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Introduction to Option Pricing in a Securities Market I: Binary Models[1]

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A general mathematical model is studied for the finite binary securities market, where only a stock and a bond are traded. Basic notions such as self-financing strategies and arbitrage opportunities are characterized. Moreover, completeness of the model is shown, hedging strategies are determined and general pricing formulas for contingent claims are derived. Several examples of binary price processes, such as the nonhomogeneous binomial model and moving average processes are included and option prices are calculated.

The tools and the results are aimed at a limiting transition, to be carried out in the forthcoming parts of these papers, where the number of trading times tends to infinity and the mesh between the trading times tends to zero.

In our efforts to keep the present papers at a low technical level, the presentation of the subjects discussed in this part I is based only on simple algebraic arguments. This contrasts with the usual treatment based on a probabilistic approach, namely on the martingale approach. We intend, however, to present in the concluding part the probabilistic background and the probabilistic interpretation of the results discussed in the earlier parts.

1. Introduction

1.1. *Outline*

The main aim of this paper is to describe a general mathematical model for the finite binary securities market. In this model asset trading takes place over

[1] This is a revised version of the lecture notes (see Dzhaparidze and Van Zuijlen [9]) of the course given by this author at the Department of Mathematics, University of Nijmegen.

a finite number (say N) of periods and new prices are announced in the market at certain trading times $t_1, t_2, \ldots, t_N$. For simplicity we restrict ourselves to a market where only a stock and a bond are traded; the situation where several stocks are traded is algebraically more complicated but the essential ideas are the same. See HARRISON AND KREPS [13], VORST [24], TAQQU AND WILLINGER [23], WILLINGER AND TAQQU [26–29] and references therein.

In the binary market the prices of the risky asset, the stock, are supposed to jump from one value to one of two possible values at every trading time. The interest rate of the riskless asset, the bond, is allowed in our model to depend on the time interval. Although we restrict ourselves for the sake of simplicity to this situation, the bond price at time t_n can also be allowed to depend on the stock prices at the previous trading times $t_1, t_2, \ldots, t_{n-1}$, as can be seen easily by checking the arguments used in the paper. In the described binary setting we will characterize in a rigorous manner several important notions such as self-financing trading strategies and arbitrage opportunities by using basic ideas of HARRISON AND PLISKA [14]; see also FÖLLMER [12] and more recent references therein. Moreover we will show completeness of the model and determine hedging strategies. It can easily be seen that the restriction to the binary tree is essential in order to guarantee completeness. Finally we derive general pricing formulas for contingent claims and in particular for options. Several examples of binary price processes, such as the nonhomogeneous binomial model and moving average processes, are included and option prices are calculated. In particular the well-known Cox-Ross-Rubinstein pricing formula for options is reproduced as a special case (cf. COX, ROSS AND RUBINSTEIN [6], COX AND RUBINSTEIN [7], MERTON [17]).

The tools used and the results formulated in this paper (part I) are aimed at a limiting transition, to be carried out in the forthcoming paper, where the number of trading times tends to infinity and the mesh between trading times tends to zero. In this way a broad spectrum of pricing formulas for general underlying random price processes for the stock will be obtained. These limiting models will be restricted first to the classical and well-known geometric Brownian motion as in the famous Black-Scholes model (see BLACK AND SCHOLES [3] or KARATZAS AND SHREVE [16]) and to the so-called Merton model defined by return processes of the Poisson type (see MERTON [17] or COX AND ROSS [5]). However, other processes with not necessarily continuous paths may also appear, cf. e.g. AASE [1], BACK [2], DUFFIE AND PROTTER [8], PAGE AND SANDERS [21]. As a result of the limiting transition, the recurrent relations (3.6.1) and the relationship (3.3.2) below will turn into a corresponding partial differential equation, which coincides with the heat equation in the special case of the Black-Scholes limiting model, and the so-called Clark's formula respectively. Cf. OCONE AND KARATZAS [19], COLWELL, ELLIOTT AND KOPP [4], ELLIOTT AND FÖLLMER [11].

We want to emphasize here that essentially no probability theory is needed in the finite theory (cf. WILLIAMS [25], Chapter 15). However, probability theory (and in particular martingale theory) turns out to be very helpful in formulating and understanding the results. Moreover, the probabilistic characterization of the essential notions in the finite theory enables one to understand the dominant role in continuous market models of probability theory and in particular martingale theory and stochastic calculus. For the ease of the reader who is more specialized in econometrics rather than in stochastic calculus we will give in the concluding part of the present lecture notes the probabilistic background and probabilistic interpretations of the results discussed in the previous parts.

The paper is organized as follows. In Section 2 the binary model will be described. In Section 3 we study portfolio and value processes and characterize self-financing strategies and completeness. In Section 4 a special class of trading strategies, the so-called hedging strategies against contingent claims, will be defined. This will lead in Section 5 to the desired option pricing formulas in different situations. Finally, in Section 6 the class of binary markets excluding arbitrage opportunities (trading strategies of making profit without any initial endowment) is characterized.

1.2. Exponentials

For convenience we present formulas for the solution of linear difference equations, which will be used in future sections. These formulas are rather simple and can help in understanding the more complicated solutions of corresponding linear stochastic equations (Doleans-Dade equations) for limiting models.

We use throughout the usual notation $\Delta X_n = X_n - X_{n-1}$ for the difference operator applied to a certain sequence. It is easily verified that

$$(1.2.1)\quad \Delta(X_nY_n) = Y_{n-1}\Delta X_n + X_n\Delta Y_n = Y_{n-1}\Delta X_n + X_{n-1}\Delta Y_n + \Delta X_n \Delta Y_n.$$

For $U_n = Y_n/X_n$ with non-zero X_n we also have

$$(1.2.2)\qquad X_n\Delta U_n = \Delta Y_n - U_{n-1}\Delta X_n.$$

For a fixed sequence $\{X_n\}_{n=0,1,\ldots}$ with $X_0 = 0$, consider the linear difference equations

$$(1.2.3)\qquad \Delta Z_n = Z_{n-1}\Delta X_n, \quad n = 1,2,\ldots,$$

subject to the initial condition $Z_0 = 1$. These difference equations are equivalent to

$$(1.2.4)\qquad Z_n = 1 + \sum_{\nu=1}^{n} Z_{\nu-1}\Delta X_\nu, \quad n = 1,2,\ldots,$$

and have unique solutions given by

$$(1.2.5)\qquad Z_n = \prod_{\nu=1}^{n}(1 + \Delta X_\nu) \equiv \mathcal{E}(X)_n, \quad n = 1,2,\ldots.$$

Assume $\Delta X_n > -1$ for all $n = 1, 2, \ldots$ to get a positive solution. The symbol $\mathcal{E}$ is borrowed from the theory of stochastic differential equations. It denotes the solution of (1.2.3) (or, equivalently, (1.2.4)) and is called the stochastic, or Doleans-Dade exponential; see for instance JACOD [15], PROTTER [20], ELLIOTT [10] or SHIRYAYEV [22]. Note the following property of the Doleans-Dade exponential

$$\mathcal{E}(X)_n \mathcal{E}(Y)_n = \mathcal{E}(X + Y + [X, Y])_n \tag{1.2.6}$$

with

$$[X, Y]_n = \sum_{\nu=1}^{n} \Delta Y_\nu \Delta X_\nu,$$

which can be verified directly or by using (1.2.1).

By also using another arbitrary sequence $H_0, H_1, H_2, \ldots$, we can generalize the difference equation in (1.2.3) to

$$\Delta Z_n = \Delta H_n + Z_{n-1} \Delta X_n, \quad n = 1, 2, \ldots, \tag{1.2.7}$$

subject to the initial condition $Z_0 = H_0$. The unique solution of (1.2.7) is given by

$$Z_n = \mathcal{E}(X)_n \left(H_0 + \sum_{\nu=1}^{n} \frac{\Delta H_\nu}{\mathcal{E}(X)_\nu} \right) \equiv \mathcal{E}_H(X)_n, \quad n = 1, 2, \ldots. \tag{1.2.8}$$

Note that in the special case of the sequence of constants $H_n \equiv 1$, for $n = 0, 1, \ldots$, we have $\mathcal{E}_H(X)_n \equiv \mathcal{E}(X)_n$.

2. A BINARY MODEL

2.1. A market with two securities: a bond and a stock

Consider a *securities market* in which two *assets* (or *securities*) are traded at successive time periods marked by $0 = t_0 < t_1 < \cdots < t_N = T$. The moment $t_0 = 0$ is interpreted as the current date and $t_N = T < \infty$ as a terminal date which is considered to be fixed. It is usually said in this case that the time horizon T is finite and the trading takes place over N periods. The moments $t_0, t_1, \ldots, t_N$ are called trading times, since these are the dates at which new prices are announced in the market.

One of these assets, the *bond*, has price B_n over the period $[t_n, t_{n+1}), n = 0, 1, \ldots, N-1$ and B_N is the price announced at the terminal date $t_N = T$. Fix for simplicity $B_0 = 1$. These prices $\{B_n\}_{n=0,1,\ldots,N}$ are usually related to interest rates over the corresponding periods as follows: for $n = 1, \ldots, N$

$$B_n = r_n B_{n-1}$$

and

$$B_n = r_1 \cdots r_n \tag{2.1.1}$$

where $r_n \geq 1$ is one plus the interest rate in the interval $[t_{n-1}, t_n)$. Since the interest rates are riskless here, we call the bond the riskless asset.

Unlike the bond, the second asset, the *stock*, is risky in the sense that its price, denoted by S_n at time t_n, is allowed to evolve in time along more then one trajectory. The set of all admissible trajectories is described as follows. Currently (at $t_0 = 0$) the stock price S_0 is fixed in state s_{10}, say. At the next trading time t_1 a new price is announced and S_1 will occupy either state s_{11} or s_{21}. Schematically, the transition from the state s_{10} of S_0 to two alternative states s_{21} or s_{11} of S_1 can be portrayed as follows:

$$s_{10} = s_{10}\begin{matrix} \nearrow s_{21} \\ \searrow s_{11}. \end{matrix} \tag{2.1.2}$$

Generally, if the stock price at the trading time t_{n-1} is in state $s_{k,n-1}$ (i.e. S_{n-1} is in state $s_{k,n-1}$), then at the next trading time t_n it will be announced either in state $s_{2k,n}$ or $s_{2k-1,n}$. In other words, $s_{2k,n}$ and $s_{2k-1,n}$ are two alternative states of S_n, provided S_{n-1} was in state $s_{k,n-1}$. Schematically,

$$s_{k,n-1}\begin{matrix} \nearrow s_{2k,n} \\ \searrow s_{2k-1,n} \end{matrix}. \tag{2.1.3}$$

This transition scheme describes a so-called *binary* market in which the stock price is allowed to evolve along one of 2^N different trajectories: at the terminal date $t_N = T$ the stock price S_N occupies one of the states $\{s_{kN}\}_{k=1,\ldots,2^N}$. For convenience we assign to the trajectories the same index as to the states of S_N. This means that the following two statements are equivalent:

"the stock price evolves along the k^{th} trajectory of states" and

"at the terminal date $t_N = T$ the stock price S_N is in state s_{kN}".

In order to describe the stock price development along a particular trajectory, we use the following notations. For any number x we denote by $[x]$ the largest integer not exceeding x. Fix a positive integer n and then an integer k that belongs to the set of 2^n integers $\{1, \ldots, 2^n\}$, i.e. $k \in \{1, \ldots, 2^n\}$. With any such couple of integers we associate the sequence of integers $\{k_\nu(k,n)\}_{\nu=0,1,\ldots,n}$, where

$$k_\nu(k,n) = 1 + \left[\frac{k-1}{2^{n-\nu}}\right], \tag{2.1.4}$$

that is, the smallest integer exceeding $\frac{k-1}{2^{n-\nu}}$.

Consider now one of the possible 2^N trajectories of the stock price development, say the k^{th} trajectory. Along this trajectory the stock price is in the states $\{s_{k_n,n}\}_{n=0,1,\ldots,N}$ with $k_n = k_n(k,N)$ defined by (2.1.4). That is, at the trading time t_n the variable S_n is in state $s_{k_n,n}$.

In every state s_{kn} at time t_n with $n \in \{0, 1, \ldots, N\}$ the stock price takes on some numerical value, say $f(s_{kn})$. Since in different states the stock price may have a common numeric value, we see that although the state trajectories are all different, the value trajectories may overlap. Therefore a clear distinction is necessary between the states s_{kn} and the numerical values $f(s_{kn})$, see for instance Section 2.2 below. However, in order not to complicate the notations we shall always suppress the function f. This should cause no ambiguity, since it will be always clear from the context whether the states or the values of the stock price are meant.

In the sequel we will always assume that $f(s_{10}) = s > 0$ or simply $s_{10} > 0$. Also $f(s_{2k,n}) > f(s_{2k-1,n}) > 0$ or simply $s_{2k,n} > s_{2k-1,n} > 0$.

2.2. Binomial model

In this section we consider the following special cases of the so-called *binomial* model.

EXAMPLE 2.2.1. HOMOGENEOUS CASE. In this model the prices on the bond and stock are assumed to develop homogeneously in time in the sense described as follows.

Consider the special case of bond pricing (2.1.1) with $B_n = r^n$ for $n = 0, 1, \ldots, N$ where $r \geq 1$ is one plus the interest rate which remains constant, i.e. in (2.1.1) we have $r = r_1 = \cdots = r_N$.

Hence the *rate of returns* on the bond, defined at the consecutive trading times $t_1, \ldots, t_N$ as the sequence

$$\left\{ \frac{\Delta B_n}{B_{n-1}} \right\}_{n=1,\ldots,N}$$

is in fact the sequence of constant interest rates: for $n \in \{1, \ldots, N\}$

$$\frac{\Delta B_n}{B_{n-1}} = r - 1.$$

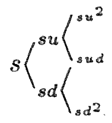
Similarly, we assume that the *rate of returns* on the stock, defined at the consecutive trading times $t_1, \ldots, t_N$ by

$$\left\{ \frac{\Delta S_n}{S_{n-1}} \right\}_{n=1,\ldots,N}$$

remains homogeneous as well: independent of $n \in \{1, \ldots, N\}$ the rate $\Delta S_n / S_{n-1}$ takes on one of the two values $u - 1$ or $d - 1$, with $u > d > 0$. Thus, if the stock price at the trading time t_{n-1} is S_{n-1}, then at the end of the following period it will be either $S_{n-1}u$ or $S_{n-1}d$, so that at the trading time t_n we have $S_n = S_{n-1}u$ or $S_{n-1}d$:

$$S_{n-1} \begin{matrix} \nearrow S_{n-1}u \\ \searrow S_{n-1}d. \end{matrix} \tag{2.2.1}$$

For example for $n = 2$ we obtain $s_{42} = su^2$, $s_{32} = s_{22} = sud$ and $s_{12} = sd^2$:



Generally, after n such displacements along 2^n different trajectories we arrive at one of the states s_{kn} with $k = 1, \ldots, 2^n$. In this case, we say that S_n is in state s_{kn} or S_n occupies state s_{kn}. Suppose now that the stock price evolved along one of the $\binom{n}{k}$ trajectories with k upward and $n - k$ downward displacements. Then the numerical value taken on by the variable S_n is $su^k d^{n-k}$. Hence only $n + 1$ different values of S_n may occur, namely the values $su^k d^{n-k}$ with $k = 0, 1, \ldots, n$.

EXAMPLE 2.2.2. NONHOMOGENEOUS CASE. Let the rate of returns on the bond be not necessarily homogeneous. Namely, let the bond prices evolve according to (2.1.1) with the corresponding interest rates

$$\frac{\Delta B_n}{B_{n-1}} = r_n - 1, \quad n = 1, \ldots, N. \tag{2.2.2}$$

Similarly, at the trading time t_n with $n = 1, \ldots, N$ the rate on the stock is defined by $\Delta S_n / S_{n-1}$, which takes on one of two values, either $u_n - 1$ or $d_n - 1$ with $u_n > d_n > 0$. Thus, the n^{th} displacement is described by the scheme (2.2.1) with u_n and d_n instead of u and d. For each $n = 1, 2, \ldots N$ and $k = 1, \ldots, 2^{n-1}$ we have (with $s = s_{10}$)

$$u_n = \frac{s_{2k,n}}{s_{k,n-1}}$$

and

$$d_n = \frac{s_{2k-1,n}}{s_{k,n-1}}.$$

2.3. Return processes

We define the *return process* on the bond $\mathcal{R} = \{\mathcal{R}_n\}_{n=0,\ldots,N}$ as follows. Set $\mathcal{R}_0 = 0$. At the trading time t_n with $n = 1, \ldots, N$ let $\mathcal{R}_n$ be the sum up to t_n of all interest rates on the bond, i.e.

$$\mathcal{R}_n = \sum_{\nu=1}^{n} \frac{\Delta B_\nu}{B_{\nu-1}}.$$

By definition $\Delta \mathcal{R}_n = \Delta B_n / B_{n-1}$ which equals the interest rate $r_n - 1$ over the interval $[t_{n-1}, t_n)$. Thus

$$\mathcal{R}_n = \sum_{\nu=1}^{n} (r_\nu - 1). \tag{2.3.1}$$

Since by (2.1.1) the bond price B_n at the trading time t_n with $n = 1, \ldots, N$ is related to the return process $\mathcal{R}$ as follows

$$B_n = \prod_{\nu=1}^{n} (1 + \Delta\mathcal{R}_\nu),$$

the sequence $\{B_n\}_{n=1,\ldots,N}$ satisfy the equations (1.2.4) with $Z = B$, $X = \mathcal{R}$ and $n = 1, \ldots, N$, i.e.

$$B_n = B_0 \mathcal{E}(\mathcal{R})_n. \tag{2.3.2}$$

Similarly, the rate $\Delta S_n/S_{n-1}$ on the stock over the interval $[t_{n-1}, t_n)$ defines the return process on the stock $R = \{R_n\}_{n=0,\ldots,N}$: set $R_0 = 0$ and $\Delta R_n = \Delta S_n/S_{n-1}$ so that for $n = 1, \ldots, N$

$$R_n = \sum_{\nu=1}^{n} \Delta R_\nu = \sum_{\nu=1}^{n} \frac{\Delta S_\nu}{S_{\nu-1}} = \sum_{\nu=1}^{n} \left(\frac{S_\nu}{S_{\nu-1}} - 1 \right).$$

With the notations of Section 1.2, we have again the inverse relationship

$$S_n = S_0 \mathcal{E}(R)_n.$$

It will be often convenient to use the notation $Z_n := 1 + \Delta R_n = S_n/S_{n-1}$ so that

$$R_n = \sum_{\nu=1}^{n} (Z_\nu - 1) \tag{2.3.3}$$

and

$$S_n = S_0 Z_1 \ldots Z_n. \tag{2.3.4}$$

REMARK 2.3.1. If at the trading time t_n the stock price S_n is in state s_{kn} for some $k = 1, \ldots, 2^n$, then the interest rate on the stock equals $\frac{s_{kn}}{s_{k_{n-1},n-1}} - 1$ with

$$k_{n-1} = k_{n-1}(k, n) = \left[\frac{k+1}{2} \right], \tag{2.3.5}$$

cf (2.1.4). In the present case Z_n with $n \in \{1, \ldots, N\}$ is in state

$$z_{kn} \equiv \frac{s_{kn}}{s_{k_{n-1},n-1}} \tag{2.3.6}$$

and R_n in state

$$\sum_{\nu=1}^{n} \left(\frac{s_{k_\nu,\nu}}{s_{k_{\nu-1},\nu-1}} - 1 \right),$$

since S_ν for $\nu \in \{1, \dots, n\}$ is in state $s_{k_\nu,\nu}$ with $k_\nu = k_\nu(k, n)$ given by (2.1.4).

EXAMPLE 2.3.2. BINOMIAL MODEL (CONTINUATION). The description of the binomial model in Section 2.2 is simplified by the fact that the 2^n states $z_{kn} = s_{kn}/s_{k_{n-1},n-1}$ of Z_n take on either the value u_n or d_n, depending on whether state s_{kn} has an even or an odd index k.

It will be useful to work with the *discounted* stock prices $\grave{S}_n = S_n/B_n$, which generate the return process $\grave{R} = \{\grave{R}_n\}_{n=0,\dots,N}$ with $\grave{R}_0 = 0$ and $\Delta\grave{R}_n = \Delta\grave{S}_n/\grave{S}_{n-1}$ for $n = 1, \dots, N$.

Similarly to (2.3.4) we have for $n = 1, \dots, N$

$$\grave{S}_n = S_0 \grave{Z}_1 \dots \grave{Z}_n = S_0 \mathcal{E}(\grave{R})_n$$

with

$$\grave{Z}_n = 1 + \Delta\grave{R}_n = \frac{\grave{S}_n}{\grave{S}_{n-1}}.$$

The states of the discounted variables are defined by the same considerations as in Remark 2.3.1. In particular, if at the trading time t_n with $n = 1, \dots, N$, the stock price S_n is in state s_{kn} for some $k = 1, \dots, 2^n$, then $\grave{Z}_n$ is in state

$$\grave{z}_{kn} \equiv \frac{\grave{s}_{kn}}{\grave{s}_{k_{n-1},n-1}} = \frac{z_{kn}}{r_n}.$$

Note finally that using the definitions of $\Delta\mathcal{R}_n$, ΔR_n and $\Delta\grave{R}_n$ we easily relate the return process on the discounted stock with the return processes on the bond and stock: for $n = 1, \dots, N$

$$\Delta\grave{R}_n = \frac{\Delta(R - \mathcal{R})_n}{1 + \Delta\mathcal{R}_n} = \frac{1}{r_n}\Delta(R - \mathcal{R})_n.$$

This relationship is equivalent to

$$\Delta R_n = \Delta\mathcal{R}_n + \Delta\grave{R}_n + \Delta\mathcal{R}_n \Delta\grave{R}_n$$

and therefore it can also be verified by using (1.2.6) which indeed yields

$$\mathcal{E}(R)_n = \frac{S_n}{S_0} = B_n \frac{\grave{S}_n}{S_0} = \mathcal{E}(\mathcal{R})_n \mathcal{E}(\grave{R})_n = \mathcal{E}(\mathcal{R} + \grave{R} + [\mathcal{R}, \grave{R}])_n.$$

2.4. Difference operators in a binary market

Apart from the difference operator in time, denoted throughout by Δ, it is also useful to define the difference operator D in the state space which applies to S_n and Z_n according to

DEFINITION 2.4.1. Fix the trading time t_n with $n \in \{1, \dots, N\}$ and let $k \in \{1, \dots, 2^{n-1}\}$. The claim

"S_{n-1} is in state $s_{k,n-1}$"

is equivalent to

"DS_n is in state $D_k(S_n) \equiv s_{2k,n} - s_{2k-1,n}$"

or

"DZ_n is in state $D_k(Z_n) \equiv D_k(S_n)/s_{k,n-1}$".

We can thus write $DZ_n = DS_n/S_{n-1}$.

DEFINITION 2.4.2. Fix the trading time t_n with $n \in \{2, \ldots, N\}$ and let $k \in \{1, \ldots, 2^{n-2}\}$. The claim

"S_{n-2} is in state $s_{k,n-2}$"

is equivalent to

"D^2S_n is in state $D_k^2(S_n) \equiv D_{2k}(S_n) - D_{2k-1}(S_n) = s_{4k,n} - s_{4k-1,n} - s_{4k-2,n} + s_{4k-3,n}$"

or

"D^2Z_n is in state $D_k^2(Z_n) \equiv D_{2k}(Z_n) - D_{2k-1}(Z_n) = \frac{D_{2k}(S_n)}{s_{2k,n-1}} - \frac{D_{2k-1}(S_n)}{s_{2k-1,n-1}}$".

The notions just introduced are particularly simple in case of the binomial model.

EXAMPLE 2.4.3. BINOMIAL MODEL. It is seen in Example 2.3.2 of the binomial model that at the trading time t_n with $n = 1, \ldots, N$ the 2^n states $\{z_{kn} = s_{kn}/s_{k_{n-1},n-1}\}_{k=1,\ldots,2^n}$ of Z_n take on either the value u_n or d_n, depending on whether state s_{kn} has even or odd index k. Since $DS_n = (u_n - d_n)S_{n-1}$, the following statement holds true:

STATEMENT 2.4.4. At each trading time t_n with $n = 1, \ldots, N$ the variable DZ_n is constant in the state space: all its 2^{n-1} statestake on the same numerical value $u_n - d_n$. Thus

$$DZ_n = u_n - d_n, \tag{2.4.1}$$

independent of the states $\{s_{k,n-1}\}_{k=1,\ldots,2^{n-1}}$ of the variable S_{n-1}.

Moreover, at each trading time t_n with $n = 2, \ldots, N$ the variable D^2Z_n is constant in the state space: all its 2^{n-2} states vanish, i.e.

$$D^2Z_n = 0 \tag{2.4.2}$$

for all the states $\{s_{k,n-2}\}_{k=1,\ldots,2^{n-2}}$ of the variable S_{n-2}.

2.5. Moving averages model

In order to describe a class of models which retain the property of the binomial model formulated in Statement 2.4.4, we introduce the variables $\epsilon_1, \epsilon_2, \dots, \epsilon_N$ as follows: ϵ_n is a variable which may occupy one of the following states $\{e_{kn}\}_{k=1,\dots,2^n}$ where for given numbers u_n and d_n with $u_n > d_n > 0$

$$e_{kn} = \begin{cases} u_n & \text{if } k \text{ is even} \\ d_n & \text{if } k \text{ is odd.} \end{cases}$$

Consider now the class of special models for the stock price development that is defined by (2.3.4) with the sequence $\{Z_n\}_{n=1,\dots,N}$ formed as follows. Set $Z_1 = \epsilon_1$ and for $n = 2, \dots, N$

$$Z_n = \epsilon_n + f_n(\epsilon_1, \dots, \epsilon_{n-1}), \tag{2.5.1}$$

with certain functions f_n of $n-1$ arguments. By our conventions this means that for $n = 1, \dots, N$ the states $\{z_{kn}\}_{k=1,\dots,2^n}$ of Z_n and the states $\{e_{k\nu}\}_{k=1,\dots,2^\nu}$ of ϵ_ν for $\nu = 1, \dots, n$ are related as follows:

$$z_{kn} = e_{k_n,n} + f_n(e_{k_1,1}, \dots, e_{k_{n-1},n-1}). \tag{2.5.2}$$

Note that $k_n = k$. Clearly, this model is portrayed by the following transition scheme: for $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$

$$s_{k,n-1} \begin{matrix} \nearrow s_{k,n-1}\, z_{2k,n} \\ \searrow s_{k,n-1}\, z_{2k-1,n} \end{matrix}$$

where

$$z_{2k,n} = u_n + f_n(e_{k_2,1}, \dots, e_{k_n,n-1}) \tag{2.5.3}$$

and

$$z_{2k-1,n} = d_n + f_n(e_{k_2,1}, \dots, e_{k_n,n-1}) \tag{2.5.4}$$

according to (2.5.2). It is easily verified that Statement 2.4.4 extends to the present case: we have again (2.4.1) and (2.4.2).

In the present paper we focus our attention on the concrete case of (2.5.1) where the functions f_n are linear. It means that

$$f_n(x_1, \dots, x_{n-1}) = \alpha_{n-1}x_1 + \cdots + \alpha_1 x_{n-1}$$

with some real parameters $\alpha_1, \alpha_2, \dots, \alpha_{N-1}$, i.e. $Z_1 = \epsilon_1$ and for $n = 2, 3, \dots, N$

$$Z_n = \epsilon_n + \alpha_{n-1}\epsilon_1 + \cdots + \alpha_1\epsilon_{n-1}. \tag{2.5.5}$$

Consider the following special examples.

2.6. Examples

EXAMPLE 2.6.1. 1^{st} ORDER MOVING AVERAGES MODEL. Consider the special case of model (2.5.1) and (2.5.5) with $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_{n-1} = 0$. Fix again the current stock price $S_0 = s$ and at the first trading time t_1 assume the transition (2.1.2) with $s_{21} = su_1$ and $s_{11} = sd_1$ (cf. the nonhomogeneous binomial model). Moreover, for $n = 2, 3, \ldots, N$ assume

$$s_{k,n-1}\begin{array}{l} \nearrow s_{k,n-1}(\alpha u_{n-1} + u_n) \\ \searrow s_{k,\, n-1}(\alpha u_{n-1}+d_n) \end{array} \quad \text{if } k \text{ is even}$$

and

$$s_{k,n-1}\begin{array}{l} \nearrow s_{k,n-1}(\alpha d_{n-1} + u_n) \\ \searrow s_{k,n-1}(\alpha d_{n-1} + d_n) \end{array} \quad \text{if } k \text{ is odd.}$$

As in the binomial case (see Example 2.4.3), it is useful to describe the present 1^{st} order moving averages model via its return process, since the variables $Z_n = S_n/S_{n-1}$ are quite simple: $Z_1 = \epsilon_1$ and

$$Z_n = \alpha\epsilon_{n-1} + \epsilon_n \tag{2.6.1}$$

for $n = 2, 3, \ldots, N$, according to (2.5.5). This means that depending on the state of the stock price S_n at the trading time t_n, Z_n takes on only one of 4 numeric values:

$(\alpha u_{n-1} + u_n)$ if S_n is in state $s_{4j+4,n}$

$(\alpha u_{n-1} + d_n)$ if S_n is in state $s_{4j+3,n}$

$(\alpha d_{n-1} + u_n)$ if S_n is in state $s_{4j+2,n}$

$(\alpha d_{n-1} + d_n)$ if S_n is in state $s_{4j+1,n}$

for all $j = 0, 1, \ldots, 2^{n-2} - 1$. Note the properties (2.4.1) and (2.4.2) for the present model.

EXAMPLE 2.6.2. 1^{st} ORDER AUTOREGRESSIVE MODEL. Consider another special case of the model (2.5.1) and (2.5.5) with $\alpha_k = \alpha^k$ for some parameter α. This model is called autoregressive because the sequence Z_n satisfies the following difference equations: $Z_1 = \epsilon_1$ and for $n = 2, 3, \ldots, N$

$$Z_n = \alpha Z_{n-1} + \epsilon_n. \tag{2.6.2}$$

3. PORTFOLIO AND VALUE PROCESS

3.1. Self-financing strategies

Suppose that one invests an amount $v \geq 0$ in the two assets described in Section 2.1. Let Ψ_n and Φ_n denote the number of shares of the bond and stock, respectively, owned by the investor at the trading time t_n, $n = 0, 1, \ldots, N$. The couple $\pi_n = (\Psi_n, \Phi_n)$ is called the investor's *portfolio* at time t_n, and the whole sequence $\pi = \{\pi_n\}_{n=0,1,\ldots,N}$ the *trading strategy*. Since the investor selects his portfolio at time t_n on the basis of the history of the price development in the market, the number of shares Ψ_n and Φ_n of the bond and stock he owns at time t_n may depend on the prices B_ν and S_ν with $\nu < n$, but not on the prices not yet announced, e.g. B_n and S_n. Observe that the components Ψ_n and Φ_n of a portfolio may become negative, which has to be interpreted as short-selling the bond or the stock.

The initial endowment has been $v \geq 0$, so that $v = \Psi_0 B_0 + \Phi_0 S_0$. Denote by $V_0(\pi)$ the right-hand side of the last equation, so that $v = V_0(\pi)$. Generally, the investor's wealth at time t_n with $n \in \{0, 1, \ldots, N\}$ is

$$V_n(\pi) \equiv \Psi_n B_n + \Phi_n S_n \tag{3.1.1}$$

or, equivalently,

$$\grave{V}_n(\pi) \equiv \Psi_n + \Phi_n \grave{S}_n$$

where $\grave{V}_n(\pi) = V_n(\pi)/B_n$ and $\grave{S}_n = S_n/B_n$ are the discounted wealth and stock price. The sequence $V = \{V_n(\pi)\}_{n=0,1,\ldots,N}$ is usually called the *value process* for a trading strategy π, since $V_n(\pi)$ represents the market value of the portfolio at time t_n held just before any changes are made in the portfolio. In the present paper we are interested in so-called *self-financing* strategies defined as follows:

DEFINITION 3.1.1. A trading strategy π is said to be *self-financing*, if the corresponding portfolio satisfies for $n = 1, \ldots, N$ the condition

$$B_{n-1}\Delta\Psi_n + S_{n-1}\Delta\Phi_n = 0 \tag{3.1.2}$$

or, equivalently,

$$\Delta\Psi_n + \grave{S}_{n-1}\Delta\Phi_n = 0.$$

It means that the construction is founded only on the initial endowment so that all changes in the portfolio values are due to capital gains during the trading and no infusion or withdrawal of funds is required.

DEFINITION 3.1.2. If the investor's wealth remains nonnegative, i.e. $V_n(\pi) \geq 0$ for all $n = 0, 1, \ldots, N$, then the self-financing trading strategy π is called *admissible*.

REMARK 3.1.2. As was noted above, a portfolio $\pi_n = (\Psi_n, \Phi_n)$ depends only on the history of the price development before the trading time t_n. In a binary market this means that it depends on the state of S_{n-1}: if S_n is in state s_{kn} for some $k \in \{1, \ldots, 2^n\}$, then Ψ_n and Φ_n are in the states $\Psi_n(s_{k_{n-1},n-1})$ and $\Phi_n(s_{k_{n-1},n-1})$, respectively. According to the discounted version of (3.1.1) the investor's discounted wealth at time t_n is then in state

$$\grave{v}_{kn}(\pi) = \Psi_n(s_{k_{n-1},n-1}) + \Phi_n(s_{k_{n-1},n-1})\grave{s}_{kn}. \tag{3.1.3}$$

Since S_ν for $\nu \in \{1, \dots, n\}$, is in state $s_{k_\nu, \nu}$ with $k_\nu = k_\nu(k,n)$ as in (2.1.4), the self-financing condition (3.1.2) in its discounted form means that

$$\begin{aligned} \Psi_n(s_{k_{n-1},n-1}) \quad &- \quad \Psi_{n-1}(s_{k_{n-2},n-2}) = \\ &- \quad -\grave{s}_{k_{n-1},n-1}\{\Phi_n(s_{k_{n-1},n-1}) - \Phi_{n-1}(s_{k_{n-2},n-2})\}. \end{aligned}$$

Recall that $k_{n-1} = k_{n-1}(k,n) = [\frac{k+1}{2}]$.

3.2. Integral representation

The property of a portfolio $\pi_n = (\Psi_n, \Phi_n)$ with $n = 1, \dots, N$ to depend only on the history of the price development before the trading time t_n, i.e. on the state of S_{n-1}, is called the *predictability* of a portfolio. This term is borrowed from the theory of stochastic calculus, where *predictable processes* play the rôle of integrands in stochastic integrals. The reader familiar with this theory, as well as with the theory of continuous trading in the spirit of, e.g. HARRISON AND PLISKA [14], Section 3, could trace the analogy of the representation (3.2.1) below and the integral representation of the discounted value process for a self-financing strategy as a stochastic integral with respect to the discounted stock price process. But again, a clear distinction has to be made between the states and the values of the processes involved.

PROPOSITION 3.2.1. *Let π be a self-financing strategy. Then the corresponding value process $V = \{V_n(\pi)\}_{n=0,1,\dots,N}$ with the initial endowment $v = V_0(\pi) \geq 0$ has the following representation: for all $n = 1, \dots, N$*

$$\grave{V}_n(\pi) = v + \sum_{\nu=1}^{n} \Phi_\nu \Delta \grave{S}_\nu. \tag{3.2.1}$$

PROOF. It suffices to prove that the discounted version of equation (3.1.2) is equivalent to

$$\Delta \grave{V}_n(\pi) = \Phi_n \Delta \grave{S}_n \quad \text{for} \quad n = 1, \dots, N. \tag{3.2.2}$$

But this is easily seen since by applying (1.2.1) to the discounted version of (3.1.1) we obtain

$$\Delta \grave{V}_n(\pi) = \Delta \Psi_n + \Delta(\Phi_n \grave{S}_n) = \Delta \Psi_n + \Delta \Phi_n \grave{S}_{n-1} + \Phi_n \Delta \grave{S}_n,$$

which equals $\Phi_n \Delta \grave{S}_n$ by the discounted version of equation (3.1.2). □

REMARK 3.2.2. The nondiscounted version of (3.2.2) is

$$\Delta V_n(\pi) = \Psi_n \Delta B_n + \Phi_n \Delta S_n.$$

REMARK 3.2.3. Fix $n \in \{1, \ldots, N\}$ and suppose that S_n occupies state s_{kn}. Since S_ν, for $\nu \in \{1, \ldots, n\}$, is in state $s_{k_\nu,\nu}$ with $k_\nu = k_\nu(k, n)$ as in (2.1.4), the representation (3.2.1) means that investor's discounted wealth at time t_n is in state

$$\grave{v}_{kn}(\pi) = v + \sum_{\nu=1}^{n} \Phi_\nu(s_{k_{\nu-1},\nu-1})(\grave{s}_{k_\nu,\nu} - \grave{s}_{k_{\nu-1},\nu-1}).$$

Arguing as in Section 3.1, one may interpret $\Psi_n B_{n-1} + \Phi_n S_{n-1}$ as the market value of the portfolio (Ψ_n, Φ_n) just after it has been selected at the trading time t_{n-1} with $n \in \{1, \ldots, N\}$. Therefore if no infusion or withdrawal of funds takes place, then apart from (3.1.1) one may expect that also the following relation holds:

$$V_{n-1}(\pi) = \Psi_n B_{n-1} + \Phi_n S_{n-1} \tag{3.2.3}$$

or, equivalently,

$$\grave{V}_{n-1}(\pi) = \Psi_n + \Phi_n \grave{S}_{n-1}.$$

It will be shown next that this is indeed true under the self-financing condition (3.1.2).

COROLLARY 3.2.4. *Under the self-financing condition* (3.1.2) *equations* (3.2.3) *holds for each* $n \in \{1, \ldots, N\}$.

PROOF. Equation (3.2.3) in its discounted form is obtained by subtracting (3.2.2) from the discounted version of (3.1.1). □

3.3. Clark's formula

In this section further analogy with the theory of continuous trading will be emphasized. In the latter theory the discounted value process, corresponding to a self-financing strategy, is represented as a stochastic integral with respect to the discounted stock price process, in which the integrand - the stock component of the portfolio - is of a special form, namely given by Clark's formula. See HARRISON AND PLISKA [14], formula (1.9), or OCONE AND KARATZAS [19]. As is seen in Corollary 3.3.2 to Proposition 3.3.1 below, the analogous formula in the case of a binary market is quite elementary. It is based on the simple use of the difference operators in the state space as defined in Section 2.4 (cf. the case of continuous trading where certain Malliavin derivatives occur, which are functional derivatives in the state space; see NUALART [18], Section 1.3.3).

PROPOSITION 3.3.1. *Under the self-financing condition* (3.1.2) *the components of the portfolio* $\pi_n = (\Psi_n, \Phi_n)$ *at trading time* t_n *with* $n = 1, \ldots, N$ *are given by*

$$\Psi_n B_n = \frac{V_n(\pi) DS_n - S_n DV_n(\pi)}{DS_n} \tag{3.3.1}$$

and

$$\Phi_n = \frac{DV_n(\pi)}{DS_n}. \tag{3.3.2}$$

PROOF. Fix $n \in \{1, \dots, N\}$. At the trading time t_{n-1} let S_{n-1} be in state $s_{k,n-1}$ with some $k \in \{1, \dots, 2^{n-1}\}$. By (3.1.3) the investor's discounted wealth at the next trading time t_n may then be in one of the following alternative states:

$$\grave{v}_{2k,n}(\pi) = \Psi_n(s_{k,n-1}) + \Phi_n(s_{k,n-1})\grave{s}_{2k,n}$$

or

$$\grave{v}_{2k-1,n}(\pi) = \Psi_n(s_{k,n-1}) + \Phi_n(s_{k,n-1})\grave{s}_{2k-1,n}.$$

By solving these equations with respect to $\Psi_n(s_{k,n-1})$ and $\Phi_n(s_{k,n-1})$, we get

$$\Psi_n(s_{k,n-1}) = \frac{\grave{v}_{2k-1,n}(\pi)\grave{s}_{2k,n} - \grave{v}_{2k,n}(\pi)\grave{s}_{2k-1,n}}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}} \tag{3.3.3}$$

and

$$\Phi_n(s_{k,n-1}) = \frac{\grave{v}_{2k,n}(\pi) - \grave{v}_{2k-1,n}(\pi)}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}}. \tag{3.3.4}$$

In non-discounted form we have

$$\Psi_n(s_{k,n-1})B_n = \frac{v_{2k-1,n}(\pi)s_{2k,n} - v_{2k,n}(\pi)s_{2k-1,n}}{s_{2k,n} - s_{2k-1,n}}$$

and

$$\Phi_n(s_{k,n-1}) = \frac{v_{2k,n}(\pi) - v_{2k-1,n}(\pi)}{s_{2k,n} - s_{2k-1,n}}.$$

By Definition 2.4.1 this is equivalent to (3.3.1) and (3.3.2). The proof is complete. □

Due to the Propositions 3.2.1 and 3.3.1 we obtain

COROLLARY 3.3.2. *Under the self-financing condition* (3.1.2) *we have for* $n = 1, \dots, N$

$$\grave{V}_n(\pi) = v + \sum_{\nu=1}^{n} \frac{DV_\nu(\pi)}{DS_\nu}\Delta\grave{S}_\nu$$

and

$$\Delta\grave{V}_n(\pi) = \frac{DV_n(\pi)}{DS_n}\Delta\grave{S}_n.$$

3.4. Risk neutral probabilities

Fix a trading time t_n for some $n = 1, \dots, N$ and consider the particular branch of the discounted price tree for the stock

$$\grave{s}_{k,n-1} \begin{cases} \grave{s}_{2k,n} \\ \grave{s}_{2k-1,n} \end{cases}$$

for some $k = 1, \dots, 2^{n-1}$. It is easy to express $\grave{s}_{k,n-1}$ as a linear combination of the next states $\grave{s}_{2k,n}$ and $\grave{s}_{2k-1,n}$ by solving the equation

$$\grave{s}_{k,n-1} = x\grave{s}_{2k,n} + (1-x)\grave{s}_{2k-1,n}$$

with respect to the unknown x. The solution is $x = (\grave{s}_{k,n-1} - \grave{s}_{2k-1,n})/(\grave{s}_{2k,n} - \grave{s}_{2k-1,n})$. In order to exhibit the dependence of it on the time and state indices, we use throughout the following notations:

$$p_{2k,n} = \frac{\grave{s}_{k,n-1} - \grave{s}_{2k-1,n}}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}} \tag{3.4.1}$$

and

$$p_{2k-1,n} = \frac{\grave{s}_{2k,n} - \grave{s}_{k,n-1}}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}}. \tag{3.4.2}$$

So, for any trading time t_n and any state of $\grave{S}_n$, the numeric values of $p_{2k,n}$ and $p_{2k-1,n}$ satisfy

$$p_{2k,n} + p_{2k-1,n} = 1. \tag{3.4.3}$$

With these notations we get

$$\grave{s}_{k,n-1} = p_{2k,n}\grave{s}_{2k,n} + p_{2k-1,n}\grave{s}_{2k-1,n} \tag{3.4.4}$$

for all $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$.

REMARK 3.4.1. In Section 6 a class of markets is considered in which the possibilities $\grave{s}_{k,n-1} \le \grave{s}_{2k-1,n}$ or $\grave{s}_{k,n-1} \ge \grave{s}_{2k,n}$ are excluded by certain arguments having clear economical meaning. Under these circumstances the numerical values of $p_{2k,n}$ and $p_{2k-1,n}$ are positive. Due to (3.4.3) this couple of numbers will be interpreted in the concluding part of this paper as certain (conditional) probabilities and the right hand side of (3.4.4) will be interpreted as the corresponding (conditional) expectation. Equations (3.4.4) will then ensure a martingale property of the sequence $\{\grave{S}_n\}_{n=0,\dots,N}$ relative to the above probabilities. Moreover, it will be shown in Section 3.6 below that every discounted value process $\grave{V}(\pi)$ corresponding to a self-financing strategy π satisfies (3.4.4) (with $\grave{s}$ substituted by $\grave{v}$, cf. (3.6.1)) and thus constitutes a martingale with respect to the same probabilities. Due to this property, the numerical values of (3.4.1) and (3.4.2) are usually called *risk neutral probabilities.*

Meanwhile we turn back to the general situation in which negative values of $p_{2k,n}$ or $p_{2k-1,n}$ are not excluded, still calling them risk neutral probabilities, however.

3.5. Risk neutral probabilities for the moving averages model

It is useful here to rewrite (3.4.1) and (3.4.2) in the following alternative form: for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$

$$p_{2k,n} = \frac{r_n - z_{2k-1,n}}{z_{2k,n} - z_{2k-1,n}} \tag{3.5.1}$$

and

$$p_{2k-1,n} = \frac{z_{2k,n} - r_n}{z_{2k,n} - z_{2k-1,n}} \tag{3.5.2}$$

where r_n is one plus the interest rate as in (2.1.1) and $\{z_{kn}\}_{k=1,\ldots,2^n}$ are the states of Z_n as in (2.3.6). In the case of the moving averages model the latter states are specified by (2.5.3) and (2.5.4), so that (3.5.1) and (3.5.2) yield

$$p_{2k,n} = \frac{r_n - d_n - \sum_{\nu=1}^{n-1} \alpha_{n-\nu} e_{k_{\nu+1},\nu}}{u_n - d_n} \tag{3.5.3}$$

and

$$p_{2k-1,n} = \frac{u_n - r_n + \sum_{\nu=1}^{n-1} \alpha_{n-\nu} e_{k_{\nu+1},\nu}}{u_n - d_n} \tag{3.5.4}$$

for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$. Consider two special examples.

EXAMPLE 3.5.1. BINOMIAL MODEL. For each $n = 1, \ldots, N$ substitute $\alpha_1 = \cdots = \alpha_{n-1} = 0$ in (3.5.3) and (3.5.4). In this special case we find that $p_{2k,n}$ and $p_{2k-1,n}$ take on the same numerical value for all $k = 1, \ldots, 2^{n-1}$, namely

$$p_{2k,n} = \frac{r_n - d_n}{u_n - d_n} \tag{3.5.5}$$

and

$$p_{2k-1,n} = \frac{u_n - r_n}{u_n - d_n}. \tag{3.5.6}$$

EXAMPLE 3.5.2. 1^{st} ORDER MOVING AVERAGES MODEL. For $n = 1, \ldots, N$ put $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_{n-1} = 0$ in (3.5.3) and (3.5.4). We obtain (since $k_n = k$)

$$p_{2k,n} = \frac{r_n - d_n - \alpha e_{k,n-1}}{u_n - d_n} \tag{3.5.7}$$

and

$$p_{2k-1,n} = \frac{u_n - r_n + \alpha e_{k,n-1}}{u_n - d_n} \tag{3.5.8}$$

for all $k = 1, \dots, 2^{n-1}$. So, for even indices the latter equations yield

$$p_{4k,n} = \frac{r_n - d_n - \alpha u_{n-1}}{u_n - d_n}$$

and

$$p_{4k-1,n} = \frac{u_n - r_n + \alpha u_{n-1}}{u_n - d_n},$$

while for odd indices they yield

$$p_{4k-2,n} = \frac{r_n - d_n - \alpha d_{n-1}}{u_n - d_n}$$

and

$$p_{4k-3,n} = \frac{u_n - r_n + \alpha d_{n-1}}{u_n - d_n}$$

with $k = 1, \dots, 2^{n-2}$.

3.6. Recurrent equations

In the next proposition it will be proved that the states of the discounted value process satisfy certain equations similar to (3.4.4).

PROPOSITION 3.6.1. *In a binary market a trading strategy π is self-financing if and only if for every $n \in \{1, \dots, N\}$ the states $\{\grave{v}_{kn}(\pi)\}_{k=1,\dots,2^n}$ of the discounted value process $\grave{V} = \{\grave{V}_n(\pi)\}_{n=0,1,\dots,N}$ at the trading time t_n and the states $\{\grave{v}_{k,n-1}(\pi)\}_{k=1,\dots,2^{n-1}}$ at the previous trading time t_{n-1} are related by the equation*

$$\grave{v}_{k,n-1}(\pi) = p_{2k,n}\grave{v}_{2k,n}(\pi) + p_{2k-1,n}\grave{v}_{2k-1,n}(\pi), \tag{3.6.1}$$

where $p_{2k,n}$ and $p_{2k-1,n}$ are given by (3.4.1) *and* (3.4.2).

PROOF. (i) By the discounted version of (3.2.3)

$$\grave{v}_{k,n-1}(\pi) = \Psi_n(s_{k,n-1}) + \Phi_n(s_{k,n-1})\grave{s}_{k,n-1} \tag{3.6.2}$$

for $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$. These equations and (3.3.3)-(3.3.4) yield

$$\begin{aligned} \grave{v}_{k,n-1}(\pi) &= \frac{\grave{v}_{2k-1,n}(\pi)\grave{s}_{2k,n} - \grave{v}_{2k,n}(\pi)\grave{s}_{2k-1,n}}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}} \\ &+ \frac{\grave{v}_{2k,n}(\pi) - \grave{v}_{2k-1,n}(\pi)}{\grave{s}_{2k,n} - \grave{s}_{2k-1,n}}\grave{s}_{k,n-1}. \end{aligned} \tag{3.6.3}$$

It can be easily verified that the coefficients of $\grave{v}_{2k,n}(\pi)$ and $\grave{v}_{2k-1,n}(\pi)$ are indeed given by (3.4.1) and (3.4.2).

(ii) Conversely, (3.6.1), (3.4.1) and (3.4.2) imply (3.6.3), and hence (3.6.2). By Definition (3.1.1), (3.6.2) implies (3.1.3), i.e. the strategy in question is indeed self-financing. □

The equations (3.6.1) are *recurrent* in the sense that if the states of the value process are given at the terminal date $t_N = T$, then working backwards one can determine the states at the previous trading times step by step, each time using the equations (3.6.1). For more details on the solution of these equations, see Section 4.1 below. Thus Proposition 3.6.1 tells us that the integral transformation (3.2.1) of the discounted stock price process $\grave{S}$ to the discounted value processes $\grave{V}(\pi)$ corresponding to self-financing strategies π, preserves the recurrence. It is perhaps interesting to note the following simple consequence of this property, though we will make use of it only in the forthcoming part II.

COROLLARY 3.6.2. *Let π be a self-financing strategy and let $\grave{V}(\pi)$ be its discounted value process. Then for each $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$*

$$p_{2k,n} = \frac{\grave{v}_{k,n-1}(\pi) - \grave{v}_{2k-1,n}(\pi)}{\grave{v}_{2k,n}(\pi) - \grave{v}_{2k-1,n}(\pi)}$$

and

$$p_{2k-1,n} = \frac{\grave{v}_{2k,n}(\pi) - \grave{v}_{k,n-1}(\pi)}{\grave{v}_{2k,n}(\pi) - \grave{v}_{2k-1,n}(\pi)}.$$

PROOF. Put $p_{2k-1,n} = 1 - p_{2k,n}$ in (3.6.1) and solve this equation with respect to $p_{2k,n}$ to obtain the first equation. The second one is obvious. □

4. HEDGING STRATEGY

4.1. Solution of the recurrent equations for the value process

Recall that if at the trading time t_n with a fixed $n \in \{1, \dots, N\}$ the stock price S_n is in state s_{kn} for some $k \in \{1, \dots, 2^n\}$, then S_ν for $\nu = 1, \dots, n$ is in state $s_{k_\nu,\nu}$ where $k_\nu = k_\nu(k, n)$ is given by (2.1.4). For each such n and k, and for $\nu < n$ define

$$P_{n|\nu}(k) = p_{k_{\nu+1},\nu+1} \cdots p_{k_n,n}. \tag{4.1.1}$$

In particular we have

$$P_{n|0}(k) \equiv P_{kn} = p_{k_1,1} \cdots p_{k_n,n} \tag{4.1.2}$$

where p_{kn} is defined by (3.4.1) if k is even and by (3.4.2) if k is odd. Note that $p_{kn} = P_{n|n-1}(k)$.

Define a variable $\grave{H}_N$ which is allowed to occupy one of the 2^N states $\{\grave{h}_{kN}\}_{k=1,\dots,2^N}$ by means of a certain function H (cf. the *payoff function* of Section 5.2) so that

$$\grave{H}_N = \frac{H(S_N)}{B_N},$$

i.e.

$$\grave{h}_{kN} = \frac{H(s_{kN})}{B_N}, \quad k = 1, \ldots, 2^N.$$

Consider then the system of recurrent equations (cf. (3.4.4) and (3.6.1))

$$\grave{x}_{k,n-1} = p_{2k,n}\grave{x}_{2k,n} + p_{2k-1,n}\grave{x}_{2k-1,n} \tag{4.1.3}$$

for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$, subject to the boundary conditions

$$\grave{x}_{kN} = \grave{h}_{kN}, \quad k = 1, \ldots, 2^N. \tag{4.1.4}$$

In order to get all the solution to this system of equations, start with substituting $n = N$ in (4.1.3) and determine $\{\grave{x}_{k,N-1}\}_{k=1,\ldots,2^{N-1}}$. Working backwards in this manner, after n such steps the solutions $\{\grave{x}_{k,N-n}\}_{k=1,\ldots,2^{N-n}}$ are obtained with

$$\grave{x}_{k,N-n} = \sum_{2^n(k-1)<j\leq 2^n k} P_{N|N-n}(j)\grave{h}_{jN}. \tag{4.1.5}$$

This procedure is terminated after N steps and yields

$$\grave{x}_{10} = \sum_{k=1}^{2^N} P_{kN}\grave{h}_{kN}. \tag{4.1.6}$$

Consider two simple applications of this formula.

EXAMPLE 4.1.1. In case of the boundary conditions $\grave{x}_{kN} \equiv 1$ for $k = 1, \ldots, 2^N$ we have by (3.4.3) that

$$\sum_{k=1}^{2^N} P_{kN} = 1. \tag{4.1.7}$$

EXAMPLE 4.1.2. Consider now the boundary conditions $\grave{x}_{kN} = \grave{s}_{kN}$ for $k = 1, \ldots, 2^N$. Then by using (3.4.4) repeatedly, we get

$$\sum_{k=1}^{2^N} P_{kN}\grave{s}_{kN} = s. \tag{4.1.8}$$

REMARK 4.1.3. In the situation mentioned in Remark 3.4.1, we have $0 < p_{2k,n} < 1$ and $0 < p_{2k-1,n} < 1$ for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$. Then by the Definitions (4.1.1) and (4.1.2) and by (3.4.3) we have that $P_{n|\nu}(k)$ and P_{kn} are positive numbers with the sum over $k = 1, \ldots, 2^n$ equal to 1. Therefore in the concluding part of this course they will be interpreted as certain probabilities, and the sums in (4.1.5) - (4.1.8) as corresponding expectations.

4.2. Examples of the solution

In this section we present two examples of the solution to the system of equations (4.1.3), subject to the boundary conditions (4.1.4) with $\grave{h}_{kN} = H(s_{kN})/B_N$ for $k = 1, \dots, 2^N$ and some function H.

EXAMPLE 4.2.1. BINOMIAL MODEL. Consider first the homogeneous case. For given numbers u, d and r, denote

$$p_u = \frac{r-d}{u-d}$$

and

$$p_d = \frac{u-r}{u-d}$$

which satisfy $p_u + p_d = 1$; cf. (3.5.5) and (3.5.6). Define the sequence of functions $\{f_n\}_{n=0,1,\dots,N}$ with

$$f_n(x) = \sum_{j=0}^{n} \binom{n}{j} p_u^j p_d^{n-j} H(xu^j d^{n-j}).$$

Note that $f_0(x) = H(x)$. With these notations (4.1.5) and (4.1.6) are reduced to

$$\grave{x}_{k,N-n} = r^{-N} f_n(s_{k,N-n}), \quad k = 1, \dots, 2^{N-n}, \tag{4.2.1}$$

and

$$\grave{x}_{10} = r^{-N} f_N(s). \tag{4.2.2}$$

Thus, after n steps backwards we arrive at the set of the solutions

$$\{\grave{x}_{k,N-n}\}_{k=1,\dots,2^{N-n}}$$

which are the states of $r^{-N} f_n(S_{N-n})$.

Consider next the nonhomogeneous case. For given numbers u_n, d_n and r_n with $n = 1, \dots, N$ denote

$$p_{u_n} = \frac{r_n - d_n}{u_n - d_n}$$

and

$$p_{d_n} = \frac{u_n - r_n}{u_n - d_n}$$

which satisfy $p_{u_n} + p_{d_n} = 1$; cf. (3.5.5) and (3.5.6). Define the sequence of functions $\{f_n\}_{n=1,\dots,N}$ with $f_0(x) = H(x)$ and

$$f_n(x) = \sum_{\varepsilon_{N-n+1} \cdots \varepsilon_N} p_{\varepsilon_{N-n+1}} \cdots p_{\varepsilon_N} H(x\varepsilon_{N-n+1} \cdots \varepsilon_N) \tag{4.2.3}$$

for $n = 1, \ldots, N$, where the summation extends over the n indices

$$\varepsilon_{N-n+1}, \ldots, \varepsilon_N$$

that are binary: ε_j takes on the value u_j or the value d_j. Then the states $\{\mathring{x}_{k,N-n}\}_{k=1,\ldots,2^{N-n}}$ of $f_n(S_{N-n})$ again satisfy (4.2.1) but now with the function (4.2.3) divided by $r_1 \cdots r_N$ instead of r^N:

$$\mathring{x}_{k,N-n} = \frac{f_n(s_{k,N-n})}{r_1 \cdots r_N}.$$

In particular

$$\mathring{x}_{10} = \sum_{\varepsilon_1 \cdots \varepsilon_N} \frac{p_{\varepsilon_1} \cdots p_{\varepsilon_N}}{r_1 \ldots r_N} H(x\varepsilon_1 \cdots \varepsilon_N). \tag{4.2.4}$$

EXAMPLE 4.2.2. 1^{st} ORDER MOVING AVERAGES MODEL. We retain here the notations of the previous case of nonhomogeneous model. Furthermore, denote

$$p_{u_n|\varepsilon_n} = \frac{r_n - d_n - \alpha\varepsilon_n}{u_n - d_n}$$

and

$$p_{d_n|\varepsilon_n} = \frac{u_n - r_n + \alpha\varepsilon_n}{u_n - d_n}$$

so that $p_{u_n|\varepsilon_n} + p_{d_n|\varepsilon_n} = 1$; cf. (3.5.7) and (3.5.8). For $n = 1, \ldots, N$ define the following functions of two variables

$$\begin{aligned} f_n(x, y) = \quad & \sum_{\varepsilon_{N-n+1} \cdots \varepsilon_N} p_{\varepsilon_{N-n+1}|y} p_{\varepsilon_{N-n+2}|\varepsilon_{N-n+1}} \cdots p_{\varepsilon_N|\varepsilon_{N-1}} \times \\ & H(x(\varepsilon_{N-n+1} + \alpha y)(\varepsilon_{N-n+2} + \alpha\varepsilon_{N-n+1}) \cdots (\varepsilon_N + \alpha\varepsilon_{N-1})). \end{aligned}$$

Then the solutions $\{\mathring{x}_{k,N-n}\}_{k=1,\ldots,2^{N-n}}$ coincide with the states of

$$f_n(S_{N-n}, \epsilon_{N-n}).$$

Thus

$$\mathring{x}_{2k,N-n} = \frac{f_n(s_{2k,N-n}, u_{N-n})}{r_1 \cdots r_N}$$

and

$$\mathring{x}_{2k-1,N-n} = \frac{f_n(s_{2k-1,N-n}, d_{N-n})}{r_1 \cdots r_N}$$

for $k = 1, \ldots, 2^{n-1}$. In particular

$$\begin{aligned} \mathring{x}_{10} = \quad & \textstyle\sum_{\varepsilon_1 \cdots \varepsilon_N} \frac{p_{\varepsilon_1} p_{\varepsilon_2|\varepsilon_1} \cdots p_{\varepsilon_N|\varepsilon_{N-1}}}{r_1 \cdots r_N} H(s(\varepsilon_1 + \alpha s)(\varepsilon_2 + \alpha\varepsilon_1) \\ & \cdots (\varepsilon_N + \alpha\varepsilon_{N-1})). \end{aligned} \tag{4.2.5}$$

4.3. Completeness of a binary market

Under the circumstances described in Section 3, consider an investor who i willing to invest now (at $t = 0$) in the bond and the stock in order to attai at the terminal date $t_N = T$ a certain wealth, say W_N, by trading over I periods without infusion or withdrawal of funds. Knowing the conditions in th market, i.e. knowing the 2^N possible trajectories of the stock price developmen up to the terminal date $t_N = T$ (which correspond as usual to the state $\{s_{kN}\}_{k=1,\ldots,2^N}$ of the stock price S_N), the investor determines the wealth h desires to attain at the terminal date $t_N = T$ by evaluating each of thes possibilities. In this way W_N is interpreted as a variable which is in one of th 2^N possible states: in state w_{kN} say, if the stock price is in state s_{kN}. In othe words, W_N is a certain function of S_N, say $W_N = W(S_N)$ and $w_{kN} = W(s_{kN}$ for $k = 1, \ldots, 2^N$.

DEFINITION 4.3.1. A binary market is called *complete* if there exists a sel financing trading strategy which attains any desired wealth $W_N = W(S_N)$ wit a certain initial endowment.

It will be shown in this section that a binary market is complete and tha for a fixed W_N the required initial endowment and self-financing strategy determined.

Fix a wealth W_N desired at the terminal date $t_N = T$ by fixing its state $\{w_{kN}\}_{k=1,\ldots,2^N}$. For the discounted wealth $\grave{W}_N = W_N/B_N$ with the coı responding states $\{\grave{w}_{kN} = w_{kN}/B_N\}_{k=1,\ldots,2^N}$, solve the recurrent equatior (4.1.3), subject to the boundary conditions

$$\grave{x}_{kN} = \grave{w}_{kN}, \quad k = 1, \ldots, 2^N. \tag{4.3.1}$$

To this end, use formula (4.1.5) with $\{\grave{h}_{kN}\}_{k=1,\ldots,2^N}$ substituted by $\{\grave{w}_{kN}\}_{k=1,\ldots,2^N}$. Denote the solutions by $\{\grave{w}_{kn}\}_{k=1,\ldots,2^n}$ for $n = 0, 1, \ldots, N-$ In particular we obtain

$$w_{10} = \sum_{k=1}^{2^N} P_{kN} \grave{w}_{kN} \tag{4.3.2}$$

with P_{kN} defined by (4.1.2); cf. (4.1.6). Finally, for each $n = 0, 1, \ldots, N$ denot by $\grave{W}_n$ the variable with states $\{\grave{w}_{kn}\}_{k=1,\ldots,2^n}$.

We are now in a position to describe a particular trading strategy whic attains the desired wealth $W_N$with the initial endowment (4.3.2).

DEFINITION 4.3.2. A specific trading strategy π whose value process $\grave{V}(\pi)$: $\{\grave{V}_n(\pi)\}_{n=0,1,\ldots,N}$ is such that $\grave{V}_N(\pi) = W_N$ is called the *hedging strate* against the desired wealth $W_N = W(S_N)$.

This strategy is uniquely determined by the following procedure:

- Given the price development of the bond and stock, determine the numerical values of all $\{p_{kn}\}_{k=1,\dots,2^n}$ for $n = 1,\dots,N$ by the formulas (3.4.1) and (3.4.2).

- Given the wealth W_N, determine recurrently the numerical values of the $\{\grave{w}_{kn}\}_{k=1,\dots,2^n}$ for $n = 0,1,\dots,N$ by formula (4.1.5), as described above.

- Determine, in particular, the numerical values of the $\{P_{kn}\}_{k=1,\dots,2^n}$ for $n = 0,1,\dots,N$ by formula (4.1.2), and then the initial endowment w_{10} by formula (4.3.2).

- Invest currently the amount w_{10} in Ψ_0 and Φ_0 shares of the bond and the stock respectively, where Ψ_0 and Φ_0 are calculated as follows. Determine the numeric values of Ψ_1 and Φ_1 by the formulas (3.3.3) and (3.3.4) with $n = k = 1, \grave{v}_{21}(\pi) = \grave{w}_{21}$ and $\grave{v}_{11}(\pi) = \grave{w}_{11}$ and identify (Ψ_0, Φ_0) with (Ψ_1, Φ_1).

- During the first period keep the portfolio unchanged, i.e. keep Ψ_1 shares of the bond and Φ_1 shares of the stock, in order to get the wealth $\grave{V}_1(\pi)$ determined by formula (3.1.1) with $n = 1$, which coincides with $\grave{W}_1$.

- If at the trading time t_1 the stock price s_{21} is announced, then during the second period keep $\Psi_2(s_{21})$ shares of the bond and $\Phi_2(s_{21})$ shares of the stock. These numbers of shares are determined by the formulas (3.3.3) and (3.3.4) with $n = 2, k = 2, \grave{v}_{42}(\pi) = \grave{w}_{42}$ and $\grave{v}_{32}(\pi) = \grave{w}_{32}$.
However, if the announced stock price is s_{11}, then keep $\Psi_2(s_{11})$ shares of the bond and $\Phi_2(s_{11})$ shares of the stock, again determined by the formulas (3.3.3) and (3.3.4) but now with $n = 2, k = 1, \grave{v}_{22}(\pi) = \grave{w}_{22}$ and $\grave{v}_{12}(\pi) = \grave{w}_{12}$. Then the wealth $\grave{V}_2(\pi)$ attained at the trading time t_2 is determined by formula (3.1.3) with $n = 2$. It coincides with $\grave{W}_2$.

- If during the forthcoming trading periods the portfolio will be held which is always determined by the same formulas (3.3.3) and (3.3.4) with $\{\grave{v}_{kn}(\pi) = \grave{w}_{kn}\}_{k=1,\dots,2^n}$ for the integers n increasing up to N, then the value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\dots,N}$ will develop in such a way that $\grave{V}_n(\pi)$ will coincide with $\grave{W}_n$ for $n = 1,\dots,N$. In particular, at the terminal date $t_N = T$ the wealth W_N will be attained.

The trading strategy just described is indeed the hedging strategy against the wealth W_N and the construction is unique. Clearly, this strategy is applicable to any desired wealth of the type $W_N = W(S_N)$. This proves the completeness of a binary market. We have

PROPOSITION 4.3.3. *A binary market is complete: any wealth $W_N = W(S_N)$, desired at the terminal date $t_N = T$, is attainable with an initial endowment uniquely defined by* (4.3.2). *If for $n = 0,1,\dots,N$ the states $\{\grave{w}_{kn}\}_{k=1,\dots,2^n}$ of $\grave{W}_n = W(S_n)$ are the solutions of the recurrent equations* (4.1.3), *subject to the boundary condition* (4.3.1), *then the so-called hedging strategy against W_N is uniquely determined by selecting the portfolio according to* (3.3.3) *and* (3.3.4) *with $\{\grave{v}_{kn}(\pi) = \grave{w}_{kn}\}_{k=1,\dots,2^n}$ for $n = 0,1,\dots,N$.*

REMARK 4.3.4. Consider the situation mentioned in Remark 4.1.3, where all the weights $\{P_{kN}\}_{k=1,\dots,2^N}$ in the sum in (4.3.2) are positive numbers. Then this sum is positive and the possibility is excluded of attaining a positive wealth at the terminal date $t_N = T$ with a nonpositive initial endowment; more details will be given in the concluding part of this course. Besides, in that case the hedging strategy of the present section is not only self-financing but also admissible in the sense of Definition 3.1.2, since the corresponding value process – the solution of the above recurrent equations – cannot be negative. It will be shown in the concluding part of this set of papers that the hedging strategy against W_N is also optimal in a certain sense.

REMARK 4.3.5. It is not hard to see that a strategy of selecting a constant portfolio (a portfolio selected at the current date $t_0 = 0$ and kept unchanged over consecutive periods of trading) is the hedging strategy against some W_N, if and only if W_N is representable as a linear combination of the bond and stock prices B_N and S_N at $t_N = T$, which means that there are constants a and b such that

$$W_N = bB_N + aS_N. \tag{4.3.3}$$

Indeed, in order to attain at $t_N = T$ the wealth W_N of the form (4.3.3), one has to invest the amount $v = b + as$ ($s = S_0$ as usual) by buying currently, at $t_0 = 0$, b shares of the bond and a shares of the stock. In other words, the investor has to select at $t_0 = 0$ the portfolio $\pi_0 = (b, a)$. If this portfolio is kept unchanged, i.e. $\pi_n = (b, a)$ for $n = 0, 1, \dots, N$, then the corresponding value process $V(\pi) = \{V_n(\pi)\}_{n=0,1,\dots,N}$ is given by

$$V_n(\pi) = bB_n + aS_n \tag{4.3.4}$$

so that we also have the desired equality $V_N(\pi) = W_N$.

Note that according to the assertion of Proposition 4.3.3 the discounted value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\dots,N}$ with $\grave{V}_n(\pi) = b + a\grave{S}_n$ solves the recurrent equations (4.1.3) subject to the boundary conditions $\grave{x}_{kN} = b + a\grave{s}_{kN}$, $k = 1, \dots, 2^N$. Hence the strategy $\pi = \{\pi_n\}_{n=0,1,\dots,N}$ of holding the constant portfolio $\pi_n = (b, a)$ is the hedging strategy against W_N as in (4.3.3).

5. OPTION PRICING

5.1. *European call option*

Suppose that today, at time $t = 0$, we sign a contract giving us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time*. If the stock price S_T is below the exercise price at maturity, i.e. $S_T \le K$, then the contract is worthless to us. On the other hand, if $S_T > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price S_T. Thus this option, called the *European call option*, yields a profit at maturity T equal to

$$\max\{0, S_T - K\} = (S_T - K)^+. \tag{5.1.1}$$

The function (5.1.1) of the stock price S_T at maturity T, is called the *payoff function* for the European call option.

Now, how much would we be willing to pay at time $t = 0$ for a ticket which gives the right to buy at maturity $t = T$ one share of stock with exercise price K? To put this in another way, what is a fair price to pay at time $t = 0$ for the ticket? This question has a direct answer only in the trivial case where the exercise price K lays outside the range of all possible values of S_T. If, for instance, the exercise price is too high (exceeding all possible values of S_T), then clearly the contract is worthless and the fair price of the ticket is 0. On the other hand, consider another extreme situation in which the exercise price K is too low; i.e. below all possible values of S_T, so that $S_T \geq K$. Obviously, the payoff function (5.1.1) then reduces to

$$S_T - K. \tag{5.1.2}$$

In order to determine the fair price to pay at time $t = 0$ for the ticket which entitles us to the payoff (5.1.2), suppose that instead of buying the ticket we act as follows. Currently, at $t = 0$, we borrow an amount $\grave{K}$ and buy one stock, where $\grave{K} = K/B_T$ is the exercise price, discounted by the bond price B_T at maturity $t = T$. In terms of Section 4, we select the special portfolio $\pi_0 = (-\grave{K}, 1)$ by investing an amount $v = V_0(\pi) = s - \grave{K}$, where as usual $s = S_0 > 0$ is the stock price at $t = 0$. We keep consequently this portfolio unchanged over N periods of trading i.e. we hold the constant portfolio $\pi_n = (-\grave{K}, 1)$ for all $n = 0, 1, \ldots, N$. Compare with Remark 4.3.5 where $a = 1$ and $b = -\grave{K}$. Our wealth at maturity $t_N = T$ will then amount to $V_T(\pi) = S_T - K$, which equals to the payoff (5.1.2); see (4.3.3) and (4.3.4) with $a = 1$ and $b = -\grave{K}$ (for convenience, we identified $V_N(\pi)$, S_N and B_N with $V_T(\pi)$, S_T and B_T). In other words, holding the call with payoff (5.1.2) is exactly equivalent to holding the above mentioned portfolio which requires, as we have seen, the investment of the amount $v = S_0 - \grave{K}$. Any option price different from $S_0 - \grave{K}$ would enable either the option seller or the option buyer to make a sure profit without any risk (in terms of Section 6 - there would be an arbitrage opportunity or a "free lunch"). It is natural, therefore, to conclude that the fair price of the equivalent call is $S_0 - \grave{K}$.

The discounted value process $\grave{V}$ for the present strategy π of keeping the constant portfolio $(-\grave{K}, 1)$ is determined by substituting in the discounted version of (4.3.4) $a = 1$ and $b = -\grave{K}$, which yields

$$\grave{V}_n(\pi) = \grave{S}_n - \grave{K}, \quad n = 0, 1, \ldots, N.$$

It solves the recurrent equations (4.1.3), subject to the boundary conditions

$$\grave{x}_{kN} = \grave{s}_{kN} - \grave{K} = \frac{s_{kN} - K}{B_N}, \quad k = 1, \ldots, 2^N,$$

see Remark 4.3.5. Hence the strategy of holding the portfolio $\pi_n = (-\grave{K}, 1)$ for all $n = 0, 1, \ldots, N$ is the *hedging strategy* against the payoff (5.1.2).

REMARK 5.1.1. The fair price of a call will be denoted throughout by C. Thus if the exercise price K in (5.1.1) is too high (exceeding all possible values of S_T), then $C = 0$. On the other hand, if the exercise price K is too low (below all possible values of S_T), then $C = S_0 - \grave{K}$. This is strictly positive if at least one of the states of $\grave{S}_T$ is strictly below $S_0 = s$, so that also $\grave{K} < S_0$.

Although we have used above the "no arbitrage" principle to obtain fair option prices, we want to emphasize here that we will not need that principle in the next section where we give a formal definition of the fair price of a contingent claim.

5.2. Pricing a contingent claim

A contract with some fixed payoff function H_N, where H_N is a nonnegative variable with possible states $\{h_{kN} = H(s_{kN})\}_{k=1,\ldots,2^N}$ (not necessarily of form (5.1.1) or (5.1.2)) is called a *contingent claim.* The European call option is thus a special contingent claim with payoff (5.1.1).

The fair price of a contingent claim is defined by the same considerations as in the special case of the linear payoff (5.1.2). The procedure used in the preceding section can be described as follows:

(i) construct the hedging strategy against the contingent claim in question, which duplicates the payoff;

(ii) determine the initial wealth needed for construction in (i);

(iii) equate this initial wealth to the fair price of the contingent claim.

According to Proposition 4.3.3 the hedging strategy against the contingent claim with a payoff function H_N consists in holding the portfolio with the components (3.3.3) and (3.3.4), where $\{\grave{v}_{kn}(\pi)\}_{k=1,\ldots,2^n}$ for $n = 0, 1, \ldots, N$ are determined by solving the recurrent equations (4.1.3), subject to the boundary conditions (4.1.4) with

$$\left\{\grave{h}_{kN} = \frac{H(s_{kN})}{B_N}\right\}_{k=1,\ldots,2^N}. \tag{5.2.1}$$

This strategy indeed duplicates the payoff, since $V_N(\pi) = H_N$. It requires the initial wealth $V_0(\pi)$ which is calculated according to (4.1.6):

$$V_0(\pi) = \sum_{k=1}^{2^N} P_{kN}\grave{h}_{kN}$$

with P_{kN} and $\grave{h}_{kN}$ defined by (4.1.2) and (5.2.1), respectively. The fair price $C = C(H_N)$ of the contingent claim with the payoff function H_N is thus defined by

$$C(H_N) = \sum_{k=1}^{2^N} P_{kN}\grave{h}_{kN}. \tag{5.2.2}$$

The European call option (5.1.1), in particular, has a special payoff function depending only on the stock price at maturity $t_N = T$ and its fair price is

$$C = \sum_{k=1}^{2^N} P_{kN}(\grave{s}_{kN} - \grave{K})^+. \tag{5.2.3}$$

REMARK 5.2.1. In the situation described in Remark 4.3.4 we have $C \geq s - \grave{K}$, so that the fair price of the European call option is not less then the fair price of the contingent claim with linear payoff (5.1.2) which has been determined in Remark 5.1.1.

REMARK 5.2.2. *The call-put parity.* Suppose that today, at time $t = 0$, we sign a contract which gives us the right to sell, at specified time T one share of a stock at a specified price K. If the stock price S_T is above the exercise price at maturity, i.e. $S_T \geq K$, the contract is worthless. On the other hand, if $S_T < K$, we can exercise our option: we can sell one share of the stock at the fixed price K and then buy it immediately in the market for the price S_T. Thus this option, called the *European put option*, yields the following profit at maturity T:

$$\max\{0, K - S_T\} = (K - S_T)^+.$$

This function $H(S_T) = (K - S_T)^+$ of the stock price S_T at maturity T, is called the *payoff function* for the European put option. The fair price of a put is denoted by P, and according to our theory we have

$$P = \sum_{k=1}^{2^N} P_{kN}(\grave{K} - \grave{s}_{kN})^+.$$

Since

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K,$$

we obtain by (4.1.7) and (4.1.8) the so-called call-put parity relationship:

$$C - P = \sum_{k=1}^{2^N} P_{kN}(\grave{s}_{kN} - \grave{K}) = s - \grave{K}.$$

REMARK 5.2.3. *Option strategies.* We can easily use the linearity of the summation operator in (5.2.3) to evaluate options formed as certain linear combinations of contingent claims. For instance prices of *spreads* in the exercise price are obtained as linear combinations of individual option prices. Also the price of for instance a *strangle* with payoff function

$$H(S_T) = |S_T - K|$$

turns out to be $C + P$ since

$$(S_T - K)^+ + (K - S_T)^+ = |S_T - K|.$$

5.3. Examples of option pricing

To conclude, the general option pricing formula (5.2.3) is applied to the special models of Section 4.2.

EXAMPLE 5.3.1. BINOMIAL MODEL. Consider first the homogeneous case in which formula (4.1.6) reduces to equation (4.2.2). Thus in this case the fair price of the European call option equals

$$C = \frac{1}{r^N} \sum_{j=0}^{N} \binom{N}{j} p_u^j p_d^{N-j} (su^j d^{N-j} - K)^+ \tag{5.3.1}$$

which is the well-known Cox-Ross-Rubinstein option pricing formula; see COX, ROSS AND RUBINSTEIN [6].

As for the nonhomogeneous case, the fair price of the European call option is determined by applying formula (4.2.4). So it equals

$$C = \sum_{\varepsilon_1 \ldots \varepsilon_N} \frac{p_{\varepsilon_1} \cdots p_{\varepsilon_N}}{r_1 \ldots r_N} (s\varepsilon_1 \ldots \varepsilon_N - K)^+. \tag{5.3.2}$$

EXAMPLE 5.3.2. 1st ORDER MOVING AVERAGE. In this case we have (4.2.5) so that the fair price of the European call option equals

(5.3.3)

$$C = \sum_{\varepsilon_1 \ldots \varepsilon_N} \frac{p_{\varepsilon_1} p_{\varepsilon_2|\varepsilon_1} \cdots p_{\varepsilon_N|\varepsilon_{N-1}}}{r_1 \ldots r_N} (s(\varepsilon_1 + \alpha s)(\varepsilon_2 + \alpha\varepsilon_1) \ldots (\varepsilon_N + \alpha\varepsilon_{N-1}) - K)^+.$$

6. MARKETS EXCLUDING ARBITRAGE OPPORTUNITIES

6.1. Arbitrage opportunities

It will be shown in Proposition 6.1.2 below that under natural restrictions on the stock price development in a binary market the possibility is excluded of making a profit without any initial endowment. More precisely, the possibility is excluded of constructing a self-financing trading strategy $\pi = \{\pi_n\}_{n=0,1,\ldots,N}$ with an initial endowment

$$v = V_0(\pi) = \Psi_0 + \Phi_0 S_0 = 0 \tag{6.1.1}$$

and with a value process which attains at t_N only states with nonnegative values, i.e. $V_N(\pi) \geq 0$, and at least one state with a strictly positive value. A strategy of selecting such a portfolio is called an *arbitrage opportunity*. Thus, an arbitrage opportunity represents a riskless plan of making a profit without any investment: there is no threat of loss, since $V_N(\pi) \geq 0$ and moreover, there is a chance of a pure gain in case the stock price develops along one of those trajectories for which $V_N(\pi)$ attains one of the states with a strictly positive value. It is, therefore, economically meaningful (and mathematically useful, as we will see below), to treat separately the security markets which exclude arbitrage opportunities. These markets also allow for a certain economic equilibrium as will be seen in the concluding part of the present course.

REMARK 6.1.1. Since negative values of the components of the portfolio $\pi_0 = (\Psi_0, \Phi_0)$ at $t = 0$ are not excluded, there is no real reason for keeping the initial endowment $v = V_0(\pi) = \Psi_0 + \Phi_0 S_0$ nonnegative, as in Section 3.1. We may drop this assumption and take into consideration the possibility of an arbitrary initial endowment v, not necessarily $v = V_0(\pi) \geq 0$. The equality to 0 in (6.1.1) has to be replaced then by the sign $\leq$. Note also that the trivial strategy with $\pi_n \equiv (0, 0)$ is not an arbitrage opportunity.

PROPOSITION 6.1.2. *A binary market excludes arbitrage opportunities if and only if the states* $\{\grave{s}_{kn}\}_{k=1,\dots,2^n}$ *of the discounted stock prices* $\grave{S}_n, n = 1, \dots, N$, *take on values which satisfy the inequalities*

$$\grave{s}_{2k-1,n} < \grave{s}_{k,n-1} < \grave{s}_{2k,n} \tag{6.1.2}$$

for $k = 1, \dots, 2^{n-1}$ *and* $n = 1, \dots, N$.

It is instructive to prove the assertion of Proposition 6.1.2 first in the special case of the one-period model where $N = 1$, because one can easily trace in that case the main idea behind the proof.

LEMMA 6.1.3. (i) *If the transition* (2.1.2) *of the stock price is such that*

$$\grave{s}_{11} < s < \grave{s}_{21}, \tag{6.1.3}$$

then for every portfolio $\pi_0 = \pi_1 = (\psi, \phi)$, *whose value process satisfies*

$$\{\grave{v}_{21}(\pi) > 0 \text{ and } \grave{v}_{11}(\pi) \geq 0\} \tag{6.1.4}$$

or

$$\{\grave{v}_{21}(\pi) \geq 0 \text{ and } \grave{v}_{11}(\pi) > 0\},$$

we have $v = V_0(\pi) > 0$, *so that there is no arbitrage opportunity.*
(ii) *Conversely, if* (6.1.3) *is violated, so that* $s \leq \grave{s}_{11}$ *or* $s \geq \grave{s}_{21}$, *then the trading strategy of selecting portfolio*

$$\pi_0 = \pi_1 = (-\phi s, \phi) \tag{6.1.5}$$

with some

$$\begin{cases} \phi > 0 & if\ s \leq \grave{s}_{11} \\ \phi < 0 & if\ s \geq \grave{s}_{21} \end{cases}$$

is an arbitrage opportunity.

PROOF. (i) Suppose that (6.1.3) and (6.1.4) hold. If $\phi = 0$, then $V_0(\pi) = \grave{V}_1(\pi) = \psi$ is strictly positive by (6.1.4). We get the desired strict inequality also for $\phi \neq 0$, since

$$V_0(\pi) = \psi + \phi s > \begin{cases} \psi + \phi\grave{s}_{21} = \grave{v}_{21}(\pi) \geq 0 & \text{if } \phi < 0 \\ \psi + \phi\grave{s}_{11} = \grave{v}_{11}(\pi) \geq 0 & \text{if } \phi > 0. \end{cases}$$

(ii) To see that the portfolio (6.1.5) yields an arbitrage opportunity, observe that $v = V_0(\pi) = -\phi s + \phi s = 0$ and that $\grave{V}_1(\pi) = \phi(\grave{S}_1 - s)$ can be in two alternative states: either $\grave{v}_{21}(\pi) = \phi(\grave{s}_{21} - s)$ or $\grave{v}_{11}(\pi) = \phi(\grave{s}_{11} - s)$. By the definition of ϕ these states satisfy condition (6.1.4). Hence, the portfolio (6.1.5) yields an arbitrage opportunity. □

COROLLARY 6.1.4. *The assertion in Proposition 6.1.2 holds in the special case where $N = 1$.*

PROOF. We note that the notion of self-financing is empty in case $N = 1$. (i) The sufficiency of (6.1.3) for excluding arbitrage opportunities is indeed reduced to the statement (i) that under (6.1.3) and (6.1.4) we have $v = V_0(\pi) > 0$ whatever the components ψ and ϕ of a portfolio $\pi_0 = \pi_1 = (\psi, \phi)$, so that the self-financing strategy attaining (6.1.4) cannot be an arbitrage opportunity. □

6.2. The proof of Proposition 6.1.2

Let us turn back to the general multi-period model of a binary market. It is useful to extend first in a separate lemma the arguments used in the course of proving assertion (i) of Lemma 6.1.3.

LEMMA 6.2.1. *Let the discounted value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,\ldots,N}$ of a self-financing strategy π be such that for some trading time $t_n, n \in \{1, \ldots, N\}$, we have $\grave{V}_n(\pi) \geq 0$ and at least one of the states $\{\grave{v}_{kn}(\pi)\}_{k=1,\ldots,2^n}$ takes on a strictly positive value. Then under condition (6.1.2) $\grave{V}_{n-1}(\pi)$ is of the same type: $\grave{V}_{n-1}(\pi) \geq 0$ and at least one of the states $\{\grave{v}_{k,n-1}(\pi)\}_{k=1,\ldots,2^{n-1}}$ takes on a strictly positive value.*

PROOF. Recall that under the self-financing condition (3.1.2) the states $\{\grave{v}_{k,n-1}(\pi)\}_{k=1,\ldots,2^{n-1}}$ at the fixed trading time t_{n-1} satisfy

$$\grave{v}_{k,n-1}(\pi) = \Psi_n(s_{k,n-1}) + \Phi_n(s_{k,n-1})\grave{s}_{k,n-1}, \tag{6.2.1}$$

cf. (3.6.2). If $\Phi_n(s_{j,n-1}) = 0$ for some j, then

$$\grave{v}_{j,n-1}(\pi) = \Psi_n(s_{j,n-1}) = \grave{v}_{2j,n}(\pi) = \grave{v}_{2j-1,n}(\pi) \geq 0,$$

with strict inequality if $\grave{v}_{2j,n}(\pi) = \grave{v}_{2j-1,n}(\pi) > 0$. If $\Phi_n(s_{j,n-1}) \neq 0$, then by assumption (6.1.2) and by (6.2.1)

$$\grave{v}_{j,n-1}(\pi) > \begin{cases} \Psi_n(s_{j,n-1}) + \Phi_n(s_{j,n-1})\grave{s}_{2j,n} & = \quad \grave{v}_{2j,n}(\pi) \geq 0 \\ & \quad\ \text{if } \Phi_n(s_{j,n-1}) < 0 \\ \Psi_n(s_{j,n-1}) + \Phi_n(s_{j,n-1})\grave{s}_{2j-1,n} & = \quad \grave{v}_{2j-1,n}(\pi) \geq 0 \\ & \quad\ \text{if } \Phi_n(s_{j,n-1}) > 0. \end{cases}$$

The lemma is proved. □

Proof of Proposition 6.1.2. (i) Suppose that condition (6.1.2) holds and suppose we have a self-financing strategy for which $\grave{V}_N(\pi) \geq 0$, with at least one strictly positive state. We will show that this strategy requires a positive investment $v = V_0(\pi) > 0$ and therefore cannot be an arbitrage opportunity. To prove this claim we apply Lemma 6.2.1. If $\grave{V}_N(\pi)$ is of the above type, then $\grave{V}_{N-1}(\pi)$ is of the same type, and so on. Hence the states of $\grave{V}_1(\pi)$ satisfy (6.1.4) and by Lemma 6.1.3 (i) we have $v = V_0(\pi) > 0$.
(ii) As in the special case of $N = 1$ (cf. Lemma 6.1.3 (ii)), the necessity of (6.1.2) for $N > 1$ will be proved by contradiction: it will be shown that there is an arbitrage opportunity, provided (6.1.2) is violated. Suppose that for some $m \in \{1, \ldots, N\}$ we have

(6.2.2)
$$\grave{s}_{j,m-1} \leq \grave{s}_{2j-1,m}$$

or

$$\grave{s}_{j,m-1} \geq \grave{s}_{2j,m}$$

for some $j \in \{1, \ldots, 2^{m-1}\}$. Consider the self-financing strategy of selecting the following portfolio. At the trading times preceding t_m we take the trivial portfolio:

$$\pi_n = (\Psi_n, \Phi_n) = (0, 0), \quad \text{for } \ n < m.$$

At t_m the portfolio $\pi_m = (\Psi_m, \Phi_m)$ is selected according to

(6.2.3)
$$(\Psi_m(s_{k,m-1}), \Phi_m(s_{k,m-1})) = \begin{cases} (0,0) & \text{if } k \neq j \\ (-\phi s_{j,m-1}, \phi) & \text{if } k = j \end{cases}$$

where $\phi > 0$ if $\grave{s}_{j,m-1} \leq \grave{s}_{2j-1,m}$ and $\phi < 0$ if $\grave{s}_{j,m-1} \geq \grave{s}_{2j,m}$.

Next, at t_{m+1} the portfolio $\pi_{m+1} = (\Psi_{m+1}, \Phi_{m+1})$ is nonzero only when $k = j$ and

$$(6.2.4)\quad \begin{cases} (\Psi_{m+1}(s_{2j,m}), \Phi_{m+1}(s_{2j,m})) &= (\phi(\grave{s}_{2j,m} - \grave{s}_{j,m-1}), 0) \\ (\Psi_{m+1}(s_{2j-1,m}), \Phi_{m+1}(s_{2j-1,m})) &= (\phi(\grave{s}_{2j-1,m} - \grave{s}_{j,m-1}), 0) \end{cases}$$

Finally, for $n > m+1$ we do not change the portfolio $\pi_n = (\Psi_n, \Phi_n)$ anymore (note that this strategy is self-financing). Choosing for this strategy, the investor is not taking any risk before and after the trading time t_{m-1}. Awaiting the stock price announcement at the trading time t_{m-1}, the investor acts only if state $s_{j,m-1}$ occurs, by selecting the portfolio according to (6.2.3) and choosing the appropriate sign of ϕ. It will be shown that this strategy is an arbitrage opportunity. Observe that the corresponding value process evolves as follows: $V_n(\pi) = 0$ for all $n = 0, 1, \ldots, m-1$, while $\grave{V}_m(\pi) \geq 0$ is in one of the following states. Fix $k \in \{1, \ldots, 2^{m-1}\}$. If $k \neq j$, then both $\grave{v}_{2k,m}$ and $\grave{v}_{2k-1,m}$ vanish. If $k = j$ we have

$$\grave{v}_{2k,m}(\pi) = \phi(\grave{s}_{2j,m} - \grave{s}_{j,m-1})$$

and

$$\grave{v}_{2k-1,m}(\pi) = \phi(\grave{s}_{2j-1,m} - \grave{s}_{j,m-1}).$$

Therefore either

$$\{\grave{v}_{2j,m}(\pi) > 0 \text{ and } \grave{v}_{2j-1,m}(\pi) \geq 0\}$$

or

$$\{\grave{v}_{2j,m}(\pi) \geq 0 \text{ and } \grave{v}_{2j-1,m}(\pi) > 0\},$$

depending on whether $\grave{s}_{j,m-1} \leq \grave{s}_{2j-1,m}$ or $\grave{s}_{j,m-1} \geq \grave{s}_{2j,m}$. Hence, there is no threat of loss. Moreover, if at t_{m-1} the stock price is in state $s_{j,m-1}$, then a pure gain is attained (unless the next state is either $s_{2j-1,m}$ and $\grave{s}_{j,m-1} = \grave{s}_{2j-1,m}$, or $s_{2j,m}$ and $\grave{s}_{j,m-1} = \grave{s}_{2j,m}$). Since in the subsequent trading intervals no risk is taken, the investor's wealth remains nonnegative, and thus the above strategy is indeed an arbitrage opportunity. The proof of Proposition 6.1.2 is complete. □

REMARK 6.2.2. Condition (6.1.2) can be written in the following alternative way: for $n = 1, \ldots, N$

$$(6.2.5)\quad \max\{z_{2k-1,n} : k = 1, \ldots, 2^{n-1}\} < r_n < \min\{z_{2k,n} : k = 1, \ldots, 2^{n-1}\}$$

where $z_{kn} = s_{kn}/s_{k_{n-1},n-1}$ (cf. (2.3.6)).

6.3. No arbitrage for moving averages models

The condition of no arbitrage is simply tractable in the following

EXAMPLE 6.3.1. BINOMIAL MODEL. Since $z_{2k-1,n} = d_n$ and $z_{2k,n} = u_n$ whatever the index k (cf. Example 2.4.3), the condition (6.2.5) for no arbitrage opportunities reduces to

$$d_n < r_n < u_n \quad \text{for} \quad n = 1, 2, \ldots, N.$$

For the general moving averages model of Section 2.5, however, the condition of no arbitrage opportunities is quite complicated, since (6.2.5) means that

$$\begin{cases} d_1 < r_1 < u_1, \\ \max\{\alpha_{n-1} e_{k_1,1} + \cdots + \alpha_1 e_{k_{n-1},n-1}\} + d_n < r_n < \\ \min\{\alpha_{n-1} e_{k_1,1} + \cdots + \alpha_1 e_{k_{n-1},n-1}\} + u_n \\ \qquad\qquad\qquad\qquad \text{for } n = 2, 3, \ldots, N, \end{cases}$$

(cf. (2.5.2) and (2.5.5)) where the maximum and the minimum are taken over all possible values of the sequence $\{e_{k_\nu,\nu}\}_{\nu=1,\ldots,n}$.

EXAMPLE 6.3.2. 1^{st} ORDER MOVING AVERAGES MODEL. In the present model of a market with $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_{n-1} = 0$ (cf. (2.6.1)) arbitrage opportunities are excluded only if

$$|\alpha| < \frac{u_n - d_n}{u_{n-1} - d_{n-1}} \quad \text{for} \quad n = 2, 3, \ldots, N.$$

In this case the above condition of no arbitrage opportunities reduces to the following conditions:

$$\begin{cases} d_1 < r_1 < u_1 \\ \alpha u_{n-1} + d_n < r_n < \alpha d_{n-1} + u_n \quad \text{for } n = 2, 3, \ldots, N \end{cases}$$

if α is positive and

$$\begin{cases} d_1 < r_1 < u_1 \\ \alpha d_{n-1} + d_n < r_n < \alpha u_{n-1} + u_n \quad \text{for } n = 2, 3, \ldots, N \end{cases}$$

if α is negative.

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