Printed at the Mathematical Centre, Kruislaan 413, Amsterdam, The Netherlands.

The Mathematical Centre, founded 11 February 1946, is a non-profit institution for the promotion of pure and applied mathematics and computer science. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).
DYNAMIC FEEDBACK IN FINITE- AND INFINITE-DIMENSIONAL LINEAR SYSTEMS

J.M. SCHUMACHER

SECOND PRINTING

MATHEMATISCH CENTRUM AMSTERDAM 1984
ACKNOWLEDGEMENTS

The research leading to this work was stimulated by discussions with Jan C. Willems, Ruth F. Curtain, Malo Hautus, Jan van Schuppen, Peter Janssen, Henk Nijmeijer, Arjan van der Schaft, Freek van Schagen, Harm Bart, M.A.Kaashoek, Aart van Harter and G.Y.Nieuwland. Special thanks go to Rien van Veldhuizen for his help and advice in numerical and computational matters. I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts and all those at the Mathematical Centre who have contributed to its technical realization.
CONTENTS

INTRODUCTION

1. CONTROL AND OBSERVATION
   1.1 Introduction to linear systems 6
   1.2 Basic results on pole placement 6
   1.3 Controlled and conditioned invariance 12
   1.4 Pole placement under restrictions 15
   1.5 Stabilizability and detectability 17

2. THE COMPENSATOR PROBLEM
   2.1 Introduction 22
   2.2 Problem statement and main results 29
   2.3 The construction lemma 33
   2.4 Methods of compensator construction 42
      2.4.1 The full-order compensator 45
      2.4.2 The reduced-order compensator 45
      2.4.3 The Brasch-Pearson compensator 47
      2.4.4 Zero-order compensators 48
      2.4.5 An iterative approach 50
   2.5 Separating subspaces 51

3. TRACKING, REGULATION AND DISTURBANCE LOCALIZATION
   3.1 Problem statement 54
   3.2 Preliminary results 54
   3.3 Main theorem 61
   3.4 Comparison with other work 67
   3.5 Final remarks 73

4. BASIC CONCEPTS OF INFINITE-DIMENSIONAL SYSTEMS
   4.1 Semigroup theory 80
   4.2 Composite systems 80
   4.3 Stabilizability, detectability and the spectral decomposition 84
   4.4 Remarks on the scope of the theory 91
   4.5 Introduction to the examples 97

5. FINITE-DIMENSIONAL COMPENSATORS FOR INFINITE-DIMENSIONAL SYSTEMS
   5.1 Introduction 106
   5.2 The basic theorem 106
   5.3 The existence result 111
   5.4 The design procedure 115
   5.5 Example I: A diffusion system 118
   5.6 Example II: A delay system 122

6. TRACKING AND REGULATION IN INFINITE DIMENSIONS
   6.1 Introduction 130
   6.2 Problem formulation 130
   6.3 The basic theorem 131
   6.4 The existence result 135
   6.5 The design procedure 138
   6.6 Example III: A heat regulator 144
   6.7 Example IV: Protecting a delay system against a constant disturbance 146
   6.8 Example V: The moving hot spot 150
INTRODUCTION

In the beginning of his celebrated 1868 paper "On governors", James Clerk Maxwell made a careful distinction between two types of regulators. One type consists of the so-called 'moderators', which are only able to diminish the effect of the disturbances. After describing a few mechanisms of this sort, Maxwell writes:

"In all these contrivances an increase of driving-power produces an increase of velocity, though a much smaller increase than would be produced without the moderator. But if the part acted on by centrifugal force, instead of acting directly on the machine, sets in motion a contrivance which continually increases the resistance as long as the velocity is above its normal value, and reverses its action when the velocity is below that value, the governor will bring the velocity to the same normal value whatever variation (within the working limits of the machine) be made in the driving-power or the resistance.

I propose at present, without entering into any details of mechanism, to direct the attention of engineers and mathematicians to the dynamical theory of such governors." (MAXWELL (1868; p.271))

In today's terminology, the second type of regulators to which Maxwell refers is said to provide dynamic feedback. The offset is eliminated via an integration of the error, which means that the controller brings its own dynamics into the feedback loop.

The concept of dynamic feedback (or integral control) has given rise to important theoretical developments from its very inception on. Second-order models were used to study the motion of engines, but the combination with an integral controller led Maxwell to consider third-, fourth- or even fifth-order equations. The resulting question of giving verifiable necessary and sufficient conditions for a polynomial of arbitrary degree to have only roots with negative real parts led to the well-known theorems of E.J.Routh and A.Hurwitz. An interesting survey of the developments around the stability of polynomials has been given by BENNETT (1979), and a detailed account can be found in BATEMAN (1945).

In the fast and sweeping development of mathematical control theory
in the last twenty or so years, dynamic feedback has not been a central concept. Rather, one can say that a key role has been played by the notion of state feedback. In a rough paraphrase of the system-theoretic definition, the state of a system at a given time consists of all parameters that determine the system's future behavior. So when studying state feedback, one assumes that all relevant information about the system is available to the controller.

This idealization turned out to be extremely useful for the description of fundamental properties of systems such as controllability (introduced by R.E. Kalman at the end of the fifties). Moreover, the theory of state feedback was used in dual form to obtain an observer theory, which answered the question of how to reconstruct the state from the given measurements. Thus, a feasible controller could be derived by combining a state feedback law with an observer. The solution obtained in this way could in fact even be proved to be the optimal solution for the standard regulator problem of stochastic control; this is the so-called separation principle.

The method of reducing a two-sided problem (input/output) to two easier to solve one-sided problems (input/state and state/output) has been one of the main successes of the state space approach to control theory. Not surprisingly therefore, the method is also strongly present in the 'geometric' approach to linear multivariable systems. This approach was developed from 1969 on by G. Basile, G. Marro, A.S. Morse, W.M. Wonham and many others; it is characterized by its consistent formulation of results in terms of subspaces of the state space. In Wonham's trend-setting book (Wonham (1974)), a systematic pattern is discernible which proceeds from 'restricted' to 'extended' problems. Problems of the first kind have to be solved using state feedback, while an observer is added to study the 'extended' problem in which the control has to be based on the observations. The reliance on duality and the resulting one-sidedness of the theory are clearly illustrated by the fact that the concept of 'controlled invariant subspace' (i.e., a subspace that can be made invariant by state feedback) is used throughout Wonham's book, whereas the dual concept of 'conditioned invariant subspace' is only mentioned in one of the exercises.

In the present monograph, we want to show that it is possible to go one step further and to develop a two-sided approach. This means that we are going to make a direct study of dynamic feedback. The basic idea is the introduction of the concept of a compensator couple (Section 2.2), which is used to describe the subspaces that can be made invariant by dynamic feed-
back. The two-sidedness of our method is reflected in the fact that a compensator couple is a pair of subspaces, one of which is controlled invariant and the other conditioned invariant.

The price one has to pay for the use of the separation method is particularly clear in the context of systems with an infinite-dimensional state space (distributed parameter systems). One has been able, for instance, to solve the linear-quadratic optimal control problem for several important classes of infinite-dimensional systems, but the resulting controllers are of infinite order and therefore impractical. To design finite-dimensional compensators, one has to leave the idea of separation. In Chapter 5, we shall present a method of compensator design based on a certain adjustment between the 'feedback' and the 'observer' parts (Section 5.4). This method is able to produce controllers of low order, as will be illustrated in various examples. Moreover, we shall prove the existence of a stabilizing compensator of finite order for a large class of infinite-dimensional systems, including those described by diffusion and delay equations (Thm.5.2).

The theory of Ch.5 is based on that of Ch.2, in which we treat the stabilization problem for finite-dimensional systems. A general construction theorem is given (Thm.2.4), which is based on the notion 'compensator couple'. We prove that the traditional state-space techniques of compensator design can all be described within this framework (Section 2.4). We also investigate the question of when a separation between 'feedback' and 'observer' action is possible for a given compensator. We propose a precise definition of this separation, taken here in a more general sense than that of the 'separation principle' (Section 2.2), and we prove that any compensator for which such a separation is possible can be constructed by our method.

Although in Wonham's book the disturbance decoupling problem more or less bears the banner of the 'geometric approach', the "DDP" is only solved for state feedback, whereas the natural extension to output feedback is not even mentioned. Indeed, one needs a compensator couple to solve the general problem. We give the solution in Ch.3 as a special case of a theorem (Thm.3.3) that attains a level of generality which seems to be new. Disturbance decoupling is combined with a regulation task, and not only the solution of the problem of disturbance decoupling by observation feedback but also many other results are recovered as special cases (Section 3.4). Even at this level of generality, the solution we obtain is completely constructive (Section 3.3).

In the final Chapter 6, we consider tracing and regulation problems for infinite-dimensional systems. Again, we give an existence theorem for finite-
dimensional controllers (Thm.6.2) and an algorithm for the actual design of such controllers (Section 6.3). The practicability of the method is shown by several examples. One of these examples concerns a delay systems which is to be protected against constant disturbances (Section 6.7). The ability of a regulator to eliminate the effect of a constant disturbance is cited by Maxwell as a typical distinction between 'moderators' and 'governors' (MAXWELL (1868; p.274)). Also, one may note that delay effects were at least partially responsible for the 'hunting' of steam engines (slow oscillatory variations in the work of the engine), one of the main energy problems of the nineteenth century. Our solution is simple enough to have been implementable by the methods of Maxwell's time.

Admittedly, this solution is a bit on the late side. Our final example (Section 6.8) is perhaps more modern in that it has been inspired by a problem in nuclear reactor design.

Organization

This work is divided in two parts, each consisting of three chapters. The first part is concerned with finite-dimensional systems, the second with systems having an infinite-dimensional state space. Both parts are organized in the same way: the first chapter (Ch.1 and Ch.4) contains introductory material, the second chapter (Ch.2 and Ch.5) is concerned with stabilization problems, and regulation problems are treated in the third chapter of each part (Ch.3 and Ch.6).

The reader who is interested in the finite-dimensional theory and who is reasonably well acquainted with the 'geometric' approach to linear multivariable systems could skim over Ch.1 to pick up the new results and then proceed directly to Ch.2 or Ch.3. The third chapter is fairly independent from the second, except for the definition of a "compensator couple" that is given in Section 2.2. The reader who is not initiated in the 'geometric' theory could read Ch.1 as an introduction, but it should be said that the primary purpose of this chapter has not been to give an extensive motivation for the presented theory from a more general control point of view.

Although some of the main ideas of the second part are already present in the first part, both parts are mathematically independent and so the reader who is interested in control of distributed parameter systems could start immediately with Ch.4. Most of the material in this chapter will be familiar to those versed in the 'semigroup' approach to infinite-dimensional
systems. Although in principle Ch.6 is independent from Ch.5, the idea of constructing a finite-dimensional controller is developed in Ch.5 in a somewhat easier context, so the reader is advised to read these chapters in their natural order.

All chapters are divided into sections. At the beginning of each chapter, a brief description of the organization of the sections is given.
CHAPTER 1

CONTROL AND OBSERVATION

We start our treatise by briefly explaining some of the central notions of linear systems theory, such as: the concept of state, the notions of stability, controllability and observability, and the relationship between decomposition of systems and invariant subspaces. The problem of changing the dynamics of a given system by state feedback is fundamental. On one hand, we have at our disposal the solution of the "pole placement problem", which allows us to relocate eigenvalues. On the other hand, the structural change in a system brought about by state feedback may be captured in the concept of "controlled invariant subspace". Combining the two results we can also consider pole placement problems under certain structural restrictions on the nature of the feedback. Motivated by our needs in later chapters, we give a rather extensive treatment of this type of problems and present some new results. The theory will also be needed in its dualized form, and so many results will be given together with their mirror images obtained by dualization.

The chapter contains five sections. The first section gives a brief introduction into linear systems and explains some notation and terminology. In the second section, we give the basic standard results on pole placement. Controlled and conditioned invariant subspaces are defined in section 3, and section 4 considers pole placement with certain restrictions on the feedback map. Finally, in section 5 we discuss stabilizability and detectability properties related to invariant subspaces.

1.1. Introduction to linear systems

Let us suppose that we have a process which evolves in time according to a set of differential equations, which in the first part of this thesis, we shall assume to be ordinary differential equations. Furthermore, we shall always assume that the equations are linear and have constant coefficients. If we take all equations together and write them in a first-order form, then the evolution of the process under consideration is described by an equation
where \( x(\cdot) \) is a function of time with values in some finite-dimensional linear space \( X \) called the state space, and \( A \) is a linear mapping from \( X \) into \( X \) to which we shall refer as the system matrix. The vector \( x(t) \) is called the state of the system at time \( t \); if the state at some time \( t_0 \) is given, then the future behaviour of the system is completely determined by the equation (1.1). The 'state space approach' will be used throughout in this work.

In the real world, the evolution of a process is always affected by disturbances. There are many ways in which such disturbances can be modeled, and we shall encounter some of them further on. The simplest model for a disturbance is obtained by setting a non-zero initial condition for the equation (1.1). This represents a sudden departure of the system from the origin, which is thought of as representing the nominal operating mode of the system. It is a well-known fact that the state will return to the origin regardless of the initial condition, if and only if the eigenvalues of the mapping \( A \) are in the left half of the complex plane:

\[ \sigma(A) \subseteq \mathbb{C}_{-} = \{ \lambda \in \mathbb{C} | \Re \lambda < 0 \}. \]

(This result, and in fact all the necessary material on ordinary differential equations, is covered in textbooks like HIRSCH & SMALE (1974) or ARNOLD (1973)). Moreover, the rate of convergence will be faster according as the eigenvalues of \( A \) are situated further to the left. In general, we shall say that the mapping \( A \) is stable if its eigenvalues are contained in some prescribed part of the complex plane denoted by \( \mathbb{C}_{R} \). Here, "R" means "good". The complement of \( \mathbb{C}_{R} \) will be denoted by \( \mathbb{C}_{B} \), where "B" stands for "bad".

Very broadly speaking, the purpose of control is to make the controlled process satisfy certain performance specifications which would not be satisfied if the system was left to itself. For instance, the uncontrolled system (1.1) may not be stable in the above sense. Of course, we have to make certain assumptions about how the control enters into the system. We shall assume that the control is a function of time with values in a finite-dimensional linear space \( U \), and which acts on the system through a linear mapping \( B \) from \( U \) to \( X \), in the following way:
(1.2) \[ x'(t) = Ax(t) + Bu(t). \]

The operator \( B \) is called the input mapping; clearly, it is no restriction to assume that this operator is injective, and we shall always do so. If we want to talk about feedback, then the action of the control is to be based on some observation of the state. We shall assume that this observation is provided by some linear operator \( C \) mapping \( X \) into another finite-dimensional linear space \( V \):

(1.3) \[ y(t) = Cx(t). \]

The operator \( C \) is called the output mapping, \( y(t) \in V \) is called the observation at time \( t \). Without loss of generality, we can assume that \( C \) is surjective.

A fundamental notion describing the relation between the system matrix \( A \) and the input mapping \( B \) is that of controllability. Suppose that the system (1.1) has been obtained by combining two sets of linear differential equations, of the following form:

(1.4.1) \[ x'_1(t) = A_{11}x_1(t) + A_{12}x_2(t) \]

(1.4.2) \[ x'_2(t) = A_{22}x_2(t). \]

Also suppose that the input mapping is such that the control \( u(\cdot) \) only affects the first set of state variables, so that (1.2) reads:

(1.5.1) \[ x'_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \]

(1.5.2) \[ x'_2(t) = A_{22}x_2(t). \]

Then it is clear that the behaviour of \( x_2(\cdot) \) is not influenced by the control function. In particular, if \( A_{22} \) is unstable, then there is no way in which the system as a whole can be stabilized by a suitable choice of \( u(\cdot) \).

Pictorially, the situation can be described as follows.
Whether this phenomenon occurs or not depends on the mappings $A$ and $B$. The pair $(A,B)$ is said to be *controllable* if there is no decomposition of (1.2) in the form (1.5).

A similar situation can arise with respect to the observation. If the system (1.1) is decomposed as in (1.4) and if the output equation (1.3) is actually of the form

\[(1.6) \quad y(t) = C_2 x_2(t)\]

then one sees that the observation $y(t)$ does not contain any information about $x_1(t)$. We can illustrate this in a diagram:

![Diagram](image)

**Fig. 1.2. Unobservable system**

This phenomenon depends on the mappings $A$ and $C$. The pair $(C,A)$ is called *observable* if the equations (1.1) and (1.3) do not decompose into (1.4) and
(1.6). The concepts of controllability and observability also play a crucial role in realization theory, in which a state space description is constructed from a system's input-output behaviour (see for instance Kalman, Falb & Arbib (1969)).

One may note that the figures 1.1 and 1.2 can be obtained from each other by reversing the arrows (and renaming the parts). In the language of linear algebra, this duality just comes down to transposition of matrices. In such a way, statements concerning a pair consisting of a system matrix and an input mapping can be turned into statements about a pair consisting of a system matrix and an output mapping, and vice versa. In the rest of this chapter, we shall omit proofs of dual results with reference to this principle of duality.

If we reconsider the system matrix $A$ and the input mapping $B$ as they appear in the uncontrollable system (1.5), we see that these mappings can be written down in a "block matrix" format in the following way:

$$\begin{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \\ \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \end{pmatrix}.$$

From this, it is clear that the controllable vector $x_1(t)$ corresponds to a (non-trivial) subspace of the state space $X$ which is invariant for $A$ and which contains the range of the input mapping $B$. So we can say: The pair of mappings $(A,B)$ is controllable if and only if the only $A$-invariant subspace containing $\text{Im } B$ (the range of $B$) is the trivial one, namely the whole state space $X$ on which $A$ acts. If we write

$$<A \mid V> = n \{ \omega | A \omega \subset \omega, \omega \supset V \}$$

for the smallest $A$-invariant subspace containing $V$, then the condition for controllability is:

$$<A \mid \text{Im } B> = X.$$

Dually, a decomposition in the form (1.4) and (1.6) can be made if and only if there is a non-trivial $A$-invariant subspace contained in $\text{Ker } C$ (the kernel of $C$). The largest $A$-invariant subspace in $\text{Ker } C$ will be denoted by

$$<V \mid A> = \Sigma \{ \omega | A \omega \subset \omega, \omega \subset V \}.$$
Then the condition for observability is

\[(1.11) \quad \langle \ker C \mid A \rangle = 0.\]

(We use the symbol \( \cap \) for the trivial subspace consisting of only the zero element.)

This discussion of controllability and observability also shows the importance of invariant subspaces in describing structural properties of linear systems. Below, we shall introduce the concepts of "controlled invariant subspace" and "conditioned invariant subspace", which arise naturally in this context. The approach to linear systems that makes strong use of these invariant subspaces has been initiated by G. BASILE & G. MARRO (1969b) and by W.M. WONHAM & A.S. MORSE (1970); it is now commonly described as the 'geometric' approach (WONHAM (1974,1979)).

Let us conclude this section by briefly reviewing the basic notation and terminology that will be used in the sequel. In the first part of this work (chapters 1 to 3), all spaces will be finite-dimensional real vector spaces. The standard complexifications of these vector spaces will be used occasionally without change of notation. Spaces and subspaces will be denoted by script capitals, some of which have a standard meaning: \( X \) will always be the state space and its dimension will always be denoted by \( n \). \( U \) will be the input space (of dimension \( m \)), and \( V \) is the output space (of dimension \( p \)). The elements of these vector spaces will as a rule be denoted by the corresponding lower case letters: \( x \in X, u \in U \) etc. Roman capitals will be used for linear mappings, and the following denotations will be standard: \( A: X \to X \) is the system matrix, \( B: U \to X \) is the input mapping, and \( C: X \to V \) is the output mapping. The words "matrix", "mapping" and "operator" will all be used as synonyms for "linear mapping". If \( V \subset X \) is an invariant subspace for \( T \), then we shall write

\[(1.12) \quad T: V = \text{restriction of } T \text{ to } V\]

\[(1.13) \quad T: X/V = \text{quotient mapping induced by } T \text{ on } X/V.\]

(Quotient mappings and all of the further material on linear algebra we shall need can be found, for instance, in GANTMACHER (1959)). More generally, if \( V_1 \) and \( V_2 \) are \( T \)-invariant subspaces and \( V_1 \subset V_2 \), then we shall write
(1.14) \[ T: \mathcal{V}_2/\mathcal{V}_1 = \text{restriction of } T: \mathcal{X}/\mathcal{V}_1 \text{ to } \mathcal{V}_2/\mathcal{V}_1 = \text{quotient mapping induced by } T:\mathcal{V}_2 \text{ on } \mathcal{V}_2/\mathcal{V}_1. \]

We have already used the notation \( \sigma(T) \) for the set of eigenvalues of \( T \). We shall want to add spectra of different operators "counting multiplicities". To be able to state this properly, we introduce the **spectral multiplicity function** which is the following function from \( \mathbb{F} \) to \( \mathbb{Z} \).

(1.15) \[ \sigma(T)(\lambda) = k \text{ if } \lambda \text{ is an eigenvalue of } T \text{ with multiplicity } k \]
\[ = 0 \text{ if } \lambda \text{ is not an eigenvalue of } T. \]

Such functions inherit a natural additive structure and a partial ordering from \( \mathbb{Z} \). We shall freely use simple properties of the spectral multiplicity function, such as

(1.16) \[ \sigma(T; (\mathcal{V}_1 + \mathcal{V}_2)/\mathcal{V}_1) = \sigma(T; \mathcal{V}_2/(\mathcal{V}_1 \cap \mathcal{V}_2)) \]

where \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) are \( T \)-invariant subspaces.

### 1.2. Basic results on pole placement

The eigenvalues of the system matrix are often called "poles" in the control literature; this terminology stems from the transfer function approach to linear systems, in which these eigenvalues appear as the zeros of the denominator polynomial of the transfer function. The problem of pole placement by state feedback is: Given a system matrix \( A \) and an input mapping \( B \), find a *feedback mapping* \( F \) such that the eigenvalues of \( A + BF \) have a prescribed location. A celebrated theorem says that if the pair \((A,B)\) is controllable, then this can always be done subject only to the restrictions imposed by the dimension of the state space and by the use of the real number field. The version we shall present is a slight variation of the one given by Wonham (1967); other versions were proved by J. Bertram in 1959 (according to Kalman, Falb and Arbib (1969; p.49)), Rissanen (1960) and Popov (1964).

To get a proper statement of the result, let us introduce some terminology. A *multiplicity function* will be a nonnegative function from \( \mathbb{F} \) to \( \mathbb{Z} \) that assumes nonzero values only at finitely many points of \( \mathbb{F} \). The
total multiplicity of such a function is defined by

\[ m(f) = \sum_{\lambda \in \mathbb{C}} f(\lambda). \]

We shall say that a multiplicity function \( f \) is symmetric if \( f(\lambda) = f(\bar{\lambda}) \) for all \( \lambda \in \mathbb{C} \). Now we can state:

**Theorem 1.1** Let a system matrix \( A: X \to X \) and an input mapping \( B: U \to X \) be given, and suppose that the pair \((A,B)\) is controllable. For any symmetric multiplicity function \( f \) of total multiplicity \( n = \dim X \), there exists a real feedback mapping \( F: X \to U \) such that \( \sigma(A+BF) = f \).

**Proof** See Wonham (1979; p. 50)

More detailed results are known (see Rungenbrock (1970)), but this form of the theorem will suffice for our purposes. One may ask what happens if the pair \((A,B)\) is not controllable; the answer follows almost immediately from the theorem.

**Corollary 1.2** Let a system matrix \( A: X \to X \) and an input mapping \( B: U \to X \) be given. For all \( F: X \to U \), the subspace \( \langle A | \text{Im } B \rangle \) is \((A+BF)\)-invariant, and we have

\[ \sigma(A: X/\langle A | \text{Im } B \rangle) = \sigma(A + BF: X/\langle A | \text{Im } B \rangle). \]

On the other hand, given any symmetric multiplicity function \( f \) with total multiplicity equal to \( \dim \langle A | \text{Im } B \rangle \), there exists an \( F: X \to U \) such that

\[ \sigma(A + BF: \langle A | \text{Im } B \rangle) = f. \]

**Proof** Take \( x \in \langle A | \text{Im } B \rangle \) and \( F: X \to U \); then \( Ax \in \langle A | \text{Im } B \rangle \) because \( \langle A | \text{Im } B \rangle \) is \( A \)-invariant and \( BF \in \langle A | \text{Im } B \rangle \) because \( BF \in \text{Im } B \). So \( (A+BF)x \in \langle A | \text{Im } B \rangle \) and we see that \( \langle A | \text{Im } B \rangle \) is \((A+BF)\)-invariant for all \( F \). The actions of \( A \) and \( A + BF \) modulo \( \langle A | \text{Im } B \rangle \) are the same because \( \text{Im}(A-(A+BF)) \in \text{Im } B \); hence the equality (1.18). The final statement of the corollary follows by applying Theorem 1.1 to the pair of mappings obtained by restriction to \( \langle A | \text{Im } B \rangle \).
We shall say that the pair \((A, B)\) is *stabilizable* if there exists a feedback mapping \(F\) such that \(\sigma(A + BF) \subseteq \mathbb{E}_g\).

**COROLLARY 1.3** The pair \((A, B)\) is stabilizable if and only if

\[(1.20) \quad \sigma(A: X/<A | \text{Im } B>) \subseteq \mathbb{E}_g.\]

**PROOF.** This is immediate from the foregoing corollary. \(\Box\)

We shall also need the dual results with respect to a pair consisting of a system matrix \(A\) and an output mapping \(C\). The problem is then to assign the eigenvalues of \(A + GC\) by a suitable choice of the *injection matrix* \(G: Y \rightarrow X\). By dualization of Corollary 1.2, we obtain:

**COROLLARY 1.4** Let a system matrix \(A: X \rightarrow X\) and an output mapping \(C: X \rightarrow Y\) be given. For all \(G: Y \rightarrow X\), the subspace \(<\text{Ker } C \mid A>\) is \((A + GC)-\)invariant, and we have

\[(1.21) \quad \sigma(A: <\text{Ker } C \mid A>) = \sigma(A + GC: <\text{Ker } C \mid A>).\]

On the other hand, given any symmetric multiplicity function \(f\) with total multiplicity equal to \(\dim X/<\text{Ker } C \mid A>\), there exists a \(G: Y \rightarrow X\) such that

\[(1.22) \quad \sigma(A + GC: X/<\text{Ker } C \mid A>) = f.\]

The pair \((C, A)\) is said to be *detectable* if there exists an injection mapping \(G\) such that \(\sigma(A + GC) \subseteq \mathbb{E}_g\).

**COROLLARY 1.5** The pair \((C, A)\) is detectable if and only if

\[(1.23) \quad \sigma(A: <\text{Ker } C \mid A>) \subseteq \mathbb{E}_g.\]

The pole placement results of Corollary 1.2 and Corollary 1.4 can be expressed by means of simple lattice diagrams. The word "free" indicates parts on which eigenvalues can be freely assigned (these are \(\sigma(A + BF: <A \mid \text{Im } B>)\) in Corollary 1.4, and \(\sigma(A + GC: X/<\text{Ker } C \mid A>)\) in Corollary 1.6), whereas the word "fixed" refers to parts on which the poles are fixed (which are, respectively, \(\sigma(A + BF: X/<A \mid \text{Im } B>)\) and \(\sigma(A + GC: <\text{Ker } C \mid A>)\)).
These diagrams should be compared to the block diagrams in Figs. 1.1 and 1.2. Further on, we shall use more complicated diagrams of the above kind.

1.3. Controlled and conditioned invariance

We do not only want to consider the change in eigenvalues that can be brought about by going from $A$ to $A + BF$; we also need to have information about the subspaces which can be made invariant by feedback. This may be necessary, for instance, to see if certain structural features can be assigned by a suitable choice of the control law. So let us suppose that a system matrix $A$, acting on the state space $X$, and an input mapping $B$ are given. We are led to the following definition: A subspace $V$ of $X$ is a \textit{controlled invariant subspace} if there exists a feedback mapping $F$ such that $(A + BF)V \subset V$.

There are many ways to characterize controlled invariant subspaces; for instance, one can show that $V$ is controlled invariant if and only if for every $x_0 \in V$ there is a piecewise continuous control function $u(\cdot)$ such that the solution of the controlled equation (1.2) remains in $V$ for all time (see BASILE & MARRO (1969)). One characterization that is particularly convenient, because it is stated purely in terms of the pair $(A, B)$ and the subspace $V$, is the following:

\textbf{PROPOSITION 1.6} A subspace $V$ of $X$ is controlled invariant if and only if

\begin{equation}
AV \subset V + \text{Im } B.
\end{equation}

\textbf{PROOF} The proof is found in WONHAM (1979; p.88).
Controlled invariant subspaces are also called \((A \mod B)\)-invariant subspaces, or \((A, B)\)-invariant subspaces; the reason for this terminology is given by the above proposition. The term "controlled invariant" stems from BASTILE & MARRO (1969); the adjective "\((A, B)\)-invariant" has been introduced by WONHAM & MORSE (1970).

Dually, we define: A subspace \(T\) of \(X\) is a **conditioned invariant subspace** (with respect to a given pair \((C, A)\)) if there exists an injection mapping \(G\) such that \((A + GC)T \subseteq T\). We have the following characterization:

**PROPOSITION 1.7** A subspace \(T\) of \(X\) is conditioned invariant if and only if

\[
A(T \cap \text{Ker} C) \subseteq T.
\]

It is immediately clear from Prop. 1.6 that the set of controlled invariant subspaces is closed under the operation of subspace addition. This implies that, with every subspace \(K\) of \(X\), there is a largest controlled invariant subspace contained in \(K\); this subspace will be denoted by \(V^*(K)\). We have the following algorithm to compute \(V^*(K)\) for any \(K\).

**PROPOSITION 1.8** Let a pair of mappings \((A, B)\) and a subspace \(K\) of \(X\) be given. Define the sequence \(V^k(k = 0, 1, 2, \ldots)\) according to

\[
\begin{align*}
V^0 &= K \\
V^{k+1} &= \{ x \in K \mid Ax \in V^k + \text{Im} B \}.
\end{align*}
\]

Then the sequence \(V^k\) is decreasing, and the limit subspace (which is reached after a number of steps at most equal to \(\dim V\)) is \(V^*(K)\).

**PROOF** See WONHAM (1979; p.91).

Again, other interpretations of \(V^*(K)\) are possible; for instance, this subspace can also be considered as the set of initial values such that the solution of the equation \((1.2)\) can be kept inside \(K\) by a piecewise continuous control function \(u(\cdot)\) (see HAUTUS (1980)). If one uses a discrete time parameter, then this idea naturally leads to the algorithm \((1.26)\).

From Prop. 1.7, one sees that the set of conditioned invariant subspaces is closed under intersection. Consequently, given a subspace \(E\) of \(X\) there is a smallest conditioned invariant subspace containing \(E\), which
is denoted by $T^*(E)$, and which can be computed by an algorithm dual to the one given above.

1.4. Pole placement under restrictions

In sections 1.2 and 1.3, we have discussed the effect of state feedback on the eigenvalues and on the invariant subspaces of the system matrix. We now want to combine these two aspects. One important question is: How much freedom is left in the assignment of eigenvalues of $A + BF$, if it is required that a given subspace should be made invariant?

Let us introduce, for a controlled invariant subspace $V$,

$$ F(V) = \{F : X \rightarrow U \mid (A+BF)V \subset V\}. $$

Dually, we write for a conditioned invariant subspace $T$:

$$ G(T) = \{G : Y \rightarrow X \mid (A+GC)T \subset T\}. $$

Our problem is now to determine the amount of freedom one has in choosing the eigenvalues of $A + BF$ when $F$ is restricted to $F(V)$. This can be solved completely.

**Theorem 1.9** Let a system matrix $A : X \rightarrow X$ and an input mapping $B : U \rightarrow X$ be given; let $V$ be a controlled invariant subspace with respect to the pair $(A,B)$. For all $F : X \rightarrow U$, the subspace $<A \mid \text{Im } B + V>$ is $(A+BF)$-invariant, and we have

$$ g(A : X/(<A \mid \text{Im } B + V>)) = g(A + BF : X/(<A \mid \text{Im } B + V>)). $$

Moreover, if we define the subspace $R_F$ for $F \in F(V)$ by

$$ R_F = \langle A + BF \mid \text{Im } B \cap V >, $$

then for all $F_1, F_2 \in F(V)$ we have $R_{F_1} = R_{F_2} = R$ and

$$ g(A + BF_1 : V/R) = g(A + BF_2 : V/R). $$

On the other hand, given any symmetric multiplicity
function $f_1$ with total multiplicity equal to $\dim \mathcal{R}$, and another symmetric multiplicity function $f_2$ with total multiplicity equal to $\dim (\langle A \mid \text{Im } B \rangle + \mathcal{V})/\mathcal{V}$, there exists an $F \in F(\mathcal{V})$ such that

$$\sigma(A + BF; \mathcal{R}) = f_1$$

$$\sigma(A + BF; (\langle A \mid \text{Im } B \rangle + \mathcal{V})/\mathcal{V}) = f_2.$$ 

Before we turn to the proof of this theorem, let us make some remarks. The theorem says that if a feedback mapping makes $\mathcal{V}$ invariant for $A + BF$, then it must do the same with $\mathcal{R}$ and $\langle A \mid \text{Im } B \rangle + \mathcal{V}$. The three subspaces $\mathcal{R}, \mathcal{V}$ and $\langle A \mid \text{Im } B \rangle + \mathcal{V}$ form a chain, and so we can draw the following lattice diagram to describe the pole placement properties announced in the theorem:

```
X
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
  |  |
fixed
  \\
\langle A \mid \text{Im } 3 \rangle + \mathcal{V}
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
free
  \\
\mathcal{V}
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
fixed
  \\
\mathcal{R}
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
free
  \\
\mathcal{O}
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
A + BF
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
/  \\
(R \in F(\mathcal{V}))
```

Figure 1.4: Pole placement under invariance of $\mathcal{V}$.

This corresponds to the following block diagram (compare Fig. 1.1):
**Figure 1.5: A system with a prescribed invariant subspace.**

**Proof.** It follows immediately from Prop. 1.6 that $A | B > + V$ is an $A$-invariant subspace containing $\text{Im} B$; the statements concerning this subspace then follow as in the proof of Cor. 1.2. For the second part of the proof, take $F_1$ and $F_2$ from $F(V)$, and let $x \in V$. Then we must have $(A+BF_1)x = (A+BF_2)x \in \bar{V}$, but also $(A+BF_1)x = (A+BF_2)x = B(F_1 - F_2)x \in \text{Im} B$. It is now clear that $\text{Im}((A+BF_1); V) = (A+BF_2); V) \subset \text{Im} B \cap V$ for all $F_1, F_2 \in F(V)$; consequently, $F_1^* = F_2^*$ for all such $F_1$ and $F_2$, and (1.31) holds.

Now, let multiplicity functions $f_1$ and $f_2$ be given as described in the statement of the theorem. Then we must show that a feedback mapping $F$ can be constructed satisfying (1.32) and (1.33). To do this, we start by picking an arbitrary $F_0 \in F(V)$. Write $A_0 = A + BF_0$; define $U_0 = \{u \in U | Bu \in V \}$ and write $B_0$ for the restriction of $B$ to $U_0$, considered as a mapping to $V$.

Then we have $R = A_0 | \text{Im} B_0 >$, and by Cor. 1.2 there exists a mapping $F_1: V \to U_0$ such that $\sigma(A_0^*B_0F_1; R) = f_1$.

For the next step, let $P$ be the canonical projection of $X$ onto the factor space $X/V$. Write $A_2 = A + BF_0; \ X/V$ and define $B_2: U \to X/V$ by $B_2 = PB$. Denote $S = P(A | \text{Im} B > + V)$. Then we have: $S = A_2 | \text{Im} B_2 >$.

For, from the fact that $A | B > + V$ is $A + BF_0$-invariant and contains $\text{Im} B$, it follows readily that $S$ is $A_2$-invariant and contains $\text{Im} B_2$. Let
be another \( A_2 \)-invariant subspace of \( X/V \) containing \( \text{Im} B_2 \); then
\[
\{ x \in X \mid Px \in \mathcal{S} \}
\]
is an \( A + BF_0 \)-invariant subspace of \( X \) containing both \( \text{Im} B \) and \( V \), so that we get
\[
\{ x \in X \mid Px \in \mathcal{S} \} = \langle A \mid \text{Im} B \rangle + V \quad \text{and consequently,}
\]
\[\mathcal{S} = S.\]Now we can again appeal to Cor. 1.2 and conclude that there exists a mapping \( F_2 : X/V \to U \) such that \( \sigma(A_2 + BF_2; S) = f_2. \)

We now construct our feedback matrix \( F \). Let \( F_3 \) be any extension of \( F_1 \) to a mapping from \( X \) into \( U \) such that \( \text{Im} F_3 \subseteq U_0 \), and define
\[
F_4 : X \to U \quad \text{by} \quad F_4 = F_2 P. \]Then \( F = F_0 + F_3 + F_4 \) satisfies (1.32) and (1.33):

\[
\sigma(A + BF; R) = \sigma(A + BF_0 + BF_3; R) =
\]
\[
= \sigma(A_0 + BF_3; R) = f_1
\]

\[
\sigma(A + BF; \langle A \mid \text{Im} B \rangle + V/V) =
\]
\[
= \sigma(A + BF_0 + BF_4; \langle A \mid \text{Im} B \rangle + V/V) =
\]
\[
= \sigma(A_2 + BF_2; S) = f_2.
\]

Results like this have appeared before in the literature, although not in the complete form as given above; see WONHAM (1979; p. 111) and MORSE (1973b; Appendix, Lemma A. 2). An alternative proof using matrix arguments has been given in SCHUMACHER (1980 a).

We shall also need an extension of the theorem, in which \( F \) is restricted to an intersection of two sets of the form \( F(V) \). First we mention a simple result which can easily be proved (see also WONHAM (1979; exercise 9.1)).

**LEMMA 1.10** Let \( V_1 \) and \( V_2 \) be controlled invariant subspaces with respect to the pair \( (A, B) \) and suppose that \( V_1 \subseteq V_2 \). Given any \( F_1 \in F(V_1) \), there exists an \( F_2 \in F(V_2) \cap F(V_1) \) such that \( A + BF_2; V_1 = A + BF_1; V_1 \).

In particular, the lemma shows that \( F(V_1) \cap F(V_2) \) is non-empty if \( V_1 \subseteq V_2 \). We can consider the pole placement problem when \( F \) is restricted to a set of this type; again, a complete solution is available.

**THEOREM 1.11** Let a system matrix \( A : X \to X \) and an input mapping \( B : U \to X \) be given; let \( V_1 \) and \( V_2 \) be controlled invariant subspaces with respect to the pair \( (A, B) \), and suppose that \( V_1 \subseteq V_2 \). Then \( R_1 = \langle A + BF \mid V_1 \cap \text{Im} B \rangle \).
and \( R_2: = \langle A+BF \mid V_2 \cap \text{Im } B \rangle \) do not depend on the specific choice of \( F \) from \( \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \). All subspaces in the chain

\[(1.36) \quad 0 \subset R_1 \subset V_1 \subset V_1 + R_2 \subset V_2 \subset V_2 + \langle A \mid \text{Im } B \rangle = X\]

are invariant for each \( F \in \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \). Our freedom in selecting the eigenvalues of \( A + BF \) when \( F \) is restricted to \( \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \) can be described in the following lattice diagram:

```
    X
  /   \    \  \\
 fixed  \  V_2 + \langle A \mid \text{Im } B \rangle free
      \   |   |
      \  V_2固定 free
          \   |
          \  V_1 + R_2
             \ |
             \ fixed free
               \ |
                  \ fixed free
                      \ |
                          \ free
                              \ A+BF

(F \in \mathcal{F}(V_1) \cap \mathcal{F}(V_2))
```

Figure 1.6: Pole placement under invariance of \( V_1 \) and \( V_2 \).

**Proof** The facts that \( R_1 \) and \( R_2 \) do not depend on \( F \in \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \) and that the subspaces in the chain (1.36) are invariant for any such \( F \) follow just as in the proof of Theorem 1.9. The same holds for the 'fixed' parts in the above diagram.

It remains to show that an \( F \) can be selected to obtain any desired eigenvalues (with suitable symmetry and multiplicity) in the parts marked 'free' in the lattice diagram. To do this, first consider the subspace \( V_2 \). Take an arbitrary \( F_0 \in \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \). Write \( A_0 = A + BF_0; V \); define \( U_0 = \{ u \in U \mid Bu \in V_2 \} \) and write \( B_0 \) for the restriction of \( B \) to \( U_0 \), considered as a mapping to \( V_2 \). Then \( V_1 \) is a controlled invariant subspace with
respect to \( (A_0, B_0) \) \((V_1 \text{ is even } A_0\text{-invariant}), \) and so we can apply Thm.1.9 to conclude that there exists a feedback matrix \( F_1 : V_2 \to U_0 \) such that \( A_0 + B_0 F_1 : V_2/(V_1 + <A_0 | \text{ Im } B_0>) \) and \( A_0 + B_0 F_2 : <A_0 | V_1 \cap \text{ Im } B_0> \) have prescribed multiplicity functions. Note that \( \text{ Im } B_0 = V_2 \cap \text{ Im } B \) so that 
\[
<A_0 | \text{ Im } B_0> = R_2 \quad \text{and} \quad <A_0 | V_1 \cap \text{ Im } B_0> = <A_0 | V_1 \cap \text{ Im } B> = R_1.
\]

If we extend \( F_1 \) in an arbitrary way to a mapping \( F_3 : \mathcal{X} \to \mathcal{U} \) such that \( \text{ Im } F_3 \subseteq U_0 \), then we shall have \( A + B(F_0 + F_3) : V_2/(V_1 + R_2) = A_0 + B_0 F_1 : V_2/(V_1 + R_2) \) and \( A + B(F_0 + F_3) : R_2 = A_0 + B_0 F_1 : R_2 \). Now we can proceed just as in the proof of Thm. 1.9 to find a suitable mapping \( F_2 : \mathcal{X}/V \to U \). We finally define \( F \) by \( F = F_0 + F_3 + F_4 \) where \( F_4 = F_2 P \). Then \( A + BF \) will have the desired properties.

We could go on by induction and prove a pole placement theorem for \( F_\in \mathcal{F}(V_1) \cap \ldots \cap \mathcal{F}(V_k) \) (for a chain \( V_1 \subseteq \ldots \subseteq V_k \) of controlled invariant subspaces), but perhaps this would take us a little bit too far at this point. Note that there is a striking contrast between our success in dealing with such restrictions on \( F \), and the great difficulties that arise when one tries to do pole placement for \( F \in \mathcal{F}(X \to U | K \subseteq \ker F) \), where \( K \) is an arbitrary given subspace. (This is the problem of "pole placement by output feedback"; \( F \) can be written as \( F = KC \) for some \( K \) if and only if \( \ker C \subseteq \ker F \).)

We may extend the lattice diagram in Fig. 1.6 to get a more complete picture of the situation, as follows: (see Fig. 1.7, next page)

This diagram shows that the poles that are fixed when \( F \) is restricted to \( \mathcal{F}(V_1) \cap \mathcal{F}(V_2) \) are precisely the fixed poles following from the restriction \( F \in \mathcal{F}(V_1) \) taken together with those following from the restriction \( F \in \mathcal{F}(V_2) \), with the understanding that overlapping parts are counted only once. So the restrictions \( F \in \mathcal{F}(V_1) \) and \( F \in \mathcal{F}(V_2) \) do not interfere, in the sense that combining them does not add any extra fixed poles to those that must be present due to each restriction taken separately.

1.5. Stabilizability and detectability

Thm. 1.9 leads easily to a characterization of those controlled invariant subspaces \( V \) for which there exists an \( F \in \mathcal{F}(V) \) such that \( \sigma(A+BF; V) \subseteq C \). These subspaces are called stabilizability subspaces (HAUTUS (1980)). The name controllability subspace is used in the special case where the eigenvalues of \( A + BF; V \) can freely be assigned by a suitable choice of \( F \in \mathcal{F}(V) \).
This term is much older (WONHAM & MORSE (1975)). For alternative interpretations of these concepts, we refer again to HAUTUS (1980).

Sometimes, it will be important to know something about \( o(A+BF; X/V) \). We shall say that the controlled invariant subspace \( V \) is outer-stabilizable if there exists an \( F \in \mathcal{F}(V) \) such that \( A + BF; X/V \) is stable. Correspondingly, stabilizability subspaces will sometimes be called inner-stabilizable. We also introduce the dual terminology. If \( T \) is a conditioned invariant subspace, then we shall say that \( T \) is outer-detectable (or that \( T \) is a detectability subspace) if there exists a \( G \in \mathcal{G}(T) \) such that \( A + GC; X/T \) is stable, and we shall say that \( T \) is inner-detectable if there exists a \( G \in \mathcal{G}(T) \) such that \( A + GC; T \) is stable.

From Thm. 1.9, we immediately obtain the following characterization of inner- and outer-stabilizable controlled invariant subspaces.
PROPOSITION 1.12 Let $V$ be a controlled invariant subspace with respect to the pair $(A, B)$. Then $V$ is inner-stabilizable if and only if

$$\sigma(A + BF: V/(\langle A + BF \mid V \cap \text{Im } B \rangle)) \subseteq \mathbb{C}_b$$

for any $F \in \mathcal{F}(V)$, and $V$ is outer-stabilizable if and only if

$$\sigma(A: X/(V + \langle A \mid \text{Im } B \rangle)) \subseteq \mathbb{C}_b.$$

The characterization of stabilizability subspaces given by (1.37) is not always convenient, because it depends on the computation of an $F \in \mathcal{F}(V)$. This dependence is not present in the following characterization of stabilizability subspaces, which as far as the author knows is new (see also Schumacher (1980 d)).

THEOREM 1.13 Let $V$ be a controlled invariant subspace with respect to the pair $(A, B)$. Then $V$ is inner-stabilizable if and only if

$$\lambda(A) V + \text{Im } B = V - \text{Im } B \quad \text{for all } \lambda \in \mathbb{C}_b.$$ 

PROOF "only if": Suppose that the controlled invariant subspace $V$ is inner-stabilizable; then we can find $F \in \mathcal{F}(V)$ such that $A + BF: V$ is stable. No $\lambda \in \mathbb{C}_b$ is an eigenvalue of $A + BF: V$, and consequently

$$\lambda(A + BF) V = V \quad \text{for all } \lambda \in \mathbb{C}_b.$$ 

The desired conclusion now follows from noting that, for all $F \in \mathcal{F}(V)$,

$$\lambda(A + BF) V + \text{Im } B = (\lambda - A)V + \text{Im } B.$$ 

"if": Now suppose that $V$ is a controlled invariant subspace for which (1.39) holds. By Thm. 1.9, we can take $F \in \mathcal{F}(\mathcal{F})$ such that $A + BF: \mathcal{R}$ is stable, where $\mathcal{R}$ is defined by (1.30). With $F$ chosen in this way, we contend that $A + BF: V$ is stable.

Take any $\lambda \in \mathbb{C}_b$. From (1.39) and the general equality (1.41), we get

$$V = (\lambda - (A + BF)) V + \text{Im } B.$$
Because $(\lambda-(A+BF))V = V$, this is equivalent to

$$V \subseteq (\lambda-(A+BF))V + (V \cap \text{Im } B)$$

which immediately implies

$$V \subseteq (\lambda-(A+BF))V + \mathbb{R}.$$

Because it follows from $\lambda \notin \sigma(A+BF; \mathbb{R})$ that $(\lambda-(A+BF))\mathbb{R} = \mathbb{R}$, we obtain:

$$V \subseteq (\lambda-(A+BF))V + (\lambda-(A+BF))\mathbb{R} = (\lambda-(A+BF))V \subseteq V.$$

So our final conclusion is

$$V = (\lambda-(A+BF))V$$

which shows that $\lambda$ is not an eigenvalue of $A + BF$: $\Box$.

The dual result is

**Theorem 1.14** Let $T$ be a conditioned invariant subspace with respect to the pair $(C,A)$. Then $T$ is outer-detectable if and only if

$$\lambda - A)^{-1} T \cap \text{Ker } C = T \cap \text{Ker } C \quad \text{for all } \lambda \in \mathbb{C}.$$

Of course, $\lambda - A$ need not be invertible for all $\lambda \in \mathbb{C}$; by the notation $(\lambda-A)^{-1} T$, we mean $\{x \in X \mid (\lambda-A)x \in T\}$.

The condition (1.39) can be formulated somewhat differently if we note that the inclusion $(\lambda-A)V + \text{Im } B \subseteq \mathcal{V} + \text{Im } B$ holds for any $\lambda \in \mathbb{C}$ if $V$ is controlled invariant; this follows readily from Prop. 1.6. Also, the inclusion $\text{Im } B \subseteq (\lambda-A)V + \text{Im } B$ is trivial. So we get:

**Corollary 1.15** Let $V$ be a controlled invariant subspace with respect to the pair $(A,B)$. Then $V$ is inner-stabilizable if and only if

$$V \subseteq (\lambda-A)V + \text{Im } B \quad \text{for all } \lambda \in \mathbb{C}.$$

The dual version reads:
COROLLARY 1.16 Let $T$ be a conditioned invariant subspace with respect to the pair $(G,A)$. Then $T$ is outer-detectable if and only if

\[(1.49) \quad T \supseteq (\lambda - A)^{-1}T \cap \ker C \quad \text{for all } \lambda \in \mathbb{C}.\]

From Thm. 1.13, it follows immediately that the set of stabilizability subspaces is closed under the operation of subspace addition. In particular, to every given subspace $K$ in $X$ there is a largest stabilizability subspace contained in $K$. This also proved, but by a different method, in WONHAM (1979; p. 114). We shall denote the largest stabilizability subspace in $K$ by $\mathcal{V}^*_K$. Of course, there is also a smallest detectability subspace containing a given subspace $E$; this will be denoted by $\mathcal{V}^*_E$. There is a constructive procedure available to compute $\mathcal{V}^*_K$ for any given $K$; see WONHAM (1979; p. 114). A dual procedure computes $\mathcal{V}^*_E$ for any $E$. We shall denote the largest element in the set of all stabilizability subspaces by $X^\text{stab}$, and $X^\text{det}$ will be the smallest element in the set of all detectability subspaces.

The span of the generalized eigenspaces of the system matrix $A$ associated with eigenvalues in $\mathbb{C}$ will be denoted by $X^G(A)$. That is, if we factorize the characteristic polynomial $p(\lambda)$ of $A$ as $p(\lambda) = p^G(\lambda)p^b(\lambda)$ where $p^G$ has its zeros in $\mathbb{C}$ and $p^b$ has its zeros in $\mathbb{C}^b$, then

\[(1.50) \quad X^G(A) = \ker(p^G(A)).\]

Likewise, we define

\[(1.51) \quad X^b(A) = \ker(p^b(A))\]

and this is the span of the generalized eigenspaces of $A$ that are associated with eigenvalues in $\mathbb{C}^b$.

After these definitions, we are able to identify $X^\text{stab}$ as follows.

**Lemma 1.17** $X^\text{stab} = X^G(A) + <A | \text{Im } B>$. 

**Proof** Clearly, $X^G(A)$ is a stabilizability subspace; $<A | \text{Im } B>$ is even a controllability subspace. So $X^G(A) + <A | \text{Im } B>$ is a stabilizability subspace and hence we must have $X^\text{stab} \supseteq X^G(A) + <A | \text{Im } B>$. We see that
$X_{\text{stab}} \supset \text{Im } B$ and so $X_{\text{stab}}$ must be $A$-invariant. Now take $F$ such that $A + BF$:

$X_{\text{stab}}$ is stable. Then

\begin{equation}
\sigma(A + BF; X_{\text{stab}}/\langle X_{\text{g}}(A) + <A | \text{ Im } B > \rangle) = \\
= \sigma(A; X_{\text{stab}}/\langle X_{\text{g}}(A) + <A | \text{ Im } B > \rangle).
\end{equation}

On the one hand, we have

\begin{equation}
\sigma(A; X_{\text{stab}}/\langle X_{\text{g}}(A) + <A | \text{ Im } B > \rangle) \subseteq \sigma(A + BF; X_{\text{stab}}) \subseteq \mathcal{E}_{g}
\end{equation}

but on the other

\begin{equation}
\sigma(A; X_{\text{stab}}/\langle X_{\text{g}}(A) + <A | \text{ Im } B > \rangle) \subseteq \sigma(A; X/X_{\text{g}}(A)) \subseteq \mathcal{E}_{b}.
\end{equation}

This can only be so if $X_{\text{stab}}$ is in fact equal to $X_{\text{g}}(A) + <A | \text{ Im } B >$. $\square$

The dual result is

**Lemma 1.18** $X_{\text{det}} = X_{d}(A) \cap \text{Ker } C | A >$.

We can use the subspace $X_{\text{stab}}$ to give an alternative characterization of outer-stabilizability.

**Proposition 1.19** A controlled invariant subspace $V$ is outer-stabilizable if and only if

\begin{equation}
V + X_{\text{stab}} = X.
\end{equation}

**Proof** First suppose that $V$ is outer-stabilizable. Then it follows from Prop. 1.12 that $V + <A | \text{ Im } B > \supset X_{d}(A)$. Hence, $V + X_{\text{stab}} \supset X_{d}(A) + X_{g}(A) = X$.

Now suppose that (1.55) holds. Then (using (1.16)):

\begin{equation}
\sigma(A; X/(V + <A | \text{ Im } B >)) = \\
= \sigma(A; (V + <A | \text{ Im } B > + X_{g}(A))/(V + <A | \text{ Im } B >)) = \\
= \sigma(A; X_{g}(A)/(X_{g}(A) \cap (V + <A | \text{ Im } B >))) \subseteq \\
\subseteq \sigma(A; X_{g}(A)) \subseteq \mathcal{E}_{g}.
\end{equation}
This shows that $V$ is outer-stabilizable.

We can easily derive some alternative characterizations of stabilizability of a pair $(A, B)$ from the above results.

**Corollary 1.20** Let a system matrix $A$ and an input mapping $B$ be given. Then the following statements are equivalent:

(i) the pair $(A, B)$ is stabilizable;

(ii) $X^\lambda_B(A) = \langle A \mid \text{Im } B \rangle$;

(iii) $X_{stab} = X$;

(iv) $\text{Im}(\lambda-A) + \text{Im } B = X$ for all $\lambda \in \mathbb{E}_B$.

**Proof** The equivalence between (i) and (ii) is just a reformulation of Cor. 1.3. The pair $(A, B)$ is stabilizable if and only if the zero subspace is outer-stabilizable; the equivalence between (i) and (iii) thus follows from Prop. 1.19. Also, the pair $(A, B)$ is stabilizable if and only if the whole space $X$ is inner-stabilizable; then Thm. 1.13 shows that (i) and (iv) are equivalent. \[\square\]

The criterion given in (iv) is called the Hautus test for stabilizability (Hautus (1969)). The corresponding characterizations of detectability are given by our final result in this chapter.

**Corollary 1.21** Let a system matrix $A$ and an output mapping $C$ be given. Then the following statements are equivalent:

(i) the pair $(C, A)$ is detectable;

(ii) $X^\lambda_C(A) = \langle \text{Ker } C \mid A \rangle$;

(iii) $X^\lambda_{det} = 0$;

(iv) $\text{Ker}(\lambda-A) \cap \text{Ker } C = \emptyset$ for all $\lambda \in \mathbb{E}_B$.

Note that the criterion (iv) may be rephrased as follows: The output mapping $C$ does not annihilate any eigenvector of $A$ corresponding to a 'bad' eigenvalue of $A$. 
CHAPTER 2

THE COMPENSATOR PROBLEM

In this chapter, we begin our study of dynamic feedback. The problem we shall consider is a basic one: to stabilize a given system (described by a system matrix, an input mapping and an output mapping) by adding a compensator. The compensator brings its own dynamics into the feedback loop, which is why the term "dynamic feedback" is used for this type of automatic control.

It is easy to give necessary and sufficient conditions for the solvability of the problem of stabilizing a given system by dynamic feedback. Serious difficulties are encountered, however, if one tries to minimize the order of the dynamics that are introduced in the feedback loop. We shall present a new and general approach to the compensator problem, which allows for low-order solutions. The basic idea of this method is that reduction of the compensator order can be obtained if the 'feedback' and the 'observer' part of the controller are suitably adapted to each other.

We also describe the class of compensators that can be obtained by our method. It will be shown that this class is given precisely by those compensators that have a 'feedback-observer' interpretation. Comparing the usual state-space methods of compensator design with our approach, we find that all these methods are recovered as special cases of our theory. Finally, we are also able to present a systematic searching procedure for low-order compensators.

The chapter is organized as follows. In the first section, we discuss the concept of dynamic feedback and introduce some notation. The second section gives an extensive discussion of the compensator problem and of our main results on their interpretation. Proofs are given in Section 3 (for the construction theorem) and in Section 5 (for the interpretation of the class of compensators that are thus constructed). In the intermediate Section 4 we discuss several methods of compensator design, including our own proposal.

2.1. Introduction

Our starting point is the linear system given by
(2.1.1) \[ x'(t) = Ax(t) + Bu(t) \]

(2.1.2) \[ y(t) = Cx(t). \]

This system can be represented by the following block diagram:

![Block Diagram](image)

**Figure 2.1:** Open-loop system.

We now want to "close the loop" by connecting the output \( y(t) \) to the input \( u(t) \). Of course one can imagine many ways to do this, but our object of study will be linear dynamic feedback. This term is used for the form of automatic control that is provided by an additional finite-dimensional linear system (the compensator) which takes the observation \( y(t) \) as its input and which specifies the control function \( u(t) \) at its output, in the following way:

(2.2.1) \[ w'(t) = Nw(t) + My(t) \]

(2.2.2) \[ u(t) = Lw(t). \]

Combining (2.1) and (2.2) gives us the closed-loop system.

![Closed-loop System](image)

**Figure 2.1:** Closed-loop system.
The word "dynamic feedback" is used because the compensator introduces dynamics in the feedback loop; sometimes one also uses the term "integral control", because in (2.2) the control function is obtained via an integration of the observed variable y(t). One may also apply "static feedback" or "proportional control" by directly connecting the control to the observation:

(2.3) \[ u(t) = Ky(t). \]

Of course, one can easily combine the two types of feedback in one compensator equation, by replacing (2.2.2) by

(2.2.2)’ \[ u(t) = Lw(t) + Ky(t). \]

In a block diagram, this may be illustrated as follows:

![Block Diagram of Dynamic and Static Feedback Combined](image)

**Figure 2.3: Dynamic and static feedback combined.**

However, we shall prefer to consider static and dynamic feedback as separate processes. So we shall only consider compensators of the 'purely integral' form (2.2), but sometimes we shall first change the system matrix from A to A + BKC by static feedback before adding a compensator of this type. The overall result is still the same, of course, and the corresponding controller will be referred to as a PI-Compensator (proportional/integral). Our view of this type of compensator is represented by Fig.2.4 rather than by Fig.2.3.
Now let us explain some notation and terminology that will be used in the sequel. The state space of the compensator equation (2.2) will be written as $\mathcal{W}$; the dimension of this space is called the order of the compensator, for which we shall use the letter $k$. The equations (2.1) and (2.2) can be taken together to form the extended system equation:

$$\frac{d}{dt}(\begin{bmatrix} x \\ \mathcal{W} \end{bmatrix})(t) = (A \quad BL)(\begin{bmatrix} x \\ \mathcal{W} \end{bmatrix})(t).$$

This is a differential equation in the extended state space $X \oplus \mathcal{W}$ which has dimension $n + k$. The matrix

$$A_e = (\begin{bmatrix} A & BL \\ MC & N \end{bmatrix})$$

is called the extended system matrix.

There are two natural mappings between the original state space $X$ and the extended state space $X \oplus \mathcal{W}$: the natural embedding $Q: X \to X \oplus \mathcal{W}$ defined by

$$Qx = \begin{bmatrix} x \\ 0 \end{bmatrix} \quad (x \in X)$$

and the canonical projection $P: X \oplus \mathcal{W} \to X$ defined by

$$P\begin{bmatrix} x \\ \mathcal{W} \end{bmatrix} = x \quad \left(\begin{bmatrix} x \\ \mathcal{W} \end{bmatrix} \in X \oplus \mathcal{W}\right).$$
Accordingly, there are two subspaces of $X$ naturally related to any subspace $M$ of $X \oplus W$:

\begin{align*}
(2.8) & \quad Q^{-1}M = \{ x \in X \mid \langle x \rangle_Q \in M \} \\
(2.9) & \quad PM = \{ x \in X \mid \exists \tilde{w} \in \tilde{W} : \langle \tilde{x} \rangle_{\tilde{W}} \in M \}.
\end{align*}

The first subspace can be considered as the intersection of $M$ with the $X$-plane, where as the second is the projection of $M$ onto that same plane. It should be noted, however, that both subspaces are considered as subspaces of $X$; one may view the pair $(Q^{-1}M, PM)$ in some sense as the representative, within the given state space $X$, of a subspace in the extended state space $X \oplus W$ which the designer has to construct. Below, we shall give a more precise meaning to this vague idea.

2.2. Problem statement and main results

In this chapter, disturbances will be modeled simply by a non-zero initial condition for the controlled system (2.4), and the purpose of the control will be to make the system return to its nominal operating point (represented by the origin in the state space) after such a disturbance. We shall consider more elaborate disturbance models and other control objectives in Chapter 3. Our problem here can thus be formulated as follows: Given a system matrix $A$, an input mapping $B$ and an output mapping $C$, find a linear space $W$ and mappings $N : W \rightarrow W$, $L : U \rightarrow U$ and $M : Y \rightarrow W$ such that the extended system matrix $A_e$ defined in (2.5) is stable; moreover, do this using a space $W$ of lowest possible dimension. The latter requirement is added because low-order compensators are in general easier to implement and more reliable than high-order compensators.

We shall call a compensator of the form (2.2) a stabilising compensator if it is defined by mappings $L$, $M$ and $N$ that make the extended system matrix $A_e$ stable. The conditions under which such a compensator exists are simple and well-known. (See also Section 3.2 and Section 2.4).

**Theorem 2.1** There exists a stabilising compensator for the system (2.1) if and only if the pair $(A, B)$ is stabilisable and the pair $(C, A)$ is detectable.
Several methods are known to construct a stabilizing compensator if one exists; we shall discuss some of them in Section 2.4. However, the minimal compensator problem of finding the minimal order of a stabilizing compensator for a given system has not been solved (and it will not be solved here). Still we can present a further analysis of the compensator problem; this will provide more insight into the problem and, moreover, suggest a way to find low-order compensators.

Our starting point is an observation concerning the pair \((Q^{-1}W, PM)\) related to any invariant subspace \(W\) of an extended system matrix of the form (2.5). To formulate this, let us say that a pair \((T, V)\) of subspaces of \(X\) is a compensator couple (with respect to the triple \((Q, A, B)\)) if the following holds:

\begin{align*}
\text{(cc1)} & \quad T \text{ is conditioned invariant (with respect to the pair } (C, A)) \\
\text{(cc2)} & \quad V \text{ is controlled invariant (with respect to the pair } (A, B)) \\
\text{(cc3)} & \quad T \subset V \\
\text{(cc4)} & \quad AT \subset V.
\end{align*}

This concept will be very important for us. For instance, we have the following result (which also explains the nomenclature); the proof will be given in Section 3.2, as we shall not need it in this chapter.

**Theorem 2.2** Suppose that \(W \subset X \oplus W\) is an invariant subspace for an extended system matrix \(A_e\) of the form (2.5). Then \((Q^{-1}W, PM)\) is a compensator couple.

The notion "compensator couple" is introduced here for the first time, although situations in which pairs of controlled and conditioned invariant subspaces arise have been studied before, particularly in the early Italian work on the geometric approach to linear systems (see for instance BASILE & MARRO (1969a)). In SCHUMACHER (1980 d,e), the notion of "\((C, A, B)\)-pair of subspaces" has been used; this is the same as a compensator couple, except that the condition (cc4) is lacking in the definition of a \((C, A, B)\)-pair. The two concepts are related via static feedback, in the following way.

**Lemma 2.3** Let \((T, V)\) be a \((C, A, B)\)-pair. Then there exists a static feedback
mapping \( e: V \rightarrow U \) such that \((T, V)\) is a compensator couple with respect to the triple \((C, A + BKC, B)\).

This lemma is a special case of Lemma 3.6, which will be proved in Ch. 3. We shall not need the result in this chapter.

Of course, the interesting question is whether we can reverse Thm. 2.2 in some way or another. Given a compensator couple, can we construct a compensator from it? The answer is yes, if certain additional conditions hold.

**Theorem 2.4** Suppose that we have a compensator couple \((T, V)\) and mappings \( F \in F(V) \) and \( G \in G(T) \) such that

\[
\begin{align*}
\text{(2.10.1)} & \quad \text{Ker } F = T \\
\text{(2.10.2)} & \quad \text{Im } G \subseteq V \\
\text{(2.11.1)} & \quad \sigma(A + BF; V) \subseteq \mathcal{E} \quad_8 \\
\text{(2.11.2)} & \quad \sigma(A + GC; X/T) \subseteq \mathcal{E} _8.
\end{align*}
\]

Then there exists a stabilizing compensator of order \( \dim V - \dim T \).

The proof is by construction; it will be given in Section 2.3. In Section 2.4, we shall show that several well-known methods of compensator design can be derived as special cases of the above theorem. There we shall also discuss the question how the theorem can be used in a systematic search for low-order compensators. Because the pole placement properties of \( A + BF \) under a restriction of the form \((2.10.1)\) are not easily described (a similar remark holds for \( A + GC \), of course), the utility of the theorem is certainly not immediately clear.

A nice theoretical property is that the conditions of the theorem are symmetric with respect to dualization. That is, if one would try to dualize the theorem, one would only get back the same statement (of course, and as always, after suitable renaming). Note that the compensator problem itself is also symmetric with respect to dualization.

To get some intuitive feeling for the conditions of the theorem, it is perhaps best to consider first the standard procedure for full-order
compensator design. This procedure is based on the idea of applying feedback to an estimate of the state. Suppose that the system (2.1) has been given, and define \( \hat{x}(t) \) by the equation

\[
\frac{d}{dt} \hat{x}(t) = (A+GC)\hat{x}(t) - Gy(t) + Bu(t).
\]

One easily checks that \( \hat{x}(t) \) is an estimate of \( x(t) \) in the sense that

\[
\frac{d}{dt}(x(t) - \hat{x}(t)) = (A+GC)(x(t) - \hat{x}(t)).
\]

If the matrix \( A + GC \) is stable, the error \( x(t) - \hat{x}(t) \) will approach zero for any initial values \( x(0) \) and \( \hat{x}(0) \). Let us assume that we have chosen \( G \) such that \( A + GC \) is stable, and also that we have found a feedback mapping \( F \) such that \( A + BF \) is stable. Defining \( u(t) \) by

\[
u(t) = F\hat{x}(t)\]

we obtain the following form for the equation (2.1.1):

\[
x'(t) = (A+BF)x(t) - BF(x(t) - \hat{x}(t)).
\]

Together with (2.13), this shows that \( x(t) \) will approach zero, as well as \( \hat{x}(t) \), for any initial values \( x(0) \) and \( \hat{x}(0) \). In other words, the equations (2.12) and (2.14) constitute a stabilizing compensator. We can write the equations in the form (2.2):

\[
\frac{d}{dt} \hat{x}(t) = (A+BF+GC)\hat{x}(t) - Gy(t)
\]

\[
u(t) = F\hat{x}(t).
\]

The extended system matrix \( A_e \) (as defined in (2.5)) becomes:

\[
A_e = \begin{pmatrix}
A & BF \\
-GC & A+BF+GC
\end{pmatrix}.
\]

It follows from (2.13) and (2.15) that
\( (2.18) \quad \sigma(A_e) = \sigma(A+B\mathcal{F}) + \sigma(A+GC). \)

This formula is connected to a natural interpretation of the action of the compensator \((2.16)\), in which there is a 'separation' between 'feedback dynamics' represented by \(\sigma(A+B\mathcal{F})\) and the 'observer dynamics' represented by \(\sigma(A+GC)\). Now, let us note that the equation \((2.16.1)\) can also be written in the following form:

\( (2.19) \quad \frac{d}{dt}\hat{x}(t) = (A+B\mathcal{F})\hat{x}(t) - GC(x(t) - \hat{x}(t)) \)

Viewed in this way, the evolution of \(\hat{x}(t)\) is described by the dynamics of \(A+B\mathcal{F}\) together with an input signal that enters through \(\mathcal{G}\). Now suppose that \(\mathcal{V}\) is an \((A+B\mathcal{F})\)-invariant subspace such that \(\text{Im}\mathcal{G} \subset \mathcal{V}\) (as in the conditions of the theorem). Then it is clear from \((2.19)\) that \(\hat{x}(t)\) will always remain in \(\mathcal{V}\) if it starts there. Because the compensator should work for any value of \(\hat{x}(0)\), this suggests (and we shall establish the correctness of this idea in the proof of Thm.2.4) that the state space of \(\hat{x}(t)\) could be taken equal to \(\mathcal{V}\) instead of \(\mathcal{X}\). Consequently, the compensator order is reduced from \(\text{dim } \mathcal{X}\) to \(\text{dim } \mathcal{V}\).

The order reduction is possible because we are using information about the feedback in the construction of the observer. The equation \((2.12)\) has the strong property of defining an observer for \(x(t)\) without any information about the control function \(u(t)\), but as a consequence the equation has to be formulated in the complete state space \(\mathcal{X}\). (For a more precise discussion of the order of observers, see Schumacher (1980b)). In the closed-loop situation, the control function \(u(t)\) is given by \((2.14)\). We may use this information to obtain a reduced-order compensator, as indicated above. In this situation, we shall say that we are using a feedback adapted observer and the corresponding compensator will be called an FAO-compensator.

Dually, we can also consider observer-adapted feedback. This occurs when there is an \((A+GC)\)-invariant subspace \(\mathcal{T}\) such that \(\mathcal{T} \subset \text{Ker } \mathcal{F}\). Under these conditions, the control law \((2.14)\) is defined even if \(\hat{x}(t)\) is only an estimate "modulo \(T\". This means that the compensator order can be reduced from \(\text{dim } \mathcal{X}\) to \(\text{dim } \mathcal{X} - \text{dim } \mathcal{T}\). A compensator obtained in this way will be called an OAF-compensator.

Finally, it is also possible to combine the two design methods under the further condition that \(\mathcal{T} \subset \mathcal{V}\). (Note that this, together with
(A + BF)V ⊂ V and Ker F ⊢ T, implies automatically that AT ⊂ V). The compensator order is reduced to dim V - dim T. This is precisely the statement of Thm. 2.4. We may speak of "mutual adaption" in this case, and the corresponding compensators will be called MA-compensators.

The construction that will be given in Section 2.3 leads to an extended system matrix $A_e$ whose spectrum is given by

$$\sigma(A_e) = \sigma(A+BF; V) + \sigma(A+GC; X/T).$$

This should be compared to the formula (2.18) which holds for the 'classical' full-order compensator. The equation (2.20) is in line with the interpretation of our compensation method as one which is still based on "feedback applied to an estimate of the state", be it that the feedback and the observer action are adapted to each other in order to obtain the reduction of the compensator order.

An obvious question that should be answered is, which compensators of the general form (2.2) can be interpreted as MA-compensators. Stated more precisely: Given a closed-loop system of the form (2.4), when do we have a compensator couple $(T,V)$ and corresponding mappings $F$ and $G$ such that (2.10), (2.11) and (2.20) hold? The discussion above suggests that there should be a feedback-observer interpretation available if a compensator is to be of the MA-type, and a conjecture might be that all compensators that have a feedback-observer interpretation are in fact MA-compensators.

To proceed in this direction, we have to make precise what we mean by the statement that a compensator "has a feedback-observer interpretation". Consider a compensator of the general form (2.2), giving rise to the closed-loop system (2.4). As a preliminary definition, let us say that the given compensator has a feedback-observer interpretation if there exists a subspace $T$ of $X$ and a mapping $\Phi : X \Theta W \to X/T$ (which we shall call the error mapping) with the following properties:

(i) $\Phi$ can be factored as $\Phi = T(I - S)$ where $S$ is a linear mapping from $W$ to $X$ and $T$ is the canonical mapping from $X$ to $X/T$;

(ii) if $\Phi x(0)_W = 0$, then $\Phi x(t)_W = 0$ for all $t \geq 0$;

(iii) if $\Phi x(0)_W = 0$ and moreover $x(0) = 0$, then $x(t) = 0$ for all $t \geq 0$.

The mapping $\Phi$ gives an observer interpretation to the closed-loop dynamics, in the following way. To every vector $x_W$ in the extended state
space \( X \otimes W \), \( \phi \) associates the error of \( Sw \) as an estimate of \( x \) modulo \( T \); this is stated in (i) above. The condition (ii) says that the error will remain zero for all time if it is zero initially. This means that the error satisfies an autonomous differential equation. Finally, the condition (iii) seems to be natural: if in the initial situation the state is at the nominal point zero and also the estimation error is zero, then no reasonable compensator should cause an excitation.

The conditions (i) - (iii) can also be expressed in terms of \( M := \text{Ker} \phi \).

It follows easily from (i) - (iii) that \( M \) satisfies the following conditions:

(i)' \( Q^{-1}M = T \), and \( \text{dim } M = \text{dim } W + \text{dim } T \);

(ii)' \( M \) is \( A_e \)-invariant;

(iii)' \( M_w := \{(X_w) \in M | x = 0\} \) is \( A_e \)-invariant.

Note that we may just as well require

(iii)'\( M_w = 0 \)

instead of (iii)', because an \( A_e \)-invariant subspace that is fully contained in (the natural embedding of) \( W \) represents a redundant part of the compensator. If \( M_w \) is non-trivial, then the compensator order can be reduced by factoring out this subspace.

Motivated by this development, let us say that a subspace \( M \) of an extended state space \( X \otimes W \) is a separating subspace if the following is true:

\[(2.21.1) \quad \text{dim } M = \text{dim } W + \text{dim } Q^{-1}M \]

\[(2.21.2) \quad \text{dim } M = \text{dim } P_M. \]

The condition (2.21.1) is equivalent to (i)' above if we define \( T \) as \( Q^{-1}M \), and (2.21.2) is equivalent to (iii)'\( . \) So the conditions (i)', (ii)' and (iii)' imply that \( A_e \) has a separating invariant subspace. On the other hand, suppose that an extended system matrix \( A_e \) of the form (2.5) has been given, and that \( A_e \) has a separating invariant subspace \( M \). Then it is not difficult to see that there exist a subspace \( T \) of \( X \) and a mapping \( \phi : X \otimes W \rightarrow X/T \) such that (i) - (iii) above are satisfied. (Take \( T = Q^{-1}M \), by (2.21), \( \text{dim } W = \text{dim } M = \text{dim } P_M = \text{dim } Q^{-1}M \) so there exists a surjective mapping from \( P_M \) to \( W \) whose kernel is precisely \( Q^{-1}M \). Let \( S \) be a right inverse of this mapping, and define \( \phi \) by \( \phi = T(I - S) \) where \( T \) is the canonical mapping from \( X \) to \( X/T \).
So our final definition is: The compensator (2.2) for the system (2.1) has a feedback-observer interpretation if the corresponding extended system matrix $A_e$, as given by (2.5), has a separating invariant subspace. We can now state the following theorem, which will be proved in Section 2.5.

**Theorem 2.5.** Suppose that (2.2) gives a stabilizing compensator for the system (2.1) which has a feedback-observer interpretation (in the above sense). Then there exist a compensator couple $(T, V)$ and corresponding mappings $F$ and $G$ such that (2.10), (2.11) and (2.20) hold.

In other words, the conjecture that we formulated above is true: any compensator that has a feedback-observer interpretation is in fact an MA-compensator. It will follow from the discussion in Section 2.5 that each separating invariant subspace $M$ of $A_e$ gives rise to a compensator couple $T = Q^{-1}M$, $V = PM$ that satisfies the conditions of Thm.2.5. Correspondingly, the eigenvalues of $A_e : M$ can be called 'feedback poles', and the eigenvalues of $A_e : (X \oplus W)/M$ can be labeled 'observer poles'.

There is no reason to assume that the decomposition in a 'feedback' and an 'observer' part is unique. Rather, the converse is likely to be true: in general, there will be many separating invariant subspaces and so there will be many decompositions of the closed-loop spectrum in 'feedback poles' and 'observer poles'. A full discussion of this subject would take us into algebraic geometry and outside the scope of the present work, but we would like to indicate what is probably true. The conditions (2.21) describe a transversality relation in the sense of Wonham (1979; p.29) in case the following relation holds for $M$:

\[(2.22) \quad \dim W \leq \dim M \leq \dim X.\]

This implies that subspaces of $X \oplus W$ that satisfy (2.22) will 'in general' be separating. If $\dim W \leq \dim X$, there will be 'in general' many invariant subspaces of a given extended system matrix $A_e$ satisfying (2.22). So it seems not unreasonable to conjecture that there will 'in general' be many separating invariant subspaces of a given extended system matrix.

This needs proof, of course. Our point here is mainly to argue that it can not be expected that there is an intrinsic property that divides the closed-loop poles into 'feedback poles' and 'observer poles'. A given
compensator could be interpreted as a combination of 'slow' feedback with a 'fast' observer or vice versa, without any preference for one of the two interpretations. This point of view is untraditional. In many textbooks (for instance KWAKERNAAK & SIVAN (1972, p.383,387)), the designer is advised to choose the 'observer poles' somewhat faster than the 'feedback poles'. The implicit assumption that there is such an intrinsic distinction should, at least, not be taken for granted.

The question if there are compensators that are not of the "MA" type is easily answered. If \( \dim \mathcal{W} > \dim X \), then there are no separating subspaces of \( X \oplus \mathcal{W} \). Consequently, no compensator of order larger than the order of the original system can be an MA-compensator. This example is unsatisfactory insofar as it is always possible to construct a stabilizing compensator of order \( n \) if one can be constructed at all. Another example is the extended system matrix

\[
A_e = \begin{pmatrix}
-1 & 1 \\
\vdots & \ddots \\
0 & 1
\end{pmatrix}
\]

which has no invariant separating subspace. If we suppose that \( -1 \in \mathbb{G} \), then the extended system is stable but the original system is also stable, so that in this case it is unnecessary to add a compensator. If \( -1 \notin \mathbb{G} \), the original system is unstable, but then the compensator does not stabilize it. So this example is also unsatisfactory. One is tempted to think that the only examples of extended system matrices that do not have a separating invariant subspace are those for which there is a trivial reduction of the compensator order. We shall leave this as a conjecture.
2.3. The construction lemma

In this section, we approach the compensator problem from the constructive perspective. Applications will be given in the next section. Our main objective is to prove Thm. 2.4. The following construction lemma (similar lemmas will be proved in the next chapter) already provides all the material that we shall need.

**Lemma 2.6** Suppose that we are given a compensator couple \((T, \mathcal{V})\) and mappings \(F \in \mathcal{F}(\mathcal{V})\) and \(G \in \mathcal{G}(T)\) such that \(\text{Ker } F \supset T\) and \(\text{Im } G \subset \mathcal{V}\). Then we can construct an extended system matrix \(A_e\) acting on an extended state space of dimension \(n + \dim \mathcal{V} - \dim T\), such that \(A_e\) has an invariant subspace \(M\) with the following properties:

\[
\begin{align*}
(2.24.1) & \quad Q^{-1}M = T, \quad PM = \mathcal{V} \\
(2.24.2) & \quad \sigma(A_e; M) = \sigma(A+BF; \mathcal{V}) \\
(2.24.3) & \quad \sigma(A_e; (X\mathcal{W})/M) = \sigma(A+GC; X/T).
\end{align*}
\]

**Proof** Let \(\mathcal{W}\) be a linear space of dimension \(k = \dim \mathcal{V} - \dim T\), and let \(R\) be a mapping of \(\mathcal{V}\) onto \(\mathcal{W}\) such that \(\text{Ker } R = T\). Because \(R\) is surjective, there is a right inverse \(R^* : \mathcal{W} \to \mathcal{V}\) such that \(RR^* = I_{\mathcal{W}}\). We now define \(A_e\) by

\[
A_e = \begin{pmatrix}
A & BL \\
MC & N
\end{pmatrix}
\]

where the mappings \(L, M\) and \(N\) are defined as follows:

\[
\begin{align*}
(2.26.1) & \quad L = FR^* \\
(2.26.2) & \quad M = -RG \\
(2.26.3) & \quad N = R(A+BF+GC)R^*.
\end{align*}
\]

The mapping \(R^*\) is not determined uniquely (if \(T \neq 0\)); but if \(R_1^*\) and \(R_2^*\) are both right inverses to \(R\) then we must have \(\text{Im}(R_1^* - R_2^*) \subset T\), so that, in particular, \(FR_1^* = FR_2^*\). So we see that (2.26.1) uniquely defines a mapping \(L\). The definition (2.26.2) is justified because \(G\) maps into \(\mathcal{V}\) which is
precisely the domain of $R$. As to (2.26.3), it should be noted that both $T$ and $U$ are invariant for $A + BF + GC$; from this it follows that $N$ is well-defined and that $N$ is uniquely determined by (2.26.3), despite the non-uniqueness of $R^+$.

Now consider the subspace

$$H: = \{(\begin{bmatrix} X \\ X \end{bmatrix} x \in U)\}

of $X \oplus U$. It is immediately clear that $PM = V$ and $Q^{-1}M = T$. Moreover, we have for all $x \in V$

$$A_e(x) = \begin{bmatrix} (A+BF)x \\ R(A+BF)x \end{bmatrix} \in H.$$  

(This is seen by noting that $R(I-R^+R) = R - (RR^+)R = 0$ so that $I - R^+R$ maps into $\text{Ker} \ R = T$. Consequently, $(A+BF)^+x = (A+BF)x$ and $R(A+BF+GC)^+Rx - RGx = R(A+BF)x$ for all $x \in V$. From (2.21), we see that $H$ is $A_e$-invariant.

Moreover, if we denote by $\overline{P}$ the restriction of $P$ to $H$ considered as a mapping to $PM = V$, then $\overline{P}$ is an isomorphism, and the following diagram commutes:

$$\begin{array}{ccc}
H & \overset{A_e}{\longrightarrow} & H \\
\downarrow \overline{P} & & \downarrow \overline{P} \\
V & \overset{A+BF}{\longrightarrow} & V
\end{array}
$$

This gives us (2.24.2). Finally, define the mapping $\overline{Q}: X/T \to (X \oplus U)/M$ by

$$\overline{Q}[x] = [Qx]$$  

(where, of course, the first equivalence class is modulo $T$ and the second is modulo $M$). The mapping $\overline{Q}$ is well-defined because $QT = \emptyset$. Moreover, $\overline{Q}$ is an isomorphism between $X/T$ and $(X \oplus U)/M$. (To see this, note that $Q$ is injective because $Qx \in M$ implies $x \in T$, and compute: $\dim (X \oplus U)/V = (\dim X + \dim V - \dim T) - \dim \overline{V} = \dim X/T$.)

For all $x \in X$, we have
(2.31) \[ Q(A+GC)x - A_eQx = (A+GC)x - Ax = (GCx) \in M \]

so that we have the following equality in \( (X\#U)/M \):

(2.32) \[ [A_eQx] = [Q(A+GC)x]. \]

In other words, the diagram

\[
\begin{array}{ccc}
X/T & \xrightarrow{A+GC: X/T} & X/T \\
\downarrow & & \downarrow \\
\bar{Q} & & \bar{Q}
\end{array}
\]

(2.33)

\[
\begin{array}{ccc}
(X\#U)/M & \xrightarrow{A_e: (X\#U)/M} & (X\#U)/M \\
\downarrow & & \downarrow \\
\bar{Q} & & \bar{Q}
\end{array}
\]

commutes. We already noted that \( \bar{Q} \) is an isomorphism; thus (2.24.3) follows.

A graphical illustration of the lemma may be given by a set of three lattice diagrams, in which we use corresponding letters to indicate the equalities (2.24.2) and (2.24.3).

\[
\begin{array}{ccc}
X & \xrightarrow{a} & X \# W \\
\downarrow a & & \downarrow a \\
T & \xrightarrow{b} & M \\
\downarrow 0 & & \downarrow 0 \\
A+GC & & A_e
\end{array}
\]

Figure 2.5: Compensator construction.

This is fairly simple, but we shall encounter more complicated versions in the next chapter.

Of course, Thm. 2.4 as announced in the previous section follows immediately from an application of the construction lemma that we have just
proved.

2.4. Methods of compensator construction

In this section, we present several methods of compensator design which can be derived from Thm. 2.4. Four of these contain well-known results, which are now brought within a single theoretical framework; we shall also present a fifth method which is new and different in nature.

2.4.1. The full-order compensator

COROLLARY 2.7 Let the system (2.1) be given, and suppose that (A,B) is stabilizable and (C,A) is detectable. Then the system can be stabilized by a compensator of order n.

PROOF Take F such that A + BF is stable and G such that A + GC is stable. Then apply Thm. 2.4 with \( V = X \) and \( T = 0 \).

This is the standard "full-order compensator" that appears in many textbooks. (See for instance KWAKERNAAK & SIVAN (1972; §5.2))

2.4.2. The reduced-order compensator

It is possible to reduce the order of the compensator by employing static output feedback. First we make the following simple observation.

LEMMA 2.8 For any given feedback mapping \( \gamma \) and any given subspace \( T \) such that \( T \oplus \text{Ker} \ C = X \), there exists a static output feedback mapping \( K: \gamma \to \gamma \) and a feedback mapping \( F_0 \) such that \( A + BF = A + BKC + BF_0 \) and \( \text{Ker} \ F_0 \supset T \).

PROOF Define \( T: \gamma \to X \) by \( \text{Im} \ T = T \) and \( \text{CT} = I_{\gamma} \). Then TC is the projection along \( \text{Ker} \ C \) onto \( T \). Take \( K = FT \) and \( F_0 = F(I-TC) \); then \( A + BF = A + BKC + BF_0 \), and \( \text{Ker} \ F_0 \supset T \).

Of course, the idea is to use the pair \((T,X)\) as the compensator couple that is used in Thm. 2.4, after the system matrix has been changed to \( A + BKC \) by static feedback. Note that any complement \( T \) of \( \text{Ker} \ C \) is
conditioned invariant. However, we need more than that: \( T \) has to be a detectability subspace. On this, we have the following result.

**Lemma 2.9** Let a system matrix \( A \) and an output mapping \( C \) be given, and suppose that the pair \((C,A)\) is detectable. Then we can find an outer-detectable complement of \( \text{Ker} \ C \).

**Proof** Take an arbitrary complement \( T_0 \) of \( \text{Ker} \ C \), and define \( T_0' = T_0 = \text{Im} \ T_0 \) and \( C T_0' = I_y \). Then we can consider \((I - T_0')A\): \( \text{Ker} \ C \) as a system matrix acting on the state space \( \text{Ker} \ C \), and we may view \( T_0' \) as an output space and \( T_0'CA \) as an output mapping from \( \text{Ker} \ C \) to \( T_0' \). We contend that the pair \((T_0'CA, (I - T_0')A)\) is detectable. For, suppose the contrary; then it follows from Cor. 1.21 that there would exist \( \lambda \in \mathbb{C}_b \) and \( x \in \text{Ker} \ C \) (\( x \neq 0 \)) such that

\[
(I - T_0')Ax = \lambda x \quad \text{and} \quad T_0'CAx = 0.
\]

But then we would have \( Ax = \lambda x \) and \( Cx = 0 \), contradicting the assumption that the pair \((C,A)\) is detectable.

We conclude that there exists a mapping \( Z: T_0' \to \text{Ker} \ C \) such that

\[
(I - T_0')A + ZT_0'CA = (I - (I - T_0')A) \quad \text{is stable (on} \ \text{Ker} \ C \).
\]

Write \( \Gamma = T_0' - ZT_0' \), and \( \Gamma = \text{Im} \ T \). Then \( \Gamma \) is a complement of \( \text{Ker} \ C \), since \( C \Gamma = I_y \). Again proceeding by contradiction, suppose that \( \Gamma \) would not be a detectability subspace. Then, by Thm. 1.14, there would exist \( x \in \text{Ker} \ C \) (\( x \neq 0 \)) such that \((\lambda - A)x \in \Gamma \) for some \( \lambda \in \mathbb{C}_b \). But then we would also have

\[
(\lambda - (I - TC)A)x = (I - TC)(\lambda - A)x = 0,
\]

contradicting the fact that \((I - TC)A: \text{Ker} \ C \) is stable.

This result is a variant of Thm. 3.3 in WOHNAM (1979; p. 64); we have replaced observability with detectability. Using Thm. 1.14, the result may also be stated as follows: if \((\lambda - A)^{-1}T \cap \text{Ker} \ C = 0 \) for all \( \lambda \in \mathbb{C}_b \), then there exists a complement \( \Gamma \) of \( \text{Ker} \ C \) such that \((\lambda - A)^{-1}\Gamma \cap \text{Ker} \ C = 0 \) for all \( \lambda \in \mathbb{C}_b \). This formulation clearly shows that the condition of detectability is not only sufficient but also necessary for such a \( \Gamma \) to exist.

Now we are able to derive the compensator that is commonly known as the "reduced-order compensator" (LUIENBERGER (1964)). Its order is \( n-p \) where \( p = \text{dim} \ y \) (the number of outputs).

**Corollary 2.10** Let the system \((2.1)\) be given, and suppose that \((A,B)\) is stabilizable and \((C,A)\) is detectable. Then the system can be stabilized by a PI-compensator of order \( n-p \).
PROOF Take $F$ such that $A + BF$ is stable. Let $T$ be an outer-detectable complement of Ker $C$, and take $G \in G(T)$ such that $A + GC: X/T$ is stable. Define $T$ by $\text{Im } T = T$ and $CT = I_Y$. Now apply Thm. 2.4 to the system matrix $A + BFTC$ with the feedback mapping $F_0 = F(C-TC)$ and the injection mapping $G_0 = G - BFT$, using the compensator couple $(T, X)$.

The procedure can be dualized to give a compensator of order $n - m$, where $m = \dim U$ is the number of inputs; Thm. 2.4 is then applied to a compensator couple $(0, V)$ where $V$ is a suitable complement of $\text{Im } B$. This possibility was first noted by F.M. Brasch (according to Luenberger (1971)).

2.4.3. The Brasch-Pearson compensator

We can find yet another bound for the compensator order by using a lemma that has its origin in observer theory. The result, which we state here in a dualized and somewhat adapted form, is due to Luenberger (1966), with a minor correction by Wonham & Morse (1972). The lemma uses the notion of the controllability index of the pair $(A, B)$. This is one of the structural invariants of linear systems (see, for instance, Morse (1973a)), defined, for a controllable pair $(A, B)$, by

$$
\kappa_c = \min \{ k \in \mathbb{N} \mid \frac{k}{1^2} A^{i-1}(\text{Im } B) = X \}.
$$

**Lemma 2.11** Let $(A, B)$ be a controllable pair. Then, for any one-dimensional subspace $T$ of $X$, there exists a stabilizability subspace $V$, of dimension equal to or less than $\kappa_c$, such that $T \in V$.

**Proof** See Wonham (1979; § 3.9).

We want to use a pair $(T, V)$ as in the lemma to construct a compensator of order $\leq \kappa_c - 1$. This can be done using a procedure which is applicable in a more general situation.

**Lemma 2.12** Suppose that we have an outer-detectable complement $T$ of Ker $C$ and a stabilizability subspace $V$ such that $T \in V$. Then we can construct a stabilizing PI-compensator of order $\dim V - \dim T = \dim V - p$.

**Proof** Take $F \in F(V)$ such that $A + BF: V$ is stable; define $T: X \rightarrow V$ by $\text{Im } T = T$ and $CT = I_Y$. If we take $G = -AT$, then $G \in G(T)$ because
\[(A+GC)T = A(I-TC)T = 0 \in T.\] Moreover, \(A + GC; X/T\) is stable; for, if this was not the case, there would exist an \(x \in \text{Ker } C\) \((x \neq 0)\) such that \(A(I-TC)x - \lambda x = Ax - \lambda x \in T\) for some \(\lambda \in \mathbb{R}_b\), contradicting the outer-detectability of \(T\).

Now we apply Thm. 2.4 with the system matrix \(A + BFT\), the feedback mapping \(F_0 = F(I-TC)\) and the injection mapping \(C_0 = G - BFT = -(A+BF)T\), using the compensator couple \((T, \mathcal{V})\). This is possible because \((A+BFT)T = (A+BF)T = (A+BF)\mathcal{V} \subset \mathcal{V}\); also, \(\text{Ker } F_0 \supset T\) and \(\text{Im } G_0 = (A+BF)T \subset \mathcal{V}\).

From the foregoing lemmas, we now immediately have the following result.

**Corollary 2.13** Let the system \((2.1)\) be given; suppose that \((A, B)\) is controllable, \((C, A)\) is detectable, and \(p = \dim \mathcal{V} = 1\). Then the system can be stabilized by a PI-compensator of order less than or equal to \(\kappa_c - 1\).

**Proof** By Lemma 2.9, we can find an outer-detectable complement \(T\) of \(\text{Ker } C\); we shall have \(\dim T = p = 1\). Then we can apply Lemma 2.11 to find a stabilizability subspace \(\mathcal{V}\) of dimension \(\leq \kappa_c\) such that \(T \subset \mathcal{V}\). Finally, a compensator of order \(\leq \kappa_c - 1\) is then obtained from Lemma 2.12.

If \(p\) is larger than 1, the number of outputs can of course be reduced simply by replacing \(C\) by \(\tilde{C} = HC\), where \(H\) is any functional on \(\mathcal{V}\); but this has to be done in such a way that the pair \((\tilde{C}, A)\) will be detectable. Under general circumstances this can easily be done, but there are special cases in which this is not possible. If the original pair \((C, A)\) is observable (and \((A, B)\) is controllable), then one can prove that there is a preliminary static output feedback \(K_0\) such that \(A + BK_0C\) is cyclic (see Wonham (1979; p. 74)), and then again there is no problem in performing the reduction (use Cor. 1.1 in Wonham (1979; p. 43)).

Of course, there is also a dual procedure leading to a compensator of order \(\leq \kappa_0 - 1\) where \(\kappa_0\) is the observability index of the pair \((C, A)\) (see also Wonham (1979; § 3.8)). The compensator order we have found is due to Brasch & Pearson (1970).

### 2.4.4. Zero-order compensators

It is seen from Lemma 2.12 that if we can find a complement of \(\text{Ker } C\) that is at the same time a stabilizability subspace and a detectability
subspace, then it is possible to construct a stabilizing PI-compensator of order 0; that is, the system can be stabilized by static output feedback only. The following lemma implies that such complements of Ker C are 'almost always' present if the total number of inputs and outputs exceeds the state space dimension.

**Lemma 2.14** The property "V is a controllability subspace with respect to the pair (A, B)" is generic in the set of triples (A, B, V) where A is a system matrix acting on a fixed n-dimensional state space X, B is an input mapping with dim Im B equal to some fixed number m, and V ⊂ X is a subspace of dimension larger than n - m + 1.

**Proof** In the indicated set, we have generically V + Im B = X, in which case V is a controlled invariant subspace. We can then take a projection P on X such that Im P = V and Ker P ⊂ Im B. Because Im(I-P) ⊂ Im B, there exists H: X → U such that I - P = BH. Consequently, PA = A - BHA is of the form A + BF, and of course PAU ⊂ V. Also note that Im(PB) = Im B ∩ V. By genericity of controllability (see, for instance, WONHAM (1979; Thm. 1.3)), the equality <PA | Im(PB)> = V (implying that V is a controllability subspace) will hold generically.

An alternative proof based on the characterization of controllability subspaces given by the "controllability subspace algorithm" (WONHAM (1979; p. 106)) has been given in SCHUMACHER (1980 e).

**Corollary 2.15** Let the system (2.1) be given; suppose that (A, B) is stabilizable, (C, A) is detectable, and dim Im B ≥ dim Ker C + 1 (so m + p ≥ n + 1). Also suppose that c is an open set. Then it is generically (with respect to A and B) possible to stabilize the system by static output feedback only.

**Proof** By Lemma 2.9, we can find an outer-detectable complement T of Ker C; note that the property of being an outer-detectable complement of Ker C is preserved under small perturbations of T. We have dim T ≥ n - m + 1 so, generically, T is a stabilizability subspace and we can apply Lemma 2.12 to arrive at the desired conclusion.

The result is related to the material presented in WANG & DAVISON (1975) and KIMURA (1975; 1978).
2.4.5 An iterative approach

The success of the above methods in reducing the compensator order is moderate; for instance, for single-input single-output systems none of these methods gives a better estimate than \( n - 1 \). This is in sharp contrast with practical experience; as a rule, engineers are able to stabilize rather complicated systems (notably of the single-input-single-output type) using relatively simple controllers. The reason may be that most systems met in practice are "reasonably stabilizable" (in a sense that is not easily made precise), whereas the methods used above perform a "worst case analysis" in that they consider all systems of a given order, with additional information consisting only of the number of inputs and outputs, and in some cases the controllability (or observability) indices.

So one would like to see methods which give low-order compensators for some restricted class of "reasonably stabilizable" systems; but then one has the problem of how to describe such a class. Under these circumstances, it is reasonable to look for a procedure which allows one to perform a more or less systematic search for low-order compensators, without trying to give an a priori bound for the compensator order that will ultimately be found. It is indeed possible to give such a procedure, based on Thm. 2.4.

Consider the following steps.

1. Find \( F \) and \( \hat{G} \) such that \( A + BF \) and \( A + \hat{GC} \) are stable.

2. Among the subspaces of dimension \( k(\geq p) \) that are invariant for \( A + BF \), find one that is close (in the sense of some metric in the set of subspaces of \( X \)) to \( \text{Im} \hat{G} \); call this subspace \( V \).

3. Let \( G \) be a (small) perturbation of \( \hat{G} \) such that \( \text{Im} G \subset V \).

4. See if \( A + GC \) is still stable; if so, apply Thm. 2.4 with the mappings \( F \) and \( G \) that are now found, using the compensator couple \( (\tilde{0}, V) \). If not, re-initialize (Step 1) or go back to Step 2 replacing \( k \) by \( k + 1 \).

This is basically the procedure we shall use in the second half of this work to find finite-dimensional stabilizing compensators for infinite-dimensional systems. (See Ch. 5 for a detailed discussion and examples.) We shall not go into the details here; just note that we have taken \( T = 0 \) to satisfy the requirement \( \text{Ker} F \geq T \) automatically, and to avoid difficulties in Step 3 stemming from the requirement \( G \in \mathcal{G}(T) \). An alternative would be to base the procedure on Lemma 2.12 instead of Thm. 2.4; the outer-detectable...
complement $T$ of Ker $C$ then replaces Im $G$, and the compensator order is decreased by $p$.

2.5 Separating subspaces

In this section, we approach the compensator problem from the 'necessity' point of view. The conditions of Thm. 2.4 may seem rather special; are they necessary for a stabilizing compensator to exist? We now show that these conditions must hold in every situation in which the extended system matrix has an invariant subspace that satisfies a simple dimensional relation.

THEOREM 2.16 Let $A_e$ be a stable extended system matrix of the form (2.5), and suppose that $A_e$ has an invariant subspace $M$ with the following property:

$$(2.35) \quad \dim PM = \dim \mathcal{W} + \dim Q^{-1}M.$$

Then there exists a compensator couple $(T,V)$ (in the original state space $X$) such that $\dim \mathcal{W} = \dim V - \dim T$, and there exist mappings $F \in F(V)$ and $G \in G(T)$ such that

$$(2.36.1) \quad \ker F = T$$

$$(2.36.2) \quad \text{Im } G \subseteq \mathcal{V}$$

$$(2.37.1) \quad \sigma(A+BF : V) \subseteq \mathcal{E}_g$$

$$(2.37.2) \quad \sigma(A+GC : X/T) \subseteq \mathcal{E}_g.$$

In the course of the proof, we shall need the following lemma.

LEMMA 2.17 For any subspace $M$ of the extended state space $X \oplus \mathcal{W}$, the following inequalities hold:

$$(2.38) \quad \dim \mathcal{W} + \dim Q^{-1}M \geq \dim M \geq \dim PM.$$

PROOF Introduce the following operator from $M$ to $\mathcal{W}$:

$$(2.39) \quad P_M : (x_M) \mapsto x_M \quad (x_M) \in M).$$
Because \( \text{Ker } P_w \) is isomorphic to \( Q^{-1}M \) and \( \text{Im } P_w \) is contained in \( \mathcal{W} \), we get

\[
\dim M = \dim \text{Ker } P_w + \dim \text{Im } P_w \leq \dim \mathcal{W} + \dim Q^{-1}M.
\]

(2.40)

The second inequality in (2.35) is of course trivial.

PROOF (of Thm. 2.16) Take \( T = Q^{-1}M \) and \( \mathcal{V} = PM \). Then it follows immediately from (2.35) that \( \dim \mathcal{W} = \dim \mathcal{V} - \dim T \), and also it is clear that \( T \subseteq \mathcal{V} \).

From (2.35) and Lemma 2.17 it follows that \( \dim M = \dim PM \); thus, there exists a mapping \( R: \mathcal{V} \to \mathcal{W} \) such that

\[
M = \{ (x^T) | x \in \mathcal{V} \}.
\]

Of course, \( \text{Ker } R = Q^{-1}M = T \) and because \( \dim \mathcal{V} = \dim \mathcal{W} + \dim T \) it follows that \( R \) must be surjective. Consequently, \( R \) has a right inverse which we shall denote by \( R^+: \mathcal{W} \to \mathcal{V} \).

Now, define \( F: X \to \mathcal{U} \) to be any extension to \( X \) of the mapping LR (which is defined on \( V \)), and let \( G \) be equal to \(-R^+M \) considered as a mapping from \( V \) to \( X \); here, the mappings \( L \) and \( M \) are taken from the extended system matrix in its form

\[
(2.5) \quad A_e = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}.
\]

Clearly, (2.36) is satisfied. For any \( x \in \mathcal{V} \), we have

\[
(2.42) \quad \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ Rx \end{pmatrix} = \begin{pmatrix} Ax + BLx \\ MCx + NRx \end{pmatrix} \in M
\]

so that \( (A + BL) x = (A + BF) x \in V \). We see that \( V \) is controlled invariant and that \( \mathcal{F} \in \mathcal{F}(V) \). Moreover, if we let \( \tilde{P} \) denote the restriction of \( P \) to \( M \) considered as a mapping onto \( PM = V \), then \( \tilde{P} \) is an isomorphism and (2.42) shows that the diagram

\[
(2.43)
\]

is commutative. Therefore, \( \tilde{P} \) is an isomorphism.
(2.44) \( \sigma(A+BF: V) = \sigma(A_+: M) \subset \Xi_g' \).

Furthermore, for any \( x \in \mathcal{T} \) we have

\[
\begin{pmatrix}
A & BL \\
MC & N
\end{pmatrix}(O) = \begin{pmatrix}
Ax \\
MCx
\end{pmatrix} = \begin{pmatrix}
(A-R^+MC)x \\
MCx
\end{pmatrix} + \begin{pmatrix}
R^+MCx \\
MCx
\end{pmatrix} \in M.
\]

Because

\[
\begin{pmatrix}
R^+MCx \\
MCx
\end{pmatrix} = \begin{pmatrix}
R^+MCx \\
MCx
\end{pmatrix} \in M,
\]

this shows that \( \mathcal{T} \) is conditioned invariant and that \( G \in \mathcal{G}(\mathcal{T}) \). If we define \( \bar{Q}: X/\mathcal{T} \to (X\oplus W)/\mathcal{M} \) by \( \bar{Q}[x] = [Qx] \) (which is justified since \( QT \subset M \)), then \( \bar{Q} \) is an isomorphism and (2.45) shows that the diagram

\[
\begin{array}{ccc}
X/\mathcal{T} & \xrightarrow{A+GC} & X/\mathcal{T} \\
\downarrow \bar{Q} & & \downarrow \bar{Q} \\
(X\oplus W)/\mathcal{M} & \xrightarrow{A_+: (X\oplus W)/\mathcal{M}} & (X\oplus W)/\mathcal{M}
\end{array}
\]

commutes. So we obtain

(2.48) \( \sigma(A+GC: X/\mathcal{T}) = \sigma(A_+: (X\oplus W)/\mathcal{M}) \subset \Xi_g' \).

We still have to show that \( AT \subset V \). This follows, for instance, from \( AT = (A+BF)T \subset (A+BF)V \subset V \).

The theorem that has been announced in Section 2.2 (Thm. 2.5) now follows immediately from the proof, using (2.44) and (2.48).

The condition (2.35) appears to be important for the separation of the closed-loop eigenvalues into "feedback poles" and "observer poles" (according to the interpretation given in Section 2.3). This is why we have called subspaces \( X \oplus W \) that satisfy this condition separating subspaces.
CHAPTER 3

TRACKING, REGULATION AND DISTURBANCE LOCALIZATION

After having treated the fundamental problem of stabilization by dynamic feedback, we now turn to more complex problems. The complexity arises from setting new control objectives and considering more elaborate disturbance models. We shall require that some given set of state variables follows ("tracks") a signal produced by some independent finite-dimensional linear system. We shall assume that the system is affected by disturbances; for some of these disturbances we may have a dynamic model, but others may be completely unknown functions. Protection against modeled disturbances is called "regulation"; the term "disturbance localization" is used when there is no model available.

The subjects of "tracking" and "regulation" can be brought into one setting, and they have been extensively studied in this way. Disturbance localization is a problem of which the state feedback solution is well-known; the solution by dynamic output feedback has only recently been obtained. Here the two lines of research are brought together into a single framework. A very general problem is obtained but nevertheless we are able to give a completely constructive solution, in which the concept of "compensator couple" is again crucial. The main theorem, which provides this solution, enables us to derive many of the existing results both on tracking and regulation and on disturbance localization as special cases.

There are five sections. The first section provides the motivation for the problem as we state it. In the second section, we give preliminary results: an adapted version of Thm. 2.16, and an extended construction lemma. The main theorem follows in Section 3, and in Section 4 we give a number of special cases, some of which contain earlier results. Section 5 contains a brief discussion of order reduction.

3.1. Problem statement

In the previous chapter, disturbances were modeled by nonzero initial conditions. Of course, this is not always a suitable model and so we consider some alternatives. For instance, assume that a disturbance function $q(t)$
enters the system in the following way (where \( E \) is a linear mapping into \( X \)):

\[
(3.1.1) \quad \dot{x}(t) = Ax(t) + Bu(t) + Eq(t)
\]

\[
(3.1.2) \quad y(t) = Cx(t).
\]

If we now add a compensator of the form (2.2), the closed loop system becomes:

\[
(3.2) \quad \frac{d}{dt}X(t) = \begin{bmatrix} A & BL \\ MC & N \end{bmatrix} X(t) + \begin{bmatrix} E \\ 0 \end{bmatrix} q(t).
\]

Clearly, it would be asking too much if we would require that the full state vector should approach zero as \( t \to \infty \), regardless of the behaviour of the function \( q(t) \). So we suppose that there is a special set of variables in which we are interested. Call these the variables-to-be-controlled (to be denoted by \( z(t) \)), which depend on \( x(t) \) via a linear mapping \( D \):

\[
(3.3) \quad z(t) = Dx(t).
\]

The equation (3.2) becomes supplemented by the equation for the variables-to-be-controlled

\[
(3.4) \quad z(t) = (D \ 0)X(t).
\]

We may now require that the evolution of \( z(\cdot) \) is independent of \( q(\cdot) \); we shall then say that the disturbance has been \textit{localized} with respect to the variables-to-be-controlled. This means that there is a subspace of the extended state space \( \tilde{X} \oplus \tilde{W} \) with the following properties (where \( A_e \) is, as usual, the extended system matrix appearing in (3.2)):

\[
(3.5.1) \quad A_e \mathcal{M} \subseteq \mathcal{M}
\]

\[
(3.5.2) \quad \text{Im}(\begin{bmatrix} E \\ 0 \end{bmatrix}) \subseteq \mathcal{M} \subseteq \text{Ker}(D \ 0).
\]

(See also WONHAM (1979; p. 87).) Moreover, the variable \( z(t) \) will have the required stability properties if
(3.5.3) \( \sigma(\mathcal{A}_e; (\lambda_0 \mathcal{W})/\mathcal{H}) \leq \xi_\varepsilon \).

In this set-up, there is no assumption on the behaviour of the function \( q(t) \) (except for the usual regularity hypotheses, for instance piecewise continuity). In many cases however, it may be possible to give a description of the disturbance; for instance, it may be constant over long periods of time, or behave like a sine function of known frequency. Then we can re-write the equation (3.1.1) using a different notation:

(3.6.1) \( x'_1(t) = A_{11}x_1(t) + B_1u(t) + A_{12}x_2(t) \)

and we can add the model for the disturbance, which in general has the following form:

(3.6.2) \( x'_2(t) = A_{22}x_2(t) \)

So the disturbance signal is generated by the matrix \( A_{22} \); for instance, by taking \( A_{22} = 0 \) one gets a constant function whose value is determined by \( x_2(0) \). The equations (3.6.1) and (3.6.2) can be combined into one equation:

(3.7.1) \( x'(t) = Ax(t) + Bu(t) \)

where

(3.7.2) \( A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \).

Now the system equations are again in the standard form, but one sees that the pair \((A,B)\) is not controllable. Therefore, one can only hope to stabilize a set of variables-to-be-controlled:

(3.8) \( z(t) = Dx(t) = \begin{pmatrix} D_1 & D_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(t) \).

From the context, it is reasonable to assume that one would put \( D_2 = 0 \) in the matrix \( D \).

After having discussed these extended disturbance models, let us now take a look at alternative control objectives. In the previous chapter, we have worked with a setpoint at 0 to which the system should return after
an initial disturbance. However, it may be required that the system stays at a non-zero setpoint, which may be changed from time to time. More generally, one may require that some set of variables-to-be-controlled follows a reference signal, modeled as the output of an autonomous linear system. For such a situation, we can write down the following equations:

\[
\begin{align*}
(3.9.1) \quad x_1'(t) &= A_{11} x_1(t) + B_1 u(t) \\
(3.9.2) \quad x_2'(t) &= A_{22} x_2(t) \\
(3.9.3) \quad z(t) &= D_1 x_1(t) + D_2 x_2(t).
\end{align*}
\]

Here, the reference signal is given by the variable \( x_2(t) \), and the desired relation between \( x_1(t) \) and \( x_2(t) \) is expressed in (3.9.3) where \( z(t) \) should approach zero. Again, the system equations may be written down in the concise form

\[
\begin{align*}
(3.10.1) \quad x'(t) &= Ax(t) + Bu(t) \\
(3.10.2) \quad z(t) &= Dx(t)
\end{align*}
\]

where, in this case, the matrices \( A, B \) and \( D \) are given by

\[
A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 & D_2 \end{pmatrix}.
\]

We see that the pair \((A, B)\) is not controllable. In contrast with the setting we derived above, the matrix \( D_2 \) will not be equal to zero. The purpose of the control, however, is the same: \( z(t) \) has to approach zero at some prescribed rate.

We have now discussed three possible extensions of the compensator problem: localization of an unmodeled disturbance, regulation against a modeled disturbance, and tracking of a modeled reference signal. Comparing the condensed forms of the system equations in each case, it is clear that it is possible to cover all cases in one general framework, given by the following set of equations:

\[
\begin{align*}
(3.12.1) \quad x'(t) &= Ax(t) + Bu(t) + Eq(t)
\end{align*}
\]
(3.12.2) \[ y(t) = Cx(t) \]

(3.12.3) \[ z(t) = Dx(t). \]

Pictorially, the situation can be described as follows:

\[
\begin{array}{ccc}
q(t) & \rightarrow & E \\
\downarrow & & \uparrow \\
u(t) & \rightarrow & B \\
\downarrow & & \uparrow \\
\begin{bmatrix} A \end{bmatrix} & \rightarrow & \begin{bmatrix} x_1 \\
x_2 \end{bmatrix} \\
\uparrow & & \uparrow \\
y(t) & \rightarrow & C \\
\downarrow & & \downarrow \\
z(t) & \rightarrow & E
\end{array}
\]

Figure 3.1: Setting for general regulator problem.

This setting has a natural generality. There are two kinds of inputs: one is available to the controller, the other is chosen by "nature". There are also two kinds of outputs: one is the observation that can be used by the controller, the other defines the control objective. After closing the loop, only one input and one output remain.

While the formulation (3.12) has the important advantage of combining generality with simplicity, it is not immediately clear how the control objectives should be described in this framework. The "disturbance localization" requirement already has been given in (3.5), but to cover the "tracking" and "regulation" aspects we first return to the more explicit form

(3.13.1) \[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} q(t) \]

(3.13.2) \[ y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) \]

(3.13.3) \[ z(t) = \begin{bmatrix} D_1 & D_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t). \]

When a compensator of the form (2.2) is added, eqn. (3.13.1) becomes:

(3.14) \[ \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) = \begin{bmatrix} A_{11} & B_1 L & A_{12} \\ MC_1 & N & MC_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}(t) + \begin{bmatrix} E_1 \\ 0 \end{bmatrix} q(t). \]
The extended system matrix has a block-diagonal structure. If the pair \((A_1, B_1)\) is stabilizable and the pair \((C_1, A_1)\) is detectable, then it is clear that the left upper block may be stabilized by a suitable choice of \(L, M\) and \(N\); and this is certainly a reasonable design objective. On the other hand, it is clear that the eigenvalues of \(A_{22}\) cannot be shifted by dynamic output feedback.

We can formulate the requirement "left upper block stable" in coordinate-free terms, in the following way. Roughly stated, what we want is that the unstable poles of the closed-loop system are precisely the unstabilizable poles of the open-loop system. Let us write \(X_e(A_e)\) for the unstable subspace of \(A_e\) (cf. (1.5.1)). Then our requirement can be expressed as follows:

\[
(3.15) \quad \dim X_e(A_e) = \text{codim} X_{\text{stab}}
\]

where the subspace on the right-hand side is the one defined in Section 1.5 (see also Lemma 1.17). One can say that in the coordinate-free terminology, the subspace \(X_{\text{stab}}\) takes the place of the state space of \(x(t)\) in the explicit form (3.13). (This identification assumes, of course, that the pair \((A_1, B_1)\) is stabilizable. If this is not the case, then the "left upper block" cannot be made stable and the problem in the form we want to pose it is not solvable. The identification assumes also that \(A_{22}\) is completely unstable, but this is less essential.)

We now have two stability requirements, one in (3.5.3) and one in (3.15). In (3.5), the behaviour of the variables-to-be-controlled is in focus, whereas (3.15) gives a condition for all stabilizable state variables. The two stability requirements do not have to be the same; in fact, it is quite natural to assume that a slow response of a large part of the system would be satisfactory whereas the variables-to-be-controlled should behave much faster. Therefore, we should use two partitionings of the complex plane instead of one, as illustrated below.

![Figure 3.2: Two partitionings of the complex plane.](image-url)
Further on, we shall assume that two parts of the complex plane have been
given, denoted by $\mathcal{E}_f$ ("f" for "fast") and $\mathcal{E}_s$ ("s" for "slow"). We shall also
assume that $\mathcal{E}_f \subset \mathcal{E}_s$, and denote $\mathcal{E}_t = \mathcal{E} \setminus \mathcal{E}_f$ ("t" for "tardy") and $\mathcal{E}_u = \mathcal{E} \setminus \mathcal{E}_s$
("u" for "unstable"). Because of the condition $\mathcal{E}_f \subset \mathcal{E}_s$, what we obtain is
a division of the complex plane into three disjoint parts: $\mathcal{E}_f$, $\mathcal{E}_s \cap \mathcal{E}_t$ and
$\mathcal{E}_u$. We shall change our notations in an obvious way to conform to the new
situation; for instance $X^o_{\text{stab}}$ will denote the largest stabilizability sub-
space of the pair $(A, B)$ with respect to the division $\mathcal{E} = \mathcal{E}_s \cup \mathcal{E}_u$, $X^o_u(A)$
will denote the span of the generalized eigenvectors of $A$ associated with
eigenvalues in $\mathcal{E}_u$, etc..

We can now give the following problem statement: Given a system of
the form (3.12), find a compensator of the form (2.2) such that the extended system matrix $A_e$ satisfies

$$
(3.16) \quad \dim X^o_{u}(A_e) = \text{codim } X^o_{\text{stab}}
$$

and such that there is a subspace $M$ of the extended state space $X \otimes W$
satisfying

$$
(3.17.1) \quad A_e M \subset M
$$

$$
(3.17.2) \quad \text{Im}(\mathcal{E}_Q) \subset M \subset \text{Ker} (D \quad 0)
$$

$$
(3.17.3) \quad \sigma(A_e : (X \otimes W)/M) \subset \mathcal{E}_f.
$$

We shall refer to this problem as the regulator problem. The problem
will be solved completely in Section 3.3; we shall formulate necessary
and sufficient conditions for the solvability of the problem, give a con-
structive algorithm to verify these conditions, and present a method
to construct a solution if there exists one.

The regulator problem as we formulate it is a generalization of a
number of problems which have been studied separately in the past. There
are two lines of research that can be distinguished. On one hand, we have
the theory of "tracking and regulation" which has aroused considerable
interest; see for instance JOHNSON (1968, 1971), YOUNG & WILLEMS (1972),
DAVISON & GOLDENBERG (1975), WONHAM & PEARSON (1974), FRANCIS (1977),
WONHAM (1979). In this theory, one does not consider the presence of an
unmodeled disturbance (i.e., $E = 0$ in the system (3.12)). On the other
hand, such a disturbance is the central object of study in the theory of "disturbance localization". Here, one usually assumes that the pair \((A, B)\) is stabilizable. Without stability requirements, the state feedback solution of the problem is perhaps the simplest application of the idea of controlled invariant subspaces (see Wonham (1974) or, for a dual version, Basile & Marro (1969c)), but the solution by dynamic output feedback has only recently been obtained (Akashi & Imai (1979), Schumacher (1979)). For a treatment of the latter problem with stability requirements added, see Willems & Conault (1981) and Imai & Akashi (1981).

Many of the results in the above articles follow as special cases of the general theorem we shall prove. We shall perform a number of these specializations in Section 3.4. Our partitioning of the complex plane into three parts (instead of the usual number of two) is essential to obtain one general theorem, which simultaneously treats the problems with "output stability" and "internal stability" (terminology of Wonham (1979)).

3.2. Preliminary results

First, we give a result that will be helpful in proving the 'necessity' part of the main theorem. The result may be compared to Thm. 2.16. Again, we consider an invariant subspace of the extended state space and the related pair of subspaces in the original state space. But now we do not assume that the invariant subspace is separating, and so our conclusions are somewhat weaker. One remark on notation: we shall use \(\mathcal{E}_g\) which may denote either \(\mathcal{E}_d\) or \(\mathcal{E}_c\).

**Theorem 3.1** Let \(A_e\) be an extended system matrix of the form (2.5). For any \(A_e\)-invariant subspace \(M\), the pair \((\mathcal{Q}^{-1}M, PM)\) is a compensator couple. Moreover, if \(A_e\) : \(M\) is stable, then \(\mathcal{Q}^{-1}M\) is inner-detectable and PM is inner-stabilizable. Also, if \(A_e\) : \((XM0)/M\) is stable, then \(\mathcal{Q}^{-1}M\) is outer-detectable and PM is outer-stabilizable.

**Proof** Let \(M\) be \(A_e\)-invariant. Take \(x \in \mathcal{Q}^{-1}M\); then \(\overline{x}\) \(\in M\) so that \(x \in PM\) and, moreover, \(Ax \in PM\) because

\[
A_e \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ MCx \end{pmatrix} \in M.
\]

To show that \(\mathcal{Q}^{-1}M\) is conditioned invariant, take \(x \in \mathcal{Q}^{-1}M \cap \text{Ker C};\) then (3.18) shows that \(Ax \in \mathcal{Q}^{-1}M\) and we obtain the desired conclusion from
Prop. 1.7. Next, take $x \in PM$. Then there exists $w \in W$ such that $(x_w)_w \in M$, and consequently:

\begin{equation}
A_e (x_w)_w = (\begin{pmatrix} A & BL \\ MC & N \end{pmatrix}) (x_w)_w = (Ax + BLw) = (Ax + BLw) = (\lambda x + BLw) \in M.
\end{equation}

Hence, $Ax + BLw \in PM$ which implies $Ax \in PM + Im B$; we may conclude that $PM$ is controlled invariant using Prop. 1.6.

Now, let us assume that $A_e : M$ is stable. To prove that $PM$ is a stabilizability subspace, it is sufficient (in view of Cor. 1.15) to show that

\begin{equation}
PM \subset (\lambda - A)PM + Im B \text{ for all } \lambda \in \mathbb{C}_b.
\end{equation}

But because $\sigma(A_e : M) \subset \mathbb{C}_b$, we know that

\begin{equation}
M = (\lambda - A_e)M \quad \text{for all } \lambda \in \mathbb{C}_b.
\end{equation}

From this we obtain (3.20) immediately by noting that for $(x_w)_w \in M$,

\begin{equation}
P(\lambda - A_e) (x_w)_w = (\lambda - A)x - BLw \in (\lambda - A)PM + Im B.
\end{equation}

Still assuming that $A_e : M$ is stable, let us now show that $Q^{-1}M$ is inner-detectable. Suppose the contrary; then it follows from Prop. 1.19 (in dualized form) and Lemma 1.18 that $Q^{-1}M \cap Ker C \cap A \cap X^N \neq 0$. So there would exist $x \neq 0$ and $\lambda \in \mathbb{C}_b$ such that $x \in Q^{-1}M \cap Ker C$ and $Ax = \lambda x$. Then we would have $(x_w)_w \in M$ and

\begin{equation}
A_e (x_w)_w = (\begin{pmatrix} A & BL \\ MC & N \end{pmatrix}) (x_w)_w = (Ax + BLw) = (\lambda x + BLw),
\end{equation}

contradicting our assumption that $A_e : M$ is stable. Thus, $Q^{-1}M$ must be inner-detectable.

The rest of the proof follows by duality.

The fact that $PM$ is a stabilizability subspace if $A_e : M$ is stable has also been proved by DAI & AKASHI (1981); these authors used a different method requiring the computation of an $F \in F(PM)$. We did not mention the inner-detectability of $Q^{-1}M$ in Thm. 2.16; however, this follows immediately from the statement of the theorem because (using (2.36.1) and (2.37.1))
(3.24) \( o(A + \text{GC}: Q^{-1}M \cap <\text{Ker C} | A>) = \)
\( = o(A: Q^{-1}M \cap <\text{Ker C} | A>) = \)
\( = o(A + \text{BF}: Q^{-1}M \cap <\text{Ker C} | A>) = \)
\( c o(A + \text{BF}: \Phi M) = \Phi_0^c. \)

Also, Thm. 3.1 gives the 'necessity' part of the proof of Thm. 2.1 (for instance, take \( M = 0 \)). Of course, the 'sufficiency' part of this proof is provided by Cor. 2.7.

As in Ch. 2, the proof of the main theorem will be based on a construction lemma, which we now present.

**Lemma 3.2** Suppose that we are given three compensator couples \((\xi_s, \Psi_s), (\xi_f, \Psi_f),\) and \((\xi_c, \Psi_c),\) such that \(\xi_s = \xi_f\) and \(\Psi_s = \Psi_f.\) Further suppose that we are also given a mapping \(F \in \Psi_s(\Psi_s) \cap \Psi_f(\Psi_f) \cap \Psi_c(\Psi_c)\) with \(\text{Ker } F = \xi_c\) and a mapping \(G \in \Psi_s(\Psi_s) \cap \Psi_f(\Psi_f) \cap \Psi_c(\Psi_c)\) with \(\text{Im } G = \Psi_c.\) Let \(W\) be a linear space isomorphic to \(\Psi / \xi_c\) and let \(R\) be a mapping of \(\Psi_c\) onto \(W\) such that \(\text{Ker } R = \xi_c.\)

Let \(R^+\) be any right inverse of \(R,\) and define

\[
\begin{align*}
A_e &= \begin{pmatrix} A & BFR^+ \\ -GRC & R(A + BF + GC)R^+ \\ \end{pmatrix},
\end{align*}
\]

Then the subspaces of \(X \oplus W\) defined by

(3.26.1) \[ M_s = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \xi_s \right\} + \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \Psi_s \cap \Psi_c \right\} \]

(3.26.2) \[ M_f = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \mid x \in \xi_f \right\} + \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \Psi_f \cap \Psi_c \right\} \]

(3.26.3) \[ M_c = \left\{ \begin{pmatrix} x \\ Rx \end{pmatrix} \mid x \in \Psi_c \right\} \]

are all \(A_e\)-invariant, and we have the following relations:

(3.27.1) \[ \sigma(A + BF: \Psi_s \cap \Psi_c) = \sigma(A_e: M_s \cap M_c) \]

(3.27.2) \[ \sigma(A + BF: \Psi_f \cap \Psi_c) = \sigma(A^e: M_f \cap M_c / M_s \cap M_c) \]

(3.27.3) \[ \sigma(A + BF: \Psi_s \cap \Psi_f) = \sigma(A^e: M_s \cap M_f / M_c) \]
(3.27.4) \[ \sigma(A+GC: T_{s} / T_{s} \cap T_{c}) = \sigma(A^{E}; M_{s} / M_{s} \cap M_{c}) \]

(3.27.5) \[ \sigma(A+GC: T_{a} / T_{c} \cap T_{e}) = \sigma(A^{E}; M_{a} + M_{c} / M_{a} + M_{c}) \]

(3.27.6) \[ \sigma(A+GC: X / T_{f} \cap T_{c}) = \sigma(A^{E}; X^{E} / M_{f} + M_{c}) . \]

Before embarking on the proof of this lemma, let us make a few remarks.

The spectral relations in (3.27) can conveniently be summarized in the following scheme:

![Diagram](image)

Fig. 3.3. Regulator construction
Corresponding letters are used to indicate the equalities in (3.27); compare Fig. 2.5. It should be noted that \( V_s = (V_s \cap V_c) = V_s \cup V_c \) (the 'distributive law' may be applied because \( V_s \subseteq V_c \)), and that similar remarks can be made concerning the other diagrams (it follows immediately from the definitions that \( M_s = M_c \)). As a final remark note that \( M_c \) can be described similarly as \( M_s \) and \( M_f \):

\[
M_c = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in T_c \right\} + \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in T_c \right\}.
\]

To see this, note that \( Rx = 0 \) for \( x \in T_c \).

**Proof** First we have to show that the subspaces defined in (3.26) are \( A \)-invariant; so let us consider \( M_j \) with \( j \in \{s,f,c\} \). If \( x \in T_j \), then

\[
A \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax \\ -RGCx \end{bmatrix} = \begin{bmatrix} (A + GC)x \\ 0 \end{bmatrix} = \begin{bmatrix} (A + GC)x \\ RGCx \end{bmatrix} \in M_j
\]

because \( T_j \) is \((A + GC)\)-invariant, and because \( GCx = (A + GC)x - Ax \in V_j \cap V_c \) (using that \( A T_j = V_j \) and that \( Im G \subseteq V_c \)). Let us next take \( x \in V_j \cap V_c \), then

\[
A \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} Ax + BF^\ast Rx \\ R(A + BF + GC)R^\ast Rx - RGCx \end{bmatrix}.
\]

Write \( x = R^\ast Rx + \tilde{x} \), then \( Rx = 0 \) so that \( \tilde{x} \in T_c \subseteq \text{Ker } F \). Thus we have \( FR^\ast Rx = Fx \). Moreover, \( R(A + GC)R^\ast Rx = R(A + GC)x \) because \( (A + GC)T_c \subseteq T_c \). We find that \( R(A + BF + GC)R^\ast Rx - RGCx = R(A + BF + GC)x - RGCx = R(A + BF)x \). So (3.30) becomes

\[
A \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} (A + BF)x \\ R(A + BF)Rx \end{bmatrix} \in M_j \quad (x \in V_j \cap V_c)\).
\]

This completes the first part of the proof. The second part (to prove the relations (3.27)) is an exercise in finding natural isomorphisms between subspaces and constructing commutative diagrams. We shall not work this out completely; by way of example, let us prove (3.27.5).

We define a mapping \( \bar{Q} \): \( T_f + T_c / T_g + T_c \rightarrow M_f + M_c / M_g + M_c \) as follows:

\[
\bar{Q}(x) = \begin{bmatrix} Qx \\ 0 \end{bmatrix} \quad (x \in T_f + T_c)
\]
First of all we have to show that this mapping is well-defined; this follows from the fact that \( Q(T_e + T_c) \subseteq M_s + M_c \). Because \( Q(T_e + T_c) \subseteq M_s + M_c \), \( \tilde{Q} \) does indeed map into \( M_c + H_s / M_c + M_c \). Next we show that \( \tilde{Q} \) is in fact an isomorphism. If \( \tilde{Q}[x] = 0 \) for some \( x \in T_e + T_c \), then \( Qx \in M_s + M_c = QT_s + T_c \); this gives \( x \in Q^{-1}(QT_s + T_c) = T_s + T_c \). So \( \tilde{Q} \) is injective. Surjectivity of \( \tilde{Q} \) follows from the fact that \( M_c + M_c = QT_e + M_c \subset Q(T_e + T_c) + (M_s + M_c) \).

Finally, we prove that the following diagram, in which \( \Lambda + GC \) and \( \Lambda_e \) denote the induced mappings on the indicated factor spaces, commutes.

\[
\begin{array}{ccc}
T_e + T_c / T_s + T_c & \xrightarrow{\Lambda + GC} & T_e + T_c / T_s + T_c \\
\tilde{Q} \downarrow & & \tilde{Q} \downarrow \\
M_c + H_s / M_c + M_c & \xrightarrow{\Lambda_e} & M_c + H_s / M_c + M_c
\end{array}
\]

(3.34)

The proof is by direct computation: for \( x \in T_e + T_c \), we have

(3.35) \( \Lambda_e \) \( \tilde{Q}[x] = [\begin{bmatrix} Ax \\ -RGCx \end{bmatrix}] \).

and on the other hand

(3.36) \( \tilde{Q} \) \( \Lambda + GC[x] = [\begin{bmatrix} (\Lambda + GC)x \\ 0 \end{bmatrix}] \).

Both results are equal because

(3.37) \( (\Lambda + GC)x - (\Lambda - RGCx) = (GCx) \in M_c + M_c \).

The situation described by Fig. 3.3 simplifies considerably if we assume that \( T_c \subseteq T_s \subseteq T_e \) and \( \psi \subset \psi_c \subset \psi_e \). The corresponding picture is:
3.3. Main theorem

We now state the necessary and sufficient conditions for a solution of the regulator problem to exist.

**THEOREM 3.3.** Let the system (3.12) be given. Then there exists a solution of the regulator problem as formulated in (3.16) and (3.17) if and only if there exist two compensator couples \((T_s, V_s)\) and \((T_f, V_f)\) with the following properties:

\[
\begin{align*}
(3.38.1) & \quad T_s \subset T_f, \quad V_s \subset V_f \\
(3.38.2) & \quad T_f \text{ is outer-detectable and } V_f \text{ is outer-stabilizable, both with respect to } E_f \\
(3.38.3) & \quad \text{Im } E \subset T_f \subset V_f \subset \text{Ker } D \\
(3.38.4) & \quad T_s \text{ is outer-detectable with respect to } E_s, \quad \text{and } V_s \circ X^{\delta}_{\text{stab}} = X \\
(3.38.5) & \quad V_f = V_s \circ S \text{ for some inner-stabilizable (with respect to } E_s) \text{ controlled invariant subspace } S.
\end{align*}
\]

**Proof.** Necessity Assume that \(A_e\) is an extended system matrix (of the form (2.5)) satisfying (3.16), and let \(\mathcal{M}\) be a subspace of \(X \circ \mathcal{W}\) such that (3.17)
holds. Denote the unstable subspace (in the sense of \( \mathfrak{U} \)) of \( A_e \) by \( X_u^e(A_e) \), and define \( T_s = Q^{-1} X_u^e(A_e) \), \( V_s = PX_u^e(A_e) \), \( T_f = Q^{-1} H \) and \( V_f = PH \). It follows immediately from Thm. 3.1 that \((T_s, V_s)\) and \((T_f, V_f)\) are compensator couples, and that (3.38.2) holds; (3.38.4) will follow from Thm. 3.1 and Prop. 1.19 if we can also show that

\[
(3.39) \quad Px_u^e(A_e) \cap X_{\text{stab}}^s = \emptyset.
\]

But this is immediate from (3.16).

Because \( \mathfrak{U} \subset \mathfrak{U}_s \) (see Fig. 2.2) and because of (3.17.3), we have \( X_u^e(A_e) \subset X_e^e(A_e) \subset H \) and this immediately gives (3.38.1). Finally, define \( S \) by

\[
(3.40) \quad S = P(H \cap X_e^e(A_e)).
\]

Then it is clear that \( S \) is inner-stabilizable with respect to \( \mathfrak{U}_s \), so that \( S \subset X_{\text{stab}}^s \). Since \( X_u^e(A_e) \subset H \), we also have

\[
(3.41) \quad M = X_u^e(A_e) \oplus (H \cap X_e^e(A_e)).
\]

Projecting both sides into \( X \), we obtain

\[
(3.42) \quad V_f = V_s + S.
\]

But we already noted that \( V_s \cap X_{\text{stab}}^s = \emptyset \) and that \( S \subset \not{X}_{\text{stab}}^s \), so in fact we have \( V_f = V_s \oplus S \).

**Sufficiency.** Now we suppose that two compensator couples \((T_s, V_s)\) and \((T_f, V_f)\) are given for which (3.38) holds. We construct \( F \in F(V_s) \cap F(V_f) \) and \( G \in C(T_s) \cap C(T_f) \) as follows. Because \( V_s \uplus \not{X}_{\text{stab}}^s = X \), we may specify \( F \) by giving its action on \( V_s \) and \( X_{\text{stab}}^s \) separately. On \( V_s \), we define \( F \) such that \( F \in F(V_s) \). Note that \( V_s \supseteq S \supseteq X_{\text{det}}^s \), so that \( X_{\text{det}}^s \) is outer-detectable; as \( X_{\text{det}}^s \subset X_{\text{det}}^s \subset X \), we can assume that \( \ker F \supseteq X_{\text{det}}^s \). (In fact, it is not difficult to show that \( F \) is uniquely defined on \( V_s \) by the requirement \( F \in F(V_s) \), so that the relation \( X_{\text{det}}^s \subset \ker F \) holds necessarily. Also, \( \sigma(A \uplus BF; V_s) = \sigma(A; X^s_{\text{stab}}) \subset \mathfrak{U}_u \).)

As to \( X_{\text{stab}}^s \), we note that this is an \( A \)-invariant subspace containing \( \Im B \); so we can consider the pair \((A; X_{\text{stab}}^s, B)\) where \( B \) is viewed as a mapping into \( X_{\text{stab}}^s \). With respect to this pair, the subspace \( S \) is inner-
stabilizable w.r.t. $\mathcal{E}_s$ as well as outer-stabilizable w.r.t. $\mathcal{E}_f$ (the latter because it follows from $V_f + X_{\text{stab}}^f = X$ that $S + X_{\text{stab}}^s = X_{\text{stab}}^s$; note that $X_{\text{stab}}^f \subset X_{\text{stab}}^s$). So we can define $F$ on $X_{\text{stab}}^s$ such that $F \in \mathcal{F}(S)$, $\sigma(A+BF; S) \subset \mathcal{E}_s$ and $\sigma(A+BF; X_{\text{stab}}^s/S) \subset \mathcal{E}_f$. Finally, it follows from the dual of Thm. 1.11 that we can choose $G \in \mathcal{G}(T_s) \cap \mathcal{G}(T_f)$ such that $\sigma(A+GC; X/T_f) \subset \mathcal{E}_f$ and $\sigma(A+GC; T_f/X_{\text{det}}) \subset \mathcal{E}_s$.

The spectral situation for $A + BF$ and $A + GC$ can be summarized as follows:

```
\begin{align*}
\text{X} & \rightarrow \text{X} \\
\text{b} & \in \mathcal{E}_f & \text{a} & \in \mathcal{E}_f \\
\text{c} & \in \mathcal{E}_s & \text{d} & \in \mathcal{E}_s \\
\text{e} & \in \mathcal{E}_s & \text{f} & \in \mathcal{E}_u \\
\text{A+BF} & \rightarrow \text{A+GC} \\
0 & \in \mathcal{E}_u & 0 & \in \mathcal{E}_u \\
\text{X}_{\text{det}} & \rightarrow \text{X}_{\text{det}} \\
\mathcal{T}_f & \rightarrow \mathcal{T}_f \\
\mathcal{T}_s & \rightarrow \mathcal{T}_s \\
\end{align*}
```

Figure 3.5: Basis situation for the construction lemma.

Now, we apply Lemma 3.2 with $T_c = X_{\text{det}}^s$ and $V_c = X$. The conditions of the lemma are all satisfied, and so we obtain the matrix $A_e$ of (3.25) for which the scheme (3.27) holds. Comparing Fig. 3.5 with Fig. 3.4, we see immediately that $M_f$ is outer-stable w.r.t. $\mathcal{E}_s$. We have $\text{Im} = T_f \subset Q^{-1}M_f \subset \text{Im}(Q)$ and $\text{I} \subset \text{Ker}D$ so $\text{Im}(Q) \subset M_f \subset \text{Ker}D$; thus, (3.17) is satisfied. It is also clear from Fig. 3.4 and Fig. 3.5 that $X_{\text{det}}^s(A_e) = M_s \cap M_c$, and that $\sigma(A^s; X_{\text{det}}^s(A_e)) = \sigma(A+BF; V_s) = \sigma(A+BF; X/X_{\text{det}}^s) = \sigma(A; X/X_{\text{det}}^s)$ which shows that (3.16) holds.

The result as it has been stated here is given in terms of the existence of two compensator couples satisfying (3.38). So of course one may ask: Is there an algorithm by which one (knowing the system operators $A, B, C, D$ and $E$) can find out if two such pairs exist? And if this is the case, can such pairs be constructed? If these questions admit a positive answer, then we are entitled to say that the regulator problem has been solved in a constructive way. We shall show that this is indeed the case.
In the formulation of the theorem, there are four subspaces to be selected. However, this amount can readily be reduced to one.

PROPOSITION 3.4 There exist two compensator couples $(T^*_s, V_s)$ and $(T^*_f, V_f)$ satisfying (3.38) if and only if there exists a controlled invariant subspace $V$ such that

\[(3.43.1) \quad V \oplus X^g_{\text{stab}} = X\]

\[(3.43.2) \quad V = X^g_{\text{det}}\]

\[(3.43.3) \quad V \subseteq \text{Ker } D\]

\[(3.43.4) \quad V + V^s_s(\text{Ker } D) = T^*_f(\text{Im } E) + AT^*_f(\text{Im } E)\]

and if, moreover,

\[(3.44) \quad V^s_s(\text{Ker } D) + X^f_{\text{stab}} = X^g_{\text{stab}}\]

PROOF First assume that we have two compensator couples $(T^*_s, V_s)$ and $(T^*_f, V_f)$ satisfying (3.38). Then $V_f = V_s \oplus S$ where $S$ is a controlled invariant subspace that is inner-stabilizable w.r.t. $E_s$. Because $V_f \subseteq \text{Ker } D$, we also have $S \subseteq \text{Ker } D$; so $S = V^s_s(\text{Ker } D)$. From the fact that $V_f$ is outer-stabilizable it follows that $S + X^f_{\text{stab}} = X^g_{\text{stab}}$, so (3.44) certainly holds.

Now take $V = V_s$. We already noted that $V^s_s(\text{Ker } D) = S$; thus, $V + V^s_s(\text{Ker } D) \supseteq V + S = V_f$; because $T^*_f(\text{Im } E) \subseteq T^*_f$, (3.43.4) follows. Also, we have $V \supseteq T^*_s \supseteq X^s_s$; finally, $V \subseteq V_f \subseteq \text{Ker } D$, and $V \oplus X^g_{\text{stab}} = X$ by (3.38.4).

Now assume that a controlled invariant subspace $V$ that satisfies (3.43) has been given. Define $T^*_s = X^g_{\text{det}}, V_s = V, T^*_f = T^*_f(\text{Im } E)$ and $V_f = V + V^s_s(\text{Ker } D)$. Then (3.38.1) holds because $X^g_{\text{det}} \subset X^f_{\text{det}} \subset T^*_f(\text{Im } E)$.

For (3.38.2), we note that $V_f + X^f_{\text{stab}} = V + X^g_{\text{stab}} = X$ so that $V_f$ is indeed outer-stabilizable w.r.t. $E^f_f$. The other conditions in (3.38) are trivially verified.

It should be noted that the subspaces $X^g_{\text{stab}}, X^f_{\text{stab}}, X^g_{\text{det}}, V^s_s(\text{Ker } D)$ and $T^*_f(\text{Im } E)$ can all be considered as known, in view of the algorithms mentioned in Section 1.4. So the condition (3.44) can be verified immediately and the remaining question is: How can we verify the existence of a
controlled invariant subspace satisfying (3.43), and how can we find such a subspace if it exists? (Note that the two compensator couples of Theorem 3.3 can then be constructed as in the proof of the proposition.)

We shall solve this problem by introducing coordinates relative to $\mathcal{X}_{\text{stab}}$. Fix a basis $\{x_1, \ldots, x_{k'}, x_{k'+1}, \ldots, x_n\}$ for $\mathcal{X}$ such that $\{x_1, \ldots, x_k\}$ is a basis for $\mathcal{X}_{\text{stab}}$. With respect to the thus partitioned basis, let the matrices of $A, B$ and $D$ be given by

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

$$D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}.$$  

Also, we introduce basis matrices for the known subspaces $\mathcal{X}_{\text{det}}$, $\mathcal{T}_{\text{f}}(\text{Im} E) + A\mathcal{T}_{\text{f}}(\text{Im} E)$, and $\mathcal{V}_{\text{g}}(\text{Ker} D)$:

$$\mathcal{X}_{\text{det}} = \text{sp}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\mathcal{T}_{\text{f}}(\text{Im} E) + A\mathcal{T}_{\text{f}}(\text{Im} E) = \text{sp}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$\mathcal{V}_{\text{g}}(\text{Ker} D) = \text{sp}\begin{pmatrix} s_1 \\ 0 \end{pmatrix}.$$  

We state the following result.

**COROLLARY 3.5** Let the system (3.12) be given. Then there exists a solution to the regulator problem as formulated in (3.16) and (3.17) if and only if (3.44) holds and (using the notation (3.45)) there exist matrices $V, Q$ and $R$ such that

$$A_{12} = VA_{22} - A_{11}V + B_1R$$

$$X_1 = VX_2$$

$$D_2 = -D_1V$$

$$W_1 = SQ + VW_2.$$
Moreover, if these conditions are fulfilled then a compensator as required can be constructed as in the proof of Thm. 3.3, using the compensator couples \((T_s, V_s)\) and \((T_f, V_f)\) given by

\[
\begin{align*}
(3.47.1) \quad T_s &= \text{sp}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right), \quad V_s = \text{sp}\left(\begin{bmatrix} V_1 \\ I \end{bmatrix}\right) \\
(3.47.2) \quad T_f &= \text{sp}\left(\begin{bmatrix} W_1 \\ W_2 \end{bmatrix}\right), \quad V_f = \text{sp}\left(\begin{bmatrix} S & V_1 \\ 0 & I \end{bmatrix}\right).
\end{align*}
\]

**PROOF** We use Prop. 3.4. With respect to the basis used in (3.45), denote \(V\) by

\[
(3.48) \quad V = \text{sp}\left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right).
\]

Now we can start translating the conditions (3.43) into matrix terms. It follows from (3.43.1) that \(V\) should be \((n\cdot n)\)-dimensional, so that the matrix \(V_2\) in (3.48) can be taken square. Then (3.43.1) says that the \(n\times n\)-matrix

\[
(3.49) \quad \begin{pmatrix}
I_2 & V_1 \\
0 & V_2
\end{pmatrix}
\]

should be nonsingular. This will be the case if and only if \(V_2\) is invertible. But then we have

\[
(3.50) \quad \text{sp}\left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right) = \text{sp}\left(\begin{bmatrix} V_1 V_2^{-1} \\ V_2 \end{bmatrix}\right) = \text{sp}\left(\begin{bmatrix} V_1 V_2^{-1} \\ I \end{bmatrix}\right)
\]

and consequently, we shall from now on describe \(V\) by

\[
(3.51) \quad V = \text{sp}\left(\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}\right).
\]

It is required that \(V\) should be a controlled invariant subspace; this means that there should exist matrices \(\hat{Q}\) and \(I\) such that

\[
(3.52) \quad \begin{pmatrix}
A_{11} & A_{12} \\
0 & A_{22}
\end{pmatrix}\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \hat{Q} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} R
\]

It readily follows that we must have \(\hat{Q} = A_{22}\). Thus, \(V\) is controlled invariant if and only if there is a matrix \(R\) such that
(3.53) \[ A_{11}V + A_{12} = VA_{22} + B_1R. \]

Next, (3.43.2) is equivalent to the existence of a matrix \( \hat{Q} \) such that

(3.54) \[ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} V \\ \hat{Q} \end{pmatrix}. \]

This is clearly equivalent to

(3.55) \[ X_1 = VX_2. \]

To translate (3.43.3) is very simple too:

(3.56) \[ (D_1 \ D_2) \begin{pmatrix} V \\ \hat{Q} \end{pmatrix} = D_1V + D_2 = 0. \]

Finally, (3.43.4) holds if and only if there exist matrices \( Q \) and \( \hat{R} \) such that

(3.57) \[ \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = (S)Q + (\hat{T})\hat{R}. \]

We get \( \hat{R} = W_2 \), and so (3.43.4) is equivalent to the existence of matrix \( Q \) such that

(3.58) \[ W_1 = SQ + WW_2. \]

The mapping that takes \((V, Q, R)\) to \((VA_{22} - A_{11}, V + B_1R, VX_2, -D_1V, SQ + WW_2)\) is a linear mapping between linear spaces of finite dimension, and so it is in principle straightforward to verify the condition of the corollary (it is in the form "a given vector must be in the range of a given linear operator"). Thus, we have indeed obtained a fully constructive solution.

3.4. Comparison with other work

In order to compare our results with earlier work on tracking, regulation and disturbance localization problems, we have to take into account the fact that most authors allow the use of static feedback in the solution of
these problems. That is, one constructs extended system matrices not of the form (2.5) but of the form

\[(3.59) \quad A_e = \begin{pmatrix} A & B \cr C & D \end{pmatrix} \quad \text{(A+BKC)} \quad \begin{pmatrix} I & N \end{pmatrix} \quad \text{(N)}. \]

We shall refer to the regulator problem in this modified form as the relaxed regulator problem. The link between the two versions is given by the following lemma.

**Lemma 3.6** Let \((T_s, V_s)\) and \((T_f, V_f)\) be two pairs of subspaces such that \(T_s \subseteq T_f\) and \(V_s \subseteq V_f\), and suppose that both pairs satisfy the conditions (ccl-3) (see Section 2.2). Then there exists a mapping \(K\) such that \((T_s, V_s)\) and \((T_f, V_f)\) are compensator couples with respect to the triple \((C, A+BKC, B)\).

**Proof** We have to show that, under the given conditions, there exists \(K\) such that \((A+BKC)T_s \subseteq V_s\) and \((A+BKC)T_f \subseteq V_f\). Because \(T_s \subseteq T_f\), there exists a projection \(P: X \to X\) with \(\text{Im } P = \text{Ker } C\) such that \(PT_s \subseteq T_s\) and \(PT_f \subseteq T_f\). Because \(V_s \subseteq V_f\); there exists \(F \in F(V_s) \cap F(V_f)\). Write \(P = I - TC\) (with \(T: V \to X\); then \(K = PT\) will do.

To verify this statement, note that \(T_s = PT_s \cap TC S\) and \(T_f = PT_f \cap TC F\). Moreover, \(PT_s = T_s \cap \text{Ker } C\) and \(PT_f = T_f \cap \text{Ker } C\). We get

\[(3.60) \quad (A+BFTC)T_s = (A+BFTC)(T_s \cap \text{Ker } C) + (A+BFTC)TC T_s \subset \]

\[\subseteq A(T_s \cap \text{Ker } C) + (A+BF)T_s \subseteq T_s + V_s \subseteq V_s \]

and similarly for the pair \((T_f, V_f)\).

Using this lemma, it is easy to see that necessary and sufficient conditions for the solvability of the relaxed regulator problem can be given precisely as in Thm. 3.3, with only one modification: the pairs \((T_s, V_s)\) and \((T_f, V_f)\) are not required to satisfy all conditions (ccl-4), but only the conditions (ccl-3). (Compare Schumacher (1980 d), where such pairs ("(C, A, B)-pairs") were used throughout.) Accordingly, the necessary and sufficient conditions can also be given in the form of Prop. 1.4, if (3.42.4) is replaced by

\[(3.61) \quad V + U_s^*(\text{Ker } D) \supset T_f^*(\text{Im } E)\].
Note that both versions coincide if $E = 0$. This means that in this case the solvability conditions for the regulator problem and its relaxed version are the same; of course, adding static feedback may still be useful to obtain improved bounds for the compensator order (see Wonham (1979), Schumacher (1980 d), Janssen (1981)).

Following Wonham (1979), we use the term "output stability" to describe the situation in which $\xi$ has been set equal to $\xi$, so that there is only a stability requirement on the variables-to-be-controlled. In this situation, the complex plane has in effect been divided into two parts ($\xi_f$ and $\xi_e$) and we shall fall back on the old notation, writing $\xi$ instead of $\xi_f$ etc.

**Corollary 3.7** The regulator problem with output stability is solvable if and only if

\begin{align*}
(3.62.1) \quad & \mathcal{V}^*(\text{Ker } D) \supset T^*_{\xi}(\text{Im } E) \\
(3.62.2) \quad & \mathcal{V}^*(\text{Ker } D) \supset A_{\xi}^*\mathcal{V}^*(\text{Im } E) \\
(3.62.3) \quad & \mathcal{V}^*(\text{Ker } D) + X_{\text{stab}} = X.
\end{align*}

**Proof** If $\xi = \xi$, then $X_{\text{stab}}^* = X$, $X_{\text{det}}^* = 0$ and $\mathcal{V}^*(\text{Ker } D) = \mathcal{V}^*(\text{Ker } D)$. The statement now follows immediately from Thm. 3.3 and Prop. 3.4.

The necessary and sufficient conditions for the solvability of the relaxed regulator problem are, of course, given by (3.62.1) and (3.62.3); see Sonnevend (1977; Thm. 3). We can specialize further by assuming that $E = 0$; then we obtain the "Extended Regulator Problem" of Wonham (1979; p. 139). The solution is as follows (Wonham's Thm. 6.2).

**Corollary 3.8** The regulator problem with output stability and $E = 0$ is solvable if and only if

\begin{align*}
(3.63.1) \quad & X_{\text{det}} \subseteq \text{Ker } D \\
(3.63.2) \quad & X_{\text{det}}(A) \subseteq \mathcal{V}^*(\text{Ker } D) + \langle A \mid \text{Im } B \rangle.
\end{align*}

**Proof** If $E = 0$, then $T^*_{\xi}(\text{Im } E) = X_{\text{det}}$. The condition (3.62.2) is then implied by (3.62.1) which is in its turn equivalent to (3.63.1), because $X_{\text{det}}$ is
A-invariant and hence also controlled invariant. Further, (3.63.2) is equivalent to (3.62.3) by Prop. 1.12 and Prop. 1.19.

Another possible specialization is to see $\mathfrak{C}_e$ equal to $\mathfrak{C}_f$, so that there is no distinction between the stability requirements for the variables-to-be-controlled and the other controllable variables. We may call this "maximal stability". Again, we are left with a division of the complex plane into two parts; so we shall use the $\mathfrak{C}_e - \mathfrak{C}_d$-notation.

The general result in this situation is the following.

**Corollary 3.9** The regulator problem with maximal stability is solvable if and only if there exists a controlled invariant subspace $V$ such that

\begin{align}
(3.64.1) & \quad V \otimes X_{\text{stab}} = X \\
(3.64.2) & \quad X_{\text{det}} \subset V \subset \text{Ker } D \\
(3.64.3) & \quad \tau^*_e(\text{Im } E) \subset V + V^*(\text{Ker } D) \\
(3.64.4) & \quad \tau^*_g(\text{Im } E) \subset V + V^*(\text{Ker } D).
\end{align}

**Proof** Immediate from Thm. 3.3 and Prop. 3.4.

We may once more remove the "disturbance decoupling" aspect by setting $E = 0$; then the result is as follows.

**Corollary 3.10** The regulator problem with maximal stability and $E = 0$ is solvable if and only if there exists a controlled invariant subspace $V$ such that

\begin{align}
(3.65.1) & \quad V \otimes X_{\text{stab}} = X \\
(3.65.2) & \quad X_{\text{det}} \subset V \subset \text{Ker } D.
\end{align}

**Proof** Immediate from the foregoing corollary.

If we assume that $X_{\text{det}} = 0$, then our concept of "maximal stability" coincides with the notion of "internal stability" used in Wonham (1979); and if we further assume that $X_{\text{stab}} = \langle A \mid \text{Im } B \rangle$, then Cor. 3.10 specializes
to Thm. 8.1 in WONHAM (1979; p. 179).

It should be noted that it is quite reasonable to set $\hat{X}_{\text{det}}$ equal to zero beforehand (see FRANCIS (1977)). If one does not take this approach then there is a question how to deal with non-detectability of the pair $(C,A)$ in the formulation of the regulator problem. Our approach is different from that of Wonham, who calls a closed-loop system "internally stable" also in the case where unstable closed-loop poles are due to indetectability rather than instabilizability (see WONHAM (1979: p. 147)). In the author's opinion, one may doubt the naturalness of this formulation. A discussion of formulation differences for the regulator problem has been given in JANSSSEN (1981).

The "tracking and regulation" aspect is removed if we assume that $X^b_{\text{stab}} = X$ (which means that the pair $(A,B)$ is $\mathcal{E}_s$-stabilizable). In this case, the condition (3.16) simply says that the extended system matrix should be stable in the sense of $\mathcal{E}_s$. We obtain the following corollary.

**COROLLARY 3.11** Let the system (3.12) be given. Then there exists a compensator such that the extended system matrix is stable in the sense of $\mathcal{E}_s$ and has the disturbance localization property described in (3.17) if and only if $(A,B)$ is $\mathcal{E}_s$-stabilizable, $(C,A)$ is $\mathcal{E}_s$-detectable, and moreover

\begin{align*}
(3.66.1) & \quad T^*_f(\text{Im} E) \subset V^s_s(\text{Ker} D) \\
(3.66.2) & \quad AT^*_f(\text{Im} E) \subset V^s_s(\text{Ker} D) \\
(3.66.3) & \quad V^s_s(\text{Ker} D) + X^e_{\text{stab}} = X.
\end{align*}

**PROOF** Immediate from Thm. 3.3 and Prop. 3.4.

This result is for a compensator of the form (2.5); if static feedback is allowed, the condition (3.66.2) disappears. Under the further assumption that $\mathcal{E}_s = \mathcal{E}_f$, the condition (3.66.3) is also removed and we recover, in a slightly restated form, the results on "disturbance decoupling by observation feedback with stability" of WILLEMS & COMMAULT (1981) and IMAI & AKASHI (1981).

As a final specialization, let us set $\mathcal{E}_s = \mathcal{E}_f = \mathcal{E}$; this means that we do not impose any stability conditions.
COROLLARY 3.12 Let the system (3.18) be given. Then there exists an extended system matrix $A_e$ of the form (3.59) having an invariant subspace $M$ such that

\[(3.67) \quad \text{Im}(E) \subset M \subset \text{Ker}(D,0)\]

if and only if

\[(3.68) \quad T^*(\text{Im} E) \subset U^*(\text{Ker} D).\]

**PROOF** Immediate from the foregoing corollary.

This is the result on "disturbance localization by observation feedback" of AKASHI & IMAI (1979) and SCHUMACHER (1979). The 'necessity' part of the statement was already proved in BASILE & MARRO (1969 a).

3.5. Final remarks

We have formulated a very general problem in the synthesis of linear systems with specified structural features, and we have solved this problem completely. However, there are some aspects that we did not discuss. One of these aspects is order reduction. The construction lemma of Section 3.2 gives rise to a compensator order which equals $\dim V_c \cdot \dim T_c$; so there is room for low-order compensation, at least in principle. In case $E = 0$, the situation is relatively easy; we can set $T_s = X_s^{\text{det}^*}$, $T_f = X_f^{\text{det}}$, and $T_c = 0$, and then the condition $G \in G(T_s) \cap G(T_f) \cap G(T_c)$ will be satisfied automatically. It is then not difficult to extend the methods of Section 2.4 to the present case. In particular, the method suggested in subsection 2.4.5 will be used in Ch. 6 to construct low-order regulators for infinite-dimensional systems.

The problem of order reduction becomes more difficult in the presence of an unmodeled disturbance (i.e., $E \neq 0$). It has been shown in SCHUMACHER (1980 d) that the compensator order can be decreased by $p$ (the number of observation outputs) by the use of static feedback. The basic idea is the same as in subsection 2.4.2, but the extension is not trivial; one has to show that the method of order reduction is compatible with the special structural requirements of the regulator problem. A different approach has been given by WILLEMS & COMMAULT (1981), who solve the problem in three
successive steps and who obtain a bound on the compensator order which may be either larger or smaller than the order of the original system. Because the method we presented here can only produce compensators of order \( \leq n \), it seems unlikely that there is a simple relation between our method and that of Willems and Commault.
CHAPTER 4

BASIC CONCEPTS OF INFINITE-DIMENSIONAL SYSTEMS

This is the first chapter of the second part of this monograph, in which we study a wider class of linear systems. The assumption of finite-dimensionality of the state space is dropped, so that we are able to include processes of a distributed nature into our considerations.

The present chapter has an introductory character, and we do not give any essentially new results. We shall use semigroup theory as a convenient framework for studying infinite-dimensional systems. Some of the basic facts of this theory are given in Section 4.1. Just as we did in the first part, we shall make extensive use of 'block matrix' representations of operators. However, it requires a little bit more care to do this in the infinite-dimensional context and so we have collected the basic results in Section 4.2. Section 4.3 deals with the concepts of stabilizability and detectability and here we also present the important method of spectral decomposition.

In Section 4.4, we give a list of the assumptions that will be used to prove the main results in the subsequent chapters. These assumptions together define the class of systems to which our theory is applicable, and we discuss in some detail which physical systems are contained in this class. Also in Section 4.4, we try to explain briefly the key ideas in our approach to infinite-dimensional systems. Finally, some introductory remarks on the examples we shall use are given in Section 4.5.

4.1. Semigroup theory

In the first three chapters, we have always considered systems for which the state $x(t)$ is an element of a finite-dimensional linear space. In many applications, however, the assumption of finite-dimensionality is restrictive. In order to deal with systems modeled by partial or functional differential equations, we need to consider infinite-dimensional state spaces.

There is a general theory available to describe the evolution of linear deterministic time-invariant systems on a Banach space. This is the theory of semigroups, explained in great detail in HILLE & PHILLIPS (1957). We
shall restrict ourselves to the concept of a strongly continuous semigroup of bounded linear operators. By definition, this is a function $T(t)$ from $[0,\infty)$ to the space of bounded linear operators on a Banach space $X$, which has the following properties:

\begin{align}
(4.1.1) \quad T(t+s) &= T(t)T(s) \quad \forall s, t \in [0,\infty) \\
(4.1.2) \quad T(0) &= I \\
(4.1.3) \quad T(t)x &\in C([0,\infty);X) \quad \forall x \in X.
\end{align}

When we use the word "semigroup" below, we shall always mean a strongly continuous semigroup of bounded linear operators on a Banach space.

Although the semigroup property (4.1.1) was already formulated in HADAMARD (1903, 1924) (where it is referred to as the major premise of Huygens' principle), the extensive development of semigroup theory started in the late 1940's in the works of E.Hille, R.S.Phillips, K.Yosida, W.Feller, I.Miyadera and many others. The semigroup approach has been used in systems theory since FATTORINI (1964) and it has been systematically applied to control problems by many authors; see CURTAIN & PRITCHARD (1978) and the references cited therein. The main advantage of the use of semigroups to study infinite-dimensional control theory is that one obtains general results that are applicable to systems of many different types (parabolic, hyperbolic, delayed, etc.) Other approaches are also feasible, however.

In particular, one should note the strongly PDE-oriented approach of LIONS (1971) to problems of optimal control. For a feedback function approach to feedback design for some classes of infinite-dimensional systems, one may consult, for instance, CALLIER & DESOER (1978).

The semigroup $T(t)$ generalizes the fundamental matrix $\exp(tA)$ in the theory of ordinary linear differential equations. The exponential function $\exp(tA)$ is said to be generated by the matrix $A$. In general, the infinitesimal generator of a semigroup $T(t)$ is defined by specifying its domain

\begin{align}
(4.2.1) \quad D(A) &= \{x \in X \mid \lim_{t \to 0} \frac{T(t)x-x}{t} \text{ exists} \}
\end{align}

and its action on the elements of this subspace:
\[(4.2.2) \quad Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} \quad (x \in D(A)).\]

Some of the elementary properties of the infinitesimal generator are given as follows.

**Proposition 4.1** The infinitesimal generator \( A \) of a semigroup \( T(t) \) is a densely defined closed linear operator. The differential equation

\[(4.3) \quad x'(t) = Ax(t), \quad x(0) = x_0 \in D(A)\]

has a unique solution which is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), and which is given by

\[(4.4) \quad x(t) = T(t)x_0.\]

**Proof** See Curtain & Prにとっては (1978; p.14) and, for the uniqueness, Bellen-Morante (1979; p.163).

For the inhomogeneous equation, we have the following result.

**Proposition 4.2** Let \( T(t) \) be a semigroup acting on a Banach space \( X \), and let \( A \) be its infinitesimal generator. Let \( f \) be a \( C^1 \)-function with values in \( X \). Then the differential equation

\[(4.5) \quad x'(t) = Ax(t) + f(t), \quad x(0) = x_0 \in D(A)\]

has a unique solution which is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), and which is given by

\[(4.6) \quad x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds.\]

**Proof** See Curtain & Prにとっては (1978; p.29). The uniqueness of the solution follows from the same fact for the homogeneous equation.

A basic property describing the 'growth' of a semigroup is the following.

**Proposition 4.3** Let \( T(t) \) be a semigroup. Then

\[(4.7) \quad \inf \left\{ \frac{1}{t} \log \| T(t) \| \mid t > 0 \right\} = \lim_{t \to \infty} \frac{1}{t} \log \| T(t) \|\]
where both members are either finite or equal to $-\infty$.

**Proof** See Hille & Phillips (1957; p. 306).

From this proposition, and from the fact that $T(t)$ is bounded on bounded intervals (see Curtain & Prritchard (1978; p. 12)), it is clear that for each $\omega > \omega_0 := \lim_{t \to -\infty} \frac{1}{t} \log \|T(t)\|$ there exists a constant $M_\omega$ such that

$$\|T(t)\| \leq M_\omega e^{\omega t}.$$  \hspace{1cm} (4.8)

Moreover, $\omega_0$ is the smallest number in $\mathbb{R} \cup \{-\infty\}$ having this property. Therefore, $\omega_0$ is called the growth constant of the semigroup. We shall use the growth constant to describe the 'degree of stability' of a semigroup (Section 4.3).

In our constructions, we shall often use operators which are bounded perturbations of infinitesimal generators. The following lemma states that such operators are generators too, and it also gives a bound on the norm of the perturbed semigroup.

**Lemma 4.4** Let $T(t)$ be a semigroup on a Banach space $X$, and let $A$ be its infinitesimal generator. For any given bounded linear operator $B: X \to X$, $A + B$ is the generator of a semigroup, which we shall denote by $S(t)$. If the estimate

$$\|T(t)\| \leq M \exp(\omega t)$$  \hspace{1cm} (4.9)

holds for $T(t)$, then the estimate

$$\|S(t)\| \leq M \exp((\omega + M\|B\|)t)$$  \hspace{1cm} (4.10)

holds for $S(t)$. In particular, the growth constant of $S(t)$ is smaller than or equal to $\omega + M\|B\|$.

**Proof** See Curtain & Prritchard (1978; p. 38).

Following the finite-dimensional terminology, we shall say that a linear mapping between two Banach spaces is a similarity transformation if the mapping is bounded and has a bounded inverse. We have the following obvious result on modification of semigroups via such a transformation.
The simple proof will be omitted.

**Lemma 4.5** Suppose that \( H: X_1 \rightarrow X_2 \) is a similarity transformation between the Banach spaces \( X_1 \) and \( X_2 \). Let \( T(t) \) be a semigroup on \( X \) with infinitesimal generator \( \lambda \) and growth constant \( \omega_0 \). The function \( \tilde{T}(t) \) defined by

\[
(4.11) \quad \tilde{T}(t) = HT(t)H^{-1} \quad (t \geq 0)
\]

then gives a semigroup on \( X \). The infinitesimal generator of \( \tilde{T}(t) \) is the mapping \( \tilde{\lambda} = H\lambda H^{-1} \) with domain \( D(\tilde{\lambda}) = \{x \in X_2 \mid H^{-1}x \in D(\lambda)\} = H[D(\lambda)] \), and the growth constant of \( \tilde{T}(t) \) is \( \omega_0 \).

### 4.2. Composite systems

As we shall consider dynamic feedback, which means that both the original system and the feedback controller will have their own state space, we shall be concerned with composite systems. Let us first introduce some terminology in a general setting. The *direct sum* of two Banach spaces \( X_1 \) and \( X_2 \) is defined as the set of pairs \( (x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2 \), made into a linear space in the obvious way, and endowed with the norm

\[
(4.12) \quad \| (x_1, x_2) \| = \max(\|x_1\|, \|x_2\|).
\]

In this way, one obtains a new Banach space which is written as \( X_1 \oplus X_2 \).

The natural embeddings of \( X_1 \) and \( X_2 \) are closed linear subspaces of \( X_1 \oplus X_2 \), and the natural projections of \( X_1 \oplus X_2 \) onto \( X_1 \) and \( X_2 \) are continuous.

Now suppose that \( A_{11}: D(A_{11}) \rightarrow X_1, A_{12}: D(A_{12}) \rightarrow X_2, A_{21}: D(A_{21}) \rightarrow X_2 \) and \( A_{22}: D(A_{22}) \rightarrow X_2 \) are linear operators with \( D(A_{11}) \subset D(A_{21}) \subset X_1 \) and \( D(A_{22}) \subset D(A_{2}) \subset X_2 \). We shall use the notation

\[
(4.13) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}
\]

for the operator that is defined as follows:

\[
(4.14.1) \quad D(A) = \{(x_1, x_2) \mid x_1 \in D(A_{11}), x_2 \in D(A_{22})\}
\]
\[ (4.14.2) \quad A(x) = \begin{pmatrix} A_{11} x_1 + A_{12} x_2 \\ A_{21} x_1 + A_{22} x_2 \end{pmatrix}, \quad (x) \in D(A). \]

We also want to use the converse process in which a given system is decomposed. Let \( X \) be a Banach space and let \( X_1 \) and \( X_2 \) be subspaces of \( X \) such that each \( x \in X \) can be written, in a unique way, as \( x = x_1 + x_2 \) with \( x_1 \in X_1 \) and \( x_2 \in X_2 \). In this case one says that \( X \) is the direct sum of \( X_1 \) and \( X_2 \), and one writes \( X = X_1 \otimes X_2 \) (see TAYLOR & LAY (1980; p.28)). If \( X_1 \) and \( X_2 \) are both closed subspaces, then the projections of \( X \) onto \( X_1 \) along \( X_2 \) and onto \( X_2 \) along \( X_1 \) are continuous (TAYLOR & LAY (1980; p.247)) and there is an obvious identification between the concepts of 'direct sum' in one sense or the other. Below, we shall adopt the convention that the use of the expression \( X = X_1 \otimes X_2 \) implies that \( X_1 \) and \( X_2 \) are closed subspaces of \( X \) or can be considered as such.

Corresponding to a decomposition of the space \( X \), we shall want to write an operator \( A \) acting on \( X \) in the 'block' form \( (4.13) \). In the case \( A \) is not defined on all of \( X \), we have to take some care to avoid domain problems. The pertinent facts are given in the following lemma, whose simple proof will be omitted.

**Lemma 4.6**  Let \( X \) be a Banach space and let \( A \) be a linear operator mapping its domain \( D(A) \subset X \) into \( X \) assume that \( X \) has a direct sum decomposition \( X = X_1 \otimes X_2 \). Let \( P_1 \) be the projection onto \( X_1 \) along \( X_2 \), and let \( P_2 \) be the projection onto \( X_2 \) along \( X_1 \). Suppose that \( P_2 \) maps \( D(A) \) into itself. Then the same holds for \( P_1 \), and we have

\[ (4.15) \quad P_1[D(A)] = X_1 \cap D(A), \quad P_2[D(A)] = X_2 \cap D(A). \]

Moreover, if we define the mapping \( A_{ij} \) for \( i,j \in \{1,2\} \) as the restriction of \( P_i A \) to \( X_i \cap D(A) \), considered as a mapping into \( X_i \), then the following equality holds:

\[ (4.16) \quad A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}. \]

Finally, if \( D(A) \) is dense in \( X \), then \( X_1 \cap D(A) \) is dense in \( X_1 \) and \( X_2 \cap D(A) \) is dense in \( X_2 \).

A simple sufficient condition for the inclusion \( P_2[D(A)] \subset D(A) \) to
hold is that \( X_2 \) be a closed subspace contained in \( D(A) \). We shall often use this condition in the sequel.

We shall now consider situations in which the block matrix appearing in 4.13 has a triangular structure. The first result is the following.

**Proposition 4.7** Suppose that \( A_{11} \) and \( A_{22} \) are generators of semigroups \( T_1(t) \) and \( T_2(t) \) on the Banach spaces \( X_1 \) and \( X_2 \), with growth constants \( \omega_1 \) and \( \omega_2 \) respectively. Suppose also that \( A_{21} : X_1 \to X_2 \) is a bounded linear operator. Then the operator

\[
\begin{pmatrix}
A_{11} & 0 \\
A_{21} & A_{22}
\end{pmatrix}
\]

is the generator of a strongly continuous semigroup on \( X_1 \otimes X_2 \). This semigroup is given by

\[
T(t) = \begin{pmatrix}
T_1(t) & 0 \\
T_{21}(t) & T_2(t)
\end{pmatrix}
\]

where \( T_{21}(t) \) is defined by

\[
T_{21}(t)x = \int_0^t T_2(t-s) A_{21} T_1(s)x \, ds \quad (x \in X_1).
\]

Moreover, the growth constant of \( T(t) \) is equal to \( \max(\omega_1, \omega_2) \).

**Proof** One easily verifies that

\[
\tilde{T}(t) = \begin{pmatrix}
T_1(t) & 0 \\
0 & T_2(t)
\end{pmatrix}
\]

is a semigroup with infinitesimal generator

\[
\tilde{A} = \begin{pmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{pmatrix}.
\]

As \( A \) is a bounded perturbation of \( \tilde{A} \), it follows from Lemma 4.4 that \( A \) is the generator of a semigroup on \( X_1 \otimes X_2 \). To determine the form of the semigroup, we solve the differential equation
\[ \frac{d}{dt} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} \in D(A). \]

We re-write (4.22) as follows:

\[ \begin{align*}
(4.23.1) & \quad x_1'(t) = A_{11}x_1(t), & x_1(0) = x_{10} \in D(A_{11}) \\
(4.23.2) & \quad x_2'(t) = A_{21}x_1(t) + A_{22}x_2(t), & x_2(0) = x_{20} \in D(A_{22}).
\end{align*} \]

The solution of (4.23.1) is given by the semigroup generated by \( A_{11} \):

\[ x_1(t) = T_1(t)x_{10}. \]

This is a \( C^1 \)-function with values in \( X_1 \), and because \( A_{21} \) is bounded it follows that the function \( A_{21}x_1(t) \) is a \( C^1 \)-function with values in \( X_2 \). Thus we can solve (4.23.2) using Prop. 4.2:

\[ x_2(t) = T_2(t)x_{20} + \int_0^t T_2(t-s)A_{21}x_{10} ds. \]

It now follows from Prop. 4.1 that the semigroup generated by \( A \) is given by (4.18).

To prove the final assertion, let \( \omega \) be any number larger than \( \max(\omega_1, \omega_2) \). Then we can make the following estimates, in which \( \omega_1 \) and \( \omega_2 \) are numbers such that \( \omega_1 < \omega^1_1 \leq \omega, \omega_2 < \omega^1_2 \leq \omega, \) and \( \omega^1_1 \neq \omega^1_2 \):

\[ \begin{align*}
(4.26) & \quad \|T(t)x_1(t)\| = \max\left\{ \|T_1(t)x_1(t) + T_2(t)x_2(t)\|, \|T_2(t)x_2(t)\| \right\} \\
& \leq \max(M_1e^{\omega_1^1t}\|x_1\| + M_2\int_0^t 0 \omega^1_2(t-s)\|A_{21}\|e^{\omega_1^1s}\|x_2\|ds, \\
& \quad M_2e^{\omega_1^2t}\|x_2\|) \leq \\
& \quad \max(M_1e^{\omega_1^1t}\|x_1\| + M_2\|e^{-\omega_1^2t}\|\|x_2\|, M_2e^{\omega_1^2t}\|x_2\|) \leq \\
& \quad M_2e^{\omega_1^2t}\|x_2\|. 
\end{align*} \]
This shows that the growth constant of $T(t)$ is smaller than or equal to $\max(w_1, w_2)$. On the other hand, it follows immediately from the triangular form of $T(t)$ that its growth constant cannot be smaller than $w_1$ or $w_2$. Hence, equality holds, and the proof is complete. 

Now suppose that we have a semigroup $T(t)$ acting on a Banach space $X$ which has a direct sum decomposition $X = X_1 \oplus X_2$. Also suppose that $X_2$ is contained in the domain of the infinitesimal generator $A$, and that $AX_2 \subset X_2$. Then we shall denote the restriction of $A$ to $X_2$ by $A : X_2$. We can also consider the quotient space $X/X_2$, which is defined as the set of equivalence classes modulo $X_2$, endowed with the norm

$$\|[x]\| = \inf\{\|x - x_2\| \mid x_2 \in X_2\}. \tag{4.27}$$

With this norm, $X/X_2$ is a Banach space (TAYLOR & LAY (1980; p. 71)). From the fact that the canonical mapping $x \mapsto [x]$ is a continuous mapping of $X$ onto $X/X_2$, it follows that the subspace

$$D(\overline{A}) := \{[x] \mid x \in D(A)\} \tag{4.28}$$

is dense in $X/X_2$. It is easily verified that we can define a mapping

$$\overline{A} : D(\overline{A}) \to X_1$$

by

$$\overline{A}[x] = [Ax] \quad (x \in D(A)). \tag{4.29}$$

This mapping, which will sometimes also be denoted by $A : X/X_2$, is called the quotient mapping induced by $A$ on $X/X_2$.

Under the above circumstances, it is possible to define the restriction and the quotient of the semigroup $T(t)$ with respect to $X_2$. These are the contents of the next two results.

**PROPOSITION 4.8** Suppose that $A$ is the infinitesimal generator of a semigroup $T(t)$ on a Banach space $X$, and suppose that $X = X_1 \oplus X_2$ with $X_2 \subset D(A)$ and $AX_2 \subset X_2$. Then $X_2$ is $T(t)$-invariant for each $t \geq 0$, and $T(t) : X_2$ is a semigroup on $X_2$, with $A : X_2$ as its infinitesimal generator.

**PROOF** From the fact that $A$ is a closed operator it follows that $A : X_2$ is also closed. Because $A : X_2$ is defined on all of $X_2$, the closed graph
Theorem (Taylor & Lay (1980; p. 213)) shows that \( A: X_2 \) is bounded. Consequently, the unique solution of the differential equation

\[
(4.30) \quad x'(t) = Ax(t), \quad x(0) = x_0 \in X_2
\]

is given by

\[
(4.31) \quad x(t) = \exp(tA; X_2)x_0
\]

which is clearly in \( X_2 \) for all \( t \geq 0 \). It follows from Prop. 4.1 that we must have

\[
(4.32) \quad T(t)x_0 = \exp(tA; X_2)x_0 \quad (t \geq 0, \ x_0 \in X_2)
\]

showing that \( X_2 \) is \( T(t) \)-invariant for each \( t \geq 0 \), and that \( T(t): X_2 \) is a semigroup with infinitesimal generator \( A: X_2 \).

It should be noted that a subspace \( X_2 \) which is \( A \)-invariant in the sense that \( Ax \in X_2 \) for \( x \in X_2 \cap D(A) \) \( (\neq X_2) \) is not necessarily invariant for the semigroup generated by \( A \) (see Schmidt & Stern (1980)).

**Proposition 4.9** Suppose that \( A \) is the infinitesimal generator of a semigroup \( T(t) \) on a Banach space \( X_1 \) and suppose that \( X = X_1 \oplus X_2 \) with \( X_2 \subset D(A) \) and \( AX_2 \subset X_2 \). Then we can define a mapping \( \tilde{T}(t): X/X_2 \to X/X_2 \), for each \( t \geq 0 \), by

\[
(4.33) \quad \tilde{T}(t)[x] = [T(t)x].
\]

Moreover, \( \tilde{T}(t) \) is a semigroup on \( X/X_2 \) whose generator is an extension of \( A \) (defined in (4.32)).

**Proof** The correctness of the definition (4.33) follows from the \( T(t) \)-invariance of \( X_2 \) (Prop. 4.8). To show that \( \tilde{T}(t) \) is bounded let \( \omega \) be the growth constant of \( T(t) \) and let \( \omega' \in \mathbb{R} \) be such that \( \omega' > \omega \). Then there exists a constant \( M \) such that for all \( x \in X \) and \( x_2 \in X_2 \):

\[
(4.34) \quad \|\tilde{T}(t)[x]\| = \inf\{\|T(t)x - y\| \mid y \in X_2\} \leq \|T(t)x - T(t)x_2\| \leq M\omega'^t\|x_2\|.
\]
Consequently, we have for all \( x \in X \):

\[
(4.35) \quad \left\| \mathcal{T}(t)[x] \right\| \leq e^{\omega t} \inf \{ \left\| x - x_2 \right\| \mid x_2 \in X_2 \} = e^{\omega t} \left\| [x] \right\|
\]

The semigroup axioms (4.1) are easily seen to be satisfied by \( \mathcal{T}(t) \). Finally, the fact that

\[
(4.36) \quad \lim_{t \to 0} \frac{1}{t}(\mathcal{T}(t)[x] - [x]) = \tilde{A}[x] \quad ([x] \in \mathcal{D}(\tilde{A}))
\]

follows immediately from the continuity of the canonical mapping \( x \mapsto [x] \). \( \Box \)

The inequality (4.35) shows that the growth constant of the induced semigroup \( \mathcal{T}(t) \) is smaller than or equal to that of the original semigroup. Of course, the same is true of the semigroup \( T(t) : X_2 \) obtained by restriction.

To complete this section, we prove a proposition on the decomposition of composite systems. We shall say that two semigroups \( T_1(t) \) and \( T_2(t) \) acting on Banach spaces \( X_1 \) and \( X_2 \) are similar if there exists a similarity transformation \( H : X_1 \to X_2 \) such that

\[
(4.37) \quad HT_1(t) = T_2(t)H
\]

for all \( t \geq 0 \).

**Proposition 4.10** Suppose that \( A_{11} \) and \( A_{22} \) are generators of semigroups \( T_1(t) \) and \( T_2(t) \) on the Banach spaces \( X_1 \) and \( X_2 \), respectively, and assume that \( A_{22} \) is bounded. Suppose also that \( A_{21} : X_1 \to X_2 \) is a bounded linear operator. Let \( T(t) \) denote the semigroup on \( X_1 \oplus X_2 \) with generator

\[
(4.38) \quad A := \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix}
\]

Then the restricted semigroup \( T(t) : X_2 \) is similar to \( T_2(t) \), and the quotient semigroup \( \mathcal{T}(t) \) on \( X/X_2 \) is similar to \( T_1(t) \).

**Proof** The first assertion is obvious from the form of the semigroup \( T(t) \) given in (4.18). To prove the second assertion, define the mapping
H: \( X_1 \to X/X_2 \) by

(4.39) \( Hx = \begin{bmatrix} x \\ 0 \end{bmatrix} \) \( (x \in X_1) \).

This is clearly a continuous bijection from \( X_1 \) onto \( X/X_2 \). By the open mapping theorem (see Taylor & Lay (1980; p.212-213), \( H^{-1} \) is also continuous so that \( H \) provides a similarity between \( X_1 \) and \( X/X_2 \). Using the explicit form of \( T(t) \) again, we have

(4.40) \( \bar{T}(t)Hx = [T(t)x_0] = \begin{bmatrix} T_1(t)x \\ 0 \end{bmatrix} = HT_1(t)x \)

for all \( x \in X_2 \). This completes the proof.

4.3 Stabilizability, detectability and the spectral decomposition

As in the finite-dimensional case, a 'system' will be described by three operators. The main operator (or system operator) \( A \) will be the generator of a semigroup \( T(t) \) on the state space \( X \). The input operator \( B \) will be a bounded mapping from a finite-dimensional space \( U \) into \( X \). The output operator \( C \) will be a bounded mapping from \( X \) into a finite-dimensional space \( Y \).

We defined the concept of stability in the finite-dimensional situation through a partitioning of the complex plane in a part labeled 'stable' and a part labeled 'unstable'. In the infinite-dimensional context, we want to use the growth constant as an indicator of stability, and so we shall consider only partitionings in which the parts are divided by a vertical line in the complex plane. In our discussions, we shall assume that some fixed constant \( \omega \in \mathbb{R} \) has been given, and we shall say that a semigroup is exponentially stable (or simply stable) if its growth constant is smaller than or equal to \( \omega \). In applications, \( \omega \) is always negative.

We can now define the concepts of stabilizability and detectability. If \( A \) and \( B \) are mappings as described above, the pair \((A,B)\) is stabilizable if there exists a bounded mapping \( F: X \to U \) such that the semigroup generated by \( A + BF \) is stable. (Note that \( A + BF \) is indeed the generator of a semigroup; this follows from Lemma 4.4.) The pair \((C,A)\), as described above, is said to be detectable if there exists a bounded mapping \( G: Y \to X \) such that the semigroup generated by \( A + GC \) is stable. Of course, Lemma 4.4 is used again here.
In the finite-dimensional situation, the modal decomposition of the state space into a 'good' and a 'bad' \( A \)-invariant subspace was useful to us. We want to use the same idea in the present context, and this is made possible by the following result.

**Proposition 4.11** Let \( A \) be the generator of a semigroup on the Banach space \( X \). Suppose that \( \sigma_1 \) is a bounded subset of \( \sigma(A) \) and that \( \sigma_1 \) is closed and open in the relative topology of \( \sigma(A) \). Then there exists a bounded projection \( P: X \rightarrow X \) such that \( P[D(A)] \subseteq D(A) \), and with respect to the decomposition \( X = \text{Im} P \oplus \text{Ker} P \) we have

\[
A = \begin{pmatrix}
A_{11} & 0 \\
0 & A_{22}
\end{pmatrix}, \\
T(t) = \begin{pmatrix}
T_{11}(t) & 0 \\
0 & T_{22}(t)
\end{pmatrix}.
\]

Moreover, \( \sigma(A_{11}) = \sigma_1 \) and \( \sigma(A_{22}) = \sigma(A) \setminus \sigma_1 \).

**Proof** See TAYLOR & LAY (1980; pp. 321-323) and, for the decomposition of the semigroup, TRIGGANI (1975; App.2).

We shall use this spectral decomposition to derive sufficient conditions for stabilizability and detectability. With this in mind, we first have to discuss the relation between the spectrum of the generator and the growth constant of the semigroup. The following general result holds:

\[
\lim_{t \to \infty} \frac{1}{t} \log \|T(t)\| \geq \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \}
\]

(HILLE & PHILIPS (1957; p.457)). In general, equality does not have to hold in (4.42); see HILLE & PHILIPS (1957; p.665) and ZABCZYK (1975). In many cases, however, we do have equality and then the generator is said to satisfy the spectrum determined growth assumption (TRIGGANI (1975)). One of these cases is when the operator \( T(t) \) is compact for large \( t \), as is typically the case in delay equations; see HALE (1971; pp.112-115). Also, the spectrum determined growth assumption holds if there exists a \( t_0 \geq 0 \) such that \( T(t) \) is strongly differentiable for \( t > t_0 \) (see SLEMROD (1976)). This condition is typically satisfied (even with \( t_0 = 0 \)) for equations of diffusion type, and the semigroup is also certainly differentiable if the infinitesimal generator is bounded. In fact, it has been shown that differentiability of the semigroup implies compactness if the generator has a compact resolvent (FAZY (1968; Th. 3.2 and Lemma 2.1)), such as is the case
for many diffusion equations. For further discussion on the spectrum determined growth assumption, see TRIGGIANI (1975).

An infinitesimal generator $A$ is said to satisfy the spectrum decomposition assumption (with respect to some number $\omega$ denoting the desired degree of stability) if

$$\sigma_1 := \{ \lambda \in \sigma(A) \mid \text{Re} \lambda > \omega \}$$

is a bounded, closed and open subset of $\sigma(A)$ (TRIGGIANI (1975)). We can then use Prop. 4.11 to decompose $A$, and we shall in this special case use the following notation ('$u$' for 'unstable', 's' for 'stable'):

$$A = \begin{pmatrix} A_u & 0 \\ 0 & A_s \end{pmatrix}.$$  \hspace{1cm} (4.44)

Also, we shall write the corresponding decomposition of the state space $X$ as $X = X_u \oplus X_s$. In an obvious way, we can write the input operator $B: U \rightarrow X$ as

$$B = \begin{pmatrix} B_u \\ B_s \end{pmatrix}.$$  \hspace{1cm} (4.45)

The output operator $C: X \rightarrow Y$ can likewise be written in matrix form:

$$C = \begin{pmatrix} C_u & C_s \end{pmatrix}.$$  \hspace{1cm} (4.46)

We shall also use the notation $A_u^\omega$, $X_u^\omega$ etc. if we want to stress the dependence on $\omega$.

In case the subspace $X_u$ is finite-dimensional, the mappings $A_u$, $B_u$ and $C_u$ are mappings between finite-dimensional spaces and we can use the concepts of controllability and observability as defined in Ch. 1.

The spectrum decomposition assumption can be checked easily when the spectrum of the generator consists only of isolated eigenvalues. This is the case if the resolvent $(\lambda - A)^{-1}$ is compact for some $\lambda \in \rho(A)$, which applies to many differential operators appearing in the classical boundary value problems of mathematical physics (see KATO (1966; p.187)). The same type of spectrum is obtained for functional-differential operators describing delay equations; see HALE (1971; pp.38-101). In both cases, all eigenvalues have finite multiplicities (which means that the corresponding
eigenprojections have finite rank; see TAYLOR & LAY (1980; p.330)) and so the condition on the finite-dimensionality of $X_u$ comes down to requiring that there are only finitely many eigenvalues to the right of the line $\text{Re} \lambda = \omega$.

Using the spectrum decomposition assumption and the spectrum determined growth assumption, we are able to characterize stabilizability and detectability in finite-dimensional terms if the unstable subspace is finite-dimensional. The following propositions are proved constructively, and we shall use them for actual computations of feedback or injection mappings. The 'sufficiency' parts of these results are due to TRIGGIANI (1975; Thm. 6.1) and CURTAIN (1979; Thm. 3.1).

**PROPOSITION 4.12** Suppose that $A$ satisfies the spectrum decomposition assumption and that $A_s$ satisfies the spectrum determined growth assumption. Suppose also that the unstable subspace $X_u$ is finite-dimensional. Then the pair $(A, B)$ is stabilizable if and only if the pair $(A_u, B_u)$ is controllable.

**PROOF** First, let us assume that $(A_u, B_u)$ is controllable. There then exists a mapping $F: X_u \rightarrow U$ such that $A_u + B_u F_u \text{ generates a stable semigroup.}$

Define $F: X \rightarrow U$, with respect to the decomposition $X = X_u \oplus X_s$, by

\begin{equation}
F = (F_u, 0).
\end{equation}

Then we have

\begin{equation}
A + BF = \begin{pmatrix} A_u + BF_u & 0 \\ B_u & A_s \end{pmatrix}
\end{equation}

and it follows from Prop. 4.7 that the semigroup generated by $A + BF$ is stable.

Conversely, suppose that $F: X \rightarrow U$ is a bounded mapping such that $A + BF$ generates a stable semigroup, and suppose that the pair $(A_u, B_u)$ would not be controllable. Decompose $X_u$ as $X_2 \oplus X_3$, where

\begin{equation}
X_2 = \langle A_u | \text{Im } B_u \rangle
\end{equation}

(notation as in (1.8)). Write $X_1 = X_s$. Then we have, with respect to the decomposition $X = X_1 \oplus X_2 \oplus X_3$:
Here, we have written \( A_{11} \) for \( A_s \), etc. By our assumptions, the matrix \( A_{33} \) is unstable. If we now write \( F = (F_1, F_2, F_3) \), we get

\[
(4.51) \quad A + BF = \begin{pmatrix}
A_{11} + B_1 F_1 & B_1 F_2 & B_1 F_3 \\
D_2 F_1 & A_{22} + B_2 F_2 & A_{23} + B_3 F_3 \\
0 & 0 & A_{33}
\end{pmatrix}.
\]

The upper two-by-two block is the generator of a semigroup because it is a bounded perturbation of the upper two-by-two block appearing in (4.50). So we can apply Prop. 4.7 and conclude that the semigroup generated by \( A + BF \) cannot be stable. This is a contradiction. Hence, \((A_s, B_u)\) must be controllable.

Proposition 4.13: Suppose that \( A \) satisfies the spectrum decomposition assumption and that \( A_s \) satisfies the spectrum determined growth assumption. Suppose also that the unstable subspace \( X_u \) is finite-dimensional. Then the pair \((C, A)\) is detectable if and only if the pair \((C_u, A_u)\) is observable.

Proof: We first assume that \((C_u, A_u)\) is observable. Then there exists a mapping \( G_u : Y \to X_u \) such that \( A_u + G_u C_u \) generates a stable semigroup. Define \( G : Y \to X \) with respect to the decomposition \( X = X_u \oplus X_s \), by

\[
(4.52) \quad G = \begin{pmatrix} G_u \\ 0 \end{pmatrix}.
\]

Then we have

\[
(4.53) \quad A + GC = \begin{pmatrix}
A_u + G_u C_u & G_u C_s \\
0 & A_s
\end{pmatrix}
\]

and it follows from Prop. 4.7 that \( A + GC \) generates a stable semigroup.

Conversely, suppose that \( G : Y \to X \) is a bounded mapping such that \( A + GC \) is the generator of a stable semigroup, and suppose that the pair \((C_u, A_u)\)
would not be observable. Decompose \( X_u \) as \( X_2 \oplus X_3 \), with

\[
X_3 = \ker C_u | A_u
\]

(notation as in (1.10)). Write \( X_1 = X_a, A_1 = A_a \) etc.. Then we have, with respect to the decomposition \( X = X_1 \oplus X_2 \oplus X_3 \):

\[
A = \begin{pmatrix}
A_{11} & 0 & 0 \\
0 & A_{22} & 0 \\
0 & A_{32} & A_{33}
\end{pmatrix},
\]

\( C = (C_1 \ C_2 \ 0) \).

By our assumptions, the matrix \( A_{33} \) is unstable. Writing

\[
G = \begin{pmatrix}
G_1 \\
G_2 \\
G_3
\end{pmatrix},
\]

we get

\[
A + GC = \begin{pmatrix}
A_{11} + G_1 C_1 & G_1 C_1 & 0 \\
G_2 C_1 & A_{22} + G_2 C_2 & 0 \\
G_3 C_1 & A_{32} + G_3 C_2 & A_{33}
\end{pmatrix}.
\]

The two-by-two upper left block generates a semigroup because it is a bounded perturbation of the two-by-two upper left block in (4.55). Using Prop. 4.7, we see that the semigroup generated by \( A + GC \) cannot be stable. This is a contradiction, and therefore \( (C_u, A_u) \) must be observable.

Another application of the spectral decomposition is to growth estimates for the semigroup. In general, a growth estimate of the form (4.8) can only be made for \( \omega \) larger than the growth constant \( \omega_0 \). Under certain circumstances which will prevail in our examples below, it is nevertheless possible to give the estimate with \( \omega = \omega_0 \).

**Lemma 4.14** Suppose that \( T(t) \) is a semigroup with generator \( A \), and let \( \omega_0 = \sup \{ \Re \lambda \mid \lambda \in \sigma(A) \} \). Suppose that \( \sigma_1 = \{ \lambda \in \sigma(A) \mid \omega_0 - \delta < \Re \lambda < \omega_0 \} \) satisfies the assumptions of Prop. 4.11, and let the operator \( A_{22} \) satisfy the spectrum determined growth assumption. If \( \sigma_1 \) consists of finitely many eigenvalues of \( A \) which all have multiplicity one, then there exists a constant \( M \) such that the following holds:
(4.58) \[ \|T(t)\| \leq M_1 e^{\omega_0 t} \quad (t \geq 0). \]

**Proof** In the decomposition of Prop. 4.11, \( A_{11} \) is a diagonalizable matrix whose eigenvalues have real parts \( \leq \omega_0 \). Thus, there exists a constant \( M_1 \) such that \( \|T_1(t)\| \leq M_1 \exp(\omega_0 t) \) for all \( t \geq 0 \). By the assumption on \( A_{22} \) and the fact that \( \Re \lambda < \omega_0 - \delta \) for all \( \lambda \in \sigma(A_{22}) \), there is a constant \( M_2 \) such that \( \|T_2(t)\| \leq M_2 \exp(\omega_0 t) \) for all \( t \geq 0 \). The assertion of the lemma now follows directly.

4.4 Remarks on the scope of the theory

In this section, we present a list of the assumptions that determine the class of systems we shall consider in the subsequent chapters. We shall also give some comments on the assumptions and on the way we shall use them. It is assumed that a fixed number \( w \) has been given, to indicate the dividing line between the 'stable' and the 'unstable' part of the complex plane. The systems we shall study are described by three operators \( A, B \) and \( C \) under the following assumptions.

(A1) \( A \) is the generator of a semigroup \( T(t) \) on a Banach space \( X \)

(A2) \( B \) is a bounded mapping from a finite-dimensional space \( U \) into \( X \)

(A3) \( C \) is a bounded mapping from \( X \) into a finite-dimensional space \( Y \)

(A4) the spectrum of \( A \) is discrete, i.e. it consists only of isolated eigenvalues with finite multiplicities

(A5) there exists \( \delta > 0 \) such that \( \{ \lambda \in \mathbb{C} \mid \Re \lambda > \omega - \delta \} \) contains only finitely many eigenvalues of \( A \)

(A6) the operator \( A_{s}^{\omega - \delta} \) satisfies the spectrum determined growth assumption, with \( \delta \) as in (A5)

(A7) the pair \( (A, B) \) is controllable

(A8) the pair \( (C, A) \) is observable

(A9) the eigenvectors of \( A \) are complete, in the sense that \( \text{span} \{ x \in X \mid \exists \lambda \in \mathbb{C} : \exists n \in \mathbb{N} : (\lambda I - A)^n x = 0 \} \) is dense in \( X \).
To prevent confusion, let us point out here that we shall use the term "discrete spectrum" always in the sense of A (cf. KATÔ (1966; p.187)), and that the word "multiplicity" will be used for the rank of the eigen-projection (cf. KATÔ (1966; p.181)). Every non-zero vector in the range of the eigenprojection will be termed an "eigenvector" (so this includes 'generalized eigenvectors').

The assumptions (A1) and (A2) are relevant for the situation in which the control enters in the following way:

\[(4.59) \quad x'(t) = Ax(t) + Bu(t).\]

For systems described by partial differential equations, the control could also be implemented via the boundary conditions, and then a description under the assumptions (A1) and (A2) would be impossible. The specifically infinite-dimensional phenomenon of boundary control is interesting, but we shall leave it out of our present discussion. Our object of study is given by (4.59), and it is sometimes called distributed control.

The boundedness assumption on the output operator (A3) excludes point observations on an $L^2$-space. However, CURTAIN (1979) and POHJOLAINEN (1980) show that, under certain conditions, it is very well possible to deal with unbounded observations in feedback design problems. It seems not unreasonable to expect that most of our results will remain true if $C$ is relatively bounded with respect to $A$ (KATÔ (1966; p.190)).

The assumption (A4) has already been discussed in Section 4.3; we have argued that a large class of systems described by partial differential equations on a bounded domain or by functional differential equations of 'delay' type falls within the category described by (A4). Within this class, there is a distinction between those generators for which there are only finitely many eigenvalues to the right of any vertical line in the complex plane, and those for which this is not true. This distinction, which is obviously of crucial importance in connection with (A5), corresponds to a well-known classification both in partial differential equations and in functional differential equations. On one hand, we have parabolic equations (such as the heat equation) and equations of 'retarded' type (HALE (1971; p.4)). In systems described by these equations, the real parts of the eigenvalues tend to $-\infty$ and so the assumption (A5) will hold for any desired growth constant $\omega$. On the other hand, we have hyperbolic equations (such as the wave equation) and equations of 'neutral' type (HALE (1971; p.5)).
Systems of these types have infinitely many eigenvalues in a vertical strip and so they will only satisfy (A5) if this strip happens to be to the left of the prescribed value of $\omega$. (Note that the real parts of the points in the spectrum of a generator must be bounded above; this follows from (4.42) and Prop. 4.3.) In conclusion, we may say that parabolic and retarded systems will as a rule satisfy the assumption (A5), whereas hyperbolic and neutral systems will satisfy (A5) only if the system is 'basically' stable.

The assumption (A6) has also been discussed in Section 4.3. We have formulated the assumption for $A_s^\omega$ and not for $A_s$, because this is what is needed to apply Prop. 4.12 and Prop. 4.13. However, it is easily seen that if the semigroup generated by $A$ is differentiable for $t > t_0$ or compact for large $t$, then the same holds for restrictions of the semigroup to a subspace of the form $X^s$. It follows from Prop. 4.7 and assumption (A5) that the spectrum determined growth assumption holds for $A_s^\omega$ if it holds for $A_s^\omega$.

The assumptions (A7) and (A8) need little explanation; they are just as important as in the finite-dimensional situation. Finally, the assumption (A9) will be essential to prove the existence of finite-dimensional control schemes. The completeness property is quite common in partial differential equations; in fact, the classical method of solving equations by expansion in eigenfunctions is based on it. Material on completeness of eigenvectors can be found, for instance, in TRÈVES (1975; p.325), MIHOHATA (1973; pp.465-470) and AGMON (1965; pp.278-289). For completeness of eigenvectors in functional differential equations, see MANITIUS (1980) and DELFOUR & MANITIUS (1980 b).

For a given system, the main restrictions on the selection of the desired growth constant $\omega$ are given by (A5), (A7) and (A8). We see that $\omega$ may be set equal to the largest uncontrollable and/or unobservable eigenvalue of $A$. But if there is a vertical 'asymptotic line' (i.e. a line \( \{ \lambda \in \mathbb{C} \mid \text{Im} \lambda = c \} \)) such that each strip \( \{ \lambda \in \mathbb{C} \mid c - \varepsilon \leq \text{Im} \lambda \leq c + \varepsilon \} \) \( (\varepsilon > 0) \) contains infinitely many eigenvalues of $A$, then $\omega$ must be to the right of that line.

The assumptions (A1, 4-6,9) describe which operators are allowed to occur as the main operator of the systems we shall consider. The following proposition gives a large class of operators that are contained in this category.

**Proposition 4.15** Suppose that $X$ is a Hilbert space and that $A$ is a densely defined, self-adjoint linear operator on $X$. Suppose furthermore that $A$ is
bounded above (i.e., there is a constant \( c \in \mathbb{R} \) such that \((Ax,x) \leq c \) for all \( x \in D(A) \) with \( \|x\| = 1 \)). Finally, suppose that \((\lambda_0 - A)^{-1}\) is compact for some \( \lambda_0 \in \rho(A) \). Then \( A \) satisfies the assumptions (A1,4–6,9) for any \( \omega \in \mathbb{R} \).

**Proof.** Using the fact that \( A \) is closed (because \((\lambda_0 - A)^{-1}\) is closed) and the estimate \( \|\mathcal{N}(A)\| \leq M|\lambda_1|^{-1} \), which holds for \( \lambda \in \mathcal{E} \) \( c_1 \|\text{Im } \lambda \| c_2 - \text{Re } \lambda \) \( (c_1 > 0, c_2 > c) \) (Kato (1966; p. 272)), we derive from Thm. 2.28 in Curtain & Pritchard (1978; p. 33) that \( A \) generates an analytic semigroup. Assumption (A4) is fulfilled by the compactness of the resolvent; see, for instance, Kato (1966; p. 187). Also by the compactness of the resolvent, the eigenvalues of \( A \) cannot have a finite accumulation point; and because \( A \) is bounded above and self-adjoint, the eigenvalues are real and they can be numbered in order of decreasing magnitude \( \lambda_1 > \lambda_2 > \cdots \) with \( \lambda_k \to -\infty \) as \( k \to \infty \). This shows that (A5) is satisfied for any \( \omega \in \mathbb{R} \). Any operator of the form \( A_0^\omega \) will generate an analytic semigroup and so (A6) holds too. Finally, the completeness of the eigenvectors of \( A \) follows from the same fact for \((\lambda_0 - A)^{-1} \), where \( \lambda_0 \in \mathbb{R} \) and \( \lambda_0 > c \) (Kato (1966; p. 260)).

The theorem applies, for instance, to elliptic operators of the following type: Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with smooth boundary, and let \( X = L_2(\Omega) \). Define \( A \) by

\[
(4.60) \quad A\varphi = -\frac{d}{dx_1} \frac{d}{dx_j} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \varphi \right) + c(x)\varphi
\]

where the functions \( a_{ij}(x) \) are real-valued and once continuously differentiable, \( c(x) \) is real-valued, \( a_{ij}(x) = a_{ji}(x) \) for all \( i \) and \( j \), and the uniform ellipticity condition holds:

\[
(4.61) \quad \frac{d}{dx_1} \frac{d}{dx_j} a_{ij}(x) \xi_1 \xi_j \geq \eta |\xi|^2 \quad (x \in \Omega, \xi \in \mathbb{R}^d)
\]

where \( \eta \) is a positive constant. Then \( A \) satisfies the conditions of the proposition if we add Dirichlet boundary conditions (see, for instance, Mizohata (1973; Ch. 3)).

The self-adjointness assumed in the proposition is not at all essential. A large class of non-self-adjoint generators of semigroups is given by delay equations of the following type:

\[
(4.62) \quad \frac{dx(t)}{dt} = \int_{[-\beta,0]} d\eta(\theta) x(t+\theta)
\]
where \( \eta(t) \) is an \( n \times n \) matrix of real functions of bounded variation on \([-h, 0]\). To this equation, a semigroup can be associated which acts on the product space \( \mathbb{R}^n \times L_2(-h, 0) \) (see VINTER (1978)). How this association is done will be explained, in a considerably more specific setting, in Section 4.5. The resolvent of the generator turns out to be compact which shows that (A4) holds. Also the semigroup itself is compact for \( t \leq h \), and this entails that (A5) and (A6) are satisfied for any \( \omega \in \mathbb{R} \). (For, if there were infinitely many eigenvalues of \( A \) in a strip \( \{ \lambda \in \mathbb{C} \mid c_1 \leq \Re \lambda \leq c_2 \} \) then there would be infinitely many eigenvalues of \( T(h) \) in a ring-shaped domain \( \{ \lambda \in \mathbb{C} \mid e^{c_1 h} \leq |\lambda| \leq e^{c_2 h} \} \) (HILLE & PHILLIPS (1957; p.467)) and consequently the eigenvalues of \( T(h) \) would have a non-zero accumulation point, which is impossible by the compactness of \( T(h) \).) So the only question that remains is whether the eigenvectors associated with (4.62) are complete. Detailed conditions for this are given in MANITIUS (1980) and DELFOUR & MANITIUS (1980b), and their results show that the completeness is obtained in many cases, if the state space is chosen correctly.

Of course, the assumptions (A1–6) and (A9) hold in particular for finite-dimensional systems. Generally speaking, one might say that the assumptions delineate a class of systems that have those features of finite-dimensional systems that make the constructive methods of Chapters 2 and 3 applicable. Indeed, in the subsequent chapters we shall follow the lines of these chapters closely, although we shall introduce some specializations and we shall concentrate on construction methods rather than necessary conditions. Controlled invariant subspaces will be replaced by finite-dimensional subspaces which are invariant for a generator of the form \( A + BF \). The fact that we shall use finite-dimensional subspaces in our constructions is motivated by our aim to obtain controllers of finite order, but is also helps to avoid difficulties that arise in connection with infinite-dimensional spaces. This is the key to our approach in the subsequent chapters, together with the relatively simple nature of the systems we shall study and the availability of a finite-dimensional theory that gives ample opportunity for low-order controller design.
4.5 Introduction to the examples

In this section we give some basic facts about the equations we shall use in our examples. We shall use the one-dimensional heat equation as an archetype of parabolic equations, and delay equations will be represented essentially by a scalar equation with one pure delay.

The heat equation, provided with distributed control and observation, has the following form.

\[(4.63.1) \quad \frac{\partial}{\partial t} \varphi(x,t) = \frac{1}{\tau^2} \frac{\partial^2}{\partial x^2} \varphi(x,t) + \sum_{l=1}^{m} b_l(x) u_l(t) \quad (x \in [0,1], t \geq 0)\]

\[(4.63.2) \quad \frac{\partial}{\partial x} \varphi(0,t) = \frac{\partial}{\partial x} \varphi(1,t) = 0 \quad (t \geq 0)\]

\[(4.63.3) \quad \varphi(x,0) = \varphi_0(x) \quad (x \in [0,1])\]

\[(4.63.4) \quad y_i(t) = \int_0^1 c_i(x) \varphi(x,t) dx \quad (i = 1, \ldots, p)\]

As the state space for this equation, we shall take \( X = L_2(0,1) \). The operator \( A \) is then given by

\[(4.64.1) \quad D(A) = \{ \varphi \in L_2(0,1) \mid \frac{\partial^2 \varphi}{\partial x^2} \in L_2(0,1), \frac{\partial \varphi}{\partial x}(0) = \frac{\partial \varphi}{\partial x}(1) = 0 \}\]

\[(4.64.2) \quad A \varphi = \frac{1}{\tau^2} \frac{\partial^2 \varphi}{\partial x^2} \].

It is a well-known fact that this operator generates a semigroup which is differentiable (even analytic) for \( t > 0 \). Furthermore, \( A \) has a discrete spectrum with simple eigenvalues at \(-k^2(k = 0, 1, 2, \ldots)\). The corresponding normalized eigenvectors are given by

\[(4.65) \quad \varphi_k(x) = \begin{cases} 1 & (k = 0) \\ \sqrt{2} \cos k\pi x & (k = 1, 2, \ldots) \end{cases}\]

and they form a complete orthonormal set in \( L_2(0,1) \).

We shall also use another version of this equation, in which the Neumann boundary condition (4.63.2) has been replaced by the Dirichlet boundary condition.
\[ (4.63.2)' \quad \varphi(0,t) = \varphi(1,t) = 0. \]

In this case, the operator \( A \) is given by

\[
(4.66.1) \quad D(A) = \{ \varphi \in L^2(0,1) \mid \frac{d^2\varphi}{dx^2} \in L^2(0,1), \varphi(0) = \varphi(1) = 0 \}
\]

\[
(4.66.2) \quad A\varphi = \frac{1}{\pi^2} \frac{d^2\varphi}{dx^2}.
\]

Again, \( A \) has a discrete spectrum and the eigenvalues, which are now at \(-k^2 (k = 1,2,\ldots)\), are all simple. The corresponding normalized eigenvectors are given by

\[
(4.67) \quad \varphi_k(x) = \sqrt{2} \sin k\pi x \quad (k = 1,2,\ldots)
\]

and these also form a complete orthonormal set in \( L^2(0,1) \).

The input operator is given by (4.63.1) as \( B : \mathbb{R}^m \rightarrow X \), with

\[
(4.68) \quad Bu = \sum_{i=1}^{m} b_i u_i \quad (u \in \mathbb{R}^m).
\]

Of course, the input functions \( b_i \) are assumed to be in \( L^2(0,1) \). The equation (4.63.4) defines the output mapping \( C : X \rightarrow \mathbb{R}^p \) as

\[
(4.69) \quad (C\varphi)_i = \int_0^1 c_{i} \varphi(x) dx \quad (i = 1,\ldots,p),
\]

where the \( c_i \) are functions in \( L^2(0,1) \).

The delay equations we shall consider are all of the following type:

\[
(4.70.1) \quad z'(t) = A_0 z(t) + A_1 z(t-1) + B_0 u(t) \quad (z(t) \in \mathbb{R}^n)
\]

\[
(4.71.2) \quad z(t) = f(t) \quad (t \in [-1,0])
\]

\[
(4.71.3) \quad y(t) = C_0 z(t).
\]

Here, \( A_0 \) and \( A_1 \) are \( mn \times mn \) matrices, \( B_0 \) is an \( mn \times m \) matrix and \( C_0 \) is an \( pmn \times pmn \) matrix. We re-write this equation by introducing a function of two variables \( \varphi(t,\theta) \) with \( t \in [0,\infty) \) and \( \theta \in [-1,0] \), which is related to \( z(t) \) in the following way:
(4.72) \[ \phi(t, \theta) = z(t+\theta). \]

Under suitable regularity hypotheses on \( z(t) \), this relation implies

(4.73) \[ \frac{3}{3t} \phi(t, \theta) = \frac{3}{3\theta} \phi(t, \theta). \]

If \( z(t) \) satisfies (4.60.1), then we obtain for the function \( \phi \):

(4.74) \[ \frac{3}{3\theta} \phi(t, 0) = A_0 \phi(t, 0) + A_1 \phi(t, -1) + B_0 u(t). \]

This leads to the following set-up (cf. DELFOUR (1980)). Let \( M_2((-1,0); \mathbb{R}^n) \) denote the product space \( \mathbb{R}^n \times L_2((-1,0); \mathbb{R}^n) \), and let us write \( H^1([-1,0]; \mathbb{R}^n) \) for the set of \( \mathbb{R}^n \)-valued functions whose distributional derivative is in \( L_2((-1,0); \mathbb{R}^n) \) (see ADAMS (1975; p.44)). By Sobolev's lemma (ADAMS (1975; p.97)), we can consider \( H^1([-1,0]; \mathbb{R}^n) \) as a subspace of \( C([-1,0]; \mathbb{R}^n) \). In particular, the quantities \( \phi(0) \) and \( \phi(1) \) are well-defined for \( \phi \in H^1([-1,0]; \mathbb{R}^n) \). Now, define the state space by \( X = M_2((-1,0); \mathbb{R}^n) \) and define the operator \( A \) by

(4.75.1) \[ D(A) = \{ (\phi_0, \phi) \mid \phi_0 \in \mathbb{R}^n, \phi \in H^1([-1,0]; \mathbb{R}^n), \phi(0) = \phi_0 \} \]

(4.75.2) \[ A(\phi_0, \phi) = (A_0 \phi(0) + A_1 \phi(-1), \phi'). \]

It has been shown (BORISOVIC & TURBABIN (1969)) that this operator is the generator of a semigroup \( T(t) \) on \( M_2((-1,0); \mathbb{R}^n) \). See DELFOUR (1980) for a survey and a further extension of the results in this direction. The spectrum of \( A \) is discrete; this follows from Prop. 4.2 in DELFOUR & MANITIUS (1980 b). To the right of any vertical line in the complex plane, there are only finitely many eigenvalues of \( A \) (HALE (1971; p.114)). For \( t \geq 1 \), the operator \( T(t) \) is compact (DELFOUR & MITTER (1972)). The eigenvectors of \( A \) are complete in \( M_2(-1,0) \) if and only if \( A_1 \) is non-singular (MANITIUS (1980)).

The input operator \( B \) is defined as an operator from \( \mathcal{U} = \mathbb{R}^m \) to \( M_2((-1,0); \mathbb{R}^n) \) by

(4.76) \[ Bu = (B_0 u, 0) \]
and the output operator $C$ is defined as an operator from $M^n_2((-1,0); \mathbb{R}^n)$ to $V = \mathbb{R}^p$ by

$$C(\Phi_0, \phi) = C_0\phi_0.$$  

(4.77)

Clearly, $B$ and $C$ are both bounded operators.

A specific example of a delay equation that we shall use in the sequel is the scalar equation

$$x'(t) = -\frac{\pi}{2} x(t-1).$$  

(4.78)

The eigenvalues of the associated operator $A$ can be computed as the roots of the characteristic equation, which is obtained by requiring that $x(t) = e^{\lambda t}$ is a solution of (4.78):

$$\lambda + \frac{\pi}{2} e^{-\lambda} = 0.$$  

(4.79)

It is easily seen that two roots of these equation are at $\frac{\pi}{2}i$ and at $-\frac{\pi}{2}i$, and that the other roots must all be in the left half plane. Apart from $\pm \frac{\pi}{2}i$, the roots can only be calculated approximately. A simple Newton procedure is sufficient for this purpose, because a good initial guess for the $k$-th pair of roots ($k = 1, 2, \ldots$) is given by the asymptotic formula

$$\lambda_k \approx -\log(4k-3) + \frac{\pi}{2}(4k-3)i.$$  

(4.80)

Rules for deriving such formulas are given by Bellman & Cooke (1963).
CHAPTER 5

FINITE-DIMENSIONAL COMPENSATORS FOR INFINITE-DIMENSIONAL SYSTEMS

The purpose of this chapter is to describe a method for the design of stabilizing compensators of finite order for a wide class of infinite-dimensional systems. We prove the existence of a finite-dimensional compensator for systems that satisfy the assumptions of Section 4.4. Moreover, we give a design procedure that can be used to find such compensators. The procedure will be illustrated by examples.

The chapter is divided into six sections. In Section 1, we give a brief discussion of the known results on compensator design for infinite-dimensional systems, and we indicate what the differences are with the method we shall use. In Section 2, we formulate the stabilization problem and present a basic construction theorem. The conditions of this theorem are not immediately verifiable, however, and we proceed in Section 3 to prove an existence result that is applicable to any system satisfying the assumptions (A1-9) of Section 4.4. Although the proof of the existence theorem is constructive, it is preferable to use a somewhat different procedure for the actual design of stabilizing compensators. Such a procedure is given in Section 4. The design method is illustrated by examples in the final two sections of this chapter. The first example concerns a diffusion system, and in the second example we consider a system governed by a delay equation.

5.1. Introduction

The subject of stabilization for infinite-dimensional systems has been studied extensively. Many different approaches have been used, and we shall not attempt to give a survey; the reader is referred to CURTAIN & PRITCHARD (1978) for a comprehensive bibliography and an exposition of the main results that have been obtained via the semigroup approach. An even more recent review can be found in CURTAIN, PRITCHARD et al. (1981).

We shall concentrate on stabilization by dynamic output feedback. A direct translation of finite-dimensional result to infinite-dimensional systems leads to compensators of infinite order (BHAT (1976), GRESSANG &
LAMONT (1975), FUJI (1980)). In practice, such compensators can only be implemented "approximately". This procedure, which does not seem to be completely worked out yet, will probably lead to high-order dynamics in the feedback loop. The 'converse' approach is to find, in some way or another, a finite-dimensional model for the infinite-dimensional system, and then to apply the standard finite-dimensional theory. Recently, M. J. Balas has worked out a proposal to make this approach rigorous, using a singular perturbation method; but his paper (BALAS (1981)) does not contain a general existence result.

Reduced-order modeling is known to be a difficult subject, even in the finite-dimensional context, and there have been several attempts to develop a theory for finite-dimensional compensator design without reduction of the system order. However, the results which have appeared up to now (BALAS (1978), BALAS (1979), CURTAIN (1981)) are all based on a very special assumption on the operators defining the system. We shall explain this assumption, called "zero spillover", in Section 5.5. In practice, the condition of zero spillover can only approximately be met, which means that the theory of the above-mentioned papers is not really applicable in practical situations. Moreover, serious design restrictions are introduced if one tries to satisfy this condition.

Below, we shall present an approach that avoids reduced-order modeling without introducing special assumptions on the system. Under the assumptions (A1-9) of the previous chapter, which represent a wide class of infinite-dimensional systems, we are able to give a rigorous treatment of finite-dimensional compensator design and in particular, to prove the existence of a compensator of finite order. As will be illustrated by examples, our approach is moreover suited for the actual computation of low-order compensators.

5.2. The basic theorem

We consider a system whose evolution is described by the equations

\begin{align}
(5.1.1) & \quad x'(t) = Ax(t) + Bu(t) \quad x(t) \in X, u(t) \in U \\
(5.1.2) & \quad y(t) = Cx(t) \quad y(t) \in Y.
\end{align}

The spaces $X$, $U$ and $Y$ and the operators $A$, $B$ and $C$ are supposed to satisfy
at least the assumptions (A1-3) of Section 4.4; the other assumptions will be appealed to when they are needed. To the system described by (5.1) we want to add a compensator, in order to obtain improved stability properties. The compensator will be a new dynamical system taking the observation \( y(t) \) from (5.1) as its input, and specifying at its output the control function \( u(t) \) appearing in (5.1):

\[
(5.2.1) \quad w'(t) = Nw(t) + My(t) \quad \text{w(t) } \in \mathcal{W}
\]

\[
(5.2.2) \quad u(t) = Lw(t).
\]

The order of the compensator will be the dimension of its state space \( \mathcal{W} \). We shall only consider compensators of finite order, so that (5.2) will represent a finite-dimensional system. Of course, \( L, M \) and \( N \) are linear mappings between the appropriate spaces.

The systems (5.1) and (5.2) together give rise to the following set of equations:

\[
(5.3.1) \quad x'(t) = Ax(t) + BLw(t)
\]

\[
(5.3.2) \quad w'(t) = MCx(t) + NW(t)
\]

which we may also write in the extended state space \( X \oplus \mathcal{W} \):

\[
(5.4) \quad \frac{d}{dt} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix}.
\]

Here, the extended system operator

\[
(5.5) \quad A_e := \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}
\]

with domain \( D(A) \oplus \mathcal{W} \) and range space \( X \oplus \mathcal{W} \) is defined in the way explained in Section 4.2.

The stabilization problem can now be formulated as follows: Given a system (5.1) under the assumptions (A1-3), find a finite-dimensional compensator (5.2) such that the extended system operator \( A_e \) defined in (5.5) generates a stable semigroup on the extended state space \( X \oplus \mathcal{W} \). The
notion of 'stability' is assumed to be defined by some fixed (negative) number $\omega$ representing the desired growth constant; see Section 4.3. A compensator that gives a solution to the stabilization problem will be called a stabilizing compensator.

The following theorem gives sufficient conditions for a stabilizing compensator of order $k$ to exist. The proof is by construction.

**Theorem 5.1** Consider the system (5.1) under the assumptions (A1-3). Assume that there exist bounded mappings $F: X \to U$ and $G: Y \to X$, together with a finite-dimensional subspace $V \subset D(A)$, such that the following holds:

\begin{align}
(5.6.1) & \quad A + BF \text{ generates a stable semigroup} \\
(5.6.2) & \quad A + GC \text{ generates a stable semigroup} \\
(5.6.3) & \quad (A+BF)x \in V \text{ for all } x \in V \\
(5.6.4) & \quad \text{Im } G \subset V.
\end{align}

Then there exists a stabilizing compensator of order $k$, where $k = \dim V$.

**Proof** Introduce a new linear space $W$ isomorphic to $V$, and let $R: V \to W$ be the mapping that provides the isomorphism. Define a compensator of the form (5.3) by setting $L = FR^{-1}$, $M = -RG$ (well-defined by (5.6.4)) and $N = R(A+BF+GC)R^{-1}$ (well defined by (5.6.3) and (5.6.4)). We obtain the following extended system operator:

\begin{equation}
(5.7) \quad A_c = \begin{bmatrix} A & BFR^{-1} \\ -RG & R(A+BF+GC)R^{-1} \end{bmatrix}.
\end{equation}

We introduce the following subspace of $X^n := X \oplus W$:

\begin{equation}
(5.8) \quad M = \{(x_{1,n}) \mid x \in V\} = \{(R^{-1}w) \mid w \in W\}.
\end{equation}

There is an obvious isomorphism from $W$ to $M$, given by

\begin{equation}
(5.9) \quad T: w \mapsto (R^{-1}w).
\end{equation}
Considering $X$, $\mathcal{W}$ and $M$ as subspaces of $X^e$, we note that $X^e$ can be decomposed either as $X \oplus \mathcal{W}$ or as $X \oplus M$. The similarity transformation from one decomposition to the other is given by

$$H: X \oplus \mathcal{W} \rightarrow X \oplus M, \quad H(X) = (X^e R^{-1} \mathcal{W}).$$

Written in matrix format, we have

$$H = \begin{pmatrix} I & -R^{-1}T \mathcal{W} \\ 0 & T \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} I & R^{-1}T^{-1} \mathcal{W} \\ 0 & T^{-1} \end{pmatrix}.$$  

By straightforward computation, we find the following form for the extended system operator with respect to the 'basis' $X \oplus M$:

$$\mathcal{A}_e = H \mathcal{A} e H^{-1} = \begin{pmatrix} A + GC & 0 \\ -TRGC & TR(A + BF)R^{-1}T^{-1} \end{pmatrix}.$$  

The right lower block is clearly similar to $A + BF$: $\mathcal{V}$, which generates a stable semigroup by (5.6.1). Because the left upper block is also the generator of a stable semigroup according to (5.6.2), it follows from Prop. 4.7 that $\mathcal{A}_e$ generates a stable semigroup, and Lemma 4.5 shows that the same must hold for $\mathcal{A}_e$.

In our formulation of the stabilization problem, we have only allowed integral control. We could have added proportional control by letting the compensator be of the form

$$\mathcal{W}(t) = Nw(t) + My(t)$$

and

$$u(t) = Lw(t) + Ky(t).$$

In effect, proportional control allows us to change the system operator $A$ into $A + \mathcal{B}K$. It should be noted that the above theorem is applicable to the triple $(C, A + \mathcal{B}K, B)$ for any $K$, simply by absorbing the $\mathcal{B}K$-term into $A$. One may use proportional control as an independent means for improving the stability properties of the system, but we shall not discuss this aspect. For a treatment of proportional control in an infinite-dimensional context,
one is referred to POHLJAINEN (1980).

Thm. 5.1 is clearly not directly applicable, because it is not immedi-
ately clear how to find a subspace $V$ and mappings $F$ and $G$ that satisfy
the conditions (5.6). Additional material will be needed to obtain practical
results. In the next two sections, we shall present such material.

5.3. The existence result

Our aim in this section is to prove the following result.

**Theorem 5.2** Consider the system (5.1). If the assumptions (A1-3) hold, then
there exists a stabilizing compensator of finite order.

This establishes the existence of finite-dimensional stabilizing compens-
sators for a large class of infinite-dimensional systems (as discussed in
Section 4.4). An upper bound for the compensator order is not given, but the
examples at the end of this chapter suggest that in many practical cases
it will be possible to design compensators of fairly low order.

The proof of the theorem consists of a combination of the results of
two lemmas, which we shall now give.

**Lemma 5.3** Consider the system (5.1) under the assumptions (A1-3). Suppose
that there exist bounded mappings $F: X \rightarrow U$ and $G: Y \rightarrow X$ such that

\begin{align*}
(5.14.1) & \quad A + BF \text{ generates a stable semigroup } \\
(5.14.2) & \quad A + BF \text{ has a discrete spectrum, and its eigenvectors form a complete set in } X \\
(5.14.3) & \quad \text{there exists } \delta > 0 \text{ such that the semigroup generated by } \\
& \quad A + GC + \delta I \text{ is stable.}
\end{align*}

Then there exists a stabilizing compensator of finite order for the given
system.

**Proof** Let $S(t)$ denote the semigroup generated by $A + GC$. By (5.14.3), the
growth constant of $S(t)$ is less than or equal to $\omega - \delta$ (where $\omega$ denotes the
growth constant that defines our notion of 'stability'). Consequently, there
exists a constant $M$ such that
\( (5.15) \quad \| S(t) \| \leq M \exp((\omega - 6)t). \)

For any \( \hat{G} : V \to X \) such that

\( (5.16) \quad \| G - \hat{G} \| \leq \frac{1}{4} N^{-1} \| C \|^{-1} \delta \)

the operator \( A + \hat{G}C \) will generate a semigroup \( \hat{S}(t) \) with

\( (5.17) \quad \| \hat{S}(t) \| \leq M \exp(\omega t) \)

(Lemma 4.4). Let us write \( \epsilon := \| M^{-1} \| C \|^{-1} \delta. \) Pick some orthonormal basis of \( V, \)
and define \( g_i := G y_i. \) By the completeness assumption on the eigenvectors of
\( A + BF, \) there exists for every \( i = 1, \ldots, p \) a finite set \( \{ x_{i1}^1, \ldots, x_{IN_i}^1 \} \) of
generalized eigenvectors of \( A + BF \) such that

\( (5.18) \quad \| g_i - \sum_{j \in I_i} a_{ij} x_{ij} \| < \epsilon \)

for suitable numbers \( a_{ij} (i = 1, \ldots, p; j = 1, \ldots, N_i). \) To every pair of indices
\( (i, j), \) there exists a \( \lambda_{ij} \in \mathbb{C} \) and an \( n_{ij} \in \mathbb{N} \) such that
\( (\lambda_{ij} - (A + BF))^n x_{ij} = 0. \)

Now define the subspace \( V \) as follows:

\( (5.19) \quad V := \text{span} \{ (\lambda_{ij} - (A + BF))^k x_{ij} \mid i = 1, \ldots, p; j = 1, \ldots, N_i; k = 0, \ldots, n_{ij} - 1 \}. \)

Then it is clear that \( V \) is a finite-dimensional subspace contained in
\( D(A + BF) = D(A), \) and that \( V \) is invariant under \( A + BF. \)

Write

\( (5.20) \quad \hat{g}_i := \sum_{j \in I_i} a_{ij} x_{ij} \)

and define \( \hat{G} : V \to X \) by

\( (5.21) \quad \hat{G} y_i = \hat{g}_i. \)

We then have \( \text{Im} \hat{G} \subset V. \) Moreover, \( (5.16) \) holds and so \( A + \hat{G}C \) generates a
stable semigroup. We can now apply Thm. 5.1, using the subspace \( V \) and the
mappings $F$ and $\hat{G}$, to conclude that there exists a stabilizing compensator of finite order (equal to $\dim V$).

Remark. In the conditions of the lemma, it is required that the semigroup generated by $A + GC$ has a certain extra stability margin. In the proof, this is needed to allow a small perturbation of $G$ without loss of stability. In the situation where assumption $(A5)$ of Section 4.4 holds, it is possible to do a spectral decomposition with respect to the set $\sigma_1 = \{\lambda \in \sigma(A) | \Re \lambda > w - \delta\}$, and then one would want to use a finite-dimensional method to compute $G$ such that $\Re \lambda \leq w - \delta$ for all $\lambda \in \sigma(A + GC)$. However, we have not excluded that there may be unobservable eigenvalues $\omega \in \lambda$ in the strip $\{\lambda \in \sigma | w - \delta < \Re \lambda \leq \omega\}$ This would make it impossible to shift the eigenvalues of $A + GC$ to the left of the line $\Re \lambda = w - \delta$, but we can still reach our ultimate goal of slightly perturbing $G$ without destroying the stability. This is seen in the following way.

Decompose $X_{u - \delta}$ as $X_2 \otimes X_3$, where

$$X_3 = \ker \begin{pmatrix} e^{u - \delta} & \hat{A}^{u - \delta} \end{pmatrix}$$

(notation as in (1.10)). Write $X_1 = X_{u - \delta}$. Then we have, with respect to the decomposition $X = X_1 \otimes X_2 \otimes X_3$,

$$A = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & C_2 & 0 \end{pmatrix}.$$

Here, we have written $A_{11}$ for $A^u_{11}$, etc. By our assumptions, the pair $(C_2, A_{22})$ is observable, and the matrix $A_{33}$ is stable in the sense of $\omega - \delta$, but not in the sense of $w - \delta$. If we choose $G_2$ such that $(A_{22} + G_2 C_2) \subset \{\lambda \in \sigma | \Re \lambda < w - \delta\}$, and if we define $G$ by

$$G = \begin{pmatrix} G_2 \\ 0 \end{pmatrix}$$

then we shall have

$$A + GC = \begin{pmatrix} A_{11} & 0 & 0 \\ G_2 C_1 & A_{22} + G_2 C_2 & 0 \\ 0 & A_{32} & A_{33} \end{pmatrix}.$$
By Prop. 4.7, the two-by-two left upper block is the generator of a semigroup which is stable in the sense of \( \omega - \delta \). If \( \hat{G} \) is perturbed to
\[
\hat{G} = \begin{pmatrix}
\hat{G}_1^T & \hat{G}_2^T & \hat{G}_3^T
\end{pmatrix}^T,
\]
then we get
\[
A + \hat{G} = \begin{pmatrix}
A_{11} + \hat{G}_1C_1 & \hat{G}_1C_2 & 0 \\
\hat{G}_2C_1 & A_{22} + \hat{G}_2C_2 & 0 \\
\hat{G}_3C_1 & A_{32} + \hat{G}_3C_2 & A_{33}
\end{pmatrix},
\]
(5.26)

The two-by-two left upper block will generate a semigroup with growth constant \( \leq \omega \) whenever \( \|G - \hat{G}\| \) is small enough. Prop. 4.7 then shows that the semigroup generated by \( A + \hat{G} \) is stable in the sense of \( \omega \).

The conclusion of this remark is that the construction of Lemma 5.3 can be done even when there are unobservable nearly unstable eigenvalues. Of course, the basic reason for this fact is that such eigenvalues are insensitive to the choice of the injection mapping and can therefore be discarded when this mapping is manipulated.

It may be possible to verify the conditions of Lemma 5.3 immediately in some cases where a well-developed theory of completeness of eigenvectors exists; such a theory is given by MANITIUS (1980) and DELFOUR & MANITIUS (1980b) for systems described by delay equations. In general, however, only the completeness of the eigenvectors of \( A \) would be known, and we would like to infer from this that the same property holds for \( A + BF \). An extensive study of this inference for diffusion processes, based on the sufficient condition for completeness given in DUNFORD & SCHWARTZ (1963; p.1115), has been made in VAN HARTEN (1979). However, if the feedback mapping \( F \) is constructed as in the proof of Prop. 4.12, we can use a much simpler argument.

**Lemma 5.4** Consider the system (5.1). If the assumptions (A1-7) and (A8) hold, then there exists a bounded mapping \( F: X \to U \) such that \( A + BF \) generates a stable semigroup, the spectrum of \( A + BF \) is discrete and the eigenvectors of \( A + BF \) are complete.

**Proof** Define \( F \) as in the proof of Prop. 4.12, taking care that the eigenvalues of \( A, A_u, B, u \) do not coincide with those of \( A \). From the form (4.48) of \( A + BF \), we have
\[
\sigma(A +BF) = \sigma(A_B) \cup \sigma(A_uB,F_u).
\]
(5.27)
The two parts of the spectrum are separated, so there is a corresponding spectral decomposition which will be written

\[(5.28) \quad X = X_s \oplus X_n.\]

The spectrum of \(A + BF\) is clearly discrete, and it remains to show that every element \(x \in X\) can be approximated arbitrarily close by a finite linear combination of generalized eigenvectors of \(A + BF\). So take \(x \in X\), and let \(\varepsilon\) be a positive number. We can write \(x = x_s + x_n\) with \(x_s \in X_s\) and \(x_n \in X_n\). The subspace \(X_s\) is a finite-dimensional eigenspace of \(A + BF\) and so \(x_s\) is obviously a finite linear combination of eigenvectors of \(A + BF\).

By (A9) and the fact that \(AX_s \subset X_s\), there is a finite linear combination of eigenvectors of \(A\) in \(X_s\), which we shall call \(\tilde{x}_s\), such that \(\|x_s - \tilde{x}_s\| < \varepsilon\). Generalized eigenvectors of \(A\) in \(X_s\) are also generalized eigenvectors of \(A + BF\) and so \(\tilde{x}_s + x_n\) is a finite linear combination of eigenvectors of \(A + BF\). Moreover, \(\|x + (\tilde{x}_s + x_n)\| = \|x_s - \tilde{x}_s\| < \varepsilon\).

Using the results of the two lemmas, the proof of Thm. 5.2 follows almost immediately.

**Proof** (of Thm. 5.2) By Lemma 5.4, there exists a bounded mapping \(F : X \to U\) such that \(A + BF\) generates a stable semigroup, the spectrum of \(A + BF\) is discrete and the eigenvectors of \(A + BF\) are complete. By assumption (A5), there exists a \(\delta > 0\) such that \(\sigma = \{\lambda \in \sigma(A) \mid \text{Re} \lambda > \omega - \delta\}\) is a finite set to which the spectral decomposition (Prop. 4.11) can be applied. If the resulting finite-dimensional pair \((C_1, A_1)\) is observable, the procedure of the proof of Prop. 4.13 leads to an injection mapping \(G : Y \to X\) such that the semigroup generated by \(A + GC\) has a growth constant \(\leq \omega - \delta\), and appealing to Lemma 5.3 completes the proof. If the pair \((C_1, A_1)\) is not observable, a suitable modification of this procedure still leads to the same result; see the remark following the proof of Lemma 5.3.

For a discussion of the class of systems for which we now have established the existence of a finite-dimensional stabilizing compensator, the reader is referred to Section 4.4.

**5.4 The design procedure**

The proof of our existence result has been constructive, and so in principle we have obtained a method to compute solutions to the stabilization
problem. For practical purposes, however, the method suggested by the proofs above would not be very convenient. Below, we shall present an iterative procedure for the design of low-order compensators, and the final two sections of this chapter will be devoted to an illustration of this procedure by examples. We give the method as a series of steps; comment on each step will be given afterwards. It is not claimed that the procedure has a very high degree of numerical refinement, but it is good enough for our purposes and it may serve as a starting point for further numerical research. We proceed as follows, assuming that we have a system which satisfies the conditions (A1-9) of Section 4.4.

STEP 1 Find F such that A + BF has a discrete spectrum, the eigenvectors of A + BF are complete, and the semigroup generated by A + BF is stable.

STEP 2 Find G such that the growth constant of the semigroup generated by A + GC is somewhat smaller than the constant \( \omega \) that indicates the dividing line between 'stable' and 'unstable'.

STEP 3 Approximate the vectors in \( \text{Im} \ G \) by linear combinations of \( k \) selected eigenvectors of A + BF, and form the mapping \( \hat{G} \) which is close to G.

STEP 4 See if the semigroup generated by A + \( \hat{G} \) is stable. If not, select a different F and/or a different G, or repeat Step 3 with \( k \) replaced by \( k + 1 \). If the semigroup is stable, go to Step 5.

STEP 5 Construct the compensator of order \( k \) as in the proof of Thm. 5.1.

Comments The feedback mapping F may be constructed by the method of TRIGGIANI (1975) as we did in the previous section, but any other method may be used just as well. The stabilization property of F is of course fundamental. Because A generates a semigroup and BF is a bounded perturbation of finite rank, the spectrum of A + BF must be discrete; this follows from Thm. 6.2 and Thm. 6.5 in KATO (1966; Ch.IV) and from (4.42) and Prop. 4.3. The question of completeness may be less easy to settle; see Lemma 5.4 and the remarks preceding this lemma. Of course, a check on completeness is not necessary if one is willing to try the procedure without the guarantee that it will ultimately be successful.

The mapping G appearing in Step 2 may also be found by any suitable method. For the purposes of the design procedure, the effect of 'unobservable
poles' (eigenvalues with corresponding eigenvectors in Ker C) on the growth constant of the semigroup generated by $A + GC$ may be discarded, in view of the remark made after the proof of Lemma 5.3. The words "somewhat smaller" are vague, of course, but at present we do not have any better. Further research will have to show if it is possible to give any general guiding lines for selecting the growth constant of $A + GC$. The approximation of the vectors in Im $G$ has to be done in the norm for which the completeness of the eigenvectors of $A + BF$ has been established, if one wants to be assured of the ultimate success of the procedure when $k$ is increased. In many practical cases, however, the use of another norm may be easier computationally while still giving good results.

In principle, it would be possible to give an a priori estimate on the compensator order using (5.16), but this bound may be difficult to compute and it is likely to be conservative. The compensator order found by the iterative procedure can be much lower. The stability of the semigroup generated by $A + GC$ has to be verified from a computation of the eigenvalues, and it has to be shown by a direct argument that $A + GC$ satisfies the spectrum determined growth assumption.

The eigenvalue computation can conveniently be done by use of the Weinstein-Aronszajn-method (see KATZ 1966; p.244). The basic idea of this method can be described in a simple way: Suppose that $\lambda_0 \not\in \sigma(A)$ is an eigenvalue of $A + GC$ with corresponding eigenvector $x$, then we have $(A+GC)x = \lambda_0 x$ and consequently

$$x = (\lambda_0 - A)^{-1} GCx$$

which shows that the kernel of the matrix $I - C(\lambda_0 - A)^{-1} G$ contains the vector $Cx$. Because $\lambda_0 \not\in \sigma(A)$ we must have $Cx \neq 0$, and it follows that the function $\det(I - C(\lambda - A)^{-1} G)$ has a zero at $\lambda_0$. On the other hand, it is shown in KATZ 1966(1966) that a zero of this function at a point in the resolvent set of $A$ gives rise to an eigenvalue of $A + GC$. Now define the following functions from $E$ to $\mathbb{Z}$:

$$\sigma(A)(\lambda_0) = k \text{ if } \lambda_0 \text{ is an eigenvalue of } A \text{ with multiplicity } k$$

$$= 0 \text{ if } \lambda_0 \text{ is not an eigenvalue of } A$$
\[ \nu_G(\lambda_0) = \begin{cases} k & \text{if } \lambda_0 \text{ is a zero of order } k \text{ of } \det(I - C(\lambda - A)^{-1}C) \\ \infty & \text{if } \lambda_0 \text{ is a pole of order } k \text{ of } \det(I - C(\lambda - A)^{-1}C) \\ 0 & \text{if } \lambda_0 \text{ is neither a zero nor a pole} \end{cases} \]

and define \( \sigma(A + \hat{C}C) \) in the same way as \( \sigma(A) \). It is proved in Kato (1966) that the following formula holds:

\[ \sigma(A + \hat{C}C) = \sigma(A) + \nu_G \]

or else the 'singular' case occurs in which \( \gamma(A + \hat{C}C) = \mathbb{C} \). In our case, the latter possibility is excluded because \( A + \hat{C}C \) will always generate a semigroup so that \( \text{Re } \sigma(A + \hat{C}C) \) must be bounded above.

The final step of the procedure is purely a matter of computation. The numerical results that one obtains could be taken as the starting point of a process of tuning of parameters aimed at: a further improvement of the system's behaviour, but we shall leave this out of our discussion. By the results of Section 5.3, the procedure as it stands is already guaranteed to lead to a stabilizing compensator of finite order.

### 5.5 Example I: A diffusion system

The evolution of the temperature distribution on a thin, uniform, isolated rod can be described by the following parabolic partial differential equation:

\[
\begin{align*}
(5.33.1) \quad \frac{\partial}{\partial t} z(x,t) &= \frac{1}{\lambda^2} \frac{\partial^2}{\partial x^2} z(x,t) + b(x)u(t) & (t \geq 0, \ 0 \leq x \leq 1) \\
(5.33.2) \quad \frac{\partial z}{\partial x} (0,t) &= \frac{\partial z}{\partial x} (1,t) = 0 & (t \geq 0) \\
(5.33.3) \quad z(x,0) &= z_0(x) & (0 \leq x \leq 1).
\end{align*}
\]

We assume a scalar input and a scalar output, given by

\[
\begin{align*}
(5.33.4) \quad b(x) &= \sqrt{10} & (0.2 \leq x \leq 0.3) \\
&= 0 & (0 \leq x < 0.2, \ 0.3 < x \leq 1) \\
(5.33.5) \quad y(t) &= \sqrt{10} \int_0^{0.8} z(x,t)dx. & (0.2 \leq x \leq 0.3) \\
&= \int_0^{0.7} z(x,t)dx. & (0 \leq x < 0.2, \ 0.3 < x \leq 1)
\end{align*}
\]
As our state space, we take \( L_2(0,1) \). The operator \( A \) is defined by

\[
\begin{align*}
\text{(5.34.1)} & \quad D(A) = \{ \varphi \in L_2(0,1) \mid \frac{d^2\varphi}{dx^2} \in L_2(0,1), \frac{d\varphi}{dx}(0) = \frac{d\varphi}{dx}(1) = 0 \} \\
\text{(5.34.2)} & \quad A\varphi = \frac{1}{\pi^2} \frac{d^2\varphi}{dx^2} \quad (\varphi \in D(A)).
\end{align*}
\]

The input space and the output space are both equal to \( \mathbb{R} \); the mappings \( B \) and \( C \) are given by

\[
\begin{align*}
\text{(5.35)} & \quad B\alpha = ab \quad (\alpha \in \mathbb{R}) \\
\text{(5.36)} & \quad C\varphi = \sqrt{10} \int_{0.7}^{0.8} \varphi(x)dx \quad \varphi \in L_2(0,1)).
\end{align*}
\]

Both mappings have been normalized to 1, in order to obtain a clear picture of the gains that will be needed in the final design. The symmetry between input and output is inessential.

The pertinent facts about the operator \( A \) have been given in Section 4.5. We can calculate the entries of the infinite matrices of \( B \) and \( C \) with respect to the orthonormal set \( \{\varphi_0, \varphi_1, \ldots\} \) of eigenvectors of \( A \):

<table>
<thead>
<tr>
<th>( k )</th>
<th>( k\varphi_k = &lt;b, \varphi_k&gt; )</th>
<th>( Y_k = C\varphi_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.3162</td>
<td>0.3162</td>
</tr>
<tr>
<td>1</td>
<td>0.3149</td>
<td>-0.3149</td>
</tr>
<tr>
<td>2</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>3</td>
<td>-0.3047</td>
<td>0.3047</td>
</tr>
<tr>
<td>4</td>
<td>-0.4184</td>
<td>-0.4184</td>
</tr>
<tr>
<td>5</td>
<td>-0.2847</td>
<td>0.2847</td>
</tr>
<tr>
<td>6</td>
<td>0.</td>
<td>0.</td>
</tr>
<tr>
<td>7</td>
<td>0.2562</td>
<td>-0.2562</td>
</tr>
<tr>
<td>8</td>
<td>0.3385</td>
<td>0.3385</td>
</tr>
<tr>
<td>9</td>
<td>0.2209</td>
<td>-0.2209</td>
</tr>
<tr>
<td>10</td>
<td>0.</td>
<td>0.</td>
</tr>
</tbody>
</table>

Table 5.1. First entries of \( B \) and \( C \).
We set the desired growth constant $\omega$ equal to $-1$. From the remarks in Section 4.5 and the table above, it is clear that our system satisfies the assumptions (A1-9) and so Thm. 5.2 guarantees the existence of a finite-dimensional stabilizing compensator.

The assumption of "zero spillover" that was alluded to in Section 5.1 comes down to requiring that either the coefficients $\alpha_k$ or the coefficients $\gamma_k$ would be zero from a certain index $k_0$ on. In our example, this is clearly not the case. The presence of both "control spillover" and "observation spillover" is not a problem in our approach, however.

Let us follow the procedure given in Section 5.4.

**STEP 1** The unstable subspace $X_u$ is the one-dimensional eigenspace corresponding to the eigenvalue 0 of $A$. We have

\[(5.37) \quad A_u = 0, \quad B_u = 0.3162.\]

The eigenvalue at 0 is shifted to $-1.5$ by taking

\[(5.38) \quad F_u = -1.5 F_0^{-1} = -7.906.\]

If we define $F$ by $F = F_u F$ where $P$ is the orthogonal projection onto $X_u$, the operator $A + BF$ has eigenvalues at $\nu_0 = -1.5$ and at $\nu_k = -k^2$ ($k = 1, 2, \ldots$), with corresponding eigenfunctions

\[(5.39) \quad \psi_k = (\nu_0 - A)^{-1} b \quad \quad (k = 0)\]
\[= \varphi_k \quad \quad (k = 1, 2, \ldots).\]

The eigenfunction $\psi_0$ has been normalized such that $F\psi_0 = 1$.

**STEP 2** We consider the subspace $X_1 = \text{span} \{\psi_0, \varphi_1\}$ and the corresponding mappings:

\[(5.40) \quad A_{11} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}, \quad C_1 = (0.3162, -0.3149).\]

The pair $(C_1, A_{11})$ is observable. A short calculation shows that we can assign new eigenvalues of $A + GC$ at $-1.1$ and $-1.2$ by defining $G$ as follows:
(5.41.1) \( a = a g \) 
\( (a \in \mathbb{R}) \)

(5.41.2) \[ g = -1.32 \frac{1}{\sqrt{2}} \varphi_0 + 0.02 \frac{1}{\sqrt{2}} \varphi_1 = \]
\[ = -4.174 \varphi_0 - 0.064 \varphi_1. \]

STEP 3 For \( g \), let us take the orthogonal projection of \( g \) onto the subspace span \( \{ \varphi_0, \varphi_1 \} \):

(5.42) \[ \hat{g} = 27.216 \varphi_0 + 5.669 \varphi_1. \]

STEP 4 In the present case, it follows from the Weinstein-Aronszajn theory (see Section 5.4) that the eigenvalues of an operator of the form \( A + \hat{G} C \) are found as the zeros of the function

(5.43) \[ f(\lambda) = 1 - C(\lambda - A)^{-1} \hat{G} \]

together with the eigenvalues of \( A \) that are not poles of \( f(\lambda) \). If \( \hat{G} \) is given in terms of the basis \( \{ \varphi_0, \varphi_1, \ldots \} \):

(5.44.1) \[ \hat{G} a = a \hat{g} \] 
\( (a \in \mathbb{R}) \)

(5.44.2) \[ \hat{g} = \sum_{k \neq 0} a_k \varphi_k \]

then the function \( f(\lambda) \) can be written as

(5.45) \[ f(\lambda) = 1 - \prod_{k \neq 0} a_k \gamma_{-k} (\lambda - \lambda_{-k})^{-1}. \]

Using the expressions for \( \varphi_0 \) and \( \varphi_1 \) given in (5.39) we write \( \hat{g} \) in the form (5.44), and then we compute the zeros of the function \( f \) appearing in (5.45). It turns out that the first two eigenvalues of \( A + \hat{G} C \) are at \( -1.0187 \pm 0.1401i \). We also note that \( A + \hat{G} C \) satisfies the spectrum determined growth condition because the semigroup generated by \( A + \hat{G} C \) is analytic; this follows from the fact that the semigroup generated by \( A \) is analytic and from the fact that analyticity of the semigroup is preserved under bounded perturbations of the generator (Hille & Phillips (1957; p. 418)).
STEP 5 With respect to the basis \( \{ \psi_0, \psi_1 \} \), the compensator equations can now be computed as follows:

\[
(5.46.1) \quad \frac{d}{dt}(\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix})(t) = \begin{bmatrix} -1.929 & -2.779 \\ 0.119 & 27.216 \end{bmatrix}(\begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix})(t) + (\begin{bmatrix} 0.569 \end{bmatrix}) y(t)
\]

\[
(5.46.2) \quad u(t) = \psi_1(t).
\]

The eigenvalues of the extended system operator \( A_e \) consist of the eigenvalues of \( A + BF \) corresponding to \( \psi_0 \) and \( \psi_1 \), together with all eigenvalues of \( A + GC \); so we get \(-1, -1.0187 \pm 0.1401i, -1.5, -4, \ldots\). The growth constant of the compensated system is thus precisely equal to \(-1\).

Some simulation results showing the effect of the compensator are given in the Appendix, Fig.A1.

5.6 Example II: A delay system

We consider the following retarded equation, with control and observation:

\[
(5.47.1) \quad x_1'(t) = -\frac{\pi}{2} x_1(t-1) + x_2(t)
\]

\[
(5.47.2) \quad x_2'(t) = u(t)
\]

\[
(5.47.3) \quad y(t) = x_1(t).
\]

The characteristic equation of the open-loop system is given by

\[
(5.48) \quad \det \begin{bmatrix} \lambda + \frac{\pi}{2} e^{-\lambda} & -1 \\ 0 & \lambda \end{bmatrix} = 0.
\]

The characteristic function

\[
(5.49) \quad \Delta_A(\lambda) := \lambda (\lambda + \frac{\pi}{2} e^{-\lambda})
\]

has zeros at \( 0, \frac{\pi}{2} i, -\frac{\pi}{2} i \), and at infinitely many other points which are all in the left half-plane. (See the remarks in Section 4.5.) As our state space, we take \( M_2(-1,0) \oplus \mathbb{R} \) or the complexified version of this space. The operator \( A \) is defined by
\[ (5.50.1) \quad D(A) = \left\{ \left( \begin{array}{c} \varphi_0 \\ \alpha \end{array} \right) \mid \varphi_0 \in \mathbb{R}, \varphi \in H^2[-1,0], \alpha \in \mathbb{R}, \varphi(0) = \varphi_0 \right\} \]

\[ (5.50.2) \quad A \left( \begin{array}{c} \varphi_0 \\ \alpha \end{array} \right) = \left( \begin{array}{c} -\frac{\pi}{2} \varphi(-1) + \alpha, \varphi' \end{array} \right). \]

This setting for the delay equation has been explained in Section 4.5.

We let the input space \( U \) and the output space \( V \) both be equal to \( \mathbb{R} \) (or \( \mathbb{C} \)). The mappings \( B \) and \( C \) are given by

\[ (5.51) \quad Bu = \left( \begin{array}{c} 0 \\ u \end{array} \right) \quad (u \in \mathbb{R}) \]

\[ (5.52) \quad C \left( \begin{array}{c} \varphi_0 \\ \alpha \end{array} \right) = \varphi_0 \quad \left( \begin{array}{c} \varphi_0 \\ \alpha \end{array} \right) \in M_2(-1,0) \oplus \mathbb{R}. \]

The stabilizability of the pair \((A,B)\) and the detectability of the pair \((C,A)\) can be verified conveniently using the generalization of the Hautus test (HAUTUS (1969)) given by K. P. M. Bhat (BIAT (1976)). Because

\[ (5.53) \quad \text{rank} \left( \begin{array}{ccc} \lambda + \frac{\pi}{2} e^{-\lambda} & 1 & 0 \\ 0 & \lambda & 1 \end{array} \right) = 2 \quad \forall \lambda \in \mathbb{C}, \]

the pair \((A,B)\) is stabilizable with respect to any desired growth constant \( \omega \), and detectability of the pair \((C,A)\) also holds for any \( \omega \) because

\[ (5.54) \quad \text{rank} \left( \begin{array}{ccc} \lambda + \frac{\pi}{2} e^{-\lambda} & 1 \\ 0 & \lambda \\ 1 & 0 \end{array} \right) = 2 \quad \forall \lambda \in \mathbb{C}. \]

Let us set the desired growth constant \( \omega \) equal to \(-1\). It follows from the above remarks and from the remarks in Section 4.5 that the system under consideration satisfies the assumptions (AI-9) of section 4.4. So we start the design procedure.

**STEP 1** A spectral decomposition for an equation closely related to \((5.47.1)\) is given in HALE (1971; p.117). Using this, we find the following feedback mapping \( F \), which replaces the eigenvalues of \( A \) at \( 0, \frac{\pi}{2} i \) and \(-\frac{\pi}{2} i \) by eigenvalues at \(-1, -1 + \frac{\pi}{2} i \) and \(-1 - \frac{\pi}{2} i \) for \( A + BF \), while leaving all the other eigenvalues unchanged.
(5.55) \[ F\left(\begin{bmatrix} \psi_0 \\ \alpha \end{bmatrix}\right)_a = -3\psi_0 + \int_{-1}^{0} \left( \frac{\pi^2}{2} \cos \frac{\pi \theta}{2} - \frac{3\pi}{2} \sin \frac{\pi \theta}{2} \right) \psi(\theta) d\theta - 3\alpha. \]

The interested reader may wish to verify that indeed the characteristic function of \( A + BF \), which is given by

(5.56) \[ \Delta_{A+BF}(\lambda) = \det \begin{pmatrix} \lambda + \frac{\pi e^{-\lambda}}{2} & -1 \\ \frac{3-\int \left( \frac{\pi^2}{2} \cos \frac{\pi \theta}{2} - \frac{3\pi}{2} \sin \frac{\pi \theta}{2} \right) e^{\lambda \theta} d\theta}{-1} & \lambda + 3 \end{pmatrix} \]

can be calculated as

(5.57) \[ \Delta_{A+BF}(\lambda) = (\lambda + 1)((\lambda + 1)^2 + \frac{\pi^2}{4})(\lambda + e^{-\lambda})(\lambda^2 + \frac{\pi^2}{4})^{-1}. \]

All eigenvalues of \( A + BF \) are simple, and we proceed to compute the corresponding eigenvectors. If \( \mu \) is an eigenvalue of \( A + BF \) and \( \psi = \begin{bmatrix} \psi_0 \\ \alpha \end{bmatrix} \in D(A) \) is the corresponding eigenvector, then the following equations hold:

(5.58.1) \[ \mu \psi_0 = -\frac{\pi}{2} \psi(-1) + \alpha \]

(5.58.2) \[ \mu \psi = \psi' \]

(5.58.3) \[ \mu \alpha = F\psi \]

Because \( \psi \in D(A) \), we also must have \( \psi \in H^1[-1,0] \) and \( \psi(0) = \psi_0 \). Then (5.58.2) gives

(5.59) \[ \psi(\theta) = e^{i\theta} \psi_0 \quad (-\pi \leq \theta \leq 0) \]

and from (5.58.1) we obtain

(5.60) \[ \alpha = (\mu + \frac{\pi e^{-\mu}}{2})\psi_0. \]

The eigenvector will be normalized such that \( C\psi = 1 \) if we put \( \psi_0 = 1 \).
STEP 2 We proceed to compute $G$ such that $A + GC$ has its eigenvalues to
the left of the line $\text{Re} \, \lambda = -1$. It is easy to find the matrices of
$A_u$ and $C_u$ with respect to the basis
\[
\begin{pmatrix}
1, \cos \frac{\pi \theta}{2} \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0, \sin \frac{\pi \theta}{2} \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
\frac{\pi}{2}, \frac{2}{w} \\
0 \\
1
\end{pmatrix}
\]
(5.61)
of $X_u$. They are given as follows:
\[
A_u = \begin{pmatrix}
0 & \frac{1}{2} \pi & 0 \\
-\frac{1}{2} \pi & 0 & 0 \\
0 & 0 & 0
\end{pmatrix},
C_u = \begin{pmatrix}
1 & 0 & \frac{2}{w} \\
0 & 0 & 0
\end{pmatrix}.
(5.62)

By straightforward computation, one finds that $A + GC$ will have eigenvalues
at $-\frac{\pi}{2}$ (double) and $-\pi$ if we take
\[
G = \text{sgn}(s \in \mathbb{R})
\]
(5.63.1)
\[
g = -\pi \left(2, \cos \frac{\pi \theta + 2 \sin \frac{\pi \theta + 1}{2}}{2} \right)
\quad (\frac{1}{2} \pi).
(5.63.2)

STEP 3 We compute the orthogonal projection (with respect to the norm
of $L^2(-1,0) \oplus \mathbb{R}$) of $g$ onto the subspace spanned by the eigenvectors of
$A + BF$ corresponding to the eigenvalues at $-1$ and $-1 + \frac{\pi}{2} i$. This gives
\[
g = -\pi \left(0.66 \left(1, e^{-\theta} \cos \frac{\pi \theta}{2} \right) + 2.89 \left(0, e^{-\theta} \sin \frac{\pi \theta}{2} \right)
\quad \frac{1}{4} \pi (1-e)\right) + 2.66 \left(1, e^{-\theta} \right).
(5.64)

STEP 4 We need an explicit formula for the eigenvalues of an operator of
the form $A + GC$, when the range of $G$ is contained in the span of
finitely many eigenvectors of $A + BF$. It follows from the W-A theory
(see formula (5.32)) that the eigenvalues of $A + GC$ are found as
the zeros of
\[
\Delta_{A+GC} (\lambda) = \Delta_A (\lambda) (1 - C (\lambda - A)^{-1} G).
(5.65)
\]

It is seen from (5.59) and (5.60) that the eigenfunctions of $A + BF$ are of
the form

125
(5.66) \[ g = \begin{pmatrix} (1, e^{i\theta}) \\ u + \pi e^{-\mu} \end{pmatrix} \]

where \( \mu \) is a root of the characteristic function of \( A + BF \) (see (5.56) or (5.57)). In order to find \( (\lambda - A)^{-1}g \), we have to solve the following equation for \( \lambda \in \rho(A) \):

(5.67) \[ (\lambda - A) \begin{pmatrix} \varphi_0, \varphi \\ \alpha \end{pmatrix} = \begin{pmatrix} (1, e^{i\theta}) \\ u + \pi e^{-\mu} \end{pmatrix} \varphi \in H^1[-1,1], \varphi(0) = \varphi_0. \]

The equation (5.66) is equivalent to

(5.68.1) \[ \lambda \varphi_0 + \frac{\pi}{2} \varphi(-1) - \alpha = 1 \]

(5.68.2) \[ \varphi(\theta) - \varphi'(\theta) = e^{i\theta} \quad (-1 \leq \theta \leq 0) \]

(5.68.3) \[ \lambda \alpha = \mu + \frac{\pi}{2} e^{-\mu}. \]

We can immediately solve (5.68.2):

(5.69) \[ \varphi(\theta) = e^{\lambda \theta} \varphi_0 - \frac{\lambda \theta - e^{i\theta}}{\lambda - \mu}. \]

Using this in (5.68.1), we get

(5.70) \[ (\lambda + \frac{\pi}{2} e^{-\lambda}) \varphi_0 = \frac{\pi}{2} e^{-\lambda \theta} e^{-\mu} + \alpha + 1. \]

Multiplying through by \( \lambda \) and using (5.68.3), we obtain

(5.71) \[ \Delta_A(\lambda) \varphi_0 = \frac{\pi}{2} \lambda e^{-\lambda \theta} e^{-\mu} + \mu + \frac{\pi}{2} e^{-\mu} + \lambda = \frac{1}{\lambda - \mu} (\Delta_A(\lambda) - \Delta_A(\mu)). \]

We finally arrive at the following formula:
\[(5.72) \quad C(\lambda - A)^{-1} \hat{G} = \frac{1}{\lambda - \mu} (1 - (\Delta_A(\lambda))^{-1} \Delta_A(\mu)).\]

Suppose now that \(\hat{G}\) is given by

\[(5.73) \quad \hat{G} = \frac{m}{\lambda - \mu} \sum_{k=1}^{m} \alpha_k \left( e^{\frac{\mu_k \theta}{\lambda - \mu}} \right).\]

Then we have, by linearity,

\[(5.74) \quad C(\lambda - A)^{-1} \hat{G} = \frac{m}{\lambda - \mu} \sum_{k=1}^{m} \alpha_k \left( \lambda - \mu \right) (1 - (\Delta_A(\lambda))^{-1} \Delta_A(\mu)).\]

Inserting this into (5.65) gives the explicit formula that we wanted:

\[(5.75) \quad \Delta_{A+\hat{G}C}(\lambda) = \Delta_A(\lambda) - \frac{m}{\lambda - \mu} \sum_{k=1}^{m} \alpha_k \left( \Delta_A(\lambda) - \Delta_A(\mu) \right).\]

This formula enables us to compute the eigenvalues of \(A + \hat{G}C\) when \(\hat{G}\) is of the form (5.73). Of course, this has to be done numerically. For our purposes, a simple Newton procedure will be sufficient, because the known eigenvalues of \(A + GC\) give good initial guesses for the eigenvalues of \(A + \hat{G}C\) when \(\hat{G}\) is close to \(G\).

We note that \(A + \hat{G}C\) satisfies the spectrum determined growth assumption for any \(\hat{G}\). This is because \(A + \hat{G}C\) is the adjoint of an operator to which the compactness result of DELFOUR & MITTER (1972) applies.

If we define the mapping \(\hat{G}\) by (5.64), then the first eigenvalues of \(A + \hat{G}C\) can be listed as follows (compared to those of \(A + GC\)):
We see that $A + \hat{G}C$ generates a stable semigroup. Consequently, it turns out that we are able to construct a stabilizing compensator of order 3 for our system (5.47). The eigenvalues of the extended system will be those of $A + \hat{G}C$ (see the table above) together with the eigenvalues of $A + BF$ corresponding to the eigenvectors used in the approximation; these are the eigenvalues at $-1$ and $-1 + \frac{\nu}{2}$. The growth constant of the resulting closed-loop system will be exactly equal to $-1$.

**STEP 5** The form of the compensator is (in sloppy notation – restriction symbols are omitted):

\[(5.76.1) \quad w'(t) = (A+BF+\hat{G}C)w(t) + \hat{G}Cx(t)\]

\[(5.76.2) \quad u(t) = Fw(t)\]

where the state space of $w(t)$ is the three-dimensional subspace of $M_2(-1,0) \oplus K$ spanned by the vectors

\[(5.77) \quad \mathcal{W}_1 = \begin{pmatrix} (1, e^{-\theta} \cos \frac{\pi}{2} \\ -1 \end{pmatrix}, \quad \mathcal{W}_2 = \begin{pmatrix} (0, e^{-\theta} \sin \frac{\pi}{2} \\ \frac{\nu}{2} (1-e^{-\theta}) \end{pmatrix}, \quad \mathcal{W}_3 = \begin{pmatrix} (1, e^{-\theta}) \\ -1+\frac{\nu}{2} e \end{pmatrix}.\]

The coordinates of $\hat{G}$ with respect to this basis are given by (5.64):
\begin{align}
\begin{pmatrix}
0.66 \\
2.89 \\
2.66
\end{pmatrix}
\begin{pmatrix}
-1 \\
4.71 \\
-8.36
\end{pmatrix}
\end{align}

The matrices of $A + BF$ and $C$ are easily found:

\begin{align}
A + BF &= \begin{pmatrix}
-1 & \frac{1}{2} & 0 \\
-\frac{1}{2} & -1 & 0 \\
0 & 0 & -1
\end{pmatrix},
C = \begin{pmatrix}
1 & 0 & 1
\end{pmatrix}.
\end{align}

To complete the design, we need the numerical values of $Fw_1$, $Fw_2$, and $Fw_3$. These could be computed using the explicit form of $F$ given in (5.55), but it is considerably easier to combine (5.58.3) and (5.60), which gives the following formula for an eigenvector $\psi$ of $A + BF$ (normed such that $C\psi = 1$) corresponding to an eigenvalue $\mu$:

\begin{align}
F\psi = \Delta_A(\mu).
\end{align}

Taking real and imaginary parts, we get

\begin{align}
Fw_1 &= 1 - \frac{\pi^2}{4} + \frac{\pi^2}{4}e = 5.24 \\
Fw_2 &= -\pi + \frac{\pi e}{2} = 1.13 \\
Fw_3 &= 1 - \frac{\pi e}{2} = -3.27.
\end{align}

It is interesting to note that the explicit formula (5.55) has not been used at all in the design procedure. This means that it is not necessary to compute the projection corresponding to the spectral decomposition of $A$. We leave it as a topic for further research to see under what general circumstances the computation of the spectral projection can be avoided.

The compensator equations are finally found to be the following:

\begin{align}
w'(t) &= \begin{pmatrix}
1.08 & 1.57 & 2.08 \\
-10.65 & -1.08 & 9.08 \\
-8.36 & 0.00 & -9.36
\end{pmatrix}w(t) + \begin{pmatrix}
-2.08 \\
9.08 \\
8.36
\end{pmatrix}y(t) \\
u(t) &= (5.24 & 1.13 & -3.27)w(t).
\end{align}

The result of some simulations using this compensator are given in the Appendix, Fig.A2.
CHAPTER 6

TRACKING AND REGULATION IN INFINITE DIMENSIONS

In this chapter, we consider problems that arise when certain system variables are to follow a signal of a prescribed form, or when they are to be made insensitive to such a signal. These problems of 'tracking' and 'regulation' can be brought into the same mathematical framework, which we studied for finite-dimensional systems in Ch.3. Here, we shall use the same class of infinite-dimensional systems as in the previous chapter, but we shall assume that the model for the exogenous signal is finite-dimensional. It is required that the tracking or regulation task will be carried out by a compensator of finite order. We shall give an existence result and a design procedure, and examples will serve to illustrate the method.

We start with a brief introduction in Section 1. A precise formulation of the problems we shall consider is given in Section 2. The basic construction method, that will be presented in Section 3, is used in Section 4 to derive an existence theorem. A certain 'transmission' condition enters into the statement of this theorem, and we indicate an important class of systems for which this condition holds. Next, Section 5 presents a step-wise procedure to construct a (low-order) compensator for a given tracking or regulation problem. The final three sections contain examples. In Section 6, we have a diffusion system with a constant reference signal to be tracked. In Section 7 we construct a finite-dimensional compensator that protects a delay system against a constant disturbance, and finally, in Section 8, we have again a diffusion system but now the signal to be followed is not a constant but a sinusoid.

6.1 Introduction

We want to study problems in which a given infinite-dimensional system has to follow or reject a prescribed signal. The signal will be modeled by a finite-dimensional linear system, of which the initial condition is left free; this includes, for instance, a sine wave of known frequency but unknown phase and amplitude. The models of the system and the signal will be taken together in one larger system, whose state space is the direct sum of the
state space of the original system and that of the signal model. The tracking or rejection task is expressed via a linear operator, which maps the state variable onto the "variable-to-be-controlled". For instance, if the temperature at a certain point in a room is to be made equal to a constant reference variable, the variable-to-be-controlled would be the difference between the temperature at the given point and the reference value. By adding a dynamic compensator, we want to accomplish that the variable-to-be-controlled will return to zero after an initial disturbance. Moreover, we shall require that the system as a whole will be stable, with the exception of that part that represents the exogenous signal.

The essence of our approach will be to combine basic ideas from Ch. 3 with the finite-dimensional compensator design procedure of Ch. 5. In Bhat (1976) and Pohjalainen (1980), one finds other state-space approaches to the theory of tracking and regulation for infinite-dimensional systems. Bhat follows closely the lines of Wonham (1974) and gives applications to delay systems. His proposed compensators are of infinite order, which in the case of the delay examples means that they require an integration of the state variable over the delay interval. Motivated by the work of E.J.Davison and colleagues, Pohjalainen gives controllers of finite order. However, a basic assumption in his work is that the original system is stable. Pohjalainen's examples are of diffusion type, and his theorems are formulated for self-adjoint parabolic equations. However, the observation operator \( \mathbf{C} \) is not required to be bounded but only to be relatively bounded with respect to \( \mathbf{A} \).

The main benefit of our approach contrasted with Bhat's is that it leads to finite-dimensional compensators; compared with the work of Pohjalainen, our method has the advantage that it combines stabilizing and regulating ability. On the other hand, we do not consider the problem of robustness which is central both in the work of Bhat and in the work of Pohjalainen.

6.2 Problem formulation

We shall denote the state space of the "original system" by \( X_1 \), and that of the "exogenous signal" by \( X_2 \). The "variable-to-be-controlled" will be written as \( z(t) \). We consider the following system equations:

\[
(6.1.1) \quad \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) \quad x_1(t) \in X_1, u(t) \in U
\]
\begin{align}
(6.1.2) \quad x_2'(t) &= A_{22} x_2(t) \quad x_2(t) \in X_2 \\
(6.1.3) \quad y(t) &= (C_1 \ C_2) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad y(t) \in Y \\
(6.1.4) \quad z(t) &= (D_1 \ D_2) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad z(t) \in Z.
\end{align}

We can write these equations in a more concise form if we introduce

\begin{align}
(6.2.1) \quad X &= X_1 \oplus X_2, \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\
(6.2.2) \quad A &= \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \\
(6.2.3) \quad C &= (C_1 \ C_2), \quad D = (D_1 \ D_2).
\end{align}

The system (6.1) then becomes:

\begin{align}
(6.3.1) \quad x'(t) &= Ax(t) + Bu(t) \\
(6.3.2) \quad y(t) &= Cx(t) \\
(6.3.3) \quad z(t) &= Dx(t).
\end{align}

Let us describe the precise setting in which we want to use these equations. Our conditions for the original system will be the same as before, so that the results below will apply to the same class, including many systems of parabolic and delay type, and also some hyperbolic and neutral systems. Thus, \( X_1 \) is a Banach space and \( A_{11} \) generates a semigroup \( T_1(t) \) on \( X_1 \) (A1), the spectrum of \( A_{11} \) is discrete (A4), there are only finitely many eigenvalues of \( A_{11} \) to the right of some vertical line \( \text{Re} \lambda = \omega-\delta \) (A5), the operator \( A_{11}^\perp \) (obtained by spectral decomposition in the space \( X_1 \)) satisfies the spectrum determined growth assumption (A6), and the eigenvectors of \( A_{11} \) are complete (A9). The space \( X_2 \) is a finite-dimensional linear space, \( A_{22} \) is a linear mapping acting on \( X_2 \), and \( A_{12} \) is a bounded linear mapping from \( X_2 \) into \( X_1 \).

Under these circumstances, it follows from Prop. 4.7 (after interchanging the indices 1 and 2) that the conditions (A1,4-6,9) also hold for the
operator A defined in (6.2.2). Conversely, suppose that A is an operator acting on a Banach space X which has a direct sum decomposition \( X = X_1 \oplus X_2 \), where \( X_2 \) is a finite-dimensional subspace of \( D(A) \), and suppose that \( Ax \in X_1 \) for all \( x \in D(A) \cap X_1 \). Then A can be written in the form (6.2.2), and if the conditions (A1,4–6,9) hold for A then they also hold for \( A_{11} \). Thus it is equivalent to require (A1,4–6,9) to be satisfied by \( A_{11} \) or by A, and we shall use the latter alternative because it leads to a formulation that corresponds nicely to the one used in Ch. 5.

Naturally, the matrix \( A_{22} \), which represents the signal dynamics, will be unstable. It is then obvious from (6.2.2) that the pair \((A,B)\) is not stabilizable. What we do need to require, however, is that the pair \((A_{11},B_{1})\) is stabilizable. To get a verifiable condition for this, we use the spectral decomposition in \( X_1 \) in order to introduce the controllability of the finite-dimensional pair \((A_{11}^{u},B_{1}^{u})\) as an assumption.

Summarizing, we shall use the following extra conditions, which are formulated for the system (6.3):

(B1) \( X \) has a direct sum decomposition \( X = X_1 \oplus X_2 \) such that \( \dim X_2 < \infty \), 
\[ X_2 \subset D(A), \text{ and } Ax \in X_1 \text{ for all } x \in D(A) \cap X_1, \text{ so that we have} \]
\[ A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \]

(B2) \( Z \) is a finite-dimensional linear space, and \( D \) is a bounded linear mapping from \( X \) onto \( Z \)

(B3) the range of \( B \) is contained in \( X_1 \), so that we have \( B = \begin{pmatrix} B_{1} \\ 0 \end{pmatrix} \) with respect to the decomposition \( X = X_1 \oplus X_2 \)

(B4) the pair \((A_{11}^{u},B_{1}^{u})\) is controllable.

Most of these conditions follow naturally from the characteristics of the tracking and regulation problem. Only the boundedness of \( D \) could be weakened, for instance to allow point measurements in an \( L_2 \)-space.

To the system (6.3), we are going to add a compensator of the usual form:

\[ (6.4.1) \quad w'(t) = Nw(t) + My(t) \quad w(t) \in \mathcal{W} \]
(6.4.2) \( u(t) = Lw(t) \).

The dimension of \( \bar{W} \) is finite and is called the order of the compensator; \( L, M, N \) and \( K \) are linear mappings between the appropriate spaces. Combining (6.3) and (6.4), we obtain the following closed-loop system:

(6.5.1) \[
\frac{d}{dt} X_w(t) = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix} X_w(t)
\]

(6.5.2) \( z(t) = (D \ 0) X_w(t) \).

We shall write

(6.6) \( A_e = \begin{pmatrix} A & BL \\ MC & N \end{pmatrix}, \quad D_e = (D \ 0) \).

The purpose of the compensation has been described in general terms in the preceding section; we now want to make this precise. We shall say that the compensator (6.4) solves the regulation problem for the system (6.3), or that (6.4) gives a regulating compensator, if the following is true. There is a subspace \( V \subset \mathcal{D}(A_e) \) such that \( A_e V = V \) and \( \dim V = \dim X_\mathcal{S} \). The restriction of \( A_e \) to \( V \) is similar to \( A_\mathcal{S} \). The operator \( A_e \) is the generator of a semigroup \( T_e(t) \) on \( X \otimes \bar{W} \), and the induced semigroup \( T_e(t) \) on \( (X \otimes \bar{W})/V \) is stable. Finally, the subspace \( V \) is contained in \( \ker D_e \).

This says that the closed-loop system is stable 'modulo the signal dynamics' (stabilization property). The requirement "\( V \subset \ker D_e \)" means that the variable-to-be-controlled is a function only of the stable part of the system (regulation property).

The regulation problem as formulated above is rather simple compared with some of the regulation problems that we studied in the finite-dimensional context. In particular, we do not differentiate between the stability requirement for the variables-to-be-controlled and that for the stabilizable part of the system as a whole, like we did in Ch. 3. It would not be impossible to introduce this distinction in the infinite-dimensional situation, but this would only make the theory more complex, while the basic features are already present in the problem as we posed it.
6.3 The basic theorem

We now give sufficient conditions for a regulating compensator of order \( k \) to exist. The proof is by construction.

**Theorem 6.1** Consider the system \((6.3)\) under the assumptions \((A1-3)\) and \((B1-3)\). Assume that there exist bounded mappings \( F: X \to U \) and \( G: V \to X \) together with finite-dimensional subspaces \( V_s \subset D(A) \) and \( V_c \subset D(A) \), such that the following holds:

\[
\begin{align*}
(6.7.1) & \quad V_s \subset V_c \cap \text{Ker } D \\
(6.7.2) & \quad X = X_1 \oplus V_s \\
(6.7.3) & \quad (A+BF)x \in V_s \text{ for all } x \in V_s \\
(6.7.4) & \quad (A+BF)x \in V_c \text{ for all } x \in V_c \\
(6.7.5) & \quad \text{the quotient semigroup generated by } A + BF \text{ on } X/V_s \text{ is stable} \\
(6.7.6) & \quad \text{the semigroup generated by } A + GC \text{ is stable} \\
(6.7.7) & \quad \text{Im } G \subset V_c. 
\end{align*}
\]

Then there exists a regulating compensator of order \( k \), where \( k = \dim V_c \).

**PROOF** Introduce a new linear space \( W \) isomorphic to \( V_c \), and let \( R: V_c \to W \) be the mapping that provides the isomorphism. Define a compensator of the form \((6.4)\) by setting \( L = FR^{-1}, M = -RG \) (well-defined by \((6.7.7)\)) and \( N = R(A+BF+GC)R^{-1} \) (well-defined by \((6.7.4)\) and \((6.7.7)\)). We obtain the following extended system operator:

\[
(6.8) \quad A_e = \begin{pmatrix} A & BFR^{-1} \\ -RG & R(A+BF+GC)R^{-1} \end{pmatrix},
\]

Consider the following subspaces of \( X^e := X \oplus W \):

\[
(6.9) \quad V = \{(x, x) \mid x \in V_s \}
\]
(6.10) \[ M = \{ (x^c, x^e) \mid x^c \in V^c_c, x^e \in \mathbb{R}_w^e \} = \{ 0^e_w \mid w \in \Omega \} \]

Note that \( V \) is well-defined by (6.7.1). Both \( V \) and \( M \) are contained in \( D(A_e) = D(A) \otimes \Omega \). There is an obvious isomorphism from \( \Omega \) to \( M \), given by

\[ \tau : \omega \mapsto (0^e_w) \].

(6.11)

Considering \( X, \Omega \) and \( M \) as subspaces of \( X^e \), we note that \( X^e \) can be decomposed either as \( X \otimes \Omega \) or as \( X \otimes M \). The similarity transformation from one decomposition to the other is given by

\[ H : X \otimes \Omega \to X \otimes M, \quad H(x^c, x^e) = (x^c - R^{-1}_w x^e) \] \[ T \omega = (0^e_w) \].

(6.12)

Written in matrix format, we have

\[ H = \begin{pmatrix} I & -R^{-1} \\ 0 & T \end{pmatrix}, \quad H^{-1} = \begin{pmatrix} I & R^{-1}T^{-1} \\ 0 & T^{-1} \end{pmatrix} \].

(6.13)

By straightforward computation, we find the following form for the extended system operator with respect to the 'basis' \( X \otimes M \):

\[ \tilde{A}_e = H A_e H^{-1} = \begin{pmatrix} A + BC & 0 \\ 0 & TRC \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & T^{-1} \end{pmatrix}. \]

(6.14)

By Prop. 4.7 and Lemma 4.5, this shows that \( A_e \) generates a semigroup \( T_e(t) \) on \( X^e \). The mapping \( TR \) gives an isomorphism between \( V_c \) and \( M \), and it is seen from (6.14) that the following diagram commutes:

\[ \begin{array}{ccc}
V^c_c & \xrightarrow{A + BF} & V^c_c \\
\downarrow{TR} & & \downarrow{TR} \\
M & \xrightarrow{A_e} & M
\end{array} \]

(6.15)

The image of \( V^c_s \) under \( TR \) is \( V \). Hence, it follows from (6.15) and (6.7.3) that \( V \) is \( A_e \)-invariant, and that \( A_e : V = A + BF : V^c_s \). Now let \( Q : V^c_c \to V^c_c / V^c_s \) be the factor mapping, and define a mapping \( J : (X^e \Omega) / V \to X \otimes (V^c_c / V^c_s) \) by
(6.16) \( J: \left[ \begin{array}{c} x \\ w \end{array} \right] \mapsto \left[ \begin{array}{c} x - R^{-1}w \\ QR^{-1}w \end{array} \right] \).

It is easily verified that \( J \) is well-defined and that it gives a bijection between \((X\oplus W)/V\) and \(X \oplus (V_c/V_s)\). Assuming, without loss of generality, that \(R: V_c \rightarrow W\) is an isometry, we can make the following estimates for any \(x_0 \in V_s\):

(6.17.1) \[ \|x - R^{-1}w\| \leq \|x - x_0\| + \|R^{-1}w - x_0\| = \|x - x_0\| + \|w - Rx_0\| \]

(6.17.2) \[ \|QR^{-1}w\| \leq \|R^{-1}w - x_0\| = \|w - Rx_0\| \]

It follows that

(6.18) \[ \left\| \left[ \begin{array}{c} x - R^{-1}w \\ QR^{-1}w \end{array} \right] \right\| \leq 2\left\| \left[ \begin{array}{c} x \\ w \end{array} \right] - \left[ \begin{array}{c} x_0 \\ 0 \end{array} \right] \right\| \quad \text{for all} \quad x_0 \in V_s \]

Using the definition of the norm in \((X\oplus W)/V\) and (6.9), we see that

(6.19) \[ \left\| \left[ \begin{array}{c} x - R^{-1}w \\ QR^{-1}w \end{array} \right] \right\| \leq 2\left\| \left[ \begin{array}{c} x \\ w \end{array} \right] \right\| \]

or, \( J \) is bounded. By the Banach open mapping theorem (TAYLOR & LAY (1980; p.212-213)), \( J^{-1} \) must be bounded too, and so \( J \) is a similarity transformation between \((X\oplus W)/V\) and \(X \oplus (V_c/V_s)\). Let us write \( A_e \) for the mapping induced by \( A_e \) on \((X\oplus W)/V\). It is straightforward to verify that \(JD(A_e) = D(A) \oplus (V_c/V_s) \), and that the following relation holds for \( x \in D(A), v \in V_c \):

(6.20) \[ J\bar{A}_{e}^{-1}(X) = A^*GC \begin{pmatrix} 0 & A + BF: V_s / V \end{pmatrix}_{c} X_{V} \]

Using (6.7.5) and (6.7.6), we see that the operator appearing on the right-hand side of this equation generates a stable semigroup. It follows from (6.20) that the semigroup \( \bar{T}_{e}(t) \) is similar to this semigroup, and hence \( \bar{T}_{e}(t) \) is stable.

We have already shown that \( A_e: V \) is similar to \( A + BF: V_s \), so now we want to prove that \( A + BF: V_s \) is similar to \( A_{22} \). It follows from (6.7.2) that \( \dim V_s = \dim X_2 \). Define a mapping \( \bar{F}: V_s \rightarrow X_2 \) by
By (6.7.2), this mapping is a bijection. Its inverse is clearly of the form

\[(6.22) \quad \tilde{p}^{-1} x = (Sx, x) \in X = X_1 \oplus X_2\]

where \(S\) is some linear mapping from \(X_2\) into \(X_1\). Now let us compute, for \(x \in X_2\):

\[(6.23) \quad \tilde{p}(A + BF)\tilde{p}^{-1} x = \left(\begin{array}{c}
A_11 + B \cdot 1 + B \cdot 1 \\
A_22 \cdot 1 + 0
\end{array}\right) x = A_{22}x.

This gives the desired similarity between \(A + BF\): \(V_s\) and \(A_{22}\). Finally, we have to prove that \(V \subseteq \ker D_6\). This follows immediately from (6.7.1) and (6.9).

The theorem is not directly applicable; we need to know how to find mappings \(F\) and \(G\) and subspaces \(V_s\) and \(V_c\) such that the conditions of the theorem are satisfied. This question is treated in the next section.

6.4 The existence result

Our main result in this section is the following.

**Theorem 6.2** Consider the system (6.3) under the assumptions (A1-8, 8-9) and (B1-4). If there exists a linear mapping \(S: X_2 \to X_1\), such that

\[(6.24.1) \quad \text{Im } S \subseteq D(A_{11})
\]

\[(6.24.2) \quad \text{Im}(A_11 S - SA_{22} + A_{12}) \subseteq \text{Im } B_1
\]

\[(6.24.3) \quad D_1 S + D_2 = 0
\]

then there exists a regulating compensator of finite order.

The conditions (6.24) will be discussed at the end of this section. The class of systems to which the theorem applies has been described in Section 4.4. For the proof of the theorem we need the following lemma.
**Lemma 6.3** Consider the system (6.3) under the assumptions (A1-3) and (B1-3). Suppose that there exist bounded linear mappings \( F_1: X_1 \to U \) and \( G: V \to X \) and a finite-dimensional subspace \( V_s \subset D(A) \), such that the following holds:

1. \( V_s \subset \ker D \)
2. \( X = X_1 \oplus V_s \)
3. \( AV_s \subset V_s + \text{Im } B \)
4. \( A_{11} + B_{1} F_1 \) generates a stable semigroup on \( X_1 \)
5. \( A_{11} + B_{1} F_1 \) has a discrete spectrum, and its eigenvectors form a complete base in \( X \)
6. \( \delta > 0 \) such that the semigroup generated by \( A + GC + \delta I \) is stable.

Then there exists a regulating compensator of finite order.

**Proof** It follows from (6.25.2) that the mapping \( \bar{F}: V_s \to X_2 \) defined by

\[
\bar{F}(x_2) = x_2 \quad (x_2 \in V_s)
\]

is a bijection. Let us define a mapping \( S: X_2 \to X_1 \) by

\[
\bar{F}^{-1}(x) = (Sx)_x \quad (x \in X_2).
\]

Then we have

\[
V_s = \{(Sx)_x | x \in X_2 \}.
\]

Let \( \{x_1, \ldots, x_r\} \) be a basis for \( X_2 \). From (6.25.3), we see that for every \( i = 1, \ldots, r \) there exists \( u_i \in U \) such that

\[
A(x_1) + Bu_i \in V_s.
\]

Now define \( F_2: X_2 \to U \) by
Using the mapping \( F \) that is given in the statement of the theorem, we form \( F: X \to \mathcal{U} \) by putting \( F = (F_1, F_2) \). We then have, for \( i = 1, \ldots, r \):

\[
Sx_i = u_i - F_1 x_i
\]

(6.30)

\[
(A + BF)(x_i) = A(x_i) + B(F_1 x_i + F_2 x_i) = A(x_i) + Bu_i \in \mathcal{U}
\]

(6.31)

by (6.29). Thus, \( \mathcal{U}_s \) is \((A + BF)\)-invariant.

With respect to the decomposition \( X = X_1 \oplus \mathcal{U}_s \), write

\[
G = \begin{pmatrix} G_1 \\ G_s \end{pmatrix}
\]

(6.32)

Let \( S(t) \) denote the semigroup generated by \( A + GC \). By (6.25.6), the growth constant of \( S(t) \) is less than or equal to \( \omega - \delta \) (where \( \omega \) denotes the growth constant that defines our notion of 'stability'). Consequently, there exists a constant \( M \) such that

\[
\|S(t)\| \leq M \exp((\omega - \delta)t)
\]

(6.33)

If \( \hat{G}_1: \mathcal{Y} \to X_1 \) is such that

\[
\|G_1 - \hat{G}_1\| \leq M^{-1} \|G\|^{-1} \delta
\]

(6.34)

and if we define \( \hat{G}: \mathcal{Y} \to X \) by

\[
\hat{G} = \begin{pmatrix} \hat{G}_1 \\ G_s \end{pmatrix}
\]

(6.35)

then the estimate (6.34) also holds for \( \|G - \hat{G}\| \). Therefore, the operator \( A + \hat{G}C \) will generate a semigroup \( \hat{S}(t) \) with

\[
\|\hat{S}(t)\| \leq M \exp(\omega t)
\]

(6.36)

(Lemma 4.4). By following further the procedure used in the proof of Lemma 5.3, we obtain a finite-dimensional subspace \( \mathcal{V}_{cl} \subset X_1 \) with the following properties: \( \mathcal{V}_{cl} \) is invariant under \( A_{11} + B_{11} F_1 \), and there is a mapping \( \hat{G}_1: \mathcal{Y} \to X_1 \) satisfying (6.34) and \( \text{Im} \hat{G}_1 \subset \mathcal{V}_{cl} \).
Now define $\hat{G}$ by (6.35) and $V_c$ by

$V_c = V_{c1} \oplus V_s$.

Then (6.36) shows that $A + \hat{G}$ generates a stable semigroup, and furthermore we have $\text{Im} \, \hat{G} \subset V_c$ and $V_c \subset V_s$. From the fact that $V_{c1}$ is invariant under $A_{11} + B_1 F_1$, and from the form of $A + BF$ as given by

$A + BF = \begin{pmatrix} A_{11} + B_1 F_1 & A_{12} + B_1 F_2 \\ 0 & A_{22} \end{pmatrix}$

it follows immediately that $V_{c1}$ (considered as a subspace of $X$) is $(A+BF)$-invariant. We already had the $(A+BF)$-invariance of $V_s$, and so it follows from (6.37) that $V_c$ is invariant under $A + 3F$.

Finally, it is easily seen that the mapping

$H : X_1 \to X/V_s, \ H x = [x]_0$

provides a similarity transformation between $X_1$ and $X/V_s$. Moreover, the following diagram commutes:

$D(A_{11}) \xrightarrow{A_{11} + B_1 F_1} X_1$
$\downarrow H$
$\downarrow H$

$D(A+BF) \xrightarrow{A+BF;X/V_s} X/V_s$

Hence, the quotient semigroup generated by $A + BF$ on $X/V_s$ is stable.

The conditions of Thm. 6.1 are satisfied for the mappings $F$ and $\hat{G}$ and the subspaces $V_s$ and $V_c$. Consequently, a regulating compensator of finite order (equal to $\dim V_c$) exists.

We are now ready to prove the theorem.

PROOF (of Thm.6.2) Define a subspace $V_s \subset X$ by

$V_s = \{ \frac{x}{X} \mid x \in V_2 \}$.
Clearly, we have $X = X_1 \oplus V_s$. It follows from (6.24.3) that $V_s \subseteq \text{Ker } D$. Because of (6.24.1), $V_s$ is contained in $D(A)$. Moreover, we obtain from (6.24.2):

$$A \sigma S x = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} x = \begin{pmatrix} S A_{22} x \\ A_{22} x \end{pmatrix} = \begin{pmatrix} 0 \\ (A_{11} S - S A_{22} A_{12}) x + \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix} \in V_s + \text{Im } B$$

for each $x \in X_2$. This means that $A V_s \subseteq V_s + \text{Im } B$.

Applying Lemma 5.4 to the pair $(A_{11}, B_1)$, we find that there exists a mapping $F_1: U \to X_1$ such that $A_{11} + B_1 F_1$ is the generator of a stable semigroup on $X_1$, and moreover $A_{11} + B_1 F_1$ has a discrete spectrum and its eigenvectors form a complete set in $X_1$. It follows from assumptions (A4) and (A5) that there exists $\delta > 0$ such that $\{ \lambda \in \sigma(A) \mid \text{Re } \lambda \geq \omega - \delta \} = \{ \lambda \in \sigma(A) \mid \text{Re } \lambda > \omega - \delta \}$. We can do a spectral decomposition with respect to this subset of $\sigma(A)$. If the resulting finite-dimensional pair $(C_1, A_{11})$ is observable, the procedure of the proof of Prop. 4.13 leads to an injection mapping $G: V \to X$ such that the semigroup generated by $A + GC$ has a growth constant $\leq \omega - \delta$. In this case, we have satisfied all assumptions of Lemma 6.3 and so the existence of a finite-dimensional regulating compensator follows. If the pair $(C_1, A_{11})$ is not observable, we use the remark made after the proof of Lemma 5.3 to obtain the same result.

The class of systems to which the existence theorem applies is determined by several factors. The 'exogenous signal' has to be modeled by a finite-dimensional system, and the 'original system' has to satisfy the conditions discussed in Section 4.4. Moreover, we have the usual stabilizability and detectability conditions, but there is also the set of conditions (6.24) which is related to the interaction of the control function and the variables-to-be-controlled via the system dynamics. Equations of the type (6.24.2) have been studied in the finite-dimensional context in GANTMACHER (1959; p.225). In the infinite-dimensional case involving unbounded operators, KREIN (1971; p.316) gives a solution under the assumption $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$. It should be noted, however, that the equations (6.24) may have a solution even if this assumption does not hold (see the remark following Example III). We shall not go into the difficult questions that arise here. The following result is simple but still useful in many applications.
PROPOSITION 6.4 Consider the system (6.3) under the assumptions (A1-3) and (B1-3). If $A_{22}$ is a diagonalizable matrix, if $\sigma(A_{11}) \cap \sigma(A_{22}) = \emptyset$ and if the matrix

$$M(\lambda) = D_1(\lambda - A_{11})^{-1}B_1: U \to Z \quad (\lambda \in \rho(A))$$

has a right inverse $M^+(\lambda)$ for each $\lambda \in \sigma(A_{22})$, then there is a mapping $S: X_2 \to X_1$ such that (6.24) holds.

PROOF Let $\{x_1, \ldots, x_r\}$ be a basis for $X_2$ consisting of eigenvectors of $A_{22}$, and let $\{\lambda_1, \ldots, \lambda_r\}$ be the corresponding eigenvalues. Define $u_i \in U$ ($i = 1, \ldots, r$) by

$$u_i = M^+(\lambda_i)(D_1(\lambda_i - A_{11})^{-1}A_{12} + D_2)x_i.$$  

Define a linear mapping $S: X_2 \to X_1$ by

$$Sx_i = (\lambda_i - A_{11})^{-1}(A_{12}x_i - B_1u_i) \quad (i = 1, \ldots, r).$$

It is clear that $\text{Im } S \subset D(A_{11})$. For each $i = 1, \ldots, r$ we have

$$A_{11}Sx_i - SA_{22}x_i + A_{12}x_i = (A_{11} - \lambda_i)Sx_i + A_{12}x_i = B_1u_i \subset \text{Im } B_1.$$  

And finally,

$$D_1Sx_i + D_2x_i = D_1(\lambda_i - A_{11})^{-1}A_{12}x_i =$$

$$- D_1(\lambda_i - A_{11})^{-1}B_1M^+(\lambda_i)(D_1(\lambda_i - A_{11})^{-1}A_{12}x_i + D_2x_i) +$$

$$+ D_2x_i = 0 \quad (i = 1, \ldots, r).$$

Thus, the mapping $S$ satisfies all requirements.

The matrix $M(\lambda)$ will never be right invertible (that is, surjective) if $\dim U < \dim Z$, which means that the number of control inputs is smaller than the number of variables-to-be-controlled. On the other hand, $M(\lambda)$ will in general be right invertible if the number of control inputs is larger than or equal to the number of variables-to-be-controlled. This corresponds
very well to intuition, of course.

As will be illustrated in the examples, the mapping $S$ and the associated subspace $V_S$ represent the situation in which the system behaves exactly as we want it to; in this sense, they have the same meaning for the regulation problem as the state space origin has for the stabilization problem. The invertibility properties of the matrix $M(\lambda)$ are related to the amount of steering that is necessary in the 'ideal situation'. See in particular example III for this interpretation of $M(\lambda)$.

In the special case of a constant exogenous signal ($A_{22} = 0 \in \mathbb{R}^{1 \times 1}$), the condition of Prop. 6.4 means that $0 \notin \sigma(A_{11})$ and rank $B_1 A_{11}^{-1} B_1 = \dim Z$. This is the condition given in Pohjolainen (1980; Thm. 3.1).

6.5 The design procedure

Assuming that we have a system that satisfies the conditions (A1-6, 8-9) of Section 4.4 and the conditions B(1-4) of Section 6.2, the following step-wise procedure can be given to find a (low-order) regulating compensator. The first step of the procedure is to find a mapping $S: X_2 \rightarrow X_1$ such that (6.24) holds. If this can be done, then the existence theorem (Thm. 6.2) guarantees that the method will lead to a finite-dimensional compensator after a finite number of iterations.

We shall give the procedure first and then add some comments. Some parts of the procedure are similar to the method we used in Ch. 5 to construct a stabilizing compensator, and the reader is referred to Section 5.4 for the corresponding comments. The procedure gives a basic computational scheme which is certainly amenable to further numerical refinement. In the final three sections of this chapter, we shall illustrate the method by examples.

The design procedure consists of the following steps.

STEP 1 Find a mapping $S: X_2 \rightarrow X_1$ such that (6.24) is satisfied.

STEP 2 Find $F_1$ such that $A_{11} + B_1 F_1$ has a discrete spectrum, the eigenvectors of $A_{11} + B_1 F_1$ are complete, and the semigroup generated by $A_{11} + B_1 F_1$ is stable.

STEP 3 Choose a basis $\{x_1, \ldots, x_r\}$ for $X_2$, and select $u_{\lambda} (i=1, \ldots, r)$ such that $(A_{11} S A_{22} + A_{12}) x_{\lambda} = B_1 u_{\lambda}$. Determine $F_2$ by $F_2 x_\lambda = -u_\lambda - F_1 S x_\lambda (i=1, \ldots, r)$.
STEP 4 Find $G$ such that the growth constant of the semigroup generated by $A + GC$ is somewhat smaller than the desired growth constant $\omega$.

STEP 5 Write $G$ in the form $G = G_1 + G_s$ with $\text{Im } G_1 \subset X_1$ and $\text{Im } G_s \subset V_s$ (as defined in (6.28)), and approximate the vectors in $\text{Im } G_1$ by linear combinations of $k$ selected eigenvectors of $A_{11} + B_1 V_1$. Form the mapping $\hat{G}_1$ which is close to $G_1$ and whose range is spanned by these $k$ eigenvectors, and write $\hat{G} = \hat{G}_1 + G_s$.

STEP 6 See if the semigroup generated by $A - \hat{G}C$ is stable. If not, select a different $F_1$ and/or a different $G$ and start anew, or repeat Step 5 with $k$ replaced by $k + 1$. If the semigroup is stable, go to Step 7.

STEP 7 Construct the compensator of order $k$ as in the proof of Thm. 5.1.

Comments If the conditions of Prop. 6.4 hold, then the problem of Step 1 can be solved by the method used in the proof of that proposition, or by any other suitable method. In any other case, one should consult the literature (GANTMACHER (1959), KREIN (1971)) or look for a direct solution. The second step of the procedure is entirely analogous to Step 1 in the procedure of Section 5.4, and we refer to the remarks that were made there. The purpose of Step 3 is to choose $F_2$ such that the subspace $V_s$ defined in (6.28) becomes $(A + BF)$-invariant. If Prop. 6.4 is used, suitable $u_1$ are given by (6.44). The rest of the procedure is again analogous to the design procedure of Section 5.4, with the decomposition of $G$ as the only exception. If $G$ is given initially as

$$G = \begin{pmatrix} G_{11} & G_{10} \\ G_{20} & G_{22} \end{pmatrix}$$

then the decomposition we want is $G = G_1 + G_s$, with

$$G_1 = \begin{pmatrix} G_{11} - SG_{20} \\ 0 \end{pmatrix}, \quad G_s = \begin{pmatrix} 0 \\ G_{20} \end{pmatrix}.$$

So the approximation has to be done with respect to the vectors in $\text{Im } (G_{11} - SG_{20})$. 
6.6 Example III: A heat regulator

For our first example of regulator design, consider the following set of equations.

\[(6.50.1) \quad \frac{\partial}{\partial t} \phi(x, t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial x^2} \phi(x, t) + b(x) u(t) \quad (t \geq 0, \ 0 \leq x \leq 1)\]

\[(6.50.2) \quad \phi(0, t) = \phi(1, t) = 0\]

\[(6.50.3) \quad r'(t) = 0 \quad (r(t) \in \mathbb{R}, t \geq 0)\]

\[(6.50.4) \quad r(0) = r_0\]

\[(6.50.5) \quad b(x) = \begin{cases} \sqrt{10} & (0.1 \leq x \leq 0.2) \\ 0 & (0 \leq x < 0.1; 0.2 < x \leq 1) \end{cases}\]

\[(6.50.6) \quad y_1(t) = \sqrt{10} \int_{0.8}^{0.9} \phi(x, t) dx \]

\[(6.50.7) \quad y_2(t) = r(t)\]

\[(6.50.8) \quad z(t) = \sqrt{5} \int_{0.4}^{0.6} \phi(x, t) dx - r(t).\]

For instance, these equations can be interpreted in the following way. You are sitting in the middle of a one-dimensional room with windows in both walls. The heater is on one side of the room, and the thermostat is on the other. You want the temperature around your chair to become equal to the reference value \(r_0\) that you have specified, and you want this to happen reasonably fast.

Returning to the abstract framework of the previous sections, we take \(L_2(0, 1) \oplus \mathbb{R}\) as our state space \(X\). The operator \(A_{11}\) is defined on \(X_1 := L_2(0, 1)\) by

\[(6.51.1) \quad D(A_{11}) = \{\phi \in L_2(0, 1) \mid \frac{d^2\phi}{dx^2} \in L_2(0, 1), \ \phi(0) = \phi(1) = 0\}\]

\[(6.51.2) \quad A_{11}\phi = \frac{1}{\pi^2} \frac{d^2\phi}{dx^2} \quad (\phi \in L_2(0, 1)).\]
Furthermore, we have $A_{12} = 0$ and $A_{22} = 0$. The output space $V$ will be identified with $R$, and the output mapping $C$ is defined by

$$
C_{\psi}(\varphi) = \begin{pmatrix}
\sqrt{10} \
0.8
\end{pmatrix} \int_{0}^{1} \psi(x) dx
$$

$$
\begin{pmatrix}
0.9 \\
0.8
\end{pmatrix}
$$

$$
\text{for } \varphi \in L_{2}(0,1), \quad r \in \mathbb{R}.
$$

The input space $U$ is taken equal to $R$, and the definition of the input mapping $B$ can be derived from (6.50.1) in the obvious way. Finally, the mapping $D$ that gives the variable to-be-controlled is clearly given by

$$
D_{\varphi}(\psi) = \sqrt{5} \int_{0}^{1} \psi(x) dx - r
$$

$$
\text{for } \varphi \in L_{2}(0,1), \quad r \in \mathbb{R}.
$$

We said that the reference value should be reached at a reasonably fast rate, so we set the desired growth constant $\nu$ at 2. Now we are ready to enter the design procedure.

**STEP 1** In the present case, $\sigma(A_{22})$ consists only of the point 0 which is not in the spectrum of $A_{11}$. Moreover, the value of the function $M(\lambda)$ (defined in (6.43)) in $\lambda = 0$ is given by

$$
M(0) = -D_{1}^{-1} A_{11}^{-1} B_{1} = 0.1047
$$

and so Prop. 6.4 applies. Consequently, we derive the following mapping $S: \mathbb{R} = X_{1} \to L_{2}(0,1) = X_{2}$ from (6.44) and (6.45):

$$
S: 1 \mapsto -(M(0))^{-1} A_{11}^{-1} B_{1} = \tilde{\psi}.
$$

**STEP 2** Since $A_{11}^{u} = -1$ and $B_{1}^{u}$ is given by

$$
\beta_{1} = \sqrt{10} \int_{0}^{1} \sqrt{2} \sin \pi x dx = 0.2022
$$

a suitable feedback mapping $F_{1}$ that replaces the eigenvalue at -1 by one at -2 and leaves all the other eigenvalues unchanged is given by

$$
F_{1}\varphi = \frac{-1}{\beta_{1}} \sqrt{2} \int_{0}^{1} \varphi(x) \sin \pi x dx
$$
(see Prop. 4.12).

STEP 3 Following the recipe given in the design procedure, \( F_2 : X_2 = R + U = R \) is given by

\[
(6.58) \quad F_2 : 1 \mapsto (M(0))^{-1}(1 + F_1 A_1^{-1} b) = 19.105.
\]

STEP 4 We have to define a mapping \( G : Y = R^2 + X = L_2(0,1) \oplus R \) such that
\( A + GC \) is stable. A basis for the unstable subspace \( X_u \) is given by
the two vectors

\[
(6.59) \quad \begin{pmatrix} \sqrt{2} \sin \pi x, 0 \\ 0, 1 \end{pmatrix}.
\]

With respect to this basis, we have

\[
(6.60) \quad A_u = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad C_u = \begin{pmatrix} \gamma_1 & 0 \\ 0 & 1 \end{pmatrix},
\]

where

\[
(6.61) \quad \gamma_1 = \sqrt{10} \int_0^1 \sqrt{2} \sin \pi x \, dx = 0.2022.
\]

Perhaps the simplest solution is to retain the triangular form by choosing \( G \) as follows:

\[
(6.62) \quad G : \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \mapsto a_1 \begin{pmatrix} -1.5\gamma_1^{-1}\sqrt{2} \sin \pi x \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -2.1 \gamma_1^{-1} \\ 0 \end{pmatrix}.
\]

By Prop. 4.13, this choice of \( G \) will result in a shift of the eigenvalues at
\(-1\) and \(0\) to \(-2.5\) and \(-2.1\), respectively, while all other eigenvalues remain
unchanged. We have selected \( G \) in such a way that its range is spanned by two
vectors, one of which is already in \( \Psi_2 \) (as defined by (6.41)). The other
basis vector is in \( X_1 \) and so it remains to approach this one by linear
combinations of eigenvectors of \( A_{11} + B_1 F_1 \).

STEP 5 As approximating vectors, let us take the eigenvectors of \( A_{11} + B_1 F_1 \)
belonging to the eigenvalues at \(-2\) and \(-4\):

\[
(6.63) \quad \psi_1 = (-2 - A_{11})^{-1} b, \quad \psi_2(x) = \sqrt{2} \sin 2\pi x.
\]
The best approximation (in $L_2(0,1)$-sense) of $\tilde{g}_{11} := -1.5 \sqrt{\frac{1}{2}} \sin \pi x$ can be computed as follows:

\begin{equation}
\tilde{g}_{11} := 32.902 \psi_1 - 5.855 \psi_2.
\end{equation}

So the mapping $\hat{G}$ is now defined by

\begin{equation}
\hat{G} : (a_1 \ a_2) \mapsto a_1 \begin{pmatrix} \tilde{g}_{11} \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ -2.1 \end{pmatrix}.
\end{equation}

STEP 6 If we denote the mapping that takes $a_1$ to $a_1 \tilde{g}_{11}$ by $\hat{G}_{11}$, the operator $A + \hat{G}C$ can be written in matrix format as follows:

\begin{equation}
A + \hat{GC} = \begin{pmatrix} A_{11} + \hat{G}_{11}C_{11} & -2.1 \tilde{\varphi} \\ 0 & -2.1 \end{pmatrix}.
\end{equation}

So we only have to check the stability of $A_{11} + \hat{G}_{11}C_{11}$. It turns out that the first eigenvalue of this operator is at -2.508, so that the stability is indeed obtained.

STEP 7 Let us take the following basis for $\Psi$:

\begin{equation}
\rho_1 = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \rho_2 = \begin{pmatrix} \psi_2 \\ 0 \end{pmatrix}, \rho_3 = \begin{pmatrix} \tilde{\varphi} \\ 1 \end{pmatrix}.
\end{equation}

The matrix of (the restriction of) $A + BF$ with respect to this basis is of course given by

\begin{equation}
A + BF = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\end{equation}

while the $\hat{GC}$-matrix is given by

\begin{equation}
\hat{GC} = \begin{pmatrix} 32.902 & 0 \\ -5.855 & 0 \\ 0 & -2.1 \end{pmatrix} = \begin{pmatrix} C_{11}\psi_1 & C_{11}\psi_2 & C_{11}\tilde{\varphi} \\ 0 & 0 & 1 \\ -2.850 & -11.709 & -6.979 \\ 0.507 & 2.084 & 1.242 \\ 0. & 0. & -2.1 \end{pmatrix}.
\end{equation}

So the dynamic equation of our compensator becomes
\[ w'(t) = \begin{pmatrix} -4.850 & -11.709 & -6.979 \\ 0.507 & -1.916 & 1.242 \\ 0 & 0 & -2.1 \end{pmatrix} w(t) - \begin{pmatrix} 32.902 \\ -5.855 \\ 0 \end{pmatrix} y_1(t) - \begin{pmatrix} 0 \\ 0 \\ -2.1 \end{pmatrix} y_2(t). \]

To find the equation that will give the control function, we have to compute \( Fw_1, Fw_2, \) and \( Fw_3. \) We have \( Fw_1 = F_1 \Phi_1 = 1 \) and \( Fw_2 = F_1 \Phi_2 = 0. \) Finally, from the fact that \( w_3 \) is an eigenvector of \( A + BF \) belonging to the eigenvalue at 0, one easily derives that

\[ Fw_3 = (M(0))^{-1} = 9.553. \]

So, together with the equation (6.70), the compensator is defined by

\[ u(t) = w_1(t) + 9.553 w_3(t). \]

**Remark** The closed-loop system will converge to a situation which is described by the eigenvector corresponding to the eigenvalue at 0. So the asymptotic temperature profile is a multiple of \( \Phi \), determined by the reference value \( r_0. \) Moreover, in the steady-state situation the control function has a constant value \( u(t) = 9.553 r_0. \) This is necessary, of course, to compensate for the loss of heat through the boundary.

Suppose that we would replace the Dirichlet boundary conditions in (6.50.2) by Neumann boundary conditions, which correspond to complete isolation:

\[ \frac{3}{3} \Phi(0,t) = \frac{3}{3} \Phi(1,t) = 0. \]

Then the operator \( A_{11} \) would have an eigenvalue at 0 and we would not be able to use Prop. 6.4. From the interpretation of the function \( \Phi \) given above, it is however easy to guess that in this case \( \Phi \) must be a constant function. Indeed, one can verify immediately from the equations (6.24) that this guess is correct. Moreover, the steady-state value of the control function \( u(t) \) is in this case equal to 0. So one might say that we have a singular case here, but, as it appears to be, not in a troublesome sense.

Some simulation results using the compensator we designed above are given in the Appendix, Fig.A3.

**6.7 Example IV: Protecting a delay system against a constant disturbance**

We now consider the following set of equations:
\[(6.74.1)\quad x_1'(t) = -\frac{\pi}{2} x_1(t-1) + x_2(t) + u(t)\]
\[(6.74.2)\quad x_2'(t) = 0\]
\[(6.74.3)\quad z(t) = y(t) = x_1(t).\]

The basic equation is (6.74.1) in which we have a scalar variable satisfying a retarded equation, influenced both by an unknown but constant disturbance \(x_2(t)\) and by the control function \(u(t)\). Even without the disturbance the open-loop system is unstable (with eigenvalues at \(\pm \frac{\pi}{2}\)) and so the goal of the compensation is both to stabilize the system and to reject the constant disturbance.

We re-write the equations (6.74) in the usual way. As our state space, we take \(X = X_1 \oplus X_2\) with \(X_1 = \text{M}_2(-1,0) = \mathbb{R} \times L_2(-1,0)\) and \(X_2 = \mathbb{R}\). We shall also use this space in its complexified form. The mapping \(A_{11}\) is defined on \(X_1\) by

\[(6.75.1)\quad D(A_{11}) = \{ (\varphi_0,\varphi) \in \text{M}_2(-1,0) \mid \varphi \in H^1[-1,0], \varphi(0) = \varphi_0 \}\]

\[(6.75.2)\quad A_{11}(\varphi_0,\varphi) = (-\frac{\pi}{2}\varphi(-1),\varphi') \quad ((\varphi_0,\varphi) \in D(A_{11}))\]

The mapping \(A_{12}: X_2 \to X_1\) is given by

\[(6.76)\quad A_{12}: 1 \mapsto (1,0),\]

and \(A_{22}: X_2 \to X_2\) is equal to 0.

The input space \(U\), the output space \(Y\) and the space of the variable-to-be-controlled \(Z\) are all taken equal to \(\mathbb{R}\), and the mappings \(B\), \(C\) and \(D\) are defined by

\[(6.77)\quad B: 1 \mapsto \begin{pmatrix} 1 \\ 0 \end{pmatrix}\]

\[(6.78)\quad C\left(\begin{pmatrix} \varphi_0 \\ \alpha \end{pmatrix}\right) = D\left(\begin{pmatrix} \varphi_0 \\ \alpha \end{pmatrix}\right) = \varphi_0.\]

We set the desired growth constant \(\omega\) equal to \(-1\), and we start the design procedure.
STEP 1 Since \( \sigma(A_{22}) = \{0\} \) and \( 0 \not\in \sigma(A_{11}) \), we can use Prop. 6.4. To determine the value of the function \( M(\lambda) \) (defined in (6.43)) in \( \lambda = 0 \), we first have to solve the equation

\[
A_{11}(\phi_0, \phi) = (1, 0) \quad \Rightarrow \quad (\phi_0, \phi) \in D(A_{11}).
\]

One readily finds that the solution is given by \( (\phi_0, \phi) = (-\frac{2}{\pi}, -\frac{2}{\pi}) \). Consequently, we get

\[
M(0) = -D_{A^{-1}}^{1,1} (1,0) = \frac{2}{\pi}.
\]

We can now compute the mapping \( S: X_2 \rightarrow X_1 \) from (6.44) and (6.45), and it turns out that \( S \) is given by

\[
S: 1 \mapsto (0,0).
\]

This is no surprise, of course, because the origin in \( X_1 \) represents the situation we are trying to establish.

STEP 2 We can choose \( F_1 \) to create new eigenvalues at \( -1 \pm \frac{\pi}{2}i \), in a similar fashion as in Example II (Section 5.6):

\[
F_1(\phi_0, \phi) = -2\phi_0 - \int_{-1}^{0} (\cos \frac{\pi}{2} \theta + \sin \frac{\pi}{2} \theta) \phi(\theta) \, d\theta.
\]

STEP 3 The mapping \( F_2: X_2 = \mathbb{R} \mapsto \mathbb{R} = \mathbb{R} \) is immediately found as

\[
F_2: 1 \mapsto -1.
\]

STEP 4 A basis for the unstable subspace \( X_u \) is given by

\[
\phi_1 = \begin{pmatrix} (1, \cos \frac{\pi}{2}) \\ 0 \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} (0, \sin \frac{\pi}{2}) \\ 0 \end{pmatrix}, \quad \phi_3 = \begin{pmatrix} (1, 1) \end{pmatrix}.
\]

With respect to this basis, the matrices of \( A_u \) and \( B_u \) have the following form:
(6.85) \[ A_u = \left( \begin{array}{ccc} 0 & \frac{\pi}{2} & 0 \\ -\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \quad C_u = (1 \quad 0 \quad 1). \]

It is now easy to calculate a suitable injection mapping \( G \). If we take

(6.86) \[ G: \; 1 \mapsto -\bar{w} \psi_1 - 2\bar{w} \psi_2 - \bar{w} \psi_3 \]

then the eigenvalues at 0 and \( \pm \frac{\pi}{2} \) of \( A \) are shifted to eigenvalues at

\(-\frac{\pi}{2}\) (double) and \(-\pi\) of \( A + GC \), whereas the other eigenvalues of \( A \) remain unchanged.

**STEP 5** Because \( S \) is the zero mapping, the required decomposition of \( G \) is given by \( G = G_1 + G_2 \), with

(6.87) \[ G_1: \; 1 \mapsto \left( \begin{array}{c} -2\pi, -\pi \cos \frac{\pi}{2} \cos \frac{\pi}{2} - 2\pi \sin \frac{\pi}{2} - \pi \\ 0 \end{array} \right), \quad G_2: \; 1 \mapsto \left( \frac{1}{2}, \bar{z} \right). \]

So we have to approximate the function \( g_1 : (-2\pi, -\pi \cos \frac{\pi}{2} \cos \frac{\pi}{2} - 2\pi \sin \frac{\pi}{2} - \pi) \notin M_2(-1, 0) \) by linear combinations of eigenvectors of \( A_1 + B_1 F_1 \). A basis for the eigenspace of \( A_1 + B_1 F_1 \), corresponding to the eigenvalues at \(-1 \pm \frac{\pi}{2}\) is given by (cf. (5.58-60)):

(6.88) \[ \psi_1 = (1, e^{-\theta} \cos \frac{\pi}{2} \cos \frac{\pi}{2}), \quad \psi_2 = (0, e^{-\theta} \sin \frac{\pi}{2}). \]

Upon computing, we find that the best approximation (in \( L_2(-1, 0) \)-sense) of \( g_1 \) in terms of \( \psi_1 \) and \( \psi_2 \) is given by

(6.89) \[ \hat{g}_1 = -3.66 \psi_1 - 1.990 \psi_2. \]

So we arrive at

(6.90) \[ G: \; 1 \mapsto \left( \begin{array}{cc} -3.66, -3.66 e^{-\theta} \cos \frac{\pi}{2} \cos \frac{\pi}{2} - 1.990 e^{-\theta} \sin \frac{\pi}{2} \\ -4.935 \end{array} \right). \]

**STEP 6** Calculation shows that the rightmost eigenvalues of \( A + \hat{G} C \) are at

\(-1.136 \pm 1.445i\), resulting, of course, from a split-up of the double eigenvalue of \( A + GC \) at \(-1.571\). In view of the desired growth
constant $\omega = -1$, this result is satisfactory.

STEP 7 As a basis for $\dot{\mathbf{w}}$, let us take

$$w_1 = \begin{pmatrix} 1, e^{-\theta} \cos \frac{\pi \theta}{2} \\ 0 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0, e^{-\theta} \sin \frac{\pi \theta}{2} \\ 0 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 0, 0 \\ 1 \end{pmatrix}. \tag{6.91}$$

With respect to this basis, we have

$$A + BF + \mathbf{C}C = \begin{pmatrix} -1 & \frac{\pi}{2} & 0 \\ -\frac{\pi}{2} & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -3.666 \\ -1.990 \\ -4.935 \end{pmatrix} (1 \ 0 \ 0) = \begin{pmatrix} -4.666 & 1.571 & 0. \\ -3.561 & -1. & 0. \\ -4.935 & 0. & 0. \end{pmatrix} \tag{6.92}$$

We also have to compute $Fw_1$, $Fw_2$, and $Fw_3$. In the same way as in Example II, we find

$$Fw_1 = -1 \tag{6.93.1}$$

$$Fw_2 = \frac{\pi}{2} - \frac{\pi e}{2} = -2.699. \tag{6.93.2}$$

From the fact that $w_3$ is an eigenvector of $A + BF$ corresponding to the eigenvalue at 0, we also easily find

$$Fw_3 = -1. \tag{6.93.3}$$

So the final compensator equations are:

$$w'(t) = \begin{pmatrix} -4.666 & 1.571 & 0. \\ -3.561 & -1. & 0. \\ -4.935 & 0. & 0. \end{pmatrix} w(t) + \begin{pmatrix} 3.666 \\ 1.990 \\ 4.935 \end{pmatrix} y(t) \tag{6.94.1}$$

$$u(t) = -w_1(t) - 2.699w_2(t) - w_3(t). \tag{6.94.2}$$

Some simulation results of the closed-loop system are given in the Appendix, Fig.A4.
6.8 Example III: The moving hot spot

Our final example is inspired by a problem in nuclear reactor design, as explained in Owens (1980). We do not use the stronger word "motivated", because the diffusion equation we shall use bears only a very faint relation to the 'real' equation derived by Owens. Nevertheless, one should note that this latter equation is basically of the diffusion type.

The problem can roughly be described as follows. The distribution of a certain variable which is associated with the intensity of the reaction is required to move periodically about an equilibrium profile, as is indicated in the figure:

![Diagram of a one-dimensional reactor](image)

Fig. 6.1. A one-dimensional reactor.

In this way, a "hot spot" moves from left to right in the one-dimensional reactor. We shall try to simulate this type of behaviour by introducing two 'variables-to-be-controlled'; one depends on the values of the state variable at the left side of the reactor, and the other depends on the right side. Both variables should follow the same sine wave, but with a phase difference of 180°. More concretely, we consider the following set of equations.

\[(6.95.1)\]  \[\frac{\partial}{\partial t} \phi(x,t) = \frac{1}{\pi^2} \frac{\partial^2}{\partial x^2} \phi(x,t) + b_1(x)u_1(t) + b_2(x)u_2(t)\]

\[(6.95.2)\]  \[\phi(0,t) = \phi(1,t) = 0\]

\[(6.95.3)\]  \[\frac{d}{dt} \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}\]
\( (6.95.4) \quad y(t) = \sqrt{2} \int_{0}^{\frac{1}{2}} \phi(x,t)dx + r_1(t) \)

\( (6.95.5) \quad z_1(t) = \sqrt{2} \int_{0}^{\frac{1}{2}} \phi(x,t)dx + r_1(t) \)

\( (6.95.6) \quad z_2(t) = \sqrt{2} \int_{\frac{1}{4}}^{1} \phi(x,t)dx - r_1(t) \).

The input functions \( b_1(x) \) and \( b_2(x) \) are determined by

\( (6.96.1) \quad b_1(x) = \sqrt{10} \quad 0.2 \leq x \leq 0.3 \\
0 \quad 0 \leq x < 0.2, \ 0.3 < x \leq 1 \)

\( (6.92.2) \quad b_2(x) = \sqrt{10} \quad 0.7 \leq x \leq 0.8 \\
0 \quad 0 \leq x < 0.7, \ 0.8 < x \leq 1. \)

As our state space, we take \( X = X_1 \oplus X_2 = L_2(0,1) \oplus \mathbb{R}^2 \). The input space \( \mathcal{U} \) will be taken equal to \( \mathbb{R}^2 \), the output space \( \mathcal{Y} \) will be \( \mathbb{R} \), and the space of the variables to be controlled is \( \mathbb{R} \). The operator \( A_{11} \) is defined on \( X_1 \) by

\( (6.97.1) \quad D(A_{11}) = \{ \phi \in L_2(0,1) \mid \frac{d^2\phi}{dx^2} \in L_2(0,1), \ \phi(0) = \phi(1) = 0 \} \)

\( (6.97.2) \quad A_{11}\phi = \frac{1}{\pi} \frac{d^2\phi}{dx^2} . \)

The mapping \( A_{12} \) is 0, and

\( (6.98) \quad A_{22} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \)

The mappings \( B, C \) and \( D \) are given by

\( (6.99) \quad B(a_1^{a_2^b}) = \begin{pmatrix} a_1b_1 + a_2b_2 \\ 0 \end{pmatrix} \)

\( (6.100) \quad C^{(t)} = \sqrt{2} \int_{0}^{\frac{1}{2}} \phi(x,t)dx + r_1 \)
We set the desired growth constant \( \omega \) at \(-1.5\), and we follow the design procedure.

**STEP 1** We first have to find a mapping \( S: \chi_2 = \mathbb{R}^2 \to \chi_1 = L_2(0,1) \) that satisfies (6.24). We could use Prop. 6.4 for this, but we prefer to follow a somewhat different route. Write

\[
(6.102) \quad S(1) = \overset{\text{1}}{\Phi}, \quad S(0) = \overset{\text{0}}{\Phi}.
\]

Then (6.24) will be satisfied if \( \overset{\text{1}}{\Phi} \in D(A_{11}), \overset{\text{0}}{\Phi} \in D(A_{11}), \) and

\[
(6.103.1) \quad A_{11}\overset{\text{1}}{\Phi} + \overset{\text{0}}{\Phi} = a_{11}b_1 + a_{12}b_2
\]

\[
(6.103.2) \quad A_{11}\overset{\text{0}}{\Phi} - \overset{\text{1}}{\Phi} = a_{21}b_1 + a_{22}b_2
\]

for suitable \( a_{11}, a_{12}, a_{21} \) and \( a_{22} \), and moreover

\[
(6.104.1) \quad \sqrt{2} \int_0^1 \overset{\text{0}}{\Phi}(x)dx = -1
\]

\[
(6.104.2) \quad \sqrt{2} \int_0^1 \overset{\text{1}}{\Phi}(x)dx = 1
\]

\[
(6.104.3) \quad \sqrt{2} \int_0^1 \overset{\text{1}}{\Phi}(x)dx = 0
\]

\[
(6.104.4) \quad \sqrt{2} \int_0^1 \overset{\text{0}}{\Phi}(x)dx = 0.
\]

It is verified by direct computation that

\[
(6.105) \quad \begin{pmatrix} A_{11} & I \\ -I & A_{11} \end{pmatrix}^{-1} = \begin{pmatrix} (A_{11})^{-1} & -(A_{11}^2 + I)^{-1} \\ (A_{11}^2 + I)^{-1} & (A_{11}^2 + I)^{-1} \end{pmatrix}.
\]

This gives the following solution for (6.103):
(6.106.1) \[ \tilde{\varphi}_1 = A_{11}^2 (A_{11}^2 + I)^{-1} (a_{11} b_1 + a_{12} b_2) - (A_{11}^2 + I)^{-1} (a_{21} b_1 + a_{22} b_2) \]

(6.106.2) \[ \tilde{\varphi}_2 = (A_{11}^2 + I)^{-1} (a_{11} b_1 + a_{12} b_2) + A_{11} (A_{11}^2 + I)^{-1} (a_{21} b_1 + a_{22} b_2). \]

Combining this with (6.104) gives rise to four linear equations in the four unknowns \( a_{11}, a_{12}, a_{21}, \) and \( a_{22}, \) which can readily be solved. The numerical results are as follows.

(6.107) \[
\begin{align*}
a_{11} &= 7.337 \\
a_{12} &= -7.337 \\
a_{21} &= 1.884 \\
a_{22} &= -1.884.
\end{align*}
\]

The mapping \( S \) is now specified by (6.102), (6.106) and (6.107).

**STEP 2** The mappings \( A_{11}^u \) and \( B_1^u \) are given by

(6.108) \[ A_{11}^u = -I, \quad B_1^u = (0.3149 \quad 0.3149). \]

It follows that a feedback mapping \( F_1 \) that shifts the eigenvalue of \( A_{11} \) at \(-1\) to \(-1.5\), while leaving all the other eigenvalues unchanged, is given by

(6.109) \[ F_1 \varphi = \left( \frac{f_1}{f_2} \right) \sqrt{2} \int_0^1 \varphi(x) \sin \pi x \, dx \]

where \( f_1 \) and \( f_2 \) are such that

(6.110) \[ 0.3149(f_1 + f_2) = -0.5. \]

There is one degree of freedom left here, and we shall use this later on.

**STEP 3** It turns out that both \( \tilde{\varphi}_1 \) and \( \tilde{\varphi}_2 \) are perpendicular to the eigenvector \( \varphi_1 = \sqrt{2} \sin \pi x \) corresponding to the eigenvalue \(-1\) of \( A_{11} \). So we find

(6.111.1) \[ F_2^1 \tilde{\varphi}_1 = -\frac{a_{11}}{a_{12}} \tilde{\varphi}_1 = \begin{pmatrix} -7.337 \\ 7.337 \end{pmatrix} \]
(6.111.2) \( F_2 (0) = -\binom{a_{21}}{a_{22}} \)  
\( \tilde{F}_1 \tilde{\varphi}_2 = \begin{pmatrix} -1.884 \\ -1.884 \end{pmatrix} \).

STEP 4 A basis for the unstable subspace of A is given by

\[
\chi_1 = \begin{pmatrix} \sqrt{2} \sin \pi x \\ 0 \\ 0 \end{pmatrix}, \quad \chi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \chi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}
\]

With respect to this basis, we have

\[
A_u = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad C_u = \begin{pmatrix} \frac{2}{\pi} & 1 & 0 \end{pmatrix}.
\]

After some computations, we find that an injection mapping \( G \) that replaces the eigenvalues at \( \pm i \) and \(-1\) by new eigenvalues at \(-1.8\) (double) and \(-2.0\) is given by

\[
G: 1 \mapsto -0.16 \pi \chi_1 - 4.28 \chi_2 - 5.16 \chi_3 = g.
\]

STEP 5 If we define

\[
g_1 = -0.16 \pi \varphi_1 + 4.28 \tilde{\varphi}_1 + 5.16 \tilde{\varphi}_2,
\]

Then it is clear that \( g \) (as defined in (6.114)) is equal to \( g_1 + g_2 \) with \( g_1 \in \mathcal{X}_X \) and \( g_2 \in \mathcal{V}_S \) (defined by (6.28) and (6.102)). So we have to approximate the function \( g_1 \in L_2(0,1) \) by linear combinations of eigenvectors of \( A_{11} + B_1 F_1 \). These eigenvectors are the same as the eigenvectors of \( A_{11} \) except for the one corresponding to the new eigenvalue at \(-1.5\), which is given by

\[
\psi_1 := f_1 (\alpha - A_{11}^{-1})^{-1} b_1 + f_2 (\alpha - A_{11}^{-1})^{-1} b_2 = f_1 \psi_1^1 + f_2 \psi_2^1,
\]

(normalized such that \( \langle \psi_1, \varphi_1 \rangle = 1 \)). Because \( f_1 \) and \( f_2 \) can be chosen freely subject to the condition (6.110), the vectors \( \psi_1^1 \) and \( \psi_2^1 \) can be used independently in the approximation process. If we denote \( \psi_2 := \sqrt{2} \sin 2 \pi x \), then the best approximation of \( g_1 \) by a linear combination of \( \psi_1^1, \psi_1^2 \) and \( \psi_2 \) (in the sense of \( L_2(0,1) \)) is given by
(6.117) \[ \mathbf{\hat{B}}_1 = -39.100 \mathbf{\psi}_1 + 38.305 \mathbf{\psi}_2^2 + 6.645 \mathbf{\psi}_3. \]

So we obtain

(6.118) \[ \hat{\mathbf{C}} : 1 \mapsto \begin{pmatrix} \mathbf{\hat{B}}_1 \\ 0 \\ 0 \\ 0 \\ -4.28 \mathbf{\psi}_1 \\ 1 \\ -5.16 \mathbf{\bar{\psi}}_2 \\ 1 \end{pmatrix}. \]

**STEP 6** To verify the stability of the semigroup generated by \( A + \hat{\mathbf{G}} \mathbf{C} \), we find the first eigenvalues of this operator. It turns out that the rightmost eigenvalues are at \(-1.744 \pm 0.108i\), so we have a satisfactory situation.

**STEP 7** It follows from (6.110) and (6.117) that we should take

(6.119) \[ \mathbf{\psi}_1 = -78.091 \mathbf{\psi}_1 + 76.503 \mathbf{\psi}_2. \]

We then have

(6.120) \[ \mathbf{\hat{B}}_1 = 0.501 \mathbf{\psi}_1 + 6.649 \mathbf{\psi}_2. \]

We obtain a fourth-order compensator. As a basis for \( \tilde{\mathbf{W}} \), let us take the following vectors:

(6.121) \[ \mathbf{w}_1 = \begin{pmatrix} \mathbf{\psi}_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} \mathbf{\psi}_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} \mathbf{\bar{\psi}}_1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} \mathbf{\bar{\psi}}_2 \\ 0 \\ 1 \end{pmatrix}. \]

With respect to this basis, we finally get the following compensator equations.

(6.122.1) \[ \mathbf{w}'(t) = \begin{pmatrix} -9.809 & -0.319 & 0 & 0.01 \\ -10.267 & 0.233 & 0 & 0 \\ 70.980 & -2.725 & 0 & 1 \\ 85.574 & -3.285 & -1 & 0.1 \end{pmatrix} \mathbf{w}(t) - \begin{pmatrix} 0.501 \\ 6.649 \\ -4.28 \\ -5.16 \end{pmatrix} y(t) \]

(6.122.2) \[ u_1(t) = -78.091 \mathbf{w}_1(t) + 7.337 \mathbf{w}_3(t) - 1.884 \mathbf{w}_4(t) \]

(6.122.3) \[ u_2(t) = 76.503 \mathbf{w}_1(t) + 7.337 \mathbf{w}_3(t) + 1.884 \mathbf{w}_4(t). \]

The action of the compensator is illustrated in the Appendix, Fig.A5.
REFERENCES

S. AGMON (1965), Lectures on elliptic boundary value problems, Van Nostrand, Princeton.
S. BENNETT (1979), A history of control engineering, 1800-1930, IEE Control Engineering Series 8, Peter Peregrinus, Stevenage.
F. M. BRASCH & J. B. PEARSON (1970), Pole placement using dynamic compensators,


et al. (1981), Lecture notes on infinite dimensional systems, to appear.


N.DUNFORD & J.T.SCHWARTZ (1963), Linear operators (Part II), Wiley, New York.


D.H. OWENS (1980), Spatial kinetics in nuclear reactor systems, in Modelling of dynamical systems, Peter Peregrinus, Stevenage.


S. POHJOLAINEN (1980), Robust multivariable controller for distributed parameter systems, Tampere University of Technology Publications, 9.


J.M. SCHMUGHEL (1979), (C,A)-invariant subspaces: some facts and uses, report nr. 110, Wisk. Sem. VU.


(1980c), A direct approach to compensator design for distributed parameter systems, report nr.148, Wisk. Sem. VU.

(1980d), Regulator synthesis using (C,A,B)-pairs, report nr. 149, Wisk. Sem. VU.


(1979), Linear multivariable control: a geometric approach (2nd ed.), Springer-Verlag, New York.

(1972), Feedback invariants of linear multivariable systems, Automatica, Vol. 8, pp. 93-100.


APPENDIX: SIMULATION RESULTS

Fig. A1. Behaviour of the diffusion process (5.33) under the control of the compensator (5.46). The heating/cooling device is situated at the left side of the interval, the sensor at the right side. The initial condition is given by $\phi_0(x) = \cos \pi x$. In the figure are indicated the lines of equal temperature (level lines) for $t=0$ to $t=2.5$. 
Fig.A2. Behaviour of the delay system (5.47) under the control of the compensator (5.82). The initial condition is given by \( x_1(0) = 1(1 \leq b < 0), \) \( x_2(0) = 0. \) Fig.A2a shows \( x_1(t); \) the uncontrolled solution of (5.4.7) is dotted. Fig.A2b gives \( u(t), \) Fig.A2c gives \( x_2(t). \)
Fig. A3. Behaviour of the diffusion system (6.50) under the control of the compensator (6.70-72). The goal of the control is to bring the temperature in the middle of the internal to a specified value. The initial value is 0. The position of the heater is marked by dotted lines.

Fig. A4a

Fig. A4b

Fig. A4. Behaviour of the delay system (6.74) under the control of the compensator (6.94). The initial value is given by $x_1(0) = 0$ ($-1 \leq \delta \leq 0$). A constant disturbance is applied at $t=0$. Fig. A4a gives the behaviour of $x_1(t)$; the uncontrolled solution is dotted. Fig. A4b gives the control function $u(t)$. 
Fig. 4.5. Behaviour of the diffusion system (6.95) under the control of the compensator (6.122). The goal of the control is to bring about a periodic pattern of hot spots appearing left and right in the interval. The initial condition is 0. Each symbol marks a point having a certain temperature (see list.).
AUTHOR INDEX

R.A. Adams 104
S.Agmon 99
H.Akashi 61,62,77,78
M.A.Arbib 10,12
V.I.Arnold 7
M.J.Balas 107
G.Basile 2,11,15,16,34,61,78
H.Bateman 1
A.Belleni-Morante 82
R.Bellman 105
S.Bennett 1
J.Bertram 12
M.K.P.Bhat 106,123,131
J.G.Borisovic 104
F.M.Brasch 47,48

F.M.Callier 81
C.Commault 61,77,78,79
K.Cooke 105
R.F.Curtain 81,82,83,94,98,100,106,107
E.J.Davison 49,60,131
M.C.Delfour 99,101,104,114,127
G.A.Desoer 81
N.Dunford 114
P.L.Falb 10,12
H.O.Fattorini 81
W.Feller 81
B.A.Francis 60,77
N.Fuji 107
P.R.Gantmacher 11,142,145
A.Goldenberg 60
R.V.Gressang 106
J.Hadamard 81
J.Hale 92,93,98,123
A.van Harten 114
M.L.J.Hautus 16,22,23,28,123
E.Hille 80,81,83,92,101,121
M.W.Hirsch 7
A.Hurwitz 1
H.Imai 61,62,77,78
P.H.M.Janssen 75,77
C.D.Johnson 60
R.E.Kalman 2,10,12
T.Kato 93,98,100,116,117
H.Kimura 49
S.G.Krein 142,145
H.Kwakernaak 41,45
G.B.Lamont 107
D.C.Lay 85,88,89,91,92,94,137
D.G.Luenberger 46,47
A.Manitius 99,101,114
G.Marro 2,11,15,16,34,61,78
J.C.Maxwell 1,4
S.K.Mitter 104,127
I.Miyadera 81
S.Mizohata 99,100
A.S.Morse 2,11,16,20,23,47
D.H.Owens 155
A.Pazy 92
J.B.Pearson 47,48,60
K.S.Phillips 80,81,83,92,101,121
S.Pohjolainen 98,111,131,144
V.M.Popov 12
A.J.Pritchard 81,82,83,100,106
J.Rissanen 12
H.H.Rosenbrock 13
E.J.Routh 1
E.J.P.G.Schmidt 89
J.T.Schwartz 114
R.Sivan 41,45
M.Slemrod 92
S.Smale 7
Gy.Sonnevend 75
X.J.Stern 89
A.E.Taylor 85,88,89,91,92,94,137
?.Trèves 99
S.Triggiani 92,93,94
A.S.Turbabin 104
R.B.Vinter 101
S.H.Wang 49
J.C.Willems 60,61,77,78,79
A.M.Wonham 2,3,11,12,13,15,16,20,23,26,40,46,47,48,49,55,60,61,75,76,77,131
K.Yosida 81
P.C.Young 60
J.Zabczyk 92
boundary control 98
Brasch-Pearson compensator 47

(C,A,B)-pair 34,74
compensator 30,108,133
compensator couple 2,3,34,51,61,63,74
completeness of eigenvectors 97,99,103,114,116
conditioned invariant subspace 16
controllability 8,9,10,13,56
controllability index 47,50
controllability subspace 22,49
controlled invariant subspace 15
delay systems 103,122,150
detectability 16,28,91,95,123
detectability subspace 23,46
diffusion systems 118,146,155
direct sum 84,85
discrete spectrum 97,98,102,103,104,114,116
disturbance decoupling (localization) 3,55,57,61,77,78
duality 2,10,35
dynamic feedback 1,29,30
exponential stability 91
extended state space 32
extended system matrix 32
extended system operator 108
feedback-adapted observer (FAO) 37
feedback mapping 12
feedback-observer interpretation 29,38,40
full-order compensator 45
generator see infinitesimal generator
geometric approach 2,11
growth constant 83,86,99
hyperbolic systems 98,99
infinitesimal generator 81
injection mapping 14
inner-detectability 23,61
inner-stabilizability 22,24,25,61
integral control 1,31,110
internal stability 61,76,77
maximal stability 76
minimal compensator problem 34
multiplicity 12,93,98
multiplicity function 12
mutually adapted (MA) 38
neutral systems 98,99
nuclear reactors 4,155
observability 9,10,11
observability index 48,50
observer-adapted feedback (OAF) 37
order (of a compensator) 32,108,134
outer-detectability 22,25,26,61
output stability 75
parabolic systems (see also diffusion systems) 98,99,102
partitioning of the complex plane 7,59,61,97
PET-compensator 31,46,47,48,49
pole placement 6,12,15,17,18,21,22,23
proportional control 31,110
quotient mapping 88
reduced-order compensator 45,46,78
reduced-order modeling 107
regulation 1,3,54,57,60,131
regulating compensator 134,135,138
regulator problem 2,60,67,71
relaxed regulator problem 74
retarded systems (see also delay systems) 98,99
separating subspace 39,53,61
similarity transformation 83,90
spectral decomposition 92,123
spectral multiplicity function 12
spectral decomposition assumption 93
spectrum determined growth assumption 92,97,99,117,121,127
spillover 107,120
stability 1,2,91
stabilizability 14,22,28,91,96,123
stabilizability subspace 22,47,62
stabilizing compensator 33,35,51,109,111
state space 2,7,91
static (output) feedback (see also proportional control) 31,45,49,73
steam engines 4
symmetric multiplicity function 13
total multiplicity 13
tracking 3,54,57,60,131
variables-to-be-controlled 55,131
Weinstein-Aronszajn method 117,121
SYMBOL INDEX

The meaning of the symbols appearing in the first column is briefly described in the second column, while the third column refers to pages where the symbol is defined or used in a typical way. We first follow the Latin alphabet, then the Greek alphabet, and we close with composite notations.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>system matrix/operator</td>
<td>8,107</td>
</tr>
<tr>
<td>( A_e )</td>
<td>extended system matrix/operator</td>
<td>32,108</td>
</tr>
<tr>
<td>( A_u, A_s, \ldots )</td>
<td>restriction of A to a modal subspace corresponding with ( \xi_u, \xi_s, \ldots )</td>
<td>93</td>
</tr>
<tr>
<td>( A^w_s )</td>
<td>restriction of A to a modal subspace corresponding with ( \xi_s = { \lambda \in \mathbb{C} \mid \Re \lambda \leq w } )</td>
<td>93</td>
</tr>
<tr>
<td>B</td>
<td>control input mapping</td>
<td>8,107</td>
</tr>
<tr>
<td>( B_u, B_s, \ldots )</td>
<td>obtained from B by restricting its range to a modal subspace</td>
<td>93</td>
</tr>
<tr>
<td>C</td>
<td>output mapping (observation)</td>
<td>8,107</td>
</tr>
<tr>
<td>( C_u, C_s, \ldots )</td>
<td>obtained from C by restricting its domain to a modal subspace</td>
<td>93</td>
</tr>
<tr>
<td>( \xi_b, \xi )</td>
<td>parts of the complex plane defining a bipartition</td>
<td>7</td>
</tr>
<tr>
<td>( \xi_s, \xi_u, \xi_F, \xi_e )</td>
<td>parts of the complex plane defining a tripartition</td>
<td>59,60</td>
</tr>
<tr>
<td>( C([-1,0]; \mathbb{R}^n) )</td>
<td>space of continuous ( \mathbb{R}^n )-valued functions on ([-1,0])</td>
<td>104</td>
</tr>
<tr>
<td>D</td>
<td>output mapping (variables-to-be-controlled)</td>
<td>55,132</td>
</tr>
<tr>
<td>D(A)</td>
<td>domain of A</td>
<td>81</td>
</tr>
<tr>
<td>E</td>
<td>disturbance input mapping</td>
<td>55</td>
</tr>
<tr>
<td>F</td>
<td>feedback mapping (from X to U)</td>
<td>12,91</td>
</tr>
<tr>
<td>( F(V) )</td>
<td>set of all F such that V is ((A+B)V)-invariant</td>
<td>17</td>
</tr>
<tr>
<td>G</td>
<td>injection mapping (from V to X)</td>
<td>14,91</td>
</tr>
<tr>
<td>( G(T) )</td>
<td>set of all G such that T is ((A+G))-invariant</td>
<td>17</td>
</tr>
<tr>
<td>( \tilde{G} )</td>
<td>perturbation of G</td>
<td>112,116</td>
</tr>
<tr>
<td>( H^1([-1,0]; \mathbb{R}^n) )</td>
<td>Sobolev space of functions whose first derivative is in ( L_2([-1,0]; \mathbb{R}^n) )</td>
<td>104</td>
</tr>
<tr>
<td>I</td>
<td>identity mapping</td>
<td>81</td>
</tr>
<tr>
<td>( \text{Im} ; T )</td>
<td>image of the mapping T</td>
<td>10</td>
</tr>
<tr>
<td>K</td>
<td>feedthrough mapping</td>
<td>31,110</td>
</tr>
<tr>
<td>k</td>
<td>order of the compensator</td>
<td>32</td>
</tr>
<tr>
<td>Ker T</td>
<td>kernel of the mapping T</td>
<td>11</td>
</tr>
<tr>
<td>( L, M, N )</td>
<td>mappings appearing in the compensator equations</td>
<td>30</td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>$L_2$</td>
<td>space of square integrable functions</td>
<td></td>
</tr>
<tr>
<td>$M_2((-1,0);\mathbb{R}^n)$</td>
<td>product space $\mathbb{R}^n \times L_2((-1,0);\mathbb{R}^n)$</td>
<td></td>
</tr>
<tr>
<td>$M(\lambda)$</td>
<td>transmission matrix</td>
<td></td>
</tr>
<tr>
<td>$M^*(\lambda)$</td>
<td>right inverse of $M(\lambda)$</td>
<td></td>
</tr>
<tr>
<td>$m$</td>
<td>dimension of the input space $U$</td>
<td></td>
</tr>
<tr>
<td>$n$</td>
<td>dimension of the state space $X$</td>
<td></td>
</tr>
<tr>
<td>$0$</td>
<td>subspace consisting of the zero element only</td>
<td></td>
</tr>
<tr>
<td>$P$</td>
<td>projection of the extended state space onto the original state space</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>dimension of the output space $Y$</td>
<td></td>
</tr>
<tr>
<td>$Q$</td>
<td>embedding of the original state space into the extended state space</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>surjective mapping used in compensator construction</td>
<td></td>
</tr>
<tr>
<td>$R^*$</td>
<td>right inverse of $R$</td>
<td></td>
</tr>
<tr>
<td>$S$</td>
<td>(in Ch.6) mapping used to describe $V_s$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>conditioned invariant subspace</td>
<td></td>
</tr>
<tr>
<td>$T^*(E)$</td>
<td>smallest conditioned invariant subspace containing a given subspace $E$</td>
<td></td>
</tr>
<tr>
<td>$T^*_g(E)$</td>
<td>smallest detectability subspace (w.r.t. $\mathcal{E}_g$) containing a given subspace $E$</td>
<td></td>
</tr>
<tr>
<td>$t$</td>
<td>time variable</td>
<td></td>
</tr>
<tr>
<td>$T(t)$</td>
<td>semigroup generated by $A$</td>
<td></td>
</tr>
<tr>
<td>$T_e(t)$</td>
<td>semigroup generated by $A_e$</td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>input space (domain of $B$)</td>
<td></td>
</tr>
<tr>
<td>$u(t)$</td>
<td>control function</td>
<td></td>
</tr>
<tr>
<td>$V$</td>
<td>(often) controlled invariant subspace</td>
<td></td>
</tr>
<tr>
<td>$V^*_g(K)$</td>
<td>largest controlled invariant subspace contained in a given subspace $K$</td>
<td></td>
</tr>
<tr>
<td>$V^*_g(K)$</td>
<td>largest stabilizability subspace (w.r.t. $\mathcal{E}_g$) contained in a given subspace $K$</td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>state space for the compensator dynamics</td>
<td></td>
</tr>
<tr>
<td>$w(t)$</td>
<td>state of the compensator</td>
<td></td>
</tr>
<tr>
<td>$X$</td>
<td>state space</td>
<td></td>
</tr>
<tr>
<td>$x(t)$</td>
<td>state</td>
<td></td>
</tr>
<tr>
<td>$X^e$</td>
<td>extended state space ($= X \oplus W$)</td>
<td></td>
</tr>
<tr>
<td>$X_g(A), \ldots$</td>
<td>modal subspace of $A$ corresponding to $\mathcal{E}_g, \ldots$</td>
<td></td>
</tr>
<tr>
<td>Symbol</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------</td>
<td></td>
</tr>
<tr>
<td>$X^e(A_e)$</td>
<td>modal subspace of $A_e$ corresponding to $E_e$</td>
<td></td>
</tr>
<tr>
<td>$X^d_{det}$</td>
<td>smallest detectability subspace</td>
<td></td>
</tr>
<tr>
<td>$X^d_{det}$</td>
<td>id. with respect to $E_g$</td>
<td></td>
</tr>
<tr>
<td>$X_{stab}$</td>
<td>largest stabilizability subspace</td>
<td></td>
</tr>
<tr>
<td>$X_{stab}$</td>
<td>id. with respect to $E_g$</td>
<td></td>
</tr>
<tr>
<td>$X_1 \otimes X_2$</td>
<td>(in Ch.6) decomposition of $X$ in which $X_2$ represents the 'signal' state space</td>
<td></td>
</tr>
<tr>
<td>$X/X_2$</td>
<td>quotient space</td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>output space (range of $C$)</td>
<td></td>
</tr>
<tr>
<td>$y(t)$</td>
<td>observation</td>
<td></td>
</tr>
<tr>
<td>$Z$</td>
<td>space of variables-to-be-controlled (range of $D$)</td>
<td></td>
</tr>
<tr>
<td>$z(t)$</td>
<td>variables-to-be-controlled</td>
<td></td>
</tr>
<tr>
<td>$\Lambda_A(\lambda)$</td>
<td>characteristic function of $A$</td>
<td></td>
</tr>
<tr>
<td>$\Theta$</td>
<td>delay time variable</td>
<td></td>
</tr>
<tr>
<td>$\kappa$</td>
<td>controllability index</td>
<td></td>
</tr>
<tr>
<td>$\kappa_0$</td>
<td>observability index</td>
<td></td>
</tr>
<tr>
<td>$\rho(A)$</td>
<td>resolvent set of $A$</td>
<td></td>
</tr>
<tr>
<td>$\sigma(A)$</td>
<td>spectrum of $A$</td>
<td></td>
</tr>
<tr>
<td>$\sigma(A)$</td>
<td>spectral multiplicity function</td>
<td></td>
</tr>
<tr>
<td>$\omega$</td>
<td>real number indicating 'stable part' of $E$</td>
<td></td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>growth constant</td>
<td></td>
</tr>
<tr>
<td>$TV$</td>
<td>image of $V$ under $T$</td>
<td></td>
</tr>
<tr>
<td>$T^{-1}V$</td>
<td>set of all $x$ such that $Tx \in V$</td>
<td></td>
</tr>
<tr>
<td>$&lt;T</td>
<td>V&gt;$</td>
<td>smallest $T$-invariant subspace containing $V$</td>
</tr>
<tr>
<td>$&lt;U</td>
<td>V&gt;$</td>
<td>largest $T$-invariant subspace contained in $V$</td>
</tr>
<tr>
<td>$T:V$</td>
<td>restriction of $T$ to $V$</td>
<td></td>
</tr>
<tr>
<td>$T:X/V$</td>
<td>quotient mapping induced by $T$ on the factor space $X/V$</td>
<td></td>
</tr>
<tr>
<td>$T:V_2/V_1$</td>
<td>restriction of the quotient mapping $T:X/V_1$ to $V_2/V_1$</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>(sometimes) quotient mapping induced by $T$</td>
<td></td>
</tr>
<tr>
<td>$[x]$</td>
<td>image of $x$ under a factor mapping</td>
<td></td>
</tr>
<tr>
<td>$|x|$</td>
<td>norm of $x$</td>
<td></td>
</tr>
<tr>
<td>$|T|$</td>
<td>norm of $T$</td>
<td></td>
</tr>
</tbody>
</table>
TITLES IN THE SERIES MATHEMATICAL CENTRE TRACTS

(An asterisk before the MCT number indicates that the tract is under preparation).

A leaflet containing an order form and abstracts of all publications mentioned below is available at the Mathematisch Centrum, Kruislaan 413, 1098 SJ Amsterdam, The Netherlands. Orders should be sent to the same address.

MCT 7 W.R. VAN ZWET, Convex transformations of random variables, 1964. ISBN 90 6196 007 X.
MCT 13 H.A. LAUWERIER, Asymptotic expansions, 1966, out of print; replaced by MCT 54.


MCT 22 T.J. DEKKER, ALGOL 60 procedures in numerical algebra, part 1, 1968. ISBN 90 6196 029 0.


MCT 28 J. OOSTERHOFF, Combination of one-sided statistical tests, 1969. ISBN 90 6196 041 X.


MCT 42 W. VERVAT, *Success epochs in Bernoulli trials (with applications in number theory)*, 1972. ISBN 90 6196 077 0.


MCT 65 J. DE VRIES, Topological transformation groups 1 A categorical approach, 1975. ISBN 90 6196 113 0.


*MCT 87 S.G. VAN DER MEULEN & M. VELDHORST, Torris II, ISBN 90 6196 153 X.


*MCT 97 A. FEDERGRUEN, Markovian control problems; functional equations and algorithms, ISBN 90 6196 165 3.


MCT 125 R. EISING, 2-D systems, an algebraic approach, 1980. ISBN 90 6196 198 X.


MCT 156 P.M.G. APERS, *Query processing and data allocation in distributed database systems*, 1983. ISBN 90 6196 251 X.
MCT 157 H.A.W.M. KNEPPERS, The covariant classification of two-dimensional smooth commutative formal groups over an algebraically closed field of positive characteristic, 1983.


ISBN 90 6196 255 0.

ISBN 90 6196 256 0.


ISBN 90 6196 259 5.

MCT 163 H. SCHIPPERS, Multiple grid methods for equations of the second kind with applications in fluid mechanics, 1983.

MCT 164 P.A. VAN DER DUYN SCHOUTEN, Markov decision processes with continuous time parameter, 1983.

ISBN 90 6196 262 5.

MCT 166 H.B.M. JONKERS, Abstraction, specification and implementation techniques, with an application to garbage collection, 1983.


ISBN 90 6196 265 X.

ISBN 90 6196 266 8.

An asterisk before the number means "to appear"