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FROM A TO Z

PROCEEDINGS OF A SYMPOSIUM IN HONOUR OF A.C. ZAANEN

Editors:

C.B. HUIJSMANS
M.A. KAASHOEK
W.A.J. LUXEMBURG
W.K. VIETSCH

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PREFACE

This volume contains the contributions to a symposium in honour of Professor A.C. Zaanen, which was held at Leiden on July 5 - 6, 1982, on the occasion of his retirement. There were three invited lectures, the speakers being H.H. Schaefer, F. Smithies and B. Sz.-Nagy. The other authors are all among Zaanen's former Ph.D. students.

"From A to Z" is of course a play with Zaanen's initials. On the other hand it reflects Zaanen's mathematical career over more than forty years. During this period Zaanen made essential contributions to several parts of mathematics, mainly in functional analysis, integration theory and Riesz space theory. Through these four decades Zaanen established his own school of thought with its centre of gravity in Leiden and with its representatives abroad.

We do not pretend to give a survey of Zaanen's contributions to mathematics in this volume. However, all papers are in one way or another connected with Zaanen's work or his interests.

This volume is dedicated to Professor Zaanen as a token of respect and gratitude from his former Ph.D. students.

We would like to thank the Mathematical Centre for their willingness to publish these proceedings. We are also extremely grateful to Sonja Wassenaar, Netty Zuidervaart, Ellen Janssen and Len Koerts for the excellent typing work.

The editors.

CONTENTS

ORLICZ SPACES - A SURVEY OF CERTAIN ASPECTS

J.J. Grobler

I am honored to have been invited by the organizing committee to lecture on the subject of Orlicz spaces, a subject to which professor Zaanen has made fundamental contributions during his career as a mathematician. His first paper in this field dates back to 1946 [71] and the most recent paper was published with W.J. Claas in 1978 [12]. He was also one of the first to bring the theory of Orlicz spaces to the attention of a wider mathematical audience by means of his book "Linear Analysis" ([74]).

The aim of this lecture is to present some basic facts on Orlicz spaces and to survey some recent developments. In this we shall concentrate mainly on those parts of the theory to which professor Zaanen has contributed, either personally or through his students. Time and space do not permit us to survey the whole field. In fact, Orlicz spaces have become such a well established part of functional analysis that the literature on the subject is vast and it involves almost every facet of the theory of Banach spaces.

1. General theory

Orlicz spaces were introduced in 1932 by W. Orlicz [53] who discovered that with each increasing convex function M there corresponds a Banach space in much the same way as the Lebesgue space L^p corresponds to the convex function $M(t) = t^p$ $(t \geq 0,\ p > 1)$. His discovery was preceded by an investigation by W.H. Young [70] on the properties of such functions and on the properties of classes of functions satisfying a condition of the type $\int M(|f(t)|)d\mu < \infty$.

Let ϕ be a non-zero, increasing, real function defined on $[0,\infty)$ such that $\phi(0) = 0$ and such that ϕ is left continuous on $(0,\infty)$. Let ψ be its left continuous inverse. Then M and N defined on $[0,\infty)$ by the Lebesgue integrals

$$M(u) := \int_0^u \phi(t)dt, \quad N(v) := \int_0^v \psi(t)dt$$

are called complementary Young (or Orlicz) functions. For such M and N one gets

(i) $uv \le M(u) + N(v)$ for all $u,v \ge 0$

(ii) $N(v) = \sup\{uv - M(u) : u \ge 0\}$, $M(u) = \sup\{uv - N(v) : v \ge 0\}$.

Young [70] published a proof for the inequality (i), valid under certain continuity restrictions on ϕ. The first general proof (though geometric in character) is due to A.C. Zaanen [72], and W.A.J. Luxemburg [40] gave a purely analytic proof. Recently, C. Bylka and W. Orlicz published further generalizations of the inequality (i) [10]. The relations (ii) originate with S. Mandelbrojt [46].

For any measurable function f, Orlicz defined $\|f\|_M := \sup\{\int|fg|d\mu : \int N(|g|)d\mu \le 1\}$ and $L_M := \{f : \|f\|_M < \infty\}$. He then showed (at least for continuous M and N) that $(L_M, \|.\|_M)$ is a Banach space. Obviously, if $M(t) = t^p$ $(1 < p < \infty)$, then $\|f\|_M$ is equivalent to the L^p-norm of f. His definitions did not include the spaces L^1 and L^∞. This defect was remedied by Zaanen [72] who considered the case that N may jump to infinity, extended Young's inequality to this case and showed that L^1 and L^∞ are Orlicz spaces in the new context.

Luxemburg [38] defined an equivalent norm ρ_M on L_M as follows.

$$\rho_M(f) := \inf\{\alpha \mid \alpha > 0, \int M(\alpha^{-1}|f|)d\mu \le 1\}.$$

For an excellent exposition motivating the introduction of this norm, we refer to [41].

It is clear that the Young function M determines the properties of the Orlicz space L_M. Generally speaking, an Orlicz space has "nice" properties if the rate of growth of M is restricted. In his first paper on the subject [53], Orlicz considered functions M satisfying

$$M(2u) \le CM(u) \text{ for some constant } C > 0 \text{ and all } u \ge 0.$$

Such functions were previously studied by Orlicz and Birnbaum [7]. We refer to this condition as the (δ_2,Δ_2)-property. If $M(2u) \le CM(u)$ holds for large values of u, we say M satisfies the Δ_2-condition and if it holds for small values of u we say that M satisfies the δ_2-condition. These Δ-conditions and their effect on L_M were studied by many authors. (See [32], [38] and [15].) Zaanen himself showed in his first paper on Orlicz spaces that if the underlying measure is the Lebesgue measure and if the (δ_2,Δ_2)-condition holds, then L_M is separable and every continuous linear functional on L_M can be represented by an element of L_N ([71]). (See also [72].) Nowadays we know that the essential point is that these different Δ-conditions force the norm on L_M to be order continuous, i.e., $0 \le f_n \in L_M$ and $f_n \downarrow 0$ pointwise a.e. implies $\|f_n\|_M \downarrow 0$. (Which Δ-condition to use depends on the underlying measure space.) De Jonge [15] gives a rather complete picture of this situation; for example, if the measure space is $\mathbb{N}$ with the counting measure, δ_2 is used; when a finite diffuse measure space is considered, Δ_2 is used and when a space has infinite measure (δ_2,Δ_2) is required. In [32] much additional information about the Δ-conditions is collected.

From the remarks made above and the general theory of Banach function spaces as it was developed by Luxemburg and Zaanen in the period 1955-1966 [44] it follows that an Orlicz space L_M is *reflexive* if both M and its complementary function N satisfy an appropriate Δ-condition, that L_M is *separable* if the measure μ is separable and if M satisfies an appropriate Δ-condition, and that $L_M^* = L_N$ if the appropriate Δ-condition holds for M. In the latter case it was already illustrated by Orlicz [54] that the Haar-functions form a *basis* in L_M. In the case of Orlicz sequence spaces the unit vectors δ_n are a basis for L_M if M satisfies δ_2 [37]. Finally, Luxemburg [38] shows that if M is strictly convex and satisfies (δ_2,Δ_2) then (L_M,ρ_M) is *uniformly convex*.

2 . Linear functionals on Orlicz spaces

It was discovered by T. Ando [2] that singular functionals on Orlicz spaces have the remarkable property that they satisfy the triangle equality. A linear functional $F \in L_M^*$ is called *singular* whenever $g \in L_N$ and $|\int fg\,d\mu| \le |\langle f,F\rangle|$ for all $f \in L_M$ implies $g = 0$. The closed linear space of all singular linear functionals on L_M is denoted by $L_{M,s}^*$. If M satisfies an appropriate Δ-condition then $L_{M,s}^* = \{0\}$ but, whenever $\dim L_{M,s}^* > 0$, it follows that $\dim L_{M,s}^* = \infty$ [15]. De Jonge [15] proves that, in general, $L_{M,s}^*$ is an abstract L-space. This generalizes the result of Ando who proved it in a special case [2]. M.M. Rao [60] proved a similar (but less general) result.

If we denote by L_M^a the set of all $f \in L_M$ satisfying $\int M(\alpha|f|)d\mu < \infty$ for all $\alpha \in \mathbb{R}$, we have that L_M^a is a closed solid subspace of L_M with the property that the norm restricted to L_M^a is order continuous. Moreover $(L_M/L_M^a)^* = L_{M,s}^*$. (See [38].) Hence, for Orlicz spaces L_M/L_M^a is an abstract M-space. This led De Jonge to his definition of semi-M-spaces in general [15].

In [43] Luxemburg and Zaanen considered the problem of extending the modulars $m_M(f) := \int M(|f|)d\mu$ and $m_N(f) := \int N(|f|)d\mu$ and the norms $||.||_M$, ρ_M and $||.||_N$, ρ_N defined on L_M and L_N to the conjugate spaces $L_{\rho_N}^* \subset L_M$ and $L_{\rho_N}^* \subset$ $\subset L_N$ in such a way that the relations which exist between the functionals persist to hold in the larger domain.

3. Indices for Orlicz spaces

W. Matuszewska and W. Orlicz introduced in 1960 [55] as a gauge for the rate of growth of a Young function M numbers

$$\beta_M := \lim_{s\to\infty} \log [\limsup_{t\to\infty} M(st)/M(t)]/\log s$$

$$\alpha_M := \lim_{s\to\infty} \log [\liminf_{t\to\infty} M(st)/M(t)]/\log s ,$$

called indices for L_M which satisfy $0 \leq \alpha_M \leq \beta_M \leq \infty$ and $\alpha_M^{-1} + \beta_N^{-1} = 1$, $\alpha_N^{-1} + \beta_M^{-1} = 1$. In [24] I showed that if μ is a finite diffuse measure, these indices can be defined in terms of the so called ℓ_p-decomposition property and ℓ_p-composition property of the space L_M. These notions also coincide with the notions of lower-p-estimate and upper-p-estimate as used by T. Figiel-W.B. Johnson [19] and B. Maurey [48]. (See also P.G. Dodds [16].) In these terms $\beta_M = \sigma_M$ with $\sigma_M := \inf \{p : L_M$ has the ℓ_p-decomposition property$\}$ and $\alpha_M = s_M$ with $s_M := \sup \{p : L_M$ has the ℓ_p-composition property$\}$. For Orlicz spaces the indices s_M and σ_M are also the reciprocals of those defined by D.W. Boyd [9] for rearrangement invariant function spaces. In [9] Boyd calculates these indices in Orlicz spaces. In the formulas the underlying measure space plays a rôle in much the same way as it influences the choice of Δ-conditions in some of our earlier remarks. If μ is the counting measure on $\mathbb{N}$ the indices are characterized by the fact that *ℓ_p is isomorphic to a subspace of ℓ_M if and only if* $s_M \leq p \leq \sigma_M$

($\ell_p = c_0$ if $p = \infty$). This result is due to J. Lindenstrauss and L. Tzafriri [35]. It is easily seen that $1 < s_M \le \sigma_M < \infty$ implies that L_M is reflexive [24]. Lindenstrauss and Tzafriri show that the converse is true for sequence spaces if M has a continuous strictly increasing derivative [35].

Generalizing a theorem of H.R. Pitt [58], T. Ando proved in [3] that if $s_{M_1} > \sigma_{M_2}$ *then every integral operator on* L_{M_1} *into* L_{M_2} *is compact* (μ is a finite diffuse measure); Lindenstrauss and Tzafriri proved for Orlicz sequence spaces that *if moreover* M_1 *and* M_2 *satisfy* δ_2 *then every bounded linear operator from* ℓ_{M_1} *into* ℓ_{M_2} *is compact if and only if* $s_{M_1} > \sigma_{M_2}$.

In conclusion we mention that these results can be generalized to Banach lattices. (See [16], [17].)

4. The structure of Orlicz spaces

Initiating papers in this field were papers by K.J. Lindberg [34] and Lindenstrauss and Tzafriri [35, 36]. Structural questions studied are inter alia the uniqueness of symmetric bases in Orlicz sequence spaces, minimal Orlicz sequence spaces and the question which sequence spaces are isomorphic to subspaces of a given separable sequence space.

Though it is not true in general for Banach spaces, it is shown that every Orlicz sequence space has a subspace isomorphic to either c_0 or ℓ_p, $1 \le p < \infty$. (See section 3.)

A complete survey of these and related questions can be found in Lindenstrauss and Tzafriri's monograph [37]. In these investigations a significant rôle is played by the indices and also by the following non-void norm compact subsets of Young functions in $C[0,\frac{1}{2}]$

$$E_{M,\Lambda} := \overline{\{M(st)/M(s) \,:\, 0 < s < \Lambda\}},\ 0 < \Lambda \le \infty,\ E_M := \bigcap_{\Lambda>0} E_{M,\Lambda}$$

$$C_{M,\Lambda} := \overline{\text{co}}\, E_{M,\Lambda},\ C_M := \bigcap_{\Lambda>0} C_{M,\Lambda} \text{ with M a Young function.}$$

A few sample results are:

An Orlicz space ℓ_M^a *is isomorphic to a subspace of the sequence space* ℓ_N^a *if and only if* M *is equivalent to some Young function* $C_{N,1}$.

Let ℓ_M *be a reflexive Orlicz space with complementary Young function* M^*. *Then* ℓ_N *is isomorphic to a quotient space of* ℓ_M *if and only if* N^* *is equivalent to a function in* $C_{M^*,1}$.

A Young function M with δ_2 is called minimal if $E_{N,1} = E_{M,1}$ for every $N \in E_{M,1}$. It is conjectured that if M is minimal then every complemented subspace of ℓ_M is isomorphic to ℓ_M. (See [37].)

Finally we mention works of N.J. Kalton ([27, 28]), G.E. Thebucava [63], N.J. Nielsen [50], D. Dacunha-Castelle [14] and R.P. Macleev and S. Trojanski [45].

5. Representation of abstract Orlicz spaces

A well known result in the theory of L^p-spaces is that a Banach lattice with p-additive norm is isomorphic to a space $L^p(X,\Lambda,\mu)$, (X,Λ,μ) some measure space. (See [6], [8], [12] and [47].) A non-negative real function M on a Riesz space L is called a *modular* if M(f) = 0 if and only if f = 0, $|f| \leq |g|$ implies $M(f) \leq M(g)$ and if M is convex. A Riesz space equipped with a modular is called a *modulared space* [49]. The Minkowski functional of the set $\{x \in L : M(x) \leq 1\}$ is a norm on L. The modular M is called an *Orlicz modular* if $M(2f) \leq CM(f)$ for some constant $C > 0$ and all $f \in L$ and if moreover $M(f+g) = M(f) + M(g)$ for disjoint f and g. L is called an *Orlicz lattice* if (L,ρ_M) is complete. L is called *component invariant* if L has a weak order unit e and if, for every component p of e the relation $M(\alpha e)/M(e) = M(\alpha p)/M(p)$ holds for all $\alpha \geq 0$. These definitions are due to W.J. Claas and A.C. Zaanen [15] who proved that *every component invariant Orlicz lattice is isomorphic to a real Orlicz space* $L_M(X,\Lambda,\mu)$. Recently P. Kranz and W. Wnuk improved this result by showing that the Δ-condition on M, and the component invariance of L can be dropped. (See [31] and [69].)

6. Applications

From the very early days of the theory, Orlicz spaces were used in applications. In his 1946 paper Zaanen already investigated compactness properties of kernel operators on Orlicz spaces with the idea to solve linear integral equations [71]. In [73] he developed the Fredholm determinant theory for kernel operators of finite double norm in Orlicz spaces. Incidentally, a problem posed in this paper concerning determinant inequalities was only recently solved by P. Nowosad and R. Tovar [51] who generalized the Carleman inequalities for kernel operators, which were originally proved for Hilbert-Schmidt operators on L^2, to the case of operators of finite double norm on reflexive Orlicz spaces. These operators were also studied by J.J. Uhl [68] who proved Bochner integral representation theorems for them in Orlicz spaces.

Today there is hardly a sphere of linear (and non-linear) analysis in which Orlicz spaces do not find applications. For instance in the theory of *partial differential equations* the rôle played by L^p-space in the definition of Sobolev spaces is taken over by Orlicz spaces. The resulting space, called an *Orlicz-Sobolev* space is the object of study of many authors. The interested reader is referred to the following papers: [1, 11, 18, 22, 23, 26, 29, 30, 57, 61, 65]. For applications to *interpolation* theory the reader is referred to [25, 29, 64], and we also find that the *calculus of variations* is studied in the setting of Orlicz spaces. (See [5, 21].) In [52] O'Neil studies *integral transforms* in Orlicz spaces. Finally we mention the success which met the study of *non-linear integral equations* by Krasnosell'skii and Rutickii. We refer to their monograph [32] for results in this direction.

7. Concluding remarks

As was mentioned in the introduction we touched only a few subjects and without doubt there are many important contributions to the theory that we did not mention. This is because of the fact that for almost every property a Banach space may have, there is a paper investigating the property in Orlicz spaces. Thus, J.B. Turett and J.J. Uhl [66] investigate the *Radon-Nikodym property* in this connection, A.J. Pach, M.A. Smith and B. Turett [56] prove that under certain conditions an Orlicz space is *flat*. B. Turett [67] finds conditions for an Orlicz space to be *rotund* and Lindenstrauss and Tzafriri [36] show that every reflexive Orlicz space has the *uniform approximation property*. C.D. Aliprantis and O. Burkinshaw study *minimal topologies* on Orlicz spaces [4] and W. Fischer and U. Schöler [20] study the range of *vector measures* in Orlicz spaces. (See also M.S. Skaff [62].) Finally we mention also the construction of *ultraproducts of Orlicz spaces* by D. Dacunha-Castelle [13] and the study of *Hardy-Orlicz spaces* by R. Lesniewicz [33].

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ORTHOMORPHISMS

C.B. Huijsmans

The Riesz space (vector lattice) A is called a *Riesz algebra (latticc ordered algebra)* if A is a real algebra such that the ordering and the (not necessarily commutative) multiplication are compatible, i.e., $u,v \in A^+$ implies $uv \in A^+$. If A has the additional property that $u \wedge v = 0$ in A implies that $(uw) \wedge v = (wu) \wedge v = 0$ for all $w \in A^+$, then A is called an f-*algebra* (generalized function-algebra). The latter property is equivalent to saying that $f \perp g$ in A implies $fh \perp g$ and $hf \perp g$ for all $h \in A$. Furthermore, it is not difficult to show that the lattice ordered algebra A is an f-algebra if and only if $\{fg\}^{dd} \subset \{f\}^{dd} \cap \{g\}^{dd}$ for all $f,g \in A$, a characterization due to S.J. Bernau ([3], 1965). Denoting the left multiplication by h with π_h^{ℓ} and the right multiplication with π_h^{r}, these mappings have the property that $f \perp g$ implies $\pi_h^{\ell} f \perp g$ and $\pi_h^{r} f \perp g$. This observation gives rise to the definition of a so-called *orthomorphism* in a Riesz space L (which, so to say, takes over the role of the multiplication in an f-algebra A). For the sake of convenience, we assume from now on that *all Riesz spaces* involved in this paper are *Archimedean*.

DEFINITION. The order bounded linear mapping π from the Riesz space L into itself is called an orthomorphism if it follows from $f \perp g$ in L that $\pi f \perp g$ (equivalently, $u \wedge v = 0$ implies $\pi u \perp v$).

Obviously, the order bounded linear mapping π is an orthomorphism if and only if π leaves all bands invariant (i.e., π is so-called *band preserving*). Any positive orthomorphism on L is evidently a lattice homomorphism.

The first to define a notion which is more or less the same as an orthomorphism was H. Nakano ([16], 1940). Apart from a continuity condition he calls a positive linear operator on a Dedekind σ-complete Riesz space a *dilitator* whenever it commutes with all order projections. The following theorem shows that for this class of Riesz spaces such operators are precisely the positive orthomorphisms.

THEOREM. *Let* L *be a Riesz space with the principal projection property and* $\pi: L \to L$ *a positive linear operator on* L . *Then* π *is a positive orthomorphism iff* $\pi P = P\pi$ *for all order projections* P .

PROOF. Let π be an orthomorphism and P the order projection on the principal band B_u generated by some $u \in L^+$. Since for every $f \in L$ the element $(I-P)f \in B_u^d$, it follows that $\pi(I-P)f \in B_u^d$ and so $P\pi(I-P)f = 0$. Hence,

$$\pi f = \pi(I-P)f + \pi P f$$

implies that $P\pi f = P\pi P f = \pi P f$, on account of $\pi P f \in B_u$.

Conversely, take $u \in L^+$ and let P be the order projection on B_u . Since $u = Pu$, it follows that $\pi u = \pi P u = P\pi u$, so $\pi u \in B_u$. This implies immediately that $\pi(B) \subset B$ for all bands B in L .

The notion of (positive) orthomorphism according to the above definition is due to A. Bigard and K. Keimel ([6], 1969) and, independently, to P.F. Conrad and J.E. Diem ([8],1971). The latter authors use the name of *polar preserving endomorphisms*. The main result in both papers (the proof is by means of representation theory) is that the set of all orthomorphisms on a Riesz space is an Archimedean f-algebra with unit element with respect to pointwise vector space operations, pointwise ordering and the composition as multiplication. Because of its importance we state this result as a theorem.

THEOREM ([6],[8],[9],[14]). *Let* L *be an Archimedean Riesz space. Then the set* Orth(L) *of all orthomorphisms on* L *is an Archimedean* f*-algebra with unit element.*

However, this result can be proved without using any representation theory. This was done for the first time by W.A.J. Luxemburg, but not published before 1978 in [13] (see also [12] and [17]). The key to his proof is the fact that any orthomorphism π on L is order continuous (i.e., $u_\tau \downarrow 0$ implies $\inf|\pi u_\tau| = 0$) and so π can be extended uniquely to the Dedekind completion of L . After him several other authors succeeded to prove the above theorem without representation ([1],[4]).

In the above mentioned pioneering papers on orthomorphisms ([6] and [8]) there is a remarkable lack of examples. A.C. Zaanen deserves praise for filling up this gap in [19]. We list some of his examples.

EXAMPLES.

(I) For any Archimedean f-algebra A with unit element every orthomorphism of A is a multiplication by a fixed element of A . In other words, A and Orth(A) can be identified. This holds in particular for the following Archimedean unital f-algebras.

(a) $A = C(X)$, the f-algebra of all real continuous functions on some topological space X .

(b) $A = C_b(X)$, the f-algebra of all real continuous bounded functions on X .

(c) $A = M(X,\mu)$, the f-algebra of all real μ-almost everywhere finite μ-measurable functions (with identification of μ-almost equal functions) on some measure space (X,μ) .

(d) $A = L_\infty(X,\mu)$, the f-algebra of all essentially bounded functions on (X,μ) .

(e) A is one of the following sequence spaces: (s) , ℓ_∞ , (c) , the space of all eventually constant sequences and the space of all sequences with only finitely many different values.

(f) $A = C''(\mathcal{D})$, the second commutant of a commuting subset $\mathcal{D}$ of the ordered vector space of all bounded Hermitian operators on some Hilbert space.

(II) Also in cases there is not a unit element available the space of orthomorphisms can be determined.

(a) $A = (c_{00})$, the sequence space of all eventually zero sequences. Every orthomorphism on A is a coordinatewise multiplication by some arbitrary sequence.

(b) $A = (c_0)$. Every orthomorphism on A is a multiplication by some ℓ_∞-sequence.

(c) $A = C_c(X)$, the Riesz space of all real continuous functions with compact carrier on some locally compact Hausdorff space X . Every orthomorphism on A is a multiplication by some fixed continuous function, so Orth(A) can be identified with C(X) .

(d) $A = C_\infty(X)$, the Riesz space of all real continuous functions on some locally compact, σ-compact Hausdorff space X which vanish at infinity. In this case $\mathrm{Orth}(A) = C_b(X)$.

(III) Let (X,μ) be a σ-finite measure space and L be the Riesz space $L_p(X,\mu)$, $(0<p<\infty)$. Every orthomorphism on L is a multiplication by some essentially bounded function, so $\mathrm{Orth}(L) = L_\infty(X,\mu)$.

(IV) Let L be a Banach function space L_ρ (i.e., a linear subspace of the space of all real μ-measurable functions on some σ-finite measure space (X,μ) , norm complete with respect to a Riesz norm ρ) . Again, $\mathrm{Orth}(L) = L_\infty(X,\mu)$.

(V) Obviously, every scalar multiple αI of the identity mapping I on some Archimedean Riesz space L is an orthomorphism. We call these the trivial orthomorphisms. In all examples above there exist non-trivial orthomorphisms. However, there are Riesz space with only the trivial orthomorphisms. We mention one.

Let L be the Riesz space of all real continuous functions f on some interval $[a,b]$ such that f is piecewise linear (i.e., the graph of f consists of but a finite number of line segments). Then $\dim \mathrm{Orth}(L) = 1$.

One of the most interesting problems in f-algebra theory is the interplay of order properties and algebraic properties. There is a famous result of this kind, namely that *every Archimedean f-algebra is commutative*. Though the credit for this theorem is often given to G. Birkhoff and R.S. Pierce ([7], 1956), the first proof seems to go back to I. Amemiya, who gave the proof by means of spectral functions ([2], 1953). Birkhoff and Pierce prove that the inequality

$$n|fg-gf| \leq f^2 + g^2 \quad (n=1,2,\ldots)$$

holds for any f-algebra. They show this by observing that the class of lattice ordered algebras is "equationally definable" and that due to this fact the f-algebra can be treated as if it was totally ordered. Using this metamathematical theorem, the above inequality needs only a proof in the class of totally ordered f-algebras in which it is almost trivial. Obviously, the present inequality implies immediately that any Archimedean f-algebra is commutative.

Several years later S.J. Bernau ([3], 1965) published an elementary, though not very transparent proof of the inequality in question. In 1975 A.C. Zaanen showed in his paper on orthomorphisms ([19]) that the fact that all Archimedean f-algebras are commutative can also be proved by means of the theory of orthomorphisms. We sketch his proof.

THEOREM. *Any Archimedean* f*-algebra* A *is commutative.*

PROOF. The first observation to be made is that two orthomorphisms π_1 and π_2 coinciding on an order dense subset D of A are necessarily equal. Indeed, putting $\pi = \pi_1 - \pi_2$, the kernel K_π of the order continuous orthomorphism π is a band and so $K_\pi = K_\pi^{dd}$. Since $D \subset K_\pi$ and $D^{dd} = A$, we derive $K_\pi = A$, so $\pi = 0$, i.e., $\pi_1 = \pi_2$ on A.

Secondly, observe that it follows immediately from $\{fg\}^{dd} \subset \{f\}^{dd} \cap \{g\}^{dd} = \{|f| \wedge |g|\}^{dd}$ that $f \perp g$ in A implies $fg = 0$.

Finally as before we denote for any $f \in A$ the left multiplication by f with π_f^ℓ and the right multiplication by f with π_f^r; as stated before π_f^ℓ and π_f^r are orthomorphisms. It follows from the second observation above that $\pi_f^\ell g = \pi_f^r g = 0$ for all $g \in \{f\}^d$. Furthermore, $\pi_f^\ell f = \pi_f^r f = f^2$. Hence, $\pi_f^\ell = \pi_f^r$ on the order dense set $\{f\} \cup \{f\}^d$. By the first remark above we have $\pi_f^\ell = \pi_f^r$ on A. This holding for all $f \in A$, the theorem is proved.

There is some connection between the notion of an orthomorphism and the concept of a *centralizer* (generalized translation). The linear mapping operator π on some algebra A is said to be a centralizer whenever $\pi(fg) = (\pi f)g = f(\pi g)$ for all $f,g \in A$ (see [11], 1964). Other closely related notions are the notions of *normalizer*, *multiplier* and *fraction*.

THEOREM. *Let* A *be an Archimedean* f*-algebra without nonzero nilpotents and* $\pi: A \to A$ *a positive linear operator. Then* π *is an orthomorphism if and only if* π *is a centralizer.*

PROOF. Note first that in A we have $f \perp g$ if and only if $fg = 0$. As observed in an earlier stage, $|f| \wedge |g| = 0$ implies $fg = 0$. Conversely, it follows from $(|f| \wedge |g|)^2 \leq |f|.|g| = |fg|$ that $fg = 0$ implies $(|f| \wedge |g|)^2 = 0$, so, by hypothesis, $|f| \wedge |g| = 0$.

Since Orth(A) is an Archimedean f-algebra, any two orthomorphisms on A commute. Furthemore, A is commutative as well so we can write π_f instead of $\pi_f^{\ell} = \pi_f^{r}$.

Now take $0 \leq \pi \in$ Orth(A) . For all $f,g \in A$ we have $(\pi\pi_f)(g) = (\pi_f\pi)(g)$, in other words $\pi(fg) = f(\pi g)$. Similarly, $\pi(fg) = (\pi f)g$, proving that π is a centralizer.

Conversely, let π be a centralizer and suppose that $u \wedge v = 0$ in A . It follows from $uv = 0$ that $\pi(uv) = \pi(u)v = 0$ and hence $(\pi u) \wedge v = 0$, and thus π is an orthomorphism.

After the papers on orthomorphisms of the late sixties and the early seventies several new results on the subject have been published. We mention some.

Whereas the kernel K_π of an orthomorphism π is always a band (actually, $K_\pi = R_\pi^d$, where R_π denotes the range of π) , the image R_π need not be so. By way of example, let in $L = C([0,1])$ the positive orthomorphism π be defined by $\pi f = i.f$ (where $i(x) = x$ for all $0 \leq x \leq 1$). Putting $u(x) = \left|x \sin\frac{1}{x}\right|$ on $(0,1]$ and $u(0) = 0$, the continuous function u satisfies $0 \leq u \leq i$. However, $i \in R_\pi$ but $u \notin R_\pi$, showing that R_π is not even an ideal.

In [3] (1972) A. Bigard proves that in any Dedekind complete Riesz space the range of every orthomorphism is an ideal. This result is improved by C.B. Huijsmans and B. de Pagter in [10] (1982). They show in fact that if the Archimedean Riesz space L is uniformly complete and has, in addition, the property that every proper prime ideal contains a unique minimal prime ideal, the range of every orthomorphism on L is necessarily an ideal.

Another striking result was also proved by A. Bigard in [3]. He showed that if L is a Dedekind complete Riesz space, the space Orth(L) is nothing else than the band B_I generated by the identity mapping I on L in the Riesz space of all order bounded linear operators on L .

It is always a good custom to end a paper with an open problem. In the definition of an orthomorphism π on an Archimedean Riesz space L it is required that π , besides being linear and band preserving, is order bounded. This condition is not redundant, as shown by M. Meyer in [15] (1979) (see also [4] (1981)) in the following example.

Let L be the Archimedean Riesz space of all real functions f on [0,1) for which there exists a partition $0 = x_0 < \ldots < x_n = 1$ such that the restrictions $f/[x_{i-1},x_i)$ (i=1,...,n) are linear. Define πf to be the right derivative of f . It is not hard to show that π is a linear band preserving operator on L , which is not order bounded.

This suggests the problem of determining the class of all Archimedean Riesz spaces for which the band preserving property of the linear operator implies the order boundedness, by W.A.J. Luxemburg in [12] referred to as *"the automatic order boundedness problem"*. It does not seem easy to find a simple characterization for this class of Riesz spaces. For instance, A.W. Wickstead has shown in [18] (1980), that there exist Dedekind complete and universally complete Riesz spaces which admit band preserving linear operators which are not order bounded. It can be shown that the above mentioned class includes all Banach lattices.

Another approach of the problem is the following. What additional condition has to be imposed on the linear band preserving operator to be order bounded? One particular condition is given by S.J. Bernau in [4]. He actually proves that every linear band preserving operator which is relative uniformly continuous is an orthomorphism.

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EMBEDDINGS OF RIESZ SUBSPACES WITH AN APPLICATION TO MATHEMATICAL STATISTICS

E. de Jonge

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1. Introduction.

Throughout this paper, [3] will be the standard reference for Riesz space theory and terminology. From now on L will always be an Archimedean Riesz space. We shall discuss Riesz subspaces of L which are regular and/or normal (see section 2 below). In section 3 an application will be given in the theory of mathematical statistics.

To motivate the discussion from a Riesz space theoretical point of view, let $T : L \to L$ be a positive linear map such that $T^2 = T$ and such that $f \neq 0$ implies $Tf \neq 0$ (i.e., T is a strictly positive projection). Set $M=T(L)$. Then we have

THEOREM 1.1. *For* L, T *and* M *as above the following hold:*

(a) M *is a Riesz subspace of* L*;*

(b) if $f_\tau \downarrow 0$ *in* M^+, *then* $f_\tau \downarrow 0$ *in* L^+*;*

(c) if $f_\tau \in M^+$ *and* $f \in L^+$ *are such that* $f_\tau \uparrow f$ *in* L^+, *then* $f \in M$.

Hence, the image of a strictly positive projection is a Riesz subspace which is "nicely embedded". By way of an example we show how strictly positive projections occur in mathematical statistics.

EXAMPLE 1.2. (Sufficiency). Let X be a point set and let $\mathfrak{a}$ be a σ-algebra of subsets of X. Furthermore, let $\{P_\theta : \theta \in \Theta\}$ be a class of probability measures on $\mathfrak{a}$. Set

$$N = \{A \in \mathfrak{a} : P_\theta(A) = 0 \text{ for all } \theta \in \Theta\}.$$

Define $L_\infty(\mathfrak{a})$ to be the collection of all real-valued $\mathfrak{a}$-measurable functions on X and let L_o denote the ideal of $L_\infty(\mathfrak{a})$ consisting of all functions which vanish outside an N-set. Finally, set

$$L = L_\infty(\mathfrak{a}) / L_o.$$

Then L is a Dedekind σ-complete Riesz space. Next, let $\mathfrak{a}_o$ be a sub σ-algebra of $\mathfrak{a}$ such that $N \subset \mathfrak{a}_o$. The space $L_\infty(\mathfrak{a}_o)$ is defined similarly as above. Clearly $L_o \subset L_\infty(\mathfrak{a}_o)$. Hence, setting

$$M = L_\infty(\mathfrak{a}_o)/L_o,$$

it follows that M is a Riesz subspace of L. We shall say that $\mathfrak{a}_o$ is *sufficient* for $\{P_\theta : \theta \in \Theta\}$ whenever for all $f \in L$ there exists an $f_o \in M$ such that

$$\int_A f \, dP_\theta = \int_A f_o \, dP_\theta$$

holds for all $A \in \mathfrak{a}_o$ and for all $\theta \in \Theta$. Obviously, if for $f \in L$ such an $f_o \in M$ exists, then it must be unique. Hence, assuming that $\mathfrak{a}_o$ is sufficient, it follows that we can define a map $T : L \to M$ by setting $Tf = f_o$ for all $f \in L$. It is easily verified that T is linear, strictly positive and that $T^2 = T$. For the theory of sufficiency we refer to [1].

In section 3 we continue our discussion on sufficiency, but first some Riesz space theory has to be developed.

2. Riesz subspaces

Throughout this section, let M be a Riesz subspace of L. Furthermore, let $B(L)$, $B_p(L)$ and $P(L)$ denote the collection of bands, the collection of principal bands and the collection of projection bands of L respectively. Similarly we can define the classes $\mathcal{B}(M)$, $\mathcal{B}_p(M)$ and $\mathcal{P}(M)$ for the Riesz space M.

Let $B \in \mathcal{B}(M)$. Setting

$$\pi_M(B) = B^{dd} \quad (\text{in } L),$$

it follows that $\pi_M(B) \in \mathcal{B}(L)$. The map $\pi_M : \mathcal{B}(M) \to \mathcal{B}(L)$ thus defined, is easily verified to be a one-one Boolean homomorphism from $\mathcal{B}(M)$ into a principal ideal of $\mathcal{B}(L)$. Furthermore, if $B \in \mathcal{B}(M)$, then we can retrieve B from its image $\pi_M(B)$ in $\mathcal{B}(L)$ simply by

$$B = \pi_M(B) \cap M.$$

However, if $C \in \mathcal{B}(L)$ is arbitrary, then it does not automatically follow that $C \cap M$ is a band in M (although it is always an ideal). We also observe that if $B_\tau \downarrow \{0\}$ in the Boolean algebra $\mathcal{B}(M)$, then it is not necessarily true that $\pi_M(B_\tau) \downarrow \{0\}$ in $\mathcal{B}(L)$, i.e., π_M is not always order continuous.

Next, in view of theorem 1.1.b, we define

DEFINITION 2.1. We say that M is a *regular* Riesz subspace of L whenever it follows from $f_\tau \in M^+$ and $f_\tau \downarrow 0$ in M that $f_\tau \downarrow 0$ in L^+.

We note that ideals and one-dimensional Riesz subspaces are always regular. On the other hand, considering $C([0,1])$ as a Riesz subspace of $L_\infty([0,1])$ it is easily verified that $C([0,1])$ is not regular.

With respect to the introductory remarks above, we now have

THEOREM 2.2. *The following are equivalent.*

(a) M *is a regular Riesz subspace of* L*;*

(b) for all $C \in \mathcal{B}(L)$ *we have* $C \cap M \in \mathcal{B}(M)$*;*

(c) $\pi_M : \mathcal{B}(M) \to \mathcal{B}(L)$ *is order continuous;*

(d) $\pi_M : \mathcal{B}_p(M) \to \mathcal{B}_p(L)$ *is order continuous.*

Next, motivated by thoerem 1.1.c, we turn our attention to another property that M might have.

DEFINITION 2.3. We say that M is a *normal* Riesz subspace of L whenever it follows from $f_\tau \in M^+$, $f \in L^+$ and $f_\tau \uparrow f$ in L that $f \in M$ (i.e., whenever M is closed for taking arbitrary suprema).

Note that an ideal is always regular but not necessarily normal. On the other hand, normal Riesz subspaces need not always be regular as we shall see from theorem 2.5 below. Thus, regularity and normality are in general independent properties. Normal Riesz subspaces are nice in the sense that they inherit many properties. For instance

THEOREM 2.4. *Let* M *be a normal Riesz subspace of* L. *If* L *is either Dedekind σ-complete or Dedekind complete, then* M *is Dedekind σ-complete or Dedekind complete. If* L *has the (principal) projection property, then* M *has the (principal) projection property.*

Somewhat more general than above, one can show that if M is a normal Riesz subspace of L and if $B \in \mathcal{B}(M)$ is such that $C = \pi_M(B) \in \mathcal{P}(L)$, then $B \in \mathcal{P}(M)$. Furthermore, in that case, if $f \in M$, then $P_C f \in M$, where P_C denotes the band projection.

Obvious examples of normal Riesz subspaces are bands and one-dimensional Riesz subspaces. Furthermore, if we define for $B \in \mathcal{B}(L)$ and $f \in L^+$

$$M(B,f) = \{g \in L : g \in b + \alpha f,\ b \in B,\ \alpha \in \mathbb{R}\},$$

then M(B,f) is also a normal Riesz subspace. In connection with regularity we now have

THEOREM 2.5. *The following are equivalent.*

(a) L *has the projection property;*

(b) every normal Riesz subspace of L *is regular;*

(c) for all $B \in \mathcal{B}(L)$ *and for all* $f \in L^+$, M(B,f) *is regular.*

Returning to strictly positive projections, we conclude this section with the following result.

THEOREM 2.6. *Let* $T : L \to L$ *be a strictly positive projection and let* $M = T(L)$. *Then we have*

(a) M *is a regular and normal Riesz subspace of* L;

(b) if either T *is σ-order continuous or if the ideal in* L *generated by* M *equals* L *and if furthermore*

$B \in \pi_M(\mathcal{B}(M)) \cap \mathcal{P}(L)$, *then we have* $P_B Tf = TP_B f$ *for all* $f \in L$;

(c) *conversely, if* L *has the projection property and if either* T *is* σ*-order continuous or if the ideal in* L *generated by* M *equals* L , *then* $\pi_M(\mathcal{B}(M))$ *can be described as follows: for* $B \in \mathcal{B}(L)$ *we have* $B \in \pi_M(\mathcal{B}(M))$ *if and only if* $P_B Tf = TP_B f$ *for all* $f \in L$.

The proofs of all above results are given in [2].

3. Applications to sufficiency

Using the results of section 2, we are now able to make some statements on sufficient σ-algebras. Therefore, in this section, we assume the notations and terminology of example 1.2. Furthermore, we set

$$\bar{\mathfrak{a}} = \mathfrak{a}/N$$

and similarly if $\mathfrak{a}_o$ is a sub σ-algebra of $\mathfrak{a}$ containing N,we set $\bar{\mathfrak{a}}_o = \mathfrak{a}_o/N$. Clearly $\bar{\mathfrak{a}}_o$ is a Boolean sub σ-algebra of the σ-complete Boolean algebra $\bar{\mathfrak{a}}$. Also it is well-known that (for L and M as in example 1.2)

$$\bar{\mathfrak{a}} \cong \mathcal{P}(L) = \mathcal{B}_p(L) \text{ and } \bar{\mathfrak{a}}_o \cong \mathcal{P}(M) = \mathcal{B}_p(M)$$

as Boolean algebras. In view of theorems 2.2 and 2.6 the following is therefore immediate.

COROLLARY 3.1. *Assume that* $\mathfrak{a}_o$ *is sufficient for* $\{P_\theta : \theta \in \Theta\}$. *Then we have*

(a) if $A_\tau \in \bar{\mathfrak{a}}_o$ *and if* $\cap A_\tau = \{0\}$ *in* $\bar{\mathfrak{a}}_o$, *then* $\cap A_\tau = \{0\}$ *in* $\bar{\mathfrak{a}}$;

(b) if $A_\tau \in \bar{\mathfrak{a}}_o$, $A \in \bar{\mathfrak{a}}$ *and if* $A = \cup A_\tau$, *then* $A \in \bar{\mathfrak{a}}_o$.

(Intersections and unions should be taken in the Boolean algebra sense).

We note that if $\bar{\mathfrak{a}}$ has the so-called countable chain property, then (a) and (b) of corollary 3.1 are satisfied for every sub σ-algebra $\mathfrak{a}_o$ of $\mathfrak{a}$. However as soon as $\bar{\mathfrak{a}}$ lacks this property, then this does not need to be the case. Since $\bar{\mathfrak{a}}$ having the countable chain property is equivalent to the existence of a σ-finite measure μ on $\mathfrak{a}$ with the property that $\mu(A) = 0$ implies $A \in N$ (i.e., μ dominates $\{P_\theta : \theta \in \Theta\}$), the above corollary explains why the theory of sufficient σ-algebras is much more involved in the general case than in the dominated case (see [1]). For instance, corollary 3.1 above shows that if

$$N \subset \mathbf{a}_1 \subset \mathbf{a}_2 \subset \mathbf{a}$$

and if $\mathbf{a}_1$ is sufficient, then $\mathbf{a}_2$ does not need to be sufficient, because it may not satisfy (a) or (b) of corollary 3.1.

Although it is not an immediate consequence of the previous results, I would like to conclude with the following extension of the well-known Halmos-Savage theorem. An elegant proof can be given without using Radon-Nikodym derivates. Instead one should apply the Riesz space theory from [3] and some results from [2].

THEOREM 3.2. *Assume that* $\bar{\mathbf{a}}$ *is a complete Boolean algebra.*

(a) If $\mathbf{a}_o$ *is sufficient for* $\{P_\theta : \theta \in \Theta\}$ *and if* $\mathbf{a}_1$ *is a sub* σ*-algebra of* $\mathbf{a}$ *such that* $\mathbf{a}_o \subset \mathbf{a}_1$*, then* $\mathbf{a}_1$ *is sufficient for* $\{P_\theta : \theta \in \Theta\}$ *if and only if* $\mathbf{a}_1$ *satisfies (a) and (b) of corollary 3.1.*

(b) Let Σ *denote the class of all sub* σ*-algebras of* $\mathbf{a}$ *which contain* N *and which are sufficient for* $\{P_\theta : \theta \in \Theta\}$*. Then* Σ *has a smallest element. In other words,*

$$\cap\{\mathbf{a}' : \mathbf{a}' \in \Sigma\} \in \Sigma.$$

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SYMMETRISABLE OPERATORS AND MINIMAL FACTORIZATION

M.A. Kaashoek

0. Prologue

It is a pleasure to participate in this symposium honoring professor A.C. Zaanen. My own mathematical work is to a large extent a product of Zaanen's interest in functional analysis and operator theory. Naturally, as one of his students, I learned a lot from the lectures Zaanen gave. But most effective has been a two hour session Zaanen had with me on a hot day in the summer of 1960 in his room on the top floor of the former mathematical institute of Leiden University in the Vreewijkstraat. What was supposed to be an oral examination was transformed by Zaanen, maybe because of the temperature, into a private course on basic ideas concerning integral equations and linear operators, after which I decided that I should learn more about these topics. Later Zaanen introduced me to Kato's perturbation theory for nullity and deficiency of linear operators, which together with the Gohberg-Krein paper on the same subject became the starting point for my Ph.D. work under his supervision. It was also the beginning of a friendship. I mention this with great respect and gratitude.

1. Introduction

In his earlier papers on integral equations and also in "*Linear Analysis*" A.C. Zaanen has used the concept of a symmetrisable operator as a tool to give a smooth, coherent and concise treatment of selfadjoint integral equations and non-selfadjoint integral equations with kernels like Marty kernels, Pell kernels and Garbe kernels. The work Zaanen did on this subject may be regarded as a beginning of what is nowadays called the theory of selfadjoint operators on spaces with an indefinite inner product (see [4]).

In my contribution to this symposium I would like to discuss another instance in which symmetrisable operators can be employed as a useful tool.

I have in mind factorization problems. For simplicity I shall restrict the attention to one case, namely to minimal factorization of non-negative rational matrix functions. The results I shall mention are not new and concern mainly symmetrisable operators acting on finite dimensional spaces. They serve as a sample of several recent developments in which symmetrisable operators play an important role (cf., [5,6,7,8,11,14]).

2. Definition and first remarks

Throughout this section X is a complex Hilbert space with inner product (.,.) and $H : X \to X$ is a bounded selfadjoint operator on X. Following [16], Section 12 of Chapter 9, a bounded linear operator $K : X \to X$ is called *symmetrisable relative to the operator* H if HK is selfadjoint, or, equivalently, if

(1) $$HK = K^*H.$$

With the selfadjoint operator H a so-called indefinite inner product is associated (see [4] for the terminology), namely the form $[x,y] = (Hx,y)$. In terms of this indefinite inner product formula (1) can be rewritten as

(2) $$[Kx,y] = [x,Ky].$$

Thus K is symmetrisable relative to H if and only if K is selfadjoint on X endowed with the indefinite inner product [.,.]. The latter property is summarized as K is H-selfadjoint.

Let L be the kernel of H, and put $M = L^\perp$ $(= \overline{\mathrm{Im}\, H})$. The fact that K is symmetrisable relative to H implies that L is invariant under K. Thus with respect to the decomposition $X = L \oplus M$ we can write

(3) $$K = \begin{pmatrix} K_L & C \\ 0 & K_M \end{pmatrix} : L \oplus M \to L \oplus M.$$

Let H_M be the restriction of H to M. Since K is symmetrisable relative to H, the operator K_M is symmetrisable relative to H_M.

Conversely, if the operator $K_M : M \to M$ is symmetrisable relative to

H_M, then for any choice of the operators K_L and C the operator K defined by (3) will be symmetrisable relative to H. It follows that from the property of symmetrisability one cannot deduce any information concerning the operators K_L and C.

For this reason we add to the definition of symmetrisability that the *self-adjoint operator* H *is injective*. In case X is finite dimensional this implies that H is invertible.

3. Symmetrisable operators on finite dimensional spaces

When is an operator $K : X \to X$ symmetrisable? In the finite dimensional case this question can be answered completely. First of all, if X is finite dimensional, then K is symmetrisable if and only if $HK = K^*H$ for some invertible selfadjoint operator H. Thus if K is symmetrisable, then K is similar to K^*. This condition is not only necessary, but also sufficient.

To see this, let us consider some simple examples of symmetrisable operators on $\mathbb{C}^\alpha$. (In what follows an $\alpha \times \alpha$ matrix is identified with the operator induced by its action on the standard basis of $\mathbb{C}^\alpha$.) Let $J_\alpha(\lambda_0)$ be a Jordan block of order α with single eigenvalue λ_0, and let P_α be the $\alpha \times \alpha$ permutation matrix defined by $P_\alpha = [\delta_{i,\alpha-j+1}]_{i,j=1}^{\alpha}$. If λ_0 is real, then $J_\alpha(\lambda_0)$ is symmetrisable relative to P_α and to $-P_\alpha$. If λ_0 is not real, then the $2\alpha \times 2\alpha$ Jordan form

$$J_\alpha(\lambda_0,\overline{\lambda}_0) = \begin{bmatrix} J_\alpha(\lambda_0) & 0 \\ 0 & J_\alpha(\overline{\lambda}_0) \end{bmatrix}$$

is symmetrisable relative to the selfadjoint operator $P_{2\alpha}$.

Now assume that $K : \mathbb{C}^n \to \mathbb{C}^n$ is similar to its own adjoint K^*. Then the eigenvalues of K are symmetric with respect to the real axis and the partial multiplicities of a given eigenvalue λ_0 of K are equal to those of $\overline{\lambda}_0$. It follows that the Jordan normal form J of K is a direct sum of Jordan blocks $J_\alpha(\lambda_0)$ with real eigenvalues and of blocks $J_\alpha(\lambda_0,\overline{\lambda}_0)$ with non-real λ_0. Let $P_{\varepsilon,J}$ be a corresponding direct sum consisting of signed matrices P_α, one of appropriate size for each real Jordan block in J, and of blocks $P_{2\alpha}$ for each block $J_\alpha(\lambda_0,\overline{\lambda}_0)$. The sub-index ε denotes the ordered set of signs of the blocks P_α corresponding to the real eigenvalues. The operator

$P_{\varepsilon,J}$ constructed in this way is invertible and selfadjoint, and J is symmetrisable relative to $P_{\varepsilon,J}$. Let S be a similarity between K and its Jordan normal form J, i.e., $K = S^{-1}JS$, and put $H = S^*P_{\varepsilon J}S$. Then H is an invertible, selfadjoint matrix and K is symmetrisable relative to H. Thus in the finite dimensional case K is symmetrisable if and only if K is similar to its adjoint.

The remark of the previous paragraph is the beginning of a much deeper result which states that for a finite dimensional operator K any H for which K is symmetrisable may be obtained in the way described above. The following theorem holds.

THEOREM 1. *Let the operator* $K : \mathbb{C}^n \to \mathbb{C}^n$ *be symmetrisable relative to* H, *and let* J *be a Jordan normal form of* K. *Then there exists a unique ordered set of signs* ε *and an invertible operator* S *such that*

$$K = S^{-1}JS, \qquad H = S^*P_{\varepsilon,J}S.$$

The ordered set of signs ε is unique up to permutations within subsets of ε corresponding to Jordan blocks with the same eigenvalue and of the same size. The set ε is called the K-*sign characteristic* of H. Theorem 1 and its proof appear in [6] (see also [7]), but in various forms the result has been known for many years.

The sign characteristic serves as a complete set of invariants in the following sense (see [8]):

THEOREM 2. *For* $i = 1,2$ *let the operator* $K_i : \mathbb{C}^n \to \mathbb{C}^n$ *be symmetrisable relative to* H_i. *Then there exists an invertible operator* S *such that*

$$K_1 = S^{-1}K_2S, \qquad H_1 = S^*H_2S,$$

if and only if K_1 *and* K_2 *have the same Jordan normal form and the* K_1*-sign characteristic of* H_1 *is equal to the* K_2*-sign characteristic of* H_2.

The structure theorems for symmetrisable operators stated above are most useful. For example, they can be employed to construct invariant subspaces of a symmetrisable operator K which are positive, negative or neutral with respect to H. As an example I mention the paper [15], where under certain conditions on the K-sign characteristic of H a full description is given of all K-invariant subspaces of $\mathbb{C}^n$ that are H-neutral and of maximal dimension. Recall that a subspace M of $\mathbb{C}^n$ is called H-*neutral* if $(Hx,x) = 0$ for each $x \in M$.

THEOREM 3. *Let the operator* $K : \mathbb{C}^n \to \mathbb{C}^n$ *be symmetrisable relative to* H, *and assume that the signs corresponding to the Jordan blocks of* K *with the same real eigenvalue are of equal type (i.e., either all equal to* 1 *or all equal to* -1*). Put*

$$\nu = \sum_{i=1}^{r} [\tfrac{1}{2}m_i] + \tfrac{1}{2}(n - \sum_{i=1}^{r} m_i),$$

where $m_1, \ldots, m_r$ *are the sizes of the Jordan blocks of* K *with real eigenvalues and* $[\alpha]$ *is the integer part of the positive number* α. *Then* ν *is the maximal possible dimension of a* K*-invariant* H*-neutral subspace, and if* M_+ *is the spectral subspace of* K *corresponding to the eigenvalues in the open upper halfplane, then the map*

$$N \longmapsto M_+ \cap N$$

establishes a one-one correpondence between all K*-invariant* H*-neutral subspaces* N *of maximal dimension and the* K*-invariant subspaces of* M_+.

4. Minimal factorizations

The notion of minimal factorization arises in mathematical systems theory. In so-called state space representation a (time-invariant linear dynamical) system is given by two equations:

$$(1) \qquad \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

Here $u(t)$ represents the input at time t and $y(t)$ is the output at time t. The functions $u(t)$ and $y(t)$ are vector functions with values in $\mathbb{C}^m$, say. The function $x(t)$ represents the internal state of the system and takes values in $\mathbb{C}^n$, where n may be larger than m. The coefficients A,B,C and D are matrices of sizes corresponding to the lengths of the vectors $u(t)$, $x(t)$ and $y(t)$.

If the system is at rest at $t = 0$, then after Laplace transformation the system may be written as

$$(2)\qquad \begin{cases} \lambda\hat{x}(\lambda) = A\hat{x}(\lambda) + B\hat{u}(\lambda), \\ \hat{y}(\lambda) = C\hat{x}(\lambda) + D\hat{u}(\lambda). \end{cases}$$

Hence the relation between input and output is given by

$$\hat{y}(\lambda) = [D + C(\lambda I_n - A)^{-1}B]\hat{u}(\lambda) .$$

Here I_n denotes the $n \times n$ identity matrix. The function

$$(3)\qquad W(\lambda) = D + C(\lambda I_n - A)^{-1}B$$

is called the *transfer function* of the system $\theta = [A,B,C,D]$.

The transfer function of the cascade connection (or series connection) of two systems is equal to the matrix product of the corresponding transfer functions. Thus factorizations of transfer functions are important, because they can be used to construct a system with a prescribed transfer function from simpler elements. In general one is only interested in factorizations in which no pole-zero cancellation occurs. To define these so-called minimal factorizations we need the concept of degree for a transfer function.

Since a transfer function $W(\lambda)$ is analytic at infinity, its Laurent expansion at infinity is of the following form:

$$W(\lambda) = D + \frac{1}{\lambda}D_1 + \frac{1}{\lambda^2}D_2 + \dots .$$

Consider the $p \times p$ block Hankel matrix

$$T_p = \left[D_{i+j-1}\right]_{i,j=1}^{p} .$$

The rank of T_p is independent of p for p sufficiently large. By definition the *McMillan degree* of W is the number

$$\delta(W) = \sup_{p\geq 1} \operatorname{rank} T_p \quad (< +\infty).$$

The McMillan degree satisfies a certain sub-logarithmic property, that is, if

$$(4) \qquad W(\lambda) = W_1(\lambda)W_2(\lambda)$$

is a product of transfer functions, then $\delta(W) \le \delta(W_1) + \delta(W_2)$. The factorization (4) is called *minimal* if $\delta(W_1) = \delta(W_1) + \delta(W_2)$. The study of minimal factorizations is the main topic of [2,3].

To find the minimal factorizations of a given transfer function we return to the expression (3). From (3) it is clear that a transfer function is a rational matrix-valued function which is analytic at infinity. From mathematical systems theory it is well-known that the converse is also true. This is the so-called realization problem. Thus any rational matrix function $W(\lambda)$ which is analytic at infinity, can be written in the form (3). The (right hand side of the) identity (3) is called a *realization* of W.

In general, for a given rational matrix function many different realizations are possible. We call a realization *minimal* if the dimension of the state space (that is the number n in (3)) is as small as possible. Alternatively, the realization (3) is minimal if and only if

$$(5) \qquad \bigcap_{\nu=0}^{\infty} \operatorname{Ker} CA^{\nu} = (0), \qquad \operatorname{span} \bigcup_{\nu=0}^{\infty} A^{\nu}B = \mathbb{C}^n.$$

One can prove that in that case n is precisely equal to the McMillan degree of W.

If (5) holds, then the system $\theta = [A,B,C,D]$ is called *minimal*. If two minimal systems $\theta_i = [A_i,B_i,C_i,D_i]$, $i = 1,2$, have the same transfer function, then $D_1 = D_2$ and there exists a unique invertible operator S such that

$$(6) \qquad A_1 = S^{-1}A_2S, \qquad B_1 = S^{-1}B_2, \qquad C_1 = C_2S.$$

In that case the two systems are said to be *similar* and S is called a *system similarity*. Of course, conversely, if (6) holds and $D_1 = D_2$, then the systems $[A_1,B_1,C_1,D_1]$ and $[A_2,B_2,C_2,D_2]$ have the same transfer function.

<u>THEOREM 1</u>. *Let* $W(\lambda) = I_m + C(\lambda I_n - A)^{-1}B$ *be a minimal realization, and let* Π *be a projection of* $\mathbb{C}^n$ *such that*

(7) $\quad A[\mathrm{Ker}\ \Pi] \subset \mathrm{Ker}\ \Pi, \quad (A - BC)[\mathrm{Im}\ \Pi] \subset \mathrm{Im}\ \Pi.$

Consider the functions

(8a) $\quad W_1(\lambda) = I_m + C(\lambda I_n - A)^{-1}(I_n - \Pi)B,$

(8b) $\quad W_2(\lambda) = I_m + C\Pi(\lambda I_n - A)^{-1}B.$

Then $W(\lambda) = W_1(\lambda)W_2(\lambda)$ *is a minimal factorization of* W *and any minimal factorization of* W *may be obtained in this way.*

A projection Π of $\mathbb{C}^n$ with the properties (7) is called a *supporting projection* of the system $[A,B,C,I_m]$. If in Theorem 1 we consider only factors that have the value I_m at infinity, then there is a one-one correspondence between the supporting projections of the minimal system $[A,B,C,I_m]$ and the minimal factorizations of its transfer function. For the proof of Theorem 1 and more information concerning minimal factorization we refer to [2].

The operator A - BC appearing in the second part of (7) is called the *associate operator* of the system $[A,B,C,I_m]$ and will be denoted by $A^\times$.

5. Non-negative rational matrices

In this section we shall use basic information about symmetrisable operators to construct minimal factorizations of a non-negative rational matrix function $V(\lambda)$. Recall that a rational $m \times m$ matrix function V is said to be *non-negative* if

$$0 \le \langle V(\lambda)y,y\rangle \le +\infty \qquad (y \in \mathbb{C}^m,\ \lambda \in \mathbb{R}).$$

A method to obtain minimal factorizations of the form $V(\lambda) = L_*(\lambda)L(\lambda)$ will be explained. Here $L_*(\lambda) = L(\bar{\lambda})^*$.

We start with a selfadjoint rational matrix function W and assume that for W the following minimal realization is given:

(1) $\quad W(\lambda) = I_m + C(\lambda I_n - A)^{-1}B.$

First let us see how symmetrisability comes to play a role. The fact that $W(\lambda)$ is selfadjoint means that $W_*(\lambda) = W(\lambda)$. So for W we also have the following realization

(2) $$W(\lambda) = I_m + B^*(\lambda I_n - A^*)^{-1}C^*.$$

Since the realization (1) is minimal, the same is true for the realization (2), and hence we have two minimal realizations for the same function . But then there exists a unique invertible operator $H : \mathbb{C}^n \to \mathbb{C}^n$ such that

(3) $$A = H^{-1}A^*H, \quad B = H^{-1}C^*, \quad C = B^*H.$$

By taking adjoints in (3) one sees that also

(4) $$A = (H^*)^{-1}A^*H^*, \quad B = (H^*)^{-1}C^*, \quad C = B^*H^*.$$

Because of minimality the operator H is uniquely determined by the identities in (3). So from (4) we may conclude that $H = H^*$ and hence H is an invertible selfadjoint operator. Now the first identity in (3) tells us that A is symmetrisable relative to H and from all three identities in (3) it is clear that the same is true for the associate operator $A^\times = A - BC$. So both A and $A^\times$ are symmetrisable relative to H.

Next we consider a minimal factorization

(5) $$W(\lambda) = K(\lambda)L(\lambda)$$

with factors that have the value I_m at infinity. There exists a unique supporting projection Π of the minimal system $[A,B,C,I_m]$ which yields the factorization (5) (in the sense of Theorem 1 of the previous section). Since $W(\lambda)$ is selfadjoint, we have $W(\lambda) = L_*(\lambda)K_*(\lambda)$. Again this is a minimal factorization of W with factors that have the value I_m at infinity, and it can be shown that the corresponding supporting projection of the system $[A,B,C,I_m]$ is equal to $H^{-1}(I_n - \Pi^*)H$.

For a non-negative function W we are interested in minimal factorizations of the form (5) with $K = L_*$. From the remarks made in the previous paragraph it is clear that in (5) we have $K = L_*$ if and only if for the corresponding supporting projection the following identity holds:

$$(6) \qquad H\Pi H^{-1} = I_n - \Pi^*.$$

Formula (6) is equivalent to the statement that both Ker Π and Im Π are H-neutral subspaces of maximal dimension, which in this case is equal to $\frac{1}{2}n$.

Now assume that $W(\lambda)$ is non-negative. Then the sizes of the Jordan blocks of A and $A^\times$ corresponding to real eigenvalues are even and both the A-sign characteristic and the $A^\times$-sign characteristic of H consist of the integers +1 only. But then we can apply Theorem 3 in Section 2 to find all H-neutral subspaces of dimension $\frac{1}{2}n$ that are invariant under A and all H-neutral subspaces of dimension $\frac{1}{2}n$ that are invariant under $A^\times$. Let M and $M^\times$ be two such subspaces, where M is invariant under A and $M^\times$ is invariant under $A^\times$. Then automatically M and $M^\times$ form a direct sum decomposition of $\mathbb{C}^n$, i.e.,

$$\mathbb{C}^n = M \oplus M^\times,$$

and hence the projection Π of $\mathbb{C}^n$ along M onto $M^\times$ is a supporting projection. So for a non-negative function W (which has the value I_m at infinity) we have sketched a method to construct all minimal factorizations of the form $L_*(\lambda)L(\lambda)$ with $L(\infty) = I_m$. By using this method a theorem of the following type may be derived.

THEOREM 1. *Let* V *be a non-negative rational matrix function, and assume that* $\det V(\lambda)$ *does not vanish identically. Let* σ *be a set of poles and* τ *be a set of zeros of* V *which are maximal with respect to inclusion and do not contain real numbers or pairs of complex conjugate numbers. Then* V *admits a minimal factorization*

$$V(\lambda) = L_*(\lambda)L(\lambda)$$

such that the set of non-real poles of L *is* σ *and the set of non-real zeros of* L *is* τ.

To prove the above theorem, choose $\lambda_0 \in \mathbb{R}$ such that λ_0 is neither a pole nor a zero of V. Put

$$W(\lambda) = V(\lambda_0)^{-\frac{1}{2}} V(\lambda_0 + \frac{1}{\lambda}) V(\lambda_0)^{-\frac{1}{2}}.$$

Then W is a non-negative rational matrix function which is analytic at infinity and has the value I_m at infinity. Next choose a minimal system $[A,B,C,I_m]$ such that the representation (1) holds. But then we can apply the method outlined above to get the desired result. See [13,14] for the full proof and further details.

In the present paper only one example is given of the use of symmetrisable operators in factorization problems. Many others, involving symmetrisable operators acting on finite and infinite dimensional spaces, could be added. See, e.g., [10,9,12], all of which concern infinite dimensional problems, and [5,6,8], which deal mainly with finite dimensional symmetrisable operators. Further, the use of symmetrisable operators is not restricted to factorization problems, but there are several other branches of analysis and its applications where symmetrisable operators play an important role (see, e.g., [1] and [11, 15]).

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ORTHOMORPHISMS AND THE RADON-NIKODYM THEOREM REVISITED

W.A.J. Luxemburg*

1. Introduction

It is with the greatest pleasure that I present this article honoring my former teacher and friend Professor A.C. Zaanen.

I met Professor Zaanen for the first time in 1951. At that time we were both members of the Mathematics Department of the T.H. in Delft, all be it, in totally different capacities. A year or so later, it was my priviledge to become Zaanen's assistant. One of my first tasks was to help with the proof reading of the galleys of Zaanen's first major treatise entitled "Linear Analysis". I do not recall now whether I found many misprints, but I do remember vividly how much functional analysis I learned in the process. Despite the fact that I divided my time between finding misprints and learning about Banach spaces and integral equations, I must have done a reasonable job, because as a reward I could keep a copy of the last corrected proof sheets. To put them to good use, I made three hard covered books out of them by binding together each of the three parts that make up the book. I hasten to add, however, that when "Linear Analysis" finally appeared Professor Zaanen presented me, to my pleasant surprise, with an autographed copy with an inscription thanking me for my help. These books are now a treasured possession. The proofsheet copies are still frequently consulted and borrowed by my colleagues and students. The autographed copy I keep for myself in a safe place.

But "Linear Analysis" did more for me, it aroused my interest in functional analysis, and so I became Zaanen's first Ph.D. student. This association grew into a long lasting collaboration which produced numerous

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joint papers and the book on Riesz spaces. Later this year we may celebrate the thirtieth anniversary of our collaboration and friendship and I am looking forward for many more years of the same.

I may be permitted to use this forum to express my sincere feeling of respect and gratitude towards my former teacher. I thank him for the countless hours he spent in reading, improving and rewriting of my mathematical messages. From this exchange of ideas I learned, at a very early stage of my career, what mathematical research is all about.

Clearly, Professor Zaanen's mathematical contributions have been imminent. This is also reflected in the mathematical contributions of his many students. This booklet is further testimony of the influence of Zaanen's work on the development of functional analysis. During all these years, Zaanen established his own school of thought in functional analysis with its center of gravity in Leiden and with its representatives abroad.

I very much hope that my former teacher will accept this contribution as a token of my respect and gratitude and as a reminder of our many common interests.

2. "Examples of orthomorphisms" revisited

When I started to think seriously what to contribute I was drawn inevitably to the mathematical work of Zaanen for a point of contact. Soon I found one in a paper listed as number sixty entitled "Examples of Orthomorphisms".

In this paper the theory of f-algebras is firmly and elegantly established on the theory of orthomorphisms. We remind the reader that orthomorphisms are order bounded linear transformation with a strong local behavior in that they preserve disjointness. As is characteristic of Zaanen's work the results are illustrated with a number of very important worked-out examples. An important group among them are the f-algebras of orthomorphisms of $C(K)$-type spaces. This brought again to my mind the example of the abstract L-space whose elements can be written as the difference of two positive harmonic functions in the unit circle whose f-algebra of orthomorphisms was still to be determined. These considerations led to the following observations.

The reader will understand that in an article of this size and purpose there is little room for a detailed explanation of the terminology, notation and the various results that are used. Two good sources of information are the book on Riesz Spaces referred to above and my Arkansas Lecture Notes entitled "Some Aspects of the Theory of Riesz Spaces".

Let $(L, \|.\|)$ be an *abstract* L-*space* in the sense of Kakutani. This means that $(L, \|.\|)$ is a Banach lattice whose norm is additive on the cone L^+ of the non-negative elements of L. It is important to keep in mind that we assume that L is norm-complete. From this assumption it follows that the norm is order continuous and L is Dedekind complete. The linear functional $e(f) := \|f^+\| - \|f^-\|$, $f \in L$, is order continuous and strictly positive. The latter implies in particular, that L is also order separable.

The Banach dual space $(M, \|.\|) := (L, \|.\|)^*$ is an *abstract* M-*space* in the sense of Kakutani. Its norm has the dual property $\|\sup(f,g)\| = \sup(\|f\|, \|g\|)$ on the cone M^+ of the non-negative elements of M. The strictly positive linear functional e defined above is a strong order unit of M of norm one. From the order continuity of the norm of L it follows that $M = L_n^\sim$, the Riesz space of all order continuous linear functionals of L. Furthermore, $(L_n^\sim)_n^\sim = M_n^\sim = L$, that is, L is *perfect*. $M_n^\sim \neq M^*$, the dual of M, except when L is finite dimensional.

For many spaces of type M the f-algebras of its orthomorphisms are identified in Zaanen's paper referred to in the title of the section. It is a natural question to ask what can be shown about the f-algebras of orthomorphisms of abstract L-spaces? Since such spaces, as defined here, are Banach lattices, it is well-known that their f-algebras of orthomorphisms coincide with their centers. The center $Z(L)$ of an abstract L-space consists of the order ideal generated by the identity operator in the algebra of all bounded linear operators of L into itself. I remind the reader that for an abstract L-space bounded linear transformations are order bounded. Thus $\mathrm{Orth}(L)$, the f-algebra of the orthomorphisms of L, is itself an abstract M-space. Hence, the question arises whether $\mathrm{Orth}(L)$ and the dual M of L are somehow related?

As may be glanced from Zaanen's paper such is definitely the case if L is an L^1-space of integrable functions.

By using Kakutani's representation theory of abstract L- and M-spaces it is easy to conclude that M and Orth(L) are the same. However, as Zaanen and I have shown at a number of occasions, a direct analysis, bypassing representations, often reveals more about the situation, and this case, I believe, is no exception.

For a better understanding of the theorem of this section let me briefly recall the process of defining an operation of multiplication in an abstract M-space with a strong order unit such as the dual M of L .

If a and b are components of the strong order unit e of M , then we define $ab := \inf(a,b)$; analogous with the case of characteristic functions of sets. If $f := \Sigma_1^n f_i a_i$ and $g := \Sigma_1^m g_j b_j$ are two step elements of M in canonical form, i.e., the real numbers $f_1,\dots,f_n$, $g_1,\dots,g_m$ are all non-zero and $\{a_1,\dots,a_n\}$ and $\{b_1,\dots,b_m\}$ are two systems of mutually disjoint components of e , then we define $fg := \Sigma_{i,j} f_i g_j (a_i b_j)$. An application of Freudenthal's spectral theorem finishes the process.

On the basis of these preliminaries we are now in a position to prove the following theorem.

THEOREM 2.1. *If* $(L,\|.\|)$ *is an abstract* L*-space in the sense of Kakutani, then* Orth(L) , *the* f*-algebra of its orthomorphisms, is isomorphic with* M , *the dual space of* L .

PROOF. The main ingredient of the proof is the result that the adjoint of an orthomorphism is an orthomorphism. This fact enables us to assign to each $T \in \mathrm{Orth}(L)$, the uniquely determined bounded linear functional $\tau(T) := T^*e$ of L , where $e \in M$ is the strong order unit of M and T^* is the adjoint of T . If T is the identity operator E of L ,then obviously $\tau(E) = e$. It follows readily that if $T = P$ is a band projection operator of L , then $\tau(P)$ is a component of e .

The mapping τ is obviously linear and positive. To show that the mapping is one-to-one, assume that $T_1,T_2 \in \mathrm{Orth}(L)$ and $\tau(T_1) = \tau(T_2)$.

The latter condition means that the two orthomorphisms T_1^* and T_2^* take on the same value on the complete element e (i.e., $\{e\}^d = \{0\}$) and from a well-known property of orthomorphisms it follows that $T_1^* = T_2^*$; and finally $T_1 = T_2$. We shall now show that τ is onto. To this end, observe that if $\phi \in M$, then there exists a unique orthomorphism $S \in \mathrm{Orth}(M)$ such that $\phi = Se$. The adjoint S^* of S is an orthomorphism of $M^* = L^{**}$. Using the fact that L is perfect and that orthomorphisms are order continuous it follows immediately that S^* leaves $L \subset L^{**}$ invariant. From this we may conclude that if $T \in \mathrm{Orth}(L)$ denotes the restriction of S^* to L , then $T^* = S$ and $\tau(T) = T^*e = Se = \phi$. In addition we may conclude that the inverse mapping of τ is positive and, hence τ is a Riesz homomorphism of $\mathrm{Orth}(L)$ onto M .

Finally, observe that since the τ-image of a band-projection of L is a component of e it follows readily that τ preserves multiplication as well. We shall leave the details to the reader to verify this statement and remark only that if P_1, P_2 are bandprojections of L , then $\tau(P_1P_2) = \inf(\tau(P_1), \tau(P_2))$ and an application of Freudenthal's spectral theorem will finish the proof.

Perhaps it may be useful to point out here that since the algebras involved are commutative, we have that if $T_1, T_2 \in \mathrm{Orth}(L)$, then $(T_1T_2)^* = T_2^*T_1^* = T_1^*T_2^*$.

From the proof it follows also that M is isomorphic with its own f-algebra of orthomorphisms. We shall list this in the following Corollary.

<u>COROLLARY 2.2</u>. *If* L *is an abstract* L*-space in the sense of Kakutani and* M *is its dual space, then the* f*-algebras* Orth(L) , Orth(M) *and* M *are all isomorphic.*

We shall now turn to the examples.

<u>EXAMPLES</u>. 1. Let L be the Riesz space of all harmonic functions u in the unit circle $\Delta = \{z : |z| < 1\}$ which can be written as the difference of two positive harmonic functions. L can be made into a normed space by means of the definition $\|u\| := \|\mu\|$, where $\|\mu\|$ denotes the total variation of the representing measure μ of u on the boundary $T = \{z : |z| = 1\}$ of Δ . Under this norm L is an abstract L-space in the

sense of Kakutani which is Riesz isomorphic to the space $M(T)$ of all bounded Borel measures on T. Since $M(T)$ is the dual of the space $C(T)$ of all real continuous functions on T, it follows from the above Theorem that $\mathrm{Orth}(L)$ is isomorphic with $C^{**}(T)$, the second dual of $C(T)$. The structure of the f-algebra $C^{**}(T)$, although complicated, has been extensively studied, notably by S. Kaplan and his school. It is not the place here to show how these results can be exploited in the theory of harmonic functions.

2. Let $\mathcal{B}$ denote a non-degenerate Boolean algebra with smallest element o and unit element E. By $m(\mathcal{B})$ we shall denote the abstract L-space of all the real finitely additive measures μ of finite total variation $|\mu|(E) < \infty$, with norm $\|\mu\| := |\mu|(E) < \infty$. As is well-known $m(\mathcal{B})$ is the dual space of the Banach lattice $\mathcal{C}(\mathcal{B})$, the uniform completion of the Riesz space of all finitely valued place functions of $\mathcal{B}$ in the sense of Carathéodory. The reader who is not familiar with the theory of place functions may consult the book by C. Caratheodory entitled "Mass und Integral und Ihre Algebraisierung" or D.H. Fremlin's book entitled "Topological Riesz Spaces and Measure Theory". If Ω denotes the Stone representation space of $\mathcal{B}$, then the finite place functions of $\mathcal{B}$ can be represented as the step functions w.r.t. the algebra of the open and closed subsets of Ω, and $C(\mathcal{B})$ can be represented by the space $C(\Omega)$ of all real continuous functions on the compact Hausdorff space Ω. The L-space $m(\mathcal{B})$ of measures on $\mathcal{B}$ is of course isomorphic to the dual space $C^*(\Omega)$ of $C(\Omega)$ of all regular Borel measures on Ω. From the Theorem of this section it follows then readily that $\mathrm{Orth}(m(\mathcal{B}))$ is isomorphic to the second dual space $C^{**}(\Omega)$ of $C(\Omega)$.

For later use the following remark concerning $m(\mathcal{B})$ is inserted here. If the Boolean algebra $\mathcal{B}$ is a subalgebra of a Boolean algebra $\mathcal{A}$, then an element $\mu \in m(\mathcal{B})$ is called *$\mathcal{A}$-countably additive ($\mathcal{A}$-completely additive)* whenever for every decreasing sequence $\{A_n\}$, $n=1,2,\dots$, of elements of $\mathcal{B}$ (for every downward directed system $\{A_\alpha\}$, $\alpha \in \{\alpha\}$, of elements of $\mathcal{B}$) satisfying $\inf_n A_n = 0$ ($\inf_\alpha A_\alpha = 0$), where the "inf" symbol refers to the Boolean operations in $\mathcal{A}$ ($A \in \mathcal{A}$ and $A \leq A_n$ for all n implies $A = 0$, similarly for the directed system), we have $\inf_n |\mu|(A_n) = 0$ ($\inf_\alpha |\mu|(A_\alpha) = 0$).

In the case $A = B$, the A-countably additive measures on B are the *σ-order continuous measures* and the A-completely additive measures on B are the *order continuous measures (normal measures)* respectively. With this definition we shall show that for each non-degenerate Boolean algebra B there exists a Boolean algebra A containing B as a sub-algebra with the property that every $\mu \in m(B)$ is A-completely additive. Indeed, we may take for A the Boolean algebra of all subsets of the Stone representation space Ω of B , since in that case if $\{A_\alpha\}$, $\alpha \in \{\alpha\}$, is a downwards directed set of open and closed subsets of Ω satisfying $\inf_\alpha A_\alpha = 0$ in A , i.e., $\bigcap_\alpha A_\alpha = \emptyset$, then, by the compactness of Ω , we have that for some $\alpha \in \{\alpha\}$, $A_\alpha = \emptyset$. Hence, with this choice of A all the measures on B are A-completely additive in a trivial way.

We may use this observation, however, to conclude that each element $\mu \in m(B)$ may be extended uniquely, by the Caratheodory extension procedure, to a countably additive measure whose domain of definition contains the smallest σ-algebra, say Λ, generated by the open and closed subsets of Ω .

3. The Radon-Nikodym theorem revisited

In this section we shall show that the Theorem of Section 1 may also be used to prove a general Radon-Nikodym type theorem for general measures.

It is well-known that Riesz space type techniques are effective tools in dealing with Radon-Nikodym type theorems. In our book on Riesz spaces one may find such an elegant proof of the classical Radon-Nikodym theorem for σ-finite countably additive positive measures. In two very interesting papers (numbers 36 and 37) Zaanen deals with the non-σ-finite case. Here we shall not deal with this aspect of the theorem, but rather indicate what form the Radon-Nikodym Theorem takes for measures which are not necessarily countably additive.

The setting we shall need for this purpose is that defined in the example 2 of the preceding section.

We recall that if $\mu,\nu \in m(B)$ we say that ν is *absolutely continuous* with respect to μ , and we write $\nu \ll \mu$, whenever for each $\varepsilon > 0$ such that for all $A \in B$ satisfying $|\mu|(A) < \delta$ we have $|\nu|(A) < \varepsilon$.

This notion of absolute continuity, which originated with Lebesgue, can be characterized in terms of the Riesz space structure of $m(\mathcal{B})$ in the following well-known manner.

LEMMA 3.1. *If* $\mu,\nu \in m(\mathcal{B})$, *then* ν *is* μ*-absolutely continuous if and only if* ν *is in the band* P_μ *generated by* μ *in* $m(\mathcal{B})$.

PROOF. We may assume without loss of generality that μ and ν are both positive. Assume first that $\nu \in P_\mu$. Then $\nu = \sup_n \nu_n$, where $\nu_n = \inf(\nu, n\mu)$ $(n=1,2,\ldots)$. Since for each $n = 1,2,\ldots$, the measure ν_n is μ-absolutely continuous and $\|\nu_n - \nu\| \to 0$ as $n \to \infty$ we obtain that $\nu \ll \mu$.

Conversely, assume that ν is μ-absolutely continuous. Then ν may be decomposed in the form $\nu = \nu_1 + \nu_2$ with $\nu_1 \in P_\mu$ and $\nu_2 \in P_\mu^d$, the disjoint complement of P_μ in $m(\mathcal{B})$. Since both ν_1 and ν are μ-absolutely continuous it follows that ν_2 is μ-absolutely continuous as well. But $\inf(\nu_2,\mu) = 0$ implies that $\inf(\nu_2(E \setminus A)+\mu(A):A\in\mathcal{B}) = 0$. Hence, there exists a sequence $\{A_n\}$, $n=1,2,\ldots$, of $\mathcal{B}$ such that $\mu(A_n)$ tends to zero as $n \to \infty$ and $\nu_2(E\setminus A_n)$ tends to zero as $n \to \infty$. Since ν is μ-absolutely continuous it follows that $\nu_2(A_n)$ tends to zero as $n \to \infty$ as well. Hence, for all $n = 1,2,\ldots$, $\nu_2(E) = \nu_2(E\setminus A_n) + \nu_2(A_n)$ implies $\nu_2(E) = 0$, i.e., $\nu \in P_\mu$ and the proof is finished.

The reader may ask whether the elements of the ideal generated by μ can be characterized in a similar way? There is a way to do this. The reader may judge for himself whether it is the kind of answer he may have been looking for. The following condition is analogous to that for functions satisfying a Lipschitz condition.

LEMMA 3.2. *If* $\mu,\nu \in m(\mathcal{B})$, *then* $\nu \in I_\mu$, *the ideal generated by* μ *in* $m(\mathcal{B})$, *if and only if for each* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that for all finite systems* $\{A_1,\ldots,A_n\}$ *of elements of* $\mathcal{B}$ *satisfying* $\Sigma_1^n|\mu|(A_i) < \delta$ *we have* $\Sigma_1^n|\nu|(A_i) < \varepsilon$.

This result is more a curiosity and left to the reader as an exercise.

If $f \in C(\mathcal{B})$ and $\mu \in m(\mathcal{B})$, then it is obvious that the measure $\nu := f.\mu$ is μ-absolutely continuous. Conversely, however, not every μ-absolutely continuous measure is of this form. To see this the following additional notation is introduced. If $A \in \mathcal{B}$, then we shall denote by P_A the band projection of $m(\mathcal{B})$ defined as follows: $(P_A\mu)(\mathcal{B}) := \mu(\inf(A,B))$ for all $\mu \in m(\mathcal{B})$ and for all $B \in \mathcal{B}$. The measure $P_A\mu$, being the restriction of μ to the principal ideal of $\mathcal{B}$ generated by A in $\mathcal{B}$, is a special component of μ . If now $\phi \in C^{**}(\mathcal{B})$, then the measure $\nu := \phi.\mu$ defined by the formula for all $A \in \mathcal{B}$, $\nu(A) := \phi(P_A\mu) = \langle P_A\mu,\phi\rangle$ is μ-absolutely continuous. In fact, $\phi.\mu$ is even contained in the ideal I_μ generated by μ , and so satisfies a Lipschitz condition w.r.t. μ . To see this observe that for all $A \in \mathcal{B}$ we have $|\nu(A)| \leq \|\phi\|.|\mu|(A)$, where $\|\phi\|$ denotes the norm of the functional ϕ .

It is not without interest that the following converse holds.

<u>THEOREM 3.1</u>. *A measure* $\nu \in m(\mathcal{B})$ *is contained in the ideal* I_μ *of a measure* $\mu \in m(\mathcal{B})$ *if and only if there exists a bounded linear functional* $\phi \in C^{**}(\mathcal{B})$ *such that* $\nu = \phi.\mu$.

<u>PROOF</u>. We have already shown that if ν is of the form $\phi.\mu$, $\phi \in C^{**}(\mathcal{B})$, then $\nu \in I_\mu$. To prove the converse, we may assume, without loss of generality, that $0 \leq \nu \leq \mu$. Then, by Freudenthal's spectral theorem, we have that $\nu = \int_0^1 t dP_t\mu$, where $P_t\mu$ is the spectral system of ν w.r.t. μ , i.e., $P_t\mu$ is the projection of μ on the band generated by $(t\mu-\nu)^+$. The system P_t $(0 \leq t \leq 1)$ of band projections of $m(\mathcal{B})$ is a spectral system and well of the orthomorphism $T := \int_0^1 t dP_t \in \mathrm{Orth}(m(\mathcal{B}))$. Hence, by the representation theorem of the preceding section, there exists a unique positive linear functional ϕ on $m(\mathcal{B})$, i.e., $0 \leq \phi \in C^{**}(\mathcal{B})$, such that $T^*e = \phi$. Hence, for all $A \in \mathcal{B}$, we have $\nu(A) = \langle P_AT\mu,e\rangle = \langle TP_A\mu,e\rangle = \langle P_A\mu,T^*e\rangle = \langle P_A\mu,\phi\rangle =$ $= (\phi.\mu)(A)$; and the proof is finished.

A number of remarks are in order. From the definition of ϕ we observe immediately that $\phi \in ((I_\mu)^d)^\perp$ in $C^{**}(\mathcal{B})$, i.e. ϕ vanishes on all the measures in $m(\mathcal{B})$ disjoint from μ . In that sense ϕ is of course uniquely determined.

At the end of the preceding section we have shown that every $\mu \in m(\mathcal{B})$ uniquely extends to a countably-additive measure μ defined on the σ-algebra Λ generated by the open and closed subsets of the Stone representation space Ω of $\mathcal{B}$. If $0 \leq \nu \leq \mu$, then this relation is preserved under the Carathéodory extension process. Hence, by the classical Radon-Nikodym theorem for finite countably additive measures, there exists a non-negative Λ-measurable function f on Ω such that $0 \leq f \leq 1$ and $\nu = f.\mu$, i.e., $\nu(A) = \int_A f d\mu$ for all $A \in \Lambda$. The main purpose of the above theorem is to provide a representation of ν in terms of μ without referring directly to the Stone representation of $\mathcal{B}$.

It is not without interest to ask when ϕ may be represented by an element of $C(\mathcal{B})$. For this to happen it is obviously necessary and sufficient that ν must have the property that all the components $P_t\mu$, where P_t is the projection on the band generated by $(t\mu-\nu)^+$, are of the form $P_A\mu$ for some $A \in \mathcal{B}$. This in turn is equivalent to the statement that the orthomorphism $T = \int_0^1 t dP_t$ is $\sigma(m(\mathcal{B}),C(\mathcal{B}))$-continuous. Unfortunately, I do not know of any interesting intrinsic characterization of those components of a measure μ that are of the form $P_A\mu$. From the results of R.R. Phelps concerning subreflexivity contained in his paper "Some subreflexive Banach spaces", Arch. der Math. 10, p. 162-169 (1959), one may conclude that the following result holds.

<u>THEOREM 3.2</u>. *If* $0 \leq \nu$, $\mu \in m(\mathcal{B})$, *then* ν *is a component of* μ *of the form* $P_A\mu$ *for some* $A \in \mathcal{B}$ *if and only if the measure* $\mu - 2\nu$ *considered as a linear functional on* $C(\mathcal{B})$ *attains its maximum on the unit ball of* $C(\mathcal{B})$ *and* ν *is a component of* μ .

We shall only present a sketch of the proof. If ν is a component of μ , then $\mu - \nu = (\mu-2\nu)^+$ and $\nu = (\mu-2\nu)^-$. Now, if $f \in C(\mathcal{B})$ satisfies $\|\mu-2\nu\| = f.(\mu-2\nu)$, then $|\mu-2\nu| = f.(\mu-2\nu)$ and $\int (1-|f|)d|\mu-2\nu| = 0$. So except for a $|\mu-2\nu|$ -null set the place values of f are $+1$ or -1 . From the theory of place functions it follows now that there exists an element $A \in \mathcal{B}$ such that A contains the spectral set $E\backslash\{f<1\}$ and its complement $B = E\backslash A$ contains the set $\{f\leq -1\}$. Then the complementary pair A,B of elements of $\mathcal{B}$ determines a Hahn-decomposition of the measure $\mu - 2\nu$, and so $\nu = P_A\mu$.

We may draw the attention of the reader to one special case namely if $0 \leq \mu \in m(\mathcal{B})$ is totally additive or normal on $\mathcal{B}$ in the sense of the definition contained in Example 2 of Section 2. In that case, the set where μ vanishes is a complete ideal and so a principal ideal. Its disjoint complement being a principal ideal generated by say $A \in \mathcal{B}$ may be called the carrier or support of μ, because it has the property: $\mu = P_A\mu$. Since the component of a normal measure μ is normal it is of the the form $P_B\mu$ for some $B \in \mathcal{B}$. Hence, for normal measures one has the result. If μ is normal and $0 \leq \nu \leq \mu$, then there exists an element $f \in C(\mathcal{B})$ such that $\nu = f.\mu$. For further details concerning normal measures the reader should consult the author's paper "On the existence of σ-complete prime ideals in Boolean algebras" Colloq. Math. 19, p. 51-58(1968).

If $0 \leq \nu$ is μ-absolutely continuous, then it follows from Theorem 3.1 that $\nu \in P_\mu$, the band generated by the positive measure μ. Hence, in that case, $\nu = \sup_n \inf(\nu, n\mu)$. From this observation the following general Radon-Nikodym type result follows easily.

THEOREM 3.3. *If* $\nu,\mu \in m(\mathcal{B})$, *then* ν *is* μ*-absolutely continuous if and only if there exists a sequence of bounded linear functionals* $\phi_n \in C^{**}(\mathcal{B})$ $(n=1,2,\ldots)$ *such that for all* $A \in \mathcal{B}$ *we have* $\nu(A) = \lim_{n\to\infty} \phi_n(P_A\mu)$.

If ν and μ are positive, then there exists an increasing sequence $\{\phi_n\}$ of positive linear functionals of $C^{**}(\mathcal{B})$ with the property of the theorem. The reader who is familiar with the notion of the extended order dual introduced in my joint paper with J.J. Masterson, Canad. J. of Math. 19, p. 488-498(1967), will immediately recognize the following formulation of the preceding theorem.

THEOREM 3.4. *If* $0 \leq \nu$, $\mu \in m(\mathcal{B})$, *then* ν *is* μ*-absolutely continuous if and only if there exists an extended positive order continuous linear functional* $\phi \in \Gamma(m(\mathcal{B}))$ *whose order dense domain contains* μ *such that* $\nu(A) = \phi(P_A\mu)$ *for all* $A \in \mathcal{B}$.

If $\nu,\mu \in m(\mathcal{B})$ and if $1 < p < \infty$, then ν is called of *finite* p-*variation* with respect to μ, whenever

$$\|\nu\|_{p,\mu} = \sup\left(\Sigma_i \left|\frac{\nu(A_i)}{\mu(A_i)}\right|^p . |\mu(A_i)|\right)^{\frac{1}{p}} < \infty ,$$

where the sup is taken over all finite systems $\{A_1,\ldots,A_n\}$ of mutually disjoint elements of $\mathcal{B}$. Finite 1-variation means finite total variation and $\nu \in I_\mu$, the ideal generated by μ , corresponds to $p = \infty$.

If $\|\nu\|_{p,\mu} < \infty$, then $\nu << \mu$, and so, by the preceding Theorem, $\nu = \phi.\mu$ with $\phi \in \Gamma(m(\mathcal{B}))$, and ϕ may be represented by an element of $L^p(\Omega,\Lambda,\mu)$.

A similar characterization for absolutely continuous functions with L^p-derivatives is due to F. Riesz.

Finally, we remark that, by approximation, the following result is immediate. If $\nu,\mu \in m(\mathcal{B})$, then ν is μ-absolutely continuous if and only if for each $\varepsilon > 0$ there exists a finitely valued place function f such that $|\nu-f.\mu| < \varepsilon$.

ON THE RADON-NIKODYM THEOREM

P. Maritz

1. Introduction

I am much obliged to the organizers of this symposium for their kindness in inviting me, especially since this is a most suitable occasion to pay tribute to prof. dr. A.C. Zaanen, under whose guidance some of us had, and others still have, the opportunity to do some research. During a conversation in 1974, prof. Zaanen indicated to me that the notion of a *direct sum* measure is slightly stronger than the notion of a *localizable* measure. That remark had had the effect that I became more acquainted with certain aspects of Radon-Nikodým (R-N) theory, and that in turn led to some positive results at that time.

In this paper we mention a few aspects of the Radon-Nikodým theorem and the Radon-Nikodým property (RNP). It must, however, be emphasized that this is neither a comprehensive discussion nor a survey of RNP. The Radon-Nikodým theorem (sometimes called the Lebesgue-Nikodým theorem) was proved first by Lebesgue in 1904 in terms of point functions on the real line, then by Radon in 1913 for Borel measures in $\mathbb{R}^n$, and by Nikodým in 1930 in the general form.

2. Radon-Nikodým for Banach spaces

The standard reference for measure theoretic properties is [6].

T is a non-empty set on which no topological structure is required, X a Banach space and X* its topological dual. If A is a ring of subsets of T and F: $A \to X$ a measure of finite variation $|F|$ on A, then $|F|$ can be extended to a measure $|F|^*$ on the σ-algebra $P(|F|)$ of all $|F|$-measurable sets. Then $\Sigma(|F|)$ is the δ-ring of all $|F|$-integrable sets. $|F| = |F|^*|\Sigma(|F|)$. F on A can be extended to a measure, again denoted by F, on $\Sigma(|F|)$; this F is also of finite variation. If A is a σ-algebra and $|F|$ is complete on $\mathcal{A}$, then $\mathcal{A} = \Sigma(|F|) = \mathcal{P}(|F|)$.

Let $\mathcal{A}$ now be a σ-algebra and consider the finite positive measure space $(T,\mathcal{A},\mu)$. If $1 \leq p < \infty$, the symbol $L^p(T,\mathcal{A},\mu,X)$ will stand for all μ-Bochner integrable functions $f : T \to X$ with the usual norm.

2.1 DEFINITIONS.

(1) X has RNP *with respect to* $(T,\mathcal{A},\mu)$ if for each measure $F : \mathcal{A} \to X$, where $F \ll \mu$ and F is of finite variation , there exists a Bochner integrable $f : T \to X$ such that $F(E) = (\text{Bochner})\text{-} \int_E f\, d\mu$ for every $E \in \mathcal{A}$.

(2) X has RNP if X has RNP with respect to *every* finite measure space.

(3) A bounded linear operator $\Phi: L^1(T,\mathcal{A},\mu,\mathbb{R}) \to X$ is Riesz representable if there exists $g \in L^\infty(T,\mathcal{A},\mu,X)$ such that $\Phi f = \int_T fg\, d\mu$ for all $f \in L^1(T,\mathcal{A},\mu,\mathbb{R})$.

It is known that X has RNP with respect to $(T,\mathcal{A},\mu)$ if and only if each $\Phi \in \mathcal{L}(L^1(T,\mathcal{A},\mu,\mathbb{R});X)$ is Riesz representable. See [5], p. 63.

2.2 EXAMPLES ([5],p.60,61).

(1) The failure of the R-N theorem for a c_0-valued measure.

(2) The failure of the Riesz representation theorem for an operator $\Phi: L^1[0,1] \to c_0$.

(3) The failure of the R-N theorem for an $L^1(T,\mathcal{A},\mu,\mathbb{R})$-valued measure, with μ *non-atomic* on $\mathcal{A}$. See also 4.3(1) and 5.4(1).

(4) The failure of the Riesz representation theorem for the identity operator on $L^1(T,\mathcal{A},\mu,\mathbb{R})$.

The relationships between examples (1) and (2), and between (3) and (4) are by no means accidental.

The space ℓ_1 has RNP.This follows from the fact that ℓ_1 has a *boundedly complete* Schauder basis. The following theorem of Dunford-Morse is a corollary to a theorem in [11].

2.3 THEOREM. *If* X *has a boundedly complete Schauder basis* (x_n), *then* X *has* RNP.

Consequently, neither c_0 nor $L^1(T,\mathcal{A},\mu,\mathbb{R})$ has a boundedly complete Schauder basis. The space ℓ_1 plays an important role in the study of RNP.

2.4 THEOREM ([10]). *X has RNP if and only if every $\Phi \in L(L^1(T,A,\mu,\mathbb{R});X)$ factors through ℓ_1.*

In spite of theorem 2.4 above, it is not true that every space with RNP contains a copy of ℓ_1. An example of such a space is ℓ_2, which by theorem 2.3, has RNP. The notion of WRNP is useful in this respect.

2.5 DEFINITION. X has the weak RNP (WRNP) if for each (T,A,μ) and each $F: A \to X$ with $F \ll \mu$, F of finite variation, there exists a function $f: T \to X$ such that $F(E) = (\text{Pettis}) - \int_E f\, d\mu$, for all $E \in A$.

2.6 THEOREM ([18]). *If X* has WRNP, then X does not contain any isomorphic copy of ℓ_1.*

The classical results pertaining to the RNP are the following.

2.7 THEOREMS. *(1).*([7]) *Separable dual spaces have* RNP; *(2).* ([20]) *Reflexive Banach spaces have* RNP.

Grothendieck ([8]) gave a number of new directions to modern functional analysis; the theorems above were useful in doing so.

There is a close connection between separable dual spaces and reflexive spaces on the one hand, and RNP on the other.

2.8 THEOREM ([25]). *If every separable closed linear subspace of X is isomorphic to a subspace of a separable dual space, then X has RNP.*

2.9 THEOREM ([23],[25]). *X* has RNP if and only if every separable subspace of X has a separable dual.*

An example of a separable Banach space with RNP which is not isometric (but is isomorphic) to a subspace of a separable dual, is given in [16]. An example of a separable Banach space X with RNP that is not isomorphic to a subspace of a separable dual space, is given in [17]. In [9] James and Ho introduced the notion of asymptotic-norming properties (ANP). It is shown that ANP is satisfied by a larger class of Banach spaces than those that are isomorphic to subspaces of separable duals, and that ANP is a sufficient condition for RNP. To show that ANP is more general than "isomorphism with a subspace of a separable dual", James and Ho show that there is a separable Banach space that has ANP and that is not isomorphic to any subspace of a separable dual.

3. Geometrical properties

Rieffel's ([21]) "dentable theorem" led Maynard ([15]) to the first internal characterization of the RNP for a Banach space X.

3.1 THEOREM ([5] , [15]). *The following statements are equivalent:*
(1) Every bounded subset of X *is dentable.*
(2) Every bounded subset of X *is* σ*-dentable.*
(3) X *has* RNP.
If we add "closed convex" to (1) and (2) above, we have, from [26]*:*
(4) Every bounded closed convex subset of X *is weak dentable.*
Then (1) $\Rightarrow$ *(2)* $\Rightarrow$ *(3)* $\Rightarrow$ *(4)* $\Rightarrow$ *(1).*

A connection between extreme point structure and RNP is suggested by theorem 3.1.

3.2 DEFINITIONS.
(1) X is said to have Krein-Milman property (KMP) if every bounded closed convex $A \subset X$ has at least one extreme point.
(2) X is said to have the strong KMP (SKMP) if for every bounded $A \subset X$, $\overline{co}$ (A) has an extreme point in the strong closure of A.
(3) X is said to have the weak KMP (WKMP) if for every bounded $A \subset X$, $\overline{co}$ (A) has an extreme point in the weak closure of A.

3.3 THEOREM ([26]). RNP, SKMP *and* WKMP *are equivalent for Banach spaces.*

It is known that RNP $\Rightarrow$ KMP ([19]), but it is not known whether the converse is true. However, from [5], if X* has RNP, then X* has KMP, and conversely.

An interesting result is the following, with special reference to example 5.4(2).

3.4 THEOREM ([5]). *A Banach space* X *lacks* RNP *if and only if there is a bounded open convex set* B *in* X *and a norm closed subset* A *of* B *such that the closed convex hull of* A *coincides with* $\bar{B}$.

In a somewhat different setting, but still in connection with KMP:

3.5 THEOREM ([2]). *(1) A* C*-*algebra* X *is scattered if, and only if, its dual* X* *has* RNP.
(2) If M *is a Von Neumann algebra, then its predual* M_* *has* RNP *if, and only if,* M *is a direct sum of type* I *factors.*

The proof of (2) above only requires the KMP of M_* and thus: RNP ⇔ KMP in the predual of a Von Neumann algebra.

4. Ranges of measures

A. Zonoids

A *zonoid* is the range $R(F)$ of a non-atomic vector measure $F: \Sigma(|F|) \to X$. The theorem of Lyapunov (1940) and his counterexample (1946) on zonoids are well known. The following theorem was originally formulated by Uhl for X a *reflexive* or a *separable dual* space. Let A be a σ-algebra and $F: A \to X$ of finite variation.

4.1 THEOREM ([24]). *Suppose* X *has* RNP.
(1) Then $R(F)$ *is precompact in the norm topology of* X.
(2) If F *is non-atomic, then* $\overline{R(F)}$ *is convex and norm compact. (Weak Lyapunov theorem for the strong topology.)*

4.2 COROLLARY. *Under the hypotheses of 4.1:*
(1) If $R(F)$ *is closed, then it is norm compact.*
(2) Let F *be non-atomic. If* $R(F)$ *is closed, then it is norm compact and convex.*

4.3 EXAMPLES.
(1).([24]). See also 2.2(3). *A non-atomic vector measure* F *of finite variation with* $R(F)$ *closed but non-convex and non-compact.* Let $T=[0,1]$, A be the Lebesgue σ-algebra of subsets of T and λ the Lebesgue measure on A. Define $F: A \to L^1(T,A,\lambda,\mathbb{R})$ by $F(E) = \chi_E$. Then F is a non-atomic measure of finite variation $|F|$ on A and $|F| = \lambda$ on A. But $R(F)$ is not precompact and $R(F)$ is non-convex. (In Uhl's original formulation of the theorem above, this example provides another proof of the fact that the separable space $L^1(T,A,\lambda,\mathbb{R})$ is not a dual space. Of course,

this space does not have RNP.)
(2) For Lyapunov's 1946-example of an ℓ_2*-valued non-atomic vector measure of finite variation whose range is non-convex,* see [5] or [24].
(3) For a c_0-valued measure whose range is weakly compact and convex, see [5].

It is interesting to note that theorem 4.1 is a direct consequence of theorem 2 in [4].

B. Average ranges

4.4 DEFINITION. Let $\mathcal{A}^+ = \{A \in \mathcal{A} \mid \mu(A) > 0\}$. The *average range* of $F: \mathcal{A} \to X$ over $A \in \mathcal{A}^+$ with respect to μ is

$$R_A(F) = \left\{ \frac{F(B)}{\mu(B)} \mid B \subset A,\ B \in \mathcal{A}^+ \right\}.$$

The results below are examples of the usage of average ranges; the terminology is the usual.

4.5 THEOREM ([15]). *X has RNP if and only if $R_A(F)$ is σ-dentable (or, if and only if $R_A(F)$ is locally relatively weakly compact) for all $A \in \mathcal{A}^+$.*

4.6 THEOREM ([5]). *Let $F: \mathcal{A} \to X$ and let $F \ll \mu$. If $R_A(F)$ is relatively norm (weakly) compact for all $A \in \mathcal{A}^+$, then there exists a μ-measurable Pettis integrable $g: T \to X$ such that*

$$F(E) = (\text{Pettis})\text{-}\int_E g \, d\mu$$

for all $E \in \mathcal{A}$. If F is of finite variation (as usual), then g is also Bochner integrable.

Theorem 4.1(2) can be written in the following form:

4.7 THEOREM ([22]). *Let X be a quasi-complete locally convex Hausdorff space in which every bounded subset is metrizable. Let X have RNP, let $F: \mathcal{A} \to X$ be such that $F \ll \mu$, let μ be non-atomic and suppose that $R_A(F)$ is bounded. Then $\overline{R(F)}$ is convex and compact.*

5. Multifunctions

The bilinear integral is in the sense of Dinculeanu [6]. Let Y and Z be Banach spaces and consider a bilinear transformation $(x,y) \to xy$ on $X \times Y$ into Z such that $||(x,y)|| \leq ||x||.||y||$. For the terminology concerning multifunctions, see [12] . If $\Lambda : T \to Y$ is a multifunction, we have

$$\int_A \Lambda(t)dF = \left\{ \int_A f(t)dF \,|\, f \text{ is an F-integrable selector of } \Lambda \right\} \subset Z,$$

where $A \in \mathcal{P}(|F|)$.

5.1 THEOREM ([12]). *Let* $\Lambda: T \to Y$ *be a multifunction,* $A \in \Sigma(|F|)$ *and* $F : \Sigma(|m|) \to X$ *non-atomic. If* Z *is finite dimensional, then* $\int_A \Lambda(t)dF$ *is convex.*

If $|F|$ has the *direct sum* property ([6], p. 179), then $|F|$ is *localizable,* see [27], p. 180. If T is a countable union of sets of $\Sigma(|F|)$, then F has the direct sum property.

5.2 THEOREM ([13]). *If* $F : \mathcal{A} \to \mathbb{R}^p$ *has the direct sum property and* $\Lambda: T \to \mathbb{R}^n$ *is an integrably bounded point-compact convex* $|F|$*-measurable multifunction, then* $\int_A \Lambda(t)dF$ *is a convex and compact subset of* $\mathbb{R}^{np}$ *for every* $A \in \mathcal{P}(|F|)$.

If $\Lambda: T \to Y$ is a multifunction, let $(\text{ext } \Lambda)(t) = \{y \in \Lambda(t) \,|\, y \text{ is an extreme point of } \Lambda(t)\}$.

The following theorem is based on a *generalized* version of the Radon-Nikodým theorem ([6]) and also on some results in [12] .

5.3 THEOREM ([12]). *Let* T *be a countable union of sets of* $\mathcal{A}$, $\Lambda:T \to \mathbb{R}^n$ *integrably bounded, point-compact convex and* $|F|$*-measurable and let* $F: \Sigma(|F|) \to \mathbb{R}^p$ *be non-atomic. Then*

$$\int_A \Lambda(t)dF = \int_A (\text{ext } \Lambda)(t)dF,$$

for every $A \in \Sigma(|F|)$.

In dealing with ext Λ, we make freely use of the fact that a non-empty compact subset of a locally convex linear Hausdorff space has extreme points, and also of the Krein-Milman theorem.

5.4 EXAMPLES ([13],[14]).

(1) We refer to examples 2.2(3) and 4.3(1), which we use as the basis of some of our examples. Let T, $\mathcal{A}$ and F be as in 4.3(1) and put Z = $L^1(T,\mathcal{A},\mu,\mathbb{R})$.

(a) Theorem 5.1 fails if Z is infinite dimensional: define $\Lambda: T \to \mathbb{R}$ by $\Lambda(t) = \{0,1\}$ for all $t \in T$. Then $\int \Lambda(t)d(F) = \mathcal{R}(F)$, which is not convex.

(b) Theorem 5.2 fails if Z is infinite dimensional: define $\Lambda: T \to \mathbb{R}$ by $\Lambda(t) = [0,1]$ for all $t \in T$. Then F has the direct sum property and Λ satisfies all the necessary conditions. Since $\mathcal{R}(F) \subset \int \Lambda(t)dF$, it follows that $\int \Lambda(t)dF$ is not compact.

(2) Theorem 5.3 fails if Λ is not point-compact. We refer to examples 2.2(1) and 4.3(3) - the space c_0. The closed unit ball A of c_0 is non-compact and convex. Let T = [0,1], Σ be the Lebesgue σ-algebra of subsets of T and F the Lebesgue measure on T. Consider $c_0 = \mathcal{L}(\mathbb{R}, c_0)$. Define $\Lambda : T \to c_0$ by $\Lambda(t) = A$ for all $t \in T$. Then Λ is clearly $|F|$-measurable and integrably bounded. Since ext A = $\emptyset$, it follows that $\int (\text{ext } \Lambda)(t)dF = \emptyset \neq \int \Lambda(t)dF$.

6. Miscellaneous

(1) Let $(T,\mathcal{A},\mu)$ be a non-atomic measure space and E a Dedekind complete Banach lattice. In [3] De Jonge describes all E-valued measures that have R-N-derivate, for all E not containing c_0. The result is that although $L^1(T,\mathcal{A},\mu,E)$ does not have RNP, all the L^1-valued measures that have R-N derivative,are described.

(2) For the X of theorem 4.7, we have from [22]: if C is a closed bounded convex subset of X, then the following are equivalent: (a) C has RNP; (b) C is subset dentable; (c) C is subset σ-dentable. (Compare with theorem 3.1)

(3) Andrews and Uhl ([1]) developed an easy technique (originally developed by Figiel) by means of which weakly compact subsets of $L^\infty(T,\mathcal{A},\mu,\mathbb{R})$ can be studied, where $(T,\mathcal{A},\mu)$ is a finite measure space.

(4) Thus far, we have been dealing with a finite measure space. For σ-finite and non-σ-finite cases, we refer to Zaanen [27],where both the integral and measure versions of the R-N theorem (for Stieltjes-

Lebesgue integrals) are given. It is shown that if the measure μ has the finite subset property, then μ is localizable if and only if μ has the R-N property (for measures). This result is then extended to establish the equivalence between localizability and the R-N property for μ (without the finite subset property). On p. 180 in [27], we have the result that if μ has the direct sum property, then μ is localizable. *Professor Zaanen, thank you very much for that remark in 1974!*

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DUALITY IN THE THEORY OF BANACH LATTICES

B. de Pagter*

The concept of duality plays an important rôle in the theory of normed linear spaces, especially the embedding of a normed linear space in its second dual and the reflexivity of such a space. In their series of papers "Notes on Banach function spaces" W.A.J. Luxemburg and A.C. Zaanen have discussed these and related problems for normed Riesz space extensively. It is the purpose of the present note to give a survey of their contributions to this part of the theory of normed Riesz spaces. In the discussion which follows we shall restrict ourselves to Banach lattices only, although many of the results do have analogues for normed Riesz spaces.

As is the case in many parts of the theory of Banach lattices, most of the theorems concerning duals and biduals of Banach lattices have their origin in the theory of Banach function spaces. We shall discuss, therefore, the situation for these spaces first.

We assume that (X,Λ,μ) is a σ-finite measure space and we denote by $M(X,\mu)$ the collection of all μ-measureable functions which are finite almost everywhere. As usual, we identify functions which differ only on a set of measure zero. We shall restrict ourselves here to real-valued functions, although most of the theory is also valid for the complex case. Let ρ be a *function norm* on $M(X,\mu)$, i.e. ρ is a function defined on $M(X,\mu)$ with values in $\mathbb{R}^+ \cup \{\infty\}$ with the properties: (i) $\rho(f) = 0$ iff $f = 0$, $\rho(\alpha f) = |\alpha|\rho(f)$ $(\alpha \in \mathbb{R})$ and $\rho(f+g) \leq \rho(f) + \rho(g)$; (ii) $\rho(f) \leq \rho(g)$ whenever $|f| \leq |g|$. If we denote by L_ρ the

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collection of all $f \in M(X,\mu)$ for which $\rho(f) < \infty$, then L_ρ is a normed Riesz space with norm ρ. The space L_ρ is Banach iff ρ satisfies the *Riesz-Fischer condition*, i.e., $\Sigma_{n=1}^{\infty} \rho(u_n) < \infty$ $(0 \leq u_n \in M(X,\mu))$ implies that $\rho(\Sigma_{n=1}^{\infty} u_n) < \infty$. If L_ρ is Banach, then L_ρ is called a *Banach function space*. It should be pointed out that in the first papers on general Banach function spaces (e.g. [2],[3] and [4]) it was assumed that the norm ρ has some additional properties. In particular it was assumed that ρ has the *Fatou property*, i.e., if $0 \leq u_n \uparrow u$ in $M(X,\mu)$, then $\rho(u_n) \uparrow \rho(u)$. The general definition of a Banach function space as given above first appeared in [1], Section 4.

If ρ is a function norm on $M(X,\mu)$, then the *associate function norm* is defined by

$$\rho'(g) = \sup\left\{\int_X |fg|\,d\mu : f \in L_\rho, \rho(f) \leq 1\right\} .$$

The associate function norm ρ' always has the Fatou property. The collection of all $g \in M(X,\mu)$ for which $\rho'(g) < \infty$ is denoted by L'_ρ and is called the *associate space* of L_ρ. Since the Fatou property implies the Riesz-Fischer property, L'_ρ is a Banach function space. The associate function norm of ρ' is called the second associate function norm of ρ, denoted by ρ''. Since $\rho'' \leq \rho$, the inclusion $L_\rho \subset L''_\rho$ always holds. In fact, $\rho'' = \rho$ holds iff ρ has the Fatou property, a result proved by W.A.J. Luxemburg in his thesis ([4]), and also proved independently by G.G. Lorentz.

It is possible that $L_\rho = L''_\rho$, although $\rho \neq \rho''$. If $L_\rho = L''_\rho$, then L_ρ is called *perfect*. The space is perfect iff ρ has the *weak Fatou property*, i.e., $0 \leq u_n \uparrow u$ and $\sup \rho(u_n) < \infty$ in $M(X,\mu)$ implies $\rho(u) < \infty$. The notion of perfectness for Banach function spaces has been introduced by G.G. Lorentz and G.D. Wertheim ([3]).

Considering the Banach function space as a Banach lattice, there is a nice characterization of the associate space. In fact L'_ρ can be identified with the space of normal integrals (order continuous linear functionals) on L_ρ, i.e., ϕ is a linear functional on L_ρ with the property that $\phi(u_\alpha) \downarrow 0$ whenever $u_\alpha \downarrow 0$ in L_ρ iff there exists a function g in L'_ρ such that $\phi(f) = \int_X fg\,d\mu$ for all functions f in L_ρ. In other words $L'_\rho = L^*_{\rho,n}$. Similarly, the second associate space can be identified with $(L^*_{\rho,n})^*_n$. Since L_ρ and $L^*_{\rho,n}$ are Banach lattices, we have $(L^*_{\rho,n})^*_n = (L^\sim_{\rho,n})^\sim_n$, where $L^\sim$ denotes the order dual of the Riesz space L .

These last remarks show us that it is of some interest to study the relation between L and $(L^\sim_n)^\sim_n$ for an arbitrary Riesz space L . Suppose L is a Riesz space and ${}^\perp(L^\sim_n) = \{0\}$ (i.e., the normal integrals on L separate the points of L). Each element $f \in L$ can be considered as a linear functional f'' on $L^\sim_n$ by defining $f''(\phi) = \phi(f)$ for all $\phi \in L^\sim_n$. The mapping which assigns to each f the functional f'' is an embedding of L in $(L^\sim_n)^\sim_n$, which is investigated in [1], VIII, Section 28. In this situation L is always a Riesz subspace of $(L^\sim_n)^\sim_n$, whereas L is an ideal in $(L^\sim_n)^\sim_n$ iff L is Dedekind complete. If L is not Dedekind complete, then the ideal generated by L in $(L^\sim_n)^\sim_n$ is the Dedekind completion of L , which is proved in [1], X, Theorem 32.8.

Analogous to the situation for Banach function spaces, the Riesz space L is called *perfect* if $L = (L^\sim_n)^\sim_n$, and the characterization of perfect Riesz spaces is in a certain sense similar to the characterization of the perfect Banach function spaces. Indeed, L is perfect iff ${}^\perp(L^\sim_n) = \{0\}$ and if $0 \leq u_\tau \uparrow$ in L such that $\sup \phi(u_\tau) < \infty$ for all $0 \leq \phi \in L^\sim_n$ then $u_\tau \uparrow u$ for some $u \in L$. This shows in particular that $L^\sim$ and $L^\sim_n$ are perfect for any Riesz space L .

The situation becomes more interesting if we assume that L_ρ is a Banach lattice such that ${}^\perp(L^*_{\rho,n}) = \{0\}$. Again we consider the embedding of L_ρ as a Riesz subspace in $(L^*_{\rho,n})^*_n$. The norms in $L^*_{\rho,n}$ and $(L^*_{\rho,n})^*_n$ are denoted by ρ' and ρ'' respectively. The restriction of ρ'' to L_ρ (considered as a subspace of $(L^*_{\rho,n})^*_n$) is a norm on L_ρ , given by $\rho''(f) = \sup\{|\phi(f)| : 0 \leq \phi \in L^*_{\rho,n}, \rho'(u) \leq 1\}$. We always have $\rho'' \leq \rho$, but in general ρ and ρ'' are not even equivalent. By way of example, define for any real sequence $f = (f(1), f(2), \ldots)$ the function norm ρ by

$$\rho(f) = \rho_\infty(f) + \sup_k \left\{ k \limsup_{n \in N_k} |f(n)| \right\} ,$$

where $\rho_\infty(f) = \sup |f(n)|$ and N_k are countable, mutually disjoint subsets of $\mathbb{N}$ such that $\bigcup_{k=1}^\infty N_k = \mathbb{N}$. Let L_ρ be the corresponding Banach function space. Then $L^*_{\rho,n} = L'_\rho = \ell_1$ (and ρ' is the ℓ_1-norm), $(L^*_{\rho,n})^*_n = L''_\rho = \ell_\infty$ and $\rho'' = \rho_\infty$. Clearly L_ρ is properly contained in ℓ_∞ and ρ is not equivalent with ρ_∞ .

If L_ρ is a Banach lattice such that ${}^\perp(L^*_{\rho,n}) = \{0\}$, then ρ'' is a norm in L_ρ which is Fatou (i.e. $0 \leqq u_\tau \uparrow u$ implies that $\rho''(u_\tau) \uparrow \rho''(u)$), since ρ'' is the supremum of an upwards directed set of seminorms which are all Fatou. Indeed, $\rho'' = \sup\{\rho_\phi : 0 \leqq \phi \in L^*_{\rho,n}, \rho^*(\phi) \leqq 1\}$, where ρ_ϕ is defined by $\rho_\phi(f) = \phi(|f|)$. This implies that if ρ and ρ'' are equivalent, then ρ must be *weakly Fatou* (i.e., there exists a constant $k(\rho) \geqq 1$ such that $0 \leqq u_\tau \uparrow u$ implies that $\rho(u) \leqq k(\rho).\sup \rho(u_\tau)$). If L_ρ is in addition Dedekind complete, then ρ and ρ'' are equivalent iff ρ is weakly Fatou, and in this situation $\rho'' \leqq \rho \leqq k^2(\rho)\rho''$ holds. This is essentially proved in [1], XIII, Lemma 41.1. In particualar we see that $\rho = \rho''$ holds iff ρ is Fatou (in this case $k(\rho) = 1$).

If $\rho = \rho''$ holds on L_ρ , then it is *not* necessarily true that $L_\rho = (L^*_{\rho,n})^*_n$, i.e., L_ρ is not necessarily perfect. As an example, let L_ρ be the space (c_0) of all sequences converging to zero with the supremum norm. Then ρ is Fatou, but $(L^*_{\rho,n})^*_n = \ell_\infty$. Observe that if L_ρ is perfect, then ρ and ρ'' are two norms on L such that L is a Banach lattice with respect ot both norms. Therefore ρ and ρ'' are equivalent in this case, so ρ must be at least weakly Fatou. Perfect Banach lattices are completely characterized in [1], XIII, Theorem 41.1, where it is shown that L_ρ is perfect iff ${}^\perp(L^*_{\rho,n}) = \{0\}$ and L_ρ has the *weak Fatou property for directed sets* (i.e., if $0 \leqq u_\tau \uparrow$ and $\sup \rho(u_\tau) < \infty$ in L_ρ , then there exists $u \in L_\rho$ such that $u_\tau \uparrow u$). If L_ρ is a Banach lattice which has the weak Fatou property for directed sets, then L_ρ is Dedekind complete and ρ is weakly Fatou. However, if L_ρ is perfect, then we do not have $\rho = \rho''$ in general. By way of example, define for any sequence $f = (f(1),f(2),\dots)$ the function norm

$$\rho(f) = \rho_\infty(f) + \limsup |f(n)| \ ,$$

and let L_ρ be the corresponding Banach function space. Then $L_\rho = L''_\rho = \ell_\infty$, so L_ρ is perfect, but $\rho \neq \rho_\infty = \rho''$. It follows from the above that the Banach lattice L_ρ is perfect *and* $\rho = \rho''$ iff ${}^\perp(L^*_{\rho,n}) = \{0\}$ and L_ρ has the *Fatou property for directed sets* (i.e., if $0 \leqq u_\tau \uparrow$ and $\sup \rho(u_\tau) < \infty$ in L_ρ , then there exists $u \in L_\rho$ such that $u_\tau \uparrow u$ and $\rho(u_\tau) \uparrow \rho(u)$). In this situation we say that ρ is a *Fatou norm.*

The Banach dual L_ρ^* of any Banach lattice L_ρ is perfect ([1], XIII, Theorem 40.2), since L_ρ^* has the weak Fatou property for directed sets. Furthermore ρ^* is Fatou. Indeed, for any $0 \leq \phi \in L_\rho^*$ we have

$$\rho^*(\phi) = \sup\left\{\phi(u) : 0 \leq u \in L_\rho, \rho(u) \leq 1\right\} \leq$$
$$\leq \sup\left\{\Phi(\phi) : 0 \leq \Phi \in (L_{\rho,n}^*)_n^*, \rho^{**}(\Phi) \leq 1\right\} \leq \rho^*(\phi) ,$$

so $\rho^* = \sup\{\rho_\Phi^* : 0 \leq \Phi \in (L_\rho^*)_n^*, \rho^{**}(\Phi) \leq 1\}$, i.e., ρ^* is the supremum of an upwards directed set of seminorms which are all Fatou. Hence ρ^* is a Fatou norm.

Since any Banach lattice L_ρ is in particular a Banach space, we can also investigate the embedding of L_ρ in its second Banach dual L_ρ^{**} by assigning to each $f \in L_\rho$ the functional f^{**} on L_ρ^* defined by $f^{**}(\phi) = \phi(f)$. By this embedding L_ρ can be considered as a Riesz subspace of L_ρ^{**} and $\rho^{**} = \rho$ on L_ρ . Since any f^{**} is a normal integral on L_ρ^* , the embedding is actually into $(L_\rho^*)_n^*$. In general this embedding does not preserve arbitrary suprema and infima. In fact, this embedding preserves suprema and infima of arbitrary sets iff the norm ρ is *order continuous* (i.e., $u_\tau \downarrow 0$ in L_ρ implies $\rho(u_\tau) \downarrow 0$), and in this situation we have $L_\rho^* = L_{\rho,n}^*$ ([1], XII, Theorem 38.3).

The question arises under what conditions L_ρ is an ideal or a band in L_ρ^{**} , and if it is possible to find necessary and sufficient conditions such that $L_\rho = L_\rho^{**}$ (i.e., such that L_ρ is reflexive). These and related problems were studied in [1], XII, Sections 38, 39 and 40.

First of all it was proved in [1], XII, Theorem 39.1 that L_ρ is an ideal in L_ρ^{**} iff L_ρ is super Dedekind complete and ρ is order continuous. This last condition, in his turn, is equivalent to the property that any order bounded increasing sequence in L_ρ has a norm limit ([1], X, Theorem 33.4). If we replace in this condition "order bounded" by the weaker condition "norm bounded" we get the property that any norm bounded increasing sequence in L_ρ has a norm limit, which is equivalent to the weak Fatou property for directed sets together with the order continuity of ρ ([1], XI, Theorem 34.2). A Banach lattice L_ρ with order continuous norm and the weak Fatou property for directed sets is sometimes called a KB-*space* (KB = Kantorovitch-Banach). It follows now easily from the above results that KB-spaces are precisely those Banach lattices L_ρ which are a band in L_ρ^{**} . Indeed, L_ρ is a band in L_ρ^{**} iff ρ is order continuous and L_ρ is perfect, i.e., iff ρ is order continuous and L_ρ has the weak Fatou property for directed sets.

A Banach lattice L_ρ is reflexive iff L_ρ is a KB-space and $L_\rho^{**} = (L_\rho^*)_n^*$. Therefore, L_ρ is reflexive iff ρ and ρ^* are both order continuous and L_ρ has the weak Fatou property for directed sets. This is Ogasawara's theorem ([5]). An elegant proof of this result has been given in [1], XIII, Theorem 40.1. Observe that reflexive Banach lattices are precisely those Banach lattices L_ρ for which both L_ρ and L_ρ^* are KB-spaces.

As we have seen above, the Banach lattice L_ρ is a band in L_ρ^{**} iff L_ρ is perfect and the norm ρ is order continuous. In this situation it is clear that there exists a positive projection P in L_ρ^{**} onto L_ρ (take for P simply the order projection onto the band L_ρ in L_ρ^{**}). For Banach lattices L_ρ with ${}^{\perp}(L_{\rho,n}^*) = \{0\}$, the existence of such a positive projection in L_ρ^{**} onto L_ρ is in fact equivalent to perfectness of L_ρ . This result shows us once again the importance of the perfectness property for Banach lattices.

The theory of Banach lattices has developed rapidly during the last 20 years. Overlooking the above mentioned results we may conclude that the contribution of Luxemburg and Zaanen to the theory of Banach lattices is of fundamental importance for the development of this theory.

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SOME RECENT RESULTS ON POSITIVE GROUPS AND SEMI-GROUPS

H.H. Schaefer

Even though ordered Banach spaces and the related operator theory can be traced back almost to the beginning of functional analysis, the modern theory of *normed Riesz spaces* (*Banach function spaces*, or *Banach lattices*) begins much later and has much of its origin in the well known series of papers by W.A.J. Luxemburg and A.C. Zaanen [6], which contain a wealth of material. Similarly, even though there are some earlier extensions to infinite dimensions of the classical Perron-Frobenius theory, a systematic study of order bounded operators - and especially of the spectral theory of positive linear operators - was not initiated until the early 1960's.

Thereafter, the latter theory progressed rather rapidly, and a survey of the results obtained can be found in [8]. The present contribution is concerned with recent applications of the spectral theory of positive operators (or equivalently, of discrete monothetic semi-groups) to the normed structures of groups of positive operators on an arbitrary Banach lattice (Section 2), and with the extension of that theory to strongly continuous one-parameter semi-groups of positive operators (Section 3). For reasons of space, only a few results from each of these areas could be selected. However, in the interest of independent readability, those results on which our applications are based or of which they are extensions, are quoted and briefly discussed in Section 1. Each section ends with a number of open problems.

1. The peripheral spectrum of positive operators

We are concerned with the spectral behavior of positive linear operators on an arbitrary complex Banach lattice and we will use [8] as a general reference. The reader who is not familiar with the notions and techniques involved may visualize the spaces C(K) (complex continuous functions) or M(K) (complex Radon measures) on a compact space K, or the spaces $L^p(X,\Sigma,\mu)$ ($1 \le p \le \infty$) as typical examples. The simplest example of all is, of course, the space $\mathbb{C}^n$ provided with its usual absolute value (modulus) and some *lattice norm* (i.e., some norm for which $|x| \le |y|$ (modulus) implies $\|x\| \le \|y\|$).

On $\mathbb{C}^n$ a positive (linear) operator is given by an $n \times n$-matrix A with non-negative entries; such a matrix is called *irreducible* (or *indecomposable*) if A leaves no non-trivial lattice ideal I of $\mathbb{C}^n$ invariant. The complex of results available for irreducible and general positive matrices is usually referred to as the *Perron-Frobenius Theory*; a systematic account from an operator theoretic point of view is given in [8], Chapter 1. Also in [8], the reader can find bibliographical sketches of the development that led to an extension of the Perron-Frobenius Theory to infinite-dimensional Banach lattices, up to the early 1970's. In the sequel we only pick out a few central results whose bearing on groups and one-parameter semi-groups will be discussed below.

Let T be a positive linear operator on a (complex) Banach lattice E. By r we will denote the spectral radius of T, and the subset $\{\lambda \in \mathbb{C} : |\lambda| = r\} \cap \sigma(T)$ of the spectrum $\sigma(T)$ will be called the *peripheral spectrum* of T. Accordingly, the *peripheral point spectrum* of T is the set of all eigenvalues of T of modulus r. The operator T is called *irreducible* if for any closed lattice ideal I of E, $T(I) \subset I$ implies $I = \{0\}$ or $I = E$. (For example, if $E = C(K)$ and φ is an ergodic flow on K, then the operator T defined by $Tf = f \circ \varphi$ ($f \in C(K)$) is irreducible on E.)

<u>1.A THEOREM</u>. *Let* T *denote a positive irreducible operator on* E, *and suppose* $r = 1$.
(i) *If the peripheral point spectrum* Π *of* T *is nonvoid and* T' *possesses a positive fixed vector, then* Π *is a subgroup of the circle group, and each of its elements has geometric multiplicity* 1.
(ii) *If* $\lambda = 1$ *is a pole of the resolvent* $R(\lambda,T) = (\lambda - T)^{-1}$, *the peripheral spectrum of* T *consists entirely of first order poles of* $R(\lambda,T)$.

While assertion (i) is comparatively easy to prove, (ii) is much harder and highly technical; in full generality (with respect to the underlying Banach lattice) (ii) was first proved by I. Sawashima and F. Niiro (see [8], V.5.). If T is an irreducible operator with spectral radius

$r > 0$, the assumption $r = 1$ is a mere normalization; however, there are examples of irreducible operators on spaces $L^1(\mu)$ that are topologically nilpotent.

What can be said about the peripheral spectrum in general? It was proved by the author [7] that the peripheral spectrum of an irreducible Markov operator on $C(K)$ (K compact) is always a subgroup of the circle group, but not much more seems to be known in the irreducible case. On the other hand, if the irreducibility assumption is dropped, the case of an $n \times n$-matrix $A \geq 0$ shows what can at best be expected, namely: If $T \geq 0$ is an operator with spectral radius $r = 1$, then the peripheral spectrum π of T is a union of subgroups of the circle group (i.e., $\alpha \in \pi \Rightarrow \alpha^n \in \pi$ for all $n \in \mathbb{Z}$). However, in infinite dimensions even this plausible assertion could so far only be proved under additional assumptions. One of these is that T be a *lattice homomorphism* (or *Riesz homomorphism*), the other consists in placing a *growth condition* on the resolvent $R(\lambda,T)$:

1.B DEFINITION. *A positive operator* T, *with spectral radius* r, *is said to satisfy* Condition (G) *if* $(\lambda - r)R(\lambda,T)$ *is bounded as* $\lambda \downarrow r$. T *is called* (G)-solvable *if there exists a chain of closed* T-*invariant ideals:* $\{0\} = E_0 \subset E_1 \subset \ldots \subset E_n = E$, *such that for each* ν $(1 \leq \nu \leq n)$ *the operator* T_ν *induced by* T *on* $E_\nu/E_{\nu-1}$ *satisfies* (G).

Let us point out briefly how this growth condition enters the picture. If $T \geq 0$, $r = 1$ and $\alpha x = Tx$ where $x \neq 0$, $|\alpha| = 1$, then obviously $|x| \leq T|x|$; what one would like to have is equality, $|x| = T|x|$, because then it follows that $\alpha^n x^{(n)} = Tx^{(n)}$ $(n \in \mathbb{Z})$ for suitably defined vectors $x^{(n)}$. (In the case of a function lattice, $x^{(n)} = |x|.(\mathrm{sign}\ x)^n$, and this can be extended without difficulty to abstract Banach lattices.) Now if T satisfies (G) $(r = 1)$ and we have $|x| \leq T|x|$, it is possible to find a closed T-invariant ideal E_1 such that $|x| - T|x| \in E_1$ but $x \notin E_1$; then $\alpha\hat{x} = \hat{T}\hat{x}$ and $|\hat{x}| = \hat{T}|\hat{x}|$ for the element $\hat{x} \in E/E_1$ corresponding to x, hence, the group $\{\alpha^n : n \in \mathbb{Z}\}$ is in the point spectrum of the operator $\hat{T}$ on E/E_1 induced by T and, consequently, in $\sigma(T)$. This method was apparently first employed by the author in 1962; later H.P. Lotz introduced the more general notion of (G)-solvability and extended the results on the peripheral point spectrum to the peripheral spectrum through the use of ultraproducts (for details, see [8], V.4-5.). On the other hand, if T is a Riesz homomorphism, then $\alpha x = Tx$ $(|\alpha| = 1)$ implies $|x| = T|x|$ without further assumptions; however, Scheffold [10] proved that in this case, for any element $\lambda = |\lambda|\gamma \in \sigma(T)$ one has $|\lambda|\gamma^n \in \sigma(T)$ for all $n \in \mathbb{Z}$ (i.e., the entire spectrum of a Riesz homomorphism is *cyclic*). We summarize:

1.C THEOREM. *Let* T *be a positive linear operator on a Banach lattice* E.
(i) *If* T *is* (G)-*solvable, the peripheral spectrum of* T *is cyclic.*
(ii) *If* T *is a Riesz homomorphism, the entire spectrum* $\sigma(T)$ *is cyclic.*

It should be pointed out that T is (G)-solvable whenever the spectral radius r of T is a pole of the resolvent;so far, this seems to be the principal application of that concept.

1.D OPEN PROBLEMS. We conclude this section with several problems which, to the best knowledge of the author, are so far unsolved.

(i) *Is the peripheral spectrum of a positive operator on any Banach lattice necessarily cyclic?*
(ii) *Is the peripheral spectrum of an irreducible positive operator on any Banach lattice necessarily of the form* rH, *where* H *is a subgroup of the circle group?*
(iii) *Can a compact irreducible positive operator be topologically nilpotent?*

As was pointed out above, the answer for (ii) is in the affirmative for Markov operators in C(K) [7]; also, (iii) has a negative answer in certain cases (e.g., E = C(K) or $E = L^1(\mu)$; for details, see [8], V.6).

2. Groups of positive operators

The purpose of this section is to discuss the application of the spectral theory of Section 1 to the structure (in the norm topology) of groups G of positive operators on arbitrary Banach lattices. As an important auxiliary concept, we need the notion of the *center* (or *ideal center*) of a complex Banach lattice E. The algebraic theory of orthomorphisms of a Riesz space has recently been discussed in [2]; in the case of a Banach lattice, the situation is much simpler. We define the *center* C(E) of a Banach lattice E as the set of all linear operators $T : E \to E$ satisfying $|Tx| \leq k|x|$ for all $x \in E$, where k is a constant depending on T. It follows that each $T \in C(E)$ is a bounded (and even order bounded) operator on E, and the best constant k in the above estimate turns out to be $\|T\|$; but actually more is true.

2.A LEMMA. *Let* E *be any (complex) Banach lattice and denote by* C(E) *the center of* E. *Then:*
(i) C(E) *is a full subalgebra of the operator algebra* $L(E)$ *and closed in the strong operator topology.*
(ii) *With respect to the norm, order, and algebraic structures induced by* $L(E)$, C(E) *is isometrically isomorphic to the Banach algebra* $C(K_E)$ *for some compact space* K_E.

Thus if $E = C(K)$ for some compact space K, K_E can be identified with K and the operators $T \in C(E)$ viewed as multiplication by continuous functions; the situation is similar for $E = L^p(\mu)$ $(1 \le p \le \infty)$, and here K_E is the Stone space of the underlying measure algebra. For our purposes, the main conclusion from Lemma 2.A is the fact that for each $T \in C(E)$, the spectrum $\sigma(T)$ of T (in $L(E)$) coincides with the spectrum of T in $C(K_E)$, i.e., with the range of the continuous function on K_E corresponding to T.

The following spectral characterization [9] of those Riesz isomorphisms of a Banach lattice E which belong to C(E), arose from the question if a Riesz homomorphism T of E satisfying $\sigma(T) = \{1\}$ is necessarily the identity map 1_E of E.

2.B THEOREM. *For any Riesz isomorphism of a complex Banach lattice* E, *the following assertions are equivalent:*
(a) $T \in C(E)$
(b) *The spectrum* $\sigma(T)$ *is contained in* $(0,+\infty)$.
In particular, if T *is a Riesz homomorphism satisfying* $\sigma(T) = \{1\}$ *then* $T = 1_E$.

The proof of the equivalence (a) $\Leftrightarrow$ (b) is far from straightforward (see [9]). However, the final assertion of 2.B is an easy consequence: A Riesz homomorphism T satisfying $\sigma(T) = \{1\}$ is invertible, hence a Riesz isomorphism, and thus we have $T \in C(E)$; from 2.A it now follows that T is the unit of C(E). Consequently, $0 \le Tx \le x$ and $0 \le T^{-1}x \le x$ for all $0 \le x \in E$ which shows that $T = 1_E$.

The proof of 2.B does not make use of the cyclicity of the spectrum of arbitrary Riesz homomorphisms established in Theorem 1.C (ii); the combination of 1.C with 2.B yields the following interesting lemma [9].

2.C LEMMA. *Let* T *be a Riesz isomorphism of* E *and let* $b^{-1} := r(T^{-1})$. *If the spectral radius of* $1_E - T$ *satisfies* $r(1_E - T) < \sqrt{1+b+b^2}$, *then* $T \in C(E)$; *in particular, we have* $T \in C(E)$ *whenever* $\| 1_E - T \| \le 1$.

In fact, if $T \notin C(E)$ then by 2.B we have $\sigma(T) \not\subset (0,+\infty)$; thus by 1.C (ii) there exists $\mu \in \sigma(T)$, $\mu \neq 0$, such that $2\pi/3 \le \arg \mu \le 4\pi/3$ whence $-1 \le \mathrm{Re}(\mu/|\mu|) \le -\frac{1}{2}$. This implies $r(1_E - T)^2 \ge |\mu - 1|^2 \ge |\mu|^2 + |\mu| + 1$. But $\sigma(T)^{-1} = \sigma(T^{-1})$ by the spectral mapping theorem, so $|\mu|^{-1} \le r(T^{-1})$ and $|\mu| \ge b$;therefore we obtain $r(1_E - T)^2 \ge b^2 + b + 1$, contradicting our hypothesis. From the preceding argument we note the following special cases: If T is a Riesz isomorphism not in C(E), then (α) $r(T^{-1}) \le 1$ implies $r(1_E - T) \ge \sqrt{3}$ and (β) $-1 \in \sigma(T)$ implies $r(1_E - T) \ge 2$.

We now consider groups G of positive operators on Banach lattices E, with 1_E the group identity and group multiplication defined by composition. These will briefly be called *positive groups*.

2.D THEOREM. *Let* E *be any Banach lattice and let* G *be a positive group on* E. *If* G *is non-central, i.e. if* $G \cap C(E) = \{1_E\}$, *then* G *is discrete in the norm topology. Moreover we have:*

(i) $1 < \|S - T\| . \min(\|S^{-1}\|, \|T^{-1}\|)$ *whenever* $S,T \in G$, $S \neq T$.

(ii) *If* G *is bounded then* $\sqrt{3} \le \|S - T\| . \min(\|S^{-1}\|, \|T^{-1}\|)$ $(S \neq T)$.

(iii) *If* G *is a torsion-free group of isometries then* $\|S - T\| = 2$ $(S \neq T)$.

To prove (i), we observe that by hypothesis $S,T \in G$, $S \neq T$ implies $S^{-1}T \notin C(E)$, hence by 2.C: $1 < \|1_E - S^{-1}T\| \le \|S^{-1}\| \, \|S - T\|$ and, similarly, $1 < \|T^{-1}\| \, \|S - T\|$. (ii) and (iii) now result in like fashion using remarks (α) and (β) above. In fact, if G is bounded then $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$ for all $T \in G$ so (α) applies. If G is torsionfree then $\sigma(T) = \{|z| = 1\}$ for each $T \in G$, $T \neq 1_E$, by 2.C (ii); moreover we have $\|T\| = 1$ in this particular case for all $T \in G$ and hence $2 \le \|S - T\| \le 2$ whenever $S \neq T$.

2.E OPEN PROBLEMS. The following unsolved problems are closely related to the material of this section. Recall that a positive operator T on a space C(K) is called a *Markov operator* if $T1 = 1$ (1 the constant-one function on K).

(i) *Let* T *be a Markov operator on* C(K) *satisfying* $\sigma(T) = \{1\}$. *Is* T *necessarily the identity map of* C(K)?

(ii) *Let* T *be a positive contraction (i.e.,* $T \geq 0$ *and* $\|T\| \leq 1$*) on a space* $L^p(X,\Sigma,\mu)$ $(1 \leq p < +\infty)$ *and satisfying* $\sigma(T) \subset \{z \in \mathbb{C} : |z| = 1\}$*. Is* T *necessarily an isometry?*

It was shown by the author that (ii) has a positive answer whenever $L^p(X,\Sigma,\mu)$ $(1 \leq p < +\infty)$ is finite dimensional. Moreover, if the answer to (ii) is positive for every space $L^1(X,\Sigma,\mu)$, then also (i) has a positive answer. For, the adjoint T' of a Markov operator T satisfying $\sigma(T) = \{1\}$ satisfies the assumption of (ii) (for $p = 1$); but if T' is an isometry, T' is easily seen to be a Riesz isomorphism. Thus so is T and we obtain $T = 1_{C(K)}$ from Theorem 2.B.

3. Positive one-parameter semi-groups

By a *one-parameter semi-group* we understand , as usual, a homomorphism $t \to T_t$ of the additive semi-group $\mathbb{R}_+$ into the operator algebra $L(E)$ of a complex Banach space E, such that $T_0 = 1_E$ and the mapping $t \to T_t$ is continuous for the strong operator topology on $L(E)$. (We refer the interested reader to the excellent recent monograph by E.B. Davies [1].) Let us recall that every such semi-group satisfies estimates

$$\|T_t\| \leq Me^{\omega t} \qquad (t \in \mathbb{R}_+)$$

for suitable constants $M \in \mathbb{R}$, $\omega \in \mathbb{R}$. The greatest lower bound for ω is given by $a := \lim_{t\to\infty} t^{-1}\log \|T_t\|$ and is often called the *abscissa of absolute convergence* (see below).

The *generator* Z of $\{T_t\}$, defined by

$$Zx = \lim_{t\downarrow 0} t^{-1}(T_t x - x),$$

is a generally unbounded closed operator Z whose dense domain $\mathcal{D}(Z)$ consists of all $x \in E$ for which the limit exists.

In the following we are interested in the case where E is a (complex) Banach lattice. A semi-group $\{T_t\}$ on E will be called *positive* if each T_t $(t \in \mathbb{R}_+)$ is a positive operator. It should be noted that the positivity of $\{T_t\}$ is, in general, in no obvious way reflected by the generator Z; for example, if $E = L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ then the Laplacian is the generator of a positive semigroup (cf. [1], p. 9). It is all the more striking that for positive semi-groups, at least under reasonable assumptions, the spectrum of Z has symmetry properties quite analogous to those of the spectrum of a positive operator T (or equivalently, of the discrete semi-group generated by T). Those properties have recently been established by G. Greiner [4], [5], following preliminary work by Derndinger [3]. It is the purpose of this section to report on some of them, in close parallel to Section 1 above.The proofs of the theorems reported are generally sophisticated, and not routine generalizations of the discrete case. However, while the known spectral properties of bounded positive operators do not enter in most of those proofs, the methods employed are widely those developed for the discrete case. It is safe to say that without the latter, the recent results for positive non-discrete semi-groups would in all likelihood not have been discovered.

Let $\{T_t\}$ be a positive semi-group on E. It is well known (cf. [1]) that the resolvent $R(\lambda,Z) := (\lambda - Z)^{-1}$ exists in an open subset D of $\mathbb{C}$ containing the right half-plane $\{\lambda \in \mathbb{C} : \text{Re}\ \lambda > a\}$, and for Re $\lambda > a$ is given by the Laplace transform

$$(*) \qquad R(\lambda,Z)x = \int_0^{+\infty} e^{-\lambda t} T_t x\,dt \qquad (x \in E);$$

hence $R(\lambda,Z)$ is a positive operator on E for all real λ, $\lambda > a$. Thus $\sigma(Z) \subset \{z : \text{Re}\ z \leq a\}$, and $\rho := \sup\{\text{Re}\ z : z \in \sigma(Z)\}$ is called the *spectral bound* of Z (we let $\rho = -\infty$ if $\sigma(Z)$ contains no finite points). In general, one has $\rho < a$; it was Greiner's observation that nevertheless, for positive $\{T_t\}$ formula (*) continues to hold for all λ, Re $\lambda > \rho$. (The integral is then to be understood as an E-valued "improper Riemann integral" not necessarily converging absolutely.) From the validity of (*) for all λ, Re $\lambda > \rho$, we conclude at once: *For every positive semi-group one has* $\rho \in \sigma(Z)$. In particular, $R(\lambda,Z) \geq 0$ whenever $\lambda > \rho$ $(\lambda \in \mathbb{R})$.

Because of the convergence properties of the Laplace integral in (*), it is natural to call the set $\sigma(Z) \cap \{\lambda : \text{Re}\ \lambda = \rho\}$ the *peripheral spectrum* of Z (or of the semi-group $\{T_t\}$) and the subset of eigenvalues of Z having real part ρ, the *peripheral point spectrum* of Z (or of the semi-group). Also, the semi-group $\{T_t\}$ is called *irreducible* if for any closed lattice ideal I of E, $T_t(I) \subset I$ (all $t \in \mathbb{R}_+$) implies $I = \{0\}$ or $I = E$. (For this it is sufficient, but not necessary, that some T_t be irreducible on E.) An example of an irreducible semi-group is the semi-group on $L^p(\mathbb{R}^n)$

($1 \le p < +\infty$) generated by the Laplacian Δ. We now have the following complete analog [5] of Theorem 1.A.

3.A THEOREM. *Let* $\{T_t\}$ *denote a positive irreducible semi-group on* E, *and suppose* $\rho = 0$.
(i) *If the peripheral point spectrum* Π *of the generator* Z *is non-void and the adjoint semi-group* $\{T_t'\}$ *possesses a positive fixed vector, then* Π *is a subgroup of the additive group* $i\mathbb{R}$, *and each of its elements has geometric multiplicity* 1.
(ii) *If* $\rho = 0$ *is a pole of the resolvent* $R(\lambda,Z) = (\lambda - Z)^{-1}$, *the peripheral spectrum of* Z *consists entirely of first order poles of* $R(\lambda,Z)$.

Concerning the proof of these results, similar observations are valid to those made following Theorem 1.A: If $\rho > -\infty$ then the assumption $\rho = 0$ is a mere normalization; if $\rho = -\infty$ the statements of the theorem lose their meaning. Moreover, as before, (ii) is much harder to prove than (i); generally the proofs do not rely on transferring the problem to one of the operators T_t but on skillfully exploiting the properties (especially positivity) of the resolvent for $\lambda \in D$, $\mathrm{Re}\,\lambda > \rho$.

What can be said about the peripheral spectrum of Z in general? Again (with the exception of semi-groups of Riesz homomorphisms), some restriction on the growth of the resolvent, this time of $R(\lambda,Z)$, appears indispensable to obtain the desired results.

3.B DEFINITION. *A positive semi-group* $\{T_t\}$, *with spectral bound* $\rho > -\infty$, *is said to satisfy* Condition (G) *if* $(\lambda - \rho)R(\lambda,Z)$ *is bounded as* $\lambda \downarrow \rho$. $\{T_t\}$ *is called* (G)-solvable *if there exists a chain of closed* $\{T_t\}$-*invariant ideals:* $\{0\} = E_0 \subset E_1 \subset \dots \subset E_n = E$, *such that for each* ν ($1 \le \nu \le n$), *the semi-group induced by* $\{T_t\}$ *on* $E_\nu/E_{\nu-1}$ *satisfies* (G).

This growth condition enters the proof of 3.C (i) below in a fashion somewhat similar to that described following Definition 1.B above. For example, if $\rho = 0$ and if $i\alpha x = Zx$ for some $x \neq 0$, $0 \neq \alpha \in \mathbb{R}$, then by known properties of $R(\lambda,Z)$ we have $x = \lambda R(\lambda + i\alpha)x$ for all $\lambda > 0$. Now, if $|x| = \lambda R(\lambda)|x|$, it can be concluded that $x^{(k)} = \lambda R(\lambda + ik\alpha)x^{(k)}$ for all $k \in \mathbb{Z}$ which, in turn, means that $ik\alpha x^{(k)} = Zx^{(k)}$. The point is thus to obtain the equality $|x| = \lambda R(\lambda)|x|$; for example, in 3.A above this is ensured by irreducibility of $\{T_t\}$ and the (strictly) positive fixed vector of $\{T_t'\}$

whose existence is postulated. In the more general case of non-irreducible semi-groups, condition (G) serves to ensure equality in a suitable quotient, and the group property of the peripheral point spectrum is generally lost. For the still more general case of the peripheral spectrum, transition to an ultraproduct leads back to point spectrum of the operator corresponding to $R(\lambda,Z)$; however, as pointed out in [4], this operator is no longer a resolvent in general, but a mere *pseudo-resolvent*. In order to set the analogy of the following result with Theorem 1.C in clear evidence, we will call a subset M of $\mathbb{C}$ *imaginary-additively cyclic* (abbreviated *i.a. cyclic*) whenever $\alpha + i\beta \in M$ ($\alpha,\beta \in \mathbb{R}$) implies $\alpha + ik\beta \in M$ for all $k \in \mathbb{Z}$.

3.C THEOREM. *Let* $\{T_t\}$ *be a positive semi-group, with generator* Z, *on a Banach lattice* E.
(i) *If* $\{T_t\}$ *is* (G)*-solvable, the peripheral spectrum of* Z *is i.a. cyclic.*
(ii) *If* $\{T_t\}$ *is a semi-group of Riesz homomorphisms, the entire spectrum and the entire point spectrum of* Z *are i.a. cyclic.*

We point out that $\{T_t\}$ is (G)-solvable whenever ρ is a pole of $R(\lambda,Z)$, and that the results mentioned here are but a few of those proved in [4], [5]. Theorem 3.C (ii) was proved by Derndinger [3].

Let us mention an easy consequence of Lemma 2.C: *Every norm continuous semi-group of Riesz homomorphisms on* E *is contained in the center* C(E).

3.D OPEN PROBLEMS. Again in complete analogy with the discrete case, the following problems in the spectral theory of positive semi-groups on Banach lattices are so far unsolved.

(i) *Is the peripheral point spectrum of an irreducible positive semi-group with spectral bound* $\rho = 0$, *necessarily a subgroup of* $i\mathbb{R}$*?*
(ii) *Is the peripheral spectrum of every positive semi-group necessarily i.a. cyclic?*

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INTEGRAL OPERATORS

A.R. Schep*

1. Introduction

The study of linear (non-singular) integral operators was originally an important stimulus for the development of functional analysis, but it was mostly restricted to special classes of integral operators like Carleman operators, Hilbert-Schmidt operators or Hille-Tamarkin operators. In the last decade there has been a renewed interest in the theory of linear integral operators as is shown e.g. by the publication of the book [6] by Halmos and Sunder. We shall try to present a survey of most of the recent results of these investigations. Most of the results will be presented without proofs, but in the last two sections we shall present some new results with their proofs.

The paper is organized as follows. In section 2 we present some preliminary material. Section 3 deals with the order structure of order bounded integral operators. In section 4 we discuss two recent characterizations of integral operators. Both characterizations provide a solution to the so-called recognition problem as posed by Halmos and Sunder in [6]. In section 5 we derive some compactness properties of integral operators. We discuss in particular the compactness in measure and "almost-compactness" of integral operators. In section 6 we discuss the relation between integral operators and multiplication operators. The main result in this section is theorem 6.3, which has not appeared in print before.

2. Preliminaries

We begin with some simple definitions. Let $L_0(Y,\nu)$ be the vector

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space of all real ν-measurable functions on the σ-finite measure space (Y,Σ,ν) with the usual identification of ν-almost equal functions. A linear subspace L of $L_0(Y,\nu)$ is called an *ideal* (or *order ideal*) if $f \in L_0(Y,\nu)$, $g \in L$ and $|f(y)| \leq |g(y)|$ a.e. implies that $f \in L$. Let (X,Λ,μ) be another σ-finite measure space and let $T(x,y) \in L_0(X\times Y,\mu\times\nu)$. The natural (maximal) domain $D_Y = D_Y(T)$ of the integral operator induced by $T(x,y)$ is $D_Y = \{f \in L_0(Y,\nu) : \int |T(x,y)\ f(y)|\ d\nu < \infty \text{ a.e.}\}$. Clearly D_Y is an order ideal in $L_0(Y,\nu)$ and the integral operator T induced by the kernel $T(x,y)$ maps D_Y into $L_0(X,\mu)$. This T is given by

$$Tf(x) = \int T(x,y)\ f(y)\ d\nu \quad \text{a.e.}$$

for $f \in D_Y$. In what follows we shall study the collection of integral operators defined on some given ideal $L \subset L_0(Y,\nu)$ with range in some other given ideal $M \subset L_0(X,\mu)$, i.e. we shall assume that $D_Y(T) \supset L$ for each given integral operator T and that $T(L) \subset M$. Note that this implies that if $T(x,y)$ is the kernel of an integral operator from L into M then $|T(x,y)|$ is not necessarily the kernel of an integral operator from L into M, but it is the kernel of an integral operator from L into $L_0(X,\mu)$. The ideal $L \subset L_0(Y,\nu)$ is *order dense* in $L_0(Y,\nu)$ if every set E of positive measure contains a subset F of positive measure such that $\chi_F \in L$. If L is order dense in $L_0(Y,\nu)$ then by a standard measure theoretical exhaustion argument one shows that there exist disjoint Y_n with $\cup_n Y_n = Y$, $\nu(Y_n) < \infty$ such that $\chi_{Y_n} \in L$. The following theorem was proved first in [1] by N. Aronszajn and P. Szeptycki and then by pure measure theoretical arguments in [11] by W.A.J. Luxemburg and A.C. Zaanen.

<u>THEOREM 2.1.</u> *Let* $T(x,y) \in L_0(X\times Y,\mu\times\nu)$ *such that* D_Y *is order dense in* $L_0(Y,\nu)$. *Then*

(i) there exists $g_0 \in D_Y$ *with* $g_0(y) > 0$ *a.e.;*

(ii) $D_X = \{g \in L_0(X,\mu) : \int |T(x,y)\ g(x)|\ d\mu < \infty$ *a.e.}* *is order dense in* $L_0(X,\mu)$.

In the next section we shall study the order structure of the set of order bounded integral operators from an ideal L of measurable functions into an ideal M of measurable functions. We first recall that the linear operator $T : L \to M$ is said to be *positive* if T maps non-negative functions into non-negative functions and that T is called *order bounded* if $T = T_1 - T_2$

with T_1 and T_2 positive. The set of all order bounded linear operators from L into M is denoted by $L_b(L,M)$. If $M = L_0(X,\mu)$, then every integral operator from L into M is order bounded. For $T, S \in L_b(L,M)$, the operator $\sup(T,S)$ is given for $0 \leq f \in L$ by

$$\sup(T,S)(f) = \sup(Tg + Sh : g \geq 0 , h \geq 0 , g + h = f) .$$

Similarly $\inf(T,S)(f) = \inf(Tg + Sh : g \geq 0 , h \geq 0 , g + h = f)$.

3. Order structure of order bounded integral operators

In this section we consider integral operators from an ideal L in $L_0(Y,\nu)$ into an ideal M in $L_0(X,\mu)$. We shall assume throughout that L and M are order dense in $L_0(Y,\nu)$ and $L_0(X,\mu)$, respectively. Now let $K = K(L,M)$ denote the set of all order bounded integral operators from L into M. The following theorem describes how the subspace K is embedded in $L_b(L,M)$. Part (ii) of the theorem is due to W.A.J. Luxemburg and A.C. Zaanen ([11], 1971). The present proof of (ii) is much simpler than that in [11]; it was given in [17] by the author, who first proved (i) and used that in the proof of (ii).

THEOREM 3.1. *(i) If* $0 \leq S \leq T \in K$ *, then* $S \in K$.

(ii) If $T, S \in K$ *, then* $\sup(T,S) \in K$ *and* $\sup(T,S)$ *has the pointwise supremum of* $T(x,y)$ *and* $S(x,y)$ *as kernel.*

(iii) If $T_0 = \sup(T_\alpha : \alpha \in \{\alpha\})$ *in* L_b *and if* $T_\alpha \in K$ *for all* α *, then* $T_0 \in K$.

In the terminology of Riesz spaces (see [10]) one expresses the above theorem by saying that K is a *band* in L_b. For any subset A of L_b we call the band generated by A the smallest subset of L_b containing A and satisfying (i), (ii) and (iii) of the above theorem. To give another description of K we introduce

$$L^\wedge = \{g \in L_0(Y,\nu) : \int_Y |fg| \, d\nu < \infty \text{ for all } f \in L\} .$$

Then $L^\wedge$ is an order ideal in $L_0(Y,\nu)$ and elements of $L^\wedge \otimes M$ can be identified with integral operators of finite rank from L into M. The following theorem was proved in special cases by G.Ya. Lozanovskii ([8])

and by R.J. Nagel and U. Schlotterbeck ([14]), and in the general case by A.V. Buhvalov ([2]) and, more elementary, by the author ([17]).

THEOREM 3.2. *If the ideal* $L^\frown$ *is also order dense in* $L_0(Y,\nu)$, *then the band* K *of all order bounded integral operators is exactly equal to the band generated by the finite rank integral operators.*

4. Recognition of integral operators

As in the preceding sections, let L and M be ideals in $L_0(Y,\nu)$ and $L_0(X,\mu)$, respectively. We shall assume that L and $L^\frown$ are order dense in $L_0(Y,\nu)$. For $f_n \in L_0(Y,\nu)$ we write $f_n \overset{*}{\to} 0$ if every subsequence of $\{f_n : n = 1,2,\ldots\}$ contains a subsequence converging to zero a.e. on every set of finite measure. Recall that a set $H \subset L_0(X,\mu)$ is called *equimeasurable* if for all $\varepsilon > 0$ and all $X_0 \subset X$ with $\mu(X_0) < \infty$ there exists $X_1 \subset X_0$ with $\mu(X_0 - X_1) \leqq \varepsilon$ such that $\{h\chi_{X_1} : h \in H\}$ is a relatively compact subset of $L_\infty(X,\mu)$. The following theorem was partially proved by A.V. Buhvalov in [2], by W. Schachermayer in [15] and by the author in [17] and [19].

THEOREM 4.1. *Let* T *be a linear operator from* L *into* M. *Then the following are equivalent.*

(a) T *is an integral operator;*

(b) if $0 \leqq f_n \leqq f \in L$ *and* $f_n \overset{*}{\to} 0$, *then* $Tf_n(x) \to 0$ *a.e.;*

(c) if $0 \leqq f_n \leqq f \in L$ *and* $f_n \overset{*}{\to} 0$, *then* $Tf_n \overset{*}{\to} 0$ *and* T *maps order bounded sets into equimeasurable sets.*

Recently L. Weis gave in [21] a proof of (b) ⇒ (a) which does not use the order theoretic description of $K(L, L_0(X,\mu))$ discussed in Theorem 3.2. The proof of (a) ⇒ (c) depends on the following theorem for which we refer to [19].

THEOREM 4.2. *Let* $T : L_\infty(Y,\nu) \to L_\infty(X,\mu)$ *be an integral operator and assume that* $\mu(X) < \infty$. *Then for all* $\varepsilon > 0$ *there exists* $X_0 \subset X$ *with* $\mu(X - X_0) \leqq \varepsilon$ *such that* $P_{X_0} \cdot T : L_\infty(Y,\nu) \to L_\infty(X,\mu)$ *is compact, where* P_{X_0} *denotes the operator* $P_{X_0} f = \chi_{X_0} \cdot f$.

In the next section we shall discuss an extension of the above theorem to L_p-spaces. We also remark that Theorem 4.1 (a) ⇔ (b) has been studied for so-called abstract order bounded integral operators on Riesz spaces by W.K. Vietsch ([20]) and by P. van Eldik and J.J. Grobler ([4]).

5. Compactness properties of integral operators

In this section we shall discuss compactness properties of integral operators defined on a Banach function space $L_\rho = L_\rho(Y,\nu)$. Recall that L_ρ is a *Banach function space* if L_ρ is an order dense ideal in $L_0(Y,\nu)$ provided with a norm ρ such that $|f| \leq |g|$ implies $\rho(f) \leq \rho(g)$ and such that L_ρ is norm complete with respect to ρ. For our discussion we shall use the following theorem, due to B. Maurey and E.M. Nikishin, which has its origin in the theory of factorization of operators (see [13]).

THEOREM 5.1. *Let* $H \subset L_0(X,\mu)$ *be a convex set of non-negative functions which is bounded in measure. Then there exists* $0 < \phi \in L_0(X,\mu)$ *such that* $\frac{1}{\phi}\cdot H = \{\frac{h}{\phi} : h \in H\}$ *is norm bounded in* $L_1(X,\mu)$.

Recall that a function norm ρ is called *order continuous* if $f_n \in L_\rho$, $f_n(x) \downarrow 0$ a.e. implies that $\rho(f_n) \downarrow 0$. Recall also that $\rho'(g) = \sup(\int |fg|\, d\mu : \rho(f) \leq 1)$. The following theorem is due to V.B. Korotkov ([7]).

THEOREM 5.2. *Let* $L_\rho = L_\rho(Y,\nu)$ *be a Banach function space such that* ρ' *is order continiuous. Then every integral operator from* L_ρ *into* $L_0(X,\mu)$ *is compact in measure.*

PROOF. Let T be an integral operator from L_ρ into $L_0(X,\mu)$ with kernel $T(x,y)$. Then $H = \{\int |T(x,y)\, f(y)|\, d\nu : \rho(f) \leq 1\}$ is convex and bounded in measure in $L_0(X,\mu)$. Hence by above theorem there exists $0 < \phi \in L_0(X,\mu)$ such that $\frac{1}{\phi}\cdot H$ is norm bounded in $L_1(X,\mu)$. It follows that $\frac{1}{\phi}\cdot T$ is an order bounded integral operator from L_ρ into $L_1(X,\mu)$. Hence by lemma 5.2 of [12] we know that $\frac{1}{\phi}\cdot T$ is compact, so the operator $\frac{1}{\phi}\cdot T$ is compact in measure. Since $\phi > 0$ a.e. we can deduce that T is also compact in measure.

The above theorem fails in case ρ' is not order continuous, as can be seen from the following example.

EXAMPLE. Let $X = Y = [0,1]$ with Lebesgue measure and let $T(x,y) = \sum_{n=1}^{\infty} \sin(n\pi x) \cdot \chi_{[2^{-n},2^{-n+1}]}(y)$. If $f \in L_1[0,1]$, then

$\int |T(x,y)\ f(y)|\ dy \leq \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} |f(y)|\ dy = \int_0^1 |f(y)|\ dy = \|f\|_1$. Hence $T(x,y)$ is the kernel of an order bounded integral operator from $L_1[0,1]$ into $L_\infty[0,1]$. Now let $f_n = 2^n \cdot \chi_{[2^{-n},2^{-n+1}]}$. Then $\|f_n\|_1 = 1$ and $Tf_n = \sin(n\pi x)$. It is easy to see that $\{\sin(n\pi x) : n = 1,2,\ldots\}$ is not compact in measure. It follows that T is not compact in measure.

In the following theorem we denote by P_ϕ the multiplication operator $P_\phi f = \phi f$ and we denote by T' the restriction of the Banach adjoint of T to $L'_{\rho_2} \subset L^*_{\rho_2}$.

THEOREM 5.3. *Let* $L_{\rho_1} \subset L_0(Y,\nu)$ *and* $L_{\rho_2} \subset L_0(X,\mu)$ *be Banach function spaces and let* $T : L_{\rho_1} \to L_{\rho_2}$ *be an integral operator. Then there exists* $\phi > 0$ *in* $L_0(X,\mu)$ *such that* $T'\cdot P_\phi$ *is an integral operator from* L'_{ρ_2} *into* L'_{ρ_1} *which is compact in measure.*

PROOF. Assume first that (X,μ) is a finite measure space and let $\varepsilon > 0$. Let $0 \leq g_0 \in L_{\rho_1}$ such that $g_0(y) > 0$ ν-a.e. and let $T(x,y)$ denote the kernel of T. Then

$$\int |T(x,y)\ g_0(y)|\ d\nu < \infty \quad \text{a.e.}\ ,$$

and so there exists $M > 0$ such that

$$X_{\varepsilon,1} = \{x \in X : \int |T(x,y)\ g_0(y)|\ d\nu \leq M\}$$

satisfies $\mu(X^c_{\varepsilon,1}) \leq \varepsilon/2$ and $\chi_{X_{\varepsilon,1}} \in L_{\rho_2}$. Let $A_{g_0} = \{g \in L_0(Y,\nu) : |g| \leq c\ g_0$ for some $c > 0\}$. Then $P_{\chi_{X_{\varepsilon,1}}}\cdot T$ is an order bounded integral operator from A_{g_0} into $L_\infty(X_{\varepsilon,1},\mu)$. Note that $L_\infty(X_{\varepsilon,1},\mu) \subset L_{\rho_2}$. By Theorem 4.2 we know that we can find $X_\varepsilon \subset X_{\varepsilon,1}$ with $\mu(X_{\varepsilon,1}-X_\varepsilon) \leq \varepsilon/2$ such that $P_{X_\varepsilon}\cdot T$ is compact as an operator from A_{g_0} into $L_\infty(X_\varepsilon,\mu)$. Using Fubini's theorem one finds that $T'\cdot P_{X_\varepsilon} : L_1(X_\varepsilon,\mu) \to (A_{g_0})' \simeq L_1(g_0\ d\nu)$ is a compact integral operator, so also $T'\cdot P_{X_\varepsilon} : L'_{\rho_2} \to L_1(g_0\ d\nu)$ is a compact integral operator. From proposition 2.1 of [3] we conclude that $T'\cdot P_{X_\varepsilon} : L'_{\rho_2} \to L'_{\rho_1}$ is an integral operator, which by the above is compact in measure.

In case X is of infinite measure, we can write $X = \cup_n X_n$ with the X_n's disjoint and $\mu(X_n) < \infty$.By the above argument we can find $X_{\varepsilon_n} \subset X_n$ such that $T'\cdot P_{X_{\varepsilon_n}}$ is an integral operator from L'_{ρ_2} into L'_{ρ_1} . By piecing the

$\chi_{X_{\varepsilon_n}}$'s appropriately together, we find the function $\phi > 0$ with the required properties.

<u>THEOREM 5.4</u>. *Let* $L_\rho \subset L_0(Y,\nu)$ *be a Banach function space and let* $T : L_\rho \to L_0(X,\mu)$ *be an integral operator. Then there exists* $\phi > 0$ *a.e. in* $L_0(Y,\nu)$ *such that* $T\cdot P_\phi$ *is compact in measure.*

<u>PROOF</u>. As in the proof of Theorem 5.2 we can find $\chi_0 > 0$ in $L_0(X,\mu)$ such that $P_{1/\chi_0}\cdot T$ is an integral operator from L_ρ into $L_1(X,\mu)$. So by above theorem we can find $\phi_0 > 0$ in $L_0(X,\mu)$ such that $(P_{1/\chi_0}\cdot T)'\cdot P_{\phi_0} = T'\cdot P_{\phi_0/\chi_0}$ is an integral operator from $L_\infty(X,\mu)$ into L_ρ'. Applying the above theorem again we find $\phi_1 > 0$ in $L_0(Y,\nu)$ such that $P_{\phi_0/\chi_0}\cdot T''\cdot P_{\phi_1} : L_\rho'' \to L_1(X,\mu)$ is compact in measure. As in Theorem 5.2 this implies that $T\cdot P_\phi : L_\rho \to L_0(X,\mu)$ is compact in measure.

We now state an extension of Theorem 4.2 due to W. Schachermayer and L. Weis ([16], see also [21]).

<u>THEOREM 5.5</u>. *Let* $T : L_p(Y,\nu) \to L_p(X,\mu)$ $(1 < p < \infty)$ *be an integral operator and assume that* $\mu(X) < \infty$. *Then for all* $\varepsilon > 0$ *there exists* $X_0 \subset X$ *with* $\mu(X-X_0) \leqq \varepsilon$ *such that* $P_{X_0}\cdot T : L_p(Y,\nu) \to L_p(X,\mu)$ *is compact, where* P_{X_0} *denotes the operator* $P_{X_0}\cdot f = \chi_{X_0}\cdot f$.

Let $\mu(X) < \infty$. Then we call a bounded operator from $L_p(Y,\nu)$ into $L_p(X,\mu)$ *almost compact* if it satisfies the conclusion of the above theorem. In particular integral operators from $L_p(Y,\nu)$ into $L_p(X,\mu)$ $(1 < p < \infty)$ are almost compact. The following recent theorem due to L. Weis ([21]) gives the precise relation between integral operators and almost compact operators.

<u>THEOREM 5.6</u>. *A bounded linear operator* T *from* $L_p(Y,\nu)$ *into* $L_p(X,\mu)$ $(1 < p < \infty)$ *is almost compact if and only if* T *is the norm limit of a sequence of integral operators from* $L_p(Y,\nu)$ *into* $L_p(X,\mu)$.

6. Integral operators and multiplication operators

Throughout this section we shall assume that (Y,ν) does not contain any atoms, i.e., if E is a subset of Y of positive measure and if $0 < \alpha < \nu(E)$, then there exists a subset E_1 of E such that $\nu(E_1) = \alpha$. For $g \in L_0(Y,\nu)$ we denote by P_g the multiplication operator $P_g f = gf$. Our first theorem specifies the relation between order bounded integral operators and multiplication operators.

THEOREM 6.1. *Let* L *be an order ideal in* $L_0(Y,\nu)$. *Assume that* T *is an order bounded integral operator from* L *into* L *and that* P_g *is a multiplication operator from* L *into* L . *Then* T *and* P_g *are disjoint in* $L_b(L)$ *i.e.,* $\inf(|T|,|P_g|) = 0$ *in* $L_b(L)$.

PROOF. It is no loss in generality to assume that T and P_g are positive operators. Let $S = \inf(T,P_g)$. Theorem 3.1 (i) implies that S is an integral operator and similarly to [22] we can prove that S is also a multiplication operator, say $S = P_h$ with $h \geqq 0$. If $h \neq 0$, then we can find $E \subset Y$ with $\chi_E \in L$ and $h \geq c > 0$ on E. By non-atomicity of (Y,ν) we can find $E_n \subset E$ with $\nu(E_n) \to 0$ such that χ_{E_n} does not converge to zero a.e.. Since S is an integral operator we know from Theorem 4.1 (a) $\Rightarrow$ (b), that $S(\chi_{E_n})(x) \to 0$ a.e., i.e. $h\cdot\chi_{E_n}(x) \to 0$ a.e.. But $h \geq c > 0$ on all E_n's, so also $\chi_{E_n}(x) \to 0$ a.e., which is a contradiction.

For non order bounded operators we cannot use the order structure to investigate the relation between integral operators and multiplication operators. Therefore we shall consider a Banach function space instead of an arbitrary order ideal in $L_0(Y,\nu)$. Consequently, every multiplication operator P_g from L_ρ into L_ρ is given by g with $g \in L_\infty(Y,\nu)$. We first prove a technical lemma.

LEMMA 6.2. *Let* $T : L_\infty(Y,\nu) \to L_\infty(Y,\nu)$ *be an integral operator and assume that* $\nu(Y) < \infty$. *Then for all* $\varepsilon > 0$ *there exists* $E \subset Y$ *with* $\nu(E) > 0$ *such that* $|T(\chi_E)| \leqq \varepsilon$ *a.e. on* E.

PROOF. Let $\varepsilon > 0$. Then by Theorem 4.2 we can find $Y_0 \subset Y$ with $\nu(Y-Y_0) \leqq \frac{1}{2}\nu(Y)$ such that $P_{Y_0}\cdot T : L_\infty \to L_\infty$ is compact. Let $B_n \subset Y_0$ with $B_n \downarrow \emptyset$ such that $\nu(B_n) > 0$ for all n. Then by order continuity of integral operators we have $T(\chi_{B_n})(x) \to 0$ a.e.. On the other hand there exists a subsequence $\chi_{B_{n_k}}$ such that $\{P_{Y_0}\cdot T(\chi_{B_{n_k}})\}$ is convergent in the L_∞-norm. It follows that $\|P_{Y_0}\cdot T(\chi_{B_{n_k}})\|_\infty \to 0$ as $n \to \infty$. Let n_0 be such that $\|P_{Y_0}\cdot T(\chi_{B_{n_0}})\|_\infty < \varepsilon$. Then obviously $|T(\chi_{B_{n_0}})| \leqq \varepsilon$ a.e. on B_{n_0}, so take $E = B_{n_0}$.

Recall that a function norm ρ is called *Fatou* if $\rho = (\rho')'$ (cf [9]).

<u>THEOREM 6.3</u>. *Let* $L_\rho = L_\rho(Y,\nu)$ *be a Banach function space with Fatou norm* ρ *and let* $T : L_\rho \to L_\rho$ *be an integral operator. Then for all* $g \in L_\infty$ *we have* $\|T - P_g\| \geq \|g\|_\infty$.

<u>PROOF</u>. Let $\varepsilon > 0$ and let $A \subset Y$ of finite positive measure such that $g(y) \geq (1 - \varepsilon)\|g\|_\infty$ on A and $\int_A |T(x,y)|\, d\nu < \infty$ a.e. on Y. Then we can find $B \subset A$ of finite positive measure and a constant $M > 0$ such that

$$\int_A |T(x,y)|\, d\nu \leq M \quad \text{a.e. on } B .$$

In particular

$$\int_B |T(x,y)|\, d\nu \leq M \quad \text{a.e. on } B .$$

It follows that $P_B \cdot T$ maps $L_\infty(B,\nu)$ into $L_\infty(B,\nu)$. By the above lemma there exists $E \subset B$ such that $\nu(E) > 0$ and $|T(\chi_E)| \leq \varepsilon$ a.e. on E. We may assume that $\chi_E \in L_\rho$. Let $f = \frac{1}{\rho(\chi_E)} \cdot \chi_E$. Then we can find $0 \leq h_n \in L'_\rho$ with $\rho'(h_n) = 1$ such that $\int f\, h_n\, d\nu \geq 1 - \frac{1}{n}$. We may assume that $h_n = 0$ on E^c for all n. Now we have

$$\int |Tf \cdot h_n|\, d\nu \leq \varepsilon \int_E h_n\, d\nu \leq \varepsilon\, \rho(\chi_E) \quad .$$

On the other hand

$$\int |f\, g\, h_n|\, d\nu \geq (1 - \varepsilon)\|g\|_\infty (1 - \tfrac{1}{n}) \quad .$$

It follows that

$$\|T - P_g\| \geq \int |Tf - P_g f|\, h_n\, d\nu \geq \int |P_g f|\, h_n\, d\nu - \int |Tf|\, h_n\, d\nu \geq$$

$$\geq (1 - \varepsilon)\ \|g\|_\infty\ (1 - \tfrac{1}{n}) - \varepsilon\, \rho(\chi_E)$$

for all $\varepsilon > 0$ and all n. Hence $\|T - P_g\| \geq \|g\|_\infty$.

<u>REMARKS</u>.

1. Above theorem implies that $\|P_g - \lambda T\| \geq \|P_g\|$ for all $\lambda > 0$, i.e., the multiplication operator P_g is orthogonal in the sense of Birkhoff to every integral operator in the Banach space of all norm bounded operators on L_ρ.

2. In the book by Halmos and Sunder it is proved that for $L_\rho = L_2[0,1]$ we have $\|T - P_g\| \geq \|g\|_2$ for every integral operator. The above theorem therefore not only generalizes this result, but it also gives the correct lower bound, i.e., $\|g\|_\infty$ instead of $\|g\|_2$.

3. We note that some results in the present paper, like Theorem 4.1, can be specialized for subclasses of integral operators, like the Carleman integral operators. For these results we refer to the paper [5] by J.J. Uhl and N. Gretzky and to the paper [18] by the author.

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THE BACKGROUND TO CAUCHY'S DEFINITION OF THE INTEGRAL

F. Smithies

My aim in this talk is to say something about the background to Cauchy's definition of the integral and about what his new definition achieved. I do not intend to go deeply into technical details; I want rather to mention some of the difficulties raised by Cauchy's predecessors, their tentative suggestions for putting things right, and the effect of Cauchy's new approach on the situation.

Let us begin by taking a look at the concept of the definite integral as it existed before Cauchy's time. Leibniz had thought of the integral primarily as a sum of infinitesimals, but later in the eighteenth century the predominant view was that integration was the process inverse to differentiation, the indefinite integral being an antiderivative or primitive function, and the definite integral merely the difference between two values of the primitive function. A function was generally thought of as being given by an analytic expression, and it was taken for granted that it always possessed a primitive funtion, whether or not an analytic expression could be found for it.

The use of the definite integral for the evaluation of areas was of course well known. Euler, in his book on the integral calculus ([7]) considered the approximate evaluation of a definite integral by a sum of the form (in modern notation)

$$(x_1-x_0)\ f(x_0) + (x_2-x_1)\ f(x_1) + \dots + (x_n-x_{n-1})\ f(x_{n-1}) \quad ,$$

and obtained an upper bound for the error for the case where f(x) is monotonic. Lacroix ([10]) and Legendre ([11]) had refined Euler's argument, the latter even applying it to some cases where the integrand has an infinity in the interval of integration.

We must now turn to some developments in the concept of function. By Cauchy's time, the definition of a function as an analytic expression was beginning to be found inadequate. 'Arbitrary' functions appeared in the general solutions of partial differential equations. Physical considerations suggested the desirability of admitting functions other than those given by a single analytic expression; thus, to describe the initial configuration of a plucked string, one seemed to need different expressions on the two sides of the point where the string was plucked. Euler was prepared to admit such functions, calling them 'discontinuous', a 'continuous' function being one given by the same analytic expression throughout. In a letter to D'Alembert (20 December 1763), who refused to recognise such functions as being admissable in mathematical analysis, Euler remarked that 'the consideration of functions that are subject to no law of continuity opens to us a wholly new domain of analysis'.

Daniel Bernoulli ([1]) suggested that every possible initial configuration of a vibrating string could be expressed as a trigonometric series; he based this statement almost entirely on physical considerations, without any attempt at a mathematical proof. Euler ([6]) expressed scepticism about the generality of Bernoulli's solution as a superposition of normal modes of vibration, since he was unable to believe that so arbitrary a function could be expanded in a trigonometric series. Later, however, Euler seems to have modified his position somewhat, though he remained sceptical about the practical possibility of obtaining such expansions. In a paper ([2]) published in 1772, Daniel Bernoulli obtained the expansion

$$\frac{1}{2}(\pi-x) = \sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots ,$$

pointing out that equality holds only for $0 < x < 2\pi$ and that, in the interval $2(n-1)\pi < x < 2n\pi$, the sum of the series is $(n-\frac{1}{2})\pi-\frac{1}{2}x$. This result exhibits the sum of the series as a function that is simultaneously 'continuous' and 'discontinuous'. Finally, in a paper ([8]) written in 1777, but not published till 1798, Euler obtained the coefficients of a general trigonometric series by term-by-term integration, and quite explicitly insists on using the geometrical interpretation of the definite integrals

$$\int_0^{\ell} f(s) \cos \frac{k\pi s}{\ell} \, ds \quad .$$

One suspects, though Euler does not say it in so many words, that he was contemplating the admission of 'discontinuous' functions f(s) in the integrand.

In 1820 Poisson published a thought-provoking paper ([12]). In this he discusses the difficulties that arise when the integrand becomes infinite between the limits of integration. He remarks that mathematicians who have found finite values for such integrals have generally supposed that the infinities cancel out 'by the opposition of the signs + and -'. He points out that this explanation does not always work; for example, we should have

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-\frac{1}{x}\right]_{-1}^{1} = -2 \quad ,$$

whereas dx/x^2 is positive throughout the interval. On the other hand, in the integral

$$\int_{-1}^{1} \frac{dx}{x}$$

the infinities should cancel, giving the value 0, whereas the usual drill leads to

$$\int_{-1}^{1} \frac{dx}{x} = [\log x]_{-1}^{1} = -\log(-1)$$

which has the infinity of values $(2n+1)\pi\sqrt{(-1)}$; how, he asks, can the sum of the real elements dx/x have several values, all of them imaginary? He then states that, at the birth of the integral calculus, the definite integral was regarded as the sum of the values of the differential; later, therefore, it came to be felt that one had to prove that, if $dF(x) = f(x)\,dx$, then $F(b)-F(a)$ is the sum of the values of $f(x)\,dx$ when x goes from a to b by infinitely small steps dx. This proposition continues to hold if $f(x)\,dx$ changes sign in the interval, and even if it passes through imaginary values, but its proof assumes essentially that f(x) remains finite. If it passes through infinity, then, as we have seen, there are cases where it ceases to hold. To restore its validity, Poisson suggests making the change of variable

$$x = -(\cos\zeta + \surd(-1)\sin\zeta) \quad ,$$

and integrating from $\zeta = 0$ to $\zeta = (2n+1)\pi$, where n is an integer; since x does not pass through 0, the integrand no longer has an infinity. We then obtain

$$\int_{-1}^{1} \frac{dx}{x} = -\surd(-1) \int_{0}^{(2n+1)\pi} d\zeta = -(2n+1)\pi\surd(-1) \quad ,$$

the result we had before. More generally, he evaluates

$$\int_{-1}^{1} \frac{dx}{x^m}$$

for positive integral values of m; in particular, the argument explains why we obtain a negative result when $m = 2$.

Thus Poisson seems to be advocating the desirability of returning to the definition of the definite integral as being in some sense a sum of infinitesimals.

Let us now look at some early ideas of Cauchy's. What we may perhaps call Poisson's puzzle must already have been in the air before 1820, since Cauchy made some remarks on it in his famous 1814 memoir ([3]) on definite integrals, in which he laid some of the foundations for complex analysis. He begins by saying that if the function $\phi(z)$ increases or decreases in a continuous manner between the limits $z = b'$ and $z = b''$ (in other words, is a continuous function in the sense of the definition he gave in 1821), then

$$\int_{b'}^{b''} \phi'(z)\, dz = \phi(b'') - \phi(b') \quad .$$

However, if $\phi(z)$ has a jump discontinuity of amount Δ at the point Z of the interval, in the sense that

$$\phi(Z+\zeta) - \phi(Z-\zeta)$$

is close to Δ when ζ is very small, then

$$\int_{b'}^{b''} \phi'(z)\,dz = \phi(b'') - \phi(b') - \Delta \quad .$$

He establishes this by showing that, if ζ is small, then

$$\int_{b'}^{Z-\zeta} + \int_{Z+\zeta}^{b''} \phi'(\zeta)\,d\zeta = \phi(b'') - \phi(Z+\zeta) + \phi(Z-\zeta) - \phi(b') \quad ,$$

which becomes $\phi(b'') - \phi(b') - \Delta$ when ζ goes to 0. He illustrates this by considering

$$\int_{-2}^{4} \frac{dz}{z} \quad .$$

Since $\log \zeta - \log(-\zeta) = -\log(-1)$, we have $\Delta = -\log(-1)$; hence

$$\int_{-2}^{4} \frac{dz}{z} = \log 4 - \log(-2) - \Delta = \log 4 - \log 2 \quad .$$

This foreshadows Cauchy's later definition of the principal value of such an integral, which indeed he mentions in a footnote added to the memoir before its ultimate publication in 1827.

The principal value itself makes its first appearance in a paper ([4]) published in 1822, where he defines the general value of $\int_{x'}^{x''} f(x)\,dx$ for a function having an infinity at x_0 as

$$\int_{x'}^{x_0-k\alpha'} + \int_{x_0+k\alpha''}^{x''} f(x)\,dx \quad ,$$

where k is infinitely small (i.e. is a variable approaching 0); this gives

$$\int_{-1}^{1} \frac{dx}{x} = \log \frac{\alpha'}{\alpha''} \quad .$$

If, in particular, we take $\alpha' = \alpha'' = 1$, we obtain what he defines to be the *principal value* of the integral. Later in the same paper he says that every definite integral between real limits should be regarded as being the sum of the values of the differential corresponding to the values of the real variable between the limits. This, he says, applies to all cases, even when no primitive function is known, and always gives real values for the integrals of real-valued functions. On the other hand, if the integral is taken as the difference between two values of the primitive function, this fails to be true in general, e.g. if the latter is discontinuous or if we pass from one limit to the other through imaginary values. So far, he seems to be echoing Poisson, but he then brings in a new point, remarking that the primitive function approach can give either

$$\int_{-1}^{2} \frac{dx}{x} = \log 2 - \log(-1)$$

or

$$\int_{-1}^{2} \frac{dx}{x} = \int_{-1}^{2} \frac{x\,dx}{x^2} = [\tfrac{1}{2} \log x^2]_{-1}^{2} = \log 2 \quad .$$

Cauchy's formal definition of the definite integral appears first in his 1823 book [5]. We have already seen that Euler and others had used approximating sums for the evaluation of definite integrals, whose existence was taken for granted. Cauchy reverses this procedure, defining the definite integral of a continuous function as the limit of sums of the form

$$(x_1-x_0)\ f(x_0) + (x_2-x_1)\ f(x_1) + \ldots + (x_n-x_{n-1})\ f(x_{n-1})$$

as the maximum length of the subintervals approaches 0, and also giving a proof of the existence and uniqueness of the limit. It is well known that his proof is not watertight, since his definition of continuity is somewhat vague, and could be interpreted as referring to either pointwise or uniform continuity, the latter being needed for a rigorous proof. Nevertheless, he does enough to convince the reader that the definite integral of a continuous function always exists. Shortly afterwards he remarks that one could also consider sums of the more general form

$$(x_1-x_0)\ f(\xi_0) + (x_2-x_1)\ f(\xi_1) + \ldots + (x_n-x_{n-1})\ f(\xi_{n-1}) \quad ,$$

where ξ_k is an arbitrary point of the interval (x_k, x_{k+1}).

On the basis of his definition, Cauchy proves such elementary properties of the integral as

$$\int_{a}^{b} f(x+c)\ dx = \int_{a+c}^{b+c} f(x)\ dx$$

and

$$\int_0^a f(a-x)\,dx = \int_0^a f(x)\,dx \quad ,$$

establishes the linearity of the integral operation and extends his definition to complex-valued functions. He also proves that the definite integral is an additive function of intervals. He also obtains an estimate for the error of an approximating sum in evaluating the integral of a piecewise monotonic function. Further on, he defines improper integrals as limits in the way that is familiar to us all, and repeats the discussion of principal values from his 1822 paper. Finally, he links his new definition with the old one by proving that if

$$F(x) = \int_{x_0}^{x} f(x)\,dx \quad ,$$

where $f(x)$ is a continuous function, then $F'(x) = f(x)$, and that if $\tilde{\omega}'(x) = 0$, then $\tilde{\omega}(x)$ must be a constant.

We see that Cauchy's new definition of the integral has several advantages over the older one; it establishes the existence of the integral for a much wider class of functions, namely, the continuous (and indeed piecewise continuous) functions, thus opening the way for further generalisations, such as Riemann's, and it avoids some of the paradoxes to which the older theory was subject. It also made possible a proper treatment of integration along paths in the complex plane, as foreshadowed in Poisson's 1820 paper, and thus led eventually to the proof of Cauchy's theorem and its manifold consequences.

A few words about the notation for definite integrals may be of interest. Euler and his contemporaries had either used the somewhat clumsy notation

$$\int f(x)\,dx \begin{bmatrix} x = x_0 \\ x = X \end{bmatrix}$$

or had specified the limits of integration in the text. The familiar notation

$$\int_{x_0}^{X} f(x)\,dx$$

was invented by Fourier, who used it in his prize essay of 1811. Although Fourier's work did not appear in print till 1822, Cauchy had certainly seen a copy of it by 1818, and immediately adopted the new notation. He had himself used the old notation in his 1814 memoir, and didn't trouble to alter it in the published version, though the new notation appears in the footnotes. However, the editors of Cauchy's collected works did change all the definite integrals in the 1814 memoir to the new notation when they reprinted it.

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SOME LATTICE PROPERTIES OF THE SPACE L^2

B. Sz.-Nagy

1. The classical function spaces L^p $(1 \leq p \leq \infty)$, C , etc., whose concept and basic properties were established since the first decade of this century, in particular by F. Riesz, have algebraic (linearity), topological (in particular, metric) and order structures, which are compatible in some definite senses. The rapid development of the pertaining investigations, their importance within mathematics and their various applications, eventually lead to the evolution of several kinds of abstract linear spaces, based on axioms which are distilled from one or another type of structure properties of the classical function spaces. Thus, the general linear and metric properties are distilled in the axioms of Banach spaces, and in particular in the axioms of (abstract) Hilbert space; here the basic, axiomatically defined concepts are the *norm*, and (in the case of Hilbert space) the *inner product* (from which the norm derives as in the classical case L^p with $p=2$).

The bases of the abstract theories of Banach and Hilbert spaces were laid down in the 2nd and 3rd decade of this century, and opened a spectacularly rapid and succesful development in modern mathematics and in many of its applications. Axiomatization of the order properties of the classical function spaces followed somewhat later, with no, or only a slight regard to the metric properties of these spaces. F. Riesz, whose pioneering work of what is now called functional analysis concerned in particular linear metric spaces, was also the first to point out, in a short note presented to the 1928 Congress at Bologna, the perspectives of an abstract theory of ordered linear spaces. He expounded his ideas later, in a paper of 1937 in Hungarian, a translation of which appeared in 1940 in the *Annals of Math.*, see [R].
In the meantime, H. Freudenthal also elaborated and published in 1937 his independent theory of partially ordered moduli, see [F]. About the same

time, L.V. Kantorovič also began extensive investigations in this area, in which several Soviet authors took part in the sequel. Today we have a highly developed theory on (partially) ordered linear spaces (or "vector lattices"), also called - in homage of his pioneering merits - *Riesz spaces*. Dutch mathematicians, following the path of H. Freudenthal, continued to play an important part in this development, namely A.C. Zaanen and some of his colleagues and former pupils. His monograph with W.A.J. Luxemburg [LZ] is an excellent exposition of the fundamentals of this abstract theory. The authors promise to give, in a forthcoming volume of this monograph, an exposition also of some connections between the two, rather independently grown-up daughters of the classical theory of function spaces.

In the present exposition I shall deal with problems of this nature, notably connecting in some sense or other the metric structure of abstract Hilbert space with the additional vector lattice structure of the classical, functional Hilbert space L^2 .

These results are not new, I obtained them in the years 1936/37 and 1949, and published in the papers [I-III].

Papers [I] and [II] described those subsets E and P of abstract (real, separable) Hilbert space H which can be mapped unitarily onto the set $X(\alpha)$ of characteristic functions, or on the set $L^2_+(\alpha)$ of nonnegative functions, in some $L^2(\alpha)$, respectively; α means here a conveniently chosen non negative, finite Borel measure. Paper [III] also considers characterization of sets P in H which can be mapped onto an $L^2_+(\alpha)$, if not by a unitary, but at least by an "affine" transformation of H onto $L^2(\alpha)$.

2. Let us introduce a *notation*: For any two vectors $\phi,\psi \in H$,

$$\phi \prec \psi \quad \text{means} \quad \phi \perp \psi - \phi \text{ , i.e. } (\phi,\psi) = \|\phi\|^2 \text{ .}$$

<u>THEOREM I</u>. *Let* E *be a subset of* H *(with elements denoted by* e,f,g,h,... *). In order that there exists a linear isometric (i.e., unitary) map of* H *onto some space* $L^2(\alpha)$ *, which carries* E *precisely on* $X(\alpha)$ *, it is necessary and sufficient that the following conditions hold:*

(I.1) E *is complete in* H .

(I.2) *For* $e,f \in E$ *we have* $e-f \in E$ *iff* $f \prec e$.

(I.3) *For any,* $e,f \in E$ *there exists a* $g \in E$ *such that* $e+f-g \in E$ *and* $g \prec \{^e_f$.

(I.4) E *is closed in the sense that it contains the limit of every convergent "increasing" sequence* $\{e_n\}$ *(i.e. such that* $e_1 \prec e_2 \prec e_3 \prec \cdots$.

(I.5) *There exists an* $e_* \in E$ *such that* $e \prec e_*$ *for all* $e \in E$.

Necessity of these conditions is obvious. Indeed, if χ_S is the characteristic function (in $L^2(\alpha)$) of an α-measurable set S then $\chi_S \prec \chi_{S'}$ means $S \subset S'$, and in this case $\chi_{S'} - \chi_S = \chi_{S' \setminus S}$; moreover,

$$\chi_S + \chi_{S'} - \chi_{S \cap S'} = \chi_{S \cup S'}$$

holds for any S,S' with characteristic functions in $L^2(\alpha)$. Since $\chi_{S \cap S'}$ equals $\chi_S \cdot \chi_{S'}$ it is convenient to denote (any) vector g associated with e,f in the sense of (I.3) also by $e.f$.

The main problem is to establish sufficiency.

One firstly deduces, in a more or less straightforward way, the following properties (where only conditions (I.2 & 3) are used):

(a) If f,g , and $f+g$ belong to E , then $f \perp g$;

(b) $e \prec f \prec g$ implies $e \prec g$;

(c) $e.f$ is uniquely determined by e and f , and satisfies $\|e.f\|^2 = (e,f)$;

(d) The operation $e.f$ is commutative, associative, and distributive (i.e., $e.(f+g) = e.f+e.g$ whenever e,f , *and* $f+g$ belong to E);

(e) $e \perp f$ implies $e' \perp f'$ whenever $e' \prec e$, $f' \prec f$;

(f) $e \prec f$ implies $\|e\| \leq \|f\|$.

Also using conditions (I.4 & .5) one obtains, furthermore:

(g) Every "increasing" sequence $\{e_n\}$ in E is convergent to an element of E .

The next step in the proof is to associate with every $e \in E$ the subspace H_e of H which is spanned by the set $\{f : f \prec e\}$; let P_e denote the orthogonal projection operator from H onto H_e . Note that condition (I.5) implies $H_{e_*} = H$.

Using the (postulated or deduced) properties of the set E it is easy to prove that

$$P_e g = e.g \quad \text{(for every } e,g \text{);}$$

and hence, that $P_{e_*} = I$, $P_0 = 0$, and

$$P_e P_f g = P_f P_e g = P_{e.f} g \quad \text{(for every } e,f,g \text{).}$$

Using now condition (I.1) (completeness of E in H) we deduce hence that $P_e P_f = P_f P_e = P_{e.f}$. One deduces in an analogous way that $P_e + P_f = P_{e+f}$ whenever e+f also belongs to E and, again using conditions (I.4 & 5) that $P_{e_*} = I$, and $P_{e_n} \uparrow P_e$ for every increasing sequence $\{e_n\}$ converging to e in E .

At this stage we can refer to the theorem of spectral theory that every commuting system of projection operators in Hilbert space has a simultaneous spectral representation, i.e.

$$P_e = \int_0^1 \chi_e(x) dE_x$$

for some spectral measure $E(\omega)$ defined on the Boole algebra B of Borel subsets of $[0,1]$ and for the characteristic function

$$\chi_e(x) = \begin{cases} 1 & x \in S_e \\ 0 & x \in [0,1] \setminus S_e \end{cases} \quad \text{of some } S_e \in B \text{ .}$$

Let now $\alpha(\omega)$ be the scalar valued measure $\alpha(\omega) = (E(\omega)e_*, e_*)$, where e_* is the maximal element postulated by (I.5). Since $P_e e_* = e$ for every $e \in E$, we have for every finite linear combination $\Sigma\, c_i e_i$ $(e_i \in E)$:

$$\left\|\Sigma\, c_i e_i\right\|^2 = \left\|\left(\Sigma\, c_i P_{e_i}\right) e_*\right\|^2 = \int_0^1 \left|\Sigma\, c_i \chi_{e_i}(x)\right|^2 d\alpha(x) \text{ .}$$

Hence, the map

$$\Sigma\, c_i e_i \rightarrow \Sigma\, c_i \chi_{e_i}(x)$$

is isometric (and therefore uniquely defined and linear); by virtue of the completeness condition (I.1) it extends by closure to an isometric map of the whole space H into the function space $L^2(\alpha)$, every $e \in E$ being carried to $\chi_e \in L^2(\alpha)$. The map is actually *onto* i.e., unitary. It is enough to consider the characteristic function $\chi_S(x)$ of an *arbitrary* set $S \in B$ in [0,1]. This gives rise to an operator $U = \int_0^1 \chi_S(x) dE_x$ on H . The element Ue_* of H can be approximated by sums $\Sigma\, c_i e_i$ as closely as we wish. Thus, for every $\varepsilon > 0$, we can find $\Sigma\, c_i e_i$ such that $\varepsilon^2 > \|Ue_* - \Sigma\, c_i e_i\|^2 = \|(U - \Sigma\, c_i P_{e_i})e_*\|^2 = \int_0^1 |\chi_S(x) - \Sigma\, c_i \chi_{e_i}(x)|^2 \, d\alpha(x)$. This settles the fact that the map is onto $L^2(\alpha)$.

It remains only to show that actually we have $\chi_S = \chi_e$ for some $e \in E$. This can be achieved, starting from the relation above and by using properties of the sets S_e $(e \in E)$ deriving of (already known) properties of the elements e , namely $S_e \cap S_f = S_{e.f}$, $S_e \cup S_f = S_{e+f}$ (if $e+f$ also belongs to E), $S_{e_*-e} = S_{e_*} \setminus S_e = CS_e$, and

$$\bigcup_n S_{e_n} = S_e \quad \text{if} \quad S_{e_1} \subset S_{e_2} \subset \dots \quad \text{and} \quad e = \lim_n e_n \;.$$

This concludes the proof. Let us remark that if we also allow spaces $L^2(\alpha)$ with only σ-finite measure α , then condition (I.5) of the Theorem can be omitted. Indeed, the other conditions already imply that E can be splitted into countably many "orthogonal parts" each of which has a "maximal" element of type e_* .

3. Now we pass to a metric characterization of the set of positive functions in L^2 , as done in the paper [II].

<u>THEOREM II</u>. *Let P be a subset of the Hilbert space H (with elements denoted by u,v,w,...). In order that there exists a unitary map of H onto some $L^2(\alpha)$, carrying P onto the set $L^2_+(\alpha)$ of positive (i.e., nonnegative) functions in $L^2(\alpha)$ it is necessary and sufficient that the following conditions hold:*

(II.1) $(u,v) \geqq 0$ *for all* $u,v \in P$.

(II.2) $(u,\phi) \geqq 0$ *for all* $\phi \in H$ *and for all* $u \in P$ *implies* $\phi \in P$.

(II.3) ("Riesz interpolation property".) *If* $u_1 + u_2 = v_1 + v_2$ *for some* $u_i, v_k \in P$ *then there exist* $w_{ik} \in P$ *such that* $u_i = \sum_k w_{ik}$, $v_k = \sum_i w_{ik}$ $(i,k=1,2)$.
[*This property readily extends to sums of more than two terms.*]

The necessity part of the Theorem is fairly obvious. (For (II.2) set, in case $L^2_+(\alpha)$, say, $w_{11} = u_1 \wedge v_1$, $w_{12} = u_1 - w_{11}$, $w_{21} = v_1 - w_{11}$, $w_{22} = u_2 - w_{21}$.) It is also obvious that none of these three conditions is implied by the others.)

As for the sufficiency part, note first that conditions (II.1 & 2) imply that P is a closed cone in H , issued from the point 0 , and that every $\phi \in H$ admits a unique decompostion $\phi = u-v$ with $u,v \in P$, $u \perp v$. This cone implies a vector lattice structure in H ($\phi \leqq \psi$ iff $\psi - \phi \in P$) such that

$$0 \leqq u_1 \leqq u_2 \quad \text{and} \quad 0 \leqq v_1 \leqq v_2 \quad (\text{in } P) \text{ imply}$$
$$(u_1,v_1) \leqq (u_2,v_2) \ .$$

Moreover, it is easy to show that every nondecreasing, normbounded sequence $\{u_n\}$ in P is convergent.

The Riesz interpolation condition, combined with the other properties already stated, also implies the existence, for any $u,v \in P$, of a unique greatest minorant $u \wedge v$ and of a smallest majorant $u \vee v$ in P such that

$$(u \wedge v) + (u \vee v) = u + v \ .$$

Since we only consider separable Hilbert spaces H , there exists a sequence $\{u^{(n)}\}$ dense in P . Taken appropriate positive numerical factors c_n we can have convergence in $\Sigma\, c_n u^{(n)}$, the sum e_* belonging to P .

Next we consider the set

$$E = \left\{ e \in P : e \leq e_* \text{ and } e \prec e_* \right\} .$$

By its definition, this set E satisfies condition (I.5) of Theorem I. Conditions (I.2 & 3) also follow rather easily from the lattice properties of P , namely with $e_1 . e_2 = e_1 \wedge e_2$ for $e_1, e_2 \in E$. Again, Condition (I.4) is a consequence of the closure of P , and of the continuity of the inner product. More work is needed to prove that Condition (I.1) is also satisfied, i.e. that our set E is complete in P . Since P *is* complete, and since the sequence $\{u_n\}$ we have used to define the element e_* is dense in P , it is enough to show that every element u of this sequence (say $u = u^{(n)}$) can be approached in H as closely as we wish by finite combinations of elements of E .

To this effect, first observe that our u is majored by a positive multiple Λe_* of e_* (indeed, $\Lambda = 1/c_n$ does this). Thus, defining

$$u_\lambda = (\lambda e_*) \vee u \quad (\lambda \text{ a real parameter, } 0 \leq \lambda < \infty) ,$$

we have $u_0 = u$, $u_n = \lambda e_*$ if $\lambda \geq \Lambda$, and u_λ is a monotone *increasing*, while $u_\lambda - \lambda e_*$ is a monotone *decreasing* function of λ . It is easy to show that, moreover, u_λ is a *convex* function of the parameter λ , i.e.,

$$(1-\alpha)u_\lambda + \alpha u_\mu \geq u_{(1-\alpha)\lambda+\alpha\mu} \quad \text{for } \lambda,\mu \geq 0 , \text{ and } 0 \leq \alpha \leq 1 .$$

From known properties of P we infer that the quotient $\frac{1}{\mu-\lambda}(u_\mu - u_\lambda)$ belongs to P , and if $\mu \downarrow \lambda$, tends decreasingly to a limit belonging to E :

$$\lim_{\mu \downarrow \lambda} \frac{1}{\mu-\lambda}(u_\mu - u_\lambda) = e_\lambda \in E .$$

Note that $e_\lambda = e_*$ for $\lambda \geq \Lambda$. Convexity of u_λ also implies that e_λ is an increasing function of λ and that

$$(\mu-\lambda)e_\lambda \leq u_\mu - u_\lambda \leq (\mu-\lambda)e_\mu \quad \text{holds for } \mu > \lambda ;$$

whence,

$$0 \leqq (u_\mu - u_\lambda) - (\mu-\lambda)e_\lambda \leqq (\mu-\lambda)(e_\mu - e_\lambda) .$$

Set $\nu_k = k\Lambda/N$, where N is a fixed integer $\geqq 1$ and $k = 0,1,\dots,N$. Choosing $\mu = \nu_{k+1}$ and $\lambda = \nu_k$ in the preceding inequalities for $k = 0,\dots,N-1$, and adding, we get (recalling that $u_\Lambda = \Lambda e_*$, $u_0 = u$, $e_\Lambda = e_*$)

$$0 \leqq \Lambda e_* - u - \frac{\Lambda}{N} \sum_0^{N-1} e_{\nu_k} \leqq \frac{\Lambda}{N}(e_* - e_0) ,$$

and hence,

$$0 \leqq \frac{\Lambda}{N} \sum_0^{N-1} (e_* - e_{\nu_k}) - u \leqq \frac{\Lambda}{N} (e_* - e_0) \leqq \frac{\Lambda}{N} e_* .$$

This implies

$$\| (\Lambda/N) \sum_0^{N-1} (e_* - e_{\nu_k}) - u \| \leqq (\Lambda/N) \| e_* \| < \varepsilon$$

if N was chosen sufficiently large. As $e_* - e \in E$ for every $e \in E$, the proof of completeness of E is complete.

This is the way, sketched in its main points only, the proof of Theorem II was achieved in the paper [II].

4. Condition (II.1) implies for any $u,v \in P$:

$$\|u+v\|^2 = \|u\|^2 + 2(u,v) + \|v\|^2 \geqq \|u\|^2 + \|v\|^2 .$$

Also recall that, as a consequence of (II.1 & 2), every $\phi \in H$ can be decomposed in the form $\phi = u - v$ with $u,v \in P$ and $u \perp v$; hence,

$$\|\phi\|^2 = \|u\|^2 + \|v\|^2 .$$

This leads to the question what can be said about *closed convex cones* P in H satisfying the weaker conditions:

(III.1) *There exists a constant* $K > 0$ *such that*

$$\|u\|^2 + \|v\|^2 \leq K^2 \|u+v\|^2 \quad \textit{for any} \quad u,v \in P .$$

(III.2) *There exists a constant* $C > 0$ *such that every element* $\phi \in H$ *can be decomposed in the form* $\phi = u - v$ *with* $u,v \in P$ *and*

$$\|u\|^2 + \|v\|^2 \leq C^2 \|\phi\|^2 .$$

In contrast to Conditions (II.1 & 2), these conditions are invariant not only for unitary, but (with possibly different constants K and C) for affine maps of H also. (By affinity we mean a bicontinuous linear map of H onto itself, or onto another Hilbert space.) The same kind of invariance is obvious for Condition (II.3) also.

This question leads to the following result, proved in the paper [III].

THEOREM III. *Let* P *be a subset of the Hilbert space* H *(with elements denoted by* $u,v,w,\ldots$ *). In order that there exists an affine map of* H *onto some* $L^2(\alpha)$ *, carrying* P *onto* $L^2_+(\alpha)$ *, it is necessary and sufficient that the following conditions hold:*

(III.0) P *is a closed, convex cone issued at the origin* 0.

(III.1&2) *as above.*

(III.3) = (II.3) *, i.e. Riesz decomposition property.*

Necessity being obvious, the real problem is again with sufficiency. The proof is rather involved so we can give a short sketch of it only.

First we introduce the dual P^* of the cone P :

$$P^* = \left\{\phi : \phi \in H, (u,\phi) \geq 0 \text{ for all } u \in P\right\}$$

and the bidual

$$(P^*)^* = \left\{\tilde{\phi} : \tilde{\phi} \in H, (\tilde{\phi},\phi) \geq 0 \text{ for all } \phi \in P^*\right\} ,$$

and show that $P = (P^*)^*$. Using this relation we infer, relying on Riesz's method [R] that the order relation in H implied by the cone P

renders H a vector lattice with operations $\vee$ and $\wedge$ having the usual properties.

Next we consider a sequence $\{u_n\}$ of elements dense in P and choose numbers $c_n > 0$ such that $\sum_0^\infty c_n u_n$ converges; the sum e_* will also belong to P . Then we introduce the set

$$E = \left\{e : 0 \leq e \leq e_*, e \wedge (e_* - e) = 0\right\}$$

(notice the similarity to, and the difference from the definition of the set E in the preceding section).

We first observe that E is a Boole algebra, $0 \in E$; $e_* \in E$; $e, f \in E$ imply $e \wedge f$, $e \vee f \in E$; and if $e \geq f (\in E)$ then $e - f \in E$ also.

Next we consider a $u \in P$ such that $0 \leq u \leq \Lambda e_*$, and show that for any integer $N \geq 1$ we can find a finite linear combination $\Sigma \lambda_k e_k$ with coefficients $\lambda_k \geq 0$, of elements $e_k \in E$, such that

$$0 \leq \Sigma \lambda_k e_k - u \leq (\Lambda/N) e_* ,$$

and hence, by (III.1),

$$\| \Sigma \lambda_k e_k - u \| \leq K \Lambda \| e_* \| / N ,$$

the right hand side is as small as we wish if we had chosen N large enough.

This proves completeness of E in H , and moreover, let us define, for any fixed $e \in E$, a bounded linear operator Q_e on H , by first defining it as

$$Q_e \; \Sigma \lambda_k e_k = \Sigma \lambda_k (e \wedge e_k)$$

for finite linear combinations with coefficients $\lambda_k \geq 0$, of elements $e \in E$, and then extending it to H , with essential use of (III.1&2). We shall have eventually

$$\|Q_e\| \leq M \|e\| \text{ with a constant } M \text{ deriving from } c \text{ and } K$$

and $e \to Q_e$ will turn out to be a uniformly *bounded representation* of the

Boole algebra E by operators on Hilbert space; indeed, with

$$Q_0 = 0 \ , \ Q_e = I \ , \ Q_{e \wedge f} = Q_e Q_f \ ;$$

$$Q_{e-f} = Q_e - Q_f \quad \text{if} \quad f \leq e \quad (\text{in } E) \ .$$

Now, by a theorem of Dixmier [D], which in turn is an analog of some earlier results in my paper [IV] it follows that this representation of E is similar to a representation by (orthogonal) projections, i.e. there exists an affinity T in H such that

$$P_e = T^{-1} Q_e T$$

are projections. Since these projections are commuting ($P_e P_f = P_{e \wedge f} = P_{f \wedge e} = P_f P_e$) they have a simultaneous spectral representation, and the proof concludes in analogy to the proof of Theorem II.

It is an interesting question to what extent the measure α for which the affine representations $H \to L^2(\alpha)$, $P \to L^2_+(\alpha)$ are valid, are determined by further geometric properties of the cone P .

It turns out (see [III]) that α has the same number of atoms as there are extremal generatrices of the cone P , and that it is purely atomic iff P is spanned by its extremal generatrices.

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COMPACT OPERATORS

W.K. Vietsch

Almost anything we will discuss here, is contained in chapter 18 of the forthcoming book [18] by A.C. Zaanen. However, our presentation will be somewhat different, since we want to stress the fact that nearly all results are due to Prof.dr. A.C. Zaanen, his students or his students' students, and that Zaanen played an important role when these ideas originated. The 1963 paper [13] by Luxemburg and Zaanen will be starting point for our discussion.

1. Compact kernel operators

Let (X,Λ,μ) be a (totally) σ-finite measure space, i.e., X is a non-empty point set, Λ is a σ-algebra of subsets of X, and μ is a non-negative σ-additive measure on Λ such that X is the union of an at most countable number of sets of finite measure. We will assume that μ is not identically zero, and that the Carathéodory extension procedure has been applied to μ, so any subset of a measurable set of measure zero is also a measurable set of measure zero. The set of all realvalued μ-almost everywhere finitevalued μ-measurable functions on X will be denoted by $M^r(X,\mu)$. Functions in $M^r(X,\mu)$ differing only on a set of measure zero are identified and the set of the thus obtained equivalence classes is again denoted by $M^r(X,\mu)$. The set of all positive functions in $M^r(X,\mu)$ will be denoted by $M^+(X,\mu)$.

The mapping $\rho: M^+(X,\mu) \to \mathbb{R}^+ \cup \{\infty\}$ is called a *function norm* if

(a) $\rho(u) = 0$ iff $u = 0$,

(b) $\rho(au) = a\,\rho(u)$ for all $u \in M^+(X,\mu)$ and all $a \in \mathbb{R}^+$,

(c) $\rho(u+v) \le \rho(u) + \rho(v)$ for all $u, v \in M^+(X,\mu)$,

(d) $\rho(u) \le \rho(v)$ whenever $0 \le u(x) \le v(x)$ μ-a.e. on X.

The function norm ρ is extended to the whole of $M^r(X,\mu)$ by setting $\rho(f) = \rho(|f|)$ for all $f \in M^r(X,\mu)$, where $|f|(x) = |f(x)|$ μ-a.e. on X. The linear space of all $f \in M^r(X,\mu)$ satisfying $\rho(f) < \infty$ is denoted by $L_\rho(X,\mu)$;

such a space is called a *(real) normed Köthe space*. If $L_\rho(X,\mu)$ happens to be norm complete, then it is called a *(real) Banach function space*. Defining an ordering by $f \leq g$ iff $f(x) \leq g(x)$ μ-a.e. on X, we see that the Banach function space $L_\rho(X,\mu)$ can be made into a Banach lattice in a natural way.

For any μ-measurable subset E of X we will denote the characteristic function of E by χ_E. The projection P_E is defined by $(P_E f)(x) = f(x)\chi_E(x)$. The function norm ρ is called *saturated* if for any subset E of X with $\mu(E) > 0$ there exists a subset F of E with $\mu(F) > 0$ and $\rho(\chi_F) < \infty$. For any $u \in M^+(X,\mu)$ the number $\rho'(u)$ is defined by

$$\rho'(u) = \sup \{ \int_X u(x)\ v(x)\ d\mu(x) : v \in M^+(X,\mu),\ \rho(v) \leq 1\} .$$

If ρ is saturated, then ρ' is also a saturated function norm. The corresponding normed Köthe space $L'_\rho(X,\mu)$ is a Banach function space, which is called the *associate space* of $L_\rho(X,\mu)$. There is a one-one correspondence between the order continuous linear functionals on $L_\rho(X,\mu)$ and the functions in $L'_\rho(X,\mu)$.

The subset S of the Banach function space $L_\rho(X,\mu)$ with order continuous norm is said to be of *uniformly order continuous norm* if for any sequence $E_n \downarrow \emptyset$ and $\varepsilon > 0$ there exists an index N such that $\rho(f\chi_{E_n}) < \varepsilon$ for all $n \geq N$ and all $f \in S$.

Let $L_\rho(Y,\nu)$ and $M_\lambda(X,\mu)$ be Banach function spaces and let $t(x,y)$ be a realvalued $(\mu\times\nu)$-measurable function on $X\times Y$. The operator $T: L_\rho(Y,\nu) \to M_\lambda(X,\mu)$ defined by

$$(Tf)(x) = \int_Y t(x,y)\ f(y)\ d\nu(y)$$

is called an *absolute kernel operator* from $L_\rho(Y,\nu)$ into $M_\lambda(X,\mu)$ if for all $f \in L_\rho(Y,\nu)$ we have

$$\int_Y |t(x,y)\ f(y)|\ d\nu(y) \in M_\lambda(X,\mu) \qquad (+).$$

We will now assume that the norms in the Banach function spaces $M_\lambda(X,\mu)$ and $L'_\rho(Y,\nu)$ are order continuous and that the adjoint operator T^*, which is actually given by

$$(T^*g)(y) = \int_X t(x,y)\; g(x)\; d\mu(x) \;,$$

is an absolute kernel operator mapping $M'_\lambda(X,\mu)$ into $L'_\rho(Y,\nu)$. The final theorem from the paper [13] states that the operator T is compact iff one of the following conditions is satisfied:

(a) $\{Tf : \rho(f) \leq 1\}$ is of uniformly order continuous norm,

(b) $\| P_{E_n} T \| \to 0$ whenever $E_n \downarrow \emptyset$ in X,

(c) $\| T P_{F_n} \| \to 0$ whenever $F_n \downarrow \emptyset$ in Y,

(d) $\| P_{E_n} T P_{F_n} \| \to 0$ whenever $E_n \downarrow \emptyset$ in X and $F_n \downarrow \emptyset$ in Y.

As a special case we obtain the result that operators of finite double-norm, and in particular Hille-Tamarkin and Hilbert-Schmidt operators are compact. Another special case of the theorem had been obtained earlier by Andô, who proved in [2] that for Orlicz spaces condition (d) implies compactness of T.

The above mentioned theorem of Luxemburg and Zaanen has been generalized in many ways. Nowadays one knows a rather large number of results on compactness of order bounded or simply norm bounded operators on Banach lattices. We will discuss some of these results below.

2. Compactness in measure

Studying the proofs in [13] one finds a rather simple way to extend the theorem of Luxemburg and Zaanen. First of all, while by assuming (+) we restricted ourselves to order bounded operators, we may just as well consider norm bounded operators. Secondly, it is an unjustified limitation to consider only kernel operators. Indeed, as Fremlin showed in [8], there exist compact operators on Banach function spaces which cannot be represented as kernel operators.

The sequence of functions f_n in $M^r(X,\mu)$ is said to *converge in measure* on X to f if $\mu(\{x \in X: |f_n(x) - f(x)| \geq a\}) \to 0$ as $n \to \infty$ for every $a > 0$. We say that the subset S of the Banach lattice $L_\rho(X,\mu)$ is *compact in measure* if every sequence in S contains a subsequence which converges in measure on every subset E of X with $\mu(E) < \infty$. The norm bounded linear operator T from the Banach function space $L_\rho(Y,\nu)$ into the Banach function space $M_\lambda(X,\mu)$ is called *compact in measure* if the image under T of the unit ball of $L_\rho(Y,\nu)$ is compact in measure.

Assuming that $M_\lambda(X,\mu)$ has order continuous norm we can prove that the norm bounded linear operator $T: L_\rho(Y,\nu) \to M_\lambda(X,\mu)$ is compact iff one of the following conditions is satisfied:

(a) T is compact in measure and $\{Tf: \rho(f) \leq 1\}$ is of uniformly order continuous norm,

(b) T is compact in measure and $\| P_{E_n}T \| \to 0$ whenever $E_n \downarrow \emptyset$ in X.

If we assume in addition that the norm in $L'_\rho(Y,\nu)$ is order continuous and that the adjoint operator T* maps $M'_\lambda(X,\mu)$ into $L'_\rho(Y,\nu)$, we get that T is compact iff one of the following holds:

(c) T* is compact in measure and $\| TP_{F_n} \| \to 0$ whenever $F_n \downarrow \emptyset$ in Y,

(d) T and T* are compact in measure and $\| P_{E_n} TP_{F_n} \| \to 0$ whenever $E_n \downarrow \emptyset$ in X and $F_n \downarrow \emptyset$ in Y.

These results are generalizations of those in the previous section; it can easily be proved that any absolute kernel operator is compact in measure. Of course, in view of Fremlin's example, the converse is not true.

Krasnoselskii and several other Soviet mathematicians have investigated measure-compact operators on L_p-spaces with underlying measure spaces of finite Lebesgue measure; their results may be found in [10]. It should be pointed out that the case studied by them is a very special one, and that the proofs are therefore much simpler. The results for the general case were not written down until 1979 ([17]). By that time however, the whole idea had been generalized to a much more general setting.

3. Compact operators on Banach lattices

We will consider linear operators from the Banach lattice L into the Banach lattice M. Throughout we will assume that the norm in M is order continuous. By $\mathcal{L}_b(L,M)$ we denote the set of all order bounded linear operators from L into M. The well-known Riesz-Kantorovitch theorem (1936) states that $\mathcal{L}_b(L,M)$ is a Dedekind complete Riesz space.

An interesting subset of $\mathcal{L}_b(L,M)$ is the set of those operators which transform order bounded subsets of L into precompact subsets of M. We will refer to these operators as the *AM-compact* operators. Evidently, every compact linear order bounded operator is AM-compact. If L is an AM-space with unit, then any AM-compact operator is compact, which explains the terminology. The subset S of L is said to be almost order bounded if for every $\varepsilon > 0$ there exist f, g $\in$ L such that $S \subset [f,g]+\{h: \|h\| \leq \varepsilon\}$. It is easy to prove AM-compact operators do not only transform order bounded sets into precompact sets, but transform almost order bounded sets into precompact sets as well.

AM-compact operators were first studied by Dodds. In 1976 he and Fremlin proved that the AM-compact operators form a band in $\mathcal{L}_b(L,M)$.

Another interesting subset of $\mathcal{L}_b(L,M)$ is the class of operators which map norm bounded sets onto almost order bounded sets. These operators were first studied in [14] by Meyer-Nieberg, who called them L-weakly compact operators. Following the terminology of [18] we will refer to them as *semi-compact operators*. It is easy to prove that compact operators are semi-compact. Moreover, any positive operator majorized by a semi-compact operator is semi-compact.

From the properties mentioned above it follows immediately that if $0 \leq S \leq T$ in $\mathcal{L}_b(L,L)$ and the norm in L is order continuous, then compactness of T implies compactness of S^2. It is not very difficult to prove that in this situation compactness of T implies compactness of S^3 even if L does not have order continuous norm. Surprisingly, these simple consequences of the work of Dodds, Fremlin and Meyer-Nieberg were not noted until a few years later, when Aliprantis and Burkinshaw published [1]. Another result of this type, due to Van Eldik, states that, without continuity assumptions, S^4 is compact whenever $0 \leq S \leq T$ with T AM-compact as well as semi-compact.

Dodds proved that if L* and M have order continuous norms, $T \in \mathcal{L}_b(L,M)$ is compact iff T is both AM-compact and semi-compact. This is a generalization of the order bounded case of the theorem mentioned in the previous section, AM-compactness corresponding with compactness in measure and semi-compactness corresponding with the image of the unit ball being of uniformly order continuous norm. As a direct consequence we have that compactness of T implies compactness of S whenever $0 \leq S \leq T$ in $\mathcal{L}_b(L,M)$ and L* and M have order continuous norms. This corollary, the main result of [5], is known as the Dodds-Fremlin theorem.

In view of the Dodds-Fremlin theorem one might conjecture that the compact order bounded operators from L into M form a band in $\mathcal{L}_b(L,M)$ if the norms in L* and M are order continuous. Unfortunately this conjecture is false. Indeed, in [11] Krengel has given an example of a compact operator in $\mathcal{L}_b(\ell_2,\ell_2)$ with a non-compact modulus. So the class of semi-compact operators does not share the nice order theoretical properties of the class of AM-compact operators.

4. Compact operators and indices

The Banach lattice L is said to have the *strong ℓ_p-composition property* $(1 \le p < \infty)$ if there exists a finite positive constant C such that for every finite disjoint sequence $\{u_i : i = 1,2, \ldots, n\}$ in L^+ we have

$$\| u_1 + u_2 + \ldots + u_n \| \le C\, (\| u_1 \|^p + \| u_2 \|^p + \ldots + \| u_n \|^p)^{1/p}.$$

The Banach lattice L is said to have the *strong ℓ_p-decomposition property* $(1 \le p < \infty)$ if there exists a finite positive constant C such that for every finite disjoint sequence $\{u_i : i = 1,2, \ldots, n\}$ in L^+ we have

$$(\| u_1 \|^p + \| u_2 \|^p + \ldots + \| u_n \|^p)^{1/p} \le C\, \| u_1 + u_2 + \ldots + u_n \| .$$

The notions of strong ℓ_p-composition property and strong ℓ_p-decomposition property were introduced by Dodds in [4]. They coincide with the notions of upper-p-estimate and lower-p-estimate as used by Figiel and Johnson in [7] and by Lindenstrauss and Tzafriri in [12].

The Banach lattice L is said to have the *ℓ_p-composition property* $(1 \le p \le \infty)$ if

$$\sup \{\| a_1 u_1 + a_2 u_2 + \ldots. + a_n u_n \| : n = 1,2, \ldots\} < \infty$$

whenever $(a_1, a_2, \ldots) \in \ell_p^+$ and $\{u_i : i = 1,2, \ldots\}$ is a disjoint sequence in L^+ with $\| u_i \| \le 1$ for all i. The Banach lattice L is said to have the *ℓ_p-decomposition property* $(1 \le p \le \infty)$ if $(\| u_1 \| , \| u_2 \| , \ldots) \in \ell_p$ whenever $\{u_n : n = 1,2, \ldots\}$ is an order bounded sequence of disjoint positive elements in L. Dodds proved in [4] that a Banach lattice has the ℓ_p-decomposition property iff it has the strong ℓ_p-decomposition property and by a lemma of Meyer-Nieberg ([15]) we have that a Banach lattice has the ℓ_p-composition property iff it has the strong ℓ_p-composition property. The ℓ_p-composition property and ℓ_p-decomposition property were introduced by Grobler for Banach function spaces ([9]) and by Dodds for Banach lattices

([4]).

Every Banach lattice has the ℓ_∞-decomposition property, and if L has the ℓ_p-decomposition property and $r \geq p$, then L has also the ℓ_r-decomposition property. If L has the ℓ_p-decomposition property for some $p < \infty$, then the norm in L is order continuous. The *upper index* σ_L of the Banach lattice L is defined by

$$\sigma_L = \inf \{p : \text{L has the } \ell_p\text{-decomposition property}\}.$$

Every Banach lattice has the ℓ_1-composition property, and if $r \leq p$ and L has the ℓ_p-composition property, then L has also the ℓ_r-composition property. The *lower index* s_L of L is defined by

$$s_L = \sup \{p : \text{L has the } \ell_p\text{-composition property}\}.$$

It can be proved that L has the ℓ_p-decomposition property iff L* has the ℓ_q-composition property ($p^{-1} + q^{-1} = 1$) and that L has the ℓ_p-composition property iff L* has the ℓ_q-decomposition property ($p^{-1} + q^{-1} = 1$). Hence $\sigma_L^{-1} + s_{L^*}^{-1} = 1$ and $s_L^{-1} + \sigma_{L^*}^{-1} = 1$. If L is of infinite dimension, we have $1 \leq s_L \leq \sigma_L \leq \infty$.

Note that $s_L = \sigma_L = p$ for $L = L_p(X,\Lambda,\mu)$. The upper and lower indices have been calculated for a number of Banach function spaces: by Grobler ([9]) for Orlicz spaces with non-atomic underlying measure space of finite measure and for the space of Korenblyum, Krein and Levin, by Vietsch ([17]) for Beurling spaces and for the space of Gould, and by Creekmore and Vietsch ([3], [17]) for Lorentz spaces.

Now let L and M be Banach lattices of infinite dimension such that $s_L > \sigma_M$. Then any order bounded operator from L into M is semi-compact. It follows that in this case $T \in \mathcal{L}_b(L,M)$ is compact iff T is AM-compact. This result is due to Dodds ([5]). Several particular cases of the theorem appear in the literature. The oldest result of this kind is a theorem of Pitt (1936), who proved in [16] that every bounded linear operator from ℓ_p into ℓ_r is compact if $1 \leq r < p < \infty$. Andô ([2]) proved the theorem for absolute kernel operators on Orlicz spaces, and Grobler ([9]) extended Andô's result to general Banach function spaces.

5. Epilogue

We have only described a small part of the theory. For other results, e.g. about non order bounded operators on Banach lattices, and for different expositions we may refer to [5], [18], [6], [1]. It has been our purpose to emphasize Zaanen's role in the development of the theory. Many of his students and his students' students have made a contribution: Luxemburg and Zaanen himself, who wrote the inspiring paper [13], Grobler and Dodds, who established the theory of indices ([9], [4]), Dodds, who proved most of the important results ([5]), and Schep, De Pagter, Grobler, Van Eldik and Vietsch, who improved and simplified proofs and contributed additional results. Compact operators on Banach lattices, or Prof.dr. A.C. Zaanen as the head of a mathematical family.

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CURRICULUM VITAE OF A.C. ZAANEN

Name	: Zaanen, Adriaan Cornelis, born June 14, 1913 at Rotterdam (The Netherlands)
Nationality	: Dutch.
Name of wife	: Van der Woude, Ada Jacoba.
Children	: Four sons, born in 1946, 1948, 1957 and 1959.
University education	: Mathematics at the University of Leiden, 1930-1938.
Degree	: Doctor's degree in Mathematics at the University of Leiden, March 10, 1938 on a thesis entitled: *"Over reeksen van eigenfuncties van zekere randproblemen"* (thesis supervisor: Prof.dr. J. Droste).
Appointments	: 1938-1947 High school teaching. 1947-1950 Professor of Mathematics at the Faculty of Technology of the University of Indonesia (Bandung, Java, Indonesia). 1950-1956 Professor of Mathematics at the Delft Institute of Technology. 1956-1981 Professor of Mathematics at the University of Leiden. 1982 Teaching assignment at the Universtiy of Leiden. 1960-1961, 1968-1969 Visiting Professor at the California Institute of Technology.
Memberships	: Dutch Mathematical Society, since 1937; American Mathematical Society, since 1948; Royal Dutch Academy of Science, Amsterdam, since 1960.
Other positions	: 1. Examination Committee "M.O.-Wiskunde" 1951- member 1974 chairman 1975- vice-chairman. 2. Editor of "Nieuw Archief voor Wiskunde" 1953- 3. Member of the Board of the Mathematical Centre (Amsterdam) 1950-1979.

4. Chairman of the Dutch Mathematical Society 1970-1972;
   Member of the Board of the Dutch Mathematical Society 1970-1973.
5. Member of the Board of advisory editors of the series "North-Holland Mathematical Library".
6. Member of the Royal Dutch Academy of Science since 1960, occasionally secretary of Z.W.O., member of the committee of exact sciences 1965-1967.
7. Member of the Mathematics section of the "Akademische Raad" for several years.
8. Chairman of the department of Mathematics of Leiden University, 1965-1971;
   Member of the Faculty Board 1966-1973;
   Chairman of the Subfaculty Board 1976-1979;
   Member of the Subfaculty Board 1976-1980.

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W.A.J. Luxemburg	: Banach function spaces, October 12, 1955.
B.C. Strydom	: Abstract Riemann integration, May 20, 1959.
M.A. Kaashoek	: Closed linear operators on Banach spaces, February 26, 1964.
A.C. van Eijnsbergen	: Beurling spaces, a class of normed Köthe spaces, July 3, 1967.
J.J. Grobler	: Non-singular linear integral equations in Banach function spaces, May 6, 1970.
N.A. van Arkel	: Algebras of holomorphic functions of n complex variables, June 24, 1970.
C.B. Huijsmans	: Prime ideals in commutative rings and Riesz spaces, May 17, 1973.
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P. Maritz	: Integration of set-valued functions, June 11, 1975.
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