matrix with dominant eigenvalue τ and index equal to one, then

$$\lim_{t\to\infty}\{\exp(-t\tau)\exp(tQ)\}$$

exists. If this limit is denoted by Q^*, then $Q^* \geqq \underline{\underline{0}}$ and $QQ^* = Q^*Q = \tau Q^*$. If Q is irreducible, we have $Q^* > \underline{\underline{0}}$. □

A proof of proposition 6.2 is easily given by transforming Q into its Jordan canonical form (cf. PEASE [44]). The following corollary holds:

COROLLARY. $Q^*\exp(tQ) = \exp(tQ)Q^* = \exp(t\tau)Q^*$,

$$(Q^*)^2 = Q^*.$$ □

Finally, we prove a side result which will be needed in the next section.

LEMMA 6.3. Let Q be an ML-matrix and let x(.) be a differentiable vector function, satisfying

$$\frac{dx}{dt}(t) \geqq Qx(t) \qquad t \in [0,\infty).$$

Then

$$x(t) \geqq \exp(tQ)x(0) \qquad t \in [0,\infty)$$

PROOF. Define $u(.) : [0,\infty) \to \mathbb{R}^N$ by

$$u(t) := \frac{dx}{dt}(t) - Qx(t) \qquad t \in [0,\infty).$$

Then $u(t) \geq \underline{0}$ for $t \geq 0$ and

$$x(t) = \exp(tQ)x(0) + \int_0^t \exp((t-s)Q)u(s)ds \qquad t \in [0,\infty).$$

Since $u(s) \geq \underline{0}$ and $\exp((t-s)Q) \geq \underline{0}$ for $s \leq t$, the result follows. □

In the next section, we turn to continuous-time dynamic programming models and study the asymptotic behaviour of the vector function $x(t)$, for $t\to\infty$, where $x(t)$ is defined by

(6.0.4) $$\frac{dx}{dt}(t) = \max_{Q\in M} Qx(t) \qquad t \in [0,\infty);\ x(0) > \underline{0},$$

with M a finite set of ML-matrices with the product property. The following result deals with the question of existence and uniqueness of a solution $x(.)$ of (6.0.4). For a proof, see BELLMAN [5].

PROPOSITION 6.4. There exists a unique solution $x(.)$ of (6.0.4). □

In fact, Bellman established for a more general situation the existence of a continuous function $x(.) : [0,\infty) \to \mathbb{R}^N$ that satisfies (6.0.4) almost everywhere. Since the maximum of a finite number of continuous functions is again continuous, proposition 6.4 follows (cf. ZIJM [80]).

One more remark has to be made. Let $x(.)$ on $[0,\infty)$ be defined by (6.0.4) and let α be such that

(6.1.5) $$Q+\alpha I \geq \underline{0} \qquad \text{for all } Q \in M.$$

Define $y(.)$ on $[0,\infty)$ by

(6.1.6) $$y(t) = \exp(\alpha t)x(t) \qquad t \in [0,\infty).$$

Then

$$\frac{dy}{dt}(t) = \exp(\alpha t)(\alpha x(t) + \frac{dx}{dt}(t)) = \max_{Q\in M} (Q+\alpha I)y(t) \qquad t \in [0,\infty)$$

Hence, instead of (6.0.4), we may study a system which is specified by a set of <u>nonnegative</u> matrices with the product property. Sometimes, this obvious observation simplifies proofs considerably.

6.2. Systems with irreducible ML-matrices.

In this section M denotes a finite set of <u>irreducible</u> ML-matrices with the product property. Furthermore, $\hat{\tau}$ is defined by

$$(6.2.1) \qquad \hat{\tau} := \max \{\tau(Q) \mid Q \in M\},$$

where $\tau(Q)$ denotes the dominant eigenvalue of Q, $Q \in M$.

As in the preceding chapters, the analysis of systems of the kind

$$(6.0.4) \qquad \frac{dx}{dt}(t) = \max_{Q \in M} Qx(t) \qquad t \in [0,\infty);\ x(0) > \underline{0},$$

is relatively simple, if M contains only irreducible matrices. The methodology, used to prove convergence results in the general case, can be made completely transparent already in this case.

A first result is the following:

<u>LEMMA 6.5</u>. Let x(.) be given by (6.0.4) and let $\hat{\tau}$ be defined as in (6.2.1). Then the function $\exp(-t\hat{\tau})x(t)$ is bounded for $t \in [0,\infty)$.

<u>PROOF</u>. Recall that, without loss of generality, we may assume $Q \geqq \underline{0}$ for all $Q \in M$. Since, by lemma 3.1, there exist a $\hat{Q} \in M$ and a vector $\hat{u} > \underline{0}$, such that

$$(6.2.2) \qquad \hat{Q}\hat{u} = \max_{Q \in M} Q\hat{u} = \hat{\tau}\hat{u},$$

it follows from lemma 6.3 and the fact that $\exp(t\hat{Q}) > \underline{\underline{0}}$ for $t > 0$ that

$$x(t) \geqq \exp(t\hat{Q})x(0) \geqq c_1 \exp(t\hat{Q})\hat{u} = c_1 \exp(t\hat{\tau})\hat{u} \qquad t \in [0,\infty),$$

if $x(0) \geqq c_1\hat{u}$.

On the other hand, let $x(0) \leqq c_2\hat{u}$. Since x(.) is continuous, there exists for each $\varepsilon > 0$ a $t_1 = t_1(\varepsilon) > 0$ such that

(6.2.3) $\quad t_1 := \sup \{t \mid x(t) \leqq (c_2+\varepsilon)\exp(t\hat{\tau})\hat{u}\}.$

Suppose $t_1 < \infty$. Since $Q \geqq \underline{0}$ for all $Q \in M$ it follows that

$$\frac{dx}{dt}(t) = \max_{Q \in M} Qx(t) \leqq (c_2+\varepsilon)\exp(t\hat{\tau})\hat{\tau}\hat{u} \qquad t \leqq t_1.$$

Hence

$$x(t) - x(0) \leqq (c_2+\varepsilon)\exp(t\hat{\tau})\hat{u} - (c_2+\varepsilon)\hat{u} \qquad t \leqq t_1.$$

In other words, since $x(0) \leqq c_2\hat{u}$, we find

$$x(t) \leqq (c_2+\varepsilon)\exp(t\hat{\tau})\hat{u} - \varepsilon\hat{u} \qquad t \leqq t_1.$$

By the continuity of $x(.)$, we conclude that $t_1 = \infty$, hence

(6.2.4) $$x(t) \leqq (c_2+\varepsilon)\exp(t\hat{\tau})\hat{u} \qquad \text{for } t \in [0,\infty).$$

Since (6.2.4) holds for each $\varepsilon > 0$, the result follows. □

Convergence of $\exp(-t\hat{\tau})x(t)$ for $t\to\infty$ can be proved along the same lines as in the discrete case, by using proposition 6.2, its corollary and lemma 6.5. We have

<u>THEOREM 6.6</u>. Let $x(.)$ and $\hat{\tau}$ be defined as in lemma 6.5. Then

$$\lim_{t\to\infty} \exp(-t\hat{\tau})x(t)$$

exists and is strictly positive.

<u>PROOF</u>. Since $\exp(-t\hat{\tau})x(t)$ is bounded on $[0,\infty)$, we may define

$$a := \liminf_{t\to\infty} \exp(-t\hat{\tau})x(t),$$

$$b := \limsup_{t\to\infty} \exp(-t\hat{\tau})x(t).$$

Let $\hat{Q} \in M$ satisfy (6.2.2), hence $\tau(\hat{Q}) = \hat{\tau}$. For $t,s \in [0,\infty)$ with $t > s$ we have

$$\exp(-t\hat{\tau})x(t) = \exp(-(t-s)\hat{\tau})\exp(-s\hat{\tau})\, x(t) \geqq$$

$$\geqq \exp(-(t-s)\hat{\tau})\exp((t-s)\hat{Q})\exp(-s\hat{\tau})x(s).$$

Choosing a sequence $(t_1, t_2, \ldots)$ with $\lim_{k\to\infty} t_k = \infty$ and

$$\lim_{k\to\infty} \exp(-t_k\hat{\tau})x(t_k) = a,$$

we find (cf. proposition 6.2):

$$a \geqq \hat{Q}^* \exp(-s\hat{\tau})x(s).$$

By choosing a sequence $(s_1, s_2, \ldots)$ with $\lim_{k\to\infty} s_k = \infty$ and

$$\lim_{k\to\infty} \exp(-s_k\hat{\tau})x(s_k) = b,$$

we conclude

$$a \geqq \hat{Q}^* b.$$

Analogously, we find $b \geqq \hat{Q}^* a$. As in the proof of lemma 4.1, we conclude $a=b$ (note that $\hat{Q}^* > \underline{\underline{0}}$). The property $a > \underline{0}$ follows from $x(0) > \underline{0}$, $\hat{Q}^* > \underline{\underline{0}}$ and $a \geqq \hat{Q}^* x(0)$.

One may wonder whether the convergence, proved in theorem 6.6, is exponential, analogously to the geometric convergence result in the discrete-time case. If M contains only one matrix, the answer to this question is clearly affirmative. It turns out that also in general such an exponential convergence result exists for continuous-time Markov decision processes. Details will be given in appendix 6.A. The translation from the stochastic case to our more general situation is straightforward. We return to this question in the next section where we deal with systems specified by a set of (possibly) reducible ML-matrices with the product property.

6.3. Systems with ML-matrices: the general case.

In this section we formulate the continuous-time analogues of the results of section 4.2 and section 5.1. The proofs of the results in this section will not be given in detail: they proceed along the same lines as in the discrete-time case.

In this section, M denotes a set of general (i.e. possibly reducible) ML-matrices with the product property. Let τ be defined by

$$(6.3.1) \qquad \hat{\tau} := \max \{\tau(Q) \mid Q \in M\},$$

and define ν by

$$(6.3.2) \qquad \nu := \max \{\nu(Q) \mid Q \in M,\ \tau(Q) = \hat{\tau}\}$$

(cf. definition 6.4).

We are interested in the asymptotic behaviour of x(t), for $t \to \infty$, where x(t) obeys

$$(6.0.4) \qquad \frac{dx}{dt}(t) = \max_{Q \in M} Qx(t) \qquad\qquad t \in [0,\infty);\ x(0) > \underline{0}.$$

Let us first treat the case $\nu = 1$.

THEOREM 6.7. Let M, $\hat{\tau}$, ν and $x(.)$ be given as above. Suppose $\nu = 1$. Then there exists a semi-positive vector x, such that

$$\lim_{t\to\infty} \exp(-t\hat{\tau})x(t) = x.$$

PROOF. Without loss of generality, we may assume that $Q \geq \underline{\underline{0}}$, for all $Q \in M$. Let $\{D(1),D(0)\}$ be the principal partition of S with respect to M. By theorem 3.6, there exists a vector $u \geq \underline{0}$, with $u_i = 0$ for $i \in D(0)$, $u_i > 0$ for $i \in D(1)$, such that

$$(6.3.3) \qquad \max_{Q \in M} Qu = \hat{\tau}u$$

Let $Q^{(k,\ell)}$ denote the restriction of Q to $D(k) \times D(\ell)$, for $k,\ell = 0,1$. Then $Q^{(0,1)} = \underline{\underline{0}}$ for all $Q \in M$ and

$$\max_{Q\in M} (Q^{(0,0)}) < \hat{\tau}.$$

Moreover, there exist a $\lambda < \hat{\tau}$ and a strictly positive vector $w^{(0)}$, such that

$$\text{(6.3.4)} \quad \max_{Q\in M} Q^{(0,0)} w^{(0)} \leqq \lambda w^{(0)}.$$

Choose $c > 0$ such that $x(0)_i \leqq cu_i$ for $i \in D(1)$ and $x(0)_i \leqq cw_i^{(0)}$ for $i \in D(0)$. As in the proof of lemma 6.5, one can show that

$$\text{(6.3.5)} \quad x(t)_i \leqq c \exp(\lambda t) w_i^{(0)} \qquad i \in D(0),\ t \in [0,\infty),$$

which implies that

$$\lim_{t\to\infty} \exp(-t\hat{\tau}) x(t)_i = 0 \qquad i \in D(0),$$

and that this convergence is exponential (of order at least $\exp(t(\lambda-\hat{\tau}))$.

Let $u^{(1)}$ be the restriction of u to $D(1)$ (hence $u^{(1)} > \underline{0}$) and choose α such that

$$\max_{Q\in M} Q^{(1,0)} w^{(0)} \leqq \alpha u^{(1)}.$$

Again, by methods analogous to those in the proof of lemma 6.5, and by use of (6.3.5), it can be shown that

$$\text{(6.3.6)} \quad x(t)_i \leqq (\exp(t\hat{\tau}))(c + \frac{\alpha}{\hat{\tau}-\lambda} (1-\exp((\lambda-\hat{\tau})t)))u_i \qquad i \in D(1).$$

Combining (6.3.5) and (6.3.6), we have established boundedness of $\exp(-t\hat{\tau})x(t)$ on $[0,\infty)$. Convergence to a vector x can be proved in exactly the same way as in the proof of theorem 4.6, by using the results of the preceding section for the basic classes (cf. theorem 6.6). As in the proof of theorem 4.6, it follows from $x(0) > \underline{0}$ that $x_i > 0$ for $i \in D(1)$. □

In chapter 4 we showed that for discrete systems the convergence is geometric (cf. theorem 4.7), by exploiting a result from the theory of controlled Markov processes (appendix 4.A). An analogous result exists for continuous-time Markov decision processes; the main difficulty in its proof appears to be the fact that no upperbound exists for the number of changes of the maximizing matrix in a fixed, finite time interval. This implies that one has to determine a contraction factor which is independent of this number of changes. In the appendix to this chapter, we present a rather detailed treatment of the subject; here we only show how this exponential convergence result can be exploited in order to establish exponential convergence of $\exp(-t\hat{\tau})x(t)$, for $t\to\infty$, in the situation, described in theorem 6.7.

Let x denote the limit vector, obtained in theorem 6.7. Hence

$$(6.3.7)\qquad \lim_{t\to\infty} \exp(-t\hat{\tau})x(t) = x.$$

Note that

$$(6.3.8)\qquad \frac{d}{dt}(\exp(-t\hat{\tau})x(t)) + \hat{\tau}\exp(-t\hat{\tau})x(t) = \max_{Q\in M} Q\exp(-t\hat{\tau})x(t) \qquad t \in [0,\infty).$$

From (6.3.7) and (6.3.8) it follows that

$$\lim_{t\to\infty} \frac{d}{dt}(\exp(-t\hat{\tau})x(t))$$

exists. Since x is a fixed vector, we conclude

$$\lim_{t\to\infty} \frac{d}{dt}(\exp(-t\hat{\tau})x(t)) = \underline{0}.$$

Let $t\to\infty$ in (6.3.8). Then we find

$$(6.3.9)\qquad \max_{Q\in M} Qx = \hat{\tau}x.$$

Now, (6.3.8) and (6.3.9) together imply that for t sufficiently large, $t \geqq t_1$ say, we only deal with matrices $Q \in M$ such that

$$(6.3.10)\qquad Qx = \hat{\tau}x.$$

Let $M_1 \subset M$ denote the set of matrices that satisfy (6.3.10). Let $x(t)^{(i)}$ be the restriction of x(t) to D(i), and $x^{(i)}$ the restriction of x to D(i), i = 0,1. Recall that $x^{(1)} > \underline{0}$. Obviously, we have

$$(6.3.11)\qquad \frac{dx^{(1)}}{dt}(t) = \max_{Q\in M_1} \{Q^{(1,1)}x(t)^{(1)} + Q^{(1,0)}x(t)^{(0)}\} \qquad t \geqq t_1.$$

Now define $\bar{Q}^{(1,1)}$, $\bar{x}(t)$ and $\bar{r}(t,Q)$ by

$$\bar{q}_{ij}^{(1,1)} := x_i^{-1}(q_{ij} - \hat{\tau}\delta_{ij})x_j \qquad Q \in M_1;\ i,j \in D(1),$$

$$\bar{x}(t)_i^{(1)} := \exp(-t\hat{\tau})x_i^{-1}x(t)_i^{(1)} \qquad t \in [0,\infty);\ i \in D(1),$$

$$\bar{r}(t,Q)_i := \exp(-t\hat{\tau})x_i^{-1}\Big(\sum_{j\in D(0)} q_{ij}x(t)_j^{(0)}\Big) \qquad t \in [0,\infty);\ i \in D(1),$$

Using these transformations, (6.3.11) can be written as

$$\text{(6.3.12)} \quad \frac{d}{dt}(\bar{x}(t)^{(1)}) = \max_{Q\in M_1} \{\bar{r}(t,Q) + \bar{Q}^{(1,1)}\bar{x}(t)^{(1)}\} \qquad t \geqq t_1,$$

where $\{\bar{Q}^{(1,1)} \mid Q \in M_1\}$ is now a set of generators for a continuous-time Markov decision process (i.e., $\bar{Q}^{(1,1)}e = 0$, for all $Q \in M_1$). Since

$$\lim_{t\to\infty} \bar{r}(t,Q) = \underline{0} \qquad Q \in M_1,$$

the results of appendix 6.A now immediately imply exponential convergence of $\bar{x}(t)^{(1)}$ to a constant vector, for $t\to\infty$. Translating back, we obtain

THEOREM 6.8. The convergence, proved in theorem 6.7, is exponential. □

Once having obtained this exponential convergence result, the most general case (with no restrictions on the size of ν) can be analyzed along the lines of section 5.1 (compare the proof of theorem 5.1). Below we formulate the final result concerning the asymptotic behaviour of $x(t)$ for $t\to\infty$. The proof is left to the reader.

THEOREM 6.9. Let M, $\hat{\tau}$, ν and $x(.)$ be given as above. Then there exist a partition $\{D(\nu),D(\nu-1),\ldots,D(1),D(0)\}$ of the state space S and a collection of vectors $\{x(\nu),x(\nu-1),\ldots,x(1)\}$, with

$$x(k)_i > 0 \qquad \text{for } i \in D(k),\ k = \nu,\nu-1,\ldots,1,$$

and

$$x(k)_i = 0 \qquad \text{for } i \in \bigcup_{\ell=0}^{k-1} D(\ell),\ k = \nu,\nu-1,\ldots,1,$$

such that

$$x(t) = \frac{t^{\nu-1}}{(\nu-1)!} \exp(t\hat{\tau})x(\nu)+..+\frac{t}{1!} \exp(t\hat{\tau})x(2)+\exp(t\hat{\tau})x(1)+O(\exp(\lambda t))$$

for $t\to\infty$, where $\lambda < \hat{\tau}$.

Furthermore, the vectors $x(\nu),x(\nu-1),\dots,x(1)$ satisfy

$$(6.3.13.\nu) \qquad \max_{Q\in M} Qx(\nu) = \hat{\tau}x(\nu),$$

$$(6.3.13.k) \qquad \max_{Q\in M_{k+1}} Qx(k) = \hat{\tau}x(k) + x(k+1) \qquad k = \nu-1,\dots,2,1,$$

where M_k denotes the set of matrices that maximize the left-hand side in (6.3.13.k), for $k = \nu, \nu-1,\dots,1$. □

With theorem 6.9 we have completed the analysis of the asymptotic behaviour of x(t), for $t\to\infty$. It will be clear that no iteration methods for finding a matrix $Q \in M$ maximizing the growth of the system need to be developed; these methods can be copied almost directly from the results in chapter 3. We emphasize the fact that results for average reward and total discounted reward Markov decision processes with finite state space, finite action space and continuous time axis can be established immediately by applying the results of the present chapter. Also sensitive optimality criteria in continuous-time Markov decision processes can be analyzed by methods developed here (in a forthcoming publication attention will be paid to these matters). What remains to be done is the proof of exponential convergence in continuous-time Markov decision processes with the average reward criterion. Appendix 6.A will be devoted to a derivation of this result, which is indispensible in the proof of theorem 6.9 (compare also the proof of theorem 5.1).

Appendix 6.A. Exponential convergence in continuous-time Markov decision processes.

This appendix deals with undiscounted, continuous-time Markov decision processes. These processes have been studied by a large number of authors, compare e.g. HOWARD [29], MILLER [40], [41], BATHER [2], DOSHI [20], DE LEVE, FEDERGRUEN AND TIJMS [14], [15] and VAN DER DUYN SCHOUTEN [67]. In this appendix, both the state space S and the action space A are supposed to be finite. Our objective is to prove an exponential convergence result, analogous to the geometric convergence result for discrete-time Markov decision processes (cf. SCHWEITZER AND FEDERGRUEN [61] and appendix 4.A).

A detailed description of a continuous-time Markov decision process has been given in section 1.3 (examples 1.3.1.b). Here we only recall the fact that the optimal return v(t) for an interval of length t obeys

(6.a.1) $$\frac{dv}{dt}(t) = \max_{Q \in M} \{r(Q) + Qv(t)\} \qquad t \in [0,\infty),$$

with v(0) chosen arbitrarily. M is a finite set of ML-matrices, such that

(6.a.2) $$\sum_{j \in S} q_{ij} = 0 \qquad i \in S,\ Q \in M.$$

Furthermore, r(Q) denotes a reward vector, associated with Q ($Q \in M$). The set of $N \times (N+1)$ matrices

$$\{(Q, r(Q)) \mid Q \in M\}$$

is assumed to have the product property.

We are interested in the asymptotic behaviour of v(t), for $t \to \infty$. A few preliminary results are needed.

<u>LEMMA 6.a.1</u>. Let Q be an ML-matrix such that $Qe = \underline{0}$. The solution of

(6.a.3) $$\frac{dy}{dt}(t) = r(Q) + Qy(t) \qquad t \in [0,\infty)$$

(with y(0) given) can be written as

(6.a.4) $$y(t) = tg - (\exp(tQ)-I)w + (\exp(tQ))y(0),$$

where g and w satisfy

$$(6.a.5)\quad \begin{cases} Qg = \underline{0} \\ r(Q) + Qw = g \end{cases}$$

Remark. As in appendix 2.A it follows that a solution of (6.a.5) exists. The reader easily verifies that, if (g,w) is a solution of (6.a.5), then g is uniquely determined by

$$g = Q^{*}r(Q)$$

(use proposition 6.2 with $\tau = 0$). Obviously, w is not unique, since, if x is an eigenvector of Q associated with $\tau = 0$, then $w+\alpha x$ also satisfies the second equation of (6.a.5). However, since

$$(\exp(tQ)-I)x = \underline{0}$$

y(t) is still uniquely determined by (6.a.4).

PROOF OF LEMMA 6.a.1. A constructive proof of (6.a.4) can be given by matrix-theoretical methods and will not be carried out in detail here. Note that the vector $(w_1,w_2,\ldots,w_N,1)^T$ is in fact a generalized eigenvector of order 2, associated with the eigenvalue 0 of the matrix $\tilde{Q}$, defined by

$$\tilde{Q} := \begin{pmatrix} Q & r(Q) \\ 0 & 0 \end{pmatrix}$$

Exploiting the Jordan canonical form of Q, it can be shown that

$$\exp(t\tilde{Q}) = \begin{pmatrix} \exp(tQ) & tg - (\exp(tQ)-I)w \\ 0 & 1 \end{pmatrix} .$$

Since (6.a.3) can be written as

$$\begin{pmatrix} \frac{dy}{dt}(t) \\ 0 \end{pmatrix} = \begin{pmatrix} Q & r(Q) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y(t) \\ 1 \end{pmatrix} \qquad t \in [0,\infty),$$

the result follows immediately. □

LEMMA 6.a.2. There exists a unique solution v(.) of (6.a.1) on $[0,\infty)$.

PROOF. See BELLMAN [5]. □

LEMMA 6.a.3. Let v(.) be the solution of (6.a.1). Then

$$v(t) \geqq y(t,Q) \qquad Q \in M;\ t \in [0,\infty),$$

where y(.,Q) is the solution of (6.a.3) with $y(0) := v(0)$.

PROOF. The result follows from lemma 6.3, $y(0)-v(0) = \underline{0}$ and

$$\frac{d}{dt}(v(t)-y(t)) \geqq Q(v(t)-y(t)) \qquad Q \in M;\ t \in [0,\infty).$$ □

LEMMA 6.a.4. There exists a solution (g,w) of the set of equations

$$(6.a.6)\quad \begin{cases} \max\limits_{Q\in M} \{Qg\} = \underline{0} \\ \max\limits_{Q\in M_1} \{r(Q) + Qw\} = g \end{cases}$$

with $M_1 := \{Q \in M \mid Qg = \underline{0}\}$. The vector g is unique, and

$$g = \max_{Q\in M} Q^* r(Q) \qquad \text{(component-wise)}.$$

PROOF. The result follows, after some transformations, from the corresponding result for stochastic matrices (compare (4.a.6) and (4.a.10) in appendix 4.A). □

LEMMA 6.a.5. Let (g,w) be a solution of (6.a.6) and let v(.) be the solution of (6.a.1). Then there exists a constant $c > 0$ such that

$$\| v(t)-tg \| \leqq c \qquad t \in [0,\infty).$$

PROOF. Choose $\hat{Q} \in M$ such that

$$\begin{cases} \hat{Q}g & 0 \\ r(\hat{Q}) + \hat{Q}w = g. \end{cases}$$

Then

$$\exp(t\hat{Q})r(\hat{Q}) + \exp(t\hat{Q})\hat{Q}w = \exp(t\hat{Q})g = g \qquad t \in [0,\infty).$$

For $t \to \infty$ we find (cf. proposition 6.2)

$$g = \hat{Q}^* r(\hat{Q}).$$

Lemma 6.a.1 and lemma 6.a.3 now imply

$$v(t)-tg \geqq \exp(t\hat{Q})v(0) - (\exp(t\hat{Q})-I)w(\hat{Q}) =$$

$$= \exp(t\hat{Q})v(0) - (\exp(t\hat{Q})-I)w,$$

which shows that $v(t)-tg$ is bounded from below.

In order to find an upperbound, we reason as follows. Note that for β sufficiently large we have

$$(6.a.7) \qquad \max_{Q \in M} \{r(Q)+Q(w+\beta g)\} = \max_{Q \in M_1} \{r(Q)+Q(w+\beta g)\} = g.$$

Choose $\alpha > 0$ such that $Q+\alpha I \geqq \underline{0}$ for all $Q \in M$. By a simple transformation of (6.a.1), we obtain for $t \in [0,\infty)$

$$(6.a.8) \qquad \frac{d}{dt}(\exp(\alpha t)v(t)) = \max_{Q \in M} \{\exp(\alpha t)r(Q)+(Q+\alpha I)\exp(\alpha t)v(t)\}.$$

Finally choose γ such that

$$(6.a.9) \qquad v(0) \leqq w + \beta g + \gamma e$$

Combining (6.a.7), (6.a.8) and (6.a.9), it is easy to verify, by a method analogous to that in the proof of lemma 6.5, that

$$v(t) \leqq w + \beta g + \gamma e + tg \qquad t \in [0,\infty).$$

By choosing c appropriately, the result follows. □

Lemma 6.a.5 is the continuous-time analogue of lemma 4.a.2 (compare also BROWN [11]). LEMBERSKY [81] established a stronger result, i.e. the convergence of $v(t) - tg^*$ for $t \to \infty$. Another proof of this result can be found in ZIJM [80]. Below we give an outline of the method of proof, used in [80].

LEMMA 6.a.6. Let v(.) be the solution of (6.a.1) and let $g^* := \max_{Q \in M} Q^* r(Q)$. Then

$$\lim_{t\to\infty} (v(t) - tg^*)$$

exists.

PROOF. Define $w(t) := v(t) - tg^*$ and $M_1 := \{Q \in M \mid Qg^* = \underline{0}\}$. For t large enough, $t \geq t_1$ say, (6.a.1) reduces to

$$g^* + \frac{dw}{dt}(t) = \max_{Q \in M_1} \{r(Q) + Qw(t)\}.$$

Let

$$a := \liminf_{t\to\infty} w(t),$$

$$b := \limsup_{t\to\infty} w(t).$$

It can be shown (cf. ZIJM [80]) that

$$\max_{Q \in M_1} \{r(Q) + Qa\} \leq g^* \leq \max_{Q \in M_1} \{r(Q) + Qb\}$$

and the rest of the proof follows the lines of that of theorem 4.6. □

We have now established the existence of vectors g^* and w^*, with

(6.a.10) $$g^* := \max_{Q \in M} Q^* r(Q),$$

and such that

$$\lim_{t\to\infty} (v(t) - tg^*) = w^*,$$

where v(.) satisfies (6.a.1). Clearly, (g^*, w^*) is a solution of (6.a.6). Now let

$$M_2 := \{Q \in M \mid Qg^* = 0,\ r(Q)+Qw^* = g^*\},$$

and define

$$e(t) := v(t)-tg^*-w^* \qquad t \in [0,\infty).$$

For t sufficiently large, $t \geq \bar{t}$ say, (6.a.1) reduces to

$$\text{(6.a.11)} \quad \frac{de}{dt}(t) = \max_{Q\in M_2} Qe(t),$$

where

$$\text{(6.a.12)} \quad \lim_{t\to\infty} e(t) = \underline{0}.$$

Our objective is to show that this convergence is exponential. The following result, essentially due to MILLER [40], will prove to be helpful.

LEMMA 6.a.7. Let M be a finite set of ML-matrices with the product property, such that $Qe = \underline{0}$ for all $Q \in M$ Let x(.) satisfy

$$\frac{dx}{dt}(t) = \max_{Q\in M} Qx(t) \qquad t \in [0,\infty),$$

with x(0) chosen arbitrarily.

Then, for each t, there exist a positive integer n (depending on t), numbers $t_0,t_1,\ldots,t_n$ with $0 = t_0 < t_1 < \ldots. < t_{n-1} < t_n = t$, and matrices $Q(1),Q(2),\ldots,Q(n) \in M$, such that for $k = 1,2,\ldots,n$:

$$x(t_k) = \exp((t_k-t_{k-1})Q(k))x(t_{k-1}). \qquad \square$$

MILLER [40] proves that, for a continuous-time Markov decision process, there exists an optimal decision rule with only a finite number of "switching points" (i.e. time points where the optimal matrix changes) on each finite time interval. For our purpose the lemma as formulated above is sufficient.

Now, let us return to the equations (6.a.11) and (6.a.12). Define

$$C(t) := \{i \in S \mid e(t)_i > 0\} \qquad C'(t) = S\backslash C(t),$$

$$D(t) := \{i \in S \mid e(t)_i < 0\} \qquad D'(t) = S\backslash D(t).$$

Clearly, (6.a.11) and (6.a.12) together imply that $C(t) \neq S, D(t) \neq S$ for $t \geqq \bar{t}$ (since $e(t) > \underline{0}$ would imply $\lim_{t\to\infty} e(t) > \underline{0}$; compare the proof of lemma 6.5 with $\tau = 0$ and $\hat{u} = e$). The next lemma shows that $\max_{i\in S} e(t)_i$ decreases exponentially to zero.

<u>LEMMA 6.a.8</u>. There exist positive numbers α and ε such that

$$\max_{i\in S} e(t+s)_i \leqq (1-\exp(-\alpha s)\ \frac{\varepsilon s^N}{N!}\ )\max_{i\in S} e(t)_i \qquad s \geqq 0,\ t \geqq \bar{t}$$

(where N denotes the number of states in S). □

<u>PROOF</u>. If $C(\bar{t}) = \emptyset$, the lemma follows trivially. Suppose $C(\bar{t}) \neq \emptyset$. Choose α such that $\alpha+q_{ii} > 0$ for all $Q \in M_2$, $i \in S$. Define $R(n)$ recursively by $R(0) = C(\bar{t})$ and

$$R(n) := \{i \in S \mid \exists Q \in M_2 : \sum_{j\notin R(n-1)} (\alpha\delta_{ij}+q_{ij}) = 0\} \qquad n \in \mathbb{N}.$$

If $R(n) \neq \emptyset$, then, since M_2 has the product property, there exists a $Q \in M_2$ such that, simultaneously for all $i \in R(n)$, we have

$$\text{(6.a.13)} \qquad \sum_{j\notin R(n-1)} (\alpha\delta_{ij}+q_{ij}) = 0 .$$

From the choice of α it follows that $R(n) \subset R(n-1)$. If for some $n \geqq 1$ we have $R(n) = R(n-1)$ and $R(n) \neq \emptyset$ then (6.a.13) becomes

$$\sum_{j\notin R(n)} q_{ij} = 0 \qquad i \in R(n),$$

which implies (by 6.a.2)

$$\text{(6.a.14)} \qquad \sum_{j\in R(n)} (\exp(tQ))_{ij} = 1 \qquad i \in R(n);\ t \in [0,\infty).$$

But, if (6.a.14) holds for $Q \in M_2$, then, since $R(n) \subset R(0)$ and $R(0) = C(\bar{t})$, $\max_{i\in S} e(t)_i$ cannot converge to zero for $t \to \infty$. We conclude

$$R(n) \underset{\neq}{\subset} R(n-1) \qquad n \in \mathbb{N}.$$

But then, $R(n) = \emptyset$ for $n \geqq N$. By definition of $R(n)$ we find

$$\sum_{j \notin R(0)} ((\alpha I+Q(1)) \ldots (\alpha I+Q(n)))_{ij} > 0 \qquad i \in S,$$

for each n-tuple $(Q(1),Q(2),\ldots,Q(n))$ with $n \geqq N$. Since M_2 is finite, there exists a constant $\varepsilon > 0$ such that

$$\sum_{j \notin R(0)} ((\alpha I+Q(1)) \ldots (\alpha I+Q(N)))_{ij} \geqq \varepsilon \qquad i \in S;\ Q(1),\ldots,Q(N) \in M_2.$$

For $n \geqq N$ we have furthermore

$$\exp(t_1 Q(1))\exp(t_2 Q(2)) \ldots \exp(t_n Q(n)) =$$

$$= \exp(-\alpha(t_1+\ldots+t_n))\exp(t_1(\alpha I+Q(1)))\ldots\exp(t_n(\alpha I+Q(n))) \geqq$$

$$\geqq \exp(-\alpha(t_1+\ldots+t_n)) \sum_{k_1+\ldots+k_n=N} \frac{(\alpha I+Q(1))^{k_1} \ldots (\alpha I+Q(n))^{k_n} t_1^{k_1} \ldots t_n^{k_n}}{k_1! \ldots k_n!}.$$

Hence, for each $n \geqq N$ and each n-tuple $(Q(1),\ldots,Q(n))$, it follows that

$$\sum_{j \notin R(0)} (\exp(t_1 Q(1)) \ldots \exp(t_n Q(n)))_{ij} \geqq$$

$$\geqq \exp(-\alpha(t_1+\ldots+t_n)) \sum_{k_1+\ldots+k_n=N} \varepsilon \frac{t_1^{k_1} \ldots t_n^{k_n}}{k_1! \ldots k_n!} =$$

$$= \exp(-\alpha(t_1+\ldots+t_n))\, \varepsilon \frac{(t_1+\ldots+t_n)^N}{N!} \qquad i \in S.$$

Using lemma 6.a.7 and the fact that $R(0) = C(\bar{t})$, it follows that

$$\max_{i \in S} e(\bar{t}+s)_i \leqq (1-\exp(-\alpha s) \frac{\varepsilon s^N}{N!})\max_{i \in S} e(\bar{t})_i \qquad s \geqq 0.$$

Since the same result can be obtained for every starting point t, with $t \geqq \bar{t}$, the lemma now follows. □

By similar arguments, one may show (compare the second part of the proof of lemma 4.a.4):

LEMMA 6.a.9. There exist positive numbers α,δ such that

$$\min_{i\in S} e(t+s)_i \geq (1-\exp(-\alpha s)\frac{\delta s^N}{N!})\min_{i\in S} e(t)_i \qquad s \geq 0,\ t \geq \bar{t}. \quad \square$$

Now, choose $\bar{s} \in \mathbb{R}^+$ and $\beta \in \mathbb{R}^+$ such that

$$\beta := \max_{0<s<\infty} \{\min(\exp(-\alpha s)\frac{\delta s^N}{N!},\ \exp(-\alpha s)\frac{\varepsilon s^N}{N!}\}$$

$$= \min(\exp(-\alpha\bar{s})\frac{\delta\bar{s}^N}{N!},\ \exp(-\alpha\bar{s})\frac{\varepsilon\bar{s}^N}{N!}).$$

Then $\bar{s} > 0$, $\beta > 0$ and for $t \geq \bar{t}$

$$\max_{i\in S} e(t+\bar{s})_i - \min_{i\in S} e(t+\bar{s})_i \leq (1-\beta)(\max_{i\in S} e(t)_i - \min_{i\in S} e(t)_i),$$

which shows that the convergence in (6.a.12) is exponential. Combining the results, we have found:

THEOREM 6.a.10. Let $v(.)$ be the solution of (6.a.1). Then there exist vectors g^* and w^* such that

$$\lim_{t\to\infty} (v(t)-tg^*) = w^*,$$

and this convergence is exponential. $\square$

Up to now we have considered Markov decision processes with a fixed reward vector $r(Q)$, associated with each $Q \in M$. Let us now suppose that for each Q a continuous vector function $r(Q,.) : [0,\infty) \to \mathbb{R}^N$ is defined such that for some $c > 0$ and $\gamma > 0$

$$(6.a.15) \quad \| r(Q,t) - r(Q) \| \leq c \exp(-\gamma t) \qquad Q \in M,\ t \in [0,\infty).$$

Furthermore, for each $t \in [0,\infty)$, the set of $N \times (N+1)$ matrices

$$\{(Q,r(Q,t)) \mid Q \in M\}$$

is assumed to have the product property.

Consider the following nonlinear differential equation

$$\text{(6.a.16)} \quad \frac{du}{dt}(t) = \max_{Q \in M} \{r(Q,t) + Qu(t)\} \qquad t \in [0,\infty),$$

where u(0) is arbitrarily chosen.

Then, by methods similar to those used in appendix 4.A, and taking into account (6.a.15), the following theorem can be proved.

THEOREM 6.a.11. There exists a unique solution u(.) of (6.a.16). Furthermore there exist vectors g^* and w^*, with g^* defined by (6.a.10), such that

$$\lim_{t\to\infty} (u(t) - tg^*) = w^*,$$

and this convergence is exponential. □

It is this final result which has to be used in the proofs of theorem 6.8 and, especially, theorem 6.9.

PART II

COUNTABLY INFINITE-DIMENSIONAL SYSTEMS

CHAPTER 7

COUNTABLE STOCHASTIC MATRICES: STRONG ERGODICITY AND THE DOEBLIN CONDITION

This chapter is introductory. Its objective is to introduce some important characterizations of stochastic matrices of countably infinite dimension (or, shortly, countable stochastic matrices). The introduction of these characterizations is felt to be fundamental for a good appreciation of the conditions needed to establish a theory for countable nonnegative matrices, analogous to the one for finite matrices developed in chapter 2. These conditions are related to the concept of strong ergodicity in a Markov process with one recurrent class, and to the Doeblin condition in a Markov process with more than one recurrent class. In section 7.1 these concepts are introduced and some important features of Markov processes, satisfying these conditions, are discussed. In section 7.2 we pay special attention to the relationship between strong ergodicity, the Doeblin condition and the mean recurrence time. Also relationships with higher recurrence moments are investigated.

7.1. Strong ergodicity and the Doeblin condition.

We start with a brief review of the main ideas in the theory of homogeneous Markov processes on a countable state space that are important in the sequel. All these ideas are treated extensively in CHUNG [13]. Next, the concepts of strong ergodicity and the Doeblin condition are discussed. Important references with respect to these notions are DOEBLIN [18], DOOB [19], HORDIJK [27], ISAACSON AND LUECKE [31], ISAACSON AND MADSEN [32], NEVEU [43] and WIJNGAARD [72], and some of the references given there.

Consider the following stochastic process. An autonomous system may be in any state of a countable state space S. If at time t $(t \in \mathbb{N}_0)$ the system is in state i, then p_{ij} denotes the probability (independent of t) of observing the system in state j at time t+1. Assume

$$\sum_{j \in S} p_{ij} = 1 \qquad i \in S.$$

Hence, the transition law of the process is specified by a countable stochastic matrix P. Such a process is called a *homogeneous discrete-time Markov process* or (in this chapter) shortly: a *Markov process*.

As usual, $p_{ij}^{(n)}$ denotes the ij-th entry of P^n ($n \in \mathbb{N}_0$), and $p_{ij}^{(0)} := \delta_{ij}$. Note that $p_{ij}^{(n)}$ is the probability of being in state j after n transitions, when starting in state i. Furthermore, by definition

$$(7.1.1) \qquad p_{iA}^{(n)} := \sum_{j \in A} p_{ij}^{(n)} \qquad i \in S,$$

for a subset $A \subset S$.

An important concept, extremely helpful in characterizing the structure of a Markov process, is a *taboo probability*. Let H be a subset of the state space S. Define the *taboo probability* ${}_Hp_{ij}^{(n)}$ by

$$(7.1.2) \qquad {}_Hp_{ij}^{(0)} := \begin{cases} \delta_{ij} & \text{if } i \notin H,\ j \in S \\ 0 & \text{if } i \in H,\ j \in S, \end{cases}$$

and for $n \geqq 1$:

$$(7.1.3) \qquad {}_Hp_{ij}^{(n)} := \sum_{i_1,\ldots,i_{n-1} \notin H} p_{ii_1} p_{i_1 i_2} \cdots p_{i_{n-1} j} \qquad i,j \in S.$$

Hence, for $n > 0$, ${}_Hp_{ij}^{(n)}$ denotes the probability of a transition from i to j in n steps, without entering H in the meantime (note that $i,j \in H$ is allowed). H is called a *taboo set*. If H is empty, it is omitted from the notation; if H consists of a single state, $H = \{\ell\}$ say, we write ${}_\ell p_{ij}^{(n)}$ instead of ${}_{\{\ell\}}p_{ij}^{(n)}$.

Choose $j \in S$ fixed and define

$$(7.1.4) \qquad f_{ij}^{(n)} := {}_jp_{ij}^{(n)} \qquad n \in \mathbb{N}_0;\ i \in S,$$

or, more generally,

(7.1.5) $f_{iH}^{(n)} := {}_Hp_{iH}^{(n)} = \sum_{j\in H} {}_Hp_{ij}^{(n)}$ $\qquad n \in \mathbb{N}_0;\ i \in S.$

These values are often called *first-entrance probabilities*. A *first-entrance taboo probability* ${}_Hf_{ij}^{(n)}$ can be defined by

(7.1.6) ${}_Hf_{ij}^{(n)} := {}_{jH}p_{ij}^{(n)}$ $\qquad n \in \mathbb{N}_0;\ i,j \in S,$

where jH denotes $\{j\} \cup H$. Notice that

(7.1.7) $f_{iH}^{(n)} = \sum_{j\in H} {}_Hf_{ij}^{(n)}$ $\qquad n \in \mathbb{N}_0;\ i \in S.$

First-entrance probabilities are useful tools for characterizing the state space S of a Markov process. Let F_{iH} denote the probability that the system reaches the set H, when starting in state i. Then obviously

(7.1.8) $F_{iH} = \sum_{n=0}^{\infty} f_{iH}^{(n)}$ $\qquad i \in S$

(note that $f_{iH}^{(0)} = 0$). Clearly, $F_{iH} \leqq 1$, since F_{iH} is a probability.

<u>DEFINITION 7.1</u>. A state $i \in S$ is called *recurrent* if $F_{ii} = 1$, and *transient* if $F_{ii} < 1$. □

Extensions of these concepts to a collection of Markov chains are given in HORDIJK [27]. A further classification can be obtained by considering

(7.1.9) $m_{iH} := \sum_{n=1}^{\infty} nf_{iH}^{(n)}$ $\qquad i \in S.$

<u>DEFINITION 7.2</u>. A state $i \in S$ is called *positive recurrent* if $F_{ii} = 1$ and $m_{ii} < \infty$; it is called *null recurrent* if $F_{ii} = 1$ and $m_{ii} = \infty$. □

Definitions of *(ir)reducibility*, *(a)periodicity*, *classes*, etc. can be copied directly from the corresponding definitions in the finite-dimensional case. A Markov process is called *(ir)reducible*, *(a)periodic*, etc., if this is so for its corresponding stochastic matrix.

A proof of the next result can be found in CHUNG [13].

<u>PROPOSITION 7.1</u>. With respect to an irreducible Markov process either all states are recurrent or all states are transient. In case of recurrence,

either all states are positive recurrent or all states are null recurrent. □

In view of proposition 7.1 we speak of a *recurrent* and *transient* process, respectively. In case of recurrence, we speak of a *positive recurrent* and a *null recurrent* process, respectively.

If $F_{iH} > 0$ for some $i \in S$, $H \subset S$, then

$$m_{iH} / F_{iH} = \sum_{n=1}^{\infty} n(f_{iH}^{(n)} / F_{iH}),$$

i.e., $m_{iH}F_{iH}^{-1}$ can be seen as the conditional expectation of the time, necessary to reach H from i, under the condition that H will indeed be reached. If $F_{ii} = 1$ (i.e., if state i is recurrent), then m_{ii} is called the *mean recurrence time*. In this case we have

$$m_{ii} = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} f_{ii}^{(k)} = \sum_{n=0}^{\infty} (1 - \sum_{k=0}^{n} f_{ii}^{(k)}) = 1 + \sum_{n=1}^{\infty} \sum_{j \neq i} {}_{i}p_{ij}^{(n)}$$

Remark. In order to avoid possible confusion, we remark that several authors (e.g. HORDIJK [27]) define m_{iH} by

$$m_{iH} := \sum_{n=0}^{\infty} \sum_{j \notin H} {}_{H}p_{ij}^{(n)} \qquad i \in S,\ H \subset S.$$

The reader may note that this definition is equal to our definition (7.1.9) if and only if $F_{iH} = 1$ and $i \notin H$. In the sequel m_{iH} is defined as in (7.1.9).

Higher moments will also be needed in the sequel. Let

$$(7.1.10) \quad m_{iH}^{(k)} := \sum_{n=k}^{\infty} \binom{n}{k} f_{iH}^{(n+1-k)} \qquad k \in \mathbb{N}_0,\ i \in S,\ H \subset S.$$

Note that $m_{iH}^{(0)} = F_{iH}$, $m_{iH}^{(1)} = m_{iH}$ for $i \in S$, $H \subset S$. Clearly these moments can be used for more detailed classifications of the states.

Well known is the following property (cf. CHUNG [13]):

PROPOSITION 7.2. Let P be a stochastic matrix. Then

$$(7.1.11) \quad \lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} P^k$$

exists (element-wise). Let Q denote this limit and let the Markov process with transition matrix P be irreducible. If this process is transient ot null recurrent, then $Q = \underline{0}$; if it is positive recurrent, then

$$q_{ij} = m_{jj}^{-1} > 0 \qquad i,j \in S \qquad \square$$

Let P be a countable stochastic matrix. The definition of the period of a state $i \in S$ can be copied from definition 2.3. As in the finite case one can prove that all states have the same period if P is irreducible (cf. proposition 2.1). Hence we may speak of an *(a)periodic, irreducible* matrix P and of the *(a)periodic, irreducible* Markov process, induced by P. If P is irreducible and aperiodic, then the matrix Q, defined in proposition 7.2, satisfies

$$(7.1.12) \quad Q = \lim_{n\to\infty} P^n .$$

Note that Q is a matrix with equal rows.

If P is reducible, then the state space S can be partitioned uniquely into a collection of subsets $\{C(1), C(2), \ldots\}$, where $C(k)$ denotes an irreducible set of states which cannot be enlarged without destroying its irreducibility ($k \in \mathbb{N}$). As before $C(k)$ is called a *class*; we say that $C(k)$ is *final* if $p_{ij} = 0$ for all $i \in C(k)$, $j \in S\backslash C(k)$ ($k \in \mathbb{N}$). Furthermore, a class $C(k)$ is called *recurrent* if $F_{ii} = 1$ for some (and hence each) state $i \in C(k)$, otherwise $C(k)$ is called *transient*. *Positive recurrent* and *null recurrent* classes are defined analogously. Note that a recurrent class C must be final, since $p_{ij} > 0$ for some $i \in C$ and some $j \in S\backslash C$ implies $F_{ii} < 1$, contradicting the recurrence of i (recall that P is stochastic).

A proof of the next result can be found in CHUNG [13].

PROPOSITION 7.3. Let P be a countable stochastic matrix. Then

1. $\lim_{n\to\infty} p_{ij}^{(n)} = 0$ if j is transient or null recurrent ($i,j \in S$).

2. If j is positive recurrent, then

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_{ij}^{(k)} = F_{ij} m_{jj}^{-1} \qquad i,j \in S.$$

If in addition j is aperiodic, then

$$\lim_{n\to\infty} p_{ij}^{(n)} = F_{ij} m_{jj}^{-1} \qquad i,j \in S.$$

Finally, $F_{ij} = 1$ if i and j belong to the same recurrent class. $\square$

Let us take a closer look at a special type of Markov process, namely a process that has exactly one final class. Such a process is called *ergodic* if this final class is aperiodic and positive recurrent. The same adjective is used to describe the associated transition matrix P. The following, stronger, definition can be found in ISAACSON AND LUECKE [31].

DEFINITION 7.3. A stochastic matrix P is called *strongly ergodic* if

$$(7.1.13)\quad \lim_{n\to\infty} \| P^n - Q \| = 0,$$

where Q is a stochastic matrix with identical rows and $\|\cdot\cdot\|$ denotes the usual sup-norm. □

DEFINITION 7.4. The *delta coefficient* of a stochastic matrix P is defined by

$$\delta(P) := 1 - \inf_{i,k\in S} \sum_{j\in S} \min(p_{ij}, p_{kj}) \qquad \Box$$

ISAACSON AND MADSEN [32] proved

LEMMA 7.4. A stochastic matrix P is strongly ergodic if and only if $\delta(P^n) < 1$ for some $n \in \mathbb{N}$. □

Another characterization of strong ergodicity follows from

LEMMA 7.5. A stochastic matrix P is strongly ergodic if and only if there exists a state $s \in S$, a constant $\varepsilon > 0$ and an $n_0 \in \mathbb{N}$, such that

$$(7.1.14)\quad \inf_{i\in S} p_{is}^{(n)} \geqq \varepsilon \qquad n \geqq n_0.$$

PROOF. If P is strongly ergodic then

$$\sup_i |p_{is}^{(n)} - q_{ss}| \leqq \sup_i \sum_{j\in S} |p_{ij}^{(n)} - q_{ij}| = \| P^n - Q \|$$

Hence (7.1.14) holds for each state s with $q_{ss} > 0$ (take $\varepsilon = \frac{1}{2} q_{ss}$). On the other hand, if (7.1.14) holds for the triple (s, ε, n_0), then

$$\delta(P^n) \leqq 1 - \varepsilon \qquad n \geqq n_0,$$

which implies strong ergodicity, according to lemma 7.4. □

The characterization of strong ergodicity, given in lemma 7.5, will prove to be very useful in the sequel. Analogously we define *Cesaro strong ergodicity*.

DEFINITION 7.5. A stochastic matrix P is called *Cesaro strongly ergodic* if there exists a state $s \in S$, a constant $\varepsilon > 0$ and an $n_0 \in \mathbb{N}$, such that

$$\inf_{i\in S} \frac{1}{n+1} \sum_{k=0}^{n} p_{is}^{(k)} \geqq \varepsilon \qquad n \geqq n_0. \qquad \square$$

Strong ergodicity and Cesaro strong ergodicity are important tools for analyzing Markov processes with one final class. For processes with more than one final class there exists an analogue of the concept of strong ergodicity (cf. DOOB [19], NEVEU [43], HORDIJK [27]),

DEFINITION 7.6. A stochastic matrix P is said to satisfy the *Doeblin condition* if for some finite subset $A \subset S$, some $\varepsilon > 0$ and some $n \in \mathbb{N}$

$$\inf_{i\in S} p_{iA}^{(n)} \geqq \varepsilon. \qquad \square$$

Clearly, a Markov process with a transition probability matrix satisfying the Doeblin condition possesses at most a finite number of final classes. One may wonder how small the set A can be chosen. The next definition is relevant.

DEFINITION 7.7. A set of *reference states* of a Markov process is a subset $B \subset S$ which contains exactly one state from each recurrent class and no other states. □

The above-mentioned question can now be answered in the aperiodic case.

LEMMA 7.6. Let P be the transition matrix, associated with a Markov chain each recurrent class of which is aperiodic. Then P satisfies the Doeblin condition if and only if

$$\text{(7.1.15)} \qquad \inf_{i\in S} p_{iB}^{(n)} \geqq \varepsilon \qquad n \geqq n_0,$$

for some finite set of reference states B, some $\varepsilon > 0$ and some integer $n_0 > 0$.

PROOF. Obviously we only have to prove the "only if" part. Suppose P satisfies the Doeblin condition, i.e., suppose

$$(7.1.16)\quad \inf_i p_{iA}^{(k)} \geq \delta$$

for some finite set A, some $\delta > 0$ and some $k \in \mathbb{N}$. Clearly P possesses at least one, and at most a finite number of final classes C(1), C(2),...,C(r), say. Let $A(\ell) := A \cap C(\ell)$ $(\ell = 1,2,\ldots,r)$. Then

$$\sum_{j\in A(\ell)} p_{ij}^{(k+n)} = \sum_{h\in C(\ell)} p_{ih}^{(n)} \sum_{j\in A(\ell)} p_{hj}^{(k)} \geq \delta \qquad i \in C(\ell),\ n \in \mathbb{N}_0,$$

which implies that each class $C(\ell)$ is positive recurrent $(\ell = 1,2,\ldots,r)$, by proposition 7.2 and (7.1.12). Furthermore, $S\backslash(C(1) \cup \ldots \cup C(r))$ contains no recurrent class, since each recurrent class is final. Choose $t_\ell \in C(\ell)$ for $\ell = 1,2,\ldots,r$, then $B := \{t_1, t_2, \ldots, t_r\}$ is a set of reference states. Each state $i \in S$ has access to B. This certainly holds for each state in the finite set A. Hence, by the aperiodicity, there exists a constant $\rho > 0$ and an integer $\bar{n} \geq 0$ such that (cf. proposition 7.3)

$$p_{jB}^{(n)} \geq \rho \qquad j \in A,\ n \geq \bar{n}.$$

Since

$$p_{iB}^{(n+k)} \geq \sum_{j\in A} p_{ij}^{(k)}\, p_{jB}^{(n)} \qquad i \in S,\ n \in \mathbb{N}_0,$$

(7.1.15) follows immediately with $n_0 := \bar{n} + k$ and $\varepsilon = \rho\delta$. □

Lemma 7.6 shows that, in the aperiodic case, the Doeblin condition is the analogue of strong ergodicity in the case that more than one final class exists (cf. lemma 7.5). When assuming the Doeblin condition we may restrict ourselves to a set of reference states. If the aperiodicity assumption is removed, we obtain

LEMMA 7.7. A stochastic matrix P satisfies the Doeblin condition if and only if there exists a finite set of reference states B, a constant $\varepsilon > 0$

and an $\bar{n} \in \mathbb{N}$, such that

$$\inf_{i\in S} \frac{1}{n+1} \sum_{k=0}^{n} p_{iB}^{(k)} \geqq \varepsilon \qquad n \geqq \bar{n}. \tag{7.1.17}$$

PROOF. Suppose (7.1.17) holds. It is easy to construct a finite set A such that

$$p_{iA}^{(n)} \geqq \tfrac{1}{2} \qquad i \in B;\ n = 0,1,\ldots,\bar{n}. \tag{7.1.18}$$

Combining (7.1.17) and (7.1.18) we find

$$p_{iA}^{(\bar{n})} \geqq \tfrac{1}{2}\varepsilon \qquad i \in S.$$

Hence, the Doeblin condition holds. The proof of the inverse implication is similar to that of lemma 7.6 and will be left to the reader. □

Hence, also in the periodic case, we may restrict ourselves to a set of reference states when assuming the Doeblin condition. Note that (7.1.17) is the analogue of Cesaro strong ergodicity for the case of more than one final class. In the next section the notion of a set of reference states B is used to study relations between the Doeblin condition and the moments $m_{iB}^{(k)}$ (cf. (7.1.10)).

7.2. Doeblin condition and mean recurrence time.

In this section it is shown that the Doeblin condition is equivalent to uniform boundedness of the expected time, necessary to enter a set of reference states. Relations with higher moments are also investigated.

First we have:

THEOREM 7.8. Let P be a countable stochastic matrix. Then P satisfies the Doeblin condition if and only if there exists a nonempty, finite set of reference states B, such that

$$\inf_i F_{iB} = 1$$

and

$$\sup_i m_{iB} < \infty.$$

PROOF. "only if". Suppose P satisfies the Doeblin condition. By lemma 7.7 we know that there exists a nonempty, finite set of reference states B such that for some $\varepsilon > 0$ and some $n_0 \in \mathbb{N}_0$ we have

$$\inf_i \frac{1}{n+1} \sum_{k=0}^{n} p_{iB}^{(k)} \geqq \varepsilon \qquad n \geqq n_0.$$

It follows that

$$\sup_i \sum_{j \notin B} {}_Bp_{ij}^{(n)} \leqq 1-\varepsilon \qquad n \geqq n_0,$$

hence

$$\text{(7.1.19)} \quad \lim_{k\to\infty} \sup_i \sum_{j \notin B} {}_Bp_{ij}^{(kn)} \leqq \lim_{k\to\infty} (1-\varepsilon)^k = 0 \qquad n \geqq n_0.$$

Since, for each $i \in S$,

$$\sum_{j \notin B} {}_Bp_{ij}^{(n)}$$

is nonincreasing in n, it follows that

$$\lim_{n\to\infty} \sup_i \sum_{j \notin B} {}_Bp_{ij}^{(n)} = 0.$$

From

$$\sum_{k=0}^{n} f_{iB}^{(k)} + \sum_{j \notin B} {}_Bp_{ij}^{(n)} = 1 \qquad n \geqq 1,\ i \in S,$$

it now follows that $F_{iB} = 1$ for all $i \in S$. Furthermore

$$\text{(7.1.20)} \quad m_{iB} = \sum_{n=1}^{\infty} nf_{iB}^{(n)} = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} f_{iB}^{(k)} = 1 + \sum_{n=1}^{\infty} \sum_{j \notin B} {}_Bp_{ij}^{(n)} \qquad i \in S,$$

since $F_{iB} = 1$ for all $i \in S$. Using (7.1.19) and the fact that

$$\sum_{j \notin B} {}_Bp_{ij}^{(n)}$$

is nonincreasing in n, we conclude:

$$\sup_i m_{iB} < \infty .$$

"if". Suppose now that $F_{iB} = 1$ and $m_{iB} \leqq c$ for all $i \in S$ and for some constant $c > 0$ and a nonempty, finite set of reference states B. Since, for each $i \in S$,

$$\sum_{k=n+1}^{\infty} f_{iB}^{(k)}$$

is nonincreasing in n, it follows that for each δ, $0 < \delta < 1$, there exists an integer $n_0 = n_0(\delta) > 0$, such that

$$\sup_i \sum_{j \notin B} {}_B p_{ij}^{(n)} = \sup_i \sum_{k=n+1}^{\infty} f_{iB}^{(k)} \leqq \delta \qquad n \geqq n_0$$

(use (7.1.20) and the fact that $F_{iB} = 1$, $m_{iB} \leqq c$ for all $i \in S$).
It follows that

$$\inf_i \sum_{k=0}^{n} f_{iB}^{(k)} \geqq 1-\delta \qquad n \geqq n_0 .$$

In particular, for each $i \in S$ there exists a nonnegative integer $k = k(i)$, $k \leqq n_0$, such that

$$p_{iB}^{(k)} \geqq (1-\delta)/n_0 . \tag{7.1.21}$$

Furthermore, it is easy to find a finite set A, such that

$$p_{iA}^{(\ell)} \geqq \tfrac{1}{2} \qquad i \in B;\ \ell = 0,1,\ldots,n_0 . \tag{7.1.22}$$

Combining (7.1.21) and (7.1.22), we conclude

$$p_{iA}^{(n_0)} \geqq (1-\delta)/(2n_0) \qquad \text{for all } i \in S.$$

Hence, the Doeblin condition holds (cf. definition 7.6). □

The following result relates the Doeblin condition to higher moments.

<u>THEOREM 7.9</u>. Let B be a set of reference states. Let $F_{iB} = 1$ for all $i \in S$. Then

$$\sup_i m_{iB} < \infty \iff \sup_i m_{iB}^{(k)} < \infty \qquad \text{for all } k \in \mathbb{N}.$$

PROOF. Suppose

$$\sup_i m_{iB} < \infty.$$

Since $F_{iB} = 1$, this implies (cf. 7.1.20))

$$\sup_i \sum_{n=1}^{\infty} \sum_{j \notin B} {}_B p_{ij}^{(n)} < \infty.$$

Choose $\alpha \in \mathbb{N}$ and let

$$\sup_i m_{iB}^{(\alpha)} < \infty$$

Then

$$\infty > \sup_i \left(m_{iB}^{(\alpha)} + \sum_{n=1}^{\infty} \sum_{j \notin B} {}_B p_{ij}^{(n)} \, m_{jB}^{(\alpha)}\right) =$$

$$= \sup_i \left(\sum_{k=\alpha}^{\infty} \binom{k}{\alpha} f_{iB}^{(k+1-\alpha)} + \sum_{n=1} \sum_{j \notin B} {}_B p_{ij}^{(n)} \sum_{k=\alpha}^{\infty} \binom{k}{\alpha} f_{jB}^{(k+1-\alpha)} \right. =$$

$$= \sup_i \sum_{n=0}^{\infty} \sum_{k=\alpha}^{\infty} \binom{k}{\alpha} f_{iB}^{(n+k+1-\alpha)} = \sup_i \sum_{n+k=\alpha}^{\infty} \sum_{\ell=\alpha}^{n+k} \binom{\ell}{\alpha} f_{iB}^{(n+k+1-\alpha)} =$$

$$= \sup_i \sum_{n+k=\alpha}^{\infty} \binom{n+k+1}{\alpha+1} f_{iB}^{(n+k+1-\alpha)} = \sup_i \sum_{r=\alpha+1}^{\infty} \binom{r}{\alpha+1} f_{iB}^{((r+1)-(\alpha+1))} =$$

$$= \sup_i m_{iB}^{(\alpha+1)}$$

(here we used $\binom{n+1}{k} = \sum_{\ell=k-1}^{n} \binom{\ell}{k-1}$ for $k \geq 1$, $n \geq k-1$).

By induction it is proved that

$$\sup_i m_{iB}^{(k)} < \infty \qquad \text{for all } k \in \mathbb{N}.$$

The reverse implication is trivial. □

COROLLARY. A stochastic matrix P satisfies the Doeblin condition if and only if there exists a nonempty finite set of reference states B such that

$$(7.1.23) \quad F_{iB} = 1 \qquad \text{for all } i \in S,$$

and

$$(7.1.24) \quad \sup_i m_{iB}^{(k)} < \infty \qquad \text{for all } k \in \mathbb{N}.$$

PROOF. Immediately from theorem 7.8 and theorem 7.9. □

Note that a finite stochastic matrix trivially satisfies the Doeblin condition and hence (7.1.23), (7.1.24). Roughly speaking, one might say that a countable stochastic matrix satisfying the Doeblin condition behaves more or less the same as a finite stochastic matrix.

It will appear that many of the definitions given in this chapter can be extended to general (i.e. not necessarily stochastic) matrices of countably infinite dimension (compare the next two chapters). Moreover, we will meet several conditions which, translated to the stochastic case, take the form (7.1.23), (7.1.24). The notion of a "set of reference states" will play a basic role again in chapter 8. Once being familiar with the analysis of countable stochastic matrices, we hope that the results for general nonnegative matrices of countably infinite dimension will become more transparent. This has been the main reason for writing this introductory chapter, in spite of the fact that most of the results will be well known to readers familiar with stochastic analysis; as far as we know only the notion of a set of reference states, lemmas 7.6 and 7.7, and theorem 7.9 (and its corollary) are new.

CHAPTER 8

R-THEORY FOR COUNTABLE NONNEGATIVE MATRICES

In this chapter we investigate the possibility of extending the results of chapter 2 to matrices of countably infinite dimension. Apart from being interesting in itself such an extension would be important in several fields of application that require the assumption of a countable state space (e.g., in queueing theory). Moreover, once having established an extension to systems with countable state space one may hope that at least some light is shed on systems with a more general state space, i.e., systems where we deal with general nonnegative linear operators.

VERE-JONES [65], [66] showed that the Perron-Frobenius theorem can be extended to irreducible nonnegative matrices of countably infinite dimension. A basic role in his analysis is played by the parameter R, the common convergence radius of the series

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} z^n \qquad\qquad i,j \in S,$$

where P is a countable irreducible nonnegative matrix and S a countable set of states. In section 8.1 we review the fundamental results of this R-theory (SENETA [52]). Next, the theory is extended to general nonnegative matrices; we show how a "generalized eigenvector theory", analogous to the one in chapter 2, can be developed (section 8.2). The results will prove to be of particular importance in the use of certain contraction properties of nonnegative matrices and for the construction of so-called β-subinvariant vectors. The construction of generalized eigenvectors, in this chapter called generalized R-invariant vectors, requires some assumptions on the nonnegative matrix P; in section 8.3 we will give a separate discussion of these assumptions and relate them to results of the preceding chapter.

The methodology, used in chapter 8, has been deeply influenced by the treatment of countable Markov chains in CHUNG [13]. As in the preceding chapter, concepts like *first-entrance transition, recurrence*, etc. will appear to play a fundamental role. In addition, semi-probabilistic interpretations of invariant vectors and generalized invariant vectors can be given.

8.1. Countable irreducible nonnegative matrices.

This section is devoted to a brief exposition of the main ideas of Vere-Jones' treatment of countable irreducible nonnegative matrices. In addition we give some preliminary results which will be needed in the sequel.

Throughout this chapter we work under the following assumption.

Assumption 1. For each nonnegative matrix P we suppose

$$p_{ij}^{(n)} < \infty \qquad i,j \in S,\ n \in \mathbb{N}.$$

Let P be a countable irreducible nonnegative matrix. Consider the following power series

$$P_{ij}(z) := \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n \qquad i,j \in S.$$

Let R_{ij} denote the convergence radius of $P_{ij}(z)$. One can show that for all $i,j,s,t \in S$ we have

$$R_{ij} = R_{st},$$

if P is irreducible. The proof uses inequalities of the kind

$$(8.1.1) \qquad p_{ij}^{(n+k+\ell)} \geqq p_{is}^{(k)}\, p_{st}^{(n)}\, p_{tj}^{(\ell)},$$

and the fact that for some fixed k and ℓ

$$p_{is}^{(k)} > 0,\ p_{tj}^{(\ell)} > 0$$

by the irreducibility of P. Hence $R_{ij} \leqq R_{st}$. The inverse inequality is proved similarly. This shows the desired result.

It follows that the series $P_{ij}(z)$, $i,j \in S$ have a common convergence radius R. R will be called the *convergence parameter* of P. Throughout this section we assume

Assumption 2. $R > 0$.

The reader may note that for finite nonnegative matrices R is equal to the reciprocal of the spectral radius. SENETA [52] shows that $R < \infty$ by applying a theorem on supermultiplicative functions, due to KINGMAN [37]. We follow another way by introducing so-called *first-entrance transition power series*; the finiteness of R then follows as a by-product.

Let H be a subset of S. We define quantities ${}_Hp_{ij}^{(n)}$, $f_{ij}^{(n)}$, $f_{iH}^{(n)}$ and ${}_Hf_{ij}^{(n)}$ in exactly the same way as in chapter 7 (compare (7.1.2) until (7.1.6)) for $i,j \in S$, $n \in \mathbb{N}_0$. We speak of *taboo-transition values* and *first-entrance transition values* (note that the use of the word "probability" is not allowed in the present context). Now consider the power series $F_{ij}(z)$, defined by

$$F_{ij}(z) := \sum_{n=0}^{\infty} f_{ij}^{(n)} z^n \qquad i,j \in S.$$

Obviously, $F_{ij}(z)$ has convergence radius at least equal to R since $f_{ij}^{(n)} \leqq p_{ij}^{(n)}$ for all $n \in \mathbb{N}_0$, $i,j \in S$. Furthermore, it follows from

$$p_{ij}^{(n)} = \sum_{k=0}^{n} f_{ij}^{(k)} p_{jj}^{(n-k)} \qquad i,j \in S,\ n \in \mathbb{N}$$

(this is the first-entrance decomposition, cf. CHUNG [13]), that

$$(8.1.2) \qquad P_{ij}(z) = \delta_{ij} + F_{ij}(z)P_{jj}(z) \qquad i,j \in S, |z| < R.$$

In particular

$$(8.1.3) \qquad F_{ii}(z) = (P_{ii}(z)-1)/P_{ii}(z) \qquad i \in S,\ |z| < R,$$

which implies that

$$(8.1.4) \qquad F_{ii}(R-) := \lim_{z\to R} F_{ii}(z) \leqq 1.$$

The next lemma follows immediately from (8.1.4).

LEMMA 8.1. The convergence parameter R of a countable irreducible nonnegative matrix P is finite. □

We use (8.1.4) to classify the states, as follows:

DEFINITION 8.1. Let P be a countable, irreducible, nonnegative matrix with convergence parameter R. Then state $i \in S$ is called *R-recurrent* if $F_{ii}(R-) = 1$ and *R-transient* if $F_{ii}(R-) < 1$. □

It follows from (8.1.3) that i is R-transient if and only if $P_{ii}(R-) < \infty$. Using this result and inequalities of the kind (8.1.1), one easily establishes the following lemma.

LEMMA 8.2. Let P be countable and irreducible with convergence parameter R. Then, either all states are R-recurrent or all states are R-transient. The matrix P is then called *R-recurrent*, or *R-transient*, respectively. □

The reader easily verifies the following equality:

$$(8.1.5) \qquad F_{ij}(z) = z \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(z) + zp_{ij} \qquad i,j \in S, \ |z| < R.$$

Since for $0 < z < R$ the series $F_{ij}(z)$ is increasing in z for all $i,j \in S$, we have

$$F_{ij}(z) \leqq R \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(R-) + Rp_{ij} \qquad i,j \in S, \ 0 \leqq z < R,$$

hence

$$F_{ij}(R-) \leqq R \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(R-) + Rp_{ij} \qquad i,j \in S.$$

Using the same argument, we find

$$F_{ij}(R-) \geqq z \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(z) + zp_{ij} \qquad i,j \in S, \ 0 \leqq z < R,$$

hence, by Fatou's lemma

$$F_{ij}(R-) \geqq R \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(R-) + Rp_{ij} \qquad i,j \in S.$$

Combining these results yields

(8.1.6) $F_{ij}(R-) = R \sum_{\ell \neq j} p_{i\ell} F_{\ell j}(R-) + Rp_{ij}$ $\qquad i,j \in S,$

and in particular

$$1 \geq F_{jj}(R-) \geq R^n{}_j p_{j\ell}^{(n)} F_{\ell j}(R-) \qquad j,\ell \in S,\ n \in \mathbb{N}.$$

(Obviously, (8.1.6) follows immediately from (8.1.5) by the Monotone Convergence Theorem. The proof of (8.1.6) is given in detail since it is typical for several proofs to follow).

Using the irreducibility of P, we conclude

$$0 < F_{\ell j}(R-) < \infty \qquad \ell,j \in S.$$

The following result is fundamental.

THEOREM 8.3. Let P be irreducible with convergence parameter R. Fix some state $s \in S$ and define the vector u by

$$u_i := \delta_{is} + (1-\delta_{is})F_{is}(R-) \qquad i \in S.$$

Then

$$RPu \leqq u$$

with equality if and only if P is R-recurrent. If x is any semi-positive vector such that $\beta Px \leqq x$ for some $\beta > 0$, then $x > \underline{0}$ and $\beta \leqq R$. Moreover

(8.1.7) $\dfrac{x_i}{x_s} \geqq F_{is}(\beta)$ $\qquad i \in S.$

PROOF. $RPu \leqq u$ follows immediately from (8.1.6). If P is R-recurrent then $RPu = u$ since $F_{ss}(R-) = 1 = u_s$ in that case. If $\beta Px \leqq x$ for some $x \geq \underline{0}$ and $\beta > 0$, then $x > \underline{0}$ by the irreducibility of P. Suppose $\beta > R$. Choose α such that $R < \alpha < \beta$. Then

$$\sum_{n=0}^{\infty} \alpha^n P^n x \leqq \sum_{n=0}^{\infty} \alpha^n \beta^{-n} x = (1-\alpha\beta^{-1})^{-1} x,$$

hence in particular

$$\sum_{n=0}^{\infty} p_{ij}^{(n)} \alpha^n < \infty \qquad i,j \in S,$$

contradicting the fact that $\alpha > R$. Hence $\beta \leq R$.

Finally we prove (8.1.7). Clearly the result holds for $i = s$. Suppose $i \neq s$. By induction we show that for all $m \in \mathbb{N}_0$ and $i \neq s$

$$(8.1.8) \qquad \frac{x_i}{x_s} \geq \sum_{k=0}^{m} f_{is}^{(k)} \beta^k,$$

which implies the desired result. For $m = 1$ the result holds since $f_{is}^{(0)} = 0$ and

$$x_i \geq \beta p_{is} x_s = \beta f_{is}^{(1)} x_s \qquad i \in S.$$

Suppose (8.1.8) holds for $m = \bar{m} \geq 1$. Then

$$x_i \geq \beta \sum_j p_{ij} x_j \geq \beta p_{is} x_s + \beta \sum_{j \neq s} p_{ij} \sum_{k=1}^{\bar{m}} f_{js}^{(k)} \beta^k x_s =$$
$$= \beta f_{is}^{(1)} x_s + \sum_{k=2}^{\bar{m}+1} f_{is}^{(k)} \beta^k x_s = \sum_{k=1}^{\bar{m}+1} f_{is}^{(k)} \beta^k x_s,$$

hence (8.1.8) holds for all m. □

The proof of theorem 8.3 follows also from a combination of several results in SENETA [52]. The method used above will also prove to be useful in the next section.

The reader may note that the vector u plays the role of a strictly positive eigenvector, associated with R^{-1}, if P is R-recurrent (recall that for finite matrices R^{-1} is equal to the spectral radius). Theorem 8.3 also shows the nature of these eigenvectors (their elements being power series of first-entrance transition values, calculated at the convergence parameter R).

<u>DEFINITION 8.2</u>. Let P be a countable nonnegative matrix and let x be a semi-positive vector such that for some constant $\beta > 0$

$$\beta P x \leq x.$$

Then the vector x is called *β-subinvariant*. If $\beta Px = x$ then x is called *β-invariant*. □

It is well known that a strictly positive eigenvector of a finite nonnegative square matrix must be associated with its spectral radius (cf. BRAUER [10]). The reader may wonder whether it is possible for a countable irreducible nonnegative matrix P to have values in its point spectrum with modulus larger than R^{-1}. The answer is affirmative as the following example shows.

Example 8.1. Let $S := \{1,2,\ldots\}$ and let P be a stochastic matrix defined by

$$p_{i,i+1} = 1-p_{i,1} = \frac{1+p^{i+1}}{1+p^{i}} \; ; \; p_{ij} = 0 \text{ for } j \neq 1,i+1 \qquad i \in S,$$

with $0 < p < 1$. Obviously, P is irreducible and aperiodic. We have

$$f_{1,1}^{(n)} = \frac{1+p^n}{1+p}\,\frac{p^n-p^{n+1}}{1+p^n} = \frac{1-p}{1+p}\,p^n \qquad n \in \mathbb{N}.$$

$$F_{1,1}(z) = \frac{1-p}{1+p}\sum_{n=1}^{\infty}(pz)^n = \frac{(1-p)pz}{(1+p)(1-pz)} ,$$

$$P_{1,1}(z) = (1-F_{1,1}(z))^{-1} = \frac{(1+p)(1-pz)}{1+p-2pz} ,$$

hence $R = \frac{1+p}{2p}$. Furthermore, $F_{1,1}(R-) = 1$, hence P is R-recurrent. It is not hard to verify that the vector x, defined by

$$x_i = \frac{p^{i-1}+p^i}{1+p^i} \qquad i \in S,$$

is a strictly positive solution of

$$Px = R^{-1}x = \frac{2p}{1+p}\,x.$$

If $p < 1$, then $R^{-1} < 1$. Hence, there exists a strictly positive eigenvector associated with R^{-1}, although R^{-1} is smaller than the spectral radius of P, This is impossible for finite-dimensional nonnegative matrices. □

A further classification of the states can be obtained by taking into account the series $m_{ij}(z)$, defined by

$$m_{ij}(z) := \sum_{n=1}^{\infty} nf_{ij}^{(n)} z^n \qquad i.j \in S, \ |z| < R.$$

DEFINITION 8.3. Let P be an irreducible R-recurrent nonnegative matrix. Then state $i \in S$ is called *R-positive* if

$$m_{ii}(R-) = \lim_{z \to R} m_{ii}(z) < \infty,$$

otherwise i is called *R-null*. □

As before one may prove that either all states of an irreducible R-recurrent nonnegative matrix are R-positive or all states are R-null (cf. VERE-JONES [65]). We speak of an *R-positive (R-null)* matrix. The reader may verify that the matrix P of example 8.1 is R-positive for $0 < p < 1$.

For the analysis in the next section also higher moments are needed. Define $m_{ij}^{(k)}(z)$ by

$$m_{ij}^{(k)}(z) = \sum_{n=k}^{\infty} \binom{n}{k} f_{ij}^{(n+1-k)} z^{n+1-k} \qquad k \in \mathbb{N}_0 ;\ i,j \in S;\ |z| < R.$$

Note that $m_{ij}^{(0)}(z) = F_{ij}(z)$, $m_{ij}^{(1)}(z) = m_{ij}(z)$. The following result holds.

LEMMA 8.4. Let P be irreducible with convergence parameter R. Let $m_{ij}^{(k)}(R-) < \infty$ for $k \in \mathbb{N}$; $i,j \in S$. Then for $k \geq 1$

$$(8.1.9) \qquad m_{ij}^{(k)}(R-) = m_{ij}^{(k-1)}(R-) + R \sum_{\ell \neq j} p_{i\ell}\, m_{\ell j}^{(k)}(R-) \qquad i,j \in S.$$

PROOF. As in the proof of theorem 7.9 we find for $k \geq 1$

$$m_{ij}^{(k)}(z) = m_{ij}^{(k-1)}(z) + \sum_{n=1}^{\infty} \sum_{\ell \neq j} z^n\, {}_jp_{i\ell}^{(n)} m_{\ell j}^{(k-1)}(z) \qquad i,j \in S, |z| \in R,$$

from which it is immediately deduced that

$$m_{ij}^{(k)}(z) = m_{ij}^{(k-1)}(z) + z \sum_{\ell \neq j} p_{i\ell}\, m_{\ell j}^{(k)}(z) \qquad i,j \in S, |z| < R.$$

The proof is completed in the same way as the proof of (8.1.6). □

The moments $m_{ij}^{(k)}(R-)$ will play a crucial role in the construction of so-called generalized R-invariant vectors (the equivalent of generalized eigenvectors, associated with the spectral radius, in the infinite dimensional case). Moreover, they are "minimal" in a sense, similar to the quantities $F_{ij}(R-)$ (compare 8.1.8), as is shown by the next theorem.

THEOREM 8.5. Let P be irreducible with convergence parameter R and let there exist nonnegative vectors y(0), y(1), y(2),... such that for some $\beta > 0$ the following holds

$$(8.1.10)\quad \begin{cases} y(0)_i \geqq \beta \sum_{j\in S} p_{ij}\, y(0)_j & i \in S, \\ y(k)_i \geqq y(k-1)_i + \beta \sum_{j\neq s} p_{ij}\, y(k)_j & k \in \mathbb{N},\ i \in S, \end{cases}$$

where $y(0) \geq \underline{0}$ and s is some fixed chosen state in S. Then $y(k) > \underline{0}$ for $k \in \mathbb{N}_0$ and $\beta \leqq R$. Furthermore, if y(Q) is normalized in such a way that $y(0)_s = 1$, then

$$(8.1.11)\quad y(k)_i \geqq m_{is}^{(k)}(\beta) \qquad i \in S,\ k \in \mathbb{N}_0.$$

PROOF. By theorem 8.3 it follows that $y(0) > \underline{0}$ and $\beta \leqq R$. Since $y(k) \geqq y(k-1)$ for $k \geqq 1$, it follows by induction that $y(k) > \underline{0}$, $k \in \mathbb{N}_0$. By (8.1.7) we have

$$y(0)_i \geqq F_{ij}(\beta) = m_{is}^{(0)}(\beta) \qquad i \in S,$$

if $y(0)_s = 1$. Suppose (8.1.11) holds for $k = \alpha$. Iteration of (8.1.10) yields (recall that $y(\alpha+1) > \underline{0}$)

$$y(\alpha+1)_i \geqq m_{is}^{(\alpha)}(\beta) + \sum_{n=1}^{\infty} \sum_{j\neq s} \beta^n\, {}_sp_{ij}^{(n)} m_{js}^{(\alpha)}(\beta) = m_{is}^{(\alpha+1)}(\beta)$$

(compare the proof of lemma 8.4). Hence (8.1.11) holds for $k = \alpha+1$. By induction the proof is completed. □

The reader may note that, if the conditions of theorem 8.5 are fulfilled for $\beta = R$, the result implies the existence of all moments $m_{is}^{(k)}(R-)$ for $i,s \in S$, $k \in \mathbb{N}_0$. This may become important in applications where vectors y(0), y(1), y(2),... are relatively easy to obtain, whereas calculation of the moments $m_{is}^{(k)}(R-)$ may become a hard, or even impossible job. The functions $y(0)$, $y(1)$, $y(2)$,... are often called *Lyapunov functions*; HORDIJK [28] uses these functions to analyze (sets of) stochastic matrices (cf. also VAN HEE, HORDIJK AND VAN DER WAL [83]). In addition, equations of the kind (8.1.10) are usually called a *Lyapunov function criterion*.

With theorem 8.5 we conclude the analysis of countable irreducible nonnegative matrices and turn to the reducible case. In particular we will make an attempt to extend the generalized eigenvector theory of chapter 2 to the infinite-dimensional situation. The moments $m_{ij}^{(k)}(R-)$ will play a fundamental role in this analysis.

8.2. Countable reducible nonnegative matrices.

In this section we shall discuss countable reducible nonnegative matrices and give conditions guaranteeing the existence of strictly positive R-subinvariant vectors. Moreover, we discuss β-subinvariant vectors and generalized R-invariant vectors (the equivalent of the generalized eigenvectors, considered in chapter 2, for the infinite dimensional case). The formal definition follows below. Under some special conditions, related to the Doeblin condition for stochastic matrices, these generalized R-invariant vectors can be chosen nonnegative. All conditions needed appear to hold trivially in the finite case so that the results of this section extend those of chapter 2.

Before continuing, we have to specify what we mean by the convergence parameter R of a reducible nonnegative matrix P. As before, consider the power series

$$P_{ij}(z) = \sum_{n=0}^{\infty} p_{ij}^{(n)} z^n \qquad\qquad i,j \in S,$$

and let R_{ij} be the convergence radius of $P_{ij}(z)$. Define the *convergence parameter* R of P by

$$R := \inf_{i,j} R_{ij}.$$

We have seen that in the irreducible case $R_{ij} = R$ all $i,j \in S$. Obviously, this does not hold in general for reducible matrices. As in the finite case, we may partition the state space S into a number of subsets $C(1)$, $C(2)$,..., such that the restriction of P to $C(k) \times C(k)$ is irreducible and such that the C(k) cannot be enlarged without destroying this irreducibility. As before, the subsets C(k) are called *classes*; these classes are partially ordered by accessibility relations and we may speak of a common convergence radius $R^{(k)}$ of all $P_{ij}(z)$, $i,j \in C(k)$ $(k \in \mathbb{N})$. Moreover, it is easy to show, by methods similar to those of the preceding section, that for two classes C(k) and C(ℓ) with $k \neq \ell$ we have

$$R_{ij} = R_{st} \qquad\qquad i,s \in C(k);\ j,t \in C(\ell).$$

Hence we may speak of a common convergence radius $R^{(k,\ell)}$. This has been noticed by TWEEDIE [62], who also proved

$$(8.2.1) \qquad R^{(k,\ell)} \leqq \min\ (R^{(k)},\ R^{(\ell)}).$$

As usual, we say that a class C is *final* if no state in C has access to any state in S\C. Furthermore, we say that a class in C is *R-transient, R-null* or *R-positive* if this holds for some (and hence each) state in C.

In this section we will work under some restrictions concerning the structure of the nonnegative matrix P (which may be considerably relaxed; compare the last section of this chapter), in order to simplify the proofs. Assumptions 1 and 2 of the preceding section are supposed to hold, together with

Assumption 3. P partitions S into a finite number of classes.

First, the structure of β-subinvariant vectors of P is investigated. Analogous to theorem 8.3 the following result holds.

THEOREM 8.6. Let P be a countable nonnegative matrix with convergence parameter R. Let C(1), C(2),...,C(k) be the final classes with respect to P. Choose a state $t_\ell \in C(\ell)$ $(\ell = 1,2,\ldots,k)$ and suppose

$$(8.2.2) \qquad F_{it_\ell}(R-) < \infty \qquad\qquad \text{for all } i \in S,\ \ell = 1,..,k.$$

Let the vector u(R-) be defined by

$$u(R-)_i := \sum_{\ell=1}^{k} \{\delta_{it_\ell} + (1-\delta_{it_\ell})\, F_{it_\ell}(R-)\} \qquad i \in S.$$

Then

(8.2.3) $\quad RPu(R-) \leqq u(R-)$,

with strict equality if and only if all final classes are R-recurrent. Furthermore, if x is a semi-positive β-subinvariant vector with $x_{t_\ell} > 0$ ($\ell = 1,2,\ldots,k$), then $\beta \leqq R$ and

(8.2.4) $$x_i \geqq \sum_{\ell=1}^{k} \{\delta_{it_\ell} + (1-\delta_{it_\ell})F_{it_\ell}(\beta)\}\, x_{t_\ell} \qquad i \in S. \qquad \square$$

The proof of theorem 8.6 is completely analogous to that of theorem 8.3 and is left to the reader. Note that $F_{t_\ell t_n}(\beta) = 0$ for $\ell \neq n$, $\ell, n = 1,..,k$.

In the next section a separate discussion will be devoted to condition (8.2.2); in particular it will be shown that this condition, and hence the theorem, is independent of the choice of the set $\{t_1, t_2,\ldots,t_k\}$; we only must have $t_\ell \in C(\ell)$ for $\ell = 1,\ldots,k$.

The question arises what can be said if the conditions of theorem 8.6 are not fulfilled, i.e., if (8.2.2) does not hold. In the finite case this question has been answered in terms of structural properties of generalized eigenvectors (cf. theorem 2.12). In order to develop a theory for generalized eigenvectors, chain-structures of the underlying graph associated with the nonnegative matrix have been investigated. It seems reasonable to conjecture that such a structure can also be exploited in the case, where we deal with a countable state space.

Let $\{C(1), C(2),\ldots,C(n)\}$ be a set of classes of a nonnegative matrix P of countably infinite dimension, such that for each $k \in \{1,\ldots,n-1\}$ there exists a pair of states i,j (depending on k), with $i \in C(k)$, $j \in C(k+1)$ and $p_{ij} > 0$. Analogous to the finite case we call such a sequence a *chain*, that *starts* with C(1) and *ends* with C(n). The *length of a chain* is now defined as the number of R-recurrent classes it contains. As before we have

<u>DEFINITION 8.4</u>. The *index* ν of a nonnegative matrix P with convergence parameter R is defined as the length of its longest chain. The *depth of a class* C is the length of the longest chain which starts with C. The *depth* ν_i *of a state i* is the depth of the class to which i belongs. □

The notion of a *set of reference states* will appear to be useful in the forthcoming analysis. For general nonnegative matrices this concept is defined as follows:

DEFINITION 8.5. Let P be a countable nonnegative matrix with convergence parameter R. A *set of reference states* B is a subset of S, which contains exactly one state from each R-recurrent class and no other states. □

Analogous to the concept of generalized eigenvectors we have in the infinite-dimensional case

DEFINITION 8.6. A *generalized R-invariant* vector of order k ($k \in \mathbb{N}$) is a vector x such that $(I-RP)^k x = \underline{0}$ and $(I-RP)^{k-1} x \neq \underline{0}$. □

Using these definitions a theorem concerning the existence of generalized R-invariant vectors, similar to theorem 2.12, can be formulated. Before establishing this result we need the following

LEMMA 8.7. Let P be a nonnegative matrix with convergence parameter R and let $H \subset S$. Define

$$ {}_H m_{ij}^{(k)}(z) := \sum_{n=k}^{\infty} \binom{n}{k} {}_H f_{ij}^{(n+1-k)} z^{n+1-k} \qquad k \in \mathbb{N}_0 ;\ i,j \in S,\ |z| < R. $$

As usual, we write ${}_H F_{ij}(z)$ instead of ${}_H m_{ij}^{(0)}(z)$ and ${}_H m_{ij}(z)$ instead of ${}_H m_{ij}^{(1)}(z)$.

The following equalities hold (jH denotes $\{j\} \cup H$).

$$ {}_H F_{ij}(R-) = R \sum_{\ell \notin jH} p_{i\ell}\ {}_H F_{\ell j}(R-) + R p_{ij} \qquad i,j \in S, $$

$$ {}_H m_{ij}^{(k)}(R-) = {}_H m_{ij}^{(k-1)}(R-) + R \sum_{\ell \notin jH} p_{i\ell}\ {}_H m_{\ell j}^{(k)}(R-) \qquad i,j \in S,\ k \in \mathbb{N} $$

(whenever these moments exist).

PROOF. Analogous to the proof of (8.1.6) and that of lemma 8.4. □

One technical remark has to be made.

Remark. Let $H \subset S$ and let $\{t_1,\dots,t_k\} \subset H$, $\{n_1,\dots,n_k\} \subset \mathbb{N}_0$, and $\{\alpha_1,\dots,\alpha_k\} \subset \mathbb{R}$. Suppose

$${}_{H}m_{it_\ell}^{(n_\ell+1)}(R-) < \infty \qquad i \in S,\ \ell = 1,\dots,k,$$

and define the vector x by

$$x_i := \sum_{\ell=1}^{k} \alpha_\ell\, {}_{H}m_{it_\ell}^{(n_\ell)}(R-) \qquad i \in S.$$

In the proof of theorem 8.8 on generalized R-invariant vectors we will use the vector x', defined by

$$x_i' := \sum_{\ell=1}^{k} \alpha_\ell\, {}_{H}m_{it_\ell}^{(n_\ell+1)}(R-) \qquad i \in S.$$

From lemma 8.7 it is easily deduced that

$$\text{(8.2.5)} \qquad R \sum_{j \notin H} p_{ij}\, x_j' = x_i' - x_i \qquad i \in S.$$

Now we are ready to prove the generalization of theorem 2.12.

THEOREM 8.8. Let P be a countable nonnegative matrix with convergence parameter R and index ν. Let B be a set of reference states and suppose

$$\text{(8.2.6)} \qquad {}_{B}m_{it}^{(k)}(R-) < \infty \qquad i \in S,\ t \in B,\ k = 0,1,..,\nu.$$

Then there exist vectors $w(1), w(2),\dots,w(\nu)$, with $w(k) > 0$ if $\nu_i = k$ and $w(k)_i = 0$ if $\nu_i < k$ $(k = 1,2,\dots,\nu;\ i \in S)$, such that

$$\text{(8.2.7)} \qquad \begin{cases} RPw(\nu) = w(\nu) & \\ RPw(k) = w(k) + w(k+1) & k = 1,\dots,\nu-1. \end{cases}$$

PROOF. Let D(k) contain all states with depth k and let $B(k) := B \cap D(k)$ $(k = 0,1,\dots,\nu)$. Note that $B(0) = \emptyset$. Define $y(\nu,t)$ by

$$y(\nu,t)_i := {}_{B}F_{it}(R-) \qquad i \in S,\ t \in B(\nu).$$

Since ${}_{B}F_{it}(R-) = F_{it}(R-)$ and $F_{tt}(R-) = 1$ for $t \in B(\nu)$ (cf. (8.2.6)) we have

$$\text{(8.2.8)} \qquad RPy(\nu,t) = y(\nu,t) \qquad t \in B(\nu).$$

Next, define $y(\nu-1,t)$ by

$$y(\nu-1,t)_i := {}_BF_{it}(R-) - \sum_{s\in B(\nu)} \alpha_{st}\, y(\nu,s)'_i \qquad i\in S,\ t\in B(\nu-1),$$

where α_{st} is determined such that $y(\nu-1,t)_s = 0$ for $s \in B\backslash\{t\}$. Note that $y(\nu-1,t)_i = 0$ for $i \in S\backslash(D(\nu)\cup D(\nu-1))$ and for $i \in B(\nu-1)\backslash\{t\}$ and that

$$y(\nu,s)'_r = {}_Bm_{rs}(R-) = m_{rs}(R-) = 0 \qquad r,s \in B(\nu),\ r \neq s,$$

hence

$$\alpha_{st} = {}_BF_{st}(R-)/y(\nu,s)'_s \qquad s \in B(\nu),\ t \in B(\nu-1).$$

It follows from (8.2.5), ${}_BF_{tt}(R-) = 1$ for $t \in B(\nu-1)$, and the choice of α_{st} that

$$RPy(\nu-1,t) = y(\nu-1,t) + \sum_{s\in B(\nu)} \alpha_{st}\, y(\nu,s) \qquad t \in B(\nu-1).$$

Hence each $y(\nu-1,t)$ is a generalized R-invariant vector of order 2. Continuing in this way, we define for $k = \nu-1,\nu-2,\ldots,2,1$:

$$y(k,t)_i := {}_BF_{it}(R-) - \sum_{n=k+1}^{\nu} \sum_{s\in B(n)} \alpha_{st}\, y(n,s)'_i \qquad i\in S,\ t\in B(k),$$

with α_{st} chosen such that $y(k,t)_s = 0$ for $s \in B\backslash\{t\}$. Hence

$$\alpha_{st} = {}_BF_{st}(R-)/y(k+1,s)'_s \qquad s \in B(k+1),\ t \in B(k),$$

$$\alpha_{st} = \{{}_BF_{st}(R-) - \sum_{r\in B(k+1)} \alpha_{rt}\, y(k+1,r)'_r\}/y(k+2,s)'_s$$

$$s \in B(k+2),\ t \in B(k),$$

etc. From (8.2.5), ${}_BF_{tt}(R-) = 1$ for $t \in B(k)$ and the choice of α_{st}, we find

$$RPy(k,t) = y(k,t) + \sum_{n=k+1}^{\nu} \sum_{s\in B(n)} \alpha_{st}\, y(n,s) \qquad t\in B(k);\ k = 1,..,\nu-1, \tag{8.2.9}$$

which implies that $y(k,t)$ is a generalized R-invariant vector of order $\nu-k+1$. Finally, define

$$w(1) := \sum_{t \in B(1)} y(1,t),$$

and for $k = 1,2,\ldots,\nu-1$:

$$w(k+1) := RPw(k) - w(k).$$

Then the vectors $w(k)$ satisfy (8.2.7) (combine (8.2.8) and (8.2.9)). Trivially we have

$$w(k)_i = 0 \qquad\qquad i \in \bigcup_{\ell=0}^{k-1} D(\ell).$$

Furthermore,

$$w(1)_i = \sum_{t \in B(1)} {}_B F_{it}(R-) \qquad\qquad i \in D(1),$$

hence $w(1)_i > 0$ for $i \in D(1)$. Also notice that for $k = 1,2,\ldots,\nu-1$

$${}_B F_{st}(R-) = {}_s F_{st}(R-) \qquad\qquad s \in B(k+1),\ t \in B(k),$$

and since each $s \in B(k+1)$ has access to at least one $t \in B(k)$, we conclude that for such a pair (s,t)

$$\alpha_{st} = {}_s F_{st}(R-)/y(k+1,s)'_s = {}_s F_{st}(R-)/m_{ss}(R-) > 0.$$

By induction it then follows immediately that

$$w(k)_i > 0 \qquad\qquad i \in D(k);\ k = 1,2,\ldots,\nu,$$

and the proof is complete. □

The generalized R-invariant vectors, constructed in the proof of theorem 8.8, are not necessarily nonnegative. In order to establish the existence of nonnegative generalized R-invariant vectors, we need one additional assumption. First a definition:

DEFINITION 8.7. Let H and A be two subsets of S. Define

$${}_H f_{iA}^{(n)} := \sum_{j \in A} {}_{H \cup A} f_{ij}^{(n)} \qquad\qquad \text{for } i \in S,\ n \in \mathbb{N}_0$$

(compare also (7.1.7)), and let

$$ {}_{H}m_{iA}^{(k)}(z) := \sum_{n=k}^{\infty} \binom{n}{k} {}_{H}f_{iA}^{(n+1-k)} z^{n+1-k} \qquad i \in S,\ k \in \mathbb{N}_0,\ |z| < R. \square $$

The following result now holds.

THEOREM 8.9. Let $P,R,\nu,B,B(1),B(2),\ldots,B(\nu)$ be defined as in theorem 8.8 and its proof, and suppose

$$ (8.2.10) \qquad {}_{B}F_{i,B(k)}(R-) < \infty \qquad i \in S;\ k = 1,\ldots,\nu, $$

and

$$ (8.2.11) \qquad {}_{B}m_{i,B(k)}^{(\ell)}(R-) \leqq c\ {}_{B}F_{i,B(k)}(R-) \qquad i \in S;\ k,\ell = 1,\ldots,\nu, $$

where c is some positive constant.

Then the vectors $w(1),w(2),\ldots,w(\nu)$, satisfying (8.2.7), can be chosen nonnegative.

PROOF. Since $B(k) \subset B$ and since two different states in $B(k)$ do not have access to each other, it follows that

$$ {}_{B}m_{i,B(k)}^{(\ell)}(R-) = \sum_{j \in B(k)} {}_{B}m_{ij}^{(\ell)}(R-) \qquad i \in S;\ k,\ell = 1,\ldots,\nu. $$

Hence the conditions (8.2.10) and (8.2.11) certainly imply (8.2.6). Furthermore, note that (8.2.11) implies

$$ (8.2.12) \qquad {}_{B}m_{i,B(k)}^{(\ell)}(R-) \leqq c\ {}_{B}m_{i,B(k)}^{(h)}(R-) \qquad i \in S;\ h,\ell = 1,\ldots,\nu, $$

since

$$ {}_{B}m_{i,B(k)}^{(h)}(R-) \geqq {}_{B}F_{i,B(k)}(R-) \qquad i \in S,\ h \in \mathbb{N}_0. $$

Finally, if $\{w(1),w(2),\ldots,w(\nu)\}$ is a set of vectors satisfying (8.2.7), then also $\{x(1),x(2),\ldots,x(\nu)\}$, defined by

$$x(\nu) := w(\nu)$$

$$x(k) := w(k) + \alpha x(k+1) \qquad k = \nu-1, \nu-2, .., 2, 1.$$

Since (8.2.12) holds, α can be chosen so large that $x(k) \geq 0$ for $k = \nu, \nu-1, \ldots, 2, 1$ (compare section 5.2). This proves the theorem. □

Theorem 8.8 and theorem 8.9 together constitute the natural generalization of theorem 2.12 to the countably infinite dimensional case. The results also show the nature of the generalized eigenvectors, discussed in chapter 2 (i.e., each generalized eigenvector is a linear combination of certain power series, based on taboo-transition values and calculated at the convergence parameter). Again, it can be shown that the conditions do not depend on the particular choice of $B(1), B(2), \ldots, B(\nu)$. Furthermore, there exists a strong relationship between assumption (8.2.11) and the Doeblin condition for stochastic matrices. In the final section of this chapter some attention is paid to this correspondence.

8.3. Discussion of the conditions of the theorems 8.6, 8.8 and 8.9.

In this section we discuss in more detail the conditions appearing in the theorems of the preceding section. In particular, we discuss formulas (8.2.2), (8.2.6), (8.2.10), (8.2.11) and assumption 3. Our objective is to elucidate the nature of these conditions which, although looking rather artificial, are in fact natural extensions of properties which hold trivially in the finite case. Finally, some relaxations of assumption 3 are briefly discussed.

Before starting this discussion we need one important property of countable irreducible nonnegative matrices.

LEMMA 8.10. Let P be irreducible with convergence parameter R, and suppose that for some state $t \in S$ and some $k \in \mathbb{N}_0$

(8.3.1) $\quad m_{tt}^{(k)}(R-) < \infty.$

Then

(8.3.2) $m_{ij}^{(k)}(R-) < \infty$ for all $i,j \in S$.

PROOF. The proof will be given by induction with respect to k. For $k = 0$ the result follows from lemma 8.2 and repeated application of

$$F_{it}(R-) = R \sum_{j \neq t} p_{ij}F_{jt}(R-) + Rp_{it} \geqq R \sum_{j} p_{ij}F_{jt}(R-) \qquad i \in S.$$

by using the irreducibility of P (note that $F_{tt}(R-) \leqq 1$). Suppose the result holds for $n = 0,1,\ldots,k-1$ and let (8.3.1) hold. Define

$$_{\ell}F_{ij}^{(n)}(z) := \left(\frac{d}{dz}\right)^n {}_{\ell}F_{ij}(z) \qquad i,j,\ell \in S;\ n \in \mathbb{N}_0;\ |z|<R.$$

It is not hard to verify the equivalence

(8.3.3) $$_{\ell}m_{ij}^{(n)}(R-) < \infty \iff {}_{\ell}F_{ij}^{(n)}(R-) < \infty \qquad i,j,\ell \in S;\ n \in \mathbb{N}_0.$$

Also the following relationship is easily established

(8.3.4) $$F_{ij}(z) = {}_{\ell}F_{ij}(z) + {}_{j}F_{i\ell}(z)\, F_{\ell j}(z) \qquad i,j,\ell \in S.$$

Differentation of (8.3.4) (k times) yields

(8.3.5) $$F_{ij}^{(k)}(z) = {}_{\ell}F_{ij}^{(k)}(z) + \sum_{n=0}^{k} \binom{k}{n} {}_{j}F_{i\ell}^{(n)}(z)\, F_{\ell j}^{(k-n)}(z) \qquad i,j,\ell \in S.$$

Now fix $s \in S$, $s \neq t$. Take $i = j = t$ and $\ell = s$ in (8.3.5). Since

$$_{s}F_{tt}^{(k)}(R-) < F_{tt}^{(k)}(R-)$$

and

(8.3.6) $$F_{tt}^{(k)}(R-) < \infty \implies F_{tt}^{(n)}(R-) < \infty \text{ for } n < k,$$

we find from (8.3.1), (8.3.3), (8.3.5), and the induction hypothesis, that

(8.3.7) $$0 < F_{st}^{(k)}(R-) < \infty,$$

and

$$(8.3.8) \qquad 0 < {}_tF_{ts}^{(k)}(R-) < \infty.$$

Taking $i = \ell = t$, $j = s$ in (8.3.5) we find

$$(8.3.9) \qquad F_{ts}^{(k)}(z)(1-{}_sF_{tt}(z)) = {}_tF_{ts}^{(k)}(z) + \sum_{\ell=1}^{k} \binom{k}{\ell} {}_sF_{tt}^{(\ell)}(z)\, F_{ts}^{(k-\ell)}(z).$$

Since

$${}_sF_{tt}(R-) < F_{tt}(R-) \leqq 1,$$

and

$${}_sF_{tt}^{(n)}(R-) < F_{tt}^{(n)}(R-) \qquad n \in \mathbb{N},\ n \leqq k,$$

it follows from (8.3.1), (8.3.3), (8.3.6), (8.3.8), (8.3.9), and the induction hypothesis, that

$$(8.3.10) \qquad 0 < F_{ts}^{(k)} < \infty.$$

Taking $i = \ell = s$, $j = t$ in (8.3.5), we find analogously from (8.3.7), that

$${}_tF_{ss}^{(k)}(R-) < \infty,$$

and, finally, by taking $i = j = s$, $\ell = t$ in (8.3.5), we conclude to

$$(8.3.11) \qquad F_{ss}^{(k)}(R-) < \infty.$$

Now, choose $r \in S$. By taking $i = j = s$, $\ell = r$ in (8.3.5) we find

$$(8.3.12) \qquad F_{rs}^{(k)}(R-) < \infty.$$

Since r and s have been chosen arbitrarily and since (8.3.3) holds, we conclude

$$(8.3.2) \qquad m_{ij}^{(k)}(R-) < \infty \qquad \text{for all } i,j \in S. \qquad \square$$

The proof of lemma 8.10 has been given in detail as it is typical for proofs of this kind of results. Not all proofs of forthcoming results will be given in full since they use essentially the same techniques.

Next, we discuss assumption (8.2.2) of theorem 8.6. Recall that we have chosen one state t_ℓ in each final class $C(\ell)$ ($\ell = 1,\dots,k$) and that we assumed

(8.2.2) $$F_{it_\ell}(R-) < \infty \qquad \text{for all } i \in S,\ \ell = 1,\dots,k.$$

The following lemma states that this condition is independent of the choice of $t_\ell \in C(\ell)$ ($\ell = 1,\dots,k$).

<u>LEMMA 8.11</u>. Let P have convergence parameter R and let C be a final class with respect to P. Then for all $i \in S$, $s,t \in C$,

$$F_{is}(R-) < \infty \iff F_{it}(R-) < \infty$$

<u>PROOF</u>. By use of

$$F_{it}(z) = {}_sF_{it}(z) + {}_tF_{is}(z)\, F_{st}(z)$$

and

$$F_{is}(z) = {}_tF_{is}(z) + {}_sF_{it}(z)\, F_{ts}(z)$$

the result follows immediately, since $0 < F_{st}(R-) < \infty$ and $0 < F_{ts}(R-) < \infty$.□

<u>Remark</u>. If i and j are elements of an initial class C and t belongs to a final class $D \neq C$, then

$$F_{it}(R-) \geqq F_{ij}(R-)\, F_{jt}(R-).$$

Since $0 < F_{ij}(R-) < \infty$, it follows that we need (8.2.2) only for one state i in each initial class of P.

Similar remarks can be made with respect to condition (8.2.6):

<u>LEMMA 8.12</u>. Let P have convergence parameter R and index ν. Let A and B be two different sets of reference states. Suppose that for some $k \in \mathbb{N}$

(8.3.13) $\quad {}_{A}m_{is}^{(k)}(R-) < \infty$ for all $i \in S$, $s \in A$.

Then also

(8.3.14) $\quad {}_{B}m_{it}^{(k)}(R-) < \infty$ for all $i \in S$, $t \in B$.

PROOF. The proof can be given by a combination of the methods used in the proofs of lemma 8.10 and lemma 8.11. We will not present it in detail but only indicate the essential parts.

If s and t belong to the same R-recurrent class, then

$$m_{st}^{(k)}(R-) < \infty,$$

by (8.3.13). Lemma 8.10 now implies

$$m_{ts}^{(k)}(R-) < \infty.$$

Furthermore, if s and t belong to an R-recurrent class C, and r to an R-recurrent class D, such that C has access to D, then, by methods similar to those used in the proof of lemma 8.10, we obtain

$$ {}_{s,E}m_{ir}^{(k)}(R-) < \infty \quad \Rightarrow \quad {}_{t,E}m_{ir}^{(k)}(R-) < \infty \qquad \text{for all } i \in S,$$

where E is defined by

$$E := \{i \in A \mid i \neq s, \text{ i has access to s}\}.$$

Hence, E contains precisely those states in A with depth larger than the depth of s (cf. definition 8.4).

The proof of (8.3.14) now follows by consecutive substitution of a state from A by a state from B (belonging to the same R-recurrent class), until A is completely replaced by B. □

Remark. As before, it can be shown that (8.2.6) is needed only for one state i in each initial class of P.

With respect to the conditions (8.2.10) and (8.2.11) the following remarks can be made. For each $i \in S$ let ν_i denote the depth of i (cf. definition 8.4). For $k = 0,1,\dots,\nu-1$ define

$$B(k) := \{i \in B \mid \nu_i = k\}$$

$$E(k) := \{i \in S \mid \nu_i \geqq k \text{ and } i \notin B\backslash B(k)\}$$

(the sets $B(k)$ were defined in the proof of theorem 8.8 already).

If for all $i \in S$

$$_BF_{i,B(k)}(R-) > 0 \qquad k = 1,\dots,\nu,$$

then conditions (8.2.10) and (8.2.11) imply that $P^{E(k)}$ (the restriction of P to $E(k) \times E(k)$) is equivalent to a stochastic matrix satisfying the Doeblin condition. This can be seen by using the similarity transformation

$$q_{ij} := R^{-1}({}_BF_{i,B(k)}(R-))^{-1} p_{ij}({}_BF_{j,B(k)}(R-)) \qquad i,j \in E(k).$$

For $k = 1,2,\dots,\nu$ let

$$D(k) := \{i \in S \mid \nu_i = k\}.$$

Then, in particular, $P^{D(k)}$ is equivalent to a stochastic matrix satisfying the Doeblin condition. Since the Doeblin condition holds trivially in the finite case, it follows that the conditions (8.2.10) and (8.2.11) are always fulfilled with respect to finite-dimensional nonnegative matrices.

By theorem 7.8 it is sufficient to assume (instead of (8.2.11))

$$(8.3.15) \qquad {}_Bm_{i,B(k)}(R-) \leqq c\ {}_BF_{i,B(k)}(R-) \qquad i \in S;\ k = 1,\dots,\nu,$$

for some positive constant c.

However, note that for the construction of generalized R-invariant vectors assumption (8.2.6) is already sufficient. Condition (8.2.11) or (8.3.15) (which will be called the *uniform boundedness conditions*) are only needed to obtain semi-positive generalized R-invariant vectors.

Let us take once again a closer look at condition (8.2.6) and try to explain its meaning. As before, let

$$B(k) := \{t \in B \mid \nu_t = k\} \qquad k = 1,\dots,\nu.$$

Note that (8.2.6) implies that

$${}_sF_{st}(R-) < \infty \qquad s \in B(\nu),\ t \in B(\nu-1)$$

$$m_{ss}(R-) < \infty \qquad s \in B(\nu).$$

Since

$$F_{st}(z) = {}_sF_{st}(z) + F_{ss}(z)\, F_{st}(z) \qquad |z| < R,$$

or

$$F_{st}(z) = {}_sF_{st}(z)\, (1-F_{ss}(z))^{-1} \qquad |z| < R,$$

we find, multiplying with R-z and taking limits

$$\lim_{z\to R} (R-z)\, F_{st}(z) = {}_sF_{st}(R-)/F_{ss}^{(1)}(R-) < \infty$$

Here we used (8.3.3) for n = 1. Similarly

$$\lim_{z\to R} (R-z)^k\, F_{st}(z) < \infty \qquad s \in B(\nu),\ t \in B(\nu-k)\ ;\ k = 1,\dots,\nu.$$

Since

$$P_{st}(z) = F_{st}(z)\, P_{tt}(z) = F_{st}(z)(1-F_{ss}(z))^{-1} \qquad s \neq t,\ |z| < R,$$

it follows that

(8.3.16) $$\lim_{z\to R} (R-z)^k\, P_{st}(z) < \infty \qquad s \in B(\nu),\ t \in B(\nu-k+1)\ ;\ k = 1,\dots,\nu.$$

An interpretation of (8.2.16) can also be given in terms of *last-exit* and *first-entrance transition values*. *Last exit transition values* $\ell_{ij}^{(n)}$ are defined by

$$\ell_{ij}^{(n)} := {}_iP_{ij}^{(n)} \qquad i,j \in S;\ n \in \mathbb{N}_0.$$

Define $L_{ij}(z)$ by

(8.3.17) $$L_{ij}(z) := \sum_{n=0}^{\infty} \ell_{ij}^{(n)} z^n \qquad i,j \in S, \ |z| < R.$$

A complete analysis of countable irreducible nonnegative matrices can be given by means of the power series $L_{ij}(z)$ instead of $F_{ij}(z)$ (cf. SENETA [52]). In particular, it can be shown that for an irreducible R-recurrent nonnegative matrix P the vector y^T, defined by

$$y_i := L_{ti}(R-) \qquad i \in S,$$

(with t some fixed state) satisfies

$$y^T = y^T RP.$$

More general, it is easy to show that

$$_sF_{st}(R-) = R \sum_{\substack{i \in D(\nu) \\ j \in D(\nu-1)}} L_{si}(R-)\, p_{ij}\, F_{jt}(R-) \qquad s \in B(\nu),\ t \in B(\nu-1).$$

Hence, assumption (8.2.6) (which implies $_sF_{st}(R-) < \infty$ for $s \in B(\nu)$ and $t \in B(\nu-1)$ gives in fact a relationship between the nondiagonal blocks of P and the left and right R-invariant vectors of the diagonal blocks.

We conclude this section with a few remarks concerning assumption 3 in section 8.2. With respect to theorem 8.6 it is obvious that only the finiteness of the number of final classes has to be assumed. Similarly, we need in theorem 8.8 only finiteness of the number of R-recurrent classes (an infinite number of R-transient classes may be allowed). In fact, even the case $\nu = \infty$ is allowed; in that case we only assume that it is possible to define some "depth-structure" for the matrix P, i.e. to define a depth for each class.

CHAPTER 9

R-THEORY FOR SETS OF COUNTABLE NONNEGATIVE MATRICES

In this final chapter, some of the results of chapter 8 are extended to sets of countable nonnegative matrices. The reader may expect a generalization of the results obtained in chapter 3, concerning strictly positive eigenvectors and block-triangular structures. Also, a generalization of the results of section 5.2, concerning generalized eigenvectors, is formulated. The proof of this final result is rather technical; therefore it will not be given here. The techniques are based on methods used to prove the existence of solutions to the optimality equations in average reward denumerable state Markov Decision Chains (cf. ZIJM [84]).

The results of this chapter are of particular importance for the analysis of Markov decision processes with a countable state space. For instance, it is possible to find (almost) optimal contraction factors, which is useful for determining sharp bounds for the value of such a process (cf. WESSELS [71], VAN HEE AND WESSELS [70]).

In section 9.1 we treat the relatively simple case, where the system is communicating (compare definition 5.1). Some results of this section can also be found in KENNEDY [36]. More general situations are treated in section 9.2; we show how, under some restrictions, the results of section 3.2 can be generalized. Furthermore, some partial results are given concerning a possible extension of the generalized eigenvector theory for sets of nonnegative matrices (cf. section 5.2). We conclude with some comments.

9.1. Communicating systems.

In this section we deal with a set K of countable nonnegative matrices with the product property. It is assumed that K is *communicating* (cf. definition 5.1). Since this notion plays a basic role in this section, we repeat its definition.

DEFINITION 9.1. K is called *communicating* if for $i,j \in S$ there exists a matrix $P \in K$ such that i has access to j under P. □

Throughout this chapter we suppose:

Assumption 1. K is compact (regarded as a subset of $\mathbb{R}^{\infty}$ with the usual topology of element-wise convergence).

If we define an operator

$$x \longrightarrow \sup_{P \in K} Px \qquad x \in \mathbb{R}^{\infty},$$

then assumption 1, together with the fact that K has the product property, implies that for each x there exists a $\tilde{P} \in K$ such that

$$(9.1.1) \qquad \tilde{P}x = \sup_{P \in K} Px,$$

a property which is referred to as to the *optimal choice property* (cf. SENETA [52], and the introduction of chapter 3).

In order to develop an R-theory for sets of nonnegative matrices, we first have to specify what we mean by R. Define a *strategy* π as a sequence of matrices, $\pi := (P(1), P(2), \ldots)$ with $P(\ell) \in K$, $\ell \in \mathbb{N}$. The *n-th step transition value* $t_{ij}^{(n)}(\pi)$ is defined as the ij-th element of the matrix $P(1)P(2)\ldots P(n)$. Let Π denote the set of all strategies. Define

$$(9.1.2) \qquad t_{ij}^{(n)} := \sup_{\pi \in \Pi} t_{ij}^{(n)}(\pi) \qquad i,j \in S,\ n \in \mathbb{N},$$

and let $t_{ij}^{(0)} := \delta_{ij}$ for $i,j \in S$. In order to avoid trivialities, throughout this chapter we work under

Assumption 2. $t_{ij}^{(n)} < \infty$ for all $i,j \in S$, $n \in \mathbb{N}$.

Let R_{ij} denote the convergence radius of the series

$$\sum_{n=0}^{\infty} t_{ij}^{(n)} z^n \qquad i,j \in S,$$

then we define the *convergence parameter* R of K by

$$(9.1.3) \qquad R := \inf_{i,j} R_{ij}.$$

In order to avoid trivialities we suppose (compare also chapter 8):

Assumption 3. $R > 0$.

The following result can be proved by methods similar to those used in the one-matrix case (cf. KENNEDY [36]).

LEMMA 9.1. Let K be communicating. Then $R_{ij} = R$ for all $i,j \in S$. □

For each strategy π we define

$$(9.1.4) \qquad T_{ij}(\pi,z) := \sum_{n=0}^{\infty} t_{ij}^{(n)}(\pi)\ z^n \qquad i,j \in S,$$

and

$$(9.1.5) \qquad T_{ij}(z) := \sup_{\pi} T_{ij}(\pi,z) \qquad i,j \in S.$$

Now, it is easy to show (cf. KENNEDY [36]):

LEMMA 9.2. Let K be communicating and let R be defined by (9.1.3). Then R is the common convergence radius of all series $T_{ij}(z)$, $i,j \in S$.

PROOF. Let α_{ij} be the convergence radius of $T_{ij}(z)$ for some $i,j \in S$. Clearly $\alpha_{ij} \geqq R$. If $\alpha_{ij} > R$, then choose β and γ such that

$$R < \gamma < \beta < \alpha_{ij}.$$

Define a constant $c > 0$ by

$$c := \sup_{\pi} \sum_{n=0}^{\infty} t_{ij}^{(n)}(\pi)\ \beta^n.$$

Then certainly

$$t_{ij}^{(n)}\ \beta^n = \sup_{\pi} t_{ij}^{(n)}(\pi)\ \beta^n \leqq c \qquad n \in \mathbb{N}_0.$$

Hence

$$\sum_{n=0}^{\infty} t_{ij}^{(n)} \gamma^n \leqq c \sum_{n=0}^{\infty} \beta^{-n} \gamma^n = c(1-\gamma\beta^{-1})^{-1} < \infty.$$

contradicting the fact that $\gamma > R$. Hence $\alpha_{ij} = R$. □

Taboo transition values can be defined again. Let $H \subset S$ be some taboo set and let $\pi = (P(1), P(2), \ldots)$ be some strategy. For $i,j \in S$ set

(9.1.6) $$ {}_Ht_{ij}^{(0)} := \begin{cases} \delta_{ij} & \text{if } i \notin H,\ j \in S \\ 0 & \text{if } i \in H,\ j \in S, \end{cases}$$

and for $n \geqq 1$:

(9.1.7) $$ {}_Ht_{ij}^{(n)}(\pi) := \sum_{i_1,\ldots,i_{n-1}\notin H} p(1)_{ii_1} p(2)_{i_1i_2} \ldots p(n)_{i_{n-1}j} \qquad i,j \in S.$$

Again we omit the subscript H if H is empty and we write ${}_kt_{ij}^{(n)}(\pi)$ if $H = \{k\}$. Furthermore, define

(9.1.8) $$ {}_Hf_{ij}^{(n)}(\pi) := {}_{jH}t_{ij}^{(n)}(\pi) \qquad i,j \in S,\ n \in \mathbb{N}_0,$$

where jH denotes $\{j\} \cup H$. Finally, let

(9.1.9) $$ {}_HF_{ij}(z,\pi) := \sum_{n=0}^{\infty} {}_Hf_{ij}^{(n)}(\pi)\, z^n \qquad i,j \in S,$$

(9.1.10) $$ {}_HF_{ij}(z) := \sup_{\pi} {}_HF_{ij}(z,\pi) \qquad i,j \in S.$$

The next result has been proved by KENNEDY [36] in the communicating case, but it is easy to verify that it holds in general. Since the methodology is typical for several proofs to follow, we five a complete proof. We have

LEMMA 9.3. Let K have convergence parameter R. Then

(9.1.11) $$ F_{ij}(R-) := \sup_{P\in K} \{R \sum_{\ell\neq j} p_{i\ell}\, F_{\ell j}(R-) + R\, p_{ij}\} \qquad i,j \in S.$$

PROOF. For each strategy $\pi = (P(1), P(2), \ldots)$, we define $\pi^{(n)}$ by

$$(9.1.12) \qquad \pi^{(n)} := (P(n),\, P(n+1), \ldots) \qquad\qquad n \in \mathbb{N}.$$

Obviously, we have for $|z| < R$

$$F_{ij}(z) = \sup_{\pi} \sum_{n=1}^{\infty} f_{ij}^{(n)}(\pi)\, z^n =$$
$$= \sup_{\pi} \{z \sum_{\ell \neq j} p_{i\ell} \sum_{n=1}^{\infty} f_{\ell j}^{(n)}(\pi^{(2)})\, z^n + z\, p_{ij}\} \leqq$$
$$\leqq \sup_{P \in K} \{z \sum_{\ell \neq j} p_{i\ell}\, F_{\ell j}(z) + z\, p_{ij}\}.$$

On the other hand, if we define

$$F_{ij}^{(n)}(z) := \sup_{\pi} \sum_{k=1}^{n} f_{ij}^{(k)}(\pi)\, z^k \qquad\qquad i,j \in S,\ n \in \mathbb{N},$$

then by induction with respect to n, using the optimal choice property, we find for $|z| < R$:

$$F_{ij}^{(n)}(z) = \sup_{P} \{z \sum_{\ell \neq j} p_{i\ell}\, F_{\ell j}^{(n-1)}(z) + z\, p_{ij}\} \qquad n \geqq 2.$$

Since

$$\sup_{\pi} \sum_{k=1}^{\infty} f_{ij}^{(k)}(\pi)\, z^k - \sup_{\pi} \sum_{k=1}^{n} f_{ij}^{(k)}(\pi)\, z^k \leqq$$
$$\leqq \sup_{\pi} \sum_{k=n+1}^{\infty} f_{ij}^{(k)}(\pi)\, z^k \leqq \sum_{k=n+1}^{\infty} t_{ij}^{(k)}\, z^k,$$

and since the last term tends to zero for $n \to \infty$, we conclude by Fatou's lemma that for all $P \in K$:

$$F_{ij}(z) \geqq z \sum_{\ell \neq j} p_{i\ell}\, F_{\ell j}(z) + z\, p_{ij}. \qquad\qquad i,j \in S,\ |z| < R.$$

Combining the results, we conclude

$$F_{ij}(z) = \sup_{P \in K} \{z \sum_{\ell \neq j} p_{i\ell}\, F_{\ell j}(z) + z\, p_{ij}\} \qquad\qquad i,j \in S,\ |z| < R.$$

That this result holds for $z = R$ follows now in the same way as in the proof of (8.1.6). □

In the proof of the next lemma we will meet *stationary* strategies. A strategy π is called *stationary* if $\pi = (P, P, P, \ldots)$, with $P \in K$. If π is stationary, we usually write $p_{ij}^{(n)}$ instead of $t_{ij}^{(n)}(\pi)$, $f_{ij}^{(n)}(P)$ instead of $f_{ij}^{(n)}(\pi)$, $F_{ij}(z,P)$ instead of $F_{ij}(z,\pi)$, etc.

With respect to the power series $F_{ij}(z)$ the following holds:

<u>LEMMA 9.4</u>. Let K be communicating with convergence parameter R. Then for all $i,j \in S$

$$F_{ii}(R-) \leqq 1$$

and

$$F_{ij}(R-) < \infty.$$

<u>PROOF</u>. Choose $j \in S$ fixed. In the proof of lemma 9.3 we found that

$$F_{ij}(z) = \sup_{P\in K} \{z \sum_{\ell\neq j} p_{i\ell}\, F_{\ell j}(z) + z\, p_{ij}\} \qquad i \in S,\ |z| < R.$$

Hence, by the optimal choice property, there exists a $P \in K$ such that

$$F_{ij}(z) = z \sum_{\ell\neq j} p_{i\ell}\, F_{\ell j}(z) + z\, p_{ij} \qquad i \in S,\ |z| < R.$$

Iteration yields

$$F_{ij}(z) = \sum_{k=1}^{n-1} f_{ij}^{(k)}(P)\, z^k + z^n \sum_{\ell\neq j} {}_j p_{i\ell}^{(n)}\, F_{\ell j}(z).$$

Now, for $|z| < \alpha < R$, we find

$$\alpha^n \sum_{\ell\neq j} {}_j p_{i\ell}^{(n)}\, F_{\ell j}(\alpha) \leqq F_{ij}(\alpha) < \infty$$

Hence

$$|z^n \sum_{\ell\neq j} {}_j p_{i\ell}^{(n)} F_{\ell j}(z)| \leqq |z|^n \sum_{\ell\neq j} {}_j p_{i\ell}^{(n)}\, F_{\ell j}(\alpha) \leqq (|z|\alpha^{-1})^n\, F_{ij}(\alpha),$$

which tends to zero for $n\to\infty$. We conclude

$$(9.1.13) \qquad F_{ij}(z) = \sum_{k=1}^{\infty} f_{ij}^{(k)}(P)\, z^k = F_{ij}(z,P) \qquad i \in S,\ |z| < R.$$

Now, since

$$F_{jj}(z,P) < 1 \qquad\qquad |z| < R,$$

we find $F_{jj}(z) < 1$ for $|z| < R$, hence $F_{jj}(R-) \leqq 1$. Since j was chosen arbitrarily, this holds for each $j \in S$. By iteration of (9.1.11) and by the communicatingness of K, it now follows immediately that $F_{ij}(R-) < \infty$ for all $i,j \in S$. □

COROLLARY. $R < \infty$. □

The results of lemma 9.4 suggest a classification of the states, similar to the one given in chapter 8.

DEFINITION 9.2. State $i \in S$ is called *R-recurrent* if $F_{ii}(R-) = 1$ and *R-transient* if $F_{ii}(R-) < 1$. □

We point out that the analogue of lemma 8.2 does not hold in general, i.e., in a communicating system we may have both R-recurrent and R-transient states. However, the following result is easily verified:

LEMMA 9.5. Let K be communicating with convergence parameter R. Let $s \in S$ be R-recurrent and let $K_1 \subset K$ denote the set of matrices P that obey

$$F_{is}(R-) = R \sum_{j \neq s} p_{i\ell} F_{\ell s}(R-) + R\, p_{is} \qquad i \in S.$$

If K_1 is still communicating, then all states are R-recurrent. □

KENNEDY [36] proved lemma 9.5 under the condition that K contains only irreducible nonnegative matrices. His proof remains valid under our (weaker) condition.

The next theorem is fundamental. It gives a characterization of the convergence parameter R and of β-(sub)invariant vectors, similar to the one presented in section 8.1.

THEOREM 9.6. Let K be communicating with convergence parameter R. Choose $s \in S$ and define the vector $u > \underline{0}$ by

(9.1.14) $\quad u_i := \delta_{is} + (1-\delta_{is})\, F_{is}(R-) \qquad\qquad i \in S.$

Then

(9.1.15) $\quad R \sup_P P u \leqq u,$

with equality if and only if s is R-recurrent.

If x is a semi-positive vector such that for some $\beta > 0$

(9.1.16) $\quad \beta \sup_P P x \leqq x,$

then $x > \underline{0}$ and $\beta \leqq R$. Furthermore

(9.1.17) $\quad \frac{x_i}{x_j} \geqq F_{ij}(\beta) \qquad\qquad i,j \in S.$

PROOF. As in chapter 8. Compare also KENNEDY [36] and HORDIJK [27], ch. 8.

The vector x satisfying (9.1.16) is called a β-*subinvariant* vector for the set K. If strict equality holds in (9.1.16) we speak of a β-*invariant* vector.

Note that for $|z| < R$:

$$F_{ij}(z) = \sup_{P \in K} F_{ij}(z,P) \qquad\qquad i,j \in S$$

(compare (9.1.13)). Analogous to the proof of (8.1.6) one can show that

(9.1.18) $\quad F_{ij}(R-) = \sup_{P \in K} F_{ij}(R-,P) \qquad\qquad i,j \in S.$

Take $j \in S$ fixed. One may wonder whether there exists a $P \in K$ such that

$$F_{ij}(R-) = F_{ij}(R-,P) \qquad\qquad \text{for all } i \in S.$$

Moreover, if R(P) denotes the convergence parameter of P, does there exist a $P \in K$ such that

$$R = R(P) \;?$$

This section will be concluded with a theorem, stating that under certain conditions a positive answer to these questions exists. First, we define

$$(9.1.19) \quad m_{ij}(z,P) := \sum_{n=1} nf_{ij}^{(n)}(P)z^n \qquad i,j \in S;\ P \in K.$$

THEOREM 9.7. Let K have convergence parameter R and let all $P \in K$ be irreducible. Let $s \in S$ be R-recurrent and suppose

$$(9.1.20) \quad \sup_{P \in K} m_{ss}(R-,P) < \infty.$$

Then there exists a $P \in K$ such that

$$R(P) = R$$

and

$$(9.1.21) \quad F_{is}(R-) = F_{is}R-,P) \qquad \text{for all } i \in S.$$

PROOF. Define

$$F_{ss}^{(n)}(R-,P) = \sum_{k=1}^{n} f_{ss}^{(k)}(P)R^k \qquad n \in \mathbb{N}_0,\ P \in K.$$

Since

$$m_{ss}(R-,P) = \sum_{n=0}^{\infty} \{F_{ss}(R-,P) - F_{ss}^{(n)}(R-,P)\}$$

and since

$$F_{ss}(R-,P) - F_{ss}^{(n)}(R-,P)$$

is nonincreasing in n, for all P, it follows from (9.1.20) that $F_{ss}^{(n)}(R-,P)$ is converging to $F_{ss}(R-,P)$ uniformly in P, for $n \to \infty$. It is easy to establish the continuity of $F_{ss}^{(n)}(R-,P)$ as a function of P. Hence $F_{ss}(R-,P)$ is continuous in P. Assumption 1 and (9.1.18) now imply that there exists a P such that

$$(9.1.22) \quad F_{ss}(R-,P) = F_{ss}(R-) = 1.$$

Hence $R(P) = R$ and P is R-recurrent. Formula (9.1.21) now follows easily from theorem 6.2 in SENETA [52].

□

9.2. Sets of reducible nonnegative matrices.

In this final section the results of section 3.2 are extended to countable reducible nonnegative matrices. Furthermore, we discuss extensions of the generalized eigenvector theory for sets of nonnegative matrices (cf. section 5.2).

Let K be a set of countable nonnegative matrices with the product property and let R be its convergence parameter (cf. (9.1.3)). Assumptions 1,2 and 3 of section 9.1 are supposed to hold, together with

Assumption 4. There exists a finite set $D \subset S$ and a positive vector c such that

$$0 < F_{iD}(R-,P) < \infty \qquad \text{for all } i \in S,\ P \in K.$$

The reader may verify that under assumption 4 each P has at most $|D|$ R-recurrent classes (where $|D|$ denotes the cardinality of D). Furthermore, for each P a set of reference states $B(P)$ (cf. definition 8.5) can be chosen such that $B(P) \subset D$.

The next theorem deals with the existence of strictly positive R-(sub)invariant vectors. We have:

THEOREM 9.8. Let K have convergence parameter R and let $F_{ij}(R-) < \infty$ for all $i,j \in D$. Then there exist a positive integer n and states $s_1, s_2, \dots, s_n$ in D such that the vector u, defined by

$$(9.2.1) \qquad u_i := \sum_{\ell=1}^{n} \{\delta_{is_\ell} + (1-\delta_{is_\ell})F_{is_\ell}(R-) \qquad i \in S$$

is strictly positive and R-subinvariant for the set K. Furthermore, u is R-invariant if and only if each state s_ℓ $(\ell = 1,\dots,n)$ is R-recurrent.

Let $x > \underline{0}$ be β-subinvariant for the set K, then $\beta \leq R$ and

$$(9.2.2) \qquad x_i \geq \sum_{\ell=1}^{n} \{\delta_{is_\ell} + (1-\delta_{is_\ell})F_{is_\ell}(\beta)\}x_{s_\ell} \qquad i \in S.$$

PROOF. Clearly, the incidence matrix of K has a finite number of final classes, $C(1), C(2), \ldots, C(n)$ say. Choose $s_\ell \in C(\ell)$ for $\ell = 1, 2, \ldots, n$, then assumption 4, (9.1.18) and the condition in the theorem imply

$$F_{is_\ell}(R-) < \infty \qquad i \in S,\ \ell = 1, \ldots, n. \tag{9.2.3}$$

Note that $C(\ell)$ is a communicating class with respect to K. The results can now be proved in the same way as theorem 9.6. □

It is clear that in the situation, described in theorem 9.8, all eventually existing R-recurrent classes of a matrix P must be final.

Next, we turn to the more general case where nonfinal R-recurrent classes may exist. If K has convergence parameter R, then obviously $R \leqq R(P)$ for all $P \in K$. Let $\nu(P)$ denote the index of P (cf. definition 8.4) and let

$$\nu := \sup \{\nu(P) \mid P \in K,\ R(P) = R\}. \tag{9.2.4}$$

Obviously, assumption 4 implies that $\nu \leqq |D|$. Note that $\nu \leqq 1$ if the conditions of theorem 9.8 hold.

Before establishing the analogue of theorem 3.6, i.e. the block-triangular decomposition result, we first state formally

Assumption 5. Let $B(P)$ denote a set of reference states of P, chosen such that $B(P) \subset D$ (for all $P \in K$). For all $i \in S$ and $s \in B(P)$, let

$$\sup_{P \in K} {}_{B(P)}F_{is}(R-, P) < \infty. \tag{9.2.5}$$

The following result holds:

THEOREM 9.9. Let K have convergence parameter R and let ν be defined by (9.2.4). Suppose $\nu > 0$. Then there exist a partition $\{D(\nu), D(\nu-1), \ldots, D(0)\}$ of the state space S, and strictly positive vectors $\hat{u}^{(\nu)}, \hat{u}^{(\nu-1)}, \ldots, \hat{u}^{(1)}$, such that for some $\hat{P} \in K$

$$R \sup_{P \in K} P^{(k,k)} \hat{u}^{(k)} = R \hat{P}^{(k,k)} \hat{u}^{(k)} = \hat{u}^{(k)} \qquad k = 1, \ldots, \nu. \tag{9.2.6}$$

(As usual, $P^{(k,\ell)}$ denotes the restriction of P to $D(k) \times D(\ell)$, for all $P \in K$;

$k, \ell = 0, \ldots, \nu$).

Furthermore

$$P^{(k,\ell)} = \underline{\underline{0}} \qquad \text{for all } P \in K,\ k < \ell,$$

and $\hat{P}$ can be chosen such that each state in $D(k)$ has access to some state in $D(k-1)$ under $\hat{P}$ ($k = \nu, \nu-1, \ldots, 2$).

PROOF. Define for each $i \in S$

$$\nu_i := \sup \{\nu_i(P) \mid P \in K,\ R(P) = R\},$$

where $\nu_i(P)$ denotes the depth of i under P (cf. definition 8.4). Let

$$D(k) := \{i \in S \mid \nu_i = k\} \qquad k = 0, 1, \ldots, \nu.$$

Obviously, $D(\nu), D(\nu-1), \ldots, D(1), D(0)$ partitions S uniquely, while furthermore $P^{(k,\ell)} = \underline{\underline{0}}$ for $k < \ell$; $k, \ell = 0, 1, \ldots, \nu$.

For each k ($k = 1, 2, \ldots,$), the set of matrices $\{P^{(k,k)} \mid P \in K\}$ satisfies the assumptions of Theorem 9.8, hence (9,2,6) follows immediately. The definition of ν implies that $\hat{P}$ can be chosen such that each state in $D(k)$ has access to some state in $D(k-1)$ under $\hat{P}$ ($k = \nu, \nu-1, \ldots, 2$). This completes the proof. □

Also theorem 5.2 can be extended to the infinite dimensional case. Define for all $i, j \in S$, $H \subset S$ and $P \in K$:

$$_{H}m_{ij}^{(k)}(z,P) := \sum_{n=k}^{\infty} \binom{n}{k} {}_{H}f_{ij}^{(n+1-k)}(P)\, z^{n+1-k} \qquad k \in \mathbb{N}.$$

Next we make

Assumption 6. Let $B(P)$ be defined as in assumption 5, for each $P \in K$. For all $i \in S$ and $s \in B(P)$, let

$$\text{(9.2.7)} \qquad \sup_{P \in K} {}_{B(P)}m_{is}^{(k)}(R-,P) < \infty \qquad k = 1, \ldots, \nu,$$

where ν is defined by (9.2.4).

Then the following result can be established:

THEOREM 9.10. Let K have convergence parameter R and let ν be defined by (9.2.4). Suppose $\nu > 0$ and let $\{D(\nu), D(\nu-1), \dots, D(1), D(0)\}$ be defined as in theorem 9.9. Then there exist semi-positive vectors $w(1), w(2), \dots, w(\nu)$ such that

$$R \sup_{P \in K} Pw(1) = w(1), \tag{9.2.8}$$

$$R \sup_{P \in K_{\ell-1}} Pw(\ell) = w(\ell) + w(\ell-1) \qquad \ell = 2, \dots, \nu, \tag{9.2.9}$$

where $K_1 := \{P \mid P \in K,\ RPw(1) = w(1)\}$ and $K_\ell := \{P \mid P \in K_{\ell-1},\ RPw(\ell) = w(\ell) + w(\ell-1)\}$, $\ell = 2, \dots, \nu$. For $\ell = 1, 2, \dots, \nu$ we have furthermore

$$w(\ell)_i > 0 \qquad \text{for } i \in D(\ell),$$

$$w(\ell)_i = 0 \qquad \text{for } i \in \bigcup_{k=0}^{\ell-1} D(k).$$

□

The proof if theorem 9.10 is rather technical and will not be given here. The following elements are essential. For each $P \in K$, with $R(P) = R$ and $\nu(P) = \nu$, theorem 8.8 can be applied in order to establish the existence of generalized R-invariant vectors $w(1,P), w(2,P), \dots, w(\nu,P)$. When normalized appropriately, these vectors are continuous as functions of P by assumption 6 (compare also the proof of theorem 9.7). Exploiting this, theorem 9.10 can be established by methods similar to those used in ZIJM [84], with respect to the proof of the existence of solutions to the average cost optimality equations in a denumerable state Markov Decision Chain.

A few concluding remarks are in place. With respect to the assumptions 5 and 6 the reader may notice that these conditions are in fact natural extensions of similar conditions in the preceding chapter. All remarks made in section 8.3 can also be made with respect to these conditions.

Strictly positive β-invariant vectors are treated extensively in a paper by VAN HEE AND WESSELS [70], where they give a lower bound for β^{-1} which is, unfortunately, normdependent. The question whether ρ, defined by

$$\rho := \sup_{P \in K} \sup_{i,j \in S} \limsup_{n \to \infty} \{p_{ij}^{(n)}\}^{1/n},$$

is a lower bound for β^{-1}, was already answered in the negative by Van Hee and Wessels. It is easily shown that the right answer has to be R^{-1}, where R is defined by (9.1.3) (compare also theorem 9.8).

Finally, we note that applications of strictly positive β-subinvariant vectors can be found in WESSELS [71].

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