DYNAMIC FEEDBACK IN FINITE- AND INFINITE-DIMENSIONAL LINEAR SYSTEMS

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CONTENTS

INTRODUCTION

In the beginning of his celebrated 1868 paper "On governors", James Clerk Maxwell made a careful distinction between two types of regulators. One type consists of the so-called 'moderators', which are only able to diminish the effect of the disturbances. After describing a few mechanisms of this sort, Maxwell writes:

"In all these contrivances an increase of driving-power produces an increase of velocity, though a much smaller increase than would be produced without the moderator. But if the part acted on by centrifugal force, instead of acting directly on the machine, sets in motion a contrivance which continually increases the resistance as long as the velocity is above its normal value, and reverses its action when the velocity is below that value, the governor will bring the velocity to the same normal value whatever variation (within the working limits of the machine) be made in the driving-power or the resistance.

I propose at present, without entering into any details of mechanism, to direct the attention of engineers and mathematicians to the dynamical theory of such governors." (MAXWELL (1868; p.271))

In today's terminology, the second type of regulators to which Maxwell refers is said to provide *dynamic feedback*. The offset is eliminated via an integration of the error, which means that the controller brings its own dynamics into the feedback loop.

The concept of dynamic feedback (or *integral control*) has given rise to important theoretical developments from its very inception on. Second-order models were used to study the motion of engines, but the combination with an integral controller led Maxwell to consider third-, fourth- or even fifth-order equations. The resulting question of giving verifiable necessary and sufficient conditions for a polynomial of arbitrary degree to have only roots with negative real parts led to the well-known theorems of E.J.Routh and A.Hurwitz. An interesting survey of the developments around the stability of polynomials has been given by BENNETT (1979), and a detailed account can be found in BATEMAN (1945).

In the fast and sweeping development of mathematical control theory

in the last twenty or so years, dynamic feedback has not been a central
concept. Rather, one can say that a key role has been played by the notion
of *state feedback*. In a rough paraphrase of the system-theoretic definition,
the *state* of a system at a given time consists of a complete specification
of the parameters that determine the system's future behaviour. So when
studying state feedback, one assumes that all relevant information about
the system is available to the controller.

This idealization turned out to be extremely useful for the description
of fundamental properties of systems such as *controllability* (introduced
by R.E.Kalman at the end of the fifties). Moreover, the theory of state
feedback was used in dual form to obtain an *observer theory*, which answered
the question of how to reconstruct the state from the given measurements.
Thus, a feasible controller could be derived by combining a state feedback
law with an observer. The solution obtained in this way could in fact even
be proved to be the optimal solution for the standard regulator problem of
stochastic control; this is the so-called *separation principle*.

The method of reducing a two-sided problem (input/output) to two easier
to solve one-sided problems (input/state and state/output) has been one of
the main successes of the state space approach to control theory. Not
surprisingly therefore, the method is also strongly present in the 'geometric'
approach to linear multivariable systems. This approach was developed from
1969 on by G.Basile, G.Marro, A.S.Morse, W.M.Wonham and many others; it is
characterized by its consistent formulation of results in terms of subspaces
of the state space. In Wonham's trend-setting book (WONHAM (1974)), a
systematic pattern is discernible which proceeds from 'restricted' to
'extended' problems. Problems of the first kind have to be solved using
state feedback, while an observer is added to study the 'extended' problem
in which the control has to be based on the observations. The reliance on
duality and the resulting one-sidedness of the theory are clearly illustrated
by the fact that the concept of 'controlled invariant subspace' (i.e., a
subspace that can be made invariant by state feedback) is used throughout
Wonham's book, whereas the dual concept of 'conditioned invariant subspace'
is only mentioned in one of the exercises.

In the present monograph, we want to show that it is possible to go
one step further and to develop a *two-sided* approach. This means that we are
going to make a direct study of dynamic feedback. The basic idea is the
introduction of the concept of a *compensator couple* (Section 2.2), which is
used to describe the subspaces that can be made invariant by dynamic feed-

back. The two-sidedness of our method is reflected in the fact that a
compensator couple is a *pair* of subspaces, one of which is controlled
invariant and the other conditioned invariant.

The price one has to pay for the use of the separation method is
particularly clear in the context of systems with an infinite-dimensional
state space (ditributed parameter systems). One has been able, for instance,
to solve the linear-quadratic optimal control problem for several important
classes of infinite-dimensional systems, but the resulting controllers are of
infinite order and therefore impractical. To design finite-dimensional
compensators, one has to leave the idea of separation. In Chapter 5, we shall
present a method of compensator design based on a certain adjustment between
the 'feedback' and the 'observer' parts (Section 5.4). This method is able to
produce controllers of low order, as will be illustrated in various examples.
Moreover, we shall prove the existence of a stabilizing compensator of finite
order for a large class of infinite-dimensional systems, including those
described by diffusion and delay equations (Thm.5.2).

The theory of Ch.5 is based on that of Ch.2, in which we treat the
stabilization problem for finite-dimensional systems. A general construction
theorem is given (Thm.2.4), which is based on the notion 'compensator couple'.
We prove that the traditional state-space techniques of compensator design can
all be described within this framework (Section 2.4). We also investigate the
question of when a separation between 'feedback' and 'observer' action is
possible for a given compensator. We propose a precise definition of this
separation, taken here in a more general sense than that of the 'separation
principle' (Section 2.2), and we prove that any compensator for which such a
separation is possible can be constructed by our method.

Although in Wonham's book the disturbance decoupling problem more or less
bears the banner of the 'geometric approach', the "DDP" is only solved for
state feedback, whereas the natural extension to output feedback is not even
mentioned. Indeed, one needs a compensator couple to solve the general problem.
We give the solution in Ch.3 as a special case of a theorem (Thm.3.3) that
attains a level of generality which seems to be new. Disturbance decoupling
is combined with a regulation task, and not only the solution of the problem
of disturbance decoupling by observation feedback but also many other results
are recovered as special cases (Section 3.4). Even at this level of generality,
the solution we obtain is completely constructive (Section 3.3).

In the final Chapter 6, we consider tracking and regulation problems for
infinite-dimensional systems. Again, we give an existence theorem for finite-

dimensional controllers (Thm.6.2) and an algorithm for the actual design of
such controllers (Section 6.5). The practicability of the method is shown
by several examples. One of these examples concerns a delay systems which is
to be protected against constant disturbances (Section 6.7). The ability of
a regulator to eliminate the effect of a constant disturbance is cited by
Maxwell as a typical distinction between 'moderators' and 'governors'
(MAXWELL (1868; p.274)). Also, one may note that delay effects were at least
partially responsible for the 'hunting' of steam engines (slow oscillatory
variations in the work of the engine), one of the main energy problems of
the nineteenth century. Our solution is simple enough to have been
implementable by the methods of Maxwell's time.

Admittedly, this solution is a bit on the late side. Our final example
(Section 6.8) is perhaps more modern in that it has been inspired by a problem
in nuclear reactor design.

Organization

This work is divided in two parts, each consisting of three chapters.
The first part is concerned with finite-dimensional systems, the second with
systems having an infinite-dimensional state space. Both parts are organized
in the same way: the first chapter (Ch.1 and Ch.4) contains introductory
material, the second chapter (Ch.2 and Ch.5) is concerned with stabilization
problems, and regulation problems are treated in the third chapter of each
part (Ch.3 and Ch.6).

The reader who is interested in the finite-dimensional theory and who
is reasonably well acquainted with the 'geometric' approach to linear
multivariable systems could skim over Ch.1 to pick up the new results and
then proceed directly to Ch.2 or Ch.3. The third chapter is fairly independent
from the second, except for the definition of a "compensator couple" that is
given in Section 2.2. The reader who is not initiated in the 'geometric'
theory could read Ch.1 as an introduction, but it should be said that the
primary purpose of this chapter has not been to give an extensive motivation
for the presented theory from a more general control point of view.

Although some of the main ideas of the second part are already present
in the first part, both parts are mathematically independent and so the
reader who is interested in control of distributed parameter systems could
start immediately with Ch.4. Most of the material in this chapter will be
familiar to those versed in the 'semigroup' approach to infinite-dimensional

systems. Although in principle Ch.6 is independent from Ch.5, the idea of constructing a finite-dimensional controller is developed in Ch.5 in a somewhat easier context, so the reader is advised to read these chapters in their natural order.

All chapters are divided into sections. At the beginning of each chapter, a brief description of the organization of the sections is given.

CHAPTER 1

CONTROL AND OBSERVATION

We start our treatise by briefly explaining some of the central notions
of linear systems theory, such as: the concept of state, the notions of
stability, controllability and observability, and the relationship
between decomposition of systems and invariant subspaces. The problem of
changing the dynamics of a given system by state feedback is fundamental.
On one hand, we have at our disposal the solution of the "pole placement
problem", which allows us to relocate   eigenvalues. On the other hand, the
structural change in a system brought about by state feedback may be captured
in the concept of "controlled invariant subspace". Combining the two results
we can also consider pole placement problems under certain structural restric-
tions on the nature of the feedback. Motivated by our needs in later chapters,
we give a rather extensive treatment of this type of problems and present
some new results. The theory will also be needed in its dualized form, and
so many results will be given together with their mirror images obtained by
dualization.

The chapter contains five sections. The first section gives a brief
introduction   into linear systems and explains some notation and termi-
nology. In the second section, we give the basic standard results on pole
placement. Controlled and conditioned invariant subspaces are defined in
section 3, and section 4 considers pole placement with certain restrictions
on the feedback map. Finally, in section 5 we discuss stabilizability and
detectability properties related to invariant subspaces.

1.1. Introduction to linear systems

Let us suppose that we have a process which evolves in time according
to a set of differential equations, which in the first part of this thesis,
we shall assume to be ordinary differential equations. Furthermore, we shall
always assume that the equations are linear and have constant coefficients.
If we take all equations together and write them in a first-order form, then
the evolution of the process under consideration is described by an equation

$$(1.1) \qquad x'(t) = Ax(t)$$

where $x(\cdot)$ is a function of time with values in some finite-dimensional linear space X called the *state space*, and A is a linear mapping from X into X to which we shall refer as the *system matrix*. The vector $x(t)$ is called the *state* of the system at time t; if the state at some time t_0 is given, then the future behaviour of the system is completely determined by the equation (1.1). The 'state space approach' will be used troughout in this work.

In the real world, the evolution of a process is always affected by disturbances. There are many ways in which such disturbances can be modeled, and we shall encounter some of them further on. The simplest model for a disturbance is obtained by setting a non-zero initial condition for the equation (1.1). This represents a sudden departure of the system from the origin, which is thought of as representing the nominal operating mode of the system. It is a well-known fact that the state will return to the origin regardless of the initial condition, if and only if the eigenvalues of the mapping A are in the left half of the complex plane:

$$\sigma(A) \subset \mathbb{C}_- : = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\ \lambda < 0\}.$$

(This result, and in fact all the necessary material on ordinary differential equations, is covered in textbooks like HIRSCH & SMALE (1974) or ARNOLD (1973)). Moreover, the rate of convergence will be faster according as the eigenvalues of A are situated further to the left. In general, we shall say that the mapping A is *stable* if its eigenvalues are contained in some prescribed part of the complex plane demoted by $\mathbb{C}_g$. Here, "g" means "good". The complement of $\mathbb{C}_g$ will be denoted by $\mathbb{C}_b$, where "b" stands for "bad".

Very broadly speaking, the purpose of *control* is to make the controlled process satisfy certain performance specifications which would not be satisfied if the system was left to itself. For instance, the uncontrolled system (1.1) may not be stable in the above sense. Of course, we have to make certain assumptions about how the control enters into the system. We shall assume that the control is a function of time with values in a finite-dimensional linear space U, and which acts on the system through a linear mapping B from U to X, in the following way:

(1.2) $x'(t) = Ax(t) + Bu(t).$

The operator B is called the *input mapping*; clearly, it is no restriction to assume that this operator is injective, and we shall always do so. If we want to talk about *feedback*, then the action of the control is to be based on some observation of the state. We shall assume that this observation is provided by some linear operator C mapping X into another finite-dimensional linear space Y:

(1.3) $y(t) = Cx(t).$

The operator C is called the *output mapping*, $y(t) \in Y$ is called the *observation* at time t. Without loss of generality, we can assume that C is surjective.

A fundamental notion describing the relation between the system matrix A and the input mapping B is that of *controllability*. Suppose that the system (1.1) has been obtained by combining two sets of linear differential equations, of the following form:

(1.4.1) $x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t)$

(1.4.2) $x_2'(t) = A_{22}x_2(t).$

Also suppose that the input mapping is such that the control $u(\cdot)$ only affects the first set of state variables, so that (1.2) reads:

(1.5.1) $x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t)$

(1.5.2) $x_2'(t) = A_{22}x_2(t).$

Then it is clear that the behaviour of $x_2(\cdot)$ is not influenced by the control function. In particular, if A_{22} is unstable, then there is no way in which the system as a whole can be stabilized by a suitable choice of $u(\cdot)$. Pictorially, the situation can be described as follows.
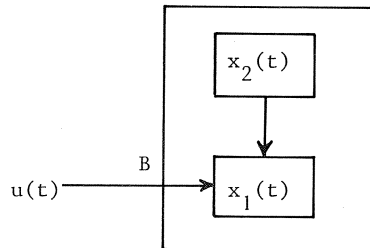
Fig. 1.1. Uncontrollable system

Whether this phenomenon occurs or not depends on the mappings A and B. The pair (A,B) is said to be *controllable* if there is no decomposition of (1.2) in the form (1.5).

A similar situation can arise with respect to the observation. If the system (1.1) is decomposed as in (1.4) and if the output equation (1.3) is actually of the form

(1.6) $y(t) = C_2 x_2(t)$

then one sees that the observation $y(t)$ does not contain any information about $x_1(t)$. We can illustrate this in a diagram:
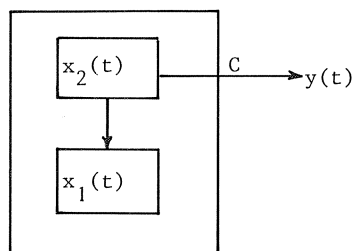

Fig. 1.2. Unobservable system

This phenomenon depends on the mappings A and C. The pair (C,A) is called *observable* if the equations (1.1) and (1.3) do not decompose into (1.4) and

(1.6). The concepts of controllability and observability also play a crucial role in *realization theory*, in which a state space description is constructed from a system's input-output behaviour (see for instance KALMAN, FALB & ARBIB (1969)).

One may note that the figures 1.1 and 1.2 can be obtained from eachother by reversing the arrows (and renaming the parts). In the language of linear algebra, this duality just comes down to transposition of matrices. In such a way, statements concerning a pair consisting of a system matrix and an input mapping can be turned into statements about a pair consisting of a system matrix and an output mapping, and vice versa. In the rest of this chapter, we shall omit proofs of dual results with reference to this *principle of duality*.

If we reconsider the system matrix A and the input mapping B as they appear in the uncontrollable system (1.5), we see that these mappings can be written down in a "block matrix" format in the following way:

$$(1.7) \qquad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad , \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}.$$

From this, it is clear that the controllable vector $x_1(t)$ corresponds to a (non-trivial) *subspace* of the state space X which is invariant for A and which contains the range of the input mapping B. So we can say: The pair of mappings (A,B) is controllable if and only if the only A-invariant subspace containing Im B (the range of B) is the trivial one, namely the whole state space X on which A acts. If we write

$$(1.8) \qquad <A \mid V> = \cap \ \{W \mid AW \subset W, \ W \supset V\}$$

for the smallest A-invariant subspace containing V, then the condition for controllability is:

$$(1.9) \qquad <A \mid \text{Im } B> = X.$$

Dually, a decomposition in the form (1.4) and (1.6) can be made if and only if there is a non-trivial A-invariant subspace contained in Ker C (the kernel of C). The largest A-invariant subspace in KerC will be denoted by

$$(1.10) \qquad <V \mid A> = \Sigma \ \{W \mid AW \subseteq W, W \subset V\}.$$

Then the condition for observability is

(1.11) $\langle \text{Ker } C \mid A \rangle = 0.$

(We use the symbol 0 for the trivial subspace consisting of only the zero
element.)

This discussion of controllability and observability also shows the
importance of invariant subspaces in describing structural properties of
linear systems. Below, we shall introduce the concepts of "controlled in-
variant subspace" and "conditioned invariant subspace", which arise natu-
rally in this context. The approach to linear systems that makes strong
use of these invariant subspaces has been initiated by G. BASILE &
G. MARRO (1969b) and by W.M. WONHAM & A.S. MORSE (1970); it is now commonly
described as the *geometric* approach (WONHAM (1974,1979)).

Let us conclude this section by briefly reviewing the basic notation
and terminology that will be used in the sequel. In the first part of this
work (chapters 1 to 3), all spaces will be finite-dimensional real vector
spaces. The standard complexifications of these vector spaces will be used
ocasionally without change of notation. Spaces and subspaces will be denoted
by script capitals, some of which have a standard meaning: X will always be
the state space and its dimension will always be denoted by n. U will be the
input space (of dimension m), and Y is the output space (of dimension p). The
elements of these vector spaces will as a rule be denoted by the corresponding
lower case letters: $x \in X$, $u \in U$ etc.. Roman capitals will be used for linear
mappings, and the following denotations will be standard: A: $X \rightarrow X$ is the
system matrix, B: $U \rightarrow X$ is the input mapping, and C: $X \rightarrow Y$ is the output
mapping. The words "matrix", "mapping" and "operator" will all be used as
synonyms for "linear mapping". If $V \subset X$ is an invariant subspace for T, then
we shall write

(1.12)   T: V = restriction of T to V

(1.13)   T: X/V = quotient mapping induced by T on X/V.

(Quotient mappings and all of the further material on linear algebra we shall
need can be found, for instance, in GANTMACHER (1959)). More generally, if
V_1 and V_2 are T-invariant subspaces and $V_1 \subset V_2$, then we shall write

(1.14) T: V_2/V_1 = restriction of T: X/V_1 to V_2/V_1 =

= quotient mapping induced by T: V_2 on V_2/V_1.

We have already used the notation $\sigma(T)$ for the set of eigenvalues of T.
We shall want to add spectra of different operators "counting multiplicities".
To be able to state this properly, we introduce the *spectral multiplicity
function* which is the following function from $\mathbb{C}$ to $\mathbb{Z}$.

(1.15) $\underline{\sigma}(T)(\lambda)$ = k if λ is an eigenvalue of T with multiplicity k

= 0 if λ is not an eigenvalue of T.

Such functions inherit a natural additive structure and a partial ordering from
$\mathbb{Z}$. We shall freely use simple properties of the spectral multiplicity func-
tion, such as

(1.16) $\underline{\sigma}(T: (V_1+V_2)/V_1) = \underline{\sigma}(T: V_2/(V_1 \cap V_2))$

where V_1 and V_2 are T-invariant subspaces.

1.2. Basic results on pole placement

The eigenvalues of the system matrix are often called "poles" in the
control literature; this terminology stems from the transfer function ap-
proach to linear systems, in which these eigenvalues appear as the zeros
of the denominator polynomial of the transfer function. The problem of pole
placement by state feedback is: Given a system matrix A and an input mapping
B, find a *feedback mapping* F such that the eigenvalues of A + BF have a
prescribed location. A celebrated theorem says that if the pair (A,B) is
controllable, then this can always be done subject only to the restrictions
imposed by the dimension of the state space and by the use of the real number
field. The version we shall present is a slight variation of the one given
by WONHAM (1967); other versions were proved by J. Bertram in 1959 (according
to KALMAN, FALB and ARBIB (1969; p.49)), RISSANEN (1960) and POPOV (1964).

To get a proper statement of the result, let us introduce some
terminology. A *multiplicity function* will be a nonnegative function from
$\mathbb{C}$ to $\mathbb{Z}$ that assumes nonzero values only at finitely many points of $\mathbb{C}$. The

total multiplicity of such a function is defined by

(1.17) $m(f) = \sum_{\lambda \in \mathbb{C}} f(\lambda)$.

We shall say that a multiplicity function f is *symmetric* if $f(\lambda) = f(\bar{\lambda})$ for all $\lambda \in \mathbb{C}$. Now we can state:

THEOREM 1.1 *Let a system matrix* A: $X \to X$ *and an input mapping* B: $U \to X$ *be given, and suppose that the pair* (A,B) *is controllable. For any symmetric multiplicity function f of total multiplicity* n (= dim X), *there exists a real feedback mapping* F: $X \to U$ *such that* $\underline{\sigma}$(A+BF) = f.

PROOF See WONHAM (1979; p. 50)   ⊠

More detailed results are known (see ROSENBROCK (1970)), but this form of the theorem will suffice for our purposes. One may ask what happens if the pair (A,B) is not controllable; the answer follows almost immediately from the theorem.

COROLLARY 1.2 *Let a system matrix* A: $X \to X$ *and an input mapping* B: $U \to X$ *be given. For all* F: $X \to U$, *the subspace* <A | Im B> *is* (A+BF)-*invariant, and we have*

(1.18) $\underline{\sigma}$(A: X/<A | Im B>) = $\underline{\sigma}$(A + BF: X/<A | Im B>).

On the other hand, given any symmetric multiplicity function f with total multiplicity equal to dim<A | Im B>, *there exists an* F: $X \to U$ *such that*

(1.19) $\underline{\sigma}$(A + BF: <A | Im B>) = f.

PROOF Take x $\in$ <A | Im B> and F: $X \to U$; then Ax $\in$ <A | Im B> because <A | Im B> is A-invariant and BFx $\in$ <A | Im B> because BFx $\in$ Im B. So (A+BF)x $\in$ <A | Im B> and we see that <A | Im B> is (A+BF)-invariant for all F. The actions of A and A + BF modulo <A | Im B> are the same because Im(A-(A+BF)) $\subseteq$ Im B; hence the equality (1.18). The final statement of the corollary follows by applying Theorem 1.1 to the pair of mappings obtained by restriction to <A | Im B>.   ⊠

We shall say that the pair (A,B) is *stabilizable* if there exists a feedback mapping F such that $\sigma(A+BF) \subset \mathbb{C}_g$.

COROLLARY 1.3 *The pair (A,B) is stabilizable if and only if*

(1.20) $\sigma(A: X/<A \mid \text{Im } B>) \subset \mathbb{C}_g$.

PROOF. This is immediate from the foregoing corollary.   ⊠

We shall also need the dual results with respect to a pair consisting of a system matrix A and an output mapping C. The problem is then to assign the eigenvalues of A + GC by a suitable choice of the *injection matrix* G: $Y \to X$. By dualization of Corollary 1.2, we obtain:

COROLLARY 1.4 *Let a system matrix A: $X \to X$ and an output mapping C: $X \to Y$ be given. For all G: $Y \to X$, the subspace <Ker C $\mid$ A> is (A+GC)-invariant, and we have*

(1.21) $\underline{\sigma}(A: <\text{Ker } C \mid A>) = \underline{\sigma}(A + GC: <\text{Ker } C \mid A>).$

On the other hand, given any symmetric multiplicity function f with total multiplicity equal to dim $X/<\text{Ker } C \mid A>$, *there exists a G: $Y \to X$ such that*

(1.22) $\underline{\sigma}(A + GC: X/<\text{Ker } C \mid A>) = f.$

The pair (C,A) is said to be *detectable* if there exists an injection mapping G such that $\sigma(A+GC) \subset \mathbb{C}_g$.

COROLLARY 1.5 *The pair (C,A) is detectable if and only if*

(1.23) $\sigma(A: <\text{Ker } C \mid A>) \subset \mathbb{C}_g$.

The pole placement results of Corollary 1.2 and Corollary 1.4 can be expressed by means of simple lattice diagrams. The word "free" indicates parts on which eigenvalues can be freely assigned (these are $\underline{\sigma}(A + BF: <A \mid \text{Im } B>)$ in Corollary 1.4, and $\underline{\sigma}(A + GC: X/<\text{Ker } C \mid A>)$ in Corollary 1.6), whereas the word "fixed" refers to parts on which the poles are fixed (which are, respectively, $\underline{\sigma}(A + BF: X/<A \mid \text{Im } B>)$ and $\underline{\sigma}(A + GC: <\text{Ker } C \mid A>)$).

```
        ┬ X                        ┬ X
  fixed │                    free  │
        ┼ <A | Im B>               ┼ <Ker C | A>
  free  │                    fixed │
        ┴ 0                        ┴ 0

        A+BF                       A+GC
       (F:X→U)                    (G:Y→X)
```
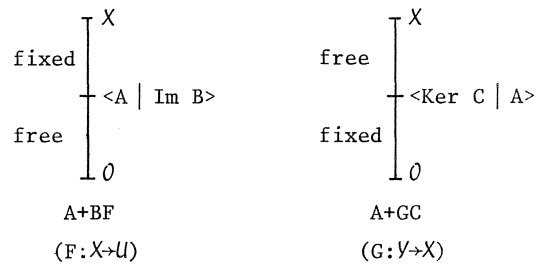
Figure 1.3: Pole placement.

These diagrams should be compared to the block diagrams in figs. 1.1 and 1.2. Further on, we shall use more complicated diagrams of the above kind.

1.3. Controlled and conditioned invariance

We do not only want to consider the change in eigenvalues that can be brought about by going from A to A + BF; we also need to have information about the subspaces which can be made invariant by feedback. This may be necessary, for instance, to see if certain structural features can be assigned by a suitable choice of the control law. So let us suppose that a system matrix A, acting on the state space X, and an input mapping B are given. We are led to the following definition: A subspace V of X is a *controlled invariant subspace* if there exists a feedback mapping F such that $(A+BF)V \subset V$.

There are many ways to characterize controlled invariant subspaces; for instance, one can show that V is controlled invariant if and only if for every $x_0 \in V$ there is a piecewise continuous control function $u(\cdot)$ such that the solution of the controlled equation (1.2) remains in V for all time (see BASILE & MARRO (1969b)). One characterization that is particularly convenient, because it is stated purely in terms of the pair (A,B) and the subspace V, is the following:

PROPOSITION 1.6 *A subspace V of X is controlled invariant if and only if*

(1.24) $AV \subset V + \text{Im B}$.

PROOF The proof is found in WONHAM (1979; p.88).   ⊠

Controlled invariant subspaces are also called (A mod B)-invariant subspaces, or (A,B)-invariant subspaces; the reason for this terminology is given by the above proposition. The term "controlled invariant" stems from BASILE & MARRO (1969); the adjective "(A,B)-invariant" has been introduced by WONHAM & MORSE (1970).

Dually, we define: A subspace T of X is a *conditioned invariant subspace* (with respect to a given pair (C,A)) if there exists an injection mapping G such that $(A+GC)T \subset T$. We have the following characterization:

PROPOSITION 1.7 *A subspace T of X is conditioned invariant if and only if*

$$(1.25) \qquad A(T \cap \text{Ker } C) \subset T.$$

It is immediately clear from Prop. 1.6 that the set of controlled invariant subspaces is closed under the operation of subspace addition. This implies that, with every subspace K of X, there is a largest controlled invariant subspace contained in K; this subspace will be denoted by $V^*(K)$. We have the following algorithm to compute $V^*(K)$ for any K.

PROPOSITION 1.8 *Let a pair of mappings* (A,B) *and a subspace K of X be given. Define the sequence* V^k($k = 0,1,2,...$) *according to*

$$(1.26.1) \qquad V^0 = K$$

$$(1.26.2) \qquad V^{k+1} = \{x \in K \mid Ax \in V^k + \text{Im } B\}.$$

Then the sequence V^k is decreasing, and the limit subspace (which is reached after a number of steps at most equal to dim K) *is* $V^*(K)$.

PROOF See WONHAM (1979; p.91).   ⊠

Again, other interpretations of $V^*(K)$ are possible; for instance, this subspace can also be considered as the set of initial values such that the solution of the equation (1.2) can be kept inside K by a piecewise continuous control function u(·) (see HAUTUS (1980)). If one uses a discrete time parameter, then this idea naturally leads to the algorithm (1.26).

From Prop. 1.7, one sees that the set of conditioned invariant subspaces is closed under intersection. Consequently, given a subspace E of X there is a smallest conditioned invariant subspace containing E, which

is denoted by $T^*(E)$, and which can be computed by an algorithm dual to the one given above.

1.4. Pole placement under restrictions

In sections 1.2 and 1.3, we have discussed the effect of state feedback on the eigenvalues and on the invariant subspaces of the system matrix. We now want to combine these two aspects. One important question is: How much freedom is left in the assignment of eigenvalues of A + BF, if it is required that a given subspace should be made invariant ?

Let us introduce, for a controlled invariant subspace V,

(1.27) $\underline{F}(V) = \{F: X \to U \mid (A+BF)V \subset V\}$.

Dually, we write for a conditioned invariant subspace T:

(1.28) $\underline{G}(T) = \{G: Y \to X \mid (A+GC)T \subset T\}$.

Our problem is now to determine the amount of freedom one has in choosing the eigenvalues of A + BF when F is restricted to $\underline{F}(V)$. This can be solved completely.

THEOREM 1.9 *Let a system matrix* A: $X \to X$ *and an input mapping* B: $U \to X$ *be given; let* V *be a controlled invariant subspace with respect to the pair* (A,B). *For all* F: $X \to U$, *the subspace* <A $\mid$ Im B + V> *is* (A+BF)-*invariant, and we have*

(1.29) $\underline{\sigma}(A: X/(<A \mid Im B> + V)) = \underline{\sigma}(A + BF: X/(<A \mid Im B> + V))$.

Moreover, if we define the subspace R_F *for* F $\in$ $\underline{F}(V)$ *by*

(1.30) $R_F = <A+BF \mid Im B \cap V>$,

then for all F_1, F_2 $\in$ $\underline{F}(V)$ *we have* $R_{F_1} = R_{F_2} := R$ *and*

(1.31) $\underline{\sigma}(A+BF_1: V/R) = \underline{\sigma}(A+BF_2: V/R)$.

On the other hand, given any symmetric multiplicity

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function f_1 *with total multiplicity equal to* dim R, *and another symmetric*
multiplicity function f_2 *with total multiplicity equal to* dim($<A \mid$ Im $B>+V)/V$,
there exists an $F \in \underline{F}(V)$ *such that*

(1.32) $\underline{\sigma}(A+BF: R) = f_1$

(1.33) $\underline{\sigma}(A+BF): (<A \mid$ Im $B> + V)/V) = f_2.$

Before we turn to the proof of this theorem, let us make some remarks.
The theorem says that if a feedback mapping makes V invariant for A + BF,
then it must do the same with R and $<A \mid$ Im $B> + V$. The three subspaces
R, V and $<A \mid$ Im $B> + V$ form a chain, and so we can draw the following
lattice diagram to describe the pole placement properties announced in the
theorem:
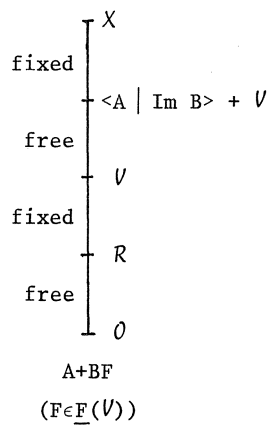
Figure 1.4: Pole placement
under invariance of V.

This corresponds to the following block diagram (compare Fig. 1.1):
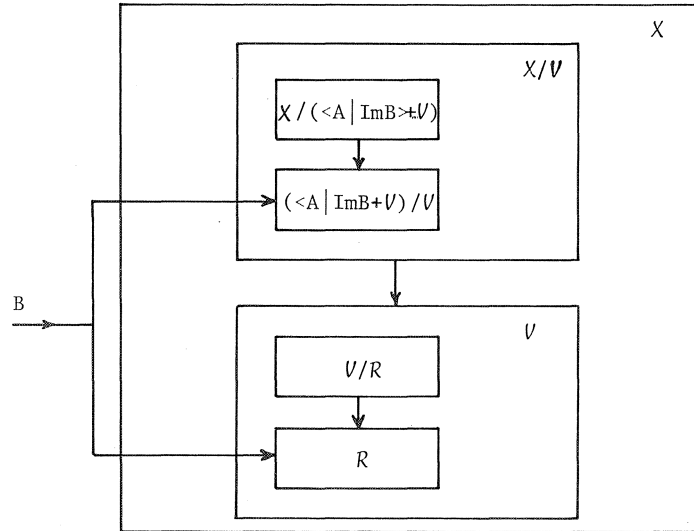
Figure 1.5: A system with a prescribed
invariant subspace.

PROOF It follows immediately from Prop. 1.6 that $<A \mid \text{Im B}> + V$ is an A-invariant subspace containing Im B; the statements concerning this subspace then follow as in the proof of Cor. 1.2. For the second part of the proof, take F_1 and F_2 from $\underline{F}(V)$, and let $x \in V$. Then we must have $(A+BF_1)x -$
$- (A+BF_2)x \in V$, but also $(A+BF_1)x - (A+BF_2)x = B(F_1-F_2)x \in \text{Im B}$. It is now clear that $\text{Im}((A+BF_1): V) - (A+BF_2): V)) \subset \text{Im B} \cap V$ for all $F_1, F_2 \in \underline{F}(V)$; consequently, $R_{F_1} = R_{F_2}$ for all such F_1 and F_2, and (1.31) holds.

Now, let multiplicity functions f_1 and f_2 be given as described in the statement of the theorem. Then we must show that a feedback mapping F can be constructed satisfying (1.32) and (1.33). To do this, we start by picking an arbitrary $F_0 \in \underline{F}(V)$. Write $A_0 = A + BF_0$: V; define $U_0 = \{u \in U \mid Bu \in V\}$ and write B_0 for the restriction of B to U_0, considered as a mapping to V. Then we have $R = <A_0 \mid \text{Im B}_0>$, and by Cor. 1.2 there exists a mapping F_1: $V \to U_0$ such that $\underline{\sigma}(A_0+B_0F_1: R) = f_1$.

For the next step, let P be the canonical projection of X onto the factor space X/V. Write $A_2 = A + BF_0$: X/V and define B_2: $U \to X/V$ by $B_2 = PB$. Denote $S = P(<A \mid \text{Im B}> + V)$. Then we have: $S = <A_2 \mid \text{Im B}_2>$. For, from the fact that $<A \mid \text{Im B}> + V$ is A + BF_0-invariant and contains Im B, it follows readily that S is A_2-invariant and contains Im B_2. Let

$\hat{S}$ be another A_2-invariant subspace of X/V containing Im B_2; then

$\{x \in X \mid Px \in \hat{S}\}$ is an $A + BF_0$-invariant subspace of X containing both Im B

and V, so that we get $\{x \in X \mid Px \in \hat{S}\} \supset <A \mid$ Im $B> + V$ and consequently,

$\hat{S} \supset S$. Now we can again appeal to Cor. 1.2 and conclude that there

exists a mapping F_2: $X/V \rightarrow U$ such that $\underline{\sigma}(A_2+B_2F_2$: $S) = f_2$.

We now construct our feedback matrix F. Let F_3 be any extension of

F_1 to a mapping from X into U such that Im $F_3 \subset U_0$, and define

F_4: $X \rightarrow U$ by $F_4 = F_2P$. Then $F = F_0 + F_3 + F_4$ satisfies (1.32) and (1.33):

(1.34) $\underline{\sigma}(A+BF$: $R) = \underline{\sigma}(A+BF_0+BF_3$: $R) =$

$= \underline{\sigma}(A_0+B_0F_1$: $R) = f_1$

(1.35) $\underline{\sigma}(A+BF$: $(<A \mid$ Im $B> + V)/V) =$

$= \underline{\sigma}(A+BF_0+BF_4$: $(<A \mid$ Im $B> + V)/V) =$

$= \underline{\sigma}(A_2+B_2F_2$: $S) = f_2$.   ⊠

Results like this have appeared before in the literature, although not in the complete form as given above; see WONHAM (1979; p. 111) and MORSE (1973b; Appendix, Lemma A. 2). An alternative proof using matrix arguments has been given in SCHUMACHER (1980 a).

We shall also need an extension of the theorem, in which F is restricted to an intersection of two sets of the form $\underline{F}(V)$. First we mention a simple result which can easily be proved (see also WONHAM (1979; exercise 9.1)).

LEMMA 1.10   *Let V_1 and V_2 be controlled invariant subspaces with respect to the pair* (A,B) *and suppose that $V_1 \subset V_2$. Given any $F_1 \in \underline{F}(V_1)$, there exists an $F_2 \in \underline{F}(V_2) \cap \underline{F}(V_1)$ such that $A + BF_2$: $V_1 = A + BF_1$: V_1.*

In particular, the lemma shows that $\underline{F}(V_1) \cap \underline{F}(V_2)$ is non-empty if $V_1 \subset V_2$. We can consider the pole placement problem when F is restricted to a set of this type; again, a complete solution is available.

THEOREM 1.11   *Let a system matrix* A: $X \rightarrow X$ *and an input mapping* B: $U \rightarrow X$ *be given; let V_1 and V_2 be controlled invariant subspaces with respect to the pair* (A,B), *and suppose that $V_1 \subset V_2$. Then R_1: $= <A+BF \mid V_1 \cap$ Im $B>$*

and R_2: = $\langle A+BF \mid V_2 \cap \text{Im } B\rangle$ *do not depend on the specific choice of* F *from* $\underline{F}(V_1) \cap \underline{F}(V_2)$. *All subspaces in the chain*

$$(1.36) \qquad 0 \subset R_1 \subset V_1 \subset V_1 + R_2 \subset V_2 \subset V_2 + \langle A \mid \text{Im } B\rangle \subset X$$

are invariant for each F $\in \underline{F}(V_1) \cap \underline{F}(V_2)$. *Our freedom in selecting the eigenvalues of* A + BF *when* F *is restricted to* $\underline{F}(V_1) \cap \underline{F}(V_2)$ *can be described in the following lattice diagram:*

$$
\begin{array}{ll}
\text{fixed} & X \\
 & V_2 + \langle A \mid \text{Im } B\rangle \\
\text{free} & \\
 & V_2 \\
\text{fixed} & \\
 & V_1 + R_2 \\
\text{free} & \\
 & V_1 \\
\text{fixed} & \\
 & R_1 \\
\text{free} & \\
 & 0
\end{array}
$$

A+BF
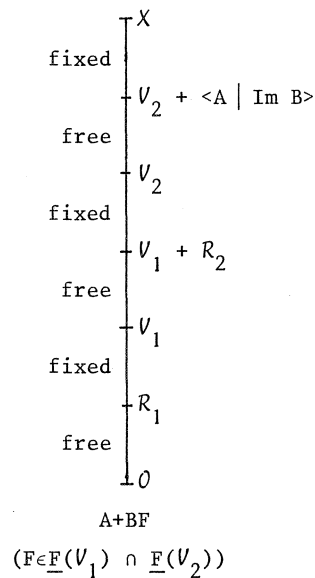
$(F \in \underline{F}(V_1) \cap \underline{F}(V_2))$

Figure 1.6: Pole placement under
invariance of V_1 and V_2.

__PROOF__ The facts that R_1 and R_2 do not depend on F $\in \underline{F}(V_1) \cap \underline{F}(V_2)$ and that the subspaces in the chain (1.36) are invariant for any such F follow just as in the proof of Theorem 1.9. The same holds for the 'fixed' parts in the above diagram.

It remains to show that an F can be selected to obtain any desired eigenvalues (with suitable symmetry and multiplicity) in the parts marked 'free' in the lattice diagram. To do this, first consider the subspace V_2. Take an arbitrary $F_0 \in \underline{F}(V_1) \cap \underline{F}(V_2)$. Write $A_0 = A + BF_0$: V; define $U_0 = \{u \in U \mid Bu \in V_2\}$ and write B_0 for the restriction of B to U_0, considered as a mapping to V_2. Then V_1 is a controlled invariant subspace with

respect to (A_0, B_0) (V_1 is even A_0-invariant), and so we can apply Thm.1.9 to conclude that there exists a feedback matrix F_1: $V_2 \to U_0$ such that $A_0 + B_0 F_1$: $V_2/(V_1 + \langle A_0 \mid \text{Im } B_0 \rangle)$ and $A_0 + B_0 F_1$: $\langle A_0 \mid V_1 \cap \text{Im } B_0 \rangle$ have prescribed multiplicity functions. Note that $\text{Im } B_0 = V_2 \cap \text{Im } B$ so that $\langle A_0 \mid \text{Im } B_0 \rangle = R_2$ and $\langle A_0 \mid V_1 \cap \text{Im } B_0 \rangle = \langle A_0 \mid V_1 \cap \text{Im } B \rangle = R_1$.

If we extend F_1 in an arbitrary way to a mapping F_3: $X \to U$ such that $\text{Im } F_3 \subset U_0$, then we shall have $A + B(F_0 + F_3)$: $V_2/(V_1 + R_2) =$ $= A_0 + B_0 F_1$: $V_2/(V_1 + R_2)$ and $A + B(F_0 + F_3)$: $R_1 = A_0 + B_0 F_1$: R. Now we can proceed just as in the proof of Thm. 1.9 to find a suitable mapping F_2: $X/V \to U$. We finally define F by $F = F_0 + F_3 + F_4$ where $F_4 = F_2 P$. Then $A + BF$ will have the desired properties. ▨

We could go on by induction and prove a pole placement theorem for $F \in \underline{F}(V_1) \cap .. \cap \underline{F}(V_k)$ (for a chain $V_1 \subset .. \subset V_k$ of controlled invariant subspaces), but perhaps this would take us a little bit too far at this point. Note that there is a striking contrast between our success in dealing with such restrictions on F, and the great difficulties that arise when one tries to do pole placement for $F \in \{F: X \to U \mid K \subset \text{Ker } F\}$, where K is an arbitrary given subspace. (This is the problem of "pole placement by output feedback"; F can be written as $F = KC$ for some K if and only if $\text{Ker } C \subset \text{Ker } F$.)

We may extend the lattice diagram in Fig. 1.6 to get a more complete picture of the situation, as follows: (see Fig. 1.7, next page)

This diagram shows that the poles that are fixed when F is restricted to $\underline{F}(V_1) \cap \underline{F}(V_2)$ are precisely the fixed poles following from the restriction $F \in \underline{F}(V_1)$ taken together with those following from the restriction $F \in \underline{F}(V_2)$, with the understanding that overlapping parts are counted only once. So the restrictions $F \in \underline{F}(V_1)$ and $F \in \underline{F}(V_2)$ do not interfere, in the sense that combining them does not add any extra fixed poles to those that must be present due to each restriction taken separately.

1.5. Stabilizability and detectability

Thm. 1.9 leads easily to a characterization of those controlled invariant subspaces V for which there exists an $F \in \underline{F}(V)$ such that $\sigma(A+BF: V) \subset \mathbb{C}_g$. These subspaces are called *stabilizability subspaces* (HAUTUS (1980)). The name *controllability subspace* is used in the special case where the eigenvalues of $A + BF$: V can freely be assigned by a suitable choice of $F \in \underline{F}(V)$.

$$X$$

fixed

$$V_2 + <A \mid \text{Im } B>$$

free fixed

$$V_2$$ $$(<A \mid \text{Im } B> + V_1)$$

fixed free

$$V_2 \cap (<A \mid \text{Im } B> + V_1)$$

fixed

$$V_1 + R_2$$

free fixed

$$V_1$$ $$R_2$$

fixed free

$$V_1 \cap R_2$$

fixed

$$R_1$$

free

$$0$$
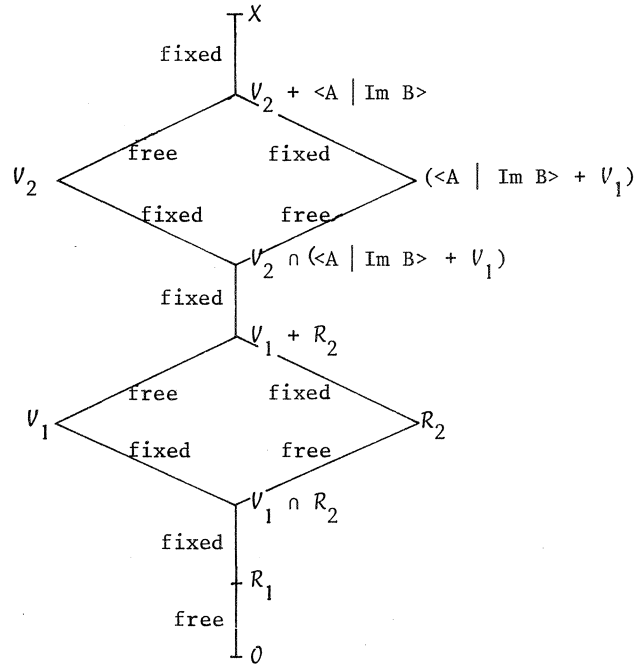
Figure 1.7: Overlap of fixed spectra.

This term is much older (WONHAM & MORSE (1970)). For alternative interpretations of these concepts, we refer again to HAUTUS (1980).

Sometimes, it will be important to know something about $\sigma(A+BF: X/V)$. We shall say that the controlled invariant subspace V is *outer-stabilizable* if there exists an $F \in \underline{F}(V)$ such that $A + BF: X/V$ is stable. Correspondingly, stabilizability subspaces will sometimes be called *inner-stabilizable*. We also introduce the dual terminology. If T is a conditioned invariant subspace, then we shall say that T is *outer-detectable* (or that T is a *detectability subspace*) if there exists a $G \in \underline{G}(T)$ such that $A + GC: X/T$ is stable, and we shall say that T is *inner-detectable* if there exists a $G \in \underline{G}(T)$ such that $A + GC: T$ is stable.

From Thm. 1.9, we immediately obtain the following characterization of inner-and outer-stabilizable controlled invariant subspaces.

PROPOSITION 1.12 *Let V be a controlled invariant subspace with respect to the pair* (A,B) . *Then V is inner-stabilizable if and only if*

(1.37) $\sigma(A+BF: V/(\langle A+BF \mid V \cap \text{Im } B\rangle)) \subset \mathbb{C}_g$

for any F ∈ F(V), *and V is outer-stabilizable if and only if*

(1.38) $\sigma(A: X/(V + \langle A \mid \text{Im } B\rangle)) \subset \mathbb{C}_g.$

The characterization of stabilizability subspaces given by (1.37) is not always convenient, because it depends on the computation of an F ∈ F(V). This dependence is not present in the following characterization of stabilizability subspaces, which as far as the author knows is new (see also SCHUMACHER (1980 d)).

THEOREM 1.13 *Let V be a controlled invariant subspace with respect to the pair* (A,B). *Then V is inner-stabilizable if and only if*

(1.39) $(\lambda - A)V + \text{Im } B = V + \text{Im } B$ *for all* $\lambda \in \mathbb{C}_b.$

PROOF *"only if"*: Suppose that the controlled invariant subspace V is inner-stabilizable; then we can find F ∈ F(V) such that A + BF: V is stable. No $\lambda \in \mathbb{C}_b$ is an eigenvalue of A + BF: V, and consequently

(1.40) $(\lambda - (A+BF))V = V$ *for all* $\lambda \in \mathbb{C}_b.$

The desired conclusion now follows from noting that, for all F ∈ F(V),

(1.41) $(\lambda - (A+BF))V + \text{Im } B = (\lambda - A)V + \text{Im } B.$

"if": Now suppose that V is a controlled invariant subspace for which (1.39) holds. By Thm. 1.9, we can take F ∈ F(V) such that A + BF: R is stable, where R is defined by (1.30). With F chosen in this way, we contend that A + BF: V is stable.

Take any $\lambda \in \mathbb{C}_b$. From (1.39) and the general equality (1.41), we get

(1.42) $V \subset (\lambda - (A+BF))V + \text{Im } B.$

Because $(\lambda-(A+BF))V \subset V$, this is equivalent to

(1.43) $V \subset (\lambda-(A+BF))V + (V \cap \text{Im } B)$

which immediately implies

(1.44) $V \subset (\lambda-(A+BF))V + R.$

Because it follows from $\lambda \notin \sigma(A+BF: R)$ that $(\lambda-(A+BF))R = R$, we obtain:

(1.45) $V \subset (\lambda-(A+BF))V + (\lambda-(A+BF))R = (\lambda-(A+BF))V \subset V.$

So our final conclusion is

(1.46) $V = (\lambda-(A+BF))V$

which shows that λ is not an eigenvalue of $A + BF: V.$ ▨

The dual result is

THEOREM 1.14 *Let T be a conditioned invariant subspace with respect to the pair (C,A). Then T is outer-detectable if and only if*

(1.47) $(\lambda-A)^{-1} T \cap \text{Ker } C = T \cap \text{Ker } C$ *for all $\lambda \in \mathbb{C}_b$.*

Of course, $\lambda - A$ need not be invertible for all $\lambda \in \mathbb{C}_b$; by the notation $(\lambda-A)^{-1} T$, we mean $\{x \in X \mid (\lambda-A)x \in T\}$.

The condition (1.39) can be formulated somewhat differently if we note that the inclusion $(\lambda-A)V + \text{Im } B \subset V + \text{Im } B$ holds for any $\lambda \in \mathbb{C}$ if V is controlled invariant; this follows readily from Prop. 1.6. Also, the inclusion $\text{Im } B \subset (\lambda-A)V + \text{Im } B$ is trivial. So we get:

COROLLARY 1.15 *Let V be a controlled invariant subspace with respect to the pair (A,B). Then V is inner-stabilizable if and only if*

(1.48) $V \subset (\lambda-A)V + \text{Im } B$ *for all $\lambda \in \mathbb{C}_b$.*

The dual version reads:

COROLLARY 1.16 *Let* T *be a conditioned invariant subspace with respect to the pair* (C,A). *Then* T *is outer-detectable if and only if*

(1.49) $\qquad T \supset (\lambda-A)^{-1}T \cap \text{Ker } C \qquad\qquad$ *for all* $\lambda \in \mathbb{C}_b$.

From Thm. 1.13, it follows immediately that the set of stabilizability subspaces is closed under the operation of subspace addition. In particular, to every given subspace K in X there is a largest stabilizability subspace contained in K. This also proved, but by a different method, in WONHAM (1979; p. 114). We shall denote the largest stabilizability subspace in K by $V_g^*(K)$. Of course, there is also a smallest detectability subspace containing a given subspace E; this will be denoted by $T_g^*(E)$. There is a constructibe procedure available to compute $V_g^*(K)$ for any given K; see WONHAM (1979; p.114). A dual procedure computes $T_g^*(E)$ for any E. We shall denote the largest element in the set of all stabilizability subspaces by X_{stab}, and X_{det} will be the smallest element in the set of all detectability subspaces.

The span of the generalized eigenspaces of the system matrix A associated with eigenvalues in $\mathbb{C}_g$ will be denoted by $X_g(A)$. That is, if we factorize the characteristic polynomial $p(\lambda)$ of A as $p(\lambda) = p_g(\lambda)\, p_b(\lambda)$ where p_g has its zeros in $\mathbb{C}_g$ and p_b has its zeros in $\mathbb{C}_b$, then

(1.50) $\qquad X_g(A) = \text{Ker}(p_g(A))$.

Likewise, we define

(1.51) $\qquad X_b(A) = \text{Ker}(p_b(A))$

and this is the span of the generalized eigenspaces of A that are associated with eigenvalues in $\mathbb{C}_b$.

After these definitions, we are able to identify X_{stab} as follows.

LEMMA 1.17 $\quad X_{stab} = X_g(A) + \langle A \mid \text{Im } B \rangle$.

PROOF Clearly, $X_g(A)$ is a stabilizability subspace; $\langle A \mid \text{Im } B \rangle$ is even a controllability subspace. So $X_g(A) + \langle A \mid \text{Im } B \rangle$ is a stabilizability subspace and hence we must have $X_{stab} \supset X_g(A) + \langle A \mid \text{Im } B \rangle$. We see that

$X_{stab} \supset$ Im B and so X_{stab} must be A-invariant. Now take F such that A + BF: X_{stab} is stable. Then

(1.52) $\sigma(A+BF: X_{stab}/(X_g(A) + <A \mid \text{Im } B>)) =$

$= \sigma(A: X_{stab}/(X_g(A) + <A \mid \text{Im } B>)).$

On the one hand, we have

(1.53) $\sigma(A: X_{stab}/(X_g(A) + <A \mid \text{Im } B>)) \subset \sigma(A+BF: X_{stab}) \subset \mathbb{C}_g$

but on the other

(1.54) $\sigma(A: X_{stab}/(X_g(A) + <A \mid \text{Im } B>)) \subset \sigma(A: X/X_g(A)) \subset \mathbb{C}_b.$

This can only be so if X_{stab} is in fact equal to $X_g(A) + <A \mid \text{Im } B>$.   ⊠

The dual result is

LEMMA 1.18 $X_{det} = X_b(A) \cap <\text{Ker } C \mid A>.$

We can use the subspace X_{stab} to give an alternative characterization of outer-stabilizability.

PROPOSITION 1.19   *A controlled invariant subspace V is outer-stabilizable if and only if*

(1.55) $V + X_{stab} = X.$

PROOF   First suppose that V is outer-stabilizable. Then it follows from Prop. 1.12 that $V + <A \mid \text{Im } B> \supset X_b(A)$. Hence, $V + X_{stab} \supset X_b(A) + X_g(A) = X.$ Now suppose that (1.55) holds. Then (using (1.16)):

(1.56) $\sigma(A: X/(V + <A \mid \text{Im } B>)) =$

$= \sigma(A: (V + <A \mid \text{Im } B> + X_g(A))/(V + <A \mid \text{Im } B>)) =$

$= \sigma(A: X_g(A)/(X_g(A) \cap (V + <A \mid \text{Im } B>))) \subset$

$\subset \sigma(A: X_g(A)) \subset \mathbb{C}_g.$

This shows that V is outer-stabilizable. ☒

We can easily derive some alternative characterizations of stabilizability of a pair (A,B) from the above results.

COROLLARY 1.20 *Let a system matrix* A *and an input mapping* B *be given. Then the following statements are equivalent:*

(i) *the pair* (A,B) *is stabilizable;*

(ii) $X_b(A) \subset \langle A \mid Im\ B \rangle$;

(iii) $X_{stab} = X;$

(iv) $Im(\lambda-A) + Im\ B = X$ *for all* $\lambda \in \mathbb{C}_b$.

PROOF The equivalence between (i) and (ii) is just a reformulation of Cor. 1.3. The pair (A,B) is stabilizable if and only if the zero subspace is outer-stabilizable; the equivalence between (i) and (iii) thus follows from Prop. 1.19. Also, the pair (A,B) is stabilizable if and only if the whole space X is inner-stabilizable; then Thm. 1.13 shows that (i) and (iv) are equivalent. ☒

The criterion given in (iv) is called the *Hautus test* for stabilizability (HAUTUS (1969)). The corresponding characterizations of detectability are given by our final result in this chapter.

COROLLARY 1.21 *Let a system matrix* A *and an output mapping* C *be given. Then the following statements are equivalent:*

(i) *the pair* (C,A) *is detectable;*

(ii) $X_g(A) \supset \langle Ker\ C \mid A \rangle;$

(iii) $X_{det} = 0;$

(iv) $Ker(\lambda-A) \cap Ker\ C = 0$ *for all* $\lambda \in \mathbb{C}_b$.

Note that the criterion (iv) may be rephrased as follows: The output mapping C does not annihilate any eigenvector of A corresponding to a 'bad' eigenvalue of A.

CHAPTER 2

THE COMPENSATOR PROBLEM

In this chapter, we begin our study of dynamic feedback. The problem
we shall consider is a basic one: to stabilize a given system (described
by a system matrix, an input mapping and an output mapping) by adding a com-
pensator. The compensator brings its own dynamics into the feedback loop,
which is why the term "dynamic feedback" is used for this type of automatic
control.

It is easy to give necessary and sufficient conditions for the solvabili-
ty of the problem of stabilizing a given system by dynamic feedback. Serious
difficulties are encountered, however, if one tries to minimize the order of
the dynamics that are introduced in the feedback loop. We shall present a new
and general approach to the compensator problem, which allows for low-order
solutions. The basic idea of this method is that reduction of the compensator
order can be obtained if the 'feedback' and the 'observer' part of the
controller are suitably adapted to each other.

We also describe the class of compensators that can be obtained by our
method. It will be shown that this class is given precisely by those com-
pensators that have a 'feedback-observer' interpretation. Comparing the
usual state-space methods of compensator design with our approach, we find
that all these methods are recovered as special cases of our theory. Finally,
we are also able to present a systematic searching procedure for low-order
compensators.

The chapter is organized as follows. In the first section, we discuss the
concept of dynamic feedback and introduce some notation. The second section
gives an extensive discussion of the compensator problem and of our main
results an their interpretation. Proofs are given in Section 3 (for the con-
struction theorem) and in Section 5 (for the interpretation of the class of
compensators that are thus constructed). In the intermediate Section 4 we
discuss several methods of compensator design, including our own proposal.

2.1. Introduction

Our starting point is the linear system given by

(2.1.1)   x'(t) = Ax(t) + Bu(t)

(2.1.2)   y(t) = Cx(t).

This system can be represented by the following block diagram:
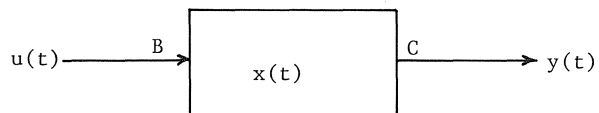
Figure 2.1: Open-loop system.

We now want to "close the loop" by connecting the output y(t) to the input u(t). Of course one can imagine many ways to do this, but our object of study will be *linear dynamic feedback*. This term is used for the form of automatic control that is provided by an additional finite-dimensional linear system (the *compensator*) which takes the observation y(t) as its input and which specifies the control function u(t) at its output, in the following way:

(2.2.1)   w'(t) = Nw(t) + My(t)

(2.2.2)   u(t) = Lw(t).

Combining (2.1) and (2.2) gives us the *closed-loop system*.
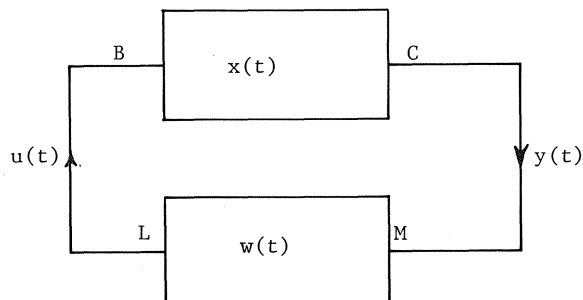
Figure 2.1: Closed-loop system.

The word "dynamic feedback" is used because the compensator introduces dynamics in the feedback loop; sometimes one also uses the term "integral control", because in (2.2) the control function is obtained via an integration of the observed variable y(t). One may also apply "static feedback" or "proportional control" by directly connecting the control to the observation:

(2.3)   u(t) = Ky(t).

Of course, one can easily combine the two types of feedback in one compensator equation, by replacing (2.2.2) by

(2.2.2)'   u(t) = Lw(t) + Ky(t).

In a block diagram, this may be illustrated as follows:
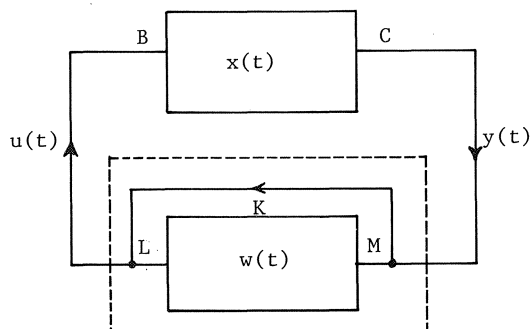
Figure 2.3: Dynamic and static feedback combined.

However, we shall prefer to consider static and dynamic feedback as separate processes. So we shall only consider compensators of the 'purely integral' form (2.2), but sometimes we shall first change the system matrix from A to A + BKC by static feedback before adding a compensator of this type. The overall result is still the same, of course, and the corresponding controller will be referred to as a PI-*compensator* (proportional/ integral). Our view of this type of compensator is represented by Fig.2.4 rather than by Fig.2.3.
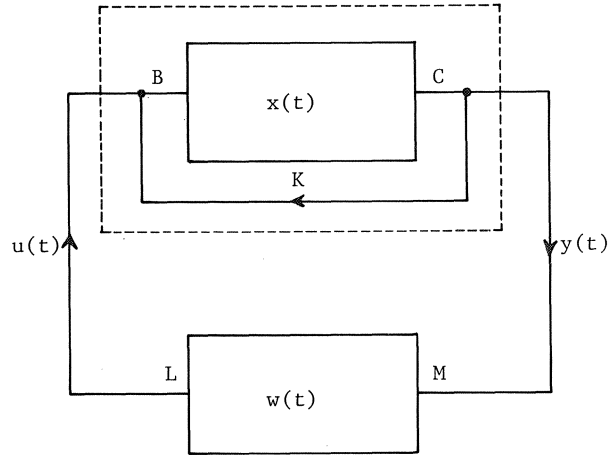
Figure 2.4: Dynamic and static feedback
separated.

Now let us explain some notation and terminology that will be used
in the sequel. The state space of the compensator equation (2.2) will be
written as W; the dimension of this space is called the *order* of the compen-
sator, for which we shall use the letter k. The equations (2.1) and (2.2)
can be taken together to form the *extended system equation:*

$$(2.4) \qquad \frac{d}{dt}\begin{pmatrix} x \\ w \end{pmatrix}(t) = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix}(t).$$

This is a differential equation in the *extended state space* $X \oplus W$ which
has dimension n + k. The matrix

$$(2.5) \qquad A_e := \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}$$

is called the *extended system matrix.*

There are two natural mappings between the original state space X and
the extended state space $X \oplus W$: the natural embedding Q: $X \to X \oplus W$ defined
by

$$(2.6) \qquad Qx = \begin{pmatrix} x \\ 0 \end{pmatrix} \qquad\qquad (x \in X)$$

and the canonical projection P: $X \oplus W \to X$ defined by

$$(2.7) \qquad P\begin{pmatrix} x \\ w \end{pmatrix} = x \qquad\qquad (\begin{pmatrix} x \\ w \end{pmatrix} \in X \oplus W).$$

Accordingly, there a two subspaces of X naturally related to any subspace M of $X \oplus W$:

$$(2.8) \qquad Q^{-1}M: = \{x \in X \mid \binom{x}{0} \in M\}$$

$$(2.9) \qquad PM = \{x \in X \mid \exists w \in W: \binom{x}{w} \in M\}.$$

The first subspace can be considered as the intersection of M with the X-plane, where as the second is the projection of M onto that same plane. It should be noted, however, that both subspaces are considered as subspaces of X; one may view the pair $(Q^{-1}M,PM)$ in some sense as the representative, within the *given* state space X, of a subspace in the extended state space $X \oplus W$ which the designer has to *construct*. Below, we shall give a more precise meaning to this vague idea.

2.2. Problem statement and main results

In this chapter, disturbances will be modeled simply by a non-zero initial condition for the controlled system (2.4), and the purpose of the control will be to make the system return to its nominal operating point (represented by the origin in the state space) after such a disturbance. We shall consider more elaborate disturbance models and other control objectives in Chapter 3. Our problem here can thus be formulated as follows: Given a system matrix A, an input mapping B and an output mapping C, find a linear space W and mappings N: $W \to W$, L: $W \to U$ and M: $Y \to W$ such that the extended system matrix A_e defined in (2.5) is stable; moreover, do this using a space W of lowest possible dimension. The latter requirement is added because low-order compensators are in general easier to implement and more reliable than high-order compensators.

We shall call a compensator of the form (2.2) a *stabilizing compensator* if it is defined by mappings L, M and N that make the extended system matrix A_e stable. The conditions under which such a compensator exists are simple and well-known. (See also Section 3.2 and Section 2.4).

THEOREM 2.1 *There exists a stabilizing compensator for the system* (2.1) *if and only if the pair* (A,B) *is stabilizable and the pair* (C,A) *is detectable.*

Several methods are known to construct a stabilizing compensator if one exists; we shall discuss some of them in Section 2.4. However, the *minimal compensator problem* of finding the minimal order of a stabilizing compensator for a given system has not been solved (and it will not be solved here). Still we can present a further analysis of the compensator problem; this will provide more insight into the problem and, moreover, suggest a way to find low-order compensators.

Our starting point is an observation concerning the pair $(Q^{-1}M, PM)$ related to any invariant subspace M of an extended system matrix of the form (2.5). To formulate this, let us say that a pair (T, V) of subspaces of X is a *compensator couple* (with respect to the triple (C,A,B)) if the following holds:

(cc1) T is conditioned invariant (with respect to the pair (C,A))

(cc2) V is controlled invariant (with respect to the pair (A,B))

(cc3) $T \subset V$

(cc4) $AT \subset V$.

This concept will be very important for us. For instance, we have the following result (which also explains the nomenclature); the proof will be given in Section 3.2, as we shall not need it in this chapter.

THEOREM 2.2 *Suppose that $M \subset X \oplus W$ is an invariant subspace for an extended system matrix A_e of the form (2.5). Then $(Q^{-1}M, PM)$ is a compensator couple.*

The notion "compensator couple" is introduced here for the first time, although situations in which pairs of controlled and conditioned invariant subspaces arise have been studied before, particularly in the early Italian work on the geometric approach to linear systems (see for instance BASILE & MARRO (1969a)). In SCHUMACHER (1980 d,e), the notion of "(C,A,B)-pair of subspaces" has been used; this is the same as a compensator couple, except that the condition (cc4) is lacking in the definition of a (C,A,B)-pair. The two concepts are related via static feedback, in the following way.

LEMMA 2.3 *Let (T, V) be a (C,A,B)-pair. Then there exists a static feedback*

mapping K: $V \to U$ *such that* (T,V) *is a compensator couple with respect to the triple* (C,A+BKC,B).

This lemma is a special case of Lemma 3.6, which will be proved in Ch. 3. We shall not need the result in this chapter.

Of course, the interesting question is whether we can reverse Thm. 2.2 in some way or another. Given a compensator couple, can we construct a compensator from it ? The answer is yes, if certain additional conditions hold.

THEOREM 2.4 *Suppose that we have a compensator couple* (T,V) *and mappings* $F \in \underline{F}(V)$ *and* $G \in \underline{G}(T)$ *such that*

(2.10.1)   Ker F $\supset$ T

(2.10.2)   Im G $\subset$ V

(2.11.1) $\sigma(A+BF: V) \subset \mathbb{C}_g$

(2.11.2) $\sigma(A+GC: X/T) \subset \mathbb{C}_g$.

Then there exists a stabilizing compensator of order dim V - dim T.

The proof is by construction; it will be given in Section 2.3. In Section 2.4, we shall show that several well-known methods of compensator design can be derived as special cases of the above theorem. There we shall also discuss the question how the theorem can be used in a systematic search for low-order compensators. Because the pole placement properties of A + BF under a restriction of the form (2.10.1) are not easily described (a similar remark holds for A + GC, of course), the utility of the theorem is certainly not immediately clear.

A nice theoretical property is that the conditions of the theorem are symmetric with respect to dualization. That is, if one would try to dualize the theorem, one would only get back the same statement (of course, and as always, after suitable renaming). Note that the compensator problem itself is also symmetric with respect to dualization.

To get some intuitive feeling for the conditions of the theorem, it is perhaps best to consider first the standard procedure for full-order

compensator design. This procedure is based on the idea of applying feed-back to an estimate of the state. Suppose that the system (2.1) has been given, and define $\hat{x}(t)$ by the equation

$$(2.12) \qquad \frac{d}{dt}\hat{x}(t) = (A+GC)\hat{x}(t) - Gy(t) + Bu(t).$$

One easily checks that $\hat{x}(t)$ is an estimate of $x(t)$ in the sense that

$$(2.13) \qquad \frac{d}{dt}(x(t) - \hat{x}(t)) = (A+GC)(x(t) - \hat{x}(t)).$$

If the matrix $A + GC$ is stable, the *error* $x(t) - \hat{x}(t)$ will approach zero for any initial values $x(0)$ and $\hat{x}(0)$. Let us assume that we have chosen G such that $A + GC$ is stable, and also that we have found a feedback mapping F such that $A + BF$ is stable. Defining $u(t)$ by

$$(2.14) \qquad u(t) = F\hat{x}(t)$$

we obtain the following form for the equation (2.1.1):

$$(2.15) \qquad x'(t) = (A+BF)x(t) - BF(x(t) - \hat{x}(t)).$$

Together with (2.13), this shows that $x(t)$ will approach zero, as well as $\hat{x}(t)$, for any initial values $x(0)$ and $\hat{x}(0)$. In other words, the equations (2.12) and (2.14) constitute a stabilizing compensator. We can write the equations in the form (2.2):

$$(2.16.1) \qquad \frac{d}{dt}\hat{x}(t) = (A+BF+GC)\,\hat{x}(t) - Gy(t)$$

$$(2.16.2) \qquad u(t) = F\hat{x}(t).$$

The extended system matrix A_e (as defined in (2.5)) becomes:

$$(2.17) \qquad A_e = \begin{pmatrix} A & BF \\ -GC & A+BF+GC \end{pmatrix}.$$

It follows from (2.13) and (2.15) that

(2.18) $\underline{\sigma}(A_e) = \underline{\sigma}(A+BF) + \underline{\sigma}(A+GC)$.

This formula is connected to a natural interpretation of the action of the compensator (2.16), in which there is a 'separation' between 'feedback dynamics' represented by $\underline{\sigma}(A+BF)$ and the 'observer dynamics' represented by $\underline{\sigma}(A+GC)$. Now, let us note that the equation (2.16.1) can also be written in the following form:

(2.19) $\dfrac{d}{dt}\hat{x}(t) = (A+BF)\hat{x}(t) - GC(x(t) - \hat{x}(t))$

Viewed in this way, the evolution of $\hat{x}(t)$ is described by the dynamics of A + BF together with an input signal that enters through G. Now suppose that V is an (A + BF)-invariant subspace such that ImG $\subset V$ (as in the conditions of the theorem). Then it is clear from (2.19) that $\hat{x}(t)$ will always remain in V if it starts there. Because the compensator should work for any value of $\hat{x}(0)$, this suggests (and we shall establish the correctness of this idea in the proof of Thm.2.4) that the state space of $\hat{x}(t)$ could be taken equal to V instead of X. Consequently, the compensator order is reduced from dim X to dim V.

The order reduction is possible because we are using informatiom about the feedback in the construction of the observer. The equation (2.12) has the strong property of defining an observer for x(t) without any information about the control function u(t), but as a consequence the equation has to be formulated in the complete state space X. (For a more precise discussion of the order of observers, see SCHUMACHER (1980b)). In the closed-loop situation, the control function u(t) is given by (2.14). We may use this information to obtain a reduced-order compensator, as indicated above. In this situation, we shall say that we are using a *feedback adapted observer* and the corresponding compensator will be called an *FAO -compensator*.

Dually, we can also consider *observer-adapted feedback*. This occurs when there is an (A + GC)-invariant subspace T such that $T \subset$ Ker F. Under these conditions, the control law (2.14) is defined even if $\hat{x}(t)$ is only an estimate "modulo T". This means that the compensator order can be reduced from dim X to dim X - dim T. A compensator obtained in this way will be called an *OAF -compensator*.

Finally, it is also possible to combine the two design methods under the further condition that $T \subset V$. (Note that this, together with

(A + BF)$V \subset V$ and Ker F $\supset T$, implies automatically that A$T \subset V$). The compensator order is reduced to dim V - dim T. This is precisely the statement of Thm. 2.4. We may speak of "mutual adaption" in this case, and the corresponding compensators will be called *MA-compensators*.

The construction that will be given in Section 2.3 leads to an extended system matrix A$_e$ whose spectrum is given by

(2.20) $\underline{\sigma}(A_e) = \underline{\sigma}(A+BF: V) + \underline{\sigma}(A+GC: X/T)$.

This should be compared to the formula (2.18) which holds for the 'classical' full-order compensator. The equation (2.20) is in line with the interpretation of our compensation method as one which is still based on "feedback applied to an estimate of the state", be it that the feedback and the observer action are adapted to eachother in order to obtain the reduction of the compensator order.

An obvious question that should be answered is, which compensators of the general form (2.2) can be interpreted as MA-compensators. Stated more precisely: Given a closed-loop system of the form (2.4), when do we have a compensator couple (T,V) and corresponding mappings F and G such that (2.10), (2.11) and (2.20) hold? The discussion above suggests that there should be a feedback-observer interpretation available if a compensator is to be of the MA-type, and a conjecture might be that *all* compensators that have a feedback-observer interpretation are in fact MA-compensators.

To proceed in this direction, we have to make precise what we mean by the statement that a compensator "has a feedback-observer interpretation". Consider a compensator of the general form (2.2), giving rise to the closed-loop system (2.4). As a preliminary definition, let us say that the given compensator has a feedback-observer interpretation if there exists a subspace T of X and a mapping $\Phi : X \oplus W \rightarrow X/T$ (which we shall call the *error mapping*) with the following properties:

(i) Φ can be factored as Φ = T(I -S) where S is a linear mapping from W to X and T is the canonical mapping from X to X/T;

(ii) if $\Phi\binom{x(0)}{w(0)} = 0$, then $\Phi\binom{x(t)}{w(t)} = 0$ for all $t \geq 0$;

(iii) if $\Phi\binom{x(0)}{w(0)} = 0$ and moreover $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$.

The mapping Φ gives an observer interpretation to the closed-loop dynamics, in the following way. To every vector $\binom{x}{w}$ in the extended state

space $X \oplus W$, Φ associates the error of Sw as an estimate of x modulo T; this
is stated in (i) above. The condition (ii) says that the error will remain
zero for all time if it is zero initially. This means that the error
satisfies an autonomous differential equation. Finally, the condition (iii)
seems to be natural: if in the initial situation the state is at the nominal
point zero and also the estimation error is zero, then no reasonable compen-
sator should cause an excitation.

The conditions (i) - (iii) can also be expressed in terms of $M := \mathrm{Ker}\ \Phi$.
It follows easily from (i) - (iii) that M satisfies the following conditions:

(i)' $\quad Q^{-1}M = T$, and dim M = dim W + dim T;

(ii)' $\quad M$ is A_e-invariant;

(iii)' $\quad M_w := \{ \binom{x}{w} \in M \mid x = 0 \}$ is A_e-invariant.

Note that we may just as well require

(iii)'' $\quad M_w = 0$

instead of (iii)', because an A_e-invariant subspace that is fully contained
in (the natural embedding of) W represents a redundant part of the compen-
sator. If M_w is non-trivial, then the compensator order can be reduced by
factoring out this subspace.

Motivated by this development, let us say that a subspace M of an ex-
tended state space $X \oplus W$ is a *separating subspace* if the following is true:

(2.21.1) $\quad$ dim M = dim W + dim $Q^{-1}M$

(2.21.2) $\quad$ dim$M$ = dim PM.

The condition (2.21.1) is equivalent to (i)' above if we define T as $Q^{-1}M$,
and (2.21.2) is equivalent to (iii)''. So the conditions (i)', (ii)' and
(iii)'' imply that A_e has a separating invariant subspace. On the other hand,
suppose that an extended system matrix A_e of the form (2.5) has been given,
and that A_e has a separating invariant subspace M. Then it is not difficult
to see that there exist a subspace T of X and a mapping $\Phi : X \oplus W \to X/T$ such
that (i) - (iii) above are satisfied. (Take $T = Q^{-1}M$. By (2.21), dim W =
= dim PM - dim $Q^{-1}M$ so there exists a surjective mapping from PM to W whose
kernel is precisely $Q^{-1}M$. Let S be a right inverse of this mapping, and
define Φ by $\Phi = T(I - S)$ where T is the canonical mapping from X to X/T).

So our final definition is: The compensator (2.2) for the system (2.1)
has a feedback-observer interpretation if the corresponding extended system
matrix A_e, as given by (2.5), has a separating invariant subspace. We can
now state the following theorem, which will be proved in Section 2.5.

THEOREM 2.5. *Suppose that* (2.2) *gives a stabilizing compensator for the
system* (2.1) *which has a feedback-observer interpretation (in the above
sense). Then there exist a compensator couple* (T,V) *and corresponding map-
pings* F *and* G *such that* (2.10), (2.11) *and* (2.20) *hold.*

In other words, the conjecture that we formulated above is true: any
compensator that has a feedback-observer interpretation is in fact an MA-
compensator. It will follow from the discussion in Section 2.5 that each
separating invariant subspace M of A_e gives rise to a compensator couple
$T = Q^{-1}M$, $V = PM$ that satisfies the conditions of Thm.2.5. Correspondingly,
the eigenvalues of $A_e : M$ can be called 'feedback poles', and the eigen-
values of $A_e : (X \oplus W)/M$ can be labeled 'observer poles'.

There is no reason to assume that the decomposition in a 'feedback' and
an 'observer' part is unique. Rather, the converse is likely to be true: in
general, there will be many separating invariant subspaces and so there will
be many decompositions of the closed-loop spectrum in 'feedback poles' and
'observer poles'. A full discussion of this subject would take us into
algebraic geometry and outside the scope of the present work, but we would
like to indicate what is probably true. The conditions (2.21) describe a
transversality relation in the sense of WONHAM (1979; p.29) in case the
following relation holds for M:

(2.22) $\dim W \leq \dim M \leq \dim X.$

This implies that subspaces of $X \oplus W$ that satisfy (2.22) will 'in general'
be separating. If $\dim W \leq \dim X$, there will be 'in general' many invariant
subspaces of a given extended system matrix A_e satisfying (2.22). So it seems
not unreasonable to conjecture that there will 'in general' be many separa-
ting invariant subspaces of a given extended system matrix.

This needs proof, of course. Our point here is mainly to argue that it
can not be expected   . that there is an intrinsic property that divides
the closed-loop poles into 'feedback poles' and 'observer poles'. A given

compensator could be interpreted as a combination of 'slow' feedback with a 'fast' observer or vice versa, without any preference for one of the two interpretations. This point of view is untraditional. In many textbooks (for instance KWAKERNAAK & SIVAN (1972, p.383,387)), the designer is advised to choose the 'observer poles' somewhat faster than the 'feedback poles'. The implicit assumption that there is such an intrinsic distinction should, at least, not be taken for granted.

The question if there are compensators that are not of the "MA" type is easily answered. If dim W > dim X, then there are no separating subspaces of $X \oplus W$. Consequently, no compensator of order larger than the order of the original system can be an MA-compensator. This example is unsatisfactory insofar as it is always possible to construct a stabilizing compensator of order n if one can be constructed at all. Another example is the extended system matrix

$$(2.23) \qquad A_e = \begin{pmatrix} -1 & \vdots & 1 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & 1 \end{pmatrix}$$

which has no invariant separating subspace. If we suppose that $-1 \in \mathbb{C}_g$, then the extended system is stable but the original system is also stable, so that in this case it is unnecessary to add a compensator. If $-1 \notin \mathbb{C}_g$ the original system is unstable, but then the compensator does not stabilize it. So this example is also unsatisfactory. One is tempted to think that the only examples of extended system matrices that do not have a separating invariant subspace are those for which there is a trivial reduction of the compensator order. We shall leave this as a conjecture.

2.3. The construction lemma

In this section, we approach the compensator problem from the construc-
tive side; applications will be given in the next section. Our main objec-
tive is to prove Thm. 2.4. The following *construction lemma* (similar lemmas
will be proved in the next chapter) already provides all the material that
we shall need.

LEMMA 2.6 *Suppose that we are given a compensator couple* (T,V) *and mappings*
$F \in F(V)$ *and* $G \in G(T)$ *such that* Ker $F \supset T$ *and* Im $G \subset V$. *Then we can construct*
an extended system matrix A_e *acting on an extended state space of dimension*
n + dim V - dim T, *such that* A_e *has an invariant subspace* M *with the*
following properties:

(2.24.1) $Q^{-1}M = T$, PM = V

(2.24.2) $\underline{\sigma}(A_e: M) = \underline{\sigma}(A+BF: V)$

(2.24.3) $\underline{\sigma}(A_e: (X \oplus W)/M) = \underline{\sigma}(A+GC: X/T)$.

PROOF Let W be a linear space of dimension k = dim V - dim T, and let R
be a mapping of V onto W such that Ker R = T. Because R is surjective, there
is a right inverse R^+: $W \to V$ such that $RR^+ = I_W$. We now define A_e by

(2.25) $A_e = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}$

where the mappings L, M and N are defined as follows:

(2.26.1)   L = FR^+

(2.26.2)   M = $-RG$

(2.26.3)   N = $R(A+BF+GC)R^+$.

The mapping R^+ is not determined uniquely (if $T \neq 0$); but if R_1^+ and
R_2^+ are both right inverses to R then we must have Im$(R_1^+ - R_2^+) \subset T$, so that,
in particular, $FR_1^+ = FR_2^+$. So we see that (2.26.1) uniquely defines a mapping
L. The definition (2.26.2) is justified because G maps into V which is

precisely the domain of R. As to (2.26.3), it should be noted that both T and V are invariant for A + BF + GC; from this it follows that N is well-defined and that N is uniquely determined by (2.26.3), despite the non-uniqueness of R^+.

Now consider the subspace

$$(2.27) \qquad M: = \{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V \}$$

of $X \oplus W$. It is immediately clear that $PM = V$ and $Q^{-1}M = T$. Moreover we have for all $x \in V$

$$(2.28) \qquad A_e \begin{pmatrix} x \\ Rx \end{pmatrix} = \begin{pmatrix} (A+BF)x \\ R(A+BF)x \end{pmatrix} \in M.$$

(This is seen by noting that $R(I-R^+R) = R -(RR^+)R = 0$ so that $I - R^+R$ maps in-to Ker R = T. Consequently, $(A+BFR^+R)x = (A+BF)x$ and $R(A+BF+GC)R^+Rx - RGCx = = R(A+BF)x$ for all $x \in V$.) From (2.21), we see that M is A_e-invariant.

Moreover, if we denote by $\bar{P}$ the restriction of P to M considered as a mapping to $PM = V$, then $\bar{P}$ is an isomorphism, and the following diagram commutes:

$$(2.29)$$

$$
\begin{array}{ccc}
 & A_e:M & \\
M & \longrightarrow & M \\
\bar{P} \downarrow & & \downarrow \bar{P} \\
V & \longrightarrow & V \\
 & A+BF:V &
\end{array}
$$

This gives us (2.24.2). Finally, define the mapping $\bar{Q}$: $X/T \to (X\oplus W)/M$ by

$$(2.30) \qquad \bar{Q}[x] = [Qx]$$

(where, of course, the first equivalence class is modulo T and the second is modulo M). The mapping $\bar{Q}$ is well-defined because $QT \subset M$. Moreover, $\bar{Q}$ is an isomorphism between X/T and $(X\oplus W)/M$. (To see this, note that Q is injective because $Qx \in M$ implies $x \in T$, and compute: $\dim(X\oplus W)/V = = (\dim X + \dim V - \dim T) - \dim V = \dim X/T$.)

For all $x \in X$, we have

$$(2.31) \qquad Q(A+GC)x - A_e Qx = \binom{(A+GC)x}{0} - \binom{Ax}{-RGCx} =$$

$$= \binom{GCx}{RGCx} \in M$$

so that we have the following equality in $(X \oplus W)/M$:

$$(2.32) \qquad [A_e Qx] = [Q(A+GC)x].$$

In other words, the diagram

$$(2.33)$$

commutes. We already noted that $\bar{Q}$ is an isomorphism; thus (2.24.3) follows.   ⊠

A graphical illustration of the lemma may be given by a set of three lattice diagrams, in which we use corresponding letters to indicate the equalities (2.24.2) and (2.24.3).
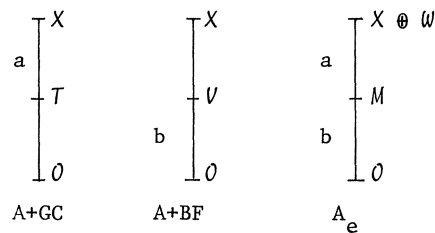

Figure 2.5: Compensator construction.

This is fairly simple, but we shall encounter more complicated versions in the next chapter.

Of course, Thm. 2.4 as announced in the previous section follows immediately from an application of the construction lemma that we have just

proved.

2.4. Methods of compensator construction

In this section, we present several methods of compensator design which can be derived from Thm. 2.4. Four of these contain well-known results, which are now brought within a single theoretical framework; we shall also present a fifth method which is new and different in nature.

2.4.1. The full-order compensator

COROLLARY 2.7 *Let the system* (2.1) *be given, and suppose that* (A,B) *is stabilizable and* (C,A) *is detectable. Then the system can be stabilized by a compensator of order* n.

PROOF Take F such that A + BF is stable and G such that A + GC is stable. Then apply Thm. 2.4 with $V = X$ and $T = 0$. ⊠

This is the standard "full-order compensator" that appears in many textbooks. (See for instance KWAKERNAAK & SIVAN (1972; §5.2))

2.4.2. The reduced-order compensator

It is possible to reduce the order of the compensator by employing static output feedback. First we make the following simple observation.

LEMMA 2.8 *For any given feedback mapping* F *and any given subspace* T *such that* $T \oplus$ Ker $C = X,$ *there exists a static output feedback mapping* $K: Y \to U$ *and a feedback mapping* F_0 *such that* A + BF = A + BKC + BF_0 *and* Ker $F_0 \supset T$.

PROOF Define T: $Y \to X$ by Im T = T and CT = I_Y. Then TC is the projection along Ker C onto T. Take K = FT and F_0 = F(I-TC); then A + BF = = A + BKC + BF_0, and Ker $F_0 \supset T$. ⊠

Of course, the idea is to use the pair (T,X) as the compensator couple that is used in Thm. 2.4, after the system matrix has been changed to A + BKC by static feedback. Note that any complement T of Ker C is

conditioned invariant. However, we need more than that: T has to be a detectability subspace. On this, we have the following result.

LEMMA 2.9 *Let a system matrix* A *and an output mapping* C *be given, and suppose that the pair* (C,A) *is detectable. Then we can find an outer-detectable complement of* Ker C.

PROOF Take an arbitrary complement T_0 of Ker C, and define T_0 by Im T_0 = = T_0 and $CT_0 = I_y$. Then we can consider $(I-T_0C)A$: Ker C as a system matrix acting on the state space Ker C, and we may view T_0 as an output space and T_0CA as an output mapping from Ker C to T_0. We contend that the pair $(T_0CA,(I-T_0C)A)$ is detectable. For, suppose the contrary; then it follows from Cor. 1.21 that there would exist $\lambda \in \mathbb{C}_b$ and $x \in$ Ker C ($x\neq0$) such that $(I-T_0C)Ax = \lambda x$ and $T_0CAx = 0$. But then we would have $Ax = \lambda x$ and $Cx = 0$, contradicting the assumption that the pair (C,A) is detectable.

We conclude that there exists a mapping Z: $T_0 \to$ Ker C such that $(I-T_0C)A + ZT_0CA = (I-(T_0-ZT_0)C)A$ is stable (on Ker C). Write T: = = $T_0 - ZT_0$, and T: = Im T. Then T is a complement of Ker C, since CT = = $CT_0 = I_y$. Again proceeding by contradiction, suppose that T would not be a detectability subspace. Then, by Thm. 1.14, there would exist $x \in$ Ker C ($x \neq 0$) such that $(\lambda-A)x \in T$ for some $\lambda \in \mathbb{C}_b$. But then we would also have $(\lambda-(I-TC)A)x = (I-TC)(\lambda-A)x = 0$, contradicting the fact that (I-TC)A: Ker C is stable. ◻

This result is a variant of Thm. 3.3 in WONHAM (1979; p. 64); we have replaced observability with detectability. Using Thm. 1.14, the result may also be stated as follows: if $(\lambda-A)^{-1}0 \cap$ Ker C = 0 for all $\lambda \in \mathbb{C}_b$, then there exists a complement T of Ker C such that $(\lambda-A)^{-1}T \cap$ Ker C = 0 for all $\lambda \in \mathbb{C}_b$. This formulation clearly shows that the condition of detectability is not only sufficient but also necessary for such a T to exist.

Now we are able to derive the compensator that is commonly known as the "reduced-order compensator" (LUENBERGER (1964)). Its order is n-p where p = dim y (the number of outputs).

COROLLARY 2.10 *Let the system* (2.1) *be given, and suppose that* (A,B) *is stabilizable and* (C,A) *is detectable. Then the system can be stabilized by a* PI-*compensator of order* n-p.

PROOF Take F such that A + BF is stable. Let T be an outer-detectable complement of Ker C, and take G $\in$ $\underline{G}(T)$ such that A + GC: X/T is stable. Define T by Im T = T and CT = I_y. Now apply Thm. 2.4 to the system matrix A + BFTC with the feedback mapping F_0: = F(I-TC) and the injection mapping G_0:= G - BFT, using the compensator couple (T,X). ⊠

The procedure can be dualized to give a compensator of order n-m, where m = dim U is the number of inputs; Thm. 2.4 is then applied to a compensator couple (O,V) where V is a suitable complement of Im B. This possibility was first noted by F.M.Brasch (according to LUENBERGER (1971)).

2.4.3. The Brasch-Pearson compensator

We can find yet another bound for the compensator order by using a lemma that has its origin in observer theory. The result, which we state here in a dualized and somewhat adapted form, is due to LUENBERGER (1966), with a minor correction by WONHAM & MORSE (1972). The lemma uses the notion of the *controllability index* of the pair (A,B). This is one of the structural invariants of linear systems (see, for instance, MORSE (1973a)), defined, for a controllable pair (A,B), by

$$(2.34) \qquad \kappa_c = \min\{k \in \mathbb{N} \mid \sum_{i=1}^{k} A^{i-1}(\text{Im B}) = X\}.$$

LEMMA 2.11 *Let* (A,B) *be a controllable pair. Then, for any one-dimensional subspace T of X, there exists a stabilizability subspace V, of dimension equal to or less than* κ_c, *such that $T \subset V$.*

PROOF See WONHAM (1979; § 3.9). ⊠

We want to use a pair (T,V) as in the lemma to construct a compensator of order $\leq \kappa_c$ - 1. This can be done using a procedure which is applicable in a more general situation.

LEMMA 2.12 *Suppose that we have an outer-detectable complement T of* Ker C *and a stabilizability subspace V such that $T \subset V$. Then we can construct a stabilizing* PI-*compensator of order* dim V - dim T = dim V - p.

PROOF Take F $\in$ $\underline{F}(V)$ such that A + BF: V is stable; define T: $X \to Y$ by Im T = T and CT = I_y. If we take G = -AT, then G $\in$ $\underline{G}(T)$ because

$(A+GC)T = A(I-TC)T = 0 \subset T$. Moreover, A + GC: X/T is stable; for, if this was not the case, there would exist an x $\in$ Ker C (x $\neq$ 0) such that $A(I-TC)x - \lambda x = Ax - \lambda x \in T$ for some $\lambda \in \mathbb{C}_b$, contradicting the outer-detectability of T.

.   Now we apply Thm. 2.4 with the system matrix A + BFTC, the feedback mapping F_0: = F(I-TC) and the injection mapping G_0: = G - BFT = -(A+BF)T, using the compensator couple (T,V). This is possible because $(A+BFTC)T = (A+BF)T \subset (A+BF)V \subset V$; also, Ker $F_0 \supset T$ and Im $G_0 = (A+BF)T \subset V$.   ⊠

From the foregoing lemmas, we now immediately have the following result.

<u>COROLLARY 2.13</u> *Let the system* (2.1) *be given; suppose that* (A,B) *is controllable,* (C,A) *is detectable, and* p = dim Y = 1. *Then the system can be stabilized by a* PI-*compensator of order less than or equal to* κ_c - 1.

<u>PROOF</u> By Lemma 2.9, we can find an outer-detectable complement T of Ker C; we shall have dim T = p = 1. Then we can apply Lemma 2.11 to find a stabilizability subspace V of dimension $\leq \kappa_c$ such that $T \subset V$. Finally, a compensator of order $\leq \kappa_c$ - 1 is then obtained from Lemma 2.12.   ⊠

If p is larger than 1, the number of outputs can of course be reduced simply by replacing C by $\tilde{C}$: = HC, where H is any functional on Y; but this has to be done in such a way that the pair $(\tilde{C},A)$ will be detectable. Under general circumstances this can easily be done, but there are special cases in which this is not possible. If the original pair (C,A) is observable (and (A,B) is controllable), then one can prove that there is a preliminary static output feedback K_0 such that A + BK_0C is cyclic (see WONHAM (1979; p. 74)), and then again there is no problem in performing the reduction (use Cor. 1.1 in WONHAM (1979; p. 43)).

Of course, there is also a dual procedure leading to a compensator of order $\leq \kappa_0$ - 1 where κ_0 is the *observability index* of the pair (C,A) (see also WONHAM (1979; § 3.8)). The compensator order we have found is due to BRASCH & PEARSON (1970).

2.4.4. Zero-order compensators

It is seen from Lemma 2.12 that if we can find a complement of Ker C that is at the same time a stabilizability subspace and a detectability

subspace, then it is possible to construct a stabilizing PI-compensator of order 0; that is, the system can be stabilized by static output feedback only. The following lemma implies that such complements of Ker C are 'almost always' present if the total number of inputs and outputs exceeds the state space dimension.

LEMMA 2.14 *The property "V is a controllability subspace with respect to the pair (A,B)" is generic in the set of triples (A,B,V) where A is a system matrix acting on a fixed n-dimensional state space X, B is an input mapping with dim Im B equal to some fixed number m, and V ⊂ X is a subspace of dimension larger than n - m + 1.*

PROOF In the indicated set, we have generically V + Im B = X, in which case V is a controlled invariant subspace. We can then take a projection P on X such that Im P = V and Ker P ⊂ Im B. Because Im(I-P) ⊂ Im B, there exists H: X → U such that I - P = BH. Consequently, PA = A - BHA is of the form A + BF, and of course PAV ⊂ V. Also note that Im(PB) = Im B ∩ V. By genericity of controllability (see, for instance, WONHAM (1979; Thm. 1.3)), the equality <PA | Im(PB)> = V (implying that V is a controllability subspace) will hold generically.   ⊠

An alternative proof based on the characterization of controllability subspaces given by the "controllability subspace algorithm" (WONHAM (1979; p. 106)) has been given in SCHUMACHER (1980 e).

COROLLARY 2.15 *Let the system (2.1) be given; suppose that (A,B) is stabilizable, (C,A) is detectable, and dim Im B ≥ dim Ker C + 1 (so m + p ≥ n + 1). Also suppose that $\mathbb{C}_g$ is an open set. Then it is generically (with respect to A and B) possible to stabilize the system by static output feedback only.*

PROOF By Lemma 2.9, we can find an outer-detectable complement T of Ker C; note that the property of being an outer-detectable complement of Ker C is preserved under small perturbations of T. We have dim T ≥ n - m + 1 so, generically, T is a stabilizability subspace and we can apply Lemma 2.12 to arrive at the desired conclusion.   ⊠

The result is related to the material presented in WANG & DAVISON (1975) and KIMURA (1975; 1978).

2.4.5 An iterative approach

The success of the above methods in reducing the compensator order is moderate; for instance, for single-input single-output systems none of these methods gives a better estimate than n - 1. This is in sharp contrast with practical experience; as a rule, engineers are able to stabilize rather complicated systems (notably of the single-input-single-output type) using relatively simple controllers. The reason may be that most systems met in practice are "reasonably stabilizable" (in a sense that is not easily made precise), whereas the methods used above perform a "worst case analysis" in that they consider *all* systems of a given order, with additional information cansisting only of the number of inputs and outputs, and in some cases the controllability (or observability) indices.

So one would like to see methods which give low-order compensators for some restricted class of "reasonably stabilizable" systems; but then one has the problem of how to describe such a class. Under these circumstances, it is reasonable to look for a procedure which allows one to perform a more or less systematic search for low-order compensators, without trying to give an a priori bound for the compensator order that will ultimately be found. It is indeed possible to give such a procedure, based on Thm. 2.4. Consider the following steps.

1. Find F and $\hat{G}$ such that A + BF and A + $\hat{G}$C are stable.

2. Among the subspaces of dimension k($\geq$ p) that are invariant for A + BF, find one that is close (in the sense of some metric in the set of subspaces of X) to Im $\hat{G}$; call this subspace V.

3. Let G be a (small) perturbation of $\hat{G}$ such that Im G $\subset$ V.

4. See if A + GC is still stable; if so, apply Thm. 2.4 with the mappings F and G that are now found, using the compensator couple $(0,V)$. If not, re-initialize (Step 1) or go back to Step 2 replacing k by k + 1.

This is basically the procedure we shall use in the second half of this work to find finite-dimensional stabilizing compensators for infinite-dimensional systems. (See Ch. 5 for a detailed discussion and examples.) We shall not go into the details here; just note that we have taken $T = 0$ to satisfy the requirement Ker F $\supset$ T automatically, and to avoid difficulties in Step 3 stemming from the requirement G $\in$ $\underline{G}(T)$. An alternative would be to base the procedure on Lemma 2.12 instead of Thm. 2.4; the outer-detectable

complement T of Ker C then replaces Im G, and the compensator order is decreased by p.

2.5 Separating subspaces

In this section, we approach the compensator problem from the 'necessity' point of view. The conditions of Thm. 2.4 may seem rather special; are they necessary for a stabilizing compensator to exist ? We now show that these conditions must hold in every situation in which the extended system matrix has an invariant subspace that satisfies a simple dimensional relation.

THEOREM 2.16   *Let* A_e *be a stable extended system matrix of the form* (2.5), *and suppose that* A_e *has an invariant subspace* M *with the following property:*

(2.35)   dim PM = dim W + dim $Q^{-1}M$.

Then there exists a compensator couple (T,V) *(in the original state space* X*) such that* dim W = dim V - dim T, *and there exist   mappings* F $\in$ $\underline{F}(V)$ *and* G $\in$ $\underline{G}(T)$ *such that*

(2.36.1)   Ker F $\supset$ T

(2.36.2)   Im G $\subset$ V

(2.37.1) σ(A+BF: V) $\subset$ $\mathbb{C}_g$

(2.37.2) σ(A+GC: X/T) $\subset$ $\mathbb{C}_g$.

In the course of the proof, we shall need the following lemma.

LEMMA 2.17   *For any subspace* M *of the extended state space* X $\oplus$ W*, the following inequalities hold:*

(2.38)   dim W + dim $Q^{-1}M$ $\geq$ dim M $\geq$ dim PM.

PROOF   Introduce the following operator from M to W:

(2.39) P_W: $\binom{x}{w}$ $\mapsto$ w $\left(\binom{x}{w} \in M \right)$.

Because Ker P_w is isomorphic to $Q^{-1}M$ and Im P_w is contained in W, we get

$$(2.40) \qquad \dim M = \dim \text{Ker } P_w + \dim \text{Im } P_w \le \dim W + \dim Q^{-1}M.$$

The second inequality in (2.35) is of course trivial. ⊠

PROOF (of Thm. 2.16) Take $T = Q^{-1}M$ and $V = PM$. Then it follows immediately from (2.35) that $\dim W = \dim V - \dim T$, and also it is clear that $T \subset V$. From (2.35) and Lemma 2.17 it follows that $\dim M = \dim PM$; thus, there exists a mapping R: $V \to W$ such that

$$(2.41) \qquad M = \{ (\begin{smallmatrix} x \\ Rx \end{smallmatrix}) \mid x \in V \}.$$

Of course, Ker $R = Q^{-1}M = T$ and because $\dim V = \dim W + \dim T$ it follows that R must be surjective. Consequently, R has a right inverse which we shall denote by R^+: $W \to V$.

Now, define F: $X \to U$ to be any extension to X of the mapping LR (which is defined on V), and let G be equal to $-R^+M$ considered as a mapping from Y to X; here, the mappings L and M are taken from the extended system matrix in its form

$$(2.5) \qquad A_e = (\begin{smallmatrix} A & BL \\ MC & N \end{smallmatrix}).$$

Clearly, (2.36) is satisfied. For any $x \in V$, we have

$$(2.42) \qquad (\begin{smallmatrix} A & BL \\ MC & N \end{smallmatrix})(\begin{smallmatrix} x \\ Rx \end{smallmatrix}) = (\begin{smallmatrix} Ax+BLRx \\ MCx+NRx \end{smallmatrix}) \in M$$

so that $(A+BLR)x = (A+BF)x \in V$. We see that V is controlled invariant and that $F \in \underline{F}(V)$. Moreover, if we let $\bar{P}$ denote the restriction of P to M considered as a mapping onto $PM = V$, then $\bar{P}$ is an isomorphism and (2.42) shows that the diagram

(2.43)

commutes. Hence, we obtain

$$(2.44) \qquad \sigma(A+BF: V) = \sigma(A_e: M) \subset \mathbb{C}_g.$$

Furthermore, for any $x \in T$ we have

$$(2.45) \qquad \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ MCx \end{pmatrix} = \begin{pmatrix} (A-R^+MC)x \\ 0 \end{pmatrix} + \begin{pmatrix} R^+MCx \\ MCx \end{pmatrix} \in M.$$

Because

$$(2.46) \qquad \begin{pmatrix} R^+MCx \\ MCx \end{pmatrix} = \begin{pmatrix} R^+MCx \\ RR^+MCx \end{pmatrix} \in M,$$

this shows that T is conditioned invariant and that $G \in \underline{G}(T)$. If we define $\bar{Q}: X/T \to (X{\oplus}W)/M$ by $\bar{Q}[x] = [Qx]$ (which is justified since $QT \subset M$), then $\bar{Q}$ is an isomorphism and (2.45) shows that the diagram

$$(2.47)$$

commutes. So we obtain

$$(2.48) \qquad \sigma(A+GC: X/T) = \sigma(A_e:(X{\oplus}W)/M) \subset \mathbb{C}_g.$$

We still have to show that $AT \subset V$. This follows, for instance, from $AT = (A+BF)T \subset (A+BF)V \subset V$. ⊠

The theorem that has been announced in Section 2.2 (Thm. 2.5) now follows immediately from the proof, using (2.44) and (2.48).

The condition (2.35) appears to be important for the separation of the closed-loop eigenvalues into "feedback poles" and "observer poles" (according to the interpretation given in Section 2.3). This is why we have called subspaces $X \oplus W$ that satisfy this condition *separating subspaces.*

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CHAPTER 3

TRACKING, REGULATION AND DISTURBANCE LOCALIZATION

After having treated the fundamental problem of stabilization by
dynamic feedback, we now turn to more complex problems. The complexity
arises from setting new control objectives and considering more elaborate
disturbance models. We shall require that some given set of state variables
follows ("tracks") a signal produced by some independent finite-dimensional
linear system. We shall assume that the system is affected by disturbances;
for some of these disturbances we may have a dynamic model, but others may
be completely unknown functions. Protection against modeled disturbances
is called "regulation"; the term "disturbance localization" is used when
there is no model available.

The subjects of "tracking" and "regulation" can be brought into one
setting, and they have been extensively studied in this way. Disturbance loca-
lization is a problem of which the state feedback solution is well-known;
the solution by dynamic output feedback has only recently been obtained.
Here the two lines of research are brought together into a single frame-
work. A very general problem is obtained but nevertheless we are able to
give a completely constructive solution, in which the concept of "compen-
sator couple" is again crucial. The main theorem, which provides
this solution, enables us to derive many of the existing results both on
tracking and regulation and on disturbance localization as special cases.

There are five sections. The first section provides the motivation for
the problem as we state it. In the second section, we give preliminary
results: an adapted version of Thm. 2.16, and an extended construction
lemma. The main theorem follows in Section 3, and in Section 4 we give a
number of special cases, some of which contain earlier results. Section 5
contains a brief discussion of order reduction.

3.1. Problem statement

In the previous chapter, disturbances were modeled by nonzero initial
conditions. Of course, this is not always a suitable model and so we consider
some alternatives. For instance, assume that a disturbance function $q(t)$

enters the system in the following way (where E is a linear mapping into X):

(3.1.1) $x'(t) = Ax(t) + Bu(t) + Eq(t)$

(3.1.2) $y(t) = Cx(t)$.

If we now add a compensator of the form (2.2), the closed loop system becomes:

(3.2) $\frac{d}{dt}\binom{x}{w}(t) = \binom{A \quad BL}{MC \quad N}\binom{x}{w}(t) + \binom{E}{0}q(t)$.

Clearly, it would be asking too much if we would require that the full state vector should approach zero as $t \to \infty$, regardless of the behaviour of the function $q(t)$. So we suppose that there is a special set of variables in which we are interested. Call these the *variables-to-be-controlled* (to be denoted by $z(t)$), which depend on $x(t)$ via a linear mapping D:

(3.3) $z(t) = Dx(t)$.

The equation (3.2) becomes supplemented by the equation for the variables-to-be-controlled

(3.4) $z(t) = (D \quad 0)\binom{x}{w}(t)$.

We may now require that the evolution of $z(\cdot)$ is independent of $q(\cdot)$; we shall then say that the disturbance has been *localized* with respect to the variables-to-be-controlled. This means that there is a subspace of the extended state space $X \oplus W$ with the following properties (where A_e is, as usual, the extended system matrix appearing in (3.2)):

(3.5.1) $A_e M \subset M$

(3.5.2) $\text{Im}\binom{E}{0} \subset M \subset \text{Ker}(D \quad 0)$.

(See also WONHAM (1979; p. 87).) Moreover, the variable $z(t)$ will have the required stability properties if

(3.5.3) $\sigma(A_e : (X \oplus W)/M) \subset \mathbb{C}_g$.

In this set-up, there is no assumption on the behaviour of the function
q(t) (except for the usual regularity hypotheses, for instance piecewise
continuity). In many cases however, it may be possible to give a description
of the disturbance; for instance, it may be constant over long periods of
time, or behave like a sine function of known frequency. Then we can re-write
the equation (3.1.1) using a different notation:

(3.6.1) $x_1'(t) = A_{11}x_1(t) + B_1 u(t) + A_{12}x_2(t)$

and we can add the model for the disturbance, which in general has the
following form:

(3.6.2) $x_2'(t) = A_{22}x_2(t)$

So the disturbance signal is generated by the matrix A_{22}; for instance,
by taking $A_{22} = 0$ one gets a constant function whose value is determined
by $x_2(0)$. The equations (3.6.1) and (3.6.2) can be combined into one equa-
tion:

(3.7.1) $x'(t) = Ax(t) + Bu(t)$

where

(3.7.2) $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$.

Now the system equations are again in the standard form, but one sees
that the pair (A,B) is *not* controllable. Therefore, one can only hope to
stabilize a set of variables-to-be-controlled:

(3.8) $z(t) = Dx(t) = (D_1 \quad D_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t)$.

From the context, it is reasonable to assume that one would put $D_2 = 0$ in
the matrix D.

After having discussed these extended disturbance models, let us now
take a look at alternative control objectives. In the previous chapter, we
have worked with a setpoint at 0 to which the system should return after

an initial disturbance, However, it may be required that the system stays at a non-zero setpoint, which may be changed from time to time. More generally, one may require that some set of variables-to-be-controlled follows a reference signal, modeled as the output of an autonomous linear system. For such a situation, we can write down the following equations:

(3.9.1) $x_1'(t) = A_{11}x_1(t) + B_1u(t)$

(3.9.2) $x_2'(t) = A_{22}x_2(t)$

(3.9.3) $z(t) = D_1x_1(t) + D_2x_2(t).$

Here, the reference signal is given by the variable $x_2(t)$, and the desired relation between $x_1(t)$ and $x_2(t)$ is expressed in (3.9.3) where $z(t)$ should approach zero. Again, the system equations may be written down in the concise form

(3.10.1) $x'(t) = Ax(t) + Bu(t)$

(3.10.2) $z(t) = Dx(t)$

where, in this case, the matrices A, B and D are given by

(3.11) $A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad D = (D_1 \quad D_2).$

We see that the pair (A,B) is not controllable. In contrast with the setting we derived above, the matrix D_2 will not be equal to zero. The purpose of the control, however, is the same: $z(t)$ has to approach zero at some prescribed rate.

We have now discussed three possible extensions of the compensator problem: *localization* of an unmodeled disturbance, *regulation* against a modeled disturbance, and *tracking* of a modeled reference signal. Comparing the condensed forms of the system equations in each case, it is clear that it is possible to cover all cases in one general framework, given by the following set of equations:

(3.12.1) $x'(t) = Ax(t) + Bu(t) + Eq(t)$

(3.12.2) $y(t) = Cx(t)$

(3.12.3) $z(t) = Dx(t)$.

Pictorially, the situation can be described as follows:
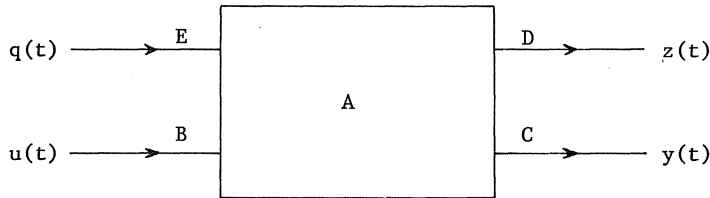

Figure 3.1: Setting for general regulator problem.

This setting has a natural generality. There are two kinds of inputs: one is available to the controller, the other is chosen by "nature". There are also two kinds of outputs: one is the observation that can be used by the controller, the other defines the control objective. After closing the loop, only one input and one output remain.

While the formulation (3.12) has the important advantage of combining generality with simplicity, it is not immediately clear how the control objectives should be described in this framework. The "disturbance localization" requirement already has been given in (3.5), but to cover the "tracking" and "regulation" aspects we first return to the more explicit form

(3.13.1) $\dfrac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) + \begin{pmatrix} B_1 \\ 0 \end{pmatrix}u(t) + \begin{pmatrix} E_1 \\ E_2 \end{pmatrix}q(t)$

(3.13.2) $y(t) = (C_1 \quad C_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t)$

(3.13.3) $z(t) = (D_1 \quad D_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t)$.

When a compensator of the form (2.2) is added, eqn. (3.13.1) becomes:

(3.14) $\dfrac{d}{dt}\begin{pmatrix} x_1 \\ w \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} A_{11} & B_1 L & A_{12} \\ MC_1 & N & MC_2 \\ 0 & 0 & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ w \\ x_2 \end{pmatrix}(t) + \begin{pmatrix} E_1 \\ 0 \\ E_2 \end{pmatrix}q(t)$.

The extended system matrix has a block-diagonal structure. If the pair (A_{11}, B_1) is stabilizable and the pair (C_1, A_{11}) is detectable, then it is clear that the left upper block may be stabilized by a suitable choice of L, M and N; and this is certainly a reasonable design objective. On the other hand, it is clear that the eigenvalues of A_{22} cannot be shifted by dynamic output feedback.

We can formulate the requirement "left upper block stable" in coordinate-free terms, in the following way. Roughly stated, what we want is that the unstable poles of the closed-loop system are precisely the unstabilizable poles of the open-loop system. Let us write $X_b^e(A_e)$ for the unstable subspace of A_e (cf. (1.5.1)). Then our requirement can be expressed as follows:

$$(3.15) \qquad \dim X_b^e(A_e) = \operatorname{codim} X_{stab}$$

where the subspace on the right-hand side is the one defined in Section 1.5 (see also Lemma 1.17). One can say that in the coordinate-free terminology, the subspace X_{stab} takes the place of the state space of $x_1(t)$ in the explicit form (3.13). (This identification assumes, of course, that the pair (A_{11}, B_1) is stabilizable. If this is not the case, then the "left upper block" cannot be made stable and the problem in the form we want to pose it is not solvable. The identification assumes also that A_{22} is completely unstable, but this is less essential.)

We now have two stability requirements, one in (3.5.3) and one in (3.15). In (3.5), the behaviour of the variables-to-be-controlled is in focus, whereas (3.15) gives a condition for all stabilizable state variables. The two stability requirements do not have to be the same; in fact, it is quite natural to assume that a slow response of a large part of the system would be satisfactory whereas the variables-to-be-controlled should behave much faster. Therefore, we should use two partitionings of the complex plane instead of one, as illustrated below.
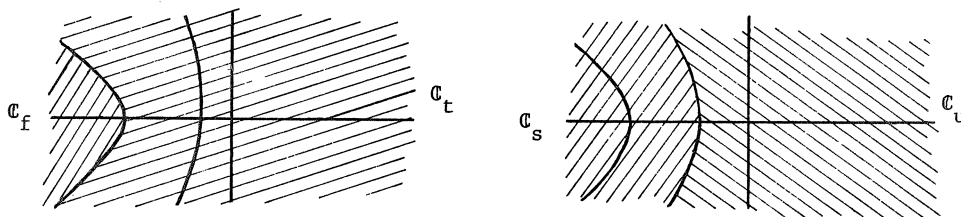

Figure 3.2: Two partitionings of the complex plane.

Further on, we shall assume that two parts of the complex plane have been given, denoted by $\mathbb{C}_f$ ("f" for "fast") and $\mathbb{C}_s$ ("s" for "slow"). We shall also assume that $\mathbb{C}_f \subset \mathbb{C}_s$, and denote $\mathbb{C}_t = \mathbb{C} \backslash \mathbb{C}_f$ ("t" for "tardy") and $\mathbb{C}_u = \mathbb{C} \backslash \mathbb{C}_s$ ("u" for "unstable"). Because of the condition $\mathbb{C}_f \subset \mathbb{C}_s$, what we obtain is a division of the complex plane into three disjoint parts: $\mathbb{C}_f$, $\mathbb{C}_s \cap \mathbb{C}_t$ and $\mathbb{C}_u$. We shall change our notations in an obvious way to conform to the new situation; for instance X^s_{stab} will denote the largest stabilizability sub-space of the pair (A,B) with respect to the division $\mathbb{C} = \mathbb{C}_s \cup \mathbb{C}_u$, $X_u(A)$ will denote the span of the generalized eigenvectors of A associated with eigenvalues in $\mathbb{C}_u$, etc..

We can now give the following problem statement: Given a system of the form (3.12), find a compensator of the form (2.2) such that the exten-ded system matrix A_e satisfies

$$(3.16) \qquad \dim X^e_u(A_e) = \operatorname{codim} X^s_{stab}$$

and such that there is a subspace M of the extended state space $X \oplus W$ satisfying

$$(3.17.1) \qquad A_e M \subset M$$

$$(3.17.2) \qquad \operatorname{Im}\begin{pmatrix}E\\0\end{pmatrix} \subset M \subset \operatorname{Ker} (D \quad 0)$$

$$(3.17.3) \qquad \sigma(A_e: (X \oplus W)/M) \subset \mathbb{C}_f.$$

We shall refer to this problem as the *regulator problem*. The problem will be solved completely in Section 3.3; we shall formulate necessary and sufficient conditions for the solvability of the problem, give a con-structive algorithm to verify these conditions, and present a method to construct a solution if there exists one.

The regulator problem as we formulate it is a generalization of a number of problems which have been studied separately in the past. There are two lines of research that can be distinguished. On one hand, we have the theory of "tracking and regulation" which has aroused considerable interest; see for instance JOHNSON (1968, 1971), YOUNG & WILLEMS (1972), DAVISON & GOLDENBERG (1975), WONHAM & PEARSON (1974), FRANCIS (1977), WONHAM (1979). In this theory, one does not consider the presence of an unmodeled disturbance (i.e., E = 0 in the system (3.12)). On the other

hand, such a disturbance is the central object of study in the theory of "disturbance localization". Here, one usually assumes that the pair (A,B) is stabilizable. Without stability requirements, the state feedback solution of the problem is perhaps the simplest application of the idea of controlled invariant subspaces (see WONHAM (1974) or, for a dual version, BASILE & MARRO(1969c)), but the solution by dynamic output feedback has only recently been obtained (AKASHI & IMAI (1979), SCHUMACHER (1979)). For a treatment of the latter problem with stability requirements added, see WILLEMS & COMMAULT (1981) and IMAI & AKASHI (1981).

Many of the results in the above articles follow as special cases of the general theorem we shall prove. We shall perform a number of these specializations in Section 3.4. Our partitioning of the complex plane into three parts (instead of the usual number of two) is essential to obtain one general theorem, which simultaneously treats the problems with "output stability" and "internal stability" (terminology of WONHAM (1979)).

3.2. Preliminary results

First, we give a result that will be helpful in proving the 'necessity' part of the main theorem. The result may be compared to Thm. 2.16. Again, we consider an invariant subspace of the extended state space and the related pair of subspaces in the original state space. But now we do not assume that the invariant subspace is separating, and so our conclusions are somewhat weaker. One remark on notation: we shall use $\mathbb{C}_g$ which may denote either $\mathbb{C}_f$ or $\mathbb{C}_s$.

THEOREM 3.1 *Let* A_e *be an extended system matrix of the form* (2.5). *For any* A_e*-invariant subspace* M, *the pair* $(Q^{-1}M, PM)$ *is a compensator couple. Moreover, if* $A_e: M$ *is stable, then* $Q^{-1}M$ *is inner-detectable and* PM *is inner-stabilizable. Also, if* $A_e: (X \oplus W)/M$ *is stable, then* $Q^{-1}M$ *is outer-detectable and* PM *is outer-stabilizable.*

PROOF Let M be A_e-invariant. Take $x \in Q^{-1}M$; then $\binom{x}{0} \in M$ so that $x \in PM$ and, moreover, $Ax \in PM$ because

$$(3.18) \qquad A_e \binom{x}{0} = \binom{A \quad BL}{MC \quad N} \binom{x}{0} = \binom{Ax}{MCx} \in M.$$

To show that $Q^{-1}M$ is conditioned invariant, take $x \in Q^{-1}M \cap \text{Ker } C$; then (3.18) shows that $Ax \in Q^{-1}M$ and we obtain the desired conclusion from

Prop. 1.7. Next, take $x \in PM$. Then there exists $w \in W$ such that $\binom{x}{w} \in M$, and consequently:

$$(3.19) \qquad A_e \binom{x}{w} = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \binom{x}{w} = \binom{Ax+BLw}{MCx+Nw} \in M.$$

Hence, $Ax + BLw \in PM$ which implies $Ax \in PM + \text{Im } B$; we may conclude that PM is controlled invariant using Prop. 1.6.

Now, let us assume that $A_e : M$ is stable. To prove that PM is a stabilizability subspace, it is sufficient (in view of Cor. 1.15) to show that

$$(3.20) \qquad PM \subset (\lambda-A)PM + \text{Im } B \quad \text{for all } \lambda \in \mathbb{C}_b.$$

But because $\sigma(A_e : M) \subset \mathbb{C}_g$, we know that

$$(3.21) \qquad M = (\lambda-A_e)M \qquad \text{for all } \lambda \in \mathbb{C}_b.$$

From this we obtain (3.20) immediately by noting that for $\binom{x}{w} \in M$,

$$(3.22) \qquad P(\lambda-A_e)\binom{x}{w} = (\lambda-A)x - BLw \in (\lambda-A)PM + \text{Im } B.$$

Still assuming that $A_e : M$ is stable, let us now show that $Q^{-1}M$ is inner-detectable. Suppose the contrary; then it follows from Prop. 1.19 (in dualized form) and Lemma 1.18 that $Q^{-1}M \cap \langle \text{Ker } C \mid A \rangle \cap X_b(A) \neq 0$. So there would exist $x \neq 0$ and $\lambda \in \mathbb{C}_b$ such that $x \in Q^{-1}M \cap \text{Ker } C$ and $Ax = \lambda x$. Then we would have $\binom{x}{0} \in M$ and

$$(3.23) \qquad A_e \binom{x}{0} = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \binom{x}{0} = \binom{Ax}{0} = \lambda \binom{x}{0},$$

contradicting our assumption that $A_e : M$ is stable. Thus, $Q^{-1}M$ must be inner-detectable.

The rest of the proof follows by duality. ☒

The fact that PM is a stabilizability subspace if $A_e : M$ is stable has also been proved by IMAI & AKASHI (1981); these authors used a different method requiring the computation of an $F \in \underline{F}(PM)$. We did not mention the inner-detectability of $Q^{-1}M$ in Thm. 2.16; however, this follows immediately from the statement of the theorem because (using (2.36.1) and (2.37.1))

(3.24) $\sigma(A+GC: Q^{-1}M \cap <\text{Ker } C \mid A>) =$

= $\sigma(A: Q^{-1}M \cap <\text{Ker } C \mid A>) =$

= $\sigma(A+BF: Q^{-1}M \cap <\text{Ker } C \mid A>) \subset$

$\subset \sigma(A+BF: PM) \subset \mathbb{C}_g$.

Also, Thm. 3.1 gives the 'necessity' part of the proof of Thm. 2.1 (for instance, take $M = 0$). Of course, the 'sufficiency' part of this proof is provided by Cor. 2.7.

As in Ch. 2, the proof of the main theorem will be based on a construction lemma, which we now present.

LEMMA 3.2 *Suppose that we are given three compensator couples* (T_s, V_s), (T_f, V_f) *and* (T_c, V_c), *such that* $T_s \subset T_f$ *and* $V_s \subset V_f$. *Further suppose that we are also given a mapping* $F \in \underline{F}(V_s) \cap \underline{F}(V_f) \cap \underline{F}(V_c)$ *with* $\text{Ker } F \supset T_c$ *and a mapping* $G \in \underline{G}(T_s) \cap \underline{G}(T_f) \cap \underline{G}(T_c)$ *with* $\text{Im } G \subset V_c$. *Let* W *be a linear space isomorphic to* V_c/T_c *and let* R *be a mapping of* V_c *onto* W *such that* $\text{Ker } R = T_c$. *Let* R^+ *be any right inverse of* R, *and define*

(3.25) $A_e = \begin{pmatrix} A & BFR^+ \\ -RGC & R(A+BF+GC)R^+ \end{pmatrix}.$

Then the subspaces of $X \oplus W$ *defined by*

(3.26.1) $M_s = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in T_s\} + \{\begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V_s \cap V_c\}$

(3.26.2) $M_f = \{\begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in T_f\} + \{\begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V_f \cap V_c\}$

(3.26.3) $M_c = \{\begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V_c\}$

are all A_e-*invariant, and we have the following relations:*

(3.27.1) $\underline{\sigma}(A+BF: V_s \cap V_c) = \underline{\sigma}(A_e: M_s \cap M_c)$

(3.27.2) $\underline{\sigma}(A+BF: V_f \cap V_c/V_s \cap V_c) = \underline{\sigma}(A^e: M_f \cap M_c/M_s \cap M_c)$

(3.27.3) $\underline{\sigma}(A+BF: V_f + V_c/V_f) = \underline{\sigma}(A^e: M_f + M_c/M_f)$

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(3.27.4) $\underline{\sigma}(A+GC: T_s/T_s \cap T_c) = \underline{\sigma}(A^e: M_s/M_s \cap M_c)$

(3.27.5) $\underline{\sigma}(A+GC: T_f+T_c/T_s+T_c) = \underline{\sigma}(A^e: M_f+M_c/M_s+M_c)$

(3.27.6) $\underline{\sigma}(A+GC: X/T_f+T_c) = \underline{\sigma}(A^e: X^e/M_f+M_c)$.

Before embarking on the proof of this lemma, let us make a few remarks. The spectral relations in (3.27) can conveniently be summarized in the following scheme:
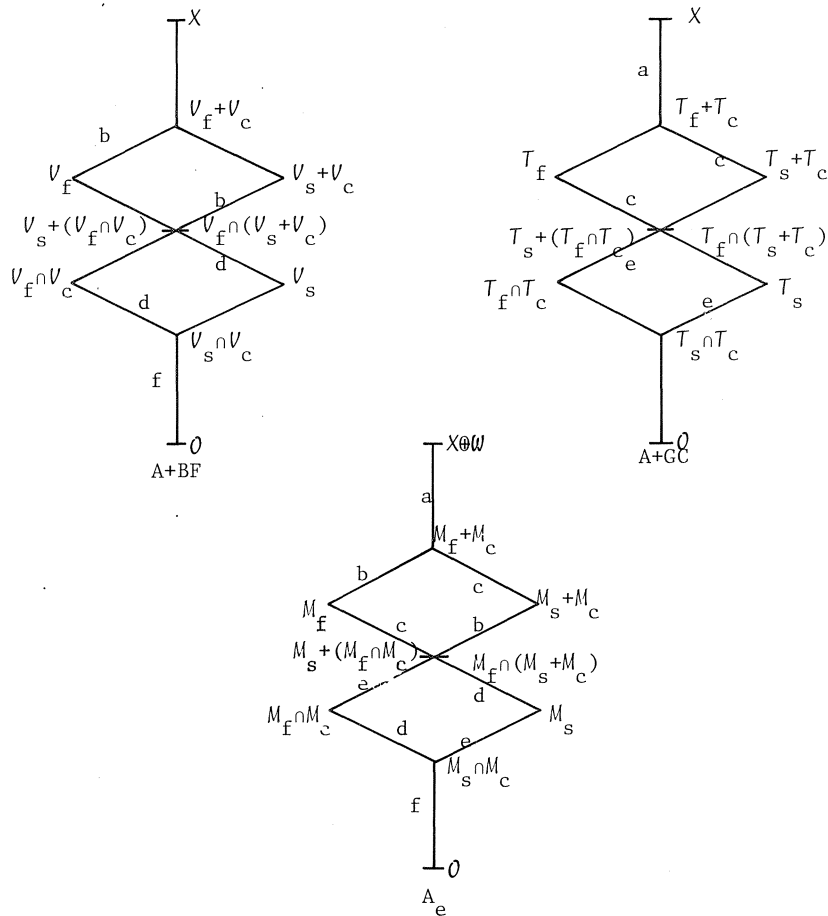

Fig. 3.3. Regulator construction

Corresponding letters are used to indicate the equalities in (3.27); compare Fig. 2.5. It should be noted that $V_s + (V_f \cap V_c) = V_f \cap (V_s + V_c)$ (the 'distributive law' may be applied because $V_s \subset V_f$), and that similar remarks can be made concerning the other diagrams (it follows immediately from the definitions that $M_s \subset M_f$). As a final remark note that M_c can be described similarly as M_s and M_f:

$$(3.28) \qquad M_c = \{ \binom{x}{0} \mid x \in T_c \} + \{ \binom{x}{Rx} \mid x \in V_c \}.$$

To see this, note that $Rx = 0$ for $x \in T_c$.

PROOF First we have to show that the subspaces defined in (3.26) are A_e-invariant; so let us consider M_j with $j \in \{s,f,c\}$. If $x \in T_j$, then

$$(3.29) \qquad A_e \binom{x}{0} = \binom{Ax}{-RGCx} = \binom{(A+GC)x}{0} - \binom{GCx}{RGCx} \in M_j$$

because T_j is $(A+GC)$-invariant, and because $GCx = (A+GC)x - Ax \in V_j \cap V_c$ (using that $AT_j \subset V_j$ and that $\text{Im } G \subset V_c$). Let us next take $x \in V_j \cap V_c$, then

$$(3.30) \qquad A_e \binom{x}{Rx} = \binom{Ax+BFR^+Rx}{R(A+BF+GC)R^+Rx-RGCx}.$$

Write $x = R^+Rx + \bar{x}$, then $R\bar{x} = 0$ so that $\bar{x} \in T_c \subset \text{Ker } F$. Thus we have $FR^+Rx = Fx$. Moreover, $R(A+GC)R^+Rx = R(A+GC)x - R(A+GC)\bar{x} = R(A+GC)x$ because $(A+GC)\bar{x} \in T_c$. We find that $R(A+BF+GC)R^+Rx - RGCx = R(A+BF+GC)x - RGCx = = R(A+BF)x$. So (3.30) becomes

$$(3.32) \qquad A_e \binom{x}{Rx} = \binom{(A+BF)x}{R(A+BF)x} \in M_j \qquad\qquad (x \in V_j \cap V_c).$$

This completes the first part of the proof. The second part (to prove the relations (3.27)) is an exercise in finding natural isomorphisms between subspaces and constructing commutative diagrams. We shall not work this out completely; by way of example, let us prove (3.27.5).

We define a mapping $\bar{Q}: T_f + T_c / T_s + T_c \to M_f + M_c / M_s + M_c$ as follows:

$$(3.33) \qquad \bar{Q}[x] = [Qx] \qquad\qquad\qquad (x \in T_f + T_c)$$

First of all we have to show that this mapping is well-defined; this follows from the fact that $Q(T_s+T_c) \subset M_s + M_c$. Because $Q(T_f+T_c) \subset M_f + M_c$, $\bar{Q}$ does indeed map into M_f+M_c/M_s+M_c. Next we show that $\bar{Q}$ is in fact an isomorphism. If $\bar{Q}[x] = 0$ for some $x \in T_f + T_c$, then $Qx \in M_s + M_c = QT_s + T_c$; this gives $x \in Q^{-1}(QT_s+M_c) = T_s + T_c$. So $\bar{Q}$ is injective. Surjectivity of $\bar{Q}$ follows from the fact that $M_f + M_c = QT_f + M_c \subset Q(T_f+T_c) + (M_s+M_c)$.

Finally, we prove that the following diagram, in which $\overline{A + GC}$ and $\overline{A}_e$ denote the induced mappings on the indicated factor spaces, commutes.

(3.34)

$$
\begin{array}{ccc}
T_f+T_c/T_s+T_c & \xrightarrow{\ \overline{A+GC}\ } & T_f+T_c/T_s+T_c \\
\bar{Q} \downarrow & & \downarrow \bar{Q} \\
M_f+M_c/M_s+M_c & \xrightarrow{\ \overline{A}_e\ } & M_f+M_c/M_s+M_c
\end{array}
$$

The proof is by direct computation: for $x \in T_f + T_c$ we have

(3.35) $\quad \overline{A}_e \, \bar{Q}[x] = [\begin{pmatrix} Ax \\ -RGCx \end{pmatrix}]$.

and on the other hand

(3.36) $\quad \bar{Q} \, \overline{A + GC}[x] = [\begin{pmatrix} (A+GC)x \\ 0 \end{pmatrix}]$.

Both results are equal because

(3.37) $\quad \begin{pmatrix} (A+GC)x \\ 0 \end{pmatrix} - \begin{pmatrix} Ax \\ -RGCx \end{pmatrix} = \begin{pmatrix} GCx \\ RGCx \end{pmatrix} \in M_c \subset M_s + M_c$. $\quad\boxtimes$

The situation described by Fig. 3.3 simplifies considerably if we assume that $T_c \subset T_s \subset T_f$ and $V_s \subset V_f \subset V_c$. The corresponding picture is:
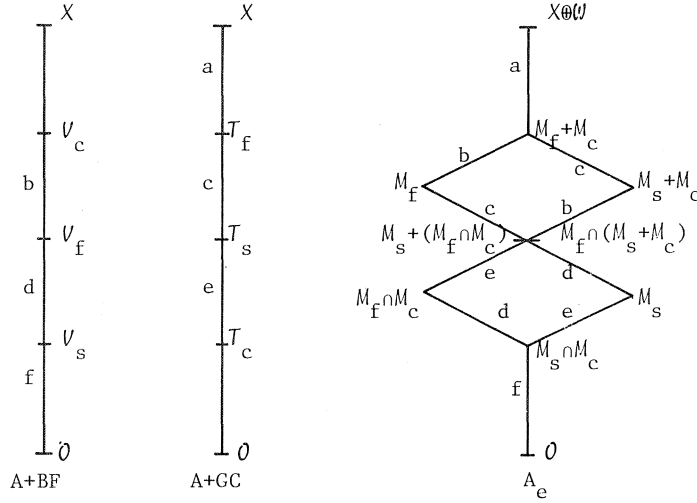
Figure 3.4: Regulator construction, simplified version.

3.3. Main theorem

We now state the necessary and sufficient conditions for a solution of the regulator problem to exist.

THEOREM 3.3. *Let the system* (3.12) *be given. Then there exists a solution of the regulator problem as formulated in* (3.16) *and* (3.17) *if and only if there exist two compensator couples* (T_s, V_s) *and* (T_f, V_f) *with the following properties:*

(3.38.1) $T_s \subset T_f, \quad V_s \subset V_f$

(3.38.2) T_f *is outer-detectable and* V_f *is outer-stabilizable, both with respect to* $\mathbb{C}_f$

(3.38.3) $\operatorname{Im} E \subset T_f \subset V_f \subset \operatorname{Ker} D$

(3.38.4) T_s *is outer-detectable with respect to* $\mathbb{C}_s$, *and* $V_s \oplus X^s_{stab} = X$

(3.38.5) $V_f = V_s \oplus S$ *for some inner-stabilizable (with respect to* $\mathbb{C}_s$) *controlled invariant subspace* S.

PROOF Necessity Assume that A_e is an extended system matrix (of the form (2.5)) satisfying (3.16), and let M be a subspace of $X \oplus W$ such that (3.17)

holds. Denote the unstable subspace (in the sense of $\mathbb{C}_u$) of A_e by $X_u^e(A_e)$, and define $T_s = Q^{-1}X_u^e(A_e)$, $V_s = PX_u^e(A_e)$, $T_f = Q^{-1}M$ and $V_f = PM$. It follows immediately from Thm. 3.1 that (T_s, V_s) and (T_f, V_f) are compensator couples, and that (3.38.2) holds; (3.38.4) will follow from Thm. 3.1 and Prop. 1.19 if we can also show that

$$(3.39) \qquad PX_u^e(A_e) \cap X_{stab}^s = 0.$$

But this is immediate from (3.16).

Because $\mathbb{C}_u \subset \mathbb{C}_t$ (see Fig. 3.2) and because of (3.17.3), we have $X_u^e(A_e) \subset X_t^e(A_e) \subset M$ and this immediately gives (3.38.1). Finally, define S by

$$(3.40) \qquad S = P(M \cap X_s^e(A_e)).$$

Then it is clear that S is inner-stabilizable with respect to $\mathbb{C}_s$, so that $S \subset X_{stab}^s$. Since $X_u^e(A_e) \subset M$, we also have

$$(3.41) \qquad M = X_u^e(A_e) \oplus (M_f \cap X_s^e(A_e)).$$

Projecting both sides into X, we obtain

$$(3.42) \qquad V_f = V_s + S.$$

But we already noted that $V_s \cap X_{stab}^s = 0$ and that $S \subset X_{stab}^s$, so in fact we have $V_f = V_s \oplus S$.

    <u>Sufficiency.</u> Now we suppose that two compensator couples (T_s, V_s) and (T_f, V_f) are given for which (3.38) holds. We construct $F \in \underline{F}(V_s) \cap \underline{F}(V_f)$ and $G \in \underline{G}(T_s) \cap \underline{G}(T_f)$ as follows. Because $V_s \oplus X_{stab}^s = X$, we may specify F by giving its action on V_s and X_{stab}^s separately. On V_s, we define F such that $F \in \underline{F}(V_s)$. Note that $V_s \supset T_s \supset X_{det}^s$ because T_s is outer-detectable; as $AX_{det}^s \subset X_{det}^s \subset V_s$, we can assume that Ker $F \supset X_{det}^s$. (In fact, it is not difficult to show that F is uniquely defined on V_s by the requirement $F \in \underline{F}(V_s)$, so that the relation $X_{det}^s \subset$ Ker F holds necessarily. Also, $\sigma(A+BF: V_s) = \sigma(A: X/X_{stab}^s) \subset \mathbb{C}_u.$)

As to X_{stab}^s, we note that this is an A-invariant subspace containing Im B; so we can consider the pair $(A: X_{stab}^s, B)$ where B is viewed as a mapping into X_{stab}^s. With respect to this pair, the subspace S is inner-

stabilizable w.r.t. $\mathbb{C}_s$ as well as outer-stabilizable w.r.t. $\mathbb{C}_f$ (the latter because it follows from $V_f + X^f_{stab} = X$ that $S + X^f_{stab} = X^s_{stab}$; note that $X^f_{stab} \subset X^s_{stab}$). So we can define F on X^s_{stab} such that $F \in \underline{F}(S)$, $\sigma(A+BF: S) \subset \subset \mathbb{C}_s$ and $\sigma(A+BF: X^s_{stab}/S) \subset \mathbb{C}_f$. Finally, it follows from the dual of Thm. 1.11 that we can choose $G \in \underline{G}(T_s) \cap \underline{G}(T_f)$ such that $\sigma(A+GC: X/T_f) \subset \mathbb{C}_f$ and $\sigma(A+GC: T_f/X^s_{det}) \subset \mathbb{C}_s$.

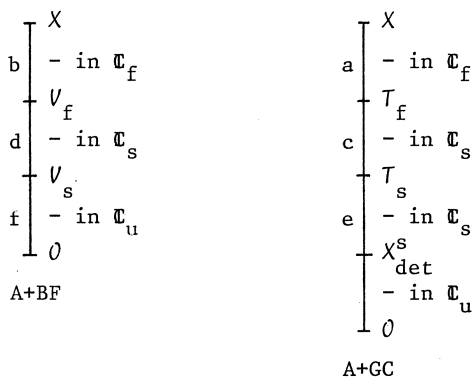The spectral situation for A + BF and A + GC can be summarized as follows:

Figure 3.5: Basis situation for the construction lemma.

Now, we apply Lemma 3.2 with $T_c = X^s_{det}$ and $V_c = X$. The conditions of the lemma are all satisfied, and so we obtain the matrix A_e of (3.25) for which the scheme (3.27) holds. Comparing Fig. 3.5 with Fig. 3.4, we see immediately that M_f is outer-stable w.r.t. $\mathbb{C}_f$. We have Im $E \subset T_f \subset Q^{-1}M_f \subset \subset PM_f \subset V_f \subset$ Ker D so Im$\binom{E}{0} \subset M_f \subset$ Ker(D 0); thus, (3.17) is satisfied. It is also clear from Fig. 3.4 and Fig. 3.5 that $X^e_u(A^e) = M_s \cap M_c$, and that $\underline{\sigma}(A^e: X^e_u(A^e)) = \underline{\sigma}(A+BF: V_s) = \underline{\sigma}(A+BF: X/X^s_{stab}) = \underline{\sigma}(A: X/X^s_{stab})$ which shows that (3.16) holds. ⊠

The result as it has been stated here is given in terms of the existence of two compensator couples satisfying (3.38). So of course one may ask: Is there an algorithm by which one (knowing the system operators A,B, C,D and E) can find out if two such pairs exist ? And if this is the case, can such pairs be constructed ? If these questions admit a positive answer, then we are entitled to say that the regulator problem has been solved in a constructive way. We shall show that this is indeed the case.

In the formulation of the theorem, there are four subspaces to be selected. However, this amount can readily be reduced to one.

PROPOSITION 3.4 *There exist two compensator couples* (T_s, V_s) *and* (T_f, V_f) *satisfying* (3.38) *if and only if there exists a controlled invariant subspace* V *such that*

(3.43.1) $V \oplus X_{stab}^s = X$

(3.43.2) $V \supset X_{det}^s$

(3.43.3) $V \subset \text{Ker } D$

(3.43.4) $V + V_s^*(\text{Ker } D) \supset T_f^*(\text{Im } E) + AT_f^*(\text{Im } E)$

and if, moreover,

(3.44) $V_s^*(\text{Ker } D) + X_{stab}^f = X_{stab}^s.$

PROOF First assume that we have two compensator couples (T_s, V_s) and (T_f, V_f) satisfying (3.38). Then $V_f = V_s \oplus S$ where S is a controlled invariant subspace that is inner-stabilizable w.r.t. $\mathbb{C}_s$. Because $V_f \subset \text{Ker } D$, we also have $S \subset \text{Ker } D$; so $S \subset V_s^*(\text{Ker } D)$. From the fact that V_f is outer-stabilizable it follows that $S + X_{stab}^f = X_{stab}^s$; so (3.44) certainly holds. Now take $V = V_s$. We already noted that $V_s^*(\text{Ker } D) \supset S$; thus, $V + V_s^*(\text{Ker } D) \supset$ $\supset V + S = V_f$; because $T_f^*(\text{Im } E) \subset T_f$, (3.43.4) follows. Also, we have $V \supset T_s \supset X_{det}^s$. Finally, $V \subset V_f \subset \text{Ker } D$, and $V \oplus X_{stab}^s = X$ by (3.38.4).

Now assume that a controlled invariant subspace V that satisfies (3.43) has been given. Define $T_s = X_{det}^s$, $V_s = V$, $T_f = T_f^*(\text{Im } E)$ and $V_f = V + V_s^*(\text{Ker } D)$. Then (3.38.1) holds because $X_{det}^s \subset X_{det}^f \subset T_f^*(\text{Im } E)$. For (3.38.2), we note that $V_f + X_{stab}^f = V + X_{stab}^s = X$ so that V_f is indeed outer-stabilizable w.r.t. $\mathbb{C}_f$. The other conditions in (3.38) are trivially verified. $\boxtimes$

It should be noted that the subspaces X_{stab}^s, X_{stab}^f, X_{det}^s, $V_s^*(\text{Ker } D)$ and $T_f^*(\text{Im } E)$ can all be considered as known, in view of the algorithms mentioned in Section 1.4. So the condition (3.44) can be verified immediately and the remaining question is: How can we verify the existence of a

controlled invariant subspace satisfying (3.43), and how can we find such a subspace if it exists ? (Note that the two compensator couples of Theorem 3.3 can then be constructed as in the proof of the proposition.)

We shall solve this problem by introducing coordinates relative to X_{stab}^s. Fix a basis $\{x_1, \ldots, x_\ell, x_{\ell+1}, \ldots, x_n\}$ for X such that $\{x_1, \ldots, x_\ell\}$ is a basis for X_{stab}^s. With respect to the thus partitioned basis, let the matrices of A,B and D be given by

$$(3.45.1) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

$$(3.45.2) \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$$

$$(3.45.3) \quad D = (D_1 \ D_2).$$

Also, we introduce basis matrices for the known subspaces X_{det}^s, $T_f^*(\text{Im } E) + AT_f^*(\text{Im } E)$, and $V_s^*(\text{Ker } D)$:

$$(3.45.4) \quad X_{det}^s = \text{sp} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

$$(3.45.5) \quad T_f^*(\text{Im } E) + AT_f^*(\text{Im } E) = \text{sp} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

$$(3.45.6) \quad V_s^*(\text{Ker } D) = \text{sp} \begin{pmatrix} S \\ 0 \end{pmatrix}.$$

We state the following result.

COROLLARY 3.5 *Let the system (3.12) be given. Then there exists a solution to the regulator problem as formulated in (3.16) and (3.17) if and only if (3.44) holds and (using the notation (3.45)) there exist matrices* V,Q *and* R *such that*

$$(3.46.1) \quad A_{12} = VA_{22} - A_{11}V + B_1R$$

$$(3.46.2) \quad X_1 = VX_2$$

$$(3.46.3) \quad D_2 = -D_1V$$

$$(3.46.4) \quad W_1 = SQ + VW_2.$$

Moreover, if these conditions are fulfilled then a compensator as required can be constructed as in the proof of Thm. 3.3, using the compensator couples (T_s, V_s) *and* (T_f, V_f) *given by*

(3.47.1) $\quad T_s = sp\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad , \quad V_s = sp\begin{pmatrix} V \\ I \end{pmatrix}$

(3.47.2) $\quad T_f = sp\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \quad , \quad V_f = sp\begin{pmatrix} S & V \\ 0 & I \end{pmatrix}.$

PROOF We use Prop.3.4. With respect to the basis used in (3.45), denote V by

(3.48) $\quad V = sp\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}.$

Now we can start translating the conditions (3.43) into matrix terms. It follows from (3.43.1) that V should be $(n-\ell)$-dimensional, so that the matrix V_2 in (3.48) can be taken square. Then (3.43.1) says that the nxn-matrix

(3.49) $\quad \begin{pmatrix} I_\ell & V_1 \\ 0 & V_2 \end{pmatrix}$

should be nonsingular. This will be the case if and only if V_2 is invertible. But then we have

(3.50) $\quad sp\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = sp\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} V_2^{-1} = sp\begin{pmatrix} V_1 V_2^{-1} \\ I \end{pmatrix}$

and consequently, we shall from now on describe V by

(3.51) $\quad V = sp\begin{pmatrix} V \\ I \end{pmatrix}.$

It is required that V should be a controlled invariant subspace; this means that there should exist matrices $\hat{Q}$ and R such that

(3.52) $\quad \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} V \\ I \end{pmatrix} \hat{Q} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} R$

It readily follows that we must have $\hat{Q} = A_{22}$. Thus, V is controlled invariant if and only if there is a matrix R such that

(3.53) $A_{11}V + A_{12} = VA_{22} + B_1R.$

Next, (3.43.2) is equivalent to the existence of a matrix $\tilde{Q}$ such that

(3.54) $\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V \\ I \end{pmatrix}\tilde{Q}.$

This is clearly equivalent to

(3.55) $X_1 = VX_2.$

To translate (3.43.3) is very simple too:

(3.56) $(D_1 \quad D_2)\begin{pmatrix} V \\ I \end{pmatrix} = D_1V + D_2 = 0.$

Finally, (3.43.4) holds if and only if there exist matrices Q and $\hat{R}$ such that

(3.57) $\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} S \\ 0 \end{pmatrix}Q + \begin{pmatrix} V \\ I \end{pmatrix}\hat{R}.$

We get $\hat{R} = W_2$, and so (3.43.4) is equivalent to the existence of matrix Q such that

(3.58) $W_1 = SQ + VW_2.$ ⊠

The mapping that takes (V,Q,R) to $(VA_{22}-A_{11}V+B_1R, \; VX_2, \; -D_1V, \; SQ+VW_2)$ is a linear mapping between linear spaces of finite dimension, and so it is in principle straightforward to verify the condition of the corollary (it is in the form "a given vector must be in the range of a given linear operator"). Thus, we have indeed obtained a fully constructive solution.

3.4. Comparison with other work

In order to compare our results with earlier work on tracking, regulation and disturbance localization problems, we have to take into account the fact that most authors allow the use of static feedback in the solution of

these problems. That is, one constructs extended system matrices not of
the form (2.5) but of the form

$$(3.59) \qquad A_e = \begin{pmatrix} A+BKC & BL \\ MC & N \end{pmatrix}.$$

We shall refer to the regulator problem in this modified form as the
relaxed regulator problem. The link between the two versions is given by
the following lemma.

<u>LEMMA 3.6</u> *Let* (T_s, V_s) *and* (T_f, V_f) *be two pairs of subspaces such that*
$T_s \subset T_f$ *and* $V_s \subset V_f$, *and suppose that both pairs satisfy the conditions*
(cc1-3) (see Section 2.2). Then there exists a mapping K *such that* (T_s, V_s)
and (T_f, V_f) *are compensator couples with respect to the triple* (C,A+BKC,B).

<u>PROOF</u> We have to show that, under the given conditions, there exists K
such that $(A+BKC)T_s \subset V_s$ and $(A+BKC)T_f \subset V_f$. Because $T_s \subset T_f$, there exists
a projection P: $X \to X$ with Im P = Ker C such that $PT_s \subset T_s$ and $PT_f \subset T_f$.
Because $V_s \subset V_f$, there exists F $\in$ $\underline{F}(V_s) \cap \underline{F}(V_f)$. Write P = I $-$ TC (with
T: $Y \to X$); then K: = FT will do.

To verify this statement, note that $T_s = PT_s \oplus TCT_s$ and $T_f =$
$= PT_f \oplus TCT_f$. Moreover, $PT_s = T_s \cap$ Ker C and $PT_f = T_f \cap$ Ker C. We get

$$(3.60) \qquad (A+BFTC)T_s = (A+BFTC)(T_s \cap \text{Ker } C) + (A+BFTC)TCT_s \subset$$

$$\subset A(T_s \cap \text{Ker } C) + (A+BF)T_s \subset T_s + V_s \subset V_s$$

and similarly for the pair (T_f, V_f). ■

Using this lemma, it is easy to see that necessary and sufficient
conditions for the solvability of the relaxed regulator problem can be given
precisely as in Thm. 3.3, with only one modification: the pairs (T_s, V_s) and
(T_f, V_f) are not required to satisfy all conditions (cc1-4), but only the
conditions (cc1-3). (Compare SCHUMACHER (1980 d), where such pairs ("(C,A,B)-
pairs") were used throughout.) Accordingly, the necessary and sufficient
conditions can also be given in the form of Prop. 1.4, if (3.42.4) is
replaced by

$$(3.61) \qquad V + V_s^*(\text{Ker } D) \supset T_f^*(\text{Im } E).$$

Note that both versions coincide if E = 0. This means that in this case the solvability conditions for the regulator problem and its relaxed version are the same; of course, adding static feedback may still be useful to obtain improved bounds for the compensator order (see WONHAM (1979), SCHUMACHER (1980 d), JANSSEN (1981)).

Following WONHAM (1979), we use the term "output stability" to describe the situation in which $\mathbb{C}_s$ has been set equal to $\mathbb{C}$, so that there is only a stability requirement on the variables-to-be-controlled. In this situation, the complex plane has in effect been divided into two parts ($\mathbb{C}_f$ and $\mathbb{C}_t$) and we shall fall back on the old notation, writing $\mathbb{C}_g$ instead of $\mathbb{C}_f$ etc..

COROLLARY 3.7 *The regulator problem with output stability is solvable if and only if*

(3.62.1) $V^*(\text{Ker } D) \supset T_g^*(\text{Im } E)$

(3.62.2) $V^*(\text{Ker } D) \supset AT_g^*(\text{Im } E)$

(3.62.3) $V^*(\text{Ker } D) + X_{\text{stab}} = X.$

PROOF If $\mathbb{C}_s = \mathbb{C}$, then $X_{\text{stab}}^s = X$, $X_{\text{det}}^s = 0$ and $V_s^*(\text{Ker } D) = V^*(\text{Ker } D)$. The statement now follows immediately from Thm. 3.3 and Prop. 3.4. ☒

The necessary and sufficient conditions for the solvability of the relaxed regulator problem are, of course, given by (3.62.1) and (3.62.3); see SONNEVEND (1977; Thm. 3). We can specialize further by assuming that E = 0; then we obtain the "Extended Regulator Problem" of WONHAM (1979; p. 139). The solution is as follows (Wonham's Thm. 6.2).

COROLLARY 3.8 *The regulator problem with output stability and* E = 0 *is solvable if and only if*

(3.63.1) $X_{\text{det}} \subset \text{Ker } D$

(3.63.2) $X_b(A) \subset V^*(\text{Ker } D) + <A \mid \text{Im } B>.$

PROOF If E = 0, then $T_g^*(\text{Im } E) = X_{\text{det}}$. The condition (3.62.2) is then implied by (3.62.1) which is in its turn equivalent to (3.63.1), because X_{det} is

A-invariant and hence also controlled invariant. Further, (3.63.2) is equivalent to (3.62.3) by Prop. 1.12 and Prop. 1.19. ⊠

Another possible specialization is to set $\mathbb{C}_s$ equal to $\mathbb{C}_f$, so that there is no distrinction between the stability requirements for the variables-to-be-controlled and the other controllable variables. We may call this "maximal stability". Again, we are left with a division of the complex plane into two parts; so we shall use the $\mathbb{C}_g$ - $\mathbb{C}_b$-notation.
The general result in this situation is the following.

COROLLARY 3.9 *The regulator problem with maximal stability is solvable if and only if there exists a controlled invariant subspace V such that*

(3.64.1) $V \oplus X_{stab} = X$

(3.64.2) $X_{det} \subset V \subset \text{Ker } D$

(3.64.3) $T_g^*(\text{Im } E) \subset V + V_g^*(\text{Ker } D)$

(3.64.4) $AT_g^*(\text{Im } E) \subset V + V_g^*(\text{Ker } D)$.

PROOF   Immediate from Thm. 3.3 and Prop. 3.4. ⊠

We may once more remove the "disturbance decoupling" aspect by setting $E = 0$; then the result is as follows.

COROLLARY 3.10 *The regulator problem with maximal stability and E = 0 is solvable if and only if there exists a controlled invariant subspace V such that*

(3.65.1) $V \oplus X_{stab} = X$

(3.65.2) $X_{det} \subset V \subset \text{Ker } D$.

PROOF   Immediate from the foregoing corollary. ⊠

If we assume that $X_{det} = 0$, then our concept of "maximal stability" coincides with the notion of "internal stability" used in WONHAM (1979); and if we further assume that $X_{stab} = <A \mid \text{Im } B>$, then Cor. 3.10 specializes

to Thm. 8.1 in WONHAM (1979; p. 179).

It should be noted that it is quite reasonable to set X_{det} equal to zero beforehand (see FRANCIS (1977)). If one does not take this approach then there is a question how to deal with non-detectability of the pair (C,A) in the formulation of the regulator problem. Our approach is different from that of Wonham, who calls a closed-loop system "internally stable" also in the case where unstable closed-loop poles are due to indetectability rather than instabilizability (see WONHAM (1979; p. 147)). In the author's opinion, one may doubt the naturalness of this formulation. A discussion of formulation differences for the regulator problem has been given in JANSSEN (1981).

The "tracking and regulation" aspect is removed if we assume that $X^s_{stab} = X$ (which means that the pair (A,B) is $\mathbb{C}_s$-stabilizable). In this case, the condition (3.16) simply says that the extended system matrix should be stable in the sense of $\mathbb{C}_s$. We obtain the following corollary.

COROLLARY 3.11 *Let the system (3.12) be given. Then there exists a compensator such that the extended system matrix is stable in the sense of* $\mathbb{C}_s$ *and has the disturbance localization property described in (3.17) if and only if* (A,B) *is* $\mathbb{C}_s$-*stabilizable,* (C,A) *is* $\mathbb{C}_s$-*detectable, and moreover*

(3.66.1) $T^*_f(\text{Im } E) \subset V^*_s(\text{Ker } D)$

(3.66.2) $AT^*_f(\text{Im } E) \subset V^*_s(\text{Ker } D)$

(3.66.3) $V^*_s(\text{Ker } D) + X^f_{stab} = X.$

PROOF Immediate from Thm. 3.3 and Prop. 3.4.   ⊠

This result is for a compensator of the form (2.5); if static feedback is allowed, the condition (3.66.2) disappears. Under the further assumption that $\mathbb{C}_s = \mathbb{C}_f$, the condition (3.66.3) is also removed and we recover, in a slightly restated form, the results on "disturbance decoupling by observation feedback with stability" of WILLEMS & COMMAULT (1981) and IMAI & AKASHI (1981).

As a final specialization, let us set $\mathbb{C}_s = \mathbb{C}_f = \mathbb{C}$; this means that we do not impose any stability conditions.

COROLLARY 3.12 *Let the system (3.12) be given. Then there exists an extended system matrix* A_e *of the form (3.59) having an invariant subspace M such that*

(3.67) $Im(\begin{smallmatrix} E \\ 0 \end{smallmatrix}) \subset M \subset Ker\ (D\ \ 0)$

if and only if

(3.68) $T^*(Im\ E) \subset V^*(Ker\ D).$

PROOF Immediate from the foregoing corollary.   ⊠

This is the result on "disturbance localization by observation feedback" of AKASHI & IMAI (1979) and SCHUMACHER (1979). The 'necessity' part of the statement was already proved in BASILE & MARRO (1969 a).

3.5. Final remarks

We have formulated a very general problem in the synthesis of linear systems with specified structural features, and we have solved this problem completely. However, there are some aspects that we did not discuss.
One of these aspects is *order reduction*. The construction lemma of Section 3.2 gives rise to a compensator order which equals dim V_c-dim T_c; so there is room for low-order compensation, at least in principle. In case E = 0, the situation is relatively easy; we can set $T_s = X^s_{det}$, $T_f = X^f_{det}$ and $T_c = 0$, and then the condition G ∈ $\underline{G}(T_s)$ ∩ $\underline{G}(T_f)$ ∩ $\underline{G}(T_c)$ will be satisfied automatically. It is then not difficult to extend the methods of Section 2.4 to the present case. In particular, the method suggested in subsection 2.4.5 will be used in Ch. 6 to construct low-order regulators for infinite-dimensional systems.

The problem of order reduction becomes more difficult in the presence of an unmodeled disturbance (i.e., E ≠ 0). It has been shown in SCHUMACHER (1980 d) that the compensator order can be decreased by p (the number of observation outputs) by the use of static feedback. The basic idea is the same as in subsection 2.4.2, but the extension is not trivial; one has to show that the method of order reduction is compatible with the special structural requirements of the regulator problem. A different approach has been given by WILLEMS & COMMAULT (1981), who solve the problem in three

successive steps and who obtain a bound on the compensator order which may be either larger or smaller than the order of the original system. Because the method we presented here can only produce compensators of order $\leq$ n, it seems unlikely that there is a simple relation between our method and that of Willems and Commault.

CHAPTER 4

BASIC CONCEPTS OF INFINITE-DIMENSIONAL SYSTEMS

This is the first chapter of the second part of this monograph, in which we study a wider class of linear systems. The assumption of finite-dimensionality of the state space is dropped, so that we are able to include processes of a distributed nature into our considerations.

The present chapter has an introductory character, and we do not give any essentially new results. We shall use semigroup theory as a convenient framework for studying infinite-dimensional systems. Some of the basic facts of this theory are given in Section 4.1. Just as we did in the first part, we shall make extensive use of 'block matrix' representations of operators. However, it requires a little bit more care to do this in the infinite-dimensional context and so we have collected the basic results in Section 4.2. Section 4.3 deals with the concepts of stabilizability and detectability, and here we also present the important method of spectral decomposition.

In Section 4.4, we give a list of the assumptions that will be used to prove the main results in the subsequent chapters. These assumptions together define the class of systems to which our theory is applicable, and we discuss in some detail which physical systems are contained in this class. Also in Section 4.4, we try to explain briefly the key ideas in our approach to infinite-dimensional systems. Finally, some introductory remarks on the examples we shall use are given in Section 4.5.

4.1. Semigroup theory

In the first three chapters, we have always considered systems for which the state x(t) is an element of a finite-dimensional linear space. In many applications, however, the assumption of finite-dimensionality is restrictive. In order to deal with systems modeled by partial or functional differential equations, we need to consider infinite-dimensional state spaces.

There is a general theory available to describe the evolution of linear deterministic time-invariant systems on a Banach space. This is the theory of *semigroups*, explained in great detail in HILLE & PHILLIPS (1957). We

shall restrict ourselves to the concept of a *strongly continuous semigroup of bounded linear operators*. By definition, this is a function T(t) from $[0,\infty)$ to the space of bounded linear operators on a Banach space X, which has the following properties:

(4.1.1) $T(t+s) = T(t)T(s)$ $\forall s,\ t \in [0,\infty)$

(4.1.2) $T(0) = I$

(4.1.3) $T(\cdot)x \in C([0,\infty);X)$ $\forall x \in X.$

When we use the word "semigroup" below, we shall always mean a strongly continuous semigroup of bounded linear operators on a Banach space.

Although the semigroup property (4.1.1) was already formulated in HADAMARD (1903,1924) (where it is referred to as *the major premise of Huygens' principle*), the extensive development of semigroup theory started in the late 1940's in the works of E.Hille, R.S.Phillips, K.Yosida, W.Feller, I.Miyadera and many others. The semigroup approach has been used in systems theory since FATTORINI (1964) and it has been systematically applied to control problems by many authors; see CURTAIN & PRITCHARD (1978) and the references cited therein. The main advantage of the use of semigroups to study infinite-dimensional control theory is that one obtains general results that are applicable to systems of many different types (parabolic, hyperbolic, delayed, etc.) Other approaches are also feasible, however. In particular, one should note the strongly PDE-oriented approach of LIONS (1971) to problems of optimal control. For a transfer function approach to feedback design for some classes of infinite-dimensional systems, one may consult, for instance, CALLIER & DESOER (1978).

The semigroup T(t) generalizes the fundamental matrix exp(tA) in the theory of ordinary linear differential equations. The exponential function exp(tA) is said to be *generated* by the matrix A. In general, the *infinitesimal generator* of a semigroup T(t) is defined by specifying its domain

(4.2.1) $D(A) = \{x \in X \mid \lim_{t \downarrow 0} \frac{T(t)x-x}{t} \text{ exists}\}$

and its action on the elements of this subspace:

$$(4.2.2) \qquad Ax = \lim_{t \downarrow 0} \frac{T(t)x-x}{t} \qquad (x \in D(A)).$$

Some of the elementary properties of the infinitesimal generator are given as follows.

PROPOSITION 4.1 *The infinitesimal generator* A *of a semigroup* T(t) *is a densely defined closed linear operator. The differential equation*

$$(4.3) \qquad x'(t) = Ax(t), \; x(0) = x_0 \in D(A)$$

has a unique solution which is continuous on [0,∞) *and differentiable on* (0,∞), *and which is given by*

$$(4.4) \qquad x(t) = T(t)x_0.$$

PROOF See CURTAIN & PRITCHARD (1978; p.14) and, for the uniqueness, BELLENI-MORANTE (1979; p.163). ⊠

For the inhomogeneous equation, we have the following result.

PROPOSITION 4.2 *Let* T(t) *be a semigroup acting on a Banach space* X, *and let* A *be its infinitesimal generator. Let* f *be a* C^1-*function with values in* X. *Then the differential equation*

$$(4.5) \qquad x'(t) = Ax(t) + f(t), \qquad x(0) = x_0 \in D(A)$$

has a unique solution which is continuous on [0,∞) *and differentiable on* (0,∞), *and which is given by*

$$(4.6) \qquad x(t) = T(t)x_0 + \int_0^t T(t-s) \; f(s)ds.$$

PROOF See CURTAIN & PRITCHARD (1978; p.29). The uniqueness of the solution follows from the same fact for the homogeneous equation. ⊠

A basic property describing the 'growth' of a semigroup is the following.

PROPOSITION 4.3 *Let* T(t) *be a semigroup. Then*

$$(4.7) \qquad \inf\{\tfrac{1}{t} \log\|T(t)\| \mid t > 0\} = \lim_{t \to \infty} \tfrac{1}{t} \log\|T(t)\|$$

where both members are either finite or equal to $-\infty$.

<u>PROOF</u> See HILLE & PHILLIPS (1957; p.306).   ⊠

From this proposition, and from the fact that $T(t)$ is bounded on bounded intervals (see CURTAIN & PRITCHARD (1978; p.12)), it is clear that for each $\omega > \omega_0 := \lim_{t\to\infty} \frac{1}{t} \log \|T(t)\|$ there exists a constant M_ω such that

$$(4.8) \qquad \|T(t)\| \leq M_\omega\, e^{\omega t}.$$

Moreover, ω_0 is the smallest number in $\mathbb{R} \cup \{-\infty\}$ having this property. Therefore, ω_0 is called the *growth constant* of the semigroup. We shall use the growth constant to describe the 'degree of stability' of a semigroup (Section 4.3).

In our constructions, we shall often use operators which are bounded perturbations of infinitesimal generators. The following lemma states that such operators are generators too, and it also gives a bound on the norm of the perturbed semigroup.

<u>LEMMA 4.4</u> *Let* $T(t)$ *be a semigroup on a Banach space* X, *and let* A *be its infinitesimal generator. For any given bounded linear operator* $B: X \to X$, $A + B$ *is the generator of a semigroup, which we shall denote by* $S(t)$. *If the estimate*

$$(4.9) \qquad \|T(t)\| \leq M \exp(\omega t)$$

holds for $T(t)$, *then the estimate*

$$(4.10) \qquad \|S(t)\| \leq M \exp((\omega + M\|B\|)t)$$

holds for $S(t)$. *In particular, the growth constant of* $S(t)$ *is smaller than or equal to* $\omega + M\|B\|$.

<u>PROOF</u> See CURTAIN & PRITCHARD (1978; p.38).   ⊠

Following the finite-dimensional terminology, we shall say that a linear mapping between two Banach spaces is a *similarity transformation* if the mapping is bounded and has a bounded inverse. We have the following obvious result on modification of semigroups via such a transformation.

The simple proof will be omitted.

LEMMA 4.5 *Suppose that* H: $X_1 \to X_2$ *is a similarity transformation between the Banach spaces* X_1 *and* X_2 . *Let* T(t) *be a semigroup on* X_1 *with infinitesimal generator* A *and growth constant* ω_0. *The function* $\tilde{T}(t)$ *defined by*

(4.11) $\tilde{T}(t) = HT(t)H^{-1}$ $(t \geq 0)$

then gives a semigroup on X_2. *The infinitesimal generator of* $\tilde{T}(t)$ *is the mapping* $\tilde{A} := HAH^{-1}$ *with domain* $D(\tilde{A}) = \{x \in X_2 \mid H^{-1}x \in D(A)\} = H[D(A)]$, *and the growth constant of* $\tilde{T}(t)$ *is* ω_0.

4.2. Composite systems

As we shall consider dynamic feedback, which means that both the original system and the feedback controller will have their own state space, we shall be concerned with composite systems. Let us first introduce some terminology in a general setting. The *direct sum* of two Banach spaces X_1 and X_2 is the defined as the set of pairs $\{\binom{x_1}{x_2} \mid x_1 \in X_1, x_2 \in X_2\}$, made into a linear space in the obvious way, and endowed with the norm

(4.12) $\left\| \binom{x_1}{x_2} \right\| = \max(\|x_1\|, \|x_2\|)$.

In this way, one obtains a new Banach space which is written as $X_1 \oplus X_2$. The natural embeddings of X_1 and X_2 are closed linear subspaces of $X_1 \oplus X_2$, and the natural projections of $X_1 \oplus X_2$ onto X_1 and X_2 are continuous.

Now suppose that $A_{11}: D(A_{11}) \to X_1$, $A_{12}: D(A_{12}) \to X_2$, $A_{21}: D(A_{21}) \to X_2$ and $A_{22}: D(A_{22}) \to X_2$ are linear operators with $D(A_{11}) \subset D(A_{21}) \subset X_1$ and $D(A_{22}) \subset D(A_{12}) \subset X_2$. We shall use the notation

(4.13) $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$

for the operator that is defined as follows:

(4.14.1) $D(A) = \{\binom{x_1}{x_2} \mid x_1 \in D(A_{11}), x_2 \in D(A_{22})\}$

$$(4.14.2) \quad A\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix} \quad (\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(A)).$$

We also want to use the converse process in which a given system is decomposed. Let X be a Banach space and let X_1 and X_2 be subspaces of X such that each $x \in X$ can be written, in a unique way, as $x = x_1 + x_2$ with $x_1 \in X_1$ and $x_2 \in X_2$. In this case one says that X is the *direct sum* of X_1 and X_2, and one writes $X = X_1 \oplus X_2$ (see TAYLOR & LAY (1980; p.28)). If X_1 and X_2 are both closed subspaces, then the projections of X onto X_1 along X_2 and onto X_2 along X_1 are continuous (TAYLOR & LAY (1980; p.247)) and there is an obvious identification between the concepts of 'direct sum' in one sense or the other. Below, we shall adopt the convention that the use of the expression $X = X_1 \oplus X_2$ implies that X_1 and X_2 are closed subspaces of X or can be considered as such.

Corresponding to a decomposition of the space X, we shall want to write an operator A acting on X in the 'block' form (4.13). In the case A is not defined on all of X, we have to take some care to avoid domain problems. The pertinent facts are given in the following lemma, whose simple proof will be omitted.

LEMMA 4.6 *Let X be a Banach space and let A be a linear operator mapping its domain $D(A) \subset X$ into X. Assume that X has a direct sum decomposition $X = X_1 \oplus X_2$. Let P_1 be the projection onto X_1 along X_2, and let P_2 be the projection onto X_2 along X_1. Suppose that P_2 maps $D(A)$ into itself. Then the same holds for P_1, and we have*

$$(4.15) \quad P_1[D(A)] = X_1 \cap D(A), \quad P_2[D(A)] = X_2 \cap D(A).$$

Moreover, if we define the mapping A_{ij} for $i,j \in \{1,2\}$ as the restriction of P_iA to $X_j \cap D(A)$, considered as a mapping into X_i, then the following equality holds:

$$(4.16) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Finally, if $D(A)$ is dense in X, then $X_1 \cap D(A)$ is dense in X_1 and $X_2 \cap D(A)$ is dense in X_2.

A simple sufficient condition for the inclusion $P_2[D(A)] \subset D(A)$ to

hold is that X_2 be a closed subspace contained in D(A). We shall often use this condition in the sequel.

We shall now consider situations in which the block matrix appearing in 4.13 has a triangular structure. The first result is the following.

PROPOSITION 4.7 *Suppose that* A_{11} *and* A_{22} *are generators of semigroups* $T_1(t)$ *and* $T_2(t)$ *on the Banach spaces* X_1 *and* X_2, *with growth constants* ω_1 *and* ω_2 *respectively. Suppose also that* $A_{21}: X_1 \to X_2$ *is a bounded linear operator. Then the operator*

$$(4.17) \qquad A := \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$$

is the generator of a strongly continuous semigroup on $X_1 \oplus X_2$. *This semigroup is given by*

$$(4.18) \qquad T(t) = \begin{pmatrix} T_1(t) & 0 \\ T_{21}(t) & T_2(t) \end{pmatrix}$$

where $T_{21}(t)$ *is defined by*

$$(4.19) \qquad T_{21}(t)x = \int_0^t T_2(t-s) \, A_{21} \, T_1(s)x \, ds \qquad (x \in X_1).$$

Moreover, the growth constant of $T(t)$ *is equal to* $\max(\omega_1, \omega_2)$.

PROOF One easily verifies that

$$(4.20) \qquad \tilde{T}(t) = \begin{pmatrix} T_1(t) & 0 \\ 0 & T_2(t) \end{pmatrix}$$

is a semigroup with infinitesimal generator

$$(4.21) \qquad \tilde{A} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}.$$

As A is a bounded perturbation of $\tilde{A}$, it follows from Lemma 4.4 that A is the generator of a semigroup on $X_1 \oplus X_2$. To determine the form of the semigroup, we solve the differential equation

(4.22) $\quad \dfrac{d}{dt} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t), \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(0) = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \in D(A).$

We re-write (4.22) as follows:

(4.23.1) $\quad x_1'(t) = A_{11}x_1(t), \qquad\qquad x_1(0) = x_{10} \in D(A_{11})$

(4.23.2) $\quad x_2'(t) = A_{21}x_1(t) + A_{22}x_2(t), \; x_2(0) = x_{20} \in D(A_{22}).$

The solution of (4.23.1) is given by the semigroup generated by A_{11}:

(4.24) $\quad x_1(t) = T_1(t)x_{10}.$

This is a C^1-function with values in X_1, and because A_{21} is bounded it follows that the function $A_{21} x_1(t)$ is a C^1-function with values in X_2. Thus we can solve (4.23.2) using Prop. 4.2:

(4.25) $\quad x_2(t) = T_2(t)x_{20} + \displaystyle\int_0^t T_2(t-s)A_{21} T_1(s)x_{10} \, ds.$

It now follows from Prop. 4.1 that the semigroup generated by A is given by (4.18).

To prove the final assertion, let ω be any number larger than $\max(\omega_1,\omega_2)$. Then we can make the following estimates, in which ω_1' and ω_2' are numbers such that $\omega_1 < \omega_1' \le \omega$, $\omega_2 < \omega_2' \le \omega$, and $\omega_1' \ne \omega_2'$:

(4.26) $\quad \left\| T(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\| = \max\left(\left\| T_1(t)x_1 + T_{12}(t)x_2\right\|, \; \left\| T_2(t)x_2\right\|\right)$

$\qquad \le \max\Big(M_1 e^{\omega_1' t}\|x_1\| + M_3 \displaystyle\int_0^t e^{\omega_2'(t-s)}\|A_{21}\| e^{\omega_1' s}\|x_2\| ds,$

$\qquad\qquad M_2 e^{\omega_2' t}\|x_2\|\Big) \le$

$\qquad \le \max\Big(M_1 e^{\omega_1' t}\|x_1\| + M_4 \left| e^{\omega_2' t} - e^{\omega_1' t}\right| \, \|x_2\|, \; M_2 e^{\omega_2' t}\|x_2\|\Big) \le$

$\qquad \le M_5 e^{\omega t} \left\|\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|.$

This shows that the growth constant of T(t) is smaller than or equal to $\max(\omega_1, \omega_2)$. On the other hand, it follows immediately form the triangular form of T(t) that its growth constant cannot be smaller than ω_1 or ω_2. Hence, equality holds, and the proof is complete. ◻

Now suppose that we have a semigroup T(t) acting on a Banach space X which has a direct sum decompoisition $X = X_1 \oplus X_2$. Also suppose that X_2 is contained in the domain of the infinitesimal generator A, and that $AX_2 \subset X_2$. Then we shall denote the *restriction of* A *to* X_2 by A: X_2. We can also consider the quotient space X/X_2, which is defined as the set of equivalence classes modulo X_2, endowed with the norm

$$(4.27) \qquad \|[x]\| = \inf\{\|x - x_2\| \mid x_2 \in X_2\}.$$

With this norm, X/X_2 is a Banach space (TAYLOR & LAY (1980; p. 71)). From the fact that the canonical mapping $x \mapsto [x]$ is a continuous mapping of X onto X/X_2, it follows that the subspace

$$(4.28) \qquad D(\bar{A}) := \{[x] \mid x \in D(A)\}$$

is dense in X/X_2. It is easily verified that we can define a mapping $\bar{A}: D(\bar{A}) \to X_1$ by

$$(4.29) \qquad \bar{A}[x] = [Ax] \qquad\qquad (x \in D(A)).$$

This mapping, which will sometimes also be denoted by A: X/X_2, is called the *quotient mapping* induced by A on X/X_2.

Under the above circumstances, it is possible to define the restriction and the quotient of the semigroup T(t) with respect to X_2. These are the contents of the next two results.

PROPOSITION 4.8 *Suppose that* A *is the infinitesimal generator of a semigroup* T(t) *on a Banach space* X, *and suppose that* $X = X_1 \oplus X_2$ *with* $X_2 \subset D(A)$ *and* $AX_2 \subset X_2$. *Then* X_2 *is* T(t)-*invariant for each* $t \geq 0$, *and* T(t): X_2 *is a semigroup on* X, *with* A: X_2 *as its infinitesimal generator.*

PROOF From the fact that A is a closed operator it follows that A: X_2 is also closed. Because A: X_2 is defined on all of X_2, the closed graph

theorem (TAYLOR & LAY (1980; p.213)) shows that $A: X_2$ is bounded. Consequently, the unique solution of the differential equation

(4.30) $x'(t) = Ax(t)$, $x(0) = x_0 \in X_2$

is given by

(4.31) $x(t) = \exp(tA: X_2)x_0$

which is clearly in X_2 for all $t \geq 0$. It follows from Prop. 4.1 that we must have

(4.32) $T(t)x_0 = \exp(tA: X_2)x_0$ $(t \geq 0, \; x_0 \in X_2)$

showing that X_2 is $T(t)$-invariant for each $t \geq 0$, and that $T(t): X_2$ is a semigroup with infinitesimal generator $A: X_2$.   ⊠

It should be noted that a subspace X_2 which is A-invariant in the sense that $Ax \in X_2$ for $x \in X_2 \cap D(A)$ $(\neq X_2)$ is not necessarily invariant for the semigroup generated by A (see SCHMIDT & STERN (1980)).

PROPOSITION 4.9 *Suppose that* A *is the infinitesimal generator of a semigroup* $T(t)$ *on a Banach space* X, *and suppose that* $X = X_1 \oplus X_2$ *with* $X_2 \subset D(A)$ *and* $AX_2 \subset X_2$. *Then we can define a mapping* $\bar{T}(t): X/X_2 \to X/X_2$, *for each* $t \geq 0$, *by*

(4.33) $\bar{T}(t)[x] = [T(t)x]$.

Moreover, $\bar{T}(t)$ *is a semigroup on* X/X_2 *whose generator is an extension of* $\bar{A}$ *(defined in (4.29)).*

PROOF The correctness of the definition (4.33) follows from the $T(t)$-invariance of X_2 (Prop.4.8). To show that $\bar{T}(t)$ is bounded let ω be the growth constant of $T(t)$ and let $\omega' \in \mathbb{R}$ be such that $\omega' > \omega$. Then there exists a constant M such that for all $x \in X$ and $x_2 \in X_2$:

(4.34) $\|\bar{T}(t)[x]\| = \inf\{\|T(t)x - y\| \mid y \in X_2\} \leq$

            $\leq \|T(t)x - T(t)x_2\| \leq Me^{\omega' t}\|x - x_2\|$.

Consequently, we have for all x ∈ *X*:

(4.35) $\|\bar{T}(t)[x]\| \le Me^{\omega't} \inf\{\|x - x_2\| \mid x_2 \in X_2\} =$

$= Me^{\omega't} \|[x]\|.$

The semigroup axioms (4.1) are easily seem to be satisfied by $\bar{T}(t)$. Finally, the fact that

(4.36) $\lim_{t\downarrow 0} \frac{1}{t}(\bar{T}(t)[x] - [x]) = \bar{A}[x] \qquad ([x] \in D(\bar{A}))$

follows immediately from the continuity of the canonical mapping x ↦ [x]. ◻

The inequality (4.35) shows that the growth constant of the induced semigroup $\bar{T}(t)$ is smaller than or equal to that of the original semigroup. Of course, the same is true of the semigroup T(t): X_2 obtained by restriction.

To complete this section, we prove a proposition on the decomposition of composite systems. We shall say that two semigroups $T_1(t)$ and $T_2(t)$ acting on Banach spaces X_1 and X_2 are *similar* if there exists a similarity transformation H: $X_1 \to X_2$ such that

(4.37) $HT_1(t) = T_2(t)H$

for all t ≥ 0.

PROPOSITION 4.10 *Suppose that* A_{11} *and* A_{22} *are generators of semigroups* $T_1(t)$ *and* $T_2(t)$ *on the Banach spaces* X_1 *and* X_2, *respectively, and assume that* A_{22} *is bounded. Suppose also that* A_{21}: $X_1 \to X_2$ *is a bounded linear operator. Let* T(t) *denote the semigroup on* $X_1 \oplus X_2$ *with generator*

(4.38) $A := \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}$

Then the restricted semigroup T(t): X_2 *is similar to* $T_2(t)$, *and the quotient semigroup* $\bar{T}(t)$ *on* X/X_2 *is similar to* $T_1(t)$.

PROOF The first assertion is obvious from the form of the semigroup T(t) given in (4.18). To prove the second assertion, define the mapping

H: $X_1 \to X/X_2$ by

(4.39)   Hx = $[\binom{x}{0}]$ $(x \in X_1)$.

This is clearly a continuous bijection from X_1 onto X/X_2. By the open mapping theorem (see TAYLOR & LAY (1980; p.212-213), H^{-1} is also continuous so that H provides a similarity between X_1 and X/X_2. Using the explicit form of T(t) again, we have

(4.40) $\bar{T}(t)Hx = [T(t)\binom{x}{0}] = \left[\begin{matrix}T_1(t)x\\0\end{matrix}\right] = HT_1(t)x$

for all $x \in X_2$. This completes the proof.   ⊠

4.3   Stabilizability, detectability and the spectral decomposition

As in the finite-dimensional case, a 'system' will be described by three operators. The *main operator* (or *system operator*) A will be the generator of a semigroup T(t) on the state space X. The *input operator* B will be a bounded mapping from a finite-dimensional space U into X. The *output operator* C will be a bounded mapping from X into a finite-dimensional space Y.

We defined the concept of stability in the finite-dimensional situation through a partitioning of the complex plane in a part labeled 'stable' and a part labeled 'unstable'. In the infinite-dimensional context, we want to use the growth constant as an indicator of stability, and so we shall consider only partitionings in which the parts are divided by a vertical line in the complex plane. In our discussions, we shall assume that some fixed constant $\omega \in \mathbb{R}$ has been given, and we shall say that a semigroup is *exponentially stable* (or simply *stable*) if its growth constant is smaller than or equal to ω. In applications, ω is always negative.

We can now define the concepts of stabilizability and detectability. If A and B are mappings as described above, the pair (A,B) is *stabilizable* if there exists a bounded mapping F: $X \to U$ such that the semigroup generated by A + BF is stable. (Note that A + BF is indeed the generator of a semigroup; this follows from Lemma 4.4.) The pair (C,A), as described above, is said to be *detectable* if there exists a bounded mapping G: $Y \to X$ such that the semigroup generated by A + GC is stable. Of course, Lemma 4.4 is used again here.

In the finite-dimensional situation, the modal decomposition of the state space into a 'good' and a 'bad' A-invariant subspace was useful to us. We want to use the same idea in the present context, and this is made possible by the following result.

PROPOSITION 4.11 *Let* A *be the generator of a semigroup on the Banach space* X. *Suppose that* σ_1 *is a bounded subset of* $\sigma(A)$ *and that* σ_1 *is closed and open in the relative topology of* $\sigma(A)$. *Then there exists a bounded projection* P: $X \rightarrow X$ *such that* $P[D(A)] \subset D(A)$, *and with respect to the decomposition* $X = \text{Im } P \oplus \text{Ker } P$ *we have*

$$(4.41) \qquad A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad T(t) = \begin{pmatrix} T_{11}(t) & 0 \\ 0 & T_{22}(t) \end{pmatrix}.$$

Moreover, $\sigma(A_{11}) = \sigma_1$ *and* $\sigma(A_{22}) = \sigma(A)\backslash\sigma_1$.

PROOF See TAYLOR & LAY (1980; pp. 321-323) and, for the decomposition of the semigroup, TRIGGIANI (1975; App.2). ☒

We shall use this *spectral decomposition* to derive sufficient conditions for stabilizability and detectability. With this in mind, we first have to discuss the relation between the spectrum of the generator and the growth constant of the semigroup. The following general result holds:

$$(4.42) \qquad \lim_{t \to \infty} \frac{1}{t} \log\|T(t)\| \geq \sup \{\text{Re } \lambda \mid \lambda \in \sigma(A)\}$$

(HILLE & PHILIPS (1957; p.457)). In general, equality does not have to hold in (4.42); see HILLE & PHILIPS (1957; p.665) and ZABCZYK (1975). In many cases, however, we do have equality and then the generator is said to satisfy the *spectrum determined growth assumption* (TRIGGIANI (1975)). One of these cases is when the operator T(t) is compact for large t, as is typically the case in delay equations; see HALE (1971; pp.112-115). Also, the spectrum determined growth assumption holds if there exists a $t_0 \geq 0$ such that T(t) is strongly differentiable for $t > t_0$ (see SLEMROD (1976)). This condition is typically satisfied (even with $t_0 = 0$) for equations of diffusion type, and the semigroup is also certainly differentiable if the infinitesimal generator is bounded. In fact, it has been shown that differentiability of the semigroup implies compactness if the generator has a compact resolvent (PAZY (1968; Th. 3.2 and Lemma 2.1)), such as is the case

for many diffusion equations. For further discussion on the spectrum determined growth assumption, see TRIGGIANI (1975).

An infinitesimal generator A is said to satisfy the *spectrum decomposition assumption* (with respect to some number ω denoting the desired degree of stability) if

$$(4.43) \qquad \sigma_1 := \{\lambda \in \sigma(A) \mid \text{Re } \lambda > \omega\}$$

is a bounded, closed and open subset of $\sigma(A)$ (TRIGGIANI (1975)). We can then use Prop. 4.11 to decompose A, and we shall in this special case use the following notation ('u' for 'unstable', 's' for 'stable'):

$$(4.44) \qquad A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix}.$$

Also, we shall write the corresponding decomposition of the state space X as $X = X_u \oplus X_s$. In an obvious way, we can write the input operator $B: U \to X$ as

$$(4.45) \qquad B = \begin{pmatrix} B_u \\ B_s \end{pmatrix}.$$

The output operator C: $X \to Y$ can likewise be written in matrix form:

$$(4.46) \qquad C = (C_u \quad C_s).$$

We shall also use the notation A_s^ω, X_s^ω etc. if we want to stress the dependence on ω.

In case the subspace X_u is finite-dimensional, the mappings A_u, B_u and C_u are mappings between finite-dimensional spaces and we can use the concepts of controllability and observability as defined in Ch. 1.

The spectrum decomposition assumption can be checked easily when the spectrum of the generator consists only of isolated eigenvalues. This is the case if the resolvent $(\lambda-A)^{-1}$ is compact for some $\lambda \in \rho(A)$, which applies to many differential operators appearing in the classical boundary value problems of mathematical physics (see KATO (1966; p.187)). The same type of spectrum is obtained for functional-differential operators describing delay equations; see HALE (1971; pp.98-101). In both cases, all eigenvalues have finite multiplicities (which means that the corresponding

eigenprojections have finite rank; see TAYLOR & LAY (1980; p.330)) and so
the condition on the finite-dimensionality of X_u comes down to requiring
that there are only finitely many eigenvalues to the right of the line
Re $\lambda = \omega$.

Using the spectrum decomposition assumption and the spectrum deter-
mined growth assumption, we are able to characterize stabilizability and
detectability in finite-dimensional terms if the unstable subspace is finite-
dimensional. The following propositions are proved constructively, and we
shall use them for actual computations of feedback or injection mappings.
The 'sufficiency' parts of these results are due to TRIGGIANI (1975; Thm.
6.1) and CURTAIN (1979; Thm.3.1).

PROPOSITION 4.12 *Suppose that* A *satisfies the spectrum decomposition
assumption and that* A_s *satisfies the spectrum determined growth assump-
tion. Suppose also that the unstable subspace* X_u *is finite-dimensional.
Then the pair* (A,B) *is stabilizable if and only if the pair* (A_u, B_u) *is
controllable.*

PROOF First, let us assume that (A_u, B_u) is controllable. There then exists
a mapping $F_u: X_u \rightarrow U$ such that $A_u + B_u F_u$ generates a stable semigroup.
Define F: $X \rightarrow U$, with respect to the decomposition $X = X_u \oplus X_s$, by

(4.47) $F = (F_u \quad 0)$.

Then we have

(4.48) $A + BF = \begin{pmatrix} A_u + B_u F_u & 0 \\ B_s F_u & A_s \end{pmatrix}$

and it follows from Prop.4.7 that the semigroup generated by A + BF is
stable.

Conversely, suppose that F: $X \rightarrow U$ is a bounded mapping such that A + BF
generates a stable semigroup, and suppose that the pair (A_u, B_u) would not
be controllable. Decompose X_u as $X_2 \oplus X_3$, where

(4.49) $X_2 = <A_u \mid \text{Im } B_u>$

(notation as in (1.8)). Write $X_1 = X_s$. Then we have, with respect to the
decomposition $X = X_1 \oplus X_2 \oplus X_3$:

$$(4.50) \qquad A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \\ 0 \end{pmatrix}.$$

Here, we have written A_{11} for A_s, etc.. By our assumptions, the matrix A_{33} is unstable. If we now write $F = (F_1 \ F_2 \ F_3)$, we get

$$(4.51) \qquad A + BF = \begin{pmatrix} A_{11}+B_1F_1 & B_1F_2 & B_1F_3 \\ B_2F_1 & A_{22}+B_2F_2 & A_{23}+B_2F_3 \\ 0 & 0 & A_{33} \end{pmatrix}.$$

The upper two-by-two block is the generator of a semigroup because it is a bounded perturbation of the upper two-by-two block appearing in (4.50). So we can apply Prop. 4.7 and conclude that the semigroup generated by $A + BF$ cannot be stable. This is a contradiction. Hence, (A_u, B_u) must be controllable. ⊠

PROPOSITION 4.13 *Suppose that* A *satisfies the spectrum decomposition assumption and that* A_s *satisfies the spectrum determined growth assumption. Suppose also that the unstable subspace* X_u *is finite-dimensional. Then the pair* (C,A) *is detectable if and only if the pair* (C_u, A_u) *is observable.*

PROOF We first assume that (C_u, A_u) is observable. Then there exists a mapping $G_u: Y \to X_u$ such that $A_u + G_u C_u$ generates a stable semigroup. Define $G_u: Y \to X$, with respect to the decomposition $X = X_u \oplus X_s$, by

$$(4.52) \qquad G = \begin{pmatrix} G_u \\ 0 \end{pmatrix}.$$

Then we have

$$(4.53) \qquad A + GC = \begin{pmatrix} A_u+G_u C_u & G_u C_s \\ 0 & A_s \end{pmatrix}$$

and it follows from Prop. 4.7 that $A + GC$ generates a stable semigroup.

Conversely, suppose that $G: Y \to X$ is a bounded mapping such that $A + GC$ is the generator of a stable semigroup, and suppose that the pair (C_u, A_u)

would not be observable. Decompose X_u as $X_2 \oplus X_3$, with

(4.54) $X_3 = \langle \mathrm{Ker}\ C_u \mid A_u \rangle$

(notation as in (1.10)). Write $X_1 = X_s$, $A_{11} = A_s$ etc.. Then we have, with respect to the decomposition $X = X_1 \oplus X_2 \oplus X_3$:

(4.55) $A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}$, $C = (C_1\ C_2\ 0)$.

By our assumptions, the matrix A_{33} is unstable. Writing

(4.56) $G = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$,

we get

(4.57) $A + GC = \begin{pmatrix} A_{11}+G_1C_1 & G_1C_1 & 0 \\ G_2C_1 & A_{22}+G_2C_2 & 0 \\ G_3C_1 & A_{32}+G_3C_2 & A_{33} \end{pmatrix}$.

The two-by-two upper left block generates a semigroup because it is a bounded perturbation of the two-by-two upper left block in (4.55). Using Prop. 4.7, we see that the semigroup generated by $A + GC$ cannot be stable. This is a contradiction, and therefore (C_u, A_u) must be observable.   ☒

Another application of the spectral decomposition is to growth estimates for the semigroup. In general, a growth estimate of the form (4.8) can only be made for ω larger than the growth constant ω_0. Under certain circumstances which will prevail in our examples below, it is nevertheless possible to give the estimate with $\omega = \omega_0$.

LEMMA 4.14 *Suppose that* $T(t)$ *is a semigroup with generator* A, *and let* $\omega_0 = \sup \{\mathrm{Re}\ \lambda \mid \lambda \in \sigma(A)\}$. *Suppose that* $\sigma_1 := \{\lambda \in \sigma(A) \mid \omega_0 - \delta \leq \mathrm{Re}\ \lambda \leq \omega_0\}$ *satisfies the assumptions of Prop. 4.11, and let the operator* A_{22} *satisfy the spectrum determined growth assumption. If* σ_1 *consists of finitely many eigenvalues of* A *which all have multiplicity one, then there exists a constant* M *such that the following holds:*

$$(4.58) \qquad \|T(t)\| \leq Me^{\omega_0 t} \qquad\qquad (t \geq 0).$$

PROOF In the decomposition of Prop. 4.11, A_{11} is a diagonalizable matrix whose eigenvalues have real parts $\leq \omega_0$. Thus, there exists a constant M_1 such that $\|T_1(t)\| \leq M_1 \exp(\omega_0 t)$ for all $t \geq 0$. By the assumption on A_{22} and the fact that Re $\lambda < \omega_0 - \delta$ for all $\lambda \in \sigma(A_{22})$, there is a constant M_2 such that $\|T_2(t)\| \leq M_2 \exp(\omega_0 t)$ for all $t \geq 0$. The assertion of the lemma now follows directly. ▨

4.4 Remarks on the scope of the theory

In this section, we present a list of the assumptions that determine the class of systems we shall consider in the subsequent chapters. We shall also give some comments on the assumptions and on the way we shall use them. It is assumed that a fixed number ω has been given, to indicate the dividing line between the 'stable' and the 'unstable' part of the complex plane. The systems we shall study are described by three operators A, B and C under the following assumptions.

(A1) A is the generator of a semigroup $T(t)$ on a Banach space X

(A2) B is a bounded mapping from a finite-dimensional space U into X

(A3) C is a bounded mapping from X into a finite-dimensional space Y

(A4) the spectrum of A is discrete, i.e. it consists only of isolated eigenvalues with finite multiplicities

(A5) there exists $\delta > 0$ such that $\{\lambda \in \mathbb{C} \mid \text{Re } \lambda > \omega - \delta\}$ contains only finitely many eigenvalues of A

(A6) the operator $A_s^{\omega-\delta}$ satisfies the spectrum determined growth assumption, with δ as in (A5)

(A7) the pair (A_u, B_u) is controllable

(A8) the pair (C_u, A_u) is observable

(A9) the eigenvectors of A are complete, in the sense that
$$\text{span }\{x \in X \mid \exists \lambda \in \mathbb{C}: \exists n \in \mathbb{N}: (\lambda I - A)^n x = 0\} \text{ is dense in } X.$$

To prevent confusion, let us point out here that we shall use the term *"discrete spectrum"* always in the sense of A4 (cf. KATO (1966; p.187)), and that the word *"multiplicity"* will be used for the rank of the eigen-projection (cf. KATO (1966; p.181)). Every non-zero vector in the range of the eigenprojection will be termed an *"eigenvector"* (so this includes 'generalized eigenvectors').

The assumptions (A1) and (A2) are relevant for the situation in which the control enters in the following way:

$$(4.59) \qquad x'(t) = Ax(t) + Bu(t).$$

For systems described by partial differential equations, the control could also be implemented via the boundary conditions, and then a description under the assumptions (A1) and (A2) would be impossible. The specifically infinite-dimensional phenomenon of *boundary control* is interesting, but we shall leave it out of our present discussion. Our object of study is given by (4.59), and it is sometimes called *distributed control.*

The boundedness assumption on the output operator (A3) excludes point observations on an L_2-space. However, CURTAIN (1979) and POHJOLAINEN (1980) show that, under certain conditions, it is very well possible to deal with unbounded observations in feedback design problems. It seems not unreasonable to expect that most of our results will remain true if C is relatively bounded with respect to A (KATO (1966; p.190)).

The assumption (A4) has already been discussed in Section 4.3; we have argued that a large class of systems described by partial differential equations on a bounded domain or by functional differential equations of 'delay' type falls within the category described by (A4). Within this class, there is a distinction between those generators for which there are only finitely many eigenvalues to the right of any vertical line in the complex plane, and those for which this is not true. This distinction, which is obviously of crucial importance in connection with (A5), corresponds to a well-known classification both in partial differential equations and in functional differential equations. On one hand, we have parabolic equations (such as the heat equation) and equations of 'retarded' type (HALE (1971; p.4)). In systems described by these equations, the real parts of the eigenvalues tend to $-\infty$ and so the assumption (A5) will hold for any desired growth constant ω. On the other hand, we have hyperbolic equations (such as the wave equation) and equations of 'neutral' type (HALE (1971; p.5)).

Systems of these types have infinitely many eigenvalues in a vertical strip and so they will only satisfy (A5) if this strip happens to be to the left of the prescribed value of ω. (Note that the real parts of the points in the spectrum of a generator must be bounded above; this follows from (4.42) and Prop.4.3.) In conclusion, we may say that parabolic and retarded systems will as a rule satisfy the assumption (A5), whereas hyperbolic and neutral systems will satisfy (A5) only if the system is 'basically' stable.

The assumption (A6) has also been discussed in Section 4.3. We have formulated the assumption for $A_s^{\omega-\delta}$ and not for A, because this is what is needed to apply Prop. 4.12 and Prop. 4.13. However, it is easily seen that if the semigroup generated by A is differentiable for $t > t_0$ or compact for large t, then the same holds for restrictions of the semigroup to a subspace of the form X_s. It follows from Prop. 4.7 and assumption (A5) that the spectrum determined growth assumption holds for A_s^{ω} if it holds for $A_s^{\omega-\delta}$.

The assumptions (A7) and (A8) need little explanation; they are just as important as in the finite-dimensional situation. Finally, the assumption (A9) will be essential to prove the existence of finite-dimensional control schemes. The completeness property is quite common in partial differential equations; in fact, the classical method of solving equations by expansion in eigenfunctions is based on it. Material on completeness of eigenvectors can be found, for instance, in TREVES (1975; p.325), MIZOHATA (1973; pp.465-470) and AGMON (1965; pp.278-289). For completeness of eigenvectors in functional differential equations, see MANITIUS (1980) and DELFOUR & MANITIUS (1980 b).

For a given system, the main restrictions on the selection of the desired growth constant ω are given by (A5), (A7) and (A8). We see that ω may be set equal to the largest uncontrollable and/or unobservable eigenvalue of A. But if there is a vertical 'asymptotic line' (i.e. a line $\{\lambda \in \mathbb{C} \mid \operatorname{Im} \lambda = c\}$ such that each strip $\{\lambda \in \mathbb{C} \mid c - \varepsilon \leq \operatorname{Im} \lambda \leq c + \varepsilon\}$ ($\varepsilon > 0$) contains infinitely many eigenvalues of A), then ω must be to the right of that line.

The assumptions (A1, 4-6,9) describe which operators are allowed to occur as the main operator of the systems we shall consider. The following proposition gives a large class of operators that are contained in this category.

PROPOSITION 4.15 *Suppose that* X *is a Hilbert space and that* A *is a densely defined, self-adjoint linear operator on* X. *Suppose furthermore that* A *is*

bounded above (i.e., there is a constant c ∈ ℝ *such that* (Ax,x) ≤ c *for all* x ∈ D(A) *with* ‖x‖ = 1). *Finally, suppose that* $(\lambda_0-A)^{-1}$ *is compact for some* λ_0 ∈ ρ(A). *Then* A *satisfies the assumptions* (A1,4-6,9) *for any* ω ∈ ℝ.

PROOF Using the fact that A is closed (because $(\lambda_0-A)^{-1}$ is closed) and the estimate $\|(\lambda-A)^{-1}\| \le M|\lambda|^{-1}$, which holds for $\lambda \in \{\lambda \in \mathbb{C} \mid c_1|\text{Im }\lambda| > c_2 - \text{Re }\lambda\}$ $(c_1 > 0, c_2 > c)$ (KATO(1966; p.272)), we derive from Thm. 2.28 in CURTAIN & PRITCHARD (1978; p.33) that A generates an analytic semigroup. Assumption (A4) is fulfilled by the compactness of the resolvent; see, for instance, KATO (1966; p.187). Also by the compactness of the resolvent, the eigenvalues of A cannot have a finite accumulation point; and because A is bounded above and self-adjoint, the eigenvalues are real and they can be numbered in order of decreasing magnitude $\lambda_1 > \lambda_2 > ..$, with $\lambda_k \to -\infty$ as k → ∞. This shows that (A5) is satisfied for any ω ∈ ℝ. Any operator of the form A_s^ω will generate an analytic semigroup and so (A6) holds too. Finally, the completeness of the eigenvectors of A follows from the same fact for $(\lambda_0-A)^{-1}$, where λ_0 ∈ ℝ and λ_0 > c (KATO (1966; p.260)). ☒

The theorem applies, for instance, to elliptic operators of the following type: Let Ω be a bounded domain in $\mathbb{R}^d$ with smooth boundary, and let X = $L_2(\Omega)$. Define A by

$$(4.60) \qquad A\varphi = -\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\varphi\right) + c(x)\varphi$$

where the functions $a_{ij}(x)$ are real-valued and once continuously differentiable, c(x) is real-valued, $a_{ij}(x) = a_{ji}(x)$ for all i and j, and the uniform ellipticity condition holds:

$$(4.61) \qquad \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}(x)\,\xi_i\xi_j \ge \eta|\xi|^2 \qquad (x \in \Omega, \xi \in \mathbb{R}^d)$$

where η is a positive constant. Then A satisfies the conditions of the proposition if we add Dirichlet boundary conditions (see, for instance, MIZOHATA (1973; Ch.3)).

The self-adjointness assumed in the proposition is not at all essential. A large class of non-self-adjoint generators of semigroups is given by delay equations of the following type:

$$(4.62) \qquad \frac{dx}{dt}(t) = \int_{[-h,0]} d\eta(\theta)\,x(t+\theta)$$

where $\eta(\theta)$ is an nxn–matrix of real functions of bounded variation on
[-h,0]. To this equation, a semigroup can be associated which acts on the
product space $\mathbb{R}^n \times L_2(-h,0)$ (see VINTER (1978)). How this association is
done will be explained, in a considerably more specific setting, in Section
4.5. The resolvent of the generator turns out to be compact which shows that
(A4) holds. Also the semigroup itself is compact for $t \leq h$, and
this entails that (A5) and (A6) are satisfied for any $\omega \in \mathbb{R}$. (For, if there
were infinitely many eigenvalues of A in a strip $\{\lambda \in \mathbb{C} \mid c_1 \leq \mathrm{Re}\ \lambda \leq c_2\}$
then there would be infinitely many eigenvalues of T(h) in a ring-shaped
domain $\{\lambda \in \mathbb{C} \mid e^{c_1 h} \leq |\lambda| \leq e^{c_2 h}\}$ (HILLE & PHILLIPS (1957; p.467)) and
consequently the eigenvalues of T(h) would have a non-zero accumulation
point, which is impossible by the compactness of T(h).) So the only question
that remains is whether the eigenvectors associated with (4.62) are complete.
Detailed conditions for this are given in MANITIUS (1980) and DELFOUR &
MANITIUS (1980b), and their results show that the completeness is obtained
in many cases, if the state space is chosen correctly.

Of course, the assumptions (A1-6) and (A9) hold in particular for
finite-dimensional systems. Generally speaking, one might say that the
assumptions delineate a class of systems that have those features of finite-
dimensional systems that make the constructive methods of Chapters 2 and 3
applicable. Indeed, in the subsequent chapters we shall follow the lines of
these chapters closely, although we shall introduce some specializations
and we shall concentrate on construction methods rather than necessary condi-
tions. Controlled invariant subspaces will be replaced by finite-dimensional
subspaces which are invariant for a generator of the form A + BF. The fact
that we shall use finite-dimensional subspaces in our constructions is moti-
vated by our aim to obtain controllers of finite order, but is also helps
to avoid difficulties that arise in connection with infinite-dimensional
spaces. This is the key to our approach in the subsequent chapters, together
with the relatively simple nature of the systems we shall study and the
availability of a finite-dimensional theory that gives ample opportunity
for low-order controller design.

4.5 Introduction to the examples

In this section we give some basic facts about the equations we shall use in our examples. We shall use the one-dimensional heat equation as an archetype of parabolic equations, and delay equations will be represented essentially by a scalar equation with one pure delay.

The heat equation, provided with distributed control and observation, has the following form.

$$(4.63.1) \quad \frac{\partial}{\partial t} \varphi(x,t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial x^2} \varphi(x,t) + \sum_{i=1}^{m} b_i(x) u_i(t) \qquad (x \in [0,1], \ t \geq 0)$$

$$(4.63.2) \quad \frac{\partial}{\partial x} \varphi(0,t) = \frac{\partial}{\partial x} \varphi(1,t) = 0 \qquad (t \geq 0)$$

$$(4.63.3) \quad \varphi(x,0) = \varphi_0(x) \qquad (x \in [0,1])$$

$$(4.63.4) \quad y_i(t) = \int_0^1 c_i(x) \ \varphi(x,t) dx \qquad (i = 1,..,p).$$

As the state space for this equation, we shall take $X = L_2(0,1)$. The operator A is then given by

$$(4.64.1) \quad D(A) = \{\varphi \in L_2(0,1) \ | \ \frac{d^2\varphi}{dx^2} \in L_2(0,1), \ \frac{\partial\varphi}{\partial x}(0) = \frac{\partial\varphi}{\partial x}(1) = 0\}$$

$$(4.64.2) \quad A\varphi = \frac{1}{\pi^2} \frac{d^2\varphi}{dx^2} .$$

It is a well-known fact that this operator generates a semigroup which is differentiable (even analytic) for $t > 0$. Furthermore, A has a discrete spectrum with simple eigenvalues at $-k^2 (k = 0,1,2,..)$. The corresponding normalized eigenvectors are given by

$$(4.65) \quad \varphi_k(x) = 1 \qquad (k = 0)$$
$$= \sqrt{2} \cos k\pi x \qquad (k = 1,2,..)$$

and they form a complete orthonormal set in $L_2(0,1)$.

We shall also use another version of this equation, in which the Neumann boundary condition (4.63.2) has been replaced by the Dirichlet boundary condition

(4.63.2)' $\varphi(0,t) = \varphi(1,t) = 0.$

In this case, the operator A is given by

(4.66.1) $D(A) = \{\varphi \in L_2(0,1) \mid \frac{d^2\varphi}{dx^2} \in L_2(0,1), \varphi(0) = \varphi(1) = 0\}$

(4.66.2) $A\varphi = \frac{1}{\pi^2} \frac{d^2\varphi}{dx^2} .$

Again, A has a discrete spectrum and the eigenvalues, which are now at $-k^2(k = 1,2,..)$, are all simple. The corresponding normalized eigenvectors are given by

(4.67) $\varphi_k(x) = \sqrt{2} \sin k\pi x$ $(k = 1,2,..)$

and these also form a complete orthonormal set in $L_2(0,1)$.

The input operator is given by (4.63.1) as $B : \mathbb{R}^m \to X$, with

(4.68) $Bu = \sum_{i=1}^{m} b_i u_i$ $(u \in \mathbb{R}^m).$

Of course, the input functions b_i are assumed to be in $L_2(0,1)$. The equation (4.63.4) defines the output mapping $C : X \to \mathbb{R}^p$ as

(4.69) $(C\varphi)_i = \int_0^1 c_i(x)\varphi(x)dx$ $(i = 1,..,p),$

where the c_i are functions in $L_2(0,1)$.

The delay equations we shall consider are all of the following type:

(4.70.1) $z'(t) = A_0 z(t) + A_1 z(t-1) + B_0 u(t)$ $(z(t) \in \mathbb{R}^n)$

(4.71.2) $z(t) = f(t)$ $(t \in [-1,0])$

(4.71.3) $y(t) = C_0 z(t).$

Here, A_0 and A_1 are nxn-matrices, B_0 is an nxm-matrix and C_0 is an pxn-matrix. We re-write this equation by introducing a function of two variables $\varphi(t,\theta)$ with $t \in [0,\infty)$ and $\theta \in [-1,0]$, which is related to $z(t)$ in the following way:

(4.72) $\varphi(t,\theta) = z(t+\theta)$.

Under suitable regularity hypotheses on $z(t)$, this relation implies

(4.73) $\frac{\partial}{\partial t} \varphi(t,\theta) = \frac{\partial}{\partial \theta} \varphi(t,\theta)$.

If $z(t)$ satisfies (4.60.1), then we obtain for the function φ:

(4.74) $\frac{\partial}{\partial \theta} \varphi(t,0) = A_0\varphi(t,0) + A_1\varphi(t,-1) + B_0 u(t)$.

This leads to the following set-up (cf. DELFOUR (1980)). Let $M_2((-1,0);\mathbb{R}^n)$ denote the product space $\mathbb{R}^n \times L_2((-1,0);\mathbb{R}^n)$, and let us write $H^1([-1,0];\mathbb{R}^n)$ for the set of $\mathbb{R}^n$-valued functions whose distributional derivative is in $L_2((-1,0);\mathbb{R}^n)$ (see ADAMS (1975; p.44)). By Sobolev's lemma (ADAMS (1975; p.97)), we can consider $H^1([-1,0];\mathbb{R}^n)$ as a subspace of $C([-1,0];\mathbb{R}^n)$. In particular, the quantities $\varphi(0)$ and $\varphi(1)$ are well-defined for $\varphi \in H^1([-1,0];\mathbb{R}^n)$. Now, define the state space by $X = M_2((-1,0);\mathbb{R}^n)$ and define the operator A by

(4.75.1) $D(A) = \{ (\varphi_0,\varphi) \mid \varphi_0 \in \mathbb{R}^n, \varphi \in H^1([-1,0];\mathbb{R}^n), \varphi(0) = \varphi_0 \}$

(4.75.2) $A(\varphi_0,\varphi) = (A_0\varphi(0) + A_1\varphi(-1), \varphi')$.

It has been shown (BORISOVIC & TURBABIN (1969)) that this operator is the generator of a semigroup $T(t)$ on $M_2((-1,0);\mathbb{R}^n)$. See DELFOUR (1980) for a survey and a further extension of the results in this direction. The spectrum of A is discrete; this follows from Prop. 4.2 in DELFOUR & MANITIUS (1980 b). To the right of any vertical line in the complex plane, there are only finitely many eigenvalues of A (HALE (1971; p.114)). For $t \geq 1$, the operator $T(t)$ is compact (DELFOUR & MITTER (1972)). The eigenvectors of A are complete in $M_2(-1,0)$ if and only if A_1 is non-singular (MANITIUS (1980)).

The input operator B is defined as an operator from $U = \mathbb{R}^m$ to $M_2((-1,0);\mathbb{R}^n)$ by

(4.76) $Bu = (B_0 u, 0)$

and the output operator C is defined as an operator from $M_2((-1,0);\mathbb{R}^n)$ to $y = \mathbb{R}^p$ by

(4.77) $C(\varphi_0,\varphi) = C_0\varphi_0.$

Clearly, B and C are both bounded operators.

A specific example of a delay equation that we shall use in the sequel is the scalar equation

(4.78) $x'(t) = -\frac{\pi}{2} x(t-1).$

The eigenvalues of the associated operator A can be computed as the roots of the characteristic equation, which is obtained by requiring that $x(t) = e^{\lambda t}$ is a solution of (4.78):

(4.79) $\lambda + \frac{\pi}{2} e^{-\lambda} = 0.$

It is easily seen that two roots of these equation are at $\frac{\pi}{2}i$ and at $-\frac{\pi}{2}i$, and that the other roots must all be in the left half plane. Apart from $\pm \frac{\pi}{2}i$, the roots can only be calculated approximately. A simple Newton procedure is sufficient for this purpose, because a good initial guess for the k-th pair of roots (k = 1,2,..) is given by the asymptotic formula

(4.80) $\lambda_k \simeq -\log(4k-3) \pm \frac{\pi}{2}(4k-3)i.$

Rules for deriving such formulas are given by BELLMAN & COOKE (1963).

CHAPTER 5

FINITE-DIMENSIONAL COMPENSATORS FOR INFINITE-DIMENSIONAL SYSTEMS

The purpose of this chapter is to describe a method for the design of stabilizing compensators of finite order for a wide class of infinite-dimensional systems. We prove the existence of a finite-dimensional compensator for systems that satisfy the assumptions of Section 4.4. Moreover, we give a design procedure that can be used to find such compensators. The procedure will be illustrated by examples.

The chapter is divided into six sections. In Section 1, we give a brief discussion of the known results on compensator design for infinite-dimensional systems, and we indicate what the differences are with the method we shall use. In Section 2, we formulate the stabilization problem and present a basic construction theorem. The conditions of this theorem are not immediately verifiable, however, and we proceed in Section 3 to prove an existence result that is applicable to any system satisfying the assumptions (A1-9) of Section 4.4. Although the proof of the existence theorem is constructive, it is preferable to use a somewhat different procedure for the actual design of stabilizing compensators. Such a procedure is given in Section 4. The design method is illustrated by examples in the final two sections of this chapter. The first example concerns a diffusion system, and in the second example we consider a system governed by a delay equation.

5.1. Introduction

The subject of stabilization for infinite-dimensional systems has been studied extensively. Many different approaches have been used, and we shall not attempt to give a survey; the reader is referred to CURTAIN & PRITCHARD (1978) for a comprehensive bibliography and an exposition of the main results that have been obtained via the semigroup approach. An even more recent review can be found in CURTAIN, PRITCHARD et al. (1981).

We shall concentrate on stabilization by dynamic output feedback. A direct translation of finite-dimensional result to infinite-dimensional systems leads to compensators of infinite order (BHAT (1976), GRESSANG &

LAMONT (1975), FUJI (1980)). In practice, such compensators can only be
implemented "approximately". This procedure, which does not seem to be
completely worked out yet, will probably lead to high-order dynamics in the
feedback loop. The 'converse' approach is to find, in some way or another,
a finite-dimensional model for the infinite-dimensional system, and then to
apply the standard finite-dimensional theory. Recently, M.J.Balas has worked
out a proposal to make this approach rigorous, using a singular perturbation
method;   but his paper (BALAS (1981)) does not contain a general existence
result.

Reduced-order modeling is known to be a difficult subject, even in the
finite-dimensional context, and there have been several attempts to develop a
theory for finite-dimensional compensator design without reduction of the
system order. However, the results which have appeared up to now (BALAS
(1978), BALAS (1979), CURTAIN (1981)) are all based on a very special assump-
tion on the operators defining the system. We shall explain this assumption,
called "zero spillover", in Section 5.5. In practice, the condition of
zero spillover can only approximately be met, which means that the theory
of the above-mentioned papers is not really applicable in practical situations.
Moreover, serious design restrictions are introduced if one tries to satisfy
this condition.

Below, we shall present an approach that avoids reduced-order modeling
without introducing special assumptions on the system. Under the assump-
tions (A1-9) of the previous chapter, which represent a wide class of
infinite-dimensional systems, we are able to give a rigorous treatment of
finite-dimensional compensator design and in particular, to prove the exis-
tence of a compensator of finite order. As will be illustrated by examples,
our approach is moreover suited for the actual computation of low-order com-
pensators.

5.2. The basic theorem

We consider a system whose evolution is described by the equations

$$(5.1.1) \quad x'(t) = Ax(t) + Bu(t) \qquad x(t) \in X, \ u(t) \in U$$

$$(5.1.2) \quad y(t) = Cx(t) \qquad y(t) \in Y.$$

The spaces X, U and Y and the operators A, B and C are supposed to satisfy

at least the assumptions (A1-3) of Section 4.4; the other assumptions will be appealed to when they are needed. To the system described by (5.1) we want to add a *compensator*, in order to obtain improved stability properties. The compensator will be a new dynamical system taking the observation y(t) from (5.1) as its input, and specifying at its output the control function u(t) appearing in (5.1):

(5.2.1) $w'(t) = Nw(t) + My(t)$ $w(t) \in W$

((5.2.2) $u(t) = Lw(t)$.

The *order* of the compensator will be the dimension of its state space W. We shall only consider compensators of finite order, so that (5.2) will represent a finite-dimensional system. Of course, L, M and N are linear mappings between the appropriate spaces.

The systems (5.1) and (5.2) together give rise to the following set of equations:

(5.3.1) $x'(t) = Ax(t) + BLw(t)$

(5.3.2) $w'(t) = MCx(t) + Nw(t)$

which we may also write in the extended state space $X \oplus W$:

(5.4) $\dfrac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix}(t) = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}(t)$.

Here, the *extended system operator*

(5.5) $A_e := \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}$

with domain $D(A) \oplus W$ and range space $X \oplus W$ is defined in the way explained in Section 4.2.

The *stabilization problem* can now be formulated as follows: Given a system (5.1) under the assumptions (A1-3), find a finite-dimensional compensator (5.2) such that the extended system operator A_e defined in (5.5) generates a stable semigroup on the extended state space $X \oplus W$. The

notion of 'stability' is assumed to be defined by some fixed (negative) number ω representing the desired growth constant; see Section 4.3. A compensator that gives a solution to the stabilization problem will be called a *stabilizing compensator*.

The following theorem gives sufficient conditions for a stabilizing compensator of order k to exist. The proof is by construction.

THEOREM 5.1 *Consider the system (5.1) under the assumptions (A1-3). Assume that there exist bounded mappings* $F: X \rightarrow U$ *and* $G: Y \rightarrow X$*, together with a finite-dimensional subspace* $V \subset D(A)$*, such that the following holds:*

(5.6.1)   A + BF generates a stable semigroup

(5.6.2)   A + GC generates a stable semigroup

(5.6.3) $(A+BF)x \in V$ for all $x \in V$

(5.6.4)   Im $G \subset V$.

Then there exists a stabilizing compensator of order k, where k = dim V.

PROOF Introduce a new linear space W isomorphic to V, and let $R: V \rightarrow W$ be the mapping that provides the isomorphism. Define a compensator of the form (5.3) by setting $L = FR^{-1}$, $M = -RG$ (well-defined by (5.6.4)) and $N = R(A+BF+GC)R^{-1}$ (well defined by (5.6.3) and (5.6.4)). We obtain the following extended system operator:

$$(5.7) \qquad A_e = \begin{pmatrix} A & BFR^{-1} \\ -RGC & R(A+BF+GC)R^{-1} \end{pmatrix}.$$

We introduce the following subspace of $X^e := X \oplus W$:

$$(5.8) \qquad M = \{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V \} = \{ \begin{pmatrix} R^{-1}w \\ w \end{pmatrix} \mid w \in W \}.$$

There is an obvious isomorphism from W to M, given by

$$(5.9) \qquad T: w \mapsto \begin{pmatrix} R^{-1}w \\ w \end{pmatrix}.$$

Considering X, W and M as subspaces of X^e, we note that X^e can be decomposed either as $X \oplus W$ or as $X \oplus M$. The similarity transformation from one decomposition to the other is given by

$$(5.10) \qquad H: X \oplus W \to X \oplus M, \quad H\begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} x - R^{-1}w \\ Tw \end{pmatrix}.$$

Written in matrix format, we have

$$(5.11) \qquad H = \begin{pmatrix} I & -R^{-1} \\ 0 & T \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} I & R^{-1}T^{-1} \\ 0 & T^{-1} \end{pmatrix}.$$

By straightforward computation, we find the following form for the extended system operator with respect to the 'basis' $X \oplus M$:

$$(5.12) \qquad \tilde{A}_e = HA_e H^{-1} = \begin{pmatrix} A+GC & 0 \\ -TRGC & TR(A+BF)R^{-1}T^{-1} \end{pmatrix}$$

The right lower block is clearly similar to $A + BF: V$, which generates a stable semigroup by (5.6.1). Because the left upper block is also the generator of a stable semigroup according to (5.6.2), it follows from Prop. 4.7 that $\tilde{A}_e$ generates a stable semigroup, and Lemma 4.5 shows that the same must hold for A_e.   ⊠

In our formulation of the stabilization problem, we have only allowed integral control. We could have added *proportional control* by letting the compensator be of the form

$$(5.13.1) \qquad w'(t) = Nw(t) + My(t)$$

$$(5.13.2) \qquad u(t) = Lw(t) + Ky(t).$$

In effect, proportional control allows us to change the system operator A into $A + BKC$. It should be noted that the above theorem is applicable to the triple $(C, A+BKC, B)$ for any K, simply by absorbing the BKC-term into A. One may use proportional control as an independent means for improving the stability properties of the system, but we shall not discuss this aspect. For a treatment of proportional control in an infinite-dimensional context,

one is referred to POHJOLAINEN (1980).

Thm. 5.1 is clearly not directly applicable, because it is not imme-
diately clear how to find a subspace V and mappings F and G that satisfy
the conditions (5.6). Additional material will be needed to obtain practical
results. In the next two sections, we shall present such material.

5.3. The existence result

Our aim in this section is to prove the following result.

THEOREM 5.2 *Consider the system (5.1). If the assumptions (A1-9) hold, then
there exists a stabilizing compensator of finite order.*

This establishes the existence of finite-dimensional stabilizing compen-
sators for a large class of infinite-dimensional systems (as discussed in
Section 4.4). An upper bound for the compensator order is not given, but the
examples at the end of this chapter suggest that in many practical cases
it will be possible to design compensators of fairly low order.

The proof of the theorem consists of a combination of the results of
two lemmas, which we shall now give.

LEMMA 5.3 *Consider the system (5.1) under the assumptions (A1-3). Suppose
that there exist bounded mappings* F: $X \to U$ *and* G: $Y \to X$ *such that*

(5.14.1)   A + BF *generates a stable semigroup*

(5.14.2)   A + BF *has a discrete spectrum, and its eigenvectors form a
complete set in* X

(5.14.3)   *there exists* $\delta > 0$ *such that the semigroup generated by*
A + GC + δI *is stable.*

*Then there exists a stabilizing compensator of finite order for the given
system.*

PROOF Let S(t) denote the semigroup generated by A + GC. By (5.14.3), the
growth constant of S(t) is less than or equal to $\omega-\delta$ (where ω denotes the
growth constant that defines our notion of 'stability'). Consequently, there
exists a constant M such that

(5.15) $\|S(t)\| \leq M \exp((\omega - \tfrac{1}{2}\delta)t)$.

For any $\widehat{G}: Y \rightarrow X$ such that

(5.16) $\|G - \widehat{G}\| \leq \tfrac{1}{2} M^{-1} \|C\|^{-1} \delta$

the operator $A + \widehat{G}C$ will generate a semigroup $\widehat{S}(t)$ with

(5.17) $\|\widehat{S}(t)\| \leq M \exp(\omega t)$

(Lemma 4.4). Let us write $\varepsilon := \tfrac{1}{2} M^{-1} \|C\|^{-1} \delta$. Pick some orthonormal basis of Y, and define $g_i := Gy_i$. By the completeness assumption on the eigenvectors of $A + BF$, there exists for every $i = 1,..,p$ a finite set $\{x_{i1},..,x_{iN_i}\}$ of generalized eigenvectors of $A + BF$ such that

(5.18) $\left\| g_i - \sum\limits_{j=1}^{N_i} \alpha_{ij} x_{ij} \right\| < p^{-\frac{1}{2}}$

for suitable numbers α_{ij} ($i=1,..,p$; $j=1,..,N_i$). To every pair of indices (i,j), there exists a $\lambda_{ij} \in \mathbb{C}$ and an $n_{ij} \in \mathbb{N}$ such that $(\lambda_{ij}-(A+BF))^{n_{ij}} x(i,j) = 0$. Now define the subspace V as follows:

(5.19) $V := \text{span } \{(\lambda_{ij}-(A+BF))^k x_{ij} \mid i=1,..,p; \ j=1,..,N_i; \ k=0,..,n_{ij}-1\}$.

Then it is clear that V is a finite-dimensional subspace contained in $D(A+BF) = D(A)$, and that V is invariant under $A + BF$.

Write

(5.20) $\widehat{g}_i := \sum\limits_{j=1}^{N_i} \alpha_{ij} x_{ij}$

and define $\widehat{G}: Y \rightarrow X$ by

(5.21) $\widehat{G}y_i = \widehat{g}_i$.

We then have $\text{Im } \widehat{G} \subset V$. Moreover, (5.16) holds and so $A + \widehat{G}C$ generates a stable semigroup. We can now apply Thm. 5.1, using the subspace V and the

mappings F and $\hat{G}$, to conclude that there exists a stabilizing compensator of finite order (equal to dim V).   ⊠

Remark In the conditions of the lemma, it is required that the semigroup generated by A + GC has a certain extra stability margin. In the proof, this is needed to allow a small perturbation of G without loss of stability. In the situation where assumption (A5) of Section 4.4 holds, it is possible to do a spectral decomposition with respect to the set $\sigma_1 = \{\lambda \in \sigma(A) \mid \text{Re } \lambda >$ $> \omega - \delta\}$, and then one would want to use a finite-dimensional method to compute G such that Re $\lambda \le \omega - \delta$ for all $\lambda \in \sigma(A+GC)$. However, we have not excluded that there may be unobservable eigenvalues of A in the strip $\{\lambda \in \mathbb{C} \mid \omega - \delta <$ $< \text{Re } \lambda \le \omega\}$. This would make it impossible to shift the eigenvalues of A + GC to the left of the line Re $\lambda = \omega - \delta$, but we can still reach our ultimate goal of slightly perturbing G without destroying the stability. This is seen in the following way.

Decompose $X_u^{\omega-\delta}$ as $X_2 \oplus X_3$, where

$$(5.22) \qquad X_3 = <\text{Ker } C_u^{\omega-\delta} \mid A_u^{\omega-\delta}>$$

(notation as in (1.10)). Write $X_1 = X_s^{\omega-\delta}$. Then we have, with respect to the decomposition $X = X_1 \oplus X_2 \oplus X_3$;

$$(5.23) \qquad A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}, \quad C = (C_1 \quad C_2 \quad 0).$$

Here, we have written A_{11} for $A_s^{\omega-\delta}$, etc.. By our assumptions, the pair (C_2, A_{22}) is observable, and the matrix A_{33} is stable in the sense of ω but not in the sense of $\omega - \delta$. If we choose G_2 such that $(A_{22} + G_2 C_2) \subset$ $\subset \{\lambda \in \mathbb{C} \mid \text{Re } \lambda < \omega - \delta\}$, and if we define G by

$$(5.24) \qquad G = \begin{pmatrix} 0 \\ G_2 \\ 0 \end{pmatrix}$$

then we shall have

$$(5.25) \qquad A + GC = \begin{pmatrix} A_{11} & 0 & 0 \\ G_2 C_1 & A_{22} + G_2 C_2 & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$

By Prop. 4.7, the two-by-two left upper block is the generator of a semigroup which is stable in the sense of $\omega-\delta$. If G is perturbed to $\hat{G} = (\hat{G}_1^t \quad \hat{G}_2^t \quad \hat{G}_3^t)^t$, then we get

$$(5.26) \qquad A + \hat{G}C = \begin{pmatrix} A_{11}+\hat{G}_1C_1 & \hat{G}_1C_2 & 0 \\ \hat{G}_2C_1 & A_{22}+\hat{G}_2C_2 & 0 \\ \hat{G}_3C_1 & A_{32}+\hat{G}_3C_2 & A_{33} \end{pmatrix}$$

The two-by-two left upper block will generate a semigroup with growth constant $\leq \omega$ whenever $\|G - \hat{G}\|$ is small enough. Prop. 4.7 then shows that the semigroup generated by $A + \hat{G}C$ is stable in the sense of ω.

The conclusion of this remark is that the construction of Lemma 5.3 can be done even when there are unobservable nearly unstable eigenvalues. Of course, the basic reason for this fact is that such eigenvalues are insensitive to the choice of the injection mapping and can therefore be discarded when this mapping is manipulated.

It may be possible to verify the conditions of Lemma 5.3 immediately in some cases where a well-developed theory of completeness of eigenvectors exists; such a theory is given by MANITIUS (1980) and DELFOUR & MANITIUS (1980b) for systems described by delay equations. In general, however, only the completeness of the eigenvectors of A would be known, and we would like to infer from this that the same property holds for A + BF. An extensive study of this inference for diffusion processes, based on the sufficient condition for completeness given in DUNFORD & SCHWARTZ (1963; p.1115), has been made in VAN HARTEN (1979). However, if the feedback mapping F is constructed as in the proof of Prop. 4.12, we can use a much simpler argument.

LEMMA 5.4 *Consider the system (5.1). If the assumptions (A1-7) and (A9) hold, then there exists a bounded mapping* F: $X \rightarrow U$ *such that* A + BF *generates a stable semigroup, the spectrum of* A + BF *is discrete and the eigenvectors of* A + BF *are complete.*

PROOF Define F as in the proof of Prop. 4.12, taking care that the eigenvalues of $A_u + B_u F_u$ do not coincide with those of A. From the form (4.48) of A + BF, we have

$$(5.27) \qquad \sigma(A+BF) = \sigma(A_s) \cup \sigma(A_u+B_uF_u).$$

The two parts of the spectrum are separated, so there is a corresponding
spectral decomposition which will be written

(5.28) $X = X_s \oplus X_n$.

The spectrum of A + BF is clearly discrete, and it remains to show that
every element x $\in$ X can be approximated arbitrarily close by a finite
linear combination of generalized eigenvectors of A + BF. So take x $\in$ X,
and let ε be a positive number. We can write x = x_s + x_n with $x_s \overset{r}{\in} X_s$ and
$x_n \in X_n$. The subspace X_n is a finite-dimensional eigenspace of A + BF
and so x_n is obviously a finite linear combination of eigenvectors of A + BF.
By (A9) and the fact that $AX_s \subset X_s$, there is a finite linear combination of
eigenvectors of A in X_s, which we shall call $\tilde{x}_s$, such that $\|x_s - \tilde{x}_s\| < \varepsilon$.
Generalized eigenvectors of A in X_s are also generalized eigenvectors of
A + BF and so $\tilde{x}_s + x_n$ is a finite linear combination of eigenvectors of
A + BF. Moreover, $\|x + (\tilde{x}_s + x_n)\| = \|x_s - \tilde{x}_s\| < \varepsilon$.   ⊠

Using the results of the two lemmas, the proof of Thm. 5.2 follows
almost immediately.

PROOF (of Thm. 5.2) By Lemma 5.4, there exists a bounded mapping F: $X \to U$
such that A + BF generates a stable semigroup, the spectrum of A + BF is
discrete and the eigenvectors of A + BF are complete. By assumption (A5),
there exists a $\delta > 0$ such that $\sigma_1 = \{\lambda \in \sigma(A) \mid Re \lambda > \omega-\delta\}$ is a finite set
to which the spectral decomposition (Prop. 4.11) can be applied. If the
resulting finite-dimensional pair (C_1, A_{11}) is observable, the procedure of
the proof of Prop. 4.13 leads to an injection mapping G: $Y \to X$ such that
the semigroup generated by A + GC has a growth constant $\leq \omega-\delta$, and appealing
to Lemma 5.3 completes the proof. If the pair (C_1, A_{11}) is not observable,
a suitable modification of this procedure still leads to the same result;
see the remark following the proof of Lemma 5.3.   ⊠

For a discussion of the class of systems for which we now have esta-
blished the existence of a finite-dimensional stabilizing compensator, the
reader is referred to Section 4.4.

5.4 The design procedure

The proof of our existence result has been constructive, and so in
principle we have obtained a method to compute solutions to the stabilization

problem. For practical purposes, however, the method suggested by the proofs above would not be very convenient. Below, we shall present an iterative procedure for the design of low-order cempensators, and the final two sections of this chapter will be devoted to an illustration of this procedure by examples. We give the method as a series of steps; comment on each step will be given afterwards. It is not claimed that the procedure has a very high degree of numerical refinement, but it is good enough for our purposes and it may serve as a starting point for further numerical research. We proceed as follows, assuming that we have a system which satisfies the conditions (A1-9) of Section 4.4.

STEP 1   Find F such that A + BF has a discrete spectrum, the eigenvectors of A + BF are complete, and the semigroup generated by A + BF is stable.

STEP 2   Find G such that the growth constant of the semigroup generated by A + GC is somewhat smaller than the constant ω that indicates the dividing line between 'stable' and 'unstable'.

STEP 3   Approximate the vectors in Im G by linear combinations of k selected eigenvectors of A + BF, and form the mapping $\hat{G}$ which is close to G.

STEP 4   See if the semigroup generated by A + $\hat{G}$C is stable. If not, select a different F and/or a different G, or repeat Step 3 with k replaced by k + 1. If the semigroup is stable, go to Step 5.

STEP 5   Construct the compensator of order k as in the proof of Thm. 5.1.

Comments   The feedback mapping F may be constructed by the method of TRIGGIANI (1975) as we did in the previous section, but any other method may be used just as well. The stabilization property of F is of course fundamental. Because A generates a semigroup and BF is a bounded perturbation of finite rank, the spectrum of A + BF must be discrete; this follows from Thm. 6.2 and Thm. 6.5 in KATO (1966; Ch.IV) and from (4.42) and Prop. 4.3. The question of completeness may be less easy to settle; see Lemma 5.4 and the remarks preceding this lemma. Of course, a check on completeness is not necessary if one is willing to try the procedure without the guarantee that it will ultimately be succesful.

The mapping G appearing in Step 2 may also be found by any suitable method. For the purposes of the design procedure, the effect of 'unobservable

poles' (eigenvalues with corresponding eigenvectors in Ker C) on the growth constant of the semigroup generated by A + GC may be discarded, in view of the remark made after the proof of Lemma 5.3. The words "somewhat smaller" are vague, of course, but at present we do not have any better. Further research will have to show if it is possible to give any general guiding lines for selecting the growth constant of A + GC. The approximation of the vectors in Im G has to be done in the norm for which the completeness of the eigenvectors of A + BF has been extablished, if one wants to be assured of the ultimate sucess of the procedure when k is increased. In many practical cases, however, the use of another norm may be easier computationally while still giving good results.

In principle, it would be possible to give an a priori estimate on the compensator order using (5.16), but this bound may be difficult to compute and it is likely to be conservative. The compensator order found by the iterative procedure can be much lower. The stability of the semigroup generated by A + $\hat{G}$C has to be verified from a computation of the eigenvalues, and it has to be shown by a direct argument that A + $\hat{G}$C satisfies the spectrum determined growth assumption.

The eigenvalue computation can conveniently be done by use of the Weinstein-Aronszajn-method (see KATO (1966; p.244)). The basic idea of this method can be described in a simple way: Suppose that $\lambda_0 \notin \sigma(A)$ is an eigenvalue of A + $\hat{G}$C with corresponding eigenvector x, then we have $(A+\hat{G}C)x = \lambda_0 x$ and consequently

$$(5.29) \qquad x = (\lambda_0 - A)^{-1}\hat{G}Cx$$

which shows that the kernel of the matrix $I - C(\lambda_0 - A)^{-1}\hat{G}$ contains the vector Cx. Because $\lambda_0 \notin \sigma(A)$ we must have $Cx \neq 0$, and it follows that the function $\det(I-C(\lambda-A)^{-1}\hat{G})$ has a zero at λ_0. On the other hand, it is shown in KATO (1966) that a zero of this function at a point in the resolvent set of A gives rise to an eigenvalue of A + $\hat{G}$C. Now define the following functions from $\mathbb{C}$ to $\mathbb{Z}$:

$$(5.30) \qquad \underline{\sigma}(A)(\lambda_0) = k \text{ if } \lambda_0 \text{ is an eigenvalue of A with multiplicity k}$$
$$= 0 \text{ if } \lambda_0 \text{ is not an eigenvalue of A}$$

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(5.31) $\nu_G(\lambda_0)$ = k if λ_0 is a zero of order k of $\det(I-C(\lambda-A)^{-1}\hat{G})$

　　　　　　 = -k if λ_0 is a pole of order k of $\det (I-C(\lambda-A)^{-1}\hat{G})$

　　　　　　 = 0 if λ_0 is neither a zero nor a pole

and define $\underline{\sigma}(A+\hat{G}C)$ in the same way as $\underline{\sigma}(A)$. It is proved in KATO (1966) that the following formula holds:

(5.32) $\underline{\sigma}(A+\hat{G}C) = \underline{\sigma}(A) + \nu_G$

or else the 'singular' case occurs in which $\sigma(A+\hat{G}C) = \mathbb{C}$. In our case, the latter possibility is excluded because $A + \hat{G}C$ will always generate a semi-group so that Re $\sigma(A+\hat{G}C)$ must be bounded above.

The final step of the procedure is purely a matter of computation. The numerical results that one obtains could be taken as the starting point of a process of tuning of parameters aimed at a further improvement of the system's behaviour, but we shall leave this out of our discussion. By the results of Section 5.3, the procedure as it stands is already guaranteed to lead to a stabilizing compensator of finite order.

5.5 Example I: A diffusion system

The evolution of the temperature distribution on a thin, uniform, isolated rod can be described by the following parabolic partial differential equation:

(5.33.1) $\dfrac{\partial}{\partial t} z(x,t) = \dfrac{1}{\pi^2} \cdot \dfrac{\partial^2}{\partial x^2} z(x,t) + b(x)u(t)$ $(t \geq 0,\ 0 \leq x \leq 1)$

(5.33.2) $\dfrac{\partial z}{\partial x} (0,t) = \dfrac{\partial z}{\partial x} (1,t) = 0$ $(t \geq 0)$

(5.33.3) $z(x,0) = z_0(x)$ $(0 \leq x \leq 1).$

We assume a scalar input and a scalar output, given by

(5.33.4) $b(x) = \sqrt{10}$ $(0.2 \leq x \leq 0.3)$

　　　　　　 = 0 $(0 \leq x < 0.2,\ 0.3 < x \leq 1)$

(5.33.5) $y(t) = \sqrt{10} \displaystyle\int_{0.7}^{0.8} z(x,t)dx.$

As our state space, we take $L_2(0,1)$. The operator A is defined by

(5.34.1) $D(A) = \{\varphi \in L_2(0,1) \mid \dfrac{d^2\varphi}{dx^2} \in L_2(0,1), \dfrac{d\varphi}{dx}(0) = \dfrac{d\varphi}{dx}(1) = 0\}$

(5.34.2) $A\varphi = \dfrac{1}{\pi^2}\dfrac{d^2\varphi}{dx^2}$ $(\varphi \in D(A))$.

The input space and the output space are both equal to $\mathbb{R}$; the mappings B and C are given by

(5.35) $B\alpha = \alpha b$ $(\alpha \in \mathbb{R})$

(5.36) $C\varphi = \sqrt{10} \displaystyle\int_{0.7}^{0.8} \varphi(x)dx$ $(\varphi \in L_2(0,1))$.

Both mappings have been normalized to 1, in order to obtain a clear picture of the gains that will be needed in the final design. The symmetry between input and output is inessential.

The pertinent facts about the operator A have been given in Section 4.5. We can calculate the entries of the infinite matrices of B and C with respect to the orthonormal set $\{\varphi_0, \varphi_1, \ldots\}$ of eigenvectors of A:

k	$\beta_k = <b,\varphi_k>$	$\gamma_k = C\varphi_k$
0	0.3162	0.3162
1	0.3149	-0.3149
2	0.	0.
3	-0.3047	0.3047
4	-0.4184	-0.4184
5	-0.2847	0.2847
6	0.	0.
7	0.2562	-0.2562
8	0.3385	0.3385
9	0.2209	-0.2209
10	0.	0.

Table 5.1. First entries of B and C.

We set the desired growth constant ω equal to -1. From the remarks in Section 4.5 and the table above, it is clear that our system satisfies the assumptions (A1-9) and so Thm. 5.2 guarantees the existence of a finite-dimensional stabilizing compensator.

The assumption of "zero spillover" that was alluded to in Section 5.1 comes down to requiring that either the coefficients β_k or the coefficients γ_k would be zero from a certain index k_0 on. In our example, this is clearly not the case. The presence of both "control spillover" and "observation spillover" is not a problem in our approach, however.

Let us follow the procedure given in Section 5.4.

STEP 1   The unstable subspace X_u is the one-dimensional eigenspace corresponding to the eigenvalue 0 of A. We have

(5.37) $A_u = 0$, $B_u = 0.3162$.

The eigenvalue at 0 is shifted to -1.5 by taking

(5.38) $F_u = -1.5 \, \beta_0^{-1} = -7.906$.

If we define F by $F = F_u P$ where P is the orthogonal projection onto X_u, the operator $A + BF$ has eigenvalues at $\mu_0 = -1.5$ and at $\mu_k = -k^2$ $(k = 1,2,\ldots)$, with corresponding eigenfunctions

(5.39) $\psi_k = (\mu_0 - A)^{-1} b$ $(k = 0)$

               $= \varphi_k$ $(k = 1,2,\ldots)$.

The eigenfunction ψ_0 has been normalized such that $F\psi_0 = 1$.

STEP 2   We consider the subspace $X_1 = \text{span} \{\varphi_0, \varphi_1\}$ and the corresponding mappings:

(5.40) $A_{11} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$, $C_1 = (0.3162, -0.3149)$.

The pair (C_1, A_{11}) is observable. A short calculation shows that we can assign new eigenvalues of $A + GC$ at -1.1 and -1.2 by defining G as follows:

(5.41.1) $G\alpha = \alpha g$ $(\alpha \in \mathbb{R})$

(5.41.2) $g = -1.32 \ \gamma_0^{-1} \ \varphi_0 + 0.02 \ \gamma_1^{-1}\varphi_1 =$

$= -4.174 \ \varphi_0 - 0.064 \ \varphi_1 .$

STEP 3   For $\hat{g}$, let us take the orthogonal projection of g onto the subspace
span $\{\psi_0, \psi_1\}$:

(5.42) $\hat{g} = 27.216 \ \psi_0 + 5.649 \ \psi_1 .$

STEP 4   In the present case, it follows from the Weinstein-Aronszajn theory
(see Section 5.4) that the eigenvalues of an operator of the form
$A + \hat{G}C$ are found as the zeros of the function

(5.43) $f(\lambda) = 1 - C(\lambda-A)^{-1} \ \hat{G}$

together with the eigenvalues of A that are not poles of $f(\lambda)$. If $\hat{G}$ is
given in terms of the basis $\{\varphi_0, \varphi_1, \ldots\}$:

(5.44.1) $\hat{G}\alpha = \alpha\hat{g}$ $(\alpha \in \mathbb{R})$

(5.44.2) $\hat{g} = \sum_{k=0}^{\infty} \alpha_k \ \varphi_k$

then the function $f(\lambda)$ can be written as

(5.45) $f(\lambda) = 1 - \sum_{k=0}^{\infty} \alpha_k \gamma_k (\lambda-\lambda_k)^{-1} .$

Using the expressions for ψ_0 and ψ_1 given in (5.39) we write $\hat{g}$ in the
form (5.44), and then we compute the zeros of the function f appearing in
(5.45). It turns out that the first two eigenvalues of $A + \hat{G}C$ are at $-1.0187\pm$
$\pm 0.1401i$.   We also note that $A + \hat{G}C$ satisfies the spectrum determined
growth condition because the semigroup generated by $A + \hat{G}C$ is analytic;
this follows from the fact that the semigroup generated by A is analytic
and from the fact that analyticity of the semigroup is preserved under
bounded perturbations of the generator(HILLE & PHILLIPS(1957; p.418)).

STEP 5   With respect to the basis $\{\psi_0, \psi_1\}$, the compensator equations can now be computed as follows:

(5.46.1) $\dfrac{d}{dt}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(t) = \begin{pmatrix} -1.929 & -8.570 \\ 0.119 & -2.779 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}(t) + \begin{pmatrix} 27.216 \\ 5.649 \end{pmatrix} y(t)$

(5.46.2) $u(t) = w_1(t).$

The eigenvalues of the extended system operator A_e consist of the eigenvalues of A + BF corresponding to ψ_0 and ψ_1, together with all eigenvalues of A + GC; so we get -1, $-1.0187 \pm 0.1401i$, -1.5, -4,.... The growth constant of the compensated system is thus precisely equal to -1.

Some simulation results showing the effect of the compensator are given in the Appendix, Fig.A1.

5.6   Example II: A delay system

We consider the following retarded equation, with control and observation:

(5.47.1) $x_1'(t) = -\dfrac{\pi}{2} x_1(t-1) + x_2(t)$

$x_2'(t) = u(t)$

(5.47.2) $y(t) = x_1(t).$

The characteristic equation of the open-loop system is given by

(5.48) $\det\begin{pmatrix} \lambda + \dfrac{\pi}{2} e^{-\lambda} & -1 \\ 0 & \lambda \end{pmatrix} = 0.$

The characteristic function

(5.49) $\Delta_A(\lambda) := \lambda\left(\lambda + \dfrac{\pi}{2} e^{-\lambda}\right)$

has zeros at 0, $\dfrac{\pi}{2} i$, $-\dfrac{\pi}{2} i$, and at infinitely many other points which are all in the left half-plane. (See the remarks in Section 4.5.) As our state space, we take $M_2(-1,0) \oplus \mathbb{R}$ or the complexified version of this space. The operator A is defined by

$$(5.50.1) \quad D(A) = \left\{ \begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} \ \Big| \ \varphi_0 \in \mathbb{R}, \ \varphi \in H^1[-1,0], \ \alpha \in \mathbb{R}, \ \varphi(0) = \varphi_0 \right\}$$

$$(5.50.2) \quad A\begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} = \begin{pmatrix} (-\frac{\pi}{2} \varphi(-1)+\alpha, \ \varphi') \\ 0 \end{pmatrix}.$$

This setting for the delay equation has been explained in Section 4.5.

We let the input space U and the output space Y both be equal to $\mathbb{R}$ (or $\mathbb{C}$). The mappings B and C are given by

$$(5.51) \quad Bu = \begin{pmatrix} 0 \\ u \end{pmatrix} \qquad\qquad (u \in \mathbb{R})$$

$$(5.52) \quad C\begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} = \varphi_0 \qquad\qquad (\begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} \in M_2(-1,0) \oplus \mathbb{R}).$$

The stabilizability of the pair (A,B) and the detectability of the pair (C,A) can be verified conveniently using the generalization of the Hautus test (HAUTUS (1969)) given by K.P.M.Bhat (BHAT (1976)). Because

$$(5.53) \quad \text{rank} \begin{pmatrix} \lambda + \frac{\pi}{2} e^{-\lambda} & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = 2 \qquad \forall \lambda \in \mathbb{C},$$

the pair (A,B) is stabilizable with respect to any desired growth constant ω, and detectability of the pair (C,A) also holds for any ω because

$$(5.54) \quad \text{rank} \begin{pmatrix} \lambda + \frac{\pi}{2} e^{-\lambda} & 1 \\ 0 & \lambda \\ 1 & 0 \end{pmatrix} = 2 \qquad \forall \lambda \in \mathbb{C}.$$

Let us set the desired growth constant ω equal to -1. It follows from the above remarks and from the remarks in Section 4.5 that the system under consideration satisfies the assumptions (A1-9) of section 4.4. So we start the design procedure.

STEP 1   A spectral decomposition for an equation closely related to (5.47.1) is given in HALE (1971; p.117). Using this, we find the following feedback mapping F, which replaces the eigenvalues of A at $0, \frac{\pi}{2}i$ and $-\frac{\pi}{2}i$ by eigenvalues at -1, $-1 + \frac{\pi}{2}i$ and $-1 - \frac{\pi}{2}i$ for A + BF, while leaving all the other eigenvalues unchanged.

$$(5.55) \quad F\begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} = -3\varphi_0 + \int_{-1}^{0} ((\frac{\pi^2}{2}-1)\cos\frac{\pi}{2}\theta - \frac{3\pi}{2}\sin\frac{\pi}{2}\theta)\varphi(\theta)d\theta - 3\alpha.$$

The interested reader may wish to verify that indeed the characteristic function of A + BF, which is given by

$$(5.56) \quad \Delta_{A+BF}(\lambda) = \det\begin{pmatrix} \lambda + \frac{\pi}{2}e^{-\lambda} & -1 \\ 3 - \int_{-1}^{0} ((\frac{\pi^2}{2}-1)\cos\frac{\pi}{2}\theta - \frac{3\pi}{2}\sin\frac{\pi}{2}\theta)e^{\lambda\theta}d\theta & \lambda+3 \end{pmatrix}$$

can be calculated as

$$(5.57) \quad \Delta_{A+BF}(\lambda) = (\lambda+1)((\lambda+1)^2 + \frac{\pi^2}{4})(\lambda + \frac{\pi}{2}e^{-\lambda})(\lambda^2 + \frac{\pi^2}{4})^{-1}.$$

All eigenvalues of A + BF are simple, and we proceed to compute the corresponding eigenvectors. If μ is an eigenvalue of A + BF and $\psi = \begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} \in D(A)$ is the corresponding eigenvector, then the following equations hold:

$$(5.58.1) \quad \mu\varphi_0 = -\frac{\pi}{2}\varphi(-1) + \alpha$$

$$(5.58.2) \quad \mu\varphi = \varphi'$$

$$(5.58.3) \quad \mu\alpha = F\psi$$

Because $\psi \in D(A)$, we also must have $\varphi \in H^1[-1,0]$ and $\varphi(0) = \varphi_0$. Then (5.58.2) gives

$$(5.59) \quad \varphi(\theta) = e^{\mu\theta}\varphi_0 \qquad\qquad (-1 \le \theta \le 0)$$

and from (5.58.1) we obtain

$$(5.60) \quad \alpha = (\mu + \frac{\pi}{2}e^{-\mu})\varphi_0.$$

The eigenvector will be normalized such that $C\psi = 1$ if we put $\varphi_0 = 1$.

STEP 2   We proceed to compute G such that A + GC has its eigenvalues to the left of the line Re λ = -1. It is easy to find the matrices of A_u and C_u with respect to the basis

$$(5.61) \qquad \left\{ \begin{pmatrix} (1,\cos\frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} (0,\sin\frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \begin{pmatrix} (\frac{2}{\pi},\frac{2}{\pi}) \\ 1 \end{pmatrix} \right\}$$

of X_u. They are given as follows:

$$(5.62) \qquad A_u = \begin{pmatrix} 0 & \frac{1}{2}\pi & 0 \\ -\frac{1}{2}\pi & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_u = (1 \quad 0 \quad \frac{2}{\pi}).$$

By straightforward computation, one finds that A + GC will have eigenvalues at $-\frac{\pi}{2}$ (double) and $-\pi$ if we take

$$(5.63.1) \qquad G\alpha = \alpha g \qquad\qquad (\alpha \in \mathbb{R})$$

$$(5.63.2) \qquad g = -\pi \begin{pmatrix} (2,\cos\frac{\pi}{2}\theta+2\sin\frac{\pi}{2}\theta+1) \\ \frac{1}{2}\pi \end{pmatrix}.$$

STEP 3   We compute the orthogonal projection (with respect to the norm of $L_2(-1,0) \oplus \mathbb{R}$) of g onto the subspace spanned by the eigenvectors of A + BF corresponding to the eigenvalues at -1 and $-1 \pm \frac{\pi}{2}i$. This gives

$$(5.64) \qquad \hat{g} = -\pi(0.66 \begin{pmatrix} (1,e^{-\theta}\cos\frac{\pi}{2}\theta) \\ -1 \end{pmatrix} + 2.89 \begin{pmatrix} (0,e^{-\theta}\sin\frac{\pi}{2}\theta) \\ \frac{1}{2}\pi(1-e) \end{pmatrix} + 2.66 \begin{pmatrix} (1,e^{-\theta}) \\ -1+\frac{1}{2}\pi e \end{pmatrix}).$$

STEP 4   We need an explicit formula for the eigenvalues of an operator of the form A + $\hat{G}$C, when the range of $\hat{G}$ is contained in the span of finitely many eigenvectors of A + BF. It follows from the W-A theory (see formula (5.32)) that the eigenvalues of A + $\hat{G}$C are found as the zeros of

$$(5.65) \qquad \Delta_{A+\hat{G}C}(\lambda) = \Delta_A(\lambda)(1-C(\lambda-A)^{-1}\hat{G}).$$

It is seen from (5.59) and (5.60) that the eigenfunctions of A + BF are of the form

$$(5.66) \qquad \bar{g} = \begin{pmatrix} (1, e^{\mu\theta}) \\ \mu + \frac{\pi}{2} e^{-\mu} \end{pmatrix}$$

where μ is a root of the characteristic function of $A + BF$ (see (5.56) or (5.57)). In order to find $(\lambda-A)^{-1}\hat{G}$, we have to solve the following equation for $\lambda \in \rho(A)$:

$$(5.67) \qquad (\lambda-A) \begin{pmatrix} (\varphi_0, \varphi) \\ \alpha \end{pmatrix} = \begin{pmatrix} (1, e^{\mu\theta}) \\ \mu + \frac{\pi}{2} e^{-\mu} \end{pmatrix} \quad (\varphi \in H^1[-1, 0], \varphi(0) = \varphi_0).$$

The equation (5.66) is equivalent to

$$(5.68.1) \qquad \lambda\varphi_0 + \frac{\pi}{2} \varphi(-1) - \alpha = 1$$

$$(5.68.2) \qquad \lambda\varphi(\theta) - \varphi'(\theta) = e^{\mu\theta} \qquad\qquad (-1 \leq \theta \leq 0)$$

$$(5.68.3) \qquad \lambda\alpha = \mu + \frac{\pi}{2} e^{-\mu}.$$

We can immediately solve (5.68.2):

$$(5.69) \qquad \varphi(\theta) = e^{\lambda\theta}\varphi_0 - \frac{e^{\lambda\theta} - e^{\mu\theta}}{\lambda-\mu}.$$

Using this in (5.68.1), we get

$$(5.70) \qquad (\lambda + \frac{\pi}{2} e^{-\lambda})\varphi_0 = \frac{\pi}{2} \frac{e^{-\lambda} - e^{-\mu}}{\lambda-\mu} + \alpha + 1.$$

Multiplying through by λ and using (5.68.3), we obtain

$$(5.71) \qquad \Delta_A(\lambda)\, \varphi_0 = \frac{\pi}{2} \lambda \frac{e^{-\lambda} - e^{-\mu}}{\lambda-\mu} + \mu + \frac{\pi}{2} e^{-\mu} + \lambda =$$

$$= \frac{\lambda}{\lambda-\mu} (\lambda + \frac{\pi}{2} e^{-\lambda} - (\mu + \frac{\pi}{2} e^{-\mu})) + \mu + \frac{\pi}{2} e^{-\mu} =$$

$$= \frac{1}{\lambda-\mu} (\Delta_A(\lambda) - \Delta_A(\mu)).$$

We finally arrive at the following formula:

$$(5.72) \qquad C(\lambda-A)^{-1}\bar{g} = \frac{1}{\lambda-\mu}(1-(\Delta_A(\lambda)^{-1}\Delta_A(\mu)).$$

Suppose now that $\hat{G}$ is given by

$$(5.73) \qquad \hat{G}1 = \sum_{k=1}^{m} \alpha_k \begin{pmatrix} (1,e^{\mu_k\theta}) \\ \mu_k + \frac{\pi}{2}e^{-\mu_k} \end{pmatrix}.$$

Then we have, by linearity,

$$(5.74) \qquad C(\lambda-A)^{-1}\hat{G} = \sum_{k=1}^{m} \alpha_k(\lambda-\mu)^{-1}(1-(\Delta_A(\lambda))^{-1}\Delta_A(\mu_k)).$$

Inserting this into (5.65) gives the explicit formula that we wanted:

$$(5.75) \qquad \Delta_{A+\hat{G}C}(\lambda) = \Delta_A(\lambda) - \sum_{k=1}^{m} \frac{\alpha_k}{\lambda-\mu}(\Delta_A(\lambda) - \Delta_A(\mu)).$$

This formula enables us to compute the eigenvalues of $A + \hat{G}C$ when $\hat{G}$ is of the form (5.73). Of course, this has to be done numerically. For our purposes, a simple Newton procedure will be sufficient, because the known eigenvalues of $A + GC$ give good initial guesses for the eigenvalues of $A + \hat{G}C$ when $\hat{G}$ is close to G.

We note that $A + \hat{G}C$ satisfies the spectrum determined growth assumption for any $\hat{G}$. This is because $A + \hat{G}C$ is the adjoint of an operator to which the compactness result of DELFOUR & MITTER (1972) applies.

If we define the mapping $\hat{G}$ by (5.64), then the first eigenvalues of $A + \hat{G}C$ can be listed as follows (compared to those of $A + GC$):

roots of $\Delta_{A+GC}(\lambda)$	roots of $\Delta_{A+\widehat{G}C}(\lambda)$
-1.571 (double)	-1.491 $\pm$ 0.288 i
-3.142	-3.401
-1.604 $\pm$ 7.647 i	-1.609 $\pm$ 7.854 i
-2.198 $\pm$ 13.98 i	-2.197 $\pm$ 14.14 i
-2.567 $\pm$ 20.29 i	-2.565 $\pm$ 20.42 i
-2.835 $\pm$ 26.60 i	-2.833 $\pm$ 26.70 i
-3.046 $\pm$ 32.89 i	-3.045 $\pm$ 32.99 i
-3.220 $\pm$ 39.19 i	-3.219 $\pm$ 39.27 i
-3.368 $\pm$ 45.48 i	-3.367 $\pm$ 45.55 i
-3.497 $\pm$ 51.77 i	-3.497 $\pm$ 51.84 i
-3.612 $\pm$ 58.06 i	-3.611 $\pm$ 58.12 i

Table 5.2. Effects of perturbation of G.

We see that A + $\widehat{G}$C generates a stable semigroup. Consequently, it turns out that we are able to construct a stabilizing compensator of order 3 for our system (5.47). The eigenvalues of the extended system will be those of A + $\widehat{G}$C (see the table above) together with the eigenvalues of A + BF corresponding to the eigenvectors used in the approximation; these are the eigenvalues at -1 and $-1 \pm \frac{\pi}{2}$i. The growth constant of the resulting closed-loop system will be exactly equal to -1.

STEP 5   The form of the compensator is (in sloppy notation - restriction symbols are omitted):

(5.76.1) $w'(t) = (A+BF+\widehat{G}C)w(t) - \widehat{G}Cx(t)$

(5.76.2) $u(t) = Fw(t)$

where the state space of $w(t)$ is the three-dimensional subspace of $M_2(-1,0) \oplus \mathbb{R}$ spanned by the vectors

(5.77) $w_1 = \begin{pmatrix} (1, e^{-\theta} \cos \frac{\pi}{2}\theta) \\ -1 \end{pmatrix}$, $w_2 = \begin{pmatrix} (0, e^{-\theta} \sin \frac{\pi}{2}\theta) \\ \frac{1}{2}\pi (1-e) \end{pmatrix}$, $w_3 = \begin{pmatrix} (1, e^{-\theta}) \\ -1+\frac{1}{2}\pi e \end{pmatrix}$.

The coordinates of $\widehat{G}$ with respect to this basis are given by (5.64):

$$(5.78) \qquad \hat{G} = -\pi \begin{pmatrix} -0.66 \\ 2.89 \\ 2.66 \end{pmatrix} = \begin{pmatrix} 2.08 \\ -9.08 \\ -8.36 \end{pmatrix}.$$

The matrices of $A + BF$ and C are easily found:

$$(5.79) \qquad A + BF = \begin{pmatrix} -1 & \frac{1}{2}\pi & 0 \\ -\frac{1}{2}\pi & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad C = (1 \quad 0 \quad 1).$$

To complete the design, we need the numerical values of Fw_1, Fw_2 and Fw_3. These could be computed using the explicit form of F given in (5.55), but it is considerably easier to combine (5.58.3) and (5.60), which gives the following formula for an eigenvector ψ of $A + BF$ (normed such that $C\psi = 1$) corresponding to an eigenvalue μ:

$$(5.80) \qquad F\psi = \Delta_A(\mu).$$

Taking real and imaginary parts, we get

$$(5.81.1) \qquad Fw_1 = 1 - \frac{\pi^2}{4} + \frac{\pi^2}{4}e = 5.24$$

$$(5.81.2) \qquad Fw_2 = -\pi + \frac{\pi e}{2} = 1.13$$

$$(5.81.3) \qquad Fw_3 = 1 - \frac{\pi e}{2} = -3.27.$$

It is interesting to note that the explicit formula (5.55) has not been used at all in the design procedure. This means that it is not necessary to compute the projection corresponding to the spectral decomposition of A. We leave it as a topic for further research to see under what general circumstances the computation of the spectral projection can be avoided.

The compensator equations are finally found to be the following:

$$(5.82.1) \qquad w'(t) = \begin{pmatrix} 1.08 & 1.57 & 2.08 \\ -10.65 & -1. & -9.08 \\ -8.36 & 0. & -9.36 \end{pmatrix} w(t) + \begin{pmatrix} -2.08 \\ 9.08 \\ 8.36 \end{pmatrix} y(t)$$

$$(5.82.2) \qquad u(t) = (5.24 \quad 1.13 \quad -3.27)w(t).$$

The result of some simulations using this compensator are given in the Appendix, Fig.A2.

CHAPTER 6

TRACKING AND REGULATION IN INFINITE DIMENSIONS

In this chapter, we consider problems that arise when certain system
variables are to follow a signal of a prescribed form, or when they are to
be made insensitive to such a signal. These problems of 'tracking' and 'regu-
lation' can be brought into the same mathematical framework, which we studied
for finite-dimensional systems in Ch.3. Here, we shall use the same class
of infinite-dimensional systems as in the previous chapter, but we shall
assume that the model for the exogenous signal is finite-dimensional. It is
required that the tracking or regulation task will be carried out by a com-
pensator of finite order. We shall give an existence result and a design
procedure, and examples will serve to illustrate the method.

We start with a brief introduction in Section 1. A precise formulation
of the problems we shall consider is given in Section 2. The basic construc-
tion method, that will be presented in Section 3, is used in Section 4 to
derive an existence theorem. A certain 'transmission' condition enters into
the statement of this theorem, and we indicate an important class of systems
for which this condition holds. Next, Section 5 presents a step-wise proce-
dure to construct a (low-order) compensator for a given tracking or regulation
problem. The final three sections contain examples. In Section 6, we have
a diffusion system with a constant reference signal to be tracked. In Section
7 we construct a finite-dimensional compensator that protects a delay system
against a constant disturbance, and finally, in Section 8, we have again a
diffusion system but now the signal to be followed is not a constant but a
sinusoid.

6.1 Introduction

We want to study problems in which a given infinite-dimensional system
has to follow or reject a prescribed signal. The signal will be modeled by
a finite-dimensional linear system, of which the initial condition is left
free; this includes, for instance, a sine wave of known frequency but unknown
phase and amplitude. The models of the system and the signal will be taken
together in one larger system, whose state space is the direct sum of the

state space of the original system and that of the signal model. The tracking or rejection task is expressed via a linear operator, which maps the state variable onto the "variable-to-be-controlled". For instance, if the temperature at a certain point in a room is to be made equal to a constant reference variable, the variable-to-be-controlled would be the difference between the temperature at the given point and the reference value. By adding a dynamic compensator, we want to accomplish that the variable-to-be-controlled will return to zero after an initial disturbance. Moreover, we shall require that the system as a whole will be stable, with the exception of that part that represents the exogenous signal.

The essence of our approach will be to combine basic ideas from Ch. 3 with the finite-dimensional compensator design procedure of Ch. 5. In BHAT (1976) and POHJOLAINEN (1980), one finds other state-space approaches to the theory of tracking and regulation for infinite-dimensional systems. Bhat follows closely the lines of WONHAM (1974) and gives applications to delay systems. His proposed compensators are of infinite order, which in the case of the delay examples means that they require an integration of the state variable over the delay interval. Motivated by the work of E.J.Davison and colleagues, Pohjolainen gives controllers of finite order. However, a basic assumption in his work is that the original system is stable. Pohjolainen's examples are of diffusion type, and his theorems are formulated for self-adjoint parabolic equations. However, the observation operator C is not required to be bounded but only to be relatively bounded with respect to A.

The main benefit of our approach contrasted with Bhat's is that it leads to finite-dimensional compensators; compared with the work of Pohjolainen, our method has the advantage that it combines stabilizing and regulating ability. On the other hand, we do not consider the problem of robustness which is central both in the work of Bhat and in the work of Pohjolainen.

6.2 Problem formulation

We shall denote the state space of the "original system" by X_1, and that of the "exogenous signal" by X_2. The "variable-to-be-controlled" will be written as $z(t)$. We consider the following system equations:

$$(6.1.1) \qquad x_1'(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \qquad x_1(t) \in X_1, \; u(t) \in U$$

(6.1.2) $x_2'(t) = A_{22}x_2(t)$ $x_2(t) \in X_2$

(6.1.3) $y(t) = (C_1 \quad C_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $y(t) \in Y$

(6.1.4) $z(t) = (D_1 \quad D_2)\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ $z(t) \in Z.$

We can write these equations in a more concise form if we introduce

(6.2.1) $X = X_1 \oplus X_2, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

(6.2.2) $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$

(6.2.3) $C = (C_1 \quad C_2), \quad D = (D_1 \quad D_2).$

The system (6.1) then becomes:

(6.3.1) $x'(t) = Ax(t) + Bu(t)$

(6.3.2) $y(t) = Cx(t)$

(6.3.3) $z(t) = Dx(t).$

Let us describe the precise setting in which we want to use these equations. Our conditions for the original system will be the same as before, so that the results below will apply to the same class, including many systems of parabolic and delay type, and also some hyperbolic and neutral systems. Thus, X_1 is a Banach space and A_{11} generates a semigroup $T_1(t)$ on X_1 (A1), the spectrum of A_{11} is discrete (A4), there are only finitely many eigenvalues of A_{11} to the right of some vertical line Re $\lambda = \omega-\delta$ (A5), the operator A_{11}^s (obtained by spectral decomposition in the space X_1) satisfies the spectrum determined growth assumption (A6), and the eigenvectors of A_{11} are complete (A9). The space X_2 is a finite-dimensional linear space, A_{22} is a linear mapping acting on X_2, and A_{12} is a bounded linear mapping from X_2 into X_1.

Under these circumstances, it follows from Prop. 4.7 (after interchanging the indices 1 and 2) that the conditions (A1,4-6,9) also hold for the

operator A defined in (6.2.2). Conversely, suppose that A is an operator acting on a Banach space X which has a direct sum decomposition $X = X_1 \oplus X_2$, where X_2 is a finite-dimensional subspace of D(A), and suppose that Ax $\in X_1$ for all x $\in$ D(A) $\cap X_1$. Then A can be written in the form (6.2.2), and if the conditions (A1,4-6,9) hold for A then they also hold for A_{11}. Thus it is equivalent to require (A1,4-6,9) to be satisfied by A_{11} or by A, and we shall use the latter alternative because it leads to a formulation that corresponds nicely to the one used in Ch. 5.

Naturally, the matrix A_{22}, which represents the signal dynamics, will be unstable. It is then obvious from (6.2.2) that the pair (A,B) is not stabilizable. What we do need to require, however, is that the pair (A_{11}, B_1) is stabilizable. To get a verifiable condition for this, we use the spectral decomposition in X_1 in order to introduce the controllability of the finite-dimensional pair (A_{11}^u, B_1^u) as an assumption.

Summarizing, we shall use the following extra conditions, which are formulated for the system (6.3):

(B1) X has a direct sum decomposition $X = X_1 \oplus X_2$ such that dim $X_2 < \infty$, $X_2 \subset$ D(A), and Ax $\in X_1$ for all x $\in$ D(A) $\cap X_1$, so that we have

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

(B2) Z is a finite-dimensional linear space, and D is a bounded linear mapping from X onto Z

(B3) the range of B is contained in X_1, so that we have $B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ with respect to the decomposition $X = X_1 \oplus X_2$

(B4) the pair (A_{11}^u, B_1^u) is controllable.

Most of these conditions follow naturally from the characteristics of the tracking and regulation problem. Only the boundedness of D could be weakened, for instance to allow point measurements in an L_2-space.

To the system (6.3), we are going to add a compensator of the usual form:

(6.4.1) $w'(t) = Nw(t) + My(t)$ $w(t) \in W$

(6.4.2) $u(t) = Lw(t)$.

The dimension of W is finite and is called the order of the compensator; L, M and N are linear mappings between the appropriate spaces. Combining (6.3) and (6.4), we obtain the following closed-loop system:

(6.5.1) $\dfrac{d}{dt} \begin{pmatrix} x \\ w \end{pmatrix}(t) = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}(t)$

(6.5.2) $z(t) = (D \quad 0) \begin{pmatrix} x \\ w \end{pmatrix}(t)$.

We shall write

(6.6) $A_e = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}$, $D_e = (D \quad 0)$.

The purpose of the compensation has been described in general terms in the preceding section; we now want to make this precise. We shall say that the compensator (6.4) solves the *regulation problem* for the system (6.3), or that (6.4) gives a *regulating compensator*, if the following is true. There is a subspace $V \subset D(A_e)$ such that $A_e V \subset V$ and dim V = dim X_2. The restriction of A_e to V is similar to A_{22}. The operator A_e is the generator of a semigroup $T_e(t)$ on $X \oplus W$, and the induced semigroup $\overline{T}_e(t)$ on $(X \oplus W)/V$ is stable. Finally, the subspace V is contained in Ker D_e.

This says that the closed-loop system is stable 'modulo the signal dynamics' (stabilization property). The requirement "$V \subset$ Ker D_e" means that the variable-to-be-controlled is a function only of the stable part of the system (regulation property).

The regulation problem as formulated above is rather simple compared with some of the regulation problems that we studied in the finite-dimensional context. In particular, we do not differentiate between the stability requirement for the variables-to-be-controlled and that for the stabilizable part of the sytem as a whole, like we did in Ch. 3. It would not be impossible to introduce this distinction in the infinite-dimensional situation, but this would only make the theory more complex, while the basic features are already present in the problem as we posed it.

6.3 The basic theorem

We now give sufficient conditions for a regulating compensator of order k to exist. The proof is by construction.

Theorem 6.1 *Consider the system (6.3) under the assumptions (A1-3) and (B1-3). Assume that there exist bounded mappings* $F:X \to U$ *and* $G: Y \to X$ *together with finite-dimensional subspaces* $V_s \subset D(A)$ *and* $V_c \subset D(A)$, *such that the following holds:*

(6.7.1) $V_s \subset V_c \cap \text{Ker } D$

(6.7.2) $X = X_1 \oplus V_s$

(6.7.3) $(A+BF)x \in V_s$ for all $x \in V_s$

(6.7.4) $(A+BF)x \in V_c$ for all $x \in V_c$

(6.7.5)   the quotient semigroup generated by $A + BF$ on X/V_s is stable

(6.7.6)   the semigroup generated by $A + GC$ is stable

(6.7.7) $\text{Im } G \subset V_c$.

Then there exists a regulating compensator of order k, *where* $k = \dim V_c$.

PROOF   Introduce a new linear space W isomorphic to V_c, and let R: $V_c \to W$ be the mapping that provides the isomorphism. Define a compensator of the form (6.4) by setting $L = FR^{-1}$, $M = -RG$ (well-defined by (6.7.7)) and $N = R(A+BF+GC)R^{-1}$ (well-defined by (6.7.4) and (6.7.7)). We obtain the following extended system operator:

(6.8) $A_e = \begin{pmatrix} A & BFR^{-1} \\ -RGC & R(A+BF+GC)R^{-1} \end{pmatrix}$.

Consider the following subspaces of $X^e := X \oplus W$:

(6.9) $V = \{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in V_s \}$

$$(6.10) \qquad M = \{ (\begin{smallmatrix} x \\ Rx \end{smallmatrix}) \mid x \in V_c \} = \{ (\begin{smallmatrix} R^{-1}w \\ w \end{smallmatrix}) \mid w \in W \}$$

Note that V is well-defined by (6.7.1). Both V and M are contained in $D(A_e) = D(A) \oplus W$. There is an obvious isomorphism from W to M, given by

$$(6.11) \qquad T: w \mapsto (\begin{smallmatrix} R^{-1}w \\ w \end{smallmatrix}).$$

Considering X, W and M as subspaces of X^e, we note that X^e can be decomposed either as $X \oplus W$ or as $X \oplus M$. The similarity transormation from one decomposition to the other is given by

$$(6.12) \qquad H: X \oplus W \to X \oplus M, \quad H(\begin{smallmatrix} x \\ w \end{smallmatrix}) = (\begin{smallmatrix} x-R^{-1}w \\ Tw \end{smallmatrix}).$$

Written in matrix format, we have

$$(6.13) \qquad H = (\begin{smallmatrix} I & -R^{-1} \\ 0 & T \end{smallmatrix}), \quad H^{-1} = (\begin{smallmatrix} I & R^{-1}T^{-1} \\ 0 & T^{-1} \end{smallmatrix}).$$

By straightforward computation, we find the following form for the extended system operator with respect to the 'basis' $X \oplus M$:

$$(6.14) \qquad \tilde{A}_e = HA_e H^{-1} = (\begin{smallmatrix} A+GC & 0 \\ -TRGC & TR(A+BF)R^{-1}T^{-1} \end{smallmatrix}).$$

By Prop. 4.7 and Lemma 4.5, this shows that A_e generates a semigroup $T_e(t)$ on X^e. The mapping TR gives an isomorphism between V_c and M, and it is seen from (6.14) that the following diagram commutes:

$$(6.15)$$

$$
\begin{CD}
V_c @>{A+BF : V_c}>> V_c \\
@V{TR}VV @VV{TR}V \\
M @>>{A_e : M}> M
\end{CD}
$$

The image of V_s under TR is V. Hence, it follows from (6.15) and (6.7.3) that V is A_e-invariant, and that A_e: $V = A + BF$: V_s. Now let Q: $V_c \to V_c/V_s$ be the factor mapping, and define a mapping J: $(X \oplus W)/V \to X \oplus (V_c/V_s)$ by

(6.16) $J: [\begin{pmatrix} x \\ w \end{pmatrix}] \mapsto \begin{pmatrix} x-R^{-1}w \\ QR^{-1}w \end{pmatrix}.$

It is easily verified that J is well-defined and that it gives a bijection between $(X \oplus W)/V$ and $X \oplus (V_c/V_s)$. Assuming, without loss of generality, that $R: V_c \to W$ is an isometry, we can make the following estimates for any $x_0 \in V_s$:

(6.17.1) $\|x-R^{-1}w\| \leq \|x-x_0\| + \|R^{-1}w-x_0\| = \|x-x_0\| + \|w-Rx_0\|$

(6.17.2) $\|QR^{-1}w\| \leq \|R^{-1}w-x_0\| = \|w-Rx_0\|.$

It follows that

(6.18) $\left\|\begin{pmatrix} x-R^{-1}w \\ QR^{-1}w \end{pmatrix}\right\| \leq 2\left\|\begin{pmatrix} x \\ w \end{pmatrix} - \begin{pmatrix} x_0 \\ Rx_0 \end{pmatrix}\right\|$ for all $x_0 \in V_s.$

Using the definition of the norm in $(X \oplus W)/V$ and (6.9), we see that

(6.19) $\left\|\begin{pmatrix} x-R^{-1}w \\ QR^{-1}w \end{pmatrix}\right\| \leq 2\left\|[\begin{pmatrix} x \\ w \end{pmatrix}]\right\|$

or, J is bounded. By the Banach open mapping theorem (TAYLOR & LAY (1980; p.212-213)), J^{-1} must be bounded too, and so J is a similarity transformation between $(X \oplus W)/V$ and $X \oplus (V_c/V_s)$. Let us write $\bar{A}_e$ for the mapping induced by A_e on $(X \oplus W)/V$. It is straightforward to verify that $J[D(\bar{A}_e)] = D(A) \oplus (V_c/V_s)$, and that the following relation holds for $x \in D(A)$, $v \in V_c$:

(6.20) $J\bar{A}_e J^{-1} \begin{pmatrix} x \\ Qv \end{pmatrix} = \begin{pmatrix} A+GC & 0 \\ -QGC & A+BF:V_c/V_s \end{pmatrix}\begin{pmatrix} x \\ Qv \end{pmatrix}.$

Using (6.7.5) and (6.7.6), we see that the operator appearing on the right-hand side of this equation generates a stable semigroup. It follows from (6.20) that the semigroup $\bar{T}_e(t)$ is similar to this semigroup, and hence $\bar{T}_e(t)$ is stable.

We have already shown that $A_e: V$ is similar to $A + BF: V_s$, so now we want to prove that $A + BF: V_s$ is similar to A_{22}. It follows from (6.7.2) that $\dim V_s = \dim X_2$. Define a mapping $\bar{P}: V_s \to X_2$ by

$$(6.21) \qquad \bar{P}\binom{x_1}{x_2} = x_2 \qquad\qquad (\binom{x_1}{x_2} \in V_s \subset X = X_1 \oplus X_2).$$

By (6.7.2), this mapping is a bijection. Its inverse is clearly of the form

$$(6.22) \qquad \bar{P}^{-1}x = \binom{Sx}{x} \in X = X_1 \oplus X_2$$

where S is some linear mapping from X_2 into X_1. Now let us compute, for $x \in X_2$:

$$(6.23) \qquad \bar{P}(A+BF)\bar{P}^{-1}x = \bar{P}\begin{pmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_1 \\ 0 & A_{22} \end{pmatrix}\binom{Sx}{x} = A_{22}x.$$

This gives the desired similarity between A + BF: V_s and A_{22}. Finally, we have to prove that $V \subset \mathrm{Ker}\ D_e$. This follows immediately from (6.7.1) and (6.9). ☒

The theorem is not directly applicable; we need to know how to find mappings F and G and subspaces V_s and V_c such that the conditions of the theorem are satisfied. This question is treated in the next section.

6.4 The existence result

Our main result in this section is the following.

THEOREM 6.2 *Consider the system (6.3) under the assumptions (A1-6,8-9) and (B1-4). If there exists a linear mapping* S: $X_2 \to X_1$, *such that*

$$(6.24.1) \qquad \mathrm{Im}\ S \subset D(A_{11})$$

$$(6.24.2) \qquad \mathrm{Im}(A_{11}S-SA_{22}+A_{12}) \subset \mathrm{Im}\ B_1$$

$$(6.24.3) \qquad D_1S + D_2 = 0$$

then there exists a regulating compensator of finite order.

The conditions (6.24) will be discussed at the end of this section. The class of systems to which the theorem applies has been described in Section 4.4. For the proof of the theorem we need the following lemma.

LEMMA 6.3 *Consider the system (6.3) under the assumptions (A1-3) and (B1-3)*
Suppose that there exist bounded linear mappings $F_1: X_1 \to U$ *and* $G: Y \to X$
and a finite-dimensional subspace $V_s \subset D(A)$, *such that the following holds:*

(6.25.1) $V_s^i \subset \text{Ker } D$

(6.25.2) $X = X_1 \oplus V_s$

(6.25.3) $AV_s \subset V_s + \text{Im } B$

(6.25.4) $A_{11} + B_1 F_1$ generates a stable semigroup on X_1

(6.25.5) $A_{11} + B_1 F_1$ has a discrete spectrum, and its eigenvectors form
a complete base in X

(6.25.6)   there exists $\delta > 0$ such that the semigroup generated by
$A + GC + \delta I$ is stable.

Then there exists a regulating compensator of finite order.

PROOF It follows from (6.25.2) that the mapping $\bar{P}: V_s \to X_2$ defined by

(6.26) $\bar{P}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2$ $\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in V_s\right)$

is a bijection. Let us define a mapping $S: X_2 \to X_1$ by

(6.27) $\bar{P}^{-1}x = \begin{pmatrix} Sx \\ x \end{pmatrix}$ $(x \in X_2).$

Then we have

(6.28) $V_s = \left\{ \begin{pmatrix} Sx \\ x \end{pmatrix} \mid x \in X_2 \right\}.$

Let $\{x_1, .., x_r\}$ be a basis for X_2. From (6.25.3), we see that for every
$i = 1, .., r$ there exists $u_i \in U$ such that

(6.29) $A\begin{pmatrix} Sx_i \\ x_i \end{pmatrix} + Bu_i \in V_s.$

Now define $F_2: X_2 \to U$ by

$$(6.30) \qquad F_2 x_i = u_i - F_1 S x_i \qquad\qquad (i=1,..,r).$$

Using the mapping F_1 that is given in the statement of the theorem, we form F: $X \to U$ by putting $F = (F_1 \quad F_2)$. We then have, for $i = 1,..,r$:

$$(6.31) \qquad (A+BF)\begin{pmatrix} Sx_i \\ x_i \end{pmatrix} = A\begin{pmatrix} Sx_i \\ x_i \end{pmatrix} + B(F_1 Sx_i + F_2 x_i) = A\begin{pmatrix} Sx_i \\ x_i \end{pmatrix} + Bu_i \in V_s$$

by (6.29). Thus, V_s is (A+BF)-invariant.

With respect to the decomposition $X = X_1 \oplus V_s$, write

$$(6.32) \qquad G = \begin{pmatrix} G_1 \\ G_s \end{pmatrix}.$$

Let S(t) denote the semigroup generated by A + GC. By (6.25.6), the growth constant of S(t) is less than or equal to $\omega - \delta$ (where ω denotes the growth constant that defines our notion of 'stability'). Consequently, there exists a constant M such that

$$(6.33) \qquad \|S(t)\| \leq M \exp((\omega - \tfrac{1}{2}\delta)t)$$

If $\hat{G}_1 : Y \to X_1$ is such that

$$(6.34) \qquad \|G_1 - \hat{G}_1\| \leq \tfrac{1}{2}M^{-1} \|C\|^{-1}\delta$$

and if we define $\hat{G}: Y \to X$ by

$$(6.35) \qquad \hat{G} = \begin{pmatrix} \hat{G}_1 \\ G_s \end{pmatrix},$$

then the estimate (6.34) also holds for $\|G - \hat{G}\|$. Therefore, the operator A + $\hat{G}$C will generate a semigroup $\hat{S}(t)$ with

$$(6.36) \qquad \|\hat{S}(t)\| \leq M \exp(\omega t)$$

(Lemma 4.4). By following further the procedure used in the proof of Lemma 5.3, we obtain a finite-dimensional subspace $V_{c1} \subset X_1$ with the following properties: V_{c1} is invariant under $A_{11} + B_1 F_1$, and there is a mapping $\hat{G}_1 : Y \to X_1$ satisfying (6.34) and Im $\hat{G}_1 \subset V_{c1}$.

Now define $\hat{G}$ by (6.35) and V_c by

(6.37) $V_c = V_{c1} \oplus V_s.$

Then (6.36) shows that $A + \hat{G}C$ generates a stable semigroup, and furthermore we have $\text{Im } \hat{G} \subset V_c$ and $V_c \subset V_s$. From the fact that V_{c1} is invariant under $A_{11} + B_1F_1$, and from the form of $A + BF$ as given by

(6.38) $A + BF = \begin{pmatrix} A_{11}+B_1F_1 & A_{12}+B_1F_2 \\ 0 & A_{22} \end{pmatrix}$

it follows immediately that V_{c1} (considered as a subspace of X) is $(A+BF)$-invariant. We already had the $(A+BF)$-invariance of V_s, and so it follows from (6.37) that V_c is invariant under $A + BF$.

Finally, it is easily seen that the mapping

(6.39) $H: X_1 \to X/V_s, \quad Hx = [\begin{pmatrix} x \\ 0 \end{pmatrix}]$

provides a similarity transformation between X_1 and X/V_s. Moreover, the following diagram commutes:

(6.40)

$$
\begin{array}{ccc}
D(A_{11}) & \xrightarrow{A_{11}+B_1F_1} & X_1 \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle H} \\
D(A+BF) & \xrightarrow[A+BF:X/V_s]{} & X/V_s
\end{array}
$$

Hence, the quotient semigroup generated by $A + BF$ on X/V_s is stable.

The conditions of Thm. 6.1 are satisfied for the mappings F and $\hat{G}$ and the subspaces V_s and V_c. Consequently, a regulating compensator of finite order (equal to $\dim V_c$) exists. $\boxtimes$

We are now ready to prove the theorem.

PROOF (of Thm.6.2) Define a subspace $V_s \subset X$ by

(6.41) $V_s = \{\begin{pmatrix} Sx \\ x \end{pmatrix} \mid x \in V_2\}.$

Clearly, we have $X = X_1 \oplus V_s$. It follows from (6.24.3) that $V_s \subset \text{Ker } D$. Because of (6.24.1), V_s is contained in $D(A)$. Moreover, we obtain from (6.24.2):

$$(6.42) \qquad A\begin{pmatrix} Sx \\ x \end{pmatrix} = \begin{pmatrix} A_{11}Sx+A_{12}x \\ A_{22}x \end{pmatrix} = \begin{pmatrix} SA_{22}x \\ A_{22}x \end{pmatrix} + \begin{pmatrix} (A_{11}S-SA_{22}-A_{12})x \\ 0 \end{pmatrix} \in V_s + \text{Im } B$$

for each $x \in X_2$. This means that $AV_s \in V_s + \text{Im } B$.

Applying Lemma 5.4 to the pair (A_{11},B_1), we find that there exists a mapping $F_1: U \to X_1$ such that $A_{11} + B_1F_1$ is the generator of a stable semigroup on X_1, and moreover $A_{11} + B_1F_1$ has a discrete spectrum and its eigenvectors form a complete set in X_1. It follows from assumptions (A4) and (A5) that there exists $\delta > 0$ such that $\{\lambda \in \sigma(A) \mid \text{Re } \lambda \geq \omega\} = \{\lambda \in \sigma(A) \mid \text{Re } \lambda > \omega-\delta\}$. We can do a spectral decomposition with respect to this subset of $\sigma(A)$. If the resulting finite-dimensional pair (C_1,A_{11}) is observable, the procedure of the proof of Prop. 4.13 leads to an injection mapping $G: Y \to X$ such that the semigroup generated by $A + GC$ has a growth constant $\leq \omega-\delta$. In this case, we have satisfied all assumptions of Lemma 6.3 and so the existence of a finite-dimensional regulating compensator follows. If the pair (C_1,A_{11}) is not observable, we use the remark made after the proof of Lemma 5.3 to obtain the same result. $\boxtimes$

The class of systems to which the existence theorem applies is determined by several factors. The 'exogenous signal' has to be modeled by a finite-dimensional system, and the 'original system' has to satisfy the conditions discussed in Section 4.4. Moreover, we have the usual stabilizability and detectability conditions, but there is also the set of conditions (6.24) which is related to the interaction of the control function and the variables-to-be-controlled via the system dynamics. Equations of the type (6.24.2) have been studied in the finite-dimensional context in GANTMACHER (1959; p.225). In the infinite-dimensional case involving unbounded operators, KREIN (1971; p.316) gives a solution under the assumption $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$. It should be noted, however, that the equations (6.24) may have a solution even if this assumption does not hold (see the remark following Example III). We shall not go into the difficult questions that arise here. The following result is simple but still useful in many applications.

PROPOSITION 6.4 *Consider the system (6.3) under the assumptions (A1-3)*
and (B1-3). If A_{22} *is a diagonalizable matrix, if* $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$ *and*
if the matrix

$$(6.43) \qquad M(\lambda) = D_1(\lambda-A_{11})^{-1}B_1 : U \rightarrow Z \qquad (\lambda \in \rho(A))$$

has a right inverse $M^+(\lambda)$ *for each* $\lambda \in \sigma(A_{22})$*, then there is a mapping*
$S: X_2 \rightarrow X_1$ *such that (6.24) holds.*

PROOF Let $\{x_1,..,x_r\}$ be a basis for X_2 consisting of eigenvectors of A_{22},
and let $\{\lambda_1,..,\lambda_r\}$ be the corresponding eigenvalues. Define $u_i \in U$ (i=1,..,r)
by

$$(6.44) \qquad u_i = M^+(\lambda_i)(D_1(\lambda_i-A_{11})^{-1}A_{12} + D_2)x_i.$$

Define a linear mapping $S: X_2 \rightarrow X_1$ by

$$(6.45) \qquad Sx_i = (\lambda_i-A_{11})^{-1}(A_{12}x_i - B_1u_i) \qquad (i=1,..,r).$$

It is clear that Im $S \subset D(A_{11})$. For each i = 1,..,r we have

$$(6.46) \qquad A_{11}Sx_i - SA_{22}x_i + A_{12}x_i = (A_{11}-\lambda_i)Sx_i + A_{12}x_i = B_1u_i \in \text{Im } B_1.$$

And finally,

$$(6.47) \qquad D_1Sx_i + D_2x_i = D_1(\lambda_i-A_{11})^{-1}A_{12}x_i -$$
$$- D_1(\lambda_i-A_{11})^{-1}B_1M^+(\lambda_i)(D_1(\lambda_i-A_{11})^{-1}A_{12}x_i + D_2x_i) +$$
$$+ D_2x_i = 0 \qquad (i=1,..,r).$$

Thus, the mapping S satisfies all requirements. ⊠

The matrix $M(\lambda)$ will never be right invertible (that is, surjective)
if dim U < dim Z, which means that the number of control inputs is smaller
than the number of variables-to-be-controlled. On the other hand, $M(\lambda)$ will
in general be right invertible if the number of control inputs is larger
than or equal to the number of variables-to-be-controlled. This corresponds

very well to intuition, of course.

As will be illustrated in the examples, the mapping S and the associated subspace V_s represent the situation in which the system behaves exactly as we want it to; in this sense, they have the same meaning for the regulation problem as the state space origin has for the stabilization problem. The invertibility properties of the matrix $M(\lambda)$ are related to the amount of steering that is necessary in the 'ideal situation'. See in particular example III for this interpretation of $M(\lambda)$.

In the special case of a constant exogenous signal $(A_{22} = 0 \in \mathbb{R}^{1 \times 1})$, the condition of Prop. 6.4 means that $0 \notin \sigma(A_{11})$ and rank $D_1 A_{11}^{-1} B_1 = \dim Z$. This is the condition given in POHJOLAINEN (1980; Thm. 3.1).

6.5 The design procedure

Assuming that we have a system that satisfies the conditions (A1-6, 8-9) of Section 4.4 and the conditions B(1-4) of Section 6.2, the following step-wise procedure can be given to find a (low-order) regulating compensator. The first step of the procedure is to find a mapping S: $X_2 \to X_1$ such that (6.24) holds. If this can be done, then the existence theorem (Thm.6.2) guarantees that the method will lead to a finite-dimensional compensator after a finite number of iterations.

We shall give the procedure first and then add some comments. Some parts of the procedure are similar to the method we used in Ch. 5 to construct a stabilizing compensator, and the reader is referred to Section 5.4 for the corresponding comments. The procedure gives a basic computational scheme which is certainly amenable to further numerical refinement. In the final three sections of this chapter, we shall illustrate the method by examples.

The design procedure consists of the following steps.

STEP 1 Find a mapping S: $X_2 \to X_1$ such that (6.24) is satisfied.

STEP 2 Find F_1 such that $A_{11} + B_1 F_1$ has a discrete spectrum, the eigen-vectors of $A_{11} + B_1 F_1$ are complete, and the semigroup generated by $A_{11} + B_1 F_1$ is stable.

STEP 3 Choose a basis $\{x_1, \ldots, x_r\}$ for X_2, and select u_i (i=1,..,r) such that $(A_{11}S - SA_{22} + A_{12})x_i = B_1 u_i$. Determine F_2 by $F_2 x_i = -u_i - F_1 S x_i$ (i=1,..,r).

STEP 4 Find G such that the growth constant of the semigroup generated by A + GC is somewhat smaller than the desired growth constant ω.

STEP 5 Write G in the form $G = G_1 + G_s$ with Im $G_1 \subset X_1$ and Im $G_s \subset V_s$ (as defined in (6.28)), and approximate the vectors in Im G_1 by linear combinations of k selected eigenvectors of $A_{11} + B_1F_1$. Form the mapping $\hat{G}_1$ which is close to G_1 and whose range is spanned by these k eigenvectors, and write $\hat{G} = \hat{G}_1 + G_s$.

STEP 6 See if the semigroup generated by A + $\hat{G}$C is stable. If not, select a different F_1 and/or a different G and start anew, or repeat Step 5 with k replaced by k + 1. If the semigroup is stable, go to Step 7.

STEP 7 Construct the compensator of order k as in the proof of Thm. 5.1.

Comments If the conditions of Prop. 6.4 hold, then the problem of Step 1 can be solved by the method used in the proof of that proposition, or by any other suitable method. In any other case, one should consult the literature (GANTMACHER (1959), KREIN (1971)) or look for a direct solution. The second step of the procedure is entirely analogous to Step 1 in the procedure of Section 5.4, and we refer to the remarks that were made there. The purpose of Step 3 is to choose F_2 such that the subspace V_s defined in (6.28) becomes (A+BF)-invariant. If Prop. 6.4 is used, suitable u_i are given by (6.44). The rest of the procedure is again analogous to the design procedure of Section 5.4, with the decomposition of G as the only exception. If G is given initially as

$$(6.48) \qquad G = \binom{G_{10}}{G_{20}}$$

then the decomposition we want is $G = G_1 + G_s$, with

$$(6.49) \qquad G_1 = \binom{G_{10}-SG_{20}}{0}, \quad G_s = \binom{SG_{20}}{G_{20}}.$$

So the approximation has to be done with respect to the vectors in $\text{Im}(G_{10}-SG_{20})$.

6.6 Example III: A heat regulator

For our first example of regulator design, consider the following set of equations.

(6.50.1) $\quad \dfrac{\partial}{\partial t} \varphi(x,t) = \dfrac{1}{\pi^2} \dfrac{\partial^2}{\partial x^2} \varphi(x,t) + b(x) u(t) \qquad (t \geq 0,\ 0 \leq x \leq 1)$

(6.50.2) $\quad \varphi(0,t) = \varphi(1,t) = 0$

(6.50.3) $\quad r'(t) = 0 \hspace{5cm} (r(t) \in \mathbb{R}, t \geq 0)$

(6.50.4) $\quad r(0) = r_0$

(6.50.5) $\quad b(x) = \begin{array}{ll} \sqrt{10} & (0.1 \leq x \leq 0.2) \\ 0 & (0 \leq x < 0.1;\ 0.2 < x \leq 1) \end{array}$

(6.50.6) $\quad y_1(t) = \sqrt{10} \displaystyle\int_{0.8}^{0.9} \varphi(x,t)dx$

(6.50.7) $\quad y_2(t) = r(t)$

(6.50.8) $\quad z(t) = \sqrt{5} \displaystyle\int_{0.4}^{0.6} \varphi(x,t)dx - r(t).$

For instance, these equations can be interpreted in the following way. You are sitting in the middle of a one-dimensional room with windows in both walls. The heater is on one side of the room, and the thermostat is on the other. You want the temperature around your chair to become equal to the reference value r_0 that you have specified, and you want this to happen reasonably fast.

Returning to the abstract framework of the previous sections, we take $L_2(0,1) \oplus \mathbb{R}$ as our state space X. The operator A_{11} is defined on $X_1 := L_2(0,1)$ by

(6.51.1) $\quad D(A_{11}) = \{\varphi \in L_2(0,1) \mid \dfrac{d^2\varphi}{dx^2} \in L_2(0,1),\ \varphi(0) = \varphi(1) = 0\}$

(6.51.2) $\quad A_{11}\varphi = \dfrac{1}{\pi^2} \dfrac{d^2\varphi}{dx^2} \hspace{3cm} (\varphi \in L_2(0,1)).$

Furthermore, we have $A_{12} = 0$ and $A_{22} = 0$. The output space Y will be identified with $\mathbb{R}$, and the output mapping C is defined by

$$(6.52) \qquad C\binom{\varphi}{r} = \begin{pmatrix} \sqrt{10} \displaystyle\int_{0.8}^{0.9} \varphi(x)dx \\ r \end{pmatrix} \qquad (\varphi \in L_2(0,1), \; r \in \mathbb{R}).$$

The input space U is taken equal to $\mathbb{R}$, and the definition of the input mapping B can be derived from (6.50.1) in the obvious way. Finally, the mapping D that gives the variable to-be-controlled is clearly given by

$$(6.53) \qquad D\binom{\varphi}{r} = \sqrt{5} \int_{0.4}^{0.6} \varphi(x)dx - r \qquad (\varphi \in L_2(0,1), \; r \in \mathbb{R}).$$

We said that the reference value should be reached at a reasonably fast rate, so we set the desired growth constant ω at -2. Now we are ready to enter the design procedure.

STEP 1   In the present case, $\sigma(A_{22})$ consists only of the point 0 which is not in the spectrum of A_{11}. Moreover, the value of the function $M(\lambda)$ (defined in (6.43)) in $\lambda = 0$ is given by

$$(6.54) \qquad M(0) = -D_1 A_{11}^{-1} B_1 = 0.1047$$

and so Prop. 6.4 applies. Consequently, we derive the following mapping $S: \mathbb{R} = X_1 \to L_2(0,1) = X_2$ from (6.44) and (6.45):

$$(6.55) \qquad S: 1 \mapsto -(M(0))^{-1}A_{11}^{-1}b := \bar{\varphi}.$$

STEP 2   Since $A_{11}^u = -1$ and B_1^u is given by

$$(6.56) \qquad \beta_1 = \sqrt{10} \int_{0.1}^{0.2} \sqrt{2} \, \sin \pi x \, dx = 0.2022$$

a suitable feedback mapping F_1 that replaces the eigenvalue at -1 by one at -2 and leaves all the other eigenvalues unchanged is given by

$$(6.57) \qquad F_1\varphi = \frac{-1}{\beta_1} \sqrt{2} \int_0^1 \varphi(x) \, \sin \pi x \, dx$$

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(see Prop. 4.12).

STEP 3 Following the recipe given in the design procedure, F_2: $X_2 =$
        $= \mathbb{R} \to U = \mathbb{R}$ is given by

(6.58) F_2: $1 \mapsto (M(0))^{-1}(1 + F_1 A_{11}^{-1} b) = 19.105$.

STEP 4 We have to define a mapping G: $Y = \mathbb{R}^2 \to X = L_2(0,1) \oplus \mathbb{R}$ such that
        A + GC is stable. A basis for the unstable subspace X_u is given by
        the two vectors

(6.59) $\begin{pmatrix} \sqrt{2} \sin \pi x \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

With respect to this basis, we have

(6.60) $A_u = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, $C_u = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix}$,

where

(6.61) $\gamma_1 = \sqrt{10} \int\limits_{0.8}^{0.9} \sqrt{2} \sin \pi x \, dx = 0.2022$.

Perhaps the simplest solution is to retain the triangular form by choosing
G as follows:

(6.62)   G: $\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \mapsto \alpha_1 \begin{pmatrix} -1.5\gamma_1^{-1}\sqrt{2} \sin \pi x \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -2.1\bar{\varphi} \\ -2.1 \end{pmatrix}$

By Prop. 4.13, this choice of G will result in a shift of the eigenvalues at
-1 and 0 to -2.5 and -2.1, respectively, while all other eigenvalues remain
unchanged. We have selected G in such a way that its range is spanned by two
vectors, one of which is already in V_s (as defined by (6.41)). The other
basis vector is in X_1 and so it remains to approach this one by linear
combinations of eigenvectors of $A_{11} + B_1 F_1$.

STEP 5 As approximating vectors, let us take the eigenvectors of $A_{11} + B_1 F_1$
        belonging to the eigenvalues at -2 and -4;

(6.63) $\psi_1 = (-2 - A_{11})^{-1} b$, $\psi_2(x) = \sqrt{2} \sin 2\pi x$.

The best approximation (in $L_2(0,1)$-sense) of $g_{11} := -1.5\gamma_1^{-1}\sqrt{2} \sin \pi x$ can be computed as follows:

(6.64) $\hat{g}_{11} := 32.902 \ \psi_1 - 5.855 \ \psi_2.$

So the mapping $\hat{G}$ is now defined by

(6.65) $\hat{G}: \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \mapsto \alpha_1 \begin{pmatrix} \hat{g}_{11} \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} -2.1\bar{\varphi} \\ -2.1 \end{pmatrix}.$

STEP 6   If we denote the mapping that takes α_1 to $\alpha_1\hat{g}_{11}$ by $\hat{G}_{11}$, the operator $A + \hat{G}C$ can be written in matrix format as follows:

(6.66) $A + \hat{G}C = \begin{pmatrix} A_{11} + \hat{G}_{11}C_{11} & -2.1\bar{\varphi} \\ 0 & -2.1 \end{pmatrix}.$

So we only have to check the stability of $A_{11} + \hat{G}_{11}C_{11}$. It turns out that the first eigenvalue of this operator is at -2.508, so that the stability is indeed obtained.

STEP 7   Let us take the following basis for $\mathcal{W}$:

(6.67) $w_1 = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \ w_2 = \begin{pmatrix} \psi_2 \\ 0 \end{pmatrix}, \ w_3 = \begin{pmatrix} \bar{\varphi} \\ 1 \end{pmatrix}.$

The matrix of (the restriction of) $A + BF$ with respect to this basis is of course given by

(6.68) $A + BF = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$

while the $\hat{G}C$-matrix is given by

(6.69) $\hat{G}C = \begin{pmatrix} 32.902 & 0 \\ -5.855 & 0 \\ 0 & -2.1 \end{pmatrix} \begin{pmatrix} C_{11}\psi_1 & C_{11}\psi_2 & C_{11}\bar{\varphi} \\ 0 & 0 & 1 \end{pmatrix} =$

$= \begin{pmatrix} -2.850 & -11.709 & -6.979 \\ 0.507 & 2.084 & 1.242 \\ 0. & 0. & -2.1 \end{pmatrix}.$

So the dynamic equation of our compensator becomes

$$(6.70) \qquad w'(t) = \begin{pmatrix} -4.850 & -11.709 & -6.979 \\ 0.507 & -1.916 & 1.242 \\ 0. & 0. & -2.1 \end{pmatrix} w(t) - \begin{pmatrix} 32.902 & 0. \\ -5.855 & 0. \\ 0. & -2.1 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}.$$

To find the equation that will give the control function, we have to compute Fw_1, Fw_2 and Fw_3. We have $Fw_1 = F_1\psi_1 = 1$ and $Fw_2 = F_1\psi_2 = 0$. Finally, from the fact that w_3 is an eigenvector of $A + BF$ belonging to the eigenvalue at 0, one easily derives that

$$(6.71) \qquad Fw_3 = (M(0))^{-1} = 9.553.$$

So, together with the equation (6.70), the compensator is defined by

$$(6.72) \qquad u(t) = w_1(t) + 9.553 \, w_3(t).$$

Remark The closed-loop system will converge to a situation which is described by the eigenvector corresponding to the eigenvalue at 0. So the asymptotic temperature profile is a multiple of $\bar{\varphi}$, determined by the reference value r_0. Moreover, in the steady-state situation the control function has a constant value $u(t) = 9.553 \, r_0$. This is necessary, of course, to compensate for the loss of heat through the boundary.

Suppose that we would replace the Dirichlet boundary conditions in (6.50.2) by Neumann boundary conditions, which correspond to complete isolation:

$$(6.73) \qquad \frac{\partial}{\partial x} \varphi(0,t) = \frac{\partial}{\partial x} \varphi(1,t) = 0.$$

Then the operator A_{11} would have an eigenvalue at 0 and we would not be able to use Prop. 6.4. From the interpretation of the function $\bar{\varphi}$ given above, it is however easy to guess that in this case $\bar{\varphi}$ must be a constant function. Indeed, one can verify immediately from the equations (6.24) that this guess is correct. Moreover, the steady-state value of the control function $u(t)$ is in this case equal to 0. So one might say that we have a singular case here, but, as it appears to be, not in a troublesome sense.

Some simulation results using the compensator we designed above are given in the Appendix, Fig.A3.

6.7 Example IV: Protecting a delay system against a constant disturbance

We now consider the following set of equations:

(6.74.1) $x_1'(t) = -\frac{\pi}{2} x_1(t-1) + x_2(t) + u(t)$

(6.74.2) $x_2'(t) = 0$

(6.74.3) $z(t) = y(t) = x_1(t).$

The basic equation is (6.74.1) in which we have a scalar variable satisfying a retarded equation, influenced both by an unknown but constant disturbance $x_2(t)$ and by the control function $u(t)$. Even without the disturbance the open-loop system is unstable (with eigenvalues at $\pm \frac{\pi}{2}i$) and so the goal of the compensation is both to stabilize the system and to reject the constant disturbance.

We re-write the equations (6.74) in the usual way. As our state space, we take $X = X_1 \oplus X_2$ with $X_1 = M_2(-1,0) = \mathbb{R} \times L_2(-1,0)$ and $X_2 = \mathbb{R}$. We shall also use this space in its complexified form. The mapping A_{11} is defined on X_1 by

(6.75.1) $D(A_{11}) = \{(\varphi_0,\varphi) \in M_2(-1,0) \mid \varphi \in H^1[-1,0], \varphi(0) = \varphi_0\}$

(6.75.2) $A_{11}(\varphi_0,\varphi) = (-\frac{\pi}{2}\varphi(-1),\varphi')$ $((\varphi_0,\varphi) \in D(A_{11}))$

The mapping $A_{12}: X_2 \to X_1$ is given by

(6.76) $A_{12}: 1 \mapsto (1,0),$

and $A_{22}: X_2 \to X_2$ is equal to 0.

The input space U, the output space Y and the space of the variable-to-be-controlled Z are all taken equal to $\mathbb{R}$, and the mappings B, C and D are defined by

(6.77) $B: 1 \mapsto \begin{pmatrix} (1,0) \\ 0 \end{pmatrix}$

(6.78) $C\begin{pmatrix} (\varphi_0,\varphi) \\ 0 \\ \alpha \end{pmatrix} = D\begin{pmatrix} (\varphi_0,\varphi) \\ 0 \\ \alpha \end{pmatrix} = \varphi_0.$

We set the desired growth constant ω equal to -1, and we start the design procedure.

STEP 1 Since $\sigma(A_{22}) = \{0\}$ and $0 \notin \sigma(A_{11})$, we can use Prop.6.4. To determine the value of the function $M(\lambda)$ (defined in (6.43)) in $\lambda = 0$, we first have to solve the equation

(6.79) $\qquad A_{11}(\varphi_0, \varphi) = (1,0) \qquad\qquad ((\varphi_0, \varphi) \in D(A_{11})).$

One readily finds that the solution is given by $(\varphi_0, \varphi) = (-\frac{2}{\pi}, -\frac{2}{\pi})$. Consequently, we get

(6.80) $\qquad M(0) = -D_1 A_{11}^{-1} (1,0) = \frac{2}{\pi}.$

We can now compute the mapping S: $X_2 \to X_1$ from (6.44) and (6.45), and it turns out that S is given by

(6.81) $\qquad S: 1 \mapsto (0,0).$

This is no surprise, of course, because the origin in X_1 represents the situation we are trying to establish.

STEP 2 We can choose F_1 to create new eigenvalues at $-1 \pm \frac{\pi}{2}i$, in a similar fashion as in Example II (Section 5.6):

(6.82) $\qquad F_1(\varphi_0, \varphi) = -2\varphi_0 - \int_{-1}^{0} (\cos \frac{\pi}{2}\theta + \pi \sin \frac{\pi}{2}\theta) \, \varphi(\theta) \, d\theta.$

STEP 3 The mapping F_2: $X_2 = \mathbb{R} \mapsto U = \mathbb{R}$ is immediately found as

(6.83) $\qquad F_2: 1 \mapsto -1.$

STEP 4 A basis for the unstable subspace X_u is given by

(6.84) $\qquad \varphi_1 = \begin{pmatrix} (1, \cos \frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \quad \varphi_2 = \begin{pmatrix} (0, \sin \frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \quad \varphi_3 = (\begin{smallmatrix} (1,1) \\ \frac{1}{2}\pi \end{smallmatrix}).$

With respect to this basis, the matrices of A_u and C_u have the following form:

$$(6.85) \qquad A_u = \begin{pmatrix} 0 & \frac{\pi}{2} & 0 \\ -\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad C_u = (1 \quad 0 \quad 1).$$

It is now easy to calculate a suitable injection mapping G. If we take

$$(6.86) \qquad G: 1 \mapsto -\pi\varphi_1 - 2\pi\varphi_2 - \pi\varphi_3$$

then the eigenvalues at 0 and $\pm \frac{\pi}{2}i$ of A are shifted to eigenvalues at $-\frac{\pi}{2}$ (double) and $-\pi$ of A + GC, whereas the other eigenvalues of A remain unchanged.

STEP 5   Because S is the zero mapping, the required decomposition of G is
given by $G = G_1 + G_s$ with

$$(6.87) \qquad G_1: 1 \mapsto \begin{pmatrix} (-2\pi, -\pi \cos \frac{\pi}{2}\theta - 2\pi \sin \frac{\pi}{2}\theta - \pi) \\ 0 \end{pmatrix}, \quad G_s: 1 \mapsto \begin{pmatrix} (0,0) \\ -\frac{1}{2}\pi^2 \end{pmatrix}.$$

So we have to approximate the function $g_1 := (-2\pi, -\pi\cos\frac{\pi}{2}\theta - 2\pi\sin\frac{\pi}{2}\theta - \pi)$ $\in M_2(-1,0)$ by linear combinations of eigenvectors of $A_{11} + B_1F_1$. A basis for the eigenspace of $A_{11} + B_1F_1$ corresponding to the eigenvalues at $-1 \pm \frac{\pi}{2}i$ is given by (cf.(5.58-60)):

$$(6.88) \qquad \psi_1 = (1, e^{-\theta}\cos\frac{\pi}{2}\theta), \quad \psi_2 = (0, e^{-\theta}\sin\frac{\pi}{2}\theta).$$

Upon computing, we find that the best approximation (in $L_2(-1,0)$-sense) of g_1 in terms of ψ_1 and ψ_2 is given by

$$(6.89) \qquad \hat{g}_1 = -3.666 \, \psi_1 - 1.990 \, \psi_2.$$

So we arrive at

$$(6.90) \qquad \hat{G}: 1 \mapsto \begin{pmatrix} (-3.66, -3.66e^{-\theta}\cos\frac{\pi}{2}\theta - 1.990e^{-\theta}\sin\frac{\pi}{2}\theta) \\ -4.935 \end{pmatrix}.$$

STEP 6   Calculation shows that the rightmost eigenvalues of A + $\hat{G}$C are at
$-1.136 \pm 1.445i$, resulting, of course, from a split-up of the double
eigenvalue of A + GC at -1.571. In view of the desired growth

constant $\omega = -1$, this result is satisfactory.

STEP 7   As a basis for $\mathcal{W}$, let us take

$$(6.91) \qquad w_1 = \begin{pmatrix} (1, e^{-\theta}\cos\frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} (0, e^{-\theta}\sin\frac{\pi}{2}\theta) \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} (0,0) \\ 1 \end{pmatrix}.$$

With respect to this basis, we have

$$(6.92) \quad A + BF + \hat{G}C = \begin{pmatrix} -1 & \frac{\pi}{2} & 0 \\ -\frac{\pi}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -3.666 \\ -1.990 \\ -4.935 \end{pmatrix} (1 \quad 0 \quad 0) =$$

$$= \begin{pmatrix} -4.666 & 1.571 & 0. \\ -3.561 & -1. & 0. \\ -4.935 & 0. & 0. \end{pmatrix}$$

We also have to compute Fw_1, Fw_2 and Fw_3. In the same way as in Example II, we find

$$(6.93.1) \qquad Fw_1 = -1$$

$$(6.93.2) \qquad Fw_2 = \frac{\pi}{2} - \frac{\pi e}{2} = -2.699.$$

From the fact that w_3 is an eigenvector of $A + BF$ corresponding to the eigenvalue at 0, we also easily find

$$(6.93.3) \qquad Fw_3 = -1.$$

So the final compensator equations are:

$$(6.94.1) \qquad w'(t) = \begin{pmatrix} -4.666 & 1.571 & 0. \\ -3.561 & -1. & 0. \\ -4.935 & 0. & 0. \end{pmatrix} w(t) + \begin{pmatrix} 3.666 \\ 1.990 \\ 4.935 \end{pmatrix} y(t)$$

$$(6.94.2) \qquad u(t) = -w_1(t) - 2.699 w_2(t) - w_3(t).$$

Some simulation results of the closed-loop system are given in the Appendix, Fig.A4.

6.8 Example III: The moving hot spot

Our final example is inspired by a problem in nuclear reactor design, as explained in OWENS (1980). We do not use the stronger word "motivated", because the diffusion equation we shall use bears only a very faint relation to the 'real' equation derived by Owens. Nevertheless, one should note that this latter equation is basically of the diffusion type.

The problem can roughly be described as follows. The distribution of a certain variable which is associated with the intensity of the reaction is required to move periodically about an equilibrium profile, as is indicated in the figure:
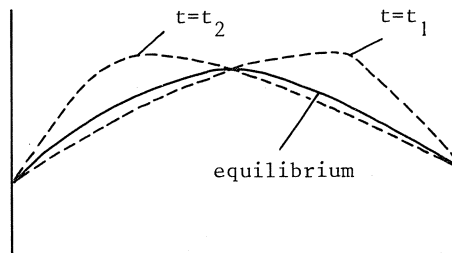
Fig.6.1. A one-dimensional reactor.

In this way, a "hot spot" moves from left to right in the one-dimensional reactor. We shall try to simulate this type of behaviour by introducing two 'variables-to-be-controlled'; one depends on the values of the state variable at the left side of the reactor, and the other depends on the right side. Both variables should follow the same sine wave, but with a phase difference of $180°$. More concretely, we consider the following set of equations.

(6.95.1) $\quad \frac{\partial}{\partial t} \varphi(x,t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial x^2} \varphi(x,t) + b_1(x)u_1(t) + b_2(x)u_2(t)$

(6.95.2) $\quad \varphi(0,t) = \varphi(1,t) = 0$

(6.95.3) $\quad \frac{d}{dt}\begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$

(6.95.4) $\quad y(t) = \sqrt{2} \int\limits_0^{\frac{1}{2}} \varphi(x,t)dx + r_1(t)$

(6.95.5) $\quad z_1(t) = \sqrt{2} \int\limits_0^{\frac{1}{2}} \varphi(x,t)dx + r_1(t)$

(6.95.6) $\quad z_2(t) = \sqrt{2} \int\limits_{\frac{1}{2}}^1 \varphi(x,t)dx - r_1(t).$

The input functions $b_1(x)$ and $b_2(x)$ are determined by

(6.96.1) $\quad b_1(x) = \sqrt{10} \quad\quad 0.2 \le x \le 0.3$

$\quad\quad\quad\quad\quad\quad 0 \quad\quad\quad 0 \le x < 0.2, \ 0.3 < x \le 1$

(6.92.2) $\quad b_2(x) = \sqrt{10} \quad\quad 0.7 \le x \le 0.8$

$\quad\quad\quad\quad\quad\quad 0 \quad\quad\quad 0 \le x < 0.7, \ 0.8 < x \le 1.$

As our state space, we take $X = X_1 \oplus X_2 = L_2(0,1) \oplus \mathbb{R}^2$. The input space U will be taken equal to $\mathbb{R}^2$, the output space Y will be $\mathbb{R}$, and the space of the variables-to-be-controlled is $\mathbb{R}^2$. The operator A_{11} is defined on X_1 by

(6.97.1) $\quad D(A_{11}) = \{\varphi \in L_2(0,1) \mid \dfrac{d^2\varphi}{dx^2} \in L_2(0,1), \ \varphi(0) = \varphi(1) = 0\}$

(6.97.2) $\quad A_{11}\varphi = \dfrac{1}{\pi^2} \dfrac{d^2\varphi}{dx^2} .$

The mapping A_{12} is 0, and

(6.98) $\quad A_{22} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$

The mappings B, C and D are given by

(6.99) $\quad B\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 b_1 + \alpha_2 b_2 \\ 0 \end{pmatrix}$

(6.100) $\quad C\begin{pmatrix} \varphi \\ r \end{pmatrix} = \sqrt{2} \int\limits_0^{\frac{1}{2}} \varphi(x)dx + r_1$

$$(6.101) \qquad D\binom{\varphi}{r} = \begin{pmatrix} \sqrt{2} \int\limits_{0}^{\frac{1}{2}} \varphi(x)dx + r_1 \\ \sqrt{2} \int\limits_{\frac{1}{2}}^{1} \varphi(x)dx - r_1 \end{pmatrix}$$

We set the desired growth constant ω at -1.5, and we follow the design procedure.

STEP 1   We first have to find a mapping $S: X_2 = \mathbb{R}^2 \to X_1 = L_2(0,1)$ that satisfies (6.24). We could use Prop.6.4 for this, but we prefer to follow a somewhat different route. Write

$$(6.102) \qquad S\binom{1}{0} = \bar{\varphi}_1, \quad S\binom{0}{1} = \bar{\varphi}_2.$$

Then (6.24) will be satisfied if $\bar{\varphi}_1 \in D(A_{11})$, $\bar{\varphi}_2 \in D(A_{11})$, and

$$(6.103.1) \qquad A_{11}\bar{\varphi}_1 + \bar{\varphi}_2 = \alpha_{11}b_1 + \alpha_{12}b_2$$

$$(6.103.2) \qquad A_{11}\bar{\varphi}_2 - \bar{\varphi}_1 = \alpha_{21}b_1 + \alpha_{22}b_2$$

for suitable $\alpha_{11}, \alpha_{12}, \alpha_{21}$ and α_{22}, and moreover

$$(6.104.1) \qquad \sqrt{2} \int\limits_{0}^{\frac{1}{2}} \bar{\varphi}_1(x)dx = -1$$

$$(6.104.2) \qquad \sqrt{2} \int\limits_{\frac{1}{2}}^{1} \bar{\varphi}_1(x)dx = 1$$

$$(6.104.3) \qquad \sqrt{2} \int\limits_{0}^{\frac{1}{2}} \bar{\varphi}_2(x)dx = 0$$

$$(6.104.4) \qquad \sqrt{2} \int\limits_{\frac{1}{2}}^{1} \bar{\varphi}_2(x)dx = 0.$$

It is verified by direct computation that

$$(6.105) \qquad \begin{pmatrix} A_{11} & I \\ -I & A_{11} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}(A_{11}^2+I)^{-1} & -(A_{11}^2+I)^{-1} \\ (A_{11}^2+I)^{-1} & A_{11}(A_{11}^2+I)^{-1} \end{pmatrix}.$$

This gives the following solution for (6.103):

$$(6.106.1) \quad \bar{\varphi}_1 = A_{11}^2 (A_{11}^2 + I)^{-1} (\alpha_{11} b_1 + \alpha_{12} b_2) - (A_{11}^2 + I)^{-1} (\alpha_{21} b_1 + \alpha_{22} b_2)$$

$$(6.106.2) \quad \bar{\varphi}_2 = (A_{11}^2 + I)^{-1} (\alpha_{11} b_1 + \alpha_{12} b_2) + A_{11} (A_{11}^2 + I)^{-1} (\alpha_{21} b_1 + \alpha_{22} b_2).$$

Combining this with (6.104) gives rise to four linear equations in the four unknowns α_{11}, α_{12}, α_{21} and α_{22}, which can readily be solved. The numerical results are as follows.

$$(6.107) \quad \begin{aligned} \alpha_{11} &= 7.337 \\ \alpha_{12} &= -7.337 \\ \alpha_{21} &= 1.884 \\ \alpha_{22} &= -1.884. \end{aligned}$$

The mapping S is now specified by (6.102), (6.106) and (6.107).

STEP 2   The mappings A_{11}^u and B_1^u are given by

$$(6.108) \quad A_{11}^u = -1, \quad B_1^u = (0.3149 \quad 0.3149).$$

It follows that a feedback mapping F_1 that shifts the eigenvalue of A_{11} at -1 to -1.5, while leaving all the other eigenvalues unchanged, is given by

$$(6.109) \quad F_1 \varphi = \binom{f_1}{f_2} \sqrt{2} \int_0^1 \varphi(x) \sin \pi x \, dx$$

where f_1 and f_2 are such that

$$(6.110) \quad 0.3149(f_1 + f_2) = -0.5.$$

There is one degree of freedom left here, and we shall use this later on.

STEP 3   It turns out that both $\bar{\varphi}_1$ and $\bar{\varphi}_2$ are perpendicular to the eigenvector $Q_1 := \sqrt{2} \sin \pi x$ corresponding to the eigenvalue -1 of A_{11}. So we find

$$(6.111.1) \quad F_2 \binom{1}{0} = -\binom{\alpha_{11}}{\alpha_{12}} - F_1 \bar{\varphi}_1 = \binom{-7.337}{7.337}$$

$$(6.111.2) \quad F_2\binom{0}{1} = -\binom{\alpha_{21}}{\alpha_{22}} - F_1\bar{\varphi}_2 = \binom{-1.884}{1.884}.$$

STEP 4 A basis for the unstable subspace of A is given by

$$(6.112) \quad \chi_1 = \begin{pmatrix} \sqrt{2}\ \sin\ \pi x \\ 0 \\ 0 \end{pmatrix}, \ \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

With respect to this basis, we have

$$(6.113) \quad A_u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \ C_u = (\tfrac{2}{\pi}\ \ 1\ \ 0).$$

After some computations, we find that an injection mapping G that replaces the eigenvalues at $\pm i$ and -1 by new eigenvalues at -1.8 (double) and -2.0 is given by

$$(6.114) \quad G: 1 \mapsto -0.16\pi\ \chi_1\ -4.28\chi_2\ -5.16\chi_3 := g.$$

STEP 5 If we define

$$(6.115) \quad g_1 = -0.16\pi\ \varphi_1\ +\ 4.28\bar{\varphi}_1\ +\ 5.16\bar{\varphi}_2,$$

Then it is clear that g (as defined in (6.114)) is equal to $g_1 + g_s$ with $g_1 \in X_1$ and $g_s \in V_s$ (defined by (6.28) and (6.102)). So we have to approximate the function $g_1 \in L_2(0,1)$ by linear combinations of eigenvectors of $A_{11} + B_1 F_1$. These eigenvectors are the same as the eigenvectors of A_{11} except for the one corresponding to the new eigenvalue at -1.5, which is given by

$$(6.116) \quad \psi_1 := f_1(-1.5-A_{11})^{-1}b_1\ +\ f_2(-1.5-A_{11})^{-1}b_2 := f_1\psi_1^1\ +f_2\psi_1^2.$$

(normalized such that $\langle\psi_1,\varphi_1\rangle = 1$). Because f_1 and f_2 can be chosen freely subject to the condition (6.110), the vectors ψ_1^1 and ψ_1^2 can be used independently in the approximation process. If we denote $\psi_2 := \sqrt{2}\ \sin\ 2\pi x$, then the best approximation of g_1 by a linear combination of ψ_1^1, ψ_1^2 and ψ_2 (in the sense of $L_2(0,1)$) is given by

(6.117) $\hat{g}_1 := -39.100 \; \psi_1^1 + 38.305 \; \psi_1^2 + 6.649 \; \psi_2.$

So we obtain

(6.118) $\hat{G} : 1 \mapsto \begin{pmatrix} \hat{g}_1 \\ 0 \\ 0 \end{pmatrix} \; -4.28 \begin{pmatrix} \bar{\varphi}_1 \\ 1 \\ 0 \end{pmatrix} \; -5.16 \begin{pmatrix} \bar{\varphi}_2 \\ 0 \\ 1 \end{pmatrix}.$

STEP 6   To verify the stability of the semigroup generated by $A + \hat{G}C$, we
the first eigenvalues of this operator. It turns out that the right-
most eigenvalues are at $-1.744 \pm 0.108i$, so we have a satisfactory
situation.

STEP 7   It follows from (6.110) and (6.117) that we should take

(6.119) $\psi_1 = -78.091 \; \psi_1^1 + 76.503 \; \psi_1^2.$

We then have

(6.120) $\hat{g}_1 = 0.501 \; \psi_1 + 6.649 \; \psi_2.$

We obtain a fourth-order compensator. As a basis for W, let us take the
following vectors:

(6.121) $w_1 = \begin{pmatrix} \psi_1 \\ 0 \\ 0 \end{pmatrix}, \; w_2 = \begin{pmatrix} \psi_2 \\ 0 \\ 0 \end{pmatrix}, \; w_3 = \begin{pmatrix} \bar{\varphi}_1 \\ 1 \\ 0 \end{pmatrix}, \; w_4 = \begin{pmatrix} \bar{\varphi}_2 \\ 0 \\ 1 \end{pmatrix}.$

With respect to this basis, we finally get the following compensator equa-
tions.

(6.122.1) $w'(t) = \begin{pmatrix} -9.809 & -0.319 & 0. & 0. \\ -110.267 & 0.233 & 0. & 0. \\ 70.980 & -2.725 & 0. & 1. \\ 85.574 & -3.285 & -1. & 0. \end{pmatrix} w(t) - \begin{pmatrix} 0.501 \\ 6.649 \\ -4.28 \\ -5.16 \end{pmatrix} y(t)$

(6.122.2) $u_1(t) = -78.091 \; w_1(t) - 7.337 \; w_3(t) - 1.884 \; w_4(t)$

(6.122.3) $u_2(t) = 76.503 \; w_1(t) + 7.337 \; w_3(t) + 1.884 \; w_4(t).$

The action of the compensator is illustrated in the Appendix, Fig.A5.

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APPENDIX: SIMULATION RESULTS
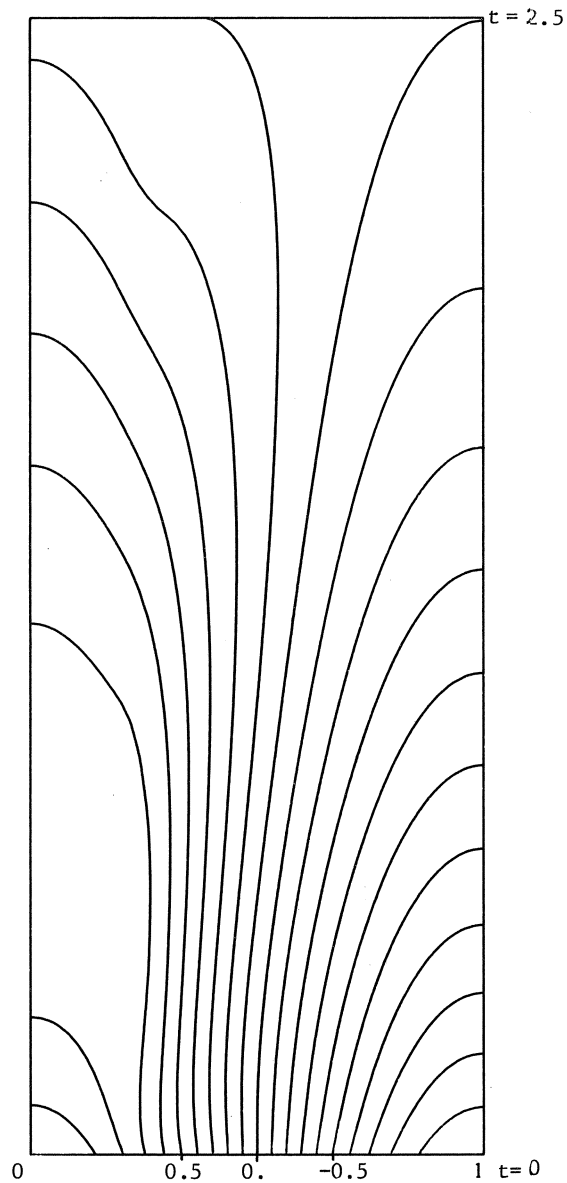
Fig. A1. Behaviour of the diffusion process (5.33) under the control of the compensator (5.46). The heating/cooling device is situated at the left side of the interval, the sensor at the right side. The initial condition is given by $\phi_0(x) = \cos \pi x$. In the figure are indicated the lines of equal temperature (level lines) for t=0. to t=2.5.
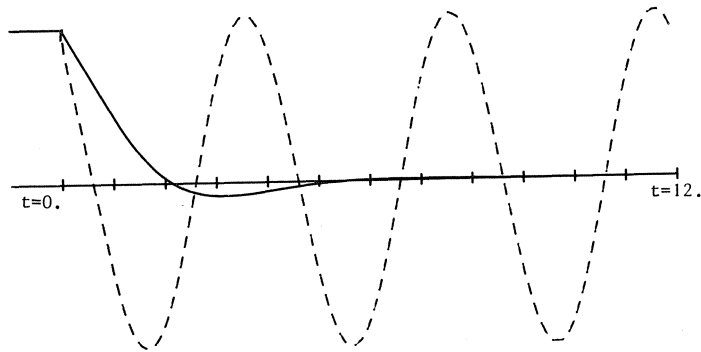
Fig.A2a
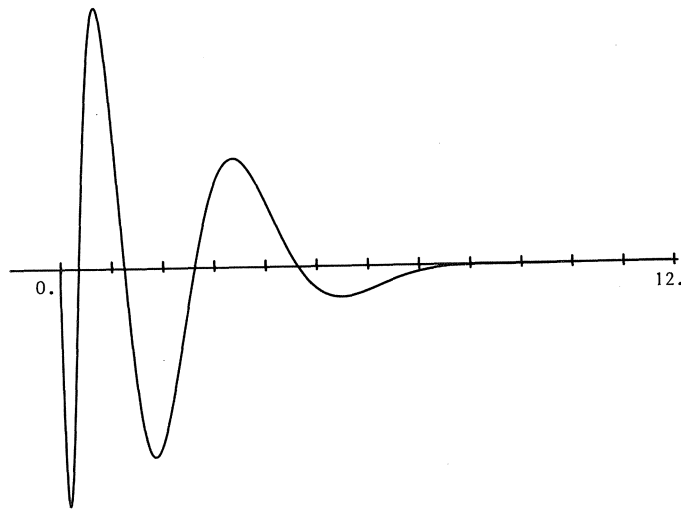
Fig.A2b
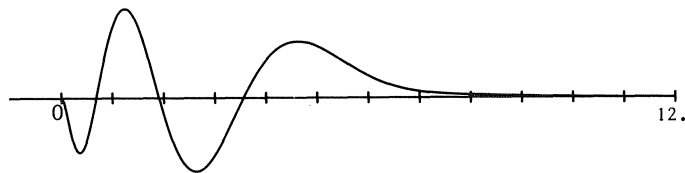
Fig.A2c

Fig.A2. Behaviour of the delay system (5.47) under the control of the compensator (5.82). The initial condition is given by $x_1(\theta) = 1(-1\leq\theta\leq0)$, $x_2(0) = 0$. Fig.A2a shows $x_1(t)$; the uncontrolled solution of (5.4.7) is dotted. Fig.A2b gives u(t), Fig.A2c gives $x_2(t)$.
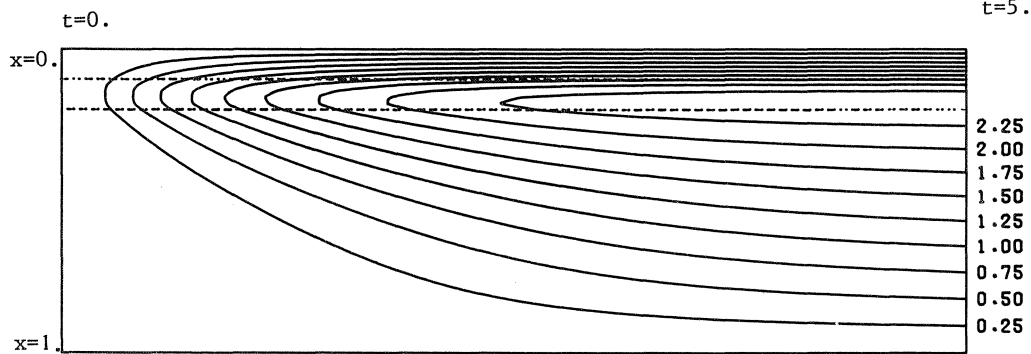
Fig. A3. Behaviour of the diffusion system (6.50) under the control of the compensator (6.70-72). The goal of the control is to bring the temperature in the middle of the internal to a specified value. The initial value is 0. The position of the heater is marked by dotted lines.
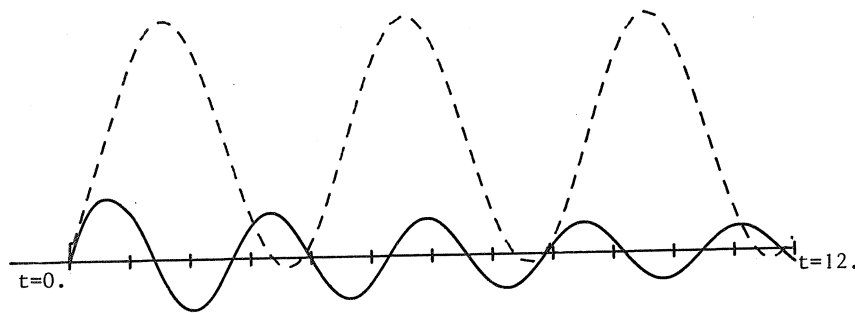
Fig. A4a
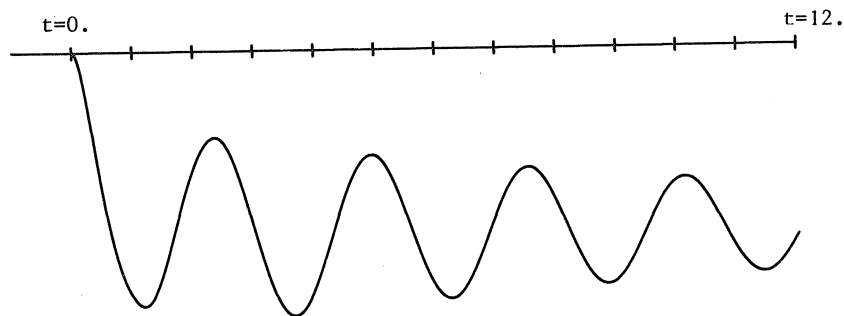
Fig. A4b

Fig.A4. Behaviour of the delay system (6.74) under the control of the compensator (6.94). The initial value is given by $x_1(\theta) = 0$ $(-1 \leq \theta \leq 0)$. A constant disturbance is applied at t=0. Fig.A4a gives the behaviour of $x_1(t)$; the uncontrolled solution is dotted. Fig.A4b gives the control function u(t).
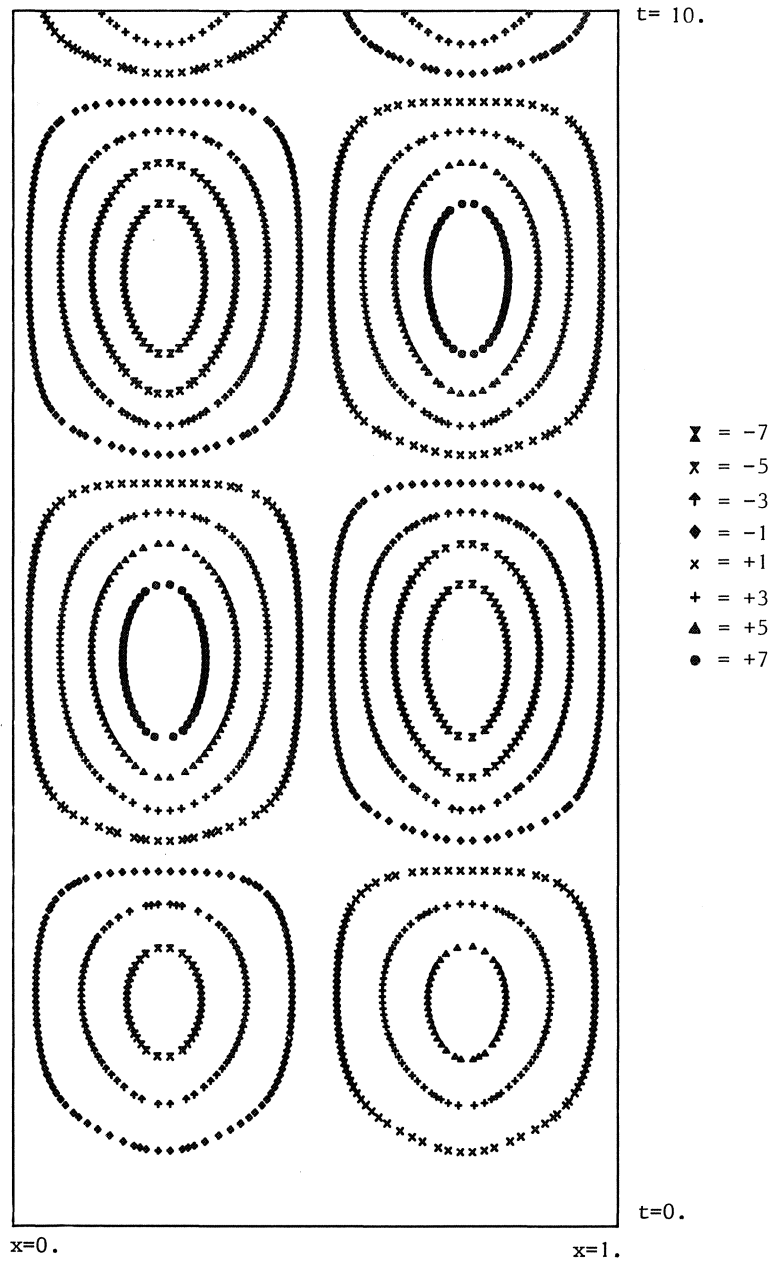
Fig.4.5. Behaviour of the diffusion system (6.95) under the control of the compensator (6.122). The goal of the control is to bring about a periodic pattern of hot spots appearing left and right in the interval. The initial condition is 0. Each symbol marks a point having a certain temperature (see list.).

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