Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11th February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

MATHEMATICAL CENTRE TRACTS 157

THE COVARIANT CLASSIFICATION OF TWO-DIMENSIONAL SMOOTH COMMUTATIVE FORMAL GROUPS OVER AN ALGEBRAICALLY CLOSED FIELD OF POSITIVE CHARACTERISTIC

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MATHEMATISCH CENTRUM AMSTERDAM 1983

1980 Mathematics subject classification: 14L05

ISBN 90 6196 252 8

Acknowledgements

I am very much indebted to Bert Ditters, who handed me the subject of this monograph and taught me how to make research.

I also would like to thank Michiel Hazewinkel and Jan Stienstra, for reading the manuscript and making suggestions.

I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts.

ABSTRACT

This monograph starts with a brief introduction of some generalities in the covariant formal group theory. In this theory, the central place is taken by the group of p-typical curves in a formal group.

The covariant technique gives rise to some new isomorphism invariants of finite-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic. For dimension two, an explicit classification *up to isomorphism* is obtained. The methods which have led to this result may in principle be extended to higher dimensions.

The next subject in this monograph is the comparison of the *covariant* classification of the two-dimensional formal groups with the existing *contravariant* classification of the same class of formal groups, due to Manin. The covariant classification turns out to be finer and more explicit then its contravariant counterpart. Nevertheless, the covariant techniques are relatively elementary, the performation of the classification is less laborious and the arrangement of the results is more clear.

The concept of a *special* module, which plays an important role in the contravariant theory, is generalized. This generalization leads to a refinement of the contravariant classification. It seems hard, however, to make the contravariant classification more explicit.

Finally, an application of the covariant classification theory in algebraic geometry is given: The computation of the isomorphism invariants of the formal group arising from the Jacobian of a generic curve of genus two over an algebraically closed field of characteristic 2. These isomorphism invariants are, of course, birational invariants.

CONTENTS

0. INTRODUCTION AND SUMMARY.

Let k be a commutative ring with identity element and $G(X,Y) = (G_1(X,Y), \ldots, G_n(X,Y))$ an n-tuple of elements of the ring $k[[X_1, \ldots, X_n, Y_1, \ldots, Y_n]]$ of formal power series in $2n$ indeterminates over k. G is called a *formal grouplaw* of dimension n if the following equalities of n-tuples hold:

$$G(X,0) = G(0,X) = X,$$

$$G(G(X,Y),Z) = G(X,G(Y,Z)),$$

where $X = (X_1, \ldots, X_n)$, etc. G is called *commutative* if moreover

$$G(X,Y) = G(Y,X).$$

Let H be another formal grouplaw, of dimension m. An m-tuple $f = (f_1, \ldots, f_m)$ of power series in n indeterminates is a *homomorphism* $f: G \rightarrow H$ of formal grouplaws if

$$f(G(X,Y)) = H(f(X),f(Y)).$$

With these definitions, the formal grouplaws over k form a category.
The interest of formal grouplaw theory has grown rapidly in recent years. It has a wide range of applications in many areas, for instance in number theory, algebraic geometry and algebraic topology.
One of the main subjects of this treatise is the 25 years old question how to give a classification *up to isomorphism* of commutative formal grouplaws over a field of positive characteristic. Citing Dieudonné: "Il semble que la classification de ces groupes pour la relation *d'isomorphie* soit une question dont la complexité défie l'analyse" (Groupes de Lie et hyperalgèbres de Lie sur un corps de caractéristique $p > 0$ (VII), Math. Annalen 134 (1957) pp 114-133).
For the study of this subject we shall use *covariant Dieudonné theory*, which is based on the following concepts:
Let G be an n-dimensional commutative formal grouplaw over a field k of characteristic $p > 0$. Let A be a linearly compact k-algebra, i.e. a complete topological k-algebra whose topology has a basis of neighbourhoods of zero consisting of ideals of finite codimension. Rings of formal power series over k, supplied with the formal power series topology, are examples of such algebras. G provides the set $Alc_k(k[[X_1, \ldots, X_n]],A)$ of continuous k-algebra morphisms: $k[[X_1, \ldots, X_n]] \rightarrow A$ with a group structure as follows: for two elements ϕ and ψ of $Alc_k(k[[X_1, \ldots, X_n]],A)$ define $\phi * \psi$ by $\phi * \psi(X_i) = G_i(\phi(X_1), \ldots, \phi(X_n), \psi(X_1), \ldots, \psi(X_n))$. This group is denoted $G(A)$. In a canonical way, a continuous k-algebra morphism $f: A \rightarrow B$ induces a group homomorphism $G(f)$: $G(A) \rightarrow G(B)$. Thus the functor $G(-) = Alc_k(k[[X_1, \ldots, X_n]],-)$ is a *formal group*, i.e. a covariant representable functor from the category Alc_k of linearly compact k-algebras to the category of groups.
Consider the subset of the group $G(k[[t]]) = Alc_k(k[[X_1, \ldots, X_n]],k[[t]])$ consisting of those elements which commute with the augmentation morphisms: $k[[X_1, \ldots, X_n]] \rightarrow k$ and $k[[t]] \rightarrow k$ (i.e. the k-algebra morphisms which map X_i $(1 \leqslant i \leqslant n)$ resp. t onto 0). This set is a subgroup of $G(k[[t]])$, it is called the group of *curves* in G, denoted $C(G)$.
In § 1.2 and § 1.3 we shall give the definition of a subgroup $C_p(G)$ of $C(G)$, called the group of *p-typical curves* in G. We have the action of the standard operators F (Frobenius), V (Verschiebung) and, for an element λ of k, $[\lambda]$ (homothety) on $C_p(G)$.

The commutation properties of these operators make it possible to provide $C_p(G)$ with the structure of a left module over the V-completion D^V of the *Dieudonné ring* $D = W(k)[F,V]$. ($W(k)$ is the Witt ring over k, D has multiplication rules $FV = VF = p$, $Fa = a^\sigma F$ and $aV = Va^\sigma$ where $a \in W(k)$ and σ is the Frobenius endomorphism of $W(k)$).
The importance of the left D^V-module $C_p(G)$ is established by the theorem, stated by Cartier, which contends that the functor $G \to C_p(G)$ gives an equivalence between the category of commutative formal grouplaws over k and a category of left D^V-modules which satisfy certain given properties (see § 1.7). The module $C_p(G)$ is called the *covariant Dieudonné module* of G.
This theorem gives a reformulation of the classification problem.
The D^V-module $C_p(G)$ is completely determined by an *F-type* of G (see § 1.8). Roughly speaking, an F-type of G gives the basic relations between the actions of F and V on $C_p(G)$. F-types always have the form

$$F = \sum_{k=0}^{\infty} V^k L_k,$$

the L_k being $n \times n$-matrices with entries in $W(k)$. Given an F-type of G, there are two basic methods to transform it into a new F-type (lemmas 1.9.1 and 1.9.2). The classification problem boils down to the following question: Transform an arbitrary F-type into an F-type which has a "normal form". We shall study this question for algebraically closed k. First, we shall construct an F-type with $L_k = 0$ $(0 \leqslant k < h_1)$, where h_1 is either ∞ or such that the matrix L_{h_1} cannot be transformed to zero. The next step consists of finding an F-type in which the matrix L_{h_1} itself has a normal form (see propositions 1.10.1 and 1.10.2).
Making optimal use of the two transformation lemmas, we find that for the two-dimensional case each F-type can be transformed either into: (by convention, $V^\infty = 0$)

$$F = V^{h_1}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + V^{h_1+h_2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}$$

with $0 \leqslant h_1,h_2 \leqslant \infty$ (G has such an F-type if and only if G is isomorphic to the direct sum of two one-dimensional formal grouplaws), or into

$$F = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \sum_{k=0}^{n-1} V^{h_1+h_2+k}\begin{bmatrix}0 & 0\\ 0 & d_k\end{bmatrix} + V^{h_1+h_2+h_3}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

with $0 \leqslant h_1 < \infty$, $0 < h_2 \leqslant \infty$, $0 \leqslant h_3 \leqslant \infty$, $n = \min\{h_2,h_3\}$, all d_k in the image of the Teichmüller map $T\colon k \to W(k)$ and (if $h_2 < \infty$, $h_3 > 0$) $d_0 \neq 0$. So if G does not split into two one-dimensional formal grouplaws and $h_2 = \infty$ it has F-type

$$F = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

If this is the case, G is a grouplaw of "streak" type (cf. Dieudonné, Lie groups and Lie hyperalgebras over a field of characteristic $p > 0$ (IV) § 10. Am. J. of Math. 77 (1955), pp. 429-452). In the case $h_2 < \infty$ and $h_3 = \infty$ we shall see that d_0 can be transformed to 1 and the d_k $(k > 0)$ to 0. The resulting F-type is

$$F = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + V^{h_1+h_2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

In this case G is said to be of "tree" type (cf. loc. cit.). Another situation in which the d_k do not occur in the F-type of a non-split G arises in the case $h_2 < \infty$ and $h_3 = 0$:

$$F = V^{h_1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + V^{h_1+h_2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

This is what Dieudonné called the "braid" type (cf. loc. cit.).
Modulo the action of a finite group, these "normalized" F-types are uniquely determined by the isomorphism class of G. This finite group is described explicitly in § 2.3.

Using *contravariant Dieudonné theory*, Manin already has given an explicit classification of two-dimensional commutative formal grouplaws over an algebraically closed field of characteristic $p > 0$ (cf. The Theory of Commutative Formal Groups over Fields of Finite Characteristic. Russ. Math. Survey 18 (1963) 1-83). In order to compare this classification to the one we have obtained by means of covariant Dieudonné theory, we derive a formula which gives the direct relation between the covariant and the contravariant Dieudonné module of a connected formal group over a perfect field (see § 3.2). Further, we shall give a translation between the two classification lists. At this stage, the covariant classification list turns out to be more detailed then the contravariant one. However, after having made a natural generalization of the concept *special* in the contravariant theory, we shall obtain a refinement of the contravariant classification (§ 3.4 and § 4.4A). Making essential use of this refinement, we shall find the explicit expressions of the discrete contravariant parameters into the discrete covariant parameters and vice versa. For a review of the results, see § 4.7.

The comparison of the two classifications gives rise to some useful remarks in connection with the important question which of the two theories (covariant and contravariant Dieudonné theory) has the preference for the study of formal grouplaws. First of all, if the base field k is algebraically closed, the covariant reformulation of the classification problem, namely finding normalized F-types, finds a solution in elementary (though tedious) calculations. The contravariant formulation of the same problem, however, is more complex. A variety of methods is needed in order to obtain the desired results.
Another point of difference appears in the final results of the two classifications. The covariant list is separated into the split and the non-split case. In each of the two cases, the whole classification is captured by one formula. In particular the components of the module space which have nonzero dimension, are all described by one formula. The necessary and sufficient conditions for two formal groups to be isomorphic are given explicitly in terms of the covariant parameters. Thus we come to an explicit description of the module space. The contravariant classification list is separated into six parts, each part having its own character. In three of these cases continuous parameters arise. No conditions are given for two formal groups to be isomorphic and it seems hard, if not impossible, to find such conditions.

A third difference between covariant and contravariant Dieudonné theory is demonstrated in chapter 5, where the covariant classification theorems for dimension two are applied to algebraic geometry as follows: Given an algebraic curve Γ of genus two, determine the isomorphism invariants of the completion of the Jacobian variety of Γ.

We shall describe an algorithm which determines an F-type for such a formal group. This F-type can be normalized by means of the two transformation lemmas. As an example, we shall carry this out completely for characteristic 2.
As far as the contravariant parameters of this same class of formal groups are concerned, only the *isogeny* parameters seem to be computable (cf. Manin, loc. cit.).

The covariant classification theorem of the non-split case has an interesting application to algebraic geometry: Let G be the completion at the origin of an abelian surface. Assume that G does not split into the direct sum of two one-dimensional formal groups. The well-known fact that the height of G is between 2 and 4, combined with the consequences of Poincaré duality, imply that G has a normalized F-type of the form

$$F = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + V \begin{pmatrix} 0 & 0 \\ 0 & d_0 \end{pmatrix} + V^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

where d_0 may be zero now. (For more details, see the comments on chapter 5). The parameter d_0 is a birational invariant. It seems to be an interesting question, what the geometric interpretation of this invariant is.

1. BASIC CONCEPTS

This chapter is meant to give the definitions and properties which we need in the following chapters. Except possibly for propositions 1.10.1 and 1.10.2 and theorem 1.10.3 it does not contain new results. We only give proofs if they are needed for the understanding of what comes. Let p be a prime number, fixed once and for all. k is either the field Q of rational numbers or a field of characteristic p. Alc is the category of commutative linearly compact k-algebras. G is a formal group in the sense of Dieudonné [5] chap. 1 with covariant bialgebra H and contravariant bialgebra H^*.

§ **1.1.** A *curve* (resp. a *curve of order i)* in G is an element ϕ of $G(k[[t]]) = Alc(H^*,k[[t]])$ (resp. of $G(k[t]/(t^{i+1})) = Alc(H^*,k[t]/(t^{i+1}))$) with the property: $\epsilon\circ\phi = \epsilon_{H^*}$ (ϵ_{H^*} is the counit of H^*, ϵ is defined by $\epsilon(t) = 0$). The curves (resp. the curves of order i) form a subgroup of $G(k[[t]])$ (resp. $G(k[t]/(t^{i+1}))$). Given a curve ϕ, linear continuous maps $\phi_j: H^* \to k$ ($j \geqslant 0$) are defined by:

$$\phi(x) = \sum_{j=0}^{\infty}\phi_j(x)t^j. \tag{1.1.1}$$

Considering the ϕ_j as elements of H, we have an injective group homomorphism from the group of curves into the group $1 + tH[[t]]$ (the latter one having multiplication of power series as group structure). An element $\sum_{j=0}^{\infty}\phi_j t^j$ of $1 + tH[[t]]$ corresponds to a curve if and only if $\Delta_H\phi_j = \sum_{a+b=j}\phi_a\otimes\phi_b$ for all j. (Δ_H is the comultiplication of H.) In an analogous way the group of curves of order i may be identified with a subgroup of $1+tH[t]/(t^{i+1})$. For proofs, see [6] section 1, [7] chap. II.2 and [12] section 36.3.

We shall always assume that G is commutative, so the group of curves also is commutative.

§ **1.2.** Suppose $k = Q$. An element ϕ of $1+tH[[t]]$ then may uniquely be written in the form

$$\phi = \exp\sum_{m=1}^{\infty} m^{-1}\sigma_m(\phi)t^m$$

and ϕ is a curve if and only if all $\sigma_m(\phi)$ are primitive (i.e. $\Delta_H(\sigma_m(\phi)) = \sigma_m(\phi)\otimes 1 + 1\otimes\sigma_m(\phi)$). ϕ is called a *p-typical curve* if moreover $\sigma_m(\phi) = 0$ for each m which is not a power of p. The p-typical curves form a subgroup, denoted $C_p = C_p(G)$, of the group of all curves. For a generalization of this concept to S-typical, S being an arbitrary set of primes, see [7] chap. II.7 and [15] chap. IV.7. For p-typical curves we shall write $\tau_j(\phi)$ instead of $\sigma_{p^j}(\phi)$. The well-known operators F (Frobenius), V (Verschiebung) and $[\lambda]$ for some element λ of k (homothety) may in this context be defined by: $\tau_j(F\phi) = \tau_{j+1}(\phi)$, $\tau_j(V\phi) = p\tau_{j-1}(\phi)$ ($\tau_{-1}(\phi) := 0$) and $\tau_j([\lambda]\phi) = \lambda^{p^j}\tau_j(\phi)$. See also [7] chap. III.6 and 7.

It is well-known (see [2]) that there exist polynomials $E_j(X)$ in $Z_{(p)}[X_0,X_1,\cdots]$, isobaric of weight j, $E_{p^k} = X_k$ for all $k \geqslant 0$, such that any p-typical curve ϕ has the form

$$\phi = \sum_{j=0}^{\infty}E_j(\xi_0,\xi_1,\cdots)t^j = \sum_{j=0}^{\infty}E_j(\xi)t^j, \tag{1.2.1}$$

the ξ_k being elements of H (cf. [7] chap. II.6 and [12] sections 38.4.3 and 38.4.4). These E_j are the *hyperexponential polynomials* of Dieudonné (see also [5] chap. III.3.1). So we have a correspondence between p-typical curves in G and sequences $(\xi_0,\xi_1,\cdots)$ in H with the property:

$$\Delta_H \xi_j = \sum_{a+b=p^j} E_a(\xi)\otimes E_b(\xi) \tag{1.2.2}$$

for all $j \geqslant 0$. We shall always identify ϕ with the sequence it corresponds to. If $\phi = (\xi_0,\xi_1,\cdots)$ then $V\phi = (0,\xi_0,\xi_1,\cdots)$, $[\lambda]\phi = (\lambda\xi_0,\lambda^p\xi_1,\cdots)$ and there exist isobaric polynomials $P_j(X)$ in $Z_{(p)}[X_0,X_1,\cdots]$ $(j \geqslant 0)$ such that $F\phi = (\xi_0^p + pP_0(\xi),\xi_1^p + pP_1(\xi),\cdots)$.

§ **1.3.** Assume k is a field of characteristic p. Consider the polynomials $E_j(X)$ as elements of $k[X_0,X_1,\cdots]$. We now *define* a p-typical curve as a curve ϕ in G which satisfies (1.2.1) (hence (1.2.2)). The group of p-typical curves will again be denoted by $C_p = C_p(G)$. Let r be a nonnegative integer. A p-typical curve *of length r* is a curve of order $p^r - 1$ which has the form

$$\phi = \sum_{j=0}^{p^r-1} E_j(\xi_0,\xi_1,\cdots)t^j.$$

The group of p-typical curves of length r will be denoted $C_{p,r} = C_{p,r}(G)$. As we did with curves, we shall identify p-typical curves of length r with sequences $(\xi_0,\ldots,\xi_{r-1})$ in H which satisfy (1.2.2).
We *define* the Frobenius, Verschiebung and homothety on C_p and $C_{p,r}$ as follows: let $\phi = (\xi_0,\xi_1,\cdots)$ (resp. $(\xi_0,\ldots,\xi_{r-1})$) be a p-typical curve (resp. of length r). Then $F\phi = (\xi_0^p,\xi_1^p,\cdots)$ (resp. $(\xi_0^p,\ldots,\xi_{r-1}^p)$), $V\phi = (0,\xi_0,\xi_1,\cdots)$ (resp. $(0,\xi_0,\ldots,\xi_{r-2})$) and $[\lambda]\phi = (\lambda\xi_0,\lambda^p\xi_1,\cdots)$ (resp. $(\lambda\xi_0,\ldots,\lambda^{p^{r-1}}\xi_{r-1})$).
By these definitions both C_p and $C_{p,r}$ have the structure of a $Cart_p$-module (cf. [12] section 26.2 and [15] chap. IV.2 and VI.1). The canonical projection: $k[t]/(t^{p^{r+1}}) \to k[t]/(t^{p^r})$ induces a group homomorphism: $C_{p,r+1} \to C_{p,r}$. The image of $(\xi_0,\ldots,\xi_r)$ under this homomorphism is $(\xi_0,\ldots,\xi_{r-1})$, and it is easily seen to be a morphism of $Cart_p$-modules. The canonical isomorphism $G(k[[t]]) = \varprojlim G(k[t]/(t^{p^r}))$ induces an isomorphism $C_p = \varprojlim C_{p,r}$ of $Cart_p$-modules.
Let $W = W(k)$ be the Witt ring over k, $D = D(k)$ the Dieudonné ring (i.e. $D = W[F,V]$ with the usual commutation rules: $FV = VF = p$ and if $a \in W$ then $Fa = a^\sigma F$ and $aV = Va^\sigma$, σ being the Frobenius endomorphism of W) and D^V the V-completion of D. We give C_p and $C_{p,r}$ the structure of a D^V-module as follows: let $a = (\alpha_0,\alpha_1,\cdots)$ be an element of W, then $a\phi = \sum_{j=0}^{\infty} V^j[\alpha_j]F^j\phi$ (see [12] section 28.1.8 and [15] chap. VI.1.8). Of course, the isomorphism $C_p = \varprojlim C_{p,r}$ is an isomorphism of D^V-modules.

§ **1.4.** Again, let k be either Q or a field of characteristic p. Let U and U_r be the *hyperexponential bialgebras,* i.e. as algebras we have $U = k[X_0,X_1,\cdots]$ and $U_r = k[X_0,\ldots,X_{r-1}]$, the comultiplications of both U and U_r are defined by:

$$\Delta X_j = \sum_{a+b=p^j} E_a(X)\otimes E_b(X)$$

(cf. [3] section 5). Let Alg be the category of commutative k-algebras and $Bialg$ the category of cocommutative cogroup objects in *Alg*. In other words: H is an object of *Bialg* if and only if $Sp\ H$ is a commutative affine k-group. We have the canonical representations $C_p = Bialg(U,H)$ and $C_{p,r} = Bialg(U_r,H)$ (as groups!). So a p-typical curve is the same as a *hyperexponential vector* as defined by Dieudonné in [5] chap. III.3. Obviously, the canonical isomorphism $U = \varinjlim U_r$ corresponds to the isomorphism $C_p = \varprojlim C_{p,r}$.

§ **1.5.** Let A and A_r be the additive Witt bialgebras, i.e. as algebras we have $A = k[X_0,X_1,\cdots]$ and $A_r = k[X_0,\ldots,X_{r-1}]$, the comultiplications of A and A_r are defined by the additive Witt polynomials (see [3] section 7). Dieudonné has shown that A (resp. A_r) is isomorphic to U (resp. U_r) as a bialgebra (see [2] section 8 or [3] section 7). This means that we also have: $C_p = Bialg(A,H)$ and $C_{p,r} = Bialg(A_r,H)$. It will be convenient for the following chapters to give an explicit isomorphism between U and A (resp. U_r and A_r). First of all, consider the case $k = \mathbf{Q}$. Let $m \geqslant 0$. Observe that by the very definition of the additive Witt polynomials the element $w_m(X) := \sum_{k=0}^{m} p^k X_k^{p^{m-k}}$ is a primitive element of A. Consider the p-typical curve

$$1 + \sum_{j=1}^{\infty} F_j(X)t^j := \exp \sum_{m=0}^{\infty} p^{-m} w_m(X)t^{p^m} \tag{1.5.1}$$

in the formal group $Spf\ A^*$, i.e. the formal group which has A as its covariant bialgebra. It is an elementary exercise to show that $\exp \sum_{m=0}^{\infty} p^{-m} w_m(X)t^{p^m} = \prod_{j=0}^{\infty} E_p(X_j t^{p^j})$, where $E_p(Y) = \exp \sum_{i=0}^{\infty} p^{-i} Y^{p^i}$ is the Artin-Hasse exponential with respect to the prime p. It follows that the polynomials F_j defined by (1.5.1) are in $\mathbf{Z}_{(p)}[X_0,X_1,\cdots]$, isobaric of weight j and F_{p^k} is of the form $X_k + Q_k(X_0,\ldots,X_{k-1})$. Using the Verschiebung operator and the fact that $w_m(0,X_0,X_1,\cdots) = pw_{m-1}(X)$, we find that $F_{p^k}(0,X_0,X_1,\cdots) = F_{p^{k-1}}(X)$. (We shall need this property in chapter 3.) One easily verifies that the algebra morphism $\Phi\colon U \to A$ defined by

$$\Phi(X_k) = F_{p^k}(X)$$

is an isomorphism of bialgebras.
Next consider the case that k is a field of characteristic p: consider the polynomials $F_j(X)$ in $\mathbf{Z}_{(p)}[X_0,X_1,\cdots]$ as elements of $k[X_0,X_1,\cdots]$. Then the algebra morphisms $\Phi\colon U \to A$ and $\Phi_r\colon U_r \to A_r$ defined in the same way as in the case $k = \mathbf{Q}$ are isomorphisms of bialgebras.

§ **1.6.** Suppose k is a field of characteristic p. One more remark about $C_{p,r}$ will be useful. Let H_r be the kernel of V_H^r (V_H is the Verschiebung in H, cf. [1] chap. II.5). Then the linear dual H_r^* of H_r is the cokernel of $F_{H^*}^r$ (F_{H^*} is the Frobenius in H^*, cf. [1] chap. II.5), hence $H_r^* = H^*/I_r$ where I_r is the ideal in H^* generated by $\{x^{p^r} \mid \epsilon_{H^*}(x) = 0\}$ (cf. [1] chap. II.6). Let $\phi = (\xi_0,\ldots,\xi_{r-1})$ be an element of $C_{p,r}$. Regarding ϕ as a continuous k-algebra morphism: $H^* \to k[t]/(t^{p^r})$, it is clear that ϕ has a unique factorization through H_r^*. So the $\phi_j\colon H^* \to k$ defined by (1.1.1) factorize through H_r^*, or equivalently, regarding the ϕ_j as elements of H, they are in H_r.

Since ϕ is p-typical, this is equivalent to saying that the $\xi_k = \phi_{p^k}$ are in H_r. We conclude that $C_{p,r} = Bialg(U_r, H_r)$.

From now on we shall assume that k is a field of characteristic p.

§ 1.7. The D^V-module C_p gives rise to an equivalence of categories. To be precise:

Theorem: (Cartier)
The functor which attaches to a formal group G its D^V-module of p-typical curves gives an equivalence between the category of smooth n-dimensional commutative formal groups and the category of D^V-modules M with the properties:
(i) M has no V-torsion,
(ii) M is complete with respect to the V-adic topology,
(iii) M / VM has length n.

Remark: the D^V-module $C_p(G)$ will be called the *covariant Dieudonné module* of G.

For a proof of this theorem see [15] chap. IV.7.12 combined with 8.1 and chap. VI sections 1,2 and 3, [12] section 27.7.14 or [7] chap. III.3.

§ 1.8. Assume G is smooth of dimension n, in other words: as an algebra $H^* = k[[X_1, \ldots, X_n]]$. We may assume that the generators $X_1, \ldots, X_n$ are such that the curves $\phi^{(j)}$ $(1 \leqslant j \leqslant n)$, defined by $\phi^{(j)}(X_i) = \delta_{ij} t$ (Kronecker δ_{ij}), are all p-typical (see [7] chap. III lemma 1.1). The element $\phi = (\phi^{(1)}, \ldots, \phi^{(n)})$ of C_p^n has the following important property: An arbitrary element ψ of C_p^n can be written as

$$\psi = \sum_{i=0}^{\infty} V^i \Lambda_i \phi \tag{1.8.1}$$

where the Λ_i are elements of $M_n(W)$ (see [7], loc. cit.).
Any element of C_p^n having this property will be called a *basic element* of C_p^n. Note that the matrices Λ_i in (1.8.1) are not uniquely determined. There are, however, unique matrices Λ_i *with entries in Im(T)* which satisfy (1.8.1). (T is the Teichmüller map: $k \to W(k)$). So an element $\phi = (\phi^{(1)}, \ldots, \phi^{(n)})$ of C_p^n is a basic element if and only if the set $\{\phi^{(1)}, \ldots, \phi^{(n)}\}$ constitutes a V-basis of C_p in the sense of [12] section (16.1.10). The ψ in (1.8.1) also is a basic element if and only if the matrix Λ_0 is invertible (see [7], loc. cit.).
Let ϕ be a basic element of C_p^n. Then we may apply property (1.8.1) to $F\phi$: There exist elements L_k of $M_n(W)$ such that $F\phi = \sum_{k=0}^{\infty} V^k L_k \phi$. Now C_p is completely determined by these L_k. To be more precise: C_p is isomorphic to the D^V-module $(D^V)^n / (D^V)^n (F - \sum_{k=0}^{\infty} V^k L_k)$. (see [5] chap. III.4 prop. 7).

Definition: An expression of the form

$$F = \sum_{k=0}^{\infty} V^k L_k$$

is called an *F-type* of the formal group G if there exists a basic element ϕ of $C_p^n(G)$ such that

$$F\phi = \sum_{k=0}^{\infty} V^k L_k \phi$$

(cf. [15] chap. VI.2.15).

By Cartier's theorem, G is determined by any of its F-types. Of course, an F-type of G is not uniquely determined by G, since it depends on the choice of a basic element ϕ.
Our aim is to classify G by finding in the set of all its F-types an element with the "nicest" properties. It will appear that for two-dimensional G over an algebraically closed field this boils down to: all L_k are zero except for finite number and the nonzero L_k have their entries in $\mathrm{Im}(T)$.

§ 1.9. We shall now derive the two fundamental tools to construct F-types of G with nice properties. The first one results from the choice of a new basic element of C_p^n, as we shall see in the proof.

Lemma 1.9.1:
Let $F = \sum_{k=0}^{\infty} V^k L_k$ be an F-type of G. Let Λ_i $(i \geqslant 0)$ be elements of $M_n(W)$ with Λ_0 invertible. Define matrices L'_k recursively by:

$$\sum_{i=0}^{k} (\Lambda_i^{\sigma^{k-i+1}} L_{k-i} - L'^{\sigma^i}_{k-i} \Lambda_i) = 0. \tag{1.9.1}$$

(For an element N of $M_n(W)$, N^σ is obtained by applying the Frobenius endomorphism of W to the entries of N.) Then $F = \sum_{k=0}^{\infty} V^k L'_k$ also is an F-type of G.

Proof: Let ϕ be a basic element of C_p^n such that $F\phi = \sum_{k=0}^{\infty} V^k L_k \phi$. Define $\psi = \sum_{i=0}^{\infty} V^i \Lambda_i \phi$. Then ψ also is a basic element of C_p^n and $F\psi = \sum_{i=0}^{\infty} V^i \Lambda_i^\sigma F\phi = \sum_{i=0}^{\infty} V^i \Lambda_i^\sigma \sum_{k=0}^{\infty} V^k L_k \phi = \sum_{k=0}^{\infty} V^k L'_k \sum_{i=0}^{\infty} V^i \Lambda_i \phi = \sum_{k=0}^{\infty} V^k L'_k \psi$. ***

The relation $p = VF$ permits us to find an F-type of G which has entries in $\mathrm{Im}(T)$. This is made explicit by

Lemma 1.9.2:
Let $F = \sum_{k=0}^{\infty} V^k L_k$ be an F-type of G. Let N be an element of $M_n(W)$ and $s \geqslant 0$. Define matrices L'_k by: $L'_k = L_k$ for $0 \leqslant k < s$, $L'_s = L_s - pN$ and $L'_k = L_k + N^{\sigma^{k-s}} L_{k-s-1}$ for $k > s$. Then $F = \sum_{k=0}^{\infty} V^k L'_k$ also is an F-type of G.

Proof: Let ϕ be a basic element of C_p^n such that $F\phi = \sum_{k=0}^{\infty} V^k L_k \phi$. Then $F\phi = \sum_{k=0}^{s} V^k L'_k \phi + V^s pN\phi + \sum_{k=s+1}^{\infty} V^k L_k \phi$ and $V^s pN\phi = V^{s+1} N^\sigma F\phi = \sum_{k=s+1}^{\infty} V^k N^{\sigma^{k-s}} L_{k-s-1} \phi$. ***

§ **1.10.** It is convenient to have the following notation:

$$T(k,i) = \Lambda_i^{\sigma^{k-i+1}} L_{k-i} - L'^{\sigma^i}_{k-i}\Lambda_i$$

for $k \geqslant i$. For $k < i$ we define $T(k,i) = 0$.

The lemmas 1.9.1 and 1.9.2 enable us to find at once isomorphism invariants of G over an algebraically closed k:

Proposition 1.10.1:
Let k be an algebraically closed field of characteristic $p > 0$. Then G has an F-type of the form

$$F = \sum_{k=h_1}^{\infty} V^k L_k,$$

where $0 \leqslant h_1 \leqslant \infty$ (by convention, $V^\infty = 0$) and (if $h_1 < \infty$) the transpose of L_{h_1} has a nonzero normal form in the sense of Hasse-Witt. (cf. [11], p. 489.)
Proof: Let $F = \sum_{k=0}^{\infty} V^k L_k$ be an F-type of G. Let $s > 0$ and assume $L_k = 0$ for $0 \leqslant k < s$. If $L_s \equiv 0 \bmod p$ then apply lemma 1.9.2 with $pN = L_s$. Iteration of this procedure shows that we may assume that for some h_1 with $0 \leqslant h_1 \leqslant \infty$ we have: $L_k = 0$ for $0 \leqslant k < h_1$ and if $h_1 < \infty$ then L_{h_1} is nonzero mod p. If $h_1 = \infty$ we are done. Suppose $h_1 < \infty$. Then there exists an invertible matrix Λ in $M_n(W)$ such that the transpose of $\Lambda^{\sigma^{h_1+1}} L_{h_1}\Lambda^{-1}$ has a Hasse-Witt normal form modulo p (cf. [11] section 3 Satz 11 for the case $h_1 = 0$, the general case follows from an easy generalization of this theorem). Take $\Lambda_0 = \Lambda$ and $\Lambda_i = 0$ for $i > 0$ in formula (1.9.1). Then this formula says: $L'_k\Lambda = \Lambda^{\sigma^{k+1}}L_k$. Clearly, $L'_k = 0$ for $0 \leqslant k < h_1$ and by choice of Λ the transpose of L'_{h_1} has a Hasse-Witt normal form modulo p. An obvious application of lemma 1.9.2 completes the proof. ***

It is a well-known fact that $h_1 = \infty$ if and only if G is isomorphic to the direct sum of n copies of the additive group.

Proposition 1.10.2:
Under the assumptions of proposition 1.10.1, the number h_1 and the matrix L_{h_1} are isomorphism invariants.
Proof: Let H be another formal group and assume H is isomorphic to G. Then there exists an isomorphism Φ: $C_p(H)^n \to C_p(G)^n$. Let ϕ be a basic element of $C_p(G)^n$ such that $F\phi = \sum_{k=h_1}^{\infty} V^k L_k\phi$, the L_k having the properties of proposition 1.10.1. Let ψ be a basic element of $C_p(H)^n$ such that $F\psi = \sum_{k=h'_1}^{\infty} V^k L'_k\psi$. Assume that, if $h'_1 < \infty$, the transpose of $L'_{h'_1}$ has a nonzero Hasse-Witt normal form. In view of proposition 1.10.1 such a ψ exists. Without loss of generality we may assume $h_1 \leqslant h'_1$. If $h_1 = \infty$ then nothing has to be shown, so assume $h_1 < \infty$. Identify ψ with its image under Φ (since Φ is an isomorphism of D^V-modules, this does not change the L'_k). Since ϕ and ψ are basic elements, there exist matrices Λ_i, $i \geqslant 0$, with entries in

$\mathrm{Im}(T)$ and Λ_0 invertible such that $\psi = \sum_{i=0}^{\infty} V^i \Lambda_i \phi$. On one hand we have $F\psi = \sum_{i=0}^{\infty} V^i \Lambda_i^\sigma F\phi$ and on the other hand $F\psi = \sum_{k=h_1}^{\infty} V^k L'_k \psi$, hence

$$\sum_{k=h_1}^{\infty} V^k \sum_{i=0}^{k-h_1} T(k,i)\phi = 0. \tag{1.10.1}$$

It follows that the coefficient of V^{h_1} in this expression, which is $T(h_1,0) = \Lambda_0^{\sigma^{h_1+1}} L_{h_1} - L'_{h_1}\Lambda_0$, is zero mod p. In view of the unicity of the Hasse-Witt normal form it follows that $L'_{h_1} = L_{h_1}$, so we also have $h'_1 = h_1$. ***

Theorem 1.10.3:
Let k be an algebraically closed field of characteristic $p > 0$. Assume $h_1 < \infty$. If L_{h_1} has stable rank $m > 0$, i.e. $L_{h_1}^n$ has rank m, then G is isomorphic to the direct sum of two smooth formal groups G_1 and G_2, where G_1 has dimension m, $h_1(G_1) = h_1$ and G_1 has F-type $F = V^{h_1}$. Moreover, if $m < n$ then $h_1(G_2) \geqslant h_1$.

Proof: In fact we have to show that G has an F-type $F = \sum_{k=h_1}^{\infty} V^k L_k$ with the properties of proposition 1.10.1 and moreover if $k > h_1$ then, writing $L_k = (a_{ij})$, the a_{ij} must be zero for all i,j with $i \leqslant m$ or $j \leqslant m$. Let $s > 0$ and assume we already have this property for $h_1 < k < h_1 + s$. With the help of lemma 1.9.1 we are able to show that we may also assume that L_{h_1+s} has this property: take the identity matrix for Λ_0 and take $\Lambda_i = 0$ if $i > 0$ and $i \neq s$. Then (1.9.1) says: $L'_k = L_k + T(k,s)$. It follows that $L'_k = L_k$ for $0 \leqslant k < h_1 + s$ and $L'_{h_1+s} = L_{h_1+s} + T(h_1+s,s)$. We finish the proof of the theorem by

Lemma 1.10.4:
Λ_s can be chosen such that L'_{h_1+s} has the required properties.

Proof: Write $(t_{ij}) := T(h_1+s,s)$, $(\lambda_{ij}) := \Lambda_s$ and $(a_{ij}) := L_{h_1+s}$. If both i and j are $\leqslant m$ then $t_{ij} = \lambda_{ij}^{\sigma^{h_1+1}} - \lambda_{ij}$. In view of [5] chap. III.5 lemma 1 the λ_{ij} can be chosen such that $t_{ij} = -a_{ij}$. If $m = n$ the proof is done, so assume $m < n$. First, consider the case $i \leqslant m$ and $j > m$. If the j-th column of L_{h_1} is zero, which is the case if for instance $j = m + 1$, then $t_{ij} = -\lambda_{ij}$. If the j-th column of L_{h_1} is nonzero, it is of the form ${}^t(0, \ldots, 1, \ldots, 0)$, where the 1 stands at place k for some k with $m < k < j$. We then have $t_{ij} = \lambda_{ik}^{\sigma^{h_1+1}} - \lambda_{ij}$. It follows that we may take the λ_{ij} such that $t_{ij} = -a_{ij}$. (First take $\lambda_{i,m+1} = a_{i,m+1}$, then determine $\lambda_{i,m+2}$ and so on.) Finally, consider the case $i > m$ and $j \leqslant m$. If the i-th row of L_{h_1} is zero, which is the case if for instance $i = n$, we have $t_{ij} = \lambda_{ij}^{\sigma^{h_1+1}}$. If the i-th row of L_{h_1} is nonzero, it has the form $(0, \ldots, 1, \ldots, 0)$, where the 1 stands at place k for some $k > i$. Then $t_{ij} = \lambda_{ij}^{\sigma^{h_1+1}} - \lambda_{kj}$. So we may take the λ_{ij} such that $t_{ij} = -a_{ij}$. (First take $\lambda_{nj} = -a_{nj}^{\sigma^{-h_1-1}}$, then determine $\lambda_{n-1,j}$ and so on.) ***

The theorem gives at once the following well-known result:

Corollary 1.10.5:
Two one-dimensional smooth formal groups G and H are isomorphic if and only if $h_1(G) = h_1(H)$.

In the one-dimensional case we have the isomorphism $C_p = D^V / D^V(F - V^{h_1})$ (see 1.8). It follows that the *height* of G, i.e. the rank of C_p as a W-module, is $h_1 + 1$.

So we see that the one-dimensional smooth formal groups are classified up to isomorphism by the invariant h_1. The two-dimensional case will be worked out completely in the following chapter.

Comments:
(i) For a treatment of the noncommutative case see [7] and [12], section 38.
(ii) Taking $W(k)$ as base ring instead of k, one may also use F-types for the study of formal groups (see [8] and [9]).
(iii)The two-dimensional case has been treated earlier by Manin (see [16] chap. III.8) with a completely different method, using the *contravariant* Dieudonné module. We shall call this method the *contravariant classification.* The classification which is based on the covariant Dieudonné module will be called the *covariant classification.* As we shall see in chapters 2 and 4, the results of the two methods look entirely different. In chapter 4 the two results will be compared with each other.

2. THE COVARIANT CLASSIFICATION

In this chapter we shall use the technique of chapter 1 in order to obtain an explicit classification up to isomorphism of the two-dimensional smooth commutative formal groups over an algebraically closed field k. Let G be such a group.

§ 2.1. The Classification Theorems

As we saw in chapter 1, G has an F-type of the form

$$F = \sum_{k=h_1}^{\infty} V^k L_k$$

with $0 \leqslant h_1 \leqslant \infty$ and (if $h_1 < \infty$) L_{h_1} is either $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

2.1.1. If $L_{h_1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then according to theorem 1.10.3

$$F = V^{h_1}$$

is an F-type of G. In that case G is isomorphic to the direct sum of two copies of the one-dimensional formal group with F-type $F = V^{h_1}$.

If $L_{h_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ then theorem 1.10.3 shows that G is the direct sum of two one-dimensional formal groups G_1 and G_2, G_1 having F-type $F = V^{h_1}$ and G_2 having F-type $F = V^{h_1+h_2}$ for some h_2 with $0 \leqslant h_2 \leqslant \infty$ (in fact we can have only $h_2 > 0$ in dimension 2).

The F-types we have met so far in this chapter can be summarized by:

$$F = V^{h_1}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V^{h_1+h_2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.1.1}$$

with $0 \leqslant h_1,h_2 \leqslant \infty$ and we have seen that a formal group with F-type (2.1.1) is isomorphic to the direct sum of two one-dimensional formal groups.

Conversely, suppose G is isomorphic to the direct sum of two one-dimensional formal groups, say G_1 and G_2. We may assume $h_1(G_1) \leqslant h_1(G_2)$. Then G has F-type (2.1.1) with $h_1 = h_1(G_1)$ and $h_2 = h_1(G_2) - h_1(G_1)$. We now have shown

Theorem 2.1.1 (classification theorem, split case):
Let G be a smooth two-dimensional commutative formal group over an algebraically closed field of positive characteristic. Then G is isomorphic to the direct sum of two one-dimensional formal groups if and only if G has F-type (2.1.1). If this is the case, then G is isomorphic to the direct sum of G_1 and G_2 where $h_1(G_1) = h_1$ and $h_1(G_2) = h_1 + h_2$.

2.1.2. Suppose G is not isomorphic to the direct sum of two one-dimensional formal groups. It follows from theorem 2.1.1 that $h_1 < \infty$ and $L_{h_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Unless otherwise stated, the matrices L_k and L'_k occuring in F-types will always be

made explicit by $L_k = \begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix}$ and $L'_k = \begin{bmatrix} a'_k & b'_k \\ c'_k & d'_k \end{bmatrix}$. Furthermore, the matrices Λ_i occuring in formula (1.9.1) will be written $\Lambda_i = \begin{bmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{bmatrix}$. In the sequel it will often happen that we do not need all matrices Λ_i. Therefore we shall make the *convention:* if for some i the value of the matrix Λ_i is not given then it is zero if $i > 0$ and the identity matrix if $i = 0$.
We shall transform the F-type step by step. First of all we shall make the matrices L_{h_1+s} $(s > 0)$ zero for as far as possible.

Lemma 2.1.2:

Let $s > 0$ and assume G has an F-type $F = \sum_{k=0}^{\infty} V^k L_k$ with $L_k = 0$ for $0 \leqslant k < h_1$ and for $h_1 < k < h_1+s$ and $L_{h_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If $a_{h_1+s}^{\sigma^{h_1+1}} + d_{h_1+s} \equiv c_{h_1+s} \equiv 0 \bmod p$ then we may also assume $L_{h_1+s} = 0$.

Proof: Take $\Lambda_s = \begin{bmatrix} 0 & 0 \\ a_{h_1+s} & b_{h_1+s} \end{bmatrix}$ in formula (1.9.1). We then find a new F-type by $L'_k = L_k + T(k,s)$. So for $0 \leqslant k < h_1+s$ we have $L'_k = L_k$. Furthermore, a straightforward verification shows that the entries of L'_{h_1+s} are zero mod p. Application of lemma 1.9.2 with $pN = L'_{h_1+s}$ completes the proof. ***

An immediate consequence of this lemma is:

Corollary 2.1.3:

There exists an h_2 with $0 < h_2 \leqslant \infty$ such that G has an F-type $F = \sum_{k=0}^{\infty} V^k L_k$ with $L_k = 0$ for $0 \leqslant k < h_1$ and for $h_1 < k < h_1+h_2$, $L_{h_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and (if $h_2 < \infty$) $c_{h_1+h_2}$ and $a_{h_1+h_2}^{\sigma^{h_1+1}} + d_{h_1+h_2}$ are not both zero mod p.

Proposition 2.1.4:
h_2 is an isomorphism invariant.
Proof: Let H be another formal group and assume H is isomorphic to G. Let $F = \sum_{k=0}^{\infty} V^k L_k$ be an F-type of G with the properties of corollary 2.1.3. Applying propositions 1.10.1 and 1.10.2 and corollary 2.1.3 to the formal group H we find the existence of an h'_2 such that H has an F-type $F = \sum_{k=0}^{\infty} V^k L'_k$ with $L'_k = 0$ for $0 \leqslant k < h_1$ and for $h_1 < k < h_1+h'_2$, furthermore $L'_{h_1} = L_{h_1}$ and (if $h'_2 < \infty$) $c'_{h_1+h'_2}$ and $a'^{\sigma^{h_1+1}}_{h_1+h'_2} + d'_{h_1+h'_2}$ are not both in pW. Without loss of generality we may assume that $h_2 \leqslant h'_2$. If $h_2 = \infty$ the proof is done. Assume $h_2 < \infty$. As we saw in the proof of proposition 1.10.2, there exists a basic element ϕ in $C_p(G)^2$ and matrices Λ_i $(i \geqslant 0)$ with entries in $\mathrm{Im}(T)$ and Λ_0 invertible such that (1.10.1) holds. Using the

fact that $L_k = L'_k = 0$ for $h_1 < k < h_1+h_2$, (1.10.1) boils down to

$$\sum_{k=h_1}^{\infty} V^k \{ \sum_{i=0}^{k-h_1-h_2} T(k,i) + T(k,k-h_1)\}\phi = 0. \tag{2.1.2}$$

As we already noted in the proof of proposition 1.10.2, the coefficient of V^{h_1}, which is $T(h_1,0)$, is zero mod p. Recalling that the entries of Λ_0 are in Im(T), an explicit calculation shows that this means: $T(h_1,0) = 0$, in other words: $\gamma_0 = 0$ and $\delta_0 = \alpha_0^{\sigma^{h_1+1}}$. Since Λ_0 is an invertible matrix, α_0 must be an invertible element of W.

Claim: $T(k,k-h_1) = 0$ (in other words: $\gamma_{k-h_1} = 0$ and $\delta_{k-h_1} = \alpha_{k-h_1}^{\sigma^{h_1+1}}$) for $h_1 \leqslant k < h_1+h_2$.

The claim can be proven by induction: let $0 < s < h_2$ and assume that it is true for $h_1 \leqslant k < h_1+s$. Then the lowest power of V which occurs in (2.1.2) is h_1+s. So the coefficient of V^{h_1+s}, which is $T(h_1+s,s)$, is zero mod p. An explicit calculation of the same kind as the one we just mentioned, shows that this implies $T(h_1+s,s) = 0$. This completes the proof of the claim. It follows that (2.1.2) boils down to:

$$\sum_{k=h_1+h_2}^{\infty} V^k \{ \sum_{i=0}^{k-h_1-h_2} T(k,i) + T(k,k-h_1)\}\phi = 0. \tag{2.1.3}$$

The first consequence of (2.1.3) is that $T(h_1+h_2,0) + T(h_1+h_2,h_2)$ is zero mod p. Another explicit calculation shows that this means:

$$c_{h_1+h_2} \equiv 0 \bmod p \text{ if and only if } c'_{h_1+h_2} \equiv 0 \bmod p. \tag{2.1.4}$$

(Here we have used the fact that α_0 is invertible in W.) So if $c_{h_1+h_2}$ is nonzero mod p we have: $h_2 = h'_2$. If $c_{h_1+h_2}$ is zero mod p then $a_{h_1+h_2}^{\sigma^{h_1+1}} + d_{h_1+h_2}$ is nonzero mod p. The explicit calculation we just mentioned, now shows that $a'^{\sigma^{h_1+1}}_{h_1+h_2} + d'_{h_1+h_2}$ cannot be zero mod p, hence again we have $h_2 = h'_2$. ***

If $h_2 = \infty$ the F-type has a form which cannot be reduced any further. If $h_2 < \infty$ we shall transform the first column of $L_{h_1+h_2+s}$ ($s \geqslant 0$) to zero for as far as possible.

Lemma 2.1.5:
Assume $h_2 < \infty$. Let $s \geqslant 0$ and suppose G has an F-type with the properties of corollary 2.1.3 and moreover $a_k = c_k = 0$ for $h_1+h_2 \leqslant k < h_1+h_2+s$. If $c_{h_1+h_2+s} \equiv 0 \bmod p$ then we may also assume that $a_{h_1+h_2+s} = c_{h_1+h_2+s} = 0$.

Proof: Take $\Lambda_{h_2+s} = \begin{bmatrix} 0 & 0 \\ a_{h_1+h_2+s} & 0 \end{bmatrix}$ in formula (1.9.1). Then we find a new F-type by $L'_k = L_k + T(k,h_2+s)$. Clearly $L'_k = L_k$ for $0 \leqslant k < h_1+h_2+s$. Furthermore, $a'_{h_1+h_2+s} = 0$ and $c'_{h_1+h_2+s} \equiv 0 \bmod p$. Application of lemma 1.9.2 at place h_1+h_2+s with a suitable N completes the proof. ***

Corollary 2.1.6:
If $h_2 < \infty$ then there exists an h_3 with $0 \leqslant h_3 \leqslant \infty$ such that G has an F-type with the properties of corollary 2.1.3, $a_k = c_k = 0$ for $h_1+h_2 \leqslant k < h_1+h_2+h_3$ and (if

$h_3 < \infty$) $c_{h_1+h_2+h_3}$ is nonzero mod p.

Proof: If $c_{h_1+h_2}$ is nonzero mod p then $h_3 := 0$, else we reach our goal by iteration of lemma 2.1.5. ***

Note that $h_3 > 0$ implies $a_{h_1+h_2} = 0$, hence in that case $d_{h_1+h_2}$ is nonzero mod p (see corollary 2.1.3).

Proposition 2.1.7:
h_3 is an isomorphism invariant.

Proof: Let all notations be the same as in the proof of proposition 2.1.4 and assume $a_k = c_k = 0$ for $h_1+h_2 \leqslant k < h_1+h_2+h_3$ and $a'_k = c'_k = 0$ for $h_1+h_2 \leqslant k < h_1+h_2+h'_3$, where $c'_{h_1+h_2+h'_3}$ is nonzero mod p (if $h'_3 < \infty$). Without loss of generality we may assume that $h_3 \leqslant h'_3$. If $h_3 = \infty$ we are done, so assume $h_3 < \infty$. (2.1.4) implies that $h_3 = 0$ if and only if $h'_3 = 0$. So assume $h_3 > 0$ (hence $a_{h_1+h_2} = a'_{h_1+h_2} = 0$). Making the relation $T(h_1+h_2,0) + T(h_1+h_2,h_2) \equiv 0 \bmod p$ explicit we find that $\gamma_{h_2} = 0$ and that the first column of $T(h_1+h_2,0) + T(h_1+h_2,h_2)$ is zero (recall that $\gamma_0 = 0$).

Claim: For $h_1+h_2 \leqslant k < h_1+h_2+h_3$ the $\gamma_{k-h_1} = 0$ and the first column of $\sum_{i=0}^{k-h_1-h_2} T(k,i) + T(k,k-h_1)$ is zero.

In order to prove our claim we first note the following: Let N be an element of $M_2(W)$ which is zero mod p, then (using $p\phi = VF\phi$):

$$N\phi \equiv \sum_{j>h_1} V^j \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \phi \bmod V^{h_1+h_2+h_3+1}\phi, \tag{2.1.5}$$

the *'s stand for elements of W for which we need no further specification. We may now rewrite (2.1.3) in the following way: Take for N the coefficient of the lowest power of V which occurs in (2.1.3) (hence N is zero mod p) and apply (2.1.5). Iteration of this procedure gives:

$$\sum_{k=h_1+h_2+s}^{\infty} V^k \{ \sum_{i=0}^{k-h_1-h_2} T(k,i) + T(k,k-h_1) + \begin{bmatrix} 0 & * \\ 0 & * \end{bmatrix} \}\phi \equiv 0 \tag{2.1.6}$$

modulo $V^{2h_1+2h_2+h_3+1}\phi$ for $0 \leqslant s \leqslant h_1+h_2+h_3$. It follows that the first column of the coefficient of $V^{h_1+h_2+s}$, which is $\sum_{i=0}^{s} T(h_1+h_2+s,i) + T(h_1+h_2+s,h_2+s)$, is zero mod p for $0 \leqslant s \leqslant h_1+h_2+h_3$. We are now able to prove our claim by induction: let $0 < s < h_3$ and assume it is true for $h_1+h_2 \leqslant k < h_1+h_2+s$. Then $\gamma_i = 0$ for at least $0 \leqslant i \leqslant s$, hence the first column of $\sum_{i=0}^{s} T(h_1+h_2+s,i)$ is zero. It follows that the first column of $T(h_1+h_2+s,h_2+s)$ is zero mod p. One easily verifies that this column is ${}^t(-\gamma_{h_2+s},0)$. Since γ_{h_2+s} is in $\mathrm{Im}(T)$, the claim now follows for $k = h_1+h_2+s$, which completes the proof of the claim. Finally we make use of the fact that the first column of $\sum_{i=0}^{h_3} T(h_1+h_2+h_3,i) + T(h_1+h_2+h_3,h_2+h_3)$ is zero mod

p. Making the (2,1)-entry of this column explicit we find that $c'_{h_1+h_2+h_3}$ is nonzero mod p, hence $h'_3 = h_3$. ***

We now are ready to state the counterpart of theorem 2.1.1 in the non-split case.

Theorem 2.1.8 (classification theorem, non-split case):
Let G be a smooth two-dimensional commutative formal group over an algebraically closed field of characteristic $p > 0$. Assume G is not isomorphic to the direct sum of two one-dimensional formal groups. Let $n := \min\{h_2, h_3\}$. Then G has an F-type of the form

$$F = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \sum_{k=0}^{n-1} V^{h_1+h_2+k}\begin{bmatrix}0 & 0\\ 0 & d_k\end{bmatrix} + V^{h_1+h_2+h_3}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$$

with all d_k in $\mathrm{Im}(T)$ and (if $h_2 < \infty$, $h_3 > 0$) $d_0 \neq 0$. If $h_2 < \infty$ and $h_3 = \infty$ then $d_0 = 1$ and $d_k = 0$ for $k > 0$. Such an F-type will be called a *normalized* F-type. If H is another formal group which does not split into the direct sum of two one-dimensional formal groups, having the same h_1, h_2, h_3 as G and parameters e_k in its normalized F-type then G and H are isomorphic if and only if there exist elements λ_i $(0 \leqslant i < n)$ in $\mathbf{F}_{p^{2h_1+h_2+h_3+2}}$ with $\lambda_0 \neq 0$ such that

$$\sum_{i+k=s} (\lambda_i^{p^{h_1+h_2+k+1}} \bar{d_k} - \lambda_i \bar{e_k}^{p^i}) = 0$$

for $0 \leqslant s < n$. (For an element a in W we write $\bar{a}$ for the element of k which corresponds to the image of a in W/pW).

Remark. The cases $h_2 = \infty$, $h_3 = \infty$ and $h_3 = 0$ give the three F-types which are mentioned by Dieudonné in [4] section 10. Here G is called resp. a group of "streak", "tree" and "braid" type.

§ 2.2. The Proof of the Classification Theorem in the Non-Split Case

If $h_2 = \infty$ then the theorem already has been proved (see corollary 2.1.3). For the remainder of this section assume $h_2 < \infty$. First of all we shall show the existence of a normalized F-type. Again the F-type will be transformed step by step. It is convenient to treat the three cases $h_3 = \infty$, $h_3 = 0$ and $0 < h_3 < \infty$ separately.

Case 1: $\mathbf{h_3} = \infty$
We have to show that G has F-type

$$F = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + V^{h_1+h_2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Claim 1.1: G has an F-type with the properties of corollary 2.1.6 and moreover:

$$L_{h_1+h_2} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

Proof: Let $F = \sum_{k=0}^{\infty} V^k L_k$ be an F-type of G with the properties of corollary 2.1.6.
Let α be an element of W which is nonzero mod p and such that

$\alpha^{\sigma^{2h_1+h_2+2}} d_{h_1+h_2} - \alpha^{\sigma^{h_1+1}} = 0$. In view of [5] ch. III.5 lemma 1 such an α exists. Furthermore, let $\delta := \alpha^{\sigma^{h_1+h_2+1}} b_{h_1+h_2}$. Take $\Lambda_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$ (note that Λ_0 is invertible) and $\Lambda_{h_2} = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$ in formula (1.9.1). Then $L'_k \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k + T(k,h_2)$. One easily verifies that $L'_k = L_k$ for $0 \leqslant k < h_1+h_2$. A straightforward calculation shows that $b'_{h_1+h_2} = 0$ and $d'_{h_1+h_2} = 1$. An easy induction argument shows that $a'_k = c'_k = 0$ for $k \geqslant h_1+h_2$, hence the properties of corollary 2.1.6 are valid for the new F-type.
***

Claim 1.2: Let $s > 0$ and assume G has an F-type with the properties of claim 1.1 and moreover $L_k = 0$ for $h_1+h_2 < k < h_1+h_2+s$. Then we may also assume $L_{h_1+h_2+s} = 0$.

Proof: Let α be an element of W such that $\alpha^{\sigma^{2h_1+h_2+2}} - \alpha^{\sigma^{h_1+1}} + d_{h_1+h_2+s} = 0$ and $\delta := b_{h_1+h_2+s}$. Take $\Lambda_s = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$ and $\Lambda_{h_2+s} = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$ in formula (1.9.1). Then $L'_k = L_k + T(k,s) + T(k,h_2+s)$. It follows that $L'_k = L_k$ for $0 \leqslant k < h_1+h_2+s$. Furthermore, an easy verification shows that $L'_{h_1+h_2+s} = 0$ and that $a'_k = c'_k = 0$ for $k > h_1+h_2+s$. ***

Iteration of claim 1.2 proves the theorem in case 1.

Case 2: $h_3 = 0$
We now have to show that G has F-type

$$F = V^{h_1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + V^{h_1+h_2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Claim 2.1: G has an F-type with the properties of corollary 2.1.6 and moreover $L_{h_1+h_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Proof: Start with an F-type which has the properties of corollary 2.1.6. Let α be an element of W which is nonzero mod p such that $\alpha^{\sigma^{2h_1+h_2+2}} c_{h_1+h_2} - \alpha = 0$. For the existence of such an α see again [5], loc. cit. (recall that $c_{h_1+h_2}$ is nonzero mod p). Furthermore, let β and γ be such that $\beta = \gamma^{\sigma^{h_1+1}} + \alpha^{\sigma^{2h_1+h_2+2}} d_{h_1+h_2}$ and $\gamma = \alpha^{\sigma^{h_1+h_2+1}} a_{h_1+h_2} + \beta^{\sigma^{h_1+h_2+1}} c_{h_1+h_2}$. For the existence of β and γ: eliminate γ and then use [5], loc. cit. in order to show the existence of a suitable β. Finally, let $\delta := \alpha^{\sigma^{h_1+h_2+1}} b_{h_1+h_2} + \beta^{\sigma^{h_1+h_2+1}} d_{h_1+h_2}$. Take $\Lambda_0 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$ (hence Λ_0 invertible) and $\Lambda_{h_2} = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix}$. Then $L'_k \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k + T(k,h_2)$. It follows that $L'_k = L_k$ for

$0 \leqslant k < h_1+h_2$ and a straightforward calculation shows that $L'_{h_1+h_2} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.
***

Claim 2.2: Let $s > 0$ and assume G has an F-type with the properties of claim 2.1 and moreover $L_k = 0$ for $h_1+h_2 < k < h_1+h_2+s$. Then we may also assume $L_{h_1+h_2+s} = 0$.
Proof: Let α be such that $\alpha^{\sigma^{2h_1+h_2+2}} - \alpha + c_{h_1+h_2+s} = 0$. Let β and γ be such that $\beta = \gamma^{\sigma^{h_1+1}} + d_{h_1+h_2+s}$ and $\gamma = \beta^{\sigma^{h_1+h_2+1}} + a_{h_1+h_2+s}$. Finally, put $\delta := b_{h_1+h_2+s}$.
Take $\Lambda_s = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$ and $\Lambda_{h_2+s} = \begin{bmatrix} 0 & 0 \\ \gamma & \delta \end{bmatrix}$. Then
$L'_k = L_k + T(k,s) + T(k,h_2+s)$. It follows that $L'_k = L_k$ for $0 \leqslant k < h_1+h_2+s$ and a straightforward calculation shows that $L'_{h_1+h_2+s} = 0$.
***

Iteration of claim 2.2 gives the proof of the theorem in case 2.

Case 3: $0 < \mathbf{h}_3 < \infty$
Claim 3.1: G has an F-type with the properties of corollary 2.1.6 and moreover $c_{h_1+h_2+h_3} = 1$ and $d_{h_1+h_2} \in \mathrm{Im}(T)$.
Proof: Again we start with an F-type which has the properties of corollary 2.1.6. Let α be an element of W which is nonzero mod p and such that
$\alpha^{\sigma^{2h_1+h_2+h_3+2}} c_{h_1+h_2+h_3} - \alpha = 0$. Take $\Lambda_0 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$. Then $L'_k \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k$ and a straightforward verification shows that the new F-type has the properties of corollary 2.1.6 and that $c'_{h_1+h_2+h_3} = 1$. In order to obtain the property $d_{h_1+h_2} \in \mathrm{Im}(T)$ we apply lemma 1.9.2 at place h_1+h_2 with a suitable N of the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$. *Note* that the properties of corollary 2.1.6 and the property $c'_{h_1+h_2+h_3} = 1$ are not disturbed by this application of lemma 1.9.2. ***

Claim 3.2: G has an F-type with the properties of claim 3.1 and moreover:
$c_k = 0 \quad (h_1+h_2+h_3 < k < h_1+h_2+h_3+n)$ and
$d_k \in \mathrm{Im}(T) \quad (h_1+h_2 \leqslant k < h_1+h_2+n)$.
Proof: Let $0 < s < n$ and assume we already have $c_k = 0$ for $h_1+h_2+h_3 < k < h_1+h_2+h_3+s$ and $d_k \in \mathrm{Im}(T)$ for $h_1+h_2 \leqslant k < h_1+h_2+s$.
Let α be an element of W such that $\alpha^{\sigma^{2h_1+h_2+h_3+2}} - \alpha + c_{h_1+h_2+h_3+s} = 0$. Take $\Lambda_s = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$. Then $L'_k = L_k + T(k,s)$. It follows that $L'_k = L_k$ for $0 \leqslant k < h_1+h_2+s$ (in particular: $d'_k = d_k$ is in $\mathrm{Im}(T)$ for $h_1+h_2 \leqslant k < h_1+h_2+s$). Furthermore, $a'_k = a_k$ and $c'_k = c_k$ for $h_1+h_2+s \leqslant k < h_1+h_2+h_3+s$, so the properties of claim 3.1 are still valid and we

also have $c'_k = 0$ for $h_1+h_2+h_3 < k < h_1+h_2+h_3+s$. Finally, $c'_{h_1+h_2+h_3+s} = 0$.
In order to obtain the property $d_{h_1+h_2+s} \in \mathrm{Im}(T)$ we apply lemma 1.9.2 at place h_1+h_2+s with a suitable N of the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$. Note that this application does not disturb the properties we already have reached. ***

We shall see that the d_k for $h_1+h_2 \leqslant k < h_1+h_2+n$ will be left untouched by all further transformation steps. So when we want to find the isomorphism invariants of a given formal group, we only need an F-type with the properties of claim 3.2.

Claim 3.3: Let $s \geqslant 0$ and assume G has an F-type with the properties of claim 3.2 and moreover:

$a_k = 0 \quad (h_1+h_2+h_3 \leqslant k < h_1+h_2+h_3+s)$,
$b_k = 0 \quad (h_1+h_2 \leqslant k < h_1+h_2+s)$,
$c_k = 0 \quad (h_1+h_2+h_3+n \leqslant k < h_1+h_2+h_3+n+s)$ and
$d_k = 0 \quad (h_1+h_2+n \leqslant k < h_1+h_2+n+s)$.

Then we may also assume $a_{h_1+h_2+h_3+s} = b_{h_1+h_2+s} = c_{h_1+h_2+h_3+n+s} = d_{h_1+h_2+n+s} = 0$.

Proof: First assume $h_3 < h_2$. Then $n = h_3$. Let α, β, γ and δ be such that they satisfy the following four equations ($d := d_{h_1+h_2}$):

$$\alpha = \alpha^{\sigma^{2h_1+h_2+h_3+2}} + c_{h_1+h_2+2h_3+s}, \tag{2.2.1}$$

$$\beta = \alpha^{\sigma^{2h_1+h_2+2}} d - \alpha^{\sigma^{h_1+1}} d^{\sigma^{h_3+s}} + \gamma^{\sigma^{h_1+1}} + d_{h_1+h_2+h_3+s}, \tag{2.2.2}$$

$$\gamma = \beta^{\sigma^{h_1+h_2+h_3+1}} + a_{h_1+h_2+h_3+s}, \tag{2.2.3}$$

$$\delta = \beta^{\sigma^{h_1+h_2+1}} d + b_{h_1+h_2+s}. \tag{2.2.4}$$

With the help of [5], loc. cit. it is not hard to see that such a choice of α, β, γ and δ is possible.

If $s = 0$ take $\Lambda_0 = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$, if $s > 0$ then $\Lambda_s := \begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$. Furthermore, take $\Lambda_{h_3+s} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{\sigma^{h_1+1}} \end{bmatrix}$, $\Lambda_{h_2+s} = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$ and $\Lambda_{h_2+h_3+s} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$.

If s $= 0$ then (1.9.1) says:

$$L'_k \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k + T(k,h_3) + T(k,h_2) + T(k,h_2+h_3),$$

if $s > 0$ then

$$L'_k = L_k + T(k,s) + T(k,h_3+s) + T(k,h_2+s) + T(k,h_2+h_3+s).$$

One easily verifies (by induction, if necessary) the following statements:
Suppose $s = 0$ (resp. $s > 0$).
For $0 \leqslant k < h_1+h_2+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is equal to L_k

and $T(k,h_3+s) = T(k,h_2+s) = T(k,h_2+h_3+s) = 0$, hence $L'_k = L_k$.
For $k = h_1+h_2+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)) =$
$\begin{bmatrix} 0 & \beta^{\sigma^{h_1+h_2+1}} d + b_k \\ c_k & d_k \end{bmatrix}$, furthermore $T(k,h_2+s)\Lambda_0^{-1} = \begin{bmatrix} 0 & -\delta \\ 0 & 0 \end{bmatrix}$ and $T(k,h_3+s) =$ $T(k,h_2+h_3+s) = 0$. In view of (2.2.4) this implies: $L'_k = \begin{bmatrix} 0 & 0 \\ c_k & d_k \end{bmatrix}$.
For $h_1+h_2+s < k < h_1+h_2+h_3+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) $= \begin{bmatrix} 0 & * \\ c_k & d_k \end{bmatrix}$ and $T(k,h_3+s) = T(k,h_2+s) = T(k,h_2+h_3+s) = 0$, hence $L'_k =$ $\begin{bmatrix} 0 & * \\ c_k & d_k \end{bmatrix}$. We now have seen that $d'_k = d_k$ for $h_1+h_2 \leqslant k < h_1+h_2+n$.
For $k = h_1+h_2+h_3+s$ we have: The first column of the matrix $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is ${}^t(a_k + \beta^{\sigma^{h_1+h_2+h_3+1}}, c_k)$ and its (2,2)-entry is $d_k - \beta$. Furthermore, the first column of $T(k,h_3+s)\Lambda_0^{-1}$ is zero and its (2,2)-entry is $\alpha^{\sigma^{2h_1+h_2+2}} d - \alpha^{\sigma^{h_1+1}} d^{\sigma^{h_3+s}}$.
The matrix $T(k,h_2+s)$ is zero and $T(k,h_2+h_3+s)\Lambda_0^{-1}$ has first column ${}^t(-\gamma,0)$ and (2,2)-entry $\gamma^{\sigma^{h_1+1}}$. In view of (2.2.3) and (2.2.2) it follows that L'_k has first column ${}^t(0,c_k)$ and (2,2)-entry 0.
For $h_1+h_2+h_3+s < k < h_1+h_2+2h_3+s$ the (2,1)-entries of the matrices $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$), $T(k,h_3+s)\Lambda_0^{-1}$ and $T(k,h_2+s)\Lambda_0^{-1}$ are zero. Furthermore, $T(k,h_2+h_3+s)$ is zero, hence the (2,1)-entry of L'_k is 0.
Finally, for $k = h_1+h_2+2h_3+s$: The matrix $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) has (2,1)-entry c_k, furthermore $T(k,h_3+s)\Lambda_0^{-1}$ has (2,1)-entry $\alpha^{\sigma^{2h_1+h_2+h_3+2}} - \alpha$, $T(k,h_2+s)\Lambda_0^{-1}$ has (2,1)-entry 0 and $T(k,h_2+h_3+s)$ is zero. In view of (2.2.1) it follows that L'_k has (2,1)-entry 0. This completes the proof of claim 3.3 for the case $h_3 < h_2$.

Next let us assume $h_2 \leqslant h_3$, hence $n = h_2$. The choice of α, β, γ and δ depends on the question whether h_2 is equal to h_3 or not.
If $h_2 < h_3$ we choose them such that they satisfy (2.2.3) and:

$$\alpha = -\gamma d^{\sigma^{h_2+h_3+s}} + \delta^{\sigma^{h_1+h_2+h_3+1}} + c_{h_1+2h_2+h_3+s}, \tag{2.2.5}$$

$$\delta = \alpha^{\sigma^{h_1+1}} + \beta^{\sigma^{h_1+h_2+1}} d + b_{h_1+h_2+s}, \tag{2.2.6}$$

$$\delta^{\sigma^{h_1+h_2+1}} d - \delta d^{\sigma^{h_2+s}} + d_{h_1+2h_2+s} = 0. \tag{2.2.7}$$

These four equations are solvable: first fix a δ which satisfies (2.2.7). Then use (2.2.3) in order to eliminate γ in (2.2.5) and then (2.2.5) to eliminate α. Then (2.2.6) has become an equation in β which is solvable. After having fixed a value for β which satisfies this equation, the α and γ are also fixed.
If $h_2 = h_3$ things are more complicated. The equations that are to be satisfied are (2.2.3), (2.2.5), (2.2.6) and:

$$\beta = \gamma^{\sigma^{h_1+1}} + \delta^{\sigma^{h_1+h_2+1}} d - \delta d^{\sigma^{h_2+s}} + d_{h_1+2h_2+s}. \tag{2.2.8}$$

For the solvability of these equations: first eliminate α with (2.2.5) and γ with (2.2.3). After this elimination the equations (2.2.6) and (2.2.8), which now contain only the variables β and δ, are equivalent with (2.2.9) $:= d^{\sigma^{h_2+s}}$(2.2.6) $-$ (2.2.8) and (2.2.10) $:=$ (2.2.9) $- \, d^{1-\sigma^{h_1+2h_2+s+1}}(2.2.8)^{\sigma^{h_1+h_2+1}}$. Now (2.2.10) has the form $d'\delta^{\sigma^{2h_1+2h_2+2}}$ = an expression in β which is not constant modulo p (the highest σ-power of β which occurs in this expression is $3(h_1 + h_2 + 1)$ with a coefficient which is nonzero mod p), where either $d' = 0$ or d' is nonzero mod p. If $d' = 0$ we may take a β which satisfies (2.2.10) and then take a δ which satisfies (2.2.9). If d' is nonzero mod p then we can eliminate δ in $(2.2.9)^{\sigma^{h_1+h_2+1}}$ and the resulting equation in β has a solution (the highest σ-power of β which occurs is $4(h_1 + h_2 + 1)$ with a coefficient which is nonzero mod p).

Take $\Lambda_s = \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix}$ if $s = 0$ and $\begin{bmatrix} 0 & \beta \\ 0 & 0 \end{bmatrix}$ if $s > 0$. Furthermore, take $\Lambda_{h_2+s} = \begin{bmatrix} \alpha & 0 \\ 0 & \delta \end{bmatrix}$ and $\Lambda_{h_2+h_3+s} = \begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$. If $s = 0$ then (1.9.1) says:

$$L'_k \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k + T(k,h_2) + T(k,h_2+h_3),$$

if $s > 0$ then

$$L'_k = L_k + T(k,s) + T(k,h_2+s) + T(k,h_2+h_3+s).$$

We now have the following properties:
Assume $s = 0$ (resp. $s > 0$).
For $0 \leqslant k < h_1+h_2+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is equal to L_k and $T(k,h_2+s) = T(k,h_2+h_3+s) = 0$, hence $L'_k = L_k$.
For $k = h_1+h_2+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) $=$ $\begin{bmatrix} 0 & \beta^{\sigma^{h_1+h_2+1}} d + b_k \\ c_k & d_k \end{bmatrix}$, furthermore $T(k,h_2+s)\Lambda_0^{-1} = \begin{bmatrix} 0 & \alpha^{\sigma^{h_1+1}} - \delta \\ 0 & 0 \end{bmatrix}$ and $T(k,h_2+h_3+s) = 0$. In view of (2.2.6) this implies: $L'_k = \begin{bmatrix} 0 & 0 \\ c_k & d_k \end{bmatrix}$.
For $h_1+h_2+s < k < h_1+2h_2+s$ we have: $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) $=$ $\begin{bmatrix} 0 & * \\ c_k & d_k \end{bmatrix}$ and $T(k,h_2+s) = T(k,h_2+h_3+s) = 0$, hence $L'_k = \begin{bmatrix} 0 & * \\ c_k & d_k \end{bmatrix}$. We now have seen that $d'_k = d_k$ for $h_1+h_2 \leqslant k < h_1+h_2+n$.
For $k = h_1+2h_2+s$ we have: The first column of the matrix $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is ${}^t(a_k + \beta^{\sigma^{h_1+h_2+h_3+1}}, c_k)$ if $h_2 = h_3$ and ${}^t(0,c_k)$ if $h_2 < h_3$. Its (2,2)-entry is $d_k - \beta$ if $h_2 = h_3$, it is d_k if $h_2 < h_3$. Furthermore, the first column of $T(k,h_2+s)\Lambda_0^{-1}$ is zero and its (2,2)-entry is $\delta^{\sigma^{h_1+h_2+1}} d - \delta d^{\sigma^{h_2+s}}$. If $h_2 = h_3$ the matrix $T(k,h_2+h_3+s)\Lambda_0^{-1}$ has first column ${}^t(-\gamma,0)$ and (2,2)-entry $\gamma^{\sigma^{h_1+1}}$, if $h_2 < h_3$ it is the zero matrix. It follows that L'_k has first column ${}^t(0,c_k)$ (if $h_2 = h_3$ we use (2.2.3) here). The (2,2)-entry of L'_k is 0 (if $h_2 = h_3$ this follows from (2.2.8), if $h_2 < h_3$ it follows from (2.2.7)).
For $h_1+2h_2+s < k < h_1+h_2+h_3+s$ (assuming, of course, that $h_2 < h_3$) the first

column of $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is ${}^t(0,c_k)$, furthermore $T(k,h_2+s)\Lambda_0^{-1}$ has first column zero and $T(k,h_2+h_3+s)$ is zero. It follows that L'_k has first column ${}^t(0,c_k)$.
For $k = h_1+h_2+h_3+s$ we already have treated the case $h_2 = h_3$, so for this value of k we may assume $h_2 < h_3$. The first column of $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) is ${}^t(\beta^{\sigma^{h_1+h_2+h_3+1}} + a_k, c_k)$, furthermore the first column of $T(k,h_2+s)\Lambda_0^{-1}$ is zero and $T(k,h_2+h_3+s)\Lambda_0^{-1}$ has first column ${}^t(-\gamma,0)$. Using (2.2.3) we find that L'_k has first column ${}^t(0,c_k)$.
For $h_1+h_2+h_3+s < k < h_1+2h_2+h_3+s$ the (2,1)-entries of the matrices $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) and $T(k,h_2+s)\Lambda_0^{-1}$ are zero and the matrix $T(k,h_2+h_3+s)$ is zero, so the (2,1)-entry of L'_k is zero.
Finally, for $k = h_1+2h_2+h_3+s$: The matrix $\Lambda_0^{\sigma^{k+1}} L_k \Lambda_0^{-1}$ (resp. $L_k + T(k,s)$) has (2,1)-entry c_k, furthermore $T(k,h_2+s)\Lambda_0^{-1}$ has (2,1)-entry $\delta^{\sigma^{h_1+h_2+h_3+1}} - \alpha$ and $T(k,h_2+h_3+s)\Lambda_0^{-1}$ has (2,1)-entry $-\gamma d^{\sigma^{h_2+h_3+s}}$. In view of (2.2.5) it follows that L'_k has (2,1)-entry 0. ***

Iteration of claim 3.3 gives the proof in case 3.

We now have shown the existence of a normalized F-type. Next consider the formal groups G and H which are mentioned in the theorem. If $h_3 = 0$ *or* ∞ then G and H have the same normalized F-type, hence nothing has to be shown. For the remainder of this section assume $0 < h_3 < \infty$.

Suppose G and H are isomorphic. Let $F = \sum_{k=0}^{\infty} V^k L_k$ and $F = \sum_{k=0}^{\infty} V^k L'_k$ be the given normalized F-types of G resp. H. As we observed in the proof of proposition 1.10.2, there exists a basic element ϕ of $C_p(G)^2$ and matrices Λ_i with entries in Im(T) and invertible Λ_0 such that (1.10.1) holds. In the proof of proposition 2.1.7 we have found that these matrices satisfy the relation (2.1.6). Let $0 \leqslant j < n$. Taking $s = h_3 + j$ in (2.1.6) we find that the first column of $\sum_{i=0}^{h_3+j} T(h_1+h_2+h_3+j,i) + T(h_1+h_2+h_3+j,h_2+h_3+j)$ is zero mod p. Using the fact that $\delta_k = \alpha_k^{\sigma^{h_1+1}}$ and $\gamma_k = 0$ for $0 \leqslant k < h_2$ (see the claim in the proof of proposition 2.1.4) and that $\gamma_k = 0$ for $h_2 \leqslant k < h_2+h_3$ (see the claim in the proof of proposition 2.1.7), a straightforward calculation shows that the (2,1)-entry in this column is $\alpha_j^{\sigma^{2h_1+h_2+h_3+2}} - \alpha_j$. Since α_j is in Im(T) this means: α_j is in $W(F_{p^{2h_1+h_2+h_3+2}})$. In order to find the relations between the d_k and the e_k we first rewrite (2.1.3) by the following procedure: let N be the coefficient of the lowest power of V which occurs in (2.1.3). Then $N \equiv 0 \bmod p$ and in the claim of proposition 2.1.7 we have seen that the first column of N is zero. Using the relation $p\phi = VF\phi$ and the given F-type of G we find

$$N\phi \equiv 0 \bmod V^{h_1+h_2+1}\phi.$$

Let $0 < s < n$. Then this procedure may be repeated s times. The result looks as follows:

$$\sum_{k=h_1+h_2+s}^{\infty} V^k \{ \sum_{i=0}^{k-h_1-h_2} T(k,i) + T(k,k-h_1)\}\phi \equiv 0 \bmod V^{2h_1+2h_2+1}\phi.$$

It follows that for $0 \leq s < n$ the matrix $\sum_{i=0}^{s} T(h_1+h_2+s,i) + T(h_1+h_2+s,h_2+s)$ is zero mod p. A straightforward calculation shows that the (2,2)-entry of this matrix is

$$\sum_{i+k=s} (\alpha_i^{\sigma^{2h_1+h_2+k+2}} d_k - \alpha_i^{\sigma^{h_1+1}} e_k^{\sigma^i}).$$

Taking $\lambda_i = \alpha_i^{p^{h_1+1}}$ for $0 \leq i < n$ we have the required relation between the d_k and the e_k. Conversely, given these relations between the d_k and the e_k, we have to show that G and H are isomorphic, or, which amounts to the same, that the given F-type of H also is an F-type of G. Let $\alpha_i := T(\lambda_i^{p^{-h_1-1}})$ and take $\Lambda_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & \alpha_i^{\sigma^{h_1+1}} \end{bmatrix}$ for $0 \leq i < n$ in formula (1.9.1). Note that α_0 is an invertible element of W, hence Λ_0 is an invertible matrix. Let $F = \sum_{k=0}^{\infty} V^k L_k$ be the given normalized F-type of G. Then formula (1.9.1) gives a new F-type of G by:

$$L_k' \Lambda_0 = \Lambda_0^{\sigma^{k+1}} L_k + \sum_{i=1}^{n-1} T(k,i).$$

A straightforward verification shows that the L_k' have the following properties: $L_k' = L_k$ for $0 \leq k < h_1+h_2$, L_k' is $\begin{bmatrix} 0 & 0 \\ 0 & e_{k-h_1-h_2} \end{bmatrix}$ mod p for $h_1+h_2 \leq k < h_1+h_2+n$ and that L_k' has the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$ for $h_1+h_2 \leq k < h_1+h_2+h_3$, $\begin{bmatrix} 0 & 0 \\ 1 & * \end{bmatrix}$ for $k = h_1+h_2+h_3$ and $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$ for $h_1+h_2+h_3 < k < h_1+h_2+h_3+n$.

Application of lemma 1.9.2 at place h_1+h_2 makes it possible to change $L'_{h_1+h_2}$ modulo p. We want to choose N such that the new F-type of G, denoted $F = \sum_{k=0}^{\infty} V^k L_k''$, has the property $L''_{h_1+h_2} = \begin{bmatrix} 0 & 0 \\ 0 & e_0 \end{bmatrix}$. So N must have the form $\begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$, which implies $L_k'' = L_k'$ for $h_1+h_2 < k < h_1+h_2+n$. Further a straightforward verification shows that $L_k'' = L_k' + \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}$ for $h_1+h_2+n \leq k < h_1+h_2+h_3+n$.

Repeating this procedure n times we get an F-type $F = \sum_{k=0}^{\infty} V^k L_k$ which has the properties of claim 3.2. Moreover, $L_k = \begin{bmatrix} 0 & 0 \\ 0 & e_{k-h_1-h_2} \end{bmatrix}$ for $h_1+h_2 \leq k < h_1+h_2+n$.

However, this F-type does not necessarily have a normal form, so we have to apply claim 3.3. As we noticed after the proof of claim 3.2 the matrices L_k $(h_1+h_2 \leq k < h_1+h_2+n)$ are left invariant by this claim, so that we finally reach the required result.

§ 2.3. The Module Space

In this section we shall give an explicit description of the algebraic structure of the module space arising from theorem 2.1.8.

Consider the noncommutative ring $k[T]$ with commutation rule $Tx = x^p T$, x being an element of k. The normalized F-types in the sense of theorem 2.1.8 correspond bijectively to the elements of the group $\Delta = (k[T]/(T^n))^*$ of invertible elements of the ring $k[T]/(T^n)$ as follows: Given a normalized F-type with continuous parameters $d_0, \ldots, d_{n-1}$, then it corresponds to the element $\sum_{k=0}^{n-1} \bar{d}_k T^k$. (Note that an element of $k[T]/(T^n)$ is invertible if and only if its constant term is nonzero.)

Let σ be the automorphism of $k[T]$ defined by $T^\sigma = T$ and $x^\sigma = x^p$ for $x \in k$. Furthermore, put $h := 2h_1+h_2+h_3+2$. (Later we shall see that h is the *height* of the formal group G.) Let Λ be the finite subgroup $(F_{p^h}[T]/(T^n))^*$ of Δ. We define a right action of Λ on Δ as follows: let $\lambda \in \Lambda$ and $d \in \Delta$ then $d * \lambda := \lambda^{-1} d \lambda^{\sigma^{h_1+h_2+1}}$.

Let $d = \sum_{k=0}^{n-1} d_k T^k$ and $e = \sum_{k=0}^{n-1} e_k T^k$ be elements of Δ. Then d and e are in the same orbit under the action of Λ if and only if there exists an element $\lambda = \sum_{i=0}^{n-1} \lambda_i T^i$ of Λ such that $\lambda e = d \lambda^{\sigma^{h_1+h_2+1}}$, or equivalently: $\sum_{i+k=s} \lambda_i e_k^{p^i} = \sum_{i+k=s} d_k \lambda_i^{p^{h_1+h_2+k+1}}$ for $0 \leqslant s < n$.

We conclude that two normalized F-types are in the same orbit under the action of Λ if and only if the formal groups to which they belong are isomorphic.

So the module space is isomorphic to the quotient of the variety $\{(x_0, \ldots, x_{n-1}) \mid x_0 \neq 0\}$ in k^n under the action of the group Λ which is described above explicitly. Since Λ is finite, this space has dimension n.

Comments.

With these techniques it is evident how to give a classification of *three*-dimensional smooth commutative formal groups. In view of theorem 1.10.3, only the cases $L_{h_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $L_{h_1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ deserve a thorough analysis.

Added in proof: The verifications which must be made in § 2.2 can be shortened by working with matrices over the Hilbert ring $W_{\sigma^{-1}}[[V]]$ (i.e. the noncommutative ring of power series over W with multiplication rule $aV = Va^\sigma (a \in W)$), instead of sequences of matrices over W. The fundamental steps, however, do not change by this approach.

3. PRELIMINARIES TO THE RELATION BETWEEN THE TWO CLASSIFICATIONS

In chapter 1 we have defined the covariant Dieudonné module of a smooth commutative formal group of finite dimension. This definition gave rise to an explicit classification up to isomorphism of the two-dimensional smooth commutative formal groups over an algebraically closed field: the *covariant* classification. On the other hand, the contravariant Dieudonné module (we shall summarize the definition below) also gave rise to a classification: the *contravariant* classification (see Manin [16] ch. III.8). At first sight these two classification lists seem to be entirely different, but of course there must be a one-to-one correspondence between them. In this chapter we shall give the basic tools which are needed in order to understand this relation and in the next chapter we shall describe the correspondence in full detail.

§ 3.1. Some Definitions and Notations

First we shall fix some notations and terminology. Throughout this chapter let k be a perfect field of characteristic $p > 0$. In section 3.4, k will assumed to be algebraically closed, but in the first three sections of this chapter there is no need for this condition. An algebra, coalgebra or bialgebra is called *finite* if it is finite-dimensional as a k-vector space. For an algebra A the *affine scheme Sp A* is the functor $Alg(A,-)$ from Alg to the category of sets. The affine scheme $Sp\ A$ is called finite if A is finite. For a coalgebra C, the linear dual C^* of C has a natural structure of *profinite* algebra, i.e. C^* is a topological algebra and it is the projective limit of finite quotient algebras (having the discrete topology), or, which amounts to the same, C^* is an object in the category Alc. The *formal scheme Spf* C^* is the functor $Alc(C^*,-)$ from the category Alc to the category of sets. It is called *finite* if C is finite. If C^* is the projective limit of finite quotients C_i^*, a standard topological argument shows that

$$Spf\ C^*\ (R) = \lim_{\rightarrow} Spf\ C_i^*\ (R),$$

R being an object of Alc.
Let i be an integer and V a k-vector space. Then the k-vector space $V^{(i)}$ is defined as follows: as an abelian group $V^{(i)} = V$, and if α is an element of k and v an element of $V^{(i)}$ then $\alpha * v := \alpha^{p^{-i}} v$. For an algebra, resp. bialgebra A we define $A^{(i)}$ in the same way, the multiplication and comultiplication of $A^{(i)}$ being the same as in A. With these notations, the map F_A which sends an element a of an algebra A to a^p is an algebra morphism from $A^{(1)}$ to A, called the *Frobenius* morphism of A. If A is a bialgebra, the Frobenius morphism is a bialgebra morphism (cf. [1] ch.II.5), if A is in Alc, the Frobenius is continuous.
For a bialgebra B, the topological bialgebra which is dual to B will be denoted by B^* (cf. [5] ch.1.2). A straightforward verification shows that $(B^*)^{(i)} \simeq (B^{(i)})^*$. The *Verschiebung* morphism $V_B \colon B \to B^{(1)}$ of B is the morphism which is dual to the Frobenius of B^* and the Verschiebung morphism of B^* is the one which is dual to the Frobenius of B. For a formal group $G = Spf\ B^*$ put $G^{(i)} = Spf\ (B^*)^{(i)}$. The Frobenius of G is the morphism $F_G := Spf\ F_{B^*} \colon G \to G^{(1)}$. The Verschiebung of G is the morphism $V_G := Spf\ V_{B^*} \colon G^{(1)} \to G$. (see [1] ch. II.5).
If M is a W-module and i an integer then we define $M^{(i)}$ in an analogous way as we did for k-vector spaces, using the frobenius automorphism σ of W instead of p-th powers. For a D-module M we define $M^{(i)}$ as follows: Take the W-module $M^{(i)}$ and let F and V act on $M^{(i)}$ in the same way as they do on M. Put $K = K(k)$ for the

quotient field of $W(k)$. For a D-module M define the D-module M^* as follows: as a W-module, $M^* = Mod_W(M,K / W)$, and if f is an element of M^* then Ff and Vf are defined by $Ff(m) = f(Vm)^\sigma$ resp. $Vf(m) = f(Fm)^{\sigma^{-1}}$. In an analogous way, the W-module $Mod_W(M,W)$ has a D-module structure. It is a straightforward verification to show that $(M^*)^{(i)} = (M^{(i)})^*$ and that for a submodule N of M we have $(M / N)^{(i)} \simeq M^{(i)} / N^{(i)}$. A D-module M which has finite length as a W-module, has the property $M^{**} = M$ (see [1] ch. III.6).

§ 3.2. Explicit Relation between Covariant and Contravariant Dieudonné Module

First of all we shall summarize the definition and the properties of the contravariant Dieudonné module of a formal group, as far as we need them. For a more complete treatment on this subject see [1] or [16]. We also shall derive a formula which gives a direct relation between the covariant and the contravariant Dieudonné module of a connected formal group of finite type.

3.2.1. Let B be a bialgebra and assume that the underlying algebra of B^* is local. The affine group $Sp\ B$ is then called *unipotent* and the formal group $Spf\ B^*$ is called *connected* (cf. [1] ch. II.9). Define the abelian group

$$M(Sp\ B) = \varinjlim Bialg(A_i,B),$$

where A_i is the i-th additive Witt bialgebra. (As an algebra, $A_i = k[X_0, \ldots, X_{i-1}]$ and $W_i = Sp\ A_i$ is the i-th Witt group, cf. [1] ch. III.2). The inductive limit is taken over the arrows which are induced by $t: A_{i+1} \to A_i$, $t(X_j) = X_{j-1}$ if $j > 0$ and $t(X_0) = 0$. Define a D-module structure on $Bialg(A_i,B)$ as follows: the actions of F and V are induced by the Frobenius resp. the Verschiebung morhism of A_i and if α is an element of Im(T), say $\alpha = T(a)$, and f is an element of $Bialg(A_i,B)$ then αf is defined by $\alpha f(X_j) = f(a^{p^{1-i+j}} X_j)$. With these definitions, the transition maps in $\varinjlim Bialg(A_i,B)$ are morphisms of D-modules, hence $M(Sp\ B)$ has the structure of a D-module. Under the canonical induction morphism, $Bialg(A_i,B)$ is identified with $\{x \in M(Sp\ B) \mid V^i x = 0\}$ (see [1] ch. III.5). So if V^rM(Sp B) $= 0$ for some r, $Bialg(A_r,B)$ is identified with $M(Sp\ B)$.
The functor $Sp\ B \to M(Sp\ B)$ gives an antiequivalence between the category of unipotent affine groups and the category of D-modules of V-torsion. $Sp\ B$ is finite if and only if the W-module $M(Sp\ B)$ has finite length. The Frobenius morphism of B induces a morphism F: $M(Sp\ B^{(1)}) = M(Sp\ B)^{(1)} \to M(Sp\ B)$ which is precisely the action of F on $M(Sp\ B)$. The action of V on $M(Sp\ B)$ is induced by the Verschiebung morphism of B (see [1],loc. cit.).

3.2.2. Let L be a finite bialgebra and assume that the underlying algebra of L is local. In this case, $Sp\ L$ is called an *infinitesimal* affine group. Then $Sp\ L^*$ is unipotent, hence we may consider $M(Sp\ L^*)$ as defined in the preceding subsection. Now define $M(Sp\ L) := M(Sp\ L^*)^*$. Since the W-module $M(Sp\ L)$ has finite length, we have the property: $M(Sp\ L^*) = M(Sp\ L)^*$ (cf. [1] ch. III.6). So if $Sp\ L$ is both unipotent and infinitesimal, the two definitions of $M(Sp\ L)$ are compatible. With these definitions, the functor $Sp\ L \to M(Sp\ L)$ gives an antiequivalence between the category of infinitesimal affine groups and the category of D-modules which have finite length as a W-module and are killed by a power of F (see [1], second theorem in ch. III.6).

3.2.3. Let H be a bialgebra and assume that H^* is local and noetherian. The formal group $G = Spf\ H^*$ is then called a connected formal group *of finite type.* Let $r > 0$ and put $H_r = \text{Ker}\ V_H^r$ (cf. section 1.6). As we already noted in section 1.6, we have $H_r^* = \text{Coker}\ F_{H^*}^r = H^* / I_r$, where I_r is the ideal in H^* generated by $\{x^{p^r} \mid \epsilon_{H^*}(x) = 0\}$. The bialgebras H_r^* are finite and local. Furthermore,

$$H^* = \varprojlim H_r^*$$

(see [1] ch. II.7). Since the functor $Sp\ L \to M(Sp\ L)$, defined in subsection 3.2.2, is an antiequivalence of categories, the canonical projections from H_{r+1}^* to H_r^* give rise to surjective D-module morphisms from $M(Sp\ H_{r+1}^*)$ to $M(Sp\ H_r^*)$. With respect to these morphisms define the D^F-module $M(G)$ by:

$$M(G) = \varprojlim M(Sp\ H_r^*).$$

$M(G)$ is called the *contravariant Dieudonné module* of the formal group G. The functor $G \to M(G)$ gives an antiequivalence between the category of connected formal groups of finite type and the category of D^F-modules N such that N / FN has finite length. G is finite if and only if $F^s M(G) = 0$ for some $s \geqslant 0$, G is smooth if and only if the action of F on M is injective (see [1] ch. III.9). Using the fact that the functor $G \to M(G)$ is an antiequivalence of categories and the fact that the action of F on $M(G)$ is induced by F_{H^*}, we get the relation

$$M(Sp\ H_r^*) = M(G) / F^r M(G).$$

3.2.4. Let $G = Spf\ H^*$ be a connected formal group of finite type. Put $M = M(G)$ and $M_r = M / F^r M$, so that $M_r = M(Sp\ H_r^*)$. Since the affine group $Sp\ H_r^*$ is infinitesimal, $M(Sp\ H_r^*)$ is by definition equal to $M(Sp\ H_r)^*$, hence $M(Sp\ H_r) = M_r^*$ (see subsection 3.2.2). Consequently, the triviality $F^r M_r = 0$ implies that $V^r M(Sp\ H_r) = 0$. Now $Sp\ H_r$ is a unipotent affine group, so we have $M(Sp\ H_r) = Bialg(A_r, H_r)$ (see subsection 3.2.1). We conclude that $M_r^* = Bialg(A_r, H_r)$.
Consider the isomorphism of bialgebras $\Phi_r \colon U_r \to A_r$ which is described explicitly in section 1.5. Write Φ_r^* for the group isomorphism from $Bialg(A_r, H_r)$ to $Bialg(U_r, H_r)$ which is induced by Φ_r. Then Φ_r^* is a group isomorphism from M_r^* to $C_{p,r}$ (see section 1.6). Next we shall describe the behaviour of Φ_r^* with respect to the D-module structure.
Let h be an element of $Bialg(A_r, H_r)$. Write $h = (h_0, \ldots, h_{r-1})$ where $h_k = h(X_k)$. Put $\phi = (\xi_0, \ldots, \xi_{r-1})$ for the image of h under Φ_r^*. Using the polynomials $F_i(X)$ which define Φ_r (see section 1.6), we get $\xi_k = F_{p^k}(h_0, \ldots, h_k)$. Now $Fh = (h_0^p, \ldots, h_{r-1}^p)$ (see [1], chap. III.3). Since the coefficients of the polynomials $F_i(X)$ are in the prime field of k, we have $\Phi_r^*(Fh) = (\xi_0^p, \ldots, \xi_{r-1}^p)$, which is $F\phi$ by definition (see section 1.2). So Φ_r^* commutes with F. Furthermore, $Vh = (0, h_0, \ldots, h_{r-2})$ (see [1], loc. cit.). From the properties of the polynomials $F_{p^k}(X)$ (see section 1.5) it follows that $\Phi_r^*(Vh) = (0, \xi_0, \ldots, \xi_{r-2})$, which is $V\phi$. So Φ_r^* also commutes with V. Finally, putting $\alpha = T(a)$, $a \in k$, it follows from the definitions in subsection 3.2.1 that $\alpha h = (a^{p^{1-r}} h_0, \ldots, a h_{r-1})$. Since the polynomials $F_i(X)$ are isobaric of weight i, we get $\Phi_r^*(\alpha h) = (a^{p^{1-r}} \xi_0, \ldots, a \xi_{r-1}) = [a^{p^{1-r}}]\phi$. We conclude that Φ_r^* induces an isomorphism of D-modules:

$$\Phi_r^* \colon M_r^{(1-r)*} \to C_{p,r}.$$

Lemma 3.2.1:
The action of F on M induces a morphism of D-modules $F\colon M_{r+1}^{(1-r)} \to M_{r+1}^{(-r)}$, which in turn induces an $f\colon M_r^{(1-r)} \to M_{r+1}^{(-r)}$. The morphism $f^*\colon M_{r+1}^{(-r)*} \to M_r^{(1-r)*}$ corresponds to the canonical map: $C_{p,r+1} \to C_{p,r}$.
Proof: In view of the explicit description of Φ_r^* given above, it is enough to show that $f^*(h) = (h_0, \ldots, h_{r-1})$ for an element $h = (h_0, \ldots, h_r)$ of $M_{r+1}^* = Bialg(A_{r+1}, H_{r+1})$.
The morphism $F_{H_{r+1}^*}\colon (H_{r+1}^*)^{(1-r)} \to (H_{r+1}^*)^{(-r)}$ factorizes through $(H_r^*)^{(1-r)}$, giving a morphism F' of bialgebras. So, putting p_r for the canonical projection: $(H_{r+1}^*)^{(1-r)} \to (H_r^*)^{(1-r)}$, we have:

$$F_{H_{r+1}^*} = F' \circ p_r.$$

Successive application of the functors * (i.e. taking the linear dual), Sp and M to this relation will give the proof of the lemma.
First of all, $F_{H_{r+1}^*}^* = V_{H_{r+1}}$ and $M(Sp\ V_{H_{r+1}})$ is the morphism $V\colon M_{r+1}^{(-r)*} \to M_{r+1}^{(1-r)*}$. Furthermore, $M(Sp\ F') = f$ and using the fact that * commutes with the composition $M \circ Sp$ (see subsection 3.2.2) we get $M(Sp\ F'^*) = f^*$. So we have the relation:

$$V = M(Sp\ p_r^*) \circ f^*. \tag{3.2.1}$$

In order to determine $M(Sp\ p_r^*)$ we first notice the following: Let $Sp\ B$ be a unipotent affine group such that $V^r M(Sp\ B) = 0$. As we already said, $M(Sp\ B)$ is isomorphic to $Bialg(A_r, B)$. But we also have $V^{r+1}M(Sp\ B) = 0$, hence $M(Sp\ B)$ also is isomorphic to $Bialg(A_{r+1}, B)$. Consequently, the transition morphism: $Bialg(A_r, B) \to Bialg(A_{r+1}, B)$ in the inductive limit which defines $M(Sp\ B)$ (see subsection 3.2.1), is an isomorphism. The image of an element $h = (h_0, \ldots, h_{r-1})$ under this isomorphism is $(0, h_0, \ldots, h_{r-1})$.
We apply this to $B = H_r$. The action of $M(Sp\ p_r^*)$, regarded as a morphism: $Bialg(A_{r+1}, H_r) \to Bialg(A_{r+1}, H_{r+1})$ is obvious. We conclude that $M(Sp\ p_r^*)$ maps an element $(h_0, \ldots, h_{r-1})$ of $M_r^* = Bialg(A_r, H_r)$ onto the element $(0, h_0, \ldots, h_{r-1})$ of M_{r+1}^*. So if we start with an element $h = (h_0, \ldots, h_r)$ of M_{r+1}^*, relation (3.2.1) gives: $(0, h_0, \ldots, h_{r-1}) = M(Sp\ p_r^*) \circ f^*(h)$. Since $M(Sp\ p_r^*)$ is injective, it follows that $f^*(h) = (h_0, \ldots, h_{r-1})$. ***

We have shown:

Theorem 3.2.2:
Let G be a connected formal group of finite type over a perfect field of characteristic $p > 0$, with covariant Dieudonné module C_p and contravariant Dieudonné module M. Then

$$C_p \simeq \varprojlim (M / F^r M)^{(1-r)*},$$

the transition maps in the projective limit being induced by the action of F on M.

Remarks: 1) For the smooth case, this relation between C_p and M may also be found in [10] (proposition 3.3 in the "complêments" combined with the theory of ch. III.5).
2) If G is a *truncated* formal group, i.e. its contravariant bialgebra H^* has the form

$k[X_1, \ldots, X_n]/(X_1^{p^{r_1}}, \ldots, X_n^{p^{r_n}})$, it follows from the definitions that the module $C_p(G)$ is zero. In this case, the module $C_{p,r}(G)$ of p-typical curves of length $r = \max\{r_i \mid 1 \leqslant i \leqslant n\}$ plays the decisive role for classification. This illustrates the importance of the fact that the isomorphism Φ_r^* in subsection 3.2.4 holds for all connected formal groups and not only for smooth ones.

Proposition 3.2.3:
Assume G is smooth of finite type and the Verschiebung morphism $V_G\colon G^{(1)} \to G$ has a finite kernel. Then $C_p = Mod_W(M^{(1)}, W)$.

Remarks:
1) In this case G is a connected *p-divisible* group (cf. [1], the proposition in ch II.11), hence M has no W-torsion (cf. [1], chap. III.8).
2) The proof of this proposition is essentially given in [10] (proposition 3.4 in the "compléments"). We shall adapt this proof to our situation.

Proof: First of all, note that Ker V_G is connected: Ker $V_G = Spf$ Coker V_{H^*} and Coker $V_{H^*} = (H^*)^{(1)}/I$, where I is the ideal in $(H^*)^{(1)}$ generated by $\{Vx \mid \epsilon_{H^*}(x) = 0\}$ (cf. [1] ch. II.6 or [5] ch. 1.3.5), hence as an algebra, Coker V_{H^*} is local. It follows that the definition of the contravariant Dieudonné module, given in subsection 3.2.3, applies to Ker V_G. Since the functor $G \to M(G)$ is an antiequivalence of categories, we have $M(\text{Ker } V_G) = (M / VM)^{(1)}$. Using the assumption that Ker V_G is finite, we get the existence of an integer s such that $F^{s-1}M(\text{Ker } V_G) = 0$ (see subsection 3.2.3). It follows that $F^s M$ is contained in pM. We define the map

$$\Psi\colon Mod_W(M^{(1)}, W) \to \varprojlim (M / F^r M)^{(1-r)*}$$

as follows: let f be an element of $Mod_W(M^{(1)}, W)$. For $r > 0$ we define the map $g_r\colon M / F^r M \to K / W$ as follows: If x is an element of M then g_r maps the class of x modulo $F^r M$ onto the class of $p^{-r} f(V^r x)$ modulo W. Note that g_r is well-defined, that it is an element of $(M / F^r M)^{(1-r)*}$ and that $g_{r+1} \circ F = g_r$. Consequently, $\Psi(f) := (g_1, g_2, \cdots)$ is an element of the projective limit of the $(M / F^r M)^{(1-r)*}$. Using the definitions of the D-module structures given in section 3.1, a straightforward verification shows that Ψ is a morphism of D-modules.
We shall complete the proof by showing that Ψ is bijective. Suppose $\Psi(f) = 0$, in other words $g_r = 0$ for all $r > 0$. Then $f[V^r M]$ is contained in $p^r W$ for all $r > 0$. In view of the fact that $F^s M$ is contained in pM, we find for all $k > 0$: $p^{ks} f[M] = f[V^{ks} F^{ks} M]$ is contained in $p^k f[V^{ks} M]$, which in turn is contained in $p^{k+ks} W$. Consequently, $f[M]$ is in $p^k W$ for all $k > 0$. It follows that f is 0, hence Ψ is injective. For the surjectivity of Ψ: let $g = (g_1, g_2, \cdots)$ be an element of $\varprojlim (M / F^r M)^{(1-r)*}$, so $g_{r+1} \circ F = g_r$. For an element x of M we define elements x_k $(k \geqslant 1)$ of M by $p^k x_k = F^{ks} x$. Since M has no p-torsion, the x_k are unique. Here we have made essential use of the assumption that G is p-divisible. From the unicity of the x_k it follows that $p x_{k+1} = F^s x_k$ for all $k > 0$. For each k, let a_k be an element of K such that, writing $\bar{x_k}$ for the class of x_k, $g_{ks}(\bar{x_k})$ is the class of $p^{-k} a_k$ in K / W. Obviously, a_k is unique modulo $p^k W$. Furthermore we have: $p^k g_{ks}(\bar{x_k}) = g_{ks}(p^k \bar{x_k}) = g_{ks} F^{ks}(\bar{x}) = 0$ in K / W, hence a_k even is in W. The class of $p^{-k} a_{k+1}$ in K / W is

$pg_{(k+1)s}(\overline{x_{k+1}}) = g_{(k+1)s}(F^s \overline{x_k}) = g_{ks}(\overline{x_k})$ (recall the property $g_{r+1} \circ F = g_r$), hence $a_{k+1} \equiv a_k \mod p^k W$. So the sequence of the a_k converges to an element a of W. We define the map $f: M \to W$ by $f(x) = a$. A straightforward verification shows that f is an element of $Mod_W(M^{(1)}, W)$. We claim that $\Psi(f) = g$, in other words: $g_r(\overline{x})$ is the class of $p^{-r}f(V^r x)$ in K / W for all $r > 0$. In order to prove this claim, we first notice the following: let $y \in M$ and let $y_r \in$ M be such that $F^{rs}y = p^r y_r$. Put $b = f(y)$ and let $b_r \in W$ be such that $g_{rs}(\overline{y_r})$ is the class of $p^{-r}b_r$ in K / W. It then follows from the definition of f that $b \equiv b_r \mod p^r W$, hence $g_{rs}(\overline{y_r}) =$ the class of $p^{-r}f(y)$ in K / W. Apply this to $y = V^r x$: then $y_r = V^r x_r$, hence the class of $p^{-r}f(V^r x)$ in K / W is equal to $g_{rs}(V^r \overline{x_r}) = g_{rs+r} \circ F^r(V^r \overline{x_r}) = g_{rs+r}(p^r \overline{x_r}) = g_{rs+r}(F^{rs}\overline{x}) = g_r(\overline{x})$. ***

§ 3.3. Application of the Formulas obtained in § 3.2

Let $0 < n < \infty$ and $0 < m \leqslant \infty$. Put $M = D^F / D^F(F^m - V^n)$. Then M is the contravariant Dieudonné module of a smooth n-dimensional formal group. This formal group is often denoted $G_{n,m}$ (see e.g. [16] chap. II.4). It is well-known that $G_{n,m}$ has F-type

$$F = L_0 + V^m L_m,$$

where L_0 is a superdiagonal matrix: it has $(i, i+1)$-entry 1 for $1 \leqslant i < n$ and all other entries 0, L_m has $(n,1)$-entry 1 and all other entries 0 (see [5] chap. III.5 and [13] section 5 examples 5.1 and 5.2). So if $\phi = (\phi^{(1)}, \ldots, \phi^{(n)})$ is a basic element of $C_p^n(G_{n,m})$ with $F\phi = L_0\phi + V^m L_m \phi$ then $F\phi^{(i)} = \phi^{(i+1)}$ for $1 \leqslant i < n$ and $F^n\phi^{(1)} = F\phi^{(n)} = V^m\phi^{(1)}$. It is an easy exercise to show that the D^V-module $C_p(G_{n,m})$ is isomorphic to $D^V / D^V(F^n - V^m)$. This result can also be obtained by the application of theorem 3.2.2 and proposition 3.2.3. In order to give an illustration how these formulas work, we shall carry this out explicitly.

3.3.1. The case $m = \infty$, hence $M = D^F / D^F V^n$. In this case we are dealing with $G_{n,\infty}$, the Witt group of length n (cf. [5], loc. cit. or [13], loc. cit.). Since Ker V_G is not finite, we use theorem 3.2.2 in this case.

Lemma 3.3.1:
An element of $D^F / D^F V^n$ may be represented by an element γ of D^F having the form:

$$\gamma = \sum_{i=1}^{n-1} c_{-i} V^i + \sum_{i=0}^{\infty} c_i F^i$$

where the c_i are elements of W, c_{-i} is unique mod $p^{n-i} W$ for $0 < i < n$ and c_i is unique mod $p^n W$ for $i \geqslant 0$.
Proof: Clearly, an element of $D^F / D^F V^n$ can be represented by such a γ. Assume γ is in $D^F V^n$, say $\gamma = \delta V^n$. Writing $\delta = \sum_{i>0} d_{-i} V^i + \sum_{i=0}^{\infty} d_i F^i$ we have $\gamma = \sum_{i \geqslant n} d_{n-i} V^i + \sum_{i=1}^{n-1} p^{n-i} d_{n-i} V^i + \sum_{i=0}^{\infty} p^n d_{n+i} F^i$, hence $c_{-i} = p^{n-i} d_{n-i}$ for $0 \leqslant i < n$ and $c_i = p^n d_{n+i}$ for $i \geqslant 0$. Consequently, $c_{-i} \equiv 0 \mod p^{n-i} W$ for $0 \leqslant i < n$ and $c_i \equiv 0 \mod p^n W$ for $i \geqslant 0$. ***

Lemma 3.3.2:
Let $r > n$. An element of $M \,/\, F^r M$ may be represented by an element γ of M having the form

$$\gamma = \sum_{i=1}^{n-1} c_{-i} V^i + \sum_{i=0}^{r-1} c_i F^i, \qquad (3.3.1)$$

the coefficients c_i being elements of W. c_{-i} is unique mod $p^{i-n} W$ for $0 < i < n$, c_i is unique mod $p^n W$ for $0 \leqslant i \leqslant r - n$ and unique mod p^{r-i} for $r - n < i < r$.
Proof: Clearly, an element of $M \,/\, F^r M$ can be represented by (3.3.1). Assume that γ is in $F^r M$, say $\gamma = F^r \delta$ for some δ in M. Writing $\delta = \sum_{i=1}^{n-1} d_{-i} V^i + \sum_{i=0}^{\infty} d_i F^i$ and using lemma 3.3.1 we find: $c_{-i} \equiv 0 \bmod p^{n-i} W$ for $0 < i < n$, $c_i \equiv 0 \bmod p^n W$ for $0 \leqslant i \leqslant r - n$ and $c_i \equiv 0 \bmod p^{r-i} W$ for $r - n < i < r$. ***

For the remainder of this section assume that $r > 2n$. We define the map $\phi_r \colon M \,/\, F^r M \to K \,/\, W$ as follows: Let γ be an element of $M \,/\, F^r M$ represented in the form (3.3.1). Then $\phi_r(\gamma) := p^{-n} c_{r-n}^{\sigma^{1-r}}$. Note that ϕ_r is well-defined (see lemma 3.3.2) and that ϕ_r is an element of $(M \,/\, F^r M)^{(1-r)^*} = C_{p,r}$. Using the definition of the D-module structure on $(M \,/\, F^r M)^{(1-r)^*}$ (see section 3.1) we find: $V^j \phi_r(\gamma) = \phi_r(F^j \gamma)^{\sigma^{-j}} = p^{-n} c_{r-n-j}^{\sigma^{1-r}}$ for $0 \leqslant j \leqslant r - n$. In the same way we find: $V^j \phi_r(\gamma) = p^{j-r} c_{r-n-j}^{\sigma^{1-r}}$ for $r - n < j < r$ and $F^j \phi_r(\gamma) = p^{j-n} c_{r-n+j}^{\sigma^{1-r}}$ for $0 \leqslant j < n$. Let ψ be an arbitrary element of $C_{p,r}$. Then for $0 < i < n$ we have $p^{n-i}\psi(V^i) = 0$ hence there exists an element x_{-i} in W such that $\psi(V^i) = p^{n-i} x_{-i}$. In an analogous way we find elements x_i of W such that $\psi(F^i) = p^{-n} x_i$ for $0 \leqslant i \leqslant r - n$ and $p^{i-r} x_i$ for $r - n < i < r$. A straightforward verification shows that $\psi = \sum_{j=0}^{r-1} x_{r-n-j} V^j \phi_r + \sum_{j=1}^{n-1} x_{r-n+j} F^j \phi_r$. It follows that $C_{p,r}$ is generated as a W-module by

$$\{F^j \phi_r \mid 1 \leqslant j < n\} \bigcup \{V^j \phi_r \mid 0 \leqslant j < r\}.$$

Clearly, for any element ψ of $C_{p,r}$ we have $F^n \psi = 0$ and $V^r \psi = 0$.

Lemma 3.3.3:
$C_{p,r}$ is isomorphic to $D \,/\, (DF^n + DV^r)$.
Proof: Let the D-module morphism Φ: $D \,/\, (DF^n + DV^r) \to C_{p,r}$ be defined by $\Phi(1) = \phi_r$. Note that Φ is well-defined and surjective. Suppose $\Phi(\xi) = 0$ for some element $\xi = \sum_{j=0}^{r-1} x_{-j} V^j + \sum_{j=1}^{n-1} x_j F^j$ of $D \,/\, (DF^n + DV^r)$. Let $\psi = \Phi(\xi)$. Then for $0 \leqslant k \leqslant r - n$ we have $\psi(F^k) = p^{-n} x_{n-r+k} = 0$ in $K \,/\, W$, hence for $0 \leqslant j \leqslant r - n$ the x_{-j} is in $p^n W$ and so $x_{-j} V^j = 0$. In the same way $\psi(F^k) = 0$ for $r - n < k < r$ implies that $x_j F^j = 0$ for $0 < j < n$ and $\psi(V^k) = 0$ for $0 < k < n$ implies that $x_{-j} V^j = 0$ for $r - n < j < r$. Consequently, $\xi = 0$, so Φ is injective. ***

The only thing left to be shown is that the transition morphism: $f^* \colon (M \,/\, F^{r+1} M)^{(-r)^*} \to (M \,/\, F^r M)^{(1-r)^*}$ (cf. lemma 3.2.1) corresponds to the canonical morphism: $D \,/\, (DF^n + DV^{r+1}) \to D \,/\, (DF^n + DV^r)$, in other words that ϕ_{r+1} is mapped onto ϕ_r. Let ψ be the image of ϕ_{r+1} under the transition morphism.

In order to find $\psi(\gamma)$ for some element γ of $M\,/\,F^r M$ we must consider $F\gamma$ as an element of $M\,/\,F^{r+1}M$ and take its image under ϕ_{r+1}. It then is a straightforward verification to show that indeed $\psi = \phi_r$.
It now follows that $C_p = \varprojlim C_{p,r} = D^V / D^V F^n$.

3.3.2. If m is finite then M is a free W-module having

$$\{F^i \mid 0 \leqslant i \leqslant m\} \bigcup \{V^i \mid 1 \leqslant i < n\}$$

as a W-basis: indeed, let ξ be an element of M then using the relations $V^n = F^m$ and $F^{m+1} = pV^{n-1}$ we find that ξ may be written in the form

$$\xi = \sum_{i=0}^{m} x_i F^i + \sum_{i=1}^{n-1} x_{-i} V^i \tag{3.3.2}$$

and if $\xi = 0$, in other words for some γ in D^F we have the relation $\xi = \gamma(F^m - V^n)$ in D^F, then an explicit calculation shows that all coefficients x_i are 0.

We use proposition 3.2.3 in order to determine C_p.
Let the map ϕ: $M \rightarrow W$ be defined as follows: if the element ξ of M is written in the form (3.3.2) then $\phi(\xi) := x_m^\sigma$. Note that ϕ is an element of $Mod_W(M^{(1)}, W) = C_p$. Then for $1 \leqslant j \leqslant n$ we have $F^j\phi(\xi) = \phi(V^j\xi)^{\sigma^j} = x_{j-n}^\sigma$. *(Warning:* $V^j\xi$ first must be written in the form (3.3.2)!) In the same way we find for $0 \leqslant j \leqslant m$ that $V^j\phi(\xi) = x_{m-j}^\sigma$. So the set

$$\{F^j\phi \mid 1 \leqslant j < n\} \bigcup \{V^j\phi \mid 0 \leqslant j \leqslant m\}.$$

is a W-basis of C_p, it is dual (in the sense of semilinear maps) to the W-basis of M mentioned above. Furthermore we have $F^n\phi = V^m\phi$. We conclude that C_p is isomorphic to $D^V / D^V(F^n - V^m)$.

§ 3.4. Basic Concepts of the Contravariant Classification Theory

In this section the groundfield k is assumed to be algebraically closed. For convenience of the reader we describe the fundamental concepts which are used for the contravariant classification in [16]. The concept of a *special* module, however, will be generalized. The basic properties of special modules will turn out to be true also for this greater class of modules. It will give one more discrete invariant in the classification of the 2-dimensional smooth commutative formal groups, as we shall see in chapter 4. This new invariant does not depend on the discrete invariants that we get without this generalization. In higher dimensions the generalization may even give more information about the module space.

3.4.1. A morphism $f: G_1 \rightarrow G_2$ of formal groups is called an *isogeny* if both its kernel and its cokernel are finite. If such a morphism exists, the formal groups G_1 and G_2 are said to be *isogenous.* The contravariant Dieudonné modules $M(G_1)$ and $M(G_2)$ of two connected finite type formal groups G_1 and G_2 are called isogenous if G_1 and G_2 are so. Let D_F be the noncommutative ring of Laurent series $W((F))$ with multiplication rule $Fa = a^\sigma F$ (a being an element of W). Then D_F is a right D^F-module by defining $1.V = pF^{-1}$. The *F-localization* of a D^F-module M is the D_F-module $M_F := D_F \otimes_{D^F} M$. Note that the F-localization of D^F is isomorphic to D_F. If M and N are Dieudonné modules of connected finite type formal groups then M and N are isogenous if and only if M_F and N_F are isomorphic D_F-modules(cf. [16]

ch. II prop. 2.1). A submodule N of M is said to be *dense* in M if N_F is isomorphic to M_F.
For the remainder of this section let M be the contravariant Dieudonné module of a smooth n-dimensional formal group G. Then there exist integers t, m_i, n_i, d_i $(1 \leq i \leq t)$ with $0 \leq m_i \leq \infty$, $1 \leq n_i < \infty$, $\gcd(m_i, n_i) = 1$ (by convention $gcd(\infty, n_i) = 1$) and $\sum_{i=0}^{t} n_i d_i = n$ such that M is isogenous to $\bigoplus_{i=1}^{t} (D^F / D^F(F^{m_i} - V^{n_i}))^{d_i}$, called the *isogeny type* of M. If $t = 1$ M is called *homogeneous.* In that case we shall write m for m_1, n for n_1 and d for d_1. The dimension of the k-vector space M/pM, which is $n + \sum_{i=1}^{t} m_i d_i$, is called the *height* of M. M has finite height if and only if G is p-divisible and then M is a free W-module, its W-rank equals the height of M (cf. [1] ch. III.8). (In that case, using the terminology of [1] ch. IV, M is an F-lattice having slopes $\dfrac{n_i}{m_i + n_i}$ and multiplicities $d_i(m_i + n_i)$ $(1 \leq i \leq t)$). M is said to be *isosimple* if its isogeny type consists of only one direct summand, otherwise M is called *decomposable.*

3.4.2. Suppose M is homogeneous of finite height. Let j be an integer such that $1 \leq j \leq d$. We shall call M *j-special* if $F^{mj}M = V^{nj}M$. An element x of M is called *j-special* if $F^{mj}x = V^{nj}x$.

Lemma 3.4.1: (generalization of [16] lemma 3.3)
If M is homogeneous of finite height then M is j-special if and only if, as a W-module, it has a basis consisting of j-special elements.
Proof: Let $\{x_1, x_2, \ldots, x_{m+n}\}$ be a W-basis of M consisting of j-special elements and let $w = \sum_{i=1}^{m+n} a_i x_i$ be an element of M. Then $F^{mj}w = \sum_{i=1}^{m+n} a_i^{\sigma^{mj}} V^{nj} x_i \in V^{nj}M$ and $V^{nj}w = \sum_{i=1}^{m+n} a_i F^{mj} x_i = F^{mj} \sum_{i=1}^{m+n} a_i^{\sigma^{-mj}} x_i \in F^{mj}M$. It follows that $F^{mj}M = V^{nj}M$.
Conversely, suppose M is j-special. Then the map $\Phi: M \to M$ defined by $V^{nj}\phi(w) = F^{mj}w$ is $\sigma^{(m+n)j}$-semilinear and bijective. Let A be the matrix of Φ with respect to a given W-basis of M. We have to show the existence of another basis, such that the matrix of Φ with respect to this new basis is the identity matrix. In other words: we have to show the existence of an invertible matrix T in $M_{m+n}(W)$ such that $T^{-1}AT^{\sigma^{(m+n)j}} = 1$.
A straightforward generalization of [10], Satz 11 (we already mentioned this generalization in the proof of proposition 1.10.1), shows the existence of an invertible T_1 such that $T_1^{-1}AT_1^{\sigma^{(m+n)j}} \equiv 1 \bmod p$. Let $s > 0$ and assume we have an invertible matrix T_s such that $T_s^{-1}AT_s^{\sigma^{(m+n)j}} = 1 + p^s C$ for some $C = (c_{ik})$ in $M_{m+n}(W)$. From [5] ch. III.5 lemma 1 it follows that for each i,k the equation $X^{\sigma^{(m+n)j}} - X + c_{ik} = 0$ has a solution λ_{ik} in W. Put $\Lambda = (\lambda_{ik})$, then $C + \Lambda^{\sigma^{(m+n)j}} = \Lambda$. Take $T_{s+1} = T_s(1 + p^s\Lambda)$. Then T_{s+1} is invertible and $AT_{s+1}^{\sigma^{(m+n)j}} = T_s(1 + p^sC)(1 + p^s\Lambda^{\sigma^{(m+n)j}}) \equiv T_{s+1} \bmod p^{s+1}$. The sequence of matrices T_s converges to the desired matrix T. ***

Next let us drop the assumption that M is homogeneous. Let $\mathrm{j} = (j_1, \ldots, j_t)$ be a t-tuple of integers such that $1 \leqslant j_i \leqslant d_i$ for $1 \leqslant i \leqslant t$. M is called *j-special* if M is, as a D^F-module, isomorphic to the direct sum of homogeneous, j_i-special D^F-modules M_i $(1 \leqslant i \leqslant t)$. If $\mathrm{j} = (1, \ldots, 1)$ we shall use the term *special* instead of j-special. By this convention the term special has the same meaning as it has in Manin's article [16].

Theorem 3.4.2: (generalization of [16] theorem 3.1)
Suppose M has finite height. Let $\mathrm{j} = (j_1, \ldots, j_t)$ be a t-tuple of integers such that $1 \leqslant j_i \leqslant d_i$ for $1 \leqslant i \leqslant t$. Then M has a unique maximal j-special sub-D^F-module M_j. This M_j is isogenous to M.
Proof: First of all, assume that M is homogeneous. Then the isomorphism between M_F and the F-localization of $(D^F / D^F(F^m - V^n))^d$ defines a D^F-module embedding of M into $(D_F / D_F(F^{m+n} - p^n))^d$ (recall that F acts injectively on M, so that M can be embedded in M_F). We identify M with its image under this embedding. Let M_j be the sub-W-module of M generated by the j-special elements of M. One easily verifies that M_j is a sub-D^F-module of M. In view of lemma 3.4.1, M_j is the maximal special sub-D^F-module of M. Since the element $(1,0, \ldots, 0)$ of $(D_F / D_F(F^{m+n} - p^n))^d$ is an element of M_F there must be a $k_1 \geqslant 0$ such that $(F^{k_1},0, \ldots, 0)$ is in M. Since this is a j-special element it is even in M_j. With the same arguments we find elements of the form $(0, \ldots, F^{k_i}, \ldots, 0)$ $(1 \leqslant i \leqslant d)$ in M_j. But then $(M_j)_F$ is isomorphic to M_F, hence M_j is isogenous to M.
Next let us assume that M is not homogeneous. In the same way as in the homogeneous case, we identify M with a sub-D^F-module of $\bigoplus_{i=1}^{t} (D_F / D_F(F^{m_i+n_i} - p^{n_i}))^{d_i}$. Let M_{j_i} be the maximal j_i-special sub-D^F-module of the intersection of M and $(D_F / D_F(F^{m_i+n_i} - p^{n_i}))^{d_i}$ (note that this intersection is homogeneous). Define $M_j = \bigoplus_{i=1}^{t} M_{j_i}$. Obviously, M_j is the maximal j-special sub-D^F-module of M and it is isogenous to M. ***

3.4.3. In this subsection assume that M is isosimple of finite height. Let a and b be integers such that $am - bn = 1$. Define the noncommutative ring $E_{n,m} = W(\mathbf{F}_{p^{m+n}})[\theta]$ with multiplication rules $\theta c = c^{\sigma^{-(a+b)}}\theta$ $(c \in W(\mathbf{F}_{p^{m+n}}))$ and $\theta^{m+n} = p$. Note that with these definitions we have $\theta^n c = c^\sigma \theta^n$ and $c\theta^m = \theta^m c^\sigma$. Further define $K_{n,m} = K(\mathbf{F}_{p^{m+n}}) \otimes_{W(\mathbf{F}_{p^{m+n}})} E_{n,m}$ (note that $K_{n,m}$ may be regarded as the quotient field of $E_{n,m}$), the ring $R_{n,m} = W \otimes_{W(\mathbf{F}_{p^{m+n}})} E_{n,m}$ and finally put $M_{n,m} = K \otimes_{W(\mathbf{F}_{p^{m+n}})} E_{n,m}$. $R_{n,m}$ and $M_{n,m}$ have the following D^F-module structure: if $w \otimes x$ is an element of $R_{n,m}$ or $M_{n,m}$ then $F(w \otimes x) = w^\sigma \otimes \theta^n x$ and $V(w \otimes x) = w^{\sigma^{-1}} \otimes \theta^m x$. With these definitions $M_{n,m}$ also has the structure of a D_F-module (identify F^{-1} with $p^{-1}V$) and $(R_{n,m})_F$ is isomorphic to $M_{n,m}$. It is easily verified that the morphism Φ: $D_F / D_F(F^{m+n} - p^n) \rightarrow M_{n,m}$ defined by $\Phi(1) = 1$ is an isomorphism, hence M_F is isomorphic to $M_{n,m}$. Note that as a D_F-module $M_{n,m}$ is generated by 1. Furthermore, one easily sees that a nonzero element x of $M_{n,m}$ uniquely can be written in the form

$$x = \sum_{i=i_0}^{\infty} a_i \theta^i$$

where the a_i are elements of Im(T) and a_{i_0} is nonzero. Clearly, $F^m x = V^n x$ if and only if all a_i are in $W(F_{p^{m+n}})$, which is the case if and only if x is in $K_{n,m}$. Define the *order* $\nu(x)$ of x to be the integer i_0. If $x = 0$ then define $\nu(x) = \infty$. It is clear that $\nu(Fx) = \nu(x) + n$ and $\nu(Vx) = \nu(x) + m$ for all x in $M_{n,m}$.
Identify $R_{n,m}$ with the set of those elements of $M_{n,m}$ which have a nonnegative order.

Proposition 3.4.3:
The D^F-module endomorphisms of $M_{n,m}$ (resp. $R_{n,m}$) are the right multiplication by elements of $K_{n,m}$ (resp. $E_{n,m}$).
Proof:
Note that for any element x of $M_{n,m}$ there exists an element ξ of D_F such that $x = \xi.1$. Let Φ be an endomorphism of $M_{n,m}$. Note that because D_F is the F-localization of D^F, Φ also is D_F-linear, hence $\Phi(x) = \xi\Phi(1)$ and $V^n\Phi(1) = F^m\Phi(1)$ hence $\Phi(1)$ is in $K_{n,m}$. Conversely, right multiplication by an element of $K_{n,m}$ is an endomorphism of $M_{n,m}$. Φ defines an endomorphism of $R_{n,m}$ if and only if $\Phi(1)$ is in $E_{n,m}$. Since an endomorphism of $R_{n,m}$ extends uniquely to an endomorphism of $M_{n,m}$ the proof is now done. ***

Lemma 3.4.4:
There exists an embedding of D^F-modules Ψ: $M \to M_{n,m}$ such that $\min\{\nu(\Psi(x)) \mid x \in M\} = 0$.
Proof: We already have noticed that the D_F-modules M_F and $M_{n,m}$ are isomorphic. An isomorphism from M_F to $M_{n,m}$ gives an embedding Φ: $M \to M_{n,m}$ of D^F-modules. Let $\{x_1, \ldots, x_{m+n}\}$ be a W-basis of M, and let $s :=$ $\min\{\nu(\Phi(x_i)) \mid 1 \leqslant i \leqslant m + n\}$. It follows that $\min\{\nu(\Phi(x)) \mid x \in M\} = s$. The map Ψ: $M \to M_{n,m}$ defined by $\Psi(x) = \Phi(x)\theta^{-s}$ is the desired embedding. ***

We are now ready to define the *order* of an element x of M.

Proposition 3.4.5:
Let x be an element of M, and let Ψ: $M \to M_{n,m}$ be an embedding such that $\min\{\nu(\Psi(x)) \mid x \in M\} = 0$. Then the order of x is well-defined by $\nu(x) := \nu(\Psi(x))$.
Proof: Suppose Φ: $M \to M_{n,m}$ is another embedding of D^F-modules such that $\min\{\nu(\Phi(x)) \mid x \in M\} = 0$. Then $\Phi\Psi^{-1}$ is a D^F-module isomorphism between $\Psi[M]$ and $\Phi[M]$, which may uniquely be extended to a D^F-module automorphism of $M_{n,m}$. Consequently, there exists an element z in $K_{n,m}$ such that $\Phi(x) = \Psi(x)z$ for all x in M (see [1] IV.3). From the fact that $\min\{\nu(\Phi(x)) \mid x \in M\} = 0$ and the similar property of Ψ it follows that $\nu(z) = 0$, hence $\nu(\Phi(x)) = \nu(\Psi(x))$ for all x in M. ***

The following properties are immediately verified :
a) $\nu(x+y) \geqslant \min\{\nu(x),\nu(y)\}$ and if $\nu(x) \neq \nu(y)$ then equality holds.
b) $\nu(\alpha x) \geqslant \nu(x)$, where x and y are elements of M and α is an element of D^F.

The set $J(M)$ is by definition the set $\{\nu(x) \mid x \in M\}$ (see [16], ch. III.3). Since M is a Dieudonné module, the set $J(M)$ is invariant under translations of the form $i \to i + cm + dn$, c and d being nonnegative integers. From proposition 3.4.5 it follows that for two isomorphic Dieudonné modules M and N we have $J(M) = J(N)$,

hence the set $J(M)$ is an isomorphism invariant.
Define $\bar{J}(M)$ to be the set $N \setminus J(M)$. It is an easy exercise to show that $\bar{J}(M)$ is a finite set, its largest member will be denoted $\max\bar{J}(M)$.

For the remainder of this section we identify M with its image under an embedding Ψ such that $\min\{\nu(\Psi(x)) \mid x \in M\} = 0$. Note that any sub-$D^F$-module N of $R_{n,m}$ such that $\min\{\nu(x) \mid x \in N\} = 0$ always is an isosimple D^F-module with isogeny type $D^F / D^F(F^m - V^n)$: indeed, N is a sub-W-module of the free finitely generated W-module $R_{n,m}$, hence N itself is free and finitely generated over W. Since N contains nonzero elements, $N_F = M_{n,m}$.
Furthermore, note that M is complete with respect to the θ-adic topology: indeed, M is a finitely generated W-module, hence it is complete with respect to the p-adic topology, and since $\theta^{m+n} = p$ the p-adic and θ-adic topologies are the same.

Proposition 3.4.6: (cf. [16] lemma 3.9)
a) For each $s \in J(M)$ there exists a unique z_s in M having the form

$$z_s = \theta^s + \sum_{k \notin J(M),\, k>s} x_{sk}\,\theta^k ,$$

with all $x_{sk} \in \mathrm{Im}(T)$.
b) M is generated as a D^F-module by $\{z_s \mid s-m \notin J(M) \text{ and } s-n \notin J(M)\}$.
c) M is special if and only if all z_s are special (in other words: all z_s are in $K_{n,m}$).
d) The x_{sk} are almost uniquely determined (i.e. unique up to a finite number of possibilities).
Proof:
a) For each s in $J(M)$ let w_s be an element of M having order s. As we already observed, there exist elements y_{sk} in Im(T) such that $w_s = \sum_{k=0}^{\infty} y_{sk}\,\theta^k$. Since w_s has order s, $y_{ss} \neq 0$. We may even assume that $y_{ss} = 1$ for if not so, we take $y_{ss}^{-1}w_s$ instead of w_s. For given s let k_0 be the smallest integer in $J(M)$ such that $k_0 > s$ and $y_{sk_0} \neq 0$. Let $w_s' = w_s - y_{sk_0}w_{k_0}$. By repeating this step we find a sequence in M which converges to z_s with respect to the θ-adic topology.

b) From a) it follows that for $s > \max\bar{J}(M)$ we have $\theta^s = z_s$, hence θ^s is in M. From the completeness of M it follows that all elements of $M_{n,m}$ having order greater than $\max\bar{J}(M)$ are in M. Let w be an element of M. Let r be the greatest integer for which $p^{-r}w$ is in M. Then $\nu(p^{-r-1}w) \leqslant \max\bar{J}(M)$, hence $\nu(p^{-r}w) \leqslant \max\bar{J}(M) + m + n$. Let c, d and s be nonnegative integers with the following properties: s is an element of $J(M)$, $s - m$ and $s - n$ are not in $J(M)$ and $\nu(p^{-r}w) = s + cm + dn$. (If $n = 0, m = 1$ we take $c = 0$.) It follows that both c and d are bounded by $\max\bar{J}(M) + m + n$. Write $w = \sum_{k=k_0}^{\infty} x_k\theta^k$ with $k_0 = \nu(w)$ and all x_k in Im(T). Define $w^{(1)} = w - x_{k_0}p^rF^cV^dz_s$. Then $\nu(w^{(1)}) > \nu(w)$ and $w^{(1)}$ is in M. By repeating this step we find a sequence $w^{(i)}$ in M which converges to 0. $w - w^{(i)}$ is of the form $\sum\alpha_s^{(i)}z_s$, the summation being restricted to the numbers s such that $s - m$ and $s - n$ are not in $J(M)$. The $\alpha_s^{(i)}$ are elements of D^F. From the construction of the $w^{(i)}$ it follows that the sequences $(\alpha_s^{(i)})_{i\geqslant 1}$ converge p-adically in D^F.

c) Suppose that all z_s are special. By an argument which is similar to the first part of the proof of lemma 3.4.1 it follows that M is special. Conversely, suppose that M is special. Let s be an element of $J(M)$. Then there exists an element z' of M such that $F^m z_s = V^n z'$. It follows that $z' = \theta^s + \sum_{k>s,\, k \notin J(M)} x_{sk}^{\sigma^{m+n}} \theta^k$ (see part a of this proposition). From the unicity of z_s it follows that z' $= z_s$, hence z_s is special.

d) Let V be the set of sub-D^F-modules N of $M_{n,m}$ such that N is isomorphic to M and $\min\{\nu(x) \mid x \in N\} = 0$. For given $N \in V$ and $s \in J(M)\ (= J(N))$ let $z_s^{(N)}$ be the unique element of N having the same form as z_s (again see part a)). Let Γ be the multiplicative group $\{\alpha \in K_{n,m} \mid \nu(\alpha) = 0\}$. Let s be an element of $J(M)$. One easily verifies that Γ acts transitively on the set $\{z_s^{(N)} \mid N \in V\}$ in the following way: if α is an element of Γ then it maps $z_s^{(N)}$ onto $z_s^{(N\alpha)}$. The proof will be finished by showing that the stabilizer of z_s has finite index in Γ. Let Γ_0 be the subgroup of Γ consisting of the elements of the form $1 + \sum_{k>\max\bar{J}(M)} y_k \theta^k$. Γ_0 obviously has finite index in Γ. Let α be an element of Γ_0. Then $z_s\alpha = z_s + w$ for some w with $\nu(w) > \max\bar{J}(M)\ (= \max\bar{J}(M\alpha))$. As we already noticed in the proof of part b), w is an element of $M\alpha$. It follows that z_s is in $M\alpha$ and from the unicity of $z_s^{(M\alpha)}$ in $M\alpha$ it follows that $z_s = z_s^{(M\alpha)}$. Consequently, Γ_0 is contained in the stabilizer of z_s, hence this stabilizer has finite index in Γ. ***

Corollary 3.4.7:
If $n = 1$ or $m = 1$ then $M = R_{n,m}$.
Proof: Immediate from the fact that $\bar{J}(M)$ is empty. ***

Corollary 3.4.8:
If $n = 2$ or $m = 2$ then M is special if and only if z_0 is special.
Proof: M is generated as a D^F-module by $\{z_0, \theta^{2i+1}\}$ for some nonnegative integer i. ***

Corollary 3.4.9:
If $n = 2$ or if $m = 2$ then two special D^F-modules M and N are isomorphic if and only if $J(M) = J(N)$.
Proof: Suppose that M and N are special and $J(M) = J(N)$. Let $\{z_0, \theta^{2i+1}\}$ be the generators of M and $\{z_0', \theta^{2i+1}\}$ the generators of N. From corollary 3.4.8 it follows that both z_0 and z_0' are in $K_{n,m}$. The map Ψ: $M \to N$ defined by $\Psi(w) = wz_0^{-1}z_0'$ is an isomorphism of D^F-modules. ***

4. THE RELATION BETWEEN THE TWO CLASSIFICATIONS

We are now able to describe the relation between the covariant and the contravariant classification. We shall work as follows: first we shall determine the contravariant Dieudonné module in terms of the covariant isomorphism invariants. Having found the contravariant Dieudonné module, we shall determine at which place in the contravariant classification list it occurs. In order to keep things self-contained, we shall review at the same time the contravariant classification. Except for propositions 4.4.9 and 4.4.10, this classification can be found in [16] chap. III.8. Propositions 4.4.9 and 4.4.10, however, are new. They give rise to a new discrete invariant in the contravariant classification, which is independent of those which already were known.
As we noticed in section 1.8, an F-type of an n-dimensional smooth formal group G can be regarded as a defining relation of the D^V-module $C_p(G)$. It will be useful to have an analogue of this for the D^F-module $M(G)$: a relation

$$V = \sum_{k=0}^{\infty} L_k F^k,$$

the L_k being elements of $M_n(W)$, is called a *defining relation* of $M(G)$ if

$$M(G) = (D^F)^n / (D^F)^n (V - \sum_{k=0}^{\infty} L_k F^k).$$

From the left-D^F-module analogue of [5] chap. III.4 prop. 7, it follows that such a defining relation always exists.
For the remainder of this chapter let G be a smooth 2-dimensional formal group, $M = M(G)$ and $C_p = C_p(G)$. We shall write e_1 and e_2 for the classes of (1,0) resp. (0,1) in $(D^F)^2 / (D^F)^2 (V - \sum_{k=0}^{\infty} L_k F^k)$.

§ 4.1. The Contravariant Dieudonné Module in Terms of the Covariant Isomorphism Parameters

4.1.1. In this subsection we shall give a defining relation of M in terms of the covariant isomorphism invariants of G.
Proposition 4.1.1:
Assume M has defining relation

$$V = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} F^{h_1} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} F^{h_1+h_2},$$

where $0 \leqslant h_1, h_2 \leqslant \infty$. Then G has F-type

$$F = V^{h_1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V^{h_1+h_2} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Proof: Obviously, the D^F-module M is isomorphic to

$$D^F / D^F (V - F^{h_1}) \oplus D^F / D^F (V - F^{h_1+h_2}).$$

From section 3.3 it follows that C_p is isomorphic to

$$D^V / D^V (F - V^{h_1}) \oplus D^V / D^V (F - V^{h_1+h_2}),$$

which completes the proof. ***

Proposition 4.1.2:
Assume M has defining relation

$$V = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} F^{h_1} + \sum_{i=0}^{n-1} \begin{pmatrix} 0 & 0 \\ 0 & d_i^{\sigma^{-h_1-2}} \end{pmatrix} F^{h_1+h_2+i} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} F^{h_1+h_2+h_3},$$

with $0 \leqslant h_1 < \infty$, $0 < h_2 \leqslant \infty$, $0 \leqslant h_3 \leqslant \infty$, $n = \min\{h_2,h_3\}$. Furthermore, all d_i are in $\mathrm{Im}(T)$, d_0 is nonzero (if $h_3 > 0$) and if $h_3 = \infty$ then $d_0 = 1$ and $d_i = 0$ for $i > 0$.
Then G has F-type

$$F = V^{h_1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \sum_{i=0}^{n-1} V^{h_1+h_2+i} \begin{pmatrix} 0 & 0 \\ 0 & d_i \end{pmatrix} + V^{h_1+h_2+h_3} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The proof is separated into the cases $h_2 = \infty$, $h_2 < h_3 = \infty$ and $h_2,h_3 < \infty$.

4.1.2. The case $\mathbf{h_2 = \infty}$

Let the subset M' of $D^F / D^F V^2$ be defined as follows: an element $\gamma = c_{-1}V + \sum_{i=0}^{\infty} c_i F^i$ of $D^F / D^F V^2$ is in M' if $c_i \equiv 0 \bmod p$ for $0 \leqslant i < h_1$ (so if $h_1 = 0$ then $M' = D^F / D^F V^2$). One easily verifies that M' is a sub-D^F-module of $D^F / D^F V^2$.

Lemma 4.1.3:
M and M' are isomorphic D^F-modules.
Proof: Let the morphism Φ: $M \to D^F / D^F V^2$ be defined by $\Phi(e_1) = F^{h_1}$ and $\Phi(e_2) = V$. Note that Φ is compatible with the defining relation of M. Let α be an element of M. Using the defining relation of M we may write

$$\alpha = \sum_{i=0}^{\infty} a_i F^i e_1 + \sum_{i=0}^{\infty} b_i F^i e_2,$$

with all a_i, b_i in $\mathrm{Im}(T)$. Then

$$\Phi(\alpha) = \sum_{i=h_1}^{\infty} a_{i-h_1} F^i + \sum_{i=0}^{\infty} p b_{i+1} F^i + b_0 V.$$

It follows that $\mathrm{Im}(\Phi)$ is contained in M'. Conversely, let $\gamma \in M'$. Then γ is the image of $\sum_{i=0}^{\infty} c_{h_1+i} F^i e_1 + c_{-1} e_2 + \sum_{i=1}^{h_1} p^{-1} c_{i-1} F^i e_2$ under Φ. The only thing left to be shown is the injectivity of Φ, so suppose $\Phi(\alpha) = 0$. From lemma 3.3.1 it follows that $b_0 \equiv 0 \bmod p$, $pb_{i+1} \equiv 0 \bmod p^2$ for $0 \leqslant i < h_1$ and $a_{i-h_1} + pb_{i+1} \equiv 0 \bmod p^2$ for $i \geqslant h_1$. But all a_i, b_i are in $\mathrm{Im}(T)$, hence they are 0 and so $\alpha = 0$. ***

Corollary 4.1.4:
M is isogenous to $D^F / D^F V^2$. If $h_1 = 0$ then M even is isomorphic to $D^F / D^F V^2$.
Proof: Obviously, $F^{h_1}(D^F / D^F V^2)$ is contained in M', hence M'_F is isomorphic to the F-localization of $D^F / D^F V^2$. ***

For the remainder of this subsection we shall identify M with M'.

Lemma 4.1.5:
Let $r > 0$ and $\gamma = c_{-1}V + \sum_{i=0}^{\infty} c_i F^i$ be an element of $D^F / D^F V^2$. Then γ is in $F^r M$ if and only if $c_{-1} \equiv 0 \bmod p$, $c_i \equiv 0 \bmod p^2$ for $0 \leqslant i < r - 1$ and $c_i \equiv 0 \bmod p$ for $r - 1 \leqslant i < r + h_1$.

Proof: Let $\alpha = a_{-1}V + \sum_{i=0}^{\infty} a_i F^i$ be an element of M such that $F^r\alpha = \gamma$. Now $F^r\alpha = pa_{-1}^{\sigma^r}F^{r-1} + \sum_{i=r}^{\infty} a_{i-r}^{\sigma^r}F^i$. From lemma 3.3.1 it follows that $c_{-1} \equiv 0 \bmod p$, $c_i \equiv 0 \bmod p^2$ for $0 \leqslant i < r - 1$ and (using the fact that α is in M) $c_i \equiv 0 \bmod p$ for $r - 1 \leqslant i < h_1 + r$. Conversely, if γ satisfies these conditions then $\alpha := p^{-1}c_{r-1}^{\sigma^{-r}}V + \sum_{i=0}^{\infty} c_{i+r}^{\sigma^{-r}}F^i$ is an element of M and $F^r\alpha = \gamma$. ***

Corollary 4.1.6:
An element of $M / F^r M$ may be represented by an element γ of M having the form

$$\gamma = c_{-1}V + \sum_{i=0}^{h_1+r-1} c_i F^i. \tag{4.1.1}$$

The coefficient c_{-1} is unique mod p, the c_i are unique mod p^2 for $0 \leqslant i < r - 1$ and they are unique mod p for $r - 1 \leqslant i < r + h_1$.

Let $r > h_1 + 1$. We define the σ^{1-r}-semilinear maps

$$\phi_r^{(i)}: M / F^r M \to K / W$$

for $i = 1,2$ as follows: Let γ be an element of $M / F^r M$, represented in the form (4.1.1), then $\phi_r^{(1)}(\gamma) := p^{-2}c_{r-2}^{\sigma^{1-r}}$ and $\phi_r^{(2)}(\gamma) := p^{-1}c_{r+h_1-1}^{\sigma^{1-r}}$. In view of corollary 4.1.6 these maps are well-defined. Notice: indeed the $\phi_r^{(i)}$ are σ^{1-r}-semilinear. Next regard $\phi_r^{(1)}$ and $\phi_r^{(2)}$ as elements of $C_{p,r}$. Then $(V^j\phi_r^{(1)})(\gamma) = \phi_r^{(1)}(F^j\gamma)^{\sigma^{-j}} = p^{-2}c_{r-2-j}^{\sigma^{1-r}}$ for $0 \leqslant j < r - 1$ and $p^{-1}c_{-1}^{\sigma^{1-r}}$ for $j = r - 1$. Furthermore, $V^j\phi_r^{(2)}(\gamma) = p^{-1}c_{r+h_1-1-j}^{\sigma^{1-r}}$ for $0 \leqslant j \leqslant h_1$. Let $\psi: M / F^r M \to K / W$ be an arbitrary σ^{1-r}-semilinear map. Then $p\psi(V) = \psi(pV) = 0$, hence there exists an x_{-1} in W such that $\psi(V) = p^{-1}x_{-1}$. In an analogous way we find elements x_i in W such that $\psi(pF^i) = p^{-1}x_i$ for $0 \leqslant i < h_1$, $\psi(F^i) = p^{-2}x_i$ for $h_1 \leqslant i < r - 1$ and $\psi(F^i) = p^{-1}x_i$ for $r - 1 \leqslant i < h_1 + r$. A straightforward verification shows that $\psi = \sum_{j=0}^{r-1} x_{r-2-j}V^j\phi_r^{(1)} + \sum_{j=0}^{h_1} x_{h_1+r-1-j}V^j\phi_r^{(2)}$. It follows that $C_{p,r}$ is generated as a W-module by $\{V^j\phi_r^{(1)} \mid 0 \leqslant j < r\} \cup \{V^j\phi_r^{(2)} \mid 0 \leqslant j \leqslant h_1\}$. In particular, taking $\psi = F\phi_r^{(1)}$ resp. $\psi = F\phi_r^{(2)}$, we find the relations $F\phi_r^{(1)} = V^{h_1}\phi_r^{(2)}$ resp. $F\phi_r^{(2)} = 0$. Obviously, $V^r\psi = 0$ for any element ψ of $C_{p,r}$.

Lemma 4.1.7:
Let N_r be the sub-D^V-module of $(D^V)^2$ defined by:

$$N_r = (D^V)^2(F - V^{h_1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) + (D^V)^2V^r.$$

Then the module $C_{p,r}$ of p-typical curves of length r is isomorphic to $(D^V)^2 / N_r$.
Proof: The following relations in $(D^V)^2 / N_r$ are easily verified: $p(V^i,0) = 0$ for $r - h_1 - 1 \leqslant i < r$, $p(0,V^i) = 0$ for all $i \geqslant 0$ and $p^2 x = 0$ for all x. Let the morphism Ψ: $(D^V)^2 / N_r \to C_{p,r}$ be defined by $\Psi(1,0) = \phi_r^{(1)}$ and $\Psi(0,1) = \phi_r^{(2)}$. Obviously, Ψ is well-defined and surjective. Suppose $\Psi(x) = 0$ for some element x of $(D^V)^2 / N_r$. From the definition of N_r it follows that x may be written as $x = \sum_{i=0}^{r-1} a_i(V^i,0) + \sum_{i=0}^{h_1} b_i(0,V^i)$ (all a_i, b_i in W). Let $\psi = \Psi(x)$. Then $p^{-1}a_{r-1} = \psi(V) = 0$ in K / W, hence a_{r-1} is in pW and so $a_{r-1}(V^{r-1},0) = 0$. In the same way, $\psi(pF^k) = 0$ for $0 \leqslant k < h_1$ implies that $a_i(V^i,0) = 0$ for $r - h_1 - 1 \leqslant i < r - 1$. Furthermore, from $\psi(F^k) = 0$ for $h_1 \leqslant k < r - 1$ it follows that for $0 \leqslant i < r - h_1 - 1$ we have $a_i \in p^2 W$, hence $a_i(V^i,0) = 0$. Finally, $\psi(F^k) = 0$ for $r - 1 \leqslant k < h_1 + r$ implies that b_i is in pW, hence $b_i(0,V^i) = 0$ for $0 \leqslant i \leqslant h_1$. We conclude that $x = 0$, so Ψ is injective. ***

The only thing left to be verified, is that the transition morphism f^*: $(M / F^{r+1}M)^{(-r)^*} \to (M / F^r M)^{(1-r)^*}$, induced by the action of F on M (cf. lemma 3.2.1), corresponds to the canonical projection: $(D^V)^2 / N_{r+1} \to (D^V)^2 / N_r$. In other words: we have to show that f^* maps $\phi_{r+1}^{(i)}$ onto $\phi_r^{(i)}$ ($i = 1,2$). Let γ be an element of $M / F^r M$, written in the form (4.1.1). Then

$$F\gamma = c_{-1}^{\sigma} p + \sum_{i=1}^{h_1+r} c_{i-1}^{\sigma} F^i .$$

It follows straight from the definitions that $\phi_{r+1}^{(i)}(F\gamma) = \phi_r^{(i)}(\gamma)$ (i = 1,2).
Although we have made the restriction $r > h_1 + 1$, we now are able to determine the projective limit of the $C_{p,r}$. The result is that G has F-type

$$F = V^{h_1} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

4.1.3. The case $h_2 < \infty, h_3 = \infty$

Although this case is more complex than the case $h_2 = \infty$, an analogous treatment is possible.
Let M' be the subset of $D^F / D^F V \oplus D^F / D^F(V - F^{h_1+h_2})$ defined as follows: Let $\gamma = \sum_{i=0}^{\infty} c_i F^i$ be an element of $D^F / D^F V$ and $\eta = \sum_{i=0}^{h_1+h_2} y_i F^i$ an element of $D^F / D^F(V - F^{h_1+h_2})$. Then (γ,η) is in M' if $c_i \equiv y_i \bmod p$ for $0 \leqslant i < h_2$. Note that M' is a D^F-submodule.

Lemma 4.1.8:
The D^F-modules M and M' are isomorphic.
Proof: We define the morphism Φ: $M \to D^F / D^F V \oplus D^F / D^F(V - F^{h_1+h_2})$ by $\Phi(e_1) = (1,1)$ and $\Phi(e_2) = (0,F^{h_2})$. Then Φ is well-defined. Let $\alpha = \sum_{i=0}^{\infty} a_i F^i e_1 + \sum_{i=0}^{\infty} b_i F^i e_2$ (all a_i, b_i in $\mathrm{Im}(T)$) be an element of M. Then

$$\Phi(\alpha) = \Big(\sum_{i=0}^{\infty} a_i F^i , \sum_{i=0}^{h_2-1} a_i F^i + \sum_{i=h_2}^{\infty} (a_i + b_{i-h_2}) F^i \Big).$$

It follows that $\text{Im}(\Phi)$ is contained in M'. Conversely, an element (γ,η) of M' is the image of $\sum_{i=0}^{h_2-1} y_i F^i e_1 + \sum_{i=h_2}^{\infty} c_i F^i e_1 + \sum_{i=h_2}^{\infty} (y_i - c_i) F^{i-h_2} e_2$ under Φ. Suppose $\Phi(\alpha) = (0,0)$. Then $a_i \equiv 0 \bmod p$ for all $i \geqslant 0$, but since all a_i are $\text{Im}(T)$ this means that they are 0. In view of the fact that any element η of $D^F / D^F(V - F^{h_1+h_2})$ may uniquely be written in the form $\eta = \sum_{i=0}^{\infty} y_i F^i$ with all y_i in $\text{Im}(T)$, we also have that $b_i = 0$ $(i \geqslant 0)$. Thus we have shown the injectivity of Φ, so Φ is an isomorphism. ***

Corollary 4.1.9:
M is isogenous to $D^F / D^F V \oplus D^F / D^F(V - F^{h_1+h_2})$.
Proof: $F^{h_2}(D^F / D^F V \oplus D^F / D^F(V - F^{h_1+h_2}))$ is contained in M'. ***

Again, identify M with M'. Let $q > 0$ and put $r := q(h_1 + h_2 + 1)$.

Lemma 4.1.10:
Let $\gamma = \sum_{i=0}^{\infty} c_i F^i$ be an element of $D^F / D^F V$ and $\eta = \sum_{i=0}^{h_1+h_2} y_i F^i$ an element of $D^F / D^F(V - F^{h_1+h_2})$. Then (γ,η) is in $F^r M$ if and only if $c_i \equiv 0 \bmod p$ for $0 \leqslant i < r$, $y_i \equiv p^q c_{i+r} \bmod p^{q+1}$ for $0 \leqslant i < h_2$ and $y_i \equiv 0 \bmod p^q$ for $h_2 \leqslant i \leqslant h_1 + h_2$.
Proof: Let $\alpha = \sum_{i=0}^{\infty} a_i F^i \in D^F / D^F V$ and $\beta = \sum_{i=0}^{h_1+h_2} b_i F^i \in D^F / D^F(V - F^{h_1+h_2})$ be such that $(\alpha,\beta) \in M$ and $F^r(\alpha,\beta) = (\gamma,\eta)$. Now $F^r \alpha = \sum_{i=r}^{\infty} a_{i-r}^{\sigma^r} F^i$ and $F^r \beta = \sum_{i=0}^{h_1+h_2} p^q b_i^{\sigma^r} F^i$. It follows that $c_i \equiv 0 \bmod p$ for $0 \leqslant i < r$, $c_i \equiv a_{i-r}^{\sigma^r} \bmod p$ for $i \geqslant r$ and $y_i = p^q b_i^{\sigma^r}$ for $0 \leqslant i \leqslant h_1 + h_2$. So $y_i \equiv 0 \bmod p^q$ in this range and for $0 \leqslant i < h_2$, using the fact that (α,β) is in M, we even have $y_i \equiv p^q c_{i+r} \bmod p^{q+1}$.
Conversely, if (γ,η) satisfies these conditions, we take $\alpha := \sum_{i=0}^{\infty} c_{i+r}^{\sigma^{-r}} F^i$ and $\beta := \sum_{i=0}^{h_1+h_2} p^{-q} y_i^{\sigma^{-r}} F^i$. Then (α,β) is an element of M and $F^r(\alpha,\beta) = (\gamma,\eta)$. ***

Corollary 4.1.11:
An element of $M / F^r M$ may be represented by an element (γ,η) of M such that

$$(\gamma,\eta) = (\sum_{i=0}^{r-1} c_i F^i, \sum_{i=0}^{h_1+h_2} y_i F^i). \tag{4.1.2}$$

The coefficients c_i are unique mod p $(i \geqslant 0)$, the y_i are unique mod p^{q+1} for $0 \leqslant i < h_2$ and they are unique mod p^q for $h_2 \leqslant i \leqslant h_1 + h_2$.
Proof: $(F^i, p^q F^{i-r})$ is an element of $F^r M$ for $r \leqslant i < h_2 + r$ and $(F^i, 0)$ is in $F^r M$ for $i \geqslant h_2 + r$. The uniqueness properties follow immediately from lemma 4.1.10.
***

It follows that $M / F^r M$ is generated as a W-module by $\{(F^i,F^i) \mid 0 \leqslant i < h_2\} \bigcup \{(0,F^i) \mid h_2 \leqslant i \leqslant h_1 + h_2\} \bigcup \{(F^i,0) \mid h_2 \leqslant i < r\}$.

We define the σ^{1-r}-semilinear maps

$$\phi_r^{(i)}: M / F^r M \to K / W$$

for $i = 1,2,3$ as follows: Let (γ,η) be an element of $M / F^r M$ represented in the form (4.1.2). Then $\phi_r^{(1)}(\gamma,\eta) := p^{-q-1} y_{h_2-1}^{\sigma^{1-r}}$ (recall that $h_2 > 0$), $\phi_r^{(2)}(\gamma,\eta) := p^{-q} y_{h_1+h_2}^{\sigma^{1-r}}$ and $\phi_r^{(3)}(\gamma,\eta) := p^{-1} c_{r-1}^{\sigma^{1-r}}$. Then for $0 \leqslant j < h_2$ we have $(V^j \phi_r^{(1)})(\gamma,\eta) = \phi_r^{(1)}(F^j\gamma,F^j\eta)^{\sigma^{-j}} = p^{-q-1} y_{h_2-j-1}^{\sigma^{1-r}}$. (*Warning:* the element $(F^j\gamma,F^j\eta)$ first must be written in the form (4.1.2).) In the same way for $0 \leqslant j \leqslant h_1+h_2$ we find $(V^j \phi_r^{(2)})(\gamma,\eta) = p^{-q} y_{h_1+h_2-j}^{\sigma^{1-r}}$ and for $0 \leqslant j < r$ we have $(V^j \phi_r^{(3)})(\gamma,\eta) = p^{-1} c_{r-1-j}^{\sigma^{1-r}}$. Using similar arguments as in the preceding subsection it follows that $C_{p,r}$ is generated as a W-module by $\{V^j\phi_r^{(1)} \mid 0 \leqslant j < h_2\} \bigcup \{V^j\phi_r^{(2)} \mid 0 \leqslant j \leqslant h_1\} \bigcup \{V^j\phi_r^{(3)} \mid 0 \leqslant j < r - h_2\}$. In particular $F\phi_r^{(1)} = V^{h_1}\phi_r^{(2)}$, $F\phi_r^{(2)} = V^{h_1+h_2}\phi_r^{(2)}$ and, using $(F^i,0) = (0,-p^q F^{i-r})$ for $i \geqslant r$ in $M / F^r M$, $V^{h_2}\phi_r^{(1)} = \phi_r^{(2)} - \phi_r^{(3)}$. Of course, $V^r\psi = 0$ for any ψ in $C_{p,r}$. We claim that $C_{p,r}$ is generated as a W-module by

$$\{V^j\phi_r^{(1)} \mid 0 \leqslant j < r\} \bigcup \{V^j\phi_r^{(2)} \mid 0 \leqslant j \leqslant h_1\}.$$

Indeed: $V^j\phi_r^{(3)} = V^j(\phi_r^{(2)} - V^{h_2}\phi_r^{(1)})$ and if $j > h_1$ then $V^j\phi_r^{(2)} = \bar{p}V^{j-h_1-1}\phi_r^{(1)}$.

Lemma 4.1.12:
Let N_r be the sub-D^V-module of $(D^V)^2$ defined by

$$N_r = (D^V)^2(F - V^{h_1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - V^{h_1+h_2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) + (D^V)^2 V^r.$$

Then $C_{p,r}$ is isomorphic to $(D^V)^2 / N_r$.
Proof: From the definition of N_r it follows that $(D^V)^2 / N_r$ is generated as a W-module by $\{(V^i,0) \mid 0 \leqslant i < r\} \bigcup \{(0,V^i) \mid 0 \leqslant i \leqslant h_1\}$. We claim that $(D^V)^2 / N_r$ is also generated as a W-module by $\{(V^i,0) \mid 0 \leqslant i < h_2\} \bigcup \{(0,V^i) \mid 0 \leqslant i \leqslant h_1\} \bigcup \{(-V^{i+h_2},V^i) \mid 0 \leqslant i < r - h_2\}$: indeed, $(V^{j+h_2},0) = (0,V^j) - (-V^{j+h_2},V^j)$. Furthermore, $(0,V^j) = (0,p^k V^{j'})$ for some $k \geqslant 0$ and $0 \leqslant j' \leqslant h_1 + h_2$. If $j' \leqslant h_1$ we are done and if $j' > h_1$ we use the relation $(0,V^{j'}) = (pV^{j'-h_1-1},0)$. It follows that any element x of $(D^V)^2 / N_r$ may be written in the form $x = \sum_{i=0}^{h_2-1} a_i(V^i,0) + \sum_{i=0}^{h_1} b_i(0,V^i) + \sum_{i=0}^{r-h_2-1} c_i(-V^{i+h_2},V^i)$. The remainder of the proof is analogous to the proof of lemma 4.1.7, identifying $\phi_r^{(1)}$ with $(1,0)$, $\phi_r^{(2)}$ with $(0,1)$ and $\phi_r^{(3)}$ with $(-V^{h_2},1)$. ***

Recall that we have $r = q(h_1+h_2+1)$. Put $s := (q+1)(h_1+h_2+1)$. We claim that the canonical projection: $(D^V)^2 / N_s \to (D^V)^2 / N_r$ corresponds to the morphism: $(M / F^s M)^{(1-s)*} \to (M / F^r M)^{(1-r)*}$ which is induced by the action of $F^{h_1+h_2+1}$ on M. Let (γ,η) be an element of $M / F^r M$ written in the form (4.1.2). Then

$$F^{h_1+h_2+1}\gamma = \sum_{i=h_1+h_2+1}^{s-1} c_{i-h_1-h_2-1}^{\sigma^{h_1+h_2+1}} F^i \text{ and}$$

$$F^{h_1+h_2+1}\eta = \sum_{i=0}^{h_1+h_2} py_i^{\sigma^{h_1+h_2+1}} F^i.$$

It now follows from the definitions that $\phi_s^{(i)}(F^{h_1+h_2+1}(\gamma,\eta)) = \phi_r^{(i)}(\gamma,\eta)$ $(i = 1,2,3)$, which proves our claim. This enables us to determine the projective limit of the $C_{p,r}$. It follows that G has F-type

$$F = V^{h_1}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + V^{h_1+h_2}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

4.1.4. The case $h_2, h_3 < \infty$

An immediate consequence of the defining relation of M is:

$$Ve_1 = F^{h_1}e_2. \tag{4.1.3}$$

It also follows from the defining relation that $Ve_2 = \sum_{i=0}^{n-1} d_i^{\sigma^{-h_1-2}} F^{h_1+h_2+i} e_2 + F^{h_1+h_2+h_3}e_1$. Using (4.1.3) and $VF = p$, we get:

$$Ve_2 = p\sum_{i=0}^{n-1} d_i^{\sigma^{-h_1-2}} F^{h_2+i-1} e_1 + F^{h_1+h_2+h_3}e_1. \tag{4.1.4}$$

It follows that both $F^{h_1+h_2+h_3}e_1$ and $F^{h_1+h_2+h_3}e_2$ are in VM, hence $F^{h_1+h_2+h_3}M$ is contained in VM. In other words: $F^{h_1+h_2+h_3}M(\text{Ker } V_G) = 0$. Consequently, Ker V_G is finite, hence G is p-divisible (see proposition 3.2.3). We conclude that M has no W-torsion. Since W is a principal ideal ring, this means that M is a free W-module.

Proposition 4.1.13:
The set

$$\{F^i e_1 \mid 0 \leqslant i \leqslant h_1+h_2+h_3\} \cup \{F^i e_2 \mid 0 \leqslant i \leqslant h_1\}$$

constitutes a W-basis of M.

Remark: Since this is the only W-basis of M which we shall use in this subsection, we shall refer to it as *the* W-basis of M.

Proof: In (4.1.3) and (4.1.4) we have given Ve_1 resp. Ve_2 as linear combinations of elements of this set. For $F^{h_1+1}e_2$ and $F^{h_1+h_2+h_3+1}e_1$ such combinations may be obtained as follows: Application of F to (4.1.3) gives

$$F^{h_1+1}e_2 = pe_1 \tag{4.1.5}$$

and application of F to (4.1.4) gives:

$$F^{h_1+h_2+h_3+1}e_1 = pe_2 - p\sum_{i=0}^{n-1} d_i^{\sigma^{-h_1-1}} F^{h_2+i} e_1. \tag{4.1.6}$$

It follows that this set generates M as a W-module. Next assume $\sum_{i=0}^{h_1+h_2+h_3} x_i F^i e_1 + \sum_{i=0}^{h_1} y_i F^i e_2 = 0$. Using the defining relation of M, we find the existence of elements α

and β in D^F such that:

$$\sum_{i=0}^{h_1+h_2+h_3} x_i F^i = \alpha V - \beta F^{h_1+h_2+h_3} \text{ and}$$

$$\sum_{i=0}^{h_1} y_i F^i = \beta(V - \sum_{i=0}^{n-1} d_i^{\sigma^{-h_1-2}} F^{h_1+h_2+i}) - \alpha F^{h_1}.$$

Note that α and β are power series in F without a constant term (this may be seen by multiplying the second of these two equations by V and then using the first equation in order to eliminate α). Multiply both equations by F. Taking them respectively modulo $p, p^2, \cdots$ we find that $\alpha = \beta = 0$, hence all x_i and y_i are 0. ***

$C_p = Mod_W(M^{(1)}, W)$ has a W-basis which is dual (in the sense of semilinear maps) to the basis of M: a σ-semilinear map δ: $M \to W$ belongs to this dual basis if it maps one of the basis elements of M onto 1 and all other basis elements onto 0. Let $\delta_i^{(k)}$ be the basis element of C_p which maps $F^i e_k$ onto 1. A straightforward verification shows that for any σ-semilinear map ξ: $M \to W$ the following relation holds:

$$\xi = \sum_{i=0}^{h_1+h_2+h_3} \xi(F^i e_1)\delta_i^{(1)} + \sum_{i=0}^{h_1} \xi(F^i e_2)\delta_i^{(2)}. \tag{4.1.7}$$

In particular, taking $\xi = V\delta_j^{(2)}$, we have $V\delta_j^{(2)} = \sum_{i=0}^{h_1+h_2+h_3} \delta_j^{(2)}(F^{i+1}e_1)^{\sigma^{-1}}\delta_i^{(1)} + \sum_{i=0}^{h_1} \delta_j^{(2)}(F^{i+1}e_2)^{\sigma^{-1}}\delta_i^{(2)}$, hence

$$V\delta_j^{(2)} = \delta_{j-1}^{(2)} \tag{4.1.8}$$

for $0 < j \leqslant h_1$. In order to get a similar expression for $V\delta_j^{(1)}$ we use formula (4.1.6). The result is:

$$V\delta_j^{(1)} = \delta_{j-1}^{(1)} - pd_{j-h_2}^{\sigma^{-h_1-1}} \delta_{h_1+h_2+h_3}^{(1)} \tag{4.1.9a}$$

for $h_2 \leqslant j < h_2 + n$ and

$$V\delta_j^{(1)} = \delta_{j-1}^{(1)} \tag{4.1.9b}$$

for $0 < j < h_2$ and for $h_2 + n \leqslant j \leqslant h_1 + h_2 + h_3$.
Remark: If $h_3 = 0$ (in which case $n = 0$) only (4.1.9b) has a meaning. All further arguments in this chapter will be valid also for this case. In the following sections we shall see more about the special role which is played by the case $h_3 = 0$.
Put $\phi^{(1)} := \delta_{h_1+h_2+h_3}^{(1)}$ and $\phi^{(2)} := \delta_{h_1}^{(2)}$. It follows from (4.1.8) and (4.1.9) that

$$\{V^i\phi^{(1)} \mid 0 \leqslant i \leqslant h_1 + h_2 + h_3\} \cup \{V^i\phi^{(2)} \mid 0 \leqslant i \leqslant h_1\}$$

is a W-basis for C_p.
From (4.1.7) it follows that $F\phi^{(1)} = \delta_0^{(2)}$, so in view of (4.1.8) we have:

$$F\phi^{(1)} = V^{h_1}\phi^{(2)}. \tag{4.1.10a}$$

By induction one can derive from (4.1.9a) that for $0 \leqslant k < n$ we have $\delta_{h_2+n-k-2}^{(1)} =$

$V^{k+1}\delta^{(1)}_{h_2+n-1} + p\sum_{i=0}^{k} V^{k-i} d^{\sigma^{-h_1-1}}_{n-i-1}\,\phi^{(1)}$ and from (4.1.9b) it follows that $\delta^{(1)}_{h_2+n-1} = V^{h_1+h_3-n+1}\phi^{(1)}$ and that $\delta^{(1)}_0 = V^{h_2-1}\delta^{(1)}_{h_2-1}$. Using (4.1.7) we find that $F\phi^{(2)} = \delta^{(1)}_0$. Consequently:

$$F\phi^{(2)} = p\sum_{i=0}^{n-1} V^{h_2+i-1} d_i^{\sigma^{-h_1-1}}\phi^{(1)} + V^{h_1+h_2+h_3}\phi^{(1)}. \tag{4.1.10b}$$

Since C_p is a free W-module we find that G has F-type

$$F\phi = V^{h_1}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \sum_{i=0}^{n-1} V^{h_1+h_2+i}\begin{bmatrix}0 & 0\\ 0 & d_i\end{bmatrix} + V^{h_1+h_2+h_3}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Comments: For each of the occuring normalized F-types we have given a defining relation of the corresponding contravariant Dieudonné module. This defining relation also has a normal form. It follows that for a given formal group G, the contravariant Dieudonné module $M(G)$ always has a normalized defining relation. Of course, this can also be shown directly: We already noticed the existence of a defining relation and with techniques which are similar to those of chapter 2, a given defining relation can be normalized.

§ 4.2. The Isogeny Type in Terms of the Covariant Parameters

In this section we shall determine the *isogeny type* of M (cf. 3.4.1) in terms of the covariant isomorphism invariants of G. If G splits into the direct sum of two one-dimensional formal groups, M is isomorphic to $D^F/D^F(F^{h_1} - V)\oplus D^F/D^F(F^{h_1+h_2} - V)$ (see theorem 2.1.1 and proposition 4.1.1).

Proposition 4.2.1:
Suppose G does not split into the direct sum of two one-dimensional formal groups. If $h_2 = \infty$ then M has isogeny type

$$D^F/D^F V^2,$$

if $h_2 < h_3 = \infty$ then M has isogeny type

$$D^F/D^F(F^{h_1+h_2} - V)\oplus D^F/D^F V,$$

if $h_2 < h_3 < \infty$ then M is nonhomogeneous of finite height, having isogeny type

$$D^F/D^F(F^{h_1+h_2} - V)\oplus D^F/D^F(F^{h_1+h_3} - V),$$

if $h_3 \leqslant h_2 < \infty$ and h_2+h_3 is even then M is homogeneous decomposable of finite height, having isogeny type

$$(D^F/D^F(F^{h_1+\frac{1}{2}(h_2+h_3)} - V))^2,$$

If $h_3 \leqslant h_2 < \infty$ and h_2+h_3 is odd then M is isosimple of finite height, having isogeny type

$$D^F/D^F(F^{2h_1+h_2+h_3} - V^2).$$

Proof: For the case $h_2 = \infty$ see corollary 4.1.4, for the case $h_2 < h_3 = \infty$ see

corollary 4.1.9.
Assume $h_2, h_3 < \infty$. Then M is free of rank $2h_1 + h_2 + h_3 + 2$ (see proposition 4.1.13). In terms of [1], chap. IV.1, M is an F-lattice. Define the D^F-module $N := D^F / D^F\gamma$ where

$$\gamma = F^{2h_1+h_2+h_3} + \sum_{i=0}^{n-1} pd_i^{\sigma^{-2}} F^{h_1+h_2+i-1} - V^2.$$

Then N is the contravariant Dieudonné module of a p-divisible group (see [1], second remark of ch. IV.1). Note that N has the same W-rank as M, in other words: the p-divisible group which belongs to N has the same height as G. Let $\Phi: N \to M$ be defined by $\Phi(\alpha) = \alpha e_1$, α being an element of D^F. In view of (4.1.3) and (4.1.4), Φ is well-defined. Now $F^{h_1} e_2 = Ve_1$ is an element of Im(Φ) and of course $F^{h_1} e_1$ is in Im(Φ). It follows that $F^{h_1} M$ is contained in Im(Φ), hence Coker(Φ) is finite and consequently Φ is an isogeny (see [1], the lemma in ch. IV.1). Notice: if $h_1 = 0$ then Φ even is an isomorpism. Put $\delta := F^2\gamma$, then δ is in $W[F]$ (regarded as a subring of D). To be explicit:

$$\delta = F^{2h_1+h_2+h_3+2} + \sum_{i=0}^{n-1} pd_i F^{h_1+h_2+i+1} - p^2.$$

If $h_2 < h_3$ then the Newton polygon of δ has slopes $\frac{1}{h_1+h_2+1}$ and $\frac{1}{h_1+h_3+1}$ with multiplicities $h_1 + h_2 + 1$ resp. $h_1 + h_3 + 1$. If $h_2 \geqslant h_3$, the Newton polygon of δ has slope $\frac{2}{2h_1+h_2+h_3+2}$ with multiplicity $2h_1 + h_2 + h_3 + 2$. In view of [5] chap. III.5 parts 1 and 2, the proof is finished. ***

The following four sections will each consist of two parts, A and B. In part A we shall always describe the relevant part of Manin's classification, in part B we shall express the contravariant invariants of G, found in part A, into the covariant invariants of G, found in chapter 2.
Since G has dimension 2, it follows that the isogeny type of M is either $D^F / D^F(F^{2m+1} - V^2)$ (the isosimple case), $(D^F / D^F(F^m - V))^2$ (the homogeneous decomposable case) or $D^F / D^F(F^{m_1} - V) \oplus D^F / D^F(F^{m_2} - V)$, where $m_1 < m_2$ (the nonhomogeneous case).

§ 4.3. The Isosimple Case (Finite Height)

A. In this section M has isogeny type $D^F / D^F(F^{2m+1} - V^2)$. Let $\Psi: M \to M_{2,2m+1}$ be an embedding of D^F-modules such that $\min\{\nu(\Psi(x)) \mid x \in M\} = 0$ (see lemma 3.4.4). In view of proposition 3.4.6, $\Psi[M]$ is as a D^F-module generated by $\{z_0, \theta^{2i+1}\}$ where i is the smallest integer such that $2i + 1$ is in $J(M)$ (hence $0 \leqslant i \leqslant m$) and z_0 is of the form

$$z_0 = 1 + \sum_{k=1}^{i} x_k \theta^{2k-1},$$

the $x_k \in \mathrm{Im}(T)$ being almost uniquely determined. If $i = m$ then $\Psi[M]$ is even generated by z_0. M is special if and only if all x_k are in $W(F_{p^{2m+3}})$. (These are all direct consequences of proposition 3.4.6).
Let j be the smallest nonnegative integer such that $x_1, \ldots, x_{i-j}$ are in $W(F_{p^{2m+3}})$. If x_1 is not in $W(F_{p^{2m+3}})$ then $j := i$. So M is special if and only if $j = 0$. In view of

proposition 3.4.3, Ψ may be chosen such that $x_k = 0$ for $1 \leqslant k \leqslant i - j$: indeed, put $\alpha := 1 + \sum_{k=1}^{i-j} x_k \theta^{2k-1}$, then the map which maps x onto $\Psi(x)\alpha^{-1}$ is the desired embedding of M into $M_{2,2m+1}$. So from now on assume that z_0 has the form

$$z_0 = 1 + \sum_{k=i-j+1}^{i} x_k \theta^{2k-1}.$$

Proposition 4.3.1: (cf. [16] theorem 3.12b)
Let M_1 be the maximal special sub-D^F-module of M. Then $\min\{\nu(x) \mid x \in M_1\} = 2j$.

Remark: it follows that j is an isomorphism invariant (cf. [16] theorem 3.12a) and that $\bar{J}(M_1) = \{1,3,5, \ldots, 2(i-j)-1\}$. In view of corollary 3.4.9, M_1 is uniquely determined by i and j.

Proof: First of all, $F^j z_0 \equiv \theta^{2j}$ modulo an element of order $\geqslant 2i + 1$. Since M contains all elements of order $\geqslant 2i + 1$, it follows that the special element θ^{2j} is in M. Let z be a special element of M and assume that $\nu(z) < 2j$. Then $\nu(z) = 2s$ for some $s < j$ (recall that $j \leqslant i$ hence $\nu(z)$ must be even). We may assume that $z \equiv \theta^{2s}$ modulo an element of order $\geqslant 2s + 1$: indeed, if the first term of z is $c\theta^{2s}$ with c in $\mathrm{Im}(T)$ then c must be in $W(F_{p^{2m+3}})$, hence $c^{-1}z$ is also a special element of M. Let α and β be elements of D^F such that $\alpha z_0 + \beta\theta^{2i+1} = z$. From the shape of z and z_0 it follows that $\alpha \equiv F^s \bmod (D^F F^{s+1} + D^F V)$, hence $z \equiv \theta^{2s} + x_{i-j+1}^{\sigma^s} \theta^{2(i-j+s)+1}$ modulo an element of order $\geqslant 2(i - j + s) + 2$. Since x_{i-j+1} is not in $W(F_{p^{2m+3}})$, this is in contradiction with the fact that z is special.
In view of lemma 3.4.1, the proof is now done. ***

Conclusion: The isomorphism class of M has three discrete invariants m, i, j. Furthermore, M gives rise to a set $\{x_k \mid i - j + 1 \leqslant k \leqslant i\}$. Modulo the action of a finite group, this set is uniquely determined (see the proof of proposition 3.4.6d). These isomorphism invariants are subject to the following conditions:
$0 \leqslant i \leqslant j \leqslant m$, all x_k are in $\mathrm{Im}(T)$ and x_{i-j+1} is not in $W(F_{p^{2m+3}})$. For any set of parameters satisfying these conditions we have a corresponding isomorphism class: An element of this class is found by defining z_0 as above and then taking the D^F-submodule of $M_{2,2m+1}$ generated by $\{z_0, \theta^{2i+1}\}$. In [16] theorem 3.12c, Manin shows that for given m, i and j we have a module space of dimension j.

B. It follows from section 4.2 that M is isosimple of finite height if and only if G does not split into the direct sum of two one-dimensional formal groups and $h_3 \leqslant h_2 < \infty$, $h_2 + h_3$ is odd. From proposition 4.2.1 it follows that $2m + 1 = 2h_1 + h_2 + h_3$, hence

$$m = h_1 + \frac{1}{2}(h_2 + h_3 - 1).$$

In order to find the value of the invariant i we have to determine the smallest odd number in the set $J(M)$. As a D^F-module, M is generated by $\{e_1, e_2\}$. Let $x = \alpha e_1 + \beta e_2$ (with α, β *in* D^F) be an element of M. Then $\nu(x) = \nu(\alpha e_1 + \beta e_2) \geqslant$

$\min\{\nu(\alpha e_1),\nu(\beta e_2)\} \geqslant \min\{\nu(e_1),\nu(e_2)\}$. By definition of the order of an element of M, M contains elements of order zero, which implies: $\min\{\nu(e_1),\nu(e_2)\} = 0$. From relation (4.1.3) it follows that: $\nu(e_1)+2h_1+h_2+h_3 = \nu(e_2)+2h_1$, hence $\nu(e_1)+h_2+h_3 = \nu(e_2)$. Consequently we have $\nu(e_1) = 0$ and $\nu(e_2) = h_2+h_3$ (recall that $h_2 > 0$). Observe that $\nu(\alpha e_1)$ is either an even number or $\geqslant 2h_1+h_2+h_3$ (which can be seen by noticing that $\nu(F^s e_1) = 2s$ and $\nu(Ve_1) = 2h_1+h_2+h_3$). Furthermore, $\nu(\beta e_2) \geqslant \nu(e_2)=h_2+h_3$, hence $\nu(x)$ can only be smaller than h_2+h_3 if $\nu(x)$ is even. h_2+h_3 itself is in $J(M)$ (it is the order of e_2). Consequently, the smallest odd number that occurs in $J(M)$ is h_2+h_3, hence $2i + 1 = h_2+h_3$ and so

$$i = \frac{1}{2}(h_2 + h_3 - 1).$$

In order to find the value of the invariant j, we determine the maximal special sub-D^F-module of M.

Proposition 4.3.2:
The maximal special sub-D^F-module M_1 of M has W-basis:

$$B = \{pF^s e_1 \mid 0 \leqslant s < h_3\} \bigcup \{F^s e_1 \mid h_3 \leqslant s \leqslant h_1+h_2+h_3\} \bigcup$$
$$\bigcup \{F^s e_2 \mid 0 \leqslant s \leqslant h_1\}.$$

Notice: This proposition implies that M is special if and only if $h_3 = 0$.
Proof: Let N be the W-submodule of M generated by B. It follows from the relations (4.1.3) - (4.1.6) that N is stable under the actions of both F and V, hence N is a sub-D^F-module of M. Let

$$x = \sum_{s=0}^{h_1+h_2+h_3} x_s F^s e_1 + \sum_{s=0}^{h_1} y_s F^s e_2$$

be a special element of M, in other words: $F^{2m+1}x = V^2 x$. Since $m = h_1 + \frac{1}{2}(h_2+h_3-1)$, we have $F^{2h_1+h_2+h_3}x = V^2 x$, hence $F^{2h_1+h_2+h_3+2}x \equiv 0 \bmod p^2 M$. From (4.1.6) it follows that for $0 \leqslant s \leqslant h_1+h_2+h_3$ we have: (recall that $h_3 \leqslant h_2$)

$$F^{2h_1+h_2+h_3+2}(F^s e_1) = pF^{h_1+s+1}(e_2 - \sum_{i=0}^{h_3-1} d_i^{\sigma^{-h_1-1}} F^{h_2+i} e_1)$$

and in view of (4.1.5) this implies:

$$F^{2h_1+h_2+h_3+2}(F^s e_1) \equiv$$

$$\equiv -p \sum_{i=0}^{h_3-s-1} d_i^{\sigma^s} F^{h_1+h_2+s+i+1} e_1 \bmod p^2 M. \tag{4.3.1}$$

Using (4.1.5), we get for $0 \leqslant s \leqslant h_1$:

$$F^{2h_1+h_2+h_3+2}(F^s e_2) = pF^{h_1+h_2+h_3+s+1} e_1$$

and so (see (4.1.6)):

$$F^{h_1+h_2+h_3+2}(F^s e_2) \equiv 0 \bmod p^2 M. \tag{4.3.2}$$

Suppose x is not in N. Let k be the smallest integer such that x_k is not in pW. Then the assumption that x is not in N implies that $k < h_3$. Now on the one hand we have: $F^{2h_1+h_2+h_3+2}x \equiv 0 \bmod p^2M$. So if we express $F^{2h_1+h_2+h_3+2}x$ as a linear combination of the W-basis of M (see proposition 4.1.13), then all coefficients in this expression are zero mod p^2W. On the other hand, however, using (4.3.1) and (4.3.2), we find that the coefficient of $F^{h_1+h_2+k+1}e_1$ in this expression is congruent to $-pd_0^{\sigma^k}x_k^{\sigma^{2h_1+h_2+h_3+2}} \bmod p^2W$. Since both x_k and d_0 are nonzero modulo pW, this is a contradiction.

In view of lemma 3.4.1 we now have shown that the maximal special sub-D^F-module of M is contained in N. We shall complete the proof of the proposition by showing that N is special, or equivalently, showing that $F^{2h_1+h_2+h_3}N = V^2N$. First of all, observe that (4.1.3) and (4.1.4) imply that $F^{h_1+h_2+h_3}e_1 = V(e_2 - \sum_{s=0}^{h_3-1} d_s^{\sigma^{-h_1-1}} F^{h_2+s}e_1)$, which is in VN. It follows that $F^{2h_1+h_2+h_3}e_2 = VF^{h_1+h_2+h_3}e_1$ is in V^2N. Furthermore, $F^{2h_1+h_2+h_3}pe_1 = F^{h_1+1}(F^{2h_1+h_2+h_3}e_2)$ is in V^2N. Finally, $F^{2h_1+h_2+h_3}(F^{h_3}e_1) = V(VF^{h_3}e_1 - \sum_{s=0}^{h_3-1} d_s^{\sigma^{h_3-1}} F^s(F^{h_1+h_2+h_3}e_1))$ is in V^2N. In view of the fact that N is as a D^F-module generated by the three elements pe_1, $F^{h_3}e_1$ and e_2 it follows that $F^{2h_1+h_2+h_3}N$ is contained in V^2N. By a similar argument it follows that V^2N is contained in $F^{h_1+h_2+h_3}N$. Since N is isogenous to M, it has the same W-rank as M, hence the set B is W-linearly independent. ***

Corollary 4.3.3:
The invariant j is equal to h_3.
Proof: It follows immediately from proposition 4.3.2 that $\min\{\nu(x) \mid x \in M_1\} = 2h_3$. In view of proposition 4.3.1 we are done. ***

Summarizing the results we have found so far: M is isosimple of finite height if and only if G does not split and $h_3 \leqslant h_2 < \infty$, h_2+h_3 is odd. Furthermore,

$$m = h_1 + \frac{1}{2}(h_2 + h_3 - 1),$$

$$i = \frac{1}{2}(h_2 + h_3 - 1),$$

$$j = h_3.$$

Remark: On the one hand, G corresponds to a point on a h_3-dimensional component of the covariant module space (cf. section 2.3). On the other hand, G corresponds to a point on a j-dimensional component of the contravariant module space. Considering this, the relation $j = h_3$ is remarkable.

We shall now study the relations between the continuous parameters. Let $\Psi: M \to M_{2,2m+1}$ be an embedding of D^F-modules such that $\min\{\nu(\Psi(x)) \mid x \in M\} = 0$. Recall that $\nu(e_1) = 0$. Write

$$\Psi(e_1) = \sum_{s=0}^{\infty} a_s\theta^s,$$

all $a_s \in \mathrm{Im}(T)$. From (4.1.3) it follows that

$$\Psi(e_2) = \sum_{s=0}^{\infty} a_s^{\sigma^{-h_1-1}} \theta^{s+h_2+h_3}.$$

Substituting this in (4.1.4) and equating coefficients of powers of θ, it follows that for $0 \leqslant s < h_2-h_3$ the coefficient a_s must be in $W(F_{p^{2m+3}})$. We claim that Ψ can be chosen such that $a_0 = 1$ and $a_s = 0$ for $0 < s < h_2-h_3$: if not so, put $\alpha = \sum_{s=0}^{h_2-h_3} a_s \theta^{p^s}$ and define a new embedding Ψ' by $\Psi'(x) = \Psi(x)\alpha^{-1}$. (recall that $h_3 \leqslant h_2$ and h_2+h_3 is odd, hence $h_3 < h_2$). If we start with $\Psi(e_1)$ for the construction of z_0 (see the proof of proposition 3.4.6a) we find relations between the x_k and the a_s. For instance, we find that $x_{i-j+1} = a_{h_2-h_3}$. Furthermore, (4.1.4) gives relations between the d_k and the a_s, for instance $d_0 \equiv a_{h_2-h_3} - a_{h_2-h_3}^{\sigma^{2m+3}} \bmod pW$. We conclude that

$$\bar{d_0} = \bar{x}_{i-j+1} - \bar{x}_{i-j+1}^{p^{2m+3}}.$$

Notice: the fact that d_0 is nonzero is equivalent to the fact that $\bar{x}_{i-j+1}$ is not in $F_{p^{2m+3}}$. Of course, the explicit relations for the higher parameters are more complex. In the case $i = j = m = 1$, necessary and sufficient conditions for two formal groups to be isomorphic are given in [16] formula (3.15). In this case there is only one continuous parameter. The condition looks as follows: if G and H are formal groups with contravariant discrete parameters $i = j = m = 1$ and contravariant continuous parameters $\bar{x_1}$ resp. $\bar{y_1}$, then G and H are isomorphic if and only if there exist elements β_0 and β_1 in F_{p^5} with $\beta_0 \neq 0$ such that

$$\bar{y_1} = \beta_0^{p^3-1}\bar{x_1} + \beta_1.$$

We shall compare this condition with the condition given by theorem 2.1.8. The covariant discrete parameters are: $h_3 = j = 1$, $h_2 = 2i - h_3 + 1 = 2$ and $h_1 = m - \frac{1}{2}(h_2+h_3-1) = 0$. The covariant continuous parameters of G and H are $\bar{d_0} = \bar{x_1} - \bar{x_1}^{p^5}$ resp. $\bar{e_0} = \bar{y_1} - \bar{y_1}^{p^5}$. Theorem 2.1.8 states that G and H are isomorphic if and only if there exists a nonzero element λ of F_{p^5} such that

$$\lambda^{p^3}\bar{d_0} = \lambda\bar{e_0}.$$

It is now readily verified that the contravariant condition is equivalent to the covariant condition, the relation between the β_0, β_1 on the one hand and λ on the other hand being given by $\beta_0 = \lambda$ and $\beta_1 = -\lambda^{p^3-1}\bar{x_1} + \bar{y_1}$.

§ 4.4. The Homogeneous Decomposable Case (Finite Height)

A. In this section M has isogeny type $(D^F / D^F(F^m - V))^2$ for some nonnegative integer m.

Proposition 4.4.1:
There exists an embedding

$$\Psi: M \to M_{1,m}^2$$

such that $\Psi[M]$:

a) contains $R^2_{1,m}$,
b) contains no nonzero elements of the form $(x\theta^{-1}, y\theta^{-1})$ where x and y are in the intersection of $W(F_{p^{m+1}})$ and $\mathrm{Im}(T)$.

Proof: Since M and $R^2_{1,m}$ are isogenous, there exists an embedding $\Psi\colon M \to M^2_{1,m}$ such that M contains $R^2_{1,m}$ (cf. [16], the remark after lemma 3.1). Assume $(x\theta^{-1}, y\theta^{-1})$ is a nonzero element of $\Psi[M]$ with x and y in the intersection of $W(F_{p^{m+1}})$ and $\mathrm{Im}(T)$. If x is nonzero we may assume that $x = 1$. Define the automorphism Φ of the D^F-module $M^2_{1,m}$ by $\Phi(w_1,w_2) = (w_1\theta, w_2 - w_1y^{\sigma})$ and let $\Psi' := \Phi\Psi$. One easily verifies that $\Psi'[M]$ satisfies a). If $x = 0$ we define Φ by $\Phi(w_1,w_2) = (w_1,w_2\theta)$ and again $\Psi' := \Phi\Psi$ is such that $\Psi'[M]$ satisfies a). If Ψ' does not satisfy b), we repeat this step. Since M is finitely generated over D^F, the orders of the w_1 and w_2 with (w_1,w_2) in M have a finite minimum. Note that the minimal order of the first component does not decrease under the step $\Psi' := \Phi\Psi$. If $-k$ is the minimal order of the second component, then after at most $k + 1$ steps the minimal order of the first component has increased. It follows that a finite number of steps is needed. ***

In view of this proposition we may from now on assume that M is a sub-D^F-module of $M^2_{1,m}$ having properties a) and b).

Proposition 4.4.2:
M contains no elements of the form $(w,0)$ or $(0,w)$ unless w is in $R_{1,m}$.

Proof: Suppose $(w,0)$ is in M and $\nu(w) < 0$. Then $F^{-\nu(w)-1}(w,0) = (F^{-\nu(w)-1}w,0)$ is an element of M. But then $(\theta^{-1},0)$ is also in M, which is a contradiction. ***

Lemma 4.4.3:
Let $w = (w_1,w_2)$ be an element of M. Then $\nu(w_1) < 0$ if and only if $\nu(w_2) < 0$.

Proof: Suppose $\nu(w_1) \geqslant 0$. Then $w_1 \in R_{1,m}$, hence $(w_1,0)$ is in M, which implies that $(0,w_2)$ is in M. In view of proposition 4.4.2, w_2 must be in $R_{1,m}$, hence $\nu(w_2) \geqslant 0$. For the same reasons $\nu(w_2) \geqslant 0$ implies $\nu(w_1) \geqslant 0$. ***

Corollary 4.4.4:
Let $w = (w_1,w_2)$ be an element of M. If w is not in $R^2_{1,m}$ then $\nu(w_1) = \nu(w_2)$.

Proof: From lemma 4.4.3 it follows that both $\nu(w_1)$ and $\nu(w_2)$ must be < 0. Note that $F^{-\nu(w_2)}(0,w_2)$ is in $R^2_{1,m}$, hence it is in M. It follows that $F^{-\nu(w_2)}(w_1,0) = F^{-\nu(w_2)}w - F^{-\nu(w_2)}(0,w_2)$ is an element of M, hence (see proposition 4.4.2) $F^{-\nu(w_2)}w_1$ is in $R_{1,m}$, in other words $\nu(w_1) \geqslant \nu(w_2)$. In the same way we find $\nu(w_2) \geqslant \nu(w_1)$. ***

Proposition 4.4.5: (cf. [16] lemma 3.14b)
Let $s > 0$ and suppose M contains an element $w = (w_1,w_2)$ such that $\nu(w_1) = -s$. Then M contains a unique element of the form

$$z_s = (\theta^{-s}, \sum_{k=1}^{s} x_{sk}\theta^{-k}),$$

with all x_{sk} in $\mathrm{Im}(T)$. Moreover, x_{ss} not in $W(F_{p^{m+1}})$.

Proof: Let $w = (w_1,w_2)$ be an element of M with $\nu(w_1) = -s$. In view of corollary 4.4.4 we also have $\nu(w_2) = -s$. Write $w_1 \equiv \sum_{k=1}^{s} a_k\theta^{-k} \bmod R_{1,m}$, with all a_k in

$\operatorname{Im}(T)$. Then $a_s \neq 0$. a_s may be assumed to be 1, for if not so, we take $a_s^{-1}w$ instead of w. Let $1 \leq r < s$ and suppose $a_i = 0$ for $r+1 \leq i < s$. Then $v :=$ $w - a_r F^{s-r} w$ is in M and $v = (v_1, v_2)$ with $v_1 \equiv \theta^{-s} + \sum_{k=1}^{r-1} b_k \theta^{-k} \bmod R_{1,m}$. After $s - 1$ steps we find an element $w = (w_1, w_2)$ in M with $w_1 \equiv \theta^{-s} \bmod R_{1,m}$ and $w_2 \equiv \sum_{k=1}^{s} x_k \theta^{-k} \bmod R_{1,m}$, all x_k in $\operatorname{Im}(T)$. Since $R_{1,m}^2$ is contained in M, it follows that M contains an element of the desired form. From the relation $F^{s-1}z_s \equiv (\theta^{-1}, x_{ss}^{\sigma^{s-1}}\theta^{-1}) \bmod R_{1,m}^2$ it follows that x_{ss} cannot be an element of $W(F_{p^{m+1}})$ in view of property b) of proposition 4.4.1.

For unicity: suppose $z' = (\theta^{-s}, \sum_{k=1}^{s} y_k \theta^{-k})$ is also in M, all y_k being in $\operatorname{Im}(T)$. Then $z_s - z' = (0, \sum_{k=1}^{s} (x_{sk} - y_k)\theta^{-k})$ is an element of M, hence (see proposition 4.4.2) $\sum_{k=1}^{s} (x_{sk} - y_k)\theta^{-k}$ is in $R_{1,m}$. It follows that $x_{sk} = y_k$ $(1 \leq k \leq s)$. ***

Corollary 4.4.6: (cf. [16] lemma 3.14a)
Let $w = (w_1, w_2)$ be an element of M. Then $\nu(w_1) \geq -m$.
Proof: Suppose $\nu(w_1) < -m$. Let $s := -\nu(w_1)$, then $s > m$. Using proposition 4.4.2 and the fact that $F^m z_s - Vz_s = (0, \sum_{k=1}^{s} (x_{sk}^{\sigma^m} - x_{sk}^{\sigma^{-1}})\theta^{-k+m})$ is in M, it follows that $x_{ss}^{\sigma^m} = x_{ss}^{\sigma^{-1}}$, hence x_{ss} is in $W(F_{p^{m+1}})$, which is a contradiction (see proposition 4.4.5). ***

Corollary 4.4.7:
$R_{1,m}^2$ is the maximal special sub-D^F-module of M.
Proof: Let M_1 be the maximal special sub-D^F-module of M. Since $R_{1,m}^2$ is special, it is contained in M_1. Consequently, M_1 satisfies the properties a) and b) of proposition 4.4.1. Suppose M_1 contains an element (w_1, w_2) with $\nu(w_1) < 0$. Let $s := -\nu(w_1)$. Application of proposition 4.4.5 to M_1 shows that $z_s \in M_1$. Since M_1 is special, it contains an element z' such that $F^m z_s = Vz'$. This implies that z' has the same form as z_s, hence $z' = z_s$ and so z_s is special. This is in contradiction with the fact that x_{ss} is not in $W(F_{p^{m+1}})$. ***

Let $w = (w_1, w_2)$ be an element of M. We define the *F-height* of w to be the smallest nonnegative integer s such that $F^s w$ is an element of $M_1 = R_{1,m}^2$. If the F-height of w is positive then it is obviously equal to $-\nu(w_1) = -\nu(w_2)$.
We define the *F-height* of M to be the smallest nonnegative integer h such that $F^h M$ is contained in M_1. In other words: The F-height of M is the largest natural number h such that M contains an element $w = (w_1, w_2)$ with $\nu(w_1) = -h$. Clearly M is special if and only if $h = 0$. From corollary 4.4.6 it follows that the F-height h of M is $\leq m$. From the definition it is clear that h is an isomorphism invariant.

Proposition 4.4.8: (cf. [16] lemma 3.14b)
M is as a D^F-module generated by z_h and $R_{1,m}^2$.
Proof: Let $w = (w_1, w_2)$ be an element of M and suppose $\nu(w_1) = -s < 0$, say $w_1 \equiv a_s \theta^{-s}$ modulo an element of order $\geq 1 - s$. Then consider $v = (v_1, v_2) :=$

$w - a_s F^{h-s} z_h$. In view of the definition of h we have $s \leqslant h$, hence v is in M. Clearly $\nu(v_1) > \nu(w_1)$ and so (see corollary 4.4.4) $\nu(v_2) > \nu(w_2)$. If $\nu(v_1) < 0$ we repeat this step. After at most s steps we have the required result. ***

Notice: From the unicity of the elements z_s in M it follows that for $s \leqslant h$ we have $F^{h-s} z_h \equiv z_s \mod R^2_{1,m}$.

We shall now introduce an invariant j, which does not occur in Manin's classification. Its definition and its properties, however, are entirely analogous to the definition and the properties of the invariant j which arose in the isosimple case. (See section 4.3.) Let M_2 be the maximal 2-special sub-D^F-module of M. Then M_2 contains $M_1 = R^2_{1,m}$, which follows immediately from the definitions of M_1 and M_2. Consequently, M_2 satisfies the properties a) and b) of proposition 4.4.1. If $M_2 = M$, in other words M is 2-special, then there exists an element z' in M such that $F^{2m} z_h = V^2 z'$. But then z' has the same form as z_h. Now from the unicity of z_h it follows that $z' = z_h$, hence z_h is a 2-special element. So if M is 2-special then all x_{hk} $(1 \leqslant k \leqslant h)$ are in $W(\mathbf{F}_{p^{2m+2}})$.
In general, let j be the smallest natural number such that $x_{hh}, x_{h,h-1}, \ldots, x_{h,j+1}$ are in $W(\mathbf{F}_{p^{2m+2}})$. If x_{hh} is not in $W(\mathbf{F}_{p^{2m+2}})$ then $j := h$. (So M is 2-special if and only if $j = 0$.) As we already have noticed, $z_{h-j} \equiv F^j z_h \mod R^2_{1,m}$, hence all $x_{h-j,k}$ $(1 \leqslant k \leqslant h-j)$ are in $W(\mathbf{F}_{p^{2m+2}})$ and so z_{h-j} is a 2-special element.

Proposition 4.4.9:
The F-height of M_2 is equal to $h - j$.
Proof: Let s be the F-height of M_2. Since z_{h-j} is 2-special, it is an element of M_2, hence $h - j \leqslant s$. Application of proposition 4.4.5 to M_2 shows that $z_s \in M_2$. Since M_2 is 2-special, all x_{sk} are in $W(\mathbf{F}_{p^{2m+2}})$. Since $F^{h-s} z_h \equiv z_s \mod R^2_{1,m}$, we see that $x_{hh}, x_{h,h-1}, \ldots, x_{h,h-s+1}$ are in $W(\mathbf{F}_{p^{2m+2}})$ and so $j \leqslant h - s$. This completes the proof of the proposition. ***

From this proposition it follows that j is an isomorphism invariant.

Proposition 4.4.10:
Suppose $j < h$. Let $c \in W(k)$ be an arbitrary element which is in the intersection of $\mathrm{Im}(T)$ and $W(\mathbf{F}_{p^{2m+2}})$ but not in $W(\mathbf{F}_{p^{m+1}})$. Then there exists an embedding Ψ of M into $M^2_{1,m}$ satisfying the conditions a) and b) of proposition 4.4.1 such that the unique element z_h in $\Psi[M]$ has the form:

$$z_h = (\theta^{-h}, c\theta^{-h} + \sum_{k=1}^{j} x_{hk}\theta^{-k}),$$

all x_{hk} in $\mathrm{Im}(T)$. In other words: $x_{hh} = c$ and $x_{h,h-1} = \cdots = x_{h,j+1} = 0$.
Proof: From the fact that the endomorphisms of $M_{1,m}$ are the right multiplication by elements of $K_{1,m}$ it follows that any endomorphism Φ of $M^2_{1,m}$ has the form $\Phi(w_1, w_2) = (w_1 t_1 + w_2 t_2, w_1 t_3 + w_2 t_4)$, where t_1, t_2, t_3 and t_4 are arbitrary elements of $K_{1,m}$. Let the map $f : \mathbf{F}_{p^{m+1}} \times \mathbf{F}^*_{p^{m+1}} \to \mathbf{F}_{p^{2m+2}} \setminus \mathbf{F}_{p^{m+1}}$ be defined by $f(\alpha,\beta) := \alpha + \beta\bar{x}_{hh}$. Note that f is well-defined. f is injective: indeed, suppose $f(\alpha_1,\beta_1) = f(\alpha_2,\beta_2)$, then $\alpha_1 - \alpha_2 = \bar{x}_{hh}(\beta_2 - \beta_1)$ and since $\bar{x}_{hh}$ is not in $\mathbf{F}_{p^{m+1}}$ this means that $\alpha_1 = \alpha_2$ and

$\beta_1 = \beta_2$. Since both $F_{p^{m+1}} \times F^*_{p^{m+1}}$ and $F_{p^{2m+2}} \setminus F_{p^{m+1}}$ have $p^{m+1}(p^{m+1} - 1)$ elements, f is also surjective. Let α and β be such that $f(\alpha,\beta) = \bar{c}$ and let $a = T(\alpha)$ and $b = T(\beta)$. Then $a + bx_{hh} \equiv c \mod pW(F_{p^{2m+2}})$. Note that b is nonzero modulo pW. Let us define the endomorphism Ψ of $M^2_{1,m}$ by:
$\Psi(w_1,w_2) = (w_1,w_1a^{\sigma^h} + w_2b^{\sigma^h})$. Since b is nonzero modulo pW, Ψ even is an automorphism. Using this, an easy verification shows that $\Psi[M]$ satisfies the conditions a) and b) of proposition 4.4.1. Furthermore we have: $\Psi(z_h) \equiv$
$(\theta^{-h},c\theta^{-h} + \sum_{k=1}^{h-1} x_{hk}\theta^{-k}b^{\sigma^h}) \mod R^2_{1,m}$.
Next let us assume that the embedding is chosen such that
$z_h = (\theta^{-h},c\theta^{-h} + \sum_{k=1}^{s} x_{hk}\theta^{-k})$, where $1 + j \leqslant s \leqslant h - 1$. Put $u = \dfrac{x_{hs} - x_{hs}^{\sigma^{m+1}}}{c^{\sigma^{m+1}} - c}$
and note that u is well-defined: its denominator is not zero, since c is not in $W(F_{p^{m+1}})$. Since both x_{hs} and c are elements of $W(F_{p^{2m+2}})$, we at once verify that $u^{\sigma^{m+1}} = u$, hence u is in $W(F_{p^{m+1}})$. Put $t = -(x_{hs} + uc)$. A straightforward verification shows that $t^{\sigma^{m+1}} = t$, hence t is also in $W(F_{p^{m+1}})$. Let us define the automorphism Ψ of $M^2_{1,m}$ by: $\Psi(w_1,w_2) = (w_1,w_1t^{\sigma^h}\theta^{h-s} + w_2(1 + u^{\sigma^h}\theta^{h-s}))$. Then $\Psi[M]$ satisfies the conditions a) and b) of proposition 4.4.1 and
$\Psi(z_h) \equiv (\theta^{-h},c\theta^{-h} + \sum_{k=1}^{s-1} x'_{hk}\theta^{-k}) \mod R^2_{1,m}$, where the x'_{hk} are elements of Im(T).
Since j is an isomorphism invariant, we still have: $x'_{hk} \in W(F_{p^{2m+2}})$ for $j + 1 \leqslant k \leqslant s - 1$.
Repeating this step $h - j - 1$ times we get the desired result. ***

Conclusion: The isomorphism class of M has three discrete invariants m, h, j. Furthermore, M gives rise to a set $\{x_{hk} \mid 1 \leqslant k \leqslant j\}$. Modulo the action of a finite group, this set is uniquely determined (this can be seen in analogous way as in the isosimple case.) These isomorphism invariants are subject to the following conditions: $0 \leqslant j \leqslant h \leqslant m$, all x_k are in Im(T) and x_j is not in $W(F_{p^{2m+2}})$. For any set of parameters satisfying these conditions we have a corresponding isomorphism class. An element of this class is found by defining z_h as above and then taking the D^F-submodule of $M^2_{1,m}$ generated by z_h and $R^2_{1,m}$. So for given m, h and j we have a module space $S_{m,h,j}$ of dimension j. The module spaces which are given by Manin are found by taking all formal groups with given m and h. This gives an h-dimensional space $S_{m,h}$. So the invariant j gives a *refinement* of Manin's classification: the module space $S_{m,h}$ is split into the disjoint union $\bigcup_{j=0}^{h} S_{m,h,j}$.

B. First, assume that G splits into the direct sum of two one-dimensional formal groups. Then M is homogeneous decomposable of finite height if and only if $h_1 < \infty$ and $h_2 = 0$, in other words M is isomorphic to $(D^F / D^F(F^{h_1} - V))^2$ (see section 4.2). It follows that

$$m = h_1.$$

Clearly M is special, hence M is also 2-special and so

$$h = j = 0.$$

Next assume that G does not split into the direct sum of two one-dimensional formal groups. It follows from proposition 4.2.1 that M is homogeneous decomposable of finite height if and only if $h_3 \leqslant h_2 < \infty$ and h_2+h_3 is even. It also follows that

$$m = h_1 + \frac{1}{2}(h_2 + h_3).$$

In order to determine the value of the invariant h, we determine explicitly the maximal special sub-D^F-module of M.

Proposition 4.4.11:
The maximal special sub-D^F-module M_1 of M has W-basis

$$A = \{pF^s e_1 \mid 0 \leqslant s < \frac{1}{2}(h_2+h_3)\} \cup$$
$$\cup \{F^s e_1 \mid \frac{1}{2}(h_2+h_3) \leqslant s \leqslant h_1+h_2+h_3\} \cup \{F^s e_2 \mid 0 \leqslant s \leqslant h_1\}.$$

Proof: Let N be the W-submodule of M generated by A. Then N is a sub-D^F-module of M: indeed, the relations (4.1.3) - (4.1.6) imply that N is invariant under F and V. Let

$$x = \sum_{s=0}^{h_1+h_2+h_3} x_s F^s e_1 + \sum_{s=0}^{h_1} y_s F^s e_2$$

be a special element of M, so $F^m x = Vx$, hence $F^{m+1}x \equiv 0 \bmod pM$. On the other hand, using (4.1.5) and (4.1.6), we find that

$$F^{m+1}x \equiv \sum_{s=0}^{\frac{1}{2}(h_2+h_3)-1} x_s^{\sigma^{m+1}} F^{s+m+1} e_1 \bmod pM.$$

It follows that $x_s \equiv 0 \bmod pW$ for $0 \leqslant s < \frac{1}{2}(h_2+h_3)$, hence x is in N. In view of lemma 3.4.1 it follows that M_1 is contained in N. We shall complete the proof by showing that N is special. First of all, $F^m e_2 = VF^{\frac{1}{2}(h_2+h_3)} e_1$, hence $F^m e_2$ is in VN. Furthermore, $F^m pe_1 = F^{m+1}Ve_1 = F^{h_1+1}(F^m e_2)$, hence $F^m pe_1$ is also in VN. Finally, $F^m(F^{\frac{1}{2}(h_2+h_3)} e_1) = F^{h_1+h_2+h_3} e_1 = V(e_2 - \sum_{s=0}^{h_3-1} d_s^{\sigma^{-h_1-1}} F^{h_2+s} e_1)$ (see the defining relation of M) which is an element of VN. (Notice: Since $h_3 \leqslant h_2$ we have $F^{h_2} e_1 \in N$.) Since N is as a D-module generated by $\{pe_1, F^{\frac{1}{2}(h_2+h_3)} e_1, e_2\}$, we have shown that $F^m N$ is contained in VN. In an analogous way one may verify that VN is contained in $F^m N$. So $F^m N = VN$, in other words N is special.
Since M and M_1 are isogenous they have the same W-rank, hence the set A is linearly independent over W. ***

Corollary 4.4.12:
M has F-height

$$h = \frac{1}{2}(h_2 + h_3).$$

In particular M is not special.

Proof: In view of the W-basis of M given in proposition 4.1.13 this is immediate.
***

Proposition 4.4.13:
The maximal 2-special sub-D^F-module M_2 of M has W-basis:

$$B = \{pF^s e_1 \mid 0 \leqslant s < h_3\} \bigcup \{F^s e_1 \mid h_3 \leqslant s \leqslant h_1+h_2+h_3\} \bigcup$$
$$\bigcup \{F^s e_2 \mid 0 \leqslant s \leqslant h_1\}.$$

Notice: This proposition implies that M is 2-special if and only if $h_3 = 0$.
Proof: Since $m = h_1 + \frac{1}{2}(h_2+h_3)$, M_2 is the maximal sub-D^F-module of M such that $F^{2h_1+h_2+h_3}M_2 = V^2 M_2$. The proof of this proposition is exactly the same as the proof of proposition 4.3.2. Note that in the proof of proposition 4.3.2 we did not use the fact that h_2+h_3 was odd. ***

Corollary 4.4.14:
The invariant j is equal to h_3.
Proof: Clearly M_2 has F-height $\frac{1}{2}(h_2-h_3)$ (see propositions 4.4.11 and 4.4.13) and we already have found that $h = \frac{1}{2}(h_2+h_3)$ (cor. 4.4.12). The corollary now follows from proposition 4.4.9. ***

Summarizing, we have the following results: If G splits then M is homogeneous decomposable of finite height if and only if $h_1 < \infty$ and $h_2 = 0$. Furthermore,

$$m = h_1,$$
$$h = 0,$$
$$j = 0.$$

If G does not split then M is homogeneous decomposable of finite height if and only if $h_2 \leqslant h_3 < \infty$ and h_2+h_3 is even. Furthermore,

$$m = h_1 + \frac{1}{2}(h_2 + h_3),$$
$$h = \frac{1}{2}(h_2 + h_3),$$
$$j = h_3.$$

Remark: As we already noticed, G lies on a h_3-dimensional component of the covariant module space. On the other hand, G lies on a h-dimensional component of Manin's contravariant module space, where $h = \frac{1}{2}(h_2+h_3)$. After the refinement we have made, G lies on a j-dimensional component of the contravariant module space, where $j = h_3$. So the new invariant j gives rise to a similar situation as in the isosimple case.

Next we shall study the relations between the continuous parameters. Let Ψ: $M \rightarrow M^2_{1,m}$ be an embedding of D^F-modules as described in proposition 4.4.1. Let $u = (u_1,u_2)$ and $v = (v_1,v_2)$ be the images of e_1 resp. e_2 under Ψ. From (4.1.3) it follows that $\nu(u_1) + m = \nu(v_1) + h_1$. Since $m > h_1$ this implies that $\nu(u_1) < \nu(v_1)$. Using the fact that M is generated as a D^F-module by e_1 and e_2 and proposition 4.4.8, it follows that $\nu(u_1) = -h$. In view of corollary 4.4.4 we also have $\nu(u_2) = -h$. (Recall that $h = \frac{1}{2}(h_2+h_3) > 0$.) Write

$$u_1 = \sum_{s=-h}^{\infty} a_s \theta^s$$

and

$$u_2 = \sum_{s=-h}^{\infty} b_s \theta^s$$

with all a_s and b_s in $\mathrm{Im}(T)$. Then (4.1.3) gives

$$v_1 = \sum_{s=-h}^{\infty} a_s^{\sigma^{-h_1-1}} \theta^{s+h}$$

and

$$v_2 = \sum_{s=-h}^{\infty} b_s^{\sigma^{-h_1-1}} \theta^{s+h}.$$

Substituting these expressions in (4.1.4) and equating coefficients of powers of θ, we find relations between the a_s, b_s and d_i. Suppose $h_2 \neq h_3$, or equivalently $h > j$. Then (4.1.4) implies that for $-h \leqslant s < -j$ the coefficients a_s and b_s are in $W(\boldsymbol{F}_{p^{2m+2}})$. We claim that Ψ may be chosen such that $a_{-h} = 1$, $b_{-h} = c$ (the same c as in proposition 4.4.10) and $a_s = b_s = 0$ for $-h < s < -j$. We shall describe automorphisms Φ of $M^2_{1,m}$, respecting a) and b) of proposition 4.4.1, which enable us to reach this result step by step. If a_{-h} is in $W(\boldsymbol{F}_{p^{m+1}})$ we define $\Phi(w_1,w_2) = (w_1 a_{-h}^{\sigma^h}, w_2)$. If a_{-h} is not in $W(\boldsymbol{F}_{p^{m+1}})$ we may choose elements f and g of $W(\boldsymbol{F}_{p^{m+1}})$ such that $a_{-h}f + b_{-h}g \equiv 1 \bmod pW$: indeed, in view of property b) of proposition 4.4.1, $\frac{a_{-h}}{b_{-h}}$ is not in $W(\boldsymbol{F}_{p^{m+1}})$ so we may use the same argument as in the proof of proposition 4.4.10. Note that g is nonzero mod pW. We now define $\Phi(w_1,w_2) = (w_1 f^{\sigma^h} + w_2 g^{\sigma^h}, w_1)$. We have reached that a_{-h} may be supposed to be 1. Next assume that $a_{-h+1} = \cdots = a_{-k-1} = 0$ with $j < k < h$. If a_{-k} is in $W(\boldsymbol{F}_{p^{m+1}})$ we define $\Phi(w_1,w_2) = (w_1(1 - a_{-k}^{\sigma^h}\theta^{h-k}), w_2)$. If not so, we take f and g in $W(\boldsymbol{F}_{p^{m+1}})$ such that $f + b_{-h}g \equiv -a_{-k} \bmod pW$ (see again the proof of proposition 4.4.10) and define $\Phi(w_1,w_2) = (w_1(1 + f^{\sigma^h}\theta^{h-k}) + w_2 g^{\sigma^h}\theta^{h-k}, w_2)$. The proof that b_{-h} may be assumed to be c and that the b_s may be assumed to be 0 for $-h < s < -j$ is exactly the same as the proof of proposition 4.4.10. This completes the verification of our claim.

Next let us drop the assumption that $h_2 > h_3$. In order to construct the element z_h (see the proof of proposition 4.4.5), we start with $\Psi(e_1)$. This gives relations between the x_{hk} on the one hand and the a_s and b_s on the other hand. If $h_2 > h_3$ we get for instance $x_{hj} \equiv b_{-j} - a_{-j}c^{\sigma^{h-j}} \bmod pW$, if $h_2 = h_3$ (in other words $h = j$) we get $x_{hj} = \frac{b_{-j}}{a_{-j}}$. Furthermore, (4.1.4) gives relations between the a_s and b_s on the one hand and the d_k on the other hand. For instance: if $h_2 > h_3$ then $a_{-j} \equiv d_0 + a_{-j}^{\sigma^{2m+2}}$

and $b_{-j} \equiv d_0 c^{\sigma^{m+h-j+1}} + b_{-j}^{\sigma^{2m+2}} \bmod pW$, if $h_2 = h_3$ then $a_{-j} \equiv d_0 a_{-j}^{\sigma^{m+1}} + a_{-j}^{\sigma^{2m+2}}$ $\bmod pW$ and $b_{-j} \equiv d_0 b_{-j}^{\sigma^{m+1}} + b_{-j}^{\sigma^{2m+2}} \bmod pW$. If $h_2 > h_3$ one easily verifies the following direct relation:

$$\bar{x}_{hj} - \bar{x}_{hj}^{p^{2m+2}} = \bar{d}_0(\bar{c}^{p^{m+1}} - \bar{c})^{p^{h-j}}.$$

Note that the condition $d_0 \neq 0$ is equivalent to the condition that $\bar{x}_{hj}$ is not in $F_{p^{2m+2}}$. If $h_2 = h_3$, it is a bit more difficult to find a direct relation between d_0 and x_{hj}. It is found as follows: put $\alpha = \bar{a}_{-j}$, $\beta = \bar{b}_{-j}$, $\delta = \bar{d}_0$ and $\xi = \bar{x}_{hj}$. Hence we have the relations $\xi = \dfrac{\beta}{\alpha}$,

$$\delta\alpha^{p^{m+1}} = \alpha - \alpha^{p^{2m+2}}, \tag{4.4.1}$$

$$\delta\beta^{p^{m+1}} = \beta - \beta^{p^{2m+2}}. \tag{4.4.2}$$

From (4.4.1) and (4.4.2) it follows that $\alpha^{p^{m+1}}(\beta - \beta^{p^{2m+2}}) = \beta^{p^{m+1}}(\alpha - \alpha^{p^{2m+2}})$, which implies that $\alpha^{p^{m+1}}\beta - \alpha\beta^{p^{m+1}} = -(\alpha^{p^{m+1}}\beta - \alpha\beta^{p^{m+1}})^{p^{m+1}}$. Using this relation, one easily verifies that

$$\xi - \xi^{p^{m+1}} = -\frac{\alpha^{p^{2m+2}}}{\alpha}(\xi - \xi^{p^{m+1}})^{p^{m+1}}. \tag{4.4.3}$$

Furthermore, a straightforward verification shows that $(\xi - \xi^{p^{m+1}})\alpha^{p^{m+1}}\alpha\delta =$ $\beta(\alpha^{p^{m+1}}\delta) - \alpha(\beta^{p^{m+1}}\delta)$ and using (4.4.1) and (4.4.2) we find that this is equal to $\beta(\alpha - \alpha^{p^{2m+2}}) - \alpha(\beta - \beta^{p^{2m+2}}) = -\alpha\alpha^{p^{2m+2}}(\xi - \xi^{p^{2m+2}})$, hence:

$$\delta = \frac{-\alpha^{p^{2m+2}}(\xi - \xi^{p^{2m+2}})}{\alpha^{p^{m+1}}(\xi - \xi^{p^{m+1}})}. \tag{4.4.4}$$

Recall that $\bar{x}_j = \bar{x}_h$ is not in $F_{p^{m+1}}$, hence $\xi - \xi^{p^{m+1}}$ is nonzero. Applying (4.4.3) to the denominator of (4.4.4) and raising the resulting equation to the power p^{m+1} gives the following equation:

$$\delta^{p^{m+1}} = \frac{-\alpha^{p^{m+1}}(\xi - \xi^{p^{2m+2}})^{p^{m+1}}}{\alpha^{p^{2m+2}}(\xi - \xi^{p^{m+1}})^{p^{2m+2}}}. \tag{4.4.5}$$

Multiplication of (4.4.4) and (4.4.5) gives the desired direct relation between $\xi = \bar{x}_{hj}$ and $\delta = \bar{d}_0$:

$$\frac{(\bar{x}_{hj} - \bar{x}_{hj}^{p^{2m+2}})^{1+p^{m+1}}}{(\bar{x}_{hj} - \bar{x}_{hj}^{p^{m+1}})^{1+p^{2m+2}}} = -\bar{d}_0^{1+p^{m+1}}.$$

Note that again the condition $d_0 \neq 0$ is equivalent to the condition that $\bar{x}_j$ is not in $F_{p^{2m+2}}$.

§ 4.5. The Nonhomogeneous Case (Finite Height)

A. In this case M has isogeny type $D^F / D^F(F^{m_1} - V) \oplus D^F / D^F(F^{m_2} - V)$ with $m_1 < m_2 < \infty$. The way in which we shall treat this case, is partly analogous to the way we treated the homogeneous decomposable case. The essential difference is that there are no D^F-module morphisms from $D^F / D^F(F^{m_1} - V)$ to $D^F / D^F(F^{m_2} - V)$ since $m_1 \neq m_2$. (see [1] ch.IV.3D).

Proposition 4.5.1:
There exists an embedding Ψ: $M \rightarrow M_{1,m_1} \oplus M_{1,m_2}$ such that the maximal special sub-D^F-module of $\Psi[M]$ is $R_{1,m_1} \oplus R_{1,m_2}$.
Proof: Let M_1 be the maximal special sub-D^F-module of M. Then M_1 is isogenous to $R_{1,m_1} \oplus R_{1,m_2}$. By definition M_1 is a direct sum of two homogeneous special modules hence (see corollary 3.4.7) M_1 even is isomorphic to $R_{1,m_1} \oplus R_{1,m_2}$. Let Ψ': $M \rightarrow M_{1,m_1} \oplus M_{1,m_2}$ be an embedding of D^F-modules and let Φ: $\Psi'[M_1] \rightarrow R_{1,m_1} \oplus R_{1,m_2}$ be an isomorphism. Then Φ extends uniquely to an automorphism of $M_{1,m_1} \oplus M_{1,m_2}$ which we shall also denote Φ. The required embedding is $\Psi := \Phi\Psi'$. ***

From now on we may assume that M is a sub-D^F-module of $M_{1,m_1} \oplus M_{1,m_2}$ and the maximal special sub-D^F-module of M is $R_{1,m_1} \oplus R_{1,m_2}$.

Proposition 4.5.2:
M contains no elements of the form $(w,0)$ resp. $(0,v)$ unless w is in R_{1,m_1} resp. v is in R_{1,m_2}.
Proof: Suppose $(w,0)$ is an element of M and $\nu(w) < 0$, then $F^{-\nu(w)-1}(w,0)$ is in M, hence $(\theta^{-1},0)$ is also in M. But the D^F-submodule of M generated by $(\theta^{-1},0)$ is special and not contained in $R_{1,m_1} \oplus R_{1,m_2}$, which is a contradiction. For the same reasons M contains no elements of the form $(0,v)$ unless v is in R_{1,m_2}. ***

Lemma 4.5.3:
Let $w = (w_1,w_2)$ be an element of M. Then $\nu(w_1) < 0$ if and only if $\nu(w_2) < 0$.
Proof: Analogous to the proof of lemma 4.4.3. ***

Corollary 4.5.4:
Let $w = (w_1,w_2)$ be an element of M. If w is not in $R_{1,m_1} \oplus R_{1,m_2}$ then $\nu(w_1) = \nu(w_2)$.
Proof: Analogous to the proof of corollary 4.4.4. ***

Proposition 4.5.5:
Let $s > 0$ and suppose M contains an element $w = (w_1,w_2)$ such that $\nu(w_1) = -s$. Then M contains a unique element of the form

$$z_s = (\theta^{-s}, \sum_{k=1}^{s} x_{sk}\theta^{-k}),$$

with all x_{sk} in Im(T). Moreover, $x_{ss} \neq 0$.
Proof: The only difference with proposition 4.4.5 is the condition on x_{ss}. This is due to the fact that we have no analogue to property b) of proposition 4.4.1. The assumption $x_{ss} = 0$ would imply that $(\theta^{-1},0)$ is an element of M, which is a contradiction. The remainder of the proof is analogous to the proof of the corresponding statement in proposition 4.4.5. ***

Corollary 4.5.6: (cf. [16] lemma 3.13a)
Let $w = (w_1,w_2)$ be an element of M. Then $\nu(w_1) \geqslant -m_1$.

Proof: Suppose $\nu(w_1) < -m_1$. Let $s := -\nu(w_1)$, then $s > m_1$. From the facts that $F^{m_1} z_s - V z_s = (0, \sum_{k=1}^{s} x_{sk}^{q^{m_1}} \theta^{-k+m_1} - \sum_{k=1}^{s} x_{sk}^{q^{-1}} \theta^{-k+m_2})$ is an element of M and $m_1 < m_2$ it follows that $x_{ss} = 0$, which is a contradiction (see proposition 4.5.5).
***

Proposition 4.5.7: (cf. [16] lemma 3.13b)
Let h be the F-height of M. M is as a D^F-module generated by z_h and $R_{1,m_1} \oplus R_{1,m_2}$ (cf. [16] lemma 3.13c).
Proof: Analogous to the proof of proposition 4.4.8. ***

Conclusion: The isomorphism class of M has three discrete invariants m_1, m_2 and h. Furthermore, M gives rise to a set $\{x_{hk} \mid 1 \leqslant k \leqslant h\}$. Modulo the action of a finite group, this set is uniquely determined (this can be seen in analogous way as in the isosimple case.) These parameters are subject to the following conditions: $0 \leqslant h \leqslant m_1 < m_2$, all x_k are in $\mathrm{Im}(T)$ and $x_h \neq 0$. For any set of parameters satisfying these conditions we have a corresponding isomorphism class. An element of this class is found by defining z_h as above and then taking the sub-D^F-module of $M_{1,m_1} \oplus M_{1,m_2}$ generated by z_h and $R_{1,m_1} \oplus R_{1,m_2}$. Thus for given m_1, m_2 and h we have a module space of dimension h.

B. If G splits into the direct sum of two one-dimensional formal groups then M is nonhomogeneous of finite height if and only if $h_1 < \infty$ and $0 < h_2 < \infty$. In that case we have

$$m_1 = h_1 \text{ and } m_2 = h_1 + h_2.$$

(Cf. section 4.2.) Clearly M is special, hence $h = 0$.

For the remainder of this section let us assume that G does not split into the direct sum of two one-dimensional formal groups. From proposition 4.2.1 it follows that M is nonhomogeneous of finite height if and only if $h_2 < h_3 < \infty$ and that

$$m_1 = h_1 + h_2 \text{ and } m_2 = h_1 + h_3.$$

Let $\Psi: M \to M_{1,m_1} \oplus M_{1,m_2}$ be an embedding of D^F-modules such that $R_{1,m_1} \oplus R_{1,m_2}$ is the maximal special sub-D^F-module of $\Psi[M]$. Let us write $u = (u_1, u_2)$ and $v = (v_1, v_2)$ for the images of e_1 resp. e_2 under Ψ. Since $Ve_1 = F^{h_1} e_2$ we have $\nu(u_1) + h_2 = \nu(v_1)$ and $\nu(u_2) + h_3 = \nu(v_2)$.

Proposition 4.5.8:
M has F-height $h = h_2$.
Proof: First of all we shall show that M has F-height $-\nu(u_1)$. Since G does not split into the direct sum of two one-dimensional formal groups, M is not special. (Recall that by definition a nonhomogeneous special module splits into homogeneous special modules.) Let $w = (w_1, w_2)$ be an element of M which is not in $R_{1,m_1} \oplus R_{1,m_2}$. In view of corollary 4.5.4 we have $\nu(w_1) = \nu(w_2) < 0$. Since $\Psi[M]$ is as a D^F-module generated by u and v, there exist α and β in D^F such that $w = \alpha u + \beta v$. It follows that $0 > \nu(w_1) \geqslant \min\{\nu(\alpha u_1), \nu(\beta v_1)\} \geqslant \min\{\nu(u_1), \nu(v_1)\} = \nu(u_1)$ (for $\nu(v_1) - \nu(u_1) = h_2 > 0$). Consequently, $F^{-\nu(u_1)} \Psi[M]$ is contained in $R_{1,m_1} \oplus R_{1,m_2}$,

hence the F-height of M is $\leqslant -\nu(u_1)$. On the other hand $F^{-\nu(u_1)-1}u$ is not in $R_{1,m_1} \oplus R_{1,m_2}$, hence the F-height of M must be equal to $-\nu(u_1)$.
We shall now show that $\nu(u_1) = -h_2$. From corollary 4.5.4 and the fact that $\nu(u_1) < 0$ we have $\nu(u_1) = \nu(u_2)$. Since $\nu(v_1) = \nu(u_1) + h_2 \neq \nu(u_1) + h_3 = \nu(v_2)$ it also follows from corollary 4.5.4 that v is in $R_{1,m_1} \oplus R_{1,m_2}$, consequently $\nu(v_1) \geqslant 0$ hence $-\nu(u_1) \leqslant h_2$.
Since (0,1) is an element of $R_{1,m_1} \oplus R_{1,m_2}$ it is also in $\Psi[M]$, hence there exist α and β in D^F such that $\alpha u + \beta v = (0,1)$. From $\alpha u_1 + \beta v_1 = 0$ it follows that $\nu(\alpha u_1) = \nu(\beta v_1)$ hence $\nu(\alpha.1) + \nu(u_1) = \nu(\beta.1) + \nu(v_1)$, which implies $\nu(\alpha.1) = \nu(\beta.1) + h_2$. Consequently we have $\nu(\alpha.1) \geqslant h_2$. From $\alpha u_2 + \beta v_2 = 1$ it follows that $\nu(\alpha u_2 + \beta v_2) = 0$. Now $\nu(\alpha u_2) = \nu(\alpha.1) + \nu(u_2)$. Using $\nu(\alpha.1) = \nu(\beta.1) + h_2$ we get $\nu(\alpha u_2) = \nu(\beta.1) + \nu(u_2) + h_2 < \nu(\beta.1) + \nu(u_2) + h_3 = \nu(\beta v_2)$. It follows that $0 = \nu(\alpha u_2 + \beta v_2) = \nu(\alpha u_2) = \nu(\alpha.1) + \nu(u_2)$, hence $\nu(u_2) = -\nu(\alpha.1) \leqslant -h_2$. Since $\nu(u_1) = \nu(u_2)$ the proof is finished. ***

Summarizing our results in the nonhomogeneous finite height case we have:
If G splits into the direct sum of two one-dimensional formal groups then M is nonhomogeneous of finite height if and only if $h_1 < \infty$ and $0 < h_2 < \infty$. Furthermore,

$$m_1 = h_1,$$
$$m_2 = h_1 + h_2,$$
$$h = 0.$$

If G does not split into the direct sum of two one-dimensional formal groups then M is nonhomogeneous of finite height if and only if $h_2 < h_3 < \infty$. Furthermore,

$$m_1 = h_1 + h_2,$$
$$m_2 = h_1 + h_3,$$
$$h = h_2.$$

Remark: Again we are in the situation that the component of the covariant module space on which G lies has the same dimension as the component of the contravariant module space on which G lies.

Next we shall study the relations between the continuous parameters. Write

$$u_1 = \sum_{s=-h}^{\infty} a_s \theta^s$$

and

$$u_2 = \sum_{s=-h}^{\infty} b_s \theta^s$$

with all a_s and b_s in Im(T). Like in the homogeneous decomposable case, the element z_h may be constructed from $u = (u_1,u_2)$. This gives relations between the x_k and the a_s and b_s, for instance $x_h = \dfrac{b_{-h}}{a_{-h}}$. In view of (4.1.3) we have

$$v_1 = \sum_{s=-h}^{\infty} a_s^{\sigma^{-h_1-1}} \theta^{s+h_2}$$

and

$$v_2 = \sum_{s=-h}^{\infty} b_s^{\sigma^{-h_1-1}} \theta^{s+h_3}.$$

Again, (4.1.4) gives relations between the a_s and b_s and the d_k, for instance $a_{-h} \equiv d_0 a_{-h}^{\sigma^{m_1+1}}$ and $d_0 b_{-h}^{\sigma^{m_1+1}} + b_{-h}^{\sigma^{m_1+m_2+2}} \equiv 0 \bmod pW$. A straightforward verification shows that we have the following direct relation:

$$x_h^{-1+p^{m_1+m_2+2}-p^{m_1+1}-p^{m_2+1}} = d_0^{p^{m_2+1}-p^{-m_1-1}}.$$

Note that the condition $d_0 \neq 0$ is equivalent to the condition $x_h \neq 0$.

§ 4.6. The Remaining Cases (Infinite Height)

A1. Isogeny type $D^F / D^F V^2$.

Proposition 4.6.1: (cf [16] theorem 3.13)
Suppose M is isogenous to $D^F / D^F V^2$. Then M is as a D^F-module generated by two elements $\{x_0, x_1\}$ with $Vx_0 = F^h x_1$ for some $h \geqslant 0$ and $V^2 x_0 = 0$. The isomorphism class of M is uniquely determined by h.
Proof: Let Ψ: $M \to D^F / D^F V^2$ be an embedding of D^F-modules. One easily verifies that Ψ may be chosen such that $\Psi[M]$ is not contained in $F(D^F / D^F V^2)$. Let j be the smallest integer such that $F^j(D^F / D^F V^2)$ is contained in $\Psi[M]$ (the existence of such an integer follows from the fact that Ψ is an isogeny). If $j = 0$ then Ψ is an isomorphism. In that case the proposition is trivial. If $j > 0$ then all elements of the form $\sum_{i=i_0}^{\infty} x_i F^i$ in $\Psi[M]$ with $i_0 < j$ must satisfy $x_{i_0} \equiv 0 \bmod p$, for else there would be an element α of D^F such that $\alpha \sum_{i=i_0}^{\infty} x_i F^i = F^{i_0}$, which would imply that $F^{i_0} \in \Psi[M]$. Since $\Psi[M]$ is not contained in $F(D^F / D^F V^2)$, it follows that there is an element $w = x_{-1} V + \sum_{i=0}^{\infty} x_i F^i$, with x_{-1} nonzero mod p, in $\Psi[M]$. We may assume that $x_{-1} = 1$, for if not so, take $x_{-1}^{-1} w$ instead of w. Moreover, all x_i may be assumed to be in Im(T): put $x_0 = x_0' + p x_0''$ with x_0' in Im(T) and define $w' = w - x_0'' F w$. Repeat this procedure for $x_1, x_2, \cdots$. In view of the fact that F^j is in $\Psi[M]$ we find the existence of an element

$$z = V + \sum_{i=0}^{j-1} x_i F^i,$$

with all x_i in Im(T), in $\Psi[M]$. One easily verifies that $\Psi[M]$ is generated as a D^F-module by $\{z, F^j\}$. If $x_i = 0$ $(0 \leqslant i < j)$ we have $V \in \Psi[M]$. If not so, let k be the smallest integer such that $x_k \neq 0$, hence $k < j$. Then there exists an element β in D^F such that $\beta \sum_{i=k}^{j-1} x_i F^i = F^k$. Consequently, $\beta z = F^k + \beta V$, hence $F^{2k} = (F^k - \beta^{\sigma^k} V)\beta z$ is in $\Psi[M]$. In view of the minimality of j we have $2k \geqslant j$. Put $\gamma = (V - \sum_{i=k}^{j-1} x_i^{\sigma^{-1}} F^i) F^{-j}$. A straightforward calculation shows the existence of an α in

D^F such that $\alpha z\gamma = F^{2k-j}$. Hence γ is a unit in the F-localization of $D^F / D^F V^2$. Define Ψ' by $\Psi'(x) = \Psi(x)\gamma$. Using the fact that $\Psi[M]$ is generated by $\{z, F^j\}$ and that $2k \geqslant j$, a straightforward verification shows that $\Psi'[M]$ is contained in $D^F / D^F V^2$ but not in $F(D^F / D^F V^2)$. Moreover, $F^{2k-j} = \alpha z\gamma$ is in $\Psi'[M]$. So if j' is the smallest integer such that $F^{j'}(D^F / D^F V^2)$ is in $\Psi'[M]$ then $j' \leqslant 2k - j$. Since $k < j$ this means that $j' < j$.
If V is not in $\Psi'[M]$ we repeat the whole procedure. After at most j steps we come to an embedding such that the image of M does contain V.
We conclude that there exists an h such that M is isomorphic to the submodule of $D^F / D^F V^2$ generated by $\{V, F^h\}$. ***

B1. Obviously, G does not split into the direct sum of two one-dimensional formal groups. In view of proposition 4.2.1, M is isogenous to $D^F / D^F V^2$ if and only if $h_2 = \infty$. Taking $x_0 = e_1$ and $x_1 = e_2$ it is clear that

$$h = h_1.$$

A2. Isogeny type $D^F / D^F V \oplus D^F / D^F(F^m - V)$.

Proposition 4.6.2: (cf. [16] theorem 3.16)
Suppose M is isogenous to $D^F / D^F V \oplus D^F / D^F(F^m - V)$ for some $m \geqslant 0$. Then M is as a D^F-module generated by elements $\{x_0, x_1, x_2\}$ with $Vx_0 = 0$, $F^m x_1 = Vx_1$ and $F^h x_2 = x_0 + x_1$ for some h with $0 \leqslant h \leqslant m$. The isomorphism class of M is uniquely determined by m and h.
Proof: We claim that there exists an embedding Ψ: $M \to (D^F / D^F V)_F \oplus M_{1,m}$ such that:
a) the sub-D^F-module of $\Psi[M]$ generated by the elements x of $\Psi[M]$ with $F^m x = Vx$, is equal to $\{(0, x_2) \mid x_2 \in R_{1,m}\}$,
b) $D^F / D^F V$ is contained in $\Psi[M]$,
c) $(F^{-1}, 0)$ is not in $\Psi[M]$.
First we show that Ψ can be chosen such that it satisfies part a) of our claim: Let $x = (x_1, x_2)$ be an element of $\Psi[M]$ such that $F^m x = Vx$. Then $x_1 = 0$ and x_2 is special in $M_{1,m}$. From the fact that $\Psi[M]$ is finitely generated over D^F and that $\Psi[M]_F$ is equal to $(D^F / D^F V)_F \oplus M_{1,m}$ it follows that the special elements x_2 of $M_{1,m}$ for which $(0, x_2)$ is in $\Psi[M]$, generate a D^F-module which is isomorphic to $R_{1,m}$ (cf. corollary 3.4.7). The remainder of the verification of a) is analogous to the proof of proposition 4.5.1. Property b) can be obtained by multiplication on the right of the first component by a suitable power of F. If $(F^{-1}, 0)$ is in $\Psi[M]$ then multiplication on the right of the first component by F is an automorphism of $(D^F / D^F V)_F \oplus M_{1,m}$ which does not disturb a) and b). Since M is finitely generated over D^F, a finite number of steps is needed in order to reach our result.
Let us identify M with $\Psi[M]$ and put $M_1 := D^F / D^F V \oplus R_{1,m}$. Let h be the smallest integer such that $F^h M$ is contained in M_1. In a way completely analogous to the preceding two sections (cf. propositions 4.4.5 and 4.5.5) one may see that M is generated over D^F by M_1 and an element z_h of the form

$$z_h = \left(\sum_{k=1}^{h} a_k F^{-k}, \theta^{-h}\right),$$

where all a_k are in $\mathrm{Im}(T)$ and $a_h \neq 0$. Now let α be an element of D^F such that $\sum_{k=1}^{h} a_k F^{-k} \alpha = F^{-h}$. Then define the automorphism Φ of $(D^F / D^F V)_F \oplus M_{1,m}$ by

$\Phi(x_1,x_2) = (x_1\alpha,x_2)$. Note that $\Phi[M]$ still satisfies a), b) and c). The image of z_h is (F^{-h},θ^{-h}). It is now clear that M is determined by h. From the definition of h it follows that h is an isomorphism invariant. Clearly x_0, x_1 and x_2 must be chosen such that $\Phi(x_0) = (1,0)$, $\Phi(x_1) = (0,1)$ and $x_2 = z_h$. ***

B2. If G splits into the direct sum of two one-dimensional formal groups then M has the relevant isogeny type if and only if $h_1 < \infty$ and $h_2 = \infty$ (cf. section 4.2). Furthermore .

$$m = h_1.$$

Taking $x_0 = e_2$, $x_1 = e_1$ and $x_2 = e_1 + e_2$ it follows that

$$h = 0.$$

If G does not split into the direct sum of two one-dimensional formal groups, proposition 4.1.2 implies that M has the relevant isogeny type if and only if $h_2 < \infty$ and $h_3 = \infty$. Furthermore ,

$$m = h_1 + h_2.$$

Taking $x_0 = F^{h_2}e_1 - e_2$, $x_1 = e_2$ and $x_2 = e_1$ we find that

$$h = h_2.$$

A3. Isogeny type $(D^F / D^F V)^2$.

Proposition 4.6.3:
If M is isogenous to $(D^F / D^F V)^2$ then M is isomorphic to $(D^F / D^F V)^2$.
Proof: Note that the automorphisms of $(D^F / D^F V)^2_F$ are of the form $\Phi(x_1,x_2) := (x_1\alpha_1 + x_2\alpha_2, x_1\alpha_3 + x_2\alpha_4)$ where $\alpha_1,\alpha_2,\alpha_3$ and α_4 are elements of $(D^F)_F$ such that $\alpha_1\alpha_4 - \alpha_2\alpha_3$ is nonzero mod $V(D^F)_F$. We may now show in an analogous way as we did in the proof of proposition 4.4.1 that there exists an embedding
$\Psi: M \to (D^F / D^F V)^2_F$ such that $\Psi[M]$ contains $(D^F / D^F V)^2$, but does not contain nonzero elements of the form (aF^{-1},bF^{-1}). Consequently, $\Phi[M] = (D^F / D^F V)^2$.
***

B3. In view of proposition 4.2.1, G splits into the direct sum of two one-dimensional formal groups and $h_1 = \infty$.

Note that all possible normalized F-types have now taken their turn.

§ 4.7. Translation from Contravariant to Covariant

Given the covariant discrete parameters of a formal group G, we have found the contravariant discrete parameters. The relations we have found, make it possible to do the converse: given the contravariant discrete parameters, we can derive from them the covariant discrete parameters. The results, which may easily be verified, are the following:

If M is isosimple of finite height then G does not split into the direct sum of two one-dimensional formal groups and

$$h_1 = m - i,$$

$$h_2 = 2i - j + 1,$$

$$h_3 = j.$$

If M is homogeneous decomposable of finite height and $h = 0$ then G splits and

$$h_1 = m,$$

$$h_2 = 0.$$

If M is homogeneous decomposable of finite height and $h > 0$ then G does not split and

$$h_1 = m - h,$$

$$h_2 = 2h - j,$$

$$h_3 = j.$$

If M is nonhomogeneous of finite height and $h = 0$ then G splits and

$$h_1 = m_1,$$

$$h_2 = m_2 - m_1.$$

If M is nonhomogeneous of finite height and $h > 0$ then G does not split and

$$h_1 = m_1 - h,$$

$$h_2 = h,$$

$$h_3 = m_2 - m_1 + h.$$

If M is isogenous to $D^F / D^F V^2$ then G does not split and

$$h_1 = h,$$

$$h_2 = \infty.$$

If M is isogenous to $D^F / D^F V \oplus D^F / D^F(F^m - V)$ and $h = 0$ then G splits and

$$h_1 = m,$$

$$h_2 = \infty.$$

If M is isogenous to $D^F / D^F V \oplus D^F / D^F(F^m - V)$ and $h > 0$ then G does not split and

$$h_1 = m - h,$$

$$h_2 = h,$$

$$h_3 = \infty.$$

If G is isogenous (hence isomorphic) to $(D^F / D^F V)^2$ then G splits and

$$h_1 = \infty.$$

5. APPLICATION TO CURVES OF GENUS TWO

A curve of genus 2 gives rise to a two-dimensional smooth commutative formal group G: the completion of its jacobian variety. In [14], Igusa gives a normal form for the curves of genus 2 over a field of arbitrary characteristic. Given a curve Γ in such a normal form, we shall in this chapter give an algorithm to determine the covariant isomorphism type of G, i.e. a normalized F-type of G. For characteristic 2 we shall even give the explicit normalized F-type of G, *without specifying* the curve Γ, in other words: we shall explicitly determine how the isomorphism type of G depends on the parameters of Igusa's normal form.

§ 5.1. Preliminaries

Let Γ be the plane curve with affine equation

$$F(X,Y) = XY + Y^3 + aXY^2 + bX^2Y + cX^2Y^2 + dX^3Y + X^4 = 0,$$

where the coefficients a,b,c and d are such that (0,0) is the only singular point of Γ. Note that the affine equation given in [14], is: $Y^4F(\frac{X}{Y},\frac{1}{Y}) = 0$ (i.e. interchanging the Y-axis and the line at infinity).
Let P_1 and P_2 be the places centered at (0,0) with tangent lines $X = 0$ resp. $Y = 0$. Put π_1 and π_2 for the classes of Y resp. X in the function field of Γ. So π_i is a local parameter of the place P_i $(i = 1,2)$. Obviously, the adjoint curves of degree 1 are the lines passing through the origin. In view of the well-known correspondence between the adjoint curves of degree 1 and the differentials of the first kind on Γ, the effective canonical divisors are the intersection divisors of lines through the origin with Γ minus the divisor $P_1 + P_2$. In particular, taking the line $Y = 0$, we find that $2P_2$ is a canonical divisor. Consequently, the divisor $P_1 + P_2$ is nonspecial.
Let ω_1 and ω_2 be the differentials of the first kind which correspond to $Y = 0$ resp. $X = 0$. Then it is obvious that ω_1 and ω_2 constitute a basis for the space of differentials of the first kind. It is an elementary exercise in algebraic geometry to show that ω_1 is locally defined by

$$\omega_1 = \frac{\pi_1}{\delta F / \delta X(\pi_2,\pi_1)}d\pi_1 = \frac{-\pi_1}{\delta F / \delta Y(\pi_2,\pi_1)}d\pi_2 \tag{5.1.1}$$

and that ω_2 is locally defined by

$$\omega_2 = \frac{\pi_2}{\delta F / \delta Y(\pi_2,\pi_1)}d\pi_2 = \frac{-\pi_2}{\delta F / \delta X(\pi_2,\pi_1)}d\pi_1. \tag{5.1.2}$$

§ 5.2. Lifting to Characteristic Zero

In this section we shall consider $F(X,Y)$ as an element of $W[X,Y]$. Identify a,b,c and d with their images under T. Let $\xi(t)$ be the power series in $W[[t]]$ of order 2 such that $F(\xi(t),t) = 0$. It is easily verified that such a $\xi(t)$ exists and is unique. Then the reduction of $\xi(t) \bmod p$ is the power series expansion of the function π_2 at the place P_1 in the local parameter π_1. Furthermore, let $\eta(t)$ be the element of $W[[t]]$ having order 3 and such that $F(t,\eta(t)) = 0$. So the reduction of $\eta(t) \bmod p$ is the expansion of π_1 at P_2 in the parameter π_2. Next put $\beta_{11}(t) = \frac{t}{\delta F / \delta X(\xi(t),t)}$, $\beta_{21}(t) = -t^{-1}\xi(t)\beta_{11}(t)$, furthermore $\beta_{22}(t) = \frac{t}{\delta F / \delta Y(t,\eta(t))}$ and $\beta_{12}(t) =$

$-t^{-1}\eta(t)\beta_{22}(t)$. Note that the $\beta_{ij}(t)$ are elements of $W[[t]]$ and that the reduction of $\beta_{ij}(t) \bmod p$ is the power series expansion of the differential ω_i at the place P_j in the local parameter π_j (cf. [11]). The power series $\beta_{ij}(t)$ themselves have the following interpretation: (5.1.1) and (5.1.2) define elements ω_1 and ω_2 in the completion of the sheaf of differentials on the scheme $Spec(W[X,Y]/(F(X,Y)))$. Let π_1 resp. π_2 be the classes of X resp. Y in $W[X,Y]/(F(X,Y))$. Then

$$\omega_i = \beta_{ij}(\pi_j)d\pi_j$$

in the stalk at the point which corresponds to the (maximal) ideal generated by π_j and p.
Let B be the matrix $(\beta_{ij}(t))$ in $M_2(W[[t]])$ and put

$$B = \sum_{i=1}^{\infty} B_i t^{i-1},$$

the B_i being elements of $M_2(W)$. One easily verifies that B_1 is the identity matrix. It follows that the reduction of $B_p \bmod p$ is the Hasse-Witt matrix of Γ (cf. [11]). The matrices $B_{p^j} (j \geqslant 0)$ determine an F-type of G as follows: Define matrices L_k $(k \geqslant 0)$ recursively by

$$B_{p^{k+1}} = \sum_{i=0}^{k} p^i B_{p^{k-i}} {}^t L_i^{\sigma^{k-i}}.$$

Then the L_k have their entries in W and $F = \sum_{k=0}^{\infty} V^k L_k$ is an F-type of G (cf. [9] formula (2.7) and section 6 p. 58). Once having an F-type, application of the lemmas 1.9.1 and 1.9.2 will give a normalized F-type.

§ 5.3. Computation of the Normalized F-types in Characteristic Two

Given the fact that the orders of $\xi(t)$ and $\eta(t)$ are 2 resp. 3, the equations $F(\xi(t),t) = 0$ and $F(t,\eta(t)) = 0$ give a recursive definition of $\xi(t)$ resp. $\eta(t)$. Furthermore, from the definitions we have recursive formulas to compute the $\beta_{ij}(t)$. Thus we have an algorithm to compute the B_i without specifying the parameters a,b,c and d in the equation of Γ. Of course, it is a heavy, if not impossible, job to do these computations by hand. Using formula pascal, as developed by F. Teer [17], we have found the B_i for $i \leqslant 18$. The entries of these matrices, being elements of $\mathbf{Z}[a,b,c,d]$, grow rapidly. (Each entry of B_{18} takes a whole page!)

Assume $p = 2$. Then we find $L_0 = {}^t B_2 = \begin{pmatrix} -a & 1 \\ 0 & -b \end{pmatrix}$. It follows that we always have $h_1 = 0$ in characteristic 2 (cf. proposition 1.10.1). Now if both a and b are nonzero, Γ has an invertible Hasse-Witt matrix. So L_0 can be transformed into the identity matrix by (1.9.1), hence in view of theorem 1.10.3 we find

Proposition 5.3.1:
If both a and b are nonzero then G has F-type

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In order to study the remaining cases we need to know that

$$B_4 = \begin{bmatrix} 2c - a^3 - 6ab & -2b \\ 3a^2 + 3b & 2a + 2bd - b^3 \end{bmatrix}.$$

Now L_1 is found by $2^t L_1 = B_4 - B_2^t L_0^\sigma$. The result is:

$$L_1 = \begin{bmatrix} -a^3 - 3ab + c & 2a^2 + 2b \\ -b & a + bd - b^3 \end{bmatrix}.$$

Notice: indeed, this matrix has its entries in W. Assume $a \neq 0$ and $b = 0$. Then $L_0 = \begin{bmatrix} -a & 1 \\ 0 & 0 \end{bmatrix}$ and $L_1 = \begin{bmatrix} c - a^3 & 2a^2 \\ 0 & a \end{bmatrix}$. Let α be a nonzero root of the equation $\alpha + \alpha^\sigma a = 0$. In view of [5] chap. III.5 lemma 1, such an α exists. Note that α is invertible in W. Put $\Lambda_0 = \begin{bmatrix} \alpha & \alpha^\sigma \\ 0 & 1 \end{bmatrix}$ and apply (1.9.1). Then we get a new F-type $F = \sum_{k=0}^{\infty} V^k L_k'$ with $L_0' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $L_1' = \begin{bmatrix} * & * \\ 0 & a \end{bmatrix}$. Using the fact that $a \neq 0$, one easily verifies that by means of the matrices Λ_0 and Λ_1 in (1.9.1) we may find a new F-type $F = \sum_{k=0}^{\infty} V^k L_k$ with $L_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $L_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. It follows that G splits into the direct sum of two one-dimensional formal groups and that $h_2 = 1$ (cf theorem 1.10.3 and subsection 2.1.1). In view of theorem 2.1.1 we have

Proposition 5.3.2:
If $a \neq 0$ and $b = 0$ then G has F-type

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next consider the case $a = 0$ and $b \neq 0$. Then $L_0 = \begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}$ and $L_1 = \begin{bmatrix} c & 2b \\ -b & bd - b^3 \end{bmatrix}$. Let β be a nonzero root of the equation $\beta + \beta^\sigma b = 0$. Put $\Lambda_0 = \begin{bmatrix} 0 & \beta \\ 1 & b^{-1/2} \end{bmatrix}$. Application of (1.9.1) gives $L_0' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $L_1' = \begin{bmatrix} * & * \\ * & c - b^{-1} \end{bmatrix}$. In our situation (i.e. $a = 0$, $b \neq 0$ and $p = 2$), the fact that (0,0) is the only singularity of Γ is equivalent to: $c - b^{-1} \neq 0$. In the same way as in the preceding case, we find

Proposition 5.3.3:
If $a = 0$ and $b \neq 0$ then G has F-type

$$F = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + V \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Finally, consider the case $a = b = 0$. Then $L_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $L_1 = \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix}$. So if

$c \neq 0$ then $h_2 = 1$ (cf. corollary 2.1.3). We now also need to know L_2. From the relation $4^t L_2 = B_8 - B_4^t L_0^{\sigma^2} - 2B_2^t L_1^{\sigma}$ and the fact that (for $a = b = 0$) $B_8 = \begin{bmatrix} -20cd & -4 \\ 10c^2 & 12d \end{bmatrix}$ it follows that $L_2 = \begin{bmatrix} -5cd & 2c^2 \\ -1 & 3d \end{bmatrix}$. Taking $\Lambda_1 = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}$ in (1.9.1) we get $L_0' = L_0$, $L_1' = \begin{bmatrix} 0 & 0 \\ 0 & c^2 \end{bmatrix}$ and $L_2' = L_2$. So if $c \neq 0$ then $h_3 = 1$ (cf. corollary 2.1.6) and if $c = 0$ then $h_2 = 2$ and $h_3 = 0$ (cf. corollaries 2.1.3 and 2.1.6).

Proposition 5.3.4:
If $a = b = 0$ then G has F-type

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + V\begin{bmatrix} 0 & 0 \\ 0 & c^2 \end{bmatrix} + V^2\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Proof: If $c = 0$ this is a direct consequence of theorem 2.1.8, if $c \neq 0$ see claims 3.1-3.3 in section 2.2. ***

Summarizing: In characteristic 2, we have $h_1 = 0$. Furthermore, if a and b are not both zero then G splits into the direct sum of two one-dimensional formal groups. If moreover a and b are both nonzero then $h_2 = 0$, hence G has slope 0 with multiplicity 2 and G has height 2 (cf. subsection 3.4.1 and section 4.2). If either a or b is zero then $h_2 = 1$, hence in that case G has slopes 0 and $\frac{1}{2}$, with multiplicities 1 resp. 2 and G has height 3. If $a = b = 0$ then G does not split into the direct sum of two one-dimensional formal groups. If moreover $c = 0$ then $h_2 = 2$ and $h_3 = 0$, if $c \neq 0$ then $h_2 = h_3 = 1$ and $\bar{d_0} = c^2$. So if $a = b = 0$ then G has slope $\frac{1}{2}$ with multiplicity 4 and G has height 4 (cf. proposition 4.3.1).
Remark: As was to be expected, the height of G is between 2 and 4 and the slopes of G are allowed by Poincaré duality.

Of course, two algebraic curves which are birationally equivalent have isomorphic formal groups. Now let Γ' be another curve in Igusa's normal form, with parameters a', b', c' and d', and assume Γ and Γ' are birationally equivalent, in other words assume that there exists an element ζ in k with $\zeta^5 = 1$ such that $(a', b', c', d') = (\zeta a, \zeta^2 b, \zeta^3 c, \zeta^4 d)$ (cf. [14]). Again suppose $p = 2$. If a and b are not both zero or if $a = b = c = 0$, G has the same normalized F-type as the formal group G' of Γ'. If $a = b = 0$ and $c \neq 0$ then G has continuous invariant $d_0 = c^2$ and G' has continuous invariant $e_0 = c'^2$. In view of theorem 2.1.8 there must exist a nonzero element λ of F_{16} such that $\lambda^4 \bar{d_0} = \lambda \bar{e_0}$. Take $\lambda = \zeta^2$: then $\lambda^{16} = \zeta^{32}$ and since $\zeta^5 = 1$ we find that $\lambda^{16} = \lambda$, so indeed $\lambda \in F_{16}$. Moreover, $\lambda^4 \bar{d_0} = \zeta^8 \bar{c}^2 = \zeta^2 \bar{c}'^2 = \lambda \bar{e_0}$, as was expected.

Comments. Let G be the completion at the origin of an abelian surface over an algebraically closed field k of characteristic $p > 0$ and assume that G is not isomorphic to the direct sum of two one-dimensional formal groups. It follows from proposition 4.2.1 that G has height $2h_1 + h_2 + h_3 + 2$. In view of the fact that the height of G must be between 2 and 4, h_1 must be 0. Since $h_2 > 0$, the following three situations are possible: $h_2 = 1$ and $h_3 = 0$, $h_2 = h_3 = 1$ or $h_2 = 2$ and $h_3 = 0$. The assumption $h_2 = 1$ and $h_3 = 0$ would imply that G has slope $\frac{1}{3}$ with multiplicity 3

(cf. proposition 4.2.1), which is impossible in view of the Poincaré duality.
If $h_2 = h_3 = 1$ or if $h_2 = 2$ and $h_3 = 0$, G has slope $\frac{1}{2}$ with multiplicity 4. In both of these cases the normalized F-type of G has the form

$$F = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + V \begin{bmatrix} 0 & 0 \\ 0 & d_0 \end{bmatrix} + V^2 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

where $d_0 = 0$ corresponds to the situation in which $h_2 = 2$ and $h_3 = 0$. Thus we have found an element $\bar{d_0}$ of k which is a birational invariant of the abelian surface. Probably it is an interesting question what geometric interpretation this (new) invariant has.

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