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APPLICATION OF THE THEORY OF BOUNDARY VALUE PROBLEMS IN THE ANALYSIS OF A QUEUEING MODEL WITH PAIRED SERVICES

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PREFACE

This monograph is a slightly revised edition of my doctoral thesis which has been submitted to the University of Utrecht (September 22, 1982).
The main changes in comparison with the thesis are: in the sections II.8, III.8 and III.9 some inequalities have been added or tightened; the proof of lemma II.5.9 has been rewritten; in section IV.2 a remark on the conditions on which the generalized model is ergodic, has been added; and section IV.3 has been rewritten, now including a description of the numerical procedures and providing other and more examples.

I express my gratitude to my thesis advisor, Prof.Dr.Ir. J.W. Cohen, and to Dr.Ir. O.J. Boxma for their valuable comments, and to Drs. F.M. Elbertsen for carrying out a part of the numerical evaluations.
I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts.

CONTENTS

GENERAL INTRODUCTION

The analysis of queueing models which differ only slightly from the classical M/G/1 model, for instance if two types of customers have to be distinguished, is much more complicated than that of the M/G/1 model itself. Such models often give rise to the problem of solving a functional equation of the form

$$K(p_1,p_2)\ \Phi(p_1,p_2) = A(p_1,p_2)\ \Phi(p_1,0) + B(p_1,p_2)\ \Phi(0,p_2) + C(p_1,p_2)\ \Phi(0,0),$$
$$|p_1| \leqslant 1,\ |p_2| \leqslant 1; \qquad (0.1)$$

here K, A, B and C are known functions and $\Phi(p_1,p_2)$ is an unknown function which should be a bivariate generating function in p_1 and in p_2 of a proper probability distribution with support the set $\{0,1,2,..\}\times\{0,1,2,..\}$.
An important role in the analysis of this type of functional equations is played by the kernel $K(p_1,p_2)$, because zeros (p_1,p_2) of the kernel $K(p_1,p_2)$ in the region $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, where the function $\Phi(p_1,p_2)$ - being a generating function of a probability distribution - is finite, lead to a functional relation between the regular functions $\Phi(p_1,0)$ and $\Phi(0,p_2)$.
Recently, a method has been developed to formulate the inherent problem for the determination of the functions $\Phi(p_1,0)$ and $\Phi(0,p_2)$ as boundary value problems for regular functions.
During the last decennia some functional equations of the type (0.1) have been solved in literature. For functional equations with a kernel being a polynomial in p_1 and in p_2 of sufficiently low degree the technique of uniformisation has been applied. The most recent of this approach was given

by FLATTO & MCKEAN [11].

An important step forward in the analysis of functional equations of the type (0.1) has been initiated by FAYOLLE [08] & IASNOGORODSKI [16], see also [09]. The essential point in their work is that once a relation describing the zeros of the kernel $K(p_1,p_2)$ has been obtained the determination of the functions $\Phi(p_1,0)$ and $\Phi(0,p_2)$ could be formulated as a Riemann-Hilbert boundary value problem (we shall use the terminology of MUSKHELISHVILI [20] in the theory of boundary value problems). This observation enabled them to solve a number of queueing problems which gave rise to functional equations as described by (0.1). However, in their approach they needed an explicit description of the zeros (p_1,p_2) of the kernel $K(p_1,p_2)$ in the whole domain $\mathbb{C} \times \mathbb{C}$. This feature implies that the analysis as proposed by FAYOLLE & IASNOGORODSKI [09] can only be applied if the kernels are of sufficiently simple type, which implies that a simple service time distribution has to be chosen (negative exponential).

The investigations of COHEN & BOXMA [04] showed that the determination of $\Phi(p_1,0)$ and $\Phi(0,p_2)$ in (0.1) could also be reduced to the solution of two Riemann-Hilbert boundary value problems for kernels of a rather general character, i.e. for kernels of the form

$$K(p_1,p_2) = p_1p_2 - \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right), \tag{0.2}$$

here $\beta(.)$ is the Laplace-Stieltjes transform of a probability distribution with support on $(0,\infty)$, and c_1, c_2, α are positive constants, $c_1 + c_2 = 1$. In this approach the service time distributions do not need to be specified, so that the generality of this method is comparable with that for the basic M/G/1 model. Recently, a still more general technique has been developed which makes it possible to formulate the determination of $\Phi(p_1,0)$ and $\Phi(0,p_2)$ as a boundary value problem for the case of the left-continuous two-dimensional random walk in the first quadrant (see a forthcoming report of Cohen & Boxma).

The technique developed by COHEN & BOXMA [04] can be characterized as follows. The zeros (p_1,p_2) of the kernel (0.2) are described by means of a two-valued analytic function of the parameter $\delta := c_1p_1 + c_2p_2$. At first the domain of this function is restricted by the conditions $|p_1| \leqslant 1$ and $|p_2| \leqslant 1$, in which region the function $\Phi(p_1,p_2)$ is known to be regular. But on account of the properties of the Laplace-Stieltjes transform $\beta(.)$ this two-valued analytic function of δ can be continued analytically into the region Re $\delta \leqslant 1$. It turns out that this continued function has exactly two branch points in the region Re $\delta \leqslant 1$. These two branch points are real. At the line segment joining them the two values of the analytic function are complex conjungate, and this line segment is mapped onto a smooth contour L.

By means of the analytic continuation of the function of δ, describing the zeros of the kernel (0.2), also the relation between $\Phi(p_1,0)$ and $\Phi(0,p_2)$ can be extended into the region Re $\delta \leqslant 1$ (principle of permanence). Hence, this relation can be considered on the line segment joining the above mentioned branch points. This leads to a relation between $\Phi(w/2c_1,0)$ and $\Phi(0,\bar{w}/2c_2)$ for $w \in L$. Moreover it can be proved that the functions $\Phi(w/2c_1,0)$ and $\Phi(0,w/2c_2)$ are regular for w in the interior of the contour L. Therefore, by taking real and imaginary parts of the relation on L two Riemann-Hilbert boundary value problems can be formulated. For the solution of these boundary value problems the conformal mapping of the unit disk onto the interior of the contour L is introduced.

Once these Riemann-Hilbert problems have been formulated the functions $\Phi(p_1,0)$ and $\Phi(0,p_2)$ can be determined uniquely inside the unit circle by means of analytic continuation from the interior of L. Finally, by substitution of $\Phi(p_1,0)$ and $\Phi(0,p_2)$ in the original functional equation (0.1) the function $\Phi(p_1,p_2)$ is obtained.

Hence the following procedure for the solution of a functional equation of the form (0.1) with kernel (0.2) has become available:

i. describe the zeros (p_1,p_2) of the kernel (0.2) for which the generating function $\Phi(p_1,p_2)$ is finite by means of a two-valued analytic function;

ii. determine the two branch points of the extended analytic function in the half plane Re $\delta \leqslant 1$, and determine the contour L on which the line segment between those branch points is mapped by this analytic function;

iii. take the real and imaginary parts of the relation between $\Phi(w/2c_1,0)$ and $\Phi(0,\bar{w}/2c_2)$ which holds for $w \in L$; these boundary conditions define two Riemann-Hilbert problems for the contour L;

iv. transform these two boundary value problems for the contour L into two Riemann-Hilbert problems for the unit circle by means of the conformal mapping of the unit disk onto the interior of L;

v. from the solutions of these Riemann-Hilbert problems (given in literature) and some side conditions the functions $\Phi(p_1,0)$ and $\Phi(0,p_2)$ can be determined; next substitute these functions in the functional equation (0.1) which gives the function $\Phi(p_1,p_2)$.

A functional equation of the type (0.1) occurs by the determination of the stationary distribution of an imbedded Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)), n=0,1,..\}$ which characterizes a certain queueing model. The existence of such a stationary distribution has actually to be proved and in most cases this requires the analysis of the time dependent behaviour of the Markov chain, in particular of its asymptotic behaviour as $n \to \infty$.

The most important part of the present investigation is the question whether the time dependent behaviour of the imbedded Markov chain can be analysed with the same techniques as discussed above for the stationary case, and if so, how the asymptotic analysis of the Markov chain as $n \to \infty$ should be carried through.

The second aim of this study can be characterized as follows. The Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$ is in many situations an imbedded chain of a process $\{(\underline{y}_1(t),\underline{y}_2(t)),t>0\}$, the latter characterizes the behaviour of the number of customers in continuous time. Although from the description of the imbedded chain important information can be obtained concerning the behaviour of the queueing system, ultimately the full description of the process in continuous time is needed. So the problem arises whether we can apply the above mentioned solution method also in the analysis of the time dependent behaviour of the process $\{(\underline{y}_1(t),\underline{y}_2(t)),t>0\}$ and whether its behaviour as $t\to\infty$ can be analysed.
It turns out that both questions on the applicability of the solution method of Cohen & Boxma can be answered affirmatively. Moreover, it is shown by using the literature on the boundary behaviour of a conformal mapping of the unit disk onto a given domain that the above mentioned asymptotic behaviour of the imbedded chain ($n\to\infty$) as well as that of the process in continuous time ($t\to\infty$) can be handled.
In the time dependent case the contour L and hence also the conformal mapping g(r;z) of the unit disk onto the interior of L depend on the time variable r; the analysis of the asymptotic behaviour of the process requires the investigation of limits of the form

$$\lim_{r\uparrow 1}(1-r)\int_{|z|=1}\frac{\phi(r;z)}{[1-g(r;z)][1-g(r;\frac{1}{z})]}\,dz,$$

here $\phi(r;z)$ is finite for $|z|=1, |r|\leqslant 1$, and $g(r;z)=1$ if and only if $z=1$, $r=1$.

Because the M/G/1 model is a basic model in queueing theory and because this model allows a complete analytic description without specification of the service time distribution we have chosen for the discussion of the above questions the simplest generalization of the M/G/1 model which leads to a functional equation of the type (0.1) and a kernel of the form (0.2). Characteristic for the analysis of the M/G/1 model is the recurrence relation

$$\underline{x}(n) = [\underline{x}(n-1) - 1]^+ + \underline{\xi}(n), \quad n = 1,2,.., \tag{0.3}$$

for the imbedded Markov chain $\{\underline{x}(n), n=0,1,..\}$ with stationary transition probabilities and state space $\{0,1,2,..\}$; here $\underline{\xi}(n)$, $n=1,2,..$, is a sequence of independent identically distributed random variables. Actually, $\underline{x}(n)$ represents the number of customers left behind in the system at the n^{th} departure instant, whereas $\underline{\xi}(n)$ denotes the number of arrivals during the n^{th} service ($n = 1,2,..$).
The generalization of this system is described by the recurrence relations

$$\underline{x}_j(n) = [\underline{x}_j(n-1) - 1]^+ + \underline{\xi}_j(n), \quad n = 1,2,.., \quad j = 1,2, \tag{0.4}$$

for the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)), n=0,1,..\}$ with stationary transition probabilities and state space $\{0,1,2,..\} \times \{0,1,2,..\}$; here $(\underline{\xi}_1(n),\underline{\xi}_2(n))$, $n = 1,2,..$, is a sequence of independent identically distributed random vectors. Note that in general $\underline{\xi}_1(n)$ and $\underline{\xi}_2(n)$ are not independent ($n=1,..$). This model may be interpreted as follows. Two types of customers arrive independently with negative exponentially distributed interarrival times at a single server facility. Customers of different types are served in pairs if possible, otherwise customers are served individually. Successive service times are independent random variables with a common - unspecified- distribution (see section II.0 for a detailed description of the model). Then for $j = 1,2$, the variable $\underline{x}_j(n)$ represents the number of type j customers left behind in the system at the n^{th} departure instant, while $\underline{\xi}_j(n)$ denotes the number of arrivals of type j customers during the n^{th} service ($n = 1,2,..$).

The organization of the present study is as follows.
In chapter I we shall summarize concepts, definitions and theorems from the theory of functions of a complex variable and of conformal mappings, and from the theory of boundary value problems. This chapter has been incorporated to give a review of these theories in order to make this study self-contained.
Chapter II is devoted to a detailed analysis of the time dependent behaviour of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n=0,1,..\}$. In section II.2 a functional equation for the generating function of this Markov chain is derived. This functional equation is analysed with the method of Cohen & Boxma in the sections II.3,..,II.6. In section II.6 two Riemann-Hilbert boundary value problems are formulated, and the solution of them is fully described. In section II.7 it is shown that the solution method can be simplified by formulating a single Hilbert boundary value problem instead of the two Riemann-Hilbert problems. In this section we shall also discuss the analytic continuation of the solution across the contour L. Section II.8 is devoted to the asymptotic behaviour of the Markov chain as $n \to \infty$. Conditions are derived on which the Markov chain is ergodic, null-recurrent or transient. In the ergodic case the stationary distribution and its first and second order moments are determined.
In chapter III the continuous time parameter process $\{(\underline{y}_1(t),\underline{y}_2(t)),\ t > 0\}$ of the same queueing model is investigated. Two supplementary variables are introduced in order to define a continuous time Markov process. With a relatively simple procedure (section III.2,III.3) the analysis of this time dependent process can be reduced to the solution of a functional equation of the type (0.1) with a kernel of the form (0.2). The analysis of this functional equation proceeds along the same lines as in chapter II and is described - without repeating every detail - in the sections III.4-III.6. The asymptotic behaviour of the process as $t \to \infty$ is discussed in section

III.8. It turns out that the stationary distribution of this process in continuous time is different from that of the imbedded Markov chain at departure instants. Once the solution of the continuous time process is obtained it is possible to describe other characteristic phenomena of the queueing model such as the virtual waiting time process and the workload of the server. Section III.7 is devoted to the time dependent behaviour of these phenomena, whereas in section III.9 their stationary distributions and first moments are determined.

Chapter IV contains three sections. In the first section the analysis of the imbedded Markov chain considered in chapter II is extended by including a random variable describing the instant of the n^{th} departure. Here also the busy period will be discussed. The second section is devoted to a variant of the present model. It is shown that in the case that the distributions of the duration of individual services differ from that of paired services also a Hilbert boundary value problem can be formulated, however of a more intricate type. In the third section some numerical examples for moments and probabilities of several distributions obtained in this study are presented. For obtaining these values it is necessary to evaluate the relevant conformal mapping and some of its derivatives numerically.

For a more detailed review of the various sections the reader is referred to the introductions of these sections.

Throughout, symbols indicating random variables are underlined.

Inside a chapter formulas, theorems, etc., are referred to just by their number, whereas references outside a chapter are prefixed by a roman numeral indicating the chapter, e.g. (I.3.2) refers to formula (3.2) in chapter I.

CHAPTER I

BOUNDARY VALUE PROBLEMS, A SUMMARY

I.0. Introduction

In this chapter we shall summarize concepts, definitions and theorems from the theory of functions of a complex variable, in particular those which are used in the theory of boundary value problems.

In section I.1 definitions concerning regions and curves in the complex plane will be given as well as definitions and basic properties of analytic functions of a complex variable.

Section I.2 deals with the Hölder conditions for functions defined on a contour.

In section I.3 integrals of the Cauchy type, and in particular their behaviour near the contour of integration will be discussed.

The Hilbert and the Riemann-Hilbert boundary value problems and their solutions will be subsequently described in the sections I.4 and I.5.

Finally section I.6 deals with the conformal mapping of a domain in the complex plane onto the unit disk. In particular, theorems on the boundary behaviour of such conformal mappings will be listed, and the method of Theodorsen for the determination of conformal mappings of the unit disk onto simply connected convex domains will be described.

I.1. Sets and Functions

For the definitions given below we refer to the books of EVGRAFOV [07], chapter II & III, and of MARKUSHEVICH [18], vol. I, chapter 4.

DEFINITION 1.1. *A set* E *is said to be connected if given any decomposition of* E *into two nonempty disjoint sets* E_1 *and* E_2 $(E_1 \cup E_2 = E)$ *at least one of the sets* E_1 *and* E_2 *contains a limit point of the other.*
An open connected set is called a domain.

A complex valued function $z = f(t)$ of a real variable which is defined, single-valued and continuous in a closed interval $t_1 \leqslant t \leqslant t_2$ is said to define a (continuous) *curve* L. The curve is said to be *closed* if $f(t_1) = f(t_2)$. Otherwise the points $f(t_1)$ and $f(t_2)$ are called the *end points* of the curve.
The *positive direction* of a curve L is chosen to be that direction which corresponds to an increase of the parameter t.
If the same point z corresponds to more than one parameter value in one of the half open intervals $t_1 \leqslant t < t_2$ or $t_1 < t \leqslant t_2$ we say that z is a *multiple* point of the curve $z = f(t)$, $t_1 \leqslant t \leqslant t_2$. A curve with no multiple points is called a *Jordan curve*. A closed Jordan curve will also be called a *contour*. For closed Jordan curves we have :

LEMMA 1.1. *A closed Jordan curve separates the complex plane into two distinct domains, both of which have the curve as their boundary (see* DIENES [06] , *chapter* VI).

DEFINITION 1.2. *Let* L *be a contour (a closed Jordan curve). The part of the plane that is on the left if* L *is traversed in positive direction will be denoted by* L^+, *the other part by* L^-.
Throughout this chapter the parametric equations $z = f(t)$ of the contours L will be chosen such that the domain L^+ is finite (the inner part) and the domain L^- is infinite (the outer part).

EXAMPLE. Throughout we shall denote the *unit circle* $\{z; |z| = 1\}$ by the symbol C. Its parametric equation will be

$$z = \cos t + i \sin t, \quad -\pi \leqslant t \leqslant \pi.$$

The positive direction on C is then counter clockwise, and

$$C^{+} = \{z; |z| < 1\}, \qquad C^{-} = \{z; |z| > 1\}. \qquad \square$$

Integration along a contour is always in the positive direction. By the choice of the parametric equation made above this direction will be counter clockwise. See for the concept of integrals of complex functions MARKUSHEVICH [18], volume I, §§61,62.

From MUSKHELISHVILI [20], §1, we adapt the concept of smooth contours.

DEFINITION 1.3. *A contour in the complex plane with parametric equation* $z = f(t) = x(t) + i\,y(t)$, $t_1 \leqslant t \leqslant t_2$, *is said to be smooth if the real valued functions* $x(t)$ *and* $y(t)$ *have continuous first derivatives for* $t_1 < t < t_2$, *righthand derivatives in* t_1, *lefthand derivatives in* t_2 *with* $x'(t_1+) = x'(t_2-)$, $y'(t_1+) = y'(t_2-)$, *and for no* $t, t_1 \leqslant t \leqslant t_2$, *the derivatives* $x'(t)$ *and* $y'(t)$ *are simultaneously zero.*

For the concept of an analytic function of a complex variable we shall use the definitions given by EVGRAFOV [07], chapter II & III. Functions may be single-valued or multiple-valued.

By a *neighborhood* of a point z_0 in the complex plane will be meant a circle with its center at z_0 and with some positive radius.

DEFINITION 1.4. *A single-valued function* $f(z)$ *defined in a neighborhood of a point* z_0 *is said to be differentiable at the point* z_0 *if the limit*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists (independent of the way z *tends to* z_0*). The limit is called the* *derivative of the complex function* f(z) *at the point* z_0 *and will be denoted by* $f'(z_0)$ *or* $\frac{d}{dz} f(z)|_{z=z_0}$.

DEFINITION 1.5. *A single-valued function* f(z) *is said to be regular at the point* z_0 *if it can be represented by a power series*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n,$$

that converges in a neighborhood of the point z_0. *A single-valued function* f(z) *is said to be regular in the domain* D *if it is defined and regular at each of its points.*

REMARK 1.1. A function f(z) that is regular at a point z_0 is infinitely differentiable at this point. A function that is differentiable in a domain is regular in this domain, cf. MARKUSHEVICH [18], vol. I, theorem 16.7.

DEFINITION 1.6. *Let* D *be a domain and* L *its boundary. A (single-valued) function* f(z) *that is regular in* D *and continuous in* D *up to its boundary* L *is said to belong to the class* RCB(D).

DEFINITION 1.7. *Let* D *be a domain,* E *a subset of* D, *and* f(z) *a function defined on* E. *A function* F(z) *which is regular in the domain* D *and coincides with* f(z) *on the set* E *is called an analytic continuation of the function* f(z) *into the domain* D.

In EVGRAFOV [07] , § II.5 it has been proved :

LEMMA 1.2. *(The principle of analytic continuation). Let* D *be a domain and* E *a subset of* D *containing at least one limit point of* D. *Then a function* f(z) *defined on* E *has at most one analytic continuation into the domain* D.

Thus starting with a function f(z) defined on a set E containing a limit point, this function can be extended as a regular function to any domain containing E in at most one way. However, generally speaking it will not be possible to extend f(z) regularly to every domain containing E. There may be points whereto f(z) cannot be extended regularly. Such points may form an essential bound for the function f(z), or the analytic continuation of the function f(z) may loose its uniqueness at such points, i.e. if the function f(z) is continued analytically along a closed curve around such a point, its values at the starting point and (the same) end point are different.

By enlarging the concept of (single-valued) functions to multiple-valued functions the function f(z) can also be extended analytically around such "branch" points as regular branches of a multiple-valued function. For a description of analytic continuation of a function into a multiple-valued function see e.g. EVGRAFOV [07], chapter III.

DEFINITION 1.8. *Let a function* f(z) *be given in a neighborhood of the point* z_0 *in the domain* D. *Suppose that this function can be continued analytically along any curve not crossing the boundary of* D. *Then the totality of all such continuations is called an analytic function (multiple-valued) in the domain* D.

REMARK 1.2. An analytic function is in general multiple-valued. A regular function is always single-valued. If an analytic function is single-valued in a domain, then it is also regular in that domain.

DEFINITION 1.9. *Let* F(z) *be a function that is analytic in the annulus* $0 < |z-z_0| < \varepsilon$, $\varepsilon > 0$. *If* F(z) *is not a regular function in this annulus (i.e. if* F(z) *is not single-valued) then we shall say that the point* $z = z_0$ *is an (isolated) branch point.*

If the number of distinct branches of $F(z)$ at each point of the annulus $0 < |z-z_0| < \varepsilon$ is finite and equals n, then we shall call z_0 a *branch point of order* $n-1, n \geqslant 2$.

For a *simply connected domain* (i.e. a domain of which the boundary is a connected set) the following result called the *monodromy theorem* holds (see e.g. EVGRAFOV [07], §III.3).

LEMMA 1.3. *A function analytic in a simply connected domain is regular (and thus single-valued) in this domain.*

Finally we state a result that is frequently used in queueing theory. For its proof see e.g. EVGRAFOV [07], §IV.6.

LEMMA 1.4. *(Rouché's theorem). Let* D *be a domain and let the functions* $f(z)$ *and* $g(z)$ *belong to the class* RCB(D) *. If the inequality* $|f(z)| < |g(z)|$ *holds on the boundary of* D *then the functions* $g(z) - f(z)$ *and* $g(z)$ *have the same number of zeros in the domain* D *(each zero being counted according to its multiplicity).*

I.2. The Hölder condition

An important class of functions defined on a contour in the complex plane consists of functions satisfying a Hölder condition :

DEFINITION 2.1. *Let there be given a contour* L *and a function* $\phi(t)$ *defined on* L. *The function* $\phi(t)$ *is said to satisfy a Hölder condition on* L, *if there exist positive constants* A *and* μ *such that for any two points* t_1, t_2 *of* L,

$$|\phi(t_2) - \phi(t_1)| \leqslant A|t_2 - t_1|^{\mu}.$$

The constant μ *is called the Hölder index.*

REMARK 2.1. A function which satisfies a Hölder condition is clearly continuous on the contour L. MUSKHELISHVILI[20], §3, shows that if $\mu > 1$ in definition 2.1 then ϕ is constant on L. Therefore it will be assumed that $0 < \mu \leq 1$.

DEFINITION 2.2. *Let* L *be a contour. By H(L) we denote the class of functions* $\phi(t)$ *defined on* L, *which satisfy a Hölder condition with any index* $\mu, 0 < \mu \leq 1$, *on the contour* L.

LEMMA 2.1. *Let* $\phi(t) = Kt^n, t \in C$, *for* $K \in \mathbb{C}, n = 0,1,\ldots$. *Then* $\phi(t) \in H(C)$.

PROOF. From the following inequality it is readily seen that the function $\phi(t)$ satisfies on C the Hölder condition with index 1 : for $n = 0,1,\ldots,$ for $t_1,t_2 \in C$,

$$|\phi(t_1)-\phi(t_2)| = |K|\ |t_1-t_2|\ |t_1^{n-1} + t_2t_1^{n-2}+\ldots+t_2^{n-1}| \leq n|K|\ |t_1-t_2|. \quad \square$$

LEMMA 2.2. *Let* L *be a contour and* $c \notin L$. *Then the functions*

$$\phi_n(t) = \frac{t^n}{c-t}, \qquad t \in L, \qquad n = 0,1,2,\ldots,$$

belong to the class H(L).

PROOF. As $c \notin L$ and L is a closed set we have for some positive ε independent of t,

$$|c-t| \geq \varepsilon, \qquad t \in L.$$

Further, L is bounded thus $|t| < M$ for some positive constant M. These two inequalities imply for any two points t_1,t_2 on L, for $n = 0,1,2,\ldots,$

$$|\phi_n(t_1) - \phi_n(t_2)| = \left|\frac{ct_1^n-t_1^nt_2-ct_2^n+t_1t_2^n}{(c-t_1)(c-t_2)}\right| \leq$$

$$\leq \frac{1}{\varepsilon^2}\{c|t_1^n-t_2^n| + |t_1t_2^n-t_1^nt_2|\} \leq \frac{1}{\varepsilon^2}\{c\ n\ M^{n-1} + |n-1|M^n\}|t_1-t_2|.$$

Hence $\phi_n(t)$ satisfies on L the Hölder condition with index $1 (n = 0,1,\ldots)$.

□

LEMMA 2.3. *Let* L *be a smooth contour.*

1. *If* $\phi_1 \in H(L)$ *and* $\phi_2 \in H(L)$ *then also* $\phi_1 \times \phi_2 \in H(L)$.
2. *If* $\phi \in H(L)$ *then also* $\mathrm{Re}\ \phi \in H(L)$ *and* $\mathrm{Im}\ \phi \in H(L)$.
3. *If* $\phi_1 \in H(L)$ *and the real axis is an axis of symmetry of the contour* L, *then also* $\phi_2 \in H(L)$ *where* $\phi_2(t) = \phi_1(\overline{t})$ *for* $t \in L$.

PROOF. These simple statements follow from :

1. the second criterium in MUSKHELISHVILI [20], §6;
2. the relation

$$|\phi(t_1) - \phi(t_2)|^2 = |\mathrm{Re}\ \phi(t_1) - \mathrm{Re}\ \phi(t_2)|^2 + |\mathrm{Im}\ \phi(t_1) - \mathrm{Im}\ \phi(t_2)|^2;$$

3. the fact that $|\overline{t}_1 - \overline{t}_2| = |t_1 - t_2|$. □

I.3. Integrals of the Cauchy type

Let L be a contour and $\phi(t)$ a function defined on L and integrable on L. Then the integral

$$\Phi(z) := \frac{1}{2\pi i} \int_L \frac{\phi(t)}{t-z} dt, \qquad z \notin L, \tag{3.1}$$

is called a *Cauchy integral*. Clearly, $\Phi(z)$ is regular at every point $z \notin L$, and vanishes at infinity. In certain cases the Cauchy integral (3.1) can also be given a definite meaning when $z \in L$ (cf. MUSKHELISHVILI [20], §12).

DEFINITION 3.1. *Let* L *be a smooth contour,* $t_0 \in L$, ℓ_ε *the part of* L *inside a circle around* t_0 *with radius* ε, *and let* $\phi(t)$ *be a function integrable on* L. *If the limit*

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{L-\ell_\varepsilon} \frac{\phi(t)}{t-t_0}\, dt,$$

exists, then this limit is called the principle value of the singular Cauchy integral (at t_0), *and it will be denoted by*

$$\frac{1}{2\pi i} \int_L \frac{\phi(t)}{t-t_0}\, dt. \tag{3.2}$$

LEMMA 3.1. *Let* L *be a smooth contour. If* $\phi \in H(L)$ *then the principle value of the Cauchy integral* (3.1) *exists at every point* $t_0 \in L$.

The proof of this lemma can be found in MUSKHELISHVILI [20], §12.
Of great importance is the behaviour of the Cauchy integral (3.1) near the line of integration. From MUSKHELISHVILI [20], §16, we have

LEMMA 3.2. *Let* L *be a smooth contour. If* $\phi(t) \in H(L)$ *then the Cauchy integral* $\Phi(z)$, cf. (3.1), *belongs to the classes* $RCB(L^+)$ *and* $RCB(L^-)$.

DEFINITION 3.2. *Let* L *be a smooth contour and* $\Psi(z)$ *a function belonging to the classes* $RCB(L^+)$ *and* $RCB(L^-)$. *Then* $\Psi(z)$ *will be called a sectionally regular (holomorphic) function (with respect to* L).

DEFINITION 3.3. *Let* L *be a smooth contour and* $\Psi(z)$ *a sectionally regular function with respect to* L. *Then for* $t \in L$:

$$\Psi^+(t) := \lim_{z \to t} \Psi(z), \qquad z \in L^+,$$
$$\Psi^-(t) := \lim_{z \to t} \Psi(z), \qquad z \in L^-. \qquad \square$$

LEMMA 3.3. *Let* L *be a smooth contour and* $\phi(t) \in H(L)$. *The limiting values of the Cauchy integral (3.1) are given by : for* $t_0 \in L$,

$$\Phi^+(t_0) = \tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i}\int_L \frac{\phi(t)}{t-t_0}\,dt,$$

$$\Phi^-(t_0) = -\tfrac{1}{2}\phi(t_0) + \frac{1}{2\pi i}\int_L \frac{\phi(t)}{t-t_0}\,dt. \tag{3.3}$$

REMARK 3.1. The integrals in (3.3) are singular, cf. definition 3.1. The formulas in lemma 3.3 are called the *Plemelj formulas* (MUSKHELISHVILI [20], §17, or the *Sochozki-Plemelj formulas* (PRÖSSDORF [22], §3.4.1), or the *Sochozki formulas* (GAKHOV [13], §4.2). These formulas can also be put in the form : for $t_0 \in L$,

$$\Phi^+(t_0) - \Phi^-(t_0) = \phi(t_0), \tag{3.4}$$

$$\Phi^+(t_0) + \Phi^-(t_0) = \frac{1}{\pi i}\int_L \frac{\phi(t)}{t-t_0}\,dt. \tag{3.5}$$

A well-known property of regular functions is (see e.g. MUSKHELISHVILI [20], §15) :

LEMMA 3.4. *Let* L *be a smooth contour,* $\Psi(z)$ *a sectionally regular function and let* $\Psi^+(t) = \Psi^-(t)$ *hold for every* $t \in L$. *Then* $\Psi(z)$ *is regular in* $\mathbb{C}$.

With lemma 3.4 and relation (3.4) the following problem can be solved (cf. MUSKHELISHVILI [20], §26) : Let L be a smooth contour and let $\phi(t) \in H(L)$. It is required to find a sectionally regular function $\Psi(z)$ vanishing at infinity and satisfying the boundary condition

$$\Psi^+(t) - \Psi^-(t) = \phi(t), \qquad t \in L. \tag{3.6}$$

LEMMA 3.5. *The above formulated boundary value problem has a unique solution and this solution can be represented by the Cauchy integral* (3.1).

REMARK 3.2. The solution of the above formulated boundary value problem having finite non-negative degree k at infinity is given by :

$$\Psi(z) = \frac{1}{2\pi i}\int_L \frac{\phi(t)}{t-z}dt + P_k(z), \qquad z \notin L, \tag{3.7}$$

here $P_k(z)$ is an arbitrary polynomial of degree k.

For later reference we conclude this section with an extension of lemma 3.3.

<u>LEMMA 3.6.</u> *Let* $\phi(r;t)$, $t \in C$, *be a family of functions for* r *in a real interval* I, *which are integrable over* $t \in C$, *which satisfy the inequality*

$$|\phi(r;t) - \phi(r;1)| \leqslant M|t - 1|^{\mu}, \qquad t \in C, r \in I, \tag{3.8}$$

where M *and* μ *are positive constants (independent of* r *and* t*); and which are continuous from the left at a point* $r_0 \in I$ *for all* $t \in C$, *i.e.*

$$\lim_{r \uparrow r_0} \phi(r;t) = \phi(r_0;t), \qquad t \in C. \tag{3.9}$$

Further, let $t_0(r)$ *be a strictly decreasing function on* I *with* $\lim_{r\uparrow r_0} t_0(r) = 1$. *Then*

$$\lim_{r\uparrow r_0} \frac{1}{2\pi i}\int_C \frac{\phi(r;t)}{t-t_0(r)}dt = -\tfrac{1}{2}\,\phi(r_0;1) + \frac{1}{2\pi i}\int_C \frac{\phi(r_0;t)}{t-1}dt, \tag{3.10}$$

$$\lim_{r\uparrow r_0} \frac{1}{2\pi i}\int_C \frac{\phi(r;t)}{t-\frac{1}{t_0(r)}}dt = \tfrac{1}{2}\,\phi(r_0;1) + \frac{1}{2\pi i}\int_C \frac{\phi(r_0;t)}{t-1}dt. \tag{3.11}$$

<u>PROOF</u>. Write for $r \in I$, $r < r_0$,

$$\int_C \frac{\phi(r;t)}{t-t_0(r)}dt = \int_C \frac{\phi(r;t)-\phi(r;1)}{t-t_0(r)}dt + \phi(r;1)\int_C \frac{dt}{t-t_0(r)}. \tag{3.12}$$

As $t_0(r) > 1$ for $r < r_0$ the last term vanishes as $r \uparrow r_0$, and we have for $r \leqslant r_0$,

$$|t - t_0(r)| \geqslant |t - 1|, \qquad t \in C,$$

so that with (3.8) it is obtained that for $r \leqslant r_0$,

$$\left|\frac{\phi(r;t)-\phi(r;1)}{t-t_0(r)}\right| \leq M|t-1|^{\mu-1}, \qquad t \in C.$$

Because for $\mu > 0$ the function $|t-1|^{\mu-1}$ is integrable over C in the ordinary sense it follows with (3.9) by the dominated convergence theorem (cf. BURRILL [02], theorem 7-4A) that

$$\lim_{r \uparrow r_0} \int_C \frac{\phi(r;t)-\phi(r;1)}{t-t_0(r)} dt = \int_C \frac{\phi(r_0;t)-\phi(r_0;1)}{t-1} dt.$$

The inequality (3.8) implies that $\phi(r_0;t) \in H(C)$, so that by lemma 3.1 the principle value of the singular integral

$$\int_C \frac{\phi(r_0;t)}{t-1} dt$$

exists. This principle value can be written as

$$\int_C \frac{\phi(r_0;t)}{t-1} dt = \int_C \frac{\phi(r_0;t)-\phi(r_0;1)}{t-1} dt + \pi i\, \phi(r_0;1).$$

This proves (3.10).

Next write for $r \in I$, $r < r_0$,

$$\int_C \frac{\phi(r;t)}{t-\frac{1}{t_0(r)}} dt = \int_C \frac{\phi(r;t)-\phi(r;1)}{t-\frac{1}{t_0(r)}} dt + \phi(r;1) \int_C \frac{dt}{t-\frac{1}{t_0(r)}}.$$

As $t_0(r) > 1$ for $r < r_0$ the last term tends to $2\pi i \phi(r_0;1)$ as $r_0 \uparrow r$. Further it is readily verified that for $r \leq r_0$,

$$\left|t - \frac{1}{t_0(r)}\right| \geq \tfrac{1}{2}\left[1 + \frac{1}{t_0(r)}\right]|t-1| \geq \tfrac{1}{2}|t-1|, \qquad t \in C$$

Then (3.11) can be proved in a similar way as (3.10). □

I.4. The Hilbert Problem

Throughout this section L stands for a smooth contour, and L^+ and L^-

are defined as in definition 1.2. It is assumed that $0 \in L^+$.
We consider the following boundary value problem.

PROBLEM 4.1. Let G(t) be a non-vanishing function on L, $G(t) \in H(L)$. Let also $g(t) \in H(L)$. It is required to find a sectionally regular function $\Psi(z)$ having finite degree at infinity and satisfying the boundary condition

$$\Psi^+(t) = G(t)\, \Psi^-(t) + g(t), \qquad t \in L. \tag{4.1}$$

REMARK 4.1. The problem formulated above is called the *Hilbert boundary value problem, homogeneous* if $g(t) \equiv 0$ on L, and *non-homogeneous* otherwise (cf. MUSKHELISHVILI [20], §§34,37). Note that GAKHOV [13], §14.1, uses the name Riemann problem. It is also known as coupling problem.

MUSKHELISHVILI [20], §§34-37, gives the following solution method for problem 4.1. First the homogeneous case ($g(t) \equiv 0$) is considered. The *index* κ of the Hilbert problem is defined to be

$$\kappa := \frac{1}{2\pi i}\,[\log G(t)]_L = \frac{1}{2\pi}\,[\arg G(t)]_L, \tag{4.2}$$

where $[..]_L$ denotes the increment of the expression in the brackets as the result of one circuit around L. The index may be any integer, but we restrict the discussion to the case $\kappa \geqslant 0$.
Since $0 \in L^+$ the argument of $t^{-\kappa}\, G(t)$ will return to its initial value after any circuit around L; hence $\log t^{-\kappa} G(t)$ can be defined as a single-valued continuous function on L belonging to the class H(L).
The homogeneous Hilbert problem then can be reduced to a problem with boundary condition (3.6) of which the solution is given by lemma 3.5. This leads to :

LEMMA 4.1. *The general solution of the homogeneous Hilbert problem is* *given by*

$$\Psi(z) = X(z)\ P(z), \qquad z \notin L, \tag{4.3}$$

here X(z) *is the fundamental solution (i.e. the solution which vanishes nowhere in the finite plane) of the homogeneous Hilbert problem :*

$$\begin{aligned} X(z) &= e^{\Gamma(z)}, \qquad z \in L^+ \\ &= z^{-\kappa} e^{\Gamma(z)}, \qquad z \in L^-, \end{aligned} \tag{4.4}$$

with, for a proper choice of the function $\log[t^{-\kappa}G(t)]$, $t \in L$ *(note that the function* X(z) *does not depend on this choice),*

$$\Gamma(z) := \frac{1}{2\pi i} \int_L \log[t^{-\kappa}G(t)]\ \frac{dt}{t-z}, \qquad z \notin L\ ; \tag{4.5}$$

and P(z) *stands for an arbitrary polynomial.* □

In order to solve the non-homogeneous Hilbert problem it is noted that the fundamental solution (4.4) of the homogeneous problem satisfies

$$X^-(t)\ G(t) = X^+(t), \qquad t \in L. \tag{4.6}$$

Substitution of (4.6) in (4.1) gives the boundary condition

$$\frac{\Psi^+(t)}{X^+(t)} - \frac{\Psi^-(t)}{X^-(t)} = \frac{g(t)}{X^+(t)}, \qquad t \in L, \tag{4.7}$$

for the sectionally regular function $\Psi(z)/X(z)$. The boundary condition (4.7) is of the form of(3.6) so that the solution follows from (3.7) :

<u>LEMMA 4.2</u>. *If* $\kappa \geqslant 0$ *then the general solution of the non-homogeneous Hilbert problem* (4.1)*, bounded at infinity, is given by*

$$\Psi(z) = \frac{X(z)}{2\pi i} \int_L \frac{g(t)}{X^+(t)}\ \frac{dt}{t-z} + X(z)\ P_\kappa(z), \qquad z \notin L,$$

here X(z) *is the fundamental solution* (4.4) *of the associated homogeneous problem,* κ *is the index of the problem, cf.* (4.2)*, and* $P_\kappa(z)$ *stands for an* *arbitrary polynomial of degree not greater than* κ. □

In the present study we are particularly interested in Hilbert problems with boundary conditions of the form

$$\Psi^{+}(t) + t^{n}\,\Psi^{-}(t) = g(t), \qquad t \in C, \tag{4.8}$$

with n = 0,1 or 2, here the smooth contour L is the unit circle, and $G(t) = -t^{n}$, cf.(4.1).

LEMMA 4.3. *The solution, bounded at infinity, of the Hilbert problem with boundary condition* (4.8) *is given by (for* n = 0,1,...*):*

$$\begin{aligned}\Psi(z) &= \frac{1}{2\pi i}\int_C \frac{g(t)}{t-z}\,dt - P_n(z), && z \in C^{+},\\ &= \frac{-1}{2\pi i z^{n}}\int_C \frac{g(t)}{t-z}\,dt + \frac{1}{z^{n}}P_n(z), && z \in C^{-}.\end{aligned}$$

PROOF. The function $G(t) = -t^{n}$ does not vanish on C. From lemma 2.1 we have $G(t) \in H(C)$. Thus the Hilbert problem (4.8) is well defined and we can apply lemma 4.2, if $\kappa \geqslant 0$. On C we can write $t = e^{i\phi}, -\pi \leqslant \phi \leqslant \pi$, so that, cf. (4.2),

$$\kappa = \frac{1}{2\pi}[\arg G(t)]_C = \frac{1}{2\pi}\{\pi + n\phi\}_{\phi=-\pi}^{\phi=\pi} = \frac{2n\pi}{2\pi} = n.$$

Thus $\kappa \geqslant 0$. For the choice $\log(-1) = i\pi$ the function $\Gamma(z)$, cf.(4.5), becomes

$$\begin{aligned}\Gamma(z) &= \frac{1}{2\pi i}\int_C \frac{\log(-1)}{t-z}\,dt = i\pi, && z \in C^{+},\\ &= 0, && z \in C^{-},\end{aligned}$$

so that the fundamental solution of (4.8) is given by, cf. (4.4),

$$\begin{aligned}X(z) &= -1, && z \in C^{+},\\ &= z^{-n}, && z \in C^{-}.\end{aligned}$$

Hence $X^+(t) = -1$, $|t| = 1$. The assertion now follows from lemma 4.2. □

I.5. The Riemann-Hilbert problem

In this section we consider another type of boundary value problems, which however is closely related to the Hilbert problem (section I.4).

PROBLEM 5.1. Let L be a smooth contour and L^+ the domain to the left of L. Let F(t), $t \in L$, be a complex valued function, non-vanishing on L, and f(t), $t \in L$, a real valued function, both belonging to the class H(L). It is required to find a function $\Omega(z) \in \mathrm{RCB}(L^+)$ satisfying the boundary condition

$$\mathrm{Re}\{F(t)\ \Omega^+(t)\} = f(t), \qquad t \in L, \tag{5.1}$$

here Re{...} denotes the real part of the expression in the brackets.

REMARK 5.1. The boundary value problem formulated above is called a *Riemann-Hilbert problem, homogeneous* if $f(t) \equiv 0$ on L and *non-homogeneous* otherwise (cf. MUSKHELISHVILI [20], §§39,40; MICHLIN & PRÖSSDORF [19], §VII, 2). GAKHOV [13], §27.1, uses the name Hilbert problem.

REMARK 5.2. Problem 5.1 has been solved by I.N. Vekua. In his paper the Riemann-Hilbert problem is transformed into an equivalent singular integral equation. See MICHLIN & PRÖSSDORF [19], §VII.2.

For the case that the boundary condition (5.1) is given on the unit circle MUSKHELISHVILI [20],§§39-40, transforms the Riemann-Hilbert problem into a Hilbert problem (problem 4.1) in the following way.
From now on let L = C, the unit circle. Rewrite (5.1) as

$$F(t)\ \Omega^+(t) + \overline{F(t)\Omega^+(t)} = 2f(t), \qquad t \in C. \tag{5.2}$$

The function $\Omega(z)$, $z \in C^+$, is extended as a sectionally regular function (definition 3.2) by defining it for $z \in C^-$ as

$$\Omega(z) := \overline{\Omega(1/\bar{z})}, \qquad z \in C^-. \tag{5.3}$$

MUSKHELISHVILI [20] , §38, shows that $\Omega(z) \in RCB(C^-)$ and that for $t \in C$,

$$\Omega^-(t) = \overline{\Omega^+(1/\bar{t})} = \overline{\Omega^+(t)}. \tag{5.4}$$

With this the boundary condition (5.2) can be formulated as

$$\Omega^+(t) + \frac{\overline{F(t)}}{F(t)}\,\Omega^-(t) = \frac{2f(t)}{F(t)}, \qquad t \in C. \tag{5.5}$$

Hence, we have obtained a Hilbert boundary value problem for the function $\Omega(z)$ with boundary condition (5.5), cf. (4.1), if we take

$$G(t) = -\frac{\overline{F(t)}}{F(t)}, \quad g(t) = \frac{2f(t)}{F(t)}, \qquad t \in C \tag{5.6}$$

The *index* of a Riemann-Hilbert problem for the unit circle is defined to be equal to that of the corresponding Hilbert problem, cf. (5.5) and (4.2). From (5.6) and (4.2) it is seen that the index of a Riemann-Hilbert problem is an even number.

REMARK 5.3. It should be noted that not every solution of the Hilbert problem (5.5) is also a solution of the Riemann-Hilbert problem (5.1). A solution of the latter has to satisfy the additional relation (5.3). However, if a sectionally regular function $\Omega(z)$ satisfies (5.5) then $\frac{1}{2}[\Omega(z) + \overline{\Omega(1/\bar{z})}]$ satisfies both (5.3) and (5.5), cf. MUSKHELISHVILI [20], §40. Hence, from every solution of the Hilbert problem (5.5) a solution of problem 5.1 can be constructed.

We shall not go further into the details of the general solution of the

Riemann-Hilbert problem. We focus our attention on the case that $F(t) = t^{-n}$, $t \in C$, $n = 0$ or $n = 1$, which we shall meet later. Then (5.1) becomes :

$$\mathrm{Re}\{t^{-n}\Omega^+(t)\} = f(t), \qquad t \in C . \tag{5.7}$$

<u>LEMMA 5.1.</u> *The solution of the Riemann-Hilbert problem with boundary condition* (5.7), *cf. problem* 5.1, *is given by : for* $z \in C^+$,

$$\Omega(z) = z^n\Big[\frac{1}{\pi i}\int_C \frac{f(t)}{t-z}dt - \frac{1}{2\pi i}\int_C f(t)\frac{dt}{t} + id_0 + \sum_{k=1}^{n}\{d_k z^k - \bar{d}_k\, z^{-k}\}\Big],$$

here d_0 *is a real constant, and* $d_k, k = 1,\dots,n,$ *are complex constants.*

<u>PROOF.</u> Extending the function $\Omega(z)$, $z \in C^+$, by (5.3) to a sectionally regular function the Hilbert problem (5.5) becomes for $F(t) = t^{-n}, t \in C$,

$$\Omega^+(t) + t^{2n}\,\Omega^-(t) = 2t^n\, f(t), \qquad t \in C. \tag{5.8}$$

This boundary condition is of the same type as (4.8). Hence, the index of the Riemann-Hilbert problem (5.7) is 2n, cf. lemma 4.3.

Moreover, from (5.3) it follows that $\Omega(z)$ is bounded at infinity, so that the solution of the Hilbert problem (5.8) is given by lemma 4.3 :

$$\begin{aligned}\Omega(z) &= \frac{1}{2\pi i}\int_C \frac{2t^n f(t)}{t-z}dt - P_{2n}(z), && z \in C^+,\\ &= -\frac{1}{2\pi i z^{2n}}\int_C \frac{2t^n f(t)}{t-z}\,dt + \frac{1}{z^{2n}}P_{2n}(z), && z \in C^-.\end{aligned} \tag{5.9}$$

Rewrite the integrals in (5.9) as follows : for $z \notin C$,

$$\begin{aligned}\int_C \frac{t^n f(t)}{t-z}\,dt &= z^n\int_C \frac{f(t)}{t-z}dt + \int_C \frac{t^n - z^n}{t-z}\,f(t)dt = \\ &= z^n\int_C \frac{f(t)}{t-z}dt + \sum_{k=1}^{n} z^{k-1}\int_C t^{n-k}\,f(t)\,dt.\end{aligned}$$

Define for $z \in \mathbb{C}\backslash\{0\}$,

$$Q_n(z) := \frac{1}{z^n}\left[\sum_{k=1}^{n} \frac{z^{k-1}}{\pi i}\int_C t^{n-k} f(t)\,dt - P_{2n}(z)\right],$$

then (5.9) can be written as

$$\begin{aligned}\Omega(z) &= z^n\left[\frac{1}{\pi i}\int_C \frac{f(t)}{t-z}\,dt + Q_n(z)\right], && z \in C^+,\\ &= \frac{-1}{z^n}\left[\frac{1}{\pi i}\int_C \frac{f(t)}{t-z}dt + Q_n(z)\right], && z \in C^-. \end{aligned} \tag{5.10}$$

In order to be a solution of the Riemann-Hilbert problem (5.7) the function $\Omega(z)$ must satisfy (5.3). Substitution of (5.10) in (5.3) leads after some calculations to the relation : for $z \in C^-$,

$$Q_n(z)+\overline{Q_n(1/\bar{z})} + \frac{1}{\pi i}\int_C f(t)\,\frac{dt}{t} = 0. \tag{5.11}$$

Because $P_{2n}(z)$ is an arbitrary polynomial of degree 2n we may write

$$Q_n(z) = q_0 + \sum_{k=1}^{n}[q_k z^k + q_{-k}z^{-k}],\ z \in \mathbb{C}\backslash\{0\}, \tag{5.12}$$

with q_i, $-n \leqslant i \leqslant n$, arbitrary complex constants. Then (5.11) implies :

$$\begin{aligned} &q_0 + \bar{q}_0 + \frac{1}{\pi i}\int_C f(t)\,\frac{dt}{t} = 0,\\ &q_k + \bar{q}_{-k} = 0,\quad k = 1,2,\ldots,n. \end{aligned} \tag{5.13}$$

Inserting (5.13) and (5.12) in (5.10) gives the stated solution of the Riemann-Hilbert problem (5.7). □

<u>REMARK 5.4.</u> The expression for the solution of the Riemann-Hilbert problem (5.7) in lemma 5.1 is directly given by the formulas in GAKHOV [13], §29.3, where problem 5.1 has been solved by the concept of regularization.

<u>REMARK 5.5.</u> For $F(t) \equiv 1$ on L the Riemann-Hilbert problem (5.1) is equivalent to the Dirichlet problem. For n = 0 lemma 5.1 gives the

solution of the Dirichlet problem for the unit circle. It can be rewritten as the Schwarz formula which relates a regular function in C^+ to the values of its real part on the boundary C : with d_0 a real constant,

$$\Omega(z) = \frac{1}{2\pi i} \int_C \mathrm{Re}\{\Omega^+(t)\} \frac{t+z}{t-z} \frac{dt}{t} + id_0, \qquad z \in C^+. \tag{5.14}$$

REMARK 5.6. In this section a solution method has been outlined for a Riemann-Hilbert problem for the unit circle. A Riemann-Hilbert problem for an arbitrary smooth contour L may be reduced to that for the unit circle by mapping L^+ conformally onto C^+. Conformal mappings will be discussed in the next section.

I.6. Conformal mapping

In section I.5 a solution of the Riemann-Hilbert boundary value problem has been given for the case that the boundary is the unit circle. However, we shall meet Riemann-Hilbert boundary value problems in which the boundary L is a smooth contour but not a circle. Such problems can be reduced to that of a circle by mapping the domain L^+ conformally onto the unit disk C^+. In this section some properties of conformal mappings will be summarized.

A mapping by a continuous function is said to be *conformal at a point* z_0 when it preserves angles between curves passing through z_0. We shall only consider conformal mappings which preserve the magnitude of the angles as well as the direction in which the angles are measured. By MARKUSHEVICH [18], vol. I, §31, such mappings are called conformal mappings of the first kind. A mapping is said to be *conformal in a domain* D if it is conformal at every point of D. In NEHARI [21], page 152, it is shown that a conformal mapping is associated with an analytic function.

In the concept of conformal mapping we shall confine ourselves to one-to-one continuous mappings. If a function f(z) is a one-to-one continuous function in a domain D then the image f(D) is again a domain (cf. MARKUSHEVICH [18], vol.I, §26, theorem 6.1).

Summarizing we have :

DEFINITION 6.1. *A continuous mapping of one domain onto another will be called a conformal mapping when it is a one-to-one mapping which preserves the magnitude of the angles between intersecting curves as well as the direction in which the angles are measured.*

NEHARI [21], pp. 149,150, shows :

LEMMA 6.1. *A function* f(z) *regular at a point* z_0 *is conformal at* z_0 *if and only if* $f'(z_0) \neq 0$.

In MARKUSHEVICH [18], vol.III, §2, theorem 1.2, it is proved :

LEMMA 6.2. *(Riemann's mapping theorem). Every simply connected domain in the extended complex plane whose boundary contains more than one point can be mapped conformally onto a disk with its center at the origin.*

This fundamental theorem in the theory of conformal mapping implies that every bounded simply connected domain (see above lemma 1.3) can be mapped conformally onto the unit disk. Moreover, it is proved in MARKUSHEVICH [18], vol. III, §2, theorem 1.3 :

LEMMA 6.3. *(Uniqueness theorem for conformal mapping). Let D be a simply connected domain in the extended complex plane whose boundary consists of more than one point, and let* w_0 *be an arbitrary finite point of D. Then there exists a unique function* z = f(w) *which maps* D *conformally onto the unit disk* C^+ *and satisfies the conditions*

$$f(w_0) = 0, \qquad f'(w_0) > 0. \tag{6.1}$$

As a conformal mapping f(w) is defined to be a one-to-one continuous mapping of a domain D onto the domain f(D) there exists an inverse $f^{-1}(z)$ and the inverse function is a conformal mapping of the domain f(D) onto D (cf. TITCHMARSH [25], §6.41). From lemma 6.2 and 6.3 it follows that for every bounded simply connected domain D there exists a conformal mapping g(z) of C^+ onto D which is uniquely determined by the conditions

$$g(0) = w_0, \qquad g'(0) > 0, \tag{6.2}$$

here w_0 is an arbitrary point of D.

For the behaviour of a conformal mapping of a domain D onto f(D) at the boundary L of the domain D we state the following results, the first can be found in MARKUSHEVICH [18], vol. III, §8, theorem 2.24.

LEMMA 6.4. *(Boundary correspondence theorem). Let* L *be a contour (a closed Jordan curve, see section* I.1*), and let* f(w) *be any conformal mapping of* L^+ *onto the unit disk* C^+*. Then* f(w) *establishes a one-to-one continuous mapping between* $L \cup L^+$ *and* $C \cup C^+$*, and hence between* L *and* C.

The next result, concerning the boundary behaviour of the derivative of a conformal mapping, can be found in TSUJI [26], theorem IX.7.

LEMMA 6.5. *(Kellogg's theorem). Let* L *be a contour with parametric equation* $w = w(s), 0 \leq s \leq s_0$*, where* s_0 *is the length of* L *and* s *the arc-length of* L*, measured from a fixed point, such that if* s *varies from* 0 *to* s_0*, then* w(s) *makes one turn on* L *in the positive sense.*
It is supposed that L *has a tangent at every point, that this tangent varies continuously, and that* w'(s) *satisfies the following Hölder condition : for constants* $M_1, M_1 > 0$*, and* μ*,* $0 < \mu < 1$*, for every*

$s_1, s_2,\ 0 \leqslant s_1 \leqslant s_2 \leqslant s_0$,

$$|w'(s_1) - w'(s_2)| \leqslant M_1 |s_1 - s_2|^{\mu}.$$

If we map the unit disk C^+ *conformally onto* L^+ *by* $w = g(z)$, *then* $g'(z) \neq 0$ *exists in* $C^+ \cup C$ *and satisfies the Hölder condition with the same index, i.e. for a constant* $M_2, M_2 > 0$, *for every* $\theta_1, \theta_2,\ -\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|g'(e^{i\theta_1}) - g'(e^{i\theta_2})| \leqslant M_2 |\theta_1 - \theta_2|^{\mu}. \qquad \square$$

In the case that $w'(s)$ does not satisfy a Hölder condition we shall use the following result.

<u>LEMMA 6.6.</u> *Let* L *be a closed Jordan curve which passes through* $w = 1$ *and touches the line* $\operatorname{Re} w = 1$, *and whose inner normal at* $w = 1$ *coincides with the real axis in negative direction. It is assumed that in a neighborhood of* $w = 1$ *the contour* L *is represented by*

$$w = \xi + i\eta, \qquad \xi = 1 - \lambda(\eta),$$

here $\lambda(\eta) \geqslant 0$ *is a continuous function of* η *which is decreasing for* $\eta \leqslant 0$ *and increasing for* $\eta \geqslant 0$.
Let $w = g(z)$, $g(1) = 1$, *be a conformal mapping of the unit disk* C^+ *onto the domain* L^+; *then for* $z \in C^+ \cup C$,

$$\lim_{z\to 1} \frac{1-g(z)}{1-z} = g'(1) > 0$$

exists if and only if for some $\delta > 0$,

$$\int_{-\delta}^{\delta} \frac{\lambda(\eta)}{\eta^2} d\eta < \infty. \tag{6.3}$$

Otherwise

$$\lim_{z\to 1} \frac{1-g(z)}{1-z} = \infty, \quad z \in C^+ \cup C,$$

but for every $\varepsilon > 0$ *there exists a constant* $M > 0$ *such that*

$$|1-g(z)| < M|1-z|^{1-\varepsilon},$$
$$|g'(z)| < M|1-z|^{-\varepsilon},$$

for $z \to 1$ *and* z *in a sector* $|\arg(1-z)| \leqslant \phi < \frac{1}{2}\pi$.

PROOF. Define

$$\tilde{g}(z) := i[1-g(\tfrac{i-z}{i+z})], \qquad \operatorname{Im} z \geqslant 0.$$

Because $z \to \frac{i-z}{i+z}$ maps the upper half plane $\operatorname{Im} z > 0$ conformally onto the unit disk C^+ it is readily verified that $\tilde{g}(z)$ is a conformal mapping of $\operatorname{Im} z > 0$ onto a domain which is obtained from the domain L^+ by the conformal mapping $w \to i[1-w]$, so that this domain touches the real axis at $w = 0$ and its inner normal coincides with the positive imaginary axis. Moreover, after the transformation $w \to i[1-w]$ the contour L is represented by

$$w = x + iy, \qquad y = \lambda(x).$$

Application of TSUJI [26], theorem IX.10, gives that, for $\operatorname{Im} z \geqslant 0$,

$$\lim_{z\to 0} \frac{\tilde{g}(z)}{z} = \tilde{g}'(1)$$

exists as a positive finite number if and only if condition (6.3) is satisfied, and that if (6.3) does not hold then for $\operatorname{Im} z \geqslant 0$,

$$\lim_{z\to 0} \frac{\tilde{g}(z)}{z} = \infty.$$

TSUJI [26], theorem IX.12, states that in general for every $\varepsilon > 0$ and for z in a sector $|\arg z - \frac{1}{2}\pi| \leqslant \phi < \frac{1}{2}\pi$,

$$\text{const.}|z|^{1+\varepsilon} \leqslant |\tilde{g}(z)| \leqslant \text{const.}|z|^{1-\varepsilon}, \qquad z \to 0,$$
$$\text{const.}|z|^{\varepsilon} \leqslant |\tilde{g}'(z)| \leqslant \text{const.}|z|^{-\varepsilon}, \qquad z \to 0.$$

Because of the above we may put $\varepsilon = 0$ in the lower bounds. By the relations

$$\frac{1-g(z)}{1-z} = -i \, \frac{\tilde{g}(i\,\frac{1-z}{1+z})}{1-z}, \qquad z \in C^+ \cup C,$$

$$g'(z) = \frac{2}{(1+z)^2}\, \tilde{g}'(i\,\frac{1-z}{1+z}), \qquad z \in C^+ \cup C,$$

the above properties of the conformal mapping $\tilde{g}(z)$ are readily translated into those of the conformal mapping $g(z)$ as stated in the lemma. □

There exist several techniques for determining conformal mappings, see e.g. [01]. For the present study the method of Theodorsen which will be discussed below is very important. A description of this method can be found in GAIER [12], chapter II.

Consider a contour L with the origin in its interior L^+ which can be represented as

$$L = \{w \; ; \; w = \rho(\theta)e^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi\}, \tag{6.4}$$

here $\rho(\theta)$ is a positive continuous function with $\rho(-\pi) = \rho(\pi)$.

Let $g(z)$ be the conformal mapping of the unit disk C^+ onto L^+ such that, cf. lemma 6.3 and the remark below it,

$$g(0) = 0, \qquad g'(0) > 0. \tag{6.5}$$

Because of lemma 6.4 the conformal mapping $g(z)$ may be considered on the boundary C. Every point $e^{i\phi}$, $-\pi \leqslant \phi \leqslant \pi$, will be mapped onto a point $\rho(\theta)e^{-i\theta}$ on L. As a consequence of lemma 6.4 there must exist a one-to-one continuous mapping $\theta = \theta(\phi)$, $-\pi \leqslant \phi \leqslant \pi$, with $\theta(\pi) = \theta(-\pi) + 2\pi$, such that

$$g(e^{i\phi}) = \rho(\theta(\phi))\; e^{i\theta(\phi)}, \qquad -\pi \leqslant \phi \leqslant \pi. \tag{6.6}$$

<u>LEMMA 6.7.</u> *The function* $\theta(\phi)$ *which determines the conformal mapping* $g(z)$ *of* C^+ *onto* L^+ *for* $z \in C$ *by* (6.6) *satisfies the non-linear singular integral equation*

$$\theta(\phi) = \phi - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho(\theta(\omega)) \cot \frac{\omega-\phi}{2}\, d\omega, \quad -\pi \leqslant \phi \leqslant \pi. \tag{6.7}$$

Equation (6.7) is called after Theodorsen. See for its deduction GAIER [12], chapter II, §1.2.

From lemma 6.2 and 6.4 it follows that the singular integral equation (6.7) must have at least one solution which is continuous and strictly increasing on $[-\pi,\pi]$. In GAIER [12], chapter II, §1.2.c, it has been proved:

LEMMA 6.8. *The singular integral equation* (6.7) *of Theodorsen has in the class of continuous, strictly increasing functions on* $[-\pi,\pi]$ *exactly one solution.*

In general it will not be possible to construct the solution of equation (6.7) explicitly. In GAIER [12], chapter II, some techniques are described for obtaining the function $\theta(\phi)$ numerically. By COHEN & BOXMA [04] such a numerical method has been elaborated, and some approximations are discussed, for the type of contours which we shall meet in the present study.

When the continuous and strictly increasing solution of equation (6.7) has been obtained the conformal mapping $g(z), z \in C^+$, is determined by Cauchy's integral formula, i.e. cf. (6.6), for $z \in C^+$,

$$g(z) = \frac{1}{2\pi i}\int_{C} \frac{g(t)}{t-z}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\rho(\theta(\phi))}{e^{i\phi}-z}\, e^{i\theta(\phi)+i\phi}d\phi.$$

GAIER [12], chapter II, §1.2.b, gives another representation of $g(z), z \in C^+$, by applying Schwarz' formula, cf. (5.14), to the function $\log g(z)/z$:

$$g(z) = z \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho(\theta(\phi))\, \frac{e^{i\phi}+z}{e^{i\phi}-z}\, d\phi\}, \qquad z \in C^+. \tag{6.8}$$

Next we shall prove a property of conformal mappings of the unit disk C^+ onto domains which have the real axis as an axis of symmetry. This property will play an important role in our analysis.

LEMMA 6.9. *Let* L *be a contour which can be represented by* (6.4) *and which has the real axis as an axis of symmetry. Let* g(z) *be the conformal mapping of* C^+ *onto* L^+ *determined by the conditions* (6.5). *Then for* $|z| \leqslant 1$,

$$g(\bar{z}) = \overline{g(z)}.$$

PROOF. Because the real axis is an axis of symmetry of the contour L, it is readily seen that in the representation (6.4) of L,

$$\rho(\theta) = \rho(-\theta), \qquad -\pi \leqslant \theta \leqslant \pi. \tag{6.9}$$

Let $\theta(\phi)$ be the unique continuous, strictly increasing solution of equation (6.7), cf. lemma 6.8 . Then for $-\pi \leqslant \phi \leqslant \pi$,

$$-\theta(-\phi) = -(-\phi) + \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho(\theta(\omega)) \cot \frac{\omega+\phi}{2}\, d\omega =$$

$$= \phi - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho(-\theta(-\omega)) \cot \frac{\omega-\phi}{2}\, d\omega,$$

i.e. $-\theta(-\phi)$ satisfies the same equation (6.7). Clearly, $-\theta(-\phi)$ is also continuous and strictly increasing, and thus

$$\theta(\phi) = -\theta(-\phi), \qquad -\pi \leqslant \phi \leqslant \pi. \tag{6.10}$$

By (6.6) and (6.9) it then follows that for $-\pi \leqslant \phi \leqslant \pi$,

$$g(e^{-i\phi}) = \rho(\theta(-\phi))\, e^{i\theta(-\phi)} = \rho(-\theta(\phi))\, e^{-i\theta(\phi)} = \rho(\theta(\phi))\, e^{-i\theta(\phi)} = \overline{g(e^{i\phi})}.$$

Finally, the assertion follows from (6.8), (6.10) and (6.9) :

$$g(\bar{z}) = \bar{z} \exp\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log \rho(\theta(\phi)) \frac{e^{i\phi}+\bar{z}}{e^{i\phi}-\bar{z}}\, d\phi\} =$$

$$= \overline{z \exp\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \rho(\theta(-\phi)) \frac{e^{i\phi}+z}{e^{i\phi}-z} d\phi\}} = \overline{g(z)}. \qquad \square$$

Later it will be convenient to consider conformal mappings which do not satisfy the conditions (6.5), but which map the point $z = 0$ onto an arbitrary point w_0 in the domain L^+. This case is easily reduced to the case discussed above by noting that every bilinear transform (cf. TITCHMARSH [25], §6.24),

$$f(z) = e^{i\lambda} \frac{z-\bar{\gamma}}{\gamma z-1}, \tag{6.11}$$

where λ is real and $|\gamma| < 1$, maps the unit disk C^+ conformally onto itself.

LEMMA 6.10. *Let* L *be a contour which can be represented by* (6.4) *and which has the real axis as an axis of symmetry. Let* $w_0 \in L^+$ *be a point on the real axis. Then the conformal mapping of* C^+ *onto* L^+ *determined by the conditions*

$$e(0) = w_0, \; e'(0) > 0, \tag{6.12}$$

is given by, for $z \in C^+ \cup C$,

$$e(z) = g\left(\frac{z+z_0}{z_0 z+1}\right),$$

here $g(z)$ *is the conformal mapping of* C^+ *onto* L^+ *satisfying the conditions* (6.5), *and* z_0 *is the (real) point in* C^+ *for which* $g(z_0) = w_0$.

PROOF. The function $e(z)$ is a conformal mapping because it is the composition of two conformal mappings and as such it is a mapping of C^+ onto L^+. The conditions (6.12) are easily verified. That z_0 is real follows from lemma 6.9 which implies that real points are mapped onto real points by the conformal mapping $g(z)$. □

We proceed with considering sequences of domains and the conformal mappings of these domains onto the unit disk.

Let $\{D_n; n = 1,2,\ldots\}$ be a sequence of simply connected domains and let the boundary Γ_n of the domain D_n be a closed Jordan curve, $n = 1,2,\ldots$. It is assumed that a fixed disk Δ_1 with its center at the origin is contained in every domain D_n, $n = 1,2,\ldots$, and that all the domains $D_n, n = 1,2,\ldots$, are contained in another fixed disk Δ_2.

Moreover we shall confine ourselves to *strictly monotonic* sequences $\{D_n, n = 1,2,\ldots\}$, i.e.

$$D_{n+1} \supset D_n \cup \Gamma_n \qquad \text{for every } n \in \mathbb{N},$$

or

$$D_{n+1} \cup \Gamma_{n+1} \subset D_n, \qquad \text{for every } n \in \mathbb{N}.$$

Obviously such sequences converge. Let D be the set of all points z with a fixed neighborhood contained in all the domains D_n starting from some value of n (depending on z). It is clear that D is nonempty because $\Delta_1 \subset D$, and that D is open.

For the above described strictly monotonic sequences $\{D_n, n = 1,2,\ldots\}$ we have as a special case of *Carathéodory's mapping theorem*, cf. MARKUSHEVICH [18], vol. III, §4, theorem 2.1 :

<u>LEMMA 6.11.</u> *Let* $\{D_n; n = 1,2,\ldots\}$ *be a strictly monotonic sequence of domains bounded by a contour, containing a fixed disk* Δ_1 *with its center at the origin, and contained in another fixed disk* Δ_2. *For* $n = 1,2,\ldots$, *let* $z = f_n(w)$ *be the conformal mapping of* D_n *onto the unit disk* C^+ *which satisfies*

$$f_n(0) = 0, \qquad f_n'(0) > 0,$$

and let $w = g_n(z)$ *be the inverse of* $z = f_n(w)$.

Further, let $z = f(w)$ *be the conformal mapping of* D *(defined above) onto* C^+ *satisfying*

$$f(0) = 0, \qquad f'(0) > 0,$$

and let $w = g(z)$ *be its inverse.*
Then the sequence of functions $\{f_n(w); n = 1,2,\ldots\}$ *converges uniformly in* D *to the function* $f(w)$, *and the sequence* $\{g_n(z), n = 1,2,\ldots\}$ *converges uniformly in* C^+ *to the function* $g(z)$.

CHAPTER II

A QUEUEING MODEL WITH TWO TYPES OF CUSTOMERS AND PAIRED SERVICES: THE QUEUES AT DEPARTURE INSTANTS

II.0. Introduction, the model

In this and the next chapter we shall analyse the following queueing model. Two types of customers arrive independently at a single service facility. An arriving customer who finds the system empty is immediately taken into service. If the server is busy then he joins queue 1 or 2 depending on his type. All arriving customers are admitted to the service system.

For each type of customers the interarrival times are independent stochastic variables with a common negative exponential distribution.

At the service facility customers are provided by the server with a service time. Successive service times are independent stochastic variables with a common distribution function B(t). The service times are independent of the arrival processes.

The *service discipline* is as follows. As soon as a service has been completed a new service is started if there are any customers present.

In general a couple of two customers of different type is simultaneously served. If after the completion of a service there are only customers of one type present a customer of this type is served individually. If at a service completion time there are no customers at all present the service facility stays empty until a new customer arrives.

In each queue customers are served in order of their arrival. But a customer of one type may be served before an earlier arrived customer of the other type.

Denote by $\underline{x}_i(n)$, $n = 0,1,2,\ldots$; $i = 1,2$, the number of type i customers left behind in the service system after the completion of the n^{th} service. The stochastic vector process $(\underline{x}_1(n),\underline{x}_2(n))$ turns out to be a discrete time Markov chain with a two dimensional discrete state space. This chapter concerns the analysis of this Markov chain. It is effected by introducing the generating function of the joint distribution of the stochastic variables $\underline{x}_1(n)$ and $\underline{x}_2(n)$. This generating function satisfies a functional equation. It will be shown that the analysis of this functional equation can be reduced by a method developed by COHEN & BOXMA [04] to that of two Riemann- Hilbert boundary value problems. Moreover it will be shown that this analysis can also be reduced to that of one Hilbert boundary value problem.

Once the solution of the time dependent Markov chain is obtained it will be proved under which conditions the Markov chain is positive recurrent, and the generating function of the stationary distribution will be given. Finally the first and second order moments of this stationary distribution will be determined.

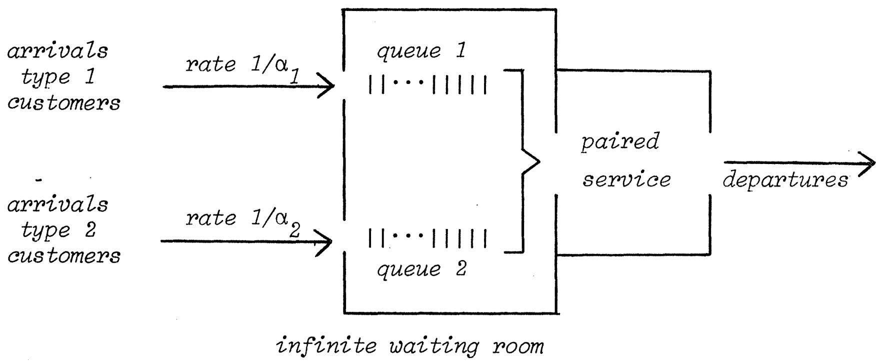


Figure 0.1. Queueing system with two independent arrival streams and paired services

II.1. Definitions

As already stated in the introduction we consider a service facility which provides two types of customers with service. For each type of customers the *interarrival times* form a sequence of independent identically distributed stochastic variables. For type i customers, $i = 1,2$, the interarrival times are *negative exponentially* distributed with *mean* α_i, $\alpha_i > 0$. The arrival processes of the two types of customers are two independent Poisson processes. As a consequence the *total arrival process* of all customers is also a Poisson process, with mean α where

$$\frac{1}{\alpha} := \frac{1}{\alpha_1} + \frac{1}{\alpha_2}. \tag{1.1}$$

Introducing

$$c_i := \frac{\alpha}{\alpha_i}, \qquad i = 1,2, \tag{1.2}$$

so that from (1.1),

$$c_1 + c_2 = 1, \tag{1.3}$$

it can be said that at each arrival epoch in the total arrival process the arriving customer has type i with probability c_i, $i = 1,2$.

Let $\underline{\tau}_n, n = 1,2,\dots$, be the *duration of the* n *th service*. The stochastic variables $\underline{\tau}_n$, $n = 1,2,\dots$, are assumed to be independent, identically distributed with general distribution,

$$\Pr\{\underline{\tau}_n < t\} = B(t), \qquad n = 1,2,\dots, \qquad B(0+) = 0, \tag{1.4}$$

and moments

$$\beta_j := \int_0^\infty t^j \, d\, B(t), \qquad j = 1,2,\dots . \tag{1.5}$$

It is throughout assumed that the first moment $\beta := \beta_1$ of the service time distribution is finite.

Further we introduce the Laplace-Stieltjes transform of the distribution $B(t)$,

$$\beta(s) := \int_0^\infty e^{-st}\, d\, B(t), \qquad \mathrm{Re}\ s \geq 0. \tag{1.6}$$

The quantity a defined by

$$a := \frac{\beta}{\alpha} \tag{1.7}$$

will be called the *traffic offered to the system*, whereas a_i,

$$a_i := \frac{\beta}{\alpha_i} = c_i a, \qquad i = 1,2, \tag{1.8}$$

will denote the *traffic offered by type i customers.*

DEFINITION 1.1. *For* i = 1,2, *let* $\underline{\xi}_i(n)$, n = 1,2..., *be the number of type* i *customers arriving during the* n^{th} *service.*

Note that the number of customers that arrive during a service time does not depend on the past interarrival time prior to the instant that this service starts, because the interarrival times are negative exponentially distributed.

LEMMA 1.1. *The vector variables* $(\underline{\xi}_1(n), \underline{\xi}_2(n))$, n = 1,2,..., *are identically distributed with joint distribution*

$$\Pr\{\underline{\xi}_1(n) = k_1, \underline{\xi}_2(n) = k_2\} = \int_0^\infty \frac{(t/\alpha_1)^{k_1}}{k_1!} \frac{(t/\alpha_2)^{k_2}}{k_2!} e^{-t/\alpha}\, d\, B(t),$$

$$k_1, k_2 = 0,1,2,\ldots .$$

From the remark above the lemma and the fact that the service times are identically distributed it follows that the vectors $(\underline{\xi}_1(n), \underline{\xi}_2(n))$

are identically distributed for $n = 1,2,\dots$.

Given that a service has duration t the number of arriving type 1 customers and the number of arriving type 2 customers are independent and have a Poisson distribution with parameter t/α_1 respectively t/α_2. This implies the formula for the joint distribution of $(\underline{\xi}_1(n),\underline{\xi}_2(n))$, $n = 1,2,\dots$ □

For the generating function of the joint distribution of $\underline{\xi}_1(n)$ and $\underline{\xi}_2(n)$ we have:

<u>LEMMA 1.2.</u> *For* $|p_1| \leq 1$, $|p_2| \leq 1$, $n = 1,2,\dots$:

$$E\left\{p_1^{\underline{\xi}_1(n)} p_2^{\underline{\xi}_2(n)}\right\} = \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right). \tag{1.9}$$

For fixed p_2 *this generating function possesses an analytic continuation as function of* p_1, *and for fixed* p_1 *as function of* p_2, *into the domains where*

$$\mathrm{Re}\{c_1p_1 + c_2p_2\} < 1.$$

<u>PROOF.</u> For $n = 1,2,\dots$, it follows from lemma 1.1 that for $|p_1| \leq 1, |p_2| \leq 1$,

$$E\left\{p_1^{\underline{\xi}_1(n)} p_2^{\underline{\xi}_2(n)}\right\} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_1^{k_1} p_2^{k_2} \int_0^{\infty} \frac{(t/\alpha_1)^{k_1}}{k_1!} \frac{(t/\alpha_2)^{k_2}}{k_2!} e^{-t/\alpha}\, dB(t).$$

Because this expression is finite for $|p_1| \leq 1$, $|p_2| \leq 1$, we may change summation and integration (dominated convergence, see e.g. BURRILL [02], §7.2.), so that

$$E\left\{p_1^{\underline{\xi}_1(n)} p_2^{\underline{\xi}_2(n)}\right\} = \int_0^{\infty} e^{p_1 t/\alpha_1} e^{p_2 t/\alpha_2} e^{-t/\alpha}\, dB(t),$$

$$|p_1| \leq 1, \ |p_2| \leq 1.$$

By using (1.2) this gives the Laplace-Stieltjes transform (1.6) of the distribution B(t) with argument $s = (1-c_1p_1-c_2p_2)/\alpha$. Noting that the transform $\beta(s)$ is regular in the domain Re $s > 0$ the analytic continuation follows, cf. section I.1, definition I.1.7. □

DEFINITION 1.2. *Let* $\underline{x}_i(n)$, $n = 0,1,2,\ldots$; $i = 1,2$, *denote the number of type* i *customers left behind in the system after the completion of the* n^{th} *service, cf. the introduction section II.0.*
It will be assumed that $t = 0$ *can be considered as the "zeroth" service completion epoch and* x_i, $i = 1,2$, *will represent the number of customers of type* i *present in the system at time* $t = 0+$.
Further, let for $x_1,x_2 = 0,1,2,\ldots$, $|p_1| \leq 1$, $|p_2| \leq 1$, $|r| < 1$,

$$\Phi_x(r;p_1,p_2) = \sum_{n=0}^{\infty} r^n \, E\left\{p_1^{\underline{x}_1(n)} p_2^{\underline{x}_2(n)} \mid \underline{x}_1(0) = x_1,\ \underline{x}_2(0) = x_2\right\}, \tag{1.10}$$

here x *stands for the vector* (x_1,x_2).

II.2. Formulation of the mathematical problem

In queueing theory one is interested in distributions or moments of queue lengths, waiting times, busy periods etc, in particular the stationary distributions of these quantities and conditions on which a queueing process possesses a stationary distribution. In this section the inherent mathematical problem for the queueing model described in the sections II.0 and II.1 will be formulated. The queueing process is considered at departure epochs because this embedded process defines a Markov chain. First we shall show recurrence relations for the series $\{\underline{x}_i(n),\ n = 0,1,2,\ldots\}$, $i = 1,2$, and then it will be proved that the generating function $\Phi_x(r;p_1,p_2)$, cf. definition I.2, satisfies a functional equation and possesses regularity properties.

Denote for real y,

$$[y]^+ := \max\{0,y\}.$$

<u>THEOREM 2.1.</u> *For* i = 1,2, *the series* $\{\underline{x}_i(n),\ n = 0,1,2,\ldots\}$ *satisfies the relations*

$$\underline{x}_i(n) = [\underline{x}_i(n-1) - 1]^+ + \underline{\xi}_i(n), \qquad n = 1,2,\ldots;\ \underline{x}_i(0) = x_i. \tag{2.1}$$

For given (x_1,x_2) *the probability distribution of* $(\underline{x}_1(n),\underline{x}_2(n))$ *is uniquely determined by this relation,* $n = 0,1,2,\ldots$.

<u>PROOF</u>. The relations for n = 0 follow from the assumption made in definition 1.2.

The recurrence relation for n > 0 will be proved for i = 1; for i = 2 it can be proved similarly.

Let n be fixed, n > 0. Three cases have to be distinguished.

i. $\underline{x}_1(n-1) > 0$. Then immediately after the completion of the $n\text{-}1^{\text{th}}$ service a new service will start in which a type 1 customer is served. The number of type 1 customers decreases by one and increases by $\underline{\xi}_1(n)$, cf. definition 1.1, during the n^{th} service, independent whether this service is a paired or an individual one. Hence, (2.1) holds in this case.

ii. $\underline{x}_1(n-1) = 0$ and during the n^{th} service a type 2 customer is served. Then $\underline{x}_1(n)$ is equal to the number of type 1 customers which arrive during this service time, i.e. $\underline{\xi}_1(n)$, so that (2.1) holds.

iii. $\underline{x}_1(n-1) = 0$ and during the n^{th} service a type 1 customer is served. This can only occur if $\underline{x}_2(n-1) = 0$ and the first arriving customer after the $n\text{-}1^{\text{th}}$ service completion epoch has type 1. Then the number of type 1 customers increases by $1 + \underline{\xi}_1(n)$ and decreases by one, so that again

$\underline{x}_1(n) = \underline{\xi}_1(n)$, i.e. (2.1) holds.
Because the distribution of the vector $(\underline{\xi}_1(n),\underline{\xi}_2(n))$ is known for $n = 1,2,\dots$, cf. lemma 1.1, it follows with induction that the probability distribution of the vector $(\underline{x}_1(n),\underline{x}_2(n))$ is uniquely determined by (2.1) for given (x_1,x_2), for every n, $n = 0,1,2,\dots$. □

THEOREM 2.2. *The stochastic process* $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\dots\}$ *is a discrete time Markov chain with two-dimensional state space* $\{0,1,2\dots\} \times \{0,1,2,\dots\}$ *which is irreducible, aperiodic, and which has stationary transition probabilities.*

PROOF. Because the vector $(\underline{\xi}_1(n),\underline{\xi}_2(n))$ is independent of the vectors $(\underline{x}_1(m),\underline{x}_2(m))$, $m < n$, for every n, $n = 1,2,\dots$, it follows from relation (2.1) that the process $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\dots\}$ possesses the Markov property. Moreover, it is obvious from theorem 2.1 and lemma 1.1 that this Markov chain has the state space $\{0,1,2,\dots\} \times \{0,1,2,\dots\}$, that it is irreducible, aperiodic, and that it has stationary transition probabilities, cf. FELLER [10], § XV. □

THEOREM 2.3. *The generating function* $\Phi_x(r;p_1,p_2)$ *of the Markov chain* $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\dots\}$ *has the following properties:*

i. *it satisfies for* $|r| < 1, |p_1| \leq 1, |p_2| \leq 1$, *the functional equation*

$$\left[p_1p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\right]\Phi_x(r;p_1,p_2) = p_1^{x_1+1}\,p_2^{x_2+1} +$$

$$+\ r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\Big[(p_2-1)\Phi_x(r;p_1,0) + (p_1-1)\Phi_x(r;0,p_2) +$$

$$+\ (p_1-1)(p_2-1)\Phi_x(r;0,0)\Big]; \tag{2.2}$$

ii. *for* p_1 *and* p_2 *fixed in* $C^+ \cup C$ *it is a regular function of* r *in the unit disk* $|r| < 1$; *as a function of* p_1 *it belongs to the class* $RCB(C^+)$, cf. definition I.1.6, *for* $|r| < 1, |p_2| \leqslant 1$; *and similarly as a function of* p_2 *it belongs to the class* $\mathrm{RCB}(C^+)$ *for* $|r| < 1$, $|p_1| \leqslant 1$.

PROOF. For $n > 0$ it follows from theorem 2.1 that for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$E\left\{p_1^{\underline{x}_1(n)} p_2^{\underline{x}_2(n)}\right\} = E\left\{p_1^{[\underline{x}_1(n-1)-1]^+ + \underline{\xi}_1(n)} p_2^{[\underline{x}_2(n-1)-1]^+ + \underline{\xi}_2(n)}\right\}.$$

Using the fact that the vectors $(\underline{\xi}_1(n),\underline{\xi}_2(n))$ and $(\underline{x}_1(n-1),\underline{x}_2(n-1))$ are independent, and distinguishing four disjunct cases we obtain for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$E\left\{p_1^{\underline{x}_1(n)} p_2^{\underline{x}_2(n)}\right\} = E\left\{p_1^{\underline{\xi}_1(n)} p_2^{\underline{\xi}_2(n)}\right\} \left[E\left\{(\underline{x}_1(n-1) = 0, \underline{x}_2(n-1) = 0)\right\} +\right.$$

$$+ \frac{1}{p_1} E\left\{p_1^{\underline{x}_1(n-1)} (\underline{x}_1(n-1) > 0, \underline{x}_2(n-1) = 0)\right\} +$$

$$+ \frac{1}{p_2} E\left\{p_2^{\underline{x}_2(n-1)} (\underline{x}_1(n-1) = 0, \underline{x}_2(n-1) > 0)\right\} +$$

$$\left. + \frac{1}{p_1 p_2} E\left\{p_1^{\underline{x}_1(n-1)} p_2^{\underline{x}_2(n-1)} (\underline{x}_1(n-1) > 0, \underline{x}_2(n-1) > 0)\right\}\right] =$$

$$= \frac{1}{p_1 p_2} \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\left[E\left\{p_1^{\underline{x}_1(n-1)} p_2^{\underline{x}_2(n-1)}\right\} +\right.$$

$$(p_2-1)E\left\{p_1^{\underline{x}_1(n-1)} (\underline{x}_2(n-1) = 0)\right\} + (p_1-1)E\left\{p_2^{\underline{x}_2(n-1)} (\underline{x}_1(n-1) = 0)\right\} +$$

$$\left.(p_1-1)(p_2-1)E\left\{(\underline{x}_1(n-1) = 0, \underline{x}_2(n-1) = 0)\right\}\right].$$

Here we have used lemma 1.2. For each $n > 0$ we multiply the above equation by $p_1 p_2 r^n$. By summing those equations over $n > 0$ and by using the initial conditions of theorem 2.1 for $n = 0$ the functional equation (2.2) follows.

The stated regularity properties of the function $\Phi_x(r;p_1,p_2)$ are well-known properties of generating functions. □

In the next sections it will be shown that the properties of the function $\Phi_x(r;p_1,p_2)$ which have been proved in the above theorem suffice to determine this generating function uniquely for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$ (cf. remark 6.1).

For fixed r, $|r| < 1$, equation (2.2) relates the bivariate function $\Phi_x(r;p_1,p_2)$ to two univariate functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ and a constant $\Phi_x(r;0,0)$. In the analysis of this functional equation a central role is played by the *kernel*

$$p_1p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right), \tag{2.3}$$

because if for a pair $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ this kernel vanishes then the righthand side of equation (2.2) must be zero because of the second condition of theorem 2.3. This provides us with a relation between the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$. Therefore we shall examine the kernel (2.3) and its zeros in the next section.

II.3. Analysis of the kernel

It will be shown that for fixed r there exist pairs $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$, for which the kernel (2.3) of equation (2.2) vanishes. In order to describe the set of all pairs of zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (2.3) we shall follow a method introduced by COHEN & BOXMA [04], §5, in the analysis of the M/G/1 queueing system with alternating service where a similar kernel appears. According to this method a parameter δ and two-valued functions $p_1(r;\delta)$ and $p_2(r;\delta)$ will be introduced for the description of these pairs of zeros. The analytic properties of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ will be studied.

Throughout this section r *is assumed to be a fixed complex number,* $0 < |r| < 1$.

Consider the equation, cf. (2.3),

$$p_1 p_2 - r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) = 0. \tag{3.1}$$

LEMMA 3.1. *For* $|r| \leqslant |p_1| \leqslant 1$ *equation* (3.1) *has exactly one root* $p_2 \in C^+$. *Similarly, for* $|r| \leqslant |p_2| \leqslant 1$ *equation* (3.1) *has exactly one root* $p_1 \in C^+$.

PROOF. Because for $|p_1| \leqslant 1$ and $p_2 \in C^+$,

$$|c_1p_1 + c_2p_2| \leqslant c_1|p_1| + c_2|p_2| < c_1 + c_2 = 1,$$

and the Laplace-Stieltjes transform $\beta(s)$ is regular for Re $s > 0$ and continuous for Re $s \geqslant 0$, the function $\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)$ belongs for $|p_1| \leqslant 1$ as a function of p_2 to the class RCB(C^+).

On the unit circle $|p_2| = 1$ we have for $|r| \leqslant |p_1| \leqslant 1$ the inequalities

$$\left|r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\right| \leqslant |r| \leqslant |p_1| = |p_1p_2|.$$

The first inequality is strict unless $p_1 = 1$, while for $p_1 = 1$ the second inequality is strict. Application of Rouché's theorem (see lemma I.1.4) to $\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)$ and p_1p_2 as functions of p_2 with contour C leads to the first assertion. The second assertion can be proved similarly. □

This lemma shows the existence of pairs $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ for which the kernel (2.3) vanishes. For such a pair of zeros the parameter δ is defined to be

$$\delta := c_1p_1 + c_2p_2. \tag{3.2}$$

Further we introduce

$$w := 2c_1p_1, \tag{3.3}$$

so that by (3.2) and (3.3),

$$2\ c_2p_2 = 2\ \delta - w. \tag{3.4}$$

Substitution of (3.3) and (3.4) in (3.1) leads to the equation

$$w^2 - 2\ \delta w + 4\ c_1c_2\ r\ \beta\left(\frac{1-\delta}{\alpha}\right) = 0. \tag{3.5}$$

Because this equation is quadratic in w it defines a two-valued function $w(r;\delta)$ which is given by

$$w(r;\delta) = \delta \pm \sqrt{\delta^2 - 4c_1c_2\ r\ \beta\left(\frac{1-\delta}{\alpha}\right)}. \tag{3.6}$$

The two-valued functions $p_1(r;\delta)$ and $p_2(r;\delta)$ are defined to be, cf. (3.3), (3.4),

$$p_1(r;\delta) := \frac{1}{2c_1}\ w(r;\delta),$$

$$p_2(r;\delta) := \frac{1}{2c_2}\ [2\delta - w(r;\delta)]. \tag{3.7}$$

<u>THEOREM 3.1.</u> *Every pair of zeros* $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ *of the kernel* (2.3) *can be described by* $p_1 = p_1(r;\delta)$, $p_2 = p_2(r;\delta)$ *for one of the two values of the function* $w(r;\delta)$, *cf.* (3.7) *and* (3.6), *and for* $\delta \in C \cup C^+$.

<u>PROOF:</u> For every pair of zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (2.3) the parameter δ can be defined by (3.2) and as above it follows that for one of the values of the function $w(r;\delta)$, cf. (3.7), (3.6),

$$p_1 = \frac{1}{2c_1}\ w(r;\delta), \qquad p_2 = \frac{1}{2c_2}\ [2\delta - w(r;\delta)].$$

Because $p_1 \in C^+ \cup C$ and $p_2 \in C^+ \cup C$ it is obtained from (3.2) that

$$|\delta| \leqslant c_1|p_1| + c_2|p_2| \leqslant c_1 + c_2 = 1. \qquad \square$$

In general we need both branches of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ for the description of all pairs of zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (2.3). This can be seen by noting that if (p_1,p_2) is a root of equation (3.1) then $\left(\frac{c_2}{c_1} p_2, \frac{c_1}{c_2} p_1\right)$ is a root of the same equation. Assume that $c_2 \leqslant c_1$. Take p_1 on the circle $|p_1| = |r|$. By lemma 3.1 there exists a root (p_1,p_2) of equation (3.1) with $p_2 \in C^+$. Then for $|r| \leqslant \frac{c_2}{c_1}$,

$$\left|\frac{c_2}{c_1} p_2\right| < 1, \qquad \left|\frac{c_1}{c_2} p_1\right| = \frac{c_1}{c_2} |r| \leqslant 1.$$

Hence both (p_1,p_2) and $\left(\frac{c_2}{c_1} p_2, \frac{c_1}{c_2} p_1\right)$ are roots of equation (3.1) and belong to $(C^+ \cup C) \times (C^+ \cup C)$. Because these two roots define the same value of δ, cf. (3.2), they must be described by the two values of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$.

Next the discriminant of equation (3.5) will be considered.

<u>LEMMA 3.2.</u> *The discriminant* $\delta^2 - 4\, c_1 c_2 r\, \beta\!\left(\frac{1-\delta}{\alpha}\right)$ *of equation* (3.5) *has exactly two zeros, say* $\delta_1(r)$ *and* $\delta_2(r)$, *in the domain* $\mathrm{Re}\ \delta < 1$. *Both zeros are bounded in absolute value by one.*

<u>PROOF.</u> The function $\beta\!\left(\frac{1-\delta}{\alpha}\right)$ is regular in the domain $\mathrm{Re}\ \delta < 1$, and continuous and bounded in absolute value by one for $\mathrm{Re}\ \delta \leqslant 1$, because it is the Laplace-Stieltjes transform of a probability distribution. Let R be a positive number, $R \geqslant 1$, and consider the contour

$$\{\delta;\ \mathrm{Re}\ \delta = 1,\ |\delta| < R\} \cup \{\delta;\ \mathrm{Re}\ \delta \leqslant 1,\ |\delta| = R\}.$$

On this contour we have the inequality, for every $R \geqslant 1$,

$$\left|4c_1c_2 r\, \beta\!\left(\frac{1-\delta}{\alpha}\right)\right| \leqslant 4c_1c_2|r| < 1 \leqslant |\delta^2|.$$

Application of Rouché's theorem (cf. lemma I.1.4) to the functions $4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right)$ and δ^2 leads for $R \to \infty$ to the first assertion, and for $R = 1$ to the second assertion. □

Let $E(r)$ denote the set

$$E(r) := \{\delta; \mathrm{Re}\,\delta < 1\} \setminus \{\delta_1(r), \delta_2(r)\}. \tag{3.8}$$

<u>THEOREM 3.2</u> *The two-valued functions* $p_1(r;\delta)$ *and* $p_2(r;\delta)$ *are analytic in the domain* $E(r)$, *and they are continuous in this domain up to its boundary.*

<u>PROOF.</u> Because the function $\beta\left(\frac{1-\delta}{\alpha}\right)$ is regular for $\mathrm{Re}\,\delta < 1$ and continuous for $\mathrm{Re}\,\delta \leqslant 1$ it follows from lemma 3.2 that the function

$$M(r;\delta) := \frac{\delta^2 - 4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right)}{[\delta-\delta_1(r)][\delta-\delta_2(r)]}, \tag{3.9}$$

is regular and non-vanishing for $\mathrm{Re}\,\delta < 1$ and continuous for $\mathrm{Re}\,\delta \leqslant 1$. Hence, also $\sqrt{M(r;\delta)}$ is regular for $\mathrm{Re}\,\delta < 1$. Clearly, the function

$$\sqrt{[\delta-\delta_1(r)][\delta-\delta_2(r)]}, \tag{3.10}$$

is a two-valued analytic function, cf. definition I.1.8, with two first order branch points at $\delta = \delta_1(r)$ and $\delta = \delta_2(r)$, cf. definition I.1.9. This implies that $w(r;\delta)$, cf. (3.6), and hence by (3.7) also $p_1(r;\delta)$ and $p_2(r;\delta)$ are two-valued analytic functions in the domain $E(r)$, which are continuous in $E(r)$ up to its boundary. □

Let $\gamma(r)$ be the line segment joining the branch points $\delta_1(r)$ and $\delta_2(r)$, and let

$$\tilde{E}(r) := E(r) \setminus \gamma(r). \tag{3.11}$$

<u>COROLLARY 3.1.</u> *The analytic functions* $p_1(r;\delta)$ *and* $p_2(r;\delta)$ *both have two regular branches in the domain* $\tilde{E}(r)$.

<u>PROOF.</u> This statement follows from theorem 3.2 and the fact that every closed Jordan curve lying entirely in the domain $\tilde{E}(r)$ contains in its interior either both of the branch points $\delta_1(r)$ and $\delta_2(r)$ or none of them (cf. MARKUSHEVICH [18], vol. I, § 55). □

It is readily seen that also the function $w(r;\delta)$ has two regular branches in the domain $\tilde{E}(r)$. These branches will be denoted by $w_1(r;\delta)$ and $w_2(r;\delta)$, $\delta \in \tilde{E}(r)$. Because these branches are the two roots of the quadratic equation (3.5) they satisfy for $\delta \in \tilde{E}(r)$,

$$w_1(r;\delta) + w_2(r;\delta) = 2\,\delta,$$

$$w_1(r;\delta)w_2(r;\delta) = 4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right). \tag{3.12}$$

It is seen that for $|\delta| \to \infty$, $\delta \in \tilde{E}(r)$, the sum of $w_1(r;\delta)$ and $w_2(r;\delta)$ tends to infinity while their product remains bounded in absolute value by one. This enables us to determine the branches $w_1(r;\delta)$ and $w_2(r;\delta)$ unambiguously by putting

$$\lim_{|\delta|\to\infty} |w_1(r;\delta)| = \infty, \quad \lim_{|\delta|\to\infty} |w_2(r;\delta)| = 0, \quad \delta \in \tilde{E}(r). \tag{3.13}$$

With (3.9) we can write, cf. (3.6), for $\delta \in \tilde{E}(r)$,

$$w_1(r;\delta) = \delta + \sqrt{M(r;\delta)}\sqrt{|\delta-\delta_1(r)||\delta-\delta_2(r)|}\; e^{\frac{1}{2}i\,\arg[\delta-\delta_1(r)] + \frac{1}{2}i\,\arg[\delta-\delta_2(r)]},$$

$$w_2(r;\delta) = \delta - \sqrt{M(r;\delta)}\sqrt{|\delta-\delta_1(r)||\delta-\delta_2(r)|}\; e^{\frac{1}{2}i\,\arg[\delta-\delta_1(r)] + \frac{1}{2}i\,\arg[\delta-\delta_2(r)]}, \tag{3.14}$$

here the arguments of $\delta-\delta_1(r)$ and $\delta-\delta_2(r)$ are between $-\pi$ and π.
The two regular branches $p_{ij}(r;\delta)$, $j = 1,2$, of the functions $p_i(r;\delta)$, $i = 1,2$, are defined according to (3.7) and (3.14), for $\delta \in \tilde{E}(r)$,

$$p_{11}(r;\delta) := \frac{1}{2c_1} w_1(r;\delta), \qquad p_{21}(r;\delta) := \frac{1}{2c_2}[2\delta - w_1(r;\delta)],$$

$$p_{12}(r;\delta) := \frac{1}{2c_1} w_2(r;\delta), \qquad p_{22}(r;\delta) := \frac{1}{2c_2}[2\delta - w_2(r;\delta)]. \tag{3.15}$$

Note that from (3.15) and (3.12) it follows that for $\delta \in \tilde{E}(r)$,

$$2c_1p_{11}(r;\delta) = 2c_2p_{22}(r;\delta) = w_1(r;\delta),$$

$$2c_1p_{12}(r;\delta) = 2c_2p_{21}(r;\delta) = w_2(r;\delta). \tag{3.16}$$

<u>LEMMA 3.3.</u> *For* $j = 1,2$ *and for every* $\delta \in \tilde{E}(r)$ *either* $p_{1j}(r;\delta)$ *or* $p_{2j}(r;\delta)$ *is bounded in absolute value by one.*

<u>PROOF.</u> From (3.16) and (3.12) it is obtained that for $j = 1,2$, for $\delta \in \tilde{E}(r)$,

$$p_{1j}(r;\delta)p_{2j}(r;\delta) = r\,\beta\left(\frac{1-\delta}{\alpha}\right).$$

Hence, for $j = 1,2$, for $\delta \in \tilde{E}(r)$,

$$|p_{1j}(r;\delta)p_{2j}(r;\delta)| \leq |r| < 1,$$

thus either $|p_{1j}(r;\delta)| < 1$ or $|p_{2j}(r;\delta)| < 1$. □

On the line segment $\gamma(r)$ we choose as positive direction that from $\delta_1(r)$ to $\delta_2(r)$, and with respect to this direction we speak of lefthand and righthand limits on $\gamma(r)$. As a consequence of theorem 3.2 the lefthand and righthand limits of the functions $p_{ij}(r;\delta)$, $i,j = 1,2$, and $w_j(r;\delta)$, $j = 1,2$, on $\gamma(r)$ exist, but they are not equal except at the end points $\delta_1(r)$ and $\delta_2(r)$.

The lefthand limits on $\gamma(r)$ will be denoted by "+", and the righthand limits by "-". For $\delta \in \gamma(r)$ we have, cf. (3.14), (3.16),

$$w_1^+(r;\delta) = w_2^-(r;\delta), \qquad w_1^-(r;\delta) = w_2^+(r;\delta),$$

$$p_{i1}^+(r;\delta) = p_{i2}^-(r;\delta), \qquad p_{i1}^-(r;\delta) = p_{i2}^+(r;\delta), \qquad i = 1,2. \tag{3.17}$$

In lemma 3.3 it has been shown that for every $\delta \in \tilde{E}(r)$ either $p_{1j}(r;\delta)$ or $p_{2j}(r;\delta)$ is bounded in absolute value by one, but from (3.13) and (3.16) it is clear that not for every $\delta \in \tilde{E}(r)$ both of them are bounded in absolute value by one ($j = 1,2$). Therefore we introduce the sets, for $j = 1,2$,

$$\Delta_j(r) := \{\delta;\ \text{Re}\ \delta \leqslant 1,\ |p_{1j}(r;\delta)| \leqslant 1,\ |p_{2j}(r;\delta)| \leqslant 1\}, \tag{3.18}$$

where for $\delta \in \gamma(r)$ the lefthand limits of the functions $p_{ij}(r;\delta)$, $i,j = 1,2$, have to be understood.

LEMMA 3.4. *At least one of the sets* $\Delta_j(r)$, $j = 1,2$ *is non-empty and its intersection with the domain* $\tilde{E}(r)$ *contains limiting points. The sets* $\Delta_j(r)$, $j = 1,2$, *are contained in* $C^+ \cup C$.

PROOF. In lemma 3.1 it has been shown that equation (3.1) has roots $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$. With theorem 3.1 this implies that at least one of the sets $\Delta_j(r)$, $j = 1,2$, is non-empty. From theorem 3.1 it also follows that the sets $\Delta_j(r)$, $j = 1,2$, are contained in $C^+ \cup C$.
From lemma 3.1 it follows that equation (3.1) defines implicitly a function $p_2 = f(r;p_1)$ which is regular for $|r| < |p_1| < 1$. The image of the annulus $|r| < |p_1| < 1$ is a domain, and hence by (3.2), (3.11) and (3.18) the intersection of $\tilde{E}(r)$ with at least one of the sets $\Delta_j(r)$, $j = 1,2$, must contain limiting points. □

This section will be concluded with the discussion of some properties of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$, $\delta \in E(r)$, which only hold for real positive values of the variable r.

<u>THEOREM 3.3.</u> *Let* r *be real,* $0 < r < 1$. *Then the two branch points* $\delta_1(r)$ *and* $\delta_2(r)$ *of the functions* $p_1(r;\delta)$ *and* $p_2(r;\delta)$ *are real, and they can be chosen such that*

$$-1 < \delta_1(r) < 0 < \delta_2(r) < 1. \tag{3.19}$$

On the real interval $\gamma(r)$ *the limiting values of the branches* $2c_1p_{1j}(r;\delta)$ *and* $2c_2p_{2j}(r;\delta)$ *are complex conjungate, i.e. for* $\delta \in \gamma(r)$, $j = 1,2$,

$$2c_1p_{1j}^{+}(r;\delta) = \overline{2c_2p_{2j}^{+}(r;\delta)}, \qquad 2c_1p_{1j}^{-}(r;\delta) = \overline{2c_2p_{2j}^{-}(r;\delta)}. \tag{3.20}$$

<u>PROOF.</u> For r real, $0 < r < 1$, we consider the continuous function

$$\delta^2 - 4c_1c_2r\,\beta\!\left(\frac{1-\delta}{\alpha}\right), \tag{3.21}$$

on the real interval $-1 \leqslant \delta \leqslant 1$. Noting that this function is positive at $\delta = 1$ and $\delta = -1$, and negative at $\delta = 0$, and knowing that this function can only have two zeros (lemma 3.2), it follows that both zeros must be real, one on the interval $-1 < \delta < 0$, the other on the interval $0 < \delta < 1$, which proves (3.19). Further, on the real interval $\gamma(r) = [\delta_1(r),\delta_2(r)]$ the function (3.21) is negative, so that, cf. (3.14), for $\delta \in \gamma(r)$,

$$w_1^{+}(r;\delta) = w_2^{-}(r;\delta) = \delta + i\sqrt{4c_1c_2r\,\beta\!\left(\frac{1-\delta}{\alpha}\right) - \delta^2},$$
$$w_1^{-}(r;\delta) = w_2^{+}(r;\delta) = \delta - i\sqrt{4c_1c_2r\,\beta\!\left(\frac{1-\delta}{\alpha}\right) - \delta^2}. \tag{3.22}$$

Hence, for $\delta \in \gamma(r)$,

$$w_1^{+}(r;\delta) = \overline{w_1^{-}(r;\delta)}, \qquad w_2^{+}(r;\delta) = \overline{w_2^{-}(r;\delta)}, \tag{3.23}$$

and (3.20) follows from (3.23) and (3.16). □

COROLLARY 3.2. *For* $0 < r < 1$, $\delta \in \gamma(r)$, $i,j = 1,2$,

$$|2c_i p_{ij}^{+}(r;\delta)^2| = |2c_i p_{ij}^{-}(r;\delta)|^2 = 4c_1 c_2 r\, \beta\left(\frac{1-\delta}{\alpha}\right). \tag{3.24}$$

PROOF. These relations follow from (3.22) and (3.16). □

LEMMA 3.5. *For* $0 < r < 1$, $\delta \in \tilde{E}(r)$,

$$|2c_2 p_{21}(r;\delta)|^2 < 4c_1 c_2 r\left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| < |2c_1 p_{11}(r;\delta)|^2,$$

$$|2c_1 p_{12}(r;\delta)|^2 < 4c_1 c_2 r\left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| < |2c_2 p_{22}(r;\delta)|^2. \tag{3.25}$$

PROOF. Consider the quotient (cf.(3.14))

$$\frac{w_2(r;\delta)}{w_1(r;\delta)}, \qquad \delta \in \tilde{E}(r). \tag{3.26}$$

As $|\delta| \to \infty$, $\delta \in \tilde{E}(r)$, this quotient vanishes because of (3.13).
On the line $\mathrm{Re}\,\delta = 1$, where

$$\left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| \leq 1 \leq |\delta|,$$

it is readily seen that

$$|\delta| - 1 < \left|\sqrt{\delta^2 - 4c_1 c_2 r\, \beta\left(\frac{1-\delta}{\alpha}\right)}\right| < |\delta| + 1,$$

and that this square root is contained in a circle of radius one around δ. Hence, for $\mathrm{Re}\,\delta = 1$,

$$|w_2(r;\delta)| < 1 < |w_1(r;\delta)|.$$

On the interval $\gamma(r)$ the boundary values of the functions $w_1(r;\delta)$ and $w_2(r;\delta)$ are complex conjungate, cf. (3.22), so that the absolute value of their quotient (3.26) equals one.
Thus on the boundary of $\tilde{E}(r)$, cf. (3.11), we have

$$\left|\frac{w_2(r;\delta)}{w_1(r;\delta)}\right| \leqslant 1. \tag{3.27}$$

The two branches $w_1(r;\delta)$ and $w_2(r;\delta)$ of the analytic function $w(r;\delta)$ are regular in the domain $\tilde{E}(r)$, cf. corollary 3.1. Moreover, the branch $w_1(r;\delta)$ does not vanish in $\tilde{E}(r)$, because a root $w = 0$ of equation (3.5) corresponds to a zero δ, Re $\delta < 1$, of $\beta\left(\frac{1-\delta}{\alpha}\right)$, and for such a zero δ it follows from (3.14) that, since $\beta\left(\frac{1}{\alpha}\right) \neq 0$,

$$w_2(r;\delta) = 0, \qquad w_1(r;\delta) = 2\delta \neq 0.$$

Hence the quotient (3.26) is regular in $\tilde{E}(r)$. By (3.27) and by the maximum modulus principle it follows that for every $\delta \in \tilde{E}(r)$,

$$|w_2(r;\delta)| < |w_1(r;\delta)|. \tag{3.28}$$

Further, the second relation of (3.12) yields for every $\delta \in \tilde{E}(r)$,

$$|w_1(r;\delta)|\,|w_2(r;\delta)| = 4c_1c_2r \, \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right|,$$

which implies together with (3.28), for every $\delta \in \tilde{E}(r)$,

$$|w_2(r;\delta)|^2 < 4c_1c_2r \, \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| < |w_1(r;\delta)|^2. \tag{3.29}$$

Finally, with (3.16) the stated inequalities (3.25) follow from (3.29). □

II.4. Analysis of the functional equation

In this section we shall use the results of the preceding section in obtaining a functional relation between the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$. In the first place this relation is valid on the sets $\Delta_j(r)$, $j = 1,2$. It will be extended by analytic continuation of the functions $\Phi_x(r;p_1(r;\delta),0)$ and $\Phi_x(r;0,p_2(r;\delta))$ into the domain $E(r)$, and in particular to the line segment $\gamma(r)$.

Let r be fixed, $0 < |r| < 1$. Later on the discussion will be restricted to real positive values of r.

THEOREM 4.1. *For* $j = 1,2$, $\delta \in \Delta_j(r)$, $p_1 = p_{1j}(r;\delta)$, $p_2 = p_{2j}(r;\delta)$,

$$\frac{\Phi_x(r;p_1,0)}{1-p_1} + \frac{\Phi_x(r;0,p_2)}{1-p_2} = \Phi_x(r;0,0) + \frac{p_1^{x_1}\, p_2^{x_2}}{(1-p_1)(1-p_2)}, \tag{4.1}$$

except for the values of δ *corresponding to the cases* $p_1 = 1$ *and* $p_2 = 1$. *Those values of* δ *lead to the relations*

$$\Phi_x(r;0,1) = \frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}, \qquad \Phi_x(r;1,0) = \frac{[\mu_2(r)]^{x_2}}{1-\mu_2(r)}, \tag{4.2}$$

with $p_i = \mu_i(r)$ *the unique solution of the equation*

$$p_i = r\, \beta\left(\frac{1-p_i}{\alpha_i}\right), \qquad p_i \in C^+, \qquad \text{for } i = 1,2. \tag{4.3}$$

PROOF. As the generating function $\Phi_x(r;p_1,p_2)$ is finite for $|p_1| \leqslant 1$ and $|p_2| \leqslant 1$ (cf. theorem 2.3.ii) it follows from equation (2.2) that

$$p_1^{x_1+1} p_2^{x_2+1} + r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\Big[(p_2-1)\Phi_x(r;p_1,0) +$$

$$+ (p_1-1)\Phi_x(r;0,p_2) + (p_1-1)(p_2-1)\Phi_x(r;0,0)\Big] = 0,$$

for zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (2.3)(which exist according to lemma 3.1).

Substituting p_1p_2 for $r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)$ the above equation may be divided by p_1p_2 because $\beta(s)$, $\mathrm{Re}\, s \geqslant 0$, has only isolated zeros (see also lemma 3.1). Then we insert $p_1 = p_{1j}(r;\delta)$, $p_2 = p_{2j}(r;\delta)$, $j = 1$ or 2 for the zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (3.2), cf. theorem 3.1 and (3.16).

For later convenience we divide by $(1-p_1)(1-p_2)$ and we obtain equation (4.1).

From lemma 3.1 however it follows that there is a j, j = 1 or 2, and a $\delta \in \Delta_j(r)$ such that $p_{2j}(r;\delta) = 1$. Then we are not allowed to divide by $(1-p_1)(1-p_2)$ but the relation reduces to

$$p_1^{x_1} + (p_1-1)\Phi_x(r;0,1) = 0, \qquad p_1 = p_{1j}(r;\delta), \qquad p_{2j}(r;\delta) = 1,$$
$$\delta \in \Delta_j(r),\ j = 1,2.$$

By lemma 3.1 this j and δ must be unique. Further equation (3.1) reduces for $p_2 = 1$ to a wellknown equation from the theory of the M/G/1-queueing system,

$$p_1 - r\,\beta\left(\frac{1-p_1}{\alpha_1}\right) = 0, \qquad |r| < 1,\ |p_1| \leq 1.$$

This proves the first relations of (4.2) and (4.3). The other relations follow similarly. □

<u>REMARK 4.1.</u> The relation (4.2) and (4.3) can also be deduced directly from the functional equation (2.2). For instance, putting $p_2 = 1$ in (2.2) gives

$$\left[p_1 - r\,\beta\left(\frac{1-p_1}{\alpha_1}\right)\right]\Phi_x(r;p_1,1) = p_1^{x_1+1} + r\,\beta\left(\frac{1-p_1}{\alpha_1}\right)(p_1-1)\,\Phi_x(r;0,1),$$
$$|p_1| \leq 1.$$

This equation is similar to that for the generating function of the queue length process at departure epochs in an M/G/1-queueing system with mean interarrival time α_1 and service time distribution B(t), cf. COHEN [03], p. 240. From the theory of the M/G/1 – queue it is known that the kernel

$$p_1 - r\,\beta\left(\frac{1-p_1}{\alpha_1}\right)$$

has for $|r| < 1$ exactly one zero $p_1 = \mu_1(r)$ inside the unit circle, where $\mu_1(r)$ is equal to the generating function of the distribution of the number of customers served during a busy period in this M/G/1-queueing system.

In this way we also obtain the relations (4.2) and (4.3). Moreover, we find the generating functions of the marginal distributions of the series $\{\underline{x}_i(n), n = 0,1,2,\dots\}$, $i = 1,2$, e.g. for $|p_1| \leq 1$,

$$\Phi_x(r;p_1,1) = p_1^{x_1} + r\,\beta\Big(\frac{1-p_1}{\alpha_1}\Big)\frac{p_1^{x_1}[1-\mu_1(r)] - (1-p_1)[\mu_1(r)]^{x_1}}{\Big[p_1 - r\,\beta\Big(\frac{1-p_1}{\alpha_1}\Big)\Big][1-\mu_1(r)]}.$$

Theorem 4.2. *The functions* $\Phi_x(r;p_{1j}(r;\delta),0)$ *and* $\Phi_x(r;0,p_{2j}(r;\delta))$, $\delta \in \Delta_j(r)$, $j = 1,2$, *possess analytic continuations into the domain* $\tilde{E}(r)$.

PROOF. From theorem 3.2 we know that the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ are two-valued analytic functions in the domain $E(r)$. Further, the generating functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ should be regular in the unit disks $|p_1| < 1$ respectively $|p_2| < 1$, cf. theorem 2.3.ii. Thus starting from a non-empty set $\Delta_j(r)$, $j = 1$ or 2, see lemma 3.4, the function $\Phi_x(r;p_{1j}(r;\delta),0)$ is regular at all points δ for which $p_{1j}(r;\delta) \in C^+$ and possesses an analytic continuation as a two-valued function in the region of points δ for which the other branch of the function $p_1(r;\delta)$ is bounded in absolute value by one. The same procedure can be applied to the function $\Phi_x(r;0,p_{2j}(r;\delta))$, $j = 1$ or 2.

In lemma 3.3 it has been proved that on both branches at least one of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ has an absolute value bounded by one. Using equation (4.1) the function $\Phi_x(r;p_1(r;\delta),0)$ can also be continued as an analytic function where one of the branches $p_{1j}(r;\delta)$, $j = 1,2$ has an absolute value exceeding one by defining for such δ, cf. (4.1),

$$\Phi_x(r;p_{1j}(r;\delta),0) = [p_{1j}(r;\delta)]^{x_1+1}\frac{[p_{2j}(r;\delta)]^{x_2+1}}{1-p_{2j}(r;\delta)} + [1-p_{1j}(r;\delta)]\Big[\Phi_x(r;0,0) - \frac{\Phi_x(r;0,p_{2j}(r;\delta))}{1-p_{2j}(r;\delta)}\Big],\quad j = 1,2.$$

From the above it follows that if $|p_{1j}(r;\delta)| > 1$ then the righthand side of this equation is regular, and thus the analytic continuation of the function $\Phi_x(r;p_{1j}(r;\delta),0)$, $\delta \in \Delta_j(r)$, $j = 1,2$, into the domain $E(r)$ is well defined by this relation (cf. chapter I, definition I.1.7 up to definition I.1.8). Note that if $p_{2j}(r;\delta) = 1$ for some δ then by (3.13) $|p_{1j}(r;\delta)| < 1$, thus for such δ the function $\Phi_x(p_{1j}(r;\delta),0)$ is regular ($j = 1,2$).
The analytic continuation of the function $\Phi_x(r,0,p_{2j}(r;\delta))$, $\delta \in \Delta_j(r)$, $j = 1,2$ into the domain $E(r)$ can be defined similarly.
The functions $\Phi_x(r;p_{1j}(r;\delta),0)$ and $\Phi_x(r;0,p_{2j}(r;\delta))$, $\delta \in \Delta_j(r)$, $j = 1,2$, and their analytic continuation into the domain $E(r)$ will be denoted by the same symbol. □

With the above defined analytic continuations $\Phi_x(r;p_1(r;\delta),0)$ and $\Phi_x(r;0,p_2(r;\delta))$ the relation (4.1) holds for $p_1 = p_1(r;\delta)$, $p_2 = p_2(r;\delta)$, $\delta \in E(r)$. From this single functional equation we have to determine the two functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$. FAYOLLE [08] and IASNOGORODSKI [16], see also [09], consider in problems of this kind the relation between the unknown functions on the line segment joining the two branch points, and then are able to formulate Riemann-Hilbert boundary value problems (cf. section I.5).
Guided by this idea we consider equation (4.1) on the line segment $\gamma(r)$, which is possible by theorem 4.2.
As we want to apply theorem 3.3 we confine ourselves to real positive values of r.

From now on, let r *be a fixed real number,* $0 < r < 1$.

At every point on the line segment $\gamma(r)$ except at the end points $\delta_1(r)$ and $\delta_2(r)$ the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ take each two different values (cf. (3.4) and (3.5)).

In order to describe equation (4.1) on $\gamma(r)$ for both of these values of the functions $p_1(r;\delta)$ and $p_2(r;\delta)$ we shall introduce in the following a suitable parameter equation of the line segment $\gamma(r)$.
It is recalled that by definition the branch points $\delta_1(r)$ and $\delta_2(r)$ are the two roots of the equation (cf. lemma 3.2)

$$\delta^2 - 4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right) = 0, \qquad \text{Re}\ \delta < 1.$$

LEMMA 4.1. *For real* t, $-1 < t < 1$, *the equation*

$$\delta = 2t\sqrt{c_1c_2\ \beta\left(\frac{1-\delta}{\alpha}\right)}, \tag{4.4}$$

has exactly one root in the domain Re $\delta < 1$. *This root is real.*

PROOF. Replacing r by t^2 in lemma 3.2 it follows that for $-1 < t < 1$, $t \neq 0$, the equation

$$\delta^2 = 4c_1c_2t^2\ \beta\left(\frac{1-\delta}{\alpha}\right),$$

has exactly two roots in the domain Re $\delta < 1$ which are both real, while one of them is positive and the other is negative, cf. theorem 3.3.
This implies that the equations (4.4) and

$$\delta = -2t\sqrt{c_1c_2\ \beta\left(\frac{1-\delta}{\alpha}\right)},$$

each have one root in the domain Re $\delta < 1$ for $-1 < t < 1$, $t \neq 0$. For $t = 0$ equation (4.4) has the single root $\delta = 0$. □

This lemma enables us to define a function $\delta = h(t)$, $-1 < t < 1$, as the unique solution of equation (4.4) in the domain Re $\delta < 1$. Thus,

$$h(t) = 2t\sqrt{c_1c_2\ \beta\left(\frac{1-h(t)}{\alpha}\right)}, \qquad h(t) < 1, \qquad -1 < t < 1. \tag{4.5}$$

LEMMA 4.2. *The function* h(t), $-1 < t < 1$, *is bounded in absolute value* *by* $2|t|\sqrt{c_1c_2}$, *it is differentiable and strictly increasing. Further*

$$\lim_{t\uparrow 1} h(t) = 1, \qquad \text{if } c_1 = c_2 = \tfrac{1}{2} \text{ and } a \leq 2,$$
$$< 1, \qquad \text{otherwise},$$

and

$$\lim_{t\uparrow 1} h'(t) = \frac{1}{1-\frac{1}{2}a}, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2} \text{ and } a < 2,$$
$$= \infty, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2} \text{ and } a = 2,$$
$$< \infty, \qquad \text{otherwise},$$

while

$$\lim_{t\uparrow 1} \sqrt{1-t}\, h'(t) = \{\beta_2/\alpha^2 - 2\}^{-\frac{1}{2}}, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2},\ a = 2,\ \beta_2 < \infty,$$
$$\lim_{t\uparrow 1} \frac{1-t}{1-h(t)} = 0, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2},\ a = 2,$$

with β_2 *the second moment of the service time distribution* $B(t)$, cf.(1.5). *The limit of* $h'(t)$ *as* $t \downarrow -1$ *exists and is positive for all parameter values.*

The second derivative $h''(t)$ *is continuous for* $-1 \leq t \leq 1$, *except in the cases* $c_1 = c_2 = \frac{1}{2}$, $a = 2$ *and* $c_1 = c_2 = \frac{1}{2}$, $\beta_2 = \infty$ *at the point* $t = 1$, *while*

$$\lim_{t\uparrow 1} (1-t)\, h''(t) = 0, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2},\ a < 2, \beta_2 = \infty,$$
$$\lim_{t\uparrow 1} (1-t)^{\frac{3}{2}}\, h''(t) = \tfrac{1}{2}\{\beta_2/\alpha^2 - 2\}^{-\frac{1}{2}}, \qquad \text{for } c_1 = c_2 = \tfrac{1}{2},\ a = 2,\ \beta_2 < \infty.$$

<u>PROOF.</u> Because $h(t) < 1$, $-1 < t < 1$, we have $\beta\left(\frac{1-h(t)}{\alpha}\right) < 1$ so that from (4.5),

$$|h(t)| \leq 2|t|\sqrt{c_1c_2} < 2\sqrt{c_1c_2}, \qquad -1 < t < 1. \tag{4.6}$$

Differentiation of (4.5) gives formally

$$h'(t) = \frac{2\sqrt{c_1c_2\,\beta\left(\frac{1-h(t)}{\alpha}\right)}}{1 + t\sqrt{c_1c_2}\,\frac{\beta'\left(\frac{1-h(t)}{\alpha}\right)}{\alpha\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}}}. \tag{4.7}$$

Since $\beta\left(\frac{1-h(t)}{\alpha}\right) > 0$ and $\beta'\left(\frac{1-h(t)}{\alpha}\right) < 0$ for $-1 < t < 1$, it is seen that $h'(t)$ exists and $h'(t) > 0$ for $-1 < t \leq 0$.

Let $0 < t < 1$ and consider on the real interval $\delta \leq 1$ the two functions

$$\delta, \qquad \text{and} \qquad 2t\sqrt{c_1c_2\,\beta\left(\frac{1-\delta}{\alpha}\right)}.$$

Both functions are strictly increasing for $\delta \leq 1$. Because $\beta\left(\frac{1-\delta}{\alpha}\right)$ is positive for $\delta \leq 1$ it follows that at the point of intersection $\delta = h(t)$, cf. lemma 4.1, of these two functions the derivative of δ has to be the greater (see figure 4.1), i.e.

$$1 > 2t\sqrt{c_1c_2}\,\frac{-\beta'\left(\frac{1-\delta}{\alpha}\right)}{2\alpha\sqrt{\beta\left(\frac{1-\delta}{\alpha}\right)}}, \qquad \delta = h(t), \qquad 0 < t < 1. \tag{4.8}$$

Note that we cannot have equality in (4.8) because the root $\delta = h(t)$ of equation (4.4) is simple for $0 < t < 1$. From (4.8) and (4.7) it follows that $h'(t) > 0$ exists for $0 < t < 1$. Hence, the function $h(t)$ is differentiable and strictly increasing for $-1 < t < 1$.

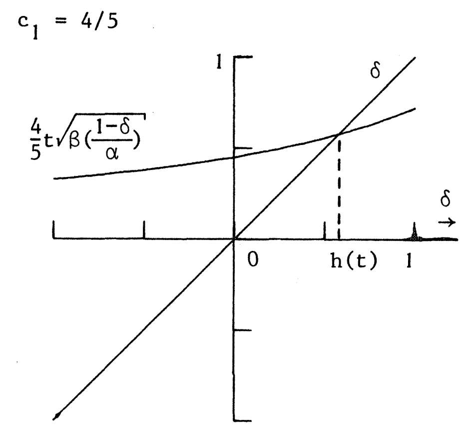


Figure 4.1. $h(t)$ for $t>0$

Because the function $h(t)$ is uniformly bounded, cf. (4.6), and strictly increasing for $-1 < t < 1$, the limiting values of this function exist as $t \downarrow -1$ and $t \uparrow 1$. The inequality (4.6) implies that $h(1) < 1$ if

$c_1 \neq c_2$. If $c_1 = c_2 = \frac{1}{2}$ then $\delta = 1$ is a root of equation (4.4) with $t = 1$. For $\delta = 1$ to be the smallest root of this equation it is necessary that, cf. (4.8),

$$1 \geqslant \tfrac{1}{2}a.$$

Thus if $c_1 = c_2 = \frac{1}{2}$ then $h(1) < 1$ for $a > 2$ and $h(1) = 1$ for $a \leqslant 2$. If $h(1) < 1$ then $h(1)$ is a simple root of equation (4.4) with $t = 1$, which implies that $h'(1)$ is finite, cf. (4.8), for $c_1 \neq c_2$ and for $c_1 = c_2 = \frac{1}{2}$, $a > 2$. If $h(1) = 1$, thus $c_1 = c_2 = \frac{1}{2}$, $a \leqslant 2$, it follows from (4.7) that

$$\begin{aligned} h'(1) &= \frac{1}{1-\frac{1}{2}a} < \infty, &\quad &\text{for } a < 2, \\ &= \infty, &\quad &\text{for } a = 2. \end{aligned}$$

In the latter case, $c_1 = c_2 = \frac{1}{2}$, $a = 2$, we have for $\delta \to 1$,

$$\sqrt{\beta\left(\frac{1-\delta}{\alpha}\right)} = 1 - \tfrac{1}{2}a(1-\delta) + o((1-\delta)) = \delta + o((1-\delta)), \tag{4.9}$$

so that equation (4.5) leads to

$$h(t) = t\,h(t) + t\,o([1-h(t)]), \qquad h(t) \to 1.$$

Since $h(1) = 1$ this implies

$$\lim_{t\uparrow 1} \frac{1-t}{1-h(t)} = 0.$$

Further, if $\beta_2 < \infty$, we have for $\delta \to 1$,

$$\sqrt{\beta\left(\frac{1-\delta}{\alpha}\right)} = 1 - \tfrac{1}{2}a(1-\delta) + \tfrac{1}{4}\left[\beta_2/\alpha^2 - \tfrac{1}{2}a^2\right](1-\delta)^2 + o((1-\delta)^2), \tag{4.10}$$

so that for $a = 2$, $c_1 = c_2 = \frac{1}{2}$, equation (4.5) leads to

$$h(t) = t\left[h(t) + \tfrac{1}{4}\left[\beta_2/\alpha^2 - 2\right](1-h(t))^2 + o((1-h(t))^2)\right], \qquad h(t) \to 1,$$

from which it is readily seen $\left(\beta_2/\alpha^2 \geqslant 4 \text{ for } a = 2\right)$ that

$$\lim_{t\uparrow 1} \frac{1-h(t)}{\sqrt{1-t}} = 2\left\{\beta_2/\alpha^2-2\right\}^{-\frac{1}{2}}, \tag{4.11}$$

which implies

$$\lim_{t\uparrow 1} \sqrt{1-t}\; h'(t) = \left\{\beta_2/\alpha^2-2\right\}^{-\frac{1}{2}}. \tag{4.12}$$

The existence of $h'(-1)$ is obvious from (4.7).

Finally we consider the second derivative $h''(t)$. Differentiation of (4.7) shows that the denominator of $h''(t)$ only contains powers of

$$1 + t\sqrt{c_1c_2}\,\frac{\beta'\left(\frac{1-h(t)}{\alpha}\right)}{\alpha\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}},$$

so that it follows from the above that $h''(t)$ exists and is continuous for $-1 \leqslant t < 1$, and that $h''(1)$ is finite for $c_1 \neq c_2$, for $c_1 = c_2 = \frac{1}{2}$, $a > 2$, (in these cases $h(1) < 1$), and for $c_1 = c_2 = \frac{1}{2}$, $a < 2$, $\beta_2 < \infty$.
If $c_1 = c_2 = \frac{1}{2}$ the above leads to

$$\left[1+\tfrac{1}{2}\frac{t}{\alpha}\,\frac{\beta'\left(\frac{1-h(t)}{\alpha}\right)}{\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}}\right] h''(t) = -\,\frac{h'(t)}{\alpha}\left[\frac{\beta'\left(\frac{1-h(t)}{\alpha}\right)}{\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}} - \right.$$

$$\left. -\,\frac{th'(t)}{2\alpha}\left\{\frac{\beta''\left(\frac{1-h(t)}{\alpha}\right)}{\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}} - \tfrac{1}{2}\,\frac{\left[\beta'\left(\frac{1-h(t)}{\alpha}\right)\right]^2}{\left[\beta\left(\frac{1-h(t)}{\alpha}\right)\right]^{3/2}}\right\}\right]. \tag{4.13}$$

Hence for $a < 2$, $\beta_2 < \infty$ we have

$$\lim_{t\uparrow 1} h''(t) = \frac{\frac{1}{2}\beta_2/\alpha^2+a-\frac{3}{4}a^2}{(1-\frac{1}{2}a)^2}. \tag{4.14}$$

If $a < 2$, $\beta_2 = \infty$ this limit becomes infinite, but since $\beta'(0) = -\beta$ is finite we must have

$$\lim_{s \to 0} s\beta''(s) = 0.$$

With this it follows from (4.13) that for $a < 2$, $\beta_2 = \infty$,

$$\lim_{t \uparrow 1} (1-t)h''(t) = 0.$$

Finally, if $a = 2$, $\beta_2 < \infty$ it follows readily from (4.13), (4.12) and (4.7) that

$$\lim_{t \uparrow 1} (1-t)^{3/2} h''(t) = \tfrac{1}{2}\left\{\beta_2/\alpha^2 - 2\right\}^{-\frac{1}{2}}. \qquad \square$$

Dividing (4.5) by t and differentiating it we obtain the useful relation

$$\frac{th'(t)-h(t)}{t^2} = -\frac{h'(t)}{\alpha}\sqrt{c_1c_2}\,\frac{\beta'\left(\frac{1-h(t)}{\alpha}\right)}{\sqrt{\beta\left(\frac{1-h(t)}{\alpha}\right)}}, \quad -1 < t < 1, \ t \neq 0. \tag{4.15}$$

<u>LEMMA 4.3.</u> *The mapping* $\delta = h(t\sqrt{r})$ *is a one-to-one correspondence between the real intervals* $-1 \leqslant t \leqslant 1$ *and* $\gamma(r)$.

<u>PROOF.</u> By lemma 4.2 the function $h(t)$, $-1 < t < 1$, is strictly increasing. Consequently the mapping $\delta = h(t\sqrt{r})$ is a one-to-one correspondence between the intervals $-1 \leqslant t \leqslant 1$ and $h(-\sqrt{r}) \leqslant \delta \leqslant h(\sqrt{r})$. From (4.5) and the definition of the branch points $\delta_1(r)$ and $\delta_2(r)$, cf. lemma 3.2, it follows that

$$h(-\sqrt{r}) = \delta_1(r), \qquad h(\sqrt{r}) = \delta_2(r). \tag{4.16}$$

This implies that $\delta = h(t\sqrt{r})$ maps the interval $-1 \leqslant t \leqslant 1$ onto $\gamma(r)$. $\square$

As a consequence of this lemma, by putting

$$\delta = h(\cos\theta\,\sqrt{r}), \qquad -\pi \leqslant \theta \leqslant \pi, \tag{4.17}$$

the interval $\gamma(r)$ is covered twice if θ increases from $-\pi$ to π. Inserting (4.17) into equation (4.1) leads to the following result.

THEOREM 4.3. *The functions* $\Phi_x(r;p_1,0)$ *and* $\Phi_x(r;0,p_2)$ *satisfy the equation*

$$\frac{\Phi_x(r;\frac{w}{2c_1},0)}{1-\frac{w}{2c_1}}+\frac{\Phi_x(r;0,\frac{\bar{w}}{2c_2})}{1-\frac{\bar{w}}{2c_2}}=\Phi_x(r;0,0)+\frac{\left(\frac{w}{2c_1}\right)^{x_1}\left(\frac{\bar{w}}{2c_2}\right)^{x_2}}{(1-\frac{w}{2c_1})(1-\frac{\bar{w}}{2c_2})},\qquad (4.18)$$

for

$$w=\frac{h(\cos\theta\sqrt{r})}{\cos\theta}\,e^{i\theta},\qquad -\pi\leqslant\theta\leqslant\pi.\qquad (4.19)$$

PROOF. In theorem 4.2 relation (4.1) has been extended from the regions $\Delta_j(r)$, $j = 1,2$, to the domain $\tilde{E}(r)$, for $j = 1,2$. When δ tends to the line segment $\gamma(r)$ from the left or from the right the limiting values of the functions $2c_1p_{1j}(r;\delta)$, $j = 1,2$, are, cf. (3.16), (3.22),

$$\delta\pm i\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right)-\delta^2},\qquad \delta\in\gamma(r).\qquad (4.20)$$

By lemma 4.3 every $\delta\in\gamma(r)$ can be described by $\delta = h(t\sqrt{r})$ for some t, $-1\leqslant t\leqslant 1$. Substituting this in (4.20) the limiting values become, using the definition of $h(t)$, cf. (4.5),

$$h(t\sqrt{r})\pm i\sqrt{h^2(t\sqrt{r})\left[\frac{1}{t^2}-1\right]},\qquad -1\leqslant t\leqslant 1.$$

Inserting $t = \cos\Theta$ we obtain, since $h(t)/t > 0$, $-1\leqslant t\leqslant 1$, for $-\pi\leqslant\theta\leqslant 0$,

$$h(\cos\theta\sqrt{r})-i\sqrt{h^2(\cos\theta\sqrt{r})\,\tan^2\theta}=h(\cos\theta\sqrt{r})\,[1+i\tan\theta]=\frac{h(\cos\theta\sqrt{r})}{\cos\theta}e^{i\theta},$$

and for $0\leqslant\theta\leqslant\pi$,

$$h(\cos\theta\sqrt{r})+i\sqrt{h^2(\cos\theta\sqrt{r})\,\tan^2\theta}=h(\cos\theta\sqrt{r})\,[1+i\tan\theta]=\frac{h(\cos\theta\sqrt{r})}{\cos\theta}e^{i\theta}.$$

Hence, for every θ, $-\pi \leqslant \theta \leqslant \pi$, the value of w defined by (4.19) is a limiting value of $2c_1p_{1j}(r;\delta)$, $j = 1,2$, on $\gamma(r)$, and every limiting value of $2c_1p_{1j}(r;\delta)$, $j = 1,2$, on $\gamma(r)$ is described by (4.19), cf. (4.20). This implies that we can substitute in relation (4.1):

$$p_1 = \frac{w}{2c_1}, \qquad w = \frac{h(\cos\theta\sqrt{r})}{\cos\theta}\, e^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi,$$

for the limiting values on $\gamma(r)$. By theorem 3.3 we must have then $p_2 = \frac{\bar{w}}{2c_2}$. This proves the assertion. □

When θ increases from $-\pi$ to π the variable w defined by (4.19) describes a closed curve. This curve will be denoted by:

$$L(r) := \{w;\ w = \frac{h(\cos\theta\sqrt{r})}{\cos\theta}\, e^{i\theta},\ -\pi \leqslant \theta \leqslant \pi\}. \tag{4.21}$$

From equation (4.5) it follows that $h(0) = 0$ and that

$$\lim_{t\to 0} \frac{h(t\sqrt{r})}{t} = 2\sqrt{c_1c_2r\ \beta\left(\frac{1}{\alpha}\right)}, \tag{4.22}$$

$$\frac{h(\cos\theta\sqrt{r})}{\cos\theta} = 2\sqrt{c_1c_2r\ \beta\left(\frac{1-h(\cos\theta\sqrt{r})}{\alpha}\right)} > 0, \qquad -\pi \leqslant \theta \leqslant \pi. \tag{4.23}$$

This implies that the curve L(r) is bounded also when $\cos\theta$ vanishes and that it is non-intersecting. Hence, L(r) is a *contour*, cf. section I.1. By lemma 4.3 the contour L(r) may also be represented as, cf. (4.20),

$$L(r) = \{w;\ w = \delta \pm i\sqrt{4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2},\quad \delta \in \gamma(r)\}. \tag{4.24}$$

Before we shall deduce boundary value problems from relation (4.18) in the sections II.6 and II.7 first some properties of the contour L(r) and of the conformal mapping of its interior onto the unit disk will be discussed in the next section.

II.5. The contour L(r) and its interior

Some properties of the closed curve L(r) defined in (4.21) and of the conformal mapping of the unit disk C^+ onto the domain $L^+(r)$ will be discussed. It will be shown that this conformal mapping is regular on C, and that its inverse can be continued as a regular function into the disk $|w| < 1 + \sqrt{1-4c_1c_2r}$, except possibly for a finite number of poles. Further it will be investigated on which conditions $2c_2 \in L^+(r)$. Finally it will be proved that the functions $\Phi_x(r;\frac{w}{2c_1},0)$ and $\Phi_x(r;0,\frac{w}{2c_2})$ are regular in the domain $L^+(r)$.

Unless stated otherwise r *is assumed to be fixed and real,* $0 < r < 1$.

LEMMA 5.1. *The contour* L(r) *defined in* (4.21) *is smooth.*

PROOF. From the definition (4.21) it is seen that L(r) has the parametric equation

$$w = x(\theta) + iy(\theta),$$
$$x(\theta) = h(\cos\theta\sqrt{r}), \quad y(\theta) = \tan\theta\ h(\cos\theta\sqrt{r}), \quad -\pi \leqslant \theta \leqslant \pi. \tag{5.1}$$

By lemma 4.2 the function $h(\cos\theta\sqrt{r})$, and thus $x(\theta)$, is differentiable for every θ. For $y(\theta)$ we have for $-\pi \leqslant \theta \leqslant \pi$, $\theta \neq \pm \frac{1}{2}\pi$,

$$y'(\theta) = \frac{h(\cos\theta\sqrt{r})}{\cos^2\theta} - h'(\cos\theta\sqrt{r})\ \frac{\sin^2\theta}{\cos\theta}\sqrt{r}.$$

By using (4.15) it is obtained:

$$\lim_{\theta\to\pm\frac{1}{2}\pi} y'(\theta) = \frac{r}{\alpha}\sqrt{c_1c_2}\ \frac{\beta'\left(\frac{1}{\alpha}\right)}{\sqrt{\beta\left(\frac{1}{\alpha}\right)}}\ h'(0) = 2\frac{r}{\alpha}c_1c_2\beta'\left(\frac{1}{\alpha}\right),$$

here we have used (4.7) for h'(0). Hence also $y(\theta)$ is differentiable for all θ.

Futher, since $h'(t) > 0$, $-1 \leqslant t < 1$, cf. lemma 4.2, the derivative

$$x'(\theta) = -\sqrt{r}\,\sin\theta\;\; h'(\cos\theta\sqrt{r}),$$

only vanishes when sin θ = 0, and it can be readily seen that $y'(\theta)$ does not vanish for $\theta = -\pi, 0, \pi$.

This proves that L(r) is a smooth contour, cf. definition I.1.3. □

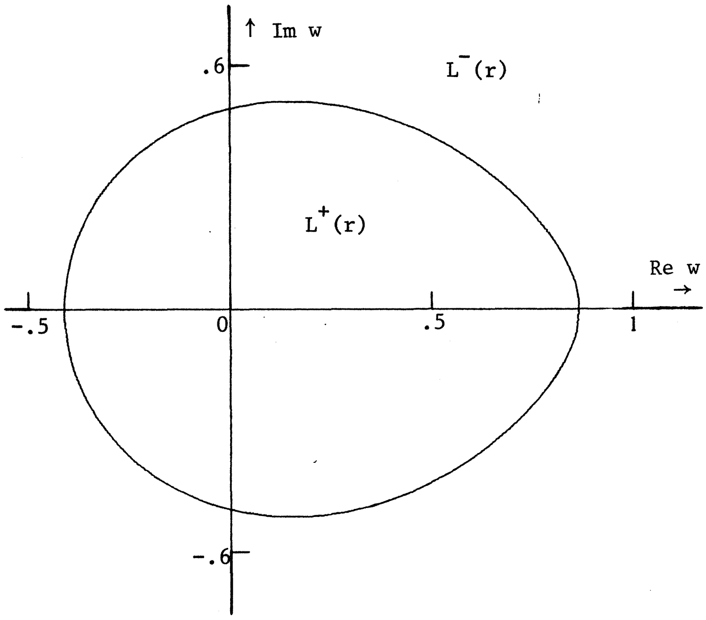


Figure 5.1.

The contour L(r) for a = 1.96, $4c_1c_2r$ = 0.96 and an Erlang-2 service time distribution

According to the parametric equation (5.1) a positive direction is defined on the contour L(r), cf. section I.1. From the proof of lemma 5.1 it is readily deduced that this direction is counter clockwise. By definition I.1.2 the interior of L(r) will be denoted by $L^+(r)$, the exterior by $L^-(r)$.

COROLLARY 5.1. *The interior* $L^+(r)$ *of the contour* L(r) *is a simply connected domain.*

PROOF. This is a consequence of the Jordan curve theorem (cf. lemma I.1.1). See also the definition above lemma I.1.3. □

LEMMA 5.2. *The contour* L(r) *is equivalent to*

$$\{w;\ |w|^2 = 4c_1c_2r\ \beta\left(\frac{1-\mathrm{Re}\,w}{\alpha}\right),\quad \mathrm{Re}\,w < 1\}.$$

PROOF. From (4.24) it follows that for every $w \in L(r)$ there exists a $\delta \in \gamma(r)$ such that $\mathrm{Re}\,w = \delta < 1$ and

$$|w|^2 = \delta^2 + 4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2 = 4c_1c_2r\ \beta\left(\frac{1-\mathrm{Re}\,w}{\alpha}\right).$$

On the other hand, let some w, $\mathrm{Re}\,w < 1$, satisfy

$$|w|^2 = 4c_1c_2r\ \beta\left(\frac{1-\mathrm{Re}\,w}{\alpha}\right), \tag{5.2}$$

or equivalently,

$$\{\mathrm{Im}\ w\}^2 = 4c_1c_2r\ \beta\left(\frac{1-\mathrm{Re}\,w}{\alpha}\right) - \{\mathrm{Re}\,w\}^2.$$

This equation has only roots if $\mathrm{Re}\ w \in \gamma(r)$, cf. lemma 3.2. Putting $\delta = \mathrm{Re}\ w$ it follows that

$$\{\mathrm{Im}\ w\}^2 = 4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2,$$

which shows that $w \in L(r)$, cf. (4.24). □

COROLLARY 5.2. *For every* $w \in L^+(r)$,

$$|w| < \delta_2(r) < 1.$$

PROOF. This is a consequence of lemma 5.2, the monotonicity of $\beta\left(\frac{1-\delta}{\alpha}\right)$ for real δ, $\delta < 1$, and lemma 3.2 which imply for $w \in L^+(r)$,

$$|w| < 2\sqrt{c_1c_2r\ \beta\left(\frac{1-\mathrm{Re}\,w}{\alpha}\right)} \leq 2\sqrt{c_1c_2r\ \beta\left(\frac{1-\delta_2(r)}{\alpha}\right)} = \delta_2(r) < 1. \quad □$$

COROLLARY 5.3. *If* $r_1 > r_2$ *then* $L(r_1) \subset L^-(r_2)$. *(See the figure on p. 238.)*

PROOF. From lemma 4.2 it follows readily that for every θ, $-\pi \leq \theta \leq \pi$,

$$\frac{d}{dr}\frac{h(\cos\theta\sqrt{r})}{\cos\theta} > 0, \qquad \text{for } 0 < r < 1.$$

With (4.21) it is then clear that the contour L(r) expands in every direction with increasing value of r, $0 < r < 1$, as asserted. □

Relation (4.23) implies that the contour L(r), cf. (4.21) has a representation of the form of formula (I.6.4), and that $0 \in L^+(r)$, so that we can apply the procedure of Theodorsen (cf. section I.6) for the determination of the conformal mapping of the unit disk C^+ onto the domain $L^+(r)$.

THEOREM 5.1. *There exists a conformal mapping* g(r;z) *of the unit disk* C^+ *onto the domain* $L^+(r)$ *which is uniquely determined by the conditions*

$$g(r;0) = 0, \qquad g'(r;0) > 0. \tag{5.3}$$

This conformal mapping is given by, for $|z| < 1$,

$$g(r;z) = z \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{h(\cos\theta(r;\varphi)\sqrt{r})}{\cos\theta(r;\varphi)}\right]\frac{e^{i\varphi}+z}{e^{i\varphi}-z}\,d\varphi\right\}, \tag{5.4}$$

where $\theta(r;\varphi)$, $-\pi \leq \varphi \leq \pi$, *is the unique strictly increasing solution of the non-linear singular integral equation of* Theodorsen *for* $-\pi \leq \varphi \leq \pi$,

$$\theta(r;\varphi) = \varphi - \frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{h(\cos\theta(r;\omega)\sqrt{r})}{\cos\theta(r;\omega)}\right]\cot\left(\frac{\omega-\varphi}{2}\right)d\omega. \tag{5.5}$$

REMARK. Throughout we shall use the notation

$$g'(r;z) := \frac{\partial}{\partial z}g(r;z), \qquad |z| \leq 1. \tag{5.6}$$

The derivative of g(r;z) as a function of r will be denoted by

$$g_r(r;z) := \frac{\partial}{\partial r}g(r;z), \qquad 0 < r < 1,\ |z| \leq 1. \tag{5.7}$$

PROOF OF THEOREM 5.1. By corollary 5.1 the domain $L^+(r)$ is simply connected, so that the existence of a conformal correspondence with the unit disk follows from Riemann's mapping theorem (lemma I.6.2). The uniqueness of the conformal mapping $g(r;z)$, $z \in C^+$, satisfying the conditions (5.3) is ensured by lemma I.6.3.

As noted above this theorem the contour $L(r)$ has a representation of the form of formula (I.6.4). Referring to section I.6 the formulas (5.4) and (5.5) follow from (I.6.8) and (I.6.7), with cf. (4.21),

$$\rho(\theta) = \frac{h(\cos\theta\sqrt{r})}{\cos\theta}, \qquad -\pi \leqslant \theta \leqslant \pi. \tag{5.8}$$

Finally, lemma I.6.8 proves the uniqueness of the solution $\theta(r;\varphi)$ of equation (5.5) in the class of continuous, strictly increasing functions on $[-\pi,\pi]$. □

COROLLARY 5.4. *The conformal mapping* $g(r;z)$ *is continuous up to the boundary* C. *It establishes a one-to-one correspondence between* $C^+ \cup C$ *and* $L^+(r) \cup L(r)$. *On the unit circle it is given by*

$$g(r;e^{i\varphi}) = \frac{h(\cos\theta(r;\varphi)\sqrt{r})}{\cos\theta(r;\varphi)}\, e^{i\theta(r;\varphi)}, \qquad -\pi \leqslant \varphi \leqslant \pi.$$

PROOF. Because $L(r)$ is a contour, lemma I.6.4 proves that $g(r;z)$ is continuous in $C^+ \cup C$ and that it establishes a one-to-one correspondence between $C^+ \cup C$ and $L^+(r) \cup L(r)$. The formula for $g(r;e^{i\varphi})$, $-\pi \leqslant \varphi \leqslant \pi$, follows from (I.6.6) and (5.8), cf. the proof of theorem 5.1. □

Next a symmetry property of the contour $L(r)$ and of the conformal mapping $g(r;z)$ will be proved, which will be of great importance in later sections. For the following result it is necessary that r is real and positive.

THEOREM 5.2. *The real axis is an axis of symmetry of the contour* L(r). *The conformal mapping* g(r;z) *of the unit circle* C^+ *onto the domain* $L^+(r)$, *given by* (5.4), *satisfies for* $z \in C^+ \cup C$,

$$g(r;\bar{z}) = \overline{g(r;z)}.$$

PROOF. Because the function cos θ is even it follows from (5.8) that $\rho(\theta) = \rho(-\theta)$ for every θ. This implies that the real axis is an axis of symmetry of the contour L(r). Moreover, lemma I.6.9 then leads to the stated property of the conformal mapping g(r;z). □

As a consequence of this theorem the conformal mapping g(r;z) maps real points onto real points. In particular, by (5.3), see also (4.21) and (4.16), for $0 < r < 1$,

$$\begin{aligned} g(r;1) &= \delta_2(r) = h(\sqrt{r}), \\ g(r;-1) &= \delta_1(r) = h(-\sqrt{r}). \end{aligned} \tag{5.9}$$

From corollary 5.4 it follows that the conformal mapping g(r;z) has an inverse mapping in $L^+(r) \cup L(r)$. This inverse function will be denoted by $g_0(r;w)$. Clearly, the function $g_0(r;w)$ performs a conformal mapping of the domain $L^+(r)$ onto the unit disk C^+, and is continuous in $L^+(r) \cup L(r)$.

THEOREM 5.3. *For every* $z \in C^+ \cup C$, g(r;z) *is a continuous function of* $r, 0 < r < 1$.
For every $r_0, 0 < r_0 < 1$, *and every* $w \in L^+(r_0) \cup L(r_0)$, $g_0(r,w)$ *is continuous at* $r = r_0$.

PROOF. In corollary 5.3 it has been shown that the domain $L^+(r)$ expands monotonicly with increasing $r, 0 < r < 1$. Let r_0 be fixed, $0 < r_0 < 1$.

Because h(t) is a continuous, strictly increasing function for $-1 < t < 1$, cf. lemma 4.2, it follows from (4.21), that

$$L^+(r_0) = \bigcup_{0 < r < r_0} L^+(r), \qquad L^+(r_0) = \bigcap_{r_0 < r < 1} L^+(r). \quad (5.10)$$

Application of lemma I.6.11 gives that as well for $r \uparrow r_0$ as for $r \downarrow r_0$,

$$g(r;z) \to g(r_0;z), \tag{5.11}$$

uniformly in C^+. It is readily verified that (5.11) then also holds on C. By the same lemma I.6.11 it follows that for $r \uparrow r_0$ and for $r \downarrow r_0$,

$$g_0(r;w) \to g_0(r_0;w), \tag{5.12}$$

uniformly in $L^+(r)$, and hence also on $L(r)$. □

Next it will be shown that the conformal mapping $g(r;z)$ can be continued as a regular function outside C^+, and that its inverse $g_0(r;w)$ can be continued as a regular function outside $L^+(r)$. For this we need some preliminary lemmas.

LEMMA 5.3. *There exists a subdomain* $S(r)$ *of the domain* $\text{Re}\,\delta < 1$, *such that* $\gamma(r) \subset S(r)$ *and such that for* $\delta \in S(r)\backslash\gamma(r)$,

$$w_1(r;\delta) \in L^-(r), \qquad w_2(r;\delta) \in L^+(r).$$

PROOF. From (3.22) it follows that for $\delta \in \gamma(r)$, $\delta \neq \delta_1(r)$, $\delta \neq \delta_2(r)$,

$$\frac{\partial}{\partial\delta} w_1^+(r;\delta) = \frac{\partial}{\partial\delta} w_2^-(r;\delta) = 1 - i\,\frac{2c_1c_2\frac{r}{\alpha}\beta'\left(\frac{1-\delta}{\alpha}\right) + \delta}{\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2}},$$

$$\frac{\partial}{\partial\delta} w_1^-(r;\delta) = \frac{\partial}{\partial\delta} w_2^+(r;\delta) = 1 + i\,\frac{2c_1c_2\frac{r}{\alpha}\beta'\left(\frac{1-\delta}{\alpha}\right) + \delta}{\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2}}. \tag{5.13}$$

This implies with (3.22), for $\delta \in \gamma(r)$, $\delta \neq \delta_1(r)$, $\delta \neq \delta_2(r)$, for $\varepsilon > 0$,

$$w_1(r;\delta+i\varepsilon) = w_1^+(r;\delta) + i\varepsilon \frac{\partial}{\partial\delta} w_1^+(r;\delta) + O(\varepsilon^2) =$$

$$= \delta + i\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2} + i\varepsilon + \varepsilon\,\frac{2c_1c_2\frac{r}{\alpha}\beta'\left(\frac{1-\delta}{\alpha}\right) + \delta}{\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2}} + O(\varepsilon^2),$$

$$\varepsilon \to 0;$$

$$w_1(r;\delta-i\varepsilon) = w_1^-(r;\delta) - i\varepsilon \frac{\partial}{\partial\delta} w_1^-(r;\delta) + O(\varepsilon^2) =$$

$$= \delta - i\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2} - i\varepsilon + \varepsilon\,\frac{2c_1c_2\frac{r}{\alpha}\beta'\left(\frac{1-\delta}{\alpha}\right) + \delta}{\sqrt{4c_1c_2r\,\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2}} + O(\varepsilon^2),$$

$$\varepsilon \to 0; \qquad (5.14)$$

Further we obtain from (3.14), for $\varepsilon > 0$ and $-\pi < \varphi < \pi$,

$$w_1(r;\delta_2(r)+\varepsilon e^{i\varphi}) = \delta_2(r) + \sqrt{\varepsilon}\, e^{\frac{1}{2}i\varphi} \sqrt{2\delta_2(r)+\frac{4}{\alpha}c_1c_2r\,\beta'\left(\frac{1-\delta_2(r)}{\alpha}\right)} + O(\varepsilon),$$

$$\varepsilon \to 0;$$

$$w_1(r;\delta_1(r)-\varepsilon e^{i\varphi}) = \delta_1(r) - \sqrt{\varepsilon}\, e^{\frac{1}{2}i\varphi} \sqrt{\left|2\delta_1(r)+\frac{4}{\alpha}c_1c_2r\,\beta'\left(\frac{1-\delta_1(r)}{\alpha}\right)\right|} + O(\varepsilon),$$

$$\varepsilon \to 0; \qquad (5.15)$$

From (5.14) and (5.15) it is seen that there exists a subdomain $S_1(r)$ of the domain $\text{Re}\,\delta < 1$ with $\gamma(r) \subset S_1(r)$ such that for every $\delta \in S_1(r)\backslash\gamma(r)$,

$$w_1(r;\delta) \in L^-(r).$$

In a similar way it can be deduced from (5.13), (3.22) and (3.14) that there exists a subdomain $S_2(r)$ of the domain $\text{Re}\,\delta < 1$ with $\gamma(r) \subset S_2(r)$ such that for every $\delta \in S_2(r)\backslash\gamma(r)$,

$$w_2(r;\delta) \in L^+(r).$$

Taking $S(r) = S_1(r) \cap S_2(r)$ the assertion follows. □

LEMMA 5.4. *For every* $\delta \in \tilde{E}(r)$ *for which* $\mathrm{Re}\; w_1(r;\delta) \leq 1$ *either* $w_1(r;\delta) \in L^+(r) \cup L(r)$ *or* $w_2(r;\delta) \in L^+(r) \cup L(r)$.

PROOF. From (3.12) it follows that for every $\delta \in \tilde{E}(r)$,

$$|w_1(r;\delta)|\,|w_2(r;\delta)| = 4c_1c_2r\left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| \leq 4c_1c_2r\;\beta\left(\frac{1-\mathrm{Re}\;\delta}{\alpha}\right),$$

$$\mathrm{Re}\; w_1(r;\delta) - \mathrm{Re}\;\delta = \mathrm{Re}\;\delta - \mathrm{Re}\; w_2(r;\delta). \tag{5.16}$$

From the Schwarz inequality, cf. BURRILL [02], theorem 9-3A, we obtain for real s, $s > 0$, and for real h, $-s \leq h \leq s$, the inequality

$$[\beta(s)]^2 \leq \beta(s-h)\beta(s+h). \tag{5.17}$$

From (3.29) it follows that for every $\delta \in \tilde{E}(r)$,

$$|w_2(r;\delta)| < 4c_1c_2r < 1.$$

Consequently (5.17) and the second relation of (5.16) imply that for every $\delta \in \tilde{E}(r)$ for which $\mathrm{Re}\; w_1(r;\delta) \leq 1$,

$$\left[\beta\left(\frac{1-\mathrm{Re}\;\delta}{\alpha}\right)\right]^2 \leq \beta\left(\frac{1-\mathrm{Re}\; w_1(r;\delta)}{\alpha}\right)\beta\left(\frac{1-\mathrm{Re}\; w_2(r;\delta)}{\alpha}\right),$$

so that with the first relation of (5.16), for every $\delta \in \tilde{E}(r)$ for which $\mathrm{Re}\; w_1(r;\delta) \leq 1$,

$$|w_1(r;\delta)|\,|w_2(r;\delta)| \leq 4c_1c_2r\sqrt{\beta\left(\frac{1-\mathrm{Re}\; w_1(r;\delta)}{\alpha}\right)\beta\left(\frac{1-\mathrm{Re}\; w_2(r;\delta)}{\alpha}\right)}. \tag{5.18}$$

Therefore, if $\delta \in \tilde{E}(r)$ and $\mathrm{Re}\; w_1(r;\delta) \leq 1$ at least one of the two relations

$$|w_1(r;\delta)|^2 \leq 4c_1c_2r\;\beta\left(\frac{1-\mathrm{Re}\; w_1(r;\delta)}{\alpha}\right),$$

$$|w_2(r;\delta)|^2 \leq 4c_1c_2r\ \beta\left(\frac{1-\text{Re } w_2(r;\delta)}{\alpha}\right)$$

must hold, which implies with lemma 5.2 that either $w_1(r;\delta) \in L^+(r) \cup L(r)$ or $w_2(r;\delta) \in L^+(r) \cup L(r)$. □

Note that it follows from (5.18) that if $w_1(r;\delta) \in L^-(r)$ for some $\delta \in \tilde{E}(r)$ for which $\text{Re } w_1(r;\delta) \leq 1$ then $w_2(r;\delta) \in L^+(r)$.

LEMMA 5.5. *For* $\delta \in \tilde{E}(r)$, $|\delta| > \delta_2(r)$,

$$|w_1(r;\delta)| > |\delta| > \delta_2(r); \tag{5.19}$$

for every $\delta \in \tilde{E}(r)$, *for which* $\text{Re } w_1(r;\delta) \leq 1$,

$$|w_2(r;\delta)| < \delta_2(r). \tag{5.20}$$

PROOF. On the real interval $\delta_2(r) < \delta < 1$ the function (3.21) is positive, cf. the proof of theorem 3.3, so that for $\delta_2(r) < \delta < 1$,

$$4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right) < \delta^2.$$

Hence for $\delta_2(r) < |\delta| < 1$,

$$4c_1c_2r\ \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| \leq 4c_1c_2r\ \beta\left(\frac{1-|\delta|}{\alpha}\right) < |\delta|^2,$$

while for $|\delta| \geq 1$, $\delta \in \tilde{E}(r)$,

$$4c_1c_2r\ \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right| \leq 4c_1c_2r < 1 \leq |\delta|^2.$$

This implies with (3.12) and (3.29) that for $\delta \in \tilde{E}(r)$, $|\delta| > \delta_2(r)$,

$$|2\delta - w_1(r;\delta)| = |w_2(r;\delta)| < 2\sqrt{c_1c_2r\ \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right|} < |\delta|. \tag{5.21}$$

Hence, (5.19) follows. Moreover, (5.19) and corollary 5.2 imply that

$w_1(r;\delta) \in L^-(r)$ for $\delta \in \tilde{E}(r)$, $|\delta| > \delta_2(r)$, so that by the remark above this lemma $w_2(r;\delta) \in L^+(r)$ for $\delta \in \tilde{E}(r)$, $|\delta| > \delta_2(r)$ and Re $w_1(r;\delta) \leqslant 1$. Again by corollary 5.2 it is obtained that (5.20) holds for $\delta \in \tilde{E}(r)$, $|\delta| > \delta_2(r)$, Re $w_1(r;\delta) \leqslant 1$.

Finally, for $\delta \in \tilde{E}(r)$, $|\delta| \leqslant \delta_2(r)$ inequality (5.20) follows from (3.29):

$$|w_2(r;\delta)| < 2\sqrt{c_1 c_2 r \left|\beta\left(\frac{1-\delta}{\alpha}\right)\right|} \leqslant 2\sqrt{c_1 c_2 r\, \beta\left(\frac{1-\delta_2(r)}{\alpha}\right)} = \delta_2(r). \qquad \square$$

<u>LEMMA 5.6.</u> *For* $\delta_2(r) < |w| \leqslant 1$ *equation (3.5) has exactly one root* $\delta = \delta_0(r;w)$ *which has an absolute value smaller than that of* w. *This root* $\delta_0(r;w)$ *is regular in the annulus* $\delta_2(r) < |w| < 1$.

<u>PROOF.</u> Let w be fixed, $\delta_2(r) < |w| \leqslant 1$. On the circle $|\delta| = |w|$ then the following inequality holds,

$$\left|w^2+4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right)\right| \leqslant |w|^2+4c_1c_2r\ \beta\left(\frac{1-|w|}{\alpha}\right) < 2|w|^2 = |2\delta w|.$$

Application of Rouché's theorem (cf. lemma I.1.4) to the functions $w^2 + 4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right)$ and $2\delta w$ which are regular functions of δ for $|\delta| < |w| \leqslant 1$, proves that equation (3.5) has exactly one root $\delta = \delta_0(r;w)$, for which

$$|\delta_0(r;w)| < |w|, \qquad \delta_2(r) < |w| \leqslant 1. \tag{5.22}$$

From the implicit function theorem and the uniqueness of the root $\delta_0(r;w)$ satisfying (5.22) it follows that the function $\delta_0(r;w)$ is regular in the annulus $\delta_2(r) < |w| < 1$. $\square$

Because $\delta_0(r;w)$ is a branch of the inverse of the function $w(r;\delta)$ we must have for every w, $\delta_2(r) < |w| < 1$,

$$w_1(r;\delta_0(r;w)) = w, \text{ or } w_2(r;\delta_0(r;w)) = w. \tag{5.23}$$

<u>COROLLARY 5.5.</u> *For* $\delta \in \tilde{E}(r)$ *with* $|\delta| > \delta_2(r)$ *and* $|w_1(r;\delta)| \leqslant 1$,

$$\delta_0(r;w_1(r;\delta)) = \delta; \tag{5.24}$$

for w *with* $\delta_2(r) < |w| \leqslant 1$ *and* $|\delta_0(r;w)| > \delta_2(r)$,

$$w_1(r;\delta_0(r;w)) = w. \tag{5.25}$$

<u>PROOF.</u> Let $\delta \in \tilde{E}(r)$ be such that $|\delta| > \delta_2(r)$ and $|w_1(r;\delta)| \leqslant 1$. Then by lemma 5.5,

$$\delta_2(r) < |\delta| < |w_1(r;\delta)| \leqslant 1.$$

Hence, δ is a root of equation (3.5) with $w = w_1(r;\delta)$ and has an absolute value smaller than $w = w_1(r;\delta)$. By the uniqueness of $\delta_0(r;w_1(r;\delta))$ with the same properties relation (5.24) follows. Next let w be such that $\delta_2(r) < |w| \leqslant 1$ and $|\delta_0(r;w)| > \delta_2(r)$. Then by (5.21) and (5.22)

$$|w_2(r;\delta_0(r;w))| < |\delta_0(r;w)| < |w|,$$

so that $w_2(r;\delta_0(r;w)) \neq w$. Hence, by (5.23) relation (5.25) follows. □

Since $w_1(r;\delta_2(r)) = \delta_2(r)$ it is seen that the set

$$D_1(r) := \{\delta; |\delta| > \delta_2(r), |w_1(r;\delta)| \leqslant 1\}, \tag{5.26}$$

is non-empty. Moreover, it follows from (5.24) as $\delta \downarrow \delta_2(r)$ that

$$\delta_0(r;\delta_2(r)) = \delta_2(r). \tag{5.27}$$

Clearly also the following set is non-empty:

$$\{w; \delta_2(r) < |w| < 1, |\delta_0(r;w)| > \delta_2(r)\} = w_1(r;D_1(r)).$$

<u>COROLLARY 5.6.</u> *For every* w *in the annulus* $A(r) := \{w; \delta_2(r) < |w| < 1\}$,

$$w_1(r;\delta_0(r;w)) = w.$$

<u>PROOF.</u> By lemma 5.6 the function $\delta_0(r;w)$ is regular in the annulus $\delta_2(r) < |w| < 1$. This annulus is mapped by $\delta_0(r;w)$ onto a domain contained in $\tilde{E}(r)$, because $\delta_0(r;w)$ is bounded in absolute value by one in this annulus, cf. lemma 5.6, and because $\delta_0(r;w)$ cannot take values on the interval $\gamma(r)$ as this interval is mapped by both $w_1(r;\delta)$ and $w_2(r;\delta)$ onto the contour $L(r)$ and as the intersection of $L(r)$ with this annulus is empty (corollary 5.2), cf. (5.23).
Hence, the function $w_1(r;\delta)$ is regular in the image of the annulus $A(r)$ under the mapping $\delta_0(r;w)$.
The assertion then follows from (5.25) by analytic continuation. □

<u>LEMMA 5.7.</u> *The function* $\delta_0(r;w)$ *can be continued as a regular function into the domain* $L^-(r) \cap C^+$. *For every* $w \in L^-(r) \cap C^+$,

$$w_1(r;\delta_0(r;w)) = w. \tag{5.28}$$

<u>PROOF.</u> As a consequence of corollary 5.6 the function $w_1(r;\delta)$ is univalent in the image $\delta_0(r;A(r))$ of the annulus $A(r)$. From (5.13) it is seen that the derivatives

$$\frac{\partial}{\partial\delta} w_1^+(r;\delta), \qquad \frac{\partial}{\partial\delta} w_1^-(r;\delta),$$

do not vanish on $\gamma(r)$. Therefore, the set $S(r)$ in lemma 5.3 can be chosen such that $w_1(r;\delta)$ is univalent in $S(r)$.
By (5.27) the intersection of $\delta_0(r;A(r))$ and $S(r)$ is such that there exists a closed Jordan curve $\Gamma(r)$ belonging to $\delta_0(r;A(r)) \cup S(r)$ and with the gap between $S(r)$ and $\delta_0(r;A(r))$ in its interior (see figure 5.2). By the above the function $w_1(r;\delta)$ is univalent on $\Gamma(r)$. Since the interior of $\Gamma(r)$ is

contained in $\tilde{E}(r)$ the function $w_1(r;\delta)$ is regular in this interior domain. But then the function $w_1(r;\delta)$ is univalent in this interior by MARKUSHEVICH [18], volume II, theorem 4.5.

By analytic continuation from the equation

$$\delta_0(r;w_1(r;\delta)) = \delta,$$

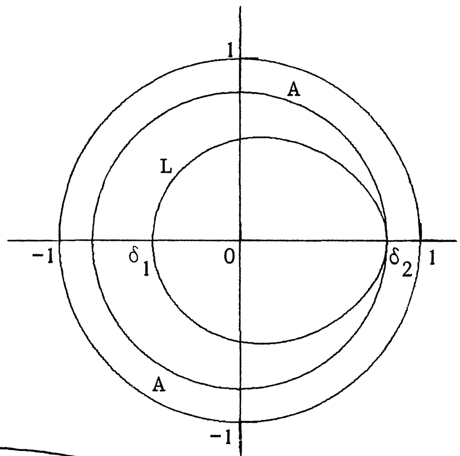
which holds in $\delta_0(r;A(r))$ by corollary 5.6 the function $\delta_0(r;w)$ can be extended as a regular function to the domain $L^-(r) \cap C^+$ by using that $w_1(r;\delta)$ is univalent in the interior of $\Gamma(r)$. It is then further clear that (5.28) holds in $L^-(r) \cap C^+$. □

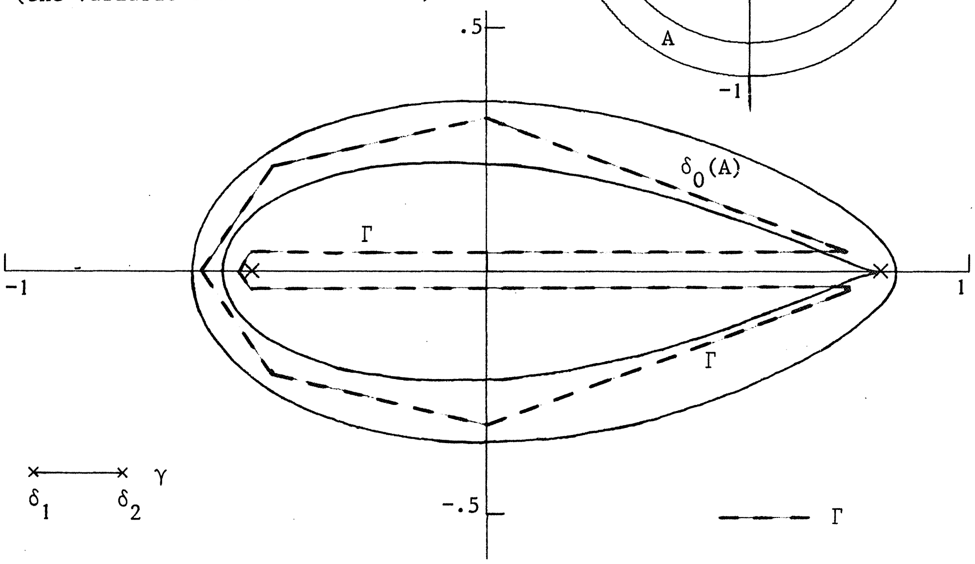
Figure 5.2

Right: the w-plane with the contour L(r) and the annulus A(r)

Below: the δ-plane with the line segment γ(r), the domain $\delta_0(r;A(r))$ and the Jordan curve Γ(r)

(the variable r has been deleted)





<u>LEMMA 5.8.</u> *The function* $\delta_0(r;w)$ *can be continued as a regular function into the domain*

$$W(r) := \{w;\ w \in L^-(r),\ |w| < 1 + \sqrt{1-4c_1c_2r}\}. \tag{5.29}$$

For every $w \in W(r)$,

$$|\delta_0(r;w)| < 1,$$
$$w_1(r;\delta_0(r;w)) = w.$$

<u>PROOF.</u> In the annulus $1-\sqrt{1-4c_1c_2r} < |w| < 1+\sqrt{1-4c_1c_2r}$ the inequality

$$|w|^2 + 4c_1c_2r < 2|w|,$$

holds. Let w belong to this annulus and let $|\delta| = 1$, then

$$|w^2+4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right)| \leq |w|^2+4c_1c_2r < 2|w| = |2w\delta|.$$

Application of Rouché's theorem (cf. lemma I.1.4) to the functions $w^2 + 4c_1c_2r\ \beta\left(\frac{1-\delta}{\alpha}\right)$ and $2\delta w$ proves that equation (3.5) has exactly one root $\delta = \delta_1(r;w)$ in the unit disk for

$$1 - \sqrt{1-4c_1c_2r} < |w| < 1 + \sqrt{1-4c_1c_2r}. \tag{5.30}$$

Moreover, this root $\delta_1(r;w)$ is regular in the annulus (5.30), cf. the proof of lemma 5.6.

By comparing this result with that of lemma 5.6 it seen that for $|w| = 1$,

$$\delta_1(r;w) = \delta_0(r;w)$$

This means that $\delta_1(r;w)$ is the analytic continuation of $\delta_0(r;w)$ into the region $1 \leq |w| < 1 + \sqrt{1-4c_1c_2r}$, cf. definition I.1.7. This analytic continuation will further be denoted by the symbol $\delta_0(r;w)$. Thus $\delta_0(r;w)$ is regular in the domain W(r). By the same arguments which has been used to prove corollary 5.6 it is seen that the domain W(r) is mapped by $\delta_0(r;w)$ into $\tilde{E}(r)$, and hence that relation (5.28) can be extended to the domain W(r).

Finally, because $\delta_0(r;w)$ is univalent in W(r), because it maps the contour L(r) onto the interval $\gamma(r) \subset C^+$ (a consequence of lemma 5.7), and because it is bounded in absolute value by one in the annulus (5.30) the function $\delta_0(r;w)$ is bounded in absolute value by one in the domain W(r). □

As an illustration the functions $w_1(r;\delta)$ and $w_2(r;\delta)$ are shown for real values of δ in figure 5.3 and in figure 5.4. For that purpose the following properties of these functions are obtained from (3.14):

$$w_1(r;\delta_1(r)) = w_2(r;\delta_1(r)) = \delta_1(r),$$
$$w_1(r;\delta_2(r)) = w_2(r;\delta_2(r)) = \delta_2(r);$$

$$w_1(r;\delta) \sim 2\delta, \qquad w_2(r;\delta) \to 0, \qquad \text{as } \delta \to -\infty;$$

$$w_1(r;1) = 1 + \sqrt{1-4c_1c_2r}, \qquad w_2(r;1) = 1 - \sqrt{1-4c_1c_2r},$$

$$w_1'(r;1) = 1 + \frac{1-2c_1c_2ra}{\sqrt{1-4c_1c_2r}}, \qquad w_2'(r;1) = 1 - \frac{1-2c_1c_2ra}{\sqrt{1-4c_1c_2r}}. \tag{5.31}$$

Further, as $w \to 1 + \sqrt{1-4c_1c_2r}$ the limit of $\delta_0(r;w)$ is the root with the smallest absolute value of the equation, cf. (3.5),

$$2\delta = 1 + \sqrt{1-4c_1c_2r} + \{1 - \sqrt{1-4c_1c_2r}\}\,\beta\left(\frac{1-\delta}{\alpha}\right). \tag{5.32}$$

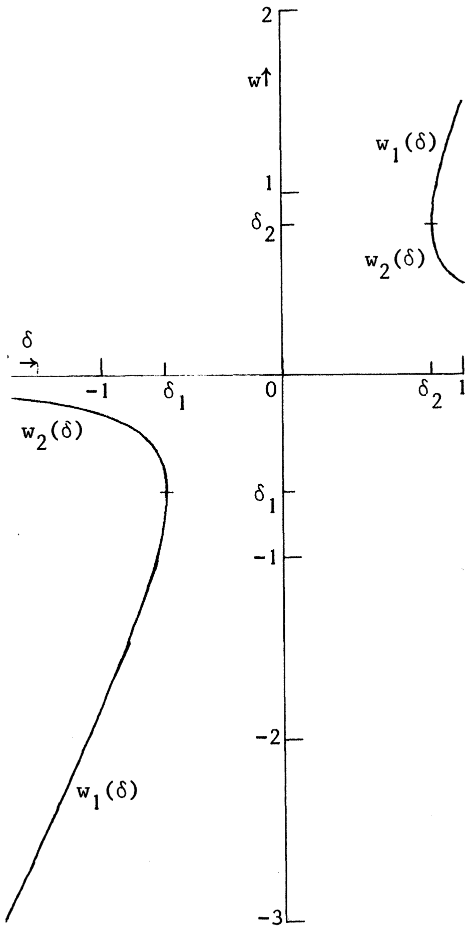
From this equation it is readily verified that

$$\begin{aligned}\delta_0(r;1+\sqrt{1-4c_1c_2r}) &= 1, \qquad \text{for } \tfrac{1}{2}a\{1-\sqrt{1-4c_1c_2r}\} \leqslant 1,\\ &< 1, \qquad \text{for } \tfrac{1}{2}a\{1-\sqrt{1-4c_1c_2r}\} > 1.\end{aligned} \tag{5.33}$$

In a similar way it is obtained that

Figure 5.3 (right)

The functions $w_1(r;\delta)$ and $w_2(r;\delta)$, δ real, $|\delta| \leq 1$, for a small value of the traffic a



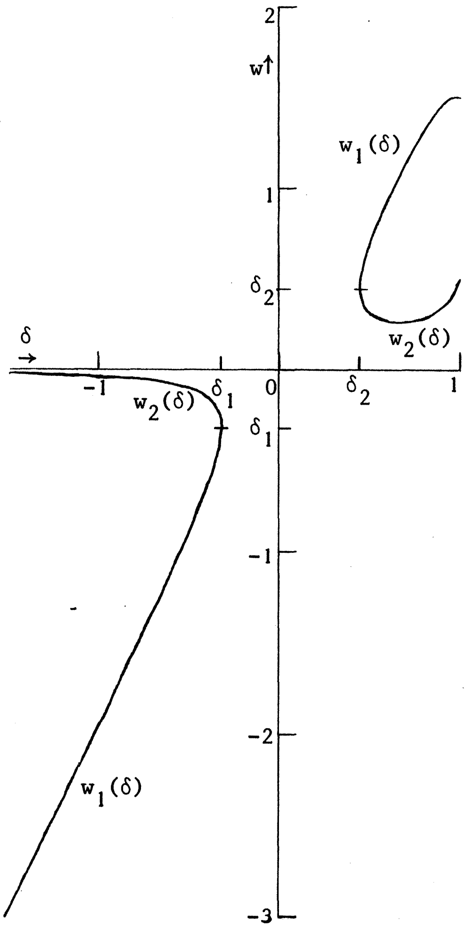


Figure 5.4 (left)

The functions $w_1(r;\delta)$ and $w_2(r;\delta)$, δ real, $|\delta| \leq 1$, for a large value of the traffic a

Note: the variable r has been deleted in the figures

$$\delta_0(r;1-\sqrt{1-4c_1c_2r}) = 1, \qquad \text{for } \tfrac{1}{2}a\{1+\sqrt{1-4c_1c_2r}\} \leqslant 1,$$
$$< 1, \qquad \text{for } \tfrac{1}{2}a\{1+\sqrt{1-4c_1c_2r}\} > 1. \tag{5.34}$$

Taking the above properties into account the functions $w_1(r;\delta)$ and $w_2(r;\delta)$ are shown in figure 5.3 for a small value of the traffic a and in figure 5.4 for a large value of the traffic a.

Having established some preliminary results in the preceding lemmas we are able to prove the following theorems on the analytic continuation of the conformal mapping $g_0(r;w)$ of $L^+(r)$ onto C^+, and of its inverse $g(r;z)$.

__THEOREM 5.4.__ *The function* $g_0(r;w), w \in L^+(r) \cup L(r)$, *possesses an analytic continuation into the domain*

$$\{w; |w| < 1+\sqrt{1-4c_1c_2r},\ \beta\left(\frac{1-\frac{1}{2}w}{\alpha}\right) \neq 0 \text{ for } w \in W(r)\}.$$

This analytic continuation which will be denoted by the same symbol $g_0(r;w)$ *has poles at points* $w \in W(r)$ *for which* $\beta\left(\frac{1-\frac{1}{2}w}{\alpha}\right) = 0$.

__PROOF.__ Because the function $g_0(r;w)$ maps the contour $L(r)$ onto the unit circle C, cf. corollary 5.4, we have for $w \in L(r)$,

$$|g_0(r;w)| = 1.$$

With theorem 5.2 this implies, for $w \in L(r)$,

$$g_0(r;w)g_0(r;\bar{w}) = 1. \tag{5.35}$$

By lemma 5.8 we have for $w \in W(r)$,

$$w_1(r;\delta_0(r;w)) = w \in W(r) \subset L^-(r),$$

so that by lemma 5.4 and the remark below it, for $w \in W(r)$, $\mathrm{Re}\, w \leqslant 1$,

$$w_2(r;\delta_0(r;w)) \in L^+(r).$$

Hence the function

$$\frac{1}{g_0(r;w_2(r;\delta_0(r;w)))}, \qquad w \in W(r),\ \mathrm{Re}\ w \leqslant 1, \tag{5.36}$$

is well defined and regular except at points $w \in W(r)$, Re $w \leqslant 1$, where $w_2(r;\delta_0(r;w)) = 0$, at which points it has poles. These poles will be discussed at the end of this proof.

If $w, w \in W(r)$, tends to the contour $L(r)$, then $\delta_0(r;w)$ tends to the interval $\gamma(r)$, cf. (5.27), so that by (3.22) the function $w_2(r;\delta_0(r;w))$ tends to (cf. lemma 5.7)

$$\overline{w_1(r;\delta_0(r;w))} = \overline{w}.$$

Hence, by (5.35) we have for $w \in L(r)$,

$$g_0(r;w) = \frac{1}{g_0(r;w_2(r;\delta_0(r;w)))}. \tag{5.37}$$

By lemma I.3.4 this implies that the function $g_0(r;w)$ is regular for $w \in L(r)$, and this shows that the function $g_0(r;w)$ possesses an analytic continuation into $\{w; w \in W(r),\ \mathrm{Re}\ w \leqslant 1,\ w_2(r;\delta_0(r;w)) \neq 0\}$. Moreover, denoting this analytic continuation by the same symbol $g_0(r;w)$ relation (5.37) holds for $w \in W(r)$, Re $w \leqslant 1$ (principle of permanence, see NEHARI [21], p. 107).

Next, for $w \in W(r)$, Re $w > 1$, we have by lemma 5.8 that $|\delta_0(r;w)| < 1$. Hence, by (3.29), for $w \in W(r)$, Re $w > 1$,

$$|w_2(r;\delta_0(r;w))| < 1,$$

so that by the above the function $g_0(r;w_2(r;\delta_0(r;w)))$ is well defined and regular for $w \in W(r)$, Re $w > 1$, except possibly for poles.

Again by (5.37) which holds for Re $w = 1$, $w \in W(r)$, it is seen that the

function $g_0(r;w)$ possesses an analytic continuation into $\{w; w \in W(r),$ $\mathrm{Re}\, w > 1,\ w_2(r;\delta_0(r;w)) \neq 0\}$, and that (5.37) holds in $W(r)$.
Thus $g_0(r;w)$ is a regular function in the disk $|w| < 1 + \sqrt{1-4c_1c_2r}$, except possibly for poles in $W(r)$, if $w_2(r;\delta_0(r;w)) = 0$.
In $L^+(r) \cup L(r)$ the conformal mapping $g_0(r;w)$ is finite and $g_0(r;w) = 0$ if and only if $w = 0$, cf. (5.3). From (5.37) it is seen that the function $g_0(r;w)$ has a pole at a point $w \in W(r)$ if and only if $w_2(r;\delta_0(r;w)) = 0$.
Equation (3.5) and the inequalities (3.29) show that for $w \in W(r)$,

$$w_2(r;\delta_0(r;w)) = 0 \Leftrightarrow \beta\left(\frac{1-\delta_0(r;w)}{\alpha}\right) = 0 \Leftrightarrow w_1(r;\delta_0(r;w)) = 2\delta_0(r;w).$$

Because $w_1(r;\delta_0(r;w)) = w$ for $w \in W(r)$, cf. lemma 5.8, it is obtained that for $w \in W(r)$,

$$w_2(r;\delta_0(r;w)) = 0 \Leftrightarrow \beta\left(\frac{1-\frac{1}{2}w}{\alpha}\right) = 0. \tag{5.38}$$

Thus the function $g_0(r;w)$, $|w| < 1 + \sqrt{1-4c_1c_2r}$, has a pole at a point w if and only if $w \in W(r)$ and $\beta\left(\frac{1-\frac{1}{2}w}{\alpha}\right) = 0$. □

Relation (5.37) which holds by the principle of permanence for $w \in W(r)$ leads to:

<u>COROLLARY 5.7.</u> *For* $w \in W(r)$, $\mathrm{Re}\, w \leqslant 1$,

$$|g_0(r;w)| > 1.$$

<u>PROOF.</u> As it has been shown in the proof of theorem 5.4 we have $w_2(r;\delta_0(r;w)) \in L^+(r)$ for $w \in W(r)$, $\mathrm{Re}\, w \leqslant 1$. Hence by the definition of the conformal mapping $g_0(r;w)$ we have for $w \in W(r)$, $\mathrm{Re}\, w \leqslant 1$,

$$g_0(r;w_2(r;\delta_0(r;w))) \in C^+.$$

By (5.37) the assertion then follows. □

REMARK 5.1. In a similar way it can be shown that for $w \in W(r)$, $\mathrm{Re}\, w > 1$,

$$|g_0(r;w)| > 1, \qquad \text{if } w_2(r;\delta_0(r;w)) \in L^+(r),$$
$$|g_0(r;w)| \leqslant 1, \qquad \text{if } w_2(r;\delta_0(r;w)) \in L(r) \cup L^-(r).$$

The latter case has not been excluded by the above results (although we did not encounter cases in which this occurs). But for instance, if the service time is constant (β) then $\beta(s) = e^{-\beta s}$. In this case the inequality (5.17) holds for any h and s (and in fact it is an equality) because

$$(e^{-\beta s})^2 = e^{-2\beta s} = e^{-\beta(s+h)}e^{-\beta(s-h)};$$

therefore lemma 5.4 holds without the restriction $\mathrm{Re}\, w_1(r;\delta) \leqslant 1$. This implies that corollary 5.7 holds for every $w \in W(r)$ if the service time is constant.

THEOREM 5.5. *The conformal mapping* $g(r;z)$ *of* C^+ *onto* $L^+(r)$ *possesses an analytic continuation into a part of* C^-; *it is regular at every point* $z \in C$, *and* $g'(r;z) \neq 0$ *on* C.

PROOF. By theorem 5.4 the function $g_0(r;w)$ possesses an analytic continuation into a part of the domain $L^-(r)$, and corollary 5.7 implies that for every $w \in L(r)$ there exists a neighborhood such that the function $g_0(r;w)$ is univalent in this neighborhood, i.e.

$$g_0'(r;w) \neq 0, \qquad \text{for } w \in L(r). \tag{5.39}$$

Hence, the function $g(r;z)$, $z \in C^+$, the inverse of $g_0(r;w)$, $w \in L^+(r)$, possesses an analytic continuation into a part of $C \cup C^-$, and it is regular at every point $z \in C$. Clearly, $g'(r;z)$ does not vanish on C. □

REMARK. By means of Kellogg's theorem (see lemma I.6.5) it can be proved directly from the definition of the contour $L(r)$, cf. (4.21), and from the properties of the function h(t), cf. lemma 4.2, that the derivative $g'(r;z)$ exists and does not vanish on C (see also the proof of (8.21) in theorem 8.2).

REMARK. In general the function $g_0(r;w)$ is not univalent in the whole disk $|w| < 1 + \sqrt{1-4c_1c_2r}$. From (5.33) it can be seen that $g_0'(r;w)$ vanishes in $W(r)$ if $w_2'(r;\delta_0(r;w))$ vanishes (see e.g. figure 5.4). This sets bounds to the analytic continuation of $g(r;z)$ into C^-.

In the rest of this section it will be shown that the functions

$$\Phi_x(r;\frac{w}{2c_1},0) \text{ and } \Phi_x(r;0,\frac{w}{2c_2}),$$

are regular in the domain $L^+(r)$. By theorem 2.3 the first of these functions is regular in the disk $|w| < 2c_1$ and the second is regular in the disk $|w| < 2c_2$. Suppose that $c_2 \leqslant \frac{1}{2} \leqslant c_1$, cf. (1.2), (1.3). Then the first of the above functions is regular in C^+, and hence by corollary 5.2 it is regular in $L^+(r)$. If $2c_2 \geqslant \delta_2(r)$ then it follows from corollary 5.2 that also the second function is regular in $L^+(r)$. However, it may happen that $2c_2 < \delta_2(r)$ as it can be seen from the following result.

From now on it is assumed that the types of customers are numbered such that $\alpha_1 \leqslant \alpha_2$, cf. section II.1, *so that*

$$0 < c_2 \leqslant \tfrac{1}{2} \leqslant c_1 < 1. \tag{5.40}$$

Because $\delta_2(r)$ is the positive zero of the discriminant of equation (3.5), cf. theorem 3.3, the equation

$$\delta_2(r) = 2c_2, \qquad 0 < r < 1,\ 0 < c_2 \leqslant \tfrac{1}{2}, \tag{5.41}$$

is equivalent to the equation

$$c_2 = (1-c_2)r\,\beta\!\left(\frac{1-2c_2}{\alpha}\right), \qquad 0 < r < 1,\ 0 < c_2 \leqslant \tfrac{1}{2}. \tag{5.42}$$

Therefore we define the function

$$R(t) := \frac{t}{(1-t)\,\beta\!\left(\frac{1-2t}{\alpha}\right)}, \qquad 0 \leqslant t \leqslant \tfrac{1}{2}. \tag{5.43}$$

THEOREM 5.6. i. *If* $a \leqslant 2$ *then* $R(c_2) < 1$ *for every* $c_2 < \frac{1}{2}$ *and* $R(\frac{1}{2}) = 1$.
ii. *If* $a > 2$ *then* $R(c_2) < 1$ *only for* $c_2 < c_{2m}(a)$; *here* $c_{2m}(a)$, $0 < c_{2m}(a) < \frac{1}{2}$, *is the constant for which* $R(c_{2m}(a)) = 1$.
iii. *For* $0 < r < \min\{1, R(c_2)\}$ *the inequality* $\delta_2(r) < 2c_2$ *holds while if* $R(c_2) < 1$ *then the inequality* $\delta_2(r) > 2c_2$ *holds for* $R(c_2) < r < 1$.

PROOF. Since $\delta_2(r) = h(\sqrt{r})$, $0 < r < 1$, cf. (5.9), it follows from lemma 4.2 that $\delta_2(r)$ is a strictly increasing function of r for $0 < r < 1$. Because the relation $r = R(c_2)$ is equivalent to (5.41) this implies statement iii. In order to investigate for which values of c_2 the inequality $R(c_2) < 1$ holds we consider for $0 < r < 1$ in the domain $\mathrm{Re}\ t < \frac{1}{2}$ the equation

$$\frac{t}{1-t} = r\,\beta\!\left(\frac{1-2t}{\alpha}\right). \tag{5.44}$$

On the line $\mathrm{Re}\ t = \frac{1}{2}$ we have

$$\left|r\,\beta\!\left(\frac{1-2t}{\alpha}\right)\right| \leqslant r < 1 = \left|\frac{t}{1-t}\right|,$$

while as $|t| \to \infty$, $\mathrm{Re}\ t \leqslant \frac{1}{2}$,

$$\left|r\,\beta\!\left(\frac{1-2t}{\alpha}\right)\right| \leqslant r < 1, \qquad \left|\frac{t}{1-t}\right| \to 1.$$

Application of Rouché's theorem, cf. lemma I.1.4, to the functions $r\,\beta\!\left(\frac{1-2t}{\alpha}\right)$ and $\frac{t}{1-t}$ in the domain $\mathrm{Re}\ t < \frac{1}{2}$ shows that equation (5.44) has exactly one root in this domain. Moreover, it is readily seen that this root belongs to the real interval $0 < t < \frac{1}{2}$ (see figure 5.5). This implies that for every r, $0 < r < 1$, there is exactly one t in the interval $(0,\frac{1}{2})$ such that $R(t) = r$. Because $R(t)$ is a continuous function for $0 < t < \frac{1}{2}$ it follows that this function increases strictly from zero to one on an interval $[0,\tau(a)]$, $0 < \tau(a) \leqslant \frac{1}{2}$, while $R(t) \geqslant 1$ for $\tau(a) \leqslant t \leqslant \frac{1}{2}$. Whether $\tau(a) = \frac{1}{2}$ or $\tau(a) < \frac{1}{2}$ depends on the derivative of the function $R(t)$ at $t = \frac{1}{2}$. From (5.43) we obtain

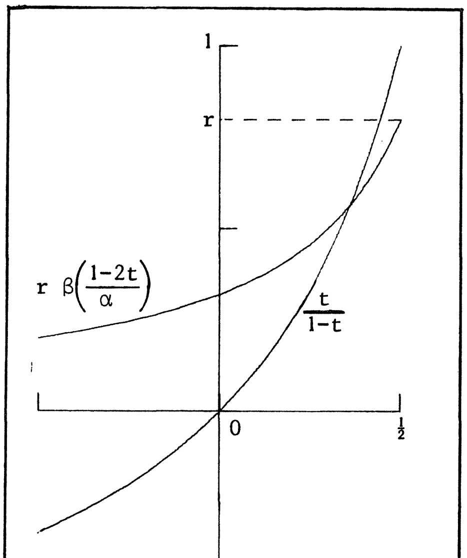


Figure 5.5

$$R'(t) = \frac{\beta\left(\frac{1-2t}{\alpha}\right) + \frac{2}{\alpha}t(1-t)\ \beta'\left(\frac{1-2t}{\alpha}\right)}{(1-t)^2\ \beta^2\left(\frac{1-2t}{\alpha}\right)}, \qquad 0 \leqslant t \leqslant \tfrac{1}{2},$$

so that

$$R'(\tfrac{1}{2}) = 4(1-\tfrac{1}{2}a).$$

If $a < 2$ then $R'(\frac{1}{2}) > 0$ and thus $\tau(a) = \frac{1}{2}$ (see figure 5.6.i). If $a = 2$ then $R'(\frac{1}{2}) = 0$. However, it appears that for $a = 2$,

$$R''(\tfrac{1}{2}) = -16\,\frac{\beta_2-\beta^2}{\beta^2} \leqslant 0,$$

because $\beta_2 - \beta^2$ is the variance of the service time distribution. Thus $\tau(a) = \frac{1}{2}$ for $a = 2$ too (in the case $\beta_2 = \beta^2$ this can be checked by determining $R'''(\frac{1}{2})$ and noting that then $\beta_3 = \beta^3$). If $a > 2$ then $R'(\frac{1}{2}) < 0$ so that $\tau(a) < \frac{1}{2}$ (see figure 5.6.ii). Noting that (5.41) is equivalent to the equation $R(c_2) = r$ it follows that (5.41) has a root on the interval

(0,1) if and only if $c_2 < \tau(a)$. Putting $c_{2m}(a) = \tau(a)$, $a > 2$, and $\tau(a) = \frac{1}{2}$, $a \leqslant 2$ and noting that $R(\tau(a)) = 1$ the proof has been completed. □

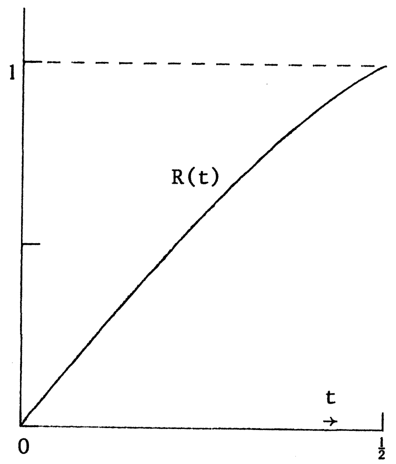


Figure 5.6.i. the case $a < 2$

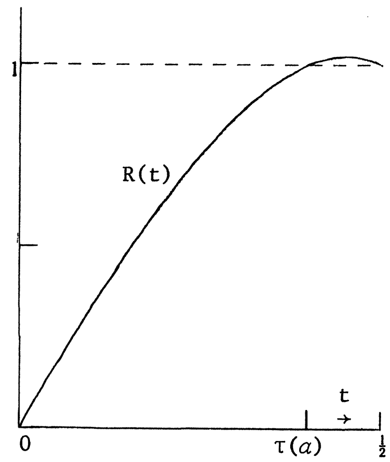


Figure 5.6.ii. the case $a > 2$

THEOREM 5.7. *The functions* $\Phi_x(r;\frac{w}{2c_1},0)$ *and* $\Phi_x(r;0,\frac{w}{2c_2})$ *belong to the class* $RCB(L^+(r))$, cf. definition I.1.6.

PROOF. As it has been noted before the function $\Phi_x(r;\frac{w}{2c_1},0)$ is regular in $L^+(r)$ for every r, $0 < r < 1$, and the function $\Phi_x(r;0,\frac{w}{2c_2})$ is regular in $L^+(r)$ for $0 < r \leqslant \min\{1,R(c_2)\}$ by assumption (5.40), theorem 2.3, corollary 5.2 and theorem 5.6.

Next suppose that $R(c_2) < 1$ and $R(c_2) < r < 1$, so that $2c_2 \in L^+(r)$. In theorem 4.2 it has been shown that the function $\Phi_x(r;0,p_2(r;\delta))$ can be continued as an analytic function on $\bar{E}(r)$. If δ tends to the branch point $\delta_2(r)$ then, cf. (3.6), (3.7),

$$p_1(r;\delta) \to \frac{1}{2c_1}\,\delta_2(r) < 1, \qquad p_2(r;\delta) \to \frac{1}{2c_2}\,\delta_2(r) > 1.$$

Hence $\Phi_x(r;\frac{1}{2c_1}\,\delta_2(r),0)$ is finite and from equation (4.1) it follows that then also

$$\Phi_x(r;0,\frac{1}{2c_2}\,\delta_2(r)) < \infty.$$

For real positive values of r the function $\Phi_x(r;0,\frac{w}{2c_2})$ has a power series expansion at $w = 0$ with positive coefficients, because it is the generating function in $\frac{w}{2c_2}$ of a probability distribution. Since this function is finite up to the positive value $\delta_2(r)$ it is regular in the disk $|w| < \delta_2(r)$, and hence by corollary 5.2 in the domain $L^+(r)$; and it is continuous in $L^+(r) \cup L(r)$. □

<u>COROLLARY 5.8.</u> *The functions*

$$\Phi_x(r;\frac{1}{2c_1}\,g(r;z),0), \qquad \Phi_x(r;0,\frac{1}{2c_2}\,g(r;z)),$$

belong to the class $RCB(C^+)$.

<u>PROOF.</u> By theorem 5.1 and corollary 5.4 the conformal mapping $g(r;z)$ of C^+ onto $L^+(r)$ belongs to the class $RCB(C^+)$. Hence, the assertion is obvious from theorem 5.7. □

For later reference this section is concluded with:

<u>LEMMA 5.9.</u> *For every a and* c_2, $c_2 \neq \frac{1}{2}$ *for* $a = 2$, *and* $c_2 \leqslant c_{2m}(a)$ *for* $a > 2$, *there exists a value* $r_0(c_2)$, $0 < r_0(c_2) < R(c_2)$, *such that for* $r_0(c_2) \leqslant r < R(c_2)$ *there exists a* $t_0(r) > 1$ *with*

$$g(r;t_0(r)) = 2c_2, \tag{5.45}$$

which is strictly decreasing for $r_0(c_2) \leqslant r < R(c_2)$ *while*

$$\lim_{r \uparrow R(c_2)} t_0(r) = 1. \tag{5.46}$$

<u>PROOF.</u> In each of the above cases we have $R(c_2) \leqslant 1$, cf. theorem 5.6. Suppose $0 < r < R(c_2)$. Then $2c_2 \in L^-(r)$, cf. theorem 5.6, and the function

$g_0(r;w)$ is defined on the interval $\delta_2(r) < w \leqslant 2c_2$ by relation (5.37). By corollary 5.6 the function $\delta_0(r;w)$ is strictly increasing on $\delta_2(r) < w < 2c_2$, and $\delta_0(r;2c_2) < 1$ by lemma 5.6. From (3.14) it follows that for $\delta_2(r) < \delta < 1$,

$$w_2'(r;\delta) = 1 - \frac{\delta + 2c_1c_2\frac{r}{\alpha}\beta'\left(\frac{1-\delta}{\alpha}\right)}{\sqrt{\delta^2 - 4c_1c_2\, r\, \beta\left(\frac{1-\delta}{\alpha}\right)}}. \tag{5.47}$$

This leads with (5.43) to:

$$w_2'(r;2c_2) = 1 - \frac{1 + c_1\frac{r}{\alpha}\beta'\left(\frac{1-2c_2}{\alpha}\right)}{\sqrt{1 - r/R(c_2)}}. \tag{5.48}$$

The numerator in (5.48) vanishes for $r \uparrow R(c_2)$ only if $\delta_2(r) = h(\sqrt{r}) = 2c_2$ is a multiple root of equation (4.5) with $r = R(c_2)$, cf. (5.43), i.e. only if $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, cf. lemma 4.2. Hence, in the cases in consideration we have $w_2'(r;2c_2) \downarrow -\infty$ as $r \uparrow R(c_2)$, cf. (4.8). This implies that for r close to $R(c_2)$ the function $w_2(r;\delta)$ is strictly decreasing on $\delta_2(r) < w \leqslant 2c_2$. With (5.37) this implies that $g_0(r;w)$ is strictly increasing on the interval $\delta_2(r) \leqslant w \leqslant 2c_2$, so that its inverse $g(r;z)$ exists on the interval $1 \leqslant z \leqslant g_0(r;2c_2)$. Now take $t_0(r) := g_0(r;2c_2)$. Since $g_0(r;2c_2)$ is a continuous function of r, cf. theorem 5.3 and (5.37), and $g_0(r;2c_2) \downarrow 1$ as $r \uparrow R(c_2)$, $r_0(c_2)$ can be chosen so close to $R(c_2)$ that $t_0(r)$ is strictly decreasing on $r_0(c_2) < r < R(c_2)$. (Note: in the case $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, it follows from (5.31) that $w_2'(r;1) = 1 - \sqrt{1-r} \to 1$ as $r \uparrow R(\frac{1}{2}) = 1$). □

II.6. Formulation as Riemann-Hilbert problems

With the aid of the conformal mapping $g(r;z)$ of the unit disk C^+ onto the domain $L^+(r)$ introduced in the preceding section two Riemann-Hilbert boundary value problems on the unit circle with index zero will be derived from equation (4.18), for $0 < r < \min\{1, R(c_2)\}$, cf. theorem 5.6. It will be shown that these two Riemann-Hilbert problems can be solved with the aid of lemma I.5.1. Together with two linear relations the solutions of these two Riemann-Hilbert problems provide us with explicit expressions for the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ in the domains $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$, for $0 < r < \min\{1, R(c_2)\}$. Moreover, they determine these two functions uniquely in the regions $|p_1| \leq 1$, $|p_2| \leq 1$, $|r| < 1$, by means of analytic continuation. As soon as the functions $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, and $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$, are determined the function $\Phi_x(r;p_1,p_2)$, $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$, is determined by equation (2.2).

It is throughout assumed that $0 < c_2 \leq \frac{1}{2}$ *and except in some remarks that* $0 < r < 1$.

In order to simplify the notation we introduce the function

$$K_x(w_1,w_2) := \frac{(w_1/2c_1)^{x_1}}{1-w_1/2c_1} \; \frac{(w_2/2c_2)^{x_2}}{1-w_2/2c_2}, \tag{6.1}$$

where x stands for the vector (x_1,x_2), cf. definition 1.2.

<u>LEMMA 6.1.</u> *The functions* $\Phi_x(r;\frac{w}{2c_1},0)$ *and* $\Phi_x(r;0,\frac{w}{2c_2})$ *satisfy on the contour* $L(r)$ *the relations*

$$\mathrm{Re}\left\{\frac{\Phi_x(r;\frac{w}{2c_1},0)}{1-w/2c_1} + \frac{\Phi_x(r;0,\frac{w}{2c_2})}{1-w/2c_2}\right\} = \mathrm{Re}\{K_x(w,\bar{w})\} + \Phi_x(r;0,0),$$

$$\mathrm{Im}\left\{\frac{\Phi_x(r;\frac{w}{2c_1},0)}{1-w/2c_1} - \frac{\Phi_x(r;0,\frac{w}{2c_2})}{1-w/2c_2}\right\} = \mathrm{Im}\{K_x(w,\bar{w})\}. \tag{6.2}$$

<u>PROOF.</u> Because $\Phi_x(r;\frac{w}{2c_1},0)$ is the generating function of a probability distribution it has for positive values of r a convergent power series expansion in w for $|w| \leq 2c_1$ with real coefficients. Consequently,

$$\Phi_x(r;\frac{\bar{w}}{2c_1},0) = \overline{\Phi_x(r;\frac{w}{2c_1},0)}, \qquad |w| \leqslant 2c_1.$$

On account of the same arguments and of theorem 5.7 we have

$$\Phi_x(r;0,\frac{\bar{w}}{2c_2}) = \overline{\Phi_x(r;0,\frac{w}{2c_2})}, \qquad |w| \leqslant \delta_2(r).$$

By taking real and imaginary parts of equation (4.18) and by using the above relations and the notation (6.1) we obtain (6.2). □

In order to apply the solution method for Riemann-Hilbert problems discussed in section I.5 we transform the equations (6.2) into an equivalent set of equations on the unit circle by means of the conformal mapping g(r;z) introduced in section II.5.

COROLLARY 6.1. *On the unit circle* $|t| = 1$,

$$\mathrm{Re}\left\{\frac{\Phi_x(r;\frac{g(r;t)}{2c_1},0)}{1-g(r;t)/2c_1} + \frac{\Phi_x(r;0,\frac{g(r;t)}{2c_2})}{1-g(r;t)/2c_2}\right\} = \mathrm{Re}\{K_x(g(r;t),\overline{g(r;t)})\} + \Phi_x(r;0,0),$$

$$\mathrm{Im}\left\{\frac{\Phi_x(r;\frac{g(r;t)}{2c_1},0)}{1-g(r;t)/2c_1} - \frac{\Phi_x(r;0,\frac{g(r;t)}{2c_2})}{1-g(r;t)/2c_2}\right\} = \mathrm{Im}\{K_x(g(r;t),\overline{g(r;t)})\}. \qquad (6.3)$$

PROOF. In theorem 5.1 it was proved that the conformal mapping g(r;z) of the unit disk C^+ onto the domain $L^+(r)$ satisfying the conditions (5.3) exists. From corollary 5.4 we know that the limiting values g(r;t), $t \in C$, exist and that they establish a one-to-one correspondence between the unit circle C and the contour L(r). Insertion of $w = g(r;t)$, $t \in C$, in the equations (6.2) thus leads to the equations (6.3). □

By corollary 5.8 the functions

$$\Phi_x(r;\frac{g(r;z)}{2c_1},0),\ \Phi_x(r;0,\frac{g(r;z)}{2c_2}),$$

are regular in the unit disk C^+. Because of the choice $c_1 \geqslant \frac{1}{2}$ and corollary 5.2 we have

$$|\frac{g(r;z)}{2c_1}| \leqslant \frac{\delta_2(r)}{2c_1} < 1, \qquad |z| \leqslant 1. \tag{6.4}$$

Hence the denominator $1 - \frac{g(r;z)}{2c_1}$ in the equations (6.3) does not vanish in the region $C^+ \cup C$. However, by theorem 5.6 it depends on the value of r whether the denominator $1 - \frac{g(r;z)}{2c_2}$ vanishes in the region $C^+ \cup C$ or not. Therefore three cases should be distinguished, cf. theorem 5.6, viz.

i. $0 < r < \min\{1;R(c_2)\}$, then $2c_2 \in L^-(r)$ and the above denominator does not vanish in $C^+ \cup C$;

ii. $r = R(c_2) < 1$, then $2c_2 \in L(r)$ and the denominator vanishes at $z = 1$;

iii. $R(c_2) < r < 1$, then $2c_2 \in L^+(r)$ and the denominator has a single zero in the region C^+.

Actually it suffices to obtain only for case i (or only for case iii) the function $\Phi_x(r;p_1,p_2)$, because if this function is known on some interval it canbe obtained for all $r \in C^+$ by analytic continuation from out that interval.

In this section for case i two Riemann-Hilbert boundary value problems with index zero will be derived and analyzed. For case iii two Riemann-Hilbert problems with index two can be derived and solved. This will be omitted. In section II.7 this case will be treated as a Hilbert problem. For case ii no Riemann-Hilbert problem as defined in section I.5 can be derived, because the denominator $1 - \frac{g(r;z)}{2c_2}$ vanishes at the boundary point $z = 1$. However, the solution for this case can be obtained by using a continuity argument, see section II.7.

<u>LEMMA 6.2.</u> *If* $2c_2 \notin L(r)$ *then the function* $K_x(g(r;t),\overline{g(r;t)})$ *and its real and imaginary parts satisfy a Hölder condition on the unit circle C.*

<u>PROOF.</u> We refer to the definitions and the results of section I.2. Because the points $2c_1$ and $2c_2$ do not lie on the contour $L(r)$, lemma I.2.2 implies that for every $x_1, x_2 = 0,1,2,\dots$, the functions

$$\frac{(w/2c_1)^{x_1}}{1-w/2c_1}, \quad \frac{(w/2c_2)^{x_2}}{1-w/2c_2},$$

satisfy the Hölder condition with index 1 on $L(r)$. Using the three statements of lemma I.2.3 it follows readily that the function $K_x(w,\bar{w})$ belongs to the class $H(L(r))$. This implies that for any $t_1, t_2 \in C$ we have the inequality

$$|K_x(g(r;t_1),\overline{g(r;t_1)}) - K_x(g(r;t_2),\overline{g(r;t_2)})| \leqslant$$

$$\leqslant A|g(r;t_1)-g(r;t_2)| \leqslant A \max_{|z|\leqslant 1} \{g'(r;z)\}|t_1-t_2|,$$

where A is a positive constant. From theorem 5.5 it follows that the derivative $g'(r;z)$ is bounded in the region $C^+ \cup C$.
This proves that the function $K_x(g(r;t),\overline{g(r;t)})$ belongs to the class $H(C)$. By lemma I.2.3(2°) also its real and imaginary parts belong to the class $H(C)$. □

For $|z| \leqslant 1$ we introduce the functions

$$\Omega_1(r;z) := \frac{\Phi_x(r;\frac{g(r;z)}{2c_1},0)}{1-g(r;z)/2c_1} + \frac{\Phi_x(r;0,\frac{g(r;z)}{2c_2})}{1-g(r;z)/2c_2} - \Phi_x(r;0,0),$$

$$\Omega_2(r;z) := -i\left\{\frac{\Phi_x(r;\frac{g(r;z)}{2c_1},0)}{1-g(r;z)/2c_1} - \frac{\Phi_x(r;0,\frac{g(r;z)}{2c_2})}{1-g(r;z)/2c_2}\right\}. \tag{6.5}$$

Each of these functions is the solution of a Riemann-Hilbert boundary value problem (cf. section I.5) as it will be shown below (see also lemma 6.2).

LEMMA 6.3. *Let* r *be fixed,* $0 < r < \min\{1, R(c_2)\}$. *Then:*

i. *the function* $\Omega_1(r;z)$ *belongs to the class* $RCB(C^+)$ *and satisfies the boundary condition: for* $t \in C$,

$$\mathrm{Re}\{\Omega_1^+(r;t)\} = \mathrm{Re}\{K_x(g(r;t), \overline{g(r;t)})\}; \tag{6.6}$$

ii. *the function* $\Omega_2(r;z)$ *belongs to the class* $RCB(C^+)$ *and satisfies the boundary condition: for* $t \in C$,

$$\mathrm{Re}\{\Omega_2^+(r;t)\} = \mathrm{Im}\{K_x(g(r;t), \overline{g(r;t)})\}. \tag{6.7}$$

PROOF. That the functions $\Omega_1(r;z)$ and $\Omega_2(r;z)$ belong to the class $RCB(C^+)$, cf. definition I.1.6, is a consequence of corollary 5.8, the inequality (6.4) and theorem 5.6. The boundary conditions (6.6) and (6.7) follow from (6.5) and corollary 6.1. □

Because the righthand sides of (6.6) and (6.7) belong to the class H(C), cf. lemma 6.2, in lemma 6.3 two *Riemann-Hilbert boundary problems* have been formulated for the functions $\Omega_1(r;z)$ and $\Omega_2(r;z)$ respectively, cf. section I.5.

THEOREM 6.1. *The functions* $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, *and* $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$, *are completely determined by (6.5) and by the solution of the two Riemann-Hilbert boundary value problems as formulated in lemma 6.3.*

PROOF. The boundary conditions (6.6) and (6.7) are of the form as described by formula (I.5.7) with n = 0, i.e. the indices of the Riemann-Hilbert boundary value problems formulated in lemma 6.3 are zero. Hence, the complete solutions of these Riemann-Hilbert problems are given by lemma I.5.1: for $0 < r < \min\{1,R(c_2)\}$, $z \in C^+$,

$$\Omega_1(r;z) = \frac{1}{2\pi i}\int_C \mathrm{Re}\{K_x(g(r;t),\overline{g(r;t)})\}\left[\frac{2}{t-z}-\frac{1}{t}\right]dt + id_{01}(r),$$

$$\Omega_2(r;z) = \frac{1}{2\pi i}\int_C \mathrm{Im}\{K_x(g(r;t),\overline{g(r;t)})\}\left[\frac{2}{t-z}-\frac{1}{t}\right]dt + id_{02}(r), \tag{6.8}$$

here $d_{01}(r)$ and $d_{02}(r)$ are real and independent of z.

For $z = 0$ the relations (6.5) read (cf. (5.3)):

$$\Omega_1(r;0) = \Phi_x(r;0,0),$$

$$\Omega_2(r;0) = 0. \tag{6.9}$$

With (6.8) this implies: for $0 < r < \min\{1,R(c_2)\}$,

$$\Phi_x(r;0,0) = \frac{1}{2\pi i}\int_C \mathrm{Re}\{K_x(g(r;t),\overline{g(r;t)})\}\frac{dt}{t} + id_{01}(r),$$

$$0 = \frac{1}{2\pi i}\int_C \mathrm{Im}\{K_x(g(r;t),\overline{g(r;t)})\}\frac{dt}{t} + id_{02}(r). \tag{6.10}$$

Because $K_x(\bar{w},w) = \overline{K_x(w,\bar{w})}$ for $w \in \mathbb{C}$, cf. (6.1), and because $\overline{g(r;t)} = g(r;\bar{t}) = g(r;\frac{1}{t})$ for $t \in C$ by theorem 5.2 it is readily verified that

$$\frac{1}{2\pi i}\int_C \mathrm{Im}\{K_x(g(r;t),\overline{g(r;t)})\}\frac{dt}{t} = 0. \tag{6.11}$$

Hence, $d_{02}(r) = 0$ for $0 < r < \min\{1,R(c_2)\}$.

Further, $\Phi_x(r;0,0)$ is real for real values of r, and also

$$\frac{1}{2\pi i}\int_C \mathrm{Re}\{K_x(g(r;t),\overline{g(r;t)})\}\frac{dt}{t} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{Re}\{K_x(g(r;e^{i\varphi}),\overline{g(r;e^{i\varphi})})\}d\varphi,$$

is real. Consequently, also $d_{01}(r) = 0$ for $0 < r < \min\{1,R(c_2)\}$.

With (6.11) the relations (6.8) and (6.10) reduce to: for $0 < r < \min\{1,R(c_2)\}$, $z \in C^+$,

$$\Omega_1(r;z) = \frac{1}{2\pi i}\int_C \mathrm{Re}\{K_x(g(r;t),\overline{g(r;t)})\}\,\frac{2dt}{t-z} - \Phi_x(r;0,0),$$

$$\Omega_2(r;z) = \frac{1}{2\pi i}\int_C \mathrm{Im}\{K_x(g(r;t),\overline{g(r;t)})\}\,\frac{2dt}{t-z},$$

$$\Phi_x(r;0,0) = \frac{1}{2\pi i}\int_C K_x(g(r;t),\overline{g(r;t)})\}\,\frac{dt}{t}. \tag{6.12}$$

By (6.12) the functions $\Omega_i(r;z)$, $z \in C^+$, $i = 1,2$, and $\Phi_x(r;0,0)$ are completely determined for $0 < r < \min\{1,R(c_2)\}$.

Next it is readily seen that the functions

$$\Phi_x(r;\frac{g(r;z)}{2c_1},0),\ z \in C^+, \qquad \Phi_x(r;0,\frac{g(r;z)}{2c_2}),\ z \in C^+,$$

can be solved from (6.5) and (6.12), for $0 < r < \min\{1,R(c_2)\}$.

Because the conformal mapping $g(r;z)$ has an inverse the function $\Phi_x(r;p_1,0)$ has then been determined for $2c_1p_1 \in L^+(r)$, $0 < r < \min\{1,R(c_2)\}$. Since $0 \in L^+(r)$, $0 < r < 1$, this includes that the power series expansion of the function $\Phi_x(r;p_1,0)$ at $p_1 = 0$ has been obtained. By theorem 2.3 the power series expansion at $p_1 = 0$ converges at least for $|p_1| \leq 1$, so that the function $\Phi_x(r;p_1,0)$ has been uniquely determined for $|p_1| \leq 1$, $0 < r < \min\{1,R(c_2)\}$. Further, for $|p_1| \leq 1$ the function $\Phi_x(r;p_1,0)$ is regular for $|r| < 1$, cf. theorem 2.3 and remark 6.2. Because this function has been obtained on the interval $0 < r < \min\{1,R(c_2)\}$ it is completely determined by analytic continuation (cf. lemma I.1.2) in the disk $|r| < 1$. The above proves that the function $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, is completely determined by (6.5) and by the solutions (6.8) of the Riemann-Hilbert problems formulated in lemma 6.3. Similar argumentation applies for the function $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$. □

<u>REMARK 6.1.</u> By substitution of $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ in equation (2.2) the function $\Phi_x(r;p_1,p_2)$ can be obtained. Hence, theorem 6.1 shows that the

properties of the generating function $\Phi_x(r;p_1,p_2)$ which have been stated in theorem 2.3 are sufficient to determine this function uniquely for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$.

<u>THEOREM 6.2.</u> *For* $0 < r < \min\{1, R(c_2)\}$ *the generating function* $\Phi_x(r;p_1,p_2)$ *is given by: for* $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$,

$$\Phi_x(r;p_1,p_2) = \frac{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2 - r\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)} \left[\frac{p_1^{x_1+1}\, p_2^{x_2+1}}{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)} - \right.$$

$$- \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\, \frac{t+g_0(r;2c_1p_1)}{t-g_0(r;2c_1p_1)}\, \frac{dt}{2t} -$$

$$\left. - \frac{1}{2\pi i}\int_C K_x(g(r;\tfrac{1}{t}),g(r;t))\, \frac{t+g_0(r;2c_2p_2)}{t-g_0(r;2c_2p_2)}\, \frac{dt}{2t}\right]. \tag{6.13}$$

<u>PROOF.</u> From (6.5) and (6.12) it follows that for $0 < r < \min\{1, R(c_2)\}$, $z \in C^+$,

$$\Phi_x(r;\frac{g(r;z)}{2c_1},0) = \left[1 - \frac{g(r;z)}{2c_1}\right] \frac{1}{2\pi i}\int_C K_x(g(r;t),\overline{g(r;t)})\, \frac{dt}{t-z},$$

$$\Phi_x(r;0,\frac{g(r;z)}{2c_2}) = \left[1 - \frac{g(r;z)}{2c_2}\right] \frac{1}{2\pi i}\int_C \overline{K_x(g(r;t),\overline{g(r;t)})}\, \frac{dt}{t-z}.$$

By inserting $z = g_0(r;2c_1p_1)$ and by theorem 5.2 the first relation leads to: for $0 < r < \min\{1, R(c_2)\}$, $2c_1p_1 \in L^+(r)$,

$$\Phi_x(r;p_1,0) = \frac{1-p_1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\, \frac{dt}{t-g_0(r;2c_1p_1)}. \tag{6.14}$$

The second relation gives by inserting $z = g_0(r;2c_2p_2)$ and by using theorem 5.2 and $K_x(\bar{w},w) = \overline{K_x(w,\bar{w})}$, cf. (6.1): for $0 < r < \min\{R(c_2)\}$, $2c_2p_2 \in L^+(r)$,

$$\Phi_x(r;0,p_2) = \frac{1-p_2}{2\pi i} \int_C K_x(g(r;\tfrac{1}{t}),g(r;t)) \frac{dt}{t-g_0(r;2c_2p_2)}. \tag{6.15}$$

The last relation of (6.12) can be rewritten as: for $0 < r < \min\{1,R(c_2)\}$,

$$\Phi_x(r;0,0) = \frac{1}{2\pi i} \int_C [K_x(g(r;t),g(r;\tfrac{1}{t})) + K_x(g(r;\tfrac{1}{t}),g(r;t))] \frac{dt}{2t}. \tag{6.16}$$

Substitution of (6.14), (6.15) and (6.16) in the functional equation (2.2) then readily gives the solution (6.13). □

REMARK 6.2. On account of the recurrence relations (2.1) the functional equation (2.2) must have at least one solution $\Phi_x(r;p_1,p_2)$ which is a generating function of a joint probability distribution in p_1 and p_2, and which is a generating function of a series with coefficients bounded in absolute value by one in r. In theorem 6.1 and 6.2 it has been proved that the functional equation (2.2) has for $0 < r < \min\{1,R(c_2)\}$ at most one solution with the above properties. This implies that (6.13) represents a function regular for $|r| < 1$ on the real interval $0 < r < \min\{1,R(c_2)\}$, and hence that the righthand side of (6.13) possesses an analytic continuation into the domain $|r| < 1$. □

In this section the functional equation (2.2) has been solved by the derivation and solution of two Riemann-Hilbert boundary value problems. In the next section it will be shown that the same result can be obtained by the derivation and solution of one Hilbert boundary value problem. Further we shall show in that section how explicit expressions for the function $\Phi_x(r;p_1,p_2)$ can be obtained for $2c_1p_1 \in L^-(r)$, $2c_2p_2 \in L^-(r)$ and/or $R(c_2) < r < 1$.

II.7. Formulation as a Hilbert problem

In the preceding section the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ have

been determined by solving two Riemann-Hilbert problems with index zero derived from equation (4.18). MUSKHELISHVILI [20], §§39-40, shows that every Riemann-Hilbert problem on the unit circle is equivalent to a Hilbert problem on the unit circle together with an additional functional relation (see also section I.5, remark 5.3).
In this section it will be shown that from equation (4.18) directly one Hilbert problem on the unit circle without additional functional relation can be deduced, again with the aid of the conformal mapping $g(r;z)$ of the unit disk C^+ onto the domain $L^+(r)$. This Hilbert problem has for $0 < r < \min\{1,R(c_2)\}$ index zero and can be solved with the aid of lemma I.4.3. Together with two linear relations the solution of this single Hilbert problem also determines the functions $\Phi_x(r;p_1,0)$, $\Phi_x(r;0,p_2)$, and hence by equation (2.2) the function $\Phi_x(r;p_1,p_2)$, completely.

Further we shall derive in this section explicit expressions for the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ in the regions $|p_1| \leq 1$, $2c_1p_1 \notin L^+(r)$, respectively $|p_2| \leq 1$, $2c_2p_2 \notin L^+(r)$, for $0 < r < \min\{1,R(c_2)\}$, with the aid of the Sochozki-Plemelj formulas and the analytic continuation of the conformal mapping $g_0(r;w)$ outside the domain $L^+(r)$.
For the case $R(c_2) < 1$ it will be shown that for $R(c_2) < r < 1$ also a Hilbert problem on the unit circle can be derived from equation (4.12). This Hilbert problem has index one and can also be solved with the aid of lemma I.4.3. Its solution, together with three linear relations, provides us with an explicit expression for the function $\Phi_x(r;p_1,p_2)$ for $R(c_2) < r < 1$, which is needed in the next section where we shall study the behaviour of the function $\Phi_x(r;p_1,p_2)$ when r tends to one.

As in the preceding sections it is assumed that $0 < c_2 \leq \frac{1}{2}$ *and except in some remarks that* $0 < r < 1$.

LEMMA 7.1. *The functions* $\Phi_x(r;p_1,0)$ *and* $\Phi_x(r;0,p_2)$ *satisfy for* $t \in C$ *the relation*

$$\frac{\Phi_x(r;\frac{1}{2c_1}\, g(r;t),0)}{1-\frac{1}{2c_1}\, g(r;t)} + \frac{\Phi_x(r;0,\frac{1}{2c_2}\, g(r;\frac{1}{t}))}{1-\frac{1}{2c_2}\, g(r;\frac{1}{t})} = K_x(g(r;t),g(r;\frac{1}{t})) + \Phi_x(r;0,0). \tag{7.1}$$

PROOF. As the conformal mapping $g(r;z)$ establishes a one-to-one correspondence between the unit circle C and the contour $L(r)$, cf. corollary 5.4, equation (7.1) follows from equation (4.18) by substituting in the latter $w = g(r;t)$, $t \in C$, and by using theorem 5.2, i.e.

$$\bar{w} = \overline{g(r;t)} = g(r;\bar{t}) = g(r;\frac{1}{t}), \qquad t \in C. \qquad \square$$

By corollary 5.8 and by (6.4) the function

$$\frac{\Phi_x(r;\frac{1}{2c_1}\, g(r;t),0)}{1-g(r;t)/2c_1},$$

belongs to the class $RCB(C^+)$ for every r, $0 < r < 1$. However, for the function

$$\frac{\Phi_x(r;0,\frac{1}{2c_2}\, g(r;\frac{1}{t}))}{1-g(r;1/t)/2c_2},$$

the same three cases as indicated in section II.6 (under formula (6.4)) should be distinguished. Only if $2c_2 \in L^-(r)$, i.e. $0 < r < \min\{1,R(c_2)\}$, (case i), the above function belongs to the class $RCB(C^-)$.

First we shall derive for this case a Hilbert boundary value problem from relation (7.1) and we shall show that its solution gives the same results as the solutions of the two Riemann-Hilbert problems formulated in lemma 6.3. Therefore we introduce the function

$$\psi(r;z) := \frac{\Phi_x(r;\frac{1}{2c_1}\, g(r;z),0)}{1 - g(r;z)/2c_1} - \Phi_x(r;0,0), \qquad z \in C^+,$$

$$\psi(r;z) := \frac{\Phi_x(r;0,\frac{1}{2c_2}\, g(r;\frac{1}{z}))}{1 - g(r;\frac{1}{z})/2c_2}, \qquad z \in C^-. \tag{7.2}$$

<u>LEMMA 7.2.</u> *Let* r *be fixed,* $0 < r < \min\{1,R(c_2)\}$. *Then the function* $\psi(r;z)$ *is a sectionally regular function (with respect to the unit circle), bounded at infinity, and satisfying the boundary condition: for* $t \in C$,

$$\psi^+(r;t) + \psi^-(r;t) = K_x(g(r;t),g(r;\tfrac{1}{t})). \tag{7.3}$$

<u>PROOF.</u> By corollary 5.8, the inequality (6.4) and theorem 5.6 the function $\psi(r;z)$ defined in (7.2) belongs to the classes $RCB(C^+)$ and $RCB(C^-)$ so that by definition I.3.2 it is a sectionally regular function.
From (7.2) and (5.3) it follows that the function $\psi(r;z)$ is bounded at infinity:

$$\lim_{z\to\infty} \psi(r;z) = \Phi_x(r;0,0). \tag{7.4}$$

The condition (7.3) for the boundary values of the function $\psi(r;z)$ at the unit circle, cf. definition I.3.3, is implied by lemma 7.1. □

Because the function $K_x(g(r;t),g(r;\frac{1}{t}))$ belongs to the class H(C) by lemma 6.2, in lemma 7.2 a *Hilbert boundary value problem* has been formulated for the function $\psi(r;z)$, cf. section I.4.

<u>THEOREM 7.1.</u> *The functions* $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, *and* $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$, *are completely determined by* (7.2) *and by the solution of the Hilbert boundary value problem (with index zero) as formulated in lemma* 7.2.

<u>PROOF.</u> The boundary condition (7.3) is of the form as described by formula

(I.4.8), with $n = 0$, i.e. the index of the Hilbert problem is zero. Hence, the complete solution of the Hilbert boundary value problem formulated in lemma 7.2 is given by lemma I.4.3: for $0 < r < \min\{1, R(c_2)\}$,

$$\psi(r;z) = \frac{1}{2\pi i} \int_C K_x(g(r;t), g(r;\tfrac{1}{t})) \frac{dt}{t-z} - p_0(r), \qquad z \in C^+,$$

$$\psi(r;z) = \frac{-1}{2\pi i} \int_C K_x(g(r;t), g(r;\tfrac{1}{t})) \frac{dt}{t-z} + p_0(r), \qquad z \in C^-, \quad (7.5)$$

where $p_0(r)$ is independent of z.

For $z = 0$ relation (7.2) reads (cf.(5.3))

$$\psi(r;0) = 0. \qquad (7.6)$$

From (7.5) it follows with (7.6) and (7.4): for $0 < r < \min\{1, R(c_2)\}$,

$$p_0(r) = \Phi_x(r;0,0) = \frac{1}{2\pi i} \int_C K_x(g(r;t), g(r;\tfrac{1}{t})) \frac{dt}{t}. \qquad (7.7)$$

With (7.7) the relations (7.5) and (7.2) lead to: for $0 < r < \min\{1, R(c_2)\}$,

$$\Phi_x(r;\frac{g(r;z)}{2c_1},0) = \left[1 - \frac{1}{2c_1} g(r;z)\right] \frac{1}{2\pi i} \int_C K_x(g(r;t), g(r;\tfrac{1}{t})) \frac{dt}{t-z}, \qquad z \in C^+,$$

$$\Phi_x(r;0,\frac{1}{2c_2} g(r;\tfrac{1}{z})) = \left[1 - \frac{1}{2c_2} g(r;\tfrac{1}{z})\right] \frac{1}{2\pi i} \int_C K_x(g(r;t), g(r;\tfrac{1}{t})) \frac{z\, dt}{t(z-t)},$$

$$z \in C^-. \qquad (7.8)$$

Consequently, for $0 < r < \min\{1, R(c_2)\}$ the function $\Phi_x(r;p_1,0)$ has been obtained for $2c_1p_1 \in L^+(r)$ and the function $\Phi_x(r;0,p_2)$ for $2c_2p_2 \in L^+(r)$. As in the proof of theorem 6.1 it follows then that these functions are completely determined for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$ (see also remark 6.1). □

We proceed with the derivation of explicit expressions for the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ with $2c_1p_1$ and $2c_2p_2$ outside the domain $L^+(r)$.

THEOREM 7.2. *In the case* $0 < r < \min\{1, R(c_2)\}$, *for* $2c_1p_1 \in L^-(r)$, $|p_1| \leq 1$, $\mathrm{Re}\ 2c_1p_1 \leq 1$,

$$\Phi_x(r;p_1,0) = (1-p_1)\Big[K_x\Big(2c_1p_1, g\Big(r;\frac{1}{g_0(r;2c_1p_1)}\Big)\Big) + \\ + \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\ \frac{dt}{t-g_0(r;2c_1p_1)}\Big], \tag{7.9}$$

and for $2c_2p_2 \in L^-(r)$, $|p_2| \leq 1$,

$$\Phi_x(r;0,p_2) = (1-p_2)\Big[K_x\Big(g\Big(r;\frac{1}{g_0(r;2c_2p_2)}\Big), 2c_2p_2\Big) + \\ + \frac{1}{2\pi i}\int_C K_x(g(r;\tfrac{1}{t}),g(r;t))\ \frac{dt}{t-g_0(r;2c_2p_2)}\Big]. \tag{7.10}$$

PROOF. By lemma (6.2) the function $K_x(g(r;t),g(r;\frac{1}{t}))$ belongs for $0 < r < \min\{1,R(c_2)\}$ to the class $H(C)$. Therefore we may apply lemma I.3.3 (the Sochozki-Plemelj formulas) to the first relation of (7.8) which gives for $t_0 \in C$,

$$\Phi_x(r;\frac{g(r;t_0)}{2c_1},0) = \Big[1-\frac{g(r;t_0)}{2c_1}\Big]\Big[\tfrac{1}{2}K_x(g(r;t_0),g(r;\tfrac{1}{t_0})) + \\ + \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\ \frac{dt}{t-t_0}\Big],$$

where the principle value of the integral has to be understood, cf. def. I.3.1. By substitution of $t_0 = g_0(r;2c_1p_1)$ we obtain for $2c_1p_1 \in L(r)$,

$$\Phi_x(r;p_1,0) = (1-p_1)\Big[\tfrac{1}{2}K_x\Big(2c_1p_1, g\Big(r;\frac{1}{g_0(r;2c_1p_1)}\Big)\Big) + \\ + \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\ \frac{dt}{t-g_0(r;2c_1p_1)}\Big]. \tag{7.11}$$

By theorem 5.4 the function $g_0(r;w)$ is regular for $w \in L^-(r)$, $|w| < 1 + \sqrt{1-4c_1c_2r}$, $\beta\Big(\frac{1-\frac{1}{2}w}{\alpha}\Big) \neq 0$, while if $\beta\Big(\frac{1-\frac{1}{2}w}{\alpha}\Big) = 0$ it has a pole.

With corollary 5.7 it follows that the righthand side of (7.9) is well-defined and regular for $2c_1p_1 \in L^-(r)$, $|p_1| \leqslant 1$, Re $2c_1p_1 \leqslant 1$. Therefore we can apply the same method as above for obtaining the limit of the righthand side of (7.9) as $2c_1p_1$ tends from $L^-(r)$ to the contour. This limit turns out to be equal to the righthand side of (7.11), which proves relation (7.9).

The proof of (7.10) is similar. Here the condition Re $2c_2p_2 \leqslant 1$ can be omitted since $2c_2 \leqslant 1$ and $|p_2| \leqslant 1$. □

<u>REMARK.</u> Relation (7.9) is also valid for Re $p_1 > \frac{1}{2c_1}$, $|p_1| \leqslant 1$, if $g_0(r;2c_1p_1) > 1$, cf. remark 5.1. In case $g_0(r;2c_1p_1) \leqslant 1$ for Re $p_1 > \frac{1}{2c_1}$, $|p_1| \leqslant 1$, lemma I.3.3 has to be applied again for obtaining the function $\Phi_x(r;p_1,0)$ at these points.

Next we shall derive in the case $R(c_2) < 1$ for $R(c_2) < r < 1$ an explicit expression for the function $\Phi_x(r;p_1,p_2)$, $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$. This will be done by formulating another Hilbert boundary value problem with boundary condition (7.1).

In the following it is assumed that $R(c_2) < 1$ *and that* r *is fixed,* $R(c_2) < r < 1$.

<u>LEMMA 7.3.</u> *If* $R(c_2) < r < 1$ *then the equation*

$$g(r;z) = 2c_2, \qquad z \in C^+, \tag{7.12}$$

has exactly one root. This root is real and positive.

<u>PROOF.</u> By theorem 5.6 we have $2c_2 \in L^+(r)$ if $R(c_2) < r < 1$. As the function $g(r;z)$ establishes a one-to-one correspondence between the domains C^+ and $L^+(r)$ equation (7.12) has exactly one root. From theorem 5.2 it follows that this root is real, and from the conditions (5.3) that it is positive. □

The root of (7.12) will be denoted by $z = z_0(r)$, i.e. for $R(c_2) < r < 1$,

$$g(r;z_0(r)) = 2c_2, \qquad 0 < z_0(r) < 1. \tag{7.13}$$

Lemma 7.3 implies that the function $\psi(r;z)$ defined in (7.2) is for $R(c_2) < r < 1$ not a sectionally regular function because it has a first order pole at $z = \frac{1}{z_0(r)}$. However in several ways the determination of the functions $\Phi_x(r;p_1,0)$ and $\Phi_x(r;0,p_2)$ can again be reduced to the solution of a Hilbert problem as may be seen as follows. One possibility is to consider the function

$$\left[z-\frac{1}{z_0(r)}\right]\psi(r;z),$$

which is again sectionally regular, cf. (7.2).
We shall use a different approach by which the pole at $z = \frac{1}{z_0(r)}$ is transferred to infinity.

Define the function

$$e(r;z) := g\left(r;\frac{z+z_0(r)}{zz_0(r)+1}\right), \qquad |z| < 1. \tag{7.14}$$

LEMMA 7.4. *The function* $e(r;z)$ *is the conformal mapping of the unit disk* C^+ *onto the domain* $L^+(r)$ *such that*

$$e(r;0) = 2c_2, \qquad e'(r;0) > 0. \tag{7.15}$$

The function $e(r;z)$ *is continuous up to the boundary* C, *and*

$$e(r;\bar{z}) = \overline{e(r;z)}, \qquad |z| \leqslant 1.$$

PROOF. From lemma I.6.10 it follows that the conformal mapping of C^+ onto $L^+(r)$ satisfying the conditions (7.15) is given by (7.14). The continuity and symmetry properties of the function $e(r;z)$ follow readily from those of the function $g(r;z)$, cf. corollary 5.4 and theorem 5.2. □

LEMMA 7.5. *The functions* $\Phi_x(r;p_1,0)$ *and* $\Phi_x(r;0,p_2)$ *satisfy for* $t \in C$ *the relation*

$$\frac{\Phi_x(r;\frac{1}{2c_1}\,e(r;t),0)}{1 - e(r;t)/2c_1} + \frac{\Phi_x(r;0,\frac{1}{2c_2}\,e(r;\frac{1}{t}))}{1 - e(r;\frac{1}{t})/2c_2} = K_x(e(r;t),e(r;\tfrac{1}{t})) + \Phi_x(r;0,0). \tag{7.16}$$

PROOF. This assertion can be shown in a similar way as that of lemma 7.1 by using the properties of the function $e(r;z)$ as described in lemma 7.4. □

We introduce the function

$$E(r;z) := \frac{\Phi_x(r;\frac{1}{2c_1}\,e(r;z),0)}{1 - e(r;z)/2c_1} - \Phi_x(r;0,0), \qquad z \in C^+,$$

$$E(r;z) := \frac{\Phi_x(r;0,\frac{1}{2c_2}\,e(r;\frac{1}{z}))}{z[1 - e(r;\frac{1}{z})/2c_2]}, \qquad z \in C^-. \tag{7.17}$$

LEMMA 7.6. *Let* r *be fixed,* $R(c_2) < r < 1$. *Then the function* $E(r;z)$ *is a sectionally regular function (with respect to the unit circle), bounded at infinity, and satisfying the boundary condition: for* $t \in C$,

$$E^+(r;t) + t\,E^-(r;t) = K_x(e(r;t),e(r;\tfrac{1}{t})). \tag{7.18}$$

PROOF. From (7.17) and (7.2) it follows with (7.14) that

$$E(r;z) = \psi\left(r;\frac{z+z_0(r)}{zz_0(r)+1}\right), \qquad z \in C^+,$$

$$E(r;z) = \frac{1}{z}\psi\left(r;\frac{z+z_0(r)}{zz_0(r)+1}\right), \qquad z \in C^-.$$

With the same arguments as in the proof of lemma 7.2 and by using lemma 7.3 it follows that the function $\psi(r;z)$ is regular in C^+ and also in C^- except at the point $z = \frac{1}{z_0(r)}$ where it has a first order pole. Because, cf. (I.6.11),

$$z \to \frac{z+z_0(r)}{zz_0(r)+1}$$

performs a conformal mapping of $C^- \cup \{\infty\}$ onto itself it follows with the above relations that the function $E(r;z)$ is sectionally regular, cf. definition I.3.2.

Further, lemma 7.4 and theorem 4.1 imply with (7.17) that

$$\lim_{z\to\infty} E(r;z) = \Phi_x(r;0,1) \lim_{z\to 0} \frac{z}{1 - e(r;z)/2c_2} = -\frac{2c_2}{e'(r;0)} \frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}. \tag{7.19}$$

Hence the function $E(r;z)$ is bounded at infinity.

Finally, the condition (7.18) for the boundary values of the function $E(r;z)$ at the unit circle, cf. definition I.3.3, follows from lemma 7.5. □

<u>REMARK.</u> It can be readily verified that the function $K_x(e(r;t),e(r;\frac{1}{t}))$ belongs to the class H(C), cf. lemma 6.2. Hence, in lemma 7.6 a *Hilbert boundary value problem* has been formulated for the function $E(r;z)$, cf. section I.4.

<u>THEOREM 7.3.</u> *The functions* $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, *and* $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$, *are completely determined by* (7.17), (7.19) *and by the solution of the Hilbert boundary value problem (with index one) as formulated in lemma* 7.6.

<u>PROOF.</u> The boundary condition (7.18) is of the form as described by formula (I.4.8), with n = 1, i.e. the index of the Hilbert problem is one. Hence, the complete solution of the Hilbert boundary value problem formulated in lemma 7.6 is given by lemma I.4.3: for $R(c_2) < r < 1$,

$$E(r;z) = \frac{1}{2\pi i} \int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\frac{dt}{t-z} - p_0(r) - zp_1(r), \qquad z \in C^+,$$

$$E(r;z) = \frac{-1}{2\pi i z} \int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\frac{dt}{t-z} + \frac{1}{z}p_0(r) + p_1(r), \qquad z \in C^-, \tag{7.20}$$

where $p_0(r)$ and $p_1(r)$ are independent of z.

For $z = -z_0(r)$ and $\frac{1}{z} = -z_0(r)$ relation (7.17) reads (cf. (7.15))

$$E(r;-z_0(r)) = 0, \qquad E\left(r;-\frac{1}{z_0(r)}\right) = -z_0(r)\ \Phi_x(r;0,0). \tag{7.21}$$

From (7.20) we obtain with (7.19) and (7.21) the following three relations: for $R(c_2) < r < 1$,

$$p_1(r) = -\frac{2c_2}{e'(r;0)} \frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)},$$

$$p_0(r) = \frac{1}{2\pi i} \int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\frac{dt}{t+z_0(r)} + z_0(r)p_1(r),$$

$$\Phi_x(r;0,0) = p_0(r) + \frac{p_1(r)}{z_0(r)} - \frac{z_0(r)}{2\pi i} \int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\ \frac{dt}{tz_0(r)+1}. \tag{7.22}$$

Clearly these relations determine $p_0(r)$, $p_1(r)$ and $\Phi_x(r;0,0)$ so that for $R(c_2) < r < 1$ the functions $\Phi_x(r;p_1,0)$, $2c_1p_1 \in L^+(r)$, and $\Phi_x(r;0,p_2)$, $2c_2p_2 \in L^+(r)$, are completely determined by (7.17), (7.20) and (7.22). As in the proof of theorem 6.1 it can be shown that then these functions are also completely determined for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$. □

For the case $R(c_2) < 1$ theorem 7.3 provides us with another approach for the determination of the functions $\Phi_x(r;p_1,0)$, $|r| < 1$, $|p_1| \leq 1$, and $\Phi_x(r;0,p_2)$, $|r| < 1$, $|p_2| \leq 1$, beside the approach of theorem 7.1. But what is more important we obtain from the procedure of theorem 7.3 expressions for these functions for $R(c_2) < r < 1$, see theorem 7.4 below. For better agreement with theorem 6.2 the results of theorem 7.3 will be transformed into expressions containing the conformal mapping $g(r;z)$ instead of $e(r;z)$.

<u>THEOREM 7.4.</u> *If* $R(c_2) < 1$ *then the generating function* $\Phi_x(r;p_1,p_2)$ *is, for* $R(c_2) < r < 1$, *given by: for* $2c_1p_1 \in L^+(r)$, $2c_2p_2 \in L^+(r)$,

$$\Phi_x(r;p_1,p_2) = \frac{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2-r\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}\left[\frac{p_1^{x_1+1}\,p_2^{x_2+1}}{r(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)} -\right.$$

$$-\frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{t+g_0(r;2c_1p_1)}{t-g_0(r;2c_1p_1)}\,\frac{dt}{2t} -$$

$$-\frac{1}{2\pi i}\int_C K_x(g(r;\tfrac{1}{t}),g(r;t))\,\frac{t+g_0(r;2c_2p_2)}{t-g_0(r;2c_2p_2)}\,\frac{dt}{2t} -$$

$$\left.-\frac{2c_2}{g'(r;z_0(r))}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}\,\frac{1-g_0(r;2c_1p_1)g_0(r;2c_2p_2)}{[1-z_0(r)g_0(r;2c_1p_1)][z_0(r)-g_0(r;2c_2p_2)]}\right], \quad (7.23)$$

where, cf. (5.6),

$$g'(r;z_0(r)) = \frac{\partial}{\partial z}\,g(r;z)\Big|_{z=z_0(r)}.$$

<u>PROOF.</u> By substituting $z = e_0(r;2c_1p_1)$ we obtain from (7.17), (7.20) and (7.22) for $R(c_2) < r < 1$, $2c_1p_1 \in L^+(r)$,

$$\Phi_x(r;p_1,0) = (1-p_1)\left[\frac{1}{2\pi i}\int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\,\frac{dt}{t-e_0(r;2c_1p_1)} -\right.$$

$$-\frac{z_0(r)}{2\pi i}\int_C K_x(e(r;t),e(r;\tfrac{1}{t}))\,\frac{dt}{tz_0(r)+1} +$$

$$\left.+\left\{e_0(r;2c_1p_1)+\frac{1}{z_0(r)}\right\}\frac{2c_2}{e'(r;0)}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}\right].$$

By using the definition of the function $e(r;z)$, cf. (7.14), which implies

$$e(r;\tfrac{1}{t}) = g\left(r;\frac{tz_0(r)+1}{t+z_0(r)}\right), \qquad |t| = 1,$$

$$e'(r;0) = [1-z_0^2(r)]g'(r;z_0(r)),$$

$$e_0(r;w) = \frac{g_0(r;w)-z_0(r)}{1-z_0(r)g_0(r;w)}, \qquad w \in L^+(r),$$

the substitution $\bar{u} = \frac{t+z_0(r)}{tz_0(r)+1}$ in the integrals above leads to: for $2c_1p_1 \in L^+(r)$, $R(c_2) < r < 1$, (writing t for u),

$$\Phi_x(r;p_1,0) = (1-p_1)\Big[\frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{dt}{t-g_0(r;2c_1p_1)} + \\ + \frac{2c_2}{z_0(r)g'(r;z_0(r))[1-z_0(r)g_0(r;2c_1p_1)]}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}\Big]. \tag{7.24}$$

In a similar way it can be obtained from (7.17), (7.20) and (7.22): for $R(c_2) < r < 1$, $2c_2p_2 \in L^+(r)$,

$$\Phi_x(r;0,p_2) = (1-p_2)\Big[\frac{1}{2\pi i}\int_C K_x(g(r;\tfrac{1}{t}),g(r;t))\,\frac{dt}{t-g_0(r;2c_2p_2)} + \\ + \frac{2c_2}{g'(r;z_0(r))[z_0(r)-g_0(r;2c_2p_2)]}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}\Big]. \tag{7.25}$$

From (7.24) and (7.25) it follows, cf. (6.16): for $R(c_2) < r < 1$,

$$\Phi_x(r;0,0) = \frac{1}{2\pi i}\int_C \{K_x(g(r;t),g(r;\tfrac{1}{t})) + K_x(g(r;\tfrac{1}{t}),g(r;t))\}\,\frac{dt}{2t} + \\ + \frac{2c_2}{z_0(r)g'(r;z_0(r))}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}. \tag{7.26}$$

Substitution of (7.24), (7.25) and (7.26) in (2.2) then leads to (7.23). □

In a similar way as in theorem 7.2 also in the case $R(c_2) < r < 1$ expressions can be derived for $\Phi_x(r;p_1,0)$, $2c_1p_1 \in L^-(r)$, $|p_1| \leqslant 1$, Re $2c_1p_1 \leqslant 1$, from (7.24), and for $\Phi_x(r;0,p_2)$, $2c_2p_2 \in L^-(r)$, $|p_2| \leqslant 1$, from (7.25). This will be omitted here.

For the sake of completeness and for later reference we show below how explicit expressions for the function $\Phi_x(r;p_1,p_2)$ can be obtained for $r = R(c_2)$ if $R(c_2) < 1$. We confine ourselves to the case $p_1 = p_2 = 0$.

THEOREM 7.5. *In the case* $R(c_2) < 1$, *for* $r = R(c_2)$,

$$\Phi_x(r;0,0) = \frac{1}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t} + \frac{c_1c_2}{(c_1-c_2)g'(r;1)} \left(\frac{c_2}{c_1}\right)^{x_1}, \tag{7.27}$$

here the function $K_x(g(r;t),g(r;\frac{1}{t}))$ *has a first order pole at* $t=1$, *and the integral has to be understood as a principle value, cf. definition* 3.1.

PROOF. Let $0 < r < R(c_2) < 1$. Then by (7.7) we have

$$\lim_{r\uparrow R(c_2)} \Phi_x(r;0,0) = \lim_{r\uparrow R(c_2)} \frac{1}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t}. \tag{7.28}$$

By theorem 5.6 the fact that $R(c_2) < 1$ implies that $c_2 < \frac{1}{2} < c_1$. From the same theorem and (5.9) it follows that

$$g(R(c_2);1) = \delta_2(R(c_2)) = 2c_2. \tag{7.29}$$

Hence the factor $1 - \frac{1}{2c_2} g(r;\frac{1}{t})$ in the denominator of $K_x(g(r;t),g(r;\frac{1}{t}))$, cf. (6.1), has for $r = R(c_2)$ a zero at $t = 1$, but no other zeros for $t \in C$. Therefore consider for $r_0(c_2) \leqslant r < R(c_2)$, $t \in C$ (cf. lemma 5.9 for $r_0(c_2)$) the function

$$K_1(r;t) := \left[t - \frac{1}{t_0(r)}\right] K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{1}{t}, \tag{7.30}$$

here $t_0(r)$ is defined by (5.45). Because $c_2 \neq c_1$ the function $K_1(r;t)$ is finite for $t \in C$ for $r_0(c_2) \leqslant r \leqslant R(c_2)$, cf. (6.1).

Since by theorem 5.5 for $0 < r < 1$ the function $g(r;t)$ is regular for $t \in C$ and its derivative does not vanish for $t \in C$, it follows readily with (5.45) and theorem 5.3 that

$$\frac{\partial}{\partial t}\left[t-\frac{r}{t_0(r)}\right]\left[1-\frac{1}{2c_2}\,g(r;\tfrac{1}{t})\right]^{-1},$$

is uniformly bounded for $t \in C$, $r_0(c_2) \leqslant r \leqslant R(c_2)$. This implies, cf. (6.1), that for some positive constant $M > 0$ the inequality

$$|K_1(r;t) - K_1(r;1)| \leqslant M|t-1|, \qquad t \in C,$$

holds uniformly for $r_0(c_2) \leqslant r \leqslant R(c_2)$.

From the above and lemma 5.9 it is seen that lemma I.3.6 can be applied to the integral

$$\frac{1}{2\pi i}\int_C K_1(r;t)\,\frac{dt}{t-\frac{1}{t_0(r)}}.$$

With (7.30), (7.28) and lemma 5.9 it is obtained from (I.3.11):

$$\lim_{r\uparrow R(c_2)} \Phi_x(r;0,0) = \tfrac{1}{2}K_1(R(c_2);1) + \frac{1}{2\pi i}\int_C K_1(R(c_2);t)\,\frac{dt}{t-1}, \qquad (7.31)$$

here the principle value of the singular integral has to be understood.

Finally, from (7.30), (7.29) and (6.1) it follows that

$$K_1(R(c_2);1) = \frac{[c_2/c_1]^{x_1}}{1-c_2/c_1}\,\frac{2c_2}{g'(R(c_2);1)},$$

so that the assertion follows from (7.31) and (7.30). □

Note that for $R(c_2) < r < 1$ the function $\Phi_x(r;0,0)$ is given by (7.26). In a similar way as above, by using $z_0(r)$, cf. (7.13), instead of $t_0(r)$, it can be shown with the aid of (I.3.10) that

$$\lim_{r\downarrow R(c_2)} \Phi_x(r;0,0) = -\frac{c_1c_2}{(c_1-c_2)g'(R(c_2);1)}\left(\frac{c_2}{c_1}\right)^{x_1} +$$

$$+\frac{1}{2\pi i}\int_C K_x(g(R(c_2);t),g(R(c_2);\tfrac{1}{t}))\,\frac{dt}{t} + \frac{2c_2}{g'(R(c_2);1)}\,\frac{[\mu_1(R(c_2))]^{x_1}}{1-\mu_1(R(c_2))}. \qquad (7.32)$$

Here $\mu_1(R(c_2))$ is the unique solution of the equation, cf. theorem 4.1,

$$p_1 = R(c_2)\ \beta\left(\frac{c_1 - c_1 p_1}{\alpha}\right), \qquad |p_1| < 1.$$

From the definition of $R(c_2)$, cf. (5.44), it follows that also $p_1 = c_2/c_1$ satisfies this equation. Hence

$$\mu_1(R(c_2)) = c_2/c_1,$$

and it is clear that (7.32) also implies (7.27), which agrees with the fact that $\Phi_x(r;0,0)$ is regular for $|r| < 1$.

II.8. Conditions for ergodicity, the stationary distribution

In theorem 2.2 it has been proved that the imbedded Markov chain $\{(\underline{x}_1(n), \underline{x}_2(n)),\ n = 0,1,2,\ldots\}$ which stands for the number of type 1 and of type 2 customers left behind in the system at the n^{th} service completion instant is irreducible and aperiodic. The states of an irreducible a-periodic Markov chain are said to be *ergodic* if they have a finite mean recurrence time; they are called *null states* if their mean recurrence time is infinite, but the recurrence time is finite with probability one; and they are called *transient* if their recurrence time is not finite with probability one. By a general theorem for irreducible Markov chains it is stated that all states are of the same type, cf. FELLER [10], §XV.6. An irreducible aperiodic Markov chain possesses a *stationary* or *invariant* distribution if and only if its states are ergodic (see FELLER [10], §XV.7, and lemma 8.1 below).

In this section it will be shown on which conditions the Markov chain $\{(\underline{x}_1(n), \underline{x}_2(n)),\ n = 0,1,2,\ldots\}$ consists of ergodic, null or transient states respectively. In the ergodic case the generating function of the joint stationary distribution of the chain will be derived.

From remark 4.1 and the theory of the M/G/1-queueing system it follows that $c_1 a < 1$ and $c_2 a < 1$ are necessary conditions for this Markov chain to be ergodic. In this section it will be proved that these conditions are also sufficient.

For n = 0,1,2,.., for x_1, x_2, k_1, k_2 = 0,1,2... denote the conditional probability that at the n^{th} service completion instant k_i customers of type i (i = 1,2) are left behind in the system if x_i customers of type i (i = 1,2) are present in the system at t = 0 by: (here $x = (x_1,x_2)$, cf. definition 1.2),

$$p_x^{(n)}(k_1,k_2) := \Pr\{\underline{x}_1(n) = k_1,\ \underline{x}_2(n) = k_2 | \underline{x}_1(0) = x_1,\ \underline{x}_2(0) = x_2\}. \tag{8.1}$$

Since by theorem 2.2 the Markov chain $\{(\underline{x}_1(n), \underline{x}_2(n)), n = 0,1,2,...\}$ is aperiodic and irreducible the following statement follows readily from the theorems in FELLER [10], §§XV.5,6,7.

<u>LEMMA 8.1.</u> *For* $k_1,k_2 = 0,1,2,...$ *the limits*

$$u(k_1,k_2) := \lim_{n\to\infty} p_x^{(n)}(k_1,k_2), \tag{8.2}$$

exist and are independent of the initial state $x = (x_1,x_2)$.
If the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)), n = 0,1,2,...\}$ *is ergodic then* $u(k_1,k_2) > 0$ *for every* $k_1,k_2 = 0,1,2,...,$ *and*

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} u(k_1,k_2) = 1, \tag{8.3}$$

while if this chain is not ergodic then $u(k_1,k_2) = 0$ *for* $k_1,k_2 = 0,1,2,....$

In the ergodic case $u(k_1,k_2)$, $k_1,k_2 = 0,1,2,...,$ is called the *stationary* or the *invariant* distribution of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),$ $n = 0,1,...\}$, and we define for $|p_1| \leq 1$, $|p_2| \leq 1$ the generating function

$$\Phi(p_1,p_2) := \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_1^{k_1} p_2^{k_2} u(k_1,k_2). \tag{8.4}$$

Noting that, cf. (1.10) and (8.1), for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\Phi_x(r;p_1,p_2) = \sum_{n=0}^{\infty} r^n \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p_1^{k_1} p_2^{k_2} p_x^{(n)}(k_1,k_2), \tag{8.5}$$

it follows from lemma 8.1 and an Abelian theorem for generating functions:

<u>COROLLARY 8.1.</u> *If the states of the Markov chain* $\{(\underline{x}_1(n),\underline{x}_2(n)),$ $n = 0,1,2,\ldots\}$ *are ergodic then for* $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\lim_{r\uparrow 1} (1-r)\Phi_x(r;p_1,p_2) = \Phi(p_1,p_2), \tag{8.6}$$

and if they are not ergodic then for $|p_1| \leq 1$, $p_1 \neq 1$, $|p_2| \leq 1$, $p_2 \neq 1$,

$$\lim_{r\uparrow 1} (1-r)\Phi_x(r;p_1,p_2) = 0, \tag{8.7}$$

both limits being independent of the initial state $x = (x_1,x_2)$.

Because all states of an irreducible Markov chain are of the same type it is sufficient to investigate the state (0,0), i.e. we have to consider

$$\lim_{r\uparrow 1} (1-r)\Phi_x(r;0,0),$$

in order to obtain the conditions on which the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),$ $n = 0,1,2,\ldots\}$ is ergodic.

For the determination of this limit from the expressions for the function $\Phi_x(r;0,0)$ which have been found in section II.6, cf. (6.16), and in section II.7, cf.(7.26), we shall subsequently study the behaviour for $r \uparrow 1$ of the contour $L(r)$, of the conformal mapping $g(r;z)$, $z \in C^+$, and of its derivative, especially at the point $z = 1$, and finally of the integral

$$\frac{1}{2\pi i} \int_C K_x(g(r;t), g(r,\frac{1}{t})) \frac{dt}{t}.$$

From lemma 5.2 it is seen that when r tends to one the contour L(r) expands (cf. corollary 5.3) to the contour

$$L(1) := \{w;\ |w|^2 = 4c_1c_2\ \beta\left(\frac{1-\text{Re } w}{\alpha}\right),\ \text{Re } w \leq 1\}. \tag{8.8}$$

This contour has the following properties.

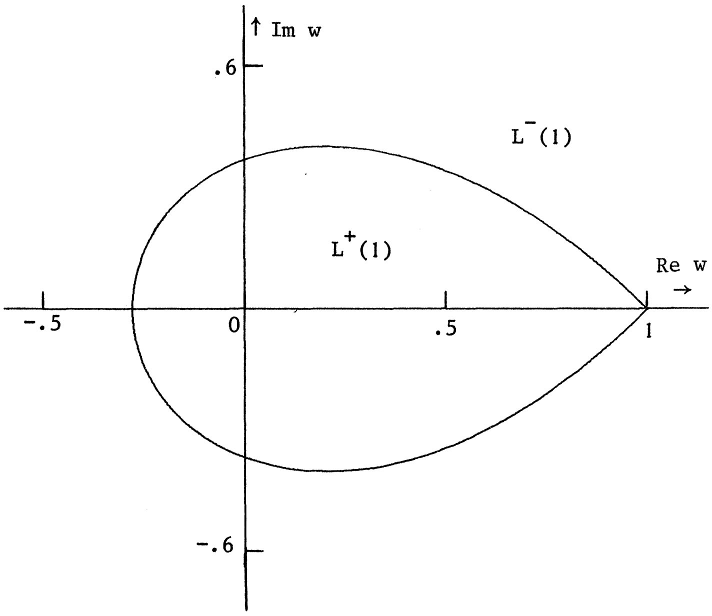


Figure 8.1.

The contour L(1) for $a = 2.00$, $c_1 = c_2 = \frac{1}{2}$, and a degenerate service time distribution; here the corner point at w = 1 has an inner angle of 90^o

<u>LEMMA 8.2.</u> *The contour* L(1) *is smooth, except in the case* $c_1 = c_2 = \frac{1}{2}$, $a = 2$, $\beta_2 < \infty$, *at the point* $w = 1$; *then it has a corner point with inner angle* $\omega\pi$, $\frac{1}{2} \leq \omega < 1$, *and*

$$\omega\pi = 2 \arctan \sqrt{\tfrac{1}{2}\beta_2/\alpha^2 - 1}. \tag{8.9}$$

<u>PROOF</u>. The contour L(1) has the parametric equation, cf. lemma 5.1,

$$w = x(\theta) + i\, y(\theta), \qquad -\pi \leqslant \theta \leqslant \pi,$$

where

$$x(\theta) = h(\cos\theta), \qquad y(\theta) = \tan\theta\; h(\cos\theta), \qquad -\pi \leqslant \theta \leqslant \pi.$$

By lemma 4.2 the derivative $h'(t)$ exists for $-1 \leqslant t \leqslant 1$ except in the case $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$. By using further the arguments in the proof of lemma 5.1 it follows that $L(1)$ is a smooth contour except in the case $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, and that in the latter case $L(1)$ is a contour with a tangent everywhere except possibly at the point $w = 1$. From lemma 4.2 we obtain in the case $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$,

$$\lim_{\theta \downarrow 0} \sin\theta\; h'(\cos\theta) = \lim_{t \uparrow 1} \sqrt{1+t}\,\sqrt{1-t}\; h'(t) = \{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}},$$

$$\lim_{\theta \uparrow 0} \sin\theta\; h'(\cos\theta) = \lim_{t \uparrow 1} -\sqrt{(1+t)}\,\sqrt{1-t}\; h'(t) = -\{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}}.$$

Consequently, because

$$x'(\theta) = -\sin\theta\; h'(\cos\theta),$$

$$y'(\theta) = \frac{h(\cos\theta)}{\cos^2\theta} - \frac{\sin^2\theta}{\cos\theta}\, h'(\cos\theta),$$

we have in the case $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$,

$$\lim_{\theta \downarrow 0} x'(\theta) = -\{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}}, \qquad \lim_{\theta \uparrow 0} x'(\theta) = \{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}},$$

$$\lim_{\theta \downarrow 0} y'(\theta) = 1, \qquad \lim_{\theta \uparrow 0} y'(\theta) = 1.$$

From this it is readily seen that the contour $L(1)$ has at the point $w = 1$ an inner angle $\omega\pi$, with

$$\omega\pi = 2 \arctan\sqrt{\tfrac{1}{2}\beta_2/\alpha^2 - 1}.$$

Because $\beta_2 \geqslant \beta^2$, and $\beta = 2\alpha$ ($\alpha = 2$), it follows that for $\beta_2 < \infty$,

$\frac{1}{2} \leqslant \omega < 1.$

In the case $\beta_2 = \infty$ we have $\omega = 1$ so that the contour L(1) has again a tangent at $w = 1$, and hence it is a smooth contour. □

THEOREM 8.1. *There exists a conformal mapping* $g(1;z)$ *of the unit disk* C^+ *onto the domain* $L^+(1)$ *which is uniquely determined by the conditions*

$$g(1;0) = 0, \qquad g'(1;0) > 0. \tag{8.10}$$

This conformal mapping is continuous up to the boundary C *and it establishes a one-to-one correspondence between* C *and* L(1).
The conformal mapping $g(1;z)$ *has the symmetry property*

$$g(1;\bar{z}) = \overline{g(1;z)}, \qquad z \in C^+ \cup C.$$

This conformal mapping is given by

$$g(1;z) = z \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{h(\cos\theta(1;\varphi))}{\cos\theta(1;\varphi)}\right] \frac{e^{i\varphi}+z}{e^{i\varphi}-z}\, d\varphi\right\}, \qquad |z| < 1,$$

$$g(1;e^{i\varphi}) = \frac{h(\cos\theta(1;\varphi))}{\cos\theta(1;\varphi)}\, e^{i\theta(1;\varphi)}, \qquad -\pi \leqslant \varphi \leqslant \pi, \tag{8.11}$$

where $\theta(1;\varphi)$, $-\pi \leqslant \varphi \leqslant \pi$, *is the unique strictly increasing solution of Theodorsen's singular integral equation*

$$\theta(1;\varphi) = \varphi - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{h(\cos\theta(1;\omega))}{\cos\theta(1;\omega)}\right] \cot\left(\frac{\omega-\varphi}{2}\right) d\omega. \tag{8.12}$$

PROOF. As $L^+(1)$ is a simply connected domain (cf. section I.1 and corollary 5.1) it follows from lemma I.6.2 that the conformal mapping $g(1;z)$ of C^+ onto $L^+(1)$ exists, and from lemma I.6.3 that it is uniquely determined by the conditions (8.10).
Because L(1) is a contour it follows from lemma I.6.4 that $g(1;z)$ is continuous up to C, and that it estabishes a one-to-one continuous mapping between C and L(1).

The representation (8.11), (8.12) for the conformal mapping $g(1;z)$ follows from lemma I.6.7 and I.6.8, cf. (I.6.6) and (I.6.8).
Finally, lemma I.6.9 shows the symmetry property. □

Next we shall investigate the boundary behaviour of the conformal mapping $g(1;z)$, $z \in C^+$, in more detail, especially at the point $z = 1$. For this we shall consider the function

$$\rho(\theta) := \frac{h(\cos\theta)}{\cos\theta}, \qquad -\pi \leq \theta \leq \pi, \tag{8.13}$$

and its derivatives.

<u>LEMMA 8.3.</u> *The derivative* $\rho'(\theta)$ *of the above defined function* $\rho(\theta)$ *is a continuous function on* $[-\pi,\pi]$ *vanishing at* $\theta = 0$, *except in the case* $c_1 = c_2 = \frac{1}{2}$, $a = 2$, $\beta_2 < \infty$, *then it is continuous on* $[-\pi,0)$ *and* $(0,\pi]$, *while*

$$\rho'(0-) := \lim_{\theta\uparrow 0} \rho'(\theta) = \{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}},$$

$$\rho'(0+) := \lim_{\theta\downarrow 0} \rho'(\theta) = -\{\tfrac{1}{2}\beta_2/\alpha^2 - 1\}^{-\frac{1}{2}}.$$

The second derivative $\rho''(\theta)$ *is continuous on* $[-\pi,\pi]$, *except in the case* $c_1 = c_2 = \frac{1}{2}$ *and* $a = 2$ *at the point* $\theta = 0$.

<u>PROOF.</u> It is readily obtained that for $-\pi \leq \theta \leq \pi$, $\theta \neq 0$,

$$\rho'(\theta) = -\frac{\sin\theta}{\cos^2\theta}[\cos\theta\, h'(\cos\theta) - h(\cos\theta)]. \tag{8.14}$$

The continuity of $\rho'(\theta)$ at the points $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$ can be shown similarly as in lemma 5.1 for the function $y'(\theta)$. By lemma 4.2 the derivative $h'(t)$ is continuous on $[-1,1]$ except in the case $c_1 = c_2 = \frac{1}{2}$, $a = 2$ at the point $t = 1$. Hence, $\rho'(\theta)$ exists and is a continuous function on $[-\pi,\pi]$ except possibly in the case $c_1 = c_2 = \frac{1}{2}$, $a = 2$ at the point $\theta = 0$.

If $c_1 = c_2 = \frac{1}{2}, \alpha = 2$, $\beta_2 < \infty$, it follows, cf. the proof of lemma 8.2, that

$$\lim_{\theta\uparrow 0} \rho'(\theta) = -\lim_{\theta\downarrow 0} \rho'(\theta) = \{\tfrac{1}{2}\beta_2/\alpha^2-1\}^{-\frac{1}{2}},$$

so that $\rho'(\theta)$ has a discontinuity at $\theta = 0$. In every other case, also when $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, $\beta_2 = \infty$, the function $\rho'(\theta)$ vanishes at $\theta = 0$, and is continuous at this point.

For the second derivative we find, for $-\pi \leq \theta \leq \pi$, $\theta \neq 0$,

$$\rho''(\theta) = -\frac{1+\sin^2\theta}{\cos^3\theta}[\cos\theta\, h'(\cos\theta)-h(\cos\theta)] + \frac{\sin^2\theta}{\cos\theta} h''(\cos\theta). \tag{8.15}$$

By using (4.5) it can be shown that $\rho''(\theta)$ is continuous at the points $\theta = \frac{1}{2}\pi$ and $\theta = -\frac{1}{2}\pi$. From lemma 4.2 it follows that $\rho''(\theta)$ is a continuous function on $[-\pi,\pi]$ except possibly in the cases $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, and $c_1 = c_2 = \frac{1}{2}$, $\beta_2 = \infty$, at the point $\theta = 0$ because $h''(t)$ does not exist at $t = 1$ in these cases. However, when $c_1 = c_2 = \frac{1}{2}$, $\alpha < 2$, $\beta_2 = \infty$, lemma 4.2 gives

$$\lim_{\theta\to 0} \sin^2\theta\, h''(\cos\theta) = \lim_{t\uparrow 1} (1-t^2)\, h''(t) = 0,$$

so that in this case $\rho''(\theta)$ is also continuous at $\theta = 0$. Only when $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$ the existence of $\rho''(\theta)$ at $\theta = 0$ is not assured. □

We proceed with introducing a class of Laplace-Stieltjes transforms $\beta(s)$ for which the corresponding function $\rho'(\theta)$ satisfies for $c_1 = c_2 = \frac{1}{2}$, $\alpha = 2$, a Hölder condition on $[-\pi,0-]$ and $[0+,\pi]$.

<u>DEFINITION 8.1.</u> *A Laplace-Stieltjes transform $\beta(s)$ of a probability distribution of a positive random variable with finite first moment is said to belong to the class* LH,

i. when $\beta_2 < \infty$ if there exists a constant $\nu_1 > 0$ such that

$$\lim_{s\to 0} |\{\beta(s)-1+\beta s-\tfrac{1}{2}\beta_2 s^2\}/s^{2+\nu_1}| < \infty, \tag{8.16}$$

ii. when $\beta_2 = \infty$ *if there exists a constant* $\nu_2 > 0$ *such that*

$$\lim_{s\to 0} |s^{2-\nu_2}/\{\beta(s)-1+\beta s\}| < \infty. \tag{8.17}$$

(Note that the limits (8.16) *and* (8.17) *vanish for* $\nu_1 = 0$ *and* $\nu_2 = 0$ *resp.)*

Lemma 8.4. *Let* $c_1 = c_2 = \frac{1}{2}$, $a = 2$. *If the Laplace-Stieltjes transform* $\beta(s) \in$ LH *then there exists a constant* $\mu > 0$ *such that the derivative* $\rho'(\theta)$ *satisfies the Hölder condition: for* $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant 0-$, *and for* $0+ \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|\rho'(\theta_1)-\rho'(\theta_2)| < M|\theta_1-\theta_2|^{\mu}, \qquad M > 0. \tag{8.18}$$

PROOF. First suppose that $\beta_2 < \infty$. Then it follows from (8.16) that

$$\frac{\beta(s)-1+\beta s-\frac{1}{2}\beta_2 s^2}{s^{2+\nu_1}},$$

remains finite for some $\nu_1 > 0$ as $s \to 0$. By putting $s = \frac{1}{\alpha}[1-h(t)]$ and by using (4.5) we obtain that

$$\frac{\frac{1}{t^2} h^2(t)-1+2[1-h(t)]-\frac{1}{2}\beta_2[1-h(t)]^2/\alpha^2}{[1-h(t)]^{2+\nu_1}},$$

remains finite as $t \uparrow 1$. This expression can be rewritten as

$$\frac{[1-h(t)]^2[1-\frac{1}{2}\beta_2 t^2/\alpha^2]+[2h(t)-1](1-t^2)}{t^2[1-h(t)]^{2+\nu_1}}.$$

In lemma 4.2 it has been shown that for $c_1 = c_2 = \frac{1}{2}$, $a = 2$, $\beta_2 < \infty$,

$$\lim_{t\uparrow 1} \frac{1-h(t)}{\sqrt{1-t}} = \frac{2}{\sqrt{\beta_2/\alpha^2-2}}.$$

Hence introducing the function $\tilde{h}(t)$ by the relation

$$h(t) = 1 - \left[\frac{2}{\sqrt{\beta_2/\alpha^2-2}} + \tilde{h}(t)\right]\sqrt{1-t},$$

it follows readily from the above that

$$\frac{\tilde{h}(t)}{(1-t)^{\frac{1}{2}\nu_1}},$$

remains finite as $t \uparrow 1$. Consequently, if $\beta(s) \in LH$ and $\beta_2 < \infty$,

$$h(t) = 1 - \frac{2\sqrt{1-t}}{\sqrt{\beta_2/\alpha^2-2}} + O\left((1-t)^{\frac{1}{2}(1+\nu_1)}\right), \qquad t \uparrow 1. \tag{8.19}$$

Next suppose that $\beta_2 = \infty$. Then it follows in a similar way from (8.17) that

$$\frac{[1-h(t)]^{2-\nu_2}}{h^2(t)/t^2+1-2h(t)},$$

remains finite as $t \uparrow 1$ for some $\nu_2 > 0$. This implies that

$$\frac{t^2[1-h(t)]^{2-\nu_2}}{[1-h(t)]^2+(1-t)(t+1)[2h(t)-1]},$$

and consequently

$$\frac{[1-h(t)]^{2-\nu_2}}{1-t},$$

remains finite as $t \uparrow 1$. Hence, if $\beta(s) \in LH$ and $\beta_2 = \infty$,

$$h(t) = 1 + O\left((1-t)^{\frac{1}{2-\nu_2}}\right), \qquad t \uparrow 1. \tag{8.20}$$

As by lemma 8.3 the second derivative exists at every $\theta \in [-\pi,\pi]$ but at $\theta = 0$, it is sufficient in order to prove (8.18) to show that there exists a $\mu > 0$ such that for some $M > 0$,

$$|\rho'(\theta) - \rho'(0+)| < M\,|\theta|^{\mu}, \qquad 0 < \theta \leqslant \pi,$$

$$|\rho'(\theta) - \rho'(0-)| < M\,|\theta|^{\mu}, \qquad -\pi \leqslant \theta < 0.$$

For this it is sufficient to show that

$$|\theta^{1-\mu}\rho''(\theta)|,$$

remains finite as $\theta \uparrow 0$ and as $\theta \downarrow 0$.

If $\beta_2 < \infty$ and $\beta(s) \in LH$ it follows from (8.15), (8.19) and the monotonicity of the function h(t), cf. lemma 4.2, that for some $\nu_1 > 0$,

$$\rho''(\theta) = O\left(|\theta|^{\nu_1-1}\right), \qquad \theta \to 0,$$

so that the inequality (8.18) is valid for every μ, $0 < \mu \leqslant \nu_1$.

If $\beta_2 = \infty$ and $\beta(s) \in LH$ it follows from (8.15), (8.20) and the monotonicity of the function h(t) that for some $\nu_2 > 0$,

$$\rho''(\theta) = O\left(|\theta|^{2\frac{\nu_2-1}{2-\nu_2}}\right), \qquad \theta \to 0,$$

so that the inequality (8.18) is valid for every μ, $0 < \mu \leqslant \frac{\nu_2}{2-\nu_2}$. □

Note that for $\beta_2 = \infty$ the function $\rho'(\theta)$ is continuous at $\theta = 0$, cf. lemma 8.3, so that (8.18) is valid for $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$.

Now we are able to prove the following theorem of the behaviour of the conformal mapping g(1;z) near the point z = 1, referring to the remark below theorem 5.1 for the meaning of g'(1;z).

<u>THEOREM 8.2.</u> *The derivative* g'(1;z) *is continuous and non-vanishing on* $C^+ \cup \underline{C}$ *except possibly at the point* z = 1 *if* $c_1 = c_2 = \frac{1}{2}$, $a = 2$.
In the neighbourhood of the point z = 1 *the conformal mapping* g(1;z) *has the following behaviour*:
for $c_1 \neq c_2$ *and for* $c_1 = c_2 = \frac{1}{2}$, $a \neq 2$, *for every* ε, $0 < \varepsilon < 1$,

$$g(1;z) = g(1;1) + (z-1)g'(1;1) + o(|1-z|^{2-\varepsilon}), \qquad z \to 1, \quad z \in C \cup C^+; \qquad (8.21)$$

for $c_1 = c_2 = \frac{1}{2}$, $a = 2$, *there exist constants* N_1 *and* N_2 *(i.e. independent of z) such that*

$$N_1|1-z| \leqslant |1-g(1;z)| \leqslant N_2\sqrt{|1-z|}, \qquad z \in C \cup C^+. \tag{8.22}$$

PROOF. The contour L(1) has the parametric equation, cf. lemma 8.2,

$$w = \rho(\theta)e^{i\theta} = \frac{h(\cos\theta)}{\cos\theta}e^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi.$$

Let $s = s(\theta)$ denote the arc length of L(1) at the point $w = \rho(\theta)e^{i\theta}$ counted from the point $w = h(-1)$, where $\theta = -\pi$; then

$$s(\theta) = \int_{-\pi}^{\theta} \sqrt{[\rho(\varphi)]^2 + [\rho'(\varphi)]^2}\, d\varphi, \qquad -\pi \leqslant \theta \leqslant \pi. \tag{8.23}$$

Further let $w = w(s)$ be the parametric equation of L(1) with its arc length as parameter, then for $-\pi \leqslant \theta \leqslant \pi$,

$$w(s(\theta)) = \rho(\theta)e^{i\theta},$$

$$\frac{d}{d\theta}w(s(\theta)) = w'(s(\theta))s'(\theta) = [\rho'(\theta) + i\,\rho(\theta)]e^{i\theta},$$

so that

$$w'(s(\theta)) = \frac{\rho'(\theta) + i\,\rho(\theta)}{\sqrt{[\rho(\theta)]^2 + [\rho'(\theta)]^2}}e^{i\theta}. \tag{8.24}$$

Suppose that $c_2 \neq c_1$ *or* $a \neq 2$. Then by lemma 8.2 the contour L(1) is smooth and by lemma 8.3 the second derivative $\rho''(\theta)$ is continuous on $[-\pi,\pi]$. Moreover, by (8.13), (4.5) and the monotocity of the functions h(t) and $\beta(s)$, s real, $s \geqslant 0$, it follows that

$$\rho(\theta) \geqslant -h(-1) > 0, \qquad -\pi \leqslant \theta \leqslant \pi.$$

Hence because $\rho''(\theta)$ is bounded and $\rho(\theta)$ is non-vanishing on $[-\pi,\pi]$ we have for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|w'(s(\theta_1)) - w'(s(\theta_2))| < \text{const.}|\theta_1-\theta_2|.$$

From (8.23) we obtain for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|s(\theta_1) - s(\theta_2)| = |\int_{\theta_1}^{\theta_2} \sqrt{[\rho(\varphi)]^2 + [\rho'(\varphi)]^2}\, d\varphi| \geqslant -h(-1)|\theta_1-\theta_2|,$$

so that for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|w'(s(\theta_1)) - w'(s(\theta_2))| < \text{const.}|s(\theta_1)-s(\theta_2)|.$$

By using the fact that $s(\theta)$ is a strictly increasing function on $[-\pi,\pi]$, cf. (8.23), it follows that for every s_1, s_2, $0 \leqslant s_1 \leqslant s_2 \leqslant s_0$ (s_0 is the length of $L(1)$),

$$|w'(s_1) - w'(s_2)| < \text{const.}|s_1-s_2| < \text{const.}|s_1-s_2|^{1-\varepsilon},$$

for every fixed ε, $0 < \varepsilon < 1$, cf. MUSKHELISHVILI [20], §3.
Application of Kellogg's theorem (lemma I.6.5) leads to the existence of $g'(1;z) \neq 0$ in $C^+ \cup C$ and to the inequality: for every ε, $0 < \varepsilon < 1$, for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|g'(1;e^{i\theta_1}) - g'(1;e^{i\theta_2})| < \text{const.}|\theta_1-\theta_2|^{1-\varepsilon}.$$

By a theorem of Hardy and Littlewood (cf. GOLUSIN [15], §IX.5, Satz 4) it then follows that for every $z \in C \cup C^+$,

$$|g'(1;z) - g'(1;1)| < \text{const.}|1-z|^{1-\varepsilon}.$$

This inequality implies (8.21).

Next consider the case $c_1 = c_2 = \frac{1}{2}$, $a = 2$. By lemma 8.2 the contour $L(1)$ then has a tangent at every point except when $\beta_2 < \infty$ at the point $w = 1$ where it has an inner angle $\omega\pi$, $\frac{1}{2} \leqslant \omega < 1$.
Let $\beta_2 < \infty$ *be fixed* and let ω be defined by (8.9). Introduce the function,

regular in $\mathbb{C}\backslash[1,\infty)$,

$$\zeta_c(w) = 1 - (1-w)^{\frac{1}{\omega}}, \qquad \zeta_c(0) = 0. \tag{8.25}$$

This function maps the domain $L^+(1)$ conformally onto a domain $Z^+(1)$, and it maps the contour $L(1)$ onto the contour $Z(1)$ which is the boundary of the domain $Z^+(1)$, and which has the parametric equation

$$\zeta = 1 - [1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}}, \qquad -\pi \leqslant \theta \leqslant \pi.$$

Let $\sigma = \sigma(\theta)$ denote the arc length of $Z(1)$ at the point $\zeta = 1 - [1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}}$ counted from the point where $\theta = -\pi$; and let $\zeta(\sigma)$ be the parametric equation of $Z(1)$ with its arc length as parameter. Then for $-\pi \leqslant \theta \leqslant \pi$,

$$\zeta(\sigma(\theta)) = 1 - [1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}},$$

$$\frac{d}{d\theta}\zeta(\sigma(\theta)) = \zeta'(\sigma(\theta))\sigma'(\theta) = \frac{1}{\omega}[1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}-1}[\rho'(\theta)+i\rho(\theta)]e^{i\theta},$$

so that

$$\zeta'(\sigma(\theta)) = \frac{[1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}-1}[\rho'(\theta)+i\rho(\theta)]}{|1-\rho(\theta)e^{i\theta}|^{\frac{1}{\omega}-1}\sqrt{[\rho'(\theta)]^2+[\rho(\theta)]^2}}\, e^{i\theta}. \tag{8.26}$$

From lemma 8.3 and the definition of ω, cf. (8.9), it follows that

$$\arg[\rho'(0+)+i] = \pi - \tfrac{1}{2}\omega\pi,$$

$$\arg[\rho'(0-)+i] = \tfrac{1}{2}\omega\pi,$$

with which we obtain from (8.26),

$$\lim_{\theta\downarrow 0}\zeta'(\sigma(\theta)) = [e^{-\frac{1}{2}\omega\pi i}]^{\frac{1}{\omega}-1}e^{(1-\frac{1}{2}\omega)\pi i} = e^{\frac{1}{2}\pi i} = i,$$

$$\lim_{\theta\uparrow 0}\zeta'(\sigma(\theta)) = [e^{\frac{1}{2}\omega\pi i}]^{\frac{1}{\omega}-1}e^{\frac{1}{2}\omega\pi i} = e^{\frac{1}{2}\pi i} = i.$$

Hence the contour $Z(1)$ has a tangent at $\zeta = 1$. As the mapping $\zeta_c(w)$, cf.

(8.25), is conformal in $\mathbb{C}\setminus[1,\infty)$ and L(1) has a tangent everywhere except at w = 1, this implies that the contour Z(1) is smooth.

Suppose that $\beta(s) \in$ LH, cf. definition 8.1. Then by lemma 8.4 there exists a constant $\mu > 0$ such that for $\theta > 0$,

$$\rho'(\theta) = \rho'(0+) + O(\theta^{\mu}), \qquad \theta \downarrow 0.$$

This implies that also, for $\theta > 0$,

$$\frac{1}{\theta}[1-\rho(\theta)e^{i\theta}] = -[\rho'(0+)+i] + O(\theta^{\mu}), \qquad \theta \downarrow 0.$$

Using this it follows from (8.26) that for $\theta > 0$,

$$\zeta'(\sigma(\theta)) = i + O(\theta^{\mu}), \qquad \theta \downarrow 0.$$

In a similar way it can be shown that the above relation is also valid for $\theta < 0$, $\theta \uparrow 0$. Because $\rho''(\theta)$ is finite on $[-\pi,\pi]\setminus\{0\}$, cf. lemma 8.3, it follows that for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|\zeta'(\sigma(\theta_1))-\zeta'(\sigma(\theta_2))| < \text{const.}\,|\theta_1-\theta_2|^{\mu}. \tag{8.27}$$

Further we have for $-\pi \leqslant \theta \leqslant \pi$,

$$\sigma'(\theta) = \left|\frac{d}{d\theta}\,\zeta(\sigma(\theta))\right| = \left|\frac{1}{\omega}[1-\rho(\theta)e^{i\theta}]^{\frac{1}{\omega}-1}[\rho'(\theta)+i\rho(\theta)]\right|,$$

so that

$$\sigma'(\theta) = O(|\theta|^{\frac{1}{\omega}-1}), \qquad \theta \to 0.$$

With this it is readily shown by using MUSKHELISHVILI [20], §6.1°, that for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|\theta_1-\theta_2| \leqslant \text{const.}\,|\sigma(\theta_1)-\sigma(\theta_2)|^{\omega}.$$

With (8.27) we obtain as before, for $0 \leqslant \sigma_1 \leqslant \sigma_2 \leqslant \sigma_0$, where σ_0 is the length of Z(1),

$$|\zeta'(\sigma_1) - \zeta'(\sigma_2)| < \text{const.}|\sigma_1 - \sigma_2|^{\mu\omega}.$$

Now let $d(z)$ be the conformal mapping of the unit disk C^+ onto the domain $Z^+(1)$ determined by the conditions

$$d(0) = 0, \qquad d'(0) > 0.$$

Then application of Kellogg's theorem proves the existence of $d'(z) \neq 0$ in $C^+ \cup C$, and the inequality, for every θ_1, θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|d'(e^{i\theta_1}) - d'(e^{i\theta_2})| < \text{const.}|\theta_1 - \theta_2|^{\mu\omega}.$$

As before this implies that for every $z \in C \cup C^+$,

$$|d'(z) - d'(1)| < \text{const.}|1-z|^{\mu\omega},$$

so that for $z \in C^+ \cup C$,

$$d(z) = 1 + (z-1)d'(1) + O((1-z)^{\mu\omega+1}), \qquad z \to 1. \tag{8.28}$$

By using the inverse mapping of $\zeta_c(w)$, cf. (8.25), and the uniqueness theorem for conformal mapping (lemma I.6.3) we obtain

$$\begin{aligned} g(1;z) &= 1 - [1-d(z)]^{\omega}, && z \in C^+ \cup C, \\ g'(1;z) &= \omega\, d'(z)[1-d(z)]^{\omega-1}, && z \in C^+. \end{aligned} \tag{8.29}$$

From the existence of $d'(z) \neq 0$ on the unit circle C it follows that $g'(1;z) \neq 0$ exists for $z \in C$, $z \neq 1$. As $z \to 1$ the derivative $g'(1;z)$ tends to infinity. With (8.28) we obtain for $z \in C^+ \cup C$,

$$g(1;z) = 1 - [d'(1)]^{\omega}(1-z)^{\omega} + o((1-z)^{\omega}), \qquad z \to 1. \tag{8.30}$$

Since $\frac{1}{2} \leqslant \omega < 1$ this proves the inequality (8.22) for the case $\beta_2 < \infty$, $\beta(s) \in LH$.

For the case $\beta_2 = \infty$, $\beta(s) \in LH$, we can prove (8.30), and thus (8.22), with the same arguments as above, with $\omega = 1$. Note that in this case $g'(1;z) \neq 0$ exists on C including $z = 1$.

Finally consider the case $\beta(s) \notin LH$, cf. definition 8.1. It can still be proved from (8.26), by using lemma 8.3, that the contour $Z(1)$ has a tangent at $\zeta = 1$, but we cannot find a $\mu > 0$ for which (8.27) holds, so that Kellogg's theorem cannot be applied. Instead we shall apply lemma I.6.6 for the case $\beta(s) \notin LH$.

From (8.8) we have for $w \in L(1)$,

$$\{\text{Im } w\}^2 = \beta\left(\frac{1-\text{Re } w}{\alpha}\right) - \{\text{Re } w\}^2, \qquad \delta_1(1) \leqslant \text{Re } w \leqslant 1.$$

Therefore consider the function, for $\delta_1(1) \leqslant \delta \leqslant 1$,

$$I(\delta) := \sqrt{\beta\left(\frac{1-\delta}{\alpha}\right) - \delta^2}. \tag{8.31}$$

It is readily verified that if $\beta_2 = \infty$ then the first derivative of this function becomes $-\infty$ as $\delta \uparrow 1$ so that the contour $L(1)$ is convex in a neighbourhood of $w = 1$. If $\beta_2 < \infty$ then the first derivative of $I(\delta)$ is negative for $\delta \uparrow 1$ and the second derivative becomes $-\infty$ as $\delta \uparrow 1$, because $\beta_3 = \infty$ in the present case, cf. definition 8.1. This implies that the contour $L(1)$ is convex in a neighbourhood of the point $w = 1$ for any $\beta(s) \notin LH$.

From the properties of the mapping $\zeta_c(w)$, cf. (8.25), it follows that also the contour $Z(1)$ is convex in a neighbourhood of $\zeta = 1$. Hence, with $\zeta = \xi + i\eta$, there exists a constant $\delta_0 > 0$ such that $Z(1)$ is represented by

$$\xi = 1 - \lambda(\eta), \qquad \lambda(\eta) \geqslant 0, \qquad -\delta_0 < \eta < \delta_0,$$

where $\lambda(\eta)$ is a continuous function, which is decreasing for $\eta \leqslant 0$ and increasing for $\eta \geqslant 0$.

Application of lemma I.6.6 to the contour Z(1) and the conformal mapping $d(z)$ of C^+ onto $Z^+(1)$ gives for $z \in C^+ \cup C$,

$$\lim_{z \to 1} \frac{1-d(z)}{1-z} > 0, \tag{8.32}$$

so that for $z \in C^+ \cup C$, since $\frac{1}{2} \leqslant \omega \leqslant 1$,

$$|1 - g(1;z)| = |1-d(z)|^{\omega} > \text{const.}|1-z|^{\omega} > \text{const.}|1-z|. \tag{8.33}$$

If the limit in (8.32) is finite, which depends on condition (I.6.3), we also have

$$|1 - g(1;z)| = |1-d(z)|^{\omega} < \text{const.}|1-z|^{\omega} < \text{const.}\sqrt{|1-z|},$$

and (8.22) has been proved. If condition (I.6.3) is not satisfied we obtain from lemma I.6.6 for every $\varepsilon > 0$ and for real z, $0 < z < 1$,

$$|d'(z)| < \frac{\text{const.}}{|1-z|^{\varepsilon}} = \frac{\text{const.}}{(1-|z|)^{\varepsilon}} .$$

By a similar result as lemma I.6.6 it can be shown that $d'(z)$ is finite for $z \in C$, $z \neq 1$, because $\rho''(\theta)$ is finite for $\theta \neq 0$, cf. lemma 8.3, so that condition (I.6.3) is satisfied. This implies that for every $\varepsilon > 0$ and every $z \in C^+$,

$$|d'(z)| < \frac{\text{const.}}{(1-|z|)^{\varepsilon}}.$$

By a theorem of Hardy and Littlewood, cf. GOLUSIN [15], §IX.5, Satz 3, it then follows that for every θ_1 and θ_2, $-\pi \leqslant \theta_1 \leqslant \theta_2 \leqslant \pi$,

$$|d(e^{i\theta_1}) - d(e^{i\theta_2})| \leqslant \text{const.}|\theta_1-\theta_2|^{1-\varepsilon}.$$

Hence, by this inequality and another theorem of Hardy and Littlewood, cf. GOLUSIN [15], §IX.5, Satz 4: for every $\varepsilon > 0$ and for every $z \in C^+ \cup C$,

$$|1-d(z)| \leqslant \text{const.}|1-z|^{1-\varepsilon},$$

and

$$|1 - g(1;z)| \leqslant \text{const.}|1-z|^{\omega(1-\varepsilon)}.$$

If $\omega \neq \frac{1}{2}$ then we can choose ε such that $\omega(1-\varepsilon) = \frac{1}{2}$, and together with (8.33) the inequalities (8.22) have been proved. If $\omega = \frac{1}{2}$ then by (8.9) we have $\beta_2 = \beta^2$. Application of a result for characteristic functions (cf. LUKÁCS [17], theorem 4.1.1) to the characteristic function

$$e^{-iu\beta}\,\beta(-iu) = 1 + \tfrac{1}{2}(\beta_2-\beta^2)u^2 + o(u^2), \qquad u \to 0,\ u \in \mathbb{R},$$

leads to

$$\beta(s) = e^{-s\beta} \in \text{LH},$$

for which class of functions (8.22) had been proved already. Now the inequalities (8.22) have been proved for all Laplace-Stieltjes transforms $\beta(s)$ with finite first moment. □

Having established this result on the conformal mapping of C^+ onto $L^+(1)$ we return to the investigation of the behaviour of the function $\Phi_x(r;0,0)$ as r tends to one. It is recalled from section II.7 that this function is given by: if $R(c_2) \geqslant 1$ for $0 < r < 1$, cf. (7.7),

$$\Phi_x(r;0,0) = \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{dt}{t}, \tag{8.34}$$

and if $R(c_2) < 1$ for $R(c_2) < r < 1$, cf. (7.24),

$$\Phi_x(r;0,0) = \frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{dt}{t} + \frac{2c_2}{z_0(r)g'(r;z_0(r))}\,\frac{[\mu_1(r)]^{x_1}}{1-\mu_1(r)}. \tag{8.35}$$

Therefore we shall first consider, cf. (6.1),

$$\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{dt}{t} = \int_C \frac{[g(r;t)/2c_1]^{x_1}}{1-g(r;t)/2c_1}\,\frac{[g(r;\frac{1}{t})/2c_2]^{x_2}}{1-g(r;\frac{1}{t})/2c_2}\,\frac{dt}{t}, \tag{8.36}$$

as r tends to one.

<u>THEOREM 8.3.</u> *The conformal mappings* $\{g(r;z);0<r<1\}$ *tend uniformly for* $z\in C^+$ *to the conformal mapping* $g(1;z)$, *while the inverses* $\{g_0(r;w);$ $0<r<1\}$ *tend uniformly for* $w\in L^+(1)$ *to the inverse* $g_0(1;w)$ *of* $g(1;z)$, *as* r *tends to one.*

<u>PROOF.</u> By corollary 5.3 the domains $L^+(r)$, $0<r<1$, are strictly monotonic (expanding), cf. section I.6, while from (8.8) and lemma 5.2 it is clear that (cf. the figure on page 238):

$$L^+(1) = \bigcup_{0<r<1} L^+(r).$$

Hence, lemma I.6.11 may be applied and leads to the assertions. □

By the continuity of $g(r;z)$ on $C^+\cup C$ for $0<r\leqslant 1$ it follows that also $g(r;z)\to g(1;z)(r\uparrow 1)$ for $z\in C$, which will be used tacitly in the next lemmas.

<u>LEMMA 8.5.</u> *If* $c_1\neq c_2$, *or when* $c_1=c_2=\frac{1}{2}$ *if* $a>2$,

$$\lim_{r\uparrow 1}\left|\frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\frac{dt}{t}\right|<\infty.$$

<u>PROOF.</u> See (8.36). As r tends to one the integrand remains finite for $|t|=1$, $t\neq 1$. Note that, cf. (5.9),

$$g(r;1)=\delta_2(r)=h(\sqrt{r}),\qquad 0<r\leqslant 1. \tag{8.37}$$

For $c_1=c_2=\frac{1}{2}$, $a>2$ we have by lemma 4.2 that $g(1;1)<1$ so that the integrand of (8.36) is also finite at $t=1$ as $r\uparrow 1$.
For $c_2<\frac{1}{2}<c_1$, $a\leqslant 2$ it follows from theorem 5.6.i, lemma 4.2 and (8.37) that

$$2c_2<g(1;1)<1<2c_1;$$

for $c_2 < c_{2m}(a)$, $a > 2$, the same inequalities are valid by theorem 5.6.ii, while by this theorem for $c_2 > c_{2m}(a)$, $a > 2$,

$$g(1;1) < 2c_2 \leqslant 2c_1.$$

Hence also for $c_2 < \frac{1}{2}$, but $c_2 \neq c_{2m}(a)$ for $a > 2$, the integrand of (8.36) remains finite at $t = 1$ as $r \uparrow 1$.

For $a > 2$, $c_2 = c_{2m}(a)$ we have $g(1;1) = 2c_2$ so that the integrand of (8.36) tends to infinity at $t = 1$ as $r \uparrow 1$, but the integral over C tends to a finite limit (this can be proved similarly as theorem 7.5). □

LEMMA 8.6. *For* $c_1 = c_2 = \frac{1}{2}$,

$$\lim_{r\uparrow 1} \left|\frac{1}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t}\right| = \infty, \qquad a \leqslant 2,$$

and

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t} = \begin{cases} \dfrac{1-\frac{1}{2}a}{g'(1;1)}, & a < 2, \\ 0, & a = 2. \end{cases}$$

PROOF. By lemma 4.2 we have for $c_1 = c_2 = \frac{1}{2}$, $a \leqslant 2$, cf. (8.37),

$$\lim_{r\uparrow 1} g(r;1) = \lim_{r\uparrow 1} h(\sqrt{r}) = 1.$$

Consequently both factors of the dominator in (8.36) vanish in this case at $t = 1$ as r tends to one, but only at $t = 1$. *First suppose* $a < 2$. As a consequence of the above remark consider for $0 < r < 1$, $t \in C$,

$$K_2(r;t) := [t-t_0(r)][t-\frac{1}{t_0(r)}]\, K_x(g(r;t),g(r;\tfrac{1}{t}))\, \frac{1}{t}, \qquad (8.38)$$

where $t_0(r)$ is defined by (5.45) and thus because $c_2 = \frac{1}{2}$,

$$g(r;t_0(r)) = 1, \qquad t_0(r) > 1, \qquad r_0(\tfrac{1}{2}) < r < 1.$$

Differentiation of this equation gives (cf. the remark below theorem 5.1)

$$\frac{d}{dr} g(r;t_0(r)) = g_r(r;t_0(r)) + t_0'(r)g'(r;t_0(r)) = 0.$$

For $a < 2$ both $g'(1;1)$, cf. theorem 8.2, and $g_r(1;1) = \frac{1}{2}h'(1)$, cf. lemma 4.2, are finite, so that

$$\lim_{r\uparrow 1} t_0'(r) = -\frac{g_r(1;1)}{g'(1;1)} = -\frac{1}{(2-a)g'(1;1)}. \tag{8.39}$$

It is readily verified that consequently for $a < 2$,

$$\lim_{r\uparrow 1}\lim_{t\to 1} \frac{t-t_0(r)}{1-g(r;t)} = \lim_{t\to 1}\lim_{r\uparrow 1} \frac{t-t_0(r)}{1-g(r;t)} = -\frac{1}{g'(1;1)}. \tag{8.40}$$

Consider for $a < 2$ the integral

$$\int_C K_2(r;t)\frac{dt}{[t-t_0(r)][t-\frac{1}{t_0(r)}]} = \frac{t_0(r)}{t_0^2(r)-1}\left[\int_C K_2(r;t)\frac{dt}{t-t_0(r)} - \int_C K_2(r;t)\frac{dt}{t-\frac{1}{t_0(r)}}\right]. \tag{8.41}$$

We want to apply lemma I.3.6 to the two integrals at the righthand side of the formula above. For that we have to show that there exist positive constants M and μ such that

$$|K_2(r;t)-K_2(r;1)| \leqslant M|t-1|^{\mu}, \qquad t \in C,\ 0 < r \leqslant 1. \tag{8.42}$$

Consider first the factor of $K_2(r;t)$,

$$T_1(r;t) := \frac{t-t_0(r)}{1-g(r;t)}.$$

From theorem 5.5 we have for $0 < r < 1$ and for $t \in C$,

$$g(r;t) = g(r;1) + (t-1)g'(r;1) + O(|t-1|^2), \qquad t \to 1.$$

This implies that for $0 < r < 1$,

$$\lim_{t\to 1} \frac{T_1(r;t)-T_1(r;1)}{t-1} = \lim_{t\to 1} \frac{t-1+t_0(r)[g(r;1)-g(r;t)]-tg(r;1)+g(r;t)}{(t-1)[1-g(r;1)][1-g(r;t)]} =$$

$$= \frac{1-g(r;1)+g'(r;1)[1-t_0(r)]}{[1-g(r;1)]^2}. \tag{8.43}$$

Let ε be a constant, $0 < \varepsilon < 1$. From theorem 8.2 we have for $a < 2$, $r = 1$, and $t \in C$,

$$g(1;t) = g(1;1) + (t-1)g'(1;1) + o(|t-1|^{2-\varepsilon}), \qquad t \to 1,$$

so that with (8.40), for $a < 2$,

$$\lim_{t\to 1} \frac{T_1(1;t)-T_1(1;1)}{(t-1)^{1-\varepsilon}} = \lim_{t\to 1} \frac{(t-1)g'(1;1)+1-g(1;t)}{(t-1)^{1-\varepsilon}[1-g(1;t)]g'(1;1)} = 0. \tag{8.44}$$

From (8.43) and (8.44) it follows readily that there exists a constant M_1 such that for $a < 2$,

$$|T_1(r;t)-T_1(r;1)| < M_1|t-1|^{1-\varepsilon}, \qquad t \in C, \qquad 0 < r \leqslant 1. \tag{8.45}$$

In a similar way it can be shown that the factor of $K_2(r;t)$, cf. (8.38),

$$T_2(r;t) := \frac{t-\frac{1}{t_0(r)}}{1-g(r;\frac{1}{t})},$$

satisfies for $a < 2$ the following inequality, where M_2 is a constant:

$$|T_2(r;t)-T_2(r;1)| < M_2|t-1|^{1-\varepsilon}, \qquad t \in C, \qquad 0 < r \leqslant 1. \tag{8.46}$$

By using (8.45) and (8.46) it is readily verified that there exists a constant M such that for $\mu = 1-\varepsilon$ the inequality (8.42) holds.

From theorem 8.3 and (8.40) it is clear that for $a < 2$,

$$K_2(1;t) := \lim_{r\uparrow 1} K_2(r;t) = \frac{(t-1)^2[g(1;t)]^{x_1}[g(1;\frac{1}{t})]^{x_2}}{[1-g(1;t)][1-g(1;\frac{1}{t})]t}, \qquad t \in C,\ t \neq 1,$$

$$= -\frac{1}{[g'(1;1)]^2}, \qquad t = 1. \tag{8.47}$$

Finally, as $t_0(r)$ is a strictly decreasing function on the interval $0 < r \leqslant 1$, and it tends to one as $r \uparrow 1$ (cf. lemma 5.9), all the conditions of lemma I.3.6 are satisfied. Application of this lemma gives, for $a < 2$,

$$\lim_{r\uparrow 1} \frac{1}{2\pi i} \int_C K_2(r;t) \frac{dt}{t-t_0(r)} = -\tfrac{1}{2}K_2(1;1) + \frac{1}{2\pi i} \int_C K_2(1;t) \frac{dt}{t-1},$$

$$\lim_{r\uparrow 1} \frac{1}{2\pi i} \int_C K_2(r;t) \frac{dt}{t-\frac{1}{t_0(r)}} = \tfrac{1}{2}K_2(1;1) + \frac{1}{2\pi i} \int_C K_2(1;t) \frac{dt}{t-1}. \quad (8.48)$$

Further, by using (8.39) we obtain for $a < 2$,

$$\lim_{r\uparrow 1} \frac{1-r}{t_0(r)-1} = -\lim_{r\uparrow 1} \frac{1}{t_0'(r)} = (2-a)g'(1;1). \quad (8.49)$$

By taking (8.38), (8.41), (8.48), (8.47) and (8.49) together we have for $a < 2$,

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t} =$$

$$\lim_{r\uparrow 1} \frac{1-r}{t_0(r)-1} \cdot \frac{t_0(r)}{t_0(r)+1} [-K_2(1;1)] = \frac{1-\frac{1}{2}a}{g'(1;1)},$$

so that the assertions for $a < 2$ have been proved.

Next suppose $a = 2$. Let ε be a constant, $0 < \varepsilon < \pi$, and write

$$\frac{1}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t})) \frac{dt}{t} =$$

$$= \frac{1}{2\pi} \int_{\varepsilon}^{2\pi-\varepsilon} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta + \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta. \quad (8.50)$$

For the last integral we have from theorem 5.2, cf. (8.36),

$$\left|\frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta\right| \geqslant \frac{1}{2\pi}\left| \int_{-\varepsilon}^{\varepsilon} \frac{\mathrm{Re}\{[g(r;e^{i\theta})]^{x_1}[g(r;e^{-i\theta})]^{x_2}\}}{|1-g(r;e^{i\theta})|^2} d\theta\right|.$$

Because $g(r;1) = h(\sqrt{r})$ is positive for $0 < r \leqslant 1$ and increases strictly to one as $r \uparrow 1$ we can choose ε_0, $\varepsilon_0 > 0$, so small that there exists a positive constant N_1 such that for $\frac{1}{2} \leqslant r \leqslant 1$ and for $-\varepsilon_0 < \theta < \varepsilon_0$,

$$\text{Re } g(r;e^{i\theta}) \geqslant N_1 > 0.$$

Hence, for $\frac{1}{2} \leqslant r \leqslant 1$,

$$\left|\frac{1}{2\pi}\int_{-\varepsilon_0}^{\varepsilon_0} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta\right| \geqslant \frac{N_1^{x_1+x_2}}{2\pi}\int_{-\varepsilon_0}^{\varepsilon_0}\frac{d\theta}{|1-g(r;e^{i\theta})|^2}.$$

From the theorems 5.5, 8.2 and 8.3 it follows that there exists a constant N_2 such that for $\frac{1}{2} \leqslant r \leqslant 1$ and for $-\varepsilon_0 < \theta < \varepsilon_0$,

$$|g(r;e^{i\theta})-g(r;1)| < N_2|\theta|^{\frac{1}{2}}.$$

This leads to: for $\frac{1}{2} \leqslant r \leqslant 1$,

$$\int_{-\varepsilon_0}^{\varepsilon_0}\frac{d\theta}{|1-g(r;e^{i\theta})|^2} \geqslant \int_{-\varepsilon_0}^{\varepsilon_0}\frac{d\theta}{\{1-g(r;1)+N_2\sqrt{|\theta|}\}^2} =$$

$$= 4\int_0^{\sqrt{\varepsilon_0}}\frac{u\,du}{\{1-g(r;1)+N_2u\}^2} =$$

$$= \frac{4}{N_2^2}\left[\log\{1-g(r;1)+N_2\sqrt{\varepsilon_0}\}-\log\{1-g(r;1)\}-\frac{N_2\sqrt{\varepsilon_0}}{1-g(r;1)+N_2\sqrt{\varepsilon_0}}\right].$$

Because ε_0 is positive and independent of r, and $g(r;1) \to 1(r\uparrow 1)$ this lower bound tends to infinity as $r \uparrow 1$; hence

$$\lim_{r\uparrow 1}\left|\frac{1}{2\pi}\int_{-\varepsilon_0}^{\varepsilon_0} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta\right| = \infty.$$

Because $g(1;t) \neq 1$ for $t \neq 1$ the first integral at the righthand side of (8.50) remains finite as $r \uparrow 1$. Consequently,

$$\lim_{r\uparrow 1}\left|\frac{1}{2\pi i}\int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\,\frac{dt}{t}\right| = \infty,$$

so that the first assertion for $a = 2$ has been proved.

At the other hand it follows from the theorems 5.5. 8.2 and 8.3, that there exists a constant $N_3 > 0$ such that for $\frac{1}{2} \leq r \leq 1$ and for $-\frac{1}{4}\pi \leq \theta \leq \frac{1}{4}\pi$,

$$|g(r;e^{i\theta})-g(r;1)| > N_3|\theta|.$$

As the angle which the line joining the points $g(r;1)$ and $g(r;e^{i\theta})$ makes with the positive real axis is obtuse for $\frac{1}{2} \leq r \leq 1$ and $-\frac{1}{4}\pi \leq \theta \leq \frac{1}{4}\pi$ it follows by the cosine rule that

$$|1-g(r;e^{i\theta})|^2 \geq |1-g(r;1)|^2 + |g(r;1)-g(r;e^{i\theta})|^2.$$

By using further $|g(r;t)| \leq 1$, $t \in C$, $0 < r \leq 1$, this leads to

$$\left|\frac{1}{2\pi}\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta\right| \leq \frac{1}{2\pi}\int_{-\frac{1}{4}\pi}^{\frac{1}{4}\pi} \frac{d\theta}{\{1-g(r;1)\}^2+\{N_3\theta\}^2} =$$

$$= \frac{1}{2\pi\{1-g(r;1)\}N_3} \arctan\left[\frac{\theta N_3}{1-g(r;1)}\right]\Bigg|_{\theta=-\frac{1}{4}\pi}^{\theta=\frac{1}{4}\pi}.$$

Because $N_3 > 0$ independent of r we have

$$\lim_{r\uparrow 1} \arctan\left[\frac{\theta N_3}{1-g(r;1)}\right]\Bigg|_{\theta=-\frac{1}{4}\pi}^{\theta=\frac{1}{4}\pi} = \pi,$$

while lemma 4.2 gives

$$\lim_{r\uparrow 1} \frac{1-r}{1-g(r;1)} = \lim_{r\uparrow 1} \frac{1-r}{1-h(\sqrt{r})} = 0.$$

Thus it has been proved that for $0 < \varepsilon \leq \frac{1}{4}\pi$,

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i}\int_{-\varepsilon}^{\varepsilon} K_x(g(r;e^{i\theta}),g(r;e^{-i\theta}))d\theta = 0.$$

By using again the fact that the first integral at the righthand side of (8.50) multiplied by $1-r$ vanishes for every $\varepsilon > 0$ as r tends to one we obtain the second assertion for $\alpha = 2$.

This completes the proof of the lemma. □

With these two preliminary lemmas we are able to prove the following theorem on the ergodicity of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,\dots\}$:

<u>THEOREM 8.4.</u> *The Markov chain* $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\dots\}$ *representing the number of customers of type 1 and of type 2 left behind in the system at the* n^{th} *service completion instant consists of ergodic states if and only if*

$$\max\{c_1,c_2\}\, a < 1. \tag{8.51}$$

If $\max\{c_1,c_2\}\, a = 1$ *this Markov chain consists of null states, and if* $\max\{c_1,c_2\}\, a > 1$ *it consists of transient states.*

<u>PROOF.</u> It is recalled from corollary 8.1 and the remark below it that the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,\dots\}$ is ergodic if and only if

$$\lim_{r\uparrow 1}\ (1-r)\Phi_{00}(r;0,0) > 0. \tag{8.52}$$

If this chain is not ergodic then, cf. FELLER [10], §XV.5, it is transient if

$$\lim_{r\uparrow 1}\ \Phi_{00}(r;0,0) = \sum_{n=0}^{\infty} p_{00}^{(n)}(0,0) < \infty, \tag{8.53}$$

and it consists of null states if

$$\lim_{r\uparrow 1}\ \Phi_{00}(r;0,0) = \sum_{n=0}^{\infty} p_{00}^{(n)}(0,0) = \infty. \tag{8.54}$$

If $R(c_2) \geqslant 1$, i.e. $a \leqslant 2$, $c_1 = c_2 = \frac{1}{2}$ or $a > 2$, $c_{2m}(a) \leqslant c_2 \leqslant \frac{1}{2}$, cf. theorem 5.6, the function $\Phi_{00}(r;0,0)$ is for $0 < r < 1$ given by (8.34). Then lemma 8.6 implies that this chain is ergodic for $c_1 = c_2 = \frac{1}{2}$, $a < 2$, and consists of null states for $c_1 = c_2 = \frac{1}{2}$, $a = 2$, cf. (8.52), (8.54), while lemma 8.5 proves that the chain is transient for $a > 2$, $c_{2m}(a) \leqslant c_2 \leqslant \frac{1}{2}$, cf. (8.53).

In the remaining cases, i.e. $a \leqslant 2$, $c_2 < \frac{1}{2} < c_1$, or $a > 2$, $c_2 < c_{2m}(a)$, we have $R(c_2) < 1$ and the function $\Phi_{00}(r;0,0)$ is given by (8.35) for $R(c_2) < r < 1$. Since $R(c_2) < 1$ it follows from theorem 5.6 that $2c_2 \in L^+(1)$. Therefore,

$$z_0(r) \to z_0 \in C^+, \qquad \text{as } r \uparrow 1,$$

where z_0 is defined by, cf. (7.13), if $2c_2 \in L^+(1)$,

$$g(1;z_0) = 2c_2, \qquad 0 < z_0 < 1. \tag{8.55}$$

Consequently, $g'(r;z_0(r)) \to g'(1;z_0)$ as $r \uparrow 1$, and $g'(1;z_0)$ is finite and not equal to zero (cf. theorem 8.3).

Further, from the theory of the M/G/1 queueing system it is known (cf. TAKÁCS [24], §1.3 lemma 1) that for the root $\mu_1(r) \in C^+$ of equation (4.3),

$$\lim_{r\uparrow 1} \mu_1(r) = 1, \text{ if } a_1 \leq 1, \qquad \lim_{r\uparrow 1} \frac{1-r}{1-\mu_1(r)} = 1-a_1, \text{ if } a_1 < 1,$$
$$< 1, \text{ if } a_1 > 1, \qquad = 0, \quad \text{if } a_1 \geq 1. \tag{8.56}$$

With lemma 8.5 the above implies (see (8.35)) that the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\ldots\}$ is ergodic for $a_1 < 1$, $c_2 < \frac{1}{2} < c_1$, consists of null states for $a_1 = 1$, $c_2 < \frac{1}{2} < c_1$, and is transient if $a_1 > 1$, $a \leq 2$, $c_2 < \frac{1}{2} < c_1$ and if $a > 2$, $c_2 < c_{2m}(a)$.

Summarizing it has been proved, under the assumption $c_2 \leq \frac{1}{2} \leq c_1$ made in (5.36), that the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\ldots\}$ is ergodic for $a_1 < 1$, consists of null states for $a_1 = 1$, and is transient for $a_1 > 1$. Clearly, if we disregard assumption (5.36) in these conditions $a_1 = c_1 a$ has to be replaced by $\max\{c_1,c_2\}a$. □

Defining $z_0 = 1$ for $c_2 = \frac{1}{2}$, $a < 2$, cf. (8.55), it is next proved:

<u>THEOREM 8.5.</u> *For* $c_2 \leq \frac{1}{2} \leq c_1$, $a_1 < 1$, *the stationary probability* ϕ_0 *that*

the queueing system described in section II.0 is left empty at a service completion instant is given by:

$$\phi_0 := \Phi(0,0) = \frac{2c_2(1-a_1)}{z_0 g'(1;z_0)}. \tag{8.57}$$

PROOF. In the case $c_2 < \frac{1}{2} < c_1$, $a_1 < 1$ this expression for ϕ_0 follows from (8.35) by using lemma 8.5 and (8.55), (8.56), and corollary 8.1.
In the case $c_1 = c_2 = \frac{1}{2}$, $a < 2$ it follows from (8.34) by using corollary 8.1 and lemma 8.6 that

$$\phi_0 = \lim_{r\uparrow 1} (1-r)\Phi_x(r;0,0) = \frac{1-\frac{1}{2}a}{g'(1;1)}, \tag{8.58}$$

which is equivalent to (8.57) with $z_0 = 1$, $2c_2 = 1$, $a_1 = \frac{1}{2}a$. □

Before an expression for the generating function of the stationary distribution of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)), n = 0,1,\dots\}$ will be derived it is first shown:

THEOREM 8.6. *The conformal mapping* $g_0(1;w)$ *of the domain* $L^+(1)$ *onto the unit circle can be continued as a regular function into the disk* $|w| < 2c_1$ *except for poles at points* $w \in L^-(1)$ *where* $\beta\left(\frac{1-\frac{1}{2}w}{\alpha}\right) = 0$. *This continuation which will be denoted by the same symbol* $g_0(1;w)$ *is continuous up to the circle* $|w| = 2c_1$.

PROOF. For $c_1 \neq c_2$ and for $c_1 = c_2 = \frac{1}{2}$, $a > 2$, we have $\delta_2(1) < 1$, cf. (4.16) and lemma 4.2, and the proof is similarly to that of theorem 5.4 and its preceding lemmas (note that $1 + \sqrt{1-4c_1c_2r} \downarrow 2c_1$ as $r \uparrow 1$).
Next let $c_1 = c_2 = \frac{1}{2}$, $a \leqslant 2$. Then $\delta_2(1) = 1$, cf. lemma 4.2 and (5.9).
In this case it can be proved in a similar way as in lemma 5.3 that there exists a subdomain $S(1)$ of the domain $\text{Re}\,\delta < 1$ such that $[\gamma(1)\backslash\{1\}] \subset S(1)$, and such that for $\delta \in S(1)\backslash\gamma(1)$ the functions $w_1(1;\delta)$ and $w_2(1;\delta)$ are univalent, and

$$w_1(1;\delta) \in L^-(1), \qquad w_2(1;\delta) \in L^+(1).$$

As in lemma 5.4 it can be proved that for $\delta \in \tilde{E}(1)$,

$$[\mathrm{Re}\ w_1(1;\delta) \leqslant 1 \text{ and } w_1(1;\delta) \in L^-(1)] \Rightarrow w_2(1;\delta) \in L^+(1). \tag{8.59}$$

In a similar way as in lemma 3.5 it can be shown, that, cf. (3.29) and (3.12), for $\delta \in \tilde{E}(1)$,

$$|2\delta - w_1(1;\delta)| = |w_2(1;\delta)| < 1. \tag{8.60}$$

Consequently, for $|\delta| = 1$, $\delta \neq 1$,

$$|w_1(1;\delta)| > 1. \tag{8.61}$$

From lemma 5.6 it can be readily deduced that for $r = 1$, $|w| = 1$, $w \neq 1$, equation (3.5) has exactly one root $\delta_0(1;w)$ inside the unit circle $|\delta| = 1$. From (8.60) and (5.23) it follows that

$$w_1(1;\delta_0(1;\omega)) = w, \qquad |w| = 1,\ w \neq 1. \tag{8.62}$$

Hence, the function $w_1(1;\delta)$ is univalent in $S(1)\backslash\gamma(1)$ and on the curve $\{\delta;\delta = \delta_0(1;w),\ |w| = 1,\ w \neq 1\}$. Further, since $a \leqslant 2$, we have, cf. (5.33),

$$\lim_{w \to 1} \delta_0(1;w) = 1.$$

Therefore we obtain by a similar argument as in the proof of lemma 5.7, that the function $w_1(1;\delta)$ is univalent for δ inside the curve $\{\delta;\delta = \delta_0(1;w),\ |w| = 1\}$ for $\delta \notin \gamma(1)$, and that the function $\delta_0(1;w)$ can be continued as a regular function into the domain $L^-(1) \cap C^+$, while (8.62) holds in this domain.
This implies with (8.59) that for $w \in L^-(1)$, $|w| \leqslant 1$,

$$w_2(1;\delta_0(1;w)) \in L^+(1). \tag{8.63}$$

With this relation the assertion can be proved by the arguments of the proof of theorem 5.4. □

As an analogue result as corollary 5.7 it is obtained:

COROLLARY 8.2. *For* $w \in L^-(1)$, $|w| \leq 2c_1$, $\mathrm{Re}\, w \leq 1$,

$$|g_0(1;w)| > 1.$$ □

LEMMA 8.7. *Let* $c_2 \leq \frac{1}{2} \leq c_1$ *and* $a_1 < 1$. *Then:*

i. *The equation* $g_0(1;w) = z_0$ *has in the region* $|w| \leq 2c_1$, $\mathrm{Re}\, w \leq 1$, *exactly one root. This root is* $w = 2c_2$.

ii. *The equation* $g_0(1;w) = \frac{1}{z_0}$ *has in the region* $|w| \leq 2c_1$ *exactly one root. This root is* $w = 2c_1$.

PROOF. For $c_2 \leq \frac{1}{2} \leq c_1$, $a_1 < 1$, theorem 5.6 implies that $2c_2 \in L^+(1) \cup L(1)$.
i. By definition we have $g_0(1;2c_2) = z_0$, cf. (8.55). The function $g_0(1;w)$ maps $L^+(1)$ conformally onto C^+, so that $2c_2$ is the only root of $g_0(1;w) = z_0$ for $w \in L^+(1) \cup L(1)$. Further, since $z_0 \leq 1$ it follows from corollary 8.2 that this equation does not have roots for $w \in L^-(1)$, $|w| \leq 2c_1$, $\mathrm{Re}\, w \leq 1$.
ii. Because $z_0 \leq 1$ and $|g_0(1;w)| < 1$ for $w \in L^+(1)$ the equation $g_0(1;w) = \frac{1}{z_0}$ has no roots for $w \in L^+(1)$. For $w \in L(1) \cup L^-(1)$, $|w| \leq 2c_1$, we have, cf. (5.37),

$$g_0(1;w) = \frac{1}{g_0(1;w_2(1;\delta_0(1;w)))}. \tag{8.64}$$

Therefore consider for $w \in L(1) \cup L^-(1)$, $|w| \leq 2c_1$, the equation

$$g_0(1;w_2(1;\delta_0(1;w))) = z_0. \tag{8.65}$$

For $w \in L(1) \cup L^-(1)$, $|w| \leq 2c_1$, we have $\mathrm{Re}\, \delta_0(1;w) \leq 1$, cf. lemma 5.8, so that $|w_2(1;\delta_0(1;w))| \leq 1$, cf. (3.29). Hence, by the first part of this lemma equation (8.65) is equivalent to: for $w \in L(1) \cup L^-(1)$, $|w| \leq 2c_1$,

$$w_2(1;\delta_0(1;w)) = 2c_2. \tag{8.66}$$

It is readily verified, cf. (5.34), that for $r = 1$, $a_1 < 1$, equation (3.5) has for $w = 2c_2$ exactly one root in the region $\text{Re}\ \delta \leqslant 1$, and that this root is $\delta = 1$. Hence, equation (8.66) is equivalent to: for $w \in L^-(1) \cup L(1)$, $|w| \leqslant 2c_1$,

$$\delta_0(1;w) = 1. \tag{8.67}$$

Finally, because $w_1(1;\delta)$ is the inverse function of $\delta_0(1;w)$ for $w \in L^-(1) \cup L(1)$, $|w| \leqslant 2c_1$, it follows that equation (8.67) is equivalent to $w = 2c_1$, since $w_1(1;1) = 2c_1$.
Summarizing, it has been proved that $w = 2c_1$ is the unique root of the equation $g_0(1;w) = \frac{1}{z_0}$ in the region $|w| \leqslant 2c_1$. □

<u>COROLLARY 8.3.</u> *For* $c_2 \leqslant \frac{1}{2} \leqslant c_1$, $a_1 < 1$, *the functions*

$$\frac{1}{1-z_0g_0(1;2c_1p)}, \qquad \frac{1}{1-g_0(1;2c_2p)/z_0}, \tag{8.68}$$

both are regular for $|p| < 1$ *and continuous for* $|p| \leqslant 1$, $p \neq 1$. *At* $p = 1$ *they have a first order pole, while*

$$\lim_{p\to1} \frac{1-p}{1-z_0g_0(1;2c_1p)} = \frac{1-a_2}{1-a_1}\frac{z_0}{2c_2}\, g'(1;z_0),$$

$$\lim_{p\to1} \frac{1-p}{1-g_0(1;2c_2p)/z_0} = \frac{z_0}{2c_2}\, g'(1;z_0). \tag{8.69}$$

<u>PROOF.</u> The first statements follow directly from theorem 8.6 and lemma 8.7 by noting that a pole of the function $g_0(1;w)$ is equivalent to a zero of the functions (8.68).
For obtaining the limits (8.69) it is noted that (8.64) and the relation $w_2(1;\delta_0(1;2c_1)) = 2c_2$, cf. the proof of lemma 8.7, imply

$$g_0'(1;2c_1) = \frac{c_2(1-a_1)}{c_1(1-a_2)}\left(\frac{1}{z_0}\right)^2 g_0'(1;2c_2), \tag{8.70}$$

and that $g(1;g_0(1;w)) = w$, $w \in L^+(1) \cup L(1)$, and (8.55) imply

$$g'(1;z_0)\cdot g_0'(1;2c_2) = 1. \tag{8.71}$$

□

Having established the preceding results we are now able to prove the following main theorem of this section on the stationary distribution of the number of customers present in the system directly after a departure instant.

<u>THEOREM 8.7.</u> *For* $\max\{c_1,c_2\}\, a < 1$ *the imbedded Markov chain* $\{(\underline{x}_1(n),\underline{x}_2(n)),$ $n = 0,1,2,\dots\}$ *representing the number of customers of type 1 and of type 2 left behind at a service completion instant in a queueing system with two types of customers and paired services (cf. section II.0) possesses a unique stationary distribution.*
In the case $c_2 \leq \frac{1}{2} \leq c_1$, $a_1 < 1$, *the generating function of this stationary distribution is given by: for* $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\Phi(p_1,p_2) = \phi_0 \frac{(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) - p_1p_2} \quad \frac{1-g_0(1;2c_1p_1)g_0(1;2c_2p_2)}{[1-z_0g_0(1;2c_1p_1)][1-g_0(1;2c_2p_2)/z_0]}, \tag{8.72}$$

here $\beta(.)$ *stands for the Laplace-Stieltjes transform of the service time distribution,* $\frac{1}{\alpha}$ *is the rate of the total Poisson arrival process,* c_i *is the proportion of the arriving customers who are of type* i $(i = 1,2)$, $g_0(1;w)$ *denotes the conformal mapping of the domain*

$$L^+(1) = \{w; |w|^2 < 4c_1c_2\, \beta\left(\frac{1-\text{Re } w}{\alpha}\right),\ \text{Re } w \leq 1\},$$

onto the unit disk C^+ *, with* $g_0(1;0) = 0$, $g_0'(1;0) > 0$; *further* $z_0 = g_0(1;2c_2)$, *and the stationary probability* ϕ_0 *that no customers are* *left behind in the system at a departure moment is given by*

$$\phi_0 = \frac{2c_2(1-a_1)}{z_0 g'(1;z_0)},$$

with $a_1 = c_1 \frac{\beta}{\alpha}$ *the traffic offered by type 1 customers and* $g'(1;z_0) = [g_0'(1;2c_2)]^{-1}$.

PROOF. By theorem 8.4 the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,\dots\}$ is ergodic for $\max\{c_1,c_2\}\ a < 1$, and hence possesses a stationary distribution (cf. lemma 8.1), the generating function of which satisfies (8.6).

First consider the case $c_2 < \frac{1}{2} < c_1$. For $a_1 < 1$ also $a < 2$ so that $R(c_2) < 1$ by theorem 5.6 and the function $\Phi_x(r;p_1,p_2)$, $2c_1p_1 \in L^+(1)$, $2c_2p_2 \in L^+(1)$, is given by (7.23) for $R(c_2) < r < 1$.

For $w \in L^+(1)$ we calculate

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i} \int_C K_x(g(r;t),g(r;\tfrac{1}{t}))\ \frac{t+g_0(r;w)}{t-g_0(r;w)}\ \frac{dt}{2t}. \tag{8.73}$$

Because the domain $L^+(1)$ is the union of the domains $L^+(r)$, $0 < r < 1$, cf. theorem 8.3, for any $w \in L^+(1)$ there exists a value $r(w) < 1$ such that $w \in L^+(r)$ for $r(w) \leqslant r \leqslant 1$. Hence, the factor $t-g_0(r;w)$ does not vanish for any $t \in C$ if $r(w) \leqslant r \leqslant 1$. Further, in the proof of lemma 8.5 it has been shown that the function $K_x(g(r;t),g(r;\frac{1}{t}))$ remains finite for $t \in C$ as $r \uparrow 1$. This implies that the limit (8.73) vanishes for every $w \in L^+(1)$. By using the above and (8.56) formula (8.72) follows readily from (7.23), for $2c_1p_1 \in L^+(1)$, $2c_2p_2 \in L^+(1)$. By theorem 8.6 it is then also valid for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, cf. corollary 8.3.

Next consider the case $c_1 = c_2 = \frac{1}{2}$. By theorem 5.6 we have $R(\frac{1}{2}) = 1$ so that the function $\Phi_x(r;p_1,p_2)$, $p_1 \in L^+(r)$, $p_2 \in L^+(r)$, is given by (6.13) for $0 < r < 1$. Again the limit (8.73) has to be investigated. For the same reasons as above the factor $t-g_0(r;w)$ does not vanish for $t \in C$ if $r(w) \leqslant r \leqslant 1$, for every $w \in L^+(1)$. However, as we have seen in the proof of lemma 8.6 the two factors in the denominator of the function $K_x(g(r;t),g(r;\frac{1}{t}))$ both vanish at $t = 1$, and only there, as $r \uparrow 1$. Consider instead of (8.38), for $w \in L^+(1)$, $r(w) \leqslant r < 1$, $t \in C$,

$$K_3(r;t,w) := [t-t_0(r)][t-\frac{1}{t_0(r)}]K_x(g(r;t),g(r;\frac{1}{t}))\,\frac{t+g_0(r;w)}{t-g_0(r;w)}\,\frac{1}{2t}.$$

In a similar way as in the proof of lemma 8.6 for $K_2(r;t)$ it can be shown that if $\alpha < 2$ then for every $w \in L^+(1)$,

$$\lim_{r\uparrow 1} \frac{1-r}{2\pi i} \int_C K_3(r;t,w) \frac{dt}{[t-t_0(r)][t-\frac{1}{t_0(r)}]} =$$

$$= -(1-\tfrac{1}{2}\alpha)\, g'(1;1)\, K_3(1;1,w) = \frac{1-\frac{1}{2}\alpha}{2\,g'(1;1)}\,\frac{1+g_0(1;w)}{1-g_0(1;w)},$$

which equals the limit (8.73).

In this way it is obtained from (6.13) and (8.6) that for $p_1 \in L^+(1)$, $p_2 \in L^+(1)$,

$$\Phi(p_1,p_2) = \frac{1-\frac{1}{2}\alpha}{g'(1;1)}\,\frac{(1-p_1)(1-p_2)\beta\left(\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)}{\beta\left(\frac{1-\frac{1}{2}p_1-\frac{1}{2}p_2}{\alpha}\right)-p_1p_2}\,\frac{1-g_0(1;p_1)g_0(1;p_2)}{[1-g_0(1;p_1)][1-g_0(1;p_2)]}. \tag{8.74}$$

With (8.58) it is seen that this expression is equivalent to (8.72). By theorem 8.6 it holds also for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$. □

<u>REMARK 8.1.</u> The zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the factor in the denominator in (8.72),

$$\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)-p_1p_2,$$

(the kernel, cf. lemma 3.1) are compensated by zeros of the factor

$$1-g_0(1;2c_1p_1)g_0(1;2c_2p_2),$$

which relation has been used for the analytic continuation of the conformal mapping $g_0(1;w)$ into $L^-(1)$ in theorem 8.6, cf. theorem 5.4, formula (5.33). The other factors in the denominator of (8.72),

$$1-z_0 g_0(1;2c_1p_1), \qquad 1-g_0(1;2c_2p_2)/z_0,$$

only vanish at $p_1 = 1$ for $p_1 \in C^+ \cup C$ respectively at $p_2 = 1$ for $p_2 \in C^+ \cup C$, and these zeros are compensated by those of $(1-p_1)(1-p_2)$, cf. corollary 8.3. □

With the aid of (8.69) the generating functions of the stationary marginal distributions of the Markov chains $\{\underline{x}_i(n),\ n = 0,1,2,\ldots\}$, $i = 1,2$, can be obtained from (8.57) and (8.72): for $|p| \leq 1$,

$$\Phi(p,1) = (1-a_1)\,\frac{(1-p)\beta\left(\frac{1-p}{\alpha_1}\right)}{\beta\left(\frac{1-p}{\alpha_1}\right)-p}, \tag{8.75}$$

$$\Phi(1,p) = (1-a_2)\,\frac{(1-p)\beta\left(\frac{1-p}{\alpha_2}\right)}{\beta\left(\frac{1-p}{\alpha_2}\right)-p}. \tag{8.76}$$

Note that these generating functions do not depend on the conformal mapping $g_0(1;w)$, and that they are similar to the generating function of the stationary distribution of the number of customers left behind at a departure instant in an M/G/1-queueing system with arrival rate $\frac{1}{\alpha_1}$ respectively $\frac{1}{\alpha_2}$ and service time distribution B(t), cf. COHEN [03], p.238. Therefore, the following results are obvious. Let $(\underline{x}_1,\underline{x}_2)$ be a stochastic vector of which the joint distribution has the generating function (8.72). Then for $i = 1,2$,

$$\Pr\{\underline{x}_i=0\} = 1-a_i,$$

$$E\{\underline{x}_i\} = a_i + \frac{a_i^2}{1-a_i}\,\frac{\beta_2}{2\beta^2},$$

$$E\{\underline{x}_i^2\} = a_i + \frac{a_i^2}{1-a_i}\frac{\beta_2}{2\beta^2}\left[3+\frac{a_i^2}{1-a_i}\frac{\beta_2}{\beta^2}\right] + \frac{a_i^3}{1-a_i}\frac{\beta_3}{3\beta^3}. \tag{8.77}$$

REMARK 8.2. It is worth noting that the generating functions (8.75) and (8.76) can be obtained without the determination of the function $\Phi(p_1,p_2)$, see remark 4.1. This shows also that in the case $c_2 < \frac{1}{2} < c_1$ the limit

$$\Phi(1,p) = \lim_{r\uparrow 1}(1-r)\Phi_x(r;1,p),$$

exists not only for $a_1 < 1$ but also for $a_2 < 1 \leq a_1$. Hence, the second component of the vector valued Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\ldots\}$ possesses a stationary marginal distribution for $a_2 < 1$. □

Clearly, the moments of the marginal distribution of $\underline{x}_1$ and of that of $\underline{x}_2$ are independent of the conformal mapping $g(1;z)$. But in general the n-th order moments of the joint distribution of $\underline{x}_1$ and $\underline{x}_2$, cf. (8.72), depend on the first n+1 derivatives of the conformal mapping $g(1;z)$ at the point $z = z_0$ and on z_0 itself, $n = 2,3,\ldots$. This will be illustrated by:

THEOREM 8.8. *For* $c_2 \leq \frac{1}{2} \leq c_1$, $a_1 < 1$, $\beta_3 < \infty$,

$$\begin{aligned}E\{\underline{x}_1\cdot\underline{x}_2\} = {} & (2c_2)^2\frac{1-a_1}{1-a_2}\left\{\frac{g'''(1;z_0)}{6[g'(1;z_0)]^3} - \frac{[g''(1;z_0)]^2}{4[g'(1;z_0)]^4}\right\} + \\ & + \frac{c_1c_2}{(1-a_1)(1-a_2)}\left\{\frac{\beta_3}{6\alpha^3}(1-2c_1c_2a) + \frac{\beta_2}{2\alpha^2}(1-a) + \frac{1}{2}\left[\frac{\beta_2}{2\alpha^2}\right]^2\right\} - \\ & - \frac{c_2(c_1-c_2)^2}{(1-a_1)(1-a_2)^2}\left\{\frac{\beta_3}{6\alpha^3} + \frac{\beta_2}{2\alpha^2}\frac{1}{c_1-c_2} + \frac{1}{2}\left[\frac{\beta_2}{2\alpha^2}\right]^2\left[c_1(1-a_2) + \frac{2c_2}{1-a_2}\right]\right\}.\end{aligned} \tag{8.78}$$

PROOF. From (8.77) and Schwarz' inequality, cf. BURRILL [02], theorem 9-3A, it follows that $E\{\underline{x}_1 \cdot \underline{x}_2\}$ is finite for $\beta_3 < \infty$, $a_1 < 1$, $c_2 \leqslant \frac{1}{2} \leqslant c_1$. In order to obtain this moment it is noted that

$$E\{(\underline{x}_1+\underline{x}_2)(\underline{x}_1+\underline{x}_2-1)\} = \lim_{p\to 1} \frac{d^2}{dp^2}\,\Phi(p,p),$$

so that

$$E\{\underline{x}_1 \cdot \underline{x}_2\} = \tfrac{1}{2} \lim_{p\to 1} \frac{d^2}{dp^2}\,\Phi(p,p) - \tfrac{1}{2}E\{\underline{x}_1(\underline{x}_1-1)\} - \tfrac{1}{2}E\{\underline{x}_2(\underline{x}_2-1)\}. \tag{8.79}$$

From (8.72) it follows that for $|p| \leqslant 1$,

$$\Phi(p,p) = \phi_0 \frac{(1-p)\beta\left(\frac{1-p}{\alpha}\right)}{\beta\left(\frac{1-p}{\alpha}\right)-p^2}\left[\frac{1-p}{1-z_0g_0(1;2c_1p)} + \frac{1-p}{1-g_0(1;2c_2p)/z_0} - 1+p\right]. \tag{8.80}$$

Hence, for $|p| \leqslant 1$,

$$\frac{d^2}{dp^2}\,\Phi(p,p) = \phi_0 \frac{(1-p)\beta\left(\frac{1-p}{\alpha}\right)}{\beta\left(\frac{1-p}{\alpha}\right)-p^2}\left[\frac{d^2}{dp^2}\left(\frac{1-p}{1-z_0g_0(1;2c_1p)}\right) + \frac{d^2}{dp^2}\left(\frac{1-p}{1-g_0(1;2c_2p)/z_0}\right)\right] +$$

$$+\, 2\phi_0 \frac{d}{dp}\left(\frac{(1-p)\beta\left(\frac{1-p}{\alpha}\right)}{\beta\left(\frac{1-p}{\alpha}\right)-p^2}\right)\left[\frac{d}{dp}\left(\frac{1-p}{1-z_0g_0(1;2c_1p)}\right) + \frac{d}{dp}\left(\frac{1-p}{1-g_0(1;2c_2p)/z_0}\right) + 1\right] +$$

$$+\, \phi_0 \frac{d^2}{dp^2}\left(\frac{(1-p)\beta\left(\frac{1-p}{\alpha}\right)}{\beta\left(\frac{1-p}{\alpha}\right)-p^2}\right)\left[\frac{1-p}{1-z_0g_0(1;2c_1p)} + \frac{1-p}{1-g_0(1;2c_2p)/z_0} - 1 + p\right]. \tag{8.81}$$

It is readily verified that if $\beta_3 < \infty$, then for $|p| \leqslant 1$,

$$\frac{(1-p)\beta\left(\frac{1-p}{\alpha}\right)}{\beta\left(\frac{1-p}{\alpha}\right)-p^2} = \frac{1}{2-a}\left[1 - \left\{a + \frac{\beta_2-2\alpha^2}{2\alpha^2(2-a)}\right\}(1-p) + \left\{\frac{\beta_3}{6\alpha^3(2-a)} + \frac{(1-a)^2}{(2-a)^2} +\right.\right.$$

$$\left.\left. + \frac{\beta_2}{2\alpha^2(2-a)^2}\left(2(1-a) + \frac{\beta_2}{2\alpha^2}\right)\right\}(1-p)^2\right] + o((1-p)^2), \qquad p \to 1. \tag{8.82}$$

Assuming that the third derivative of the function $g_0(1;w)$ exists at the point $w = 2c_1$ (the rest of the proof will make clear that this 3rd derivative must exist because $E\{\underline{x}_1 \cdot \underline{x}_2\}$ is finite, for $\beta_3 < \infty$, $a_1 < 1$, $c_2 \leq \frac{1}{2} \leq c_1$) we have for $|p| \leq 1$,

$$\frac{1-p}{1-z_0 g_0(1;2c_1 p)} = \frac{1}{2c_1 z_0 g_0'(1;2c_1)} \left[1 - 2c_1 \frac{g_0''(1;2c_1)}{2g_0'(1;2c_1)}(p-1) + \right.$$

$$\left. + (2c_1)^2 (p-1)^2 \left\{\left(\frac{g_0''(1;2c_1)}{2g_0'(1;2c_1)}\right)^2 - \frac{g_0'''(1;2c_1)}{6g_0'(1;2c_1)}\right\}\right] + o((p-1)^2), \quad p \to 1; \tag{8.83}$$

$$\frac{1-p}{1-g_0(1;2c_2 p)/z_0} = \frac{z_0}{2c_2 g_0'(1;2c_2)} \left[1 - 2c_2 (p-1) \frac{g_0''(1;2c_2)}{2g_0'(1;2c_2)} + \right.$$

$$\left. + (2c_2)^2 (p-1)^2 \left\{\left(\frac{g_0''(1;2c_2)}{2g_0'(1;2c_2)}\right)^2 - \frac{g_0'''(1;2c_2)}{6g_0'(1;2c_2)}\right\}\right] + o((p-1)^2), \quad p \to 1. \tag{8.84}$$

From (8.81), (8.82), (8.83) and (8.84) it follows that

$$\frac{1}{2} \lim_{p \to 1} \frac{d^2}{dp^2} \Phi(p,p) = \frac{\phi_0}{2-a} \left[\frac{2c_1}{z_0 g_0'(1;2c_1)} \left\{\left(\frac{g_0''(1;2c_1)}{2g_0'(1;2c_1)}\right)^2 - \frac{g_0'''(1;2c_1)}{6g_0'(1;2c_1)}\right\} + \right.$$

$$+ \frac{2c_2 z_0}{g_0'(1;2c_2)} \left\{\left(\frac{g_0''(1;2c_2)}{2g_0'(1;2c_2)}\right)^2 - \frac{g_0'''(1;2c_2)}{6g_0'(1;2c_2)}\right\} + \left\{a + \frac{\beta_2 - 2\alpha^2}{2\alpha^2(2-a)}\right\} \cdot$$

$$\left\{1 - \frac{1}{2z_0} \frac{g_0''(1;2c_1)}{[g_0'(1;2c_1)]^2} - \tfrac{1}{2} z_0 \frac{g_0''(1;2c_2)}{[g_0'(1;2c_2)]^2}\right\} + \left\{\frac{\beta_3}{6\alpha^3(2-a)} + \frac{(1-a)^2}{(2-a)^2} + \right.$$

$$\left. \left. + \frac{\beta_2}{2\alpha^2(2-a)^2}\left(2(1-a) + \frac{\beta_2}{2\alpha^2}\right)\right\} \cdot \left\{\frac{1}{2c_1 z_0 g_0'(1;2c_1)} + \frac{z_0}{2c_2 g_0'(1;2c_2)}\right\}\right]. \tag{8.85}$$

This intricate formula may be simplified by using (8.64). In the case $c_2 < \frac{1}{2} < c_1$ this relation is in a neighbourhood of $w = 2c_1$ equivalent to, cf. (3.14), lemma 5.8 and corollary 8.3: for real δ, $\delta_2(1) < \delta \leq 1$,

$$g_0(1;w_1(\delta)) \cdot g_0(1;w_2(\delta)) = 1, \tag{8.86}$$

where

$$w_1(\delta) := \delta + \sqrt{\delta^2 - 4c_1 c_2\, \beta\!\left(\frac{1-\delta}{\alpha}\right)},$$

$$w_2(\delta) := \delta - \sqrt{\delta^2 - 4c_1c_2\,\beta\left(\frac{1-\delta}{\alpha}\right)}, \tag{8.87}$$

Straightforward calculations lead to:

$$\begin{aligned}
&w_1(1) = 2c_1, \qquad w_2(1) = 2c_2,\\
&w_1'(1) = \frac{2c_1(1-a_2)}{c_1-c_2}, \qquad w_2'(1) = -\frac{2c_2(1-a_1)}{c_1-c_2},\\
&w_1''(1) = -w_2''(1) = -\frac{4c_1c_2}{c_1-c_2}\left[\frac{1}{2}\frac{\beta_2}{\alpha^2} + \frac{(1-a_1)(1-a_2)}{(c_1-c_2)^2}\right],\\
&w_1'''(1) = -w_2'''(1) = \frac{12c_1c_2}{c_1-c_2}\left[\frac{1-2c_1c_2a}{(c_1-c_2)^2}\left\{\frac{1}{2}\frac{\beta_2}{\alpha^2} + \frac{(1-a_1)(1-a_2)}{(c_1-c_2)^2}\right\} - \frac{1}{6}\frac{\beta_3}{\alpha^3}\right].
\end{aligned} \tag{8.88}$$

Differentiation of both sides of (8.86) gives for $\delta_2(1) < \delta \leq 1$,

$$w_1'(\delta)g_0'(1;w_1(\delta))g_0(1;w_2(\delta)) + w_2'(\delta)g_0(1;w_1(\delta))g_0'(1;w_2(\delta)) = 0, \tag{8.89}$$

which for $\delta = 1$ implies relation (8.70), and becomes with (8.71), (8.57):

$$\frac{1}{2c_1z_0g_0'(1;2c_1)} + \frac{z_0}{2c_2g_0'(1;2c_2)} = \frac{2-a}{\phi_0}. \tag{8.90}$$

Similarly, repeated differentiation of (8.89) gives with (8.88) for $\delta = 1$:

$$1 - \frac{1}{2}\left\{\frac{1}{z_0}\frac{g_0''(1;2c_1)}{[g_0'(1;2c_1)]^2} + z_0\frac{g_0''(1;2c_2)}{[g_0'(1;2c_2)]^2}\right\} = \frac{1}{\phi_0}\left[1 + \frac{(c_1-c_2)^2}{(1-a_1)(1-a_2)}\frac{\beta_2}{2\alpha^2}\right]; \tag{8.91}$$

$$\begin{aligned}
&\sum_{i=1}^{2} \frac{2c_ig_0(1;2c_i)}{g_0'(1;2c_i)}\left\{\left(\frac{g_0''(1;2c_i)}{2g_0'(1;2c_i)}\right)^2 - \frac{g_0'''(1;2c_i)}{6g_0'(1;2c_i)}\right\} =\\
&= \frac{1}{\phi_0}\left[(2-a)(2c_2)^2\frac{1-a_1}{1-a_2}\left\{\frac{g'''(1;z_0)}{6[g'(1;z_0)]^3} - \frac{[g''(1;z_0)]^2}{4[g'(1;z_0)]^4}\right\} + \right.\\
&\left. + \frac{c_1-c_2}{(1-a_1)(1-a_2)^2}\left\{\frac{\beta_3}{6\alpha^3}(c_1-c_2)^2 + \frac{\beta_2}{2\alpha^2}(1-2c_1c_1a)\left[1 + \frac{(c_1-c_2)^2}{(1-a_1)(1-a_2)}\frac{\beta_2}{2\alpha^2}\right]\right\}\right].
\end{aligned} \tag{8.92}$$

Substitution of (8.90), (8.91) and (8.92) in (8.85) leads with (8.79) and (8.77) after some rearrangements to the expression (8.78) in the case $c_2 < \frac{1}{2} < c_1$. In the case $c_1 = c_2 = \frac{1}{2}$ relation (8.85) reads:

$$\frac{1}{2} \lim_{p \to 1} \frac{d^2}{dp^2} \Phi(p,p) = \frac{\phi_0}{2-a} \left[\frac{2}{g_0'(1;1)} \left\{ \left(\frac{g_0''(1;1)}{2g_0'(1;1)} \right)^2 - \frac{g_0'''(1;1)}{6g_0'(1;1)} + \frac{\beta_3}{6\alpha^3(2-a)} + \right.\right.$$

$$\left.\left. + \left(\frac{1-a}{2-a}\right)^2 + \frac{\beta_2}{2\alpha^2(2-a)^2} \left[2(1-a) + \frac{\beta_2}{2\alpha^2} \right] \right\} + \left\{ a + \frac{\beta_2 - 2\alpha^2}{2\alpha^2(2-a)} \right\} \left\{ 1 - \frac{g_0''(1;1)}{[g_0'(1;1)]^2} \right\} \right]. \tag{8.93}$$

With (8.58) and with the relation

$$1 - \frac{g_0''(1;1)}{[g_0'(1;1)]^2} = \frac{2g'(1;1)}{2-a} = \frac{1}{\phi_0}, \tag{8.94}$$

which will be proved in the next chapter, lemma III.9.2, it follows from (8.93) and (8.79) that in the case $c_1 = c_2 = \frac{1}{2}$:

$$E\{\underline{x}_1 \cdot \underline{x}_2\} = \frac{1}{6} \frac{g'''(1;1)}{[g'(1;1)]^3} - \frac{1}{4} \frac{[g''(1;1)]^2}{[g'(1;1)]^4} + \frac{\beta_3}{12\alpha^3(2-a)} + \frac{\beta_2}{2\alpha^2(2-a)^2} \left[1 - a + \frac{\beta_2}{4\alpha^2} \right]. \tag{8.95}$$

which is clearly equivalent to (8.78) for $c_1 = c_2 = \frac{1}{2}$. □

Finally we note that the trivial inequality

$$\Phi(0,0) = \Pr\{\underline{x}_1 = 0, \underline{x}_2 = 0\} \leq \Pr\{\underline{x}_1 = 0\} = \Phi(0,1),$$

implies with (8.57) and (8.77) the following two inequalities:

$$\phi_0 \leq 1 - a_1, \tag{8.96}$$

$$z_0 \, g'(1;z_0) \geq 2c_2. \tag{8.97}$$

CHAPTER III

THE CONTINUOUS TIME QUEUEING PROCESS

III.0. Introduction

The queueing model which has been described in detail in section II.0 will be analyzed in this chapter as a continuous time parameter process. We are concerned with the conditional joint distribution of the number of type 1 and of type 2 customers present in the system at time $t > 0$ given the number of customers of the two types at $t = 0$.

Denote by $\underline{y}_i(t)$, $t \geq 0$, $i = 1,2$, the number of type i customers present in the system at time t. In order to investigate the stochastic process $\{(\underline{y}_1(t), \underline{y}_2(t)), t \geq 0\}$ two supplementary variables have to be introduced. The stochastic variable $\underline{z}(t)$ will indicate which type(s) of customers is (are) served at time t, while the stochastic variable $\underline{r}(t)$ is defined to be the residual service time at time t; the stochastic process $\{(\underline{y}_1(t), \underline{y}_2(t), \underline{z}(t), \underline{r}(t)), t \geq 0\}$ then turns out to be a continuous time parameter Markov process. By means of the Chapman-Kolmogorov equation partial differential difference equations will be obtained for the time dependent state probabilities of this Markov process. By introducing Laplace-Stieltjes transforms and generating functions these partial differential difference equations will be converted into a set of four functional equations. The essential step in the solution of this set of equations can be reduced to the solution of a single Hilbert boundary value problem which is analogue to the Hilbert problem formulated in theorem II.7.1.

For the greater part the analysis of the Markov process $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{z}(t),\underline{r}(t)),\ t \geq 0\}$ needs the same techniques as those applied in chapter II in the analysis of the imbedded Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,2,\ldots\}$.

After establishing the transform of the joint distribution of the number of type 1 and of type 2 customers present in the system at time $t > 0$ in section III.6, other characteristic quantities of the time dependent queueing model such as the number of queueing customers, the virtual waiting time for the two types of customers, the workload of the server and the "excess number" of waiting customers will be discussed in section III.7.

In section III.8 the transform of the limiting distribution ($t \to \infty$) of the process $\{(\underline{y}_1(t),\underline{y}_2(t),\ \underline{z}(t),\underline{r}(t)),t \geq 0\}$ will be derived for the case that this queueing process is ergodic. Finally, in section III.9 the transforms of the stationary distributions and the first moments of the quantities defined in section III.7 will be determined.

Throughout this chapter sample functions are defined to be continuous from the right in the variable t.

III.1. Definitions

Two types of customers arrive independently at a single service facility. All arriving customers are admitted to the service system. An arriving customer who finds the system empty is immediately taken into service; otherwise he joins queue 1 or 2 depending on his type. As soon as a service has been completed a new service is started if any customer is present. In general a couple of two customers of different type is simultaneously served. If at a service completion instant there are only customers of one type present a customer of this type is served individually. In each queue customers are served in order of their arrival. See also section II.0.

For the definition of the Poisson arrival processes of the two types of customers, of the service time distribution and its Laplace-Stieltjes transform, and of the traffic intensities we refer to section II.1.

DEFINITION 1.1. *The stochastic variable* $\underline{z}(t)$, $t \geq 0$, *has the set* $\{0,1,2,3\}$ *as its state space; for every* $t \geq 0$,

$\underline{z}(t) = 0$, *if no customers are present in the system at time* t,

$\underline{z}(t) = 1$, *if a customer of type* 1 *is individually served at time* t,

$\underline{z}(t) = 2$, *if a customer of type 2 is individually served at time* t,

$\underline{z}(t) = 3$, *if a couple of two customers (of different type) is served at time t.*

DEFINITION 1.2. *Let* $\underline{y}_i(t), t \geq 0, i = 1,2$, *denote the number of type* i *customers present in the system at time* t, *and let* $\underline{r}(t), t \geq 0$, *denote the residual service time at time* t, *cf. section* III.0. *It will be assumed that* $t = 0$ *can be considered as a service completion instant, cf. definition* II.1.2, *so that* $\underline{r}(0) = 0$ *if* $\underline{z}(0) = 0$ *and* $\underline{r}(0) = \underline{\tau}_1$ *if* $\underline{z}(0) \neq 0$.

Further, let for $k_1,k_2,y_1,y_2 = 0,1,2,\dots$; $i = 1,2,3$; $\tau \geq 0$, $t \geq 0$,

$$Q_y^i(t;k_1,k_2,\tau) := \Pr\{\underline{y}_1(t) = k_1, \underline{y}_2(t) = k_2,\ \underline{z}(t) = i,\ \underline{r}(t) < \tau | \underline{y}(0) = y\},$$

$$Q_y^0(t) := \Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0 | \underline{y}(0) = y\}, \tag{1.1}$$

these probabilities being continuous from the right in the variable t, *cf. section* III.0; *here* $\underline{y}(0)$ *stands for the vector* $(\underline{y}_1(0),\underline{y}_2(0))$, *and* y *for* (y_1,y_2).

It should be noted that $\underline{z}(0)$ is determined by $\underline{y}_1(0)$ and $\underline{y}_2(0)$ because

$$\underline{z}(0) = 0 \Leftrightarrow \underline{y}_1(0) = 0,\ \underline{y}_2(0) = 0;$$
$$\underline{z}(0) = 1 \Leftrightarrow \underline{y}_1(0) > 0,\ \underline{y}_2(0) = 0;$$
$$\underline{z}(0) = 2 \Leftrightarrow \underline{y}_1(0) = 0,\ \underline{y}_2(0) > 0;$$
$$\underline{z}(0) = 3 \Leftrightarrow \underline{y}_1(0) > 0,\ \underline{y}_2(0) > 0. \tag{1.2}$$

It will be assumed that for $k_1,k_2 = 0,1,2,\dots$; $i = 1,2,3$; $t \geqslant 0$, the limits

$$\tilde{Q}^i_y(t;k_1,k_2) := \lim_{\tau\downarrow 0} \frac{\partial}{\partial\tau} Q^i_y(t;k_1,k_2,\tau), \tag{1.3}$$

exist, cf. remark 2.1.

The following Laplace-Stieltjes transforms and generating functions will be needed : for $i = 1,2,3$, for $\mathrm{Re}\,\rho > 0$, $|p_1| \leqslant 1, |p_2| \leqslant 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$\Omega^i_y(\rho;p_1,p_2,\sigma) := \int_0^\infty e^{-\rho t} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty p_1^{k_1} p_2^{k_2} \int_{0-}^\infty e^{-\sigma\tau} d_\tau\, Q^i_y(t;k_1,k_2,\tau)dt,$$

$$\Omega^0_y(\rho) := \int_0^\infty e^{-\rho t}\, Q^0_y(t)dt; \tag{1.4}$$

$$\tilde{\Omega}^i_y(\rho;p_1,p_2) := \int_0^\infty e^{-\rho t} \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty p_1^{k_1} p_2^{k_2}\, \tilde{Q}^i_y(t;k_1,k_2)dt; \tag{1.5}$$

$$X_y(\rho) := \int_0^\infty e^{-\rho t}[\tilde{Q}^1_y(t;1,0) + \tilde{Q}^2_y(t;0,1) + \tilde{Q}^3_y(t;1,1)]dt; \tag{1.6}$$

$$Z^1_y(\rho;p_1) := \frac{1}{p_1} \tilde{\Omega}^1_y(\rho;p_1,0) + \lim_{p_2\to 0} \frac{1}{p_2}[\frac{1}{p_1} \tilde{\Omega}^3_y(\rho;p_1,p_2) + \tilde{\Omega}^2_y(\rho;p_1,p_2)],$$

$$Z^2_y(\rho;p_2) := \frac{1}{p_2} \tilde{\Omega}^2_y(\rho;0,p_2) + \lim_{p_1\to 0} \frac{1}{p_1}[\frac{1}{p_2} \tilde{\Omega}^3_y(\rho;p_1,p_2) + \tilde{\Omega}^1_y(\rho;p_1,p_2)]. \tag{1.7}$$

For the definitions (1.7) note that for $\mathrm{Re}\,\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\tilde{\Omega}^3_y(\rho;0,p_2) = 0, \quad \tilde{\Omega}^3_y(\rho;p_1,0) = 0, \tag{1.8}$$

because there must be at least one customer of each type present in the system at a certain time t, $t \geqslant 0$, if $\underline{z}(t) = 3$, cf. definition 1.1.

Similarly, it can be seen that for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\tilde{\Omega}_y^1(\rho;0,p_2) = 0, \qquad \tilde{\Omega}_y^2(\rho;p_1,0) = 0. \tag{1.9}$$

Therefore the limits in (1.7) exist. Moreover, the relations (1.8) and (1.9) imply that the functions defined in (1.7) are finite at $p_1 = 0$ and $p_2 = 0$. In fact, it is readily verified, cf. (1.6), that for Re $\rho > 0$,

$$Z_y^1(\rho;0) = X_y(\rho), \qquad Z_y^2(\rho;0) = X_y(\rho). \tag{1.10}$$

Finally we introduce the function : for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, Re $\sigma \geq 0$,

$$\Omega_y(\rho;p_1,p_2,\sigma) := \int_0^\infty e^{-\rho t} E\{p_1^{\underline{y}_1(t)} p_2^{\underline{y}_2(t)} e^{-\sigma \underline{r}(t)} | \underline{y}(0) = y\} dt. \tag{1.11}$$

Clearly, cf.(1.1) and (1.4), for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, Re $\sigma \geq 0$,

$$\Omega_y(\rho;p_1,p_2,\sigma) = \Omega_y^0(\rho) + \sum_{i=1}^{3} \Omega_y^i(\rho;p_1,p_2,\sigma). \tag{1.12}$$

III.2. Formulation of the mathematical problem

For the state probabilities (1.1) of the stochastic process $\{(\underline{y}_1(t), \underline{y}_2(t), \underline{z}(t), \underline{r}(t)), t \geq 0\}$ partial differential difference equations will be derived. These equations will be transformed into a set of four functional equations.

Suppose that the vector $(\underline{y}_1(t), \underline{y}_2(t), \underline{z}(t), \underline{r}(t))$ is given at time $t = t_0 \geq 0$. If any customers are present at time t_0 then $\underline{r}(t_0)$ determines the first service completion instant after t_0 and $\underline{z}(t_0)$ indicates which type(s) of customer(s) will leave the system at this service completion instant. By assumption (cf. section II.0, II.1) arrival instants and the duration of services started after t_0 do not depend on the state at $t = t_0$. Hence it is readily verified that the process $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{z}(t),r(t)),$

$t \geq 0\}$ is a *continuous time parameter Markov process*. From the definition of the arrival process and the service time distribution, cf. section II.1, it is clear that this process has *stationary transition probabilities*, cf. FELLER [10], §XVII.1. As the state space of this process we define the set, cf. definition 1.1,

$$\{(0,0,0,0)\} \cup \{\{1,2,\ldots\} \times \{0,1,2,\ldots\} \times \{1\} \times T\} \cup$$
$$\cup \{\{0,1,2,\ldots\} \times \{1,2,\ldots\} \times \{2\} \times T\} \cup \{\{1,2,\ldots\} \times \{1,2,\ldots\} \times \{3\} \times T\}, \tag{2.1}$$

here $T \subset [0,\infty)$ stands for the range of the variable $\underline{r}(t)$ which depends only on the distribution function $B(t)$. Obviously, the above defined state space is minimal, so that the Markov process $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{z}(t),\underline{r}(t)),$ $t \geq 0\}$ is *irreducible*.

We proceed with the derivation of partial differential difference equations for the state probabilities (1.1). For this it is assumed that for $k_1,k_2 = 0,1,2,\ldots;\ i = 1,2,3$, the partial derivatives

$$\frac{\partial}{\partial t} Q_y^i(t;k_1,k_2,\tau),\ \frac{\partial}{\partial t} Q_y^0(t),\ \frac{\partial}{\partial \tau} Q_y^i(t;k_1,k_2,\tau), \tag{2.2}$$

exist and are continuous for $t > 0$, $\tau > 0$, and that the limits (1.3) converge uniformly in t in any finite interval.

<u>REMARK 2.1.</u> It should be noted that the above conditions are only satisfied if the distribution function $B(t)$ possesses definite smoothness properties. However the partial differential equations which will be derived below on the above conditions may be interpreted in terms of the theory of generalized functions in the case that these conditions are not satisfied. Then also the solution should be understood in the generalized sense. See GNEDENKO & KOWALENKO [14], §3.1.4. □

THEOREM 2.1. *Under the conditions formulated above the state probabilities* (1.1) *satisfy for* $t > 0$, $\tau > 0$, *the following set of partial differential equations :*

i. *for* $k_1 \geq 1$, $k_2 \geq 1$,

$$\frac{\partial}{\partial t} Q_y^3(t;k_1,k_2,\tau) = \frac{1}{\alpha_1} Q_y^3(t;k_1-1,k_2,\tau) + \frac{1}{\alpha_2} Q_y^3(t;k_1,k_2-1,\tau) - \\ - \frac{1}{\alpha} Q_y^3(t;k_1,k_2,\tau) + \frac{\partial}{\partial \tau} Q_y^3(t;k_1,k_2,\tau) - \tilde{Q}_y^3(t;k_1,k_2) + \\ + B(\tau)[\tilde{Q}_y^3(t;k_1+1,k_2+1) + \tilde{Q}_y^1(t;k_1+1,k_2) + \tilde{Q}_y^2(t;k_1,k_2+1)]; \tag{2.3}$$

ii. *for* $k_1 \geq 1$, $k_2 \geq 1$,

$$\frac{\partial}{\partial t} Q_y^1(t;k_1,k_2,\tau) = \frac{1}{\alpha_1} Q_y^1(t;k_1-1,k_2,\tau) + \frac{1}{\alpha_2} Q_y^1(t;k_1,k_2-1,\tau) - \\ - \frac{1}{\alpha} Q_y^1(t;k_1,k_2,\tau) + \frac{\partial}{\partial \tau} Q_y^1(t;k_1,k_2,\tau) - \tilde{Q}_y^1(t;k_1,k_2); \tag{2.4}$$

for $k_1 \geq 2$, $k_2 = 0$,

$$\frac{\partial}{\partial t} Q_y^1(t;k_1,0,\tau) = \frac{1}{\alpha_1} Q_y^1(t;k_1-1,0,\tau) - \frac{1}{\alpha} Q_y^1(t;k_1,0,\tau) + \\ + \frac{\partial}{\partial \tau} Q_y^1(t;k_1,0,\tau) - \tilde{Q}_y^1(t;k_1,0) + B(\tau)\,[\tilde{Q}_y^3(t;k_1+1,1) + \\ + \tilde{Q}_y^1(t;k_1+1,0) + \tilde{Q}_y^2(t;k_1,1)]\ ; \tag{2.5}$$

for $k_1 = 1$, $k_2 = 0$,

$$\frac{\partial}{\partial t} Q_y^1(t;1,0,\tau) = \frac{1}{\alpha_1} Q_y^0(t)\, B(\tau) - \frac{1}{\alpha} Q_y^1(t;1,0,\tau) + \frac{\partial}{\partial \tau} Q_y^1(t;0,1,\tau) - \\ - \tilde{Q}_y^1(t;1,0) + B(\tau)[\tilde{Q}_y^3(t;2,1) + \tilde{Q}_y^1(t;2,0) + \tilde{Q}_y^2(t;1,1)]; \tag{2.6}$$

iii. *a similar set of equations as in ii for* $\frac{\partial}{\partial t} Q_y^2(t;k_1,k_2,\tau), k_1 \geq 0, k_2 > 0$;

iv. $$\frac{\partial}{\partial t} Q_y^0(t) = -\frac{1}{\alpha} Q_y^0(t) + \tilde{Q}_y^3(t;1,1) + \tilde{Q}_y^1(t;1,0) + \tilde{Q}_y^2(t;0,1). \tag{2.7}$$

<u>PROOF i</u>. Let us consider a small time interval $(t-h,t]$. Because the arrival process is a Poisson process there are in this interval :

no arrivals... with probability $1-h/\alpha + o(h)$, $h \downarrow 0$,

one arrival... with probability $h/\alpha + o(h)$, $h \downarrow 0$,

two or more arrivals... with probability $o(h)$, $h \downarrow 0$,

while given any arrival the arriving customer is with probability c_i of type i, $i = 1,2$.

The event $\{\underline{r}(t) < \tau\}$ can occur in the following two disjunct ways :

- $h < \underline{r}(t-h) < \tau+h$, and no service has been completed during $(t-h,t]$,
- $\underline{r}(t-h) < h$, and at least one service has been completed during $(t-h,t]$.

From the assumptions formulated above on the limits (1.3) and $B(0+) = 0$, cf. (II.1.4), it follows for the probability that two or more services are completed during $(t-h,t]$, and also for the probability that one or more arrivals occur and one or more services are completed during $(t-h,t]$, that each is $o(h)$, $h \downarrow 0$, (here a paired service is counted as one service), cf. GNEDENKO & KOWALENKO [14], §3.1.4.

If one service has been completed and no customer has arrived during $(t-h,t]$ then the event $\{\underline{y}_1(t) = k_1, \underline{y}_2(t) = k_2, \underline{z}(t) = 3\}$ can only occur if $k_1 \geq 1$, $k_2 \geq 1$, cf. definition 1.1, and if

$$\underline{y}_1(t-h) = k_1+1, \quad \underline{y}_2(t-h) = k_2+1, \quad \underline{z}(t-h) = 3 \ ; \quad \text{or}$$
$$\underline{y}_1(t-h) = k_1+1, \quad y_2(t-h) = k_2, \quad \underline{z}(t-h) = 1 \ ; \quad \text{or}$$
$$\underline{y}_1(t-h) = k_1, \quad y_2(t-h) = k_2+1, \quad \underline{z}(t-h) = 2.$$

By using the above remarks and the formula for the probability of a number of disjunct events we obtain : for $k_1 \geq 1$, $k_2 \geq 1$, $\tau > 0$, $t > 0$,

$$Q^3(t;k_1,k_2,\tau) = \frac{h}{\alpha_1} Q_y^3(t-h;k_1-1,k_2,\tau+h) + \frac{h}{\alpha_2} Q_y^3(t-h;k_1,k_2-1,\tau+h) +$$
$$+ [1-\frac{h}{\alpha}][Q_y^3(t-h;k_1,k_2,\tau+h) - Q_y^3(t-h;k_1,k_2,h) +$$

$$+ \int_0^h B(\tau+h-x)\,d_x\{Q_y^3(t-h;k_1+1,k_2+1,x) + Q_y^1(t-h;k_1+1,k_2,x) +$$

$$+ Q_y^2(t-h;k_1,k_2+1,h)\}] + o(h), \qquad h \downarrow 0.$$

By substracting $Q_y^3(t-h;k_1,k_2,\tau)$ from both sides of this equation, dividing it by h and taking the limit as $h \downarrow 0$, the partial differential equations (2.3) are obtained by using the assumption on the existence of the partial derivatives (2.2), by using the notation (1.3), and by noting that, cf. (1.8), for $k = 0,1,2,\ldots, t \geq 0, \tau \geq 0$,

$$Q_y^3(t;0,k,\tau) = 0, \qquad Q_y^3(t;k,0,\tau) = 0. \tag{2.8}$$

ii, iii, iv. These partial differential equations can be derived in a similar way as the equations (2.3) have been derived. It should be noted that an individual service of a type j customer can only be started if at least one type j customer is present and no customers of the other type are present ($j = 1,2$). □

With the definitions (1.4), (1.5), (1.6) and (1.7) the partial differential difference equations in theorem 2.1 will be transformed into a set of four functional equations.

THEOREM 2.2. *The transforms* $\Omega_y^0(\rho)$ *and* $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, *of the Markov process* $\{(\underline{y}_1(t),\underline{y}_2(t), \underline{z}(t), \underline{r}(t)), t \geq 0\}$ *have the following properties :*

i. they satisfy for $\mathrm{Re}\,\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, $\mathrm{Re}\,\sigma \geq 0$, *the functional equations*

$$\left[\rho-\sigma + \frac{1-c_1p_1-c_2p_2}{\alpha}\right]\Omega_y^3(\rho;p_1,p_2,\sigma) = p_1^{\,y_1} p_2^{\,y_2}\, \beta(\sigma)\; I(y_1 > 0, y_2 > 0) -$$
$$- \tilde{\Omega}_y^3(\rho;p_1,p_2) + \beta(\sigma)\left[\frac{1}{p_1p_2}\,\tilde{\Omega}_y^3(\rho;p_1,p_2) + \frac{1}{p_1}\tilde{\Omega}_y^1(\rho;p_1,p_2) + \frac{1}{p_2}\tilde{\Omega}_y^2(\rho;p_1,p_2) -\right.$$
$$\left. - Z_y^1(\rho;p_1) - Z_y^2(\rho;p_2) + X_y(\rho)\right], \tag{2.9}$$

$$\left[\rho-\sigma+\frac{1-c_1p_1-c_2p_2}{\alpha}\right]\Omega_y^1(\rho;p_1,p_2,\sigma) = p_1^{y_1}\beta(\sigma)\, I(y_1>0,y_2=0) - \tilde{\Omega}_y^1(\rho;p_1,p_2) +$$

$$+\beta(\sigma)\left[\frac{p_1}{\alpha_1}\Omega_y^0(\rho) + Z_y^1(\rho;p_1) - X_y(\rho)\right], \tag{2.10}$$

$$\left[\rho-\sigma+\frac{1-c_1p_1-c_2p_2}{\alpha}\right]\Omega_y^2(\rho;p_1,p_2,\sigma) = p_2^{y_2}\,\beta(\sigma)I(y_1=0,y_2>0) - \tilde{\Omega}_y^2(\rho;p_1,p_2)+$$

$$+\beta(\sigma)\left[\frac{p_2}{\alpha_2}\Omega_y^0(\rho) + Z_y^2(\rho,p_2) - X_y(\rho)\right], \tag{2.11}$$

$$\left[\rho+\frac{1}{\alpha}\right]\Omega_y^0(\rho) = I(y_1=0,y_2=0) + X_y(\rho), \tag{2.12}$$

here I(E) *is the indicator function of the event* E;

ii. they are regular functions of ρ *in the domain* $\text{Re}\,\rho>0$, *for fixed* p_1,p_2,σ, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $\text{Re}\,\sigma\geqslant 0$;

iii. for ρ *fixed,* $\text{Re}\,\rho>0$, *they belong as functions of* p_1 *to the class* $\text{RCB}(C^+)$, *as functions of* p_2 *to the class* $\text{RCB}(C^+)$, *and as functions of* σ *to the class* $\text{RCB}(\{\sigma;\text{Re}\,\sigma>0\})$, *cf. definition* I.1.6, *for* $|p_1|\leqslant 1$, $|p_2|\leqslant 1$, $\text{Re}\,\sigma\geqslant 0$.

PROOF. i. The functional equation (2.9) follows from the partial differential equations (2.3) by using that for $k_1\geqslant 1$, $k_2\geqslant 1$, $\tau>0$, $t>0$, $\text{Re}\,\rho>0$, $\text{Re}\,\sigma\geqslant 0$, $|p_1|\leqslant 1$, $|p_2|\leqslant 1$,

$$\int_0^\infty e^{-\rho t}\frac{\partial}{\partial t}Q_y^3(t;k_1,k_2,\tau)dt = -Q_y^3(0;k_1,k_2,\tau) + \rho\int_0^\infty e^{-\rho t}\,Q_y^3(t;k_1,k_2,\tau)dt,$$

$$\sum_{k_1=1}^\infty\sum_{k_2=1}^\infty p_1^{k_1}\,p_2^{k_2}\int_{0-}^\infty e^{-\sigma\tau}\,d_\tau\,Q_y^3(0;k_1,k_2,\tau) = p_1^{y_1}\,p_2^{y_2}\,\beta(\sigma)I(y_1>0,y_2>0),$$

cf. definition 1.2 ;

$$\int_{0-}^\infty e^{-\sigma\tau}d_\tau\frac{\partial}{\partial\tau}Q_y^3(t;k_1,k_2,\tau) = \sigma\int_0^\infty e^{-\sigma\tau}\,d_\tau\,Q_y^3(t;k_1,k_2,\tau),$$

because $Q_y^3(t;k_1,k_2,0+) = 0$, cf. (1.3) ;

$$\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} p_1^{k_1} p_2^{k_2}\, \tilde{Q}_y^3(t;k_1+1,k_2+1) = \frac{1}{p_1p_2}\Big[\sum_{k_1=1}^{\infty}\sum_{k_2=1}^{\infty} p_1^{k_1} p_2^{k_2}\, \tilde{Q}_y^3(t;k_1,k_2) -$$

$$- \sum_{k_1=1}^{\infty} p_1^{k_1} p_2\, \tilde{Q}_y^3(t;k_1,1) - \sum_{k_2=1}^{\infty} p_1 p_2^{k_2}\, \tilde{Q}_y^3(t;1,k_2) + p_1p_2\, \tilde{Q}_y^3(t;1,1)\Big].$$

The equations (2.10) and (2.11) follow in a similar way from (2.4), (2.5) and (2.6), and their analogue equations, while equation (2.12) follows from (2.7).

ii,iii. The stated regularity properties are well-known properties of Laplace transforms, generating functions and Laplace-Stieltjes transforms. □

From this theorem and the definitions of section III.1 it is clear that the functions

$$\tilde{\Omega}_y^i(\rho;p_1,p_2),\ i = 1,2,3; \qquad Z_y^i(\rho;p_i),\ i=1,2;\ X_y(\rho),$$

are regular functions of ρ for Re $\rho > 0$, and belong to the class RCB(C^+) as functions of p_1 and of p_2, for Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$.
In the next sections it will turn out that these regularity properties and the properties of the functions $\Omega_y^0(\rho)$ and $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, which have been stated in theorem 2.2 are sufficient to determine these functions uniquely for Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$.

III.3. Reduction to a single functional equation

The regularity of the Laplace-Stieltjes transforms $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, as functions of σ in the domain Re $\sigma \geqslant 0$ will be used to obtain first from the functional equations (2.9), (2.10) and (2.11) expressions for the functions $\tilde{\Omega}_y^i(\rho;p_1,p_2)$, $i = 1,2,3$, in terms of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$, $\Omega_y^0(\rho)$ and $X_y(\rho)$, and to obtain then by substitution expressions for the functions $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, in terms of the same functions. Then the problem of the determination of the transforms $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, and $\Omega_y^0(\rho)$ will have been reduced to the problem of the determination of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$, $\Omega_y^0(\rho)$ and $X_y(\rho)$. In order to obtain the latter functions a single functional equation with the same structure as the functional equation (II.2.2) has to be solved.

THEOREM 3.1. *The functions* $\tilde{\Omega}_y^i(\rho;p_1,p_2)$, $i = 1,2,3$, *satisfy the functional equations : for* $\mathrm{Re}\,\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$,

$$\left[p_1p_2-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\right]\tilde{\Omega}_y^3(\rho;p_1,p_2)=\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\left[p_1^{y_1+1}p_2^{y_2+1}\,I(y_1>0,y_2>0)+\right.$$

$$\left.+p_2\tilde{\Omega}_y^1(\rho;p_1,p_2)+p_1\tilde{\Omega}_y^2(\rho;p_1,p_2)+p_1p_2\{X_y(\rho)-Z_y^1(\rho;p_1)-Z_y^2(\rho;p_2)\}\right]; \tag{3.1}$$

$$\tilde{\Omega}_y^1(\rho;p_1,p_2) = \beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\left[p_1^{y_1}\,I(y_1>0,y_2=0)+\frac{p_1}{\alpha_1}\Omega_y^0(\rho)+\right.$$

$$\left.+\,Z_y^1(\rho;p_1)-X_y(\rho)\right]; \tag{3.2}$$

$$\tilde{\Omega}_y^2(\rho;p_1,p_2) = \beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\left[p_2^{y_2}\,I(y_1=0,y_2>0)+\frac{p_2}{\alpha_2}\Omega_y^0(\rho)+\right.$$

$$\left.+\,Z_y^2(\rho;p_2)-X_y(\rho)\right]; \tag{3.3}$$

PROOF. For $\mathrm{Re}\,\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, the inequality

$$\mathrm{Re}\{\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\} > 0,$$

holds. Hence by theorem 2.2.iii the functions $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, are regular at the point

$$\sigma = \rho+\frac{1-c_1p_1-c_2p_2}{\alpha}, \tag{3.4}$$

if Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$. Therefore, if we substitute (3.4) in the equations (2.9), (2.10) and (2.11) the lefthand sides of these equations vanish, so that also the righthand sides of these equations have to vanish at these points. This substitution (3.4) in the functional equations (2.9), (2.10) and (2.11) leads then readily to the functional equations (3.1), (3.2) and (3.3) respectively. □

Substitution of (3.2) and (3.3) in (3.1) gives :

<u>COROLLARY 3.1.</u> *The functions* $\tilde{\Omega}_y^3(\rho;p_1,p_2)$, $Z_y^1(\rho;p_1)$ *and* $Z_y^2(\rho;p_2)$ *satisfy the functional equation : for* Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\Big[p_1p_2-\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big]\tilde{\Omega}_y^3(\rho;p_1,p_2)=\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big[p_1^{y_1+1}p_2^{y_2+1}\, I(y_1>0,y_2>0)+$$
$$+\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big\{p_1^{y_1}p_2 I(y_1>0,y_2=0)+p_1p_2^{y_2}\, I(y_1=0,y_2>0)\Big\}+$$
$$+p_2Z_y^1(\rho;p_1)\Big\{\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)-p_1\Big\}+p_1Z_y^2(\rho;p_2)\Big\{\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)-p_2\Big\}+$$
$$+X_y(\rho)\Big\{p_1p_2-(p_1+p_2)\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big\}+\frac{p_1p_2}{\alpha}\beta\Big(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Omega_y^0(\rho)\Big]. \quad (3.5)$$

□

Obviously, by the relations (3.2), (3.3) and (3.5) the functions $\Omega_y^i(\rho;p_1,p_2)$, $i = 1,2,3$, are expressed in terms of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$, $\Omega_y^0(\rho)$ and $X_y(\rho)$. Elimination of the functions $\tilde{\Omega}_y^i(\rho;p_1,p_2)$, $i = 1,2,3$, from the functional equations (2.9), (2.10) and (2.11) by means of the relations (3.2), (3.3) and (3.5), and of the function $X_y(\rho)$ by means of the relation (2.12) leads to expressions for the functions $\Omega_y^i(\rho;p_1,p_2,\sigma)$, $i = 1,2,3$, in terms of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$. In theorem 3.2 below we shall state these readily obtained expressions and afterwards we shall discuss the question of the determination of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$.

<u>THEOREM 3.2.</u> *The functions* $\Omega^{i}_{y}(\rho;p_1,p_2,\sigma)$, i = 1,2,3, *are given by: for* Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Omega^{1}_{y}(\rho;p_1,p_2,\sigma) = \frac{\beta(\sigma)-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\alpha(\rho-\sigma)+1-c_1p_1-c_2p_2}\Big[\alpha\, p_1^{y_1}\, I(y_2=0) + $$

$$+\alpha\, Z^{1}_{y}(\rho;p_1) - (\alpha\rho+1-c_1p_1)\,\Omega^{0}_{y}(\rho)\Big]; \tag{3.6}$$

$$\Omega^{2}_{y}(\rho;p_1,p_2,\sigma) = \frac{\beta(\sigma)-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\alpha(\rho-\sigma)+1-c_1p_1-c_2p_2}\Big[\alpha\, p_2^{y_2}\, I(y_1=0) + $$

$$+\alpha\, Z^{2}_{y}(\rho;p_2) - (\alpha\rho+1-c_2p_2)\,\Omega^{0}_{y}(\rho)\Big]; \tag{3.7}$$

$$\Omega^{3}_{y}(\rho;p_1,p_2,\sigma) = \frac{\beta(\sigma)-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\alpha(\rho-\sigma)+1-c_1p_1-c_2p_2}\,\alpha\Big[p_1^{y_1}p_2^{y_2} - p_1^{y_1}\,I(y_2=0) - p_2^{y_2}\,I(y_1=0) -$$

$$- Z^{1}_{y}(\rho;p_1) - Z^{2}_{y}(\rho;p_2) + (\rho+\tfrac{1}{\alpha})\,\Omega^{0}_{y}(\rho) + \frac{\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}\Big\{\rho(1-p_1-p_2)\,\Omega^{0}_{y}(\rho) +$$

$$+ p_1^{[y_1-1]^{+}+1}\, p_2^{[y_2-1]^{+}+1} + (p_2-1)\,Z^{1}_{y}(\rho;p_1) + (p_1-1)\,Z^{2}_{y}(\rho;p_2) +$$

$$+ (1-p_1)(1-p_2)\left(\frac{1}{\alpha}\,\Omega^{0}_{y}(\rho) - I(y_1=0,y_2=0)\right)\Big\}\Big]. \tag{3.8}$$

<u>PROOF.</u> Substitute (3.2), (3.3), (3.5) and (2.12) in (2.10), (2.11) and (2.9). □

With (1.12) it follows from theorem 3.2:

<u>COROLLARY 3.2.</u> *The function* $\Omega_y(\rho;p_1,p_2,\sigma)$ *defined in* (1.11) *is given by: for* Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Omega_{y}(\rho;p_1,p_2,\sigma) = \frac{\beta(\sigma)-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\alpha(\rho-\sigma)+1-c_1p_1-c_2p_2}\Big[\alpha\, p_1^{y_1}p_2^{y_2} - (\alpha\rho+1-c_1p_1-c_2p_2)\,\Omega^{0}_{y}(\rho) +$$

$$+ \frac{\alpha\,\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{p_1p_2-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}\Big\{\rho(1-p_1-p_2)\,\Omega^{0}_{y}(\rho) + (p_2-1)Z^{1}_{y}(\rho;p_1) + (p_1-1)Z^{2}_{y}(\rho;p_2) +$$

$$+\ p_1^{[y_1-1]^+ +1}\ p_2^{[y_2-1]^+ +1} + (1-p_1)(1-p_2)\Big(\frac{1}{\alpha}\,\Omega_y^0(\rho) - I(y_1=0,y_2=0)\Big)\Big\}\Big] + \Omega_y^0(\rho). \tag{3.9}$$

It remains to determine the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$. In the next sections it will be shown that these functions can be uniquely determined with the aid of :

i. the functional equation (3.5),

ii. the relations (1.10) and (2.12),

iii. the regularity properties of the functions $\tilde{\Omega}_y^3(\rho;p_1,p_2), Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$ for $\mathrm{Re}\,\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, cf. the end of section III.2.

It should be noted that the functional equation (3.5) has essentially the same structure as the functional equation (II.2.2), i.e. for fixed ρ, $\mathrm{Re}\,\rho > 0$, it relates a function of two complex variables, $\tilde{\Omega}_y^3(\rho;p_1,p_2)$, to two functions of one complex variable, $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$, and to two related constants $\Omega_y^0(\rho)$ and $X_y(\rho)$, cf. (2.12). Moreover, the *kernel* of the functional equation (3.5),

$$p_1p_2 - \beta\Big(\rho + \frac{1-c_1p_1-c_2p_2}{\alpha}\Big), \tag{3.10}$$

has similar properties as the kernel (II.2.3) of the functional equation (II.2.2). Therefore the same methods as which has been used in chapter II for the solution of the functional equation (II.2.2) can be applied in the analysis of the functional equation (3.5).

III.4. Analysis of the functional equation and its kernel

For fixed ρ, $\mathrm{Re}\,\rho > 0$, zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (3.10) of the functional equation (3.5) lead to a functional relation between the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$. The zeros (p_1,p_2)

of the kernel (3.10) can be described by a two-valued analytic function of a parameter η. This analytic function $v(\rho;\eta)$ has two branch points. By means of analytic continuation the relation between the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ for $|p_1| \leq 1$, $|p_2| \leq 1$, leads to a relation between those functions on the line segment $\kappa(\rho)$ joining the two branch points. This relation is in particular important for real values of ρ, $\rho > 0$.
Because the proofs in this section nearly always require the same arguments as the proofs of the corresponding assertions in chapter II, there is in general no need to repeat every detail of these proofs.

As in lemma II.3.1 it can be shown that for fixed ρ, Re $\rho > 0$, the kernel (3.10) possesses zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$.

THEOREM 4.1. *Let ρ be fixed,* Re $\rho > 0$. *For pairs (p_1,p_2) satisfying*

$$p_1p_2 - \beta\left(\rho + \frac{1-c_1p_1-c_2p_2}{\alpha}\right) = 0, \quad |p_1| \leq 1,\ |p_2| \leq 1, \tag{4.1}$$

the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ satisfy the functional relation

$$\frac{Z_y^1(\rho;p_1)}{1-p_1} + \frac{Z_y^2(\rho;p_2)}{1-p_2} = X_y(\rho) + \frac{p_1p_2}{(1-p_1)(1-p_2)}\left[p_1^{[y_1-1]^+}\,p_2^{[y_2-1]^+} - \rho\Omega_y^0(\rho)\right], \tag{4.2}$$

except for pairs (p_1,p_2) corresponding to $p_1 = 1$ or $p_2 = 1$. Those values lead to the relations

$$\begin{aligned} Z_y^1(\rho;1) &= \frac{\nu_2(\rho)}{1-\nu_2(\rho)}\left[\{\nu_2(\rho)\}^{[y_2-1]^+} - \rho\Omega_y^0(\rho)\right], \\ Z_y^2(\rho;1) &= \frac{\nu_1(\rho)}{1-\nu_1(\rho)}\left[\{\nu_1(\rho)\}^{[y_1-1]^+} - \rho\Omega_y^0(\rho)\right], \end{aligned} \tag{4.3}$$

where $p_i = \nu_i(\rho)$ *is the unique solution of the equation*

$$p_i = \beta\left(\rho + \frac{1-p_i}{\alpha_i}\right), \quad p_i \in C^+, \quad \textit{for } i = 1,2. \tag{4.4}$$

<u>PROOF.</u> Because the function $\tilde{\Omega}_y^3(\rho;p_1,p_2)$ is regular for $|p_1| \leq 1, |p_2| \leq 1$, the lefthand side of equation (3.5) vanishes for pairs (p_1,p_2) satisfying (4.1). This leads with (2.12) to the relation (4.2).
For $p_1 = 1$ or $p_2 = 1$ the relation (4.1) reduces to (4.4) which is a well-known relation from the theory of the M/G/1-queueing system and which readily leads to (4.3). □

The zeros (p_1,p_2) of the kernel (3.10) can be described by a two-valued analytic function of a parameter (η) as in section II.3. Therefore we introduce

$$\eta := c_1p_1 + c_2p_2, \qquad v := 2c_1p_1. \tag{4.5}$$

Substitution of (4.5) in equation (4.1) gives the equation

$$v^2 - 2\eta v + 4c_1c_2\,\beta(\rho + \frac{1-\eta}{\alpha}) = 0, \tag{4.6}$$

which defines a two-valued function of η,

$$v(\rho;\eta) := \eta \pm \sqrt{\eta^2-4c_1c_2\beta(\rho+\frac{1-\eta}{\alpha})}. \tag{4.7}$$

As in theorem II.3.1 it can be proved that for every pair (p_1,p_2) satisfying (4.1) a value of η, $|\eta| \leq 1$, exists such that

$$p_1 = \frac{1}{2c_1}\,v(\rho;\eta), \qquad p_2 = \frac{1}{2c_2}\,[2\eta-v(\rho;\eta)], \tag{4.8}$$

for one of the two brances of the function $v(\rho;\eta)$.
Further, the discriminant of equation (4.6),

$$\eta^2 - 4c_1c_2\,\beta(\rho + \frac{1-\eta}{\alpha}), \tag{4.9}$$

has exactly two zeros, say $\eta_1(\rho)$ and $\eta_2(\rho)$, in the domain Re $\eta < 1$ for every ρ, Re $\rho > 0$ (cf. lemma II.3.2). Hence, the two-valued function

$v(\rho;\eta)$, defined in (4.7) is analytic in the domain

$$G(\rho) := \{\eta;\ \mathrm{Re}\ \eta < 1\}\setminus\{\eta_1(\rho),\eta_2(\rho)\},$$

and $\eta_1(\rho)$ and $\eta_2(\rho)$ are first order branch points of this function, cf. theorem II.3.2.

Let $\kappa(\rho)$ denote the line segment joining these branch points $\eta_1(\rho)$ and $\eta_2(\rho)$. As in section II.4 the relation (4.2) which holds for pairs (p_1,p_2) satisfying (4.1) can be extended by analytic continuation of the relevant functions to a relation which holds for pairs (p_1,p_2) given by (4.8) for $\mathrm{Re}\ \eta \leqslant 1$ and in particular for $\eta \in \kappa(\rho)$.

For real values of $\rho, \rho > 0$, the branch points $\eta_1(\rho)$ and $\eta_2(\rho)$ are real, and they can be chosen such that (cf. theorem II.3.3)

$$-1 < \eta_1(\rho) < 0 < \eta_2(\rho) < 1. \tag{4.10}$$

Moreover, for real values of ρ, $\rho > 0$, the values of p_1 and p_2 defined by (4.8) are complex conjugate for $\eta \in \kappa(\rho)$, cf. theorem II.3.3. The values of the function $v(\rho;\eta)$ for $\eta \in \kappa(\rho)$, ρ real, $\rho > 0$, can be described by (cf. theorem II.4.3)

$$v = \frac{k(\rho;\cos\theta)}{\cos\theta}\, e^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi, \tag{4.11}$$

here the function $k(\rho;t)$ is uniquely defined, cf. lemma II.4.1, as the root of the equation

$$k(\rho;t) = 2t\sqrt{c_1c_2\ \beta(\rho + \frac{1-k(\rho;t)}{\alpha})},\ k(\rho;t) < 1,\ \text{for}\ \rho > 0,\ -1 \leqslant t \leqslant 1. \tag{4.12}$$

<u>LEMMA 4.1.</u> i. *For fixed real* $\rho, \rho > 0$, *the function* $k(\rho;t)$, $-1 \leqslant t \leqslant 1$, *is bounded in absolute value by* $2|t|\sqrt{c_1c_2\beta(\rho)}$, *its derivative exists and is positive.*

<u>ii.</u> *For fixed* t, $-1 \leqslant t \leqslant 1$, *the function* $k(\rho;t)/t$ *is a strictly*

decreasing function of ρ*, for real* ρ, $\rho > 0$.

iii. *For the function* $h(t)$ *defined in* (II.4.5) *holds :*

$$\lim_{\rho \downarrow 0} k(\rho;t) = h(t), \quad -1 \leq t \leq 1. \tag{4.13}$$

PROOF. i. This statement can be proved by the arguments which have been used in the proof of lemma II.4.2. Note that the derivative of $k(\rho;t)$ has a finite limit as $t \uparrow 1$ if $\rho > 0$, because $k(\rho;t)$ is then a simple root of the equation in (4.12).

ii. By differentiation of the equation in (4.12) as functions of ρ, $\rho > 0$, it is readily verified with the method used in lemma II.4.2, cf. (II.4.8), that

$$\frac{\partial}{\partial\rho} \frac{k(\rho;t)}{t} < 0, \qquad \text{for } \rho > 0,\ -1 \leq t \leq 1.$$

iii. The limit (4.13) follows from (4.12) and (II.4.5), cf. lemma II.4.1.

□

For $\eta \in \kappa(\rho)$, $\rho > 0$, the values of the function $v(\rho;\eta)$ lie on the contour, cf. (4.11),

$$\Lambda(\rho) := \{v;\ v = \frac{k(\rho;\cos\theta)}{\cos\theta} e^{i\theta},\ -\pi \leq \theta \leq \pi\}, \tag{4.14}$$

which can also be represented by, cf. (4.12) and lemma II.5.2,

$$\Lambda(\rho) = \{v;\ |v|^2 = 4c_1c_2\ \beta(\rho + \frac{1-\text{Re } v}{\alpha}),\ \text{Re } v < 1\}. \tag{4.15}$$

We summarize the above discussion in the following theorem, cf. theorem II.4.3.

THEOREM 4.2. *Let* ρ *be real, fixed,* $\rho > 0$. *For* $v \in \Lambda(\rho)$ *the functions* $Z^1_y(\rho;p_1)$ *and* $Z^2_y(\rho;p_2)$ *satisfy the functional relation*

$$\frac{Z_y^1(\rho;v/2c_1)}{1-v/2c_1}+\frac{Z_y^2(\rho;\bar{v}/2c_2)}{1-\bar{v}/2c_2}=X_y(\rho)+$$

$$+\frac{(v/2c_1)(\bar{v}/2c_2)}{(1-v/2c_1)(1-\bar{v}/2c_2)}\left[\left(\frac{v}{2c_1}\right)^{[y_1-1]^+}\left(\frac{\bar{v}}{2c_2}\right)^{[y_2-1]^+}-\rho\,\Omega_y^0(\rho)\right]. \tag{4.16}$$

III.5. The contour $\Lambda(\rho)$ and its interior

In this section some properties of the contour $\Lambda(\rho)$ defined in (4.14) and of the conformal mapping of the unit disk C^+ onto the interior domain $\Lambda^+(\rho)$ will be stated. For the omitted proofs the reader is referred to the relevant proofs in section II.5.

Throughout this section ρ *is assumed to be a fixed real number,* $\rho > 0$.

The contour $\Lambda(\rho)$ is smooth (cf. lemma II.5.1), and has a representation of the form of formula (I.6.4), cf. (4.14). Hence, there exists a conformal mapping $\gamma(\rho;z)$ of the unit disk C^+ onto the domain $\Lambda^+(\rho)$, which is uniquely determined by the conditions

$$\gamma(\rho;0) = 0, \qquad \gamma'(\rho;0) > 0, \tag{5.1}$$

and which can be obtained by the method of Theodorsen, cf. section I.6 and theorem II.5.1. Because the real axis is an axis of symmetry of the contour $\Lambda(\rho)$ the conformal mapping $\gamma(\rho;z)$ satisfies (cf. theorem II.5.2)

$$\gamma(\rho;\bar{z}) = \overline{\gamma(\rho;z)}, \qquad \text{for } z \in C^+ \cup C. \tag{5.2}$$

The inverse of $\gamma(\rho;z)$, $z \in C^+$, will be denoted by $\gamma_0(\rho;v)$, $v \in \Lambda^+(\rho)$. From lemma 4.1.ii and (4.14) it follows that the contour $\Lambda(\rho)$ expands in every direction with decreasing values of ρ, $\rho > 0$. This implies, cf. theorem II.5.3, that $\gamma(\rho;z)$, $z \in C^+ \cup C$, and $\gamma_0(\rho;v)$, $v \in \Lambda^+(\rho) \cup \Lambda(\rho)$,

both are continuous functions of ρ for $\rho > 0$.
With lemmas similarly to lemma II.5.3,...,II.5.8 it can be proved that the conformal mapping $\gamma(\rho;z)$ is regular for $z \in C$, cf. theorem II.5.5, and that its inverse $\gamma_0(\rho;v)$ possesses an analytic continuation into the domain

$$\{v; |v| < 1 + \sqrt{1-4c_1c_2\beta(\rho)},\ \beta(\rho+\frac{1-\frac{1}{2}v}{\alpha}) \neq 0 \quad \text{for } v \in \Lambda^-(\rho)\}. \tag{5.3}$$

Assuming that (II.5.40) holds, i.e. $c_2 \leq \frac{1}{2} \leq c_1$, the following statements can be proved analogously as theorem II.5.6.
The equation $\eta_2(\rho) = 2c_2$ is equivalent to the equation, cf.(II.5.42),

$$c_2 = (1-c_2)\ \beta\left(\rho + \frac{1-2c_2}{\alpha}\right), \qquad 0 < c_2 \leq \frac{1}{2},\ \rho > 0. \tag{5.4}$$

Denote by $\rho = P(c_2)$ the real positive root (if it exists) of equation (5.4) and define $P(c_2) = 0$ otherwise.

<u>THEOREM 5.1.</u> <u>i</u>. *If* $a \leq 2$ *then* $P(c_2) > 0$ *for every* $c_2 < \frac{1}{2}$ *and* $P(\frac{1}{2}) = 0$.
<u>ii</u>. *If* $a > 2$ *then* $P(c_2) > 0$ *only for* $c_2 < c_{2n}(a)$; *here* $c_{2n}(a)$, $0 < c_{2n}(a) < \frac{1}{2}$, *is the constant for which* $\rho = 0$ *is the largest real root of equation* (5.4).
<u>iii</u>. *For* $\rho > P(c_2)$ *the inequality* $\eta_2(\rho) < 2c_2$ *holds while if* $P(c_2) > 0$ *then the inequality* $\eta_2(\rho) > 2c_2$ *holds for* $0 < \rho < P(c_2)$.

For the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ which are by their definitions regular for $p_1 \in C^+$ and $p_2 \in C^+$ the following regularity properties can be proved by the same arguments as used in theorem II.5.7.

<u>THEOREM 5.2.</u> *The functions* $Z_y^1(\rho;\frac{v}{2c_1})$ *and* $Z_y^2(\rho;\frac{v}{2c_2})$ *belong to the class* $RCB(\Lambda^+(\rho))$, cf. definition I.1.6.

<u>COROLLARY 5.1.</u> *The functions*

$$Z_y^1(\rho;\frac{1}{2c_1}\,\gamma(\rho;z)),\qquad Z_y^2(\rho;\,\frac{1}{2c_2}\,\gamma(\rho;z)),$$

belong to the class RCB(C^+).

III.6. Formulation as a Hilbert problem

Because the functional relation (4.16) has the same structure as the functional relation (II.4.18) it can be solved by the same methods. Also from relation (4.16) two Riemann-Hilbert boundary value problems on the unit circle can be derived, which lead to the determination of the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$, cf. section II.6. This procedure will be omitted here. In this chapter we shall restrict the discussion to the method of section II.7, i.e. from relation (4.16) a Hilbert boundary value problem on the unit circle will be derived and solved, for $\rho > P(c_2)$, as well as for $0 < \rho < P(c_2)$ if $P(c_2) > 0$ (the case $\rho = P(c_2) > 0$ can be treated as in theorem II.7.5). As a result the transforms defined in section III.1 are all completely determined.

In this section it is assumed that $0 < c_2 \leq \frac{1}{2}$ *and unless stated otherwise that* ρ *is real,* $\rho > 0$.

In order to simplify the notation we introduce the function

$$J_y(v_1,v_2) := \frac{(v_1/2c_1)^{[y_1-1]^+ +1}}{1 - v_1/2c_1}\,\frac{(v_2/2c_2)^{[y_2-1]^+ +1}}{1 - v_2/2c_2}\,, \tag{6.1}$$

here y stands for the vector (y_1,y_2).

<u>LEMMA 6.1.</u> *The functions* $Z_y^1(\rho;p_1)$ *and* $Z_y^2(\rho;p_2)$ *satisfy the relation :* *for* $t \in C$,

$$\frac{Z_y^1(\rho;\gamma(\rho;t)/2c_1)}{1-\gamma(\rho;t)/2c_1} + \frac{Z_y^2(\rho;\gamma(\rho;\frac{1}{t})/2c_2)}{1-\gamma(\rho;1/t)/2c_2} =$$

$$= X_y(\rho) + J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})) - \rho\, \Omega_y^0(\rho)\; J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})). \qquad (6.2)$$

PROOF. This relation (6.2) follows from (4.16) by inserting $v = \gamma(\rho;t)$ and by using (5.2) and (6.1). □

The first term at the lefthand side of (6.2) belongs to the class $RCB(C^+)$, but for the second term three cases should be distinguished, cf. theorem 5.1, viz.

i. $\rho > P(c_2)$, then $2c_2 \in \Lambda^-(\rho)$ and this term belongs to the class $RCB(C^-)$;

ii. $\rho = P(c_2) > 0$, then $2c_2 \in \Lambda(\rho)$ and this term is regular in C^-, but has pole at $t = 1$;

iii. $0 < \rho < P(c_2)$, then $2c_2 \in \Lambda^+(\rho)$ and this term is regular in C^- except at a single point where it has a first order pole, and it is continuous at the boundary C.

It is sufficient to consider only case i in order to determine the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ uniquely, because if these functions are known on some interval they can be obtained for all ρ, $\mathrm{Re}\,\rho > 0$, by analytic continuation. However, in order to obtain explicit expressions for the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ on the interval $0 < \rho < P(c_2)$, case iii has to be discussed seperately.

For the case $\rho > P(c_2)$ we introduce the function

$$\Xi(\rho;z) := \frac{Z_y^1(\rho;\gamma(\rho;z)/2c_1)}{1-\gamma(\rho;z)/2c_1} - X_y(\rho), \qquad z \in C^+,$$

$$\Xi(\rho;z) := \frac{Z_y^2(\rho;\gamma(\rho;\frac{1}{z})/2c_2)}{1-\gamma(\rho;1/z)/2c_2}, \qquad z \in C^-. \qquad (6.3)$$

LEMMA 6.2. *Let ρ be fixed, $\rho > P(c_2)$. Then the function $\Xi(\rho;z)$ is a* *sectionally regular function (with respect to the unit circle), bounded at infinity, and satisfying the boundary condition : for* $t \in C$,

$$\Xi^{+}(\rho;t) + \Xi^{-}(\rho;t) = J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})) - \rho\Omega_y^0(\rho)\; J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})). \tag{6.4}$$

PROOF. The regularity property follows from corollary 5.1 and theorem 5.1, the boundedness is implied by (5.1) and (1.10), while the boundary condition (6.4) follows from (6.2) and (6.3), cf. definition I.3.3. □

As in lemma II.6.2 it can be proved that the function $J_y(\gamma(\rho;t),\gamma(\rho;\frac{1}{t}))$ satisfies a Hölder condition on the unit circle. Therefore, in lemma 6.2 a *Hilbert boundary value problem* has been formulated for the function $\Xi(\rho;z)$, cf. section I.4, if we assume that $\Omega_y^0(\rho)$ is a given constant. In the following the Hilbert boundary value problem will be solved under this assumption. Once this solution has been obtained the constant $\Omega_y^0(\rho)$ will be determined by a supplementary condition, cf. (6.6) below.

THEOREM 6.1. *The functions* $Z_y^1(\rho;p_1)$, $\mathrm{Re}\,\rho > 0$, $|p_1| \leqslant 1$, *and* $Z_y^2(\rho;p_2)$, $\mathrm{Re}\,\rho > 0$, $|p_2| \leqslant 1$, *are completely determined by* (6.3), *by the solution of the Hilbert boundary value problem (with index zero) as formulated in lemma* 6.2, *and by the conditions* (2.12) *and* (1.10).

PROOF. The proof is analogous to that of theorem II.7.1. We only note here that the Hilbert problem formulated in lemma 6.2 has the solution (cf. lemma I.4.3) : for $\rho > P(c_2)$,

$$\Xi(\rho;z) = \frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\,\frac{dt}{t-z} - p_0(\rho) -$$

$$- \frac{\rho}{2\pi i}\,\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\,\frac{dt}{t-z}, \qquad z \in C^{+},$$

$$\Xi(\rho;z) = -\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\,\frac{dt}{t-z} + p_0(\rho) +$$

$$+ \frac{\rho}{2\pi i} \Omega_y^0(\rho) \int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})) \frac{dt}{t-z}, \qquad z \in C^-, \tag{6.5}$$

where $p_0(\rho)$ is independent of z ; and that the conditions (2.12) and (1.10) lead with (6.5) and (6.3) to : for $\rho > P(c_2)$,

$$p_0(\rho) = X_y(\rho) = (\rho+\tfrac{1}{\alpha})\,\Omega_y^0(\rho) - I(y_1=0,y_2=0) =$$

$$= \frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t} - \frac{\rho}{2\pi i}\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t}. \tag{6.6}$$

From (6.3), (6.5) and (6.6) the functions $Z_y^1(\rho;p_1)$ and $Z_y^2(\rho;p_2)$ can be solved (see corollary 6.1 below). □

<u>COROLLARY 6.1.</u> *For* $\rho > P(c_2), 2c_1p_1 \in \Lambda^+(\rho),\ 2c_2p_2 \in \Lambda^+(\rho)$,

$$Z_y^1(\rho;p_1) = (1-p_1)\Big[\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t-\gamma_0(\rho;2c_1p_1)} -$$

$$- \frac{\rho}{2\pi i}\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t-\gamma_0(\rho;2c_1p_1)}\Big], \tag{6.7}$$

$$Z_y^2(\rho;p_2) = (1-p_2)\Big[\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;\tfrac{1}{t}),\gamma(\rho;t))\frac{dt}{t-\gamma_0(\rho;2c_2p_2)} -$$

$$- \frac{\rho}{2\pi i}\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;\tfrac{1}{t}),\gamma(\rho;t))\frac{dt}{t-\gamma_0(\rho;2c_2p_2)}\Big], \tag{6.8}$$

with

$$\Omega_y^0(\rho) = \frac{I(y_1=y_2=0) + \frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t}}{\frac{1}{\alpha} + \rho + \frac{\rho}{2\pi i}\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t}}. \tag{6.9}$$

<u>PROOF</u>: These relations follow from (6.3), (6.5) and (6.6), cf. the proof of theorem II.6.2. □

By the determination of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$ in theorem 6.1 and corollary 6.1 all the generating functions and transforms defined in section III.1 are also completely determined for Re $\rho > 0$, $|p_1| \leq 1$, $|p_2| \leq 1$, Re $\sigma \geq 0$, cf. theorem 3.2, corollary 3.2.

For each of these functions an explicit expression can be obtained from corollary 6.1 for real ρ, $\rho > P(c_2), 2c_1p_1 \in \Lambda^+(\rho), 2c_2p_2 \in \Lambda^+(\rho)$. As in theorem II.7.2 also expressions can be obtained for $2c_1p_1 \notin \Lambda^+(\rho)$ and/or $2c_2p_2 \notin \Lambda^+(\rho)$. If $P(c_2) > 0$ also for $0 < \rho < P(c_2)$ explicit formulas can be obtained by considering a Hilbert boundary value problem with index one, cf. lemma II.7.6. The result of this procedure will be stated below without proof, because this proof is analogous to that of the theorems II.7.3 and II.7.4

<u>THEOREM 6.2.</u> *If* $P(c_2) > 0$ *then for* $0 < \rho < P(c_2), 2c_1p_1 \in \Lambda^+(\rho), 2c_2p_2 \in \Lambda^+(\rho)$,

$$z_y^1(\rho;p_1) = (1-p_1)\Big[\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t-\gamma_0(\rho;2c_1p_1)} - \frac{\rho}{2\pi i}\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t-\gamma_0(\rho;2c_1p_1)} + \frac{\nu_1(\rho)}{1-\nu_1(\rho)}\frac{2c_2\Big[\{\nu_1(\rho)\}^{[y_1-1]^+}-\rho\Omega_y^0(\rho)\Big]}{z_1(\rho)\gamma'(\rho;z_1(\rho))[1-z_1(\rho)\gamma_0(\rho;2c_1p_1)]}\Big], \tag{6.10}$$

$$z_y^2(\rho;p_2) = (1-p_2)\Big[\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;\tfrac{1}{t}),\gamma(\rho;t))\frac{dt}{t-\gamma_0(\rho;2c_2p_2)} - \frac{\rho}{2\pi i}\Omega_y^0(\rho)\int_C J_{00}(\gamma(\rho;\tfrac{1}{t}),\gamma(\rho;t))\frac{dt}{t-\gamma_0(\rho;2c_2p_2)} + \frac{\nu_1(\rho)}{1-\nu_1(\rho)}\frac{2c_2\Big[\{\nu_1(\rho)\}^{[y_1-1]^+}-\rho\Omega_y^0(\rho)\Big]}{\gamma'(\rho;z_1(\rho))[z_1(\rho)-\gamma_0(\rho;2c_2p_2)]}\Big], \tag{6.11}$$

with

$$\Omega_y^0(\rho) = \frac{I(y_1=y_2=0)+\frac{1}{2\pi i}\int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t}+\frac{2c_2\{\nu_1(\rho)\}^{1+[y_1-1]^+}}{\{1-\nu_1(\rho)\}z_1(\rho)\gamma'(\rho;z_1(\rho))}}{\frac{1}{\alpha}+\rho+\frac{\rho}{2\pi i}\int_C J_{00}(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t}))\frac{dt}{t}+\frac{\nu_1(\rho)}{1-\nu_1(\rho)}\frac{2c_2}{z_1(\rho)\gamma'(\rho;z_1(\rho))}}; \tag{6.12}$$

here $z_1(\rho)$ *is defined by, cf.* (II.7.13),

$$\gamma(\rho;z_1(\rho)) = 2c_2, \quad 0 < z_1(\rho) < 1, \quad 0 < \rho < P(c_2); \tag{6.13}$$

$$\gamma'(\rho;z_1(\rho)) := \frac{\partial}{\partial z}\gamma(\rho;z)|_{z=z_1(\rho)}, \qquad 0<\rho<P(c_2); \tag{6.14}$$

and the function $\nu_1(\rho)$ *is defined in theorem 4.1 as the root of equation* (4.4).

III.7. Virtual waiting time and other queueing quantities

In this section we shall derive expressions for some characteristic quantities of the time dependent queueing model described in section III.1. The transforms of the distributions of these quantities will be expressed in terms of the functions $Z_y^1(\rho;p_1)$, $Z_y^2(\rho;p_2)$ and $\Omega_y^0(\rho)$, which have been obtained for real ρ in the preceding section, cf. corollary 6.1 and theorem 6.2.
First we shall determine the joint generating function for the number of queueing customers of both types at time t, $t > 0$. Then we shall study the joint distribution of the number of these customers for the case that the number of type 1 customers exceeds, is equal to or is less than the number of type 2 customers. For this we need to solve a Hilbert boundary value problem with index zero.
With the results so obtained expressions for the transforms of the following distributions will be established: the joint distribution of the virtual waiting times for both types of customers at time t, the distribution of the workload of the server at time t, and the excess number of wãiting customers, i.e. the absolute value of the difference of the queue lengths at time t.

Define $\underline{q}_i(t)$, $t \geq 0$, $i = 1,2$, as *the number of queueing type* i *customers* (i.e. customers waiting for service) at time t. The joint generating function and Laplace-Stieltjes transform of the process $\{(\underline{q}_1(t),\underline{q}_2(t),\underline{r}(t)),\ t \geq 0\}$ will be denoted by (cf. definition 1.2)

$$\Theta_y(\rho;p_1,p_2,\sigma) := \int_0^\infty e^{-\rho t}\, E\{p_1^{\underline{q}_1(t)}\, p_2^{\underline{q}_2(t)}\, e^{-\sigma\underline{r}(t)} | \underline{y}(0) = y\} dt. \qquad (7.1)$$

THEOREM 7.1. *For* Re $\rho > 0$, $|p_1| \leqslant 1, |p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Theta_y(\rho;p_1,p_2,\sigma) = \Omega_y^0(\rho) + \frac{\beta(\sigma)-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\left[\rho-\sigma+\frac{1-c_1p_1-c_2p_2}{\alpha}\right]\left[p_1p_2-\beta\left(\rho+\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\right]} \times$$

$$\times \Big[p_1^{[y_1-1]^++1}\; p_2^{[y_2-1]^++1} + (p_2-1)\, Z_y^1(\rho;p_1) + (p_1-1)\, Z_y^2(\rho;p_2) +$$

$$+ (1-p_1-p_2)\,\rho\,\Omega_y^0(\rho) + (1-p_1)(1-p_2)\{\Omega_y^0(\rho)/\alpha - I(y_1=y_2=0)\}\Big]. \qquad (7.2)$$

PROOF. Noting that for every $t \geqslant 0$,

$$\begin{aligned}
\underline{q}_1(t) &= \underline{y}_1(t) = 0, & \underline{q}_2(t) &= \underline{y}_2(t) = 0 & &\text{for } \underline{z}(t) = 0,\\
&= \underline{y}_1(t)-1, & &= \underline{y}_2(t), & &\quad " \quad = 1,\\
&= \underline{y}_1(t), & &= \underline{y}_2(t)-1, & &\quad " \quad = 2,\\
&= \underline{y}_1(t)-1, & &= \underline{y}_2(t)-1, & &\quad " \quad = 3,
\end{aligned}$$

it follows from (1.4) and (7.1) that for Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Theta_y(\rho;p_1,p_2,\sigma) = \Omega_y^0(\rho) + \frac{1}{p_1}\,\Omega_y^1(\rho;p_1,p_2,\sigma) + \frac{1}{p_2}\,\Omega_y^2(\rho;p_1,p_2,\sigma) + $$
$$+ \frac{1}{p_1p_2}\,\Omega_y^3(\rho;p_1,p_2,\sigma).$$

Substitution of (3.6), (3.7) and (3.8) in this relation gives (7.2). □

For the derivation of the transforms of other queueing quantities the following functions are needed: for Re $\rho > 0$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Theta_y^e(\rho;p_1,\sigma) := \int_0^\infty e^{-\rho t}\, E\{p_1^{\underline{q}_1(t)}\, e^{-\sigma\underline{r}(t)}(\underline{q}_1(t) = \underline{q}_2(t)) | y(0) = y\} dt$$

$$\Theta_y^g(\rho;p_1,p_2,\sigma) := \int_0^\infty e^{-\rho t}\, E\{p_1^{\underline{q}_1(t)-\underline{q}_2(t)}\, p_2^{\underline{q}_2(t)}\, e^{-\sigma\underline{r}(t)}(\underline{q}_1(t) > \underline{q}_2(t)) | \underline{y}(0) = y\} dt,$$

$$\Theta^s_y(\rho;p_1,p_2,\sigma) := \int_0^\infty e^{-\rho t}\, E\{p_1^{q_1(t)}\, p_2^{q_2(t)-q_1(t)}\, e^{-\sigma \underline{r}(t)}\,(q_1(t) < q_2(t))\,|\,|\underline{y}(0) = y\}dt. \qquad (7.3)$$

It is readily seen that in so far the variables p_1 and p_2 are concerned these generating functions belong to the class $RCB(C^+)$. These functions are given by :

<u>THEOREM 7.2.</u> *For* Re $\rho > 0$, Re $\sigma \geqslant 0$,

$$\Theta^e_y(\rho;p,\sigma) = \frac{1}{2\pi i}\int_C \Theta_y(\rho;u\sqrt{p},\tfrac{1}{u}\sqrt{p},\sigma)\,\frac{du}{u}, \quad |p| \leqslant 1; \qquad (7.4)$$

$$\Theta^g_y(\rho;p_1,p_2,\sigma) = \frac{1}{2\pi i}\int_C \Theta_y(\rho;u\sqrt{p_2},\tfrac{1}{u}\sqrt{p_2},\sigma)\,\frac{\sqrt{p_2}\,du}{u\sqrt{p_2}-p_1} - \Theta^e_y(\rho;p_2,\sigma),$$
$$|p_1| < \sqrt{|p_2|} \leqslant 1, \qquad (7.5)$$

$$\Theta^s_y(\rho;p_1,p_2,\sigma) = \frac{1}{2\pi i}\int_C \Theta_y(\rho;\tfrac{1}{u}\sqrt{p_1},u\sqrt{p_1},\sigma)\,\frac{\sqrt{p_1}\,du}{u\sqrt{p_1}-p_2} - \Theta^e_y(\rho;p_1,\sigma),$$
$$|p_2| < \sqrt{|p_1|} \leqslant 1. \qquad (7.6)$$

<u>PROOF.</u> For $|p| \leqslant 1$, $|p| \leqslant |u| \leqslant \frac{1}{|p|}$ we have the identity

$$E\{(pu)^{q_1(t)}(\tfrac{p}{u})^{q_2(t)}\} = E\{(pu)^{q_1(t)-q_2(t)}\,p^{2q_2(t)}\,(q_1(t) > q_2(t))\} +$$
$$+ E\{p^{2q_1(t)}\,(q_1(t) = q_2(t))\} + E\{p^{2q_1(t)}\,(\tfrac{p}{u})^{q_2(t)-q_1(t)}\,(q_2(t) > q_1(t))\}.$$

This leads with (7.1) and (7.3) to the following relation : for Re $\rho > 0$, $|p| \leqslant 1$, $|p| \leqslant |u| \leqslant \frac{1}{|p|}$, Re $\sigma \geqslant 0$,

$$\Theta_y(\rho;pu,\tfrac{p}{u},\sigma) = \Theta^g_y(\rho;pu,p^2,\sigma) + \Theta^e_y(\rho;p^2,\sigma) + \Theta^s_y(\rho;p^2,\tfrac{p}{u},\sigma). \qquad (7.7)$$

Let ρ,p,σ be fixed, Re $\rho > 0$, $0 < |p| \leqslant 1$, Re $\sigma \geqslant 0$. The function $\Theta^g_y(\rho;pu,p^2,\sigma)$ is regular for $|u| < 1$, the function $\Theta^s_y(\rho;p^2,\frac{p}{u},\sigma)$ is regular for $|u| > 1$ and bounded at infinity, while for $|u| = 1$ the relation (7.7) holds. This defines a *Hilbert boundary value problem*, cf. section I.4. The boundary condition (7.7) is of the form as described by formula (I.4.8) with $n = 0$. Hence the index of this Hilbert boundary value problem is zero, and its complete solution is given by lemma I.4.3 :

$$\Theta_y^g(\rho;pz,p^2,\sigma) + \Theta_y^e(\rho;p^2,\sigma) = \frac{1}{2\pi i}\int_C \Theta_y(\rho;pu,\frac{p}{u},\sigma)\frac{du}{u-z} - p_0, \qquad z \in C^+,$$

$$\Theta_y^s(\rho;p^2,\frac{p}{z},\sigma) = \frac{-1}{2\pi i}\int_C \Theta_y(\rho;pu,\frac{p}{u},\sigma)\frac{du}{u-z} + p_0, \qquad z \in C^-. \tag{7.8}$$

From (7.3) it is seen that the function $\Theta_y^g(\rho;pz,p^2,\sigma)$ vanishes at $z = 0$, and that the function $\Theta_y^s(\rho;p^2,\frac{p}{z},\sigma)$ vanishes as $|z| \to \infty$. This implies :

$$\Theta_y^e(\rho;p^2,\sigma) = \frac{1}{2\pi i}\int_C \Theta_y(\rho;pu,\frac{p}{u},\sigma)\frac{du}{u}; \qquad p_0 = 0. \tag{7.9}$$

Replacing p by $\sqrt{p}$ in (7.9) leads to (7.4). Further (7.5) follows from (7.9) and the first relation of (7.8) by taking $p^2 = p_2$, $pz = p_1$, so that $|p_1| < \sqrt{|p_2|} \leq 1$. Finally, (7.6) follows from (7.9) and the second relation of (7.8) by replacing u by $\frac{1}{u}$ in the integral and by taking $p^2 = p_1$, $\frac{p}{z} = p_2$, so that $|p_2| < \sqrt{|p_1|} \leq 1$. □

<u>REMARK.</u> The regularity of the lefthand sides of (7.4), (7.5) and (7.6) for $|p| < 1$, $|p_1| \leq 1$ and $|p_2| \leq 1$ implies that of the righthand sides. The integrals in these relations are independent of which branch of the square root is taken because e.g.

$$\int_C \Theta_y(\rho;-u\sqrt{p_2},\frac{-1}{u}\sqrt{p_2},\sigma)\frac{-\sqrt{p_2}\,du}{-\sqrt{p_2}u-p_1} = \int_C \Theta_y(\rho;u\sqrt{p_2},\frac{1}{u}\sqrt{p_2},\sigma)\frac{\sqrt{p_2}\,du}{u\sqrt{p_2}-p_1}.$$

<u>REMARK</u>. The values of the function $\Theta_y^g(\rho;p_1,p_2,\sigma)$ for $\sqrt{|p_2|} \leq |p_1| \leq 1$ can be obtained from (7.5) by means of the Sochozki-Plemelj formulas, cf. lemma I.3.3 : e.g. for Re $\rho > 0$, Re $\sigma \geq 0$,

$$\Theta_y^g(\rho;p_1,p_2,\sigma) = \Theta_y(\rho;p_1,\frac{p_2}{p_1},\sigma) - \Theta_y^e(\rho;p_2,\sigma) +$$

$$+ \frac{1}{2\pi i}\int_C \Theta_y(\rho;u\sqrt{p_2},\frac{1}{u}\sqrt{p_2},\sigma)\,\frac{\sqrt{p_2}\;du}{u\sqrt{p_2}-p_1},\quad \sqrt{|p_2|} < |p_1| \leq 1. \tag{7.10}$$

Similarly the function $\Theta_y^s(\rho;p_1,p_2,\sigma)$ can be obtained for $\sqrt{|p_1|} \leq |p_2| \leq 1$.

Next we shall consider some characteristic quantities of the queueing process described in section II.0, of which the distributions can be determined with the aid of theorem 7.2.

Let $\underline{v}_i(t)$, $t \geq 0$, $i = 1,2$, denote the *virtual waiting time* for type i customers at time t, i.e. the time a type i customer would have to wait if he arrived at time t. The Laplace-Stieltjes transform of the distribution of $(\underline{v}_1(t),\underline{v}_2(t))$ will be denoted by : for Re $\rho \geq 0$, Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$V_y(\rho;\sigma_1,\sigma_2) := \int_0^\infty e^{-\rho t}\, E\{e^{-\sigma_1\underline{v}_1(t)-\sigma_2\underline{v}_2(t)}\,|\,\underline{y}(0) = y\}dt. \tag{7.11}$$

<u>THEOREM 7.3.</u> *The Laplace Stieltjes transform of the joint distribution of the virtual waiting times* $\underline{v}_1(t)$ *and* $\underline{v}_2(t)$ *is given by: for* Re $\rho > 0$, Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$V_y(\rho;\sigma_1,\sigma_2) = \Theta_y^g(\rho;\beta(\sigma_1),\beta(\sigma_1+\sigma_2),\sigma_1+\sigma_2) + \Theta_y^e(\rho;\beta(\sigma_1+\sigma_2),\sigma_1+\sigma_2) +$$

$$+ \Theta_y^s(\rho;\beta(\sigma_1+\sigma_2),\beta(\sigma_2),\sigma_1+\sigma_2). \tag{7.12}$$

<u>PROOF.</u> The virtual waiting time for type i customers (i=1,2) at time $t \geq 0$ is the sum of the residual service time at time t and of the service times of the type i customers queueing at time t; i.e. for i = 1,2, for $t \geq 0$,

$$\underline{v}_i(t) = \underline{r}(t) + \underline{\tau}_1 + \dots + \underline{\tau}_{\underline{q}_i(t)}, \tag{7.13}$$

here $\underline{\tau}_j$, $j = 1,2,\ldots$, stands for the duration of the j^{th} service to be initiated after time t; in (7.13) an empty sum is by definition zero. From (7.13) and the fact that customers are served in pairs if possible it follows that for $t \geq 0$, Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$, if $\underline{q}_1(t) > \underline{q}_2(t)$,

$$E\left[e^{-\sigma_1\underline{v}_1(t)-\sigma_2\underline{v}_2(t)}\right] =$$
$$= E\left[e^{-(\sigma_1+\sigma_2)[\underline{r}(t)+\underline{\tau}_1+\ldots+\underline{\tau}_{\underline{q}_2(t)}]-\sigma_1[\underline{\tau}_{\underline{q}_2(t)+1}+\ldots+\underline{\tau}_{\underline{q}_1(t)}]}\right].$$

By the independence of the successive service times this leads with (7.3) to : for Re $\rho > 0$, Re $\sigma_1 \geq 0$, Re $\sigma_2 \geq 0$,

$$\int_0^\infty e^{-\rho t}\, E\{e^{-\sigma_1\underline{v}_1(t)-\sigma_2\underline{v}_2(t)}(\underline{q}_1(t) > \underline{q}_2(t))\,|\,\underline{y}(0) = y\}dt =$$
$$= \Theta_y^g(\rho;\beta(\sigma_1),\beta(\sigma_1+\sigma_2),\sigma_1+\sigma_2).$$

The events $\{\underline{q}_1(t) = \underline{q}_2(t)\}$ and $\{\underline{q}_1(t) < \underline{q}_2(t)\}$ can be treated similarly. Together these three cases lead to (7.12). □

For later reference we specify the Laplace-Stieltjes transforms of the marginal distributions of $\underline{v}_1(t)$ and $\underline{v}_2(t)$ below.

<u>COROLLLARY 7.1.</u> *The Laplace-Stieltjes transforms of the marginal distributions of the virtual waiting times* $\underline{v}_1(t)$ *and* $\underline{v}_2(t)$ *are given by : for* Re $\rho > 0$, Re $\sigma \geq 0$,

$$V_y(\rho;\sigma,0) = \Omega_y^0(\rho) + \frac{\alpha_1}{\alpha_1(\rho-\sigma)+1-\beta(\sigma)}\Big[\{\beta(\sigma)\}^{[y_1-1]^++1} - \beta(\sigma)\,\rho\,\Omega_y^0(\rho) + $$
$$+ \{\beta(\sigma)-1\}\frac{\nu_1(\rho)}{1-\nu_1(\rho)}\{[\nu_1(\rho)]^{[y_1-1]^+} - \rho\,\Omega_y^0(\rho)\}\Big], \tag{7.14}$$

$$V_y(\rho;0,\sigma) = \Omega_y^0(\rho) + \frac{\alpha_2}{\alpha_2(\rho-\sigma)+1-\beta(\sigma)}\Big[\{\beta(\sigma)\}^{[y_2-1]^++1} - \beta(\sigma)\,\rho\,\Omega_y^0(\rho) + $$
$$+ \{\beta(\sigma)-1\}\frac{\nu_2(\rho)}{1-\nu_2(\rho)}\{[\nu_2(\rho)]^{[y_2-1]^+} - \rho\,\Omega_y^0(\rho)\}\Big]. \tag{7.15}$$

PROOF. From theorem 7.3 it follows that for Re $\rho > 0$, Re $\sigma \geq 0$,

$$V_y(\rho;\sigma,0) = \Theta_y^g(\rho;\beta(\sigma),\beta(\sigma),\sigma) + \Theta_y^e(\rho;\beta(\sigma),\sigma) + \Theta_y^s(\rho;\beta(\sigma),1,\sigma).$$

Taking $p^2 = pu = \beta(\sigma)$ in (7.7) this relation leads to : for Re $\rho > 0$, Re $\sigma \geq 0$,

$$V_y(\rho;\sigma,0) = \Theta_y(\rho;\beta(\sigma),1,\sigma). \tag{7.16}$$

Then (7.2) gives for $p_1 = \beta(\sigma), p_2 = 1$, Re $\rho > 0$, Re $\sigma \geq 0$:

$$V_y(\rho;\sigma,0) = \Omega_y^0(\rho) + \frac{\alpha_1}{\alpha_1(\rho-\sigma)+1-\beta(\sigma)}\,[\{\beta(\sigma)\}^{[y_1-1]^+ +1} +$$

$$+ \{\beta(\sigma)-1\}\, Z_y^2(\rho;1) - \beta(\sigma)\rho\Omega_y^0(\rho)].$$

Finally, inserting the expression (4.3) for $Z_y^2(\rho;1)$ leads to relation (7.14) Relation (7.15) can be derived in a similar way. □

Next we shall study the *workload of the server* at time t, by which it is meant the amount of time the server needs to serve all the customers who are present in the system at time t. It is readily seen that the workload of the server at time t is equal to the maximum of the virtual waiting times for the two types of customers at time t.

THEOREM 7.4. *The workload of the server is determined by : for* Re $\rho > 0$, Re $\sigma \geq 0$,

$$\int_0^\infty e^{-\rho t}\, E\left\{e^{-\sigma \max\{\underline{v}_1(t),\underline{v}_2(t)\}} \,|\, \underline{y}(0) = y\right\} dt =$$

$$= \frac{1}{2\pi i}\int_C \Theta_y(\rho;u\sqrt{\beta(\sigma)}, \tfrac{1}{u}\sqrt{\beta(\sigma)},\sigma)\{\frac{1}{u-\sqrt{\beta(\sigma)}} - \frac{\sqrt{\beta(\sigma)}}{u\sqrt{\beta(\sigma)}-1}\}\, du \tag{7.17}$$

PROOF. Because customers are served in pairs if possible we have, cf. (7.13), for $t \geq 0$,

$$\max\{\underline{v}_1(t),\underline{v}_2(t)\} = \underline{r}(t) + \underline{\tau}_1+\ldots+\underline{\tau}_{\max\{\underline{q}_1(t),\underline{q}_2(t)\}}, \tag{7.18}$$

here an empty sum is defined to be zero.

Consider the events $\{\underline{q}_1(t) > \underline{q}_2(t)\}$, $\{\underline{q}_1(t) = \underline{q}_2(t)\}$ and $\{\underline{q}_1(t) < \underline{q}_2(t)\}$ separately as in theorem 7.3. For the event $\{\underline{q}_1(t) > \underline{q}_2(t)\}$ we obtain from (7.18) that for $t \geq 0$, $\mathrm{Re}\ \sigma \geq 0$,

$$E\left\{e^{-\sigma \max\{\underline{v}_1(t),\underline{v}_2(t)\}}\right\} = E\left\{e^{-\sigma[\underline{r}(t)+\underline{\tau}_1+\ldots+\underline{\tau}_{\underline{q}_1(t)}]}\right\};$$

while for the other events similar relations hold. Together these relations imply that, cf. (7.3), for $\mathrm{Re}\ \rho > 0$, $\mathrm{Re}\ \sigma \geq 0$,

$$\int_0^\infty e^{-\rho t} E\{e^{-\sigma \max\{\underline{v}_1(t),\underline{v}_2(t)\}} | \underline{y}(0) = y\}dt =$$

$$= \Theta_y^g(\rho;\beta(\sigma),\beta(\sigma),\sigma) + \Theta_y^e(\rho;\beta(\sigma),\sigma) + \Theta_y^s(\rho;\beta(\sigma),\beta(\sigma),\sigma) \tag{7.19}$$

Relation (7.17) then follows from (7.19) by using theorem 7.2. □

Finally, we shall derive an expression for the generating function of the *excess number of waiting customers* at time t, i.e. $|\underline{q}_1(t) - \underline{q}_2(t)|$.

<u>THEOREM 7.5.</u> *The excess number of waiting customers is determined by : for* $\mathrm{Re}\ \rho > 0$, $|p| < 1$,

$$\int_0^\infty e^{-\rho t} E\{p^{|\underline{q}_1(t)-\underline{q}_2(t)|} | \underline{y}(0) = y\}dt =$$

$$= \frac{1}{2\pi i} \int_C \Theta_y(\rho;u,\frac{1}{u},0) \left\{\frac{1}{u-p} - \frac{p}{pu-1}\right\} du. \tag{7.20}$$

<u>PROOF.</u> With the definitions (7.3) it is readily verified that for $\mathrm{Re}\ \rho > 0$, $|p| \leq 1$,

$$\int_0^\infty e^{-\rho t} E\left[p^{|\underline{q}_1(t)-\underline{q}_2(t)|} | \underline{y}(0) = y\right] dt =$$

$$= \Theta_y^g(\rho;p,1,0) + \Theta_y^e(\rho;1,0) + \Theta_y^s(\rho;1,p,0). \tag{7.21}$$

Then by using theorem 7.2 relation (7.20) follows from (7.21). □

III.8. Description of the process for $t \to \infty$

In this section the behaviour of the Markov process $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{z}(t),\underline{r}(t)),\ t \geq 0\}$ for $t \to \infty$ will be discussed. Because the analysis of the limits of the relevant Laplace transforms as $\rho \downarrow 0$ is analogue to that of the limits of the generating functions as $r \uparrow 1$ in section II.8 the details of the proofs are omitted.
For the case that the Markov process is ergodic, i.e. for $\max\{c_1,c_2\}\, a < 1$, the transform of the stationary distribution of the process and the first moments will be determined.

With the aid of the key renewal theorem, cf. COHEN [03], theorem I.6.2, it can be shown that for $i = 0,1,2,3$,

$$\lim_{t\to\infty} E\{p_1^{\underline{y}_1(t)}\, p_2^{\underline{y}_2(t)}\, e^{-\sigma\underline{r}(t)}(\underline{z}(t) = i)\,|\,\underline{y}(0) = y\},$$

exists for every initial state $y = (y_1,y_2)$, $y_1 \geq 0$, $y_2 \geq 0$, cf. definition 1.2. The proof is omitted here. The structure of the proof is similar to that used by COHEN [03], pp. 257, 246, for the M/G/1-queueing system. The renewal functions for the states of the imbedded Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),\ n = 0,1,\ldots\}$ which are required for such a proof can be found in section IV.1. Hence the existence of the relevant limits as $t \to \infty$ being assured we may apply an Abelian theorem for Laplace transforms in obtaining the values of these limits, i.e. for $i = 1,2,3$, $|p_1| \leq 1, |p_2| \leq 1$, $\mathrm{Re}\ \sigma \geq 0$,

$$\lim_{t\to\infty} E\{p_1^{\underline{y}_1(t)}\, p_2^{\underline{y}_2(t)}\, e^{-\sigma\underline{r}(t)}(\underline{z}(t) = i)\,|\,\underline{y}(0) = y\} = \lim_{\rho\downarrow 0} \rho\Omega_y^i(\rho;p_1,p_2,\sigma),$$

$$\lim_{t\to\infty} \Pr\{\underline{y}_1(t) = 0,\ \underline{y}_2(t) = 0\,|\,\underline{y}(0) = y\} = \lim_{\rho\downarrow 0} \rho\Omega_y^0(\rho). \tag{8.1}$$

The functions at the righthand sides of (8.1) have been determined in theorem 3.2, corollary 6.1 and theorem 6.2 for real ρ, $\rho > 0$. Therefore we shall first discuss the behaviour of the conformal mapping $\gamma(\rho;z)$ of the unit circle C^+ onto the domain $\Lambda^+(\rho)$ as $\rho \downarrow 0$, cf. section III.5. By comparing (4.15) with (II.8.8) it is seen that the contour $\Lambda(\rho)$ tends to the contour $L(1)$ as $\rho \downarrow 0$. Because the contour $\Lambda(\rho)$ expands in every direction with decreasing value of ρ, $\rho > 0$, cf. section III.5, this implies that

$$L^+(1) = \bigcup_{\rho>0} \Lambda^+(\rho), \tag{8.2}$$

and that the following analogue of theorem II.8.3 holds.

THEOREM 8.1. *The conformal mappings* $\{\gamma(\rho;z);\rho > 0\}$ *of the unit disk* C^+ *onto the domains* $\Lambda^+(\rho)$ *tend uniformly for* $z \in C^+$ *to the conformal mapping* $g(1;z)$ *of* C^+ *onto* $L^+(1)$, *defined by* (II.8.11) *and* (II.8.12). *The inverses* $\{\gamma_0(\rho;v);\rho > 0\}$ *tend uniformly for* $v \in L^+(1)$ *to the inverse* $g_0(1;v)$ *of* $g(1;z)$.

PROOF. This follows from (8.2) by lemma I.6.11, cf. theorem II.8.3. □

From the above it is clear that the properties of the contour L(1) and of the conformal mapping g(1;z) which have been proved in section II.8, can be used for the evaluation of the limits in (8.1).

THEOREM 8.2. *The limiting probability* $(t \to \infty)$ *of an empty system is zero if* $\max\{c_1,c_2\}\, a \geq 1$, *and if* $\max\{c_1,c_2\}a < 1$, *it is for* $c_2 \leq c_1$ *given by :*

$$\omega_0 := \lim_{\rho\downarrow 0} \rho\Omega_y^0(\rho) = \frac{2c_2(1-a_1)}{2c_2(1-a_1)+az_0\, g'(1;z_0)}, \quad a_1 < 1, \tag{8.3}$$

independent of the initial state $y = (y_1,y_2)$.

PROOF. By a similar analysis as which has been used in section II.8 it can be shown that, cf. lemma II.8.5, lemma II.8.6 and theorem 8.1,

$$\begin{aligned}\lim_{\rho\downarrow 0} \frac{\rho}{2\pi i} \int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})) \frac{dt}{t} &= \frac{1-\frac{1}{2}a}{\beta g'(1;1)}, \quad \text{for } c_1=c_2=\tfrac{1}{2}, a < 2,\\ &= 0, \qquad \text{otherwise.}\end{aligned} \tag{8.4}$$

Further it is known from the theory of the M/G/1-queueing system (cf. COHEN [03], appendix 6) that the root $\nu_1(\rho)$ of equation (4.4) has the following properties: $\nu_1(0) = 1$ if $a_1 \leqslant 1$, $\nu_1(0) < 1$ if $a_1 > 1$, and

$$\lim_{\rho \downarrow 0} \frac{\rho}{1-\nu_1(\rho)} = \frac{1-a_1}{\beta}, \qquad \text{for } a_1 < 1,$$
$$= 0, \qquad \text{for } a_1 \geqslant 1. \tag{8.5}$$

From (6.13), (II.8.55) and theorem 8.1 it is clear that, cf. theorem 5.1,

$$\lim_{\rho \downarrow 0} z_1(\rho) = z_0, \qquad \text{if } P(c_2) > 0. \tag{8.6}$$

By using (8.4), (8.5) and (8.6) the assertion follows from (6.9) for the case $P(c_2) = 0$, and from (6.12) for the case $P(c_2) > 0$, cf. theorem 5.1 and theorem II.8.4. In (8.3) by definition $z_0 = 1$ for $c_2 = \frac{1}{2}$, $a < 2$. □

From now on it is assumed that $c_2 \leqslant \frac{1}{2} \leqslant c_1$ *and* $a_1 < 1$.

By theorem 8.2 the Markov-process $\{(\underline{y}_1(t), \underline{y}_2(t), \underline{z}(t), \underline{r}(t)), t \geqslant 0\}$ is ergodic under this assumption. In the sequel we shall determine its stationary distribution. For this we need first :

<u>LEMMA 8.1.</u> *For* $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\lim_{\rho \downarrow 0} \rho\, Z_y^1(\rho; p_1) = \frac{\omega_0}{\alpha} \frac{1-p_1}{1-z_0 g_0(1; 2c_1 p_1)}, \tag{8.7}$$

$$\lim_{\rho \downarrow 0} \rho\, Z_y^2(\rho; p_2) = \frac{\omega_0}{\alpha} \frac{1-p_2}{1-g_0(1; 2c_2 p_2)/z_0}. \tag{8.8}$$

<u>PROOF.</u> Suppose first that $c_2 < \frac{1}{2}$. Then $P(c_2) > 0$ since $a_1 < 1$, cf. theorem 5.1, so that (6.10) holds for $0 < \rho < P(c_2)$, $2c_1 p_1 \in \Lambda^+(\rho)$. As in the proof of theorem II.8.7 it follows from (6.10), cf. theorem 8.1, 8.2, that for $2c_1 p_1 \in L^+(1)$,

$$\lim_{\rho \downarrow 0} \rho\, Z_y^1(\rho; p_1) = \frac{(1-a_1) 2c_2 (1-\omega_0)}{\beta\, z_0\, g'(1; z_0)} \frac{1-p_1}{1-z_0 g_0(1; 2c_1 p_1)} .$$

By corollary II.8.3 this relation can be extended to $p_1 \in C^+ \cup C$; hence it proves (8.7) by noting that, cf. (8.3),

$$\frac{2c_2(1-a_1)}{az_0 g'(1;z_0)} = \frac{\omega_0}{1-\omega_0}. \tag{8.9}$$

Next suppose that $c_2 = \frac{1}{2}$. Then $P(c_2) = 0$, cf. theorem 5.1, so that (6.7) holds for $\rho > 0$, $p_1 \in \Lambda^+(\rho)$. As in (8.4) we have, cf. theorem II.8.7, for $p_1 \in L^+(1)$,

$$\lim_{\rho\downarrow 0} \frac{\rho}{2\pi i} \int_C J_y(\gamma(\rho;t),\gamma(\rho;\tfrac{1}{t})) \frac{dt}{t-\gamma_0(\rho;p_1)} = \frac{1-\frac{1}{2}a}{\beta g'(1;1)} \frac{1}{1-g_0(1;p_1)},$$

so that it is obtained from (6.7) that for $p_1 \in L^+(1)$,

$$\lim_{\rho\downarrow 0} \rho\, Z_y^1(\rho;p_1) = (1-\omega_0) \frac{(1-\frac{1}{2}a)}{\beta g'(1;1)} \frac{1-p_1}{1-g_0(1;p_1)}.$$

Again by corollary II.8.3 this relation holds for $p_1 \in C^+ \cup C$. By using (8.9) and the convention $z_0 = 1$ for $c_2 = \frac{1}{2}$ relation (8.7) follows. With this (8.7) has been proved. The proof of (8.8) is similar. □

With the aid of this lemma the stationary distribution of the queueing process that we consider can be obtained. For this stationary distribution the following notations are introduced, cf. (8.1), (1.11) : for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$\Omega^i(p_1,p_2,\sigma) := \lim_{\rho\downarrow 0} \rho\, \Omega_y^i(\rho;p_1,p_2,\sigma), \qquad i = 1,2,3,$$

$$\Omega(p_1,p_2,\sigma) := \lim_{\rho\downarrow 0} \rho\, \Omega_y(\rho;p_1,p_2,\sigma). \tag{8.10}$$

<u>THEOREM 8.3.</u> *The transform of the stationary distribution of the Markov process* $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{z}(t),\underline{r}(t)),t \geqslant 0\}$ *defined in section* III.1 *is given by* ω_0 *(cf. theorem* 8.2*) for the empty state, and by : for* $|p_1| \leqslant 1, |p_2| \leqslant 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$\Omega^1(p_1,p_2,\sigma) = \omega_0 \frac{\beta(\sigma)-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2-\alpha\sigma}\left[\frac{1-p_1}{1-z_0g_0(1;2c_1p_1)} - 1+c_1p_1\right]; \qquad (8.11)$$

$$\Omega^2(p_1,p_2,\sigma) = \omega_0 \frac{\beta(\sigma)-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2-\alpha\sigma}\left[\frac{1-p_2}{1-g_0(1;2c_2p_2)/z_0} -1+c_2p_2\right]; \qquad (8.12)$$

$$\Omega^3(p_1,p_2,\sigma) = \omega_0 \frac{\beta(\sigma)-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2-\alpha\sigma}\left[1- \frac{1-p_1}{1-z_0g_0(1;2c_1p_1)} - \frac{1-p_2}{1-g_0(1;2c_2p_2)/z_0} +\right.$$

$$\left. + \frac{(1-p_1)(1-p_2)\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) - p_1p_2}\left\{\frac{1}{1-z_0g_0(1;2c_1p_1)} + \frac{1}{1-g_0(1;2c_2p_2)/z_0} -1\right\}\right]. \qquad (8.13)$$

PROOF. These relations follow readily from theorem 3.2 by using lemma 8.1 and theorem 8.2. □

In the sequel let $(\underline{y}_1,\underline{y}_2,\underline{z},\underline{r})$ be a stochastic vector of which the distribution has the transform given by theorem 8.3.

For $\sigma = 0$ formula (8.11) becomes : for $|p_1| \leq 1, |p_2| \leq 1$,

$$\Omega^1(p_1,p_2,0) = \omega_0 \frac{1-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2}\left[\frac{1-p_1}{1-z_0g_0(1;2c_1p_1)} - 1+c_1p_1\right].$$

With (II.8.69) this leads for $p_1 \to 1$ to : for $|p_2| \leq 1$,

$$\Omega^1(1,p_2,0) = \omega_0 \frac{1-\beta\left((1-p_2)/\alpha_2\right)}{c_2(1-p_2)}\left[\frac{1-a_2}{1-a_1}\frac{z_0}{2c_2}g'(1;z_0) - c_2\right]. \qquad (8.14)$$

COROLLARY 8.1. *The stationary distribution of the type(s) of customers which is (are) in service, cf. definition* 1.1, *is given by*

$\Pr\{\underline{z} = 0\} = \omega_0,$

$\Pr\{\underline{z} = 1\} = 1-a_2-\omega_0,$

$\Pr\{\underline{z} = 2\} = 1-a_1-\omega_0,$

$\Pr\{\underline{z} = 3\} = a+\omega_0-1.$

PROOF. For $p_2 \to 1$ relation (8.14) gives the stationary probability that a type 1 customer is served individually, i.e. $\Pr\{\underline{z}=1\}$. The expression for $\Pr\{\underline{z}=1\}$ has been rewritten with the aid of (8.9). The other probabilities follow similarly. □

From (8.3) and (II.8.57) it follows that

$$\omega_0 = \frac{\phi_0}{a+\phi_0}, \qquad \phi_0 = \frac{a\,\omega_0}{1-\omega_0}. \tag{8.15}$$

With this and (II.8.96) the following upper bound for ω_0 can be derived, while $\Pr\{\underline{z}=3\} \geqslant 0$, cf. corollary 8.1, leads to the lower bound:

$$[1-a]^+ \leqslant \omega_0 \leqslant \frac{1-a_1}{1+a_2}. \tag{8.16}$$

The above implies with (8.15) the inequalities

$$\max\{1,a\}\,\omega_0 \leqslant \phi_0 \leqslant (1+a_2)\,\omega_0. \tag{8.17}$$

Next we consider the stationary distribution of the number of customers of both types present in the system and of the residual service time irrespective of which type(s) of customer(s) is(are) served.

THEOREM 8.4. *The transform of the stationary distribution of the process* $\{(\underline{y}_1(t),\underline{y}_2(t),\underline{r}(t)),t \geqslant 0\}$ *is given by: for* $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$\Omega(p_1,p_2,\sigma) = \omega_0 + \omega_0\,\frac{\beta(\sigma)-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2-\alpha\sigma}\Bigg[c_1p_1 + c_2p_2 - 1 +$$

$$+\frac{(1-p_1)(1-p_2)\,\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)-p_1p_2}\left\{\frac{1-g_0(1;2c_1p_1)g_0(1;2c_2p_2)}{[1-z_0g_0(1;2c_1p_1)][1-g_0(1;2c_2p_2)/z_0]}\right\}\Bigg]. \tag{8.18}$$

<u>PROOF.</u> This relation follows readily from corollary 3.2 by using lemma 8.1 and theorem 8.2, cf. (8.10). □

With (II.8.69) formula (8.18) becomes for $p_2 \to 1$: for $|p_1| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Omega(p_1,1,\sigma) = \omega_0+\omega_0 \frac{\beta(\sigma)-\beta\left(\frac{1-p_1}{\alpha_1}\right)}{c_1(1-p_1)-\alpha\sigma}\left[c_1(p_1-1)+\frac{z_0}{2c_2}\, g'(1;z_0)\frac{(1-p_1)\,\beta\left(\frac{1-p_1}{\alpha_1}\right)}{\beta\left(\frac{1-p_1}{\alpha_1}\right)-p_1}\right]. \tag{8.19}$$

By letting $p_1 \to 1$ relation (8.19) leads with (8.9) to :

<u>COROLLARY 8.2.</u> *The Laplace-Stieltjes transform of the stationary distribution of the residual service time is given by : for* Re $\sigma \geqslant 0$,

$$\Omega(1,1,\sigma) = \omega_0 + (1-\omega_0)\,\frac{1-\beta(\sigma)}{\beta\sigma}\,. \qquad \square$$

This result is not surprising because the stationary distribution of the residual service time given that the system is not empty is the same as for the M/G/1-queueing system, cf. COHEN [03], p.258.

Taking $\sigma = 0$ in (8.18) leads to :

<u>COROLLARY 8.3.</u> *The generating function for the stationary distribution of the number of type 1 and of type 2 customers present in the system is given by : for* $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\Omega(p_1,p_2,0) = \omega_0\, \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)\Bigg[1 + \frac{(1-p_1)(1-p_2)}{1-c_1p_1-c_2p_2}\,\frac{1-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)-p_1p_2}\,\frac{1-g_0(1;2c_1p_1)g_0(1;2c_2p_2)}{\{1-z_0g_0(1;2c_1p_1)\}\{1-g_0(1;2c_2p_2)/z_0\}}\Bigg]. \tag{8.20}$$

It should be noted that in the present queueing model the limiting distribution $(t\to\infty)$ of the process $\{(\underline{y}_1(t),\underline{y}_2(t)),t \geqslant 0\}$ is different from the limiting distribution $(n \to \infty)$ of the imbedded Markov chain $\{(\underline{x}_1(n),$

$\underline{x}_2(n)),n = 0,1,2,...\}$ (compare (8.20) with (II.8.72)). This feature is coherent with the fact that in the present model more than one customer can be served at the same time, cf. COOPER [05], pp. 154,155.

The generating functions of the limiting distributions of the continuous time process and its imbedded process are related by : for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\Omega(p_1,p_2,0) = \omega_0\, \beta\!\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) + \frac{\omega_0}{\phi_0}\, \frac{1-\beta\!\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{1-c_1p_1-c_2p_2}\, \Phi(p_1,p_2). \tag{8.21}$$

This relation has been obtained here by comparing the results (II.8.72) and (8.20) of the analysis performed in the chapters II and III. However, we note that without knowledge of (II.8.57), (8.3), (II.8.72) and (8.20) the relations (8.15) and the following relation - equivalent to (8.21) - can be proved with the aid of Wald's theorem and stochastic mean value theorems (cf. COHEN [03], appendix 7 and §II.6.7): for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\Omega(p_1,p_2,0) = \omega_0 + (1-\omega_0)\, \frac{1-\beta\!\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{a(1-c_1p_1-c_2p_2)}\, \{\Phi(p_1,p_2)-\phi_0+(c_1p_1+c_2p_2)\phi_0\}. \tag{8.22}$$

By taking $\sigma = 0$ in (8.19) we obtain the marginal stationary distribution of the number of type 1 customers present in the system : for $|p_1| \leqslant 1$,

$$\Omega(p_1,1,0) = \omega_0\, \beta\!\left(\frac{1-p_1}{\alpha_1}\right) \left[1 + \frac{z_0\, g'(1;z_0)}{2c_1c_2}\, \frac{1-\beta\!\left(\frac{1-p_1}{\alpha_1}\right)}{\beta\!\left(\frac{1-p_1}{\alpha_1}\right)-p_1}\right]. \tag{8.23}$$

The first moment of $\underline{y}_1$ can be obtained by differentiating (8.23) and taking $p_1 = 1$. Analogously the first moment of $\underline{y}_2$ can be calculated.

COROLLARY 8.4. *The first moments of the stationary distributions of the number of type 1 and of type 2 customers present in the system are given by:*

$$E\{\underline{y}_i\} = a_i + \frac{1-\omega_0}{a_i} \frac{a_i^2}{1-a_i} \frac{\beta_2}{2\beta^2}, \quad i = 1,2. \qquad \square$$

Because $1-\omega_0 \geq a_i$, $i=1,2$, cf. corollary 8.1, it follows with (II.8.77):

$$E\{\underline{y}_i\} \geq E\{\underline{x}_i\} = a_i + \frac{a_i^2}{1-a_i} \frac{\beta_2}{2\beta^2}, \qquad i = 1,2. \tag{8.24}$$

Note that $E\{\underline{y}_1\}$ depends on α_2 only through ω_0 in the expression in corollary 8.4. If we suppose α_1 and $\beta(s)$ to be fixed then as $\alpha_2 \to \infty$, i.e. ultimately no type 2 customers can be present, we obtain the common M/G/1-queueing model for the type 1 customers. Hence $\omega_0 \to 1-a_1$ as $\alpha_2 \to \infty$ so that (8.24) becomes an equality as $\alpha_2 \to \infty$.
On the other hand, if we keep α_2 and $\beta(s)$ fixed, $\beta_2 < \infty$, then it is seen that $E\{\underline{y}_1\}$ becomes infinite as $\alpha_1 \downarrow \beta$ (i.e. $a_1 \uparrow 1$), but

$$E\{\underline{y}_2\} = a_2 + \frac{a_2}{1-a_2} \frac{\beta_2}{2\beta^2},$$

because $\omega_0 \downarrow 0$ as $a_1 \uparrow 1$, cf.theorem 8.2. Hence, $E\{\underline{y}_2\}$ is finite for $a_1 \geq 1$ provided that $a_2 < 1$, cf. remark II.8.2.

III.9. Stationary distributions

With the aid of theorem 8.2 and lemma 8.1 in which the limits of the functions $\rho\Omega_y^0(\rho), \rho Z_y^1(\rho;p_1)$ and $\rho Z_y^2(\rho;p_2)$ as $\rho \downarrow 0$ have been established we shall determine in this section the stationary distributions of the quantities discussed in section III.7 for which the transforms of the time dependent distribution have been expressed in the just mentioned functions. The existence of the stationary distributions of these quantities is assured for max $\{c_1,c_2\}\, a < 1$ by similar arguments as in the beginning of section III.8 and therefore their transforms can be obtained by using an Abelian theorem for Laplace transforms, cf. (8.1). For all quantities the first moment of the stationary distribution will be given.

Throughout this section it is assumed that $c_2 \leqslant \frac{1}{2} \leqslant c_1$ *and that* $a_1 < 1$.

First we shall discuss the stationary distribution of the number of waiting customers of both types (exclusively the customers whose service is in progress).

Denote, cf. (7.1), for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Theta(p_1,p_2,\sigma) := \lim_{\rho \downarrow 0} \rho\, \Theta_y(\rho;p_1,p_2,\sigma). \tag{9.1}$$

THEOREM 9.1. *The transform of the stationary distribution of the process* $\{(\underline{q}_1(t),\underline{q}_2(t),\underline{r}(t)),t \geqslant 0\}$ *is given by : for* $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, Re $\sigma \geqslant 0$,

$$\Theta(p_1,p_2,\sigma) = \omega_0 +$$

$$+ \omega_0 \frac{(1-p_1)(1-p_2)}{1-c_1p_1-c_2p_2-\alpha\sigma} \frac{\beta(\sigma)-\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)}{\beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right)-p_1p_2} \frac{1-g_0(1;2c_1p_1)g_0(1;2c_2p_2)}{[1-z_0g_0(1;2c_1p_1)][1-g_0(1;2c_2p_2)/z_0]}. \tag{9.2}$$

PROOF. The assertion follows readily from theorem 7.1 with the aid of theorem 8.2 and lemma 8.1. □

By taking $\sigma = 0$ in (9.2) we obtain with (8.20): for $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\Omega(p_1,p_2,0) = \beta\left(\frac{1-c_1p_1-c_2p_2}{\alpha}\right) \Theta(p_1,p_2,0). \tag{9.3}$$

Let $(\underline{q}_1,\underline{q}_2)$ be a stochastic vector of which the joint distribution has the generating function $\Theta(p_1,p_2,0)$, cf. (9.2). Then it follows readily from (9.3) and corollary 8.4 :

COROLLARY 9.1. *The first moments of the stationary distribution of the number of queueing type* 1 *and type* 2 *customers are given by :*

$$E\{\underline{q}_i\} = \frac{1-\omega_0}{a_i}\,\frac{a_i^2}{1-a_i}\,\frac{\beta_2}{2\beta^2}, \qquad i = 1,2. \qquad \square$$

Taking $\sigma = 0$ and subsequently $p_1 = p_2 = 0$; $p_1 = 0$, $p_2 = 1$; and $p_1 = 1$, $p_2 = 0$, in (9.2) we obtain with the aid of (8.9):

<u>COROLLARY 9.2.</u> *The stationary probabilities of no waiting customers are given by :*

$$\Pr\{\underline{q}_1 = 0, \underline{q}_2 = 0\} = \omega_0/\beta(\tfrac{1}{\alpha}),$$
$$\Pr\{\underline{q}_i = 0\} = \omega_0 + \frac{1-\omega_0}{a_i}(1-a_i)[1-\beta(\tfrac{1}{\alpha_i})]/\beta(\tfrac{1}{\alpha_i}), \quad i = 1,2. \qquad \square$$

Let the functions $\Theta^e(p_1,\sigma)$, $\Theta^g(p_1,p_2,\sigma)$ and $\Theta^s(p_1,p_2,\sigma)$ be defined in a similar way as (9.1), cf. (7.3). Obviously these transforms are determined by theorem 7.2, theorem 9.1 and the following :

<u>LEMMA 9.1.</u> *For* $|p| \leqslant 1$, $|z| \neq 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$\lim_{\rho \downarrow 0} \rho \int_C \Theta_y(\rho;up,\tfrac{p}{u},\sigma)\frac{du}{u-z} = \int_C \Theta(up,\tfrac{p}{u},\sigma)\frac{du}{u-z}.$$

<u>PROOF.</u> Because the function $\Theta(p_1,p_2,\sigma)$ is the transform of a probability distribution we have for $|u| = 1$, $|p| \leqslant 1$, $\mathrm{Re}\,\sigma \geqslant 0$,

$$|\Theta(up,\tfrac{p}{u},\sigma)| \leqslant 1,$$

so that the assertion follows by the dominated convergence theorem (see BURRILL [02], §7.2), cf. (9.1). $\square$

This enables us to prove :

<u>Theorem 9.2.</u> *In the case* $c_2 < \tfrac{1}{2} < c_1$,

$$\Pr\{\underline{q}_1 = \underline{q}_2\} = \omega_0\Big[1 - \frac{1}{c_2} + (\frac{1}{c_2} - \frac{1}{c_1})\frac{1}{1-z_0^2}\Big],$$

$$\Pr\{\underline{q}_1 > \underline{q}_2\} = \omega_0\left[\frac{1-a_2}{c_1-c_2}\,\frac{1-\omega_0}{a\omega_0} - \frac{1}{c_2}\,\frac{1}{1-z_0^2} + \frac{1}{c_2}\right],$$

$$\Pr\{\underline{q}_1 < \underline{q}_2\} = \omega_0\left[\frac{1-a_1}{c_2-c_1}\,\frac{1-\omega_0}{a\omega_0} + \frac{1}{c_1}\,\frac{1}{1-z_0^2}\right].$$

PROOF. From theorem 9.1 it is readily obtained that for $|u| = 1$,

$$\Theta(u,\frac{1}{u},0) = \omega_0[1 + \frac{1-u}{c_2-c_1u}\{1 - \frac{1}{1-z_0g_0(1;2c_1u)} - \frac{1}{1-g_0(1;2c_2/u)/z_0}\}]. \tag{9.4}$$

With the aid of corollary II.8.3 relation (9.4) leads to : for $|z| < 1$,

$$\frac{1}{2\pi i}\int_C \Theta(u,\frac{1}{u},0)\,\frac{du}{u-z} = \omega_0[1 + \frac{1-z}{c_2-c_1z}\{1 - \frac{1}{1-z_0g_0(1;2c_1z)}\} - \\ - \frac{1}{c_1}\,\frac{c_1-c_2}{c_2-c_1z}\{1 - \frac{1}{1-z_0^2}\} - \frac{1}{c_1}]. \tag{9.5}$$

From (7.3), (7.4) and lemma 9.1 we have

$$\Pr\{\underline{q}_1 = \underline{q}_2\} = \Theta^e(1,0) = \frac{1}{2\pi i}\int_C \Theta(u,\frac{1}{u},0)\frac{du}{u},$$

so that taking $z = 0$ in (9.5) leads to the expression for $\Pr\{\underline{q}_1 = \underline{q}_2\}$.

From (7.3), (7.5) and lemma 9.1 we have, for $z \in C^+$,

$$\Pr\{\underline{q}_1 > \underline{q}_2\} = \Theta^g(1,1,0) = \lim_{z\to 1}\frac{1}{2\pi i}\int_C \Theta(u,\frac{1}{u},0)\,\frac{du}{u-z} - \Theta^e(1,0).$$

Hence, $\Pr\{\underline{q}_1 > \underline{q}_2\}$ is obtained from (9.5) with the aid of (II.8.69) and (8.9).

Finally, $\Pr\{\underline{q}_1 < \underline{q}_2\}$ can be obtained from (7.3) and (7.6), or by using

$$\Pr\{\underline{q}_1 < \underline{q}_2\} = 1 - \Pr\{\underline{q}_1 > \underline{q}_2\} - \Pr\{\underline{q}_1 = \underline{q}_2\}. \qquad \square$$

The formulas of this theorem do not apply for the case $c_1 = c_2 = \frac{1}{2}$. In this case formula (9.4) becomes : for $|u| = 1$,

$$\Theta(u,\frac{1}{u},0) = \omega_0\,[3 - \frac{2}{1-g_0(1;u)} - \frac{2}{1-g_0(1;1/u)}]. \tag{9.6}$$

The fact that $\Theta(1,1,0) = 1$ leads to :

LEMMA 9.2. *In the case* $c_1 = c_2 = \frac{1}{2}$, $a < 2$, *the second derivative of the conformal mapping* $g(1;z)$ *of* C^+ *onto* $L^+(1)$ *is finite at the point* $z = 1$, *and*

$$g''(1;1) = g'(1;1)\left[\frac{g'(1;1)}{1-\frac{1}{2}a} - 1\right]. \tag{9.7}$$

PROOF. By theorem II.8.2 the first derivative $g'(1;z)$ of the conformal mapping $g(1;z)$ is finite at $z = 1$ if $c_1 = c_2 = \frac{1}{2}$, $a < 2$. Because $\Theta(1,1,0) = 1$, cf. theorem 9.1, it follows from (9.6) that for $|u| = 1$,

$$1 = \omega_0\left[3-2 \lim_{u\to 1} \frac{2-g_0(1;u) - g_0(1;1/u)}{[1-g_0(1;u)][1-g_0(1;1/u)]}\right] .$$

With (II.8.69) this implies that for $|u| = 1$,

$$1 = \omega_0\left[3+2\{g'(1;1)\}^2 \lim_{u\to 1} \frac{2-g_0(1;u)-g_0(1;1/u)}{(1-u)^2}\right]. \tag{9.8}$$

Consequently, if $c_1 = c_2 = \frac{1}{2}$, $a < 2$, then

$$g_0''(1;1) = \lim_{u\to 1} \frac{g_0(1;u)+g_0(1;1/u)-2}{(1-u)^2} - g_0'(1;1),$$

must be finite and satisfies the relation, cf. (9.8) and (8.3),

$$2\{g'(1;1)\}^2[g_0''(1;1) + g_0'(1;1)] = 2 - \frac{a g'(1;1)}{1-\frac{1}{2}a}.$$

By using the fact that $g(1;z)$ and $g_0(1;w)$ are inverse functions it follows from this relation that if $c_1 = c_2 = \frac{1}{2}$, $a < 2$, then $g''(1;1)$ is also finite and satisfies relation (9.7). □

By means of this lemma we can obtain the corresponding probabilities as in theorem 9.2 for the case $c_1 = c_2 = \frac{1}{2}$.

THEOREM 9.3. *In the case* $c_1 = c_2 = \frac{1}{2}$,

$$\Pr\{\underline{q}_1 = \underline{q}_2\} = \omega_0[2g'(1;1)-1],$$

$$\Pr\{\underline{q}_1 > \underline{q}_2\} = \Pr\{\underline{q}_1 < \underline{q}_2\} = \omega_0[1 + \frac{a-1}{1-\frac{1}{2}a} g'(1;1)].$$

PROOF. From (7.3), (7.4), lemma 9.1, and (9.6) we have

$$\Pr\{\underline{q}_1 = \underline{q}_2\} = \frac{\omega_0}{2\pi i} \int_C [3 - \frac{2}{1-g_0(1;u)} - \frac{2}{1-g_0(1;1/u)}] \frac{du}{u}.$$

This integral is non-singular, cf. lemma 9.2, but separately the integrals

$$\frac{1}{2\pi i} \int_C \frac{1}{1-g_0(1;u)} \frac{du}{u} = \frac{1}{2\pi i} \int_C \frac{1}{1-g_0(1;1/u)} \frac{du}{u},$$

are singular at the point $u = 1$. But as a consequence of lemma 9.2 the function

$$\frac{u-1}{1-g_0(1;u)},$$

belongs to the class H(C), cf. definition I.2.2. Hence, these singular integrals exist as principle values, cf. lemma I.3.1, and the first integral is equal to the residue at $u=0$ plus half of the residue at $u=1$ (cf. corollary II.8.3), i.e.

$$\frac{1}{2\pi i} \int_C \frac{1}{1-g_0(1;u)} \frac{du}{u} = 1-\tfrac{1}{2}g'(1;1). \tag{9.9}$$

This leads to the expression for $\Pr\{\underline{q}_1 = \underline{q}_2\}$. The expressions for $\Pr\{\underline{q}_1 > \underline{q}_2\}$ and $\Pr\{\underline{q}_1 < \underline{q}_2\}$ can be obtained from theorem 7.2 by a similar calculation which leads to

$$\Pr\{\underline{q}_1 > \underline{q}_2\} = \Pr\{\underline{q}_1 < \underline{q}_2\} = \omega_0 [2 + \frac{g''(1;1)}{g'(1;1)} - 2g'(1;1)],$$

or by noting that $\underline{q}_1$ and $\underline{q}_2$ are exchangeable variables in the case $c_1 = c_2 = \frac{1}{2}$ which implies that

$$\Pr\{\underline{q}_1 > \underline{q}_2\} = \Pr\{\underline{q}_1 < \underline{q}_2\} = \tfrac{1}{2}[1 - \Pr\{\underline{q}_1 = \underline{q}_2\}].$$

With (9.7) and (8.3) both procedures lead to the stated expression. □

What precedes leads with the inequality $\Pr\{\underline{q}_1 = \underline{q}_2\} \geqslant \Pr\{\underline{q}_1 = \underline{q}_2 = 0\}$ to the following bounds:

$$z_0 \geqslant \sqrt{\frac{c_2}{c_1}\left(\frac{c_1 + c_2\beta(1/\alpha)}{c_2 + c_1\beta(1/\alpha)}\right)}, \qquad \text{in the case } c_2 < \tfrac{1}{2} < c_1,$$

$$g'(1;1) \geqslant \tfrac{1}{2}\{1 + 1/\beta(1/\alpha)\}, \qquad \text{in the case } c_1 = c_2 = \tfrac{1}{2}.$$

We proceed with the determination of the stationary distribution of the waiting time for a type i customer ($i = 1,2$). Because customers arrive according to a Poisson process the limiting distribution ($t \to \infty$) of the virtual waiting time for type i customers is equal to the limiting distribution as $n \to \infty$ of the actual waiting time of the n^{th} arriving customer of type i, $i = 1,2$ (cf. STIDHAM [23]).

Let $v^i(\sigma)$, $\text{Re}\ \sigma \geqslant 0$, denote the Laplace-Stieltjes transform of the stationary distribution of the waiting time for type i customers, $i = 1,2$. Then by the foregoing remark holds, cf. (7.11) : for $\text{Re}\ \sigma \geqslant 0$,

$$\begin{aligned} v^1(\sigma) &= \lim_{\rho \downarrow 0} \rho\, V_y(\rho;\sigma,0), \\ v^2(\sigma) &= \lim_{\rho \downarrow 0} \rho\, V_y(\rho;0,\sigma). \end{aligned} \tag{9.10}$$

<u>THEOREM 9.4.</u> *The Laplace-Stieltjes transforms of the stationary distributions of the waiting times for type* 1 *and for type* 2 *customers are given by : for* $\text{Re}\ \sigma \geqslant 0$,

$$v^i(\sigma) = \omega_0 + \frac{1-\omega_0}{a_i}(1-a_i)\frac{1-\beta(\sigma)}{\beta(\sigma)-1+\alpha_i\sigma}, \qquad i = 1,2. \tag{9.11}$$

<u>PROOF.</u> From (7.16) it follows, cf. (9.1) and (9.10), that for $\text{Re}\ \sigma \geqslant 0$,

$$v^1(\sigma) = \Theta(\beta(\sigma),1,\sigma).$$

Theorem 9.1 implies with (II.8.69) that for $|p_1| \leqslant 1$, $\text{Re}\ \sigma \geqslant 0$,

$$\Theta(p_1,1,\sigma) = \omega_0\left[1 + \frac{\beta(\sigma)-\beta\left(\frac{1-p_1}{\alpha_1}\right)}{\beta\left(\frac{1-p_1}{\alpha_1}\right)-p_1} \frac{1-p_1}{c_1(1-p_1)-\alpha\sigma} \frac{z_0}{2c_2} g'(1;z_0)\right]. \tag{9.12}$$

By taking $p_1 = \beta(\sigma)$ in (9.12) the expression for $V^1(\sigma)$ in (9.11) is obtained with the aid of (8.9). The expression for $V^2(\sigma)$ can be derived analogously. □

Let $\underline{w}_i$, $i = 1,2$, denote a stochastic variable of which the distribution has as Laplace-Stieltjes transform $V^i(\sigma)$.
From (9.11) and (9.12) it is seen that for $|p_1| \leqslant 1$,

$$V^1\left(\frac{1-p_1}{\alpha_1}\right) = \Theta(p_1,1,0). \tag{9.13}$$

This and a similar relation for $V^2(\sigma)$ lead to the following two formulas, which are generally known as Little's formula :

$$\frac{1}{\alpha_i} E\{\underline{w}_i\} = E\{\underline{q}_i\}, \qquad i = 1,2. \tag{9.14}$$

With corollary 9.1 these formulas imply:

<u>COROLLARY 9.3.</u> *The first moment of the stationary distribution of the waiting time for type* i *customers is given by :*

$$E\{\underline{w}_i\} = \frac{1-\omega_0}{1-\alpha_i} \frac{\beta_2}{2\beta}, \qquad i = 1,2.$$ □

From (9.11) it is obtained by letting $\text{Re}\,\sigma \to \infty$:

<u>COROLLARY 9.4.</u> *The stationary probability that an arriving customer of type* i *meets an empty system is given by :*

$$\Pr\{\underline{w}_i = 0\} = \omega_0, \qquad i = 1,2.$$ □

This section will be concluded with a discussion of the stationary distribution of the workload of the server and that of the excess number of waiting customers (see section III.7 for the definition of these quantities).

THEOREM 9.5. *The Laplace-Stieltjes transform* $M(\sigma)$ *of the stationary distribution of the workload of the server is given by : for* $\mathrm{Re}\,\sigma > 0$,

$$M(\sigma) = \omega_0\Big[\frac{1}{2\pi i}\int_C \frac{1-\beta(\sigma)}{(1-\alpha\sigma)u-(c_1u^2+c_2)\sqrt{\beta(\sigma)}}\Big\{1-\frac{1}{1-z_0g_0(1;2c_1u\sqrt{\beta(\sigma)})}\Big\}\,du\, +$$

$$+\,1-\frac{1}{2\pi i}\int_C \frac{1-\beta(\sigma)}{(1-\alpha\sigma)u-(c_1+c_2u^2)\sqrt{\beta(\sigma)}}\Big\{\frac{1}{1-g_0(1;2c_2u\sqrt{\beta(\sigma)})/z_0}\Big\}\,du\Big]. \tag{9.15}$$

PROOF. From theorem 7.4 and lemma 9.1 it follows that for $\mathrm{Re}\,\sigma > 0$,

$$M(\sigma) = \frac{1}{2\pi i}\int_C \Theta(u\sqrt{\beta(\sigma)},\frac{1}{u}\sqrt{\beta(\sigma)},\sigma)\Big\{\frac{1}{u-\sqrt{\beta(\sigma)}}-\frac{\sqrt{\beta(\sigma)}}{u\sqrt{\beta(\sigma)}-1}\Big\}\,du. \tag{9.16}$$

From theorem 9.1 an expression for $\Theta(u\sqrt{\beta(\sigma)},\frac{1}{u}\sqrt{\beta(\sigma)},\sigma)$, $|u| = 1$, $\mathrm{Re}\,\sigma > 0$, can be readily obtained. Substitution of this expression in (9.16) leads after some rearrangements to formula (9.15). □

Let $\underline{m}$ be a stochastic variable of which the distribution has as Laplace-Stieltjes transform $M(\sigma)$; then :

THEOREM 9.6. *The first moment of the stationary distribution of the workload of the server is given by : in the case* $c_2 < \frac{1}{2} < c_1$,

$$E\{\underline{m}\} = (1-\omega_0)\beta\Big[\frac{1}{1-a_1}\frac{\beta_2}{2\beta^2}-\frac{c_2(1-a_1)}{a(c_1-c_2)^2}\Big\{1+(c_1-c_2)\frac{g''(1;z_0)}{[g'(1;z_0)]^2}\Big\}\Big]\, +$$

$$+\,\frac{\omega_0z_0^2}{(c_1-c_2)[1-z_0^2]}\,\beta\,; \tag{9.17}$$

and in the case $c_1 = c_2 = \frac{1}{2}$,

$$E\{\underline{m}\} = (1-\omega_0)\beta\Big[\frac{1}{2-a}\frac{\beta_2}{\beta^2}+\frac{2-a}{a}\Big\{\frac{g'''(1;1)}{6[g'(1;1)]^3}-\frac{[g''(1;1)]^2}{4[g'(1;1)]^4}\Big\}\Big]. \tag{9.18}$$

PROOF. Because $|\beta(\sigma)| < 1$ for Re $\sigma > 0$ it follows from corollary II.8.3 that the integrals in (9.15) depend only on the residues at the roots of the equation

$$(1-\alpha\sigma)\, u - (c_1 u^2 + c_2)\sqrt{\beta(\sigma)} = 0, \qquad \text{Re } \sigma > 0, \tag{9.19}$$

inside respectively outside the unit circle. In the case $c_2 < \frac{1}{2} < c_1$ the roots $u_1(\sigma)$ and $u_2(\sigma)$ of equation (9.19) satisfy for Re $\sigma > 0$, $\sigma \to 0$,

$$\begin{aligned} u_1(\sigma) &= 1 - \frac{1-\frac{1}{2}a}{2c_1-1}\alpha\sigma + o(\sigma), \\ u_2(\sigma) &= \frac{c_2}{c_1}\{1 + \frac{1-\frac{1}{2}a}{1-2c_2}\alpha\sigma + o(\sigma)\}. \end{aligned} \tag{9.20}$$

Because $a < 2$, $c_2 < \frac{1}{2} < c_1$, both roots of equation (9.19) belong to C^+ for real $\sigma, \sigma \downarrow 0$. Hence, we obtain from (9.15) that in the case $c_2 < \frac{1}{2} < c_1$ for real σ, $\sigma \downarrow 0$,

$$M(\sigma) = \omega_0\Big[1 + \frac{1-\beta(\sigma)}{c_1\sqrt{\beta(\sigma)}[u_1(\sigma)-u_2(\sigma)]}\Big\{\frac{1}{1-z_0 g_0(1;2c_1u_1(\sigma)\sqrt{\beta(\sigma)})} - \frac{1}{1-z_0 g_0(1;2c_1u_2(\sigma)\sqrt{\beta(\sigma)})}\Big\}\Big]. \tag{9.21}$$

With (9.20) it is readily verified that for real $\sigma, \sigma \downarrow 0$,

$$\frac{1-\beta(\sigma)}{c_1\sigma\sqrt{\beta(\sigma)}[u_1(\sigma)-u_2(\sigma)]} = \frac{\beta}{c_1-c_2}\Big[1 + \tfrac{1}{2}\beta\sigma - \frac{\beta_2}{2\beta}\sigma + \frac{1-\frac{1}{2}a}{(c_1-c_2)^2}\alpha\sigma\Big] + o(\sigma). \tag{9.22}$$

Further, it follows with (II.8.83) and (II.8.55) that for real σ, $\sigma \downarrow 0$,

$$\begin{aligned} &\frac{-\sigma}{1-u_1(\sigma)\sqrt{\beta(\sigma)}} \cdot \frac{1-u_1(\sigma)\sqrt{\beta(\sigma)}}{1-z_0 g_0(1;2c_1u_1(\sigma)\sqrt{\beta(\sigma)})} - \frac{\sigma}{1-z_0 g_0(1;2c_1u_2(\sigma)\sqrt{\beta(\sigma)})} = \\ &= \frac{c_1-c_2}{\alpha(1-a_2)}\Big[1 - \frac{c_2\alpha\sigma}{1-a_2}\Big\{\frac{\beta_2}{2\alpha^2} + \frac{(1-a_1)(1-a_2)}{(c_1-c_2)^2}\Big\}\Big]\frac{1}{2c_1z_0g_0'(1;2c_1)}\Big[1 + \\ &+ c_1\alpha\sigma\frac{1-a_2}{c_1-c_2}\frac{g_0''(1;2c_1)}{g_0'(1;2c_1)}\Big] - \frac{\sigma}{1-z_0^2} + o(\sigma). \end{aligned} \tag{9.23}$$

From (9.21), (9.22) and (9.23) we obtain that in the case $c_2 < \frac{1}{2} < c_1$,

$$E\{\underline{m}\} = \frac{a}{1-a_2}\,\frac{\omega_0}{2c_1 z_0 g_0'(1;2c_1)}\Big[\,\frac{c_2\alpha}{1-a_2}\Big\{\frac{\beta_2}{2\alpha^2} + \frac{(1-a_1)(1-a_2)}{(c_1-c_2)^2}\Big\} - \tfrac{1}{2}\beta + \frac{\beta_2}{2\beta} -$$

$$- \alpha\,\frac{1-\frac{1}{2}a}{(c_1-c_2)^2} - c_1\alpha\,\frac{1-a_2}{c_1-c_2}\,\frac{g_0''(1;2c_1)}{g_0'(1;2c_1)}\Big] + \frac{\beta}{c_1-c_2}\,\frac{\omega_0}{1-z_0^2}.$$

With the aid of (II.8.91), (II.8.70), (II.8.71), (8.9) and (8.15) this expression can be rewritten in the form of formula (9.17).

Next we consider the case $c_1 = c_2 = \frac{1}{2}$. In this case the discriminant of equation (9.19) is negative for real σ, $\sigma \uparrow 0$, because $a < 2$. Therefore, the roots $u_1(\sigma)$ and $u_2(\sigma)$ of equation (9.19) are complex conjungate for real σ, $\sigma \downarrow 0$. Further, it is readily seen from (9.19) that in this case for real σ, $\sigma \downarrow 0$,

$$|u_1(\sigma)| = |u_2(\sigma)| = 1,$$

so that the integrals in (9.15) are singular. This implies that the integrals in (9.15) are equal to the sum of the half of the residues at the points $u = u_1(\sigma)$ and $u = u_2(\sigma)$. Hence we obtain in the case $c_1 = c_2 = \frac{1}{2}$ for real σ, $\sigma \downarrow 0$, cf. (9.21),

$$M(\sigma) = \omega_0\Big[1 + \frac{1-\beta(\sigma)}{\frac{1}{2}\sqrt{\beta(\sigma)}[u_1(\sigma)-u_2(\sigma)]}\Big\{\frac{1}{1-g_0(1;u_1(\sigma)\sqrt{\beta(\sigma)})} - \frac{1}{1-g_0(1;u_2(\sigma)\sqrt{\beta(\sigma)})}\Big\}\Big]. \tag{9.24}$$

From equation (9.19) it is obtained that for real σ, $\sigma \downarrow 0$,

$$u_1(\sigma)\sqrt{\beta(\sigma)} = 1 + i\sqrt{(2-a)\alpha\sigma} - \alpha\sigma - \tfrac{1}{2}i\alpha\sigma\sqrt{\frac{\alpha\sigma}{2-a}}\,[1 - \beta_2/2\alpha^2] + o(\sigma^{3/2}),$$

$$u_2(\sigma)\sqrt{\beta(\sigma)} = 1 - i\sqrt{(2-a)\alpha\sigma} - \alpha\sigma + \tfrac{1}{2}i\alpha\sigma\sqrt{\frac{\alpha\sigma}{2-a}}\,[1 - \beta_2/2\alpha^2] + o(\sigma^{3/2}).$$

With these power series expansions it follows that for real σ, $\sigma \downarrow 0$,

$$\frac{1-\beta(\sigma)}{\frac{1}{2}\sqrt{\sigma}\sqrt{\beta(\sigma)}\,[u_1(\sigma)-u_2(\sigma)]} = \frac{\beta}{i\sqrt{(2-a)\alpha}}\left[1-\frac{\beta_2}{2\beta}\sigma+\frac{1}{2}\frac{\alpha\sigma}{2-a}\left\{1-\frac{\beta_2}{2\alpha^2}\right\}\right]+o(\sigma),$$

$$\frac{\sqrt{\sigma}}{1-u_1(\sigma)\sqrt{\beta(\sigma)}}\,\frac{1-u_1(\sigma)\sqrt{\beta(\sigma)}}{1-g_0(1;u_1(\sigma)\sqrt{\beta(\sigma)})} - \frac{\sqrt{\sigma}}{1-u_2(\sigma)\sqrt{\beta(\sigma)}}\,\frac{1-u_2(\sigma)\sqrt{\beta(\sigma)}}{1-g_0(1;u_2(\sigma)\sqrt{\beta(\sigma)})} =$$

$$= \frac{2ig'(1;1)}{\sqrt{(2-a)\alpha}}\left[1-\frac{1}{2}\frac{\alpha\sigma}{2-a}\left\{1+\frac{\beta_2}{2\alpha^2}\right\}+(2-a)\alpha\sigma\left\{\left(\frac{g_0''(1;1)}{2g_0'(1;1)}\right)^2-\frac{g_0'''(1;1)}{6g_0'(1;1)}\right\}\right]+o(\sigma).$$

These relations lead with (9.24) to : in the case $c_1 = c_2 = \frac{1}{2}$,

$$E\{\underline{m}\} = \omega_0\,\frac{2ag'(1;1)}{2-a}\left[\frac{\beta_2}{2\beta}+\frac{1}{2-a}\,\frac{\beta_2}{2\alpha}-(2-a)\alpha\left\{\left(\frac{g_0''(1;1)}{2g_0'(1;1)}\right)^2-\frac{g_0'''(1;1)}{6g_0'(1;1)}\right\}\right].$$

By using (8.9) and the fact that $g(1;z)$ and $g_0(1;w)$ are inverse functions this expression can be rewritten in the form of formula (9.18). □

<u>THEOREM 9.7.</u> *The generating function of the stationary distribution of the excess of the number of waiting customers is given by : in the case* $c_2 < \frac{1}{2} < c_1$, *for* $|p| < 1$,

$$E\{p^{|\underline{q}_1-\underline{q}_2|}\} = \omega_0\Big[1+\frac{1-p}{c_1-c_2p}+\frac{c_1-c_2}{1-z_0^2}\,\frac{(1+p)(1-p)}{(c_2-c_1p)(c_1-c_2p)} -$$
$$-\frac{1-p}{c_2-c_1p}\,\frac{1}{1-z_0g_0(1;2c_1p)}-\frac{1-p}{c_1-c_2p}\,\frac{1}{1-g_0(1;2c_2p)/z_0}\Big]\ ; \qquad (9.25)$$

in the case $c_1 = c_2 = \frac{1}{2}$, *for* $|p| < 1$,

$$E\{p^{|\underline{q}_1-\underline{q}_2|}\} = \omega_0\Big[3-\frac{4}{1-g_0(1;p)}+2\,\frac{1+p}{1-p}\,g'(1;1)\Big]. \qquad (9.26)$$

<u>PROOF</u>. From theorem 7.5 and lemma 9.1 it follows that for $|p| < 1$,

$$E\{p^{|\underline{q}_1-\underline{q}_2|}\} = \frac{1}{2\pi i}\int_C \Theta(u,\tfrac{1}{u},0)\left[\frac{1}{u-p}-\frac{p}{pu-1}\right]du. \qquad (9.27)$$

The function $\Theta(u,\frac{1}{u},0)$, $|u| = 1$, is given by (9.4). The evaluation of the integral in (9.27) can be performed in a similar way as in the proof of theorem 9.2 for the case $c_2 < \frac{1}{2} < c_1$, and as in the proof of theorem 9.3

for the case $c_1 = c_2 = \frac{1}{2}$, and it leads readily to the relations (9.25) and (9.26). □

COROLLARY 9.5. *The first moment of the stationary distribution of the excess of the number of waiting customers is given by : in the case* $c_2 < \frac{1}{2} < c_1$,

$$E\{|\underline{q}_1-\underline{q}_2|\} = (1-\omega_0)\Big[\frac{a(c_1-c_2)}{(1-a_1)(1-a_2)}\frac{\beta_2}{2\beta^2} - \frac{2c_2(1-a_1)}{a(c_1-c_2)^2}\Big\{1+(c_1-c_2)\frac{g''(1;z_0)}{[g'(1;z_0)]^2}\Big\}\Big] + \\ + \frac{2\omega_0 z_0^2}{(c_1-c_2)[1-z_0^2]}; \tag{9.28}$$

in the case $c_1 = c_2 = \frac{1}{2}$,

$$E\{|\underline{q}_1-\underline{q}_2|\} = 2(1-\omega_0)\frac{2-a}{a}\Big\{\frac{g'''(1;1)}{6[g'(1;1)]^3} - \frac{[g''(1;1)]^2}{4[g'(1;1)]^4}\Big\}. \tag{9.29}$$

PROOF. First suppose that $c_2 < \frac{1}{2} < c_1$. Then it follows from (9.25) that

$$E\{|\underline{q}_1-\underline{q}_2|\} = \omega_0\Big[\frac{-1}{c_1-c_2} + \frac{1}{c_1-c_2}\cdot\lim_{p\to 1}\frac{d}{dp}\Big\{\frac{1-p}{1-z_0 g_0(1;2c_1p)} - \frac{1-p}{1-g_0(1;2c_2p)/z_0}\Big\} - \\ - \frac{1}{(c_1-c_2)^2}\lim_{p\to 1}\Big\{\frac{c_1(1-p)}{1-z_0g_0(1;2c_1p)} + \frac{c_2(1-p)}{1-g_0(1;2c_2p)/z_0}\Big\} + \frac{2}{(c_1-c_2)[1-z_0^2]}\Big].$$

With the aid of (II.8.83) and (II.8.84) the limits in this expression can be calculated. The so obtained relation can be rewritten in the form of formula (9.28) by using (II.8.91), (8.9) and (8.15).

Next suppose that $c_1 = c_2 = \frac{1}{2}$. Then relation (9.26) implies :

$$E\{|\underline{q}_1-\underline{q}_2|\} = 4\omega_0\lim_{p\to 1}\Big\{\frac{g'(1;1)}{(1-p)^2} - \frac{g_0'(1;p)}{[1-g_0(1;p)]^2}\Big\}.$$

With (II.8.84), (8.9) and the relation

$$\frac{g_0'''(1;1)}{6g_0'(1;1)} - \Big[\frac{g_0''(1;1)}{2g_0'(1;1)}\Big]^2 = -\frac{g'''(1;1)}{6[g'(1;1)]^3} + \frac{[g''(1;1)]^2}{4[g'(1;1)]^4}, \tag{9.30}$$

this leads readily to (9.29). □

III.10. Conclusion

In this and the preceding chapter it has been shown that many characteristic quantities for the queueing model with two types of customers, Poisson arrival processes, paired services and general service time distribution, cf. section II.0, can be obtained after solving a functional equation by means of the formulation of a Hilbert boundary value problem.

The transforms of the time-dependent as well as those of the stationary distributions of these characteristic quantities have been expressed in terms of a class of conformal mappings of the interior of smooth contours onto the unit disk and their inverses. In order to obtain numerical values for moments or probabilities of the stationary distributions of these quantities the singular integral equation of Theodorsen, cf.(II.8.12) and lemma I.6.7, for the conformal mapping $g(1;z)$ of the unit disk onto the domain $L^+(1)$ has to be solved. In section IV.3 some numerical examples will be given. Because the contour $L(1)$ and hence also the conformal mapping $g(1;z)$ depend on the complete service time distribution it follows that the mean waiting times and the mean number of customers present in the system depend on the complete service time distribution, this in contrast with the M/G/1 model where these means depend only on the first two moments of the service time distribution.

CHAPTER IV

EXTENSIONS AND NUMERICAL EXAMPLES

IV.0. Introduction

This chapter is devoted to some extensions of the results of chapter II and to the presentation of numerical values for several quantities obtained in this study.

In section IV.1 the study of the imbedded Markov chain discussed in chapter II is extended by including a random variable representing the n^{th} departure instant. Here also the busy period will be discussed.

In section IV.2 it is shown that a generalization of the queueing model described in section II.0 leads to the formulation of a Hilbert problem with a more intricate boundary condition than we have encountered before.

Finally, in section IV.3 numerical examples are given for several combinations of the parameters of the queueing system.

IV.1. Joint distribution of queue lengths and departure instants; the busy period

For the queueing model described in section II.0 denote by $\underline{d}_n$, n=0,1,.., the n^{th} departure instant. In this section we shall discuss the joint distribution of the process $\{(\underline{x}_1(n),\underline{x}_2(n),\underline{d}_n),n=0,1,..\}$ given that $\underline{d}_0 = 0$, cf. definition II.1.2. The transform of this distribution can be determined by a similar analysis as that applied in chapter II; therefore the details of this analysis are omitted.

Having established an expression for the transform of the above distribution we shall discuss the joint distribution of the duration of a busy period and of the number of services performed during that busy period, and we shall discuss the renewal distribution for the states (k_1,k_2) of the Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$.

For the n-step transition probabilities of the process $\{(\underline{x}_1(n),\underline{x}_2(n),\underline{d}_n),n=0,1,..\}$ we shall use the following notation: for n=1,2,.., for x_1,x_2, $k_1,k_2 = 0,1,2,..$, for $t > 0$, with $x = (x_1,x_2)$,

$$p_x^{(n)}(k_1,k_2,t) := \Pr\{\underline{x}_1(n)=k_1,\ \underline{x}_2(n)=k_2,\ \underline{d}_n < t \mid \underline{x}_1(0)=x_1,\ \underline{x}_2(0)=x_2\}; \tag{1.1}$$

their transform will be denoted by: for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$, $\text{Re}\ \rho \geq 0$,

$$\Pi_x(r;p_1,p_2,\rho) := p_1^{x_1} p_2^{x_2} + \sum_{n=1}^{\infty} r^n \sum_{k_1,k_2=0}^{\infty} p_1^{k_1} p_2^{k_2} \int_0^{\infty} e^{-\rho t}\, d\, p_x^{(n)}(k_1,k_2,t). \tag{1.2}$$

For the one-step transition probabilities we have:

<u>LEMMA 1.1.</u> *For* $k_1,k_2 = 0,1,2,..$, *for* $t > 0$,

$$p_{00}^{(1)}(k_1,k_2,t) = \int_0^t \{1 - e^{-(t-u)/\alpha}\} \frac{(u/\alpha_1)^{k_1}}{k_1!} \frac{(u/\alpha_2)^{k_2}}{k_2!} e^{-u/\alpha}\, d\, B(u);$$

for $x \neq (0,0)$, *for* $k_1 \geq [x_1-1]^+$, $k_2 \geq [x_2-1]^+$, *for* $t > 0$,

$$p_x^{(1)}(k_1,k_2,t) = \int_0^t \frac{(u/\alpha_1)^{k_1-[x_1-1]^+}}{(k_1-[x_1-1]^+)!} \frac{(u/\alpha_2)^{k_2-[x_2-1]^+}}{(k_2-[x_2-1]^+)!} e^{-u/\alpha}\, d\, B(u),$$

while this probability is zero for $k_1 < [x_1-1]^+$ *or* $k_2 < [x_2-1]^+$.

<u>PROOF.</u> These probabilities are simple generalizations of the one-step transition probabilities in the M/G/1 model, cf. COHEN [03], §II.4.3. □

Because the process $\{(\underline{x}_1(n),\underline{x}_2(n),\underline{d}_n),n=0,1,..\}$ is an imbedded Markov chain, cf. theorem II.2.2, the following relation is obvious: for n = 2,3,.., for

$k_1,k_2 = 0,1,2,\ldots$, for $t > 0$,

$$p_x^{(n)}(k_1,k_2,t) = \sum_{h_1,h_2=0}^{\infty} \int_0^t p_x^{(n-1)}(h_1,h_2,t-u)\, d_u\, p_h^{(1)}(k_1,k_2,u). \tag{1.3}$$

With the aid of this relation we obtain the following functional equation for the transform (1.2):

THEOREM 1.1. *The transform* $\Pi_x(r;p_1,p_2,\rho)$ *has the following properties:*

i. it satisfies the functional equation: for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$, $\mathrm{Re}\,\rho \geq 0$,

$$\Big[p_1p_2 - r\,\beta\Big(\rho + \frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big]\, \Pi_x(r;p_1,p_2,\rho) = p_1^{x_1+1}\, p_2^{x_2+1} +$$
$$+ r\,\beta\Big(\rho + \frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big[(p_2-1)\,\Pi_x(r;p_1,0,\rho) + (p_1-1)\,\Pi_x(r;0,p_2,\rho) +$$
$$+ (1-p_1-p_2+\frac{p_1p_2}{1+\alpha\rho})\,\Pi_x(r;0,0,\rho)\Big]; \tag{1.4}$$

ii. for fixed p_1,p_2,ρ *it is a regular function of* r *for* $|r| < 1$*; as a function of* p_1 *it belongs to the class* $\mathrm{RCB}(C^+)$ *for* r,p_2,ρ *fixed; as a function of* p_2 *it belongs also to the class* $\mathrm{RCB}(C^+)$ *for* r,p_1,ρ *fixed; as a function of* ρ *it belongs to the class* $\mathrm{RCB}(\{\rho;\ \mathrm{Re}\,\rho > 0\})$ *for* r,p_1,p_2 *fixed; here always* $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$, $\mathrm{Re}\,\rho \geq 0$.

PROOF. The functional equation (1.4) follows by straightforward calculation from the definition (1.2) by using the recurrence relation (1.3) and the one-step transition probabilities as given in lemma 1.1. The regularity properties of the function $\Pi_x(r;p_1,p_2,\rho)$ are well-known properties of generating functions and Laplace-Stieltjes transforms, cf. theorem II.2.3. □

The *kernel* of equation (1.4) is

$$p_1p_2 - r\,\beta\Big(\rho + \frac{1-c_1p_1-c_2p_2}{\alpha}\Big). \tag{1.5}$$

Note that for $\rho = 0$ this kernel is equal to the kernel (II.2.3) while for $r = 1$ it is equal to the kernel (III.3.10). Therefore it is not difficult to see that the functional equation (1.4) can be analysed by the same methods as used in the analysis of the functional equations (II.2.2) and (III.3.5). The details of this analysis are omitted here.

For $|r| < 1$, Re $\rho \geqslant 0$, and for zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel (1.5) the functional equation (1.4) reads:

$$\frac{1}{1-p_1}\Pi_x(r;p_1,0,\rho) + \frac{1}{1-p_2}\Pi_x(r;0,p_2,\rho) =$$

$$= \frac{p_1^{x_1}\, p_2^{x_2}}{(1-p_1)(1-p_2)} + \left[1 - \frac{p_1\, p_2}{(1-p_1)(1-p_2)}\frac{\alpha\rho}{1+\alpha\rho}\right]\Pi_x(r;0,0,\rho). \tag{1.6}$$

For real r and ρ, $0 < r < 1$, $\rho > 0$, we introduce the contour (cf. lemma II.5.2 and (III.4.15)):

$$L(r;\rho) := \{w;\ |w|^2 = 4c_1c_2r\ \beta\left(\rho + \frac{1-\text{Re } w}{\alpha}\right),\ \text{Re } w < 1\}, \tag{1.7}$$

and the conformal mapping $g(r;\rho;z)$ of the unit disk C^+ onto the domain $L^+(r;\rho)$, cf. theorem II.5.1, determined by the conditions

$$g(r;\rho;0) = 0, \qquad g'(r;\rho;0) > 0. \tag{1.8}$$

As in chapter II it can be shown, with the notation introduced in (II.6.1), that the following assertion holds, cf. (1.6):

<u>THEOREM 1.2.</u> *For real* r *and* ρ, $0 < r < 1$, $\rho > 0$, *the functions of* z,

$$\Pi_x(r;\frac{1}{2c_1}g(r;\rho;z),0,\rho), \qquad \Pi_x(r;0,\frac{1}{2c_2}g(r;\rho;z),\rho),$$

belong to the class RCB(C^+), *and they satisfy: for* $t \in C$,

$$\frac{\Pi_x(r;\frac{1}{2c_1}g(r;\rho;t),0,\rho)}{1 - g(r;\rho;t)/2c_1} + \frac{\Pi_x(r;0,\frac{1}{2c_2}g(r;\rho;\frac{1}{t}),\rho)}{1 - g(r;\rho;1/t)/2c_2} =$$

$$= K_x(g(r;\rho;t),g(r;\rho;\tfrac{1}{t})) + \left[1 - \frac{\alpha\rho}{1+\alpha\rho} K_{11}(g(r;\rho;t),g(r;\rho;\tfrac{1}{t}))\right]\Pi_x(r;0,0,\rho). \tag{1.9}$$

Theorem 1.2 defines again a Hilbert boundary value problem on the unit circle (cf. section II.7, III.6) of which the index depends on the position of the point $2c_2$ $(c_2 \leqslant c_1)$ in relation to the contour $L(r;\rho)$. The functions $\Pi_x(r;p_1,0,\rho)$ and $\Pi_x(r;0,p_2,\rho)$ are completely determined by the solution of this Hilbert boundary value problem, cf. theorem II.7.1, and by substitution in equation (1.4) the function $\Pi_x(r;p_1,p_2,\rho)$ is completely determined, for $|r| < 1$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$, $\mathrm{Re}\ \rho \geqslant 0$.
From theorem II.5.6 and theorem III.5.1 it is readily seen that (if $c_2 \leqslant c_1$) $2c_2 \in L^-(r;\rho)$ for $r \downarrow 0$ and $\rho \to \infty$. For this case the solution of the Hilbert boundary value problem as formulated in theorem 1.2 is stated below without proof (see theorem II.7.1).

<u>THEOREM 1.3.</u> *For real* r *and* ρ*,* $0 < r < 1$*,* $\rho > 0$*, such that* $2c_2 \in L^-(r;\rho)$*, the following relations hold for* $2c_1p_1 \in L^+(r;\rho)$*,* $2c_2p_2 \in L^+(r;\rho)$*:*

$$\Pi_x(r;p_1,0,\rho) = (1-p_1)\left[\frac{1}{2\pi i}\int_C K_x(g(r;\rho;t),g(r;\rho;\tfrac{1}{t}))\,\frac{dt}{t-g_0(r;\rho;2c_1p_1)} - \right.$$

$$\left. - \frac{\alpha\rho}{1+\alpha\rho}\,\Pi_x(r;0,0,\rho)\,\frac{1}{2\pi i}\int_C K_{11}(g(r;\rho;t),g(r;\rho;\tfrac{1}{t}))\,\frac{dt}{t-g_0(r;\rho;2c_1p_1)}\right]; \tag{1.10}$$

$$\Pi_x(r;0,p_2,\rho) = (1-p_2)\left[\frac{1}{2\pi i}\int_C K_x(g(r;\rho;\tfrac{1}{t}),g(r;\rho;t))\,\frac{dt}{t-g_0(r;\rho;2c_2p_2)} - \right.$$

$$\left. - \frac{\alpha\rho}{1+\alpha\rho}\,\Pi_x(r;0,0,\rho)\,\frac{1}{2\pi i}\int_C K_{11}(g(r;\rho;\tfrac{1}{t}),g(r;\rho;t))\,\frac{dt}{t-g_0(r;\rho;2c_2p_2)}\right]; \tag{1.11}$$

with

$$\Pi_x(r;0,0,\rho) = \frac{\frac{1}{2\pi i}\int_C K_x(g(r;\rho;t),g(r;\rho;\tfrac{1}{t}))\,\frac{dt}{t}}{1 + \frac{\alpha\rho}{1+\alpha\rho}\,\frac{1}{2\pi i}\int_C K_{11}(g(r;\rho;t),g(r;\rho;\tfrac{1}{t}))\,\frac{dt}{t}}\ ; \tag{1.12}$$

here $g_0(r;\rho;w)$ *denotes the inverse conformal mapping of* $g(r;\rho;z)$.

Next we introduce the first entrance probabilities: for n=1,2,.., for $x_1,x_2,k_1,k_2=0,1,2,..$, for $t>0$,

$$f_x^{(n)}(k_1,k_2,t) := \Pr\{\underline{x}_1(n)=k_1,\ \underline{x}_2(n)=k_2;\ \underline{x}_1(m)\neq k_1,\ \underline{x}_2(m)\neq k_2,\ m=1,..,n-1;\ \underline{d}_n < t \mid \underline{x}_1(0)=x_1,\ \underline{x}_2(0)=x_2\}; \tag{1.13}$$

and the transforms: for $|r|\leqslant 1$, $\mathrm{Re}\ \rho \geqslant 0$, for $x_1,x_2,k_1,k_2=0,1,2,..$,

$$F_x(r;k_1,k_2,\rho) := \sum_{n=1}^{\infty} r^n \int_0^{\infty} e^{-\rho t}\, d\, f_x^{(n)}(k_1,k_2,t). \tag{1.14}$$

With the aid of these functions we shall first discuss the joint distribution of the duration of a busy period and of the number of services performed during that busy period (here a paired service is counted for one). By a *busy period* we mean the time interval between an instant at which an arriving customer of any type finds the system empty, and the first departure instant afterwards at which no customers are left behind in the system.

<u>THEOREM 1.4.</u> *The joint distribution of the duration* $\underline{p}$ *of a busy period and the number* $\underline{n}$ *of services performed during this busy period is determined by, cf.* (1.12)*: for* $|r|\leqslant 1$, $\mathrm{Re}\ \rho\geqslant 0$,

$$E\{r^{\underline{n}}\, e^{-\rho \underline{p}}\} = (1+\alpha\rho)\left[1-\frac{1}{\Pi_{00}(r;0,0,\rho)}\right]. \tag{1.15}$$

Further,

$$\begin{aligned} &\Pr\{\underline{n}<\infty\}=1, \quad \Pr\{\underline{p}<\infty\}=1, \quad \textit{if } \max\{c_1,c_2\}a\leqslant 1,\\ &\Pr\{\underline{n}<\infty\}<1, \quad \Pr\{\underline{p}<\infty\}<1, \quad \textit{if } \max\{c_1,c_2\}a > 1, \end{aligned} \tag{1.16}$$

and the random variables $\underline{p}$ *and* $\underline{n}$ *have a finite mean if and only if* $\max\{c_1,c_2\}a<1$ *, and in the case* $c_2\leqslant \frac{1}{2}\leqslant c_1$ *these means are given by:*

$$E\{\underline{n}\} = \frac{1}{\beta}E\{\underline{p}\} = \frac{1}{\phi_0} = \frac{z_0\, g'(1;z_0)}{2c_2(1-a_1)}, \quad \textit{for } a_1<1. \tag{1.17}$$

<u>PROOF.</u> Clearly we have, cf. (1.14), for $|r|\leqslant 1$, $\mathrm{Re}\ \rho\geqslant 0$,

$$E\{r^{\underline{n}}\, e^{-\rho \underline{c}}\} = F_{00}(r;0,0,\rho), \tag{1.18}$$

here $\underline{c}$ stands for a busy cycle, i.e. $\underline{c} = \underline{p} + \underline{i}$ and $\underline{i}$ stands for an idle period. Because the idle period $\underline{i}$ has in the present model a negative exponential distribution (with mean α) it is independent of the pair $(\underline{p},\underline{n})$. Hence, it follows from (1.18) that for $|r| \leqslant 1$, Re $\rho \geqslant 0$,

$$E\{r^{\underline{n}}\, e^{-\rho \underline{p}}\} = (1+\alpha\rho)\, F_{00}(r;0,0,\rho). \tag{1.19}$$

By a standard renewal argument it is obtained that for n=1,2,.., for $t > 0$,

$$p_{00}^{(n)}(0,0,t) = f_{00}^{(n)}(0,0,t) + \sum_{m=1}^{n-1} \int_0^t p_{00}^{(n-m)}(0,0,t-u)\, d\, f_{00}^{(m)}(0,0,u), \tag{1.20}$$

cf. (1.1) and (1.13), which implies that for $|r| < 1$, Re $\rho \geqslant 0$,

$$\Pi_{00}(r;0,0,\rho) = F_{00}(r;0,0,\rho)\, \Pi_{00}(r;0,0,\rho) + 1. \tag{1.21}$$

The relations (1.19) and (1.21) prove (1.15).
By taking $\rho = 0$ in (1.15) we obtain, cf. (1.2), (II.1.10), that for $|r| \leqslant 1$,

$$E\{r^{\underline{n}}\} = 1 - \frac{1}{\Pi_{00}(r;0,0,0)} = 1 - \frac{1}{\Phi_{00}(r;0,0)}. \tag{1.22}$$

Hence, the statements for $\Pr\{\underline{n} < \infty\}$ and $E\{\underline{n}\}$ follow readily from the results of theorem II.8.4 and theorem II.8.5.
By taking $r = 1$ in (1.15) it follows that for Re $\rho \geqslant 0$,

$$E\{e^{-\rho \underline{p}}\} = (1+\alpha\rho) \left[1 - \frac{1}{\Pi_{00}(1;0,0,\rho)}\right]. \tag{1.23}$$

Then we claim that for Re $\rho > 0$,

$$\Pi_{00}(1;0,0,\rho) = \left(\frac{1}{\alpha} + \rho\right) \Omega_{00}^{0}(\rho); \tag{1.24}$$

We prove this relation for real ρ, $\rho > P(c_2)$, cf. theorem III.5.1. The statement for Re $\rho > 0$ then follows by analytic continuation.

Hence, let ρ be real, $\rho > P(c_2)$, so that $2c_2 \in L^-(1;\rho) = \Lambda^-(\rho)$. From (1.12) it follows by putting $r = 1$ that for $\rho > P(c_2)$,

$$\Pi_{00}(1;0,0,\rho) = \frac{\frac{1}{2\pi i}\int_C K_{00}(\gamma(\rho;t),\gamma(\rho;\frac{1}{t}))\,\frac{dt}{t}}{1+\frac{\alpha\rho}{1+\alpha\rho}\frac{1}{2\pi i}\int_C K_{11}(\gamma(\rho;t),\gamma(\rho;\frac{1}{t}))\,\frac{dt}{t}}. \tag{1.25}$$

Further, by using the relation

$$\frac{p_1p_2}{(1-p_1)(1-p_2)} = \frac{1}{(1-p_1)(1-p_2)} - \frac{1}{1-p_1} - \frac{1}{1-p_2} + 1,$$

and by noting that if $2c_2 \in \Lambda^-(\rho)$,

$$\frac{1}{2\pi i}\int_C \frac{1}{1-\gamma(\rho;t)/2c_j}\,\frac{dt}{t} = 1, \quad \text{for } j = 1,2,$$

because the function $1 - \gamma(\rho;t)/2c_j$ is regular in C^+ and continuous and non-vanishing in $C^+ \cup C$, for $j=1,2$, it follows from (II.6.1) that for $\rho > P(c_2)$,

$$\frac{1}{2\pi i}\int_C K_{00}(\gamma(\rho;t),\gamma(\rho;\frac{1}{t}))\,\frac{dt}{t} = 1 + \frac{1}{2\pi i}\int_C K_{11}(\gamma(\rho;t),\gamma(\rho;\frac{1}{t}))\,\frac{dt}{t}. \tag{1.26}$$

Because $J_{00}(v_1,v_2) \equiv K_{11}(v_1,v_2)$, cf. (II.6.1),(III.6.1), relation (1.24) follows for $\rho > P(c_2)$ from (1.25), (1.26) and (III.6.9).

Thus we may substitute (1.24) in (1.23) which implies that for Re $\rho \geq 0$,

$$E\{e^{-\rho \underline{p}}\} = 1 + \alpha\rho - \alpha/\Omega_{00}^0(\rho). \tag{1.27}$$

Then the statements for $\Pr\{\underline{p} < \infty\}$ and $E\{\underline{p}\}$ follow readily from the results of theorem III.8.2. □

From (1.18), (1.21) and (1.24) we have for Re $\rho \geq 0$,

$$E\{e^{-\rho \underline{c}}\} = 1 - \frac{\alpha}{(1+\alpha\rho)\,\Omega_{00}^0(\rho)}, \tag{1.28}$$

which implies with theorem III.8.2 that the mean duration of a busy cycle is finite if and only if $\max\{c_1,c_2\}a < 1$ and that it is then given by

$$E\{\underline{c}\} = \alpha/\omega_0. \tag{1.29}$$

Finally, we consider the renewal functions for the states of the imbedded Markov chain $\{(\underline{x}_1(n),\underline{x}_2(n)),n=0,1,..\}$ which are required in section III.8. By a standard renewal argument we have, cf. (1.1), (1.13): for $n=1,2,..,$for $x_1,x_2,k_1,k_2=0,1,2,..,$ for $t > 0$,

$$p_x^{(n)}(k_1,k_2,t) = f_x^{(n)}(k_1,k_2,t) + \sum_{m=1}^{n-1} \int_0^t f_k^{(n-m)}(k_1,k_2,t-u) \; d\, p_x^{(m)}(k_1,k_2,u). \tag{1.30}$$

Defining, for $x_1,x_2,k_1,k_2=0,1,2,..,$ for $t > 0$,

$$m_x(k_1,k_2,t) := \sum_{n=1}^{\infty} p_x^{(n)}(k_1,k_2,t), \tag{1.31}$$

$$f_x(k_1,k_2,t) := \sum_{n=1}^{\infty} f_x^{(n)}(k_1,k_2,t), \tag{1.32}$$

relation (1.30) leads to: for $x_1,x_2,k_1,k_2=0,1,2,..,$ for $t > 0$,

$$m_x(k_1,k_2,t) = f_x(k_1,k_2,t) + \int_0^t f_k(k_1,k_2,t-u) \; d\, m_x(k_1,k_2,u). \tag{1.33}$$

Hence, it is seen that $m_x(k_1,k_2,t)$ represents the renewal function of the (general) renewal process with renewal distribution $f_k(k_1,k_2,t)$ and with $f_x(k_1,k_2,t)$ as the distribution of the first renewal.

For the application of the key renewal theorem in section III.8 it is necessary that $f_k(k_1,k_2,t)$ is not a lattice distribution. That this condition is fulfilled is readily seen by noting that the time interval between two successive entrances into any state (k_1,k_2) is with a positive probability equal to the sum of a number of service times and a negative exponentially distributed idle period (cf. theorem II.2.2) and hence that the distribution of this time interval contains an absolutely continuous component, cf. LUKACS [17], theorem 3.3.2.

IV.2. Generalization of the model: other types of Hilbert problems

The Hilbert boundary value problems which we have encountered in the sections II.7, III.6, III.7 and IV.1 all possess a boundary condition of the relatively simple form as described by formula (I.4.8). In this section we shall show that more intricate boundary conditions arise when the queueing model as described in section II.0 is generalized in the way that the duration of an individual service of a type j customer has a distribution $B_j(t) \not\equiv B(t)$, $j = 1,2$; here $B(t)$ still denotes the distribution of a paired service.

It is not difficult to see that for this generalized model the functional equation for the generating function $\Phi_x(r;p_1,p_2)$ of the imbedded Markov chain representing the number of customers left behind at departure instants reads, cf. (II.2.2): for $|r| < 1$, $|p_1| \leqslant 1$, $|p_2| \leqslant 1$,

$$\begin{aligned}\Big[p_1p_2 - r\,\beta\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big]\,\Phi_x(r;p_1,p_2) &= p_1^{x_1+1}\,p_2^{x_2+1} + \\ &+ r\,\beta\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big[\Phi_x(r;0,0) - \Phi_x(r;p_1,0) - \Phi_x(r;0,p_2)\Big] + \\ &+ rp_2\,\beta_1\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big[\Phi_x(r;p_1,0) + (c_1p_1-1)\,\Phi_x(r;0,0)\Big] + \\ &+ rp_1\,\beta_2\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big[\Phi_x(r;0,p_2) + (c_2p_2-1)\,\Phi_x(r;0,0)\Big];\end{aligned} \tag{2.1}$$

with

$$\beta_j(s) := \int_0^\infty e^{-st}\, d\,B_j(t), \quad \text{Re } s \geqslant 0, \quad \text{for } j = 1,2. \tag{2.2}$$

The kernel of the functional equation (2.1) is the same as the kernel (II.2.3) of the functional equation (II.2.2); hence it can be analysed as in section II.3. For zeros $(p_1,p_2) \in (C^+ \cup C) \times (C^+ \cup C)$ of the kernel

(II.2.3) equation (2.1) becomes:

$$\Big[1 - \frac{r}{p_1}\,\beta_1\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big]\Big[\Phi_x(r;p_1,0) - (1-c_1p_1)\,\Phi_x(r;0,0)\Big] +$$

$$+ \Big[1 - \frac{r}{p_2}\,\beta_2\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big]\Big[\Phi_x(r;0,p_2) - (1-c_2p_2)\,\Phi_x(r;0,0)\Big] =$$

$$= p_1^{x_1}\,p_2^{x_2} - (1-c_1p_1-c_2p_2)\,\Phi_x(r;0,0). \tag{2.3}$$

For real values of r, $0 < r < \min\{1, R(c_2)\}$, cf. theorem II.5.6, this relation (2.3) holds for (cf. theorem II.5.1),

$$2c_1p_1 = g(r;t), \qquad 2c_2p_2 = g(r;\tfrac{1}{t}), \qquad t \in C, \tag{2.4}$$

because these expressions (2.4) represent zeros $(p_1,p_2) \in (C^+ \cup C)\times(C^+ \cup C)$ of the kernel (II.2.3), cf. lemma II.5.2, corollary II.5.2 and theorem II.5.6. Moreover, the function A(r;z) defined by: for $0 < r < \min\{1, R(c_2)\}$,

$$A(r;z) := \Phi_x(r;\frac{1}{2c_1}\,g(r;z),0) - [1 - \tfrac{1}{2}g(r;z)]\;\Phi_x(r;0,0), \quad z \in C^+,$$

$$A(r;z) := \Phi_x(r;0,\frac{1}{2c_2}\,g(r;\tfrac{1}{z})) - [1 - \tfrac{1}{2}g(r;\tfrac{1}{z})]\;\Phi_x(r;0,0), \quad z \in C^-, \tag{2.5}$$

is a sectionally regular function with respect to the unit circle. Hence, the relations (2.3) and (2.4) form the boundary condition of a *Hilbert boundary value problem* for the function A(r;z), cf. section I.4, if we assume that $\Phi_x(r;0,0)$ is a given constant. For this Hilbert problem the function G(t) in (I.4.1) is equal to, cf. (2.3):

$$G(t) = -\frac{1 - \frac{r}{p_2}\,\beta_2\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)}{1 - \frac{r}{p_1}\,\beta_1\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)}\,, \tag{2.6}$$

with p_1 and p_2 given by (2.4).

Let us consider the index of this Hilbert problem for the function A(r;z), cf. (I.4.2). From theorem II.5.2 it follows that for $t \in C$ and p_1,p_2 given by (2.4), cf. (II.4.21) and lemma II.4.3,

$$c_1p_1 + c_2p_2 = \mathrm{Re}\{g(r;t)\} \in \gamma(r). \tag{2.7}$$

This implies that for $t \in C$ and p_1,p_2 given by (2.4) both

$$\beta_1\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big), \quad \text{and } \beta_2\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big),$$

are real and positive. Further, it is not difficult to see, cf. (II.4.21), that as a result of one circuit of t around C both

$$\frac{r}{p_1} = \frac{2c_1\, r}{g(r;t)}, \quad \text{and } \frac{r}{p_2} = \frac{2c_2\, r}{g(r;1/t)},$$

traverse one circuit around contours which can be represented by formula (I.6.4), the first in negative and the second in positive direction. The above implies that for $j = 1,2$, for p_1,p_2 given by (2.4),

$$\begin{aligned}\Big[\log\Big\{1 - \frac{r}{p_j}\,\beta_j\Big(\frac{1-c_1p_1-c_2p_2}{\alpha}\Big)\Big\}\Big]_C &= 0, \qquad \text{if } \frac{2c_j\, r}{\delta_2(r)}\,\beta_j\Big(\frac{1-\delta_2(r)}{\alpha}\Big) < 1,\\ &= (-1)^j, \quad \text{if } \frac{2c_j\, r}{\delta_2(r)}\,\beta_j\Big(\frac{1-\delta_2(r)}{\alpha}\Big) > 1.\end{aligned} \tag{2.8}$$

From (2.8), (2.6) and (I.4.2) it is readily seen that the index of the Hilbert problem with boundary condition (2.3), (2.4) can only have the values 0,1,2. Moreover, from (II.5.9) and (II.4.22) it is seen that

$$\delta_2(r) \sim 2\sqrt{c_1c_2\,\beta(1/\alpha)}\,\sqrt{r}, \quad \text{for } r \downarrow 0,$$

so that for $j = 1,2$,

$$\lim_{r\downarrow 0} \frac{2c_j\, r}{\delta_2(r)}\,\beta_j\Big(\frac{1-\delta_2(r)}{\alpha}\Big) = 0.$$

Hence, for small values of r the index of the Hilbert problem is equal to zero, cf. (2.8). For these sufficiently small values of r the solution of the Hilbert problem is given by lemma I.4.2 with $\kappa = 0$. Then the conditions

$$A(r;0) = 0, \qquad \lim_{|z|\to\infty} A(r;z) = 0, \tag{2.9}$$

cf. (2.5), determine the constant $\Phi_x(r;0,0)$ and the constant in the solution of lemma I.4.2.

Then by analytic continuation, cf. the proof of theorem II.6.1, the function $\Phi_x(r;p_1,p_2)$ is completely determined for $|r| < 1$, $|p_1| \leq 1$, $|p_2| \leq 1$.

REMARK 2.1. It should be noted that the analysis of the functional relation (2.3) cannot be reduced to the solution of two Riemann-Hilbert problems as in section II.6. □

In the following remark we shall use the notation $\beta^{(j)}$ for the first moment of the service time distribution $B_j(t)$, $j = 1,2$.

REMARK 2.2. An interesting question is, on which conditions the present generalized queueing system is ergodic. It is clear that the condition $\max\{c_1,c_2\}a < 1$ is not sufficient anymore, because also some bound is needed for the first moments $\beta^{(1)}$, $\beta^{(2)}$ of the service time distributions for individual services. On the other hand, it can be imagined that $\max\{c_1,c_2\}a < 1$ is also not necessary in every case, as long as $\beta^{(1)}$ and/or $\beta^{(2)}$ are small enough for compensation.

This queueing model can be characterized as a random walk on the lattice points in the first quadrant of the plane (x_1,x_2). This random walk consists of three components with different drifts. The first component is the "free" random walk on the interior points $\{(x_1,x_2);x_1>0,x_2>0\}$. Here the drift is in the direction of the vector (c_1a-1,c_2a-1). The second component, on the x_1-axis, has a drift in the direction $(c_1\beta^{(1)}/\alpha - 1, c_2\beta^{(1)}/\alpha)$, and the third one, on the x_2-axis, in the direction $(c_1\beta^{(2)}/\alpha, c_2\beta^{(2)}/\alpha - 1)$. Intuitively, it is clear that in the ergodic case the drift of the free random walk should point to one of the axes; hence we obtain the condition

$$\min\{c_1, c_2\} a < 1. \tag{2.10}$$

Further, in order that the mean return time of the empty state is finite, we should have

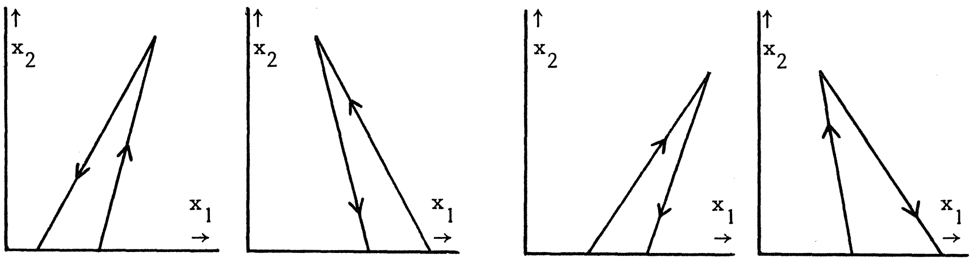
$$\beta^{(1)} < \infty, \qquad \beta^{(2)} < \infty. \tag{2.11}$$

Moreover, it is necessary that the drift on the axes is compensated by the drift of the free random walk, e.g. for the x_1-axis (see figure 2.1), if for $\lambda > 0$,

$$c_2 a - 1 + \lambda c_2 \beta^{(1)}/\alpha = 0,$$

then we should have

$$c_1 a - 1 + \lambda(c_1 \beta^{(1)}/\alpha - 1) < 0.$$



a. $c_2 a < c_1 a < 1$, $\dfrac{\beta^{(1)}}{\alpha} < \dfrac{1-c_2 a}{c_1 - c_2}$. *b.* $c_2 a < 1 < c_1 a$, $\dfrac{\beta^{(1)}}{\alpha} < \dfrac{1-c_2 a}{c_1 - c_2}$. *c.* $c_2 a < c_1 a < 1$, $\dfrac{\beta^{(1)}}{\alpha} > \dfrac{1-c_2 a}{c_1 - c_2}$. *d.* $c_2 a < 1 < c_1 a$, $\dfrac{\beta^{(1)}}{\alpha} > \dfrac{1-c_2 a}{c_1 - c_2}$.

Figure 2.1. The arrows up from the x_1-axis indicate the direction of the drift on this axis, while the arrows down to the x_1-axis indicate the direction of the drift of the free random walk. In the cases *a* and *b* the resultant drift is towards the origin, hence these cases correspond to an ergodic system; the resultant drift in the cases *c* and *d* is towards infinity, corresponding to non-ergodic systems.

This leads to the condition

$$(c_1-c_2)\ \beta^{(1)}/\alpha < 1 - c_2 a, \qquad \text{if } c_2 a < 1. \tag{2.12}$$

Similarly, for the x_2-axis we obtain the condition

$$(c_2-c_1)\ \beta^{(2)}/\alpha < 1 - c_1 a, \qquad \text{if } c_1 a < 1. \tag{2.13}$$

Together, (2.10), (2.11), (2.12), (2.13) lead to the following hypothesis for the conditions on which the generalized queueing system is ergodic:

$$\text{in the case } c_2 < \tfrac{1}{2} < c_1: \quad \text{if} \quad c_2 a < 1, \quad \beta^{(2)} < \infty, \quad \beta^{(1)}/\alpha < \frac{1-c_2 a}{c_1-c_2}; \tag{2.14}$$

$$\text{in the case } c_1 = c_2 = \tfrac{1}{2}: \quad \text{if} \quad a < 2, \quad \beta^{(1)} < \infty, \ \beta^{(2)} < \infty. \tag{2.15}$$

An interesting subject for further study would be the proof of this hypothesis with the methods used in section II.8. Finally we note that the same heuristic arguments lead to the conditions on which the model with two coupled processors studied by FAYOLLE & IASNOGORODSKI [09] is ergodic. □

IV.3. Numerical examples

In this section numerical values for several queueing quantities will be presented. For obtaining these values a computer program has been used for the calculation of the values of z_0 and of $g'(1;z_0)$ for several combinations of the parameters α, c_1, and of the service time distribution $B(t)$ (see theorem II.8.7 for the meaning of these symbols). A similar method has been used as in COHEN & BOXMA [04], §9. First the singular integral equation of Theodorsen (II.8.12) has been solved numerically by the iterative procedure: for $\phi \in [-\pi,\pi]$,

$$\theta_0(1;\phi) := \phi,$$

$$\theta_{n+1}(1;\phi) := \phi - \frac{1}{2\pi}\int_{-\pi}^{\pi} \log\left[\frac{h(\cos\theta_n(1;\omega))}{\cos\theta_n(1;\omega)}\right] \cot\left(\frac{\omega-\phi}{2}\right) d\omega, \quad n=0,1,.,(3.1)$$

and by taking 81 points ϕ on the interval $[0,\pi]$, using the symmetry, cf. theorem II.8.1, for the interval $[-\pi,0]$. The iteration has been stopped (after about 12 steps) if for some n,

$$\max_{\phi \in [-\pi,\pi]} |\theta_{n+1}(1;\phi) - \theta_n(1;\phi)| < 10^{-6}. \tag{3.2}$$

See GAIER [12], §II.1, §II.3, for an exposition on the convergence of this method. In the cases which we have considered the value of

$$\max_{\theta \in [-\pi,\pi]} \left|\frac{\rho'(\theta)}{\rho(\theta)}\right|,$$

cf. (II.8.13), was at most 0.4 (on the average 0.25), so that convergence was rather fast.

Having obtained $\theta(1;\phi)$ numerically the values of the conformal mapping $g(1;z)$ can be calculated for $|z| \leqslant 1$ with the aid of (II.8.11). This has been used to determine the value of z_0 by a standard procedure as the unique zero of the function $g(1;z) - 2c_2$ on the real interval $0 < z \leqslant 1$, cf. (II.8.55) and lemma II.7.3. Further, it follows from (II.8.11) that for $|z| < 1$,

$$g'(1;z) = g(1;z) \left[\frac{1}{z} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left[\frac{h(\cos\theta(1;\phi))}{\cos\theta(1;\phi)}\right] \frac{2e^{i\phi}}{(e^{i\phi} - z)^2}\, d\phi\right]. \tag{3.3}$$

This implies with the definition of z_0:

$$g'(1;z_0) = 2c_2 \left[\frac{1}{z_0} + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log\left[\frac{h(\cos\theta(1;\phi))}{\cos\theta(1;\phi)}\right] \frac{2e^{i\phi}}{(e^{i\phi} - z_0)^2}\, d\phi\right], \tag{3.4}$$

from which expression $g'(1;z_0)$ can be calculated.

Having calculated the values for z_0 and $g'(1;z_0)$ many queueing quantities can be numerically determined by simple arithmetic operations.

In table 1 the values of z_0 and $g'(1;z_0)$ are listed for ten combinations of the parameters c_1 and α and of the service time distribution $B(t)$. In

TABLE 1	c_1	a	β_2/β^2	β_3/β^3	z_0	$g'(1;z_0)$
Case 1	10/11	0.88	2.500	9.616	0.40017	0.47143
Case 2	5/6	0.88	2.500	9.616	0.54527	0.65872
Case 3	5/9	0.88	2.500	9.616	0.92709	1.31074
Case 4	5/9	0.18	2.500	9.616	0.90248	1.05987
Case 5	5/9	0.99	2.500	9.616	0.93071	1.35415
Case 6	5/9	1.44	2.500	9.616	0.94636	1.55275
Case 7	5/9	0.99	2.500	13.313	0.93268	1.39183
Case 8	5/9	1.44	2.500	13.313	0.94938	1.64061
Case 9	5/9	0.99	1.083	1.278	0.94379	1.64497
Case 10	5/9	0.99	1.275	1.993	0.94125	1.57774

the first six cases we have used a mixture of two negative exponential distributions for B(t); more precisely, we have chosen, cf. (II.1.6),

$$\beta(s) = \frac{1+(\theta_1+\theta_2-2)s}{(1+\theta_1 s)(1+\theta_2 s)}, \qquad \text{Re } s \geqslant 0. \tag{3.5}$$

The first moment of this distribution is $\beta = 2$, independent of the values of θ_1 and θ_2. We have taken $\theta_1 = 0.250$, $\theta_2 = 2.571$. This distribution will be characterized in the tables by $\beta_2/\beta^2 = 2.500$, $\beta_3/\beta^3 = 9.616$. In case 7 and 8 we have used the same type of distribution, cf. (3.5), however with $\theta_1 = 1.750$, $\theta_2 = 6.000$. This distribution will be indicated by $\beta_2/\beta^2 = 2.500$, $\beta_3/\beta^3 = 13.313$. In the last two cases the service time distribution has been chosen to be the convolution of the degenerate distribution (at $t = 1$) and of an Erlang-3 distribution, so that its Laplace-Stieltjes transform is given by:

$$\beta(s) = e^{-s}\,(1+\gamma s/3)^{-3}, \qquad \text{Re } s \geqslant 0. \tag{3.6}$$

This distribution has also been used in COHEN & BOXMA [04] for numerical evaluations. In case 9 we have taken $\gamma = 1$ so that $\beta = 2$, $\beta_2/\beta^2 = 1.083$,

TABLE 2	a	ϕ_0	$1-a_1$	ω_0	$(1-a_1)/(1+a_2)$
Case 4	0.18	0.836	0.900	0.823	0.833
Case 3	0.88	0.374	0.511	0.298	0.367
Case 5	0.99	0.317	0.450	0.243	0.313
Case 6	1.44	0.121	0.200	0.078	0.122

$\beta_3/\beta^3 = 1.278$; in case 10 we have chosen $\gamma = 10$, so that $\beta = 11$, $\beta_2/\beta^2 = 1.275$, $\beta_3/\beta^3 = 1.993$.

In table 2 the stationary probability ϕ_0 of an empty system at a departure instant, cf. (II.8.57), the stationary probability ω_0 of an empty system at an arbitrary instant, cf. (III.8.3), and for comparison their upper bounds given in (II.8.96) and (III.8.15) respectively have been tabulated for different values of the traffic a, with fixed c_1 and B(t). As it could have been expected, the probabilities ϕ_0 and ω_0 decrease when the traffic intensity increases.

In table 3 the same quantities as in table 2 are listed, but here for different ratios c_1/c_2 of the arrival rates of the two types of customers, with a and B(t) fixed. It is seen that the probabilities ϕ_0 and ω_0 increase according as this ratio tends to unity (i.e. $c_1 \downarrow \frac{1}{2}$). This feature can be explained by noting that the more the arrival rates are in balance the larger the proportion of paired services will be.

Next, in table 4 the dependence of the probabilities ϕ_0 and ω_0 on the second-moment of the service time distribution- for fixed c_1 and a- is shown.

TABLE 3	c_1	ϕ_0	$1-a_1$	ω_0	$(1-a_1)/(1+a_2)$
M/G/1	1	0.120	0.120	0.120	0.120
Case 1	10/11	0.193	0.200	0.180	0.185
Case 2	5/6	0.248	0.267	0.220	0.233
Case 3	5/9	0.374	0.511	0.298	0.367

TABLE 4	β_2/β^2	ϕ_0	ω_0	$(1-\omega_0)\beta_2/\beta^2$	$\bar{w}/\beta$
Case 9	1.083	0.258	0.207	0.860	0.872
Case 10	1.275	0.269	0.214	1.003	1.017
Case 5	2.500	0.317	0.243	1.893	1.920

It can be seen that ϕ_0 and ω_0 are increasing with increasing variance of the service times. Hence, the larger this variance is, the larger is the proportion of paired services, cf. corollary III.8.1. Further, the quantity $(1-\omega_0)\beta_2/\beta^2$ is given in table 4. This quantity occurs in the formulas for the mean number of customers in the system, cf. corollary III.8.4, and for the mean waiting times, cf. corollary III.9.3. Because $1-\omega_0$ decreases as β_2/β^2 increases, the dependence of the above mentioned averages on the variance of the service times is weaker in the present queueing model with paired services than in the M/G/1-queueing system. Finally, the weighted average $\bar{w}$ of the mean waiting times for the two types of customers, defined by

$$\bar{w} := c_1\ E\{\underline{w}_1\} + c_2\ E\{\underline{w}_2\} = (1-\omega_0)\ \frac{1-2c_1c_2a}{(1-a_1)(1-a_2)}\ \frac{\beta_2}{2\beta}, \tag{3.7}$$

cf. corollary III.9.3, is given in table 4.

In table 5 the dependence of the same quantities as in table 4 on the third moment of the service time distribution - with c_1, a, β_2/β^2 fixed - is illustrated, for two values of a. For $a = 0.99$ the difference in the values for ϕ_0 as well in those for ω_0 is about 3%; for $a = 1.44$ these differences are

TABLE 5	β_3/β^3	ϕ_0	ω_0	$(1-\omega_0)\beta_2/\beta^2$	$\bar{w}/\beta$
Case 5	9.616	0.317	0.243	1.893	1.920
Case 7	13.313	0.308	0.237	1.907	1.933
Case 6	9.616	0.121	0.078	2.306	4.627
Case 8	13.313	0.114	0.073	2.316	4.647

TABLE 6	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$\Pr\{\underline{z}=0\}$	0.180	0.220	0.298	0.823	0.243	0.078
$\Pr\{\underline{z}=1\}$	0.740	0.633	0.311	0.097	0.317	0.282
$\Pr\{\underline{z}=2\}$	0.020	0.047	0.213	0.077	0.207	0.122
$\Pr\{\underline{z}=3\}$	0.060	0.100	0.178	0.003	0.233	0.518

about 6%. The differences in the values for $\bar{w}/\beta$ are for $a = 0.99$ and for $a = 1.44$ less than 1%, but the differences are increasing with increasing traffic intensity.

In table 6 the stationary distribution, cf. corollary III.8.1, of the variable $\underline{z}$, see definition III.1.1, has been displayed for the first six cases of table 1. For the same six cases we have presented in table 7 the following queueing quantities: the average number of services performed during a busy period, $E\{\underline{n}\}$, cf. (1.17); the average duration of a busy cycle $E\{\underline{c}\}$, cf. (1.29); for both types of customers ($j = 1,2$) the expected number of customers present in the system just after a departure, i.e. $E\{\underline{x}_j\}$, cf. (II.8.77), as well as at an arbitrary instant, i.e. $E\{\underline{y}_j\}$, cf. corollary

TABLE 7	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$E\{\underline{n}\}$	5.18	4.03	2.67	1.20	3.15	8.26
$E\{\underline{c}\}$	12.63	10.33	7.63	13.50	8.31	17.81
$E\{\underline{x}_1\}$	4.80	3.25	1.07	0.11	1.39	4.80
$E\{\underline{y}_1\}$	4.90	3.41	1.33	0.12	1.71	5.41
$E\{\underline{w}_1\}/\beta$	5.13	3.66	1.72	0.25	2.10	5.76
$E\{\underline{x}_2\}$	0.09	0.18	0.71	0.09	0.87	2.06
$E\{\underline{y}_2\}$	0.17	0.31	0.95	0.10	1.18	2.69
$E\{\underline{w}_2\}/\beta$	1.11	1.14	1.44	0.24	1.69	3.20
$\bar{w}/\beta$	4.76	3.24	1.59	0.25	1.92	4.63

III.8.4, and the mean waiting time, i.e. $E\{\underline{w}_j\}$, cf. corollary III.9.3; and finally the weighted average $\bar{w}$ of the mean waiting times defined in (3.7). To conclude, we present in table 8 the stationary probabilities that the number $\underline{q}_1$ of queueing type 1 customers exceeds, is equal to, or is less than the number $\underline{q}_2$ of queueing type 2 customers, cf. theorem III.9.2.

TABLE 8	Case 1	Case 2	Case 3	Case 4	Case 5	Case 6
$\Pr\{\underline{q}_1 > \underline{q}_2\}$	0.671	0.578	0.267	0.019	0.318	0.580
$\Pr\{\underline{q}_1 = \underline{q}_2\}$	0.321	0.402	0.582	0.967	0.513	0.237
$\Pr\{\underline{q}_1 < \underline{q}_2\}$	0.008	0.020	0.151	0.014	0.169	0.183

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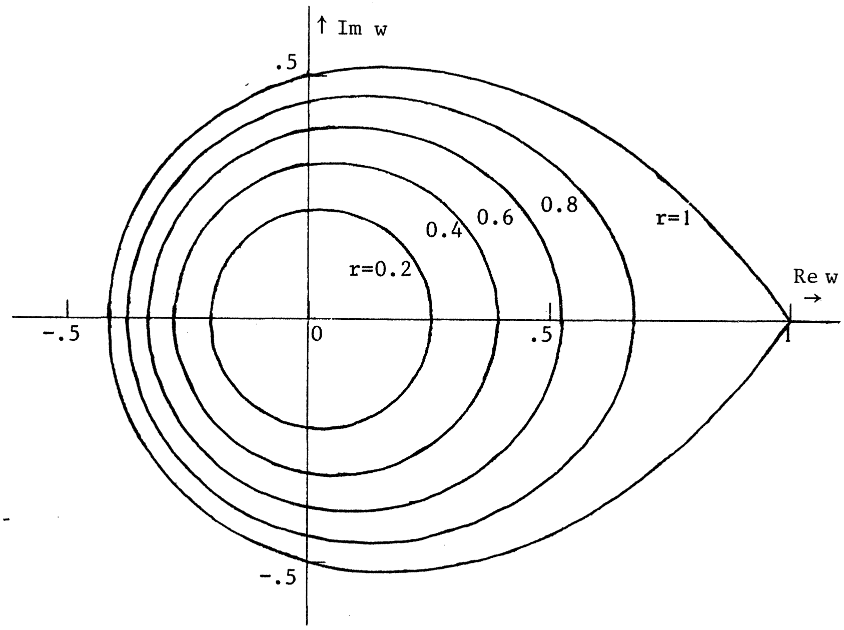

Figure. The contour L(r) in the case $c_1 = c_2 = \frac{1}{2}$, $a = 2$, and an Erlang-2 service time distribution, for increasing values of r, r = 0.2, r = 0.4, r = 0.6, r = 0.8, r = 1.0. (See corollary II.5.3)

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