Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

MATHEMATICAL CENTRE TRACTS 161

EISENSTEIN SERIES ON THE METAPLECTIC GROUP
AN ALGEBRAIC APPROACH

G.F. HELMINCK

MATHEMATISCH CENTRUM AMSTERDAM 1983

1980 Mathematics subject classification: 22E55

ISBN 90 6196 257 9

ACKNOWLEDGEMENTS

I am indebted to all who have contributed in some sense to the realization of this monograph. In particular, I am grateful to my thesis adviser Prof.dr. J.G.M. Mars for his stimulating help and the pleasant way in which he guided my wandering through the fields of mathematics. I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.

CONTENTS

INTRODUCTION

The questions that are treated in this monograph originated from the study of [9]. There Siegel considered the theta series

$$\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}, \quad \mathrm{Im}(z) > 0.$$

Clearly $\lim_{\mathrm{Im}(z)\to\infty} \theta(z) = 1$. From the behaviour of θ under substitutions of the form $z \mapsto az+b/cz+d$, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,\mathbb{Z})$, one deduces that for $a \in \mathbb{Z}$ and $c \in \mathbb{N}$ with g.c.d. $(a,c) = 1$.

$$\lim_{z\downarrow a/c} (cz-a)^{\frac{1}{2}}\theta(z) = \gamma(\tfrac{a}{c}),$$

where $\gamma(\frac{a}{c})$ equals an eigth root of unity or zero. An earlier result of Siegel implied that for $5 \le m \le 8$

$$\theta^m(z) = 1 + \sum_{a/c\in\mathbb{Q}} \gamma(\tfrac{a}{c})^m (cz-a)^{-\frac{m}{2}}.$$

However the series on the right hand side does not converge absolutely for $m = 1$. Following an idea of Hecke, Siegel introduced for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3}{2}$ the Eisenstein series

$$\theta(z,s) = 1 + \sum_{a/c\in\mathbb{Q}} \gamma(\tfrac{a}{c})(cz-a)^{-\frac{1}{2}}|cz-a|^{-s}.$$

In [9] Siegel proved that $\theta(z,s)$ as a function of s has a meromorphic continuation to $\mathbb{C}$, is a holomorphic function on $\{s \mid s \in \mathbb{C},\ \mathrm{Re}(s) > \frac{1}{2},\ s \neq 1\}$ and has a first order pole in $s = 1$. In particular it turned out that

$$\lim_{s\to 1} (s-1)\theta(z,s) = \frac{2\sqrt{2}}{\pi}\,\theta(z).$$

Finally he deduced a functional equation relating this and some other Eisenstein series.

In this paper I investigate to what extent similar results hold for other functions that resemble θ, like $\Sigma_{n \in L} n^{2k} e^{-\pi i n^2 z}$, with $k \in \mathbb{N}$ and L a sublattice of $\mathbb{Z}$. To do so, I make use of the translation of such functions to functions on the homogeneous space $S\ell(2,k)\backslash Mp(A)$ as given by WEIL in [12]. Here $Mp(A)$ denotes the so-called metaplectic group, a central extension of $S\ell(2,A)$ with the unit circle. Weil introduced a theta series $\theta(\varphi)$ for every Schwartz - Bruhat function φ of the adèle ring A. Moreover he showed that one has a natural representation of $Mp(A)$ on this space $S(A)$. All the examples given above correspond then to appropriate choices of φ. In particular all those φ belong to the K-finite elements of the restricted tensor product of the even Schwartz-Bruhat functions for a suitably chosen maximal compact subgroup K of $Mp(A)$. This subspace of $S(A)$ is a non-degenerate module for the Hecke algebra $\mathcal{H}$ of $Mp(A)$. Since I intend to exploit this $\mathcal{H}$-module structure, I confine my attention to the functions in that subspace.

For any φ as above and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{3}{2}$ one can construct an Eisenstein series $\theta(\varphi,s)$, using the zero-th Fourier coefficient of $\theta(\varphi)$ and a convergence factor. However, for fixed s, the collection of $\theta(\varphi,s)$ is no longer invariant under the action of $\mathcal{H}$. The extension of this space to an $\mathcal{H}$-module requires the introduction of more general Eisenstein series. Their meromorphic continuation and functional equation will be derived following some simple ideas, implicit already in [9]. First of all one shows that all their Fourier coefficients have a meromorphic continuation. Next one proves that their sum defines the desired meromorphic continuation. As for the functional equation, it is shown first that the zero-th Fourier coefficients of the Eisenstein series involved are equal; by using the boundedness of the other coefficients and the uniqueness of the local Whittaker models one can prove the same for the remaining ones.

The explicit expression for the Fourier coefficients furnishes us directly the place and order of the poles at the right of the critical line. By combining the expression for the residue with some local results and the well-known formula for the coefficients of a theta series, we obtain the nature of the residue. It is easy to show then that for all φ, which transform under K according to an irreducible representation, the residue of $\theta(\varphi,s)$ in $s = 1$ is proportional to $\theta(\varphi)$. This result is not true for general φ.

Some of the results derived in this paper have been announced by GELBART and SALLY in [4]. However their work is based on results of Langlands, while the present monograph can be seen as a transcription to the adèle-setting of the principles behind the calculations in [9]. Moreover it is essentially self-contained.

Let me conclude with a short description of the contents of the various chapters. In the first I recall the properties of the metaplectic group and make some necessary additional computations. The subsequent chapter gives the relation with the work of SIEGEL [9] and the framework, inside of which we will work. The local Hecke algebra modules that turn up there are analyzed in the Chapters 3 and 4. Finally, in Chapter 5, I prove the global results as stated above.

CHAPTER I

§0. NOTATIONS AND CONVENTIONS

0.1. CONVENTIONS.

(i) Let U and V be open, connected subsets of $\mathbb{C}$, with $U \subseteq V$. Let f be a holomorphic function $U \to \mathbb{C}$. Assume that f has a meromorphic continuation to V. Then I will denote this continuation also by f.

(ii) Let X be a set and take U and V as in (i). Suppose, we have for every $s \in U$ a map $f(s)\colon X \to C$ such that for every $x \in X$ the function $s \mapsto f(s)(x)$ is holomorphic on U. Assume, moreover, that all the maps $s \mapsto f(s)(x)$, $x \in X$ have a holomorphic resp. meromorphic continuation to V. Then we will say that f(s) has a holomorphic resp. meromorphic continuation to V. f(s) is said to have a pole of order h, $h \in \mathbb{Z}_{\geq 0}$, in s_0, if all the $s \mapsto f(s)(x)$ have a pole of order $\leq h$ in s_0 and at least one of them exactly a pole of order h.

(iii) All topological groups should be understood to have a Hausdorff topology and all representations to be complex.

0.2. Let K be a compact group and let dk be the Haar measure of K for which K has volume 1. I write $A(K)$ for the space of functions on K, spanned by the matrix coefficients of the irreducible continuous representations of K. If K is a closed subgroup of another locally compact group G, then $A(K)$ can be identified with a convolution algebra of measures on G via $f \to f dk$. In particular $A(K)$ is a $\star$-algebra of functions on K. Let ρ be an irreducible continuous representation of K; denote its degree by $d(\rho)$ and define the function $e(\rho)\colon K \to \mathbb{C}$ by

$$e(\rho)(k) = d(\rho) \text{ trace } (\rho(k^{-1})).$$

Every element of $A(K)$ of the form $\sum_{i=1}^{m} e(\rho_i)$, where $\{\rho_i \mid 1 \leq i \leq m\}$ is a collection of mutually non-equivalent continuous irreducible representations of

K, satisfies $\sum_{i=1}^{m} e(\rho_i) * \sum_{i=1}^{m} e(\rho_i) = \sum_{i=1}^{m} e(\rho_i)$ and is called an *elementary idempotent* of K. The collection of these elements is denoted by $\mathcal{E}(K)$. When ρ is the trivial representation of K, I will use the notation ε_K instead of $e(\rho)$.

0.3. For any set X, the identity map: $X \to X$ will be denoted by I_X. If g and $f : X \to \mathbb{C}$, are proportional, then one denotes this also by $f \approx g$. For any commutative ring R, write R^* for its group of units.

If G is a group and f a function $G \to \mathbb{C}$, then I use the notation $\check{f}$ for the function $g \mapsto f(g^{-1})$, $g \in G$. In case G is locally compact and abelian, its group of characters is denoted by $\hat{G}$. If G is a unimodular locally compact group and D a distribution on G, then the distribution $\check{D}$ on G is given by $f \mapsto D(\check{f})$. Moreover, for every $x \in G$, I denote the distribution $f \mapsto f(x)$ by δ_x.

0.4. Let k be an algebraic number field. If v is a place of k, k_v will denote the completion of k at v; v is called real, if k_v is isomorphic to $\mathbb{R}$, imaginary if k_v is isomorphic to $\mathbb{C}$, infinite in both of these cases, and finite in all other cases. I write $\mathcal{P}$ for the collection of places of k, $\mathcal{P}_\infty$ for the set of infinite places of k and $\mathcal{Q}$ for any finite set of places of k, containing $\mathcal{P}_\infty$. If v is a finite place of k, the ring of integers in k_v is denoted by $\mathcal{O}_v$, its maximal ideal by $\mathfrak{p}_v$ and a fixed generator of $\mathfrak{p}_v$ by π_v.

For every $v \in \mathcal{P}$, let $(-,-)_v$ be the Hilbert symbol on $k_v^* \times k_v^*$. If a and b are in k_v^*, this symbol is given by

$$(a,b)_v = \begin{cases} 1 & \text{if b is a norm of } k_v(\sqrt{a}) \\ -1 & \text{otherwise} \end{cases}$$

For every $a \in k_v^*$, denote the character $b \mapsto (a,b)_v$ of k_v^* by $h_v(a)$.

Let $\mathcal{A}$ be the ring of adèles of k. For $a = (a_v) \in \mathcal{A}^*$, $h(a) = \prod_{v \in \mathcal{P}} h_v(a_v)$ is a character of $\mathcal{A}^*$ and, if $a \in k^*$, $h(a) \mid k^* \equiv 1$, by quadratic reciprocity.

§1. THE METAPLECTIC GROUP

1.1. Notations being as in 0.4, X will stand in this paragraph either for k_v, $v \in \mathcal{P}$, or for $\mathcal{A}$. Let τ be a character of X such that X can be identified with its dual by means of $(x,y) \to \tau(xy)$; then dx will be the self-dual measure on X, with respect to this identification. For $Y \subset X$, denote $\{z \mid z \in X,\ \tau(zy) = 1 \text{ for all } y \in Y\}$ by $Y^\perp$.

Let $S(X)$ be the collection of Schwartz-Bruhat functions on X. Then X^* will be equipped with the natural norm $|\cdot|$ given by

$$\int_X f(x)dx = |a| \int_X f(ax)dx \qquad \text{for all } f \in S(X) \text{ and } a \in X^*.$$

The Fourier transform F will be defined as follows on $S(X)$

$$F(f)(x) = \int_X f(y)\ \tau(xy)dy.$$

1.2. Let T be $\{z | z \in \mathbb{C},\ |z| = 1\}$. Define the group A(X) as the collection $X^2 \times T$, equipped with the following multiplication structure: for $(x_i, y_i) \in X^2$ and $t_i \in T$, $i = 1,2$,

$$\{(x_1,y_1),t_1\} \cdot \{(x_2,y_2),t_2\} = \{(x_1+x_2,y_1+y_2),\ \tau(x_1y_2)t_1t_2\}.$$

For every $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S\ell(2,X)$, one defines as follows a character of the second degree of X, $f(s)$

$$f(s)(x,y) = \tau(\tfrac{1}{2}abx^2 + bcxy + \tfrac{1}{2}cdy^2).$$

From [12], section 5, we know that one can define an embedding i: $S\ell(2,X) \to \mathrm{Aut}(A(X))$ by

$$i(s)(\{(x,y),t\}) = \{(x,y)s,\ f(s)(x,y)t\}.$$

Let $\mathfrak{s}$ be the unitary representation of A(X) in $L^2(X)$ given by

$$\mathfrak{s}(\{(x,y),t\})(f)(r) = t\ \tau(yr)\ f(r+x).$$

According to [12], theorem 1, there exists for every $s \in S\ell(2,X)$, a $\mathfrak{r}(s) \in$ $\in U(L^2(X))$, the group of unitary automorphisms of $L^2(X)$, such that for all $h \in A(X)$,

$$(1.3) \qquad \mathfrak{r}(s)\ \mathfrak{s}(h)\ \mathfrak{r}(s)^{-1} = \mathfrak{s}(i(s)(h)).$$

Moreover such a $\mathfrak{r}(s)$ is determined up to an element of T. Consequently $\{(s,\mathfrak{r}(s)) | s \in S\ell(2,X),\ \mathfrak{r}(s) \in U(L^2(X))$ satisfying (1.3)$\}$ is a group, *the metaplectic group*, and it will be denoted by Mp(X).

1.4. <u>REMARK</u>. For $a \in X^*$, let $\tau^a \in \hat{X}$ be given by $x \mapsto \tau(ax)$. By starting with τ^a instead of τ, one can construct another metaplectic group. In §4, I will show that that one is isomorphic to the one constructed above.

1.5. If $S\ell(2,X)$ is equipped with its natural topology and $U(L^2(X))$ with the strong topology, then $Mp(X)$ inherits its topology from the product topology on $S\ell(2,X) \times U(L^2(X))$. In the next paragraphs we will come back to it again.

Throughout this paper, T will be considered as a subgroup of $Mp(X)$ via the embedding $t \to ((\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}), t1_{S(X)})$. If p is the natural projection: $Mp(X) \to S\ell(2,X)$ and H any subset of $S\ell(2,X)$, then one writes $\tilde{H}$ for the set $p^{-1}(H)$.

It will be convenient to introduce notations for certain elements and subsets of $S\ell(2,X)$. For $x \in X$, denote $(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix})$ by $u(x)$ and $(\begin{smallmatrix}1 & 0\\ x & 1\end{smallmatrix})$ by $n(x)$. If $a \in X^*$, then I write $d(a)$ for $(\begin{smallmatrix}a & 0\\ 0 & a^{-1}\end{smallmatrix})$ and $w(a^{-1})$ for $(\begin{smallmatrix}0 & -a\\ a^{-1} & 0\end{smallmatrix})$. I use the notation $U(X)$ for $\{u(x) \mid x \in X\}$, $N(X)$ for $\{n(x) \mid x \in X\}$ and $D(X)$ for $\{d(a) \mid a \in X^*\}$. Further, I denote the group $U(X)D(X)$ by $P(X)$ and write $\Omega(X)$ for $\{u(x)\ w(a)\ u(y) \mid x,y \in X,\ a \in X^*\}$.

1.6. Let f be in $L^2(X)$. From [12], section 13, we know that $r(u(x))$, $r(d(\alpha))$ and $r(w(\alpha))$ can be chosen in the following way:

$$\begin{aligned} r(u(x))(f)(t) &= \tau(\tfrac{1}{2}xt^2)\ f(t) \\ r(d(\alpha))(f)(t) &= |\alpha|^{\frac{1}{2}}\ f(\alpha t) \qquad (1.7) \\ r(w(\alpha))(f)(t) &= |\alpha|^{-\frac{1}{2}} F(f)(-\alpha^{-1}t). \end{aligned}$$

Now it is clear from (1.3) that we can make for $b = u(x)d(a)$ resp. $\omega = u(x)\ w(a)\ u(y)$ the choice $r(b) = r(u(x))r(d(a))$ resp. $r(\omega) = r(u(x))\ r(w(a))\ r(u(y))$. From now on, I assume that for all $g \in P(X) \cup \Omega(X)$, $r(g)$ denotes the choice given above. This choice defines a section $R : P(X) \cup \Omega(X) \to Mp(X)$. Instead of $R(d(1))$, I simply write e. In §3, one can find a choice of $r(g)$ for all $g \notin P(X) \cup \Omega(X)$.

1.8. In order to be able to give the relations that are satisfied by R, I have to introduce the function $\gamma : X^* \to T$ from [12], section 14. For every $\alpha \in X^*$, $\gamma(\alpha)$ is determined by:

$$\gamma(\alpha) \int_X f(x)dx = |\alpha|^{\frac{1}{2}} \int_X \Big(\int_X f(x-y)\tau(\tfrac{1}{2}\alpha y^2)dy\Big)dx, \text{ for all } f \in S(X).$$

In the following proposition some results concerning γ are collected. The proofs of them are either straightforward or can be found in [12], in which case the appropriate section is mentioned between brackets.

1.9. PROPOSITION.

(i) *For all* $\alpha, b \in X^*$, $\gamma(-\alpha) = \gamma(\alpha)^{-1}$ *and* $\gamma(\alpha b^2) = \gamma(\alpha)$.

(ii) *If* $X = k_v$, $v \in P$, *then one has for all* $a, b \in k_v^*$:

$$\gamma(a)\ \gamma(b) = \gamma(1)\ \gamma(ab)(a,b)_v\ . \tag{28}$$

(iii) *If* $X = \mathbb{R}$ *and* $\tau(x) = e^{2\pi i \ell x}$, *then* γ *is given by*

$$\gamma(x) = \begin{cases} e^{\pi i/4} & \text{if } \ell x > 0 \\ e^{-\pi i/4} & \text{if } \ell x < 0 \end{cases}\ . \tag{26}$$

(iv) *If* $X = \mathbb{C}$, *then* $\gamma(x) = 1$ *for all* $x \in \mathbb{C}^*$. (26)

(v) *Assume* $X = k_v$, $v \notin P_\infty$. *For every* $x \in k_v^*$, *choose a* $t \in k_v^*$ *such that for all* $y \in t.\mathcal{O}_v$: $\tau(\frac{1}{2}\, xy^2) = 1$. *Then*

$$\gamma(x) = |x|^{\frac{1}{2}} \int_{t^{-1}x^{-1}\mathcal{O}_v^{\perp}} \tau(\tfrac{1}{2}xz^2)\,dz. \tag{27}$$

(vi) *Let* X *be* $\mathbb{A}$ *and assume that* τ *is such that* $k^{\perp} = k$. *Then*

$$\gamma(\alpha) = 1 \quad \text{for all } \alpha \in k^*. \tag{30}$$

1.10. Let g_1, g_2 and $g_3 \in P(X) \cup \Omega(X)$ be such that $g_1 g_2 = g_3$. The next proposition gives the relations that are satisfied by R; the first being a consequence of [12], theorem 3, and the second a direct verification from the definitions.

1.11. PROPOSITION.

(i) *If* $g_i = u(x_i)\ w(a_i)\ u(y_i)$ *for all* i, *then*

$$R(g_1)R(g_2) = \gamma(a_1 a_2 a_3)R(g_3).$$

(ii) *In the remaining cases we have*

$$R(g_1)R(g_2) = R(g_3).$$

1.12 <u>REMARK</u>. It may happen at global considerations that, in order to avoid confusion, we have to provide the local notions with a subscript v.

§2. PROPERTIES OF $Mp(k_v)$.

2.1. For every $v \in P$, $\tilde{\Omega}(k_v)$ is open in $Mp(k_v)$. According to [12], section 34, $\tilde{\Omega}(k_v)$ is homeomorphic to $T \times k_v \times k_v^* \times k_v$ via the map $(t,x,a,y) \mapsto$ $\mapsto tR(u(x)w(a)u(y))$. Hence $Mp(k_v)$ is a locally compact group.

For infinite v, one puts, via the map given above, an analytic structure on $\Omega(k_v)$. By means of right translations this is transferred also to the rest of the group and one checks directly that, in this way, $Mp(k_v)$ becomes a Lie group. Let g be the Lie algebra of $S\ell(2,k_v)$. With the aid of the map $\alpha + X \mapsto e^{i\alpha} R(e^X w(1))$, we identify the tangent space in $R(w(1))$ with $\mathbb{R} \oplus g$ and by shifting it back to the origin one obtains an isomorphism of $\mathbb{R} \oplus g$ with the Lie algebra of $Mp(k_v)$, m. In particular we get the following expression for the exponential map:

$$\exp(\alpha+X) = e^{i\alpha}R(e^X w(1))R(w(-1)), \qquad \text{for } X \in g, \quad \alpha \in \mathbb{R}. \tag{2.2}$$

2.3. For finite v, there exists an open compact subgroup G_v of $S\ell(2,k_v)$ such that $\tilde{G}_v$ is isomorphic to $G_v \times T$. As I have to make some explicit calculations that cannot be found in [12], I will recall here shortly its construction.

Let δ_v be a generator of $O_v^{\perp}$. For every $g \in S(k_v)$ one decomposes $F(g)$ in the following way:

$$F(g)(y) = \sum_{x \in k_v/O_v} T(g)(x,y)\tau(xy),$$

where T(g) is defined by

$$T(g)(x,y) = \int_{O_v} g(x+t)\tau(ty)dt.$$

T(g) belongs to the space $S(k_v,O_v)$ of functions $f : k_v^2 \to \mathbb{C}$, with compact support that satisfy

$$f(x+z_1,y+z_2) = \tau(-z_1y)\, f(x,y) \quad \text{for } x,y \in k_v,\ z_1 \in O_v \text{ and } z_2 \in O_v^{\perp}.$$

One can put a scalar product on $S(k_v, 0_v)$ such that T becomes a pre-unitary isomorphism of $S(k_v)$ with $S(k_v, \mathcal{O}_v)$. The inverse of T is given by:

$$T^{-1}(f)(x) = |\delta_v|^{\frac{1}{2}} \sum_{y \in k_v/\mathcal{O}_v^{\perp}} f(x,y) .$$

Now take $G_v = \{g \mid g \in S\ell(2,k_v),\ (\mathcal{O}_v \times \mathcal{O}_v^{\perp})g = \mathcal{O}_v \times \mathcal{O}_v^{\perp},\ f(g) \mid \mathcal{O}_v \times \mathcal{O}_v^{\perp} \equiv 1\} =$
$= \{\left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right) \mid a,d \in \mathcal{O}_v,\ b \in \mathcal{O}_v^{\perp},\ c \in (\delta_v^{-1})$ and for all $x \in \mathcal{O}_v$ and $y \in \mathcal{O}_v^{\perp}$,

$$\tau(\tfrac{1}{2}abx^2) = \tau(\tfrac{1}{2}cdy^2) = 1\}.$$

One defines a pre-unitary representation $r^{\mathcal{O}}$ of G_v in $S(k_v, \mathcal{O}_v)$ by

$$r^{\mathcal{O}}(g)(h)(x,y) = h((x,y)g)\ f(g)(x,y).$$

By means of T, this representation is transferred from $S(k_v, \mathcal{O}_v)$ to $S(k_v)$ and extended there to a unitary one on $L^2(k_v)$. For convenience sake, this one is also denoted by $r^{\mathcal{O}}$. For all $g \in G_v$, $r^{\mathcal{O}}(g)$ satisfies (1.3) and in the sequel I will derive the relation between $r^{\mathcal{O}}(g)$ and $r(g)$.

Let h belong to $S(k_v)$. If d(a) and u(b) $\in G_v$, then we have:

$$r^{\mathcal{O}}(d(a))(h) = |\delta_v|^{\frac{1}{2}} \sum_{y \in k_v/\mathcal{O}_v^{\perp}} \int_{\mathcal{O}_v} h(ax+t)\ \tau(ta^{-1}y)dt$$
$$= h(ax)$$

$$r^{\mathcal{O}}(u(b))(h)(x) = |\delta_v|^{\frac{1}{2}} \sum_{y \in k_v/\mathcal{O}_v^{\perp}} \int_{\mathcal{O}_v} h(x+t)\ \tau(t(y+bx))\ \tau(\tfrac{1}{2}bx^2)dt$$
$$= \tau(\tfrac{1}{2}bx^2)h(x)$$

In other words, for all $b \in G_v \cap P(k_v)$

(2.4) $\qquad r^{\mathcal{O}}(b) = r(b).$

Now, let $\sigma = \left(\begin{smallmatrix}ab\\cd\end{smallmatrix}\right)$ be an element of $G_v \cap \Omega(k_v)$. Then

$$r^{O}(\sigma)(h)(x) = |\delta_v|^{\frac{1}{2}} |c| \sum_{y\in k_v/O_v^{\perp}} f(\sigma)(x,y). \int_{c^{-1}O_v} h(ax+c(y+t))\tau(tc(bx+dy))dt$$

$$= |c||\delta_v|^{\frac{1}{2}} \sum_{\rho\in c^{-1}O_v/O_v^{\perp}} \sum_{y\in k_v/O_v^{\perp}} \tau(-\frac{cd}{2}\rho^2) \int_{O_v^{\perp}} h(ax+c(y+z+\rho))f(\sigma)(x,y+z+\rho)dz =$$

$$= |c||\delta_v|^{\frac{1}{2}} \left\{\sum_{\rho\in c^{-1}O_v/O_v^{\perp}} \tau(\frac{-cd}{2}\rho^2)\right\} \int_{k_v} h(ax+cy)f(\sigma)(x,y)dy$$

By comparing this expression with formula (16) in [12], we may conclude that

$$(2.5) \qquad r^{O}(\sigma) = \left\{|\delta_v c|^{\frac{1}{2}} \sum_{\rho\in c^{-1}O_v/O_v^{\perp}} \tau(-\frac{cd}{2}\rho^2)\right\} r(\sigma)$$

2.6. If $|c| = |\delta_v|^{-1}$, then this factor equals 1. If $|c| < |\delta_v|^{-1}, |d| = 1$; hence, with proposition (1.9), (2.5) becomes

$$(2.7) \qquad r^{O}(\sigma) = \gamma(-cd)\ r(\sigma).$$

Let R_v^O be the section $G_v \to Mp(k_v)$ given by $R_v^O(g) = (g, r^O(g))$. Through R_v^O, we can consider G_v as a subgroup of $Mp(k_v)$. From [12], section 36, we know that the map $(t,g) \to tR_v^O(g)$ is a homeomorphism from $G_v \times T$ onto $\tilde{G}_v$.

§3 PROPERTIES OF Mp(A)

Let τ be $\Pi\tau_v$. It is natural to make the following

CONVENTION. If we speak in a global context of $Mp(k_v)$, then it will be tacitly assumed that this group has been built up from the identification of k_v with its dual by means of $(x,y) \mapsto \tau_v(xy)$.

3.1. For finite v and $\ell \in \mathbb{Z}$, let $\psi_v(\ell) \in S(k_v)$ be the characteristic function of p_v^{ℓ}. I denote the restricted tensor product of the $S(k_v)$, with respect to the $\{\psi_v(0) | v \notin P_\infty\}$ by $\otimes S(k_v)$. For $g \in S\ell(2,A)$, take any Q as in (0.4) such that for all $v \notin Q$, $g_v \in G_v$. Then

$$r_Q(g) = (\otimes_{v\in Q} r_v(g_v)) \otimes (\otimes_{v\notin Q} r_v^O(g_v))$$

is defined on $\otimes\, S(k_v)$ and can be extended to an element of $U(L^2(A))$, satisfying (1.3). Denote this extension also by r_Q. For the section $\Pi_{v\in Q}\, S\ell(2,k_v)\, \Pi_{v\notin Q}\, G_v \to Mp(A)$ determined by r_Q, I use the notation R_Q.

3.2. As for the topology of Mp(A), one knows from [12], section 38, that for every Q as in (0.4), $p^{-1}(\Pi_{v\in Q}\, \Omega(k_v)\, \Pi_{v\notin Q}\, G_v)$ is homeomorphic to $T \times \Pi_{v\in Q}\, \Omega(k_v)\, \Pi_{v\notin Q}\, G_v$ through $(t,g) \to tR_Q(g)$. Hence Mp(A) is a locally compact group.

If τ is such that $k = k^{\perp}$, (1.9) (vi) and (1.11) (i) imply that $S\ell(2,k)$ can be embedded into Mp(A) as a discrete subgroup.

For each v in P, there exists a natural embedding i_v of $Mp(k_v)$ into Mp(A). Assume now that one has, for $J \subset P$, a collection $\{g_v | v \in J,\ g_v \in Mp(k_v)$ and $g_v \in G_v$ for almost all $v \in J\}$. Take any Q as in (0.4) such that $g_v \in G_v$ for all $v \in J \cap (P\backslash Q)$ and embed $(g_v)_{v\in J\cap(P\backslash Q)}$ in the natural way into $\Pi_{v\notin Q}\, G_v$; then we put

$$\underset{v\in J}{\otimes} g_v = \prod_{v\in J\cap Q} i_v(g_v).R_Q((g_v)_{v\in J\cap(P\backslash Q)}).$$

If $J = P$, then I simply write $\otimes g_v$ instead of $\underset{v\in P}{\otimes} g_v$.

3.3. For Q as in (0.4), let $i_Q : \Pi_{v\in Q}\, Mp(k_v) \to Mp(A)$ be defined by $\Pi_{v\in Q}\, g_v \to \underset{v\in Q}{\otimes} g_v$. Assume now that we have a $f : i_Q(\Pi_{v\in Q}\, Mp(k_v)) \to \mathbb{C}$, a $r \in \mathbb{Z}$, and a collection $\{f_v | v \in P\backslash Q, f_v : Mp(k_v) \to \mathbb{C}\}$, satisfying:

(i) $f(tg) = t^r f(g)$ for all $t \in T$ and $g \in i_Q(\Pi_{v\in Q}\, M_p(k_v))$;
(ii) $f_v(tx_v) = t^r f_v(x_v)$ for all $v \notin Q$, $t \in T$ and $x_v \in Mp(k_v)$;
(iii) For almost all v not in Q, $f_v|G_v \equiv a_v$, with $a_v \in \mathbb{C}^*$;
(iv) Πa_v is convergent, where the product is taken over all places for which (iii) holds.

Then one can define $f \otimes \{\underset{v\notin Q}{\otimes} f_v\} : Mp(A) \to \mathbb{C}$ by

$$f \otimes \{\underset{v\notin Q}{\otimes} f_v\}\, (\otimes g_v) = f(\underset{v\in Q}{\otimes} g_v) \prod_{v\notin Q} f_v(g_v).$$

If $f \circ i_Q$ is decomposable itself, that is to say $f \circ i_Q = \Pi_{v\in Q}\, f_v$, then one simply writes $\otimes f_v$ instead of $f \otimes \{\underset{v\notin Q}{\otimes} f_v\}$.

§4 THE DEPENDENCE ON τ

4.1. Take X as in §1 and let a be an element of X^*. Call $\widetilde{Mp}(X)$ the metaplectic group that one gets by starting with τ^a instead of τ. As for the notation of notions that correspond to ones already introduced for Mp(X), I will use the symbols, used in that context, provided with a ~.

4.2. We consider first the local case. In treating the central extension $Mp(k_v)$ of $S\ell(2,k_v)$, I will follow [8]. First of all we must have a section: $N(k_v) \cup U(k_v) \to Mp(k_v)$ that is an isomorphism from $N(k_v)$ resp. $U(k_v)$ onto their respective images.

For $x \in k_v$, $y \in k_v^*$, put $\underline{u}(x) = R(u(x))$ and $\underline{n}(y) = \gamma(-y)R(n(y))$. By proposition (1.9) and the fact that $(u,1-u)_v = 1$ for all $u \in k_v^*$, $u \neq 1$, this section satisfies the required properties. Now, extend it as follows: for $x \in k_v^*$, define $\underline{w}(x) = \underline{u}(-x^{-1})\underline{n}(x)\underline{u}(-x^{-1})$ and $\underline{d}(x) = \underline{w}(-x^{-1})\underline{w}(-1)^{-1}$. From (1.11), one obtains the following relations between this section and R: for $t \in k_v^*$, $\underline{n}(t) = \gamma(-t)R(n(t))$, $\underline{w}(t) = \gamma(-t)R(w(t))$ and $\underline{d}(t) = \gamma(t)\gamma(-1)R(d(t))$. To this new section is related a Steinberg cocycle c. It is given by $c(a,b)e_v = d(a)d(b)d(ab)^{-1} = (a,b)_v e_v$, for all $a,b \in k_v^*$. Let x: $S\ell(2,k_v) \to k_v^*$ be given by: $x(d(t)u(x)) = t^{-1}$ and $x(u(x)w(t)u(y)) = t$ for all $x,y \in k_v$, $t \in k_v^*$. Now we know from [8] that the following formula defines a 2-cocycle α_v on $S\ell(2,k_v)$ with values in T:

$$\alpha_v(g_1,g_2) = c(x(g_1),x(g_2))^{-1}\, c(x(g_1g_2), -x(g_1)^{-1}x(g_2))$$

I use the notation <g,t>, $g \in S\ell(2,k_v)$, $t \in T$, for the elements of the central extension G_v of $S\ell(2,k_v)$, determined by α_v. Let j be the section: $S\ell(2,k_v) \to Mp(k_v)$ given by:

$$j(d(a)u(x)) = (a,b)_v\, \underline{d}(a)\underline{u}(x),\ j(u(x)w(a)u(y)) = \underline{u}(x)\underline{w}(a)\underline{u}(y).$$

According to [8], corollary 5.12, the map J: <g,t> $\to$ tj(g) is an isomorphism from G_v onto $Mp(k_v)$. For later use, I introduce still some notations. Put T^0 for $\{\pm 1\}$. For any closed subgroup H of $S\ell(2,k_v)$, one writes $\check{H}$ for $\tilde{H} \cap J(\{\langle g,t\rangle \mid g \in S\ell(2,k_v),\ t \in T^0\})$.

4.3. From the definitions one verifies the following relations: for all $x \in k_v$, $b \in k_v^*$,

$$\tilde{\gamma}(b) = \gamma(ba),\ \tilde{R}(u(x)) = R(u(xa)),\ \tilde{R}(w(b)) = R(w(ba)) \text{ and } \tilde{R}(d(b)) = R(d(b)).$$

Thanks to the results of section 4.2, we know that in the local case an isomorphism $A_v(a)$: $Mp(k_v) \to \widetilde{M}p(k_v)$ is given by

$$(4.4) \qquad A_v(a)(tR(d(b)u(x))) = t\,\frac{\gamma(a)\gamma(b)}{\gamma(1)\gamma(ab)}\,\widetilde{R}(d(b)u(x)) = t(a,b)_v\,\widetilde{R}(d(b)u(x))$$

$$(4.5) \qquad A_v(a)(tR(u(x)w(b)u(y))) = t\,\frac{\gamma(b)}{\gamma(ab)}\,\widetilde{R}(u(x)w(b)u(y)).$$

Note that for all $g \in S\ell(2,k_v)$, $\widetilde{r}(g) = r(\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}(g))$. From this and the fact that $\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}$ normalizes $P(k_v)$ and $\Omega(k_v)$, we see that $\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}$ can be lifted to an automorphism $I_v(a)$ of $Mp(k_v)$ in the following way:

$$(4.6) \qquad I_v(a)(tR(d(b)u(x))) = t(a,b)_v\,R(\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}(d(b)u(x)))$$

$$(4.7) \qquad I_v(a)(tR(u(x)w(b)u(y))) = t\gamma(b)\gamma(-ab)R(\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}(u(x)w(b)u(y)))$$

If $|a|_v = 1$, then one proves with the aid of (2.4), (2.6) and (2.7) that for almost all v:

$$(4.8) \qquad A_v(a)(R_v^0(g)) = \widetilde{R}_v^0(g) \qquad \text{for all } g \in G_v,$$

$$(4.9) \qquad I_v(a)(R_v^0(g)) = R_v^0(\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}(g)) \qquad \text{for all } g \in G_v.$$

These observations allow us to define for each $a \in A^*$ an isomorphism $A(a)$: $Mp(A) \to Mp(A)$ and an automorphism $I(a)$ of $Mp(A)$ by:

$$(4.10) \qquad A(a)(\otimes g_v) = \otimes A_v(a_v)(g_v),$$

$$(4.11) \qquad I(a)(\otimes g_v) = \otimes I_v(a_v)(g_v).$$

Let τ be such that $k = k^{\perp}$. If $a \in k^*$, then one concludes from (1.9) (vi), (4.4), (4.5), (4.6) and (4.7) that for all $g \in S\ell(2,k)$,

$$(4.12) \qquad A(a)(R(g)) = \widetilde{R}(g) \text{ and } I(a)(R(g)) = R(\mathrm{Int}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}(g)).$$

4.13. <u>REMARK</u>. Section 45 of [12] implies that there exists a closed subgroup $\overline{\check{S\ell}(2,A)}$ of $Mp(A)$ such that $p(\check{S\ell}(2,A)) = S\ell(2,A)$ and $\check{S\ell}(2,A) \cap T = T^0$. Such a group is used for example in [3]. The advantage of the use of $Mp(A)$ is the reduced number of manipulations with cocycles.

In the light of the foregoing results we make the following

4.14. CONVENTION. In the rest of this paper, we assume that τ, as well in the global as in the local case, is chosen as in [10].

4.15. For $m = 1+2\ell$, $\ell \in \mathbb{Z}$, the map $<g,t> \to <g,t^m>$ is an endomorphism of G_v, for all $v \in P$. Its translation $J_v(m)$ to $Mp(k_v)$ is given by

$$J_v(m)(tR(d(a)u(x))) = t^m(-1,a)_v^{\ell}\, R(d(a)u(x)) \tag{4.16}$$

$$J_v(m)(tR(u(x)w(a)u(y))) = t^m \gamma(a)^{2\ell}\, R(u(x)w(a)u(y)). \tag{4.17}$$

As before, one proves that for almost all $v \in P$, $J_v(m)|G_v = I_{G_v}$. This observation allows us to define an endomorphism $J(m)$ of $Mp(A)$ by

$$J(m)(\otimes g_v) = \otimes J_v(m)(g_v). \tag{4.18}$$

From (1.9) (vi), we see again that $J(m)$ stabilizes $S\ell(2,k)$.

CHAPTER 2

§5. THETA SERIES

5.1. Through the action of the second factor one has a unitary representation ω of $Mp(A)$ on $L^2(A)$, that stabilizes $S(A)$. For every $\varphi \in S(A)$, one defines the theta series $\theta(\varphi)$: $Mp(A) \to \mathbb{C}$ by

$$\theta(\varphi)(g) = \sum_{\xi \in k} \omega(g)(\varphi)(\xi).$$

According to [12], theorem 4, $\theta(\varphi)$ is a function on ${}_{S\ell(2,k)}\backslash Mp(A)$. Let $d\dot{x}$ be the Haar measure on A/k for which A/k has volume 1. Now, for $z \in k$, its z-th Fourier coefficient $\theta(\varphi)_z$:$Mp(A) \to \mathbb{C}$ is given by

$$\theta(\varphi)_z(g) = \int_{A/k} \left\{ \sum_{\xi \in k} \omega(g)(\varphi)(\xi)\ \tau(\tfrac{1}{2}\, x\, \xi^2) \right\} \tau(-xz)d\dot{x}$$

$$(5.2) \qquad = \begin{cases} \omega(g)(\varphi)(0) & \text{if } z = 0 \\ \omega(g)(\varphi)(\xi) + \omega(g)(\varphi)(-\xi) & \text{if } z = \frac{\xi^2}{2} \text{ with } \xi \in k^* \\ 0 & \text{for other } z. \end{cases}$$

In particular, we see that

$$(5.3) \qquad \theta(\varphi)(g) = \theta(\varphi)_0(g) + \tfrac{1}{2} \sum_{\xi \in k^*} \theta(\varphi)_{\frac{1}{2}}(R(d(\xi))g).$$

If one searches for an analogue of theorem 5 in [13], one would like to form the Eisenstein series $\sum_{\sigma \in S\ell(2,k)/P(k)} \theta(\varphi)_0(R(\sigma^{-1})g)$. However this series does not converge absolutely. Following SIEGEL [9], I will use a convergence-factor to overcome this difficulty. In our set-up it amounts to the subsequent construction.

5.4. First of all, I choose, for every $v \in P$, a compact subgroup K_v of

$S\ell(2,k_v)$. For real v, K_v will be $SO(2,\mathbb{R})$; if v is imaginary, take $K_v = SU(2,\mathbb{C})$; finally for finite v, let K_v be $S\ell(2,\mathcal{O}_v)$. Put M for $\Pi_{v\in P} K_v$. Define, for every $s \in \mathbb{C}$, the function $H(s) : Mp(A) \to \mathbb{C}$ by

$$H(s)(R(d(a)u(x))m) = |a|^s, \qquad \text{for } a \in A^*,\ x \in A,\ m \in \tilde{M}.$$

As we will see in §6, one can define, for $\varphi \in S(A)$ and $s \in \mathbb{C}$, with $Re(s) > \frac{3}{2}$, a function $\theta(\varphi,s) : Mp(A) \to \mathbb{C}$ by

$$(5.5) \qquad \theta(\varphi,s)(g) = \sum_{\sigma\in S\ell(2,k)/P(k)} \theta(\varphi)_0(R(\sigma^{-1})g)\ H(s)(R(\sigma^{-1})g).$$

Moreover $\theta(\varphi,s)$ will turn out to be holomorphic on $Re(s) > \frac{3}{2}$. Let $S(A)_e$ be $\{\varphi | \varphi \in S(A),\ \omega(R(d(t)))(\varphi) = \varphi$ for all $d(t)$ in $Z(S\ell(2,A))\}$. In [9], Siegel considers 3 explicit functions φ_1, φ_2 and φ_3 in $S(A)_e$ and proves for each of them the following results:

(5.6) $\theta(\varphi_i,s)$ has a meromorphic continuation to $\mathbb{C}$

(5.7) $\theta(\varphi_i,s)$ is holomorphic on $\{s | s \in \mathbb{C}, Re(s) > \frac{1}{2},\ s \neq 1\}$ and has a pole of order ≤ 1 in $s = 1$.

(5.8) $\lim_{s\to 1} (s-1)\ \theta(\varphi_i,s) = \lambda\theta(\varphi_i)$, with $\lambda \in \mathbb{C}^*$.

In the sequel of this paper, I will show for all φ belonging to a dense subspace of $S(A)_e$, that (5.6) and (5.7) are consequences of a more general phenomenon. It is not difficult to show for those φ that (5.8) is not true in full generality. Nevertheless, one can specify a certain subcollection, for which (5.8) still holds. This can be found in paragraph 17.

§6. EISENSTEIN SERIES

In [5], the reader can find a representation-theoretic motivation for the introduction of the spaces $C(\chi(m))$ (see section 6.1) and the construction later on, of the so-called Eisenstein series.

6.1. Let χ be a quasi-character of A^*/k^* and m an integer. I will use the notation $C(\chi(m))$ for the space of continuous functions $f : Mp(A) \to \mathbb{C}$ satisfying:

$$f(tR(d(a)u(x))g) = t^m\chi(a)f(g), \text{ for } t \in T,\ a \in A^*,\ x \in A,\ g \in Mp(A).$$

Write A_1^* for $\{a | a \in A^*, |a| = 1\}$ and put n for the degree of k over $\mathbb{Q}$. Embed $\mathbb{R}^*_{>0}$ into A^* by means of $r \mapsto (r_v)$, where $r_v = r^{1/n}$, if v is infinite, and $r_v = 1$ for all finite places of k. This map enables us to identify A^*/k^* with $\mathbb{R}^*_{>0} \times (A_1^*/k^*)$ and to decompose χ uniquely as $\nu(s)\chi_0$, with $\chi_0 \in \widehat{(A_1^*/k^*)}$ and $\nu(s)$, for $s \in \mathbb{C}$, the quasi-character $x \to |x|^s$ of A^*. Now $C(\chi(m))$ and $C(\chi_0(m))$ are isomorphic as vector spaces through the map

(6.2) $\qquad f \mapsto f.H(s) = f(s), \quad$ with $f \in C(\chi_0(m))$

6.3. Let χ be $\Pi_{v\in P}\ \chi_v$. The local analogue of $C(\chi(m))$ is the space $C(\chi_v(m))$ of continuous functions $f : Mp(k_v) \to \mathbb{C}$ satisfying

(6.4) $\ f(tR(d(a)u(x))g) = t^m\chi_v(a)f(g)$, for $t \in T,\ a \in k_v^*,\ x \in k_v,\ g \in Mp(k_v)$.

$Mp(k_v)$ acts on $C(\chi_v(m))$ by means of right translations and we use the notation $Ind(\chi_v(m))$ for this representation. Put $\nu(s_v)$ for the quasi-character $x \to |x|_v^{s_v}$, where $x \in k_v^*$ and $s_v \in \mathbb{C}$. For infinite v,
$k_v^* \cong \mathbb{R}^*_{>0} \times \{x | x \in k_v^*\ |x|_v = 1\}$ and for finite v, $k_v^* \cong \{\pi_v^r | r \in \mathbb{Z}\} \times \mathcal{O}_v^*$.
Therefore, one can decompose each χ_v uniquely as $\nu(s_v)\chi_v^o$, with χ_v^o a character of $\{x | x \in k_v^*, |x|_v = 1\}$ and $s_v \in \mathbb{C}$. As before, one obtains an isomorphism between $C(\chi_v^o(m))$ and $C(\chi_v(m))$ by

(6.5) $\qquad f \to f.(H(s_v) \circ i_v) = f(s_v), \quad$ with $f \in C(\chi_v^o(m))$.

Furthermore, it is clear from (2.4) and (6.4) that for all finite v such that χ_v is unramified and $|2|_v = 1$, there exists a unique $\varphi_v^{\mathcal{O}} \in C(\chi_v(m))$ satisfying

(6.6) $\qquad \varphi_v^{\mathcal{O}} \mid R_v^{\mathcal{O}}(G_v) \equiv 1.$

Write $\otimes\ C(\chi_v(m))$ for the restricted tensor product of the $C(\chi_v(m))$ with respect to the $\varphi_v^{\mathcal{O}}(s_v)$. By (3.3), the elements of $\otimes\ C(\chi_v(m))$ can be interpreted as functions on $Mp(A)$ and those functions belong clearly to $C(\chi(m))$.

6.7. Let C resp. Y be compact subsets of $\{s | s \in \mathbb{C}, Re(s) > 2\}$. resp. $Mp(A)$. The proof of the following proposition follows a standard method of

Godement and will therefore be left to the reader.

6.8. PROPOSITION. *Let* φ *be an element of* $C(\chi_0(m))$. *Then*

$$\sup_{s\in C}\left\{\sum_{\sigma\in S\ell(2,k)/P(k)}\{\sup_{g\in Y}|\varphi(s)(R(\sigma^{-1})g)|\}\right\}<\infty$$

We can define now for φ in $C(\chi_0(m))$ and $Re(s) > 2$ the *Eisenstein series* $E(\varphi(s),\chi(m)) : Mp(A) \to \mathbb{C}$ by

$$(6.9)\qquad E(\varphi(s),\ \chi(m))(g) = \sum_{\sigma\in S\ell(2,k)/P(k)} \varphi(s)(R(\sigma^{-1})g).$$

From (6.8) it is also clear that $E(\varphi(s),\chi(m))$ is holomorphic on $Re(s) > 2$. Since, for every $\varphi \in S(A)$, the function $\theta(\varphi)_0$ belongs to $C(\nu(\frac{1}{2})(1))$, (5.5) is a special case of (6.9).

By definition, $E(\varphi(s),\chi(m))$ is a function on $S\ell(2,k)\backslash Mp(A)$ and, if $z \in k$, its z-th Fourier coefficient $E_z(\varphi(s),\chi(m)) : Mp(A) \to C$ is given by

$$E_z(\varphi(s),\chi(m))(g) = \int_{A/k} E(\varphi(s),\chi(m))(R(u(x))g)\ \tau(-xz)d\dot{x}$$

Using the Bruhat decomposition for $S\ell(2,k)$, we get for $z \neq 0$

$$E_z(\varphi(s),\chi(m))(g) = \int_A \varphi(s)(R(w(1)u(x))g)\ \tau(-xz)dx$$

and for $z = 0$,

$$E_0(\varphi(s),\chi(m))(g) = \varphi(s)(g) + \int_A \varphi(s)(R(w(1)u(x))g)dx.$$

6.10. If one knows that $\sum_{z\in k}|E_z(\varphi(s),\chi(m))(g)| < \infty$, then

$$E(\varphi(s),\chi(m))(g) = \sum_{z\in k} E_z(\varphi(s),\chi(m))(g).$$

This fact will be used at the meromorphic continuation of the Eisenstein series.

6.11. Notations being as above, let f belong to $C(\chi(m))$, with $Re(s) > 2$. Put

$$M(\chi(m))(f)(g) = \int_{A} f(R(w(1)\ u(x))g)dx, \quad \text{with } g \in Mp(A).$$

Clearly, $M(\chi(m))(f) \in C(\nu(2)\chi^{-1}(m))$ and $M(\chi(m))$ commutes with right translations by elements of $Mp(A)$.

Analogously, one can define for all $v \in P$ and $s_v \in \{s | s \in \mathbb{C}, \mathrm{Re}(s) > 1\}$ a $M(\chi_v(m)) \in \mathrm{Hom}_{Mp(k_v)}(C(\chi_v(m)), C(\nu(2)\chi_v^{-1}(m)))$ by

$$(6.12) \qquad M(\chi_v(m))(f)(g) = \int_{k_v} f(R_v(w(1)u(x))g)dx \text{ for } f \in C(\chi_o(m)),\ g \in Mp(k_v).$$

For φ in $\otimes C(\chi_v(m))$ of the form $\otimes \varphi_v$, $\varphi_v \in C(\chi_v(m))$, it is clear that

$$(6.13) \qquad M(\chi(m))(\varphi) = \otimes M(\chi_v(m))(\varphi_v).$$

6.14. Let $S(\chi(m))$ resp. $S(\chi_v(m))$ be the subspace of $C(\chi(m))$ resp. $C(\chi_v(m))$ consisting of functions f that satisfy

there is an $\eta \in E(\tilde{M})$ resp. $E(\tilde{K}_v)$ such that $f * \eta = f$.

Obviously, $M(\chi(m))(S(\chi(m))) \subset S(\nu(2)\chi^{-1}(m))$ and for all $v \in P$, $M(\chi_v(m))(S(\chi_v(m))) \subset S(\nu(2)\chi_v^{-1}(m))$. Moreover we have

$$S(\chi(m)) = \otimes S(\chi_v(m)).$$

Indeed, suppose $\varphi \in S(\chi(m))$; then there is a sufficiently large Q such that φ is invariant under $R_Q(\Pi_{v \notin Q} G_v)$. Consequently φ has the form

$$(\varphi o i_Q) \otimes \{ \otimes_{v \notin Q} \varphi_v^o(s_v)\}.$$

Now every irreducible continuous representation of $\Pi_{v \in Q} \tilde{K}_v$ has the form $\otimes \rho_v$, with ρ_v an irreducible continuous representation of $\tilde{K}_v$, and applying this to the action of $\Pi_{v \in Q} \tilde{K}_v$ on $\varphi \circ i_Q$ gives the desired result.

6.15. Next we pay some attention to the other Fourier coefficients. For $a \in k_v^*$, $\mathrm{Re}(s_v) > 1$ and $f_v \in C(\chi_v^o(m))$ define $W(a, \chi_v(m))(f_v(s_v)) : Mp(k_v) \to \mathbb{C}$ by

$$W(a,\chi_v(m))(f_v(s_v))(g) = \int_{k_v} f_v(s_v)(R_v(w(1)u(x))g)\ \tau_v(-ax)dx$$

Let $\varphi \in C(\chi(m))$ be as in (6.13). Then we have for $\mathrm{Re}(s) > 2$

(6.16) $\qquad E_z(\varphi,\chi(m)) = \otimes\, W(z,\chi_v(m))(\varphi_v).$

For the moment, denote $W(a,\chi_v(m))(f_v(s_v))$ by f; it satisfies

(6.17) $\qquad f(R_v(u(x))g) = \tau_v(ax)f(g) \quad$ for $g \in Mp(k_v)$, $x \in k_v$

and

(6.18) $\qquad f * \check{\eta} = f \qquad\qquad$ for some $\eta \in E(\tilde{K}_v)$.

In the infinite case one can say moreover that

(6.19) $\qquad$ f is a C^∞-function on $Mp(k_v)$ and for all $t \in k_v^*$, with $|t|_v \geq 1$,

$$|f(R_v(d(t)))| \prec |t|_v^{2-\mathrm{Re}(s_v)} ;$$

the first property being a consequence of the fact that the matrix-coefficients of an irreducible continuous representation of $\tilde{K}_v$ are C^∞-functions on $\tilde{K}_v$ and the second an application of the definition.

Now, for finite v, define $S(\tau_v^a)$ as the space of $\mathbb{C}$-valued functions on $Mp(k_v)$, satisfying (6.17) and (6.18). If v is infinite, then one demands of each function h in $S(\tau_v^a)$, besides (6.17) and (6.18), that it is C^∞ and slowly increasing at infinity, that is to say there is a $N \in \mathbb{N}$ such that for all $t \in k_v^*$, with $|t|_v \geq 1$,

$$h(R_v(d(t))) \prec |t|_v^N \quad .$$

6.20. From (4.13) one sees that for even m, $C(\chi(m))$ corresponds in fact to a space of functions on $S\ell(2,A)$ and an $E(\varphi(s),\chi(m))$, with $\varphi \in C(\chi_o(m))$, to an Eisenstein series on $S\ell(2,A)$. As they are well-known and in view of the fact that I will apply it only to the situation, described in §5, I will discuss further only the genuine Eisenstein series on $Mp(A)$ i.e. those with odd m.

6.21. <u>CONVENTION</u>. Throughout the rest of this paper $m = 1 + 2\ell$, $\ell \in \mathbb{Z}$. If m equals 1, then we simplify the notations and write respectively χ, χ_o, χ_v and χ_v^o instead of $\chi(1)$, $\chi_o(1)$, $\chi_v(1)$ and $\chi_v^o(1)$.

6.22. It is my intention to show that for all $\varphi \in S(\chi_o(m))$, $E(\varphi(s),\chi(m))$ has a meromorphic continuation to $\mathbb{C}$, which is holomorphic on $\{s \mid s\in\mathbb{C},\ Re(s) > 1,\ s \neq \frac{3}{2}\}$ and has a pole of order ≤ 1 in $s = \frac{3}{2}$. To do so, I prove these assertions first for all the $E_z(\varphi(s),\chi(m))$, $z \in k$. Next I will show that $\sum_{z\in k} E_z(\varphi(s),\chi(m))$ defines the continuation I am looking for.

In order to carry this out, we have to know first if, for all $v \in P$, $f \in S(\chi_v^o(m))$ and $a \in k_v^*$, $M(\chi_v(m))(f(s_v))$ and $W(a,\chi_v(m))(f(s_v))$ have a meromorphic continuation to $\mathbb{C}$, and, if so, where the poles can occur.

As for the first property, it suffices to verify it for all $f \in S(\chi_v^o(m))$ and all $a \in k_v^*$ in the point e_v thanks to the relations:

$$\begin{aligned}(6.23)\qquad & M(\chi_v(m))(f(s_v))(R_v(d(b)u(x))h) = \\ & = \nu(2)\chi_v^{-1}(b)M(\chi_v(m))(Ind(\chi_v^o(m))(h)(f)(s_v))(e_v)\end{aligned}$$

$$\begin{aligned}(6.24)\qquad & W(a,\chi_v(m))(f(s_v))(R_v(u(x)d(b))h) = \\ & = \tau_v(ax)\nu(2)\chi_v^{-1}(b)W(ab^2,\chi_v(m))(Ind(\chi_v^o(m))(h)(f)(s_v))(e_v)\end{aligned}$$

for $b \in k_v^*$, $x \in k_v$ and $h \in \tilde{K}_v$. For the remaining local questions, I refer the reader to the chapters 3 and 4.

6.25. Note that for the infinite places, we have no longer an action of $MP(k_v)$ on $S(\chi_v(m))$. As a substitute will serve the action of a certain convolution algebra of distributions on $Mp(k_v)$ the so-called *Hecke algebra*. Also in the finite case, one can replace the action of $Mp(k_v)$ by that of a "Hecke algebra". All of them will be defined in the next paragraph.

This algebra structure of $S(\chi_v(m))$ will play a role at the determination of the nature of the residue of the Eisenstein series and at the derivation of the functional equation of these series.

§7. HECKE ALGEBRAS

7.1. THE INFINITE CASE. Denote the complexification of the real Lie algebra m by $m_{\mathbb{C}}$. The elements of the universal enveloping algebra of $m_{\mathbb{C}}$, $U(m_{\mathbb{C}})$, will be considered as distribution on $Mp(k_v)$. One easily checks that

(7.2) $$\mathcal{H}_v = \{ \sum_{i=1}^{n} \lambda_i (f_i * X_i) \mid \lambda_i \in C,\ f_i \in A(\tilde{K}_v),\ X_i \in U(m_{\mathbb{C}}) \}$$

is a convolution algebra of distributions on $Mp(k_v)$, the *Hecke algebra* of $Mp(k_v)$.

Several of the $\mathcal{H}_v$-modules E that play a role in the sequel are *non-degenerate* i.e. $E = \mathcal{H}_v.E$. This is equivalent to:

(7.3) For every $v \in E$ there exists an $\eta \in E(K_v)$ such that $\eta.v = v$.

Thanks to (7.3), we can define on every non-degenerate $\mathcal{H}_v$-module E an action of $\tilde{K}_v$ by:

$$h.v = (\delta_h * \eta).v,$$

where $h \in \tilde{K}_v$, $v \in E$ and η as in (7.3).

Hence every non-degenerate $\mathcal{H}_v$-module is a $(m,\tilde{K}_v)$-module in the sense of [11].

7.4. Examples of non-degenerate $\mathcal{H}_v$-modules are:

(i) $S(\chi_v(m))$ with the action $Ind(\chi_v(m))$ defined by

$$Ind(\chi_v(m))(h)(f) = f * \check{h} \quad \text{for } f \in S(\chi_v(m)), \quad h \in \mathcal{H}_v$$

(ii) $S(\tau_v^a)$ with the action $Ind(\tau_v^a)$ defined by

$$Ind(\tau_v^a)(h)(f) = f * \check{h} \qquad \text{for } f \in S(\tau_v^a),\ h \in \mathcal{H}_v.$$

7.5. REMARKS.

(i) It is clear that $M(\chi_v(m)) \in Hom_{\mathcal{H}_v}(S(\chi_v(m)), S(\nu(2)\chi_v^{-1}(m))$ and that $W(a,\chi_v(m)) \in Hom_{\mathcal{H}_v}(S(\chi_v(m)),S(\tau_v^a))$.

(ii) Let E be a $\mathcal{H}_v$-module. By a *Whittaker model* of E *with respect to* τ_v^a or shortly a $W(a)$-*model* of E, I mean the image of a non-zero operator in

$\mathrm{Hom}_{\mathcal{H}_v}(E,S(\tau_v^a))$.

(iii) Assume that $M(\chi_v(m))(f(s_v))$ has, for all $f \in S(\chi_v^o(m))$, a holomorphic continuation to an open connected U, with $\{s_v | s_v \in \mathbb{C}, \mathrm{Re}(s_v) > 1\} \subset U \subset \mathbb{C}$; then $M(\chi_v(m))$ commutes with the action of $\mathcal{H}_v$, for all $s_v \in U$. Evidently, the same property holds if one replaces $M(\chi_v(m))$ by $W(a,\chi_v(m))$.

7.6. One has a natural action of $Mp(k_v)$ on $L^2(k_v)$ and this representation ω_v stabilizes $S(k_v)$. For $X \in m$, define $\omega_v(X) \in \mathrm{End}(S(k_v))$ by:

$$(7.7) \qquad \omega_v(X)(f)(u) = \frac{d}{dt}\{\omega_v(\exp(tX))(f)(u)\}\Big|_{t=0}$$

for $f \in S(k_v)$ and $u \in k_v$. This turns $S(k_v)$ into a $U(m_{\mathbb{C}})$-module. The corresponding action of $A(K_v)$ on $S(\tilde{K}_v)$ is given by

$$(7.8) \qquad \omega_v(h)(f)(u) = \int_{\tilde{K}_v} h(k)\omega_v(k)(f)\, dk,$$

where $h \in A(\tilde{K}_v)$, $f \in S(k_v)$, $u \in k_v$ and dk as in (0.3). The composition of these actions, makes $S(k_v)$ into an $\mathcal{H}_v$-module. Let $S(k_v)_e$ be the subspace of even functions in $\mathcal{H}_v.S(k_v)$. This is another example of a non-degenerate $\mathcal{H}_v$-module.

7.9. Let v be imaginary in this section. From (1.9) (iv) we know that $Mp(k_v) \cong S\ell(2,k_v) \times T$. Therefore, one can define, for every function f on $S\ell(2,k_v)$ and every $r \in \mathbb{Z}$, a function f_r on $Mp(k_v)$ by: $f_r(tR_v(g)) = t^r f(g)$, with $t \in T$ and $g \in S\ell(2,k_v)$.

Write $C_o(\chi_v)$, $S_o(\chi_v)$ and $S_o(\tau_v^a)$ for the spaces that one obtains by restricting the elements of respectively $C(\chi_v(m))$, $S(\chi_v(m))$ and $S(\tau_v^a)$ to $R_v(S\ell(2,k_v))$. If we replace in (7.2) $\tilde{K}_v$ by K_v and $m_{\mathbb{C}}$ by $g \otimes_{\mathbb{R}} \mathbb{C}$, then we get a convolution algebra $\mathcal{H}_v^o$ of distributions on $S\ell(2,k_v)$. Let $\mathrm{Ind}_o(\chi_v)$ resp. $\mathrm{Ind}_o(\tau_v^a)$ be the actions of $\mathcal{H}_v^o$ on $S_o(\chi_v)$ resp. $S_o(\tau_v^a)$, given by

$$f \mapsto f * \check{h}, \text{ with } f \in S_o(\chi_v) \text{ resp. } S_o(\tau_v^a) \text{ and } h \in \mathcal{H}_v^o.$$

For $f \in S_o(\chi_v)$, put $M_o(\chi_v)(f)$ for $M(\chi_v(m))(f_m) \circ R_v$ and $W_o(a,\chi_v)(f)$ for $W(a,\chi_v(m))(f_m) \circ R_v$. Clearly, $M_o(\chi_v) \in \mathrm{Hom}_{\mathcal{H}_v^o}(S_o(\chi_v), S_o(\nu(2)\chi_v^{-1}))$ and $W_o(a,\chi_v) \in \mathrm{Hom}_{\mathcal{H}_v^o}(S_o(\chi_v), S(\tau_v^a))$. Furthermore, in order to get a meromorphic continuation of $M(\chi_v(m))(f_m)$ resp. $W(a,\chi_v(m))(f_m)$, it is sufficient to have one for $M_o(\chi_v)(f)$ resp. $W_o(a,\chi_v)(f)$.

7.10. THE FINITE CASE. $Mp(k_v)$ acts on $S(\chi_v(m))$ resp. $S(\tau_v^a)$ by means of right translations and I denote these representations by $Ind(\chi_v(m))$ resp. $Ind(\tau_v^a)$. They belong to a type that can be defined for every closed subgroup H of $Mp(k_v)$ and it will be convenient to give this general setting.

First of all, note that for such a H, either $H \cap T$ is finite or $T \leq H$, so that H possesses always open compact subgroups.

A representation σ of H on a complex vector space E is called *algebraic* if for $v \in E$ and every open compact subgroup H_0 of H, $\{\sum_{i=1}^m \lambda_i\sigma(h_i)(v) \mid \lambda_i \in \mathbb{C}, h_i \in H_0\}$ is finite-dimensional and if the action of H_0 on this subspace is continuous. Note that, if $H \cap T$ is finite, this definition agrees with the one in [1]. I will use the notation Alg(H) for the category of algebraic representations of H.

Let $\mathcal{H}(H)$ be the convolution-algebra of functions $f\colon H \to \mathbb{C}$ satisfying

(i) f has compact support

(ii) there is a H_0, as above, and an $\eta \in E(H_0)$ such that $\eta * f * \eta = f$.

Take any Haar measure dh on H. Every $(\sigma,E) \in Alg(H)$ becomes an $\mathcal{H}(H)$-module, if we define

$$\sigma(f)(v) = \int_H f(h)\ \sigma(h)(v)\ dh\ , \qquad \text{for } f \in \mathcal{H}(H) \text{ and } v \in E.$$

In particular, E is a non-degenerate $\mathcal{H}(H)$-module i.e. $\mathcal{H}(H).E = E$. On the other hand, every non-degenerate $\mathcal{H}(H)$-module E forms an object in Alg(H). Indeed, for every $v \in E$, one can take an $\eta \in E(H_0)$, with H_0 as above, such that $\eta.v = v$ and one defines the action of $g \in Mp(k_v)$ on v by

$$g.v = (\delta_g * \eta).v.$$

If $H = Mp(k_v)$, then we have to extend $\mathcal{H}(Mp(k_v))$ somewhat, in order to be able to define a global Hecke algebra. Let $\mathcal{H}_v$ be the convolution algebra $\mathcal{H}(Mp(k_v)) \oplus A(G_v)$ of distributions on $Mp(k_v)$. I call $\mathcal{H}_v$ the *Hecke algebra* of $Mp(k_v)$. Each $(\sigma,E) \in Alg(Mp(k_v))$ possesses a natural $A(G_v)$-module structure. It is given by

$$\sigma(f)(v) = \int_{G_v} f(g)\ \sigma(g)(v)dg \quad \text{for } f \in A(G_v) \text{ and } v \in E.$$

Here dg is chosen as in (0.3).

7.11. As before, let ω_v be the natural representation of $Mp(k_v)$ in $L^2(k_v)$. It stabilizes $S(k_v)$ and $S(k_v)$ is an algebraic $Mp(k_v)$-module. Write $S(k_v)_e$ for the $\mathcal{H}_v$-submodule of $S(k_v)$ consisting of even functions.

7.12. REMARKS.

(i) Also in the finite case I will use the terminology introduced in (7.5) (ii).

(ii) The properties, as stated in (7.5) (i) and (iii), are also valid for finite places.

7.13. THE GLOBAL CASE. Let the global Hecke algebra $\mathcal{H}$ be the restricted tensor product of the $\mathcal{H}_v$ with respect to the ε_{G_v}. One can define for each χ as in (6.1) and every $r \in \mathbb{Z}$ an action of $\mathcal{H}$ on $S(\chi(r))$ by

$$(\otimes h_v).(\otimes \varphi_v) = \otimes(\varphi_v * \check{h}_v), \qquad \text{with } \otimes h_v \in \mathcal{H} \text{ and } \otimes \varphi_v \in S(\chi(r)).$$

CHAPTER 3

For convenience sake, I will leave out in the chapters 3 and 4 all subscripts v.

§8 THE DECOMPOSITION OF $S(\chi(m))$ IN THE REAL CASE

8.1. For $\varphi \in [-\pi,\pi)$, put $r(\varphi) = \begin{pmatrix}\cos(\varphi) & -\sin(\varphi)\\ \sin(\varphi) & \cos(\varphi)\end{pmatrix}$. $\check{K}$ is commutative and its characters, that are non-trivial on T^0, have the form

$$\langle r(\varphi),\zeta\rangle \mapsto \zeta\, e^{i\frac{r}{2}\varphi} \quad \text{for } \zeta \in T^0,\ \varphi \in [-\pi,\pi),\ r \in 1+2\mathbb{Z}.$$

By (4.2), I can define the characters $\psi(r,m)$ of $\tilde{K}$ by

$$\psi(r,m)(tR(r(\varphi))) = \begin{cases} t^m\, \gamma(\sin(\varphi))^m\, e^{i\frac{r}{2}\varphi} & \text{if } r(\varphi) \in \Omega(k)\\ \\ t^m(\gamma(\cos(\varphi))\gamma(-1))^m\, e^{i\frac{r}{2}\varphi} & \text{if } r(\varphi) \in P(k)\end{cases}.$$

Write $\chi = \nu(s)\chi^o$ as in (6.3). Since the elements of $S(\chi(m))$ are completely determined by their restriction to $\tilde{K}$, we get:

(8.2) There exists a $g(r,m) \in S(\chi^o(m))$ such that

$$g(r,m)|\tilde{K} = \psi(r,m) \text{ if and only if } \gamma(-1)^{2m}\, e^{-i\frac{r\pi}{2}} = \chi^o(-1).$$

We will write $Z(\chi^o,m)$ for $\{r|r \in 1+2\mathbb{Z}$ and $g(r,m)$ exists$\}$. The following notations for certain elements of $\mathfrak{sl}(2,\mathbb{C})$ will be used in the sequel:

$$U = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix},\ V_+ = \begin{pmatrix}1 & -i\\ i & -1\end{pmatrix},\ V_- = \begin{pmatrix}1 & i\\ -i & -1\end{pmatrix},\ X_+ = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\ X_- = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \text{ and } Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

With (2.2) one computes that for all $r \in Z(\chi^o,m)$ and $s \in \mathbb{C}$

$$(8.3) \qquad g(r,m)(s) * \check{U} = -i\,\frac{r}{2}\,g(r,m)(s)$$

As $\mathrm{Ad}(R(r(\varphi)))(V_+) = e^{-2i\varphi}V_+$ and $\mathrm{Ad}(R(r(\varphi)))(V_-) = e^{2i\varphi}V_-$, a reasoning, like in [6], page 166, yields that for r and s as above

$$(8.4) \qquad g(r,m)(s) * \check{V}_+ = (s - \tfrac{r}{2})\,g(r-4,m)(s)$$

$$(8.5) \qquad g(r,m)(s) * \check{V}_- = (s + \tfrac{r}{2})\,g(r+4,m)(s)$$

Since an $\mathcal{H}$-submodule of $S(\chi(m))$ is spanned by the $g(r,m)(s)$ contained in it, we may conclude from (8.4) and (8.5)

8.6. <u>THEOREM</u>.
(i) $S(\chi(m))$ *is irreducible if* $s \notin \frac{1}{2} + \mathbb{Z}$,
(ii) *If* $s = \pm\frac{r}{2}$, *with* $r \in Z(\chi^o,m)$, $S(\chi(m))$ *has a unique non-trivial* $\mathcal{H}$*-submodule. For* $s = \frac{r}{2}$, *it is equal to*

$$P(\chi,m,r) = \{\sum_{i=1}^{n} \lambda_i\, g(k_i,m)(\tfrac{r}{2}) \mid \lambda_i \in C,\ k_i \in Z(\chi^o,m),\ k_i \geq r\}$$

and for $s = -\frac{r}{2}$ *to*

$$N(\chi,m,r) = \{\sum_{i=1}^{n} \lambda_i\, g(k_i,m)(-\tfrac{r}{2}) \mid \lambda_i \in C,\ k_i \in Z(\chi^o,m),\ k_i \leq r\}.$$

8.7. Next, we determine $\mathrm{Hom}_{\mathcal{H}}(S(\chi_1(m)),S(\chi_2(m)))$ and deduce from it some relations between the $P(\chi,m,r)$ and the $N(\chi,m,r)$. Let D be $X_+X_- + XX_+ + Z^2/2$; D belongs to the center of $U(m_{\mathbb{C}})$ and as in [6] one computes that for all $f \in S(\chi(m))$

$$(8.8) \qquad f * \check{D} = \frac{s(s-2)}{2}\,f.$$

From (8.8) it is clear that, if $\chi_2 \notin \{\chi,\nu(2)\chi^{-1}\}$, $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m),S(\chi_2(m)) = \{0\}$.

If $\chi_2 \in \{\chi,\nu(2)\chi^{-1}\}$ and $A(\chi,m) \in \mathrm{Hom}_{\tilde{K}}(S(\chi(m)),S(\chi_2(m)))$, then there exists for each $r \in Z(\chi^o,m)$ an $a_r(\chi,m) \in \mathbb{C}$ such that

(8.9) $$A(\chi,m)(g(r,m)(s)) = a_r(\chi,m)\begin{cases} g(r,m)(s) & \text{if } \chi_2 = \chi \\ g(r,m)(2-s) & \text{if } \chi_2 = \nu(2)\chi^{-1} \end{cases} .$$

On the other hand, any sequence $\{a_r(\chi,m) \mid r \in Z(\chi^o,m)\}$ determines by formula (8.9) an $A(\chi,m) \in \mathrm{Hom}_{\tilde{K}}(S(\chi(m)),S(\chi_2(m)))$. This $A(\chi,m)$ will belong to $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)),S(\chi_2(m)))$, if it commutes with the action of V_+ and V_-.

If $\chi = \chi_2$, this condition yields for all $r \in Z(\chi^o,m)$:

(8.10) $$(s-\tfrac{r}{2})a_r(\chi,m) = (s-\tfrac{r}{2})a_r(\chi,m) \text{ and } (s+\tfrac{r}{2})a_r(\chi,m) = (s+\tfrac{r}{2})a_r(\chi,m).$$

Since either $s \neq \frac{r}{2}$ or $s \neq -\frac{r}{2}$ for all $r \in Z(\chi^o,m)$, (8.10) implies for all χ and m that $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)), S(\chi(m))) = \{\alpha I_{S(\chi(m))} \mid \alpha \in \mathbb{C}\}$.

If $\chi_2 = \nu(2)\chi^{-1}$, then the $a_r(\chi,m)$ have to satisfy

(8.11) $$-(s+\tfrac{r-4}{2})a_r(\chi,m) = (s-\tfrac{r}{2})a_{r-4}(\chi,m), \quad \text{for all } r \in Z(\chi^o,m).$$

For $s \notin \frac{1}{2}+\mathbb{Z}$, this relation leaves the freedom to choose one $a_r(\chi,m)$ arbitrarily. If $s = \frac{r}{2}$, then it implies that $a_n(\chi,m) = 0$ for all $n \geq r$, and, for $s = -\frac{r}{2}$, all the $a_n(\chi,m)$ with $n \leq r$ have to be zero. For reasons of reference, we summarize these results in a

8.12. <u>PROPOSITION</u>. *Let* χ_2 *belong to* $\{\chi,\nu(2)\chi^{-1}\}$ *and let* $A(\chi,m)$ *be an element of* $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)),S(\nu(2)\chi^{-1}(m)))$. *Then*

(i) $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)),S(\chi_2(m)))$ *is one-dimensional.*

(ii) *If* $s = \frac{r}{2}$, *with* $r \in Z(\chi^o,m)$, *and* $A(\chi,m) \neq 0$, *then*

$$\mathrm{Ker}(A(\chi,m)) = P(\chi,m,r) \textit{ and } \mathrm{Im}(A(\chi,m)) = N(\nu(2)\chi^{-1},m,r-4) .$$

(iii) *If* $s = -\frac{r}{2}$, *with* $r \in Z(\chi^o,m)$, *and* $A(\chi,m) \neq 0$, *then*

$$\mathrm{Ker}(A(\chi,m)) = N(\chi,m,r) \textit{ and } \mathrm{Im}(A(\chi,m)) = P(\nu(2)\chi^{-1},m,r+4).$$

8.13. In this section we determine the meromorphic continuation of $M(\chi(m))(f(s))$ for $f \in S(\chi^o(m))$. Since $\tau(x) = e^{-2\pi ix}$, dx is the usual

Lebesgue measure on $\mathbb{R}$. Thanks to (6.23), it suffices to continue the following functions:

$$M(\chi(m))(g(r,m)(s))(e) = \gamma(1)^m e^{ir\frac{\pi}{4}} \int_{\mathbb{R}} (1-ix)^{-\frac{s}{2}+\frac{r}{4}}(1+ix)^{-\frac{s}{2}-\frac{r}{4}}dx$$

$$= \gamma(1)^m e^{ir\frac{\pi}{4}} \frac{\pi 2^{2-s}\Gamma(s-1)}{\Gamma(\frac{s}{2}-\frac{r}{4})\Gamma(\frac{s}{2}+\frac{r}{4})}$$

$$= \gamma(1)^m e^{ir\frac{\pi}{4}} 2^{2-s}\Gamma(s-1)\sin(\pi(\tfrac{s}{2}-\tfrac{r}{4}))\frac{\Gamma(1-\frac{s}{2}+\frac{r}{4})}{\Gamma(\frac{s}{2}+\frac{r}{4})}$$

Clearly, we can speak now for all $s \in \mathbb{C}\backslash\mathbb{Z}$ of $M(\chi(m))$ and $M(\nu(2)\chi^{-1}(m))$. By combining the expression above with (8.2) we get for all $s \in \mathbb{C}\backslash\mathbb{Z}$:

$$(8.14) \qquad M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m)) = -\ |2|\chi(-1)\Gamma(s-1)\Gamma(1-s)\cos(\pi s)I_{S(\chi(m))}.$$

8.15. We focus our attention now on the irreducible $\mathcal{H}$-submodule of $S(\nu(\frac{1}{2})h(a)(m))$. Notations being as in (7.6), I define $L \in \mathrm{Hom}_{\mathcal{H}}(S(\mathbb{R})_e, S(\nu(\frac{1}{2})))$ by:

$$(8.16) \qquad L(\varphi) = \omega(g)(\varphi)(0) \qquad \text{for } \varphi \in S(\mathbb{R})_e,\ g \in Mp(\mathbb{R})).$$

L is an injection as one can see from

$$L(\varphi)(R(w(1))R(u(x))) = \int_{\mathbb{R}} \varphi(y)\ \tau(\tfrac{1}{2}xy^2)dy$$

$$= \int_0^{\infty} \frac{\varphi(\sqrt{t})}{\sqrt{t}}\ \tau(\tfrac{1}{2}tx)dt.$$

From proposition (8.12) and the fact that ω is unitary, we may conclude that $L(S(\mathbb{R})_e) \subsetneq S(\nu(\frac{1}{2}))$. Using (4.6) and (4.16), we can state now for all $a \in k^*$ and $m \in 1+2\mathbb{Z}$:

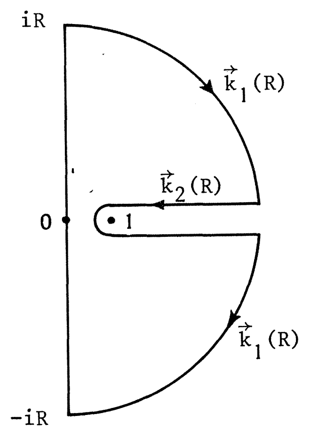
8.17. <u>PROPOSITION</u>. *The map* $\varphi \mapsto L(\varphi) \circ I(a(-1)^{\ell}) \circ J(m)$ *is an* $\mathcal{H}$*-module isomorphism between* $(\omega \circ I(a(-1)^{\ell}) \circ J(m), S(\mathbb{R})_e)$ *and the irreducible submodule of* $S(\nu(\frac{1}{2})h(a)(m))$.

§9 WHITTAKER MODELS IN THE REAL CASE

9.1. Let f be an element of $S(\chi^o(m))$. I want to show here that for all $a \in \mathbb{R}^*$ $W(a,\chi(m))(f(s))$ has a holomorphic continuation to $\mathbb{C}$. Thanks to (6.24) we have to consider only the $W(a,\chi(m))(g(r,m)(s))(e)$, for all $a \in \mathbb{R}^*$ and $r \in Z(\chi^o,m)$. Now

$$W(a,\chi(m))(g(r,m)(s))(e) \approx \int_{\mathbb{R}} (1+ix)^{-\frac{s}{2}-\frac{r}{4}}(1-ix)^{-\frac{s}{2}+\frac{r}{4}} e^{2\pi iax}dx.$$

An easy way to obtain the analytic continuation of this integral is to consider the contour



and to make the substitution u = ix, if a < 0, and u = -ix, if a > 0. One verifies that for Re(s) > 1 and a < 0,

$$\lim_{R\to\infty} \int_{\vec{k}_1(R)} (1+u)^{-\frac{s}{2}-\frac{r}{4}}(1-u)^{-\frac{s}{2}+\frac{r}{4}} e^{2\pi au} du = 0$$

and, for a > 0,

$$\lim_{R\to\infty} \int_{\vec{k}_1(R)} (1+u)^{-\frac{s}{2}+\frac{r}{4}} (1-u)^{-\frac{s}{2}-\frac{r}{4}} e^{-2\pi au} du = 0.$$

Let $\vec{k}_2$ be "$\lim\limits_{R\to\infty} \vec{k}_2(R)$". Then

$$(9.2)\quad W(a,\chi(m))(g(r,m)(s))(e) \approx \begin{cases} \displaystyle\int_{-\vec{k}_2} (1+u)^{-\frac{s}{2}-\frac{r}{4}}(1-u)^{-\frac{s}{2}+\frac{r}{4}} e^{2\pi a u}\,du & \text{if } a<0 \\ \displaystyle\int_{-\vec{k}_2} (1+u)^{-\frac{s}{2}+\frac{r}{4}}(1-u)^{-\frac{s}{2}-\frac{r}{4}} e^{-2\pi a u} & \text{if } a>0 \end{cases}.$$

Since the 2 integrals in (9.2) define holomorphic functions on $\mathbb{C}$, the assertion has been proved.

9.3. One can find on page 431 of [7] that

$$\int_{\mathbb{R}} (1-ix)^{-\frac{s}{2}+\frac{r}{4}}(1+ix)^{-\frac{s}{2}-\frac{r}{4}} e^{2\pi i a x}\,dx = \begin{cases} \dfrac{2\pi}{\Gamma(\frac{s}{2}+\frac{r}{4})}(\pi a)^{\frac{1}{2}(s-1)} W_{\frac{r}{4},\frac{1}{2}(1-s)}(4\pi a) & \text{if } a>0 \\ \dfrac{2\pi}{\Gamma(\frac{s}{2}-\frac{r}{4})}(-\pi a)^{\frac{1}{2}(s-1)} W_{-\frac{r}{4},\frac{1}{2}(1-s)}(-4\pi a) & \text{if } a<0 \end{cases}$$

with $W_{\mu,\nu}$, $\mu,\nu \in \mathbb{C}$, a so-called Whittaker function. Now, for all $a \in \mathbb{R}^*$ and $\mu \in \mathbb{C}$, the function $s \to W_{\mu,s}(|a|)$ is holomorphic on $\mathbb{C}$. Furthermore the functions $W_{\mu,\nu}$ do not vanish identically on $(0,\infty)$. Hence the formula above implies that for all $a \in \mathbb{R}^*$ and all χ as in (6.3), $W(a,\chi(m))$ is non-zero. Moreover, if $s = \frac{r}{s}$ resp. $-\frac{r}{2}$, with $r \in Z(\chi^o,m)$, then it allows us to conclude:

(9.4) $W(a,\chi(m))(P(\chi,m,r)) = \{0\}$, for $a<0$, and $W(a,\chi(m))(P(\chi,m,r)) \neq \{0\}$, if $a<0$.

(9.5) $W(a,\chi(m))(N(\chi,m,r)) = \{0\}$, for $a>0$, and $W(a,\chi(m))(N(\chi,m,r)) \neq \{0\}$, if $a<0$.

From (9.2) and (6.24) one derives the following asymptotic behaviour of the functions in $W(a,\chi(m))(S(\chi(m)))$.

9.6. <u>PROPOSITION</u>. *Let* C *resp.* Y *be compact subsets of* $\mathbb{C}$ *resp.* Mp(k) *and let* f *belong to* $S(\chi^o(m))$. *Then we have:*

(i) *For every* $N \in \mathbb{N}$,

$$\lim_{|a|\to\infty} |a|^N \Big\{ \sup_{\substack{y\in\mathbb{R}, g\in Y,\\ s\in C}} |W(a,\chi(m))(f(s))(R(u(y))g)| \Big\} = 0.$$

(ii) *There exists a* $N_o \in \mathbb{N}$ *such that*

$$\sup_{\substack{y\in\mathbb{R},\ g\in Y\\ s\in C}} |W(a,\chi(m))(f(s))(R(u(y))g)| \prec |a|^{-N_o}.$$

9.7. In this section we prove the uniqueness of the $W(a,\chi(m))$, for $a \in k^*$, $m \in 1+2\mathbb{Z}$ and χ as in (6.3). For $f \in S(\tau^a)$, define $\underline{f}:\mathbb{R}_{>o} \to \mathbb{C}$ by $\underline{f}(t) = f(R(d(t)))$. Let $\underline{f}'$ be the derivative of $\underline{f}$. Then the following relations hold:

$$(9.8)\qquad \underline{f * \check{X}_+}(t) = -2\pi i a t^2\, \underline{f}(t) \qquad \underline{f * \check{Z}}(t) = t\underline{f}'(t).$$

Moreover, if $f * \psi(r,m) = f$, then $\underline{f * \check{U}}(t) = -i\frac{r}{2}\, \underline{f}(t)$.

Let $\{\varphi(r)\,|\, r \in Z(\chi^o,m)\}$ be a collection of slowly increasing C^∞-functions on $\mathbb{R}_{>0}$. For every $r \in Z(\chi^o,m)$, define $\varphi(r,a) \in S(\tau^a)$ by

$$\varphi(r,a)(R(u(y)d(t))h) = \tau(ay)\varphi(r)(t)\psi(r,m)(h)$$

for $y \in \mathbb{R}$, $t \in \mathbb{R}_{>0}$, $h \in \tilde{K}$. Then the map $\sum_{i=1}^n \lambda_i g(r_i,m)(s) \to \sum_{i=1}^n \lambda_i \varphi(r_i,a)$ defines an operator $V(a,\chi(m)) \in \mathrm{Hom}_{\tilde{K}}(S(\chi(m)),S(\tau^a))$. It belongs to $\mathrm{Hom}_{\mathcal{JC}}(S(\chi(m)),S(\tau^a))$ if it commutes with the actions of V_+ and V_-. This condition puts the following restrictions on the $\varphi(r)$:

$$(9.9)\qquad (s-2-\tfrac{r}{2})\varphi(r)(t) = t\varphi(r+4)'(t) - (2+\tfrac{r}{2}-4\pi a t^2)\varphi(r+4)(t).$$

$$(9.10)\qquad (s+\tfrac{r}{2})\varphi(r+4)(t) = t\varphi(r)'(t) + (\tfrac{r}{2}-4\pi a t^2)\varphi(r)(t).$$

In particular one has for all $r \in Z(\chi^o,m)$:

$$(9.11)\qquad t^2\varphi(r)''(t) - t\,\varphi(r)'(t) + (4\pi a r t^2 - 16\pi^2 a^2 t^4 - s(s-2))\varphi(r)(t) = 0.$$

For any $g(r,m)$, such that $W(a,\chi(m))(g(r,m)(s)) \neq 0$ abbreviate $\underline{W(a,\chi(m))(g(r,m)(s))}$ by φ_1. If φ_2 is yet another solution of (9.11) that is slowly increasing at infinity, then $\lim_{t\to\infty} \{\varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t)\} = 0$, as both φ_1 and φ_1' are rapidly decreasing at infinity. From [2], page 83, we know that

$$\varphi_1(t)\varphi_2'(t) - \varphi_1'(t)\varphi_2(t) = \frac{t}{u}\{\varphi_1(u)\varphi_2'(u) - \varphi_1'(u)\varphi_2(u)\}$$

for $t,u \in \mathbb{R}_{>0}$. Consequently $\varphi_1\varphi_2' - \varphi_2\varphi_1' = 0$ and $\varphi_1 \approx \varphi_2$.

By applying the foregoing to a suitably chosen $g(r,m)$, we get with the help of (9.9) and (9.10):

9.12. THEOREM. *Let* χ,m *and* a *be as above. Then*

$$\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)),S(\tau^a)) = \{\lambda W(a,\chi(m)) \mid \lambda \in \mathbb{C}\}.$$

Thanks to (8.12), (9.4) and (9.5), this theorem has the following

9.13. COROLLARY. *Notations being as in* (8.6), *then*

$$\dim \mathrm{Hom}_{\mathcal{H}}(P(\chi,r,m),\ S(\tau^a)) = \begin{cases} 1 & \text{if } a > 0 \\ 0 & \text{if } a < 0 \end{cases},$$

$$\dim \mathrm{Hom}_{\mathcal{H}}(N(\chi,r,m),\ S(\tau^a)) = \begin{cases} 0 & \text{if } a > 0 \\ 1 & \text{if } a < 0 \end{cases}.$$

§10 THE IMAGINARY CASE

10.1. The notations will be as in (7.9). Let $\kappa \in \mathbb{Z}$ be such that for all $z \in \mathbb{C}^*$, with $|z| = 1$, $\chi(z) = z^\kappa$. From now on I will write χ_κ instead of χ^o.

For every $t \in \mathbb{Z}_{\geq 0}$, I denote the irreducible representation of $SU(2,\mathbb{C})$ on the homogeneous polynomials of degree t by ρ_t. Put $S_o(\chi,\rho_t)$ for $\{f \mid f \in S_o(\chi),\ f * \check{e}(\rho_t) = f\}$. The following results are contained in theorem 6.2 of [6]:

10.2. THEOREM.

(i) *If* $s \notin \{-\frac{|\kappa|}{2} - r \mid r \in \mathbb{Z}_{\geq 0}\} \cup \{2 + \frac{|\kappa|}{2} + r \mid r \in \mathbb{Z}_{\geq 0}\}$, *then* $S_o(\chi)$ *is an irreducible* $\mathcal{H}^o$*-module.*

(ii) *If* $s = -\frac{|\kappa|}{2} - r$, $r \in \mathbb{Z}_{\geq 0}$, *then* $\bigoplus_{i=0}^{r} S_o(\chi,\rho_{|\kappa|+2i})$ *is the unique non-trivial* $\mathcal{H}^o$*-submodule of* $S_o(\chi)$.

(iii) *If* $s = 2 + r + \frac{|\kappa|}{2}$, $r \in \mathbb{Z}_{\geq 0}$, *then* $\oplus_{i\geq r+1} S_o(\chi,\rho_{|\kappa|+2i})$ *is the unique non-trivial* $\mathcal{H}^o$*-submodule of* $S_o(\chi)$.

10.3. In order to carry out the meromorphic continuation of $M_o(\chi)$ $(f(s))$ and $W_o(a,\chi)$ $(f(s))$ for $f \in S_o(\chi_\kappa)$, we need a collection of functions that span $S_o(\chi)$. It will be left to the reader to verify that the subsequent one does.

10.4. For $b,t,p \in \mathbb{Z}_{\geq 0}$ such that $0 \leq t \leq |\kappa| + b$ and $0 \leq p \leq b$ one defines $\psi(b,t,p) \in S_o(\chi_\kappa)$ as the function that is given on K by

$$\psi(b,t,p)(d(\alpha)r(\theta)d(\beta)) =$$

$$= \chi_\kappa(\alpha\beta)(\sin(\theta))^{t+p}(\cos(\theta))^{2b+|\kappa|-t-p}\begin{cases}\chi_{2(t-p)}(\beta) & \text{if } \kappa \leq 0 \\ \chi_{2(p-t)}(\beta) & \text{if } \kappa > 0\end{cases}$$

Thanks to (6.23) it is sufficient to prove the assertion for the functions $s \to M_o(\nu(s)\chi_\kappa)(\psi(b,t,p))(e)$ for all b,t and p, as above. They are equal to

$$\left\{\int_0^\infty (1+r^2)^{-s-b-\frac{|\kappa|}{2}} r^{2b+|\kappa|-t-p} 2r\, dr\right\}.\begin{cases}\int_0^{2\pi} e^{i(p-t+|\kappa|)\psi}d\psi & \text{if } \kappa \leq 0 \\ \int_0^{2\pi} e^{i(t-p-\kappa)\psi}d\psi & \text{if } \kappa > 0\end{cases}.$$

Thus we are left to check still only the case $t-p = |\kappa|$. In that case

$$M_o(\nu(s)\chi_\kappa)(\psi(b,p+|\kappa|,p)(s))(e) = 2\pi\int_0^\infty (1+u)^{-2-b-|\kappa|/2}u^{b-p}du =$$

$$= 2\pi\frac{\Gamma(s+p-1+\frac{|\kappa|}{2})\Gamma(b-p+1)}{\Gamma(2+b+\frac{|\kappa|}{2})}$$

and the assertion is obvious now.

10.5. <u>CONCLUSION</u>. For all $s \notin \left\{\{1-\frac{|\kappa|}{2}-r \mid r \in \mathbb{Z}_{\geq 0}\} \cup \{1+\frac{|\kappa|}{2}+r \mid r \in \mathbb{Z}_{\geq 0}\}\right\}$ we can speak of $M_o(\chi)$ and $M_o(\nu(2)\chi^{-1})$.

10.6. For s, as in 10.5, I want to compute $M_o(\nu(2)\chi^{-1}) \circ M_o(\chi)$. To do so, I make use of a well known trick to construct elements of $S_o(\chi)$.

Consider the following action of $S\ell(2,\mathbb{C})$ on $S(\mathbb{C}^2)$:

$$(g.\Phi)(x,y) = \Phi((x,y)g),$$

where $g \in S\ell(2,\mathbb{C})$, $\Phi \in S(\mathbb{C}^2)$ and $(x,y) \in \mathbb{C}^2$. Let $\Phi \to \hat{\Phi}$ be the automorphism of $S(\mathbb{C}^2)$ defined by

$$\hat{\Phi}(a,b) = \iint_{\mathbb{C}^2} \Phi(x,y)\ \tau(bx-ay)\,dxdy.$$

It commutes with the action of $S\ell(2,\mathbb{C})$ and equals its own inverse. Now, for $\mathrm{Re}(s) > 1$ and $\Phi \in S(\mathbb{C}^2)$, one can define an $I(\Phi,\chi) \in S_o(\chi)$ by

$$I(\Phi,\chi)(g) = \int_{\mathbb{C}^*} \chi(t)(g.\Phi)(0,t)|t|^{-1}dt.$$

Call $\Phi \in S(\mathbb{C}^2)$ K-finite, if $\{\sum_{i=1}^{r}\lambda_i(h_i.\Phi)|\lambda_i \in \mathbb{C},\ h_i \in K\}$ is finite-dimensional. For $i \in \mathbb{Z}_{\geq 0}$, let $\Phi_i^{\kappa} \in S(\mathbb{C}^2)$ be given by

$$\Phi_i^{\kappa}(x,z) = e^{-2\pi(|x|+|z|)}\begin{cases} x^{\kappa+i}\bar{z}^{i} & \text{if } \kappa \leq 0. \\ \bar{x}^{\kappa+i}z^{i} & \text{if } \kappa > 0. \end{cases}$$

One verifies that Φ_i^{κ} transforms under K according to $\rho_{2i+|\kappa|}$ and that $I(\Phi_i^{\kappa},\chi) \neq 0$. This implies that $S_o(\chi) = \{I(\Phi,\chi)|\Phi$ a K-finite element of $S(\mathbb{C}^2)\}$.

From [10] we know that for every $\Phi \in S(\mathbb{C}^2)$, $I(\Phi,\chi)$ has a meromorphic continuation to $\mathbb{C}$ and is analytic on $\mathbb{C}\backslash\{-\frac{|\kappa|}{2}-r|r \in \mathbb{Z}_{\geq 0}\}$. We will denote the local zeta-functions as in [10]. Take any non-zero f in $S(\mathbb{C})$ and write $\rho(\chi)$ for $\zeta(f,\chi).\ \zeta(F(f),\nu(1)\chi^{-1})^{-1}$; then we know from [10] that $\rho(\chi)$ is independent of f, meromorphic on $\mathbb{C}$ and holomorphic on $\mathbb{C}\backslash\{-\frac{|\kappa|}{2}-r|r \in \mathbb{Z}_{\geq 0}\}$. Now, for $\mathrm{Re}(s) > 1$, we have

$$M_o(\chi)(I(\Phi,\chi))(g) = \iint_{\mathbb{C}^2} (g.\Phi)(t,x)\chi(t)|t|^{-2}dtdx$$

and for $1 < \mathrm{Re}(s) < 2$ this equals $\rho(\chi(\nu(-1)))I(\hat{\Phi},\nu(2)\chi^{-1})(g)$.

10.7. <u>CONCLUSION</u>. For all $s \notin \left\{\{1-\frac{|\kappa|}{2}-r|r\in\mathbb{Z}_{\geq 0}\} \cup \{1+\frac{|\kappa|}{2}+r|r\in\mathbb{Z}_{\geq 0}\}\right\}$

$$M_o(\nu(2)\chi^{-1}) \circ M_o(\chi) = \rho(\chi\nu(-1))\rho(\chi^{-1}\nu(1))I_{S_o(\chi)}.$$

Obviously this implies for all odd m:

$$M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m)) = \rho(\chi\nu(-1))\rho(\chi^{-1}\nu(1))I_{S(\chi(m))}$$

10.8. Notations being as in (7.6), define $L \in \mathrm{Hom}_{\mathcal{H}}(S(\mathbb{C})_e,S(\nu(\frac{1}{2})))$ by

$$L(f)(g) = \omega(g)(f)(0),$$

with $f \in S(\mathbb{C})_e$ and $g \in Mp(\mathbb{C})$. L is an injective map, as one can see from:

$$L(f)(R(w(1)u(t))) = \int_{\mathbb{C}} f(y)\ \tau(\tfrac{1}{2}ty^2)dy$$

$$= \int_{\mathbb{C}} f(\sqrt{x})\,|x|^{-\frac{1}{2}}\ \tau(\tfrac{1}{2}tx)dx.$$

By theorem 10.2, $S_o(\nu(\frac{1}{2}))$ is irreducible. Therefore L is an isomorphism and in particular $(\omega, S(\mathbb{C})_e)$ is irreducible.

10.9. In this section I will show that for all $a \in \mathbb{C}^*$ and all b,t and p, as in (10.4), $W_o(a,\chi)(\psi(b,t,p)(s))(e)$ has a holomorphic continuation to $\mathbb{C}$.

Put $a = \alpha e^{i\psi_o}$, with $\alpha \in \mathbb{R}_{>0}$. For $\mathrm{Re}(s) > 1$, $W_o(a,\chi)(\psi(b,t,p)(s))(e)$ equals

$$\int_0^\infty (1+r^2)^{-s-b-\frac{|\kappa|}{2}} r^{2b+|\kappa|-t-p} 2r.\left\{\int_0^{2\pi} e^{i\,\mathrm{sgn}(\kappa)(t-p-|\kappa|)\psi} e^{4\pi i\alpha r\cos(\psi+\psi_0)}d\psi\right\}dr$$

$$\approx \int_0^\infty (1+r^2)^{-s-b-\frac{|\kappa|}{2}} r^{2b+|\kappa|-t-p+1} J_{|t-p-|\kappa||}(4\pi\alpha r)dr = \int_0^\infty f(r,s)dr$$

Here J_n, $n \in \mathbb{Z}_{\geq 0}$, denotes the usual Bessel function. It satisfies $|J_n(r)| \leq 1$, for all $r \in (0,\infty)$ and $n \geq 0$. Hence $\int_0^1 f(r,s)dr$ is analytic on $\mathbb{C}$. As for $\int_1^\infty f(r,s)dr$, note first that for all b,t and p as in (10.4) $2b + |\kappa| - t - p \geq |t-p-|\kappa||$. Furthermore, the J_n, $n \geq 0$, satisfy the following recurrence relation: $\frac{d}{dx}(x^{n+1}J_{n+1}(x)) = x^{n+1}J_n(x)$. By applying r-times partial integration with respect to the Bessel function, one arrives at an integrand that is holomorphic in s and absolutely integrable for all s with $\mathrm{Re}(s) > 1 - r$. This proves the assertion.

10.10. Next I derive an expression for the functions in $W_o(a,\chi)(S_o(\chi))$ that will be convenient for asymptotic considerations. Let $\Phi \mapsto \tilde{\Phi}$ be the isomorphism of $S(\mathbb{C}^2)$ given by

$$\tilde{\Phi}(a,b) = \int_{\mathbb{C}} \Phi(a,y)\ \tau(-by)dy\ .$$

For $\mathrm{Re}(s) > 1$, we have

$$W_0(a,\chi)(I(\Phi,\chi))(g) = \int\int_{\mathbb{C}^2} (g.\Phi)(t,x)\ \chi(t)|t|^{-2}\ \tau(-at^{-1}x)dtdx$$

$$= \int_{\mathbb{C}} (\widetilde{g.\Phi})(t,at^{-1})\ \chi(t)|t|^{-2}\ dt$$

Since $\int_{\mathbb{C}} \Psi(t,at^{-1})\chi(t)dt$ is holomorphic on $\mathbb{C}$ for all $a \in \mathbb{C}^*$ and $\Psi \in S(\mathbb{C}^2)$, we may conclude from (10.9) that, for all χ, the functions in $W_0(a,\chi)(S_0(\chi))$ are of the form

$$g \mapsto \int_{\mathbb{C}} (\widetilde{g.\Phi})(t,at^{-1})\ \chi(t)|t|^{-2}dt$$

for some K-finite $\Phi \in S(\mathbb{C}^2)$. From this formula, one easily derives the following asymptotic behaviour of the functions in $W(a,\chi(m))(S(\chi(m)))$:

10.11. <u>PROPOSITION</u>. *Let* g *belong to* $S(\chi_K(m))$ *and let* C *be a compact subset of* $\mathbb{C}$. *The following properties hold then:*

(i) *For every* $N \in \mathbb{N}$,

$$\lim_{|a|\to\infty} |a|^N \sup_{\substack{s\in C,\, y\in\mathbb{C} \\ h\in\tilde{K}}} |W(a,\chi(m))(g(s))(R(u(y))h)| = 0$$

(ii) *There exists a* $N_0 \in \mathbb{N}$ *such that*

$$\sup_{\substack{a\in\mathbb{C}^*,\, y\in\mathbb{C} \\ s\in C,\ h\in\tilde{K}}} \{|a|^{N_0} |W(a,\chi(m))(g(s))(R(u(y))h)|\} < \infty$$

We end this chapter with a result that is a consequence of theorem 6.3 in [6].

10.12. <u>THEOREM</u>. *If* $S(\chi(m))$ *is irreducible, then it has a unique* $\mathcal{W}(a)$*-module, for all* $a \in \mathbb{C}^*$.

CHAPTER 4

§11 GENERALITIES ABOUT ALGEBRAIC REPRESENTATIONS

11.1. Let H and H_0 be as in (7.10). For the time being (σ,E) denotes an object in Alg(H). (σ,E) is called *admissible* if for each $\eta \in \mathcal{E}(H_0)$ $\dim(\sigma(\eta)(E)) < \infty$. Let σ^* be the natural representation of H on the dual space E^* of E. As in (7.10), σ^* induces a natural action of $\mathcal{A}(H_0)$ on E^*. Denote the subspace of E^* consisting of all $v \in E^*$, satisfying $\sigma^*(e)(v) = v$ for some $e \in \mathcal{E}(H_0)$, by $\tilde{E}$; it is H-stable and the algebraic representation $\tilde{\sigma}$ of H on $\tilde{E}$ is called the *contragredient* representation of σ. Put $<,>$ for the natural pairing of E and $\tilde{E}$. By a *matrix coefficient* of (σ,E) I mean a function on H of the form

$$h \mapsto <\sigma(h)v,\tilde{v}> = c_{v,\tilde{v}}(h),$$

where $h \in H$, $v \in E$ and $\tilde{v} \in \tilde{E}$.

One verifies easily that the contragredient representation of an admissible (σ,E) can be characterized by

11.2. <u>LEMMA</u>. *Let* $(\rho,V) \in \mathrm{Alg}(H)$ *be admissible. Assume that there exists a non-degenerate* H-*invariant bilinear form on* $E \times V$. *Then* $(\rho,V) \cong (\tilde{\sigma},\tilde{E})$.

11.3. <u>COROLLARY</u>. *Let* $(\sigma,E) \in \mathrm{Alg}(H)$ *be admissible. Then* $(\sigma,E) \cong (\tilde{\tilde{\sigma}},\tilde{\tilde{E}})$.

Let E' be the space E, equipped with the complex-conjugate $\mathbb{C}$-module structure. Then (11.2) has yet another consequence:

11.4. <u>COROLLARY</u>. *Let* (σ,E) *be admissible and pre-unitary. Then* $(\tilde{\sigma},\tilde{E}) \cong (\sigma,E')$.

Call (σ,E) *irreducible* if {0} and E are the only H-stable subspaces of E.

In case that H has a countable basis for its topology, Schur's lemma holds. For a proof I refer to [1], proposition (2.11).

11.5. LEMMA. *Assume* (σ,E) *is irreducible. Then*

$$\mathrm{Hom}_H(E,E) = \{\lambda I_E \mid \lambda \in \mathbb{C}\}.$$

11.6. Starting with a $(\sigma,E) \in \mathrm{Alg}(H)$, one can induce it to an algebraic representation of $\mathrm{Mp}(k)$. Indeed, let $S(\sigma)$ be the space of functions $f:\mathrm{Mp}(k) \to E$ satisfying

(i) $f(hx) = \sigma(h)(f(g))$ for all $h \in H$, $g \in \mathrm{Mp}(k)$.

(ii) There exists an open subgroup K' of G such that

$$f(gR^{o}(y)) = f(x) \qquad \text{for all } y \in K',\ g \in \mathrm{Mp}(k).$$

(iii) There exists an $\eta \in E(T)$ such that

$$f(tg) = \eta(t)\, F(x) \qquad \text{for all } t \in T,\ g \in \mathrm{Mp}(k).$$

Then $S(\sigma)$ is stable under right translations with elements of $\mathrm{Mp}(k)$ and this representation is denoted by $\mathrm{Ind}(\sigma)$. Induced representations have the following property:

11.7. LEMMA. (Frobenius reciprocity). *Notations being as above, let* (ρ,V) *be any object in* $\mathrm{Alg}(\mathrm{Mp}(k))$. *Then*

$$\mathrm{Hom}_{\mathcal{H}}(V,S(\sigma)) \cong \mathrm{Hom}_{\mathcal{H}(H)}(V,E).$$

PROOF. For $A \in \mathrm{Hom}_{\mathcal{H}}(V,S(\sigma))$ the corresponding operator $V \to E$ is given by $v \mapsto A(v)(e)$. □

11.8. Let χ be a quasi-character of k^* and r an integer. If we write $\chi(r)$ for the algebraic representation of $\tilde{P}(k)$ which is defined by

$$tR(d(a)u(x)) \mapsto t^r\ \chi(a) \quad \text{for } t \in T,\ a \in k^* \text{ and } x \in k,$$

then our notations are consistent. Since $\mathrm{Mp}(k) = \tilde{P}(k)\tilde{K}$, $S(\chi(r))$ is an admissible $\mathcal{H}$-module. Moreover every irreducible $\rho \in \mathrm{Alg}(\tilde{K})$, which occurs in $\mathrm{Ind}(\chi(r))|\tilde{K}$, does that only once.

The following lemma shows that the contragredient representation of $\mathrm{Ind}(\chi(r))$ is of the same type

11.9. LEMMA. $(\widetilde{\mathrm{Ind}}(\chi(r)), \widetilde{S}(\chi(r))) \cong (\mathrm{Ind}(\nu(2)\chi^{-1}(-r)), S(\nu(2)\chi^{-1}(-r)))$.

PROOF. For $f \in S(\chi(r))$ and $g \in S(\nu(2)\chi^{-1}(-r))$, I define

$$B(f,g) = \int_K f(R(k))g(R(k))dk.$$

Now, B is a non-degenerate Mp(k)-invariant bilinear form on $S(\chi(r) \times S(\nu(2)\chi^{-1}(-r))$. Hence the assertion is a consequence of (11.2). □

11.10. From now on, (σ,E) will denote an object in Alg(Mp(k)). Let ψ be a character of k. Then $E(\psi)$ is defined as

$$\{\sum_{i=1}^{r} \lambda_i\{\sigma(R(u(x_i)))(v_i) - \psi(x_i)v_i\} | \lambda_i \in \mathbb{C}, x_i \in k, v_i \in E, i=1,\dots,r\}.$$

It is easy to see that this equals

$$\{v | v \in E, \text{ for some } n \in \mathbb{Z} \int_{p^n} \psi(-x)\sigma(R(u(x)))(v)dx = 0\}.$$

$E_\psi = E/E(\psi)$ is the greatest quotient of E on which N(k) acts according to ψ. Following BERNSHTEIN [1], I call (σ,E) *quasi-cuspidal* if $E_\psi = \{0\}$.

Let $Z(\psi)$ be $\widetilde{D}(k)$, if $\psi = 1$, and $T.\{R(d(a)) | a^2 = 1\}$, if $\psi \neq 1$. Then we denote the canonical algebraic representation of $Z(\psi)$ on E_ψ by $\mathrm{Res}_\psi(\sigma)$. By a direct verification one proves the following

11.11. LEMMA. $(\sigma,E) \to (\mathrm{Res}_\psi(\sigma),E_\psi)$ *is an exact functor from* Alg(Mp(k)) *to* Alg(Z(ψ)).

11.12. I want to show here that the matrix coefficients of a quasi-cuspidal representation have compact support. Let Δ be $\{R(d(\pi^k)) | k \geq 0\}$. Then $\mathrm{Mp}(k) = \bigcup_{\delta \in \Delta} \widetilde{K}\delta\widetilde{K}$ and this union is disjoint. For every $i \in \mathbb{N}$, put $K^i = \{h | h \in K, h \equiv d(1) \bmod p^i\}$. One verifies that $K^i = \{n(x)d(a)u(y) | x,y \in p^i, a \in 1 + p^i\}$. For each $\widetilde{v} \in \widetilde{E}$, there exists an $i \in \mathbb{N}$ such that $K^i < G$ and $\widetilde{\sigma}(R^o(h)k_1)(\widetilde{v}) = \widetilde{\sigma}(k_1)(\widetilde{v})$, for all $h \in K^i$ and $k_1 \in \widetilde{K}$. Further one can find, for each $v \in E$, a $j \in \mathbb{N}$ such that for all $k_2 \in \widetilde{K}$

$$\int_{p^{-j}} \sigma(R(u(x))k_2)(v)dx = 0.$$

Now,

$$\int_{P^i} < \sigma(R(d(\pi^r))k_2)(v),\ \tilde{\sigma}(R^0(u(x))k_1)(\tilde{v})>dx =$$

$$\mathrm{vol}(P^i) < \sigma(R(d(\pi)^r)k_2)(v),\ \tilde{\sigma}(k_1)(\tilde{v})>,$$

but it also equals

$$|\pi|^{2r} \int_{P^{i-2r}} <\sigma(R(d(\pi^r)u(x))k_2)(v),\ \tilde{\sigma}(k_1)(\tilde{v}) > dx.$$

Hence $c_{v,\tilde{v}}$ has compact support. This enables us to prove:

11.13. LEMMA. *Let* (σ,E) *be quasi-cuspidal and irreducible. Then* (σ,E) *is admissible.*

PROOF. Thanks to (11.5) it suffices to prove for all $K^i < G$ that $\sigma(\varepsilon_{K^i})(E)$ is finite-dimensional. Take any $v \in E$, $v \neq 0$, and assume that $\sigma(\varepsilon_{K^i})(E) = \{\sum_{i=1}^r \lambda_j\sigma(\varepsilon_{K^i})\ \sigma(g_j)(v)\,|\,\lambda_j \in \mathbb{C},\ g_j \in Mp(k) \text{ for } j = 1,\dots,r\}$ is infinite-dimensional. Then one can find a linear independent collection $\{\sigma(\varepsilon_{K^i})\sigma(g_j(v)\,|\,j\in\mathbb{N}\}$ such that $g_j \in \tilde{K}\ R(d(\pi^{n}j))\tilde{K}$, with $n_j < n_{j+1}$ for all $j \in \mathbb{N}$. Next we complete it with $\{\eta_t|t \in I\}$ to a basis of $\sigma(\varepsilon_{K^i})(E)$ and define the linear form $\tilde{v}$ on $\sigma(\varepsilon_{K^i})(E)$ by: $\tilde{v}(\sigma(\varepsilon_{K^i})\sigma(g_j)(v)) = 1$ for all $j \in \mathbb{N}$ and $\tilde{v}(\eta_t) = 0$ for all $t \in I$. Now $\tilde{v} \circ \sigma(\varepsilon_{K^i})$ belongs to $\tilde{E}$, but $c_{v,\tilde{v}\ \circ\ \sigma(\varepsilon_{K^i})}$ has no compact support, which contradicts (11.12). □

If (σ,E) is quasi-cuspidal and irreducible, then it can be made pre-unitary. One simply takes a Haar measure dg on $Mp(k)$, chooses any non-zero w in $\tilde{E}$ and defines a $Mp(k)$-invariant scalar product on E by

$$(11.14) \qquad (v_1,v_2) = \int_{Mp(k)} <\sigma(g)v_1,w>\ \overline{<\sigma(g)v_2,w>}dg \qquad \text{for } v_1,v_2 \in E.$$

We can state now:

11.15. LEMMA. *Let* (σ,E) *be as above. Then there exists a constant* $d_\sigma \neq 0$ *such that*

$$\int_{Mp(k)} <\sigma(g)v_1,\tilde{v}_1>\ <\sigma(g^{-1})v_2,\tilde{v}_2>dg = d_\sigma^{-1}\ <v_1,\tilde{v}_2><v_2,\tilde{v}_1>$$

for all $v_1,v_2 \in E$, $\tilde{v}_1,\tilde{v}_2 \in \tilde{E}$.

<u>PROOF</u>. First fix a $v_1 \in E$, $v_1 \neq 0$, and a $v_2 \in E$, $v_2 \neq 0$. Then the left-hand side is a Mp(k)-invariant bilinear form on $E \times \tilde{E}$. From (11.13), (11.14) and (11.4), one sees that it is non-degenerate. Since σ is irreducible, this form differs only by a non-zero constant $\lambda(v_2,\tilde{v}_1)$ from $(v_1,\tilde{v}_2) \to \langle v_1,\tilde{v}_2\rangle$. By defining $\lambda(0,v_1) = \lambda(v_2,0) = \lambda(0,0) = 0$, we have in $(v,\tilde{v}) \to \lambda(v,\tilde{v})$ a non-degenerate Mp(k)-invariant bilinear form on $E \times \tilde{E}$. Hence $\lambda(v,\tilde{v}) = d_\sigma^{-1}\langle v,\tilde{v}\rangle$ for all $v \in E$, $\tilde{v} \in \tilde{E}$. □

Finally we will need the fact that an irreducible quasi-cuspidal (σ,E) is projective:

11.16. <u>THEOREM</u>. *Let* (ρ,V) *belong to* Alg(Mp(k)) *and let* P *be a non-zero element of* $\mathrm{Hom}_{\mathcal{H}}(V,E)$. *Then there exists a* $F \in \mathrm{Hom}_{\mathcal{H}}(E,V)$ *such that* $P \circ F = I_E$.

<u>PROOF</u>. Take any non-zero w_0 in E, and choose $\tilde{w}_0 \in \tilde{E}$ such that $\langle w_0,\tilde{w}_0\rangle = d_\sigma$. There exists a $v_0 \in V$ such that $P(v_0) = w_0$. Define $F:E \to V$ as follows:

$$F(w) = \int_{Mp(k)} c_{w,\tilde{w}_0}(g^{-1})\ \rho(g)(v_0)dg$$

Then we have for all $\tilde{w} \in \tilde{E}$

$$\langle P \circ F(w),\tilde{w}\rangle = \int_{Mp(k)} c_{w,\tilde{w}_0}(g^{-1})\ c_{w_0,\tilde{w}}(g)dg$$

$$= \langle w,\tilde{w}\rangle$$

Hence $P \circ F = I_E$. □

11.17 Let (σ,E) be irreducible and not quasi-cuspidal, that is to say $E_1 \neq \{0\}$. Then E_1 is a finitely generated $\tilde{P}(k)$-module, for it is irreducible and $Mp(k) = \tilde{P}(k)\tilde{K}$. Consequently E_1, as a $\tilde{P}(k)$-module, has an irreducible quotient of the form $\chi(r)$. In other words, by (11.7) E is isomorphic to a Mp(k)-submodule of $S(\chi(r))$. In order to determine the Mp(k)-module structure of $S(\chi(r))$ I will make use of the following

11.18. <u>THEOREM</u>. *The map* $W \to W_1$ *from the collection of* Mp(k)*-submodules of* $S(\chi(r))$ *to the collection of* $\tilde{D}(k)$*-submodules of* $S(\chi(r))_1$ *is injective.*

<u>PROOF</u>. Let V and W be Mp(k)-submodules of $S(\chi(r))$ with $V_1 = W_1$. By passing from V,W to V, V + W and by using (11.11), one sees that we may assume

$V \subseteq W$ and W/V is a finitely generated Mp(k)-module. Since $(W/V)_1 = \{0\}$, W has, if $W/V \neq \{0\}$, an irreducible quasi-cuspidal quotient. By (11.16), it occurs also as a submodule of W, but that is impossible, since, by (11.7), $S(\chi(r))$ does not have any non-trivial quasi-cuspidal submodules. Hence $V = W$. □

§12 THE $\tilde{D}(k)$-MODULE STRUCTURE OF $S(\chi(m))_1$

12.1. For every $f \in S(\chi(m))$ define $G(f) : k \to \mathbb{C}$ by

$$G(f)(x) = f(R(w(1)u(x))).$$

G(f) determines f completely. By (1.11) we know for all $x \in k^*$

$$R(w(1)u(x)) = R(d(x^{-1})u(-x))R(n(x^{-1})).$$

If $n(x^{-1}) \in G$, then by (2.7),

$$R(n(x^{-1})) = \gamma(x)R^{0}(n(x^{-1})).$$

This implies that G(f) satisfies the following property:

(12.2) There exists a $N > 0$ such that for all $x \in k^*$, $|x| > N$,

$$G(f)(x)\ \gamma(-x)^m\chi(x) = f(e).$$

On the other hand, if g is a locally constant function on k, satisfying (12.2) then one defines $\tilde{G}(g)$: Mp(k) $\to \mathbb{C}$ by

$$\tilde{G}(g)(tR(d(a)u(x))) = t^m\chi(a)c,$$

$$\tilde{G}(g)(tR(d(a)u(x)w(1)u(y))) = t^m\chi(a)g(y),$$

where $c = \lim_{|x|\to\infty} g(x)\chi(x)\gamma(-x)^m$, $t \in T$, $a \in k^*$ and $x,y \in k$.

The subsequent calculations will show that $\tilde{G}(g)$ is invariant under an open compact subgroup of G. Hence $\tilde{G}(g) \in S(\chi(m))$.

Put $q = |\pi|^{-1}$. First of all I choose a $r \in \mathbb{N}$ such that

(i) $K^r < G$

(ii) $\chi|(1+p^r) \equiv 1$ and $1 + p^r < (k^*)^2$

(iii) $g(x) = \chi(x)^{-1}\ \gamma(x)^m c$ for all $x \in k^*$, with $|x| \geq q^r$.

(iv) $g(x+y) = g(x)$ for all $x \in k$ and $y \in p^r$.

I claim now that $\tilde{G}(g)$ is invariant under K^{3r}. To prove this, it is sufficient to show for all h, belonging to a set of generators of K^{3r}, that $\tilde{G}(g)(R^o(h)) = c$ and $\tilde{G}(g)(R(w(1)u(x))R^o(h)) = g(x)$, for all $x \in k$.

First take $h = d(a)$, $a \in 1+p^{3r}$. Then $R^o(h) = R(h)$ and by (ii) $\tilde{G}(g)(R^o(h)) = c$. Since $xa^{-2} - x \in p^r$, for all $x \in p^{-r}$, the properties (ii), (iii) and (iv) imply the other equality. Next let h be of the form u(b), $b \in p^{3r}$. Again $R^o(h) = R(h)$ and the assertions follow from the definition and (iv).

Finally, take $h = n(t)$, with $0 < |t| \leq q^{-3r}$. Then we have $\tilde{G}(g)(R^o(h)) = \gamma(-t)^m\ \chi(t^{-1})\ g(t^{-1}) = c$, by (iii). If $x = -t^{-1}$, then $\tilde{G}(g)(R(w(1)u(x))R^o(h)) = \gamma(-t)^m\chi(-t)c = g(x)$. Assume, from now on, $x \neq -t^{-1}$. For those x, we have

$$\tilde{G}(g)(R(w(1)u(x))R^o(h)) = \gamma(t)^{-m}\ \gamma(t(1+tx))^m\chi(1+tx)^{-1}g(x(1+tx)).$$

If $x \in p^{-r}$, then $1+tx \in 1+p^{2r}$ and $x(1+tx)^{-1}-x \in p^r$. Hence in that case the desired result is a consequence of (ii) and (iv). Consider now the case $|x| > q^r$. Since $|x(1+tx)^{-1}| \geq |x|$, if $|x| \leq |t|^{-1}$, and $|x(1+tx)^{-1}| = |t|^{-1}$, if $|x| > |t|^{-1}$, we see from (iii) that in both cases

$$\tilde{G}(g)(R(w(1)u(x))R^o(h)) = (xt,1+tx)\ \gamma(1+tx)^{2m}\gamma(-1)^{2m}g(x) = g(x).$$

This completes the proof of the desired invariance.

The foregoing implies that for every χ and m there is a unique $f(\chi(m)) \in S(\chi(m))$ such that $G(f(\chi(m)))(x) = 0$, if $|x| \leq 1$, and $G(f(\chi(m)))(x) = \gamma(x)^m\chi(x)^{-1}$, if $|x| > 1$.

12.3. Let $\mathcal{D} : S(\chi(m)) \to \mathbb{C}$ be given by $\mathcal{D}(f) = f(e)$, $f \in S(\chi(m))$. It is clear now that $\mathrm{Ker}(\mathcal{D}) = \{f|f \in S(\chi(m)), G(f) \in S(k)\}$ and that $\mathrm{Ker}(\mathcal{D}) \supset S(\chi(m))(1)$. Let $P(0) : \mathrm{Ker}(\mathcal{D}) \to \mathbb{C}$ be defined by

$$P(0)(f) = \int_k G(f)(x)dx$$

Clearly Ker(P(0)) $\supset S(\chi(m))(1)$. On the other hand, if f $\in$ Ker(P(0)) and the support of G(f) is contained in p^r, then

$$\int_{p^r} \mathrm{Ind}(\chi(m))(R(u(x)))(f)dx = 0$$

in other words Ker(P(0)) $= S(\chi(m))(1)$. As for the action of $\tilde{D}(k)$, one verifies that the following relations hold:

$$\mathcal{D}(\mathrm{Ind}(\chi(m))(R(d(a)))(f)) = \chi(a)\mathcal{D}(f)$$

$$P(0)(\mathrm{Ind}(\chi(m))(R(d(a)))(g)) = \chi(a)^{-1}|a|^2\ P(0)(g).$$

Hence $S(\chi(m))_1$ as a $\tilde{\mathcal{D}}(k)$-module, has a Jordan-Hölder sequence of length 2 and its irreducible components are $\chi(m)$ and $\nu(2)\chi^{-1}(m)$.

If $\chi^2 \neq \nu(2)$, $S(\chi(m))_1$ can be diagonalized as a $\tilde{D}(k)$-module. Assume now that $\chi^2 = \nu(2)$. Then, by local class-field theory, $\chi = \nu(1)h(\xi)$, for some $\xi \in k^*$. From section (12.1); it is clear that $S(\chi(m))_1$ can be diagonalized if and only if for every $a \in k^*$:

(12.4) $\quad \mathrm{Ind}(\chi(m))(R(d(a)))(f(\chi(m))) - \chi(a)f(\chi(m)) \in \mathrm{Ker}(P(0)).$

Since

$$G(\mathrm{Ind}(\chi(m))(R(d(a)))(f(\chi(m))))(x) = \begin{cases} 0 & \text{if } |x| \le |a|^2 \\ \chi(a)\gamma(x)^m\ \chi(x^{-1}) & \text{if } |x| > |a|^2 \end{cases}$$

we see that it is sufficient to prove or disprove (12.4) for $a = \pi$, and in that case it amounts to $\int_{q^{-2}<|x|\le 1} \gamma(x)^m\ \chi(x^{-1})dx = 0$.

Now the left-hand side of this equation is equal to

$$\gamma(1)^{2\ell}\cdot\left\{\int_{\mathcal{O}^*} \gamma(\alpha)((-1)^{\ell}\xi,\alpha)d\alpha + \int_{\mathcal{O}^*} \gamma(\alpha\pi)((-1)^{\ell}\xi,\alpha\pi)d\alpha\right\} =$$

$$\gamma(1)^{2\ell}\gamma(1)\gamma((-1)^{\ell+1}\xi)\cdot\left\{\int_{\mathcal{O}^*} \gamma(\alpha\xi)d\alpha + \int_{\mathcal{O}^*} \gamma(\alpha\pi\xi)d\alpha\right\}.$$

As we will see in (13.8), one of the integrals in this expression is zero and the other one is equal to $|\frac{2}{\delta}|^{\frac{1}{2}}(1-q^{-1})$. Hence, if $\chi^2 = \nu(2)$, $S(\chi(m))$ cannot be diagonalized. By (11.7) and (11.18), we can summarize the obtained results as follows:

12.5. PROPOSITION. *Let* χ_1,χ_2 *be quasi-characters of* k^* *and* m_1,m_2 *odd integers.*
(i) *If there exists an* $\mathcal{H}$*-submodule* E *of* $S(\chi_1(m_1))$ *such that* $\{0\} \subsetneq E \subsetneq S(\chi_1(m_1))$, E *is irreducible and* E_1 *is one-dimensional.*
(ii) *If* $\chi_1(m_1) \notin \{\chi_2(m_2),\nu(2)\chi_2^{-1}(m_2)\}$, $\mathrm{Hom}_{\mathcal{H}}(S(\chi_1(m_1)), S(\chi_2(m_2))) = \{0\}$.
(iii) *If* $\chi_1(m_1) \in \{\chi_2(m_2),\nu(2)\chi_2^{-1}(m_2)\}$, $\mathrm{Hom}_{\mathcal{H}}(S(\chi_1(m_1)), S(\chi_2(m_2)))$ *is one-dimensional.*

12.6. REMARK. For those χ such that $\mathrm{Ind}(\chi(m))$ is pre-unitary, proposition (12.5) implies that $S(\chi(m))$ is irreducible. This is, for example, the case if $\chi = \nu(s)\chi^o$, with Re(s) = 1. A Mp(k)-invariant scalar product on $S(\chi(m)$ is then given by

$$(f,g) = B(f,\bar{g}),$$

with $f,g \in S(\chi(m))$ and B as in (11.9).

12.7. Let E be as in (12.5) (i). From (11.3) we see that $\tilde{E}$ is also irreducible. Further we know by (11.17) that there is a non-zero $A \in \mathrm{Hom}_{\mathcal{H}}(S(\nu(2)\chi^{-1}(m)),\tilde{E})$. By applying (12.7) (i) and (11.11) one concludes that $\tilde{E}$ is not a quasi-cuspidal $\mathcal{H}$-module. Therefore there exists a $\chi_2(m_2)$ such that $\tilde{E}$ is an $\mathcal{H}$-submodule of $S(\chi_2(m_2))$. Since $\mathrm{Hom}_{\mathcal{H}}(\tilde{E},S(\chi_2(m_2))) \cong \mathrm{Hom}_{\mathcal{H}}(S(\nu(2)\chi_2^{-1}(-m_2)),E)$, E is a quotient of $S(\chi_2^{-1}\nu(2)(-m_2))$ and by (12.5) $\chi_2(m_2)$ has to equal $\chi(-m)$. Furthermore, the kernel of this projection is an irreducible submodule of $S(\chi_2^{-1}\nu(2)(-m_2))$. We have arrived now at the following criterion for irreducibility of $S(\chi(m))$:

12.8. PROPOSITION. *For every* $\chi(m)$, *let* $C(\chi(m))$ *be a non-zero element of* $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)), S(\nu(2)\chi^{-1}(m)))$. *Then we have*
(i) $S(\chi(m))$ *is irreducible* $\Leftrightarrow C(\nu(2)\chi^{-1}(m)) \circ C(\chi(m)) \neq 0$.
(ii) *If* $S(\chi(m))$ *is reducible, the image of* $C(\nu(2)\chi^{-1}(m))$ *is equal to the kernel of* $C(\chi(m))$ *and is the unique non-trivial* $\mathcal{H}$*-submodule of* $S(\chi(m))$.

In the next paragraph I continue first the operators $M(\chi(m))$ to a sufficiently large open connected subset of $\mathbb{C}$ and next I compute $M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m))$.

§13 COMPUTATION OF $M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m))$.

13.1. First we show that all the $M(\chi(m))(f(s))$, $f \in S(\chi^{o}(m))$, have a meromorphic continuation to $\mathbb{C}$. Thanks to (6.23) we can restrict ourselves to that of $M(\chi(m))(f(s))(e)$, for all $f \in S(\chi^{o}(m))$. Since for all $f \in S(\chi^{o}(m))$ with $f(e) = 0$ the function $s \to \int_k G(f(s))(t)dt$ is holomorphic on $\mathbb{C}$, we have to consider still only $M(\chi(m))(f(\chi(m)))(e)$. Now,

$$\int G(f(\chi(m))(t)dt = (1-q^{2-2s})^{-1} \cdot \gamma(1)^{2\ell} q^{1-s}$$

$$\left\{ q^{1-s} \int_{\mathcal{O}^*} \gamma(\alpha)(-1,\alpha)^{\ell} \chi^{o}(\alpha^{-1}) d\alpha + (-1,\pi)^{\ell} \int_{\mathcal{O}^*} \gamma(\pi\alpha)(-1,\alpha)^{\ell} \chi^{o}(\alpha^{-1}) d\alpha \right\}$$

and therefore we can draw the following

13.2. <u>CONCLUSION</u>. For all $f \in S(\chi^{o}(m))$, the function $M(\chi(m))(f(s))$ has a meromorphic continuation to $\mathbb{C}$ and is holomorphic on $V = \mathbb{C}\setminus\{1+i\frac{\pi r}{\ell n(q)} | r \in \mathbb{Z}\}$ and, in particular, $M(\nu(s)\chi^{o}(m)) \neq 0$, for all $s \in V$.

13.3. <u>REMARK</u>. Thanks to (12.6), it poses no problem that $M(\chi(m))$ is defined only for $s \in V$.

13.4. By using (4.16) one sees that, for all χ and m, the map $f \to f \circ J(m)$ is an isomorphism between $S(\chi h(-1)^{\ell})$ and $S(\chi(m))$. Moreover we have for all $s \in V$ and every $f \in S(\chi h(-1)^{\ell})$

$$M(\chi(m))(f \circ J(m)) = \gamma(-1)^{2\ell} M(\chi h(-1)^{\ell})(f) \circ J(m).$$

Hence it will be sufficient to compute $M(\nu(2)\chi^{-1}) \circ M(\chi)$ for all $s \in V$.

13.5. There is a unique function $f_1 \in S(\chi^{o})$ such that $G(f_1) = \psi(0)$. In the sequel I will compute $M(\nu(2)\chi^{-1}) \circ M(\chi)(f_1(s))(R(w(1)))$, for all $s \in V$. First of all we have for all $x \in k^*$ and $y \in k$

$$R(w(1)u(x))R(w(1)u(y)) = \gamma(x)R(u(-x^{-1})d(x^{-1})w(1)u(y-x^{-1})).$$

This implies for $\mathrm{Re}(s) > 1$ and all $f \in S(\chi)$:

$$G(M(\chi)(f))(y) = \int_k \chi(x^{-1})\gamma(x)G(f)(y-x^{-1})dx$$

$$= \int \chi(x)\nu(-2)(x)\gamma(x)G(f)(y-x)dx.$$

In particular,

$$G(M(\chi)(f_1(s)))(y) = \begin{cases} \int_{\mathcal{O}} \chi(t)\nu(-2)(t)\gamma(t)dt = \varepsilon(\chi) & \text{for } |y| \le 1 . \\ \int_{\mathcal{O}} \chi(y+z)\gamma(y+z)dz & \text{for } |y| > 1 . \end{cases}$$

Now $\varepsilon(\chi)$ is easily seen to be equal to

$$(1-q^{2-2s})^{-1}\left\{ \int_{\mathcal{O}^*} \gamma(\alpha)\chi(\alpha)d\alpha + q^{1-s} \int_{\mathcal{O}^*} \gamma(\alpha\pi)\chi(\alpha)d\alpha \right\}.$$

Clearly, $\varepsilon(\chi)$ is analytic on V. Since $s \mapsto \int_{\mathcal{O}} \chi(y+z)\gamma(y+z)dz$ is analytic on $\mathbb{C}$, the expression for $G(M(\chi)(f_1(s)))$, given above, is valid for all $s \in V$. For $\mathrm{Re}(s) < 1$ we have again that

$$G(M(\nu(2)\chi^{-1})(M(\chi)(f_1(s))))(0) = \int_k \chi(t^{-1})\gamma(t)G(M(\chi)(f_1(s)))(-t)dt$$

$$= \varepsilon(\chi)\varepsilon(\nu(2)\chi^{-1}) + \int_{|t|>1} \chi(-t)^{-1}\gamma(-t)\nu(-2)(t)\int_{\mathcal{O}}\chi(t+z)\gamma(t+z)dzdt$$

Denote this last integral by $\Lambda(\chi)$. Later on, we will see that $\Lambda(\chi)$ does not depend on s; by analytic continuation we know then that for all $s \in V$

$$M(\nu(2)\chi^{-1}) \circ M(\chi) = \{\varepsilon(\chi)\varepsilon(\nu(2)\chi^{-1}) + \Lambda(\chi)\}.I_{S(\chi)}.$$

Before giving the explicit computation of $\varepsilon(\chi)$ and $\Lambda(\chi)$ I recall first some well-known results.

13.6. For $z \in k$, we have

$$\int_{\mathcal{O}^*} \tau(za)da = \begin{cases} 0 & \text{if } |z| > q|\delta| \\ -q^{-1}|\delta|^{-\frac{1}{2}} & \text{if } |z| = q|\delta| \\ (1-q^{-1})|\delta|^{-\frac{1}{2}} & \text{if } |z| \le |\delta| \end{cases} .$$

In case that χ^o is non-trivial with conductor p^n, I write $G(\beta)$ for $\int_{O^*}\chi^o(\alpha)\tau(\alpha\beta)d\alpha$, with $\beta\in k^*$. It satisfies:

$$G(\beta\alpha) = \chi^o(\alpha)^{-1}G(\beta), \text{ for all } \alpha\in O^*, \text{ and } G(\beta) = 0, \text{ unless } |\beta| = q^n|\delta|.$$

13.7. Next I will compute $\varepsilon(\chi)$ explicitly. Choose any $\mu\in k^*$ such that $|\mu|^2 \le |2\delta|q^{-2}$, and put

$$B_1(\chi^o) = \int_{O^*}\gamma(\alpha)\chi^o(\alpha)d\alpha = \int_{\mu^{-1}O^\perp}\int_{O^*}\tau(\tfrac{1}{2}\alpha c^2)\chi^o(\alpha)d\alpha dc$$

$$B_2(\chi^o) = \int_{O^*}\gamma(\alpha\pi)\chi^o(\alpha)d\alpha = q^{-\frac{1}{2}}\int_{\pi^{-1}\mu^{-1}O^\perp}\int_{O^*}\tau(\tfrac{1}{2}\alpha\pi c^2)\chi^o(\alpha)d\alpha dc.$$

From (13.6) one concludes that $B_1(\chi^o) = B_2(\chi^o) = 0$, unless $(\chi^o)^2 = 1$, and in that case $\chi^o = h(\xi)$ for some $\xi\in k^*$. As $B_1(h(\xi)) = \gamma(\xi)\gamma(-1)\int_{O^*}\gamma(\xi\alpha)d\alpha$ and $B_2(h(\xi)) = \gamma(\xi)\gamma(-1)\int_{O^*}\gamma(\alpha\pi\xi)d\alpha$, it suffices to compute $B_1(1)$ and $B_2(1)$.

$$B_1(1) = \left|\frac{\delta}{\mu}\right|\left\{|\delta|^{-\frac{1}{2}}(1-q^{-1})\int_{\substack{\beta\in O\\ |\beta|^2\le|\frac{2\mu^2}{\delta}|}}d\beta - |\delta|^{-\frac{1}{2}}q^{-1}\int_{\substack{\beta\in O\\ |\beta|^2=|\frac{2\mu^2}{\pi\delta}|}}d\beta\right\}$$

$$B_2(1) = q^{-\frac{1}{2}}.\left|\frac{\delta}{\mu}\right|\left\{(1-q^{-1})|\delta|^{-\frac{1}{2}}\int_{\substack{\beta\in O\\ |\beta|^2\le|\frac{2\mu^2}{\pi\delta}|}}d\beta - q^{-1}|\delta|^{-\frac{1}{2}}\int_{\substack{\beta\in O\\ |\beta^2|=|\frac{2\mu^2}{\pi^2\delta}|}}d\beta\right\}.$$

From these expressions one deduces:

$$\text{(13.8)}\qquad \begin{array}{l} \text{If } v(2\delta) \text{ is even, } B_1(1) = |\frac{2}{\delta}|^{\frac{1}{2}}(1-q^{-1}) \text{ and } B_2(1) = 0;\\ \text{if } v(2\delta) \text{ is odd, } B_2(1) = |\frac{2}{\delta}|^{\frac{1}{2}}(1-q^{-1}) \text{ and } B_1(1) = 0.\end{array}$$

(13.8) implies the following results for $\varepsilon(h(\xi))$:

$$\text{(13.9)}\qquad \begin{array}{l} \text{If } v(\xi)-v(2\delta)\in 2\mathbb{Z},\ \varepsilon(h(\xi)) = \gamma(\xi)\gamma(-1)|\frac{2}{\delta}|^{\frac{1}{2}}(1-q^{-1})(1-q^{2-2s})^{-1};\\ \text{if } v(\xi)-v(2\delta)\notin 2\mathbb{Z},\ \varepsilon(h(\xi)) = \gamma(\xi)\gamma(-1)|\frac{2}{\delta}|^{\frac{1}{2}}(1-q^{-1})q^{1-s}(1-q^{2-2s})^{-1}.\end{array}$$

13.10. In this section we focus our attention on $\Lambda(\chi)$.

$$\Lambda(\chi) = \chi(-1)\left\{\sum_{r=1}^{\infty} C(2r,\chi) + \sum_{r=0}^{\infty} C(2r+1,\chi h(\pi))\right\},$$

where $C(n,\chi)$, for $n \in \mathbb{N}$, is given by

$$\gamma(-1) \int_{\mathcal{O}^*} \left\{ \int_{1+p^n} (\alpha,\beta)\,\chi(\beta)\gamma(\beta)\,d\beta \right\} d\alpha$$

If $|2| = q^{-m_0}$ with $m_0 \in \mathbb{N}$, then it is known that $(1+p^{m_0+1})^2 = 1 + p^{2m_0+1}$. Hence, if $|2| = 1$ or $n > 2m_0$,

$$C(n,\chi) = q^{-n}(1-q^{-1})|\delta|^{-\frac{1}{2}} \int_{\mathcal{O}} \chi(1+\pi^n z)\,dz = \begin{cases} |\delta|^{-1}(1-q^{-1})q^{-n}, & \text{if } \chi^o|1+p^n = 1. \\ 0 & \text{, if } \chi^o|1+p^n \neq 1. \end{cases}$$

In order to be able to calculate the resulting $C(r,\chi)$, I make use of

(13.11) For $\alpha \in \mathcal{O}^*$, $h(\alpha)|\mathcal{O}^* = 1 \Longleftrightarrow \alpha \in (\mathcal{O}^*)^2 \cup \theta(\mathcal{O}^*)^2$,

where $\theta \in 1+p^{2m_0}$ is such that $1+p^{2m_0} = (1+p^{m_0})^2 \cup \theta(1+p^{m_0})^2$. To prove this it is sufficient to show for all $1+u \in 1+p^{2m_0}$ that $h(1+u)|\mathcal{O}^* = 1$. It is no restriction to assume $v(2\delta)$ to be even, since being ramified or not does not depend on τ. Now, we have for all $\alpha \in \mathcal{O}^*$, $u \in p^{2m_0}$

$$\begin{aligned}\gamma(\alpha)\gamma(-\alpha(1+u)) &= \left|\frac{\delta}{2}\right| \iint_{\mathcal{O}^2} \tau\left(\tfrac{1}{2}\alpha\delta^2\pi^{-v(2\delta)}(c^2-x^2-ux^2)\right)dc\,dx \\ &= \left|\frac{\delta}{2}\right| \iint_{\mathcal{O}^2} \tau\left(\tfrac{1}{2}\alpha\delta^2\pi^{-v(2\delta)}(c^2-x^2)\right)dc\,dx = 1.\end{aligned}$$

and applying this to (1.9) (ii) gives the desired result.

For the resulting $C(r,\chi)$, (13.11) implies:

$$C(2f,\chi) = \{1+\gamma(-1)\gamma(\theta)\chi(\theta)\}(1-q^{-1})|\delta|^{-\frac{1}{2}} \int_{(1+p^f)^2} \chi(t)\,dt$$

$$C(2f+1,\chi h(\pi)) = \{1-\gamma(-1)\gamma(\theta)\chi(\theta)\}(1-q^{-1})|\delta|^{-\frac{1}{2}} \int_{(1+p^{f+1})^2} \chi(t)\,dt$$

Let $k_0 \geq 1$ be minimal such that $\chi^2|1+p^{k_0} = 1$. If $k_0 > m_0$, $C(r,\chi) = C(r,\chi h(\pi)) = 0$, for all $r < m_0 + k_0$, and $\Lambda(\chi) = \chi(-1)\sum_{r\geq k_0+m_0} C(r,\chi) = \chi(-1)|2|\ |\delta|^{-1}q^{-k_0}$. If $k_0 \leq m_0$, $C_2(r,\chi) = 0$ for all $r < k_0$, and $C(2r+1,\chi h(\pi)) = 0$ for all $r < k_0-1$. Further

$$\sum_{2m_0\geq 2r\geq 2k_0} C(2r,\chi) + \sum_{2m_0\geq 2r+1\geq 2k_0-1} C(2r+1,\chi h(\pi))$$

is equal to

$$2(1-q^{-1})|\delta|^{-\frac{1}{2}} \sum_{r=k_0}^{m_0} \int_{(1+p^r)^2} dt = 2(1-q^{-1})|\delta|^{-\frac{1}{2}} \sum_{r=k_0}^{m_0} \frac{q}{2}^{-m_0-r} |\delta|^{-\frac{1}{2}},$$

so that we get for all χ:

$$\Lambda(\chi) = |2|\ |\delta|^{-1}\ \chi(-1)q^{-k_0}.$$

Summarizing the foregoing results, we have for all $s \in V$ and $m \in 1+2\mathbb{Z}$:

13.12. PROPOSITION.
(i) *Let* χ^2 *be unramified. Then*

$$M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m)) = \chi(-1)\left|\frac{2}{\delta}\right| \frac{(1-\nu(3)\chi^{-2}(\pi))(1-\nu(-1)\chi^2(\pi))}{(1-\chi^2\nu(-2)(\pi))(1-\nu(2)\chi^{-2}(\pi))} I_{S(\chi(m))}.$$

(ii) *Let* χ^2 *be ramified and* p^{k_0} *be its conductor. Then*

$$M(\nu(2)\chi^{-1}(m)) \circ M(\chi(m)) = \chi(-1)\left|\frac{2}{\delta}\right|q^{-k_0} I_{S(\chi(m))}.$$

Thanks to (12.6) and (13.12) we can state now for all quasi-characters χ and $m \in 1+2\mathbb{Z}$.

13.13. THEOREM. *Let* χ *be equal to* $\nu(\frac{1}{2})h(\xi)$ *or* $\nu(\frac{3}{2})h(\xi)$, *for some* $\xi \in k^*$. *Then* $S(\chi(m))$ *is reducible. For all other* χ, $S(\chi(m))$ *is irreducible.*

13.14. We end this paragraph with the determination of the irreducible submodule of $S(\nu(\frac{1}{2})h(a)(m))$. For $f \in S(k)_e$ define $L(f) \in S(\nu(\frac{1}{2}))$ by

(13.15) $\quad L(f)(g) = \omega(g)(f)(0).$

Clearly $L \in \mathrm{Hom}_{\mathcal{H}}(S(k)_e, S(\nu(\frac{1}{2})))$ and L is injective: suppose namely that $L(f) = 0$ and $f \neq 0$. There exists a $b \in k^*$ and a $n \in \mathbb{N}$ such that $f(b) \neq 0$ and for all $t \in p^n$, $f(b+t) = f(b)$. Since $G(L(f)) = 0$, one has for all $r \in \mathbb{Z}$

$$\int_{p^r} \tau(-\tfrac{1}{2}b^2x)\ G(L(f))(x)dx = \int_k f(y) \int_{p^r} \tau(\tfrac{1}{2}x(y^2-b^2))dxdy = 0.$$

Now, choose r so small that

$$V(r) = \{y \mid |y^2-b^2| \le |2\delta|q^r\} \subseteq \{b+p^n\} \cup \{-b+p^n\};$$

then we have $f(y) = f(b)$ for all $y \in V(r)$. However, for such r

$$\int_{p^r} \tau(\tfrac{1}{2}xb^2)G(L(f))(x)dx = f(b)q^{-r}|\delta|^{-\frac{1}{2}} \int_{V(r)} dt \neq 0$$

and we have arrived at a contradiction.

By (12.10) and (13.13), $S(\nu(\frac{1}{2}))$ has a unique non-trivial submodule. This and the fact that ω is pre-unitary imply $L(S(k)_e) \subsetneq S(\nu(\frac{1}{2}))$. Combining the foregoing with (4.4) and (4.16) we get

13.16. <u>PROPOSITION</u>. *For any* $a \in k^*$, $m \in 1+2\mathbb{Z}$, *the map* $f \to L(f) \circ I(a) \circ J(m)$ *is an* $\mathcal{H}$*-isomorphism between* $(\omega \circ I(a) \circ J(m), S(k)_e)$ *and the irreducible submodule of* $S(\nu(\frac{1}{2})h(a(-1)^{\ell})(m))$.

§14 WHITTAKER MODELS

14.1. We start by showing for every $a \in k^*$ and $f \in S(\chi^o(m))$ that $W(a,\chi(m))(f(s))$ has a holomorphic continuation to $\mathbb{C}$. Thanks to (6.24), we have to prove this only for the functions $W(a,\chi(m))(f(s))(e)$. If $f(e) = 0$, then it is clear that for all $a \in k^*$ the function

$$\int_k G(f(s))(t)\tau(-at)dt$$

is holomorphic on $\mathbb{C}$. Further $W(a,\chi(m))(f(\chi(m)))(e)$ equals

$$\gamma(1)^{2\ell}\Big\{\sum_{r=1}^{\infty} q^{2r(1-s)}G(a,2r,\chi h(-1)^{\ell}) + (-1,\pi)^{\ell}\ \gamma(\pi)\gamma(-1).$$
$$\sum_{r=0}^{\infty} q^{(2r+1)(1-s)}G(a,2r+1,\chi h(-1)^{\ell}h(\pi))\Big\}.$$

Hereby $G(b,d,\chi_1)$, for $b \in k^*$, $d \in \mathbb{Z}_{\geq 0}$ and χ_1 a quasi-character of k^*, is given by

$$(14.2) \qquad G(b,d,\chi_1) = \int_{\mathcal{O}^*} \gamma(\alpha)\chi_1(\alpha)^{-1}\tau(-\alpha b\pi^{-d})d\alpha$$

$$= \int_{\mu^{-1}\mathcal{O}^{\perp}} \int_{\mathcal{O}^*} \chi_1(\alpha)^{-1}\tau(\alpha(\frac{c^2}{2} - b\pi^{-d}))d\alpha dc$$

for all $\mu \in k^*$ with $|\mu^2| \leq |2\delta|$. From (13.6) follows that, for a fixed $a \in k^*$, only a finite number of the $G(a,d,\chi h(-1)^{\ell})$ and $G(a,d,\chi h(-1)^{\ell}h(\pi))$, $d \in \mathbb{Z}_{\geq 0}$, are non-zero. Hence $W(a,\chi(m))(f(\chi(m)))(e)$ is analytic on $\mathbb{C}$.

14.3. <u>CONCLUSION</u>. For every $a \in k^*$, $m \in 1+2\mathbb{Z}$ and χ as in 6.3, we can speak of $W(a,\chi(m))$ and in particular it is non-zero.

14.4. Next we pay some attention to the asymptotics of $W(a,\chi(m))(g(s))(e)$, for fixed $g \in S(\chi^o(m))$, s running through a compact subset C of $\mathbb{C}$ and $|a|$ tending to zero or infinity. Assume first that $g(e) = 0$. Since $G(g(s)) \in S(k)$, the same is true for its Fourier transform and one easily sees that

$$\sup_{s\in C, a\in k^*} |W(a,\chi(m))(g(s))(e)| < \infty .$$

As for $W(a,\chi(m))(f(\chi(m)))(e)$, observe first that there is a $N > 0$, independent of s, such that $W(a,\chi(m))(f(\chi(m)))(e) = 0$ for every $a \in k^*$ with $|a| > N$. Moreover, if $|a|$ is sufficiently small, one notes that $G(a,n,\chi) = 0$ for all $n \geq 2v(a)$ and this enables us to estimate $W(a,\chi(m))(f(\chi(m)))(e)$ as follows: there is a $N_0 > 0$ such that for all $a \in k^*$

$$\sup_{s\in C} |W(a,\chi(m))(f(\chi(m)))(e)| \prec \min(1,|a|)^{-N_0} .$$

By combining these results with (6.24), we get

14.5. <u>PROPOSITION</u>. *Let* g *and* C *be as above and let* Y *be a compact subset of* Mp(k). *Then*

(i) *there is a* $N \in \mathbb{N}$ *such that for all* $a \in k^*$, $|a| > N$

$$\sup_{s\in C, x\in Y} |W(a,\chi(m))(g(s))(x)| = 0,$$

(ii) *there exists a* $N_0 \in \mathbb{N}$ *such that*

$$\sup_{\substack{s \in C, x \in Y \\ a \in k^*}} |a|^{N_0} |W(a,\chi(m))(g(s))(x)| < \infty.$$

14.6. We proceed showing the uniqueness of the $\mathcal{W}(a)$-models for $S(\chi(m))$.

14.7. THEOREM. *For every* $a \in k^*$, $\mathrm{Hom}_{\mathcal{H}}(S(\chi(m)),S(\tau^a))$ *is one-dimensional.*

PROOF. By (11.7), this assertion is equivalent to: $S(\chi(m))_{\tau^a}$ is one-dimensional. Let $q(a)$ be the projection: $S(\chi(m)) \to S(\chi(m))_{\tau^a}$. For all $z \in k$ and $f \in S(\chi(m))$ it is clear that $\mathrm{Ind}(\chi(m))(R(u(z)))(f)(e) = f(e)$ and $q(a)(\mathrm{Ind}(\chi(m))(R(u(z)))(f)-f) = (\tau(az)-1)q(a)(f)$. This implies that $q(a)(S(\chi(m))) = q(a)(\mathrm{Ker}(\mathcal{D}))$ with $\mathcal{D}$ as in (12.4). Define $P(a):\mathrm{Ker}(\mathcal{D}) \to \mathbb{C}$ by

$$P(a)(f) = \int_k \tau(-ax)G(f)(x)dx$$

Then one proves analogously to (12.4) that $\mathrm{Ker}(P(a)) = \mathrm{Ker}(\mathcal{D}) \cap S(\chi(m))(\tau^a)$. Since $P(a) \neq 0$, this completes the proof of the theorem. □

In the reducible case we have

14.8. THEOREM. *Let* (σ,E) *be the non-trivial* $\mathcal{H}$*-submodule of* $S(\nu(\tfrac{1}{2})h(a)(m))$. *Then* (σ,E) *has a* $\mathcal{W}(b)$*-model if and only if* $b \in \tfrac{1}{2}(-1)^{\ell}a(k^*)^2$. *Moreover, it is unique then.*

PROOF. By analytic continuation we have for all χ and all $b \in k^*$

$$W(b,\chi(m))(f \circ J(m)) = \gamma(1)^{2\ell}W(b,\chi h(-1)^{\ell})(f) \circ J(m),$$

with $f \in S(\chi h(-1)^{\ell})$. Therefore, we can restrict ourselves to the case $m = 1$.

Assume first that $b \notin \tfrac{1}{2}a(k^*)^2$. Then there is a $r \in \mathbb{N}$ such that $\{b+p^r\} \cap \tfrac{1}{2}a(k^*)^2 = \phi$. From (13.16) and the proof of (14.7) one sees that it is sufficient to prove for all $f \in S(k)_e$ with $f(0) = 0$, that $L(f) \circ I(a) \in \mathrm{Ker}(P(b))$. Choose a $\ell_0 \in \mathbb{Z}$ such that the support of $G(L(f) \circ I(a))$ is contained in p^{ℓ_0} and $q^{-\ell_0} > |\delta|q^r$. Then

$$P(b)(L(f) \circ I(a)) \approx \int_k f(y) \int_{p^{\ell_0}} \tau(x(\tfrac{ay^2}{2} - b))dxdy = 0,$$

since $|\frac{ay^2}{2} - b| > |\delta| q^{\ell_0}$ for all $y \in k^*$.

Now take b of the form $\frac{at^2}{2}$, $t \in k^*$, and $n \in \mathbb{N}$ such that $q^{-n} < |2t|$. Clearly, it is sufficient to show $q(b)(L(f) \circ I(a)) \neq 0$ for some $f \in S(k)_e$. Put φ for the characteristic function of $\{t+p^n\} \cup \{-t+p^n\}$. As $\varphi(0) = 0$, the support of $G(L(\varphi) \circ I(a))$ is contained in p^r for some $r \in \mathbb{Z}$ and $P(b)(L(\varphi) \circ I(a))$ is proportional to

$$\int_{p^r}\int_{p^n} \tau(axtz(1+\frac{z^2}{2t}))dzdx + \int_{p^r}\int_{p^n} \tau(-axtz(1-\frac{z^2}{2t}))dzdx =$$

$$= 2q^{-r}|\delta|^{\frac{1}{2}} \int_{\substack{z\in p^n \\ |atz|\leq|\delta|q^r}} dz \neq 0$$

Finally the last assertion is a consequence of (14.7) and (11.11). □

14.9. We end this paragraph with the calculation of $W(b,\chi(m))(\varphi^0(s))(e)$ in the case that χ is unramified and $|2| = |\delta| = 1$. We will need it in the global case. Now,

$$G(b,2r,\chi h(-1)^{\ell}) = \int_{O^*} \tau(-b\pi^{-2r}\alpha)d\alpha = \begin{cases} 0 & \text{if } |b| > q^{1-2r} \\ -q^{-1} & \text{if } |b| = q^{1-2r} \\ 1-q^{-1} & \text{if } |b| \leq q^{-2r} \end{cases},$$

$$G(b,2r+1,\chi h(\pi)h(-1)^{\ell}) = \int_{O^*} (\pi,\alpha)\tau(-b\pi^{-2r-1}\alpha)d\alpha.$$

By (13.6), $G(b,2r+1,\chi h(\pi)h(-1)^{\ell}) = 0$, if $|b| \neq q^{-2r}$. If $|b| = q^{-2r}$,

$$\gamma(\pi^{-1})G(b,2r+1,\chi h(\pi)h(-1)^{\ell}) = q^{\frac{1}{2}}(\pi,2b)\int_{O_v}\int_{O_v^*}\tau(\frac{\alpha}{2\pi}(c^2-1))d\alpha dc$$

$$= q^{\frac{1}{2}}(\pi,2b)\{-q^{-1}(1-2q^{-1}) + 2q^{-1}(1-q^{-1})\}$$

$$= q^{-\frac{1}{2}}(\pi,2b).$$

By using the expression in (14.1), we obtain for $W(b,\chi(m))(\phi^{o}(s))(e)$

$$(14.10)\quad \begin{cases} 0 & \text{for } |b| > 1 \\ (1-q\chi(\pi^2))\left(\sum\limits_{t=0}^{r} (q^2\chi(\pi^2))^t\right) & \text{for } |b| = q^{-2r-1},\ r \geq 0. \\ (1-q\chi(\pi^2))\left\{\sum\limits_{t=0}^{r} (q^2\chi(\pi^2))^t + \dfrac{q^{\frac{1}{2}+2r}(\pi,(-1)^{\ell}2b)\chi(\pi)^{2r+1}}{1-q^{\frac{1}{2}}(\pi,(-1)^{\ell}2b)\chi(\pi)}\right. & \\ & \text{for } |b| = q^{-2r},\ r \geq 0 \end{cases}$$

CHAPTER 5

§15 THE MEROMORPHIC CONTINUATION OF THE FOURIER COEFFICIENTS

15.1. For χ as in (6.1), let $P(\chi)$ be the complement in P of $\{v \mid v \in P,\ v \notin P_\infty,\ |2|_v = |\delta_v|_v = 1$ and χ_v is unramified$\}$. In this chapter, Q, as in (0.4), will always be taken such that $Q \supseteq P(\chi)$. I write $S(\chi(m),Q)$ for the subspace of $S(\chi(m))$ spanned by the elements $\otimes\,\varphi_v, \varphi_v = \varphi_v^o(s_v)$ for all $v \notin Q$, and $\mathrm{Mp}(A,Q)$ for the subgroup $\{\otimes\, g_v \mid g_v \in G_v$ for all $v \notin Q\}$ of $\mathrm{Mp}(A)$.

15.2. For $\mathrm{Re}(s) > 1$ I will denote $\Pi_{v\notin Q}(1-\chi_v(\pi_v))^{-1}$ by $\zeta_Q(\chi)$. From [10] we know that $\zeta_Q(\chi)$ has a meromorphic continuation to $\mathbb{C}$, that is holomorphic on $\mathbb{C} \setminus \{1\}$ and has a pole of order ≤ 1 in $s = 1$. This pole occurs if and only if $\chi = \nu(s)$.

For all $v \notin P(\chi)$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$M(\chi_v(m))(\varphi_v^o(s_v)) = \{1 + \int_{|x|_v>1} \gamma(x)^m\, \chi_v(x^{-1})dx\}\, \varphi_v^o(2-s_v) =$$

$$= \frac{(1-\chi_v^2\nu(-1)(\pi_v))}{(1-\chi_v^2\nu(-2)(\pi_v))}\, \varphi_v^o(2-s_v).$$

For $\varphi \in S(\chi(m)),Q)$ of the form $\otimes\,\varphi_v(s_v)$, this implies

$$(15.3)\qquad M(\chi(m))(\varphi) = \frac{\zeta_Q(\chi^2\nu(-2))}{\zeta_Q(\chi^2\nu(-1))} \cdot \left\{\bigotimes_{v\in Q} M(\chi_v(m))(\varphi_v(s_v))\right\} \otimes \left\{\bigotimes_{v\notin Q}\varphi_v^o(2-s_v)\right\}$$

By combining this expression with the local results from the paragraphs 8, 10 and 13, we obtain the following

15.4. THEOREM.

(i) *For all* $\varphi \in S(\chi_0(m))$, $M(\chi(m))(\varphi(s))$ *has a meromorphic continuation to* $\mathbb{C}$, *holomorphic on* $\{s \mid s \in \mathbb{C},\ \mathrm{Re}(s) > 1,\ s \neq \frac{3}{2}\}$ *and with a pole of order* ≤ 1 *in* $s = \frac{3}{2}$. *This pole occurs if and only if* $\chi_0^3 = h(a)$ *for some* $a \in k^*$, *and we write* $R_0(\varphi(\frac{3}{2}))$ *for its residue in* $s = \frac{3}{2}$.

(ii) $\{R_0(\varphi(\frac{3}{2})) \mid \varphi \in S(\chi_0(m))\}$ *is spanned by the* $\otimes\psi_v \in S(\nu(\frac{1}{2})\chi_0(m))$ *with* ψ_v *belonging to the irreducible submodule of* $S(\nu(\frac{1}{2})h_v(a)(m))$ *for all* $v \in P$.

15.5. The results of this section are needed in the proof of the functional equation for the Eisenstein series. First of all, we note that formula (15.3) and the local results in (8.13), (10.5) and (13.2) allow us to conclude:

15.6. PROPOSITION. *For every* Q *as in* (15.1), *there exists an open connected* $U(Q)$ in $\mathbb{C}$, *with discrete complement and invariant under* $s \to 2-s$, *such that*

(i) *For all* $v \in Q$ *and* $s \in U(Q)$, $M(\chi_v(m))$ *is defined and* $S(\chi_v(m))$ *is irreducible.*

(ii) $M(\chi(m))(\varphi(s))$ *and* $M(\nu(2)\chi^{-1}(m))(\bar{\varphi}(2-s))$ *are holomorphic on* $U(Q)$, *for all* $\varphi \in S(\chi_0(m),Q)$.

Our goal in this section is to prove

15.7. THEOREM. *Let the notations be as in proposition* (15.6). *Then, for all* $s \in U(Q)$ *and* $\varphi \in S(\chi_0(m),Q)$:

$$M(\nu(2)\chi^{-2}(m)) \circ M(\chi(m))(\varphi(s)) = \varphi(s).$$

I already proved in (8.12), (10.7) and (13.12) that for all $v \in Q$ and $s_v \in U(Q)$

$$M(\nu(2)\chi_v^{-1}(m)) \circ M(\chi_v(m)) = \lambda(\chi_v) I_{S(\chi_v(m))}$$

Therefore the assertion of the theorem is equivalent to:

$$\prod_{v\in Q} \lambda(\chi_v) = \frac{\zeta_Q(\nu(3)\chi^{-2})\zeta_Q(\chi^2\nu(-1))}{\zeta_Q(\chi^2\nu(-2))\zeta_Q(\nu(2)\chi^{-2})} = \prod_{v\in Q} \rho(\chi_v^2\nu(-2))\rho(\nu(2)\chi_v^{-2})$$

with $\rho(\chi_v)$ the local factor as defined in [10].

By comparing the expressions for $\lambda(\chi_v)$ and $\rho(\chi_v)$, it will turn out that $\lambda(\chi_v) = |2|_v \chi_v(-1)\rho(\chi_v^2 \nu(-2))\rho(\nu(2)\chi_v^{-2})$ for all v in Q. This completes the proof of the theorem since $|2| = 1$ and $\chi(-1) = 1$.

In the imaginary case the desired equality is a consequence of (10.7) and $\rho(\chi_v \nu(-1))\rho(\nu(1)\chi_v^{-1}) = 4\chi_v(-1) \cdot \rho(\chi_v^2\nu(-2))\rho(\nu(2)\chi_v^{-2})$. If v is real, then one merely has to combine (8.12) and

$$\rho(\nu(2s-2)\rho(\nu(2-2s)) = \pi \frac{\Gamma(s-1)\Gamma(1-s)}{\Gamma(3/2-s)\Gamma(s-1/2)}$$

$$= -\cos(\pi s)\Gamma(1-s)\Gamma(s-1).$$

For finite v such that χ_v^2 is unramified, the assertion is clear, thanks to (13.12)(i). Finally, if χ_v^2 is ramified and the conductor of $(\chi_v^{\mathcal{O}})^2$ is p_v^f, then we have

$$\rho(\chi_v^2) = \nu(s_v)(\delta_v)\nu(-s_v f) \int_{\mathcal{O}_v^*} \chi_v^2(\alpha)\tau_v(\alpha\pi_v^{v(\delta_v)-f})d\alpha$$

$$\rho(\chi_v^2\nu(-2))\rho(\nu(2)\chi_v^{-2}) = \int_{\mathcal{O}_v^*} \chi_v^2(t)\left\{\int_{\mathcal{O}_v^*} \tau_v(\beta(1-t)\pi_v^{v(\delta_v)-f})d\beta\right\}dt$$

$$= |\delta_v|_v^{-1} q_v^{-f} .$$

Comparing with (13.12)(ii) gives the desired expression.

15.8. For $z \in k^*$ and $\varphi = \otimes \varphi_v \in S(\chi_0(m))$ we will derive now a useful expression for $E_z(\varphi(s),\chi(m))$. First choose a sufficiently large Q such that $\varphi \in S(\chi_0(m),Q)$. For $\alpha \in A^*$, $x \in A$, $\otimes g_v \in Mp(A,Q)$ and $Re(s) > 2$, $E_z(\varphi(s),\chi(m))(R(u(x)d(\alpha))\{\otimes g_v\})$ is equal to

$$(15.9) \qquad \tau(zx)\nu(2)\chi^{-2}(\alpha) \prod_{v\in Q} W(z\alpha_v^2,\chi_v(m))(\varphi_v H(s)\circ i_v)(g_v).$$

$$\prod_{v\notin Q} W(z\alpha_v^2,\chi_v(m))(\varphi_v H(s)\circ i_v)(e_v).$$

We claim that for all $v \notin Q$ and every $t \in k_v^*$ there exists a $f_v(t) \in S(k_v)$ such that

$$(15.10) \quad W(t,\chi_v(m))(\varphi_v^{\mathcal{O}}(s_v)) = (1-q_v\chi_v^2(\pi_v))\zeta_v(f_v(t),\chi_v\nu(-\tfrac{1}{2})h_v(2t(-1)^{\ell}))$$

Here and in the rest of this paper, the local and global zeta functions are denoted as in [10]. By using (14.10), one verifies directly that the subsequent choice is a right one. For $|t|_v > 1$, take $f_v(t) = 0$. If $v(t) = 2r+1$, $r \geq 0$, then we choose $f_v(t)$ as follows: $f_v(t)(x) = |x|_v^{-\frac{1}{2}}(2t(-1)^{\ell},x)_v$, if $v(x) = 2\ell_0$, $0 \leq \ell_0 \leq r$, and $f_v(t)(x) = 0$ for all other x. Finally, if $v(t) = 2r$, $r \geq 0$, then $f_v(t)$ is defined by

$$f_v(t)(x) = \begin{cases} \nu(-\tfrac{1}{2})(x) & \text{for } v(x) = 2\ell_0 \quad 0 \leq \ell_0 \leq r \\ q_v^r & \text{for } |x|_v < |t|_v \\ 0 & \text{elsewhere} \end{cases} .$$

In order to have a $f_v(t)$ for all $v \in P$ and all $t \in k_v^*$ we complement the choice made above. For infinite v, we take $f_v(t)$ equal to the function used in [10] for the calculation of $\rho(\chi_v h_v(2t(-1)^{\ell}))$, and for all finite v in Q, the function $f_v(t)$ is defined by: $f_v(t)(x) = 0$, if $|x|_v \neq 1$, and $f_v(t)(x) = (2t(-1)^{\ell},x)\chi_v(x^{-1})$, if $|x|_v = 1$.

For $a \in A^*$, let $F_Q(a) \in S(A)$ be $\otimes f_v(a_v)$. Then (15.9) equals

$$\tau(zx)\nu(2)\chi^{-2}(\alpha)\ \zeta_Q(\chi^2\nu(-1))^{-1} \prod_{v\in Q} \zeta_v(f_v(z\alpha_v^2),\chi_v\nu(-\tfrac{1}{2})h_v(2z(-1)^{\ell}))^{-1}.$$

(15.11)

$$\prod_{v\in Q} W(z\alpha_v^2,\chi_v(m))(\varphi_v H(s)\circ i_v)(g_v).\ \zeta(F_Q(z\alpha^2),\ h(2z(-1)^{\ell})\chi\nu(-\tfrac{1}{2})).$$

From this expression and the results, obtained in (9.2), (10.9) and (14.3), we may draw the following conclusion.

15.12. <u>THEOREM</u>. *Let* φ *be in* $S(\chi_0(m))$ *and* z *in* k^*. *Then*

(i) $E_z(\varphi(s),\chi(m))$ *has a meromorphic continuation to* $\mathbb{C}$; *it is holomorphic on* $\{s \mid s \in \mathbb{C}, \operatorname{Re}(s) > 1$ *and* $s \neq \frac{3}{2}\}$ *and has a pole of order* ≤ 1 *in* $s = 3/2$. *This pole can occur only if* $\chi_0 = h(2z(-1)^{\ell})$ *and we write* $R_z(\varphi(\frac{3}{2}))$ *for the residu in* $s = \frac{3}{2}$.

(ii) *Assume* $\chi_0 = h(2z(-1)^{\ell})$ *and that for all* $v \in Q$, $a_v = 1$ *and* $x_v = 0$. *Then we have for* φ, *as in* (15.8):

$$R_z(\varphi(\tfrac{3}{2}))(R(u(x)d(\alpha))\{\otimes g_v\}) \approx \tau(zx)\nu(2)\chi^{-1}(\alpha)\ F(F_Q(z\alpha^2))(0).$$

$$\prod_{v\in Q} W(z,\chi_v(m))(\varphi_v H(\tfrac{3}{2})\circ i_v)(g_v)$$

§16 THE CONVERGENCE OF THE SUM OF THE FOURIER COEFFICIENTS

16.1. We keep to the notations of (15.8). Let C resp. Y be compact subsets of $\mathbb{C}\setminus\{\frac{3}{2}\}$ resp. $Mp(A,Q)$. This paragraph will be devoted to the proof of the following results:

16.2. PROPOSITION.

(i) *For every* $g \in Mp(A,Q)$

$$\sum_{z\in k^*} |R_z(\varphi(\tfrac{3}{2}))(g)| < \infty .$$

(ii) *For* $c \in \mathbb{R}^*_{>0}$, *put* $\mathbb{R}^*_c$ *for* $\{t|t \in \mathbb{R}^* \subset A^*,\ t \geq c\}$. *Then*

$$\sup_{\substack{y\in Y\\ t\in\mathbb{R}^*_c}} \left\{\sum_{z\in k^*} \sup_{s\in C} |\zeta_Q(\chi^2\nu(-1))E_z(\varphi(s),\chi(m))(R(d(t))y)|\right\}$$

First of all, we note that by (14.10) only the z belonging to an ideal $\mathfrak{a}$ of the form $\Pi_{v\in Q\setminus P_\infty}(\mathfrak{p}_v\cap k)^{\ell_v}$ play a role in this sum. For $z \in k^*$, write $|z|_\infty$ for $\max_{v\in P_\infty}|z|_v$, and $|z|_Q$ for $\Pi_{v\notin Q}|z|_v$; then we have for all $z \in \mathfrak{a} \cap k^*$: $|z|^r_\infty \succ |z|^{-1}_Q$ and $|z|^r_\infty \succ |z|^{-1}_v$, for $v \in Q$ and r the number of infinite places of k. By applying this and the local estimates from (9.6), (10.11) and (14.5) to formula (15.11), one sees that, in order to prove (16.2)(ii), it suffices to find an $N \in \mathbb{N}$ such that for all $z \in \mathfrak{a} \cap k^*$

$$(16.3)\qquad \sup_{s\in C} |\zeta(F_Q(z),\ h(2z(-1)^{\ell})\chi\nu(-\tfrac{1}{2}))| \prec |z|^{-N}_Q .$$

The analytic continuation of $\zeta(F_Q(z),h(2z(-1)^{\ell})\chi\nu(-\frac{1}{2}))$ to $\mathbb{C}$ is given by the sum of the following 3 expressions:

$$(16.4)\qquad \int_1^\infty t^{s-\frac{1}{2}} \left\{\int_{A_1^*} F_Q(z)(xt)h(2z(-1)^{\ell})(x)\chi_o(x)d^*x\right\}d^*t$$

$$(16.5)\qquad \int_1^\infty t^{3/2-s}\left\{\int_{A_1^*} \mathcal{F}(F_Q(z))(xt)h(2z(-1)^{\ell})(x)\chi_o(x^{-1})d^*x\right\}d^*t$$

$$(16.6)\qquad \mathcal{F}(F_Q(s))(0)(s-\tfrac{3}{2})^{-1}\int_{A_1^*/k^*} h(2z(-1)^{\ell})(x)\chi_o(x^{-1})d^*\dot{x}$$

with all the measures taken as in [10]. Before starting the estimates, we remark that by adjusting Q we may assume that k^* has a compact fundamental domain X in A_1^*, with $1 \in X$ and $X \subseteq \{\alpha | \alpha \in A^*, c_v \le |\alpha_v|_v \le d_v$, for all $v \in P, c_v = d_v = 1$ for all $v \notin Q\}$.

For all $v \notin Q$, $|f_v(z)| \le |z|_v^{-\frac{1}{2}} \psi_v(0)$. Since, moreover, for all finite v $|f_v(z)| \le \psi_v(0)$, we get

$$|F_Q(z)| \le |z|_Q^{\frac{1}{2}}.\{ \underset{v\in P_\infty}{\otimes} |f_v(z)|\} \otimes \{ \underset{v\notin P_\infty}{\otimes} \psi_v(0)\}$$

and this clearly implies the desired estimate for (16.4).

Next, the Fourier transforms of the $f_v(z)$ demand our attention. For infinite v, $|F(f_v(z))| = |f_v(z)|$; if $v \in Q$ is finite, then there exists a $n_v \le 0$ such that for all $z \in k_v^*$, $x \in k_v$ and $u \in k_v^*$, with $c_v \le |u|_v \le d_v$, $|F(f_v(z))(xu)| \le \psi_v(n_v)(x).\mathrm{vol}(O_v^*)$. Finally, for $v \notin Q$, we distinguish the cases v(z) is odd resp. even. If $v(z) = 2\ell_0+1, \ell_0 \ge 0$,

$$F(f_v(z))(y) = \sum_{n=0}^{\ell_0} q^{-n} \int_{O_v^*} (\alpha,z)_v \, \tau_v(-\alpha\pi_v^{2n}y)d\alpha$$

$$= \begin{cases} q_v^n \int_{O_v^*} (z,\alpha)_v \, \tau_v(-\alpha\pi_v^{2n}y)d\alpha, & \text{if } v(y) = -2n-1,\ 0\le n\le\ell_0 \\ 0 & \text{otherwise} \end{cases} .$$

If $v(z) = 2\ell_0$, $\ell_0 \ge 0$,

$$F(f_v(z))(y) = \sum_{n=0}^{\ell_0} q_v^n\{q_v^{-2n}\psi_v(-2n)(y) - q_v^{-2n-1}\psi_v(-2n-1)(y)\} + $$
$$+ q_v^{-\ell_0-1} \psi_v(-2\ell_0-1)(y)$$

$$= \begin{cases} 1 & \text{if } |y| \le 1 \\ |y|_v^{-\frac{1}{2}} & \text{if } v(y) = -2n,\ \ell_0\ge n\ge 1 \\ 0 & \text{otherwise} \end{cases} .$$

In both cases, $|F(f_v(z))| \le \psi_v(-v(z))$ and $|F(f_v(z))(xa)| = |F(f_v(z))(x)|$ for $x \in k_v$ and $a \in O_v^*$.

For $z \in \mathfrak{a} \cap k^*$, let $\mathfrak{a}(z)$ be the ideal

$$\prod_{v\in Q\setminus P_\infty} (p_v \cap k)^{n_v} \prod_{v\notin Q} (p_v \cap k)^{-v(z)}.$$

Put $n = |k:\mathbb{Q}|$ and $\sigma = \mathrm{Re}(s)$; further, for $\xi \in k^*$, denote $\Sigma_{v\in P_\infty}\xi_v\bar{\xi}_v$ by $|\xi|^\infty$. Now, the calculations, made above, imply that one can find a $\mu > 0$ and a $L \in \mathbb{N}$, both independent of $z \in \mathfrak{a} \cap k^*$, such that

$$\int_{A^{(1)}} |F_Q(z))(tx)|d^*x \prec H(t,z) = \sum_{\xi\in\mathfrak{a}(z)\cap k^*} |\xi|_\infty^L e^{-\mu t^{2/n}|\xi|^\infty}$$

On $\mathbb{R}^*_{>0}$ we have $H(u|z|_Q^{-n},z) \le H(u,1)$. Hence

$$\int_1^\infty t^{3/2-\sigma} H(t,z)d^*t \le |z|_Q^{-n} \int_{|z|_Q^n}^\infty H(u,1)u^{\frac{1}{2}-\sigma}du$$

Since on $[|z|_Q^n,1]$,

$$\sum_{\substack{\xi\in\mathfrak{a}(1)\\ |\xi|^\infty<|z|_Q^{-4}}} |\xi|_\infty^L e^{-\mu t^{2/n}|\xi|^\infty} \prec |z|_Q^{-2nL}$$

and

$$\sum_{\substack{\xi\in\mathfrak{a}(1)\\ |\xi|^\infty\ge|z|_Q^{-4}}} |\xi|_\infty^L e^{-\mu t^{2/n}|\xi|^\infty} \prec 1,$$

we obtain the desired estimate for expression (16.5).

As for (16.6), it can occur only if $\chi_0 = h(a)$, $a \in k^*$, and

$$z \in \left\{\frac{a}{2}(-1)^\ell (k^*)^2 \cap \mathfrak{a}\right\}.$$

For those z, $\mathcal{F}(F_Q(z))(0)$ is constant and (16.6) is bounded on $\mathcal{C}$. This completes the proof of (16.3). At the same time this last observation, combined with the estimates (9.6), (10.11) and (14.5), proves assertion (16.2) (i).

Thanks to (16.2) and (6.10) we can state now

16.7. <u>THEOREM</u>. *Let* φ *belong to* $S(\chi_0(m))$. *Then* $\Sigma_{z\in k}E_z(\varphi(s),\chi(m))$ *defines a meromorphic continuation to* $\mathbb{C}$ *of* $E(\varphi(s),\chi(m))$; *it is holomorphic on* $\mathrm{Re}(s) > 1$, $s \ne \frac{3}{2}$, *and has a pole of order* ≤ 1 *in* $s = \frac{3}{2}$. *This pole can* *occur only if* $\chi_0 = h(a)$ *for some* $a \in k^*$, *and we write* $R(\varphi(\frac{3}{2}))$ *for its residue.*

§17 THE RESIDUE OF THE EISENSTEIN SERIES IN S = 3/2

17.1. From the foregoing paragraphs, it is clear that we have to consider only the case $\chi_0 = h(a(-1)^{\ell})$, $a \in k^*$. Furthermore, for $z \in k^*$ and $\varphi \in S(\chi)$, we have

$$(17.2)\qquad E_z(\varphi \circ I(a) \circ J(m), \nu(s)h(a(-1)^{\ell})(m)) = E_{za^{-1}}(\varphi, \nu(s)) \circ I(a) \circ J(m).$$

Therefore it is sufficient to calculate the residue in the case $\chi_0 = 1$, $m = 1$. This implies that only $z \in k^*$ of the form $z = \frac{1}{2}\xi^2$ will play a role. In particular we get for every $\varphi \in S(\chi_0)$ and $g \in Mp(A)$

$$(17.3)\qquad R(\varphi(\tfrac{3}{2}))(g) = R_0(\varphi(\tfrac{3}{2}))(g) + \frac{1}{2}\sum_{\xi\in k^*} R_{\frac{1}{2}}(\varphi(\tfrac{3}{2}))(R(d(\xi))g).$$

From (8.14), (10.7) and (13.16) we know that $\otimes S(k_v)_e$ is an irreducible $\mathcal{H}$-module and that the map $\varphi \to \theta(\varphi)_0$ in an injective $\mathcal{H}$-module homomorphism of $\otimes S(k_v)_e$ into $S(\nu(\frac{1}{2}))$.

17.4. For every $v \in P$ and $\psi_v \in S(k_v)_e$, define $W(\psi_v):Mp(k_v) \to \mathbb{C}$ by

$$W(\psi_v)(g) = \omega_v(g)(\psi_v)(1).$$

One verifies easily that $\psi_v \to W(\psi_v)$ is a $\mathcal{W}(\frac{1}{2})$-model of $S(k_v)_e$. By applying This and the local results in (9.13), (10.12) and (14.8) to (15.12)(ii), one concludes that $\{0\} \not\subseteq \{R_{\frac{1}{2}}(\varphi(\frac{3}{2})) \mid \varphi \in S(1)\} \subseteq \{\theta(\psi)_{\frac{1}{2}} \mid \psi \in \otimes S(k_v)_e\}$. Moreover, since $\otimes S(k_v)_e$ is an irreducible $\mathcal{H}$-module, equality must hold here. From expression (5.3) we see that $\theta(\psi)_{\frac{1}{2}} = R_{\frac{1}{2}}(\varphi(\frac{3}{2}))$ implies $\theta(\psi) - R(\varphi(\frac{3}{2})) = \theta(\psi)_0 - R_0(\varphi(\frac{3}{2}))$. By Theorem (15.4) $\theta(\psi)_0 - R(\varphi\frac{3}{2})) = \theta(\psi - \psi_1)_0$. Hence $\theta(\psi - \psi_1)_0$ is a function on $S\ell(2,k)\backslash Mp(A)$. Since clearly the same holds for $h.\theta(\psi - \psi_1)_0$ for each $h \in \mathcal{H}$, the assumption $\psi \neq \psi_1$ would lead to the conclusion that $\{\theta(\psi_2) \mid \psi_2 \in \otimes S(k_v)_e\}$ consists of functions on $S\ell(s,k)\backslash Mp(A)$. This is easily seen not to be true, so that we get

$$\{R(\varphi(\tfrac{3}{2})) \mid \varphi \in S(1)\} = \{\theta(\psi) \mid \psi \in \otimes S(k_v)_e\}.$$

17.5. For $\psi \in \otimes S(k_v)_e$, let $\Theta(\psi)$ be the residue in $s = 1$ of $\theta(\psi,s)$. $\Theta(\psi)$ belongs to $\{\theta(\varphi)|\varphi \in \otimes S(k_v)_e\}$ and the map $\psi \to \Theta(\psi)$ is a $\tilde{M}$-isomorphism. This last fact is a consequence of the following observations. For every $v \in P$ and any $\psi_v \in S(k_v)_e$, the $\tilde{K}_v$-action on ψ_v is isomorphic to that on $L(\psi_v)H_v(1)$. As every continuous irreducible representation of $\tilde{K}_v$ occurs at most once in $\mathrm{Ind}(\chi_v)|\tilde{K}_v$ and $M(\nu(\frac{3}{2}))(S(\nu(\frac{3}{2}))) = L(S(k_v)_e)$, we may conclude that $\psi_v \to M(\nu(\frac{3}{2}))(L(\psi_v)H_v(1)))$ is a $\tilde{K}_v$-isomorphism from $S(k_v)_e$ onto the irreducible submodule of $S(\nu(\frac{1}{2}))$. Since the map $\psi \to \theta(\psi)$, with ψ in $\otimes S(k_v)_e$ clearly commutes with the action of $\tilde{M}$, we arrive at the following generalization of (5.8):

17.6. <u>THEOREM</u>. *Let* $\psi \in \otimes S(k_v)_e$ *be such that* $\psi * \check{e}(\rho) = \psi$ *for some continuous irreducible representation* ρ *of* $\tilde{M}$. *Then there is a* $\lambda(\rho) \in \mathbb{C}^*$ *such that*

$$\Theta(\psi) = \lambda(\rho)\ \theta(\psi).$$

17.7. We end this paragraph with giving an example of a ψ in $\otimes S(k_v)_e$ such that $\theta(\psi)$ and $\Theta(\psi)$ are not proportional to each other. Take $k = \mathbb{Q}$. Choose $\varphi = \otimes\varphi_v$ and $\psi = \otimes\psi_v$ in $\otimes S(k_v)_e$ as follows: in the finite case $\varphi_v = \psi_v = \psi_v(0)$, and in the infinite one, $\varphi_\infty(x) = e^{-\pi x^2}$ and $\psi_\infty(x) = x^2 e^{-\pi x^2}$, for $x \in \mathbb{R}$. From [9], one can see that $\Theta(\varphi) = \mu\theta(\varphi)$. In particular, $\Theta(\varphi)_0(e) \neq 0$. As $\Theta(\psi)_0(e) = \lambda\Theta(\varphi)_0(e)$ with λ equal to

$$\frac{\int_{\mathbb{R}}\{\int_{\mathbb{R}} y^2 e^{-\pi(1+ix)y^2}dy\}(1+x^2)^{-\frac{1}{2}}dx}{\int_{\mathbb{R}}\{\int_{\mathbb{R}} e^{-\pi(1+ix)y^2}dy\}(1+x^2)^{-\frac{1}{2}}dx} = \frac{\int_{\mathbb{R}} (1+ix)^{-2}(1-ix)^{-\frac{1}{2}}dx}{\int_{\mathbb{R}} (1+ix)^{-1}(1-ix)^{-\frac{1}{2}}dx} = \frac{1}{4\pi}\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{2})},$$

$\Theta(\psi)_0(e) \neq 0$, but $\theta(\psi)_0(e) = 0$. This proves the assertion.

§18 THE FUNCTIONAL EQUATION OF THE EISENSTEIN-SERIES

18.1. Our aim in this section is to prove for every $\varphi \in S(\chi_0(m))$:

18.2. <u>THEOREM</u>. *For all* $s \in \mathbb{C}$, $E(\varphi(s),\chi(m)) = E(M(\chi(m))(\varphi(s)),\nu(2)\chi^{-1}(m))$.

First we choose a Q as in (15.1) such that $\varphi \in S(\chi_0(m),Q)$ and $Mp(A) = S\ell(2,k)Mp(A,Q)$. It suffices then to show that the two functions are equal on $Mp(A,Q)$. It is no restriction to assume that $\varphi = \otimes\varphi_v$. Let $U(Q) \subseteq \mathbb{C}$ be as in (15.6). From the uniqueness of the Whittaker models, we may conclude that for all $v \in Q$ and all $s \in U(Q)$ there is a $\lambda_v(s,z) \in \mathbb{C}^*$ such that

$$(18.3) \qquad W(z,\nu(2)\chi_v^{-1}(m)) \circ M(\chi_v(m)) = \lambda_v(s,z)W(z,\chi_v(m)).$$

Since $W(z,\chi_v(m))$ is injective for all $s \in U(Q)$ and $W(z,\chi_v(m))(\psi)$ holomorphic on $\mathbb{C}$ for every $\psi \in S(\chi_v(m))$, $\lambda_v(s,z)$ is holomorphic on $U(Q)$.

Theorem (15.7) says that $E_0(\varphi(s),\chi(m)) = E_0(M(\chi(m))(\varphi(s), \nu(2)\chi^{-1}(m))$, for all $s \in U(Q)$. Hence this equality holds on $\mathbb{C}$. By reduction theory, there exists a $c > 0$ and a compact $Y \subseteq Mp(A)$ such that $Mp(A) = \{R(\sigma d(t))y | \sigma \in S\ell(2,k),\ t \in \mathbb{R}_{>0},\ t > c \text{ and } y \in Y\}$. The estimates in proposition (16.2) show then that $E(\varphi(s),\chi(m))-E(M(\chi(m))(\varphi(s)),\nu(2)\chi^{-1}(m))$ is a bounded function on $Mp(A)$. Clearly the same holds for its Fourier coefficients. From formula (15.11) and theorem (16.7) we obtain

$$E_z(\varphi(s),\chi(m))(\otimes g_v)-E_z(M(\chi(m))(\varphi(s)))(\otimes g_v)=\lambda_Q(z,s) \prod_{v\in Q} W(z,\chi_v(m))(\varphi_v(s))(g_v)$$

with $\otimes g_v \in Mp(A,Q)$, $\varphi_v(s) = \varphi_v H(s) \circ i_v$ and $\lambda_Q(z,s)$ a holomorphic function on $U(Q)$.

For $s_0 \in U(Q)$, with $Re(s_0) > 2$, choose a δ_0, $0 < \delta_0 < Re(s_0)-2$, such that $B(s_0,\delta_0) = \{r | r \in \mathbb{C}, |r-s_0| \leq \delta_0\} \subseteq U(Q)$. Next we take for all $v \in Q$ a $g_v \in Mp(k_v)$ such that for finite v,

$$\sup_{s\in B(s_0,\delta_0)} |W(z,\chi_v(m))(\varphi_v(s))(g_v)| > 0$$

and for infinite v,

$$\sup_{s\in B(s_0,\delta_0)} |M(\chi_v(m))(\varphi_v(s))(g_v)| > 0.$$

For those g_v and $s \in B(s_0,\delta_0)$ we have then

$$\lim_{t\downarrow 0} \Big| \prod_{v\in P_\infty} W(z,\chi_v(m))(\varphi_v(s))(R(d(t))g_v) \prod_{v\in Q\setminus P_\infty} W(z,\chi_v(m))(\varphi_v(s))(g_v) \Big| = \infty.$$

This contradicts the boundedness of the Fourier coefficients, unless $\lambda_Q(z,s) = 0$ for all $s \in B(s_0,\delta_0)$. Hence $\lambda_Q(z,-) \equiv 0$ on $U(Q)$. This completes the proof of theorem (18.2).

18.4. Theorem 18.2 enables us to compute the value of $\theta(\varphi,s)$ in $s = 0$. Namely it says that

$$\theta(\varphi,s) = E(M(\nu(s+\tfrac{1}{2}))(\theta(\varphi)_0 H(s)), \nu(\tfrac{3}{2}-s)).$$

By making use of the explicit expressions (15.3) and (15.11) one sees that the right hand side is holomorphic in $s = 0$. From (17.4) we know that $\theta(\varphi,0) \in \otimes S(k_v)_e$. Applying now (15.3) and (15.7) to the zero-th Fourier coefficient of $\theta(\varphi,s)$, we obtain

18.5. THEOREM. *For all* $\varphi \in \otimes S(k_v)_e$, $\theta(\varphi,0) = \theta(\varphi)$.

REFERENCES

[1] BERNSHTEIN, I.N. & A.V. ZELEVINSKY, *Representations of the group* GL(n,F), *where* F *is a non-Archimedean local field,* Russian Mathematical Surveys 31 (1976), pages 1-64.

[2] CODDINGTON, E.A. & N. LEVINSON, *Theory of Ordinary Differential Equations,* McGraw-Hill Book Company, Inc., New York - Toronto - London, 1955.

[3] GELBART, S.S., *Weil's Representation and the Spectrum of the Metaplectic Group,* Lecture Notes in Mathematics 530 (1976).

[4] GELBART, S.S. & P. SALLY, *Intertwining Operators and Automorphic Forms for the Metaplectic Group,* Proc. Nat. Acad. Sci. USA, vol. 72, no. 4, pages 1406-1410, 1975.

[5] GODEMENT, R., *Introduction à la theorie de Langlands,* Seminaire Bourbaki no. 321, February 1967.

[6] JACQUET, H. & R.P. LANGLANDS, *Automorphic Forms on* GL(2), Lecture Notes in Mathematics 114 (1970).

[7] MAGNUS, W., F. OBERHETTINGER & R.P. SONI, *Formulas and Theorems for the Special Functions of Mathematical Physics,* third edition. Springer-Verlag, Berlin - Heidelberg - New York 1966.

[8] MATSUMOTO, H., *Sur les sous-groupes arithmétiques des groupes semi-simples déployés,* Ann. Scient. Éc. Norm. Sup., series 4, part 2 (1969), pages 1-62.

[9] SIEGEL, C.L., *Die Funktionalgleichungen einiger Dirichletscher Reihen,* Mathematische Zeitschrift 63 (1956), pages 363-373.

[10] TATE, J.T., *Fourier Analysis in Number Fields and Hecke's Zeta- Functions,* Thesis Princeton University, 1950. In Algebraic Number Theory, pages 305-347. Edited by J.W.S. Cassels and A. Fröhlich, Academic Press, London and New York 1967.

[11] WALLACH, N.R., *Representations of reductive Lie groups,* in Proceedings of Symposia in Pure Mathematics, volume 33, part 1 (1979), pages 71-86. Edited by A. Borel and W. Casselman.

[12] WEIL, A., *Sur certains groupes d'operateurs unitaires,* Acta Mathematica 111 (1964), pages 143-211.

[13] WEIL, A., *Sur la formule de Siegel dans la théorie des groupes classiques,* Acta Mathematica 113 (1965) pages 1-87.

INDEX

TABLE OF NOTATIONS

Chapter I

Chapter 2

Chapter 3

Chapter 4

Chapter 5

TITLES IN THE SERIES MATHEMATICAL CENTRE TRACTS

(An asterisk before the MCT number indicates that the tract is under preparation).

A leaflet containing an order form and abstracts of all publications mentioned below is available at the Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands. Orders should be sent to the same address.

MCT 1 T. VAN DER WALT, *Fixed and almost fixed points*, 1963. ISBN 90 6196 002 9.

MCT 2 A.R. BLOEMENA, *Sampling from a graph*, 1964. ISBN 90 6196 003 7.

MCT 3 G. DE LEVE, *Generalized Markovian decision processes, part I: Model and method*, 1964. ISBN 90 6196 004 5.

MCT 4 G. DE LEVE, *Generalized Markovian decision processes, part II: Probabilistic background*, 1964. ISBN 90 6196 005 3.

MCT 5 G. DE LEVE, H.C. TIJMS & P.J. WEEDA, *Generalized Markovian decision processes, Applications*, 1970. ISBN 90 6196 051 7.

MCT 6 M.A. MAURICE, *Compact ordered spaces*, 1964. ISBN 90 6196 006 1.

MCT 7 W.R. VAN ZWET, *Convex transformations of random variables*, 1964. ISBN 90 6196 007 X.

MCT 8 J.A. ZONNEVELD, *Automatic numerical integration*, 1964. ISBN 90 6196 008 8.

MCT 9 P.C. BAAYEN, *Universal morphisms*, 1964. ISBN 90 6196 009 6.

MCT 10 E.M. DE JAGER, *Applications of distrubutions in mathematical physics*, 1964. ISBN 90 6196 010 X.

MCT 11 A.B. PAALMAN-DE MIRANDA, *Topological semigroups*, 1964. ISBN 90 6196 011 8.

MCT 12 J.A.Th.M. VAN BERCKEL, H. BRANDT CORSTIUS, R.J. MOKKEN & A. VAN WIJNGAARDEN, *Formal properties of newspaper Dutch*, 1965. ISBN 90 6196 013 4.

MCT 13 H.A. LAUWERIER, *Asymptotic expansions*, 1966, out of print; replaced by MCT 54.

MCT 14 H.A. LAUWERIER, *Calculus of variations in mathematical physics*, 1966. ISBN 90 6196 020 7.

MCT 15 R. DOORNBOS, *Slippage tests*, 1966. ISBN 90 6196 021 5.

MCT 16 J.W. DE BAKKER, *Formal definition of programming languages with an application to the definition of ALGOL 60*, 1967. ISBN 90 6196 022 3.

MCT 17 R.P. VAN DE RIET, *Formula manipulation in ALGOL 60, part 1*, 1968. ISBN 90 6196 025 8.

MCT 18 R.P. VAN DE RIET, *Formula manipulation in ALGOL 60, part 2*, 1968. ISBN 90 6196 038 X.

MCT 19 J. VAN DER SLOT, *Some properties related to compactness*, 1968. ISBN 90 6196 026 6.

MCT 20 P.J. VAN DER HOUWEN, *Finite difference methods for solving partial differential equations*, 1968. ISBN 90 6196 027 4.

MCT 21 E. WATTEL, *The compactness operator in set theory and topology*, 1968. ISBN 90 6196 028 2.

MCT 22 T.J. DEKKER, *ALGOL 60 procedures in numerical algebra, part 1*, 1968. ISBN 90 6196 029 0.

MCT 23 T.J. DEKKER & W. HOFFMANN, *ALGOL 60 procedures in numerical algebra, part 2*, 1968. ISBN 90 6196 030 4.

MCT 24 J.W. DE BAKKER, *Recursive procedures*, 1971. ISBN 90 6196 060 6.

MCT 25 E.R. PAËRL, *Representations of the Lorentz group and projective geometry*, 1969. ISBN 90 6196 039 8.

MCT 26 EUROPEAN MEETING 1968, *Selected statistical papers, part I*, 1968. ISBN 90 6196 031 2.

MCT 27 EUROPEAN MEETING 1968, *Selected statistical papers, part II*, 1969. ISBN 90 6196 040 1.

MCT 28 J. OOSTERHOFF, *Combination of one-sided statistical tests*, 1969. ISBN 90 6196 041 X.

MCT 29 J. VERHOEFF, *Error detecting decimal codes*, 1969. ISBN 90 6196 042 8.

MCT 30 H. BRANDT CORSTIUS, *Exercises in computational linguistics*, 1970. ISBN 90 6196 052 5.

MCT 31 W. MOLENAAR, *Approximations to the Poisson, binomial and hypergeometric distribution functions*, 1970. ISBN 90 6196 053 3.

MCT 32 L. DE HAAN, *On regular variation and its application to the weak convergence of sample extremes*, 1970. ISBN 90 6196 054 1.

MCT 33 F.W. STEUTEL, *Preservation of infinite divisibility under mixing and related topics*, 1970. ISBN 90 6196 061 4.

MCT 34 I. JUHÁSZ, A. VERBEEK & N.S. KROONENBERG, *Cardinal functions in topology*, 1971. ISBN 90 6196 062 2.

MCT 35 M.H. VAN EMDEN, *An analysis of complexity*, 1971. ISBN 90 6196 063 0.

MCT 36 J. GRASMAN, *On the birth of boundary layers*, 1971. ISBN 90 6196 064 9.

MCT 37 J.W. DE BAKKER, G.A. BLAAUW, A.J.W. DUIJVESTIJN, E.W. DIJKSTRA, P.J. VAN DER HOUWEN, G.A.M. KAMSTEEG-KEMPER, F.E.J. KRUSEMAN ARETZ, W.L. VAN DER POEL, J.P. SCHAAP-KRUSEMAN, M.V. WILKES & G. ZOUTENDIJK, *MC-25 Informatica Symposium* 1971. ISBN 90 6196 065 7.

MCT 38 W.A. VERLOREN VAN THEMAAT, *Automatic analysis of Dutch compound words*, 1971. ISBN 90 6196 073 8.

MCT 39 H. BAVINCK, *Jacobi series and approximation*, 1972. ISBN 90 6196 074 6.

MCT 40 H.C. TIJMS, *Analysis of (s,S) inventory models*, 1972. ISBN 90 6196 075 4.

MCT 41 A. VERBEEK, *Superextensions of topological spaces*, 1972. ISBN 90 6196 076 2.

MCT 42 W. VERVAAT, *Success epochs in Bernoulli trials (with applications in number theory)*, 1972. ISBN 90 6196 077 0.

MCT 43 F.H. RUYMGAART, *Asymptotic theory of rank tests for independence*, 1973. ISBN 90 6196 081 9.

MCT 44 H. BART, *Meromorphic operator valued functions*, 1973. ISBN 90 6196 082 7.

MCT 45 A.A. BALKEMA, *Monotone transformations and limit laws* 1973. ISBN 90 6196 083 5.

MCT 46 R.P. VAN DE RIET, *ABC ALGOL, A portable language for formula manipulation systems, part 1: The language*, 1973. ISBN 90 6196 084 3.

MCT 47 R.P. VAN DE RIET, *ABC ALGOL, A portable language for formula manipulation systems, part 2: The compiler*, 1973. ISBN 90 6196 085 1.

MCT 48 F.E.J. KRUSEMAN ARETZ, P.J.W. TEN HAGEN & H.L. OUDSHOORN, *An ALGOL 60 compiler in ALGOL 60, Text of the MC-compiler for the EL-X8*, 1973. ISBN 90 6196 086 X.

MCT 49 H. KOK, *Connected orderable spaces*, 1974. ISBN 90 6196 088 6.

MCT 50 A. VAN WIJNGAARDEN, B.J. MAILLOUX, J.E.L. PECK, C.H.A. KOSTER, M. SINTZOFF, C.H. LINDSEY, L.G.L.T. MEERTENS & R.G. FISKER (eds), *Revised report on the algorithmic language ALGOL 68*, 1976. ISBN 90 6196 089 4.

MCT 51 A. HORDIJK, *Dynamic programming and Markov potential theory*, 1974. ISBN 90 6196 095 9.

MCT 52 P.C. BAAYEN (ed.), *Topological structures*, 1974. ISBN 90 6196 096 7.

MCT 53 M.J. FABER, *Metrizability in generalized ordered spaces*, 1974. ISBN 90 6196 097 5.

MCT 54 H.A. LAUWERIER, *Asymptotic analysis, part 1*, 1974. ISBN 90 6196 098 3.

MCT 55 M. HALL JR. & J.H. VAN LINT (eds), *Combinatorics, part 1: Theory of designs, finite geometry and coding theory*, 1974. ISBN 90 6196 099 1.

MCT 56 M. HALL JR. & J.H. VAN LINT (eds), *Combinatorics, part 2: Graph theory, foundations, partitions and combinatorial geometry*, 1974. ISBN 90 6196 100 9.

MCT 57 M. HALL JR. & J.H. VAN LINT (eds), *Combinatorics, part 3: Combinatorial group theory*, 1974. ISBN 90 6196 101 7.

MCT 58 W. ALBERS, *Asymptotic expansions and the deficiency concept in statistics*, 1975. ISBN 90 6196 102 5.

MCT 59 J.L. MIJNHEER, *Sample path properties of stable processes*, 1975. ISBN 90 6196 107 6.

MCT 60 F. GÖBEL, *Queueing models involving buffers*, 1975. ISBN 90 6196 108 4.

*MCT 61 P. VAN EMDE BOAS, *Abstract resource-bound classes, part 1*, ISBN 90 6196 109 2.

*MCT 62 P. VAN EMDE BOAS, *Abstract resource-bound classes, part 2*, ISBN 90 6196 110 6.

MCT 63 J.W. DE BAKKER (ed.), *Foundations of computer science*, 1975. ISBN 90 6196 111 4.

MCT 64 W.J. DE SCHIPPER, *Symmetric closed categories*, 1975. ISBN 90 6196 112 2.

MCT 65 J. DE VRIES, *Topological transformation groups 1 A categorical approach*, 1975. ISBN 90 6196 113 0.

MCT 66 H.G.J. PIJLS, *Locally convex algebras in spectral theory and eigenfunction expansions*, 1976. ISBN 90 6196 114 9.

*MCT 67 H.A. LAUWERIER, *Asymptotic analysis, part 2*, ISBN 90 6196 119 X.

MCT 68 P.P.N. DE GROEN, *Singularly perturbed differential operators of second order*, 1976. ISBN 90 6196 120 3.

MCT 69 J.K. LENSTRA, *Sequencing by enumerative methods*, 1977. ISBN 90 6196 125 4.

MCT 70 W.P. DE ROEVER JR., *Recursive program schemes: Semantics and proof theory*, 1976. ISBN 90 6196 127 0.

MCT 71 J.A.E.E. VAN NUNEN, *Contracting Markov decision processes*, 1976. ISBN 90 6196 129 7.

MCT 72 J.K.M. JANSEN, *Simple periodic and nonperiodic Lamé functions and their applications in the theory of conical waveguides*, 1977. ISBN 90 6196 130 0.

MCT 73 D.M.R. LEIVANT, *Absoluteness of intuitionistic logic*, 1979. ISBN 90 6196 122 X.

MCT 74 H.J.J. TE RIELE, *A theoretical and computational study of generalized aliquot sequences*, 1976. ISBN 90 6196 131 9.

MCT 75 A.E. BROUWER, *Treelike spaces and related connected topological spaces*, 1977. ISBN 90 6196 132 7.

MCT 76 M. REM, *Associations and the closure statement*, 1976. ISBN 90 6196 135 1.

MCT 77 W.C.M. KALLENBERG, *Asymptotic optimality of likelihood ratio tests in exponential families*, 1977. ISBN 90 6196 134 3.

MCT 78 E. DE JONGE & A.C.M. VAN ROOIJ, *Introduction to Riesz spaces*, 1977. ISBN 90 6196 133 5.

MCT 79 M.C.A. VAN ZUIJLEN, *Empirical distributions and rank statistics*, 1977. ISBN 90 6196 145 9.

MCT 80 P.W. HEMKER, *A numerical study of stiff two-point boundary problems*, 1977. ISBN 90 6196 146 7.

MCT 81 K.R. APT & J.W. DE BAKKER (eds), *Foundations of computer science II*, part 1, 1976. ISBN 90 6196 140 8.

MCT 82 K.R. APT & J.W. DE BAKKER (eds), *Foundations of computer science II*, part 2, 1976. ISBN 90 6196 141 6.

MCT 83 L.S. BENTHEM JUTTING, *Checking Landau's "Grundlagen" in the AUTOMATH system*, 1979. ISBN 90 6196 147 5.

MCT 84 H.L.L. BUSARD, *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?) books vii-xii*, 1977. ISBN 90 6196 148 3.

MCT 85 J. VAN MILL, *Supercompactness and Wallman spaces*, 1977. ISBN 90 6196 151 3.

MCT 86 S.G. VAN DER MEULEN & M. VELDHORST, *Torrix I, A programming system for operations on vectors and matrices over arbitrary fields and of variable size*. 1978. ISBN 90 6196 152 1.

*MCT 87 S.G. VAN DER MEULEN & M. VELDHORST, *Torrix II*, ISBN 90 6196 153 X.

MCT 88 A. SCHRIJVER, *Matroids and linking systems*, 1977. ISBN 90 6196 154 8.

MCT 89 J.W. DE ROEVER, *Complex Fourier transformation and analytic functionals with unbounded carriers*, 1978. ISBN 90 6196 155 6.

MCT 90 L.P.J. GROENEWEGEN, *Characterization of optimal strategies in dynamic games*, 1981 . ISBN 90 6196 156 4.

MCT 91 J.M. GEYSEL, *Transcendence in fields of positive characteristic*, 1979. ISBN 90 6196 157 2.

MCT 92 P.J. WEEDA, *Finite generalized Markov programming*, 1979. ISBN 90 6196 158 0.

MCT 93 H.C. TIJMS & J. WESSELS (eds), *Markov decision theory*, 1977. ISBN 90 6196 160 2.

MCT 94 A. BIJLSMA, *Simultaneous approximations in transcendental number theory*, 1978. ISBN 90 6196 162 9.

MCT 95 K.M. VAN HEE, *Bayesian control of Markov chains*, 1978. ISBN 90 6196 163 7.

MCT 96 P.M.B. VITÁNYI, *Lindenmayer systems: Structure, languages, and growth functions*, 1980. ISBN 90 6196 164 5.

*MCT 97 A. FEDERGRUEN, *Markovian control problems; functional equations and algorithms*, . ISBN 90 6196 165 3.

MCT 98 R. GEEL, *Singular perturbations of hyperbolic type*, 1978. ISBN 90 6196 166 1.

MCT 99 J.K. LENSTRA, A.H.G. RINNOOY KAN & P. VAN EMDE BOAS, *Interfaces between computer science and operations research*, 1978. ISBN 90 6196 170 X.

MCT 100 P.C. BAAYEN, D. VAN DULST & J. OOSTERHOFF (eds), *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1*, 1979. ISBN 90 6196 168 8.

MCT 101 P.C. BAAYEN, D. VAN DULST & J. OOSTERHOFF (eds), *Proceedings bicentennial congress of the Wiskundig Genootschap, part 2*, 1979. ISBN 90 6196 169 6.

MCT 102 D. VAN DULST, *Reflexive and superreflexive Banach spaces*, 1978. ISBN 90 6196 171 8.

MCT 103 K. VAN HARN, *Classifying infinitely divisible distributions by functional equations*, 1978. ISBN 90 6196 172 6.

MCT 104 J.M. VAN WOUWE, *Go-spaces and generalizations of metrizability*, 1979. ISBN 90 6196 173 4.

MCT 105 R. HELMERS, *Edgeworth expansions for linear combinations of order statistics*, 1982. ISBN 90 6196 174 2.

MCT 106 A. SCHRIJVER (ed.), *Packing and covering in combinatorics*, 1979. ISBN 90 6196 180 7.

MCT 107 C. DEN HEIJER, *The numerical solution of nonlinear operator equations by imbedding methods*, 1979. ISBN 90 6196 175 0.

MCT 108 J.W. DE BAKKER & J. VAN LEEUWEN (eds), *Foundations of computer science III*, part *1*, 1979. ISBN 90 6196 176 9.

MCT 109 J.W. DE BAKKER & J. VAN LEEUWEN (eds), *Foundations of computer science III*, part *2*, 1979. ISBN 90 6196 177 7.

MCT 110 J.C. VAN VLIET, *ALGOL 68 transput*, part *I*: *Historical review and discussion of the implementation model*, 1979. ISBN 90 6196 178 5.

MCT 111 J.C. VAN VLIET, *ALGOL 68 transput*, part *II*: *An implementation model*, 1979. ISBN 90 6196 179 3.

MCT 112 H.C.P. BERBEE, *Random walks with stationary increments and renewal theory*, 1979. ISBN 90 6196 182 3.

MCT 113 T.A.B. SNIJDERS, *Asymptotic optimality theory for testing problems with restricted alternatives*, 1979. ISBN 90 6196 183 1.

MCT 114 A.J.E.M. JANSSEN, *Application of the Wigner distribution to harmonic analysis of generalized stochastic processes*, 1979. ISBN 90 6196 184 X.

MCT 115 P.C. BAAYEN & J. VAN MILL (eds), *Topological Structures II*, part *1*, 1979. ISBN 90 6196 185 5.

MCT 116 P.C. BAAYEN & J. VAN MILL (eds), *Topological Structures II*, part *2*, 1979. ISBN 90 6196 186 6.

MCT 117 P.J.M. KALLENBERG, *Branching processes with continuous state space*, 1979. ISBN 90 6196 188 2.

MCT 118 P. GROENEBOOM, *Large deviations and asymptotic efficiencies*, 1980. ISBN 90 6196 190 4.

MCT 119 F.J. PETERS, *Sparse matrices and substructures, with a novel implementation of finite element algorithms*, 1980. ISBN 90 6196 192 0.

MCT 120 W.P.M. DE RUYTER, *On the asymptotic analysis of large-scale ocean circulation*, 1980. ISBN 90 6196 192 9.

MCT 121 W.H. HAEMERS, *Eigenvalue techniques in design and graph theory*, 1980. ISBN 90 6196 194 7.

MCT 122 J.C.P. BUS, *Numerical solution of systems of nonlinear equations*, 1980. ISBN 90 6196 195 5.

MCT 123 I. YUHÁSZ, *Cardinal functions in topology - ten years later*, 1980. ISBN 90 6196 196 3.

MCT 124 R.D. GILL, *Censoring and stochastic integrals*, 1980. ISBN 90 6196 197 1.

MCT 125 R. EISING, *2-D systems, an algebraic approach*, 1980. ISBN 90 6196 198 X.

MCT 126 G. VAN DER HOEK, *Reduction methods in nonlinear programming*, 1980. ISBN 90 6196 199 8.

MCT 127 J.W. KLOP, *Combinatory reduction systems*, 1980. ISBN 90 6196 200 5.

MCT 128 A.J.J. TALMAN, *Variable dimension fixed point algorithms and triangulations*, 1980. ISBN 90 6196 201 3.

MCT 129 G. VAN DER LAAN, *Simplicial fixed point algorithms*, 1980. ISBN 90 6196 202 1.

MCT 130 P.J.W. TEN HAGEN et al., *ILP Intermediate language for pictures*, 1980. ISBN 90 6196 204 8.

MCT 131 R.J.R. BACK, *Correctness preserving program refinements: Proof theory and applications*, 1980. ISBN 90 6196 207 2.

MCT 132 H.M. MULDER, *The interval function of a graph*, 1980. ISBN 90 6196 208 0.

MCT 133 C.A.J. KLAASSEN, *Statistical performance of location estimators*, 1981. ISBN 90 6196 209 9.

MCT 134 J.C. VAN VLIET & H. WUPPER (eds), *Proceedings international conference on ALGOL 68*, 1981. ISBN 90 6196 210 2.

MCT 135 J.A.G. GROENENDIJK, T.M.V. JANSSEN & M.J.B. STOKHOF (eds), *Formal methods in the study of language*, part *I*, 1981. ISBN 90 6196 211 0.

MCT 136 J.A.G. GROENENDIJK, T.M.V. JANSSEN & M.J.B. STOKHOF (eds), *Formal methods in the study of language*, part *II*, 1981. ISBN 90 6196 213 7.

MCT 137 J. TELGEN, *Redundancy and linear programs*, 1981. ISBN 90 6196 215 3.

MCT 138 H.A. LAUWERIER, *Mathematical models of epidemics*, 1981. ISBN 90 6196 216 1.

MCT 139 J. VAN DER WAL, *Stochastic dynamic programming, successive approximations and nearly optimal strategies for Markov decision processes and Markov games*, 1980. ISBN 90 6196 218 8.

MCT 140 J.H. VAN GELDROP, *A mathematical theory of pure exchange economies without the no-critical-point hypothesis*, 1981.
ISBN 90 6196 219 6.

MCT 141 G.E. WELTERS, *Abel-Jacobi isogenies for certain types of Fano three-folds*, 1981.
ISBN 90 6196 227 7.

MCT 142 H.R. BENNETT & D.J. LUTZER (eds), *Topology and order structures*, part 1, 1981.
ISBN 90 6196 228 5.

MCT 143 H. J.M. SCHUMACHER, *Dynamic feedback in finite- and infinite dimensional linear systems*, 1981.
ISBN 90 6196 229 3.

MCT 144 P. EIJGENRAAM, *The solution of initial value problems using interval arithmetic. Formulation and analysis of an algorithm*, 1981.
ISBN 90 6196 230 7.

MCT 145 A.J. BRENTJES, *Multi-dimensional continued fraction algorithms*, 1981. ISBN 90 6196 231 5.

MCT 146 C. VAN DER MEE, *Semigroup and factorization methods in transport theory*, 1982. ISBN 90 6196 233 1.

MCT 147 H.H. TIGELAAR, *Identification and informative sample size*, 1982.
ISBN 90 6196 235 8.

MCT 148 L.C.M. KALLENBERG, *Linear programming and finite Markovian control problems*, 1983. ISBN 90 6196 236 6.

MCT 149 C.B. HUIJSMANS, M.A. KAASHOEK, W.A.J. LUXEMBURG & W.K. VIETSCH, (eds), *From A to Z, proceeding of a symposium in honour of A.C. Zaanen*, 1982. ISBN 90 6196 241 2.

MCT 150 M. VELDHORST, *An analysis of sparse matrix storage schemes*, 1982.
ISBN 90 6196 242 0.

MCT 151 R.J.M.M. DOES, *Higher order asymptotics for simple linear Rank statistics*, 1982. ISBN 90 6196 243 9.

MCT 152 G.F. VAN DER HOEVEN, *Projections of Lawless sequences*, 1982.
ISBN 90 6196 244 7.

MCT 153 J.P.C. BLANC, *Application of the theory of boundary value problems in the analysis of a queueing model with paired services*, 1982.
ISBN 90 6196 247 1.

MCT 154 H.W. LENSTRA, JR. & R. TIJDEMAN (eds), *Computational methods in number theory, part I*, 1982.
ISBN 90 6196 248 X.

MCT 155 H.W. LENSTRA, JR. & R. TIJDEMAN (eds), *Computational methods in number theory, part II*, 1982.
ISBN 90 6196 249 8.

MCT 156 P.M.G. APERS, *Query processing and data allocation in distributed database systems*, 1983.
ISBN 90 6196 251 X.

MCT 157 H.A.W.M. KNEPPERS, *The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic*, 1983.
ISBN 90 6196 252 8.

MCT 158 J.W. DE BAKKER & J. VAN LEEUWEN (eds), *Foundations of computer science IV, Distributed systems*, part 1, 1983.
ISBN 90 6196 254 4.

MCT 159 J.W. DE BAKKER & J. VAN LEEUWEN (eds), *Foundations of computer science IV, Distributed systems*, part 2, 1983.
ISBN 90 6196 255 0.

MCT 160 A. REZUS, *Abstract automat*, 1983.
ISBN 90 6196 256 0.

MCT 161 G.F. HELMINCK, *Eisenstein series on the metaplectic group, An algebraic approach;* 1983.
ISBN 90 6196 257 9.

An asterisk before the number means "to appear"