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ON POINT PROCESSES

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P.C.T. van der Hoeven

CONTENTS

CHAPTER 1

INTRODUCTION AND SUMMARY

§1.1 Survey of the contents.

In this monograph we shall study so-called point processes; i.e. we study probability mechanisms according to which a locally finite subset of a locally compact space U with countable base can be chosen; examples of such spaces U are $\mathbb{R}^d$, (0, 1), $\mathbb{R}_+$, $\mathbb{Z}^d$, etc.; a locally finite subset of U is an unordered finite or countable set of points without condensation point. We are concerned with the probability distribution of this locally finite subset - of this finite or countable set of points. The locally finite subset chosen at random is called a simple[1)] point process.

The modern theory of point processes begins with C. Palm's paper (43). After him the name Palm distribution was given to the conditional distribution of a point process given the event that the point process contains a given point in U. Because generally this is a null-event, definition problems arise (See e.g. Kallenberg (83) - chapter 10 or Neveu (77)-II-2).

We are mainly interested in a notion which is in a way the opposite of the Palm distribution: Consider an infinitesimally small subset in U ; now we ask for the expected number of points of the point process contained in this infinitesimally small subset conditionally given the locations of all points of the point process outside this infinitesimally small subset. Because we generally have to deal with an uncountable number of null-sets, a definition problem again arises.

In chapter 4 earlier solutions to this problem are sketched. Papangelou (74) first solved this problem by a limit procedure. The object that he defined in this manner, was called by him the conditional intensity of the point process. Later Kallenberg (78) gave another proof of the existence of the conditional intensity under less restrictive conditions. Papangelou already noticed that his result gave a sort of analogue to the decomposition of Doob and Meyer in the theory of processes on $\mathbb{R}_+$.

1) In this section the word "simple" will be omitted.

The Doob-Meyer decomposition is an important result in martingale theory, in which Doob and Meyer are indeed predominant names. In the "Strassburg school" (See e.g. Dellacherie and Meyer (75) and (80)) in martingale theory an important feature is the fact that one defines σ-fields on the product of the probability space and $\mathbb{R}_+$ other than product-σ-fields on this product space, and that stochastic processes on $\mathbb{R}_+$ are regarded as measurable functions on this product space. Important to the Doob-Meyer decomposition is the notion of previsibility which is related to the previsible σ-field on $\Omega \times \mathbb{R}_+$ (chapter 3). Previsibility enables us to speak of processes whose values at any time only depend on the strict past, so that they are predictable. Here it is essential to note that this is a property which applies simultaneously at an uncountable number of times.

The fact, that previsibility helps to solve a problem analogous to the definition problem of the conditional intensity, becomes clear if we note that previsible processes at any time only depend on what happened strictly before that time, whereas the conditional intensity of a point process in an infinitesimally small set depends only on the location of the points of the point process outside this infinitesimally small set. Hence the role of the past of a moment in time corresponds in a way to the events which are determined by the part of a point process outside a given subset of U.

Our aim is to explore the analogy that we just mentioned and to use it to give a more natural definition of the conditional intensity. Because this leads to the notion of visibility - which is related to the visible σ-field on $\Omega \times U$ - we shall use the word (dual) visible projection of the point process rather than conditional intensity. The theory of visibility is developed in chapter 5 analogous to the theory of previsibility. It should be noted that visibility is defined w.r.t. the point process. A great number of the results of this chapter - visible section theorem, visible projection of processes and dual visible projection of the point process - was already present in Van der Hoeven (82).

In our context several smoothness conditions are naturally introduced. The most important of them stems from Papangelou (74). These smoothness conditions are formulated and studied in chapter 6 in a way which dovetails into the theory of visibility.

In chapter 7 expressions for visible projections are derived and in particular a limit formula is proved by which the dual visible projection indeed is identified with Papangelou's conditional intensity (cf. Van der Hoeven (82)).

The notion of martingalelike measure is defined in chapter 8. The name already indicates that this is an analogue of a martingale on $\mathbb{R}_+$. In this context we also study Papangelou kernels. The great importance of these kernels became apparent in Matthes, Warmuth and Mecke (79). They showed that the Papangelou kernel of a reasonably smooth point process determines important properties concerning the distribution of the point process. We shall discuss their results in chapter 9.

§1.2. Some generalizations and some techniques used.

The most naîve way to describe simple point processes on some space U is by defining a probability measure on the space of all locally finite subsets of U. It turns out to be more practical to associate with each locally finite subset of U the counting measure of that subset, so that we obtain a probability mechanism by which a measure on U is chosen, and hence the point process is an example of what is logically called a random measure. (It will turn out that the dual visible projection is an example of a random measure too).

The theory developed here can easily be extended to the more general compound point processes (cf. Van der Hoeven (82)). We use another form by which we are able to describe compound point processes and other generalizations. Indeed, we introduce the notion of a (simple) marked point process. In the theory of point processes on $\mathbb{R}_+$ this notion is often used (cf. e.g. Brêmaud and Jacod (77) or Jacod (79); cf. also Varsei (78)).

Marked point processes are obtained as follows: Originally we chose, according to some probability mechanism, a locally finite subset of U (or the associated counting measure), but now, in addition, each point of this locally finite subset of U is provided in some stochastic manner with a mark, i.e. an element of some space (locally compact, with countable base) K. Hence the probability mechanism chooses a subset of the product space of U and K, whose projection on U is locally finite and which is such that every point in this projection on U corresponds to exactly one point in $U \times K$. Thus we obtain a point process on U with marks in K, which can and will be described by a random measure on $U \times K$: The counting measure of the randomly chosen subset of $U \times K$, that we just described.

The space of all such counting measures may of course be used as probability space, but it turns out to be useful sometimes to consider an abstract probability space on which the (marked) point process is defined as a fundamental mapping. However, this brings with it that the notation becomes more complicated and that assertions become intuitively less clear. As a result, there are statements which are more readily understood when one assumes that the probability space consists of the counting measures on $U \times K$ described above.

It is clear that on a abstract probability space more random measures can be defined. However, visibility remains to be defined w.r.t. our basic (marked) point process.

Nonetheless, the dual visible projection (w.r.t the basic marked point process) may be defined and this definition does not give rise to any complications, in comparison with the definition of the dual visible projection of the point process itself. On the other hand, proving the existence of the conditional intensity as a limit in this more general context requires a non-trivial extension of the proof of the existence of the conditional intensity of the point process itself. Kallenberg (83) proved this extension (see §7.4).

On the space U the Borel σ-field is defined. This σ-field contains an uncountable number of sets. Since uncountable numbers are unpleasant in probability, it turns out to be useful to choose a countable sequence of countable partitions of U, which become finer and finer. (For instance, in the case $U = \mathbb{R}$, each partition may consist of lefthand open, righthand closed intervals, which are each divided in two such intervals in order to obtain the next partition). These partitions can be chosen in such a way, that their union forms a base for the topology of U. This union contains a countable number of sets, so that, if there corresponds a null-set to each of these sets, the union of these null-sets is still a null-set. Note that given a fixed Borel subset B of U, we may assume that B is an element of one of the partitions mentioned above.

CHAPTER 2

NOTATIONS

§2.1. Marked point processes.

Let U and K be two locally compact topological spaces with a countable base. Such spaces are known to be *Polish*, i.e. a metric exists for which they are separable and complete (for a proof see for instance Bauer (78) - Satz 44.1). We note that the space U × K is again locally compact with a countable base. The Borel σ-fields on the spaces U and K are denoted by $\mathcal{B}$ and $\mathcal{K}$ respectively.

There exists (and we shall choose) a sequence of partitions $\mathcal{U}_1$, $\mathcal{U}_2$, ... of U, such that $\mathcal{U} = \bigcup_i \mathcal{U}_i \subset \mathcal{B}$, such that each $V \in \mathcal{U}_i$ is the union of a bounded number of elements of $\mathcal{U}_{i+1}$ and such that $\mathcal{U}_1$ contains at most a countable number of sets, which are all bounded. This implies that $\mathcal{U}$ contains at most a countable number of sets, which are all bounded. Furthermore, we shall suppose, that for every $u \in U$ and $G \in \mathcal{B}$ open with $u \in G$ there exists a set $V \in \mathcal{U}$ such that $u \in V \subset G$; hence $\mathcal{B} = \mathcal{T}(\mathcal{U})$, $\mathcal{T}(\mathcal{U})$ denoting the σ-field generated by $\mathcal{U}$. If $V \in \mathcal{U}$, then we write $\mathcal{U}_{i,V} = \{W \in \mathcal{U}_i \mid W \subset V\}$.

We now introduce some classes of measures: If E is an arbitrary Polish space, we denote by L(E) the collection of all positive Radon measures (i.e. positive locally finite measures) on $(E,\mathcal{E})$, where $\mathcal{E}$ is the Borel σ-field on E. In the vague topology elements ρ_n converge to ρ if $\int_E f\, d\rho_n \to \int_E f\, d\rho$ for all continuous functions f on E with compact carrier. Endowed with this topology the space L(E) becomes Polish. The corresponding Borel σ-field on L(E) is also generated by the sets of the form $\{ \rho \in L(E) \mid \rho(D) < \alpha\}$ with $\alpha \geq 0$ and D a Borel subset of E (For these matters, see Kallenberg (83)).

The space $M^{\cdot}$ consists of all $\rho \in L(U)$ such that $\rho(V) \in \{0, 1, 2, \ldots, \infty\}$ for all $V \in \mathcal{B}$ and $\rho(\{u\}) \in \{0,1\}$ for all $u \in U$. Elements of $M^{\cdot}$ are called *simple point measures* on U - for the sake of completeness we note that elements of L(U) satisfying the first of these two conditions but not necessarily the second, are called point measures; the number $\rho(\{u\})$ is the multiplicity of ρ in u . The space M consists of all $\rho \in L(U \times K)$ such that $\rho(A) \in \{0, 1, 2, \ldots, \infty\}$ for all $A \in B \times K$, $\rho(\{u\} \times K) \in \{0,1\}$ for all $u \in U$ and such that $\rho(. \times K) \in L(U)$. Elements of M are called *point measures on* U *marked by elements of* K .

Choosing $K = \{1, 2, \dots\}$ or $K = (0,\infty)$ we are able to describe res-respectively non-simple and compound point measures on U by marked measures. If K reduces to one point, we see that M and $M^{\cdot}$ are isomorphic Where no confusion can arise we omit the words "simple" or "marked".

It can easily be checked that M and $M^{\cdot}$ are measurable subsets of $L(U \times K)$ and $L(U)$ respectively. The σ-field M is the trace on M of the Borel σ-field on $L(U \times K)$; writing for all $V \in B$

$$M(V) = T(\{\rho \in M \mid \rho(B \times D) \leq \alpha\} \mid \alpha \geq 0,\ B \in B,\ B \cap V = \emptyset,\ D \in K)$$

we see that $M = M(\emptyset)$.

If $B \in B$ and $\rho \in L(U \times K)$ (resp. $\rho \in L(U)$) then the measure $B\rho$ on $U \times K$ (resp. on U) will be defined by $B\rho(.) = \rho((B \times K) \cap .)$ (resp. by $B\rho(.) = \rho(B \cap .)$).

Let (Ω, A, P) be a complete probability space (hence A contains all subsets of P-null sets in A ; as no confusion can arise, we drop the letter "P" in these instances). The class of all null sets in A is denoted by N .

The main object of study in this monograph will be a measurable mapping $\mu : \Omega \to M$. This fundamental mapping is fixed in the sequel. It is called a *random (marked) point measure or a (marked) point process*[1]. We also define once and for all the (simple non-marked) random measure ξ , i.e. the measurable mapping $\xi : \Omega \to M^{\cdot}$, by $\xi(.) = \mu(. \times K)$. If K reduces to one point, then essentially $\xi = \mu$.

We define the *exterior σ-field* of an element $V \in B$ by

$$F(V) = T(\mu^{-1}(M(V)), N) ;$$

and put: $F = F(\emptyset)$. It is known (cf. e.g. Meyer and Dellacherie (75) - I - 18 and II - 31) that if $F \in F$ (resp. $F \in F(V)$, then there

[1] One might argue that the word "point process" should be reserved for the distribution of the random point measure μ . We, however, do not make this distinction.

exists a set $F^* \in M$ (resp. $F^* \in M$ (V) such that

$$P (F \Delta \{ \omega \mid \mu \in F^* \}) = 0;$$

this transition from a set in F to the corresponding one in M will in future always be indicated by an " * ".

§2.2. Some examples.

We shall give some fundamental examples of point processes. The first one - the "zero-or-one-point process" - seems very trivial; still it is important to the theory as will be seen later on. Both in theory and practice the Poisson process is encountered in many situations and it is used in the definition of many other processes. Indeed we shall define here two more processes - the Gibbs and the Cox process - using the Poisson process. In chapter 10 we shall study these and other examples in more detail.

EXAMPLE 2.2.1. The *zero-or-one-point process*. This non-marked simple point process[1)] is based on a constant $c \in [0,1]$ and a distribution ν on the state space U : with probability $1 - c$ there is no point in U at all ($\xi = \underline{0}$, $\underline{0}$ denoting the zero measure : $\underline{0}(V) = 0 \; \forall \; V \in B$) and with probability c there is exactly one point in U distributed according to ν ; i.e.:

$$\xi = Y \, \varepsilon_X,$$

where X and Y are independent r.v.'s; X is a U-valued r.v. with distribution ν and $P (Y = 1) = 1 - P (Y = 0) = c$ We denote by ε_a the unit mass in a point a : $\varepsilon_a (D) = 1_D (a)$. □

EXAMPLE 2.2.2. The *Poisson process:* let ν be a positive Radon measure on U ($\nu \in L(U)$). If ν is diffuse (atomless), that is, if $\nu(\{u\}) = 0$ for all $u \in U$, then the Poisson process[2)] with intensity ν ,

1) The expression "non marked" indicates that K contains only one point

2) The word "Poisson process" refers to a probability measure on M. The mapping $\mu : (\Omega, A, P) \to M$ such that the distribution of μ in a Poisson process, is also called the Poisson process.

denoted by Π_ν is (cf. Neveu (77) Proposition I - 6 and exercice I - 4) the unique non-marked simple point process such that:

a) for all $V \in B$ the r.v. $\xi(V)$ follows a Poisson distribution with parameter $\nu(V)$ ($\xi(V) = \infty$ a.s. if $\nu(V) = \infty$), and

b) for all $n > 1$, $V_1, \dots, V_n \in B$ disjoint, the r.v.'s $\xi(V_1), \dots, \xi(V_n)$ are independent.

Now let us suppose that ν is not diffuse; hence there exists a $u \in U$ with $\nu(\{u\}) = m > 0$. Then it is seen that the above definition becomes meaningless. Indeed, from the condition a) above it would follow that the Poisson process would have to satisfy $P(\xi(\{u\}) > 1) = 1 - e^{-m}.(1 + m) > 0$, so that there cannot possibly be a solution of a) and b) within the space of probability measures on $M^{\cdot}$.

Thus the Poisson process is now essentially non-simple, since with positive probability multiplicities bigger than one occur. To cope with this situation we have to choose $K = \{1, 2, \dots\}$ and replace the above conditions by:

a') for all $V \in B$ the r.v. $\int_{V \times K} k\mu(dv, dk)$ follows a Poisson distribution with parameter $\nu(V)$ ($\mu(V \times K) = \infty$ a.s. if $\nu(V) = \infty$), and

b') for all $n > 1$, $V_1, \dots, V_n \in B$ dispoint the r.v.'s: $\int_{V_1 \times K} k\mu(dv, dk), \dots, \int_{V_n \times K} k\mu(dv, dk)$ are independent. (Hence the multiplicities are used as marks). For each $\nu \in L(U)$ the Poisson process Π_ν is now uniquely determined; the measure ν is called the intensity of the process.

For some interesting aspects of this process, see the next example, §4.1, §10.3 and §10.6. □

<u>EXAMPLE 2.2.3.</u> The *Gibbs process* (cf. Preston (76)). The theory of Gibbs processes stems from statistical mechanics. There a Poisson process with diffuse intensity is called an ideal gas in the grand canonical ensemble. In this interpretation points correspond to particles. It is easy to imagine how the influence of an external field can be described by the intensity measure ν, but by definiton the particles are not interacting. An attempt to deal with interaction leads to the Gibbs process:

Let us consider a "box" $V \in U$ and suppose that, given the configuration of the particles outside of V, the particles inside of V are behaving like an ideal gas, i.e. a Poisson process with an intensity measure

$\nu(\mathcal{F}(V))$, depending on the locations of the particles outside of B. This dependence expresses the fact that the particles within V "feel" a field caused by the particles which are not in V. Thus we obtain for all V the conditional distribution $P(V\mu \in . \mid \mathcal{F}(V)) = \Pi_{\nu(\mathcal{F}(V))}(.)$. The system $(\Pi_{\nu(\mathcal{F}(V))})_V$ is called a specification. Of course such a system has to satisfy certain consistency conditions, but even then it is not clear whether a probability measure P exists such that the specification is indeed a system of conditional distributions with respect to this overall law P. However, conditions on the specification can be stated which ensure the existence of a Gibbs process, i.e. the above mentioned law P. Sometimes we can even find more than one distribution P fitting in with the same specification; this phenomenon represents phase-transitions. □

EXAMPLE 2.2.4. The *Cox process*. We start with a probability measure Γ on $L(U)$. According to this law we choose a random $\gamma \in L(U)$ and then construct the corresponding Poisson process Π_γ. The distribution of the element $\mu \in M$ that we obtain by this doubly stochastic procedure is called the doubly stochastic Poisson process or Cox process based on Γ [1]; it will be denoted by Π_Γ. Hence:

$$\Pi_\Gamma(.) = \int_{L(U)} \Pi_\gamma(.)\, \Gamma(d\gamma) . \qquad \square$$

§2.3. Random processes etc.

If $A \in \mathcal{A} \times \mathcal{B}$, then we call A a *random set*. The *projection* π on Ω of a random set A is defined by

$$\pi(A) = \{\omega \in \Omega \mid \exists\, u \in U : (\omega, u) \in A\} .$$

It follows from theorem A. 1 [2] that $\pi(A) \in \mathcal{A}$. A random set A will be called *evanescent* if $P(\pi(A)) = 0$.

Let Δ be a point outside U. An $\mathcal{F}$-measurable mapping $R : \Omega \to U \cup \{\Delta\}$ will be called a *random point* (Thus a random point is a $(U \cup \{\Delta\})$-valued

1) cf. the note to example 2.2.2.
2) this refers to appendix A.

F-measurable r.v.). In §2.1. we introduced the operation "*" for sets in F; now it is seen that every random point R can be identified up to an equivalence with a function R^* on M. The random set

$$\lceil R \rfloor = \{(\omega, u) \mid u = R(\omega) \in U\}$$

will be called the *graph* of R . (This definition makes sense for every $(U \cup \{\Delta\})$-valued r.v. R). If A is a random set, then the random point R is called a *section* of A, if $\lceil R \rfloor \subset A$.

A real-valued[1)] (stochastic) process is an $A \times B$-measurable mapping $X : \Omega \times U \to \mathbb{R}$ $(X : (\omega, u) \to X_u(\omega))$. We set $X_\Delta(\omega) = 0$ for all $\omega \in \Omega$. If X is a process and R a stochastic point, then the r.v. X_R is defined by $X_R(\omega) = X_{R(\omega)}(\omega)$.

An A-measurable mapping $\rho : \Omega \to L(E)$ $(\rho : \omega \to \rho_\omega)$ is called a random measure on E . If ρ is a random measure on U (resp. on $U \times K$), then we set $\rho_\omega(\{\Delta\}) = 0$ (resp. $\rho_\omega(\{\Delta\} \times K) = 0$) for all ω. If ρ is a random measure on $U \times K$, then we define the $L(K)$-valued random process $\hat{\rho}$ by:

$$\hat{\rho}_u(.)(\omega) = \rho_\omega(\{u\} \times .) ;$$

Thus for all $D \in K$ we see that $\hat{\rho}_u(D)(.) = \hat{\rho}_u(D)$ is a real-valued process. If ρ is a random measure on U , we define the process $\hat{\rho}$ by:

$$\hat{\rho}_u(\omega) = \rho_\omega(\{u\}) .$$

This is the process of atomsizes of ρ. Note that for all random measures ρ on $U \times K$, random points R and $D \in K$: $\hat{\rho}_R(D)(.) = \hat{\rho}_{R(.)}(D)(.) = \rho_{\cdot}(\{R(.)\} \times D)$ is a r.v.. Furthermore, if ρ is a random measure in K (resp. on $U \times K$) and R is a random point, then we denote by the r.v. $\rho_{\cdot}(\{R(.)\})$ (resp. $\rho_{\cdot}(\{R(.)\} \times K) = \rho_R(K)(.)$ by $\rho(R)$.

We shall not distinguish two processes X and X' such that $P(X_u = X'_u \ \forall u \in U) = 1$; if two processes X and X' are indistinguishable in this sense we write $X \doteq X'$. We also use this dot in similar cases

1) the definition of processes taking values in other measurable spaces is clear. We generally omit the word "....-valued".

for instance, if A and A' are random sets, then $A \doteq A'$ means $1_A \doteq 1_{A'}$; the meaning of $A \dot{\subset} A'$ now will be clear. If two random measures ρ and ρ' are such that $P(\rho(C) = \rho'(C)\ \forall\ C$ measurable $) = 1$, then we say $\rho = \rho'$ a.s.

We define the operation "*" for F-measurable random measures ξ in the obvious way. Furthermore, if X is an $F \times B$-measurable process, it follows from the monotone class theorem (B.2) that there exists an $M \times B$-measurable function X^* on $M \times U$ such that $X(.) \doteq X^*(\mu_.)$.

If (E,E) is some measurable space, if ρ , $\nu \in L(E)$ and if f is an E-measurable non-negative function, then the expressions: $\rho = f\nu$, $\rho(.) = f(.)\,\nu(.)$, $d\rho = f d\nu$, $\rho(de) = f(e)\,\nu(de)$, $f = \frac{d\rho}{d\nu}$ and $f(e) = \frac{\rho(de)}{\nu(de)}$ all mean that $\rho(D) = \int_D f(de)\,\nu(de)$ for all $D \in E$.

CHAPTER 3

PREVISIBILITY

As mentioned above the principal aim of this monograph is to develop a projection theory for point processes on a locally compact space with countable base, analogous to the theory which leads to the previsible projection of (point) processes on $\mathbb{R}_+$. This theory on $\mathbb{R}_+$ uses the natural order of the real numbers in an essential manner. We shall therefore refer to it by the expression "the theory of processes on $\mathbb{R}_+$" although of course theories about processes on more general spaces (as the visible one, chapter 4 sqq.) also apply to processes on $\mathbb{R}_+$.

It seems useful to gather some results of the theory of processes on $\mathbb{R}_+$ and to indicate the steps in this theory most important to our purpose. This chapter is devoted to such a sketch. No proofs are given and only little explanation. Almost everywhere we can refer to the standard work in this area, Dellacherie and Meyer, Probabilités et potentiel (75, 80). Many arguments can perhaps be guessed at from the corresponding ones in chapter 5 and 7, which is of course the wrong way round. On the other hand, appendix D, where techniques are used adopted from §5.6, enables us to state a.s. convergence in theorem 3.4.5 and hence yields a small contribution to the theory of processes on $\mathbb{R}_+$. The proof of theorem 3.4.4 is found in Neveu (77) or Jacod (79).

It should be remembered that this chapter only serves as a base for comparison in very special cases later on. This explains the sometimes rather bizarre choice of subjects and definitions. For instance, since neither martingales nor localization will be defined, the usual definition of a local martingale - a localized martingale - would make little sense.

Things will be made more clear by the example of the Poisson process, which will be elaborated in the course of this chapter. Each time we refer to it, we shall adopt automatically all notation introduced earlier in relation to it.

§3.1. Filtration: the previsible σ-field.

On our basic complete probability space (Ω, A, P) we are given an increasing *filtration* indexed by time; that is: a family of σ-fields

$(F_t)_{t \in \mathbb{R}_+}$ such that $F_t \subset F_s \subset A$ if $t \leq s$. We shall suppose that the filtration satisfies the "usual conditions"; this means that each $F_t = \bigcap_{s>t} F_s$ and contains all null-sets of A .

For instance, a filtration (F_t) is obtained when each F_t contains all the information one can observe from a system up to time t inclusive; this system could be a point process in time. When speaking of point processes on $\mathbb{R}_+$ - hence $U = \mathbb{R}_+$; for simplicity we restrict ourselves to the simple non-marked case - points may be called jumps of the process $\xi\,[0,t]$. Furthermore, the expressions "first jump" (first in time) etc. make sense. In this context one usually takes $\Delta = \infty$. Very often we use the filtration generated by ξ ; i.e. we choose $F_t = F((t,\infty))$; thus F_t contains all information on ξ until time t inclusive.

<u>EXAMPLE</u>. Let ξ be a Poisson process on $(0,\infty)$ with the Lebesque measure λ as its intensity measure. This Poisson process can also be defined as the jump process with jumps in $T_1, T_2, \ldots$, where $T_1, T_2 - T_1, T_3 - T_2, \ldots$ are independent r.v.'s all having an exponential distribution with parameter 1 . Take (F_t) to be the filtration generated by ξ . □

A process X is called *adapted* (to (F_t)) if X_t is F_t-measurable for all t. Many processes have the property that for almost all ω the function: $t \to X_t(\omega)$ is right continuous with left-hand limits, abbreviated *cadlag* (continu â droite, limitê â gauche). Combining this regularity property (cadlag) with adaptedness we obtain the notion of optionality: A process is *optional* if it is the limit of a sequence of adapted cadlag processes.

<u>EXAMPLE</u>. In the above example both the processes $X_t = \xi(0,t]$ and $Y_t = \xi(0,t)$ are optional. Indeed, X_t is clearly adapted and cadlag itself and Y_t is a limit of such processes ($Y_t = \lim_n \xi(0,t-\frac{1}{n}]$). □

A subclass of the class of optional processes is formed by the processes X which are adapted and which are such that for almost all ω the function: $t \to X_t(\omega)$ is left continuous on $(0,\infty)$ (the process is cag). Such processes and their limits are called *previsible* or predictable.

EXAMPLE. Of the two processes X and Y introduced above only the second one is previsible. □

There is another equivalent way to introduce previsibility: First we define the filtration $(F_{t-})_{t\in\mathbb{R}_+}$ by $F_{0-} = F_0$ and $F_{t-} = T(\bigcup_{s<t} F_s)$ $(t > 0)$

(N.B. often we have $F_{t-} = F_t$. EXAMPLE. In our central example this is the case since $F_t = T(F_{t-}, \{\hat{\xi}_t = 1\})$ and $\{\hat{\xi}_t = 1\}$ is a null-event. □) Now the previsible σ-field P on $\Omega \times \mathbb{R}_+$ is generated by the sets of the form $F \times [t,\infty)$ where $t \in \mathbb{R}_+$ and $F \in F_{t-}$. A process is called previsible if it is P-measurable considered as a function on $\Omega \times \mathbb{R}_+$.

An $\overline{\mathbb{R}}_+$-valued r.v. T is called a *stopping time or an optional time* if the process: $(\omega,t) \to 1_{\{T > t\}}(\omega)$ is optional. Again previsibility is a stronger property: T is called a *previsible (stopping) time* if the process $(\omega,t) \to 1_{\{T > t\}}(\omega)$ is previsible, or, equivalently, if the graph $\lceil T \rfloor$ is a previsible set. Stopping times - which we have defined in a rather unorthodox manner - will be less important to our purposes than previsible times. Note that determistic times are always previsible.

EXAMPLE. The time T_1 of the first jump of the Poisson process is optional but not previsible; $T_1 + 1$ is a previsible stopping time. □

§3.2. The previsible section theorem.

In the theory of processes in $\mathbb{R}_+$ the section theorem is a technical but important tool. We now state it.

THEOREM 3.2.1. *Let* A *be a previsible subset of* $\Omega \times \mathbb{R}_+$ *and* $\varepsilon > 0$. *Then there exists a previsible time* T *such that:*

a. $\lceil T \rfloor \subset A$, *and*

b. $P(\pi\lceil T \rfloor) > P(\pi(A)) - \varepsilon$.

(Note that by definition $\lceil T \rfloor \cap (\Omega \times \{\infty\}) = \emptyset$).

The proof of this theorem uses the ordinary section theorem (i.e. without filtration), theorem A.1.

EXAMPLE. We are again using the example of §3.1. The set $A = \{(\omega,t) \mid 0 < t \leq T_1(\omega)\}$ is previsible, its indicator being an adapted left continuous process. Since $T_1 > 0$ a.s., we see that $P(\pi(A)) = 1$. Let t_0 be a strictly positive real number and let T be the random time defined by:

$$T(\omega) = \begin{cases} t_0 & \text{if } T_1(\omega) \geqq t_0 , \\ \infty & \text{if } T_1(\omega) < t_0 . \end{cases}$$

Then T is a previsible stopping time; indeed:

$$\lceil T \rfloor = \{\omega | T_1(\omega) \geqq t_0\} \times \{t_0\} = A \cap (\Omega \times \{t_0\}) \in P$$

Clearly:

$$P(\pi\lceil T \rfloor) = P(T_1 \geqq t_0) = e^{-t_0} ,$$

which can be made arbitrary close to one; note however that it always remains strictly smaller than one. □

§3.3. The previsible projection

The section theorem of §3.2 implies among other things that a previsible process X is uniquely determined by integrals of the form EX_T where T is a previsible stopping time, as it is known in general, that a G-measurable r.v. X is determined by integrals of the form $EX1_G$, $G \in G$ ($G \subset A$ is some σ-field on Ω). This fact is used in the proof of the following theorem-definiton, which in a way, is comparable to the definition of conditional expectations.

<u>THEOREM 3.3.1.</u> *Let* X *be a non-negative or bounded process; then there exists a previsible process* ${}^pX$ *such that:*

$$E\,{}^pX_T = EX_T$$

for all previsible stopping times T *; the process* ${}^pX$ *is uniquely determined*[1] *and is called the previsible projection of* X .

N.B. Recall the unusual convention that we adopted, that $X_\infty(\omega) = 0$ ($\omega \in \Omega$) for all processes X .

For each (previsible) stopping time T define the σ-field F_{T-} of

1) By "uniquely determined" we mean up to indistinguishability (§2.3), i.e. outside an evanescent set.

events strictly anterior to T , by $F_{T-} = f_T^{-1}(P)$ where f_T is the mapping: $\omega \to (\omega, T(\omega))$. (Note that if $T = t$ a.s., then $F_{T-} = F_{t-}$). Then ${}^pX$ satisfies

$$ {}^pX_T = E(X_T | F_{T-}) $$

for all previsible stopping times T, and of course ${}^pX$ is the unique previsible process satisfying this condition. Here again the previsibility of ${}^pX$ is a necessary supplementary conditon. Indeed, in many cases X_T is F_{T-} -measurable itself for each previsible time T although the process X is not previsible. (EXAMPLE. This is the case for instance for the process X from the central example of this chapter. □)

The transition from the uncountable number of σ-fields F_{T-} , each with its own exceptional null-set, to one previsible σ-field on $\Omega \times U$ with one evanescent exceptional set is an example of the ingenuity of the previsible theory explaining its strength.

EXAMPLE. The process Y defined in the example in 3.1 is previsible itself; hence it is its own previsible projection. It also is the previsible projection of X ; this is proved by noting that Y is previsible and by showing that for all previsible times T we have $\hat{\xi}_T = 0$ a.s. and thus $X_T = Y_T$ a.s. Hence, in this case taking the previsible projection narrows down to making the process left continuous. On the other hand ${}^p 1_{\lceil T_1+1 \rfloor} = 1_{\lceil T_1+1 \rfloor}$ for, although the process $1_{\lceil T_1 + 1 \rfloor}$ is not left continuous, it still is previsible since $T_1 + 1$ is a previsible time. □

§3.4. The dual previsible projection of random measures.

A random measure ρ on $\mathbb{R}_+$ is called previsible if the process $\rho[0,t]$ is previsible. Note that this process is not left continuous in atoms of ρ ; it is cadlag and we have:

THEOREM 3.4.1. *A cadlag process* X *is previsible if and only if:*

1) *for all previsible stopping times* T *the r.v.* X_T *is* F_{T-}*-measurable, and*
2) $X_T = X_{T-0}$ *a.s. for all stopping times* T *such that* $P(S = T < \infty) = 0$ *for all previsible times* S .[1)]

1) Such stopping times T are called totally inaccessible.

Another characterization of previsible measures is given by the following statement:

THEOREM 3.4.2. *A random measure* ρ *on* $\mathbb{R}_+$ *is previsible if and only if*

$$E \int X d\rho = E \int {}^P X d\rho$$

for all non-negative processes X .

This theorem suggests the definition of an operation dual to the previsible projection of processes (§3.3). Indeed, we may introduce the *dual previsible projection* (of measures) as follows:

THEOREM 3.4.3. *Let* ρ *be a random measure. Then there exists a unique*[1] *previsible random measure, denoted by* ρ^P *and called the dual previsible projection of* ρ *, such that*

(3.4.1) $$E \int X d\rho^P = E \int X d\rho$$

for all non-negative previsible processes X .

Combining formula (3.4.1.) with theorem 3.4.2. we find that ρ^P is the unique random measure satisfying:

(3.4.2) $$E \int {}^P X d\rho = E \int X d\rho^P$$

for all non-negative processes X .

In our literature the process $\rho^P[0,t]$ is called the dual previsible projection of the process $\rho[0,t]$, which is not to be confused with ${}^P\rho[0,t]$. Since we are only speaking of dual projections of random measures, we shall drop the word "dual" sometimes.

Because the previsible σ-field is generated by sets of the form $F \times [s,t]$, $s \leq t$, $F \in \mathcal{F}_{s-}$, the dual previsible projection ρ^P is already determined among the previsible random measures by the requirement:

(3.4.3) $$E(\rho^P[s,t] \mid \mathcal{F}_{s-}) = E(\rho[s,t] \mid \mathcal{F}_{s-}) ,$$

or equivalently by:

(3.4.4) $$E((\rho - \rho^P)[s,t] \mid \mathcal{F}_{s-}) = 0$$

This means that the process $(\rho - \rho^P)[0,t]$ is a *local martingale* and this fact is also expressed by saying that $\rho^P[0,t]$ compensates $\rho[0,t]$. Now,

1) Here "unique means unique outside a null-set.

$\rho^P[0,t]$ is the unique previsible process such that $\rho[0,t] - \rho^P[0,t]$ is a local martingale. Therefore $\rho^P[0,t]$ is sometimes called the *previsible compensator* of $\rho[0,t]$.

EXAMPLE. When speaking of point processes an important random measure to project is of course the point process ξ itself. We turn to our central example of this chapter and see that in the Poisson case the previsible projection of ξ is its intensity measure, i.e. $\xi^P = \lambda$ (Because the deterministic measure λ is clearly previsible, this follows from formule (3.4.3) and the independence property of the Poisson process (property b) of example 2.2.2.)). In other terms: the previsible compensator of $\xi[0,t]$ is t . The fact that for the Poisson process $\xi[0,t] - t$ is a local martingale was already known before the theory of previsibility was developed. As a matter of fact, in the context of point processes the - in general random - dual previsible projection is considered to be a generalization of the intensity of the Poisson process (cf. e.g. Brémaud and Jacod (77)). □

For point processes we have two explicit expressions for the previsible projection of ξ :

THEOREM 3.4.4. *Let* ξ *be a simple point process and* (F_t) *the filtration generated by* ξ . *Then:*

$$\xi^P(dt) = \sum_{n=0}^{\infty} \frac{G_n(dt)}{G_n([t,\infty])} 1_{(T_n,\ T_{n+1}] \cap (0,\infty)}(t) \quad a.s. \quad where$$

$G_n(.) = P(T_{n+1} \in . \mid T_1, \dots , T_n)$. *Here* $T_1, T_2, \dots$ *denote the moment of the first second, ... jump.* $T_0 = 0$. *We assume* $T_1 > 0$ *a.s.*

Note that the set on which the countable number of conditional distributions G_n is not defined, is again a null-set. The measure $G_n(dt)/G_n([t,\infty])$ is of course the hazard rate of T_{n+1} .

THEOREM 3.4.5. *Let* ξ *be a simple point process and let* (F_t) *be the filtration generated by* ξ *; then:*

$$\xi^P[0,t) = \lim_i \sum_{k=0}^{2^i-1} E(\xi[k2^{-i}t, (k+1)2^{-i}t) \mid \mathcal{F}_{k2^{-i}t-}) \ a.s.$$

For general filtrations and random measures only a weaker form of convergence can be proved.

EXAMPLE. These theorems are of course easily checked in the case of the Poisson process (with intensity λ). For theorem 3.4.4 use the fact that the G_n are all exponential with parameter 1, thus $G_n(dt)/G_n([t,\infty]) = dt$. Theorem 3.4.5. becomes trivial thanks to the fact that for all $s \leq t$ we have

$$E(\xi[0,t) \mid \mathcal{F}_{s-}) = t - s \quad \text{a.s.} .$$

□

CHAPTER 4

SOME RESULTS ON POINT PROCESSES

In this short chapter the main object of study will be introduced intuitively. It turns out that this leads to a non-trivial definition problem. Some earlier solutions of the definition problem will be indicated.

§4.1. The intensity of the Poisson process.

In the preceding chapter we discussed the previsible projection of a point process on $\mathbb{R}_+$ and we saw that there the Poisson process formed an extremely simple example. Indeed, as a consequence of the independence property b) of example 2.2.2 the previsible projection turned out to be deterministic and to coincide with the intensity of the process. It was even noticed that in a way the dual previsible projection is a generalization of the intensity of the Poisson process because roughly speaking we have

(4.1.1) $\quad E\,((\xi - \xi^P)(dt) \mid \mathcal{F}_{t-}) = 0$

(c.f. (3.4.4)).

We now turn to point processes on an arbitrary locally compact space U with countable base and we shall see that in this case too the intensity of the Poisson process may be generalized in a certain manner[1]. To illustrate this we consider a simple non-marked Poisson process on U with (diffuse) intensity measure ν. Now the independence property of the Poisson process yields for all $B \in \mathcal{B}$:

$$\nu(B) = E(\xi(B) \mid \mathcal{F}(B)) \text{ a.s. } .$$

Replacing B by an infinitesimally small set we should obtain

$$\nu(du) = E(\xi(du) \mid \mathcal{F}(du)) \text{ .}$$

This, however, is not a mathematically meaningful formula.

1) If it happens that $U = \mathbb{R}_+$ this generalization will differ in general from the one studied in chapter 3.

Now suppose that μ is an arbitrary point process. We should like to define a random measure ζ which intuitively would have the following characterization:

$$\zeta(du) \text{ "=" } E(\xi(du) \mid \mathcal{F}(du)) .$$

Our approach to the problem of giving a correct definition of ζ will use a series of steps parallel to the results sketched in chapter 3.

§4.2. The conditional intensity of point processes.

Papangelou (74) already studied the problem outlined in §4.1, when he wanted to find out whether or not all point processes with a certain stationarity property are in fact Cox processes (cf. our theorem 10.4.2.). In order to define the, what he called, conditional intensity of a point process, Papangelou proved that

$$(4.2.1) \qquad \zeta(V) = \lim \sum_{W \in \mathcal{U}_{i,V}} E(\xi(W) \mid \mathcal{F}(W))$$

exists a.s. for all $V \in \mathcal{U}$ and that the limit determines a random measure on $\mathcal{U}$ and hence on $\mathcal{B}$ when considered as a function of V. In his proof Papangelou needed a second order integrability condition. He showed furthermore that under regularity conditions (Σ) and (Σ^*), which will be discussed in chapter 6, the limit does not depend on the choice of the sequence of partitions $(\mathcal{U}_1, \mathcal{U}_2, \ldots)$ and is a.s. diffuse. Papangelou's result is in a way comparable to theorem 3.4.5.

Kallenberg (78) proved the existence of the limit (4.2.1) under a weaker integrability condition and found another characterization of the conditional intensity. Under the conditions (Σ) and (Σ^*) mentioned above this characterization implies that ζ is the unique random measure satisfying the integral equation:

$$E \int 1_{F^*}(\{u\}^c\mu)\ 1_B(u)\ \xi(du) = E \int 1_{F^*}(\mu)\ 1_B(u)\ \zeta(du)$$

for all $F \in \mathcal{F}$, $B \in \mathcal{B}$. If $1_{F^*}(\mu)\ 1_B(u) \to 1_{F^*}(\{u\}^c\mu)\ 1_B(u)$ were some projection of the process $1_{F \times B}$, then this result would look like the characterization of the dual previsible projection by formula

(3.4.2)

Thus the analogy between parts of the theory of processes on $\mathbb{R}_+$ and the results mentioned in this chapter is clear. These results will be embedded in the "visible" theory which is to be developed in the rest of this treatise.

CHAPTER 5

VISIBILITY

§5.1. Visible σ-field, processes and points.

In §2.1. we defined for each set $B \in \mathcal{B}$ the exterior σ-field with respect to μ on Ω : $\mathcal{F}(B)$. The collection $(\mathcal{F}(B))_{B \in \mathcal{B}}$ is a decreasing filtration for the partial ordering of inclusion, i.e. $\mathcal{F}(V) \subset \mathcal{F}(W)$, whenever $W \subset V$.

We define the *visible* (from outside, w.r.t. μ) *σ-field* $\mathcal{Z}$ which is contained in $\mathcal{F} \times \mathcal{B}$. By definition $\mathcal{Z}$ is generated by the sets of the form $F \times B$, $F \in \mathcal{F}(B)$, $B \in \mathcal{B}$. In this context Kallenberg (83) uses the term "exvisible" instead of visible to express that, in fact, it implies a visibility from outside (cf. the exterior σ-fields $\mathcal{F}(B)$) in the same way as previsible phenomena can be foreseen. Note that $\mathcal{Z}$ is defined on $\Omega \times U$ as $\mathcal{P}$ is defined on $\Omega \times \mathbb{R}_+$. Still, not only are previsibility and visibility completely different - though analogous - notions, but moreover visibility depends essentially on the point process μ . Indeed, visibility is defined through the filtration $(\mathcal{F}(B))_{B\in\mathcal{B}}$ which depends on μ and, in the next section, we shall see that it is important both that the $\mathcal{F}(B)$ are generated by μ and that they contain all null-sets. In contrast with this on the other hand the results of chapter 3 up to and including theorem 3.4.3. are true for any filtration $(\mathcal{F}_t)$.

Processes - which are in fact functions on $\Omega \times U$ - are called *visible* if they are $\mathcal{Z}$-measurable. A *visible point* is a random point Z whose graph $\lceil Z \rfloor$ is a visible subset of $\Omega \times U$.

Examples of visible processes are of course indicators like $1_{F\times B}$ where $F \in \mathcal{F}(B)$ and $B \in \mathcal{B}$ or $1_{\lceil Z \rfloor}$ where Z is a visible point. Visible points are for instance all deterministic points (i.e. $\omega \to u$ a.s., $u \in U$). Another example of a visible point is obtained as follows:

<u>EXAMPLE 5.1.1.</u> Let μ be a simple non-marked point process on $U = \mathbb{R}^2$, such that $P(\xi(U) = 0) = 0$ and define Z by[1]:

1) The "*" in the sequel of this example (cf. §2.3.) may be dropped if we take $\Omega = M$.

$$Z^*(\mu) = \begin{cases} \Delta & \text{if } \xi(U) = \infty , \\ \frac{1}{\xi(U)} \int_U u\xi(du) & \text{if } \xi(U) < \infty . \end{cases}$$

Thus Z is essentially the centre of gravity of the realization. To check the visibility of Z we note that:

$$\lceil Z \rfloor = \lim_i \bigcup_{V \in U_i} (\{\xi(V^c) \neq 0,\ Z^*(V^c\mu) \in V\} \cup \{\xi(V^c) = 0\}) \times V ,$$

while clearly $\{\xi(V^c) \neq 0 , Z^*(V^c\mu) \in V\} \cup \{\xi(V^c) = 0\} \in F (V)$.

Intuitively the centre of gravity is a visible point "because its location is not affected by the absence of presence of mass of it". The requirement $P(\xi(U) = 0) = 0$ is essential; cf. the case c = 0 in example 5.2.1. □

We give another important example of visible sets: for each $V \in U$ we introduce:

$$H(V) = \{(\omega,u) \mid u \in V,\ \xi_\omega(V-\{u\}) = 0\}$$

Visibility is proved by[1]:

$$H(V) = \lim_i \bigcup_{W \in U_{i,V}} \{\xi(V-W) = 0\} \times W$$

and $\{\xi(V-W) = 0\} \in F(W)$. For all ω there exists for all $u \in U$ a set $V \in U$, $V \ni u$, such that $\xi_\omega(V-\{u\}) = 0$ and hence:

(5.1.1) $\quad P(\pi(\bigcap_{V \in U} H(V)^c)) = 0$

Note that the family U is countable.

§5.2. A visible section theorem.

To begin with we shall prove some technical lemmas.

1) The $\lim_i$ exists since it may be replaced by $\bigcap_{i>k}$ where k is such that $V \in U_k$. Something similar is usually the case when we encounter limits of sets.

LEMMA 5.2.1. *The visible σ-field is generated by the class $\mathcal{D}$ consisting of all sets of the form* $F \times V$, *where* $F \in \mathcal{F}(V)$, $V \in \mathcal{U}$.

PROOF. Let $\mathcal{C}$ denote the algebra generated by $\mathcal{U}$ and $\mathcal{E}$ the class of sets $B \in \mathcal{B}$ such that $F \times B \in \mathcal{T}(\mathcal{D})$ for all $F \in \mathcal{F}(B)$. Now we have to show that $\mathcal{E} = \mathcal{B}$. First we note that $\mathcal{E}$ is closed under countable unions since $B = \bigcup_i B_i$, $B_i \in \mathcal{E}$, and $F \in \mathcal{F}(B)$ imply that $F \in \mathcal{F}(B_i)$ for all i and hence $F \times B = \bigcup_i F \times B_i \in \mathcal{T}(\mathcal{D})$. A first consequence of this observation is that $\mathcal{E} \supset \mathcal{C}$.

Furthermore, we see that $\mathcal{E}$ is closed under monotone countable intersections; indeed, suppose $\mathcal{E} \ni B_i \downarrow B$; the σ-field $\mathcal{F}(B)$ is generated by $\mathcal{N}$ $(\subset \mathcal{F}(B_i)\ \forall\ i)$ and the sets

$$\{ \mu (A \times D) \leqq \alpha\} \qquad \alpha = 0, 1, 2, \dots , A \in \mathcal{B}, A \cap B = \emptyset , D \in \mathcal{K} ,$$

whereas

$$\{ \mu ((A \cap B_i^c) \times D) \leqq \alpha \} \times B_i \downarrow \{ \mu(A\times D) \leqq \alpha \} \times B ,$$

because

$$\{ \mu ((A\cap B_i^c) \times D) \leqq \alpha \} \downarrow \{ \mu(A\times D) \leqq \alpha \} , \text{ while}$$

$$\{ \mu ((A\cap B_i^c) \times D) \leqq \alpha \} \in \mathcal{F} (B_i)$$

Thus the class $\mathcal{E}$ satisfies the conditions of the monotone class theorem (theorem B.3), which yields: $\mathcal{E} = \mathcal{T}(\mathcal{C}) = \mathcal{B}$. □

LEMMA 5.2.2. *Let* $\varepsilon > 0$ *and* $A \in \mathcal{Z}$. *Then there exists a visible set* $A' \subset A$ *of the form*

$$A' = \bigcap_{i=1}^{\infty} \bigcup_{V \in \mathcal{U}_i} F(V) \times V ,$$

where $F(V) \in \mathcal{F}(V)$ *and* $W \subset V$ $(W, V \in \mathcal{U})$ *implies* $F(W) \subset F(V)$, *such that*

$$P (\pi(A')) > P (\pi(A)) - \varepsilon .$$

PROOF. The ordinary section theorem (theorem A.1) allows us to choose an F-measurable section R of A such that $P(\pi\lceil R\rfloor) = P(\pi(A))$. We now define the measure m on F × B by:

$$m(G) = \int I_G(\omega, R(\omega))\, P(d\omega) = P(\pi(\lceil R\rfloor \cap G))$$

for all $G \in F \times B$. Let D_s denote the class containing all finite unions of elements of D (The family D was introduced in lemma 5.2.1). The class D_s being closed under finite intersections, a well-known result of measure theory states that there exists a set A', a countable intersection of elements of D_s, such that

$$m(A') > m(A) - \varepsilon .$$

(See for instance Hahn and Rosenthal (48) 6.5.4. This result can also be considered to be a consequence of the theorem of Choquet, e.g. Meyer and Dellacherie (75) - III - 28 + 33a).

Hence:

$$P(\pi(A')) > P(\pi(A)) - \varepsilon$$

While A' has the following form:

$$A' = \bigcap_{k=1}^{\infty} A_k ,$$

where $A_k = \bigcup_{j=1}^{m_k} F_{kj} \times V_{kj}$, $F_{kj} \in F(V_{kj})$, $V_{kj} \in U$.

The number $i_k = \max\ \{ i \mid \exists\ j : V_{kj} \in U_i\}$ exists for all k. The sets V_{kj} may be divided in such a way that

$$A_k = \bigcup_{V \in U_{i_k}} F_k(V) \times V \text{ , where } F_k(V) \in F(V)$$

(indeed, take $F_k(V) = \bigcup_{j:V_{kj}\supset V} F_{kj} \in F(V)$; some $F_k(V)$ will be empty).

If $V \in U_i$, we write $F(V) = \bigcap_{k:i_k=i} F_k(V)$ $(\in F(V))$ (if $i_k \neq i$ for all k, we take $F(V) = \Omega$), then

$$A' = \bigcap_{i=1}^{\infty} \bigcup_{V \in U_i} F(V) \times V$$

It is easy to check that we can suppose without restriction that the $F(V)$ are increasing ($W \subset V \Rightarrow F(W) \subset F(V)$). □

Remark: Writing

$$(5.2.1) \quad F(u) = \bigcap_{i=1}^{\infty} \bigcup_{\substack{W \in U_i \\ W \ni u}} F(W) \quad (\in F(\{u\})).$$

We obtain:

$$A' = \bigcup_{u \in U} F(u) \times \{u\}.$$ □

The proof of the previsible section theorem (theorem 3.2.1) begins with an argument similar to the proof of lemma 5.2.2. This explains the "ε" in theorem 3.2.1. However, the proof of the previsible section theorem is completed by using the order of $\mathbb{R}_+$ and this order cannot be used to prove the visible section theorem, which now follows.

THEOREM 5.2.3. (Visible section theorem) *Let* A *be a visible non-evanescent set; then there exists a visible point whose graph is contained in* A *and is non-evanescent (a visible section of* A *with non-evanescent graph).*

Before proving this theorem we give an example.

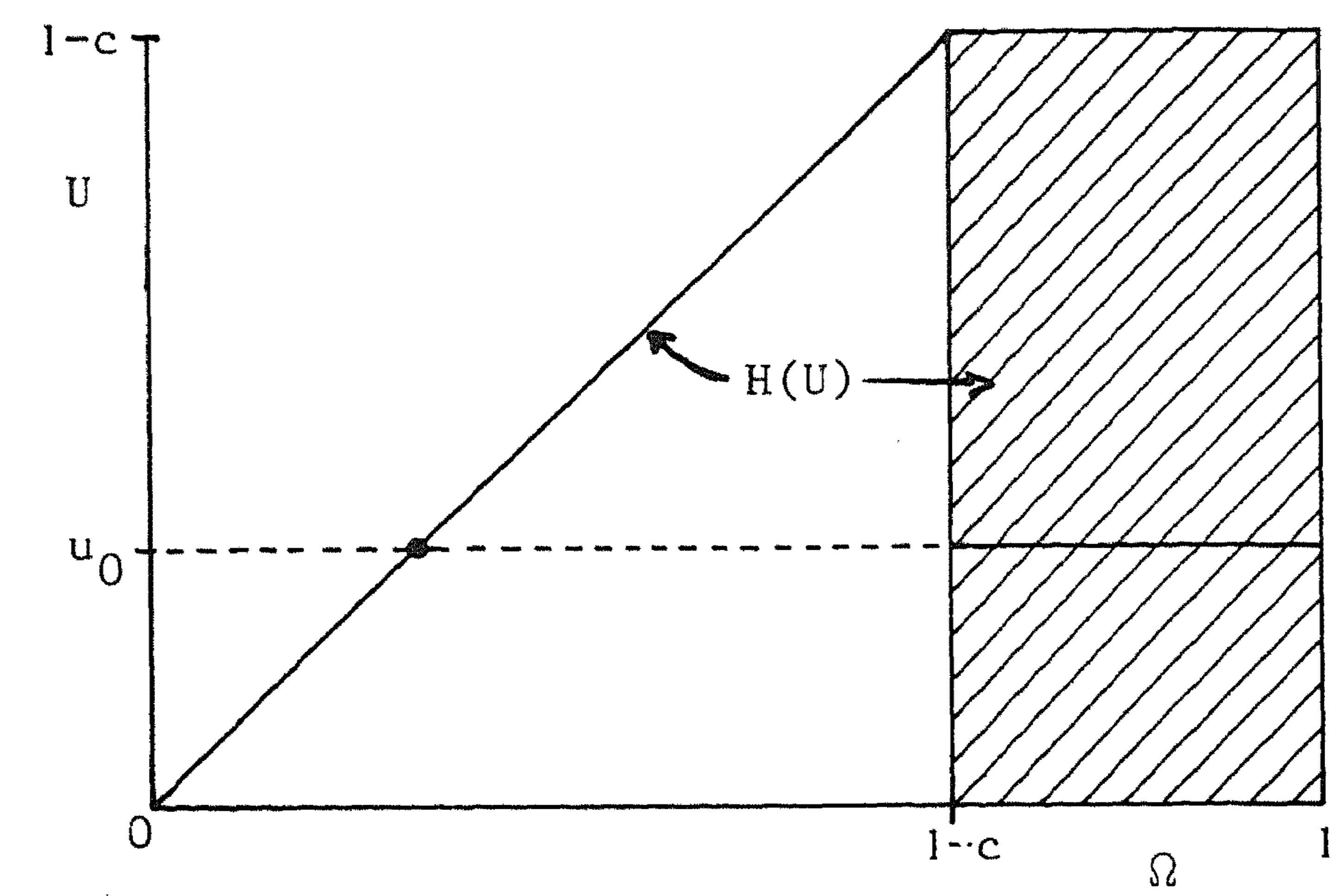
EXAMPLE 5.2.1. (This example will be elaborated in §10.1). The above theorem guarantees only the existence of a small, but non-evanescent, visible section of non-evanescent visible sets. In this sense the statement is weaker than the previsible section theorem. However, we cannot do better as will be shown in this example.

Take $0 \leq c < 1$ and $U = [0,1-c)$. The probability space will be the interval $[0,1]$ with Lebesgue measure; the simple non-marked point process μ is given by:

$$\xi_\omega(U) = \xi_\omega(\{\omega\}) = 1 \qquad \text{if } 0 \leq \omega < 1 - c,$$

$$\xi_\omega(U) = 0 \qquad \text{if not}.$$

figure 5.2.1



(Thus μ is the zero-or-one-point process (Example 2.2.1) with $U = [0,1-c)$ and ν the uniform distribution on U).
The set $H(U) = \{(\omega,\omega) \mid 0 \leq \omega < 1-c\} \cup ([1-c, 1] \times U)$ is sketched in figure 5.2.1. We know that H(U) is visible. Furthermore, $P(\pi(H(U))) = 1$. But if $c > 0$ any visible section of H(U) with non-evanescent graph must be of the following form

$$Z(\omega) = \begin{cases} u_0 & \text{if } \xi_\omega(U - \{u_0\}) = 0, \\ \Delta & \text{if not;} \end{cases} \qquad \text{a.s.}$$

for some $u_0 \in U$ (theorem 10.1.1); and then we have $P(\pi[Z]) = c$. If, on the other hand $c = 0$, then H(U) itself only differs by an evanescent set from the graph of a visible point: the a.s. existing unique point of the realization of ξ is visible itself "because if we know it is not outside a set B ($\in \mathcal{B}$), then we a.s. know it is in B". □

The distinction between the cases $c = 0$ and $c > 0$ in the above example corresponds to the one between the cases i) and ii) in the following proof.

PROOF of theorem 5.2.3. Let A be a visible set such that $P(\pi(A)) > 0$. Using lemma 5.2.2 we then see that there exists a visible set $A' \subset A$ of the form $A' = \bigcap_{i=1}^{\infty} \bigcup_{W \in \mathcal{U}_i} F(W) \times W$, where $F(W) \in \mathcal{F}(W)$ and the F(W) are increasing, such that $P(\pi(A')) > 0$. Because $\mathcal{U}$ is a countable family it follows from formula (5.1.1) that there exists a set $V \in \mathcal{U}$ such that $P(\pi(H(V) \cap A')) > 0$. Hence we may assume without loss that A is a visible set of the form:

$$A = \lim_i \bigcup_{W \in U_{i,V}} [\{\xi(V-W) = 0\} \cap F(W)] \times W ,$$

where $F(W) \in \mathcal{F}(W)$, the $F(W)$ are increasing and V is some element of U.

Now two cases are to be distinguished:

i) $P(\pi(A) \cap \{\xi(V) = 0\}) = 0$.

Then $A \cap [(\pi(A) \cap \{\xi(V) = 0\})^c \times U] \in Z$ and since $A \subset \Omega \times V$ we have $P[\pi(A \cap ((\pi(A) \cap \{\xi(V) = 0\})^c \times U))] = P(\pi(A)) > 0$, but on the other hand:

$$A \cap [(\pi(A) \cap \{\xi(V) = 0\})^c \times U] =$$

$$= \left[\lim_i \bigcup_{W \in U_{i,V}} (F(W) \cap \{\xi(V-W) = 0\}) \times W\right] \cap [(\pi(A)^c \cup \{\xi(V) = 0\}^c) \times U]$$

$$= \lim_i \bigcup_{W \in U_{i,V}} [F(W) \cap \{\xi(V-W) = 0\} \cap \{\xi(V) = 0\}^c] \times W$$

$$= \bigcup_{u \in V} [F(u) \cap \{\xi(V-\{u\}) = 0\} \cap \{\xi(V) \neq 0\}] \times \{u\}$$

$$= \bigcup_{u \in V} [F(u) \cap \{\xi(V-\{u\}) = 0\} \cap \{\hat{\xi}_u \neq 0\}] \times \{u\} = \lceil Z \rfloor ,$$

(If $u \in U$, then $F(u)$ is defined by (5.2.1)) where Z is the random point given by:

$$Z(\omega) = \begin{cases} u & \text{if } \hat{\xi}_u(\omega) \neq 0, \quad \xi_\omega(V-\{u\}) = 0, \ \omega \in F(u) , \\ \Delta & \text{if not} . \end{cases}$$

ii) $c \equiv P(\pi(A) \cap \{\xi(V) = 0\}) > 0$.

According to theorem A.1 there exists a section $\tilde{R}$ of $A \cap [\{\xi(V) = 0\} \times V]$ such that $P(\pi\lceil \tilde{R} \rfloor) = c$. We define the random point R by $R^*(\mu) = \tilde{R}^*(V^c\mu)$. The random point R is visible since its image is contained in $V \cup \{\Delta\}$ while it depends on $V\mu$ hence on what μ does outside of V. Indeed,

$$\lceil R \rfloor = \lim \bigcup_{W \in U_{i,V}} \{R \in W\} \times W$$

and

$$\{R \in W\} \in \mathcal{F}(V) \subset \mathcal{F}(W) .$$

Next we define the visible section Z of A by $\lceil Z \rfloor = \lceil R \rfloor \cap A$; then $\lceil Z \rfloor \supset \lceil \tilde{R} \rfloor$ and hence $P(\pi \lceil Z \rfloor) \geq c > 0$. □

§5.3. The visible projection

To begin this section we introduce the exterior σ-field of a random point which is to be compared with the σ-field $\mathcal{F}_{T-}$ of events strictly anterior to a stopping time T in the theory of processes on $\mathbb{R}_+$. The lemmas 5.3.1, 2 and 3 also have their analogue in that theory.

The *exterior σ-field of a random point* R is defined by:

$$\mathcal{F}(R) = \mathcal{T}(F \cap \{R \in B\} \mid F \in \mathcal{F}(B),\ B \in \mathcal{B}) .$$

Thus if $R = u$ a.s., then we have $\mathcal{F}(R) = \mathcal{F}(\{u\})$.

<u>LEMMA 5.3.1.</u> *Let* R *be a random point and let the function* $f_R : \Omega \to \Omega \times (U \cup \{\Delta\})$ *be given by* $f_R(\omega) = (\omega, R(\omega))$, *then* $\mathcal{F}(R) = f_R^{-1}(\mathcal{Z})$; *furthermore:* $f_R^{-1}(A) = \pi(A \cap \lceil R \rfloor)$.

PROOF. Obvious □

<u>LEMMA 5.3.2.</u> *Let* Z *be a visible point and* $F \in \mathcal{F}(Z)$. *Then* $(F \times U) \cap \lceil Z \rfloor \in \mathcal{Z}$ *and hence* $(F \times U) \cap \lceil Z \rfloor$ *is the graph of a visible point.*

PROOF. Lemma 5.3.1 yields that $F \in \mathcal{F}(Z)$ if and only if $F = f_Z^{-1}(A) = \pi(A \cap \lceil Z \rfloor)$ for some $A \in \mathcal{Z}$, and hence $\omega \in F$ implies $(\omega, Z(\omega)) \in A$ so that $(F \times U) \cap \lceil Z \rfloor = A \cap \lceil Z \rfloor \in \mathcal{Z}$. □

<u>LEMMA 5.3.3.</u> *Let* R *be an arbitrary random point and* Z *a visible point. Then* $\mathcal{F}(Z) \cap \{Z=R \in U\} \subset \mathcal{F}(R)$, *and hence in particular:* $\{Z = R \in U\} \in \mathcal{F}(R)$.

PROOF. The last statement of the lemma follows immediately from lemma

5.3.1, since $\{Z = R \in U\} = \pi(\lceil Z \rfloor \cap \lceil R \rfloor) = f_R^{-1}(\lceil Z \rfloor) \in \mathcal{F}(R)$.
To prove the more general assertion we note that the sets
$F \cap \{Z \in B\} \cap \{Z = R \in U\} = F \cap \{R \in B\} \cap \{Z = R \in U\}$ with $F \in \mathcal{F}(B)$, $B \in \mathcal{B}$ - which generate the σ-field $\mathcal{F}(Z) \cap \{Z = R \in U\}$ - are elements of $\mathcal{F}(R)$. □

Before coming to the projection theorem itself, we shall prove a technical lemma. The set $\sigma(\mu)$, which is defined by this lemma, is intimately related to the regularity of the point process.

LEMMA 5.3.4. *For each random measure* ρ *on* U *(resp. on* $U \times K$*) there exists a corresponding visible set* $\sigma(\rho)$ *such that:*
- *the set* $\sigma(\rho)^c$ *is the union of graphs of a finite or countable number of visible points,*
- *for all visible points* Z *with* $\lceil Z \rfloor \subset \sigma(\rho)$ *we have* $P(\rho(Z) \neq 0) = 0$, *and*
- *for all visible points* Z *with* $\lceil Z \rfloor \subset \sigma(\rho)^c$ *and* $P(\pi\lceil Z \rfloor) \neq 0$ *we have* $P(\rho(Z) \neq 0) \neq 0$.

The meaning of this lemma may not be immediately obvious. Therefore we shall first try to give the reader an intuitive feeling for it.

EXAMPLE 5.3.1. (This example will be elaborated in §10.2) We take $U = (0,1)$ and $\Omega = (0,1)$ with Lebesgue measure. The simple non-marked point process μ is defined by:

$$\xi_\omega = \begin{cases} \varepsilon_\omega & \text{if } \omega \leq \frac{1}{2} , \\ \varepsilon_{\omega-\frac{1}{2}}+\varepsilon_\omega & \text{if } \omega > \frac{1}{2} . \end{cases}$$

The points on the solid lines in figure 5.3.1 indicate the atoms of ξ_ω at the corresponding ω . The same distribution of ξ can also be obtained in another manner:

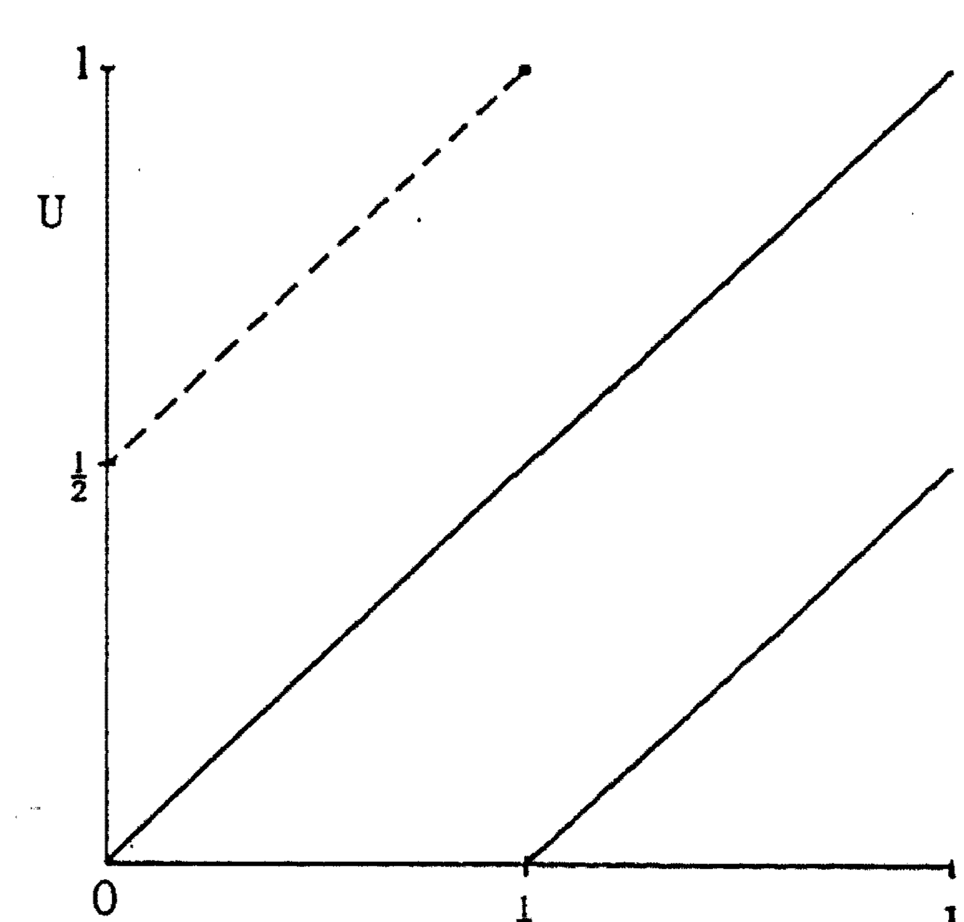


figure 5.3.1

$$\xi = \varepsilon_X + Y\varepsilon_{X+\frac{1}{2}}$$

where X and Y are independent r.v.'s; X is uniformly distributed on $(0,\frac{1}{2})$

and $P(Y=1) = P(Y=0) = \frac{1}{2}$ (take $X = \omega \bmod \frac{1}{2}$ and $Y = \begin{cases} 0 & \text{if } \omega \leqq \frac{1}{2}, \\ 1 & \text{if } \omega > \frac{1}{2}. \end{cases}$)

The set $\sigma(\xi)^c$ (of course ξ and μ are interesting random measures to put in $\sigma(.)$) now consists of the solid lines and the dotted line in figure 5.3.1; i.e. $\sigma(\xi)^c = A \equiv \{(\omega,u) \mid \omega = u \text{ or } \omega = u + \frac{1}{2} \text{ or } \omega = u - \frac{1}{2}\}$ Indeed, if $\lceil Z \rfloor \subset A^c$ then $P(\xi(Z) \neq 0) = 0$. On the other hand, since with probability one the interval $(0,\frac{1}{2})$ contains exactly one atom of ξ, we have $\lceil X \rfloor \doteq H(0,\frac{1}{2})$, so that the atom in $(0,\frac{1}{2})$ itself forms a visible point (cf. the case $c = 0$ in example 5.2.1.). The point $X+\frac{1}{2}$ is visible too since X itself can be observed from the interval $(0,\frac{1}{2})$. We have $P(\xi(X+\frac{1}{2}) = 1) = \frac{1}{2} = \frac{1}{2}P(X+\frac{1}{2} \in U)$, but we cannot leave out a non-evanescent part of the dotted line only without disturbing the visibility. □

In the above example all atoms of ξ are "in" $\sigma(\xi)^c$. However, this is not necessarily the case; for most "decent" point processes $\sigma(\xi)^c$ is even empty; for instance in the case $c > 0$ in example 5.2.1 we have $P(\xi(Z) \neq 0) = 0$ for all visible points Z.

PROOF of lemma 5.3.4. First we suppose that U (resp. $U \times K$) is bounded. We denote by V the class of all visible points. If the set $\{Z \in V \mid P(\rho(Z) > \frac{1}{2}) > \frac{1}{2}\}$ is not empty, then we choose an element Z_1^1 in it. Next we choose an element Z_2^1 in $\{Z \in V \mid \lceil Z \rfloor \cap \lceil Z_1^1 \rfloor = \emptyset,\ P(\rho(Z) > \frac{1}{2}) > \frac{1}{2}\}$ and so on. Since $P(\rho(U) < \infty) = 1$ (resp. $P(\rho(U{\times}K) < \infty) = 1$) ($U$ (resp. $U \times K$) being bounded) there exists a finite number n_1 such that
$\{Z \in V \mid \lceil Z \rfloor \cap [\bigcup_{i=1}^{n_1} \lceil Z_i^1 \rfloor] = \emptyset,\ P(\rho(Z) > \frac{1}{2}) > \frac{1}{2}\} = \emptyset$.

Now we apply the same procedure to

$\{Z \in V \mid \lceil Z \rfloor \cap [\bigcup_{i=1}^{n_1} \lceil Z_i^1 \rfloor] = \emptyset,\ P(\rho(Z) > (\frac{1}{2})^2) > (\frac{1}{2})^2\}$

and find $Z_1^2, \ldots, Z_{n_2}^2$. Etc. Proceeding in this manner we obtain a finite or countable collection of visible points Z_i^j with disjoint graphes, such that $P(\rho(Z) \neq 0) = 0$ for any visible point with $\lceil Z \rfloor \subset (\bigcup_i \bigcup_j \lceil Z_i^j \rfloor)^c$.

Next we note that for any visible point Z there exists a unique visible point $\tilde{Z}$ that such $\lceil \tilde{Z} \rfloor \subset \lceil Z \rfloor$ and $P(\rho(\tilde{Z}) \neq 0) = P(\rho(Z) \neq 0)$, which has the following property: if Z' is a visible point with $\lceil Z' \rfloor \subset \lceil \tilde{Z} \rfloor$ and $P(\pi \lceil Z' \rfloor) \neq 0$, then we have $P(\rho(Z') \neq 0) \neq 0$. (Intuitively $\tilde{Z}$ is the "smallest visible point" containing the same ρ-mass as Z").

Indeed, we may let $\tilde{Z}$ be determined by

$$\lceil \tilde{Z} \rfloor = \lceil Z \rfloor \cap (\{P(\rho(Z) \neq 0 \mid \mathcal{F}(Z)) \neq 0\} \times U) .$$

The set $(\bigcup_i \bigcup_j \lceil \tilde{Z}_i^j \rfloor)^c$ satisfies the conditions imposed on $\sigma(\rho)$. In the more general case, where the space U (resp. $U \times K$) is not bounded, make use of the fact that it is the union of a countable number of bounded sets. □

We write:

$$\sigma(\mu) = \sigma .$$

In chapter 6 we shall investigate this set σ and related matters in more detail.

<u>THEOREM 5.3.5.</u> *Let the process* X *be bounded or finite and positive; then there exists a visible process* ${}^zX$ *such that we have:*

(5.3.1) $EX_Z = E\,{}^zX_Z$

for all visible points Z . *The process* ${}^zX$ *is uniquely determined (up to indistinguishability) and is called the visible projection of* X .

PROOF. The visible section theorem (theorem 5.2.3) ensures the unicity of the visible projection, if it exists, for if ${}^1X$ and ${}^2X$ are two distinguishable visible processes, then at least one of the sets $\{(\omega, u) \mid {}^1X_u(\omega) > {}^2X_u(\omega)\}$ and $\{(\omega, u) \mid {}^1X_u(\omega) < {}^2X_u(\omega)\}$ is non-evanescent and visible and hence has a visible section Z with non-evanescent graph, so that $E\,{}^1X_Z \neq E\,{}^2X_Z$.

Furthermore, we may use the monotone class theorem (theorem B.1): Indeed, we again use the visible section theorem to prove that:

- if X and Y have visible projections ${}^zX$ and ${}^zY$ and if $X \lessapprox Y$, then ${}^zX \lessapprox {}^zY$; in particular $|X| \lessapprox c$ (c constant) implies $|{}^zX| \lessapprox c$;
- if the processes $X^{(n)}$ have visible projections ${}^zX^{(n)}$ and if (outside an evanescent set) the processes $X^{(n)}$ are uniformly bounded or positive and converge increasingly, then $\lim_n {}^zX^{(n)}$ is a version of the visible projection of $\lim_n X^{(n)}$;
- the same holds if (outside an evanescent set) the $X^{(n)}$ are bounded and converge uniformly;
- if the processes X and Y have visible projections ${}^zX$ and ${}^zY$,

then $a^zX + b^zY$ is a version of the visible projection of $aX + bY$.

According to the monotone class theorem it is enough to prove the existence of the visible projection for a class of processes which is closed under multiplication and which generates $A \times B$.
The processes

(5.3.2) $\quad X_u(\omega) = 1_{F\times B}(\omega, u)$,

where $F \in A$, $B \in B$, form such a class. Next note that clearly the processes $1_{F\times B}(\omega, u)$ and $P(F \mid F)(\omega)\ 1_B(u)$ have the same visible projection, if any. Hence it suffices to indicate the visible projection of the processes (5.3.2) with $F \in F$, $B \in B$, since these processes generate $F \times B$.

Now, let X be such an indicator process. Choose a family of visible points $\{Z_i\}$ such that $\sigma^c = \cup_i \lceil Z_i \rfloor$; this is possible according to lemma 5.3.4. On $\lceil Z_i \rfloor$ we define:

$$^zX_u(\omega) = P(F \mid F(Z_i))(\omega)\ 1_B(u) \ .$$

Lemma 5.3.2 ensures the visibility of the process $P(F \mid F(Z_i))(\omega)\ 1_{\lceil Z_i \rfloor}(\omega, u)\ 1_B(u)$. On σ we set:

$$^zX_u(\omega) = 1_{F*}(\{u\}^c\ \mu_\omega)\ 1_B(u) \ .$$

This part too is visible since on σ we may write:
$^zX_u(\omega) = \lim_i \sum_{V\in U_i} 1_{F*}(V^c\mu_\omega)\ 1_{V\cap B}(u) \in Z$, the limit existing thanks to (5.1.1). Now the visible process ^zX is defined everywhere on $\Omega \times U$.

Let Z be a visible point. Because lemma 5.3.3 implies that $\{Z = Z_i \in B\} \in F(Z_i)$, we obtain:

$$\int_{\{Z=Z_i\}} {}^zX_Z\, dP = \int_{\{Z=Z_i\in B\}} P(F \mid F(Z_i))\ dP$$
$$= \int_{\{Z=Z_i\in B\}} 1_F\ dP = \int_{\{Z=Z_i\}} X_Z\ dP$$

for all i. On the other hand $P((\pi(\lceil Z \rfloor \cap \sigma)) \cap \{\mu(Z)\} > 0\}) = 0$ hence:

$$\int_{\pi(\lceil Z \rfloor \cap \sigma)} {}^{z}X_Z \, dP = \int_{\pi(\lceil Z \rfloor \cap \sigma)} I_{F*}(\{Z\}^c \mu) \; I_B(Z) \; dP$$

$$= \int_{\pi(\lceil Z \rfloor \cap \sigma)} I_{F*}(\mu) \; I_B(Z) \; dP$$

$$= \int_{\pi(\lceil Z \rfloor \cap \sigma)} X_Z \; dP$$

(N.B. $\pi(\lceil Z \rfloor \cap \sigma) = \{\omega \mid (\omega, Z(\omega)) \in \sigma\}$). Combining the above equalities we find $E^{z}X_Z = EX_Z$.

The theorem is now proved for bounded processes. It holds for positive finite processes X since we may apply this result to $X \wedge n$ and let n tend to infinity. □

Remarks

1. Let R be a random point and Y a visible process. Then it follows from lemma 5.3.1 that Y_R is $\mathcal{F}(R)$-measurable.

2. Let Z be a visible point and X a process. Then

$${}^{z}X_Z = E(X_Z \mid \mathcal{F}(Z)) \quad \text{a.s.}.$$

Indeed, both members are $\mathcal{F}(Z)$-measurable. If $\{{}^{z}X_Z < E(X_Z \mid \mathcal{F}(Z))\}$ (or $\{{}^{z}X_Z > E(X_Z \mid \mathcal{F}(Z))\}$) is not a null-set, then the random point Z' determined by $\lceil Z' \rfloor = \lceil Z \rfloor \cap [\{{}^{z}X_Z < E(X_Z \mid \mathcal{F}(Z))\} \times U]$ is visible according to lemma 5.3.2 and $E^{z}X_{Z'} \neq EX_{Z'}$. This results in a contradiction.

3. If Y is a visible process, then we have

$${}^{z}(XY) = {}^{z}X.Y.$$

(Note that an analogous property applies to the previsible projection and also to conditional expectations).

4. If X is an $\mathcal{F} \times \mathcal{B}$-measurable process, then we have on σ:

$${}^{z}X_u(\omega) = X_u^*(\{u\}^c \mu_\omega).$$

5. In the course of the proof of theorem 5.3.5 we saw:

$$^{z}1_{F\times B} = {}^{z}(P(F \mid F)1_B)$$

for $F \in A$, $B \in B$. In a way the auxiliary process $P(F \mid F)1_B$ corresponds to the optional projection of not adapted processes in the theory of processes on $\mathbb{R}_+$.

6. We have

$$\sigma(\xi) = \sigma(\mu) = \sigma .$$

Even for arbitrary random measures ρ on $U \times K$ we have

$$\sigma(\rho(. \times K)) = \sigma(\rho) . \qquad \square$$

§5.4. Visible random measures.

We begin this section with a well-known result (cf. e.g. Kallenberg (83) lemma 2.3). We shall give an easier proof of it which uses the ordinary section theorem (theorem A.1).

LEMMA 5.4.1. *The set* $A = \{(\omega, u) \mid \hat{\xi}_u(\omega) \neq 0\}$ *is the union of a finite or countable number of graphs of random points.*

PROOF. First suppose that U is bounded. It can be checked that $A \in F \times B$. Let R_1 be an F-measurable section of A with $P(\pi\lceil R_1 \rfloor) = P(\pi(A))$ (theorem A.1). Next we look for a section R_2 of $A - \lceil R_1 \rfloor$, and so on. Because $P(\xi(U) < \infty) = 1$ (U being bounded) it is easy to verify that it will take a finite or countable number of R_i before we have $P(\pi(A - \cup_i \lceil R_i \rfloor)) = 0$.

If U is not bounded, it is the countable union of bounded sets. $\square$

We still need another small lemma.

LEMMA 5.4.2. *Let* G *be an arbitrary sub-σ-field of* A, $G \supset N$, *and let* X *be a r.v.; then the following conditions are equivalent*

i) X *is* G*-measurable;*

ii) $E(1_F X) = E(P(F \mid G)X)$ *for all* $F \in A$.

iii) $EXH = 0$ *for all r.v.'s such that* $E(H \mid G) = 0$ *a.s..*

PROOF. Obvious. □

A random measure ρ on $\mathbb{R}_+$ is uniquely determined by the non-decreasing process $\rho[0,t]$ and consequently we were able to define previsibility of random measures through the previsibility of processes. For random measures on an arbitrary locally compact space with countable base no one-to-one correspondence with processes exists, so that we cannot simply copy the definition of a visible measure from the definition of a previsible measure (§3.4). In the following definition visible random measures are defined in three equivalent manners. The first is an analogue of theorem 3.4.1. The second is intuitively (and, of course, in fact) the same as the first; note that this second definition implies that although a random measure ζ is not uniquely determined by the process $\hat{\zeta}$, this process still tells a lot about the visibility of ζ. The third definition, which corresponds to theorem 3.4.2, will be used in the definition of the dual projection.

DEFINITION 5.4.3. *A random measure* ζ *on* U *(resp. on* $U \times K$*) will be called visible if one of the three following equivalent conditions is satisfied:*

i) The random measure ζ *is* F*-measurable and we have*

(5.4.1) for each visible point Z *the real-valued (resp.* L(K)*-valued) r.v.* $\hat{\zeta}_Z$ *is* F(Z)*-measurable,*

and

(5.4.2) for each random point R *such that* $[R] \subset \sigma(\zeta)$ *we have* $\zeta(R) = 0$ *a.s..*

ii) The random measure ζ *is* F*-measurable and the real-valued (resp.* L(K)*-valued) process* $\hat{\zeta}$ *is visible.*

iii) To all non-negative processes X *the following equality applies:*

(5.4.3) $$E \int X d\zeta = E \int {}^{Z}X d\zeta$$

(resp. to all non-negative processes $\underline{X}$ *and all* $D \in K$ *the following equality applies:*

(5.4.3') $$E \int X\, 1_D d\zeta = E \int {}^{Z}X\, 1_D d\zeta \;)\,.$$

PROOF of the equivalence of the conditions i), ii) and iii). To begin with we note that the requirement of F-measurability of ζ figures in both i) and ii).

Proof "i) ⇒ ii)": First, let us suppose that ζ is a random measure on U. The set $\{(\omega, u) \mid \hat{\zeta}_u(\omega) \neq 0\} \cap \sigma(\zeta)$ is evanescent; indeed, if not, then according to theorem A.1 it would contain the graph of a random point contradicting (5.4.2). If $\{Z_i\}$ is a family of visible points, such that $\sigma(\zeta)^c = \bigcup_i \lceil Z_i \rfloor$ (cf. lemma 5.3.4), and $\alpha \geq 0$, then we have: $\{(\omega, u) \mid \hat{\zeta}_u(\omega) > \alpha\} \doteq \bigcup_i (\lceil Z_i \rfloor \cap (\{\zeta(Z_i) > \alpha\} \times U))$, which is visible according to lemma 5.3.2.

If ζ is a random measure on $U \times K$ the above argument applies to $\hat{\zeta}(D)$ for each $D \in K$.

Proof "ii) ⇒ i)": Now condition (5.4.1) is clearly satisfied thanks to lemma 5.3.1. Furthermore, the set $\sigma(\zeta) \cap \{(\omega, u) \mid \hat{\zeta}_u(\omega) \neq 0\}$ (resp. $\sigma(\zeta) \cap \{(\omega, u) \mid \hat{\zeta}_u(\omega) = \underline{0}\}$) is evanescent; indeed, if it were not, then according to theorem 5.2.3 it would have a visible section Z such that $P(\zeta(Z) \neq 0) \neq 0$, which contradicts the definition of $\sigma(\zeta)$ (lemma 5.3.4). Now condition (5.4.2) is easily verified.

Proof "i) ⇔ iii)": We shall only give a proof in the case where ζ is a random measure on U. If ζ is a random measure on $U \times K$, then the arguments of this proof apply to $\zeta(. \times D)$ for each $D \in K$

By lemma 5.4.2 F-measurability of ζ is equivalent to:

$$E \int 1_{F\times B}(\cdot, u)\, \zeta(du) = E\, 1_F\, \zeta(B)$$

$$= E\, P(F \mid F)\, \zeta(B)$$

$$= E \int P(F \mid F)\, 1_B(u)\, \zeta(du)$$

for all $F \in A$, $B \in B$. On the other hand, if $F \in A$, $B \in B$, then ${}^z 1_{F\times B} = {}^z(P(F \mid F)\, 1_B)$ (cf. §5.3 remark 5) and hence (5.4.3) yields

$$E \int 1_{F\times B}(\cdot, u)\, \zeta(du) = E \int P(F \mid F)\, 1_B(u)\, \zeta(du) .$$

Now it is seen that it is sufficient to prove the following equivalence:

(5.4.1) + (5.4.2) ⇔ (5.4.3) for all non-negative $F \times B$-measurable processes

This will be done succesively.

Proof " $\Rightarrow$ (5.4.3) for $F \times B$-measurable processes": Let X be a non-negative $F \times B$-measurable process. If Z is a visible point, then (5.4.1) and §5.3 remark 2 imply:

$$E \int {}^{Z}X 1_{\lceil Z \rfloor} d\zeta = E^{Z}X_{Z}\ \zeta(Z) = EX_{Z}\ \zeta(Z) = E \int X\ 1_{\lceil Z \rfloor} d\zeta \ .$$

On the other hand, thanks to lemma 5.4.1 and statement (5.4.2) we have:

$$P[\pi(\{(\omega,\ u)\ |\ \hat{\zeta}_{u}(\omega) \neq 0,\ \hat{\xi}_{u}(\omega) \neq 0\} \cap \sigma(\zeta))] = 0 \ .$$

Using these two results and §5.3 remark 4 we find (5.4.3). Indeed,

$$\begin{aligned} E \int {}^{Z}X d\zeta &= E \int {}^{Z}X 1_{\sigma^{c} \cup \sigma(\zeta)^{c}}\ d\zeta + E \int {}^{Z}X 1_{\sigma \cap \sigma(\zeta)} d\zeta \\ &= E \int X 1_{\sigma^{c} \cup \sigma(\zeta)^{c}} d\zeta + E \int X^{*}_{u}(\{u\}^{c}\mu)\ 1_{\sigma \cap \sigma(\zeta)}(\cdot,\ u)\ \zeta(du) \\ &= E \int X 1_{\sigma^{c} \cup \sigma(\zeta)^{c}} d\zeta + E \int X^{*}_{u}(\mu)\ 1_{\sigma \cap \sigma(\zeta)}(\cdot,\ u)\ \zeta(du) \\ &= E \int X d\zeta \ . \end{aligned}$$

Proof " $\Rightarrow$ (5.4.1)": Let Z be a visible point. To prove $F(Z)$-measurability of $\zeta(Z)$ we use characterization iii) of lemma 5.4.2: Hence we suppose that H is a r.v. such that $E(H \mid F(Z)) = 0$ a.s. and we define the process H' by $H'_{u}(\omega) = H(\omega)$ for all $u \in U,\ \omega \in \Omega$. Then ${}^{Z}H'_{Z} = E(H'_{Z} \mid F(Z)) = 0$ a.s., and thus, since $\lceil Z \rfloor \in Z$, we have:

$$\begin{aligned} E\ \zeta(Z)\ H &= E \int 1_{\lceil Z \rfloor}(\cdot,\ u)\ H'_{u}\ \zeta(du) \\ &= E \int 1_{\lceil Z \rfloor}(\cdot,\ u)\ {}^{Z}H'_{u}\ \zeta(du) = 0 \ . \end{aligned}$$

Proof " $\Rightarrow$ (5.4.2)": Let R be a random point such that $\lceil R \rfloor \subset \sigma(\zeta)$. Since $\sigma^{c} \cap \sigma(\zeta)$ is the union of a finite or countable number of graphs of visible points which are contained in $\sigma(\zeta)$, we have $E \int 1_{\sigma^{c} \cap \sigma(\zeta)}\ d\zeta = 0$, so that $P(\zeta(R) \neq 0) = 0$ if $\lceil R \rfloor \subset \sigma^{c} \cap \sigma(\zeta)$.

Next we suppose that $\lceil R \rfloor \subset \sigma \cap \sigma(\zeta)$. Thanks to (5.1.1) we now only need to consider random points R for which there exists a $V \in \mathcal{U}$ such that $\lceil R \rfloor \subset H(V)$. Then

$$\{(\omega, u) \mid u = R^*(V^c \mu_\omega),\ \xi_\omega(V-\{u\}) = 0\} =$$

$$= \{(\omega, u) \mid u = R^*(\{u\}^c \mu_\omega)\ ,\ \xi_\omega(V-\{u\}) = 0\}$$

is the graph of a visible point R' . Hence we have

$$E\ \zeta(R) = E \int 1_{\sigma(\zeta)}\ 1_{\lceil R \rfloor} d\zeta = E \int 1_{\sigma(\zeta)}\ {}^z 1_{\lceil R \rfloor}\ d\zeta = E \int 1_{\sigma(\zeta)}\ 1_{\lceil R' \rfloor} d\zeta = 0\ .$$

because ${}^z 1_{\lceil Z \rfloor} = 1_{\lceil Z' \rfloor}$ thanks to §5.3 remark 4. □

<u>Remarks</u>.

1. If ζ is a random measure on $U \times K$, then $\zeta(. \times D)$ is a random measure on U for each $D \in \mathcal{K}$. This reduction is used at several stages in the above proof and will be used in future. Note that in fact a random measure ζ on $U \times K$ is visible if and only if the random measure $\zeta(. \times D)$ on U is visible for each $D \in \mathcal{K}$.

2. If follows immediately from definition 5.4.3 ii) and lemma 5.3.2 that if ζ is a visible random measure then statement (5.4.1) also applies to non-visible random points. This can also be seen from (5.4.1), (5.4.2) and lemma 5.3.3.

3. Let ζ be a visible measure on U . Looking at the proof of the equivalence i) $\Longleftrightarrow$ ii) of definition 5.4.3 we note that we proved:
$\sigma(\zeta) \mathrel{\dot{\subset}} \{(\omega, u) \mid \hat{\zeta}_u(\omega) = 0\}$. Since on the other hand clearly
$\sigma(\zeta) \mathrel{\dot{\supset}} \{(\omega, u) \mid \hat{\zeta}_u(\omega) = 0\}$, we have

$$\sigma(\zeta) \doteq \{(\omega, u) \mid \hat{\zeta}_u(\omega) = 0\}\ .$$

If ζ is a visible measure on $U \times K$ we find

$$\sigma(\zeta) \doteq \{(\omega, u) \mid \hat{\zeta}_u(\omega) = \underline{0}\}\ .$$

(cf. §5.3 remark 6). □

We conclude this section with some more general results on random measures.

By definition a *marked process* is an $\mathcal{A} \times \mathcal{B} \times \mathcal{K}$-measurable function on $\Omega \times U \times K$. A marked process is called visible if it is $\mathcal{Z} \times \mathcal{K}$-measurable.

Now we introduce the *Doléans, Campbell or master measure* C_ρ of a random measure ρ : If ρ is a random measure on U (resp. on $U \times K$), then we define the measure C_ρ on $\Omega \times U$ (resp. on $\Omega \times U \times K$) by:

$$(5.4.4) \qquad C_\rho(Y) = E \int Y \, d\rho$$

for all non-negative processes Y (resp. for all non-negative marked processes Y). This notation will be used throughout. We already encountered the Doléans measure in definition 5.4.3 iii).

It can easily be checked that if two random measures have the same Campbell measure, then they are a.s. equal.

<u>THEOREM 5.4.4</u> *Let* ρ *and* τ *be two random measures on* U *(resp. on* $U \times K$*). Then the following three conditions are equivalent:*

i) $\rho << \tau$ *a.s.;*

ii) $C_\rho << C_\tau$*;*

iii) there exists a process (resp. a marked process) X *such that* $\rho = X\tau$ *a.s., i.e. for almost all* ω *we have* $\rho_\omega = X(\omega)\tau_\omega$

Furthermore, if ρ *and* τ *are visible, then* X *can be chosen to be visible, and conversely, if* X *can be chosen visible, and if* τ *is visible, then* ρ *has to be visible too.*

PROOF. Note that X is non-negative. The implications i) ⇒ ii) and iii) ⇒ i) are trival. If X is visible and τ is a visible measure, then the visibility of ρ follows from definition 5.4.3 iii) and §5.3 remark 3.

To proof ii) ⇒ iii), take $X = \frac{dC_\rho}{dC_\tau}$ and check that with this choice we do have $\rho = X\tau$ a.s.. If ρ and τ are visible, then by definition 5.4.3 iii) the measures C_ρ and C_τ are already determined if we only consider visible Y in formula (5.4.4); hence, then C_ρ and C_τ may be considered to be measures on $(\Omega \times U, \mathcal{Z})$ (resp. on $(\Omega \times U \times K, \mathcal{Z} \times \mathcal{K})$) so that X may be chosen to be visible. □

Note that if ρ and τ are visible random measures and $\rho = X\tau$ a.s., this does not necessarily imply that the (marked) process X is visible.

§5.5. The (dual) visible projection of random measures.

In this section the crucial notion of the dual visible projection of random measures ρ on U and on $U \times K$ is to be defined. Remember that in any case the filtration $(F(B))$ is generated by μ and we hence obtain the visible projection of an arbitrary random measure ρ w.r.t. the random measure μ.

We shall assume throughout that random measure ρ on U (resp.on $U \times K$), which are to be projected, have the following property: There exists a countable number of disjoint visible sets A_i such that $\bigcup_i A_i = \Omega \times U$ and such that $C_\rho(A_i) < \infty$ (resp. $C_\rho(A_i \times K) < \infty$).

The random measures ξ and μ itself have this property[1]: indeed, the family $(H(V))_{V \in U}$ satisfies all above conditions except that the sets are not disjoint; they can easily be made disjoint without loosing their visibility and we obtain the family:

$$\left(H(V) \cap [\bigcup_{\substack{B \in U \\ B \supset V}} H(B)]^c\right)_{V \in U} .$$

<u>THEOREM 5.5.1.</u> *Let* ρ *be a random measure on* U *; then there exists an a.s. unique visible measure* ρ^z *on* U *such that we have:*

(5.5.1) $\quad E \int X\, d\rho = E \int X d\rho^z$

for all visible non-negative processes X *; or equivalently* $C_\rho(A) = C_{\rho^z}(A)$ *for all* $A \in Z$ *. The random measure* ρ^z *is called the (dual) visible projection of* ρ *.*

PROOF. Let C_i^z be the measure on $(\Omega \times U,\ A \times B)$ which is defined for all i by $C^z_i(X) = E \int 1_{A_i}\, {}^zX d\rho$ for all non-negative processes X. If the set $A \in A \times B$ is evanescent, we have ${}^z1_A = 1_A$; hence $C^z_i(. \times V) << P$ for all i and $V \in B$. Using the same methode as the one by which the existence of conditional distributions on Polish spaces is proved (cf. e.g. Bauer (78) §56) we may show that for all i there exists an a.s. unique random measure ρ_i^z on (U,U) defined by desintegration:

$$\rho_i^z\ (V) = \frac{d\ C_i^z\ (. \times V)}{dP} .$$

1) If μ is a non-simple or compound point process described in the usual way (i.e. not by using marks), then this assumption becomes a supplementary requirement.

The unique extension of ρ_i^z on B will still be denoted by ρ_i^z. We write $\rho^z = \sum_i \rho_i^z$; thus we have $E \int {}^z 1_{F\times V}\, d\rho = E \int 1_{F\times V}\, d\rho^z$ for all $F \in A$, $V \in U$ and hence $E \int {}^z X d\rho = E \int X d\rho^z$ for all non-negative processes X. Definition 5.4.3 iii) immediately yields the visibility of the random measure ρ^z . □

THEOREM 5.5.2. *Let* ρ *be a random measure on* U *and* ρ^z *its visible projection; then:*

i) $\rho^z(Z) = E(\rho(Z) \mid F(Z))$ *a.s. for all visible points* Z .

ii) $\sigma(\rho^z) \doteq \sigma(\rho)$.

iii) $E(\rho^z(V) \mid F(V)) = E(\rho(V) \mid F(V))$ *a.s. for all* $V \in B$. *If* ρ *is integrable*[1]*, this property determines* ρ^z *among the visible random measures; it even is enough to verify it for all* $V \in U$.

iv) $\hat{\rho}^z \doteq {}^z\hat{\rho}$ [2].

PROOF.

i) Both members are $F(Z)$-measurable. If $F \in F(Z)$, then $A = (F \times U) \cap \lceil Z \rfloor \in Z$ (lemma 5.3.2) and hence:

$$E\rho^z(Z)\, 1_F = E \int 1_A(.,\, u)\rho^z(du) = E \int 1_A(.,\, u)\rho(du) = E\rho(Z)\, 1_F \, .$$

ii) This follows immediately from i).

iii) Use the fact that if $V \in B$ and $F \in F(V)$, then $F \times V \in Z$ and on the other hand the fact that Z is generated by $\{F \times V \mid V \in U,\ F \in F(V)\}$ (lemma 5.2.1).

iv) This follows from i) and definition 5.4.3 iii). □

THEOREM 5.5.3. *Let* ρ *be a random measure on* $U \times K$*; then there exists an a.s. unique visible random measure* ρ^z *on* $U \times K$ *such that we have*

(5.5.2) $\quad E \int X \times 1_D d\rho = E \int X \times 1_D d\rho^z$

for all visible non-negative processes X *and all* $D \in K$*; or equivalently* $C_\rho(A) = C_{\rho^z}(A)$ *for all* $A \in Z \times K$.

1) Integrable means: $E\rho(B) < \infty$ for all bounded $B \in B$. In fact in theorem 5.5.2 iii) $E\rho(B) < \infty$ for all $B \in U_I$ is already a sufficient condition as will be clear from the rest of the statement.
2) For convenience we write $\hat{\rho}^z$ rather than $(\rho^z)^\wedge$.

The random measure ρ^z *is called the (dual) visible projection of* ρ .

PROOF. On $\Omega \times U \times K$ resp. on $\Omega \times U$ we define the measures C^z and $\tilde{C}^z$ by

$$C^z(X \times 1_D) = E \int {}^zX \times 1_D d\rho \quad \text{and} \quad \tilde{C}^z(X) = E \int {}^zX \times 1_K d\rho$$

for all non-negative processes X and $D \in \mathcal{K}$. Now it can be proved that there exists a transition measure n from $(\Omega \times U, \mathcal{Z})$ on $(K, \mathcal{K})$ such that

$$C^z(d\omega, du, dk) = \tilde{C}^z(d\omega, du)\, n\,(\omega, u, dk)$$

(cf. the proof of the existence of ρ^z in the proof of theorem 5.5.1). Let $\tilde{\rho}^z$ denote the visible projection of $\rho(. \times K)$, which of course is a random measure on U; i.e.:

$$E \int {}^zX \times 1_K\, d\rho = E \int X d\tilde{\rho}^z \;;$$

then it can be seen that $\rho^z(du, dk) = \tilde{\rho}^z(du)\, n\,(., u, dk)$ is the required visible projection of ρ . Indeed: the fact that the measure ρ^z defined in this manner is visible, may be shown using definition 5.4.3 iii) because of all $D \in \mathcal{K}$ the mapping $(\omega, u) \to n(\omega, u, D)$ is visible; furthermore, again using the visibility of $n(., ., D)$, it becomes clear that ρ^z satisfies (5.5.2). To prove uniqueness, note that:

$$E\, 1_F \rho^z(V \times D) = E \int 1_{F\times V\times D}\, d\rho^z = E \int {}^z1_{F\times V} \times 1_D d\rho$$

for all $F \in \mathcal{A}$, $V \in \mathcal{U}$, and D in a countable semi-ring generating $\mathcal{K}$. □

Remarks.

1. Theorem 5.5.3 could have been proved without using the result of theorem 5.5.1 by direct desintegration of C^z w.r.t. ω . Later on we shall deduce an expression for the kernel n in the case $\rho = \mu$ (theorem 7.2.1).

2. Note that in the proof of theorem 5.5.2 we have

$$C^z = C_{\rho^z} \quad \text{and} \quad \tilde{C}^z = C_{(\rho(.\times K))^z} \,.$$ □

THEOREM 5.5.4. *Let* ρ *be a random measure on* $U \times K$ *and* ρ^z *its visible projection, then:*

i) $\hat{\rho}_Z(D) = E(\hat{\rho}_Z(D)) \mid F(Z))$ *a.s. for all visible points* Z *and all* $D \in K$.

ii) $\sigma(\rho^z) \doteq \sigma(\rho)$.

iii) $E(\rho^z(V\times D) \mid F(V)) = E\ (\rho(V\times D) \mid F(V))$ *a.s. for all* $V \in B$ *and* $D \in K$. *If* $\rho(. \times K)$ *is integrable, this property determines* ρ^z *among the visible random measures on* $U \times K$*; it is even enough to verify it for all* $V \in U$ *and* D *in a countable semiring generating* K .

iv) $\hat{\rho}^z(D) \doteq {}^z\hat{\rho}(D)$ *for all* $D \in K$.

v) $(\rho(.\times D))^z(B) = \rho^z(B\times D)$ *for all* $B \in B$, $D \in K$ *a.s.*.[1)]

PROOF. For i), ii), iii) and iv) see the proof of theorem 5.5.2. In v) both members are equal to $\int_B n\ (., u, D)\ \tilde{\rho}(du)$ (cf. the proof of theorem 5.5.3). □

This concludes the part devoted to the existence of the visible projection. In chapter 7 we shall see some explicit expressions for it. In chapter 10 the visible projection will be calculated in some concrete cases.

1) For all $D \in K$ the expression $\rho(. \times D)$ is a random measure on U , which may be projected according to theorem 5.5.1.

Chapter 6

THE CONDITIONS (σ) AND (Σ)

In this chapter we discuss some regularity conditions for point processes. Papangelou (74), who was the first to study the object that we call the dual visible projection, was already interested in requirements on P guaranteeing ξ^z to be diffuse. For that purpose he needed two conditions (Σ) and (Σ*).

It turns out that (Σ) is very fundamental; without this condition many things simply do not hold, for instance: Papangelou kernels (see §8.2) can only be defined when (Σ) is satisfied. Condition (Σ*) is less important, it only makes things smoother.

We shall introduce a condition (σ), which is equivalent to (Σ) and (Σ*) together. Our definition of (σ) and (Σ) will be through the visible sets σ and Σ respectively, which can be defined for every point process.

Much work in this chapter has only been done to indicate the link with results known in the literature and to simplify the study of examples. For theoretical purposes only the definition of Σ, (Σ), σ and (σ), and the theorems 6.1.1, 6.1.2, 6.1.4 i) and ii), 6.2.1 and 6.2.2 are important.

Related matters are found not only in Papangelou (74) §3 but also in Kallenberg (78) theorem 2.2, Kallenberg (83) chapter 13 and Rauchenschwandtner (80).

From this chapter on it is essential that conditional expectations of functions of random measures are considered as expectations w.r.t. a conditional distribution (cf. appendix C).

§6.1. The set Σ and the condition (Σ).

In §5.3 we defined the set σ . The following theorem is to be compared with lemma 5.3.4.

<u>THEOREM 6.1.1.</u> *There exists a visible set Σ such that:*

- *the set Σ^c is the union of graphs of a finite or countable number of visible points.*

- *for all visible points* Z *with* $\lceil Z \rfloor \subset \Sigma$ *and* $P(\pi\lceil Z \rfloor) > 0$ *we have* $P(\mu(Z) = 1) < P(\pi\lceil Z \rfloor)$, *and*
- *for all visible points* Z *with* $\lceil Z \rfloor \subset \Sigma^c$ *we have* $P(\mu(Z) = 1) = P(\pi\lceil Z \rfloor)$.

PROOF. First we suppose that U is bounded. Let V again denote the class of all visible points. If the set $\{Z \in V \mid P(\mu(Z) = 1) = P(\pi\lceil Z \rfloor) > \frac{1}{2}\}$ is not empty, we choose an element Z_1^1 in it. Then we choose an element Z_2^1 in $\{Z \in V \mid \lceil Z \rfloor \cap \lceil Z_1^1 \rfloor = \emptyset,\ P(\mu(Z) = 1) = P(\pi\lceil Z \rfloor) > \frac{1}{2}\}$ and so on. Since $P(\mu(U \times K) < \infty) = 1$ (U being bounded), there exists a finite number n_1 such that $\{Z \in V \mid \lceil Z \rfloor \cap [\bigcup_{i=1}^{n_1} \lceil Z_i^1 \rfloor] = \emptyset,\ P(\mu(Z) = 1) = P(\pi\lceil Z \rfloor) > \frac{1}{2}\} = \emptyset$.

Now we apply the same procedure to $\{Z \in V \mid \lceil Z \rfloor \cap [\bigcup_{i=1}^{n_1} \lceil Z_i^1 \rfloor] = \emptyset,\ P(\mu(Z) = 1) = P(\pi\lceil Z \rfloor) > (\frac{1}{2})^2\}$ and find $Z_1^2, \ldots, Z_{n_2}^2$; etc.

The set $(\bigcup_i \bigcup_j \lceil Z_i^j \rfloor)^c$ satisfies the conditions imposed on Σ .

If the space U is not bounded, it is the countable union of bounded sets. □

Remark: The set Σ could also have been defined in terms of ξ, because $\xi(Z) = \mu(Z)$ for all random points Z . □

EXAMPLE 6.1.1. We use example 5.3.1. In the corresponding figure 5.3.1 the set Σ^c consists of the solid oblique line segments as far as they are contained in the lower half of the square; i.e.: $\Sigma^c = \{(\omega, u) \mid u = \omega \bmod \frac{1}{2}\}$. □

Note that we may choose a version of ξ^z such that $\hat{\xi}^z \leq 1$ identically. This follows immediately from theorem 5.5.2 i).

THEOREM 6.1.2. $\Sigma^c \doteq \{(\omega, u) \mid \hat{\xi}_u^z(\omega) = 1\}$.

PROOF. If Z is a visible point whose graph is contained in Σ^c, it follows from theorem 5.5.2 i) that

$$\xi^z(Z) = E(\xi(Z) \mid F(Z)) = 1 \quad \text{a.e. on } \pi\lceil Z \rfloor .$$

On the other hand, the set $\{(\omega, u) \mid \hat{\xi}_u^z(\omega) = 1\} - \Sigma^c$ is visible.

If it were not evanescent, it would have a visible section Z with non-evanescent graph (theorem 5.2.3), which is impossible, since from $E(\xi(Z) \mid F(Z)) = \xi^Z(Z) = 1$ a.e. on $\pi\lceil Z \rfloor$ it would follow that $\xi(Z) = 1$ a.e. on $\pi\lceil Z \rfloor$. □

We shall be able to express Σ in terms of the following events $\Sigma(V)$. For all $V \in U$ we define:

$$\Sigma(V) = \{P(\xi(V) = 0 \mid F(V)) \neq 0\} .$$

Of course, $\Sigma(V) \in F(V)$. We note some more properties:
We may assume:

(6.1.1) $\quad \Sigma(V)^c \subset \{\xi(V) \neq 0\}$ for all $V \in U$.

(6.1.2) $\quad \Sigma(V)^c \cap \{\xi(V-W) = 0\} = \Sigma(W)^c \cap \{\xi(V-W) = 0\}$

for all $V, W \in U$, $W \subset V$.

The latter formula follows from theorem C.2; the first is evident.

THEOREM 6.1.3. $\quad \Sigma \doteq \lim_i \bigcup_{V \in U_i} \Sigma(V) \times V.$

PROOF. We first prove the existence of the limit. Because $(\bigcup_{V \in U_i} \Sigma(V) \times V)^c = \bigcup_{V \in U_i} \Sigma(V)^c \times V$, this means that we have to show that the set $\bar{A} \equiv \limsup \bigcup_{V \in U_i} \Sigma(V)^c \times V$ and $\underline{A} \equiv \liminf \bigcup_{V \in U_i} \Sigma(V)^c \times V$ are indistinguishable. Let $(\omega,u) \in \bar{A}$ and let $W_i \downarrow \{u\}$, $W_i \in U_i$; then there exists a number k such that $(\omega,u) \in H(W_k)$ and there exists a number $\ell \geq k$ such that $\omega \in \Sigma(W_\ell)^c$; but now formula (6.1.2) yields $\omega \in \Sigma(W_i)^c \ \forall\ i \geq k$ and hence $(\omega, u) \in \underline{A}$.

The set $A \equiv \bar{A} \doteq \underline{A}$ being visible, the set $A \cap H(V)$ is visible too for each $V \in U$. Moreover $A \cap H(V)$ is the graph of a random point Z_V:

$$Z_V : \omega \to \begin{cases} u \in V & \text{if } \xi_\omega(V-\{u\}) = 0,\ (\omega, u) \in A, \\ \\ \Delta & \text{if not.} \end{cases}$$

a.s. (The Z_V are well-defined since (6.1.1) implies that $A \subset \{(\omega, u) \mid \hat{\xi}_u(\omega) = 1\}$ and for each ω there is at most one point $u \in V$ such that $\xi_\omega(V-\{u\}) = 0$ and $\hat{\xi}_u(\omega) = 1)$. The Z_V are visible by definition. Because $\{\xi(Z_V) = 1\} = \pi\lceil Z_V \rfloor$ and $\bigcup_{V \in U} \lceil Z_V \rfloor = A$, we may conclude $A \subset \Sigma^c$.

The equality $A = \Sigma^c$ may be seen rather easily by means of theorem 7.2.2 (Corollary 7.2.4). □

We now introduce the regularity condition (Σ).

$$(\Sigma) \qquad P(\pi(\Sigma^c)) = 0 .$$

In the next theorem we shall give some equivalent ways to formulate condition (Σ) . This condition is very important in our context. Intuitively visibility has something to do with interaction between the points of the point process. For instance, in the case of the Poisson process the points are independent (not interacting) and therefore the visible projection of the Poisson process will turn out to be non-random (§10.3; cf. also §4.1): the points do not influence each other and the Poisson process satisfies (Σ). However, if (Σ) is not satisfied, something else comes in: " A point of a realization may be visible because otherwise it would be missing in the realization"; this is not a question of interaction.

Using the language of statistical mechanics (cf. §2.2): Consider an ideal gas in the grand canonical ensemble (Poisson process) then the visible projection of the process is non-random because the particles do not interact. Now consider an ideal gas in the canonical ensemble (non-mixed sample process; §10.6); here condition (Σ) is not satisfied; the realization of the process is visible itself (§10.6 remark 3^c), not because the particles interact, but because their number is fixed.

If (Σ) is not satisfied, things become visible which we do not "want" to be visible and visibility says considerably less about the distribution of the process (cf. §9.2). If (Σ) is satisfied, these problems are avoided. Note that they come from the augmentation of the filtration $(\mathcal{F}(B))$ by all null-sets.

The fact that under condition (Σ) it is less dangerous for us to rely on our intuition (in the above context) is expressed by the equivalent conditions ii) and vi) in the following theorem. Papangelou (74) who was the first to introduce (Σ), used formulation iii). For more formulations of (Σ) see Rauchenschwandtner (80) §1.2.

<u>THEOREM 6.1.4</u>. *The following statements are equivalent:*

i) Condition (Σ) *holds.*

ii) For all visible processes X *and all* $\mathcal{M} \times \mathcal{B}$*-measurable functions*

X^* on $M \times U$, such that $X \doteq X^*(\mu)$ we have: $X \doteq {}^{-}X$, where ${}^{-}X$ is the visible process defined by ${}^{-}X : (\omega,u) \to X^*_u(\{u\}^c\mu_\omega)$.

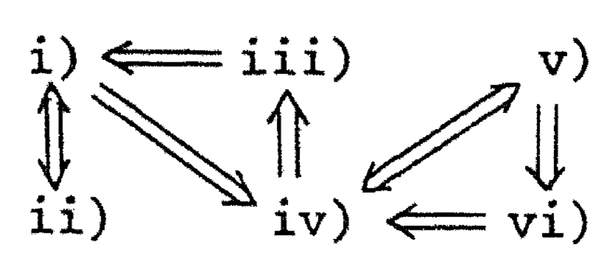
iii) For all $V \in U$ we have $P(\xi(V) = 0 \mid F(V)) > 0$ a.e. on $\{\xi(V) = 1\}$.

iv) For all $V \in U$ we have $P(\xi(V) = 0 \mid F(V)) > 0$ a.s..

v) For all $V \in U$ and $G \in F(V)$ with $P(G) > 0$ we have $P(G \cap \{\xi(V) = 0\}) > 0$.

vi) For all $V \in U$ and all $\tilde{F} \in M$ such that $\{\mu \in \tilde{F}\} \in F(V)$, we have $P(\{\mu \in \tilde{F}\} \,\Delta\, \{V^c\mu \in \tilde{F}\}) = 0$.

PROOF. We shall show the following implications:



(which is even more than necessary).

The implication "iv) ⇒ iii)" and the equivalence "iv) ⇔ v)" are obvious.

Proof "i) ⇒ ii)": Suppose that ii) is not satified; then there exists a visible process X and an $M \times B$-measurable function X^* with $X \doteq X^*(\mu)$ such that $P(\pi(A)) \neq 0$, where

$A = \{(\omega,u) \mid X^*_u(\mu_\omega) \neq X^*_u(\{u\}^c\mu_\omega)\} \doteq \{(\omega, u) \mid X_u(\omega) \neq {}^{-}X_u(\omega)\}$.

Of course A is visible. Let Z be a visible section of A with non-evanescent graph (theorem 5.2.3). Since clearly: $A \subset \{(\omega,u) \mid \hat{\xi}_u(\omega) = 1\}$ we have $\xi(Z) = 1$ a.e. on $\pi\lceil Z\rfloor$ wich contradicts (Σ).

Proof "ii) ⇒ i)": Suppose that Z is a visible point such that $P(\xi(Z) = 1) = P(\pi\lceil Z\rfloor)$. We then may assume $\{\xi(Z) = 1\} = \pi\lceil Z\rfloor$ hence ${}^{-}1_{\lceil Z\rfloor} = 0$ and because ii) implies ${}^{-}1_{\lceil Z\rfloor} \doteq 1_{\lceil Z\rfloor}$ we find $P(\pi\lceil Z\rfloor) = 0$.

Proof "iii) ⇒ i)": Condition iii) implies that $P(\Sigma(V)^c \cap \{\xi(V) = 1\}) = 0$ for all $V \in U$. Combining this fact with formula (6.1.1) and theorem 6.1.3 we find

$$\Sigma^c \doteq \lim_i \bigcup_{V \in U_i} \Sigma(V)^c \times V \,\dot{\subset}\, \lim_i \bigcup_{V \in U_i} \{\xi(V) = 1\} \times V \,\dot{\subset}\, \lim_i \bigcup_{V \in U_i} \Sigma(V) \times V \doteq$$

$\doteq \Sigma$ and this implies $\Sigma^c \doteq \emptyset$ which completes this part of the proof.

Proof "i) ⇒ iv)": Suppose iv) does not hold. Then there exists a $V \in U$, an $F \in F(V)$ and a number $m > 0$ such that $P(F) > 0$ and such that we have $P(\xi(V) = m - 1 \mid F(V)) = 0$ a.e. on F and $P(\xi(V) = m \mid F(V)) \neq 0$ a.e. on F. We may of course assume that $F \cap \{\xi(V) = m - 1\} = \emptyset$; but then:

$$A \equiv \{(\omega,u) \in F \times V \mid \xi_\omega(V-\{u\}) = m-1,\ \hat{\xi}_u(\omega) = 1\} =$$

$$= \{(\omega,u) \in F \times V \mid \xi_\omega(V-\{u\}) = m-1\}\ ,$$

which is clearly visible. Hence

$$E \int 1_A \, d\xi^z = E \int 1_A \, d\xi = m.P(F \cap \{\xi(V) = m\})\ .$$

This implies that $\hat{\xi}^z \doteq 1$ on A which contradicts (Σ) since now

$$P(\pi(\Sigma^c)) \geqq P(\pi(A)) = P(F \cap \{\xi(V) = m\}) > 0\ .$$

Proof. "v) ⇒ vi)": it is easy to check that if $V \in U$ and $\tilde{F} \in M$ such that $\{\mu \in \tilde{F}\} \in F(V)$, then $G \equiv \{\mu \in \tilde{F}\} \Delta \{V^c\mu \in \tilde{F}\} \in F(V)$ and $G \subset \{\xi(V) = 0\}$, hence $P(G \cap \{\xi(V) = 0\}) = 0$, which implies $P(G) = 0$.

Proof. "vi) ⇒ iv)": Suppose that iv) is not satisfied. Then there exists a $V \in U$ such that $P(F) > 0$, where $F = \{P(\xi(V) = 0 \mid F(V)) = 0\}$. Now take $\tilde{F} = F^* \cap \{\mu \mid \xi\ (V) \neq 0\}$. Because $F \cap \{\xi(V) = 0\}$ is a null-set, we have $\{\mu \in \tilde{F}\} \in F(V)$. On the other hand $P(V^c\mu \in \tilde{F}) = 0$; hence $P(\{\mu \in \tilde{F}\} \Delta \{V^c\mu \in \tilde{F}\}) = P(\mu \in \tilde{F}) = P(F) \neq 0$, contradicting vi). □

In some examples we shall need an assertion which is slightly more general than theorem 6.1.4.

COROLLARY 6.1.5. *Let* $B \in U$ *and* $D \in F(B)$. *Then the following statements are equivalent:*

i) $P(\pi(\Sigma^c \cap (D \times B))) = 0$

ii) *For all visible processes* X *and all* $M \times B$*-measurable functions* X^* *on* $M \times U$ *such that* $X \doteq X^*(\mu)$ *we have* $X\, 1_{D\times B} \doteq \bar{X}\, 1_{D\times B}$, *where* $\bar{X}$ *is the visible processes defined by* $\bar{X}: (\omega,u) \to X^*_u(\{u\}^c\mu_\omega)$.

iii) *For all* $V \in U$, $V \subset B$ *we have* $P(\xi(V) = 0 \mid F(V)) > 0$ *a.e. on* $\{\xi(V) = 1\} \cap D$.

iv) *For all* $V \in U$, $V \subset B$ *we have* $P(\xi(V) = 0 \mid F(V)) > 0$ *a.e. on* D.

v) *For all* $V \in U$, $V \subset B$ *and all* $G \in F(V)$ *with* $P(G \cap D) > 0$, *we have* $P(G \cap \{\xi(V) = 0\}) > 0$.

vi) *For all* $V \in U$, $V \subset B$ *and all* $\tilde{F} \in M$ *such that* $\{\mu \in \tilde{F}\} \in F(V)$ *we have* $P((\{\mu \in \tilde{F}\} \Delta \{V^c\mu \in \tilde{F}\}) \cap D) = 0$.

PROOF. We can easily adapt the proof of theorem 6.1.4. □

§6.2. The set σ and the conditions (σ) and (Σ^*).

We recall the definition of the set σ and hence restate lemma 5.3.4 in the case $\rho = \mu$.

THEOREM 6.2.1. *There exists a visible set* σ *such that:*

- *the set* σ^c *is the union of graphs of a finite or countable number of visible points,*
- *for all visible points* Z *with* $\lceil Z \rfloor \subset \sigma$ *we have* $P(\mu(Z) \neq 0) = 0$, *and*
- *for all visible points* Z *with* $\lceil Z \rfloor \subset \sigma^c$ *and* $P(\pi\lceil Z \rfloor) \neq 0$ *we have* $P(\mu(Z) \neq 0) \neq 0$.

PROOF. See lemma 5.3.4. □

The set σ has properties analogous to those of the set Σ.

THEOREM 6.2.2. $\sigma^c \doteq \{(\omega, u) \mid \hat{\xi}^z_u\ (\omega) \neq 0\}$.

PROOF. Use theorem 5.5.2 ii) and §5.4 remark 3 in order to see that $\sigma^c = \sigma^c(\xi^z) = \{(\omega,u) \mid \hat{\xi}^z_u\ (\omega) \neq 0\}$. □

For σ we find an explicit expression in terms of the sets $\Sigma(V)$ and the sets $\Sigma^*(V)$ which we defined through:

$$\Sigma^*(V)^c = \{(\omega,u) \mid u \in V,\ P(\hat{\xi}_u = \xi(V) = 1 \mid F(V))\ (\omega) > 0\}.$$

This set is an element of $F(V) \times B$ since

$$\Sigma^*(V)^c = \lim_k \lim_i \bigcup_{W \in U_{i,V}} \{P(\xi(W) = \xi(V) = 1 \mid F(V)) > \tfrac{1}{k}\} \times W .$$

THEOREM 6.2.3. $\sigma^c \doteq \lim_i \bigcup_{V \in U_i} (\Sigma(V)^c \times V) \cup \Sigma^*(V)^c$.

PROOF. We have to show that the limit exists and that is equal to σ^c. It is clear (cf. theorem 6.1.3) that we have limsup $\supset$ liminf $\supset \Sigma^c$. Hence we choose an $(\omega,u) \in \bar{\Sigma}$ (There may be an evanescent set on which the following arguments are not applicable). Let $W_i \downarrow \{u\}$, $W_i \in U_i$; then there exists a smallest number k such that $(\omega,u) \in H(W_k)$ and $\omega \in \Sigma(W_k)$. From corollary 7.2.3 we now may deduce the following equivalence:

$$\exists\, i \geqq k \text{ such that } (\omega,u) \in \Sigma^*(W_i)^c \Longleftrightarrow$$

$$\Longleftrightarrow \hat{\xi}^z_u(\omega) \neq 0 \Longleftrightarrow (\omega,u) \in \Sigma^*(W_i)^c \ \forall\, i \geqq k \, .$$

This completes the proof. □

We now introduce the condition (σ) analogous to condition (Σ) by:

(σ) $\qquad P(\pi(\sigma^c)) = 0$.

Of course (σ) is equivalent to "ξ^z is almost surely diffuse" (theorem 6.2.2). Papangelou (74) needed two conditions to ensure that ξ^z is diffuse; the condition (Σ) , which is already mentioned, and:

(Σ^*) $\qquad$ For all $V \in \mathcal{U}$ we have a.s.: $P(\hat{\xi}_u \neq 0 \mid \mathcal{F}(V)) = 0$ for all $u \in V$.

We now have:

<u>THEOREM 6.2.4.</u> *The following statements are equivalent:*

i) Condition (σ) *holds.*

ii) The sets $\Sigma(V)^c$ *and* $\Sigma^*(V)^c$ *may be taken to be empty for all* $V \in \mathcal{U}$.

iii) The conditions (Σ) *and* (Σ^*) *hold.*

PROOF. The following implications are obvious: "iii) ⇒ ii)", "ii) ⇒ i)" (use theorem 6.2.3) and "$(\sigma) \Rightarrow (\Sigma)$" (compare their definitions). We only need to prove that $(\sigma) \Rightarrow (\Sigma^*)$.

We see that (Σ^*) is equivalent to the requirement that for all $V \in \mathcal{U}$ the set

$$I(V) = \{(\omega,u) \mid u \in V,\ P(\hat{\xi}_u \neq 0 \mid \mathcal{F}(V))(\omega) > 0\}$$

is evanescent. Furthermore, we may write

$$I(V) = \bigcup_{k=1}^{\infty} I_k(V),$$

where

$$I_k(V) = \lim_i \bigcup_{W \in \mathcal{U}_{i,V}} \{P(\xi(W) \neq 0 \mid \mathcal{F}(V)) > \tfrac{1}{k}\} \times W \in \mathcal{F}(V) \times \mathcal{B} \, .$$

Suppose now that (Σ^*) is not satisfied. Then there exists a $V \in \mathcal{U}$

such that $P(\pi(I(V))) > 0$ and hence there exists a number k such that $P(\pi(I_k(V))) > 0$. Because $I_k(V) \in F(V) \times B$, there exists – according to theorem A.1 – a mapping $Z : (\Omega, F(V)) \to V \cup \{\Delta\}$ such that $\lceil Z \rfloor \subset I_k(V)$ and $P(\pi\lceil Z \rfloor) = P(\pi(I_k(V))) > 0$. Of course Z is a visible point and we have:

$$P(\xi(Z) > 0) = P\,[\lim_i \bigcup_{W \in U_i} (\{\xi(W) > 0\} \cap Z^{-1}(W))]$$

$$= E \lim_i \sum_{W \in U_i} 1_{Z^{-1}(W)} P(\xi(W) > 0 \mid F(W)) \geqq$$

$$\geqq \frac{1}{k} P(\pi\lceil Z \rfloor) > 0 ,$$

which contradicts (σ). □

In the preceding proof the sets I(V) were defined. It is clear that $\Sigma^*(V)^c \subset I(V)$; in general the inclusion is strict as is shown by the following example.

EXAMPLE 6.2.1. We choose $U = [0,\frac{1}{2})$, $\Omega = (0,1]$ with Lebesgue measure and μ is the simple non-marked point process given by

$$\xi_\omega = \begin{cases} \varepsilon_0 + \varepsilon_\omega & \text{if } \omega < \frac{1}{2} , \\ \underline{0} & \text{if } \omega \geqq \frac{1}{2} . \end{cases}$$

Take $W_i = [0, 2^{-i})$ ($i \geqq 2$) then $[\frac{1}{2}, 1] \subset \Sigma(W_i)$ and $[\frac{1}{2}, 1] \times \{0\} \subset \Sigma^*(W_i)$ for all i but on $[\frac{1}{2}, 1]$ we have $P(\hat{\xi}_0 = 1 \mid F(W_i)) = 1/(1+2^{i-1}) > 0$, hence $[\frac{1}{2}, 1] \times \{0\} \subset I(W_i)$ for all i. □

None the less we have the following result:

THEOREM 6.2.5. *Condition* (Σ^*) *holds if and only if the set* $\Sigma^*(V)^c$ *is evanescent for all* $V \in U$.

PROOF. The implication "⇒" is obvious. Assume conversely that (Σ^*) does not hold; then (cf. the proof of theorem 6.2.4) there exists a set $V \in U$ and a mapping $Z : (\Omega, F(V)) \to V \cup \{\Delta\}$ with $P(\pi\lceil Z \rfloor) > 0$ such that $P(\xi(Z) = 1 \mid F(V)) > 0$ a.e. on $\pi\lceil Z \rfloor$. Thanks to (5.1.1) there exists a set $W \in U$, $W \subset V$ such that $P(\pi(H(W) \cap \lceil Z \rfloor)) > 0$; define Z' by

$\lceil Z' \rfloor = H(W) \cap \lceil Z \rfloor$. Because $\Sigma^*(W)^c$ is evanescent we have a.s.:
$0 = P(\xi(Z') = \xi(W) = 1 \mid \mathcal{F}(W)) = P(\xi(Z') = 1 \mid \mathcal{F}(W))$ and this yields a contradiction because $P(\xi(Z') = 1 \mid \mathcal{F}(V)) = E(P(\xi(Z') = 1 \mid \mathcal{F}(W)) \mid \mathcal{F}(V))$.

□

§6.3. The condition (Σ_ν).

We mention one more regularity condition. Let ν be a Radon measure on U . Then

(Σ_ν) $\qquad \xi^z \ll \nu$ a.s. .

Under this condition the visible projection of ξ can of course be described by its density w.r.t. ν . Matthes, Warmuth and Mecke (79) introduced this kind of regularity condition. (In the definition of their condition (Σ'_ν) furthermore they require that (Σ) is satisfied. To see that apart from this their (Σ'_ν) is equivalent to our (Σ_ν) one needs their Satz 3.1 and our theorem 8.2.2 .

If ν is diffuse, then (Σ_ν) of course implies (σ) . But if (σ) is satisfied there need not exist a Radon measure ν such that (Σ_ν) holds as is seen by the following counterexample.

<u>EXAMPLE 6.3.1.</u> Take $U = (0,2) \times (0,2)$ and let μ be the single non-marked point process on U , whose distribution is given by:

$$\xi = X\,\varepsilon_{Z,Y} + X'\,\varepsilon_{Z+1,Y'} ,$$

Where X, X', Y, Y' and Z are five independent r.v.'s; Y, Y' and Z have an uniform distribution on (0,1) and $P(X=0) = P(X=1) = P(X'=0) = P(X'=1) = \frac{1}{2}$.
Then:

$$\xi^z = \begin{cases} \underline{0} & \text{if } X = X' = 1, \\ \lambda^1_{Z+1} & \text{if } X = 1,\ X' = 0, \\ \lambda^1_Z & \text{if } X = 0,\ X' = 1, \\ \lambda^2 & \text{if } X = X' = 0, \end{cases} \qquad \text{a.s.}$$

where λ_z^1 denotes the one-dimensional Lebesgue measure on the vertical line segment $\{(x,y) \mid x = z, 0 < y < 1\}$ and λ^2 the two-dimensional Lebesgue measure on U. (To verify this, note that the random measure defined above is diffuse and F-measurable and hence visible, and use theorem 5.5.2 iii).).

Because ξ^z is diffuse a.s. condition (σ) is satisfied, but there does not exist a Radon measure on U which dominates λ_z^1 for all $z \in (0,2)$. □

CHAPTER 7

EXPRESSIONS FOR VISIBLE PROJECTIONS.

In the second section of this chapter the visible projections of ξ and μ are expressed in terms of conditional distributions. On the other hand, visible projections of $F \times B$-measurable processes (§1) and of random measures (§3) can be calculated when the visible projection of μ is known.

As mentioned in chapter 4 the object we call the dual visible projection of ξ, was originally defined as a limit random measure by Papangelou (74) and Kallenberg (78) and called conditional intensity. In van der Hoeven (82) §6 this conditional intensity was identified with the dual visible projection; to that end many older arguments had to be repeated. These proofs can quite easily be extended to random measures ρ on U such that $\rho \sim \xi$ (i.e. $\rho << \xi$ and $\xi << \rho$) a.s. (cf. van der Hoeven (82)). Kallenberg (83) was the first to define the conditional (w.r.t. μ) intensity of an arbitrary random measure. In §4 techniques developed in the papers we mentioned above, are combined. The results of §4 wil not be used in the sequel.

It is not surprising that Papangelou and Kallenberg found expressions for the conditional intensity, which we derive for the dual visible projection. Indeed, corollary 7.2.3 should be compared with Papangelou (74) proposition 21 and theorem 7.3.2 with Kallenberg (83) formula (14.35).

Again, note that conditional expectations are considered as expectations w.r.t. conditional distributions of random measures (cf. Appendix C).

§7.1. The visible processes ${}^zX$, ${}^-X$ and ${}^+X$.

It is possible to find an expression for the visible projection ${}^zX$ of an $F \times B$-measurable process X in terms of the dual visible projection μ^z of μ and the visible (marked) processes ${}^-X$ and ${}^+X$.

The process ${}^-X$ has already been introduced in theorem 6.1.4 ii) for visible processes. We now define for arbitrary $F \times B$-measurable processes X the visible process ${}^-X$ by:

$$ {}^-X_u(\omega) = X^*_u(\{u\}^c \mu_\omega) . $$

We also introduce the visible marked process ${}^{+}X$ corresponding to an $\mathcal{F} \times \mathcal{B}$-measurable process X , by:

$$ {}^{+}X_{u,k}(\omega) = X^*_u(\{u\}^c \mu + \varepsilon_{u,k}) \ . $$

The processes ${}^{-}X$ and ${}^{+}X$ are indeed visible; the first because:

$$ {}^{-}X_u(\omega) = \lim_i X^*_u(W_i^c \mu_\omega) \quad \text{where} \quad W_i \in \mathcal{U}_i \ , \quad W_i \ni u \ ; $$

the second because:

$$ \{(\omega, u, k) \mid X^*_u(\{u\}^c \mu_\omega + \varepsilon_{u,k}) < \alpha\} = $$

$$ = \lim_i \bigcup_{V \in U_i} \{(\omega,u,k) \mid u \in V, \ X^*_u(V^c \mu_\omega + \varepsilon_{u,k}) < \alpha\} $$

and mapping $(\omega,u,k) \to X^*_u(V^c \mu_\omega + \varepsilon_{u,k})$ is $\mathcal{F}(V) \times (\mathcal{B} \cap V) \times \mathcal{K}$-measurable on $\Omega \times V \times K$ and hence $\mathcal{Z} \times \mathcal{K}$-measurable.

Note that the process ${}^{-}X$ is defined through X^* and that X^* is only determined up to indistinguishability: $X^*(\mu) \doteq X$. As a consequence the set on which ${}^{-}X$ is not determined may in general be non-evanescent. Fortunately we now have the following result: If X^* and X^{**} are two $\mathcal{M} \times \mathcal{B}$-measurable functions on $M \times U$ such that $X^*(\mu) \doteq X^{**}(\mu)$, then the set $A = \{(\omega,u) \mid X^*_u(\{u\}^c \mu_\omega) \neq X^{**}_u(\{u\}^c \mu_\omega\} \cap \Sigma$ is evanescent. Indeed, if not, then a visible point Z would exist with $\lceil Z \rfloor \subset A$ and $P(\pi\lceil Z \rfloor) > 0$ (Visible section theorem). From the fact that $X^*_Z(\{Z\}^c\mu) \neq X^{**}_Z(\{Z\}^c\mu)$ a.e. on $\pi\lceil Z \rfloor$ it follows that $\mu(Z) = 1$ a.e. on $\pi\lceil Z \rfloor$, which contradicts the assumption $\lceil Z \rfloor \subset A \subset \Sigma$

Something similar applies to ${}^{+}X$: Up to a C_μ-equivalence ${}^{+}X$ is the unique visible marked process satisfying the two following equivalent formulae:

$$ (7.1.1) \qquad E \int Y_{u,k} X_u \, \mu(du,dk) = E \int Y_{u,k} \, {}^{+}X_{u,k} \, \mu(du,dk) $$

for all non-negative visible marked processes Y, or:

$$ (7.1.2) \qquad (X\mu)^z = {}^{+}X\mu^z \ . $$

Formula (7.1.1) even applies to arbitrary non-negative marked process Y, because on $\{(u,k) \mid \hat{\mu}_u(\{k\}) = 1\}$ we have $\mu = \{u\}^c\mu + \varepsilon_{u,k}$, hence there ${}^+X_{u,k} = X^*_u$. To prove (7.1.2) we note that the random measure ${}^+X\mu^z$ is visible (theorem 5.4.4) and check that for every non-negative visible marked process Y we have

$$E \int Y \, d(X\mu)^z = E \int Y \, (X \times 1_K) \, d\mu$$

$$= E \int Y \, {}^+X \, d\mu$$

$$= E \int Y \, {}^+X \, d\mu^z ,$$

where the final equality is a simple consequence of (5.5.2) and the monotone class theorem B.2[1].

We already saw that ${}^+X$ is defined upto C_μ-equivalence. Hence if two versions differ on a set $N \in \mathcal{Z} \times \mathcal{K}$, then $C_\mu(N) = C_{\mu^z}(N) = 0$. Because

$$C_{\mu^z}(N) = E \int_{U\times K} 1_N(.,u,k)\mu^z(du,dk) \geqq$$

$$\geqq E \int_{U\times K} 1_N(.,u,k)\hat{\xi}^z_u \, \mu^z(du,dk) =$$

$$= E \int_K \sum_u 1_N(.,u,k)\hat{\xi}^z_u \, \hat{\mu}^z(dk) =$$

$$= E \sum_u \int_K 1_N(.,u,k)\hat{\mu}^z_u(dk) \, \hat{\xi}^z_u =$$

$$= E \int_U \int_K 1_N(.u,k) \, \hat{\mu}^z_u(dk)\xi^z(du),$$

we find $C_{\xi^z}(A) = 0$ where $A = \{(\omega,u) \mid \hat{\mu}^z_u(\{k \mid (\omega,u,k) \in N\})(\omega) \neq 0\}$. Because $\sigma^c \doteq \{(\omega,u) \mid \hat{\xi}^z_u(\omega) \neq 0\}$ (theorem 6.2.2) every C_{ξ^z}-null-subset of σ^c is evanescent. In particular, the set $A \cap \sigma^c$ is evanescent. Hence, integrals of ${}^+X$ over K w.r.t. $\hat{\mu}^z$ are uniquely determined upto indistuinguishability.

1) We shall use the monotone class theorem (corollary B.2) in more proofs where ${}^+X$ occurs. It allows us to extend properties of visible marked processes of the form $X \times 1_D$, where X is a visible process and $D \in \mathcal{K}$ to general visible marked processes.

At each special instance where we encounter the process ${}^{-}X$ and ${}^{+}X$ we can check that they are determined on the sets, where they matter. Looking for example at theorem 7.1.1 (which we shall state next) we see that we are only interested in ${}^{-}X$ on the set $\{(\omega,u) \mid 1 - \hat{\xi}^z_u(\omega) \neq 0\} = \Sigma$, whereas ${}^{+}X$ only comes in through an integral over K w.r.t. $\hat{\mu}^z$. We warn the reader that in the sequel this verification will be omitted.

From §5.3 remark 4 we may easily deduce that if (σ) holds, therefore if ξ^z is diffuse a.s., we have ${}^{z}X = {}^{-}X$ for all $F \times B$-measurable processes X . This result can now be generalized as follows.

<u>THEOREM 7.1.1.</u> *Let* X *be an* $F \times B$*-measurable process; then:*

$${}^{z}X_u = (1 - \hat{\xi}^z_u)\ {}^{-}X_u + \int_K {}^{+}X_{u,k}\ \hat{\mu}^z_u\ (dk) \ .$$

(If K *reduces to one point, then we have:*

$${}^{z}X_u = (1 - \hat{\xi}^z_u)\ {}^{-}X_u + \hat{\xi}^z_u\ {}^{+}X_u\ ).$$

PROOF. The right-hand side process is visible. This is obviously true for the first term. For the second term note that $\mu^z(D)$ is visible for all $D \in K$ and use the monotone class theorem B.2.

Furthermore let Z be a visibile point; then:

$$(1 - \hat{\xi}^z_Z) = 1 - P(\xi(Z) = 1 \mid F(Z)) = P(\mu(Z) = 0 \mid F(Z))$$

and

$$\hat{\mu}^z_Z(D) = P(\hat{\mu}_Z(D) = 1 \mid F(Z))$$

for all $D \in K$, so that we may conclude (to make this argument indisputable we again need the monotone class theorem B.2).

$$E\{(1-\hat{\xi}^z_Z)\ {}^{-}X_Z + \int_K {}^{+}X_{Z,k}\ \hat{\mu}^z_Z\ (dk)\} =$$

$$= E\ {}^{-}X_Z\ P(\mu(Z) = 0 \mid F(Z)) + E \int_K {}^{+}X_{Z,k}\ P(\hat{\mu}_Z(dk) = 1 \mid F(Z)) =$$

$$= E\ {}^{-}X_Z\ 1_{\{\mu(Z)\ =\ 0\}} + E\int_K {}^{+}X_{Z,k}\ 1_{\{\hat{\mu}_Z(dk)\ =\ 1\}}$$

$$= E\ X_Z\ 1_{\{\mu(Z)\ =\ 0\}} + E\int_K X_Z\ 1_{\{\hat{\mu}_Z(dk)\ =\ 1\}}$$

$$= E\ X_Z\ . \qquad \square$$

§7.2. Explicit expressions for ξ^z and μ^z .

In §5.5 we proved the existence of the dual visible projections of ξ and μ , but we do not know how to determine them. We now derive explicit expressions by means of which we are able to calculate the measures ξ^z and μ^z almost everywhere. But we begin by studying the transition measure n which was introduced in the proof of theorem 5.5.3.

Because in this section we restrict ourselves to the case $\rho = \mu$, we may define n here by the following properties: n is a kernel: for C_{ξ^z} - almost all (ω,u) the set function $n(\omega,u,.)$ is a probability measure on $(K,\mathcal{K})$; $n(\omega,u,.)$ is C_{ξ^z}-a.e. uniquely determined and for all $D \in \mathcal{K}$ the mapping $(\omega,u) \to n(\omega,u,D)$ is visible. Finally, $\mu^z(du,dk) = n(.,u,dk)\ \xi^z(du)$.

Let $V \in \mathcal{U}$ and $D \in \mathcal{K}$. Then it is known[1)] that there exists a measurable function $g^{V,D}$ on $M \times M^{\cdot}$ such that outside a null-event we have

$$g^{V,D}(V^c\mu_\omega,\ V\xi_\omega) = P(\mu(V \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V),\ V\xi))(\omega)\ .$$

Note that we may choose $g^{V,D} = 0$ on $M \times \{\rho \in M^{\cdot} \mid \rho(V) = 0\}$, and

$$g^{V,D}\ (V^c\mu_\omega,\varepsilon_u) = P(\mu(\{u\} \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V),V\xi))(\omega)$$

on $\{V\xi = \varepsilon_u\}$, where $u \in V$. We put

$$G_u^{V,D}\ (\omega) = g^{V,D}(V^c\mu_\omega,\varepsilon_u)$$

on $\{V\xi = \varepsilon_u\}$ hence for $(\omega,u) \in H(V) \cap (\{\xi(V) = 1\} \times V)$.

Because $C_\xi(\{\xi(V) = 0\} \times V) = 0$ it follows that

1) Cf.: If X is a r.v. and A an event, then there exists a function g such that $P(A \mid X)(\omega) = g(X(\omega))$.

$$G_u^{V,D}(\omega) = P(\mu(V \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V), V\xi))(\omega)$$

for C_ξ -almost all $(\omega,u) \in H(V)$. Note that $G^{V,D}$ is a probability measure on K as a function of D.
We want to define a function $\phi(D)$ on $\Omega \times U(D \in \mathcal{K})$. Thanks to (5.1.1) it is enough to give its value $\phi_u(D)(\omega)$ for $(\omega,u) \in H(V)$ for all $V \in \mathcal{U}$. We take for $(\omega,u) \in H(V)$:

$$\phi_u(D)(\omega) = G_u^{V,D}(\omega) \ .$$

We have to show that this definition does not depend on V. To this end choose $W, V \in \mathcal{U}$, $W \subset V$ and notice that

$$\mathcal{T}(\mathcal{F}(V),\ V\xi) \cap \{\xi(V-W) = 0\} = \mathcal{T}(\mathcal{F}(W),\ W\xi) \cap \{\xi(V-W) = 0\} \ .$$

We hence find on $\{\xi(V-W) = 0\}$:

$$\begin{aligned} P(\mu(W \times D) = 1 \mid \mathcal{T}(\mathcal{F}(W),W\xi)) &= P(\mu(W \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V),\ V\xi)) \\ &= P(\mu(V \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V),\ V\xi)) \end{aligned}$$

where the last equality is obvious because $\{\xi(V-W) = 0\} \in \mathcal{T}(\mathcal{F}(V),\ V\xi)$. Hence $G^{V,D} = G^{W,D}$ on $H(W) = H(W) \cap H(V)$.

Clearly the function $G^{V,D}$ is $\mathcal{F}(V) \times \mathcal{B}$-measurable. Thus it may be seen that the process $1_{H(V)}G^{V,D} = 1_{H(V)}\phi(D)$ is visible, so that $\phi(D)$ is visible itself; indeed for all $c \geqq 0$ we have $\{\phi(D) > c\} = \bigcup_{V \in \mathcal{U}}\{\phi(D)1_{H(V)} > c\} \in \mathcal{Z}$. Note that on $\mathcal{Z}$ the measures C_ξ and C_{ξ^z} are the same.

For C_ξ -almost all (ω,u) the set function $\phi_u(.)(\omega)$ is a probability measure on K.

Let $\{R_i\}$ be a collection of random points such that $\bigcup_i \lceil R_i \rfloor = \{(\omega,u) \mid \hat{\xi}_u(\omega) \neq 0\}$ (cf. lemma 5.4.1). Thanks to (5.1.1) we may assume that for all i there exists a set $V_i \in \mathcal{U}$ such that $\lceil R_i \rfloor \subset H(V_i)$. But if $\lceil R_i \rfloor \subset H(V_i)$ for some $V_i \in \mathcal{U}$ we see that $\mathcal{F}(R_i) \subset \mathcal{T}(\mathcal{F}(V_i),\ V_i\xi)$. Using §5.3 remark 1 we find

$$E \sum_i X_{R_i} \hat{\mu}_{R_i}(D) = E \sum_i X_{R_i} E(\hat{\mu}_{R_i}(D) \mid \mathcal{T}(\mathcal{F}(V_i), V_i\xi))$$

$$= E \sum_i X_{R_i} P(\mu(V_i \times D) = 1 \mid \mathcal{T}(\mathcal{F}(V_i), V_i\xi))$$

$$= E \sum_i X_{R_i} \phi_{R_i}(D) .$$

so that we see:

(7.2.1) $$E \int X \times 1_D \, d\mu = E \int X \, \phi(D) \, d\xi$$

for all non-negative visible processes X and all $D \in \mathcal{K}$.

Thus we proved:

<u>THEOREM 7.2.1.</u> *The mapping* $(\omega,u) \to \phi_u(.)(\omega)$ *is a version of the kernel* n *defined at the beginning of this section.* □

Intuitively ϕ_u is the conditional distribution given μ outside $\{u\} \times K$ and given there is an atom of μ in $\{u\} \times K$, of the K-position of this atom.

Now that we have theorem 7.2.1 we only need to find an expression for ξ^z (μ^z then can be calculated too). By theorem 6.1.2 we know that $\hat{\xi}^z_u(\omega) = 1$ if and only if $(\omega,u) \in \Sigma^c$ and by theorem 6.1.3 we may determine Σ^c . Combining this fact with the following theorem we obtain an expression for μ^z on the whole $U \times K$ for almost every ω . (Indeed: for all $(\omega,u) \in \Sigma$ there exists a set $V \in \mathcal{U}$ such that $(\omega,u) \in H(V)$ and $P(\xi(V) = 0 \mid \mathcal{F}(V)) > 0$). Although we need only consider ξ in the next theorem (cf. theorem 7.2.1), we prefer to derive an expression for μ^z directly.

<u>THEOREM 7.2.2.</u> *For all* $V \in \mathcal{B}$, *all non-negative visible marked processes* X *such that* $X = 0$ *on* $\{P(\xi(V) = 0 \mid \mathcal{F}(V)) = 0\} \times V \times K$, *we have*[1]*:*

$$E \int \frac{1_{H(V)}(.,u) \; X_{u,k} \; S^V(du,dk)}{P(\xi(V-\{u\}) = 0 \mid \mathcal{F}(V))} = E \int 1_{H(V)}(.,u) \; X_{u,k} \; \mu^z(du,dk) ,$$

1) with the convention $0/0 = 0$.

where S^V *is the random measure on* $V \times K$ *which is determined by* $S^V(W \times D) = P(\mu(W \times D) = \xi(V) = 1 \mid F(V))$, $W \in B$, $W \subset V$, $D \in K$.

In particular, if the process X *is not marked, we have*

$$E\int \frac{1_{H(V)}(\cdot,u)\ X_u\ S^V(du \times K)}{P(\xi(V-\{u\}) = 0 \mid F(V))} = E \int 1_{H(V)}(\cdot,u)\ X_u\ \xi^z(du)\ .$$

PROOF. First we shall consider the easier case $V = U$, $P(\xi(U) = 0) \neq 0$. Now $S \equiv S^U$ is $S(U \times K)$ times the conditional distribution, given the fact that $\xi(U) = 1$, of the unique atom of μ in $U \times K$. Hence, S is indeed a measure.

Now let $W \in B$, $D \in K$, then we see

$$E \int_D \int_W \frac{1_{H(U)}(\cdot,u)\ S(du,dk)}{P(\xi(U-\{u\}) = 0)} = \int_D \int_W \int_\Omega \frac{1_{H(U)}(\omega,u)\ P(d\omega)}{P(\xi(U-\{u\}) = 0)}\ S(du,dk)$$

$$= S(W \times D)$$

But on the other hand:

$$E \int_D \int_W 1_{H(U)}\ d\mu^z = E \int_D \int_W 1_{H(U)}\ d\mu = S(W \times D)$$

Next note that if $W \in B$ and $F \in F(W)$, then we have either $F^* \supset \{\mu \mid \xi(W^c) = 0\}$ or $F^* \cap \{\mu \mid \xi(W^c) = 0\} = \emptyset$ and hence we have $(F \times W) \cap H(U) \doteq (\Omega \times W) \cap H(U)$ or $(F \times W) \cap H(U) \doteq \emptyset$, so that the theorem is proved in the case $V = U$ for visible marked process of the form $1_{F \times W \times D}$ with $W \in B$, $F \in F(W)$ and $D \in K$, and hence for all non-negative visible marked processes.

Now we turn to the general case and choose $V \in B$ arbitrarily. It is clearly enough to consider processes of the form $1_{F' \times W \times D}$ with $W \in B$, $W \subset V$, $F' \in F(W)$, $F' \subset \{P(\xi(V) = 0 \mid F(V)) \neq 0\}$ and $D \in K$. Using an argument similar to the one just used above we see that now there exists a set $F \in F(V)$, $F \subset \{P(\xi(V) = 0 \mid F(V)) \neq 0\}$ such that $(F \times W) \cap H(V) \doteq (F' \times W) \cap H(V)$. The assertion follows from:

$$E \int \int \frac{1_{H(V)}(\cdot,u)\ 1_{F \times W \times D}(\cdot,u,k)}{P(\xi(V-\{u\}) = 0 \mid F(V))}\ S^V\ (du,dk) =$$

$$= E \int_D \int_W \int_F \frac{1_{H(V)}(\omega,u)\ P(d\omega \mid F(V))}{P(\xi(V-\{u\}) = 0 \mid F(V))} S^V(du,dk).$$

$$= E\ 1_F\ S^V(W \times D)$$

$$= E\ 1_F\ 1_{\{\mu(W\times D) = \xi(V) = 1\}}$$

$$= E \int 1_{H(V)}\ (\cdot,u)\ 1_{F\times W\times D}\ (\cdot,u,k)\ \mu(du,dk)$$

$$= E \int 1_{H(V)}\ (\cdot,u)\ 1_{F\times W\times D}\ (\cdot,u,k)\ \mu^z(du,dk)\ . \qquad \square$$

<u>COROLLARY 7.2.3.</u> *Simultaneously for all* $V \in U$ *we have a.e. on* $\{\xi(V) = 0\}$:

$$V\mu^z\ (du,dk) = \frac{S^V(du,dk)}{P(\xi(V-\{u\}) = 0 \mid F(V))}$$

and for all $V \in U$ *and all* $u \in V$ *on* $\{\xi(V-\{u\}) = 0\} \cap \{P(\xi(V) = 0 \mid F(V)) \neq 0\}$:

$$\hat{\mu}^z_u\ (dk) = \frac{S^V(\{u\} \times dk)}{P(\xi(V-\{u\}) = 0 \mid F(V))}\ . \qquad \square$$

<u>COROLLARY 7.2.4.</u> *Let* $A = \lim_i \bigcup_{V\in U_i} \Sigma(V)^c \times V$, *then* $A = \Sigma^c$.

PROOF. We still have to complete the proof of theorem 6.1.3; this will be done now. In the course of the proof of theorem 6.1.3 we already saw that the limit by which A is defined, exists and we furthermore proved that $A \subset \Sigma^c$. Hence it remains to be proved that $A^c \subset \Sigma$.

Excluding an evanescent set we may argue as follows: fix $(\omega,u) \in A^c$; then there exists a set $V \in U$, $V \ni u$ such that $P(\xi(V) = 0 \mid F(V))(\omega) \neq 0$. Without loss we may assume that $\xi_\omega(V-\{u\}) = 0$. Now it follows from corollary 7.2.3, that in ω we have:

$$\hat{\xi}^z_u = \frac{S^V(\{u\} \times K)}{P(\xi(V-\{u\}) = 0 \mid F(V))} =$$

$$= \frac{P(\xi(V) = \hat{\xi}_u = 1 \mid F(V))}{P(\xi(V) = 0 \mid F(V)) + P(\xi(V)) = \hat{\xi}_u = 1 \mid F(V))} < 1 ,$$

which completes the proof thanks to theorem 6.1.2. □

§7.3. The visible projection of arbitrary random measures.

Since visibility is defined through μ it is not surprising that the visible projection of a random measure ρ can be expressed in terms of μ and μ^z. Such expressions are stated in the following theorems. Note that the two assumptions for these theorems: " ρ is F-measurable", and: " ρ is a random measure on U " do not present serious restrictions. Indeed, if ρ is A-measurable, we may apply the theorem to $\rho' \equiv E(\rho \mid F)$ (This definition makes sense, because ρ' is determined by the countable collection of r.v.'s $E(\rho(V) \mid F)$, $V \in U$); if ρ is a random measure on $U \times K$, then we may apply the theorems to $\rho(.\times D)$, where D runs through a countable semi-ring generating K, and may use theorem 5.5.4 v).

THEOREM 7.3.1. *If ρ is an F-measurable random measure on U, then:*

$$\rho^z(du) = \rho(du) - \int_K ({}^+\hat{\rho}_{u,k} - {}^-\hat{\rho}_u)\ (\mu - \mu^z)(du,dk) \quad . \qquad (7.3.1)$$

(If K reduces to one point, we have:

$$\rho^z = \rho - ({}^+\hat{\rho} - {}^-\hat{\rho})\ (\xi - \xi^z)\ ).$$

PROOF. In this proof we encounter signed random measures, i.e.: differences of two (positive) random measures; a signed random measure is visible if it is the difference of two visible (positive) random measures.

It is easy to see that the random measure defined by the right-hand side of (7.2.1) satisfies (5.5.1). Hence, we only need to show, that it is visible.

Write:

$$\rho = (\rho - \sum_u \hat{\rho}_u\ \varepsilon_u) + \sum_u \hat{\rho}_u\ \varepsilon_u .$$

The random measure

$$(\rho - \sum_u \hat{\rho}_u \, \varepsilon_u)$$

is F-measurable and diffuse and hence visible. Theorem 5.4.4 yields the visibility of the random measure.

$$^{-}\hat{\rho} \, \xi^z = \int_K {}^{-}\hat{\rho}_{.} \; \mu^z(.,dk) \; .$$

To show the visibility of the random measure

$$\int_K {}^{+}\hat{\rho}_{.,k} \; \mu^z(.,dk)$$

we have to make use of the monotone class theorem B.2(cf. §7.1). The sum of visible random measures being visible again, we only have to consider the remaining part:

$$\sum_u \hat{\rho}_u \, \varepsilon_u - \int_K ({}^{+}\hat{\rho}_{.,k} - {}^{-}\hat{\rho}_{.}) \, \mu(.,dk) =$$

$$= \sum_u \hat{\rho}_u \, \varepsilon_u - \sum_u \int_K {}^{+}\hat{\rho}_{u,k} \, \hat{\mu}_u(dk) \, \varepsilon_u + \sum_u \int_K {}^{-}\hat{\rho}_u \, \hat{\mu}_u(dk) \, \varepsilon_u$$

$$= \sum_u \hat{\rho}_u \, \varepsilon_u - \sum_u \int_K \hat{\rho}_u \, \hat{\mu}_u(dk) \, \varepsilon_u + \sum_u {}^{-}\hat{\rho}_u \, \hat{\xi}_u \, \varepsilon_u$$

$$= \sum_u \hat{\rho}_u \, \varepsilon_u - \sum_u \hat{\rho}_u \, \hat{\xi}_u \, \varepsilon_u + \sum_u {}^{-}\hat{\rho}_u \, \hat{\xi}_u \, \varepsilon_u$$

$$= \sum_u (1 - \hat{\xi}_u) \hat{\rho}_u \, \varepsilon_u + \sum_u {}^{-}\hat{\rho}_u \, \hat{\xi}_u \, \varepsilon_u$$

$$= \sum_u (1 - \hat{\xi}_u) \; {}^{-}\hat{\rho}_u \, \varepsilon_u + \sum_u \hat{\xi}_u \; {}^{-}\hat{\rho}_u \, \varepsilon_u = \sum_u {}^{-}\hat{\rho}_u \, \varepsilon_u \; ,$$

which is an F-measurable random measure having as process of atom-sizes the visible process ${}^{-}\hat{\rho}$. This completes the proof. □

<u>THEOREM 7.3.2.</u> *If* ρ *is an F-measurable random measure on* U *, then:*

$$\rho^z = (1 - \hat{\xi}^z) \; \rho^- + \int_K {}^{+}\hat{\rho}_{.,k} \mu^z(.,dk) \; ,$$

where ρ^- *is the random measure on* U *defined by:*

$$\rho^- = (1 - \hat{\xi})\rho + {}^-\hat{\rho}\xi .$$

(If K *reduces to one point, we have:*

$$\rho^z = (1 - \hat{\xi}^z)\rho^- + {}^+\hat{\rho}\ \xi^z) .$$

PROOF. This theorem is deduced from theorem 7.3.1. (We use the fact that if τ and ψ are two arbitrary random measures on U , then we have $\hat{\tau}\psi = \hat{\psi}\tau$):

$$\rho^z = \rho - \int_K ({}^+\hat{\rho}_{.,k} - {}^-\hat{\rho}_{.})\ (\mu - \mu^z)(.,dk)$$

$$= \rho - \int_K {}^+\hat{\rho}_{.,k}\ \mu(.,dk) + \int_K {}^-\hat{\rho}_{.}\ \mu(.,dk) - \int_K {}^-\hat{\rho}\,\mu^z(.,dk) +$$

$$+ \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk)$$

$$= \rho - \hat{\rho}\xi + {}^-\hat{\rho}\xi - {}^-\hat{\rho}\xi + \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk)$$

$$= \rho - {}^-\hat{\rho}(1-\hat{\xi})\ \xi^z - \hat{\rho}\xi + {}^-\hat{\rho}\xi - {}^-\hat{\rho}\hat{\xi}\xi^z + \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk)$$

$$= \rho - \hat{\rho}(1-\hat{\xi})\xi^z - \hat{\rho}\xi + {}^-\hat{\rho}\xi - {}^-\hat{\rho}\hat{\xi}^z\xi + \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk)$$

$$= \rho - \hat{\xi}^z(1-\hat{\xi})\rho - \hat{\xi}\rho + {}^-\hat{\rho}(1-\hat{\xi}^z)\xi + \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk)$$

$$= (1-\hat{\xi}^z)((1-\hat{\xi})\rho + {}^-\hat{\rho}\xi) + \int_K {}^+\hat{\rho}_{.,k}\ \mu^z(.,dk) .$$

□

§7.4. The conditional intensity of random measures.

In this section a limit representation for random measures will be derived. Until theorem 7.4.8 inclusive ρ will denote a random measure on U satisfying

$$E\ \rho(V) < \infty \quad \text{for all} \quad V \in \mathcal{U}.$$

We first shall prove a sequence of technical lemmas.

LEMMA 7.4.1. *Let* Z *be a visible point, which has the following property:*

$$(*)\qquad \begin{cases} \textit{We may write } \lceil Z \rfloor = \lim_i \bigcup_{V\in U_i} F(V) \times V, \textit{ where } F(V) \in \mathcal{F}(V)\ , \\ F(W) \subset F(V) \textit{ if } W \subset V\ ;\ F(V) \cap F(V') = \emptyset \textit{ if } V \cap V' = \emptyset \\ \textit{and } \lim_i \bigcup_{V\in U_i} F(V) = \pi\lceil Z \rfloor\ . \end{cases}$$

Then there exists an $\mathcal{F}(Z)$*-measurable r.v.* $\zeta_1(Z)$ *such that we have* $\sum_{V\in U_i} E(\rho(V) \mid \mathcal{F}(V))\, 1_{\{Z\in V\}} \to \zeta_1(Z)$ *a.s. if* $i \to \infty$.

PROOF. The sequence
$(\sum_{V\in U_i} E(\rho(V) \mid \mathcal{F}(V))\, 1_{F(V)}\ ,\ \mathcal{T}(F(V) \cap F\ ,\ F \in \mathcal{F}(V)\ ,\ V \in U_i))_i$
forms a supermartingale. Indeed, the process clearly is adapted and if $F \in \mathcal{F}(V)$, $V \in U_i$, then we have:

$$\int_{F\cap F(V)} \sum_{W\in U_{i+1}} E(\rho(W) \mid \mathcal{F}(W)) 1_{F(W)}\, dP =$$

$$= \int_F \sum_{W\in U_{i+1,V}} \rho(W)\ 1_{F(W)}\ dP \leqq$$

$$\leqq \int_F \sum_{W\in U_{i+1,V}} \rho(V)\ 1_{F(W)}\ dP \leqq$$

$$\leqq \int_F \rho(V)\ 1_{F(V)}\ dP = \int_{F\cap F(V)} E(\rho(V) \mid \mathcal{F}(V))\ dP\ .$$

Hence $\lim_i \sum_{V\in U_i} E(\rho(V) \mid \mathcal{F}(V))\, 1_{F(V)} = \zeta_1(Z)$ exists a.s. and since $\sum_{V\in U_i} (1_{F(V)} - 1_{\{Z\in V\}}) \to 0$ as $i \to \infty$ we have

$$\zeta_1(Z) = \lim_i \sum_{V\in U_i} E(\rho(V) \mid \mathcal{F}(V))\, 1_{\{Z\in V\}} \text{ a.s.}.$$

It is clear that $\zeta_1(Z)$ is $\mathcal{F}(Z)$-measurable. □

LEMMA 7.4.2. *For all* $V \in U$ *there exists a set* $I(V) \in \mathcal{F}(V) \times \mathcal{B}$, $I(V) \subset \Omega \times V$ *, such that the set* $I(V)^c \cap (\Omega \times V)$ *is the union of a finite or countable number of graphs of visible points possessing property (*),* *and such that* $(\omega,u) \in I(V)$ *implies* $P(\hat{\beta}_u \neq 0 \mid F(V))(\omega) = 0$ *(apart from an evanescent set).*

PROOF. Define $I_k(V)$ by

$$I_k(V) = \lim_i \bigcup_{W \in U_{i,V}} \{P(\rho(W) > \tfrac{1}{k} \mid F(V)) > \tfrac{1}{k}\} \times W$$

and let Z_1^k be an $F(V)$-measurable section of $I_k(V)$ such that $P(\pi\lceil Z_1^k \rfloor) = P(\pi(I_k(V)))$; note that Z_1^k is a visible point possessing property $(*)$. We continue and find an $F(V)$-measurable section Z_2^k of $I_k(V) - \lceil Z_1^k \rfloor$; and so on. Now we check that $P(\rho(Z_i^k) > \frac{1}{k}) \geq \frac{1}{k} P(\pi\lceil Z_i^k \rfloor)$ (cf. the proof of theorem 6.2.4). From this fact it follows that after having found a countable number of Z_i^k at most, we obtain $P(\pi(I_k(V) - (\bigcup_i \lceil Z_i^k \rfloor))) = 0$, because if this were not the case, then there would exist a non-null-set on which $\rho(V) = \infty$.

The set $(\bigcup_k \bigcup_i \lceil Z_i^k \rfloor)^c \cap (\Omega \times V)$ has all properties required of $I(V)$.

□

<u>COROLARRY.</u> *Write:*

$$I = \bigcap_{V \in U} (I(V) \cup (\Omega \times V^c)) \;;$$

then the set I^c $(= \bigcup_{V \in U} (I(V)^c \cap (\Omega \times V)))$ *is the union of a finite or countable number of graphs of visible points possessing property* $(*)$ *and* $(\omega,u) \in I$ *and* $u \in V \in U$ *imply* $(\omega,u) \in I(V)$. □

<u>LEMMA 7.4.3.</u> *For all* $V \in U$ *the set* $\Sigma^c \cap H(V)$ *is the graph of a visible point possessing property* $(*)$.

PROOF. We already know, that $\Sigma^c \cap H(V) = \lceil Z_V \rfloor$ where Z_V is the visible point defined in the proof of theorem 6.1.3. We only have to check that the Z_V possess property $(*)$. To this end, using (6.1.1) and (6.1.2), we note that

$$\lceil Z_V \rfloor = \lim_i \bigcup_{W \in U_{i,V}} (\Sigma(V)^c \cap \{\xi(V-W) = 0\}) \times W$$

and that $\omega \in \lim_i \bigcup_{W \in U_{i,V}} \Sigma(V)^c \cap \{\xi(V-W) = 0\}$ implies that there exists a decreasing sequence (W_i) $(W_i \in \mathcal{U}_i)$, such that $\xi(W_i) > 0$ for all i ; hence $\xi(\bigcap_i W_i) > 0$. This yields $\bigcap_i W_i = \{u\}$ for same $u \in V$ and $u = Z_V(\omega)$. Thus we find $\omega \in \pi[Z_V]$. □

LEMMA 7.4.4. *Apart from an evanescent set,* $(\omega,u) \in \Sigma \cap I$ *implies* $\lim_i E(\rho(W_i) \mid \mathcal{F}(W_i)) = 0$, *where* $W_i \downarrow \{u\}$, $W_i \in \mathcal{U}_i$.

PROOF. Except for an evanescent set, $(\omega,u) \in \Sigma$ implies that there exists a set V_1, $u \in V_1 \in \mathcal{U}$ such that $\omega \in \Sigma(W)$ for all $W \in \mathcal{U}$, $W \subset V_1$, $W \ni u$. On the other hand it follows from (5.1.1) that there exists a set $V_2 \in \mathcal{U}$ such that $(\omega,u) \in H(V_2)$. Hence $(\omega,u) \in H(V) \cap (\Sigma(V) \times V)$, where $V = V_1 \cap V_2$.

Again excluding an evanescent set, we find, using theorem C.2:

$$E(\rho(W_i) \mid \mathcal{F}(W_i)) = \frac{E(\rho(W_i)\, 1_{\{\xi(V-W_i)=0\}} \mid \mathcal{F}(V))}{P(\xi(V-W_i) = 0 \mid \mathcal{F}(V))} \leqq$$

$$\leqq \frac{E(\rho(W_i) \mid \mathcal{F}(V))}{P(\xi(V) = 0 \mid \mathcal{F}(V))}$$

We have: $P(\hat{\rho}_u \neq 0 \mid \mathcal{F}(V))(\omega) = 0$ because $(\omega,u) \in I(V)$, hence $\rho(W_i) \to 0$ if $i \to \infty$ for $P(. \mid \mathcal{F}(V))(\omega)$ almost all ρ , so that dominated convergence yields: $E(\rho(W_i) \mid \mathcal{F}(V))(\omega) \to 0$. □

LEMMA 7.4.5. *For all* $V \in \mathcal{U}$ *we have a.e. on* $\{\xi(V) = 0\}$:

$$\sum_{W \in \mathcal{U}_{i,B}} E(\rho(W)\, 1_{\{\xi(W)=0\}} \mid \mathcal{F}(W)) \uparrow \zeta_2(B) .$$

if $i \to \infty$, *where* ζ_2 *is a random measure on the semi-ring* $\{B \in \mathcal{U} \mid B \subset V\}$.

PROOF. If we exclude one null-set, we may argue everywhere on $\{\xi(V) = 0\}$ as follows: If $W \in \mathcal{U}_i$, $W \subset V$, then, using theorem C.2, we find:

$$E(\rho(W)\, 1_{\{\xi(W)=0\}} \mid \mathcal{F}(W)) =$$

$$= \frac{E(\rho(W)\, 1_{\{\xi(V)=0\}} \mid \mathcal{F}(V))}{P(\xi(V-W) = 0 \mid \mathcal{F}(V))} =$$

$$= \sum_{W' \in U_{i+1,W}} \frac{E(\rho(W')\ 1_{\{\xi(V)=0\}} \mid F(V))}{P(\xi(V-W) = 0 \mid F(V))} \leq$$

$$\leq \sum_{W' \in U_{i+1,W}} \frac{E(\rho(W')\ 1_{\{\xi(V)=0\}} \mid F(V))}{P(\xi(V-W') = 0 \mid F(V))} =$$

$$= \sum_{W' \in U_{i+1,W}} E(\rho(W')\ 1_{\{\xi(W')=0\}} \mid F(W')) .$$

Hence the limit $\zeta_2(B)$ exists. It is easy to check that this limit is a measure in B. □

LEMMA 7.4.6. *For all* $V \in U$ *we have a.e. on* $\{\xi(V) = 0\}$:

$$\sum_{W \in U_{i,B}} E((\hat{\rho}\xi)(W)\ 1_{\{\xi(W)=1\}} \mid F(W)) \uparrow \zeta_3\ (B)$$

if $i \to \infty$ *, where* ζ_3 *is a random measure on the semi-ring* $\{B \in U \mid B \subset V\}$.

PROOF. If we exclude one null-set we may argue everywhere on $\{\xi(V) = 0\}$ as follows: If $W \in U_i$, $W \subset V$, then, again using theorem C.2, we find:

$$E((\hat{\rho}\xi)(W)\ 1_{\{\xi(W)=1\}} \mid F(W)) =$$

$$= \frac{E((\hat{\rho}\xi)(W)\ 1_{\{\xi(W)=\xi(V)=1\}} \mid F(V))}{P(\xi(V-W) = 0 \mid F(V))}$$

$$= \sum_{W' \in U_{i+1,W}} \frac{E((\hat{\rho}\xi)(W')\ 1_{\{\xi(W')=\xi(V)=1\}} \mid F(V))}{P(\xi(V-W) = 0 \mid F(V))} \leq$$

$$\leq \sum_{W' \in U_{i+1,W}} \frac{E((\hat{\rho}\xi)(W')\ 1_{\{\xi(W')=\xi(V)=1\}} \mid F(V))}{P(\xi(V-W') = 0 \mid F(V))} =$$

$$= \sum_{W' \in U_{i+1,W}} E((\hat{\rho}\xi)(W')\ 1_{\{\xi(W')=1\}} \mid F(W')) .$$

Hence the limit $\zeta_3(B)$ exists. It can easily be checked that this limit is a measure in B . □

LEMMA 7.4.7. *Let* $V \in U$*; then if* $i \to \infty$ *we have:*

$$\sum_{W \in U_{i,V}} E(\rho(W)\ 1_{\{\xi(W) > 1\}} \mid F(W))\ 1_{\{\xi(W)=0\}} \to 0 \quad a.s.$$

and

$$\sum_{W \in U_{i,V}} E((\rho - \hat{\rho}\xi)(W)\ 1_{\{\xi(W)=1\}} \mid F(W))\ 1_{\{\xi(W)=0\}} \to 0 \quad a.s..$$

PROOF. We shall first show that we may assume without restriction that $U = \mathbb{R}$, $\mathcal{U} = \{(k2^{-i+1}, (k+1)2^{-i+1}] \mid k \in \mathbb{Z},\ i = 1, 2, \ldots\}$ and $V = (0,1]$. To prove this, we assume that $V \in \mathcal{U}_1$ and that every element of $\mathcal{U}_i$ consists of two elements of $\mathcal{U}_{i+1}$; it is clear that this situation, which only simplifies notation, always can be obtained. Next note, that there exist mappings T on $\mathcal{U}$ such that $T : \mathcal{U}_i \to \{(k2^{-i+1}, (k+1)2^{-i+1}] \mid k \in \mathbb{Z}\}$, such that $T(V) = (0,1]$, such that T preserves inclusion and such that $W, W' \in \mathcal{U}$, $W \cap W' = \emptyset$ implies $T(W) \cap T(W') = \emptyset$. Choose such a mapping T. Let $u \in U$ and $W_i \downarrow \{u\}$ $(W_i \in \mathcal{U}_i)$, then either $T(W_i) \downarrow \{x\}$, where x is some real number, or $T(W_i) \downarrow \emptyset$. Note that if $u, u' \in U$, $u \neq u'$, $W_i \downarrow \{u\}$, $W_i' \downarrow \{u'\}$ $(W_i, W_i' \in \mathcal{U}_i)$ and $T(W_i) \downarrow \{x\}$, $T(W_i') \downarrow \{x'\}$, then $x \neq x'$. Note furthermore that $T(W_i) \downarrow \emptyset$ only for a countable number of $u \in U$; hence, if for all $u \in U$ such that $T(W_i) \downarrow \emptyset$ we have $P(\hat{\xi}_u \neq 0) = 0$, then the random measure μ on $U \times K$ is a.s. determined by the random measure μ' on $\mathbb{R} \times K$, defined by $\mu'(T(W) \times D) = \mu(W \times D)$ for all $W \in \mathcal{U}$, $D \in \mathcal{K}$. Thus it remains to be shown that we may choose T such that $T(W_i) \downarrow \{x\}$ $(x \in \mathbb{R})$ for all $u \in B \equiv \{u' \in U \mid P(\hat{\xi}_{u'} \neq 0) \neq 0\}$. Therefore we use a random procedure to choose T: First we define T on $\mathcal{U}_1$ in an arbitrary manner $(T(V) = (0,1])$. Every $W \in \mathcal{U}_1$ falls apart in two elements of $\mathcal{U}_2$; now there are two possibilities for these two elements of $\mathcal{U}_2$ to be mapped under T; we decide by tossing a coin; we do this for all $W \in \mathcal{U}_1$. Once T defined on $\mathcal{U}_2$ we continue by defining it on $\mathcal{U}_3$ in the same random manner. It can be checked easily that for all $u \in U$ the probability is zero that $T(W_i) \downarrow \emptyset$. Because B consists of a countable number of points at most, the probability is one that this random procedure yields a satisfactory mapping T. Hence such a satisfactory mapping T does exist.

Consequently, indeed we may assume $U = \mathbb{R}$ and $V = (0,1]$.

We now may a.s. number the atom positions of ξ by increasing sequences $(\tau_k)_{k\in\mathbb{Z}}$ and $(\tau_k')_{k\in\mathbb{Z}}$ such that $\tau_0 < 0 \leqq \tau_1$ and $\tau_{-1}' \leqq 1 < \tau_0'$ (as functions on Ω the τ_k and the τ_k' are random points) (We simplify the notation by the assumptions that the set of atom positions of almost all ξ is unbounded above and below).

Let $W \subset (a,b) \subset [0,1]$ where $W \in U$ and $a,b \in \mathbb{R}$. For the time being we write:

$$A = \{\xi((a,b) - W) = 0,\ \xi[0,a] = \ell,\ \xi[b,1] = r\}$$

$(\ell, r = 0, 1, 2, \dots)$ and

$$I = (\dots, (\tau_0, \hat{\mu}_{\tau_0}), \dots, (\tau_\ell, \hat{\mu}_{\tau_\ell}), (\tau'_r, \hat{\mu}_{\tau'_r}), \dots, (\tau'_0, \hat{\mu}_{\tau'_0}), \dots).$$

Analogous to the proof of theorem C.2 we show that on A we have:

$$E(X \mid F(W)) = \frac{E(X 1_A \mid I)}{P(A \mid I)} \qquad \text{a.s.}$$

for all non-negative r.v.'s X. From this we may conclude that on $B = \{\xi(a,b) = 0,\ \xi[0,a] = \ell,\ \xi[b,1] = r\}$ we have a.s.:

$$E(X \mid F(W)) \leqq \frac{E(X \mid I)}{P(B \mid I)},$$

which does not depend on W.

Next note that:

$$\sum_{\substack{W \in U_i \\ W \subset (a,b)}} \rho(W)\, 1_{\{\xi(W) > 1\}} \to 0$$

$$\sum_{\substack{W \in U_i \\ W \subset (a,b)}} \rho(W)\, 1_{\{\xi(W) = 1\}} \to \sum_{u \in (a,b)} \hat{\rho}_u \cdot \hat{\xi}_u$$

and

$$\sum_{\substack{W \in U_i \\ W \subset (a,b)}} (\hat{\rho}\xi)(W)\, 1_{\{\xi(W) = 1\}} \to \sum_{u \in (a,b)} \hat{\rho}_u\, \hat{\xi}_u$$

as $i \to \infty$, that all three sequences are dominated by $\rho(a,b)$, while $E(\rho(a,b) \mid I) < \infty$ a.s. and that $\rho(W) \geqq (\hat{\rho}\xi)(W)$. Furthermore, by making use of the fact that for all ω the set $(0,1] \cap \{u \mid \hat{\xi}_u(\omega) = 0\}$ is the finite union of open intervals, we prove the lemma. □

The preceding lemmas enable us to prove the main theorem of this section.

<u>THEOREM 7.4.8.</u> *Let* $B \in U$, *then:*

$$\rho^z(B) = \lim_i \sum_{W \in U_{i,B}} E(\rho(W) \mid F(W)) \quad a.s. \text{ and in } L^1.$$

PROOF. According to lemma 7.4.2 and 7.4.3 we may choose a sequence of visible points $\{Z_i\}$ possessing property (*) such that $\bigcup_i \lceil Z_i \rfloor = \Sigma^c \cup I^c$.

For almost all ω_0 we may argue as follows:
There are only finitely many $u \in B$ with $\hat{\xi}_u(\omega_0) \neq 0$; hence there exists a smallest collection of indices i_k such that:

$$\{u \in B \mid (\omega_0, u) \in \Sigma^c \cup I^c;\ \hat{\xi}_u(\omega_0) > 0\} = \bigcup_k \{Z_{i_k}(\omega_0)\}$$

Now it is clear that:

$$\lim_i \sum_{W \in U_{i,B}} E(\rho(W) \mid F(W))(\omega_0) =$$

$$= \lim_i \sum_{W \in U_i} \sum_k E(\rho(W) \mid F(W))(\omega_0)\, 1_{\{Z_{i_k} \in W\}}(\omega_0) +$$

$$+ \lim_i \sum_{W \in U_{i,B}} E(\rho(W) \mid F(W))(\omega_0)\, 1_{\bigcap_k \{Z_{i_k} \notin W\} \cap \{\xi(W) \neq 0\}}(\omega_0) +$$

$$+ \lim_i \sum_{W \in U_{i,B}} E(\rho(W)\, 1_{\{\xi(W)=0\}} \mid F(W))(\omega_0)\, 1_{\{\xi(W)=0\}}(\omega_0) +$$

$$+ \lim_i \sum_{W \in U_{i,B}} E((\hat{\rho}\xi)(W)\, 1_{\{\xi(W)=1\}} \mid F(W))(\omega_0)\, 1_{\{\xi(W)=0\}}(\omega_0) +$$

$$+ \lim_i \sum_{W \in U_{i,B}} E((\rho - \hat{\rho}\xi)(W)\, 1_{\{\xi(W)=1\}} \mid F(W))(\omega_0)\, 1_{\{\xi(W)=0\}}(\omega_0) +$$

$$+ \lim_i \sum_{W \in U_{i,B}} E(\rho(W)\, 1_{\{\xi(W)>1\}} \mid F(W))(\omega_0)\, 1_{\{\xi(W)=0\}} =$$

$$= \lim A_i(\omega_0) + \lim B_i(\omega_0) + \lim C_i(\omega_0) + \lim D_i(\omega_0) +$$

$$+ \lim E_i(\omega_0) + \lim F_i(\omega_0)$$

$$= \zeta_{\omega_0}(B) \ .$$

(say) since these limits exists thanks to respectively lemma 7.4.1, 7.4.4, 7.4.5, 7.4.6, 7.4.7 and again 7.4.7.

Using lemma 7.4.5, 7.4.6 and 7.4.7 we see that ζ_{ω_0} is a measure, because if $B_i \downarrow \emptyset$, where the B_i are unions of elements of U, there exists a number $i_0(\omega_0)$ such that $\xi(B_{i_0(\omega_0)}) = 0$. Hence ζ is a random measure. To prove that it is visible we use definition 5.4.3 ii): clearly it is F-measurable and if $\alpha \geq 0$ we have, using lemma 7.4.4:

$$\{(\omega,u) \mid \hat{\zeta}_u(\omega)>\alpha\} = \bigcup_i (\lceil Z_i \rfloor \cap (\{\zeta_1(Z_i) > \alpha\} \times U)) \in Z$$

thanks to lemma 5.3.2.

To show that $\zeta = \rho^z$ we now only need to show

$$E\ \zeta(B)\ 1_F = E\ \rho(B)\ 1_F$$

for all $B \in U$, $F \in F(B)$ (theorem 5.5.2 iii)). Since clearly

$$E\ 1_F \sum_{W\in U_{i,B}} E(\rho(W) \mid F(W)) = E\ \rho(B)\ 1_F$$

only L^1-convergence remains to be proved. To that end we write

$$\sum_{W\in U_{i,B}} E(\rho(W) \mid F(W)) =$$

$$= \sum_{W\in U_{i,B}} E(\rho(W) \mid F(W))\ 1_{\{\xi(W)\neq 0\}} +$$

$$+ \sum_{W\in U_{i,B}} E(\rho(W)\ 1_{\{\xi(W)=0\}} \mid F(W))\ 1_{\{\xi(W) = 0\}} +$$

$$+ \sum_{W\in U_{i,B}} E((\hat{\rho}\xi)(W)\ 1_{\{\xi(W)=1\}} \mid F(W))\ 1_{\{\xi(W)=0\}} +$$

$$+ \sum_{W\in U_{i,B}} E((\rho-\hat{\rho}\xi)(W)\ 1_{\{\xi(W))=1\}} \mid F(W))\ 1_{\{\xi(W)=0\}} +$$

$$+ \sum_{W \in U_{i,B}} E(\rho(W)\ 1_{\{\xi(W)>1\}} \mid F(W))\ 1_{\{\xi(W)=0\}} =$$

$$= G_i + C_i + D_i + E_i + F_i\ .$$

We now shall prove L^1-convergence of these five sequences:

Since $G_i = A_i + B_i$ and hence converges a.s., the L^1-convergence of G_i is equivalent to the assertion: For all $\varepsilon > 0$ there exists a $\delta > 0$ such that if $P(F) < \delta$ $(F \in A)$ then we have $E\ G_i\ 1_F < \varepsilon$ for all i. Hence let $\varepsilon > 0$ and choose $a > 0$ such that $E\ \rho(B)\ 1_{\{\rho(B) > a\}} < \varepsilon/2$ and $\delta > 0$ such that $P(F) < \delta$ $(F \in A)$ implies $E\ \xi(B)\ 1_F < \varepsilon/2a$. This satisfies our requirements; indeed: suppose $P(F) > \delta(F \in A)$; then:

$$E\ 1_F\ G_i =$$

$$= E\ 1_F \sum_{W \in U_{i,B}} E(\rho(W)\ 1_{\{\rho(B) \leqq a\}} \mid F(W))\ 1_{\{\xi(W) \neq 0\}} +$$

$$+ E\ 1_F \sum_{W \in U_{i,B}} E(\rho(W)\ 1_{\{\rho(B)>a\}} \mid F(W))\ 1_{\{\xi(W) \neq 0\}} \leqq$$

$$\leqq E\ 1_F\ a\ \xi(B) + E \sum_{W \in U_{i,B}} E(\rho(W)\ 1_{\{\rho(B)>a\}} \mid F(W)) <$$

$$< \varepsilon/2 + E\ \rho(B)\ 1_{\{\rho(B)>a\}} < \varepsilon$$

For C_i note that:

$$E(\rho(W)\ 1_{\{\xi(W)=0\}} \mid F(W))\ 1_{\{\xi(W)=0\}} =$$

$$= E(\rho(W) | F)\ P(\xi(W) = 0 \mid F(W))\ 1_{\{\xi(W)=0\}}\ .$$

Indeed: both members are F-measurable; we have $F \cap \{\xi(W) = 0\} = F(W) \cap \{\xi(W) = 0\}$ and for all $F \in F(W)$ we may deduce that:

$$E\ 1_F\ E(\rho(W)\ 1_{\{\xi(W)=0\}} \mid F(W))\ 1_{\{\xi(W)=0\}} =$$

$$= E\ 1_F\ E(\rho(W) \mid F)\ P(\xi(W)) = 0 \mid F(W))\ 1_{\{\xi(W)=0\}}\ .$$

Now it is easy to see, that for all i :

$$C_i \leqq E(\rho(B) \mid F)$$

and $E\,E(\rho(B) \mid F) = E\,\rho(B) < \infty$ so that we may apply the dominated convergence theorem.

If follows from lemma 7.4.6 that the D_i form an a.s. increasing sequence. They converge in L^1 because $E\,D_i \leqq E(\hat{\rho}\xi)(B) \leqq E\,\rho(B) < \infty$.

Finally we have:

$$0 \leqq E\,E_i \leqq E\,(\sum_{W \in U_{i,B}} (\rho-\hat{\rho}\xi)(W)\, 1_{\{\xi(W)=1\}}) \to 0$$

and

$$E\,F_i \leqq E\,(\sum_{W \in U_{i,B}} \rho(W)\, 1_{\{\xi(W) > 1\}}) \to 0$$

(cf. the last part of the proof of lemma 7.4.7).

This completes the proof. □

If ρ is a random measure on $U \times K$, such that $E\,\rho(V \times K) < \infty$ for all $V \in U$ we may apply theorem 7.4.8 to $\rho(. \times D)$ for all $D \in K$. Doing this we immediately obtain, thanks to theorem 5.5.4 v) :

<u>COROLLARY 7.4.9.</u> *Let* B ∈ U; *then:*

$$\sum_{W \in U_{i,B}} E(\rho(W \times .) \mid F(W)) \xrightarrow{\text{weak}} \rho^z(B \times .) \quad a.s.. \qquad \square$$

CHAPTER 8

MARTINGALELIKE MEASURES

§8.1. Definition and representation theorem.

A *martingalelike measure* on U is an F-measurable random signed measure ρ on U such that $E\,|\rho(V)| < \infty$ for all bounded $V \in B$ and such that we have:

$$(8.1.1) \qquad E(\rho(V) \mid F(V)) = 0 \quad \text{a.s.}$$

for all $V \in B$.

Formula (8.1.1) is equivalent to:

$$(8.1.2) \qquad C_\rho(X) = 0$$

for all visible processes X such that $C_\rho(|X|) < \infty$.

The notion of martingalelike measure is analogous to the familiar notion of martingales on $\mathbb{R}_+$. Indeed: if ρ is a random signed measure on $\mathbb{R}_+$ such that $E\,|\rho[0,\infty)| < \infty$, then the process $(\rho[0,.])$ is called a martingale if (we use the notation of chapter 3):

$$(8.1.3) \qquad \rho[0,t] \text{ is } F_t\text{-measurable for all } t \in \mathbb{R}_+$$

and

$$(8.1.4) \qquad E(\rho(s,t] \mid F_s) = 0 \text{ a.s. for all } s < t .$$

Formula (8.1.4) is equivalent to:

$$(8.1.5) \qquad C_\rho(X) = 0$$

for all previsible processes X such that $C_\rho(|X|) < \infty$.

Note that even on $\mathbb{R}_+$ martingales and martingalelike measures are not the same objects. A more important reason to introduce the vaguer term "martingalelike measure" is the fact that we only require F-measurability of all $\rho(V)$ which is a rather weak analogue of (8.1.3).

A martingalelike measure on $U \times K$ is a random signed measure ρ on $U \times K$ such that $\rho(. \times D)$ is a martingalelike measure on U for all $D \in \mathcal{K}$. Properties of martingalelike measures on $U \times K$ can easily be deduced from properties of martingalelike measures on U.

If ρ is an integrable random measure on U (resp. on $U \times K$) then of course $\rho - \rho^z$ is a martingalelike measure on U (resp. on $U \times K$). If R is a visible marked process, such that $C_\mu(|R| \, 1_{\Omega \times V \times K}) < \infty$ for all bounded $V \in \mathcal{B}$, then

$$(8.1.6) \qquad \rho(du) = \int_K R_{u,k} \, (\mu - \mu^z)(du,dk)$$

is clearly a martingalelike measure on U. The next theorem states that all martingalelike measures are of this form (For martingales on $\mathbb{R}_+$ w.r.t. a point process an analogous theorem holds, see Chou and Meyer (75)).

<u>THEOREM 8.1.1.</u> *Let* ρ *be a martingalelike measure on* U *; then*

$$\rho(du) = \int_K ({}^+\rho_{u,k} - {}^-\hat{\rho}_u) \, (\mu - \mu^z)(du,dk) \qquad a.s..$$

(If K *reduces to one point, we have*

$$\rho = ({}^+\hat{\rho} - {}^-\hat{\rho})(\xi - \xi^z) \qquad a.s.).$$

PROOF. Although ρ is a random signed measure, we may define its visible projection and see that $\rho^z = \underline{0}$ a.s. thanks to (8.1.2). We also may apply theorem 7.3.1 which directly yields the required result. □

An alternative proof of theorem 8.1.1 is sketched in §10.1 (theorem 10.1.3).

§8.2. Papangelou kernels.

The object that Matthes, Warmuth and Mecke (79) first called Papangelou kernel and which figures already in Kallenberg (78) turns out to be very fundamental to the study of Gibbs processes (Example 2.2.2). Indeed Matthes, Warmuth and Mecke found an explicit expression for the specification of a Gibbs process in terms of the Papangelou kernel. Most of their results will be proved in chapter 9 using a slightly different approach.

Now we shall introduce the Papangelou kernel η of a point process and find an expression for it in terms of μ and μ^z. Furthermore, to us $\mu - \eta$ yields an important example of the martingalelike measure.

Throughout this section we shall assume that μ is integrable and that (Σ) is satisfied. Then if X is an $F \times B$-measurable process, the process ${}^{-}X$ is uniquely determined upto indistinguishability (cf. §7.1). We define the measure C^{-} on $(\Omega \times K, F \times B \times K)$ by

$$C^{-}(X \times 1_D) = C_\mu({}^{-}X \times 1_D)$$

for all non-negative $F \times B$-measurable processes X and all $D \in K$. The measure C^{-} is called the *reduced Campbell measure*. Note, that if $X \doteq 0$, then we have ${}^{-}X \doteq 0$ too.
This implies that $C^{-}(. \times V \times D) << P$ for all $V \in B$, $D \in K$. Hence there exists an a.s. unique F-measurable transition measure η from Ω on $U \times K$ such that

$$C^{-}(X \times 1_D) = E \int X_u \, 1_D(k)\eta(du,dk) = C_\eta(X \times 1_D) \cdot$$

The existence of the kernel η is proved analogously to the existence of the dual visible projection and hence to the existence of conditional distributions on Polish spaces.

The transition measure η is called the *Papangelou kernel* of the point process. We shall prove that $\mu - \eta$ is a martingalelike measure. In fact we prove:

THEOREM 8.2.1. *We have*

$$(8.2.1) \qquad \mu - \eta = \frac{\mu - \mu^z}{1 - \hat{\xi}^z \times 1_K} \quad a.s..$$

(Hence we have to put $R_{u,k} = \dfrac{1_D(k)}{1 - \hat{\xi}^z_u}$ *in (8.1.6) in order to obtain* $(\mu - \eta)(. \times D)$*). Or equivalently:*

$$(8.2.2) \qquad \eta(du,dk) = \frac{\mu^z(du,dk) - \hat{\xi}^z_u \, \mu(du,dk)}{1 - \hat{\xi}^z_u} \quad a.s..$$

PROOF. We begin with some technicalities.

- First, it follows from theorem 7.1.1 that

$$^{-}X \doteq \frac{1}{1-\hat{\xi}^z} \left({}^zX - \int_K {}^{+}X_{\cdot,k}\, \hat{\mu}^z_{\cdot}\,(dk) \right) ,$$

whenever X is an $F \times B$-measurable process.

- If Y is a visible process, $D \in K$ and ρ a random measure on U, then we have, using theorem 5.5.4 iv):

$$\begin{aligned} E \iint Y_u\, 1_D(k)\, \hat{\mu}^z_u\,(dk)\, \rho(du) &= E \int Y_u\, \hat{\mu}^z_u\,(D)\, \rho(du) \\ &= E \int Y_u\, \hat{\mu}^z_u\,(D)\, \rho^z(du) \\ &= E \int Y_u\, \hat{\mu}_u\,(D)\, \rho^z(du) \\ &= E \iint Y_u\, 1_D(k)\, \hat{\mu}_u\,(dk)\, \rho^z(du) . \end{aligned}$$

Applying the monotone class theorem B.2 (cf. §7.1) we find for all non-negative $F \times B$-measurable processes **X** and all non-negative visible processes Y:

$$E \iint Y_u\, {}^{+}X_{u,k}\, \hat{\mu}^z_u\,(dk)\, \rho(du) = E \iint Y_u\, {}^{+}X_{u,k}\, \hat{\mu}_u\,(dk)\, \rho^z(du) .$$

- It is clear from the proof of theorem 5.5.3 and theorem 7.2.1 that we have $\hat{\mu}^z(.) \doteq \phi(.)\, \hat{\xi}^z$.

- Furthermore, we recall that $\hat{\rho}\tau = \hat{\tau}\rho$ a.s. for any two random measures ρ and τ on U .

Using these facts successively and finally applying formula (7.2.1) we find for any $F \times B$-measurable non-negative process X and any $D \in K$:

$$E \int X \times 1_D\, d\eta = E \int {}^{-}X \times 1_D\, d\mu =$$

$$= E \int \frac{{}^zX}{1-\hat{\xi}^z} \times 1_D\, d\mu - E \int \frac{\int_K {}^{+}X_{u,k}\, \hat{\mu}^z_u(dk)}{1-\hat{\xi}^z_u}\, \mu(du \times D)$$

$$= E \int \frac{{}^zX}{1-\hat{\xi}^z} \times 1_D\, d\mu^z - E \int \frac{\int_K {}^{+}X_{u,k}\, \hat{\mu}_u(dk)}{1-\hat{\xi}^z_u}\, \mu^z(du \times D) =$$

$$= E\int \frac{X}{1-\hat{\xi}^z} \times 1_D \, d\mu^z - E\int \frac{X_u \hat{\xi}_u}{1-\hat{\xi}^z_u} \mu^z(du \times D)$$

$$= E\int \frac{X}{1-\hat{\xi}^z} \times 1_D \, d\mu^z - E\int \frac{X_u \hat{\mu}^z_u(D)}{1-\hat{\xi}^z_u} \xi(du)$$

$$= E\int \frac{X}{1-\hat{\xi}^z} \times 1_D \, d\mu^z - E\int \frac{X_u \hat{\xi}^z_u \phi_u(D)}{1-\hat{\xi}^z_u} \xi(du)$$

$$= E\int \frac{X}{1-\hat{\xi}^z} \times 1_D \, d\mu^z - E\int \frac{X_u \hat{\xi}^z_u 1_D(k)}{1-\hat{\xi}^z_u} \mu(du,dk)$$

$$= E\int \frac{X_u 1_D(k)}{1-\hat{\xi}^z_u} (\mu^z(du,dk) - \hat{\xi}^z_u \mu(du,dk)) .$$

Because the righthand member in (8.2.2) is F-measurable, the theorem now has been proved. □

We can even deduce a more concrete expression for η in terms of μ^z and μ :

THEOREM 8.2.2. *For all* $B \in \mathcal{B}$, $D \in \mathcal{K}$ *we have:*

$$\eta(B \times D) = \mu^z(B \times D) - \sum_{u \in B} \hat{\mu}^z_u(D) + \sum_{u \in B} \frac{\hat{\mu}^z_u(D) - \hat{\xi}^z_u \hat{\mu}_u(D)}{1-\hat{\xi}^z_u}$$

In particular:

$$\eta(. \times K) = \xi^z - \sum_u \hat{\xi}^z_u \varepsilon_u + \sum_u \frac{1-\hat{\xi}_u}{1-\hat{\xi}^z_u} \hat{\xi}^z_u \varepsilon_u .$$

(Cf. Kallenberg (78) theorem 3.1 and 4.1) .

PROOF. It follows from (8.2.2) that

$$\eta = \frac{\mu^z}{1-\hat{\xi}^z \times 1_K} - \left(\frac{\hat{\xi}^z}{1-\hat{\xi}^z} \times 1_K\right)\mu$$

$$= \mu^z - \sum_u \varepsilon_u \times \hat{\mu}^z_u + \sum_u \frac{\varepsilon_u \times \hat{\mu}^z_u}{1-\hat{\xi}^z_u} - \sum_u \frac{\hat{\xi}^z_u}{1-\hat{\xi}^z_u} \varepsilon_u \times \hat{\mu}_u$$

$$= \mu^z - \sum_u \varepsilon_u \times \hat{\mu}^z_u + \sum_u \frac{\varepsilon_u \times (\hat{\mu}^z_u - \hat{\xi}^z_u \hat{\mu}_u)}{1-\hat{\xi}^z_u} .$$ □

CHAPTER 9

LOCAL UNIQUENESS OF P .

Throughout this chapter, in order to simplify the notation, we shall assume that $\Omega = M$; hence $\omega \to \mu$ is the identity; note that now the σ-field F is the P-completion of the σ-field M .

§9.1. Local conditioning.

It is clear that the visible projection of the random measure μ depends on the probability measure P . This can be expressed by writing $\mu^{z,P}$ instead of μ^{z} . Furthermore, the visible projection is a function of $\omega : \mu^{z,P}_{\omega}$.

For P-almost all ω' we know that $P(. \mid M(B))(\omega')$ for all $B \in U$ is an probability measure on (M,M). (Note that $P(. \mid M(B))$ is a version of $P(. \mid F(B))$.) Hence we may determine $\mu^{z,P(. \mid M(B))(\omega')}$, which of course is still a function on $\Omega : \omega \to \mu^{z,P(. \mid M(B))(\omega')}_{\omega}$; this latter function is measurable w.r.t. the $P(. \mid M(B))(\omega')$ - completion of M.

In section 7.2 we saw that if $B \in U$, then the visible projection of μ is determined on B by the conditional distributions $P(. \mid M(V))$ $V \in U$, $V \subset B$. Now theorem C.1 yields that on B the projections $\mu^{z,P}_{\omega}$ and $\mu^{z,P(. \mid M(B))(\omega')}_{\omega}$ are the same.

Hence in particular

<u>THEOREM 9.1.1.</u> *For* P*-almost all* ω *we have for all* $B \in U$ *:*

$$B\mu^{z,P}_{\omega} = B\mu_{\omega}^{z,P(. \mid M(B))(\omega)} .$$

The results on P , which we acquire in the rest of this chapter under the assumption $P(\xi(U) < \infty) = 1$ may be applied to $P(B\mu \in . \mid M(B))(\omega)$ for P-almost all ω and all $B \in U$, thanks to theorem 9.1.1. This fact explains the meaning of the word "local" in the heading of this chapter. A direct consequence of the above theorem is:

COROLLARY 9.1.2. *For all bounded* $B \in \mathcal{B}$ *we have for* P*-almost all* ω *:*

$$B\mu_\omega^{z,P} = B\mu_\omega^{z,P}(\cdot \mid \mathcal{M}(B))(\omega) \quad . \qquad \square$$

§9.2. Uniqueness of the distribution of μ .

In this section we shall deduce an expression for the distribution of μ in terms of its visible projection.

THEOREM 9.2.1. *Let* P *be a probability measure on* M *such that* $P(\xi(U) < \infty) = 1$ *and such that* μ *admits* ζ *as its visible projection while* ζ *satisfies* $X \equiv \hat{\zeta}(K) < 1$ *and* $^{-}X = X$ *identically (this implies that the point process satisfies condition* (Σ)*); than* P *is determined. Indeed, then for any* $n \geq 0$, $V_1, \ldots, V_n \in \mathcal{B}$ *disjoint sets and* $B_1, \ldots$ *...,* $B_n \in \mathcal{B} \times K$ *such that* $B_i \subset V_i \times K$ *for all* i *, we have*

$$P(\mu(B_1) = 1, \ldots, \mu(B_n) = 1, \ \xi(U) = n) =$$

$$= P(\xi(U) = 0) . \int_{B_n} \cdots \int_{B_1} \eta_{\varepsilon_{u_2,k_2} + \ldots + \varepsilon_{u_n,k_n}} (du_1,dk_1) \ldots \eta_{\underline{0}}(du_n,dk_n)$$

where the random measure $\eta = \frac{\zeta - X\mu}{1 - X}$ *is the Papangelou kernel and* $P(\xi(V) = 0)$ *should be considered as a normalizing constant. In the above formule* η *may clearly be replaced by* $\frac{\zeta}{1 - X}$.

PROOF. We use the notation of the proof of theorem 7.2.2. For all $B \in \mathcal{B} \times K$ we have

$$S(B) = E \, 1_{\{\mu(B) = \xi(U) = 1\}}$$

$$= E \int_B 1_{H(U) \times K} \, d\mu$$

$$= E \int_B 1_{H(U) \times K} \, d\zeta$$

$$= P(\xi(U) = 0) \int_B d\zeta_{\underline{0}} + \int_B X_u(\varepsilon_{u,k}) \, S(du,dk) \ ;$$

hence

$$\int_B (1-X_u(\varepsilon_{u,k}))\ S\ (du,dk) = P(\xi(U) = 0) \int_B \zeta_{\underline{0}}\ (du,dk),$$

so that - because $X_u(\varepsilon_{u,k}) = X_u(\underline{0})$ - we obtain:

$$S(du,dk) = P(\xi(U) = 0)\ \frac{\zeta_{\underline{0}}(du,dk)}{1-X_u(\underline{0})} .$$

Thanks to corollary 9.1.2 we may apply the same argument to $P(V\mu \in \ .\ |\ \mathcal{F}(V))$ for any $V \in \mathcal{B}$. Then we find

$$\eta^V_\mu(du,dk) = \frac{1}{1 - X_u(V^c\mu)}\ \zeta_{V^c\mu}(du,dk) . \qquad (9.2.1)$$

Note that the use of the letter "η" in the symbol "η^V" is justified by the fact that on the event $\{\xi(V) = 0\}$ the measures η and η^V coincide on V (see (8.2.2)). The theorem is now proved as follows:

$$P(\mu(B_1) = 1,\ \dots\ ,\ \mu(B_n) = 1,\ \xi(U) = n) =$$

$$\stackrel{1}{=} E\ S^{V_1}(B_1)\ 1_{\{\mu(B_2)=1,\ \dots, \mu(B_n) = 1, \xi(V_1^c) = n-1\}}$$

$$= E\ \eta_\mu^{V_1}(B_1)\ P(\xi(V_1) = 0\ |\ \mathcal{F}(V_1))\ 1_{\{\mu(B_2)=1,\dots,\mu(B_n)=1,\xi(V_1^c)=n-1\}}$$

$$\stackrel{2}{=} E\ \eta_\mu^{V_1}(B_1)\ 1_{\{\mu(B_2)=1,\dots,\mu(B_n) = 1,\ \xi(U) = n-1\}}$$

$$\stackrel{1}{=} E \int_{B_2} \eta^{V_1}_{V_2^c\mu+\varepsilon_{u_2,k_2}}(B_1)\ S^{V_2}(du_2,dk_2)\ 1_{\{\mu(B_3)=1,\dots,\mu(B_n)=1,\xi(V_2^c)=n-2\}}$$

$$\stackrel{3}{=} E \int_{B_2} \eta^{V_1}_{V_2^c\mu+\varepsilon_{u_2,k_2}}(B_1)\ \eta^{V_2}(du_2,dk_2)\ 1_{\{\mu(B_3)=1\dots,\mu(B_n)=1,\xi(U)=n-2\}}$$

$$\stackrel{4}{=} E \int_{B_n}\dots\int_{B_2} \eta^{V_1}_{\varepsilon_{u_2,k_2}+\dots+\varepsilon_{u_n,k_n}}(B_1)\ \eta^{V_2}_{\varepsilon_{u_3,k_3}+\dots+\varepsilon_{u_n,k_n}}(du_2,dk_2)\dots$$

$$\dots\eta^{V_n}_{\underline{0}}(du_n,dk_n)\ 1_{\{\xi(U)=0\}} =$$

$$= \int_{B_n} \dots \int_{B_1} \eta^{V_1}_{\varepsilon_{u_2,k_2}+\dots+\varepsilon_{u_n,k_n}}(du_1,dk_1)\dots\eta^{V_n}_{\underline{0}}(du_n,dk_n)\ P(\xi(U)=0)\,.$$

We used the following facts:

- The equalities $\overset{1}{=}$ follow from the simple observation:

(*) $\quad E\,X\,1_A = E\,X\,P(A \mid G)$ for all G-measurable non-negative r.v. X. Indeed: the measure S^V on $V \times K$ is determined by

$$S^V(du,dk) = P(\xi(V) = \mu(du,dk) = 1 \mid F(V)).$$

- To prove $\overset{2}{=}$, again use (*) .
- In order to show $\overset{3}{=}$ we first apply (9.2.1), then we argue like we did to prove $\overset{2}{=}$ and finally we use that on the event $\{\xi(V_2) = 0\}$ we have $V_2^c\mu + \varepsilon_{u_2,k_2} = \mu + \varepsilon_{u_2,k_2}$.
- Writing equalities similar to $\overset{1}{=}$ and $\overset{3}{=}$ another (n-2)-times, we obtain $\overset{4}{=}$.

This completes the proof. □

As we already mentioned, the law P is not determined by μ^z if (Σ) does not hold; this is illustrated in the following example.

EXAMPLE 9.2.1. Let $U = \{0,1\}$ and let μ be the simple non-marked point process whose distribution is determined by $P(\xi = \varepsilon_0) = 1 - P(\xi = \varepsilon_1) = p$ $(0 < p < 1)$. Then for all p we have $\xi^z = \xi$. □

§9.3. The likelihood ratio of two point processes.

Let P and Q be two probability measures on M . If P and Q are equivalent ($P \sim Q$; i.e. $P \ll Q$ and $Q \ll P$), then the r.v. $L = \frac{dQ}{dP}$ exists and is a.s. strictly positive. The process $\bar{L}$ is defined by $\bar{L}_u = L$ for all $u \in U$ ($\bar{L}_\Delta = 0$). Abusing the notation we write $L = \bar{L}$.

THEOREM 9.3.1. *Let* P *and* Q *be two equivalent probability measures on* M *and let the random measure* ζ *be a version of* $\mu^{z,P}$ *(the visible projection of* μ *under the law* P *), then* $\frac{+L}{{}^zL}\zeta$ *is a version of* $\mu^{z,Q}$ *. (Here* ${}^zL = {}^{z,P}L$ *; the visible projection of the process* L *under the law* P*).*

PROOF. First we note that the visible σ-field Z is defined unambiguously because P and Q are equivalent and that we know ${}^{z}L \gtrdot 0$ thanks to the visible section theorem. From theorem 5.4.4 it follows that $\frac{{}^{+}L}{{}^{z}L}\zeta$ is a visible measure. Suppose X is a visible non-negative marked process; then

$$E_Q \int X \, d\frac{{}^{+}L}{{}^{z}L}\zeta = E_P \int X \, L \frac{{}^{+}L}{{}^{z}L} d\zeta = E_P \int X \, {}^{z}L \frac{{}^{+}L}{{}^{z}L} d\zeta =$$

$$= E_P \int X \, {}^{+}L d\zeta = E_P \int X \, {}^{+}L \, d\mu = E_P \int XL d\mu = E_Q \int X d\mu \, . \qquad \square$$

THEOREM 9.3.2. *Suppose that* P *and* Q *are probability measures on* M *satisfying* $P(\xi(U) < \infty) = Q(\xi(U) < \infty) = 1$. *Suppose furthermore that* ζ *is a random measure on* $U \times K$ *such that* $X \equiv \hat{\zeta}(K) < 1$ *and that* A *is a strictly positive*[1] *marked stochastic process such that* $\hat{\zeta}(A)_u \equiv \int_K A_{u,k} \hat{\zeta}_u(dk) < 1$ *identically and* $\hat{\zeta} = {}^{-}\hat{\zeta}$ *and* $A = {}^{-}A$ *identically.*[2] *Finally assume that* $\mu^{z,P} = \zeta$ *and* $\mu^{z,Q} = A\zeta$ *(i.e. one can choose such versions). Then* P *and* Q *are equivalent and* $L = \frac{dQ}{dP}$ *is given by:*

$$\text{(9.3.1)} \qquad L(\varepsilon_{u_1,k_1} + \dots + \varepsilon_{u_n,k_n}) =$$

$$= L(\underline{0}) \, B_{u_1,k_1}(\underline{0}) \dots B_{u_n,k_n}(\varepsilon_{u_1,k_1} + \dots + \varepsilon_{u_{n-1},k_{n-1}}) \, ,$$

where

$$B_{u,k} = \frac{1 - X_u}{1 - \hat{\zeta}(A)_u} A_{u,k} \, ,$$

and $L(\underline{0})$ *is a normalizing constant.*

PROOF. First we prove the equivalence of P and Q . If P and Q are not equivalent, then there exists a number $n \geq 0$ such that $P(. \cap \{\xi(U) = n\})$ and $Q(. \cap \{\xi(U) = n\})$ are not equivalent. Suppose $Q(. \cap \{\xi(U) = n\}) \not\ll P(. \cap \{\xi(U) = n\})$; now we may check that there exist $V_1, \dots, V_n \in B$ disjoint and $B_1, \dots, B_n \in B \times K$

1) Here, this means $A_{u,k}(\omega) > 0$ for all ω,u,k. (Indistinguishability is not yet defined).

2) ${}^{-}A_{u,k}(\mu) = A_{u,k}(\{u\}^c\mu)$.

with $B_i \subset V_i \times K$ for all i, such that
$P(\mu(B_1) = 1, \ldots, \mu(B_n) = 1, \xi(U) = n) = 0$, but
$Q(\mu(B_1) = 1, \ldots, \mu(B_n) = 1, \xi(U) = n) \neq 0$ and this contradicts theorem 9.2.1.

Using theorem 9.3.1 we see that the set $N \equiv \{A \neq \frac{{}^{+}L}{z_L}\} \in Z \times K$ satisfies $C^P_\mu(N) = C^P_\zeta(N) = C^Q_\mu(N) = C^Q_\zeta(N) = 0$.
Theorem 7.1.1 now yields[1]:

$$^{+}L_{u,k} = A_{u,k}\,(\hat\zeta(^{+}L)_u + (1 - X_u)\,^{-}L_u)\ .$$

By integration w.r.t. $\hat\zeta$ this formula yields:

$$\hat\zeta(^{+}L)_u = \hat\zeta(A)_u\,(\hat\zeta(^{+}L)_u + (1 - X_u)\,^{-}L_u)\ ;$$

hence

$$\hat\zeta(^{+}L)_u = \frac{1 - X_u}{1 - \hat\zeta(A)_u}\cdot\hat\zeta(A)_u\cdot{}^{-}L_u\ ,$$

so that we obtain the following expression for ^{+}L :

$$^{+}L_{u,k} = \frac{1 - X_u}{1 - \hat\zeta(A)_u}\cdot A_{u,k}\cdot{}^{-}L_u = B_{u,k}\cdot{}^{-}L_u\ .$$

We thus found that if $\hat\xi_u = 0$, we have

$$L(\mu + \varepsilon_{u,k}) = B_{u,k}(\mu)\,L(\mu)\ .$$

Repeated use of this formula yields the desired expression for L . □

Remarks:

1. We have $X_u = \hat\zeta(1_{\Omega \times U \times K})_u = \hat\xi^{z,P}_u$ and $\hat\zeta(A)_u = \hat\xi^{z,Q}_u$.
2. Note that B satisfies: $B = {}^{-}B$.
3. It follows from (8.2.2) that

1) $\hat\zeta(^{+}L)_u = \int_K {}^{+}L_{u,k}\,\hat\zeta_u(dk)$.

$$B_{u,k}\eta^P(du,dk) = \eta^Q(du,dk) - \frac{A_{u,k}X_u - \hat{\zeta}(A)_u}{1 - \hat{\zeta}(A)_u}\mu(du,dk)$$

If K reduces to one point, we have $\hat{\zeta}(A)_u = A_u X_u$ so that then:

$$\frac{d\eta^Q}{d\eta^P} = B$$

everywhere.

In any case we see that if $\hat{\xi}_u = 0$, it follows that

$$\frac{\eta^Q(du,dk)}{\eta^P(du,dk)} = B_{u,k}$$

so that in formula (9.3.1) we may replace B by $d\eta^Q/d\eta^P$.

4. It should be intuitively clear that if we do not have $P(\xi(U) < \infty) = Q(\xi(U) < \infty) = 1$, equivalence of P and Q is rather exceptional.

For instance, if $U = \mathbb{R}$; λ is the Lebesgue measure and $c > 0$, then:

$$\Pi_{c\lambda}\left(\lim_{x\to\infty} \frac{\xi(-x,x)}{2cx} = 1\right) = 1 .$$

(cf. example 2.2.2 and the law of large numbers), while the events $\left\{\lim_{x\to\infty} \frac{\xi(-x,x)}{2cx} = 1\right\}$ are disjoint for different values of $c \in (0,\infty)$. □

CHAPTER 10

EXAMPLES

We finally study some examples of point processes and indicate their visible projections. Many abstract notions can be made explicit in the case of the zero-or-one-point process. In fact there are many things that can be said on visibility by only studying this elementary process; this assertion is illustrated in theorem 10.1.3 by an alternative proof of the domination of martingalelike measures by $(\xi - \xi^z)$ (cf. theorem 8.1.1).

Next we look more closely at the example to lemma 5.3.4 and theorem 6.1.1 (Example 5.3.1 and 6.1.1). The main object of this discussion is to show how everything can be checked in a concrete case.

More practical examples are formed by the Poisson process and its generalizations: the Cox process, the Gibbs process and the mixed sample process.

§10.1. The zero-or-one-point process.

We have already introduced this process in Example 2.2.1. We recall that its distribution can be obtained as follows (the process is a simple non-marked one; K reduces to one point, hence ξ is essentially equal to μ).

$$\xi = X\, \varepsilon_Y ,$$

where X and Y are independent r.v.'s, P(X=0) = 1 - P(X=1) = c ($0 \leqq c < 1$) and Y is U-valued and has a given distribution ν .

If $U = [0,1-c)$ and ν is the uniform distribution on U , then we obtain the same distribution of ξ as follows: Let (Ω, A, P) be [0,1] with the Lebesgue σ-field and the Lebesgue measure and

$$\xi_\omega = \begin{cases} \varepsilon_\omega & \text{if } 0 \leqq \omega < 1 - c , \\ \underline{0} & \text{if } 1-c \leqq \omega \leqq 1 . \end{cases}$$

(This form was used in example 5.2.1; cf. the corresponding figure 5.2.1). We shall study the example in this last form.

First assume that $0 < c < 1$. Let $\mathcal{N}$ denote the class of Lebesgue null-subsets of Ω . Now it is clear that:

$$\mathcal{F} = \mathcal{T}(\mathcal{B}[0,1-c),\mathcal{N}) \ .$$

(Of course, $\mathcal{B}(.)$ denotes the σ-field of Borel subsets of some space; note that $\mathcal{B}[0,1-c) = \mathcal{B}(U) = \mathcal{B})$, and for all $V \in \mathcal{B}$:

$$\mathcal{F}(V) = \mathcal{T}(\mathcal{B}([0,1-c) - V),\mathcal{N}) .$$

For a moment we define in addition to $\mathcal{Z}$ another σ-field on $\Omega \times U$:

$$\mathcal{Z}^o = \mathcal{T}(F \times V \mid V \in \mathcal{B},\ F \in \mathcal{B}([0,1-c) - V)) \ ,$$

which is slightly smaller than the visible σ-field. Using the monotone class theorem B.3 we see that for every visible set A there exists a set $A^o \in \mathcal{Z}^o$ such that $A \Delta A^o$ is evanescent. If $A^o \in \mathcal{Z}^o$ then we see that for all $u \in [0,1-c)$ we have:

$$(u,u) \in A^o \iff [1-c,1] \times \{u\} \subset A^o \iff \exists\ \omega \in [1-c,\ 1] \text{ such that } (\omega,u) \in A^o .$$

This in turn implies that for all visible sets A there exist null-sets N_1 and N_2, $N_1 \subset [0,\ 1-c)$ and $N_2 \subset [1-c,\ 1]$ such that for all $u \in [0,\ 1-c) - N_1$ we have

$$(u,u) \in A \iff \\ \iff ([1-c,\ 1] - N_2) \times \{u\} \subset A \iff \\ \iff \exists\ \omega \in [1-c,\ 1] - N_2 \text{ such that } (\omega,\ u) \in A . \tag{10.1.1}$$

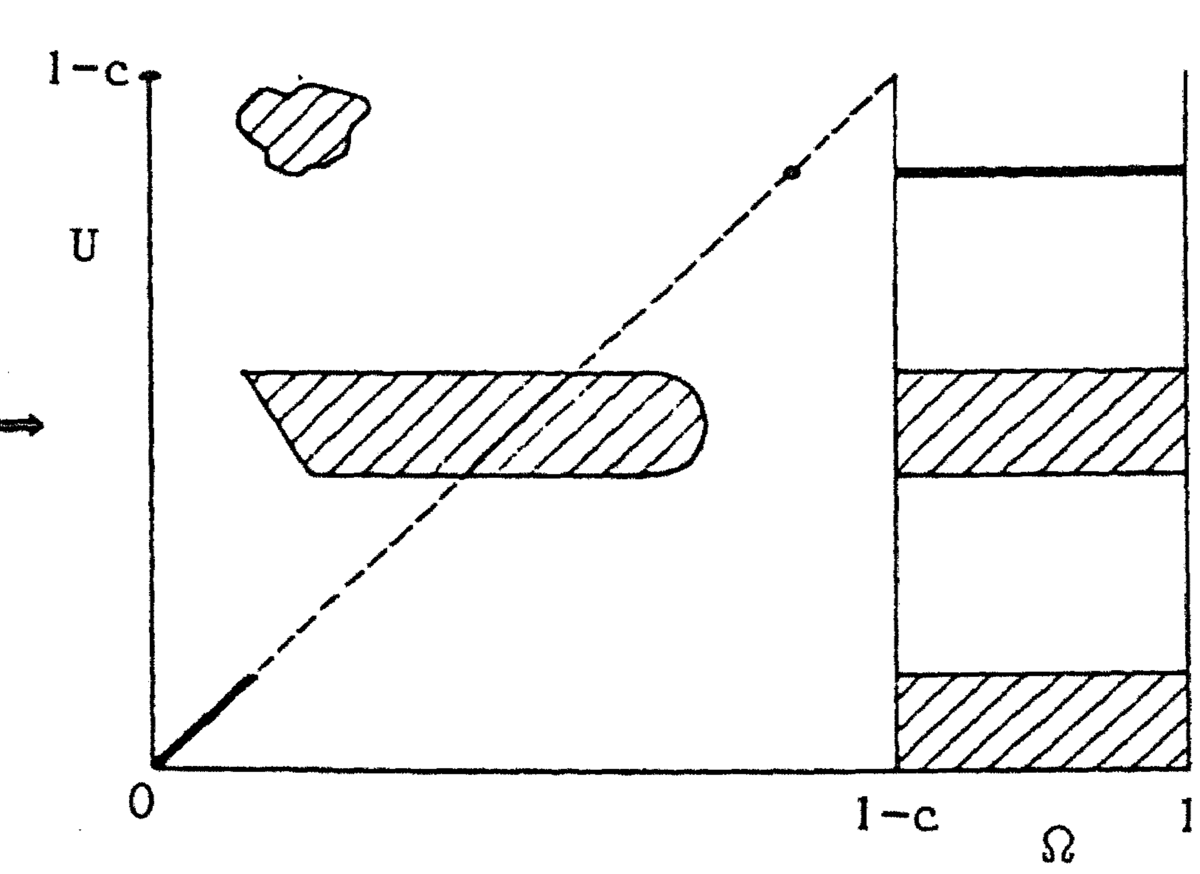


figure 10.1.1:
An element of $\mathcal{Z}^o$.

Hence , if we do not bother about evanescent sets, each visible set has the property that with each point on the diagonal $(\{(\omega,\omega) \mid \omega \in [0, 1-c)\})$ it contains the horizontal line segment at the same height above $[1-c, 1]$. For an illustration one may check that for all $V \in \mathcal{U}$ the set $H(V)$ has this property.

Random points are $U \cup \{\Delta\}$-valued $\mathcal{F}$-measurable r.v.'s. The $\mathcal{F}$-measurability implies that they are constant a.e. on $[1-c, 1]$.

<u>THEOREM 10.1.1.</u> *A random point* R *is visible if and only if*

$$P(\{\omega \mid R(\omega) = \omega\}) = 0 .$$

PROOF. Proof "⇐": One easily checks, that

$$\lceil R \rfloor \cap ([0, 1-c) \times U) \doteq \lim_i \bigcup_{V \in \mathcal{U}_i} (R^{-1}(V) \cap ([0, 1-c) - V)) \times V$$

because $R^{-1}(V) \in \mathcal{F}$ we clearly have

$$R^{-1}(V) \cap V^c \in \mathcal{T}(\mathcal{B}([0, 1-c) - V), \mathcal{N}) = \mathcal{F}(V) .$$

Furthermore, it is clear that

$$\lceil R \rfloor \cap ([1-c, 1] \times U) \doteq \begin{cases} \{\xi(U - \{u_0\}) = 0\} \times \{u_0\} \in \mathcal{Z} & \text{if } R = u_0 \text{ a.e. on } (1-c, 1], \\ \emptyset \in \mathcal{Z} \text{ if } R = \Delta \text{ a.e. on } (1-c, 1]. \end{cases}$$

Now the visibility of R is proved.

Proof "⇒": Because R is a random point, there exists a constant $u_0 \in U \cup \{\Delta\}$ such that $R = u_0$ a.e. on $[1-c, 1]$. This simple observation would contradict (10.1.1) if we had $P(\{\omega \mid R(\omega) = \omega\}) \neq 0$. □

Theorem 10.1.1 implies that the process satisfies condition (σ). For $\mathcal{F} \times \mathcal{B}$-measurable processes X we hence have ${}^{z}X \doteq {}^{-}X$. For instance: Let $V, B \in \mathcal{B}$, then the visible projection of the process $(\omega,u) \to \xi_\omega(V)\, 1_B(u)$ is: $(\omega,u) \to \xi_\omega(V - \{u\})\, 1_B(u)$. The visible projections of the $\mathcal{A} \times \mathcal{B}$-measurable process $(B \in \mathcal{B})$: $(\omega,u) \to 1_{[1-\frac{1}{2}c,\, 1]}(\omega)\, 1_B(u)$ and of the $\mathcal{F} \times \mathcal{B}$-measurable process: $(\omega,u) \to \frac{1}{2}\, 1_{\{\xi(U)=0\}}(\omega)\, 1_B(u)$ coincide and are equal to: $(\omega,u) \to \frac{1}{2}\, 1_{H(U)}(\omega,u)\, 1_B(u)$.

THEOREM 10.1.2. *Define the random measure* ζ *by:*

$$\zeta = \begin{cases} \frac{\lambda}{c} & \text{on } [1-c,\ 1] \quad \text{hence on } \{\xi = \underline{0}\}\ ; \\ \underline{0} & \text{on } [0,\ 1-c)\ , \end{cases}$$

where λ *is the Lebesgue measure on* U . *Then we have* $\xi^z = \zeta$.

PROOF. Because clearly ζ is F-measurable and a.s. diffuse, ζ is a visible random measure[1]. Furthermore, let X be a visible process. The fact that condition (Σ) holds, implies that $X \doteq {}^-X$ (theorem 6.1.4 ii)). Now we have:

$$E \int X\, d\xi = E \int {}^-X\, d\xi = \int_0^{1-c} {}^-X_\omega(\omega)\ \lambda(d\omega) = \int_0^{1-c} X^*_\omega(\underline{0})\ \lambda(d\omega)\ .$$

and on the other hand:

$$E \int X\, d\zeta = c \int_0^{1-c} X^*_u(\underline{0})\ \frac{\lambda(du)}{c} = \int_0^{1-c} X^*_u(\underline{0})\ \lambda(du)\ . \qquad \square$$

Now consider the case $c = 0$, then $U = [0,1)$. Because for all $V \in B$ the sets $\{\xi(V^c) = 0\}$ and $\{\xi(V) = 1\}$ only differ by a null-set, the graph of $Z : \omega \to \omega$ is visible. Indeed $\lceil Z \rfloor \doteq H(U)$. Note that $\lceil Z \rfloor$ is the diagonal. Using $P(\{\omega \mid \xi_\omega(\{\omega\}) = 1\}) = 1$, we find $\Sigma \subset \lceil Z \rfloor^c$ and because $E \int I_\Sigma\, d\xi = 0$ we have furthermore $\sigma = \Sigma = \lceil Z \rfloor^c$.

We shall show that now an arbitrary process X is visible and hence ${}^zX = X$ for any process X . We notice that for all $(\omega,u) \in \sigma = \Sigma$ we have $\xi_\omega = \{u\}^c\, \xi_\omega$; using this fact we find that for any real number α :

$$\{(\omega,u) \mid X_u(\omega) < \alpha\} =$$

$$= (\{(\omega,u) \mid X_u(\omega) < \alpha\} \cap \sigma) \cup (\{(\omega,u) \mid X_u(\omega) < \alpha\} \cap \sigma^c)$$

$$= (\{(\omega,u) \mid {}^-X_u(\omega) < \alpha\} \cap \sigma) \cup (\{X_Z < \alpha\} \times U) \cap \lceil Z \rfloor)$$

1) We already knew that ξ^z would be diffuse, since we proved that (σ) is satisfied.

(where again $Z: \omega \to \omega$). The first set in this union is visible because ^{-}X is visible. For the second, notice that we have $\{X_Z < \alpha\} \in A = F = F(Z)$ and use lemma 5.3.2.

Using definition 5.4.3 iii) we now see immediately that ξ is a visible random measure itself and hence is its own visible projection. In order to see that ξ is a visible measure we may use definition 5.4.3 ii) too: indeed ξ is of course F-measurable and $\hat{\xi} = 1_{\lceil Z \rfloor}$ is a visible process (again $Z: \omega \to \omega$).

For a further illustration of the basic notions we give a slightly different example of the zero-or-one-point process: Let $0 < c < \frac{1}{2}$ and $U = [c, 1-c)$, (Ω, A, P) as above and

$$\xi_\omega = \begin{cases} \varepsilon_c & \text{if } 0 \leqq \omega \leqq c, \\ \varepsilon_\omega & \text{if } c < \omega < 1-c, \text{ and} \\ \underline{0} & \text{if } 1-c \leqq \omega \leqq 1. \end{cases}$$

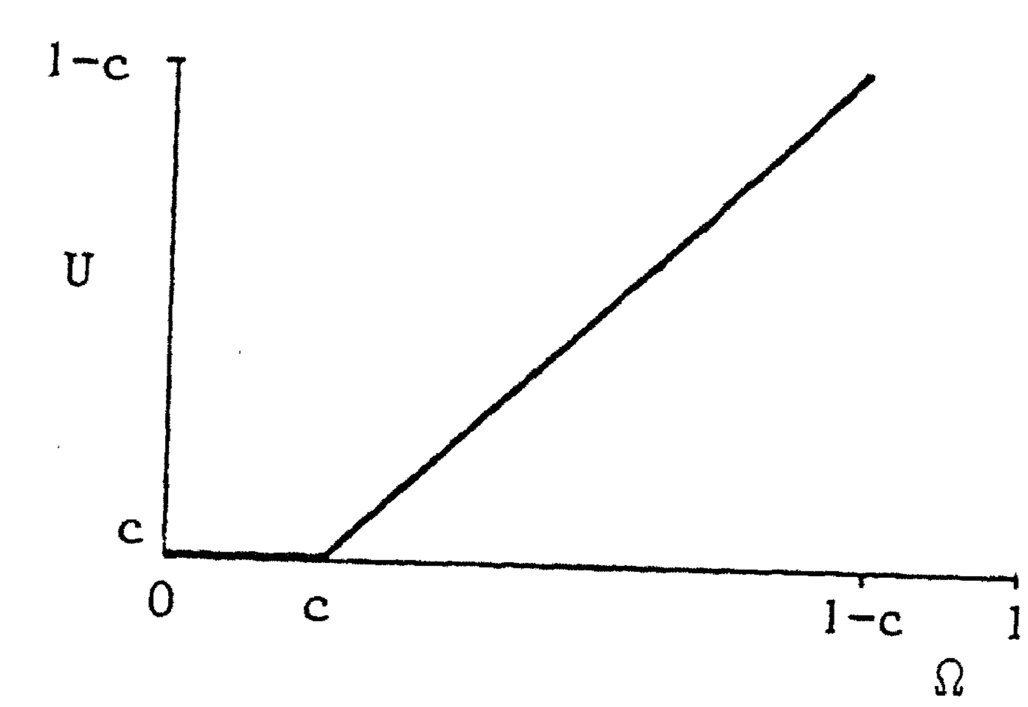


figure 10.1.2

Using the same arguments as above in the case $0 < c < 1$ we deduce:

$$F = T(B(c,1-c),[0,c],N) ;$$

for all $V \in B$ such that $c \notin V$:

$$F(V) = T(B((c, 1-c) - V), [0,c],N)$$

and

$$F(\{c\}) = T(B(c,1-c),N) .$$

Furthermore,

$$\sigma^c = \lceil Z \rfloor ,$$

where Z is the visible point:

$$Z : \omega \to \begin{cases} c & \text{if } \xi_\omega(U - \{c\}) = 0, \\ \Delta & \text{if not} \end{cases}$$

and $P(\xi(Z) > 0) = c = \frac{1}{2} P(\pi\lceil Z \rfloor)$. Condition (Σ) is satisfied.

If X an $\mathcal{F} \times \mathcal{B}$-measurable process, then

$$ {}^{z}X_c(\omega) = \begin{cases} \frac{1}{2} (X^*_c(\varepsilon_c) + X^*_c(\underline{0})) & \text{if } \omega \notin (c,1-c) , \\ X_c(\omega) & \text{if } \omega \in (c,1-c) , \end{cases} $$

and for all $u \in (c,1-c)$ we have

$$ {}^{z}X_u(\omega) = \begin{cases} X^*_u(\underline{0}) & \text{if } \omega \in [1-c,1] \cup \{u\} , \\ X_u(\omega) & \text{if } \omega \in [0,1-c) - \{u\} . \end{cases} $$

To determine ξ^z we use corollary 7.2.3. First we see that (choose $\mathcal{U}$ such that $\mathcal{U}_1 = \{ \{c\}, (c,1-c)\})$:

$$ S^{(c,1-c)} = \begin{cases} \frac{\lambda}{1-c} & \text{on } (c,1], \text{ hence if } \hat{\xi}_c = 0 , \\ \underline{0} & \text{on } [0,c], \text{ hence if } \hat{\xi}_c = 1 , \end{cases} $$

and that for all $u \in (c,1-c)$:

$$ P(\xi((c,1-c) - \{u\}) = 0 \mid \mathcal{F}(c,1-c)) = $$

$$ = P(\xi(c,1-c) = 0 \mid \mathcal{F}(c,1-c)) $$

$$ = \begin{cases} 1 & \text{on } [0,c] \text{ hence if } \hat{\xi}_c = 1 , \\ \frac{P(\xi = \underline{0})}{P(\hat{\xi}_c = 0)} = \frac{c}{1-c} & \text{on } (c,1] \text{ hence if } \hat{\xi}_c = 0 \end{cases} $$

From this it follows that

$$ {}_{(c,1-c)}\xi^z = \frac{\lambda}{c} \quad \text{on } [1-c,1] \text{ hence if } \xi = \underline{0} $$

and that for all ω we have $\hat{\xi}^z_u(\omega) = 0$ for all $u \in (c,1-c)$. Furthermore, for any $V \in \mathcal{U}$ on $\{\xi(V^c) \neq 0\}$ we have $P(\xi(V) = 1 \mid \mathcal{F}(V)) = 0$ and hence

$\xi^z(V) = 0$. Finally:

$$\xi^z(\{c\}) = P(\hat{\xi}_c = 1 \mid F(\{c\})) = \frac{P(\xi = \varepsilon_c)}{P(\xi(c,1-c) = 0)} = \frac{c}{2c} = \tfrac{1}{2}$$

on $[0,c] \cup [1-c,1]$ hence if $\xi(c,1-c) = 0$.

Combining the above facts we find:

$$\xi^z = \begin{cases} \tfrac{1}{2}\,\varepsilon_c & \text{on } [0,c]\,, \quad \text{hence if } \hat{\xi}_c = 1\,, \\ \underline{0} & \text{on } (c,1-c), \quad \text{hence if } \xi(c,1-c) = 1\,, \\ \tfrac{1}{2}\,\varepsilon_c + \frac{\lambda}{c} & \text{on } [1-c,1], \quad \text{hence if } \xi = \underline{0}\,. \end{cases}$$

Of course the fact that this formula is correct, can be proved in the same manner as theorem 10.1.2.

We now return to arbitrary point processes. As we announced at the beginning of this chapter, we shall give an alternative proof of the domination of martingalelike measures by $(\xi - \xi^z)$. For simplicity's sake we restrict ourselves here to the case of simple non-marked point processes and take $\Omega = M = M^{\cdot}$. This proof, which uses a reduction to the case of the zero-or-one-point process, gives insight into the structure of martingalelike measures.

THEOREM 10.1.3. *Let* ρ *be a martingalelike measure, then:*

$$\rho \ll \xi - \xi^z \qquad a.s..$$

PROOF. Thanks to formula (5.1.1) it is enough to check:

$$\rho^V \ll 1_{H(V)}(\xi - \xi^z) \quad \text{a.s.}$$

for all $V \in U$, where $\rho^V(du) = 1_{H(V)}(\cdot,u)\,\rho(du)$. The ρ^V are again martingalelike and have the following properties:

$$V^c\rho^V = \underline{0}$$

$$\rho^V = \underline{0} \quad \text{on } \{\xi(V) > 1\} \text{ and}$$

$$\rho^V(W) = 0 \text{ on } \{\xi(V-W) \neq 0\} \text{ for all } W \in B,\ W \subset V\ .$$

The theorem now follows by applying lemma 10.1.4, which we shall prove next, to ρ^V and $\frac{P(V\xi \in .,\ \xi(V) \leq 1 \mid F(V))}{P(\xi(V) \leq 1 \mid F(V))}$ as distribution of ξ. □

<u>LEMMA 10.1.4.</u> *Let* ξ *be the zero-or-one-point process on an arbitrary* U *with arbitrary* ν *and* $0 \leq c < 1$, *and let* ρ *be a martingalelike measure with the property that* $\rho(V) = 0$ *on* $\{\xi(V^c) \neq 0\}$, *then there exists a visible process* R *such that*

$$\rho = R(\xi - \xi^z) \ .$$

PROOF. Let the measure S on U be defined by:

$$S(V) = P(\xi(U) = \xi(V) = 1) = (1-c)\ \nu\ (V)$$

(cf. theorem 7.2.2). If c = 0, then ρ is visible and because ρ is martingalelike too, $\rho = \underline{0}$ a.s.. Hence assume that $c \neq 0$, and choose W, V $\in$ B, $W \subset V$, then:

$$0 = E\ (\rho(W) \mid \xi(V^c) = 0)$$

$$= \frac{E\ \rho(W)\ 1_{\{\xi(V^c)=0\}}}{P(\xi(V^c) = 0)}$$

$$= \frac{c.\rho_{\underline{0}}(W) + \int_V \rho_{\varepsilon_u}(W)\quad S(du)}{P(\xi(V^c) = 0)}$$

and thus the fact that ρ is martingalelike, implies:

$$c\rho_{\underline{0}}(W) + \int_V \rho_{\varepsilon_u}(W)\ S(du) = 0 \ . \tag{10.1.2}$$

Hence $\rho_{\underline{0}}(W) = 0$ implies $S(W) \neq 0$. Furthermore, the fact that $\rho_{\varepsilon_u}(W) \neq 0$ implies $u \in W$.

According to corollary 7.2.3 the visible projection of ξ is determined by:

$$\xi^z_{\underline{0}}\ (du) = \frac{S\ (du)}{c + \hat{S}_u} \quad \text{and} \quad \xi^z_{\varepsilon_u} = \frac{\hat{S}_u\ \varepsilon_u}{c + \hat{S}_u} \ ,$$

hence

$$(\xi - \xi^z)_{\underline{0}}(du) = \frac{-S(du)}{c + \hat{S}_u}$$

and

$$(\xi - \xi^z)_{\varepsilon_u} = \varepsilon_u - \frac{\hat{S}_u \varepsilon_u}{c + \hat{S}_u} = \frac{c\ \varepsilon_u}{c + \hat{S}_u} .$$

Because $\rho_{\underline{0}} << S$, there exists a function r such that:

$$\rho_{\underline{0}}(W) = - \int_W r(u)\ S(du) = \int_W (c + \hat{S}_u)\ r(u)\ (\xi - \xi^z)_{\underline{0}}\ (du) =$$

$$= \int_W 1_{H(U)}\ (\underline{0},u)\ (c + \hat{S}_u)\ r(u)\ (\xi - \xi^z)_{\underline{0}}\ (du) .$$

This suggests that $R = 1_{H(U)}\ (c + \hat{S})r$ is the right choice. In order to check, that this is the case we note that it follows from (10.1.2) that $\rho_{\varepsilon_u} = c\ r(u)\ \varepsilon_u$ or equivalently, $\rho_{\varepsilon_u}(W) = c\ r(u)\varepsilon_u(W)$ for all $W \in U$ and indeed we do have:

$$c\ r(u)\ \varepsilon_u(W) = \int_W 1_{H(U)}(\varepsilon_u,u')(c + \hat{S}_{u'})\ r(u')(\xi - \xi^z)_{\varepsilon_u}(du') .$$

□

§10.2. Example 5.3.2 and 6.1.1.

Recall that in this example (§5.3) we took $U = (0,1)$, $\Omega = (0,1)$ with Lebesgue measure and that the simple non-marked point process ξ is given by:

$$\xi_\omega = \begin{cases} \varepsilon_\omega & \text{if } \omega \leq \tfrac{1}{2} , \\ \varepsilon_{\omega - \frac{1}{2}} + \varepsilon_\omega & \text{if } \omega > \tfrac{1}{2} . \end{cases}$$

We defined $X = \omega \bmod \frac{1}{2}$, and saw

$$\xi = \varepsilon_X + 1_{\{\xi(U)=2\}}\ \varepsilon_{X+\frac{1}{2}} ,$$

and that X and $1_{\{\xi(U)=2\}}$ are independent.

It is easy to check that $F = A$ and that if $V \subset (0,\frac{1}{2}]$, we have

$$F(V) = T(B((0,1) - V), N) ,$$

and if $V \subset (\frac{1}{2}, 1)$, then

$$F(V) = T(B((0,1) - V - (V-\tfrac{1}{2})), B(V)_{\text{mod } \frac{1}{2}}, N) ,$$

where $B(V)_{\text{mod } \frac{1}{2}}$ denotes the σ-field on $V \cup (V-\frac{1}{2})$ generated by the sets $W \cup (W-\frac{1}{2})$, $W \in B(V)$ $(W - \frac{1}{2} = \{u \mid u + \frac{1}{2} \in W\})$.

Because for all $V \subset (0,\frac{1}{2}]$ we have $V \in F(V)$, the random point X is visible and furthermore $P(\xi(X) = 1) = P(\pi\lceil X\rfloor) = 1$, hence $\lceil X\rfloor \subset \Sigma^c$. On the other hand $X = \frac{1}{2}$ is a visible point too and $P(\xi(X+\frac{1}{2}) = 1) = \frac{1}{2}P(\pi\lceil X+\frac{1}{2}\rfloor) = \frac{1}{2}$ so that it is clear that $\sigma^c \supset \lceil X\rfloor \cup \lceil X + \frac{1}{2}\rfloor$. To prove that in fact $\Sigma^c = \lceil X\rfloor$ and $\sigma^c = \lceil X\rfloor \cup \lceil X + \frac{1}{2}\rfloor$ it is enough to show that for all visible points Z with $\lceil Z\rfloor \subset \lceil X + \frac{1}{2}\rfloor$ we have $P(\xi(Z) = 1) = \frac{1}{2}P(\pi\lceil Z\rfloor)$ and this is true because the above formula for the $F(V)$ $(V \subset (\frac{1}{2}, 1))$ in which the $B(V)_{\text{mod } \frac{1}{2}}$ figure, implies that

$$P((\pi\lceil Z\rfloor) \,\Delta\, (\{\xi(Z) = 1\} \cup (\{\xi(Z) = 1\} - \tfrac{1}{2}))) = 0$$

(cf. the argument in §10.1. To make this argument rigorous, define the σ-field Z^o on $\Omega \times (\frac{1}{2}, 1)$ by

$$Z^o = T(A \mid A = F{\times}V{:}F{\in}T(B((0,1){-}V{-}(V{-}\tfrac{1}{2})),\ B(V)_{\text{mod } \frac{1}{2}}),\ V{\in}B(\tfrac{1}{2},1)) .$$

For the sets A generating Z^o and hence for all sets in Z^o we have for all $u \in (\frac{1}{2}, 1)$: $(u,u) \in A \Longleftrightarrow (u-\frac{1}{2}, u) \in A$. Next note that for all visible sets $A \in \Omega \times (\frac{1}{2}, 1)$ there exists a set $A^o \in Z^o$ such that $A \doteq A^o$) .

Another way to prove that $\Sigma^c = \lceil X\rfloor$ and $\sigma^c = \lceil X\rfloor \cup \lceil X + \frac{1}{2}\rfloor$ consists of showing that

$$\xi^z = \varepsilon_X + \tfrac{1}{2}\varepsilon_{X+\frac{1}{2}} .$$

This measure indeed is visible and it satisfies the criterion of theorem 5.5.2 iii) .

§10.3. The Poisson process.

The Poisson process was already introduced in §2.2. We may recall that its distribution is determined by the intensity measure $\nu \in L(U)$. First assume that ν is diffuse; we noticed that then the process is a non-marked simple one. It has the property that for all $n > 1$ and $V_1, \dots, V_n \in \mathcal{B}$ disjoint, the r.v.'s $\xi(V_1), \dots, \xi(V_n)$ are independent From this it follows that for all $V \in \mathcal{B}$ we have:

$$E(\xi(V) \mid \mathcal{F}(V)) = E\,\xi(V) = \nu(V) .$$

Now it is clear that $\xi^z = \nu$; indeed ν being non-random obviously is a visible measure and it satisfies the criterion of theorem 5.5.2 iii). We hence have already proved one part of the following theorem.

THEOREM 10.3.1. *If ν is a diffuse Radon measure on U, then the Poisson process Π_ν is the unique simple non-marked point process such that:*

$$\xi^z = \nu .$$

PROOF. Only uniqueness remains to be proved. If $\nu(U) < \infty$ then $P(\xi(U) < \infty) = 1$ and hence uniqueness follows from theorem 9.2.1. In any case we see that if $V \in \mathcal{U}$, then according to theorem 9.1.1:

$$V\xi^{z,P(\cdot \mid \mathcal{F}(V))} = V\xi^{z,P} = V\nu ;$$

hence using theorem 9.2.1 we find that $P(V\xi \in \cdot \mid \mathcal{F}(V)) = \Pi_{V\nu}$, so that $V\xi$ is independent of $\mathcal{F}(V)$ and $\xi(V)$ has a Poisson distribution with parameter $\nu(V)$. □

If the intensity measure ν is not diffuse, then the process becomes essentially marked (§2.2). Choose $K = \{1, 2, \dots\}$. For any measure ν on U we define the measure $\tilde{\nu}$ on $U \times K$ by:

$$\tilde{\nu}(du \times \{k\}) = e^{-\hat{\nu}_u} \frac{\hat{\nu}_u^{k-1}}{k!} \nu(du)$$

with the convention $0^0 = 1$ $(\hat{\nu}_u = \nu(\{u\}))$. Hence writing

$A = \{u \in U \mid \hat{\nu}_u = 0\}$, then

$$\tilde{\nu}((A \cap .) \times \{1\}) = A\nu ,$$

$$\tilde{\nu}(A \times \{k\}) = 0 \quad (k = 2, 3, \dots) \text{ and}$$

$$\tilde{\nu}(\{u\} \times \{k\}) = e^{-\hat{\nu}_u} \frac{\hat{\nu}_u^k}{k!} \quad ((u,k) \in U \times K).$$

Now, in the same manner as theorem 10.3.1 one proves:

<u>THEOREM 10.3.2.</u> *If* ν *is a Radon measure on* U *, then the Poisson process* Π_ν *is the unique point process on* U *with marks in* $K = \{1, 2, \dots\}$ *such that*

$$\mu^z = \tilde{\nu} . \qquad \square$$

<u>Remark</u>. It follows that

$$A\xi^z = A\nu \quad \text{and} \quad \hat{\xi}^z = 1 - e^{-\hat{\nu}} . \qquad \square$$

<u>THEOREM 10.3.3.</u> *Let* ν *and* ρ *be two Radon measures on* U *with* $\nu(U) < \infty$ *and* $\rho(U) < \infty$. *Then* $\Pi_\rho << \Pi_\nu$ *if and only if* $\rho << \nu$, *and if so, then in the case where* ρ *and* ν *are diffuse, we have:*

$$\frac{d\Pi_\rho}{d\Pi_\nu}(\xi) = \exp\ (\nu(U) - \rho(U)) \prod_{i=1}^{n} g_\rho(u_i)$$

$$= \exp\ (\nu(U) - \rho(U)) \exp \int \log g_\rho(u)\ \xi(du)$$

for $\xi = \varepsilon_{u_1} + \dots + \varepsilon_{u_n}$ *and in the general case* (ν *and* ρ *not necessarily diffuse), we have:*

$$\frac{d\Pi_\rho}{d\Pi_\nu}(\mu) = \exp\ (\nu(U) - \rho(U)) \prod_{i=1}^{n} \left[\frac{1-\hat{\nu}_{u_i}}{1-\hat{\rho}_{u_i}} \exp\ (\hat{\nu}_{u_i} - \hat{\rho}_{u_i})\ g_\rho(u)^{k_i} \right]$$

for $\mu = \varepsilon_{u_1,k_1} + \dots + \varepsilon_{u_n,k_n}$. *In these expressions:*

$$g_\rho = \frac{d\rho}{d\nu} .$$

PROOF. If $\nu \sim \rho$ we only need to apply theorem 9.3.1. If $\rho << \nu$ then write $A = \{u \mid g_\rho(u) \neq 0\}$, then $\Pi_\rho(\xi(A^c) \neq 0) = 0$ and on $\{\xi(A^c) \neq 0\}$ the above formulae yield $\frac{d\Pi_\rho}{d\Pi_\nu} = 0$. On $\{\xi(A^c) = 0\}$ use the fact that both under Π_ρ and under Π_ν the random measures $A\xi$ and $A^c\xi$ are independent in order to see, that:

$$\frac{d\Pi_\rho}{d\Pi_\nu} = \frac{\Pi_\rho(\xi(A^c) = 0)}{\Pi_\nu(\xi(A^c) = 0)} \cdot \frac{d\Pi_{A\rho}}{d\Pi_{A\nu}} = e^{\nu(A^c)} \cdot \frac{d\Pi_{A\rho}}{d\Pi_{A\nu}} .$$

Next note that $A\rho \sim A\nu$ so that we may calculate $\frac{d\Pi_{A\rho}}{d\Pi_{A\nu}}$ and find the desired expression.

Assume conversely that we do not have $\rho << \nu$ then there exists a set $A \in B$ such that $\rho(A) \neq 0$, but $\nu(A) = 0$, and then $\Pi_\rho(\xi(A) \neq 0 \neq 0$ but $\Pi_\nu(\xi(A) \neq 0) = 0$, hence we do not have $\Pi_\rho << \Pi_\nu$. □

§10.4. The Cox process.

The Cox process Π_Γ on U is obtained as follows (§2.2): First we choose an element $\gamma \in L(U)$ according to a law Γ and then construct the Poisson process Π_γ :

(10.4.1) $$\Pi_\Gamma(.) = \int_{L(U)} \Pi_\gamma(.)\ \Gamma(d\gamma) .$$

If γ is atomless Γ-a.s., then the Cox process Π_Γ is non-marked simple. In the general case we choose $K = \{1, 2, \ldots\}$. If ρ is a random measure on U , then $\tilde{\rho}$ is a random measure on $U \times K$ (cf. §10.3). Note that ρ is visible if and only if $\tilde{\rho}$ is visible. Note furthermore that condition (Σ) is satisfied for every Cox process.

<u>THEOREM 10.4.1.</u> *Let* μ *be a Cox process*[1)]*and suppose* $\tilde{\zeta} = E(\tilde{\gamma} \mid F)$ *is a visible random measure; then*

$$\mu^z = \tilde{\zeta}$$

1) The letters Γ and γ have the same meaning as in (10.4.1).

PROOF. We use theorem 5.5.2 iii) or 5.5.4 iii). Let $V \in U$, $F \in F(V) \subset F$ and $D \in K$; then:

$$E\, 1_F\, \tilde{\xi}(V \times D) = E\, 1_F\, E(\tilde{\gamma}(V \times D) \mid F)$$

$$= E\, 1_F\, \tilde{\gamma}(V \times D)$$

$$= \int \Gamma(d\gamma) \int d\Pi_\gamma \quad 1_F\, \tilde{\gamma}(V \times D)$$

$$\stackrel{1}{=} \int \Gamma(d\gamma) \int d\Pi_\gamma\; 1_F\, \mu(V \times D)$$

$$= E\, 1_F\, \mu(V \times D)\ .$$

The equality $\stackrel{1}{=}$ follows from theorem 10.3.2. □

Remark: If γ is diffuse Γ-a.s., then clearly $\tilde{\xi}$ is visible. □

Now we define the infinitely remote σ-field on Ω :

$$F(\infty) = \bigcap_{\substack{V \in B \\ V \text{ bounded}}} F(V)\ .$$

THEOREM 10.4.2. *Let* μ *be a Cox process and suppose that* γ *is visible; then* γ *is* $F(\infty)$*-measurable. If, on the other hand, for some point process there exists an* $F(\infty)$*-measurable random measure* γ *on* U *such that* $\mu^z = \tilde{\gamma}$ *, then that process is the Cox process* Π_Γ *where* Γ *is the distribution of* γ *.*

PROOF. First we assume that μ is a Cox process and that γ is visible. Note that this implies that γ is F-measurable and that (theorem 10.4.1): $\mu^z = \tilde{\gamma}$. Furthermore, F-measurability of γ yields the existence of a mapping $g: M \to L(U)$ such that $g(\mu) = \gamma$ a.s. and writing $G_\gamma = g^{-1}(\{\gamma\})$ we have $\Pi_\gamma(G_\gamma) = 1$ for Γ-almost all γ ; indeed $\int\Gamma(d\gamma)\ \Pi_\gamma(G_\gamma) = \Pi_\Gamma(\{\mu \mid \mu \in G_{g(\mu)}\}) = 1$. If $B \in B$ bounded, then $\Pi_\gamma(\xi(B) = 0) > 0$ for Γ-almost all γ , and because

$$\Pi_\gamma(\{\mu \mid \mu \in G_\gamma,\ B^c\mu \in G_\gamma\}) =$$

$$= \Pi_\gamma(B^c\mu \in G_\gamma) =$$

$$= \frac{\Pi_\gamma(B^c\mu \in G_\gamma)\ \Pi_\gamma(\xi(B) = 0)}{\Pi_\gamma(\xi(B) = 0)}$$

$$\stackrel{1}{=} \frac{\Pi_\gamma(B^c\mu \in G\ ,\ \xi(B) = 0)}{\Pi_\gamma(\xi(B) = 0)}$$

$$= \frac{\Pi_\gamma(\mu \in G_\gamma,\ \xi(B) = 0)}{\Pi_\gamma(\xi(B) = 0)}$$

$$= \frac{\Pi_\gamma(\xi(B) = 0)}{\Pi_\gamma(\xi(B) = 0)} = 1\ .$$

(Equality $\stackrel{1}{=}$ holds because Π_γ is Poisson), we find $\mu \in G_\gamma \Longleftrightarrow B^c\mu \in G_\gamma$ for Π_γ-almost all μ. This implies directly that for every bounded B the random measure γ is $F(B)$-measurable so that γ is $F(\infty)$-measurable.

Suppose conversely that there exists an $F(\infty)$-measurable random measure γ on U such that $\mu^z = \tilde{\gamma}$. Choose $V \in U$, $F \in F(V)$, $D \in K$. Because $F(V) \supset F(\infty)$ and

$$E\ (\mu(V \times D)\ |\ F(V)) = E\ (\tilde{\gamma}(V \times D)\ |\ F(V)) = \tilde{\gamma}(V \times D)$$

we then have

$$\begin{aligned} E\ (\mu(V \times D)\ 1_F\ |\ \gamma) &= E\ (\ E\ (\mu(V \times D)\ 1_F\ |\ F(V))\ |\ \gamma) \\ &= E\ (\ E\ (\mu(V \times D)\ |\ F(V))\ 1_F\ |\ \gamma) \\ &= E\ (\tilde{\gamma}(V \times D)\ 1_F\ |\ \gamma) \end{aligned}$$

so that conditionally given γ, the process μ is Poisson (theorem 10.3.2) and hence unconditionally μ is Cox. □

An important class of Cox process is formed by the so-called mixed Poisson processes. These processes are obtained as follows: there exists a non-random measure ν on U and an $[0,\infty)$-valued r.v. M such that $\gamma = M\nu$. Now two cases should be distinguished:

i) $\nu(U) < \infty$; then without loss we may assume $\nu(U) = 1$. Now the process is an example of a mixed sample process with sample size distribution:

$$P(N = n) = \frac{1}{n!} Ee^{-M} M^n \qquad n = 0, 1, 2, \ldots$$

and sampled distribution ν . Mixed sample processes are studied in §10.6. At this point we only mention the obvious fact, that if ν is diffuse, then

$$\xi^z = E(M \mid \mathcal{F})\nu .$$

ii) $\nu(U) = \infty$; then, if $B_i \uparrow U$ ($B_i \in \mathcal{B}$ bounded) we find:

$$\frac{1}{\nu(B_i)} \int_K k\mu(B_i \times dk) \to M \quad \text{a.s.}$$

so that M , and hence γ , turns out to be $\mathcal{F}$-measurable. Theorem 10.4.2 now yields that:

$$\mu^z = \gamma = M\,\tilde{\nu}.$$

Kallenberg (78) (theorem 5.1) proves that conversely, if there exists a diffuse non-random measure ν on U and a $[0,\infty)$-valued r.v. M such that $\xi^z = M\nu$, then ξ is the Cox process with random intensity $\gamma = M\nu = \xi^z$. He uses the characterization of ξ^z by corollary 7.2.3.

In the mixed Poisson process case i) above only the mean number of points in U varies but their distribution remains the same. We now consider the case where the mean number of points in U is constant, hence where $\Gamma(\gamma(U) = G) = 1$ for some constant $G > 0$. We assume furthermore, that there exists a diffuse measure ν on U such that $\Gamma(\gamma \ll \nu) = 1$ and such that $\nu(U) = G$. Then theorem 10.3.3 yields for Γ-almost all γ : $\Pi_\gamma \ll \Pi_\nu$ and $\frac{d\Pi_\gamma}{d\Pi_\nu}(\xi) = g(\xi \mid \gamma)$, where $g(\varepsilon_{u_1} + \ldots + \varepsilon_{u_n} \mid \gamma) = \prod_{i=1}^{n} g_\gamma(u_i)$ with $g_\gamma = \frac{d\gamma}{d\nu}$. Using theorem 10.4.1 and Bayes' formula we find:

$$\xi^z(V) = E(\gamma(V) \mid \xi) = \frac{\int \gamma(V)\, g(\xi \mid \gamma)\, \Gamma(d\gamma)}{\int g(\xi \mid \gamma)\, \Gamma(d\gamma)} .$$

If λ is an arbitrary diffuse measure on U (not necessarily $\lambda(U) = G$), $\lambda >> \nu$, then write for Γ-almost all γ : $h_\gamma = \frac{1}{G} \frac{d\gamma}{d\lambda}$ and $h(\varepsilon_{u_i} + \dots + \varepsilon_{u_n} \mid \gamma) = \prod_{i=1}^{n} h_\gamma(u_i)$; now it follows that:

$$\xi^z(V) = \frac{\int \gamma(V)\, h(\xi \mid \gamma)\, \Gamma(d\gamma)}{\int h(\xi \mid \gamma)\, \Gamma(d\gamma)} .$$

It turns out that ξ^z has a density A w.r.t. λ . We calculate A ; because we may assume $A \in Z$, hence $A = {}^{-}A$ since (Σ) is satisfied, it is enough to calculate A in points (ω,u) such that $\hat{\xi}_u(\omega) = 0$; hence let $V \in B$. Then on $\{\xi(V) = 0\}$ we have

$$\xi^z(V) \int_{L(U)} h(\xi \mid \gamma)\, \Gamma(d\gamma) =$$

$$= \int_{L(U)} \int_V G\, h_\gamma(u)\, \lambda(du)\, h(\xi \mid \gamma)\, \Gamma(d\gamma)$$

$$= \int_V G \int_{L(U)} h_\gamma(u)\, h(\xi \mid \gamma)\, \Gamma(d\gamma)\, \lambda(du)$$

$$= \int_V G \int_{L(U)} h(\xi + \varepsilon_u \mid \gamma)\, \Gamma(d\gamma)\, \lambda(du)$$

so that on $\{\hat{\xi}_u = 0\}$ we have

$$A_u^*(\xi) = G \frac{\int h(\xi + \varepsilon_u \mid \gamma)\, \Gamma(d\gamma)}{\int h(\xi \mid \gamma)\, \Gamma(d\gamma)} .$$

Next note that h_γ is a probability density w.r.t. λ for Γ-almost all γ . Consider the U^{n+1}-valued r.v. $X = (X_0, \dots, X_n)^t$ whose distribution is obtained as follows: First one chooses γ according to the law Γ and then $X_0, \dots, X_n$ independently with density h_γ w.r.t. λ . Then according to Bayes' formula the conditional density (w.r.t. λ) of X_0 in u given $X_1 = u_1, \dots, X_n = u_n$ $(u_i \neq u;\ u_i \neq u_j,\ i \neq j)$ is the following:

$$\frac{\int h_\gamma(u) \prod_{i=1}^{n} h_\gamma(u_i)\, \Gamma(d\gamma)}{\iint h_\gamma(u') \prod_{i=1}^{n} h_\gamma(u_i)\, \Gamma(d\gamma)\, \lambda(du')} = \frac{\int h_\gamma(u) \prod_{i=1}^{n} h_\gamma(u_i)\, \Gamma(d\gamma)}{\int \prod_{i=1}^{n} h_\gamma(u_i)\, \Gamma(d\gamma)} =$$

$$= \frac{1}{G} A_u^* (\varepsilon_{u_1} + \dots + \varepsilon_{u_n}) .$$

EXAMPLE 10.4.1. Take $U = \mathbb{R}$ and λ the Lebesgue measure. The random intensity γ of the Cox process Π_Γ is given by

$$\gamma(du) = \frac{G\lambda(du)}{\sqrt{2\pi s}} \cdot \exp - \frac{(u - Y)^2}{2s} \qquad (s, G > 0)$$

where Y has a N(0,1)-distribution. (The points form a cluster of mean size G around the N(0,1) distributed centre Y). The random vector $X = (X_0, \ldots, X_n)^t$ (cf. just above) now has a N(0, E(n+1) + sI(n+1)) distribution, where E(k) is the (k × k)-matrix consisting only of ones and I(k) is the k-dimensional unit matrix. Then it is known that conditionally given $X_1 = u_1, \ldots, X_n = u_n$, the r.v. X_0 has a $N((1, \ldots, 1)\ (E(n) + sI(n))^{-1}\ (u_1, \ldots, u_n)^t,\ 1 + s - (1, \ldots, 1) \cdot (E(n) + sI(n))^{-1}(1, \ldots, 1)^t)$ distribution. Because $(E(k) + sI(k))^{-1} = \frac{1}{s}\ (I(k) - \frac{E(k)}{k+s})$ we found that in this example if $u_i \neq u$ and $u_i \neq u_j$ for $i \neq j$, then $A^*_u(\varepsilon_{u_1} + \ldots + \varepsilon_{u_n})$ is G times the density in u of a
$N(\frac{u_1 + \ldots + u_n}{n + s},\ s(1 + \frac{1}{n + s}))$ distribution. □

§10.5. The Gibbs process.

As we saw in §2.2 a Gibbs process P is determined by its specification, i.e. by the system of conditional distributions:

$$(P(V\mu \in . \mid \mathcal{F}(V)))_{V \in \mathcal{U}}$$

Generally, one assumes that $P(V\mu \in . \mid \mathcal{F}(V)) = \Pi_{\nu(\mathcal{F}(V))}$, where $\nu(\mathcal{F}(V))$ denotes an $\mathcal{F}(V)$-measurable random measure on V.

It is clear (cf. theorem 5.5.4 iii)), that the visible projection is determined by the specification. Using theorem 9.1.1 and theorem 9.2.1 we see that conversely the specification of a Gibbs process can be expressed in terms of the visible projection.

In statistical mechanics Gibbs processes are used to describe non-ideal gases. Then one takes $U = \mathbb{R}^d$ and one assumes that $\nu(\mathcal{F}(V)) << \lambda$ (λ denoting the Lebesgue measure) for all $V \in \mathcal{U}$, and more precisely for all $u \in V$:

$$\nu(F(V))(du) = \left(\int_{U-V} \phi(|u - v|)\ \xi(dv) + \Phi_u \right) \lambda(du) \ ,$$

where $\phi(r)$ indicates the interaction potential between two particles at a distance r (of course the integral $\int_{U-V} \phi(|u - v|)\ \xi(du)$ has to exist) and Φ represents the effect of some external field. Under these circumstances it follows that ξ^z has a density A w.r.t. λ and

$$A_u = \int_{U-\{u\}} \phi(|u - v|)\ \xi(dv) + \Phi_u \ .$$

This follows from corollary 7.2.3: First note that for almost all ω for all $V \in \mathcal{U}$ and all $u \in V$ we have $P(\hat{\xi}_u \neq 0 \mid F(V))(\omega) = 0$. This implies that ξ^z is a.s. diffuse and that a.s.: $P(\xi(V - \{u\}) = 0 \mid F(V)) = P(\xi(V) = 0 \mid F(V)) = e^{-\nu(F(V))(V)}$. Furthermore: $P(\xi(V) = \xi(W) = 1 \mid F(V)) = \frac{\nu(F(V))(W)}{\nu(F(V))(V)} \cdot e^{-\nu(F(V))(V)}\nu(F(V))(V) =$ $= \nu(F(V))(W)\ e^{-\nu(F(V))(V)}$. Again using corollary 7.2.3 we see that for all $V \in \mathcal{U}$ we have a.e. on $\{\xi(V) = 0\}$: $V\xi^z(du) = \frac{P(\xi(V) = \xi(du) = 1 \mid F(V))}{P(\xi(V - \{u\}) = 0 \mid F(V))} = \nu(F(V))(du)$. This proves the assertion because on $\{\xi(V) = 0\}$ of course $\xi(V- \{u\}) = 0$ for all $u \in V$.

Remarks

1. Note that we did not answer the question for which specifications there exists a Gibbs process, nor for which specifications there exists a uniquely determined Gibbs process.

2. If an arbitrary simple non-marked point process satisfies condition (Σ) we may determine the system $(P(V\mu \in \cdot \mid F(V)))_{V \in \mathcal{U}}$ (using theorem 9.1.1 and 9.2.1); hence then the process is a Gibbs process having the above system as its specification.

3. If we describe a gas in $U = \mathbb{R}^d$ consisting of n different kinds of particles, we may choose $K = \{1, \dots , n\}$. The interaction potential between two particles at a distance r , one of kind k and one of kind ℓ , now becomes $\phi_k(r,\ell)$ $(= \phi_\ell(r,k))$. The external field may influence different kinds of particles in a different way: $\Phi_{u,k}$. We assume that μ^z has a density A w.r.t. $\lambda \times \tau$ (τ denoting the counting measure on K), and

$$A_{u,k} = \int_{(U-\{u\}) \times K} \phi_k(|u - v|,\ell)\ \mu(dv,d\ell) + \Phi_{u,k} .$$

Of course the corresponding specification now can be determined. □

§10.6. Mixed sample processes.

A simple non-marked point process on U is called a *diffuse sample process* if there exist a non-negative integer n, a diffuse probability distribution ν on U and independent U-valued r.v.'s $X_1, \ldots, X_n$ with distribution ν, such that

$$\xi = \varepsilon_{X_1} + \ldots + \varepsilon_{X_n} .$$

Note that the X_i are a.s. different because ν is diffuse. If ν has atoms, there is a possitive probability that two or more X_i coincide. Hence we define:

A *sample process* on U is a simple point process on U with marks in $K = \{1, 2, \ldots\}$ for which there exists a non-negative integer n, a probability measure ν on U and independent U-valued r.v.'s X_1, $\ldots, X_n$ with distribution ν, such that

$$\int_K k\mu(.,\ dk) = \varepsilon_{X_1} + \ldots + \varepsilon_{X_n} .$$

The number n is called the *sample size*; the probability measure ν is called the *sampled distribution*.

A mixed sample process on U is a simple point process on U with marks in $K = \{1, 2, \ldots\}$ for which there exists a $\{0, 1, 2, \ldots\}$-valued r.v. N and a probability measure ν on U such that, conditionally given $N = n$ $(n = 0, 1, 2, \ldots)$ the process is a sample process on U with sample size n and sampled distribution ν. Hence there exist U-valued r.v.'s $X_1, X_2, \ldots$ with distribution ν, which are independent from N and from each other, such that:

$$\int_K k\mu(.,dk) = \sum_{n=0}^{\infty} 1_{\{N=n\}} \cdot (\varepsilon_{X_1} + \ldots + \varepsilon_{X_n}) .$$

Writing $P(N=n) = p_n$ we find for all $B \in \mathcal{B}$ and all $m = 0, 1, 2, \ldots$:

$$P\left(\int_K k\ \mu(B,dk) = m \right) = \sum_{n=0}^{\infty} p_n \binom{n}{m} \nu(B)^m \nu(B^c)^{n-m} .$$

Note that $N = \int_K k\ \mu(U,dk)$; if the sampled distribution is diffuse, then this expression reduces to $N = \xi(U)$.

<u>THEOREM 10.6.1.</u> *Let* μ *be a mixed sample process with diffuse sampled distribution* ν . *Let* I *be the set* $\{n \in \{1, 2, \dots\} \mid p_{n-1} = 0\}$. *Define the random measure* ζ *on* U *by:*

$$(10.6.1) \qquad \zeta = \xi 1_{\{N\in I\}} + (N+1) \frac{p_{N+1}}{p_N} . \nu .$$

(This defines ζ a.s., since $p_N > 0$ a.s.). *Then:* $\xi^z = \zeta$.

PROOF. First we shall determine Σ^c. From §6.1 we know that $\Sigma^c \in \{(\omega,u) \mid \hat{\xi}_u(\omega) = 1\}$. In fact we have

$$(10.6.2) \qquad \Sigma^c \doteq \bigcup_{n\in I} \{(\omega,u) \mid \hat{\xi}_u(\omega) = 1,\ N(\omega) = n\} =$$

$$\doteq \bigcup_{n\in I} \{(\omega,u) \mid \hat{\xi}_u(\omega) = 1,\ \xi_\omega(U - \{u\}) = n-1\} .$$

To prove this, fix (ω,u) such that $\hat{\xi}_u(\omega) = 1$. We clearly may assume $p_{N(\omega)} > 0$. Write $n = N(\omega)$. For all sets $V \in \mathcal{U}$ such that $(\omega,u) \in H(V)$ we have $P(\xi(V) = 0 \mid \mathcal{F}(V))(\omega) = \dfrac{P(\xi(U) = \xi(V^c) = n-1)}{P(\xi(V^c) = n-1)}$. Because $P(\xi(U) = \xi(V^c) = n-1) = 0$ if an only if $n \in I$, formula (10.6.2) follows from theorem 6.1.3.

Now it can be seen that ζ is visible; indeed, according to (10.6.1) ζ is the sum of two random measures; the first of them is visible because the indicator of its support and process of atomsizes are both equal to 1_{Σ^c}; the second is visible because it is $\mathcal{F}$-measurable and diffuse.

To complete the proof we show that ζ satisfies the criterion of theorem 5.5.2 iii). To that end we verify that for all $V \in \mathcal{U}$ and $m = 0, 1, 2, \dots$. we have on $\{\xi(V^c) = m\}$:

$$P(\xi(V^c) = m) . E(\xi(V) \mid \mathcal{F}(V)) =$$

$$= \sum_{\ell=1}^{\infty} \ell\ P(\xi(V) = \ell,\ \xi(V^c) = m) =$$

$$= \sum_{\ell=1}^{\infty} \ell \, p_{m+\ell} \binom{m+\ell}{\ell} \nu(V)^{\ell} \, \nu(V^c)^m$$

$$\stackrel{1}{=} \sum_{\substack{\ell=1 \\ m+\ell \in I}}^{\infty} \ell \;\; P(\xi(V) = \ell,\ \xi(V^c) = m) \; +$$

$$+ \sum_{\substack{\ell=1 \\ m+\ell \notin I \\ m+\ell+1 \notin I}}^{\infty} (m+\ell) \frac{p_{m+\ell}}{p_{m+\ell-1}} \nu(V) \; p_{m+\ell-1} \binom{m+\ell-1}{m} \nu(V)^{\ell-1} \nu(V^c)^m$$

$$= \sum_{\substack{\ell=1 \\ m+\ell \in I}}^{\infty} \ell \; P(\xi(V) = \ell,\ \xi(V^c) = m) \; +$$

$$+ \sum_{\substack{\ell=0 \\ m+\ell+1 \notin I \\ m+\ell+2 \notin I}}^{\infty} (m+\ell+1) \frac{p_{m+\ell+1}}{p_{m+\ell}} \nu(V) \; P(\xi(V) = \ell,\ \xi(V^c) = m)$$

$$\stackrel{2}{=} P(\xi(V^c)\ m) \,.\, E(\zeta(V) \mid F(V)).$$

At $\stackrel{1}{=}$ note that if $m+\ell+1 \in I$, then $p_{m+\ell} = 0$, at $\stackrel{2}{=}$ note that if $m+\ell+2 \in I$, then $p_{m+\ell+1} = 0$. □

Now we consider the case where ν is not diffuse.

<u>THEOREM 10.6.2.</u> *Let* ν *be the sampled distribution of a mixed sample process on* U *; write* $A = \{u \mid \hat{\nu}_u = 0\}$ *and define the random measure* ζ *on* $U \times K$ $(K = \{1, 2, \ldots\})$ *by:*

$$A\zeta(. \times \{1\}) = A\,\xi 1_{\{N \in I\}} + (N+1) \frac{p_{N+1}}{p_N} A\nu \; ,$$

$$A\zeta(\,U \times \{2, 3, \ldots.\}) = 0$$

and for $u \in A^c$ [1]*:*

$$\hat{\zeta}_u(\{\ell\}) = \frac{p_{n+\ell} \binom{n+\ell}{n} \hat{\nu}_u^{\ell}}{\sum_{i=0}^{\infty} p_{n+i} \binom{n+i}{i} \hat{\nu}_u^{i}} \quad \text{on } \{ \textstyle\int_K k \, \mu(U - \{u\}, dk) = n\} \; ,$$

where again $I = \{n \in \{1, 2, \ldots.\} \mid p_{n-1} = 0\}$. *Then* $\mu^z = \zeta$.

1) With the convention $0/0 = 0$.

PROOF. We only check the expression for $\hat{\zeta}_u$ $(u \in A^c)$, the rest being proved analogously to theorem 10.6.1. On $\{ \int_K k\, \mu(U-\{u\}),dk) = n\}$ we argue as follows:

$$\hat{\mu}_u^z(\{\ell\}) = P(\hat{\mu}_u(\{\ell\}) = 1 \mid F(\{u\}))$$

$$= \frac{P(N=n+\ell,\ \int_K k\hat{\mu}_u(dk) = \ell)}{P(\int_K k\, \mu(U-\{u\},\ dk) = n)}$$

$$= \frac{p_{n+\ell}\binom{n+\ell}{\ell}\hat{\nu}_u(1-\hat{\nu}_u)^n}{\sum_{i=0}^{\infty} p_{n+i}\binom{n+i}{n}\hat{\nu}_u^i(1-\hat{\nu}_u)^n} = \hat{\zeta}_u(\{\ell\}) .$$

□

Remarks

1. Note that if $p_n = 0$, then $\hat{\zeta}_u(K) = 1$ on $\{ \int_K k\, \mu(U-\{u\},dk) = n\}$.

2. We see that

$$\Sigma^c \doteq \{(\omega,u) \mid \int_K k\, \mu_\omega(U-\{u\},dk) + 1 \in I\} .$$

(cf. the formula for Σ^c in the special case where ν is diffuse and remark 1), and because $A\nu$ is diffuse it follows from the expression for $\hat{\zeta}_u$ $(u \in A^c)$ that

$$\hat{\xi}^z = 1_{\Sigma^c} + \sum_{u\in A^c}^{\infty} \sum_{n=0}^{\infty} \frac{\sum_{i=1}^{\infty} p_{n+i}\binom{n+i}{i}\hat{\nu}_u^i(1-\hat{\nu}_u)^n}{\sum_{i=0}^{\infty} p_{n+i}\binom{n+i}{i}\hat{\nu}_u^i(1-\hat{\nu}_u)^n} .$$

$$\cdot\, 1_{\{(\omega,u) \mid \int_K k\mu_\omega(U-\{u\},\ dk) = n\}} ,$$

and hence that

$$\sigma^c \doteq \Sigma^c \cup \{(\omega,u) \mid u \in A^c,\ \int_K k\mu_\omega(U-\{u\},dk) < m\} ,$$

where m is the smallest integer such that $P(N > m) = 0$. Hence, if $p_n \neq 0$ for all $n = 0, 1, 2, \ldots$, then $\Sigma = \Omega \times U$ and $\sigma = \Omega \times A$.

3a. The zero-or-one-point process is an example of a mixed sample process: take $p_0 = 1-c$, $p_1 = c$ and $p_n = 0$ for $n = 2, 3, \ldots$.

3b. The Poisson process with finite intensity measure is an example of a mixed sample process too: If N has a Poisson distribution with parameter m , then it is easy to check that the above formulae (theorem 10.6.2) reduce to $\mu^z = (m\nu)^\sim$.

3c. If N has a degenerate distribution - hence in the case of a non-mixed sample process - we obtain $\mu^z = \mu$.

4. As we already mentioned in §10.4, mixed Poisson processes are examples of mixed sample processes. Note that these processes always have $p_n \neq 0$ for all $n = 0, 1, 2, \ldots$.

We shall give one example: Assume that the distribution of N is obtained by mixing the Poisson distribution w.r.t. a Γ-distribution with parameters k and λ $(k,\lambda > 0)$ i.e.:

$$
\begin{aligned}
p_n = p(N=n) &= \int e^{-m} \frac{m^n}{n!} \frac{\lambda^k m^{k-1} e^{-\lambda m}}{(k-1)!} \, dm \\
&= \frac{(n+k-1)!}{n!\,(k-1)!} \frac{\lambda^k}{(\lambda+1)^{k+n}} \\
&= \binom{n+k-1}{n} p^k (1-p)^n ,
\end{aligned}
$$

where $p = \lambda/(\lambda+1)$ (For all $r > -1$ we write: $r! = \Gamma(r+1) = \int_0^\infty e^{-x} x^r \, dx$). Because of the last expression for p_n this process is called the negative binomial point process (cf. Grégoire (80)).

For simplicity we now assume that ν is diffuse $(\nu(U) = 1)$. We mention one interesting property of the negative binomial point process: For all $B \in \mathcal{B}$ and $n = 0, 1, 2, \ldots$ we have

$$
P(\xi(B) = n) = \binom{n+k-1}{n} \left(\frac{p}{p+(1-p)\nu(B)}\right)^k \left(\frac{(1-p)\nu(B)}{p+(1-p)\nu(B)}\right)^n .
$$

To derive this formula in an exercise in elementary probability. It is a much easier exercise to deduce from theorem 10.6.1 that:

$$
\xi^z = \frac{N + k}{\lambda + 1} \nu = (N + k)(1 - p)\nu \quad . \qquad \square
$$

APPENDICES

Appendix A. The section theorem without filtration.

A fundamental result, both in the theory of processes on $\mathbb{R}_+$ and in the theory developed in this treatise, is the so-called section theorem in a space without filtration.

THEOREM A.1. *Let* (Ω, G, P) *be a complete probability space and let* (U, B) *be a Polish space with its Borel* σ*-field. If* $A \in G \times B$ *(*$G \times B$ *denoting the product-*σ*-field on* $\Omega \times U$*), then:*

$$\pi(A) \in G$$

and there exists a G*-measurable mapping* $R : \Omega \to U \cup \{\Delta\}$ *such that*

$$\lceil R \rfloor \subset A$$

and such that

$$P(\pi\lceil R \rfloor) = P(\pi(A)) \ .$$

PROOF. We refer to the literature. The most difficult part of the proof consists of showing that projections are G-measurable. This result does not apply to arbitrary G, but it is true if G is complete, as in our case. In the literature many proofs are given for the case $U = \mathbb{R}_+$ (Dellacherie (72) I - T37; Dellacherie and Meyer (75) III-44); this is enough since there exists a measurable mapping f from $\mathbb{R}_+$ onto U. A direct proof of a slightly stronger assertion can be found in Dellacherie and Meyer (75)-A-IV-81). □

Appendix B. The monotone class theorem.

In several instances we need the following basic result.

THEOREM B.1. *Let* D *be a vector space of real-valued functions on some space* E . *Assume that* D *contains the constant functions, is closed* *under uniform convergence and under monotone convergence of uniformly bounded functions. Let* $C \subset D$ *be closed under multiplication. Then* D *contains all* $T(C)$*-measurable functions.*

PROOF. See for instance Dellacherie and Meyer (75)-I-21. □

We generally only need the following obvious consequence of the monotone class theorem:

COROLLARY B.2. *Let* (E_1, E_1) *and* (E_2, E_2) *be two measurable spaces. Let* D *be a vector space of real-valued functions on* $E_1 \times E_2$, *containing the constants and closed under convergence. If* $1_{D\times D'} \in D$ *for all* $D \in E_1$, $D' \in E_2$, *then* D *contains all* $E_1 \times E_2$*-measurable functions. It is enough to check* $1_{D\times D'} \in D$ *for* D *(resp.* D'*) in an* E_1*- (resp.* E_2*-) generating class of sets which is closed under intersection.* □

On one occasion we need another, perhaps more familiar form of the monotone class theorem.

THEOREM B.3. *Let* C *be an algebra of subsets of some space* E. *Let* E *be the smallest collection of subsets of* E, *which contains* C *and which is closed under countable monotone unions and countable monotone intersections. Then* $E = T(C)$.

PROOF. Meyer and Dellacherie (75)-I-19. □

Appendix C. Some results on conditioning.

It is known that it is possible to construct the conditional distribution (given some σ-field) of any r.v. taking values in a Polish space (see e.g. Bauer (78) §56). We mentioned that the space of all Radon measures on a Polish space is Polish itself. Hence we may speak of the conditional distribution of a random measure ρ and in particular of μ.

Consequently we may assume that conditional expectations of functions of one or more random measure are expectations w.r.t. the conditional distribution of this (these) random measure(s). If we speak simultaneously of the conditional expectations of an uncountable number of functions, this assumption becomes essential.

We now prove a general theorem on conditional distributions.

THEOREM C.1. *Let* (Ω, A, P) *be a probability space. Assume that the* *measurable space* (Ω, A) *is a Borel measurable Polish space. Let* B *and* C *be* σ*-fields and* $A \supset B \subset C$. *Denote the conditional distribution* $P(. \mid C)(\omega)$ *by* P_ω^C . *Then for all* $A \in A$ *for almost all* ω *the function* $P(A \mid B)(.)$ *is a version of* $P_\omega^C(A \mid B)(.)$. *Hence in particular, for almost all* ω *we have:*

$$P_\omega^C (A \mid B) (\omega) = P(A \mid B)(\omega).$$

PROOF. We have to show that $P(A \mid B)(.)$ for P-almost all ω is a version of $P_\omega^C(A \mid B)(.)$: Of course $P(A \mid B)$ is B-measurable; furthermore conditional probabilities (given B) and conditional distributions (given C) are determined by integrals; hence, let $B \in B$ and $C \in C$ and consider

$$\int_C \int_B P(A \mid B)(\omega') \, P_\omega^C(d\omega') \, P(d\omega) = \int_{C\cap B} P(A \mid B)(\omega') \, P(d\omega')$$
$$= P(A \cap B \cap C)$$
$$= \int_C P_\omega^C(A \cap B) \, P(d\omega) . \qquad \square$$

In view of theorem C.1 the following result is not surprising. It deals with the σ-fields $F(V)$ and is used very often, especially in §7.4.

<u>THEOREM C.2.</u> *Let* $V, W \in U$, $W \subset V$ *and let* X *be some non-negative r.v.; then:*

$$E(X \mid F(V)) = \frac{E(X \, 1_{\{\xi(V-W)=0\}} \mid F(V))}{P(\xi(V-W) = 0 \mid F(V))}$$

a.e. on $\{\xi(V-W) = 0\}$.

PROOF. We could easily prove this theorem by using theorem C.1 (cf. Kallenberg (83) lemma 13.9), but we prefer to copy the direct proof of Papangelou (74) proposition 1.

To prove the theorem it suffices to show:

$$E \, 1_B \, E(X \, 1_{\{\xi(V-W)=0\}} \mid F(V)) = E \, 1_B \, P(\xi(V-W) = 0 \mid F(V)) \, E(X \mid F(W))$$

for all $B \in F(W)$, $B \subset \{\xi(V-W) = 0\}$. Note that $F(V) \cap \{\xi(V - W) = 0\} = F(W) \cap \{\xi(V-W) = 0\}$. Hence there exists a set $C \in F(V)$ such that $B = C \cap \{\xi(V-W) = 0\}$. Now it is seen that

$$E\, 1_C\, 1_{\{\xi(V-W)=0\}}\, E(X\, 1_{\{\xi(V-W)=0\}} \mid F(V)) =$$

$$= E\, 1_C\, P(\xi(V-W) = 0 \mid F(V))\, X\, 1_{\{\xi(V-W)=0\}}$$

$$= E\, (E\, (1_C\, P(\xi(V-W) = 0 \mid F(V))\, X\, 1_{\{\xi(V-W)=0\}} \mid F(W)))$$

$$= E\, 1_C\, P(\xi(V-W) = 0 \mid F(V))\, E(X \mid F(V))\, 1_{\{\xi(V-W)=0\}} \,. \qquad \square$$

Appendix D. Proof of theorem 3.4.5.

In this appendix we prove theorem 3.4.5. This is a theorem on point process on $[0,\infty)$ and on previsibility. We only give a sketch of the proof. To complete this, it should be elaborated on the one hand by using techniques from §7.4 and on the other hand by using arguments from the general theory of processes on $[0,\infty)$ (see e.g. Dellacherie and Meyer (75) and (80), also see Papangelou (74)). The notation in this appendix differs in some respects from the one used so far and is introduced below:

$$U = [0,\infty)\ ,$$

$$\Delta = \infty$$

$$\mathcal{U}_i = \{[k\, 2^{-i},\ (k+1)\, 2^{-i})\}_{k=0}^{\infty}\ ;$$

if $V \in \mathcal{B}$ is of the form $V = [a,b)$, then in contravention of our usual notation, we denote by $F(V)$ the σ-field F_{a-}:

$$F(V) = F_{a-} = T(\{\mu(A) \mid A \in \mathcal{B},\ A \subset [0,\ a)\})\ .$$

if $W, V \in \mathcal{B}$, $W \subset V$, then:

$$V \wedge W = \{s \in V \mid s < s'\ \forall\, s' \in W\},$$

i.e. $V \wedge W$ is that part of V which is situated to the left of W. As always, ξ is a simple non-marked point process on U.

Now it follows, that if $V, W \in \mathcal{U}$, $W \subset V$, then for all non-negative r.v. X we have:

$$E(X \mid \mathcal{F}(W)) = \frac{E(X\, 1_{\{\xi(V \wedge W)=0\}} \mid \mathcal{F}(V))}{P(\xi(V \wedge W) = 0 \mid \mathcal{F}(V))}$$

a.e. on $\{\xi(V \wedge W) = 0\}$ (cf. theorem C.2).

For all $V \in \mathcal{U}$, the set $\{(\omega,u) \mid P(\hat{\xi}_u \neq 0 \mid \mathcal{F}(V))(\omega) \neq 0\}$ clearly is the finite or countable union of graphs of measurable mappings $T: (\Omega, \mathcal{F}(V)) \to V \cup \{\infty\}$ (cf. lemma 7.4.2) and for such mappings T (which are previsible stopping times) we have

$$\sum_{W \in \mathcal{U}_i} E(\xi(W) \mid \mathcal{F}(W))\, 1_{\{T \in W\}} \to \nu_1(T) \quad \text{a.s.}$$

as $i \to \infty$, where $\nu_1(T)$ is some r.v. (Indeed, the process is a martingale w.r.t. $\{\mathcal{T}(\{T \in W\} \cap F,\ F \in \mathcal{F}(W),\ W \in \mathcal{U}_i)\}_i$ (cf. lemma 7.4.1)). Note that

$$S^c \equiv \bigcup_{V \in \mathcal{U}} \{(\omega,u) \mid u \in V,\ P(\hat{\xi}_u \neq 0 \mid \mathcal{F}(V))(\omega) \neq 0\}$$

again is the union of a finite or countable number of graphs of mappings T.

<u>LEMMA D.1.</u> *Except for an evanescent set,* $(\omega,u) \in S$ *implies:*

$$\lim_i E(\xi(W_i) \mid \mathcal{F}(W_i))(\omega) = 0 \qquad (W_i \in \mathcal{U}_i,\ W_i \downarrow \{u\}) .$$

PROOF. Choose k such that $\xi_\omega(W_k \wedge \{u\}) = 0$. For each $V \in \mathcal{U}$ there exists a corresponding $\mathcal{F}(V)$-measurable previsible stopping time T_V given by:

$$T_V(\omega) = \begin{cases} \inf\{t \mid P(\xi(V \wedge \{t\}) = 0 \mid \mathcal{F}(V))(\omega) = 0\} \text{ if } \\ \qquad\qquad P(\xi(V) = 0 \mid \mathcal{F}(V))(\omega) = 0 ; \\ \infty \text{ if } P(\xi(V) = 0 \mid \mathcal{F}(V))(\omega) \neq 0 . \end{cases}$$

Because $(\omega,u) \in S$, $u \geqq T_{W_k}(\omega)$ implies that $P(\xi(W_k \wedge \{u\}) = 0 \mid \mathcal{F}(W_k))(\omega) = 0$; and since $\xi_\omega(W_k \wedge \{u\}) = 0$, it now can be seen that we only exclude an evanescent set by the assumption $u < T_{W_k}(\omega)$, so that $P(\xi(W_k \wedge \{u\}) = 0 \mid \mathcal{F}(W_k))(\omega) \neq 0$ and we may copy the proof of lemma 7.4.4.

□

<u>LEMMA D.2.</u> *For all* $V \in \mathcal{U}$ *we have a.e. on* $\{\xi(V) = 0\}$:

$$\sum_{W \in U_{i,B}} P(\xi(W) = 1 \mid F(W)) \uparrow \nu_2(B) \quad \text{if} \quad i \to \infty,$$

where ν_2 *is a random measure on the semiring* $\{B \in U \mid B \subset V\}$.

PROOF. Cf. lemma 7.4.5. Let $W \in U_i$, $W \subset V$, then

$$P(\xi(W) = 1 \mid F(W)) =$$

$$= \frac{P(\xi(W) = 1,\ \xi(V \wedge W) = 0 \mid F(V))}{P(\xi(V \wedge W) = 0 \mid F(V))}$$

$$= \sum_{W \in U_{i+1,W}} \frac{P(\xi(W') = 1,\ \xi(W-W') = 0,\ \xi(V \wedge W) = 0 \mid F(V))}{P(\xi(V \wedge W) = 0 \mid F(V))} \leqq$$

$$\leqq \sum_{W' \in U_{i+1,W}} \frac{P(\xi(W') = 1,\ \xi(V \wedge W') = 0 \mid F(V))}{P(\xi(V \wedge W) = 0 \mid F(V))} \leqq$$

$$\leqq \sum_{W' \in U_{i+1,W}} \frac{P(\xi(W') = 1,\ \xi(V \wedge W') = 0 \mid F(V))}{P(\xi(V \wedge W') = 0 \mid F(V))} =$$

$$= \sum_{W' \in U_{i+1,W}} P(\xi(W') = 1 \mid F(W')) .$$

It is easily checked that the limit $\nu_2(B)$ exists and is a measure in B. □

Note that $P(\xi(W) = 1 \mid F(W)) = E(\xi(W)\, 1_{\{\xi(W)=1\}} \mid F(W))$.

<u>LEMMA D.3.</u> *Let* $V \in U$; *then:*

$$\sum_{W \in U_{i,V}} E(\xi(W)\, 1_{\{\xi(W)>1\}} \mid F(W))\, 1_{\{\xi(W)=0\}} \to 0 \quad a.s..$$

PROOF. Number the atom positions of ξ: $0 \leqq \tau_1 < \tau_2 < \dots$. Choose $a > 0$, $W \in (a,\infty)$ and $\ell = 1, 2, \dots$, and write

$$A = \{\xi((a,\infty) \wedge W) = 0,\ \xi[0,a] = \ell\}$$

and

$$T = (\tau_1, \dots , \tau_\ell) ,$$

then it can be checked that on A we have for all non-negative r.v. X :

$$E(X \mid \mathcal{F}(W)) = \frac{E(X 1_A \mid T)}{P(A \mid T)}$$

and the lemma is proved in the same way as lemma 7.4.7. □

THEOREM D.4. *Let* $B \in \mathcal{U}$*; then:*

$$\sum_{k:k2^{-i}\in B} E(\xi[k2^{-i},(k+1)2^{-i}) \mid \mathcal{F}_{k2^{-i}}) =$$

$$= \sum_{W\in \mathcal{U}_{i,B}} E(\xi(W) \mid \mathcal{F}(W)) \to \xi^P(B) \quad \textit{a.s. and in} \quad L^1 .$$

PROOF. This follows in the same way as the proof of theorem 7.4.8. A.s. convergence is proved by dividing the sum in 4 parts. Mean convergence is also easily proved. The limit is a.s. a measure on $\mathcal{U}$ as a function of B ; call it ν . The random measure ν is extended to $\mathcal{B}$. It can be verified that the process $\nu[0,t]$ is previsible. Combining this fact with the mean convergence, we see that $\nu = \xi^P$. □

Remark. Theorem 3.4.5 is proved by replacing the above $\mathcal{U}$ by $\mathcal{U} = \{[k2^{-i}t, (k+1)2^{-i}t)\}_{k,i=0}^{\infty}$. □

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ZfW: *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete.*

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LIST OF SYMBOLS