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A MATHEMATICAL THEORY OF PURE EXCHANGE ECONOMIES WITHOUT THE NO-CRITICAL-POINT HYPOTHESIS

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PREFACE

The main results of this monograph concern the general structure of the set of first order equilibria and of the set of first order critical Pareto points in a pure exchange economy with ℓ commodities and m consumers. Starting from the assumption that each smooth function represents a preference relation we have defined, making use of transversality theorems, a dense set T of utility tuples.

Given $u \in T$ there is a dense set of initial endowments $r \in \mathbb{R}^{\ell m}$ for which the set of equilibria is discrete without co-called disastrous allocations and a dense set of total resources $w \in \mathbb{R}^{\ell}$ for which the critical Pareto set is a submanifold of dimension m - 1, also without disastrous allocations. Since T is dense in the C^{∞}-topology it is also dense in the C^{0}-topology. This implies that, even if the only assumption we make for the preference relations is continuity, the set of equilibria is "in general" discrete and the set of local strict Pareto optima is "in general" contained in an (m-1) dimensional submanifold.

Turning our attention to the set θ of local strict Pareto optima we observe that the so-called generalized Hessian H_z plays an important role. If each utility function satisfies the classical assumption of local strict convexity the Hessian is definite negative everywhere which implies that θ_{ex} coincides with Θ, being the intersection of an (m-1) dimensional submanifold θ_{cr} with a closed set.

The set T contains all m-tuples of utility functions satisfying local convexity.

CHAPTER 1
INTRODUCTION

1.1. Commodities, prices and preferences

In this monograph we consider a *pure exchange economy* without producers. There are ℓ durable goods and m agents. We assume $\ell \geq 2$, $m \geq 2$.
Each point $x = (x_1,\ldots,x_m) \in \mathbb{R}^{\ell m}$ represents an *allocation*, where $x_i = (x_i^1,\ldots,x_i^\ell) \in \mathbb{R}^\ell$ is the *commodity bundle* of *agent* i $(1 \leq i \leq m)$. We assume that for each agent the whole of $\mathbb{R}^\ell$ is his *consumption set*, the set of possible commodity bundles. For a more elaborate discussion of the terms "goods", "consumption sets" and the sign convention, concerning $x_i^j < 0$, $x_i^j = 0$, $x_i^j > 0$, see Debreu [1].
With each commodity, say the h-th one, is associated a real number, its *price* p^h. The price p^h may be positive (scarce commodity), null (free commodity) or negative (noxious commodity). The *price system* is the ℓ-tuple $p = (p^1,\ldots,p^\ell) \in \mathbb{R}^\ell$.
The *value* of a bundle $a = (a^1,\ldots,a^\ell)$, relative to the price system p, is the standard inner product

$$p \cdot a := \sum_{h=1}^{\ell} p^h a^h .$$

We assume $p \neq 0$. Two price systems p and q are *equivalent* if there is some positive $\lambda \in \mathbb{R}$ such that $q = \lambda p$. Hence, if we take a price system p, we always choose $p \in S^{\ell-1}$, i.e. $p \cdot p = 1$.
Each point $(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1}$ defines a *state* of the economy.
Given two bundles a and a' in $\mathbb{R}^\ell$ one and only one of the following three alternatives is assumed to hold for agent i:

(1) a *is preferred to* a';
(2) a *is indifferent to* a';
(3) a' *is preferred to* a.

It is convenient to introduce a *preference relation* $\underset{i}{\lesssim}$ on $\mathbb{R}^\ell$ for agent i. Then the foregoing alternatives read as follows:

(1) $a' \underset{i}{\lesssim} a$ and $\neg(a \underset{i}{\lesssim} a')$;

(2) $a' \underset{i}{\lesssim} a$ and $a \underset{i}{\lesssim} a'$;

(3) $a \underset{i}{\lesssim} a'$ and $\neg(a' \underset{i}{\lesssim} a)$.

The binary relation $\underset{i}{\lesssim}$ is assumed to be *reflexive* and *transitive*.
The preference relation $\underset{i}{\lesssim}$ is said to be *continuous* if for each $a' \in \mathbb{R}^\ell$ the sets $\{a \in \mathbb{R}^\ell \mid a \underset{i}{\lesssim} a'\}$ and $\{a \in \mathbb{R}^\ell \mid a' \underset{i}{\lesssim} a\}$ are closed.
If $\underset{i}{\lesssim}$ is continuous, there is a continuous function $u_i: \mathbb{R}^\ell \to \mathbb{R}$ satisfying for all a, a':

$$a' \underset{i}{\lesssim} a \quad \text{if and only if} \quad u_i(a') \le u_i(a) \quad .$$

For a proof see Debreu [1], page 56-59.
Such a function u_i is called a *utility function*, representing the preferences of agent i. In order to use the calculus of differentiable manifolds and maps we assume utility functions to be *smooth*. See Chapter 2.
In economic literature several assumptions are proposed about preference relations and, consequently, utility functions, the relevance of each of them being a matter of taste or realism. We mention here:

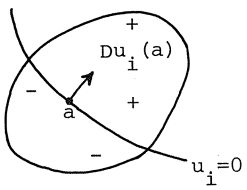
(1) *nonsatiation*, i.e. for each $a \in \mathbb{R}^\ell$ there is some $a' \in \mathbb{R}^\ell$ preferred to a;
(2) *convexity*, i.e. for each $a \in \mathbb{R}^\ell$ the set $\{a' \in \mathbb{R}^\ell \mid u_i(a') \ge u_i(a)\}$ is convex;
(3) *monotonicity*, i.e. $u_i(a') > u_i(a)$ whenever $a' \neq a$ and $a'^h \ge a^h$ for all h.

We do not make any of these assumptions. In our model the class of utility functions coincides with the class of smooth functions $\mathbb{R}^\ell \to \mathbb{R}$.

Given some utility function u_i and some bundle a, the set of points a', preferred to a by agent i, can have locally different shapes depending on the *gradient*

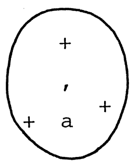
$$Du_i(a) := \left(\frac{\partial u_i}{\partial x_i^1}, \ldots, \frac{\partial u_i}{\partial x_i^\ell}\right)\bigg|_{x_i=a}$$

To get some insight in the possible situations we assume for the moment $u_i(a) = 0$ and $\ell = 2$. Then, up to degeneracies one has the following pictures:

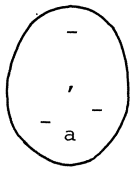
A) 

Here $Du_i(a) \neq 0$. Agent i has a clear idea in which directions to move in order to increase his utility.

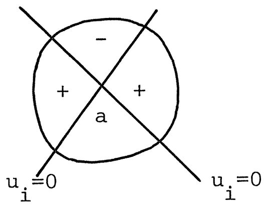
B) The same picture as (A) but $Du_i(a) = 0$. It is possible that some other utility function, representing the same preference relation, has a gradient $\neq 0$ at a.

C) 

Here $Du_i(a) = 0$ and the point a represents a local minimum for u_i. Each direction improves the position of i.

D) 

The point a represents a local maximum for u_i. This situation is described as *local satiation*.

E) 

The point a represents a *point of doubt*. If some direction improves the position of i, its opposite direction equally does.

1.2. The set of equilibria

Now we assume that agent i is endowed with some *initial bundle* $r_i \in \mathbb{R}^\ell$. Then, given some price system $p \in S^{\ell-1}$, he faces his *budget set*

$$\xi_i(p,r_i) := \{x_i \in \mathbb{R}^\ell \mid p \cdot x_i \leq p \cdot r_i\} .$$

It will be his aim to maximize his utility function u_i on this budget set. If such a maximizing bundle exists we can find it in the set $E_i(p,r_i)$ of points $x_i \in \xi_i(p,r_i)$ for which there is some $\lambda_i \geq 0$ satisfying

$$1.2.1.\quad \begin{cases} Du_i(x_i) = \lambda_i p , \\ \lambda_i p\cdot(x_i - r_i) = 0 . \end{cases}$$

Given initial bundles $r = (r_1,\ldots,r_m)$ and utility functions $u = (u_1,\ldots,u_m)$ the set $E_{ex}(r,u)$ of *extended equilibrium states* is defined by:
$(x,p) \in E_{ex}(r,u)$ if and only if

$$1.2.2.\quad \begin{cases} \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i , \\ x_i \in E_i(p,r_i)]_{i=1}^{m} . \end{cases}$$

See also Smale [15].
The first condition is inspired by the definition of pure exchange economies, in which allocations are *admissible* if and only if they can be obtained by redistribution of the *total resources* $\Sigma_{i=1}^{m} r_i$, given by the initial endowments.
The second condition is the first order criterion which implies that each agent i finds himself endowed with a bundle in his budget set which is a possible local maximum for u_i. Hence the adjective "extended".
Given an economy, defined by $(r,u) = (r_1,\ldots,r_m, u_1,\ldots,u_m)$, we also consider the set $E_{cr}(r,u) \subset \mathbb{R}^{\ell m} \times S^{\ell-1}$ of critical equilibrium states (x,p) for which the following holds:

$$1.2.3.\quad \begin{cases} \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i , \\ p\cdot x_i = p\cdot r_i]_{i=1}^{m-1} , \\ \text{there are reals } \lambda_i]_{i=1}^{m} \text{ such that } Du_i(x_i) = \lambda_i p]_{i=1}^{m} . \end{cases}$$

Then, as is easily seen, $E_{ex}(r,u) \subset E_{cr}(r,u)$.
The system (1.2.3) consists of $\ell + m - 1 + m\ell$ equations in $m + m\ell + \ell - 1$ unknowns (λ,x,p). Roughly speaking there is in general locally one solution (or none), so the set $E_{cr}(r,u)$ is in general a discrete one.
To make more exact the notion of the vague statement "in general" we use methods of *Global Analysis*, especially of *transversality theorems*. See for instance Golubitsky and Guillemin [6], Hirsch [9], or Dierker [2].

We topologize the set of economies (r,u) in a suitable way. See Chapter 2. Our first attention is to a certain set T of utility tuples. This set is introduced in Chapter 3. For this set T we are able to prove the

General result I:

T is dense and for each $u \in T$ *there is a dense set of initial endowments* r *for which* $E_{cr}(r,u)$ *is discrete.*

Since the set of smooth functions in its turn is dense in the set of continuous functions it follows that the assumption of smoothness for utility functions is not too bold.
Moreover, the question whether T is also open, is in this context less interesting, since in the set of continuous functions no neighbourhood is filled up by functions satisfying some differentiability condition, whereas openness of T by its very nature may only be studied in the set of at least twice differentiable functions. See Chapter 3.

Smale [15] restricts the consumption space of each agent to the closure of the positive orthant of $\mathbb{R}^\ell$ and, using only utility functions with non-zero gradients proves also for a certain set Y of utility tuples that it has the desired properties: there is a dense set of r for which $E_{ex}(r,u)$ is discrete. Y itself is dense.
His methods are based upon the existence of the *normed gradients*

$$\frac{Du_i(x_i)}{\|Du_i(x_i)\|}, \quad \text{where } \|v\| := (v \cdot v)^{\frac{1}{2}}, \quad v \in \mathbb{R}^\ell .$$

As will be shown in several examples there are many pairs (r,u) for which, according to our results, the set $E_{cr}(r,u)$ is discrete, and contains states (x,p) for which some bundles x_i are stationary points for the corresponding utility function u_i.
Moreover, we shall show that our methods, applied to utility functions satisfying Smale's *no-critical-point hypothesis*, lead to the same set of economies as obtained by Smale. *So, our results generalize those of Smale.*

1.3. *The set of local Pareto optima*

A second topic in pure exchange economies is the set of *local strict Pareto optima*:

Given some $w \in \mathbb{R}^{\ell}$, being the total resources in the economy, and utility functions u_i one considers the set $\theta(w,u)$ of admissible allocations $x \in \mathbb{R}^{\ell m}$ for which the following holds:

There is an open neighbourhood $\mathcal{O} \subset \mathbb{R}^{\ell m}$ of x such that for each admissible allocation $y \in \mathcal{O}$, $y \neq x$, there is at least one i such that $u_i(y_i) < u_i(x_i)$.

In this context prices, and consequently budget sets, are not involved and the only criterion for redistribution of w is non-decreasing of utility. So, given some admissible allocation r, agents are willing to accept some admissible allocation x if and only if $u_i(x_i) \geq u_i(r_i)]_{i=1}^{m}$. If we take into account that such redistribution has to be realized by exchanging small amounts of goods, it seems reasonable to assume that no trade takes place from $r \in \theta$.

It will be shown that $x \in \theta$ implies $(x,p) \in E_{ex}(x,u)$ for some $p \in S^{\ell-1}$.

Acting in the same spirit as in the definitions of equilibria we introduce sets $\theta_{ex}(w,u)$ and $\theta_{cr}(w,u)$ as follows:

(1) $x \in \theta_{ex}(w,u)$ if and only if x is admissible and $(x,p) \in E_{ex}(x,u)$ for some $p \in S^{\ell-1}$.

(2) $x \in \theta_{cr}(w,u)$ if and only if x is admissible and $(x,p) \in E_{cr}(x,u)$ for some $p \in S^{\ell-1}$.

Hence $\theta \subset \theta_{ex} \subset \theta_{cr}$.

The conditions

$$\text{1.3.1.} \quad \begin{cases} \sum_{i=1}^{m} x_i = w \,, \\ Du_i(x_i) = \lambda_i p]_{i=1}^{m} \end{cases}$$

constitute a system of $\ell + \ell m$ equations in $m + \ell m + \ell - 1$ unknowns (λ, x, p), the solutions of which determine points $x \in \theta_{cr}$. In general the set of solutions is parametrized by $m - 1$ variables.

This leads to the following *general result II*:

If $u \in T$ *there is a dense set of total resources* w, *for which* $\theta_{cr}(w,u)$ *is a submanifold of dimension* $m-1$.

For the definition of submanifold see Chapter 2.

It will be shown that θ_{ex} is the intersection of θ_{cr} with a closed set. Hence the set θ_{ex} and a fortiori the set θ may not be a submanifold. Moreover, if one restricts the consumption sets to the open or closed positive orthant in $\mathbb{R}^{\ell}$, the precise description of the structure of θ or θ_{ex} is a complicated affair. It turns out to be necessary that one invokes the theory of stratified manifolds with corners. See for instance Wan [23], Smale [17], or Schecter [11]. We do not enter into these problems. We only remark that, as for the structure of θ_{cr} our results generalize those of Smale [16] in the same sense as described in the case of equilibria.

1.4. *Disastrous allocations*

Omitting the no-critical-point hypothesis leads to some special effect. If all of the utility functions u_i have some critical point, say z_i, then $(z,p) \in E_{cr}(z,u)$ for all $p \in S^{\ell-1}$ and $z \in \theta_{cr}(w,u)$ for $w = \Sigma_{i=1}^{m} z_i$. We may consider such a point z as *disastrous* for the economy. None of the agents has some specified *short run demand*, indicated by the direction of the gradients $Du_i(z_i)$, being all zero.
It will be shown that for the set of pairs (r,u), (w,u) respectively, indicated in the general results I and II there are no admissible disastrous allocations.

1.5. *Survey of the contents of this monograph*

In Chapter 2 we give a summary of some standard material of global analysis, contained in Sections 2.1 until 2.4. Section 2.5 is devoted to a proof of a well-known theorem on local Pareto optima. This proof is based upon methods, used by the author in his paper [5]. In Section 2.6 a submanifold Γ is introduced, which plays an important role in the sequel.

Chapter 3 contains the definition of the set T and the proof that T is dense. Furthermore, a discussion about the openness of T is given in 3.6.

The proof of the general result I is the main topic of Chapter 4. Moreover, we introduce some subsets of T, which are open in the set of utilities u.

Chapter 5 is devoted to the proof of the general result II and some criteria for points in θ_{cr} on which one can decide whether they are points in θ or not, are discussed.

Chapter 6 contains some elements of *trade curves*. The introduction to this topic is postponed to the first section of Chapter 6.

1.6. Some examples

Before we end this chapter we present some standard illustrative pictures, intended to give an impression of the relationship between equilibria and Pareto optima.
We assume $\ell = m = 2$ and use the so-called *Edgeworth-box* in $\mathbb{R}^2$. See also Debreu [1], Hildenbrand-Kirman [8], Dierker [2], Smale [16], and many other authors.
The horizontal axis represents quantities of commodity I and the vertical quantities of commodity II. Let $w \in \mathbb{R}^2$ be the total resources in the economy. We measure quantities for consumer 1 from the origin and quantities for consumer 2 from w. Then each point in $\mathbb{R}^2$ represents an admissible location. For instance, the origin corresponds with the allocation $(0,w) \in \mathbb{R}^4$, whereas w represents the allocation $(w,0) \in \mathbb{R}^4$. Points within the open rectangle through 0 and w and sides parallel to the axes correspond with allocations (x_1,x_2) in the positive orthant. The closure of this rectangle is generally denoted as the Edgeworth-box. We do not confine ourselves to these allocations, so our set of allocations is the whole of $\mathbb{R}^2$.

Given utility functions u_1 and u_2, through each point in $\mathbb{R}^2$ there pass two curves, the *indifference curves* for u_1 and u_2, indicated by a solid curve for u_1 and a dotted curve for u_2. If we consider equilibria we indicate the initial bundles (r_1,r_2) by the initial bundle r_1.
Each allocation (x_1,x_2) is indicated by the bundle x_1.

(1) If u_1 and u_2 both satisfy strong convexity assumptions we have pictures as in Figure 1.

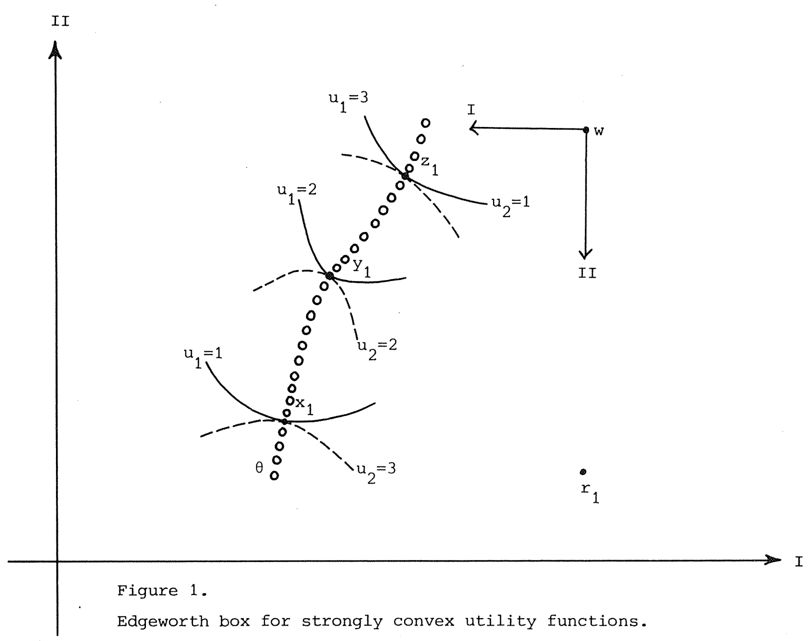


Figure 1.

Edgeworth box for strongly convex utility functions.

Obviously, each point where the indifference curves are tangent is a point of θ_{ex} and even a point of θ.

The set θ is the curve, denoted by Edgeworth's *contract curve* (o-o-o), passing through all these tangent points.

Given r_1, one finds the equilibria (x,p) by selecting those points x_1 on θ where the common tangent passes through r_1, as is the case for x_1 and y_1, but not for z_1.

(2) In Figure 2 the utility functions satisfy weak convexity conditions. The set of equilibria contains a one-dimensional set. Intuitively one sees that a slight perturbation of the utility functions breaks down the whole structure, in accordance with general result I.

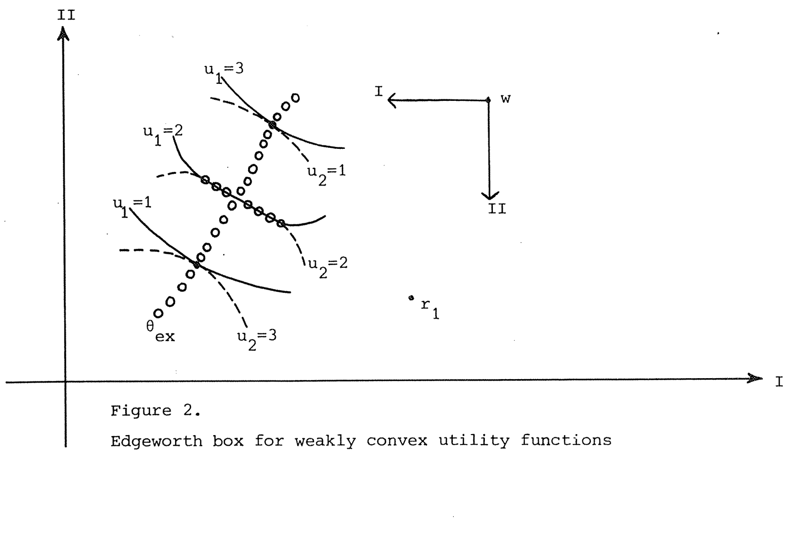


Figure 2.

Edgeworth box for weakly convex utility functions

(3) In Figure 3 the utility functions do not satisfy convexity conditions.

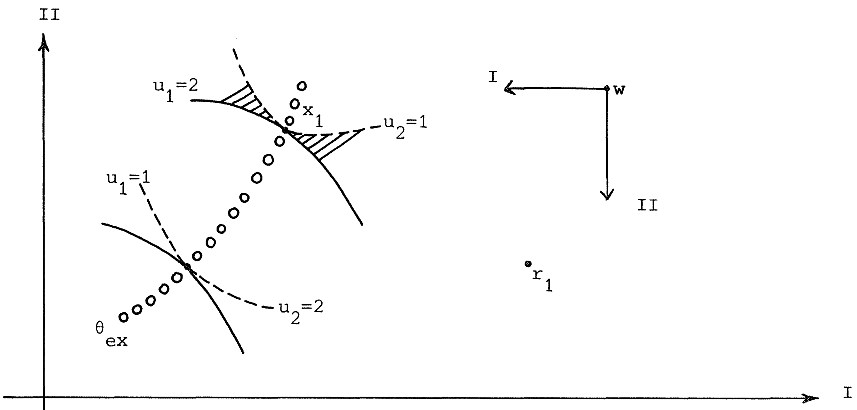


Figure 3.

Edgeworth box for concave utility functions.

The indifference curves $u_1 = 2$ and $u_2 = 1$ are tangent at x_1 and x_1 is a point of θ_{cr}. Clearly x_1 is not a point of θ since each point in the shaded region is better for both of the agents. Equally x_1 does not correspond with an equilibrium since u_1 and u_2 both increase along the line x_1r_1.

(4) In Figure 4 the function u_1 satisfies the convexity condition but u_2 does not.

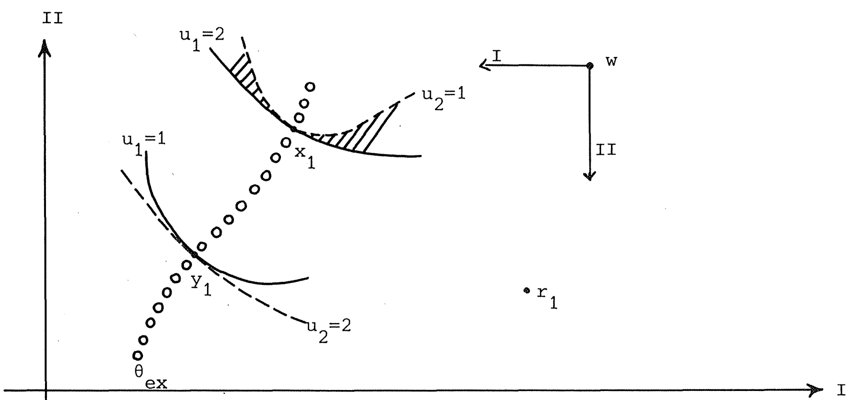


Figure 4.

Edgeworth box for u_1 convex and u_2 concave.

As one sees the situations in x_1 and y_1, both being points of θ_{ex}, are different from each other.

x_1 is not a point of θ, since all points in the shaded region are better for both of the agents. The concavity of u_2 overrules the convexity of u_1 at x_1.

y_1 is a point of θ, due to the fact that the convexity of u_1 is stronger than the concavity of u_2 at y_1.

(5) In Figure 5 we consider utility functions defined by:

$$u_i(x_i) := -\tfrac{1}{2}(x_i - a_i)\cdot(x_i - a_i)]_{i=1}^{2} ,$$

where we assume $a_1 + a_2 \neq w$.

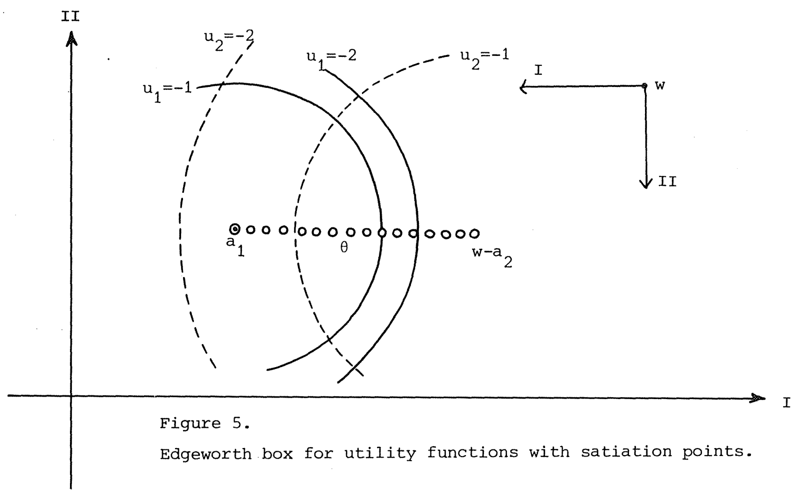


Figure 5.
Edgeworth box for utility functions with satiation points.

$\theta = \theta_{ex}$ is the closed segment between a_1 and $w - a_2$. If r_1 lies between the verticals through a_1 and $w - a_2$, the set $E_{ex}(r,u)$ consists of only one point. Otherwise $E_{ex}(r,u)$ is empty.

In case $w = a_1 + a_2$ the set θ_{ex} consists of only one point, namely a_1, but now θ_{cr} is the whole of $\mathbb{R}^2$. Moreover, for every r_1 the point (a_1,p) where $p \perp a_1 - r_1$ represents an equilibrium. Clearly the point a_1 is disastrous and we do not have the general situation as stated in general results I and II.

This example 5 will be reconsidered as an example in 4.2 and 5.4.

CHAPTER 2
PRELIMINARIES ON MANIFOLDS

Introduction

The first four sections of this chapter contain a summary of standard topics from global analysis, in a form adapted to the context they will be used in. For instance, all of the manifolds considered here are submanifolds of some Euclidean space.

In Section 2.5 we prove a well-known theorem on local Pareto optima. This theorem has been proved formerly by Smale [20], and Wan [22], but the proof given here is basically different from theirs. See also [5].

In Section 2.6 we introduce a subset Γ of $\mathbb{R}^{\ell m}$ and show that Γ is a submanifold. This set Γ plays an important role in the sequel.

Our main references for this chapter are [6] and [9]. We do not, at least not before Section 2.5, present proofs of the statements we make. They can be found in [6] or [9].

Points in $\mathbb{R}^n$ are given as a row $x = (x^1,\dots,x^n)$ or as a column $(x^1,\dots,x^n)^T$, the context making clear which form is chosen.

The topology in $\mathbb{R}^n$ will always be the metric topology, induced by the standard inner product, defined by

$$x \cdot y := \sum_{h=1}^{n} x^h y^h, \quad \|x\| := (x \cdot x)^{\frac{1}{2}} .$$

The unit sphere $S^{n-1} \subset \mathbb{R}^n$ is the set of points $x \in \mathbb{R}^n$ satisfying $\|x\| = 1$. Furthermore, given a subset U of $\mathbb{R}^n$ its interior is denoted by $\mathring{U}$ or int U, and its closure by $\bar{U}$.

2.1. *Differentiable mappings and submanifolds*

Let U be an open subset of $\mathbb{R}^n$, and k a nonnegative integer.

2.1.1. DEFINITION. $C^k(U,\mathbb{R}^m)$ *is the set of all maps* $f\colon U \to \mathbb{R}^m$ *being* k *times differentiable with all derivatives up to order* k *continuous on* U.

2.1.2. DEFINITION. $C^\infty(U,\mathbb{R}^m) := \bigcap_{k=0}^{\infty} C^k(U,\mathbb{R}^m)$ *is the set of smooth maps* $U \to \mathbb{R}^m$.

We shall extend the notion of differentiability up to order k to maps $X \to Y$, where X and Y are submanifolds, and we shall define later on the sets $C^k(X,Y)$ and $C^\infty(X,Y)$.

Let $f \in C^2(U,\mathbb{R}^m)$ and $x_0 \in U$. Then by Taylor's theorem there exists a unique linear map $\mathbb{R}^n \to \mathbb{R}^m$, denoted by $Df(x_0)$, a unique symmetric bilinear form $D^2f(x_0)\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$, and a map $\rho\colon U \to \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$ sufficiently close to x_0 the following holds:

$$f(x) = f(x_0) + Df(x_0)(x-x_0) + \tfrac{1}{2} D^2 f(x_0)(x-x_0, x-x_0) + \rho(x)$$

where

$$\lim_{x\to x_0} \frac{\|\rho(x)\|}{\|x-x_0\|^2} = 0 .$$

With respect to the coordinates $(x^1,\ldots,x^n)$ on $\mathbb{R}^n$ and $(y^1,\ldots,y^m)$ on $\mathbb{R}^m$ the *derivative* $Df(x_0)$ has the matrix

$$\left(\frac{\partial f^i}{\partial x^j}(x_0)\right)_{j=1,\ldots,n}^{i=1,\ldots,m}$$

Equally, the *second derivative* $D^2f(x_0)$ for $f\colon \mathbb{R}^n \to \mathbb{R}$ is denoted by

$$\left(\frac{\partial^2 f}{\partial x^i\, \partial x^j}\right)_{j=1,\ldots,n}^{i=1,\ldots,n} .$$

2.1.3. CHAIN RULE. *Let* $f \in C^1(U,\mathbb{R}^m)$, $g \in C^1(V,\mathbb{R}^k)$ *and* $f(U) \subset V \subset \mathbb{R}^m$. *Then the composition* $g \circ f \in C^1(U,\mathbb{R}^k)$ *and* $D(g\circ f)(x_0) = Dg(f(x_0)) \circ Df(x_0)$ *for each* $x_0 \in U$.

Before stating the implicit function theorem we need some notations. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open and $F \in C^1(U \times V, \mathbb{R}^k)$. Then, given $x_0 \in U$, $y_0 \in V$, we define

(1) $F_{y_0} \in C^1(U, \mathbb{R}^k)$ by $F_{y_0}(x) := F(x, y_0)$,

(2) $F_{x_0} \in C^1(V, \mathbb{R}^k)$ by $F_{x_0}(y) := F(x_0, y)$.

2.1.4. IMPLICIT FUNCTION THEOREM. *Let* $U \subset \mathbb{R}^n$ *and* $V \subset \mathbb{R}^m$ *be open sets. Let* $(x_0, y_0) \in U \times V$ *and* $F \in C^\infty(U \times V, \mathbb{R}^m)$ *be such that* rank $DF_{x_0}(y_0) = m$. *Then there is an open neighbourhood* $U' \subset U$ *of* x_0 *and a map* $\varphi \in C^\infty(U', \mathbb{R}^m)$ *satisfying*

(1) $\varphi(x_0) = y_0$;

(2) $F(x, \varphi(x)) = F(x_0, y_0)$ *for all* $x \in U'$;

(3) $D\varphi(x_0) = -DF_{x_0}(y_0)^{-1} \circ DF_{y_0}(x_0)$.

See [9], page 214.

As a consequence of the implicit function theorem (and vice versa) one has

2.1.5. INVERSE FUNCTION THEOREM. *Let* $U \subset \mathbb{R}^n$ *be open and* $f \in C^\infty(U, \mathbb{R}^n)$. *Let* $x_0 \in U$ *and* rank $Df(x_0) = n$. *Then there are open neighbourhoods* U_1 *of* x_0, U_2 *of* $y_0 := f(x_0)$ *and a map* $g \in C^\infty(U_2, \mathbb{R}^n)$ *satisfying*

(1) $f(U_1) = U_2$, *and* $g(U_2) = U_1$;

(2) $f(g(y)) = y$ *for all* $y \in U_2$, *and* $g(f(x)) = x$ *for all* $x \in U_1$.

See [9], page 214.

Now we come to the definition of a submanifold of dimension k as a subset of some Euclidean n-space where $n \geq k$. Intuitively a k-dimensional submanifold has locally the structure of an open subset of $\mathbb{R}^k$.

2.1.6. DEFINITION. *A subset* $Y \subset \mathbb{R}^n$ *is a submanifold of* $\mathbb{R}^n$ *of dimension* k *(or, shortly, a* k-***dimensional*** *submanifold) if for every point* $y_0 \in Y$

there exists an open neighbourhood U *of* y_0 *and a* $\varphi \in C^\infty(U,\mathbb{R}^n)$ *such that* $\varphi(y_0) = 0$, $\text{rank } D\varphi(x) = n$ *for all* $x \in U$, *and* $\varphi^{\leftarrow}(V) = Y \cap U$, *where* $V := \{(z^1,\ldots,z^n) \in \mathbb{R}^n \mid z^{k+1} = \ldots = z^n = 0\}$.
Moreover, the pair (U,φ) *is called a submanifold chart for* Y *at* y_0.

So the submanifold chart (U,φ) provides a local parametrization of Y by means of the first k coordinates of $\varphi(x)$, in a neighbourhood of y_0.
For example, each open subset of $\mathbb{R}^n$ is an n-dimensional submanifold of $\mathbb{R}^n$, and a 0-dimensional submanifold of $\mathbb{R}^n$ is a discrete set.

It should be emphasized that the topology on a submanifold of $\mathbb{R}^n$ *is the one, induced by the topology on* $\mathbb{R}^n$.

If not necessary we do not specify in the future the Euclidean space in which a submanifold is contained, nor the dimension of the submanifold.
The definition of submanifold is not always as manageable as desired in order to find out whether a subset of $\mathbb{R}^n$ is a submanifold or not. The following theorem will be useful in the sequel.

2.1.7. THEOREM. *A subset* $Y \subset \mathbb{R}^n$ *is a submanifold of* $\mathbb{R}^n$ *of dimension* k *if and only if for every point* $y_0 \in Y$ *there exists an open neighbourhood* O *of* y_0 *and a* $\psi \in C^\infty(O,\mathbb{R}^{n-k})$ *such that* $\text{rank } D\psi(x) = n-k$ *for all* $x \in O$ *and* $\psi^{\leftarrow}(0) = Y \cap O$.
See [6], page 9.

As one sees the submanifold Y is locally defined as the solution set of the equation $\psi(x) = 0$, constituting $n-k$ equations in n unknowns. Due to the implicit function theorem the set Y is locally parametrized in a smooth way by some k-tuple from the n coordinates in $\mathbb{R}^n$.
The kernel $D\psi^{\leftarrow}(y_0)(0)$ of $D\psi(y_0)$ has dimension k. On the other hand, if (U,φ) is a submanifold chart for Y at y_0 one has the k-dimensional subspace $D\varphi^{-1}(0)(V)$ of $\mathbb{R}^n$. From the definitions it follows

$$D\psi^{\leftarrow}(y_0)(0) = D\varphi^{-1}(0)(V) \quad .$$

2.1.8. DEFINITION. *Let* $Y \subset \mathbb{R}^n$ *be a* k*-dimensional submanifold. Let* y_0 *be a point in* Y *and* (U,φ) *a submanifold chart for* Y *at* y_0. *Then* $T_{y_0}Y$ *is the set of pairs* $(y_0,\delta y)$, *where* $\delta y \in D\varphi^{-1}(0)(V)$. *The set* $T_{y_0}Y$ *is* *called the tangent space to* Y *at* y_0. *It has the structure of a* k-*dimensional vector space, isomorphic to* $D\varphi^{-1}(0)(V)$.

Geometrically $T_{y_0}Y$ is the set of all tangents to smooth curves on Y, passing through y_0, considered as a Euclidean space. In general, speaking of $T_{y_0}Y$, we only give the second component of the pair $(y_0,\delta y)$, i.e. $\delta y \in \mathbb{R}^n$. Now we introduce differentiability and derivatives for maps, defined on submanifolds.

2.1.9. DEFINITION. *Let* $X_i \subset \mathbb{R}^{n_i}$ *be* k_i-*dimensional submanifolds for* $i = 1,2$. *Let* $x_i \in X_i$ *and* (U_i,φ_i) *be submanifold charts for* X_i *at* x_i $(i = 1,2)$ *and* $f\colon X_1 \to X_2$ *be a map such that* $f(x_1) = x_2$. *Furthermore,* V_1 *and* V_2 *are defined as in* 2.1.6.
Let k *be a nonnegative integer.*

(1) *The map* $f\colon X_1 \to X_2$ *is said to be of class* C^k *at* x_1 *if the map* $\varphi_2 \circ f \circ \varphi_1^{-1}\colon V_1 \to V_2$ *is of class* C^k *at* $0 \in V_1$.

(2) $C^k(X_1,X_2)$ *is the set of maps* $f\colon X_1 \to X_2$ *being of class* C^k *at every point* $x_1 \in X_1$.

(3) $C^\infty(X_1,X_2) := \bigcap_{k=0}^{\infty} C^k(X_1,X_2)$.

(4) *Given* $f \in C^1(X_1,X_2)$, $x_1 \in X_1$ *and* $(x_1,\delta x_1) \in T_{x_1}X_1$, *the map* $T_{x_1}f\colon T_{x_1}X_1 \to T_{x_2}X_2$ *is defined as follows:*

$$T_{x_1}f(x_1,\delta x_1) := (x_2, D\varphi_2^{-1}(0) \circ D(\varphi_2 \circ f \circ \varphi_1^{-1})(0) \circ D\varphi_1(x_1)\delta x_1) \;,$$

or shortly:

(5) $T_{x_1}f(x_1,\delta x_1) := (f(x_1), Df(x_1)\delta x_1)$, *where the definition of* $Df(x_1)$ *follows from* (4).

In general, speaking of the derivative $T_{x_1}f$, we only give the second part, i.e. $Df(x_1)\delta x_1$, or shortly $Df(x_1)$.

2.1.10. LEMMA. *The Cartesian product* $X_1 \times X_2$ *of two submanifolds is a submanifold and* $\dim(X_1 \times X_2) = \dim X_1 + \dim X_2$. *Furthermore*

$$T_{(x_1,x_2)}(X_1 \times X_2) = \{(x_1,x_2,\delta x_1,\delta x_2) \mid \delta x_1 \in T_{x_1}X_1,\ \delta x_2 \in T_{x_2}X_2\} .$$

See [6], page 5.

2.1.11. **DEFINITION.** *Let* $f \in C^0(X,\mathbb{R})$ *where* X *is some submanifold. Then the support* Supp f *is the closure of* $f^{\leftarrow}(\mathbb{R}\setminus\{0\})$.

2.1.12. **DEFINITION.** *Let* $\{U_\alpha\}_{\alpha\in A}$ *be a family of subsets of a submanifold* X, *such that* $\bigcup_{\alpha\in A} U_\alpha = X$. *In other words,* $\{U_\alpha\}_{\alpha\in A}$ *is a cover of* X. *Then* $\{U_\alpha\}_{\alpha\in A}$ *is said to be locally finite if for every* $x \in X$ *there is an open neighbourhood* $O \subset X$ *of* x *such that* $O \cap U_\alpha = \emptyset$ *for all but a finite number of* α*'s in* A.

2.1.13. **THEOREM** (Existence of a partition of unity, subordinate to an open cover of X.)
Let X *be a submanifold and* $\{U_\alpha\}_{\alpha\in A}$ *an open cover of* X, *i.e. all of the* U_α *are open and* $\bigcup_{\alpha\in A} U_\alpha = X$. *There is a family* $\{f_\alpha\}_{\alpha\in A}$ *of smooth maps* $X \to \mathbb{R}$ *satisfying*

(1) $f_\alpha(x) \in [0,1]$ *for all* $\alpha \in A$ *and all* $x \in X$;

(2) $\mathrm{Supp}\ f_\alpha \subset U_\alpha$ *for all* $\alpha \in A$;

(3) $\{\mathrm{Supp}\ f_\alpha\}_{\alpha\in A}$ *is a locally finite cover of* X;

(4) $\sum_{\alpha\in A} f_\alpha(x) = 1$ *for all* $x \in X$. *(This is a finite sum, due to locally finiteness. Moreover, this shows that the interiors of* $\mathrm{Supp}\ f_\alpha$ *form an open, locally finite cover of* X.*)*

See [9], page 43.

2.1.14. **COROLLARY.** *Let* X *be a submanifold. Let* U *and* V *be open subsets of* X *with* $\bar{U} \subset V$. *Then there is an* $f \in C^\infty(X,\mathbb{R})$ *such that*

$$f(x) = \begin{cases} 1 & \text{if } x \in U , \\ 0 & \text{if } x \notin V , \\ 0 \le f(x) \le 1 & \text{otherwise.} \end{cases}$$

See [6], page 17.

2.1.15. COROLLARY. *Let* C *be a closed subset of* $\mathbb{R}^n$. *Then there exists a smooth function* $f: \mathbb{R}^n \to \mathbb{R}$ *such that* $f(x) \geq 0$ *everywhere and* $C = f^{\leftarrow}(0)$.

See [6], page 17.

2.2. *Sard's Theorem*

Let $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ and $b^k > a^k$, $k = 1,\ldots,n$. Then $C(a,b)$ is the closed block consisting of all points $x \in \mathbb{R}^n$, satisfying $a^k \leq x^k \leq b^k$, $k = 1,\ldots,n$. The volume of $C(a,b)$ is $\Pi_{k=1}^n (b^k - a^k)$.

2.2.1. DEFINITION. *A subset* $S \subset \mathbb{R}^n$ *is thin in* $\mathbb{R}^n$ *if for every* $\varepsilon > 0$ *there is a countable covering of* S *with blocks in* $\mathbb{R}^n$, *the sum of whose volumes is less than* ε.

2.2.2. DEFINITION. *Let* X *be an* n-*dimensional submanifold and* S *be a subset of* X. *Then* S *is said to be thin in* X *if there exists a countable open covering* $U_1, U_2, \ldots$ *of* S *and chart-maps* $\varphi_1, \varphi_2, \ldots$, *so that* $\varphi_i(U_i \cap S)$ *is thin in* $\mathbb{R}^n$, *for all* i.

As a consequence of 2.2.2 one has the following: if S is thin in X, then S does not contain an open subset of X, so its complement is dense in X. See also 2.2.7.

2.2.3. LEMMA. *Let* $m < n$ *and* $Y \subset \mathbb{R}^m$ *be a submanifold. Then* $f(Y)$ *is thin in* $\mathbb{R}^n$, *for all* $f \in C^\infty(Y, \mathbb{R}^n)$.

See [6], page 31.

2.2.4. DEFINITION. *Let* $X \subset \mathbb{R}^n$ *be a* k-*dimensional submanifold and* $Y \subset \mathbb{R}^m$ *be a* p-*dimensional submanifold. Let* $f \in C^1(X,Y)$.

(1) $\operatorname{corank} Df(x_0) := \min(\dim X, \dim Y) - \operatorname{rank} Df(x_0)$, *for* $x_0 \in X$.

(2) x_0 *is said to be a critical point of* f *if* $\operatorname{corank} Df(x_0) > 0$. *Otherwise* x_0 *is called a regular point of* f. *The set of critical points of* f *is denoted by* $C[f]$.

(3) $y_0 \in Y$ *is said to be a critical value of* f *if* $y_0 \in f(C[f])$. *Otherwise* y_0 *is said to be a regular value of* f.

2.2.5. **REMARK.**

(1) *For the definition of* $Df(x_0)$, *see* 2.1.9.

(2) *Since, as stated in* 2.1.9, *one may interpret* $Df(x_0)$ *as a linear map* $\mathbb{R}^k \to \mathbb{R}^p$, *a critical point* x_0 *is a point where* Df *has not full rank.*

(3) *From the third part of* 2.2.4 *it follows that every point* $y \in Y$ *not being in* $f(X)$ *is a regular value of* f.

2.2.6. **THEOREM** (Sard). *Let* X *and* Y *be submanifolds and* $f \in C^\infty(X,Y)$. *Then the set of critical values of* f *is thin in* Y.
See [6], page 34.

2.2.7. **COROLLARY** (Brown). *Let* X *and* Y *be submanifolds and* $f \in C^\infty(X,Y)$. *Then the set of regular values of* f *is dense in* Y.
See [6], page 36.

2.3. *The Whitney* C^∞ *Topology*

Let A_n^k be the vector space of polynomials in n variables of degree $\leq k$, which have their constant term equal to zero. A_n^k is isomorphic to some Euclidean space.
Given $f \in C^k(\mathbb{R}^n,\mathbb{R})$, we define the continuous map

$$j^k f\colon \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R} \times A_n^k$$

as follows:

$$j^k f(x) := (x, f(x), D_k f(x)) \ ,$$

where $D_k f(x)$ is the polynomial of degree $\leq k$ given by the Taylor expansion of f at x up to order k after the first term. Since $\mathbb{R}^n \times \mathbb{R} \times A_n^k$ is isomorphic to some Euclidean space, it has the metric of that space, denoted by d_n^k.

2.3.1. DEFINITION. *Let* $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$ *and* $\delta \in C^0(\mathbb{R}^n,\mathbb{R}_+)$, *where* $\mathbb{R}_+$ *is the set of positive reals. Then*

$$B_n^k(f;\delta) := \{g \in C^\infty(\mathbb{R}^n,\mathbb{R}) \mid d_n^k(j^k g(x), j^k f(x)) < \delta(x), \text{ for all } x \in \mathbb{R}^n\}.$$

It can be shown that these sets, given n and k, form a base for a topology on $C^\infty(\mathbb{R}^n,\mathbb{R})$, called the Whitney C^k topology, or shortly W_k. So $\mathcal{O} \in W_k$ if and only if for every $f \in \mathcal{O}$ there is a $\delta \in C^0(\mathbb{R}^n,\mathbb{R}_+)$ such that $B_n^k(f;\delta) \subset \mathcal{O}$. Then $W_\ell \subset W_k$ for $\ell \le k$.

2.3.2. DEFINITION. $W := \bigcup_{k=0}^{\infty} W_k$ *is the Whitney* C^∞ *topology on* $C^\infty(\mathbb{R}^n,\mathbb{R})$.

Hence a subset $\mathcal{O} \subset C^\infty(\mathbb{R}^n,\mathbb{R})$ is open in the Whitney C^∞ topology, or shortly C^∞-open, if it is open in the Whitney C^k-topology, or shortly C^k-open, for some $k \ge 0$.
Within the same terminology: $F \subset C^\infty(\mathbb{R}^n,\mathbb{R})$ is C^∞-dense, if and only if it is C^k-dense for each $k \ge 0$.

2.3.3. LEMMA. *Let* $\{f_m\}_{m\in\mathbb{N}}$ *be a sequence of functions in* $C^\infty(\mathbb{R}^n,\mathbb{R})$. *Then* $f_m \to f$ *in the Whitney* C^k *topology, if and only if there is a compact subset* $K \subset \mathbb{R}^n$ *and an* $m_0 \in \mathbb{N}$ *such that* $f_m(x) = f(x)$ *for all* $x \notin K$, $m \ge m_0$, *and* $j^k f_m \to j^k f$ *uniformly on* K.
See [6], page 43.

2.3.4. THEOREM. $C^\infty(\mathbb{R}^n,\mathbb{R})$ *is a Baire space in the Whitney* C^∞ *topology. (So, the intersection of a countable collection of* C^∞*-open and -dense subsets of* $C^\infty(\mathbb{R}^n,\mathbb{R})$ *is* C^∞*-dense.)*
See [6], page 44.

2.3.5. THEOREM. *Given* $n,m \in \mathbb{N}$ *the set* $C^\infty(\mathbb{R}^n,\mathbb{R})^m$ *is a Baire space in the product topology, induced by the Whitney* C^∞ *topology on the factors* $C^\infty(\mathbb{R}^n,\mathbb{R})$.
See [6], page 47.

2.4. Transversality

2.4.1. DEFINITION. *Let* X, Y *and* Z *be submanifolds and* $f \in C^1(X,Y)$. *Let* $Z \subset Y$ *and* x *a point in* X. *Then* f *is said to intersect* Z *transversally at* x *(denoted by* $f \pitchfork Z$ *at* x*) if either*

(1) $f(x) \notin Z$ *or*

(2) $f(x) \in Z$ *and* $T_{f(x)}Y = T_{f(x)}Z + Df(x)(T_x X)$.

As an interpretation of this definition 2.4.1 we give the following. Since $Z \subset Y$, the tangent space to Z at $z \in Z$ is a linear subspace of the tangent space to Y at z. Now, given $x \in X$, we have:

$f \pitchfork Z$ *at* x *is equivalent to: If* $f(x) \in Z$, *then for every* $\delta y \in T_{f(x)}Y$ *there are* $\delta z \in T_{f(x)}Z$ *and* $\delta x \in T_x X$ *such that*

$$\delta y = \delta z + Df(x)\delta x \ .$$

2.4.2. DEFINITION. *Let* X, Y *and* Z *be submanifolds and* $Z \subset Y$. *Let* $f \in C^1(X,Y)$, *and* B *a subset of* Z. *Then* f *is said to intersect* Z *transversally on* B *if for every* $x \in X$ *either*

(1) $f(x) \notin B$ *or*

(2) $f(x) \in B$ *and* $T_{f(x)}Y = T_{f(x)}Z + Df(x)(T_x X)$.

2.4.3. DEFINITION. *Let* X, Y *and* Z *be submanifolds and* $Z \subset Y$. *Let* $f \in C^1(X,Y)$. *Then* f *is said to intersect* Z *transversally (denoted by* $f \pitchfork Z$*) if* $f \pitchfork Z$ *at every point* $x \in X$.

2.4.4. THEOREM. *Let* X, Y *and* Z *be submanifolds, where* $Z \subset Y$. *Let* $f \in C^\infty(X,Y)$ *and* $f \pitchfork Z$. *Then* $f^{\leftarrow}(Z) \subset X$ *is a submanifold and*

$$\dim f^{\leftarrow}(Z) = \dim X - \dim Y + \dim Z \ .$$

Furthermore, given $x \in f^{\leftarrow}(Z)$ *the tangent space to* $f^{\leftarrow}(Z)$ *consists of all* $\delta x \in T_x X$ *satisfying* $Df(x)\delta x \in T_{f(x)}Z$.
See [6], page 52.

Theorem 2.4.4 provides a powerful tool in order to construct submanifolds. We shall use it frequently in the sequel.
Now we give, in a very specialized setting, a theorem concerning transversality of a parametrized family of maps.
Let $\ell \geq 1$, $m \geq 1$ be integers and B an open subset of $\mathbb{R}^{(\ell+1)m}$. Let

$$\Phi: \mathbb{R}^{\ell m} \times B \to \mathbb{R}^{2\ell m+m}$$

be a smooth map, and W a submanifold of $\mathbb{R}^{2\ell m+m}$. We define for each $b \in B$ the smooth map

$$\Phi_b: \mathbb{R}^{\ell m} \to \mathbb{R}^{2\ell m+m}$$

by:

$$\Phi_b(x) := \Phi(x,b) \quad .$$

In this context we have:

2.4.5. THEOREM. *If* $\Phi \pitchfork W$ *the set* $\{b \in B \mid \Phi_b \pitchfork W\}$ *is dense in* B.
See [6], page 53.

Before ending this introduction to some topics of global analysis we give a theorem in which second derivatives are involved.

2.4.6. DEFINITION. *Let* $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$. *Then* f *is said to be a Morse function if* $Df(x) = 0$ *implies* $D^2f(x)$ *is nonsingular, for all* $x \in \mathbb{R}^n$.

2.4.7. THEOREM (Morse). *The set of Morse functions is open and dense in* $C^\infty(\mathbb{R}^n,\mathbb{R})$.
See [6], page 63.

2.5. *Characterization of local Pareto optima*

This section is devoted to a well-known theorem on local Pareto optima. The proof is essentially the same as in the author's paper [5], where a slightly generalized form has been presented.
Let $\ell \geq 2$, $m \geq 2$ and $u_i \in C^\infty(\mathbb{R}^\ell,\mathbb{R})$, $i = 1,\ldots,m$. Given $w \in \mathbb{R}^\ell$ we consider the set A_w consisting of the points $x \in \mathbb{R}^{\ell m}$ satisfying $\Sigma_{i=1}^m x_i = w$. So A_w is the set of admissible allocations in a pure exchange economy with total resources w.

As pointed out in 1.3, a first order condition for a point $z \in A_w$ to be a local Pareto optimum is that there are nonnegative reals λ_i, and some $p \in S^{\ell-1}$ such that $Du_i(z_i) = \lambda_i p$, $i = 1,\dots,m$. (See also Chapter 5.) We restrict ourselves here to the case that $Du_i(z_i) \neq 0$, $i = 1,\dots,m$.

2.5.1. DEFINITION. *Let* $z \in \mathbb{R}^{\ell m}$, *and* $Du_i(z_i) \neq 0$, $i = 1,\dots,m$.

$$N_z := \left\{(v_1,\dots,v_m) \in \mathbb{R}^{\ell m} \mid \sum_{i=1}^{m} v_i = 0,\ Du_i(z_i)\cdot v_i = 0,\ i = 1,\dots,m\right\}.$$

2.5.2. DEFINITION. *Let* $z \in \mathbb{R}^{\ell m}$, $w \in \mathbb{R}^{\ell}$, $z \in A_w$, $p \in S^{\ell-1}$, $Du_i(z_i) = \lambda_i p$, *with* $\lambda_i > 0$, $i = 1,\dots,m$. *Then* $H_z\colon N_z \to \mathbb{R}$ *is defined as follows:*

$$H_z(v) := \sum_{i=1}^{m} \frac{1}{\lambda_i} D^2u_i(z_i)(v_i,v_i) .$$

This map H_z acts as a generalized second derivative we use in order to establish whether a point z, satisfying the first order condition for a local strict Pareto optimum, is optimal or not. (In Chapter 5 we extend the notions N_z and H_z to the case that $Du_i(z_i) = 0$ for some i.)
We prove the following theorem, using properties of implicit functions:

2.5.3. THEOREM. *Let* $w \in \mathbb{R}^{\ell}$ *and* $z \in A_w$ *satisfy* $Du_i(z_i) = \lambda_i p]_{i=1}^{m}$, *where* $p \in S^{\ell-1}$ *and* $\lambda_i > 0$ *for all* i.

(1) *If* $H_z(v) < 0$ *for all* $v \in N_z$, $v \neq 0$, *then* z *is a local strict Pareto optimum.*

(2) *If* $H_z(v) > 0$ *for some* $v \in N_z$, *then* z *is not a local strict Pareto optimum.*

Before giving the proof of 2.5.3 we need some properties of the first and second derivatives of implicit functions.

CONVENTIONS.

(1) *Given* $x \in \mathbb{R}^{\ell}$, $\bar{x} := (x^1,\dots,x^{\ell-1}) \in \mathbb{R}^{\ell-1}$.

(2) *Given* $f \in C^\infty(\mathbb{R}^\ell, \mathbb{R})$, $\xi \in \mathbb{R}^\ell$, *we define the* $(\ell-1) \times (\ell-1)$ *matrix* $\overline{D^2 f(\xi)}$, $1 \times (\ell-1)$ *matrix* $\frac{\partial^2 f}{\partial x^\ell \, \partial \bar{x}}$, $(\ell-1) \times 1$ *matrix* $\frac{\partial^2 f}{\partial \bar{x} \, \partial x^\ell}$ *by:*

$$D^2 f(\xi) = \begin{bmatrix} \overline{D^2 f(\xi)} & \frac{\partial^2 f}{\partial \bar{x} \, \partial x^\ell}(\xi) \\ \frac{\partial^2 f}{\partial x^\ell \, \partial \bar{x}}(\xi) & \frac{\partial^2 f}{\partial x^\ell \, \partial x^\ell}(\xi) \end{bmatrix} .$$

2.5.4. LEMMA. *Let* $f \in C^\infty(\mathbb{R}^\ell, \mathbb{R})$ *and* $\xi \in \mathbb{R}^\ell$ *such that* $\frac{\partial f}{\partial x^\ell}(\xi) \neq 0$. *Then there is a neighbourhood* $\mathcal{O}$ *of* $\bar{\xi} := (\xi^1, \ldots, \xi^{\ell-1})$ *and a smooth function* $g\colon \mathcal{O} \to \mathbb{R}$ *satisfying:*

(1) $g(\bar{\xi}) = \xi^\ell$;

(2) $f(\bar{x}, g(\bar{x})) = f(\xi)$ *for all* $\bar{x} \in \mathcal{O}$.

(3) $Dg(\bar{x}) = -\frac{\partial f}{\partial x^\ell}(\bar{x}, g(\bar{x}))^{-1} \, \overline{Df(\bar{x}, g(\bar{x}))}$ *for all* $\bar{x} \in \mathcal{O}$.

(4) $$\overline{D^2 f(x)} + \frac{\partial^2 f}{\partial \bar{x} \, \partial x^\ell}(x) \, Dg(\bar{x}) + Dg(\bar{x})^T \frac{\partial^2 f}{\partial x^\ell \, \partial \bar{x}}(x) + $$
$$+ \frac{\partial^2 f}{\partial x^\ell \, \partial x^\ell}(x) \, Dg(\bar{x})^T \, Dg(\bar{x}) + \frac{\partial f}{\partial x^\ell}(x) \, D^2 g(\bar{x}) = 0$$

for all $\bar{x} \in \mathcal{O}$.

PROOF. (1), (2) and (3) follow directly from 2.1.4. Writing out (3) in components we get:

$$\frac{\partial f}{\partial x^i}(\bar{x}, g(\bar{x})) + \frac{\partial f}{\partial x^\ell}(\bar{x}, g(\bar{x})) \frac{\partial g}{\partial x^i}(\bar{x}) = 0 ,$$

for $i = 1, \ldots, \ell-1$. Differentiating with respect to x^j, $j = 1, \ldots, \ell-1$, leads to

$$\frac{\partial^2 f}{\partial x^j \, \partial x^i} + \frac{\partial^2 f}{\partial x^\ell \, \partial x^i} \frac{\partial g}{\partial x^j} + \frac{\partial^2 f}{\partial x^j \, \partial x^\ell} \frac{\partial g}{\partial x^i} + \frac{\partial^2 f}{\partial x^\ell \, \partial x^\ell} \frac{\partial g}{\partial x^j} \frac{\partial g}{\partial x^i} +$$
$$+ \frac{\partial f}{\partial x^\ell} \frac{\partial^2 g}{\partial x^j \, \partial x^i} = 0 .$$

In matrix notation this is just (4). □

Now we come to the proof of 2.5.3.

Since $p \in S^{\ell-1}$, at least one of the coordinates of p is not equal to zero. We assume $p^\ell > 0$. Other cases can be treated in a similar way. Since $\frac{\partial u_i}{\partial x_i^\ell}(z_i) = \lambda_i p^\ell > 0$, and according to the implicit function theorem there are smooth functions φ_i, defined on a neighbourhood $\mathcal{O}_i$ of $\bar{z}_i$ such that

(1) $\varphi_i(\bar{z}_i) = z_i^\ell$, $i = 1,\dots,m$;

(2) $u_i(\bar{x}_i, \varphi_i(\bar{x}_i)) = u_i(z_i)$, $i = 1,\dots,m$;

(3) $u_i(x_i) \geq u_i(z_i)$ is equivalent to $x_i^\ell \geq \varphi_i(\bar{x}_i)$ for all x_i with $\bar{x}_i \in \mathcal{O}_i$, $i = 1,\dots,m$.

Let $\mathcal{L} \in C^\infty(\mathcal{O} \cap A_{\bar{w}}, \mathbb{R})$ be defined by

$$\mathcal{L}(\bar{x}) := \sum_{i=1}^m \phi_i(\bar{x}_i) - w^\ell, \text{ where } \mathcal{O} = \mathcal{O}_1 x \dots x \mathcal{O}_m \text{ and}$$

$$A_{\bar{w}} = \{(\bar{x}, \dots \bar{x}_m) \in \mathbb{R}^{(\ell-1)m} \mid \sum_{i=1}^m \bar{x}_i = \bar{w}\}.$$

Since $\mathcal{O}_i$ is an open neighbourhood of $\bar{z}_i$ in $\mathbb{R}^{\ell-1}$, the set $\mathcal{O} \cap A_{\bar{w}}$ is an open neighbourhood of $\bar{z}$ in $A_{\bar{w}}$.

We observe that

$$\mathcal{L}(\bar{z}) = \sum_{i=1}^m \varphi_i(\bar{z}_i) - w^\ell = \sum_{i=1}^m z_i^\ell - w^\ell = 0 .$$

If $u_i(x_i) \geq u_i(z_i)$ for all i, and $\bar{x} \in \mathcal{O} \cap A_{\bar{w}}$, then $x_i^\ell \geq \varphi_i(\bar{x}_i)$ for all i and, consequently

$$\sum_{i=1}^m x_i^\ell \geq \sum_{i=1}^m \varphi_i(\bar{x}_i) ,$$

so

$$w^\ell \geq \sum_{i=1}^m \varphi_i(\bar{x}_i) \quad \text{or} \quad \mathcal{L}(\bar{x}) \leq 0 .$$

For the proof of part 1 of the theorem we assume $H_z(v) < 0$ for all $v \in N_z$, $v \neq 0$, and claim that $\mathcal{L}$ has a local strict minimum 0 at $\bar{z}$. If so, then z is a local strict Pareto optimum, since $\mathcal{L}(\bar{x}) \leq 0$, together with $u_i(x_i) \geq u_i(z_i)$ for all i, implies $\bar{x} = \bar{z}$ and $x_i^\ell \geq z_i^\ell$ for all i.

To prove our claim, we take $\delta\bar{z} = (\delta\bar{z}_1,\ldots,\delta\bar{z}_m)$ where $\delta\bar{z}_i$ is sufficiently small and $\Sigma_{i=1}^m \delta\bar{z}_i = 0$.

$$\mathcal{L}(\bar{z}+\delta\bar{z}) = \sum_{i=1}^m \varphi_i(\bar{z}_i + \overline{\delta z}_i) - w^\ell =$$

$$= \sum_{i=1}^m (\varphi_i(\bar{z}_i) + D\varphi_i(\bar{z}_i)\overline{\delta z}_i + \tfrac{1}{2} D^2\varphi_i(\bar{z}_i)(\overline{\delta z}_i,\overline{\delta z}_i) + $$

$$+\rho_i(\overline{\delta z}_i)) - w^\ell =$$

$$= \sum_{i=1}^m z_i^\ell + \sum_{i=1}^m D\varphi_i(\bar{z}_i)\overline{\delta z}_i + \tfrac{1}{2}\sum_{i=1}^m D^2\varphi_i(\bar{z}_i)(\overline{\delta z}_i,\overline{\delta z}_i) +$$

$$+\rho(\delta z) - w^\ell =$$

$$= \sum_{i=1}^m \frac{-\bar{p}}{p^\ell}\cdot\overline{\delta z}_i + \frac{1}{2}\frac{-1}{p^\ell}\sum_{i=1}^m \frac{1}{\lambda_i} D^2 u_i(z_i)(v_i,v_i) + \rho(\delta z) \;,$$

where

$$v_i = \left(\overline{\delta z}_i, \frac{-\bar{p}\cdot\overline{\delta z}_i}{p^\ell}\right) \in p^\perp \quad \text{and} \quad \lim_{\|\delta z\|\to 0} \frac{\rho(\delta z)}{\|\delta z\|^2} = 0 \;.$$

In order to derive this Taylor expansion of $\mathcal{L}$ in z we have used the derivatives of implicit functions as given in 2.5.4, together with the fact that $p\cdot p = 1$.
Since $\Sigma_{i=1}^m \delta\bar{z}_i = 0$, it follows $v \in N_z$. So the function $\mathcal{L}$ has the following properties

(1) $\mathcal{L}(\bar{z}) = 0$;

(2) $D\mathcal{L}(\bar{z})\delta z = 0$ for all δz, satisfying $\sum_{i=1}^m \delta\bar{z}_i = 0$;

(3) the second derivative $D^2\mathcal{L}(\bar{z})$ is definite positive as a quadratic form on the set $T_{\bar{z}} A_{\bar{w}}$.

This proves our claim and the first part of the theorem.
Turning to the second part, let $v \in N_z$ and $H_z(v) > 0$. Since

$$\sum_{i=1}^m \frac{1}{\lambda_i} D^2 u_i(z_i)(v_i,v_i) > 0 \;,$$

there are reals $\alpha_1,\dots,\alpha_m$ such that

$$\alpha_i + \frac{1}{\lambda_i} D^2u_i(z_i)(v_i,v_i) > 0 \text{ for all } i \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i = 0 .$$

Let $b := (b_1,\dots,b_m)$, where $b_i = \alpha_i p$. We consider the curve $x(t)$ in A_w through z:

$$x(t) := z + tv + \tfrac{1}{2} t^2 b .$$

Since u_i is smooth, $Du_i(z_i) = \lambda_i p$, $\lambda_i > 0$, and $p \cdot p = 1$, we find using Taylor's theorem:

$$\begin{aligned} u_i(z_i + tv_i + \tfrac{1}{2} t^2 b_i) &= u_i(z_i) + Du_i(z_i)(tv_i + \tfrac{1}{2} t^2 b_i) + \\ &\qquad + \tfrac{1}{2} t^2 D^2u_i(z_i)(v_i,v_i) + \rho_i(t) \\ &= u_i(z_i) + \tfrac{1}{2} t^2(\lambda_i \alpha_i + D^2u_i(z_i)(v_i,v_i)) + \rho_i(t) , \end{aligned}$$

where

$$\lim_{t\to 0} t^{-2} \rho_i(t) = 0 ,$$

since $v_i \in Du_i(z_i)^{\perp}$ and $p \cdot p = 1$. Hence it follows

$$\lim_{t\to 0} t^{-2}(u_i(z_i + tv_i + \tfrac{1}{2} t^2 b_i) - u_i(z_i)) = \tfrac{1}{2} \lambda_i \left(\alpha_i + \frac{1}{\lambda_i} D^2u_i(z_i)(v_i,v_i)\right) > 0 .$$

So there is an $\varepsilon > 0$ such that

$$u_i(x_i(t)) = u_i(z_i + tv_i + \tfrac{1}{2} t^2 b_i) > u_i(z_i)$$

for all $t \in (0,\varepsilon)$, $i = 1,\dots,m$.
Obviously this implies that z is not a local strict Pareto optimum. □

If all of the functions u_i are strictly convex at z_i, then each second derivative $D^2u_i(z_i)$ is definite negative on the kernel of the first derivative. In that case $H_z(v) < 0$ for all $v \neq 0$, $v \in N_z$, so $z \in \theta$.
If each u_i is strictly concave at z_i, then $H_z(v) > 0$ for all $v \neq 0$, and $z \notin \theta$.
If some of the u_i are convex, others concave, then convexity may dominate and H_z is definite negative, or not. See also the examples in 1.6.

2.6. The submanifold Γ

In 1.2 first order necessary conditions for an equilibrium (x,p) are formulated, one of them being $Du_i(x_i) = \lambda_i p]_{i=1}^m$. Moreover, in 1.4 we mentioned that disastrous allocations are not welcome in our model. So we are looking for allocations x, where $Du_i(x_i) = \lambda_i p]_{i=1}^m$, and where at least one λ_i is not equal to zero. This leads to the definition of the set $\Gamma \subset \mathbb{R}^{\ell m}$, consisting of those points $v \in \mathbb{R}^{\ell m}$, for which there are reals λ_i, not all zero, and some $p \in S^{\ell-1}$, satisfying $v_i = \lambda_i p]_{i=1}^m$. See 2.6.2.

2.6.1. DEFINITION.

$$\Gamma := \{(\lambda_1 p,\ldots,\lambda_m p) \mid \lambda = (\lambda_1,\ldots,\lambda_m) \neq 0,\ p \in S^{\ell-1}\} .$$

The set Γ is parametrized by $m+\ell-1$ parameters and hence the following is not unexpected.

2.6.2. LEMMA. *Γ is a submanifold of dimension $m+\ell-1$ and for each point $x = (\lambda_1 p,\ldots,\lambda_m p) \in \Gamma$ the tangent space $T_x\Gamma$ to Γ consists of all those vectors $(\delta x_1,\ldots,\delta x_m) \in \mathbb{R}^{\ell m}$ for which there are some $\delta p \in p^\perp$ and reals $\delta\lambda_1,\ldots,\delta\lambda_m$ satisfying $\delta x_i = \delta\lambda_i p + \lambda_i \delta p]_{i=1}^m$.*

PROOF. The proof is based upon 2.1.7.
Let $z = (z_1,\ldots,z_m)$ be a point of Γ. At least one of the z_i is not 0, so we assume $z_m^\ell \neq 0$. Then $x_m^\ell \neq 0$ on a neighbourhood U of z in $\mathbb{R}^{\ell m}$.
We consider the map $\psi\colon U \to \mathbb{R}^{(m-1)(\ell-1)}$ given by

$$\psi(x_1,\ldots,x_m) := (\psi_1(x_1,x_m),\ldots,\psi_{m-1}(x_{m-1},x_m)) ,$$

where

$$\psi_i(x_i,x_m) := x_m^\ell \bar{x}_i - x_i^\ell \bar{x}_m , \quad i = 1,\ldots,m-1 .$$

Here, as in 2.5, the bar denotes that the last coordinate has been skipped.
Obviously $\psi^{\leftarrow}(0) = \Gamma \cap U$.
Our claim is that rank $D\psi(x) = (m-1)(\ell-1)$ for all $x \in U$. Since $\ell m - (m+\ell-1) = (m-1)(\ell-1)$, application of 2.1.7 settles the proof.
Given $\delta x = (\delta x_1,\ldots,\delta x_m) \in \mathbb{R}^{\ell m}$ we find $D\psi(x)(\delta x)$ from

$$\psi_i(x_i+\delta x_i\ ,x_m+\delta x_m) = (x_m^\ell+\delta x_m^\ell)(\bar{x}_i+\overline{\delta x}_i) - (x_i^\ell+\delta x_i^\ell)(\bar{x}_m+\overline{\delta x}_m) =$$

$$= \psi_i(x_i, x_m) + \delta x_m^\ell \bar{x}_i + x_m^\ell \overline{\delta x_i} - x_i^\ell \overline{\delta x_m} - \delta x_i^\ell \bar{x}_m + \rho_i(\delta x) \quad ,$$

where

$$\lim_{\|\delta x\| \to 0} \frac{1}{\|\delta x\|} \|\rho_i(\delta x)\| = 0 \; .$$

So

$$D\psi_i(x_i, x_m)(\delta x_i, \delta x_m) = \delta x_m^\ell \bar{x}_i + x_m^\ell \overline{\delta x_i} - x_i^\ell \overline{\delta x_m} - \delta x_i^\ell \bar{x}_m$$

and

$$D\psi(x)(\delta x) = (D\psi_1(x_1, x_m)(\delta x_1, \delta x_m), \ldots, D\psi_{m-1}(x_{m-1}, x_m)(\delta x_{m-1}, \delta x_m)) \; .$$

With respect to Cartesian coordinates, $D\psi(x)$ is represented by the following matrix, also denoted by $D\psi(x)$:

$$D\psi(x) = \begin{bmatrix} \Delta_m & 0 & \cdots & & 0 & -\Delta_1 \\ 0 & \Delta_m & 0 & \cdots & 0 & -\Delta_2 \\ \vdots & & & \ddots & \vdots & \vdots \\ 0 & & & & \Delta_m & -\Delta_{m-1} \end{bmatrix}$$

where Δ_i is the $(\ell - 1) \times \ell$ matrix:

$$\begin{bmatrix} x_i^\ell & \cdots & 0 & -x_i^1 \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & x_i^\ell & -x_i^{\ell-1} \end{bmatrix} \quad , \quad i = 1, \ldots, m \; ,$$

and 0 is the $(\ell - 1) \times \ell$ matrix whose entries are all zero.
Since $x_m^\ell \neq 0$ in U the matrix Δ_m has rank $\ell - 1$ and consequently the $(m-1)(\ell-1) \times m\ell$ matrix $D\psi(x)$ has rank $(m-1)(\ell-1)$ for all $x \in U$. So Γ is a submanifold of dimension $m + \ell - 1$.
Since $T_x\Gamma$ is the kernel of $D\psi(x)$, it consists of all those δx satisfying

$$\Delta_m \, \delta x_i = \Delta_i \, \delta x_m \quad , \quad i = 1, \ldots, m-1 \; .$$

If $x = (\lambda_1 p, \ldots, \lambda_m p) \in U$, then $p^\ell \neq 0$ and $\lambda_m \neq 0$. So $\Delta_i = \lambda_i P$, where

$$P := \begin{bmatrix} p^\ell & 0 & \dots & 0 & -p^1 \\ \vdots & \ddots & & \vdots & \vdots \\ 0 & & & p^\ell & -p^{\ell-1} \end{bmatrix}$$

is an $(\ell - 1) \times \ell$ matrix, the kernel of which is spanned by p.
Hence $T_x\Gamma$ consists of all those $\delta x \in \mathbb{R}^{\ell m}$ for which there are reals $\alpha_i]_{i=1}^{m-1}$ such that

$$\lambda_m \, \delta x_i - \lambda_i \, \delta x_m = \alpha_i p]_{i=1}^{m-1} .$$

Given $\delta x \in T_x\Gamma$ there is a unique decomposition

$$\delta x_m = \lambda_m \, \delta p + \delta\lambda_m p , \quad \text{where} \quad \delta p \in p^\perp .$$

Then

$$\delta x_i = \lambda_i \, \delta p + \delta\lambda_i \, p , \quad \text{where} \quad \delta\lambda_i := \lambda_m^{-1} (\lambda_i \, \delta\lambda_m + \alpha_i)]_{i=1}^{m-1} .$$

On the other hand, given $\delta p \in p^\perp$ and $\delta\lambda_1, \dots, \delta\lambda_m$, the vector $\delta x = (\delta x_1, \dots, \delta x_m)$, with $\delta x_i = \lambda_i \, \delta p + \delta\lambda_i \, p$, satisfies $\delta x \in T_x\Gamma$, as is easily seen. □

The proof of 2.6.2 is based upon the assumption that at least one of the λ_i is nonzero.
The set $\{(\lambda_1 p, \dots, \lambda_m p) \mid \lambda \in \mathbb{R}^m, \, p \in S^{\ell-1}\}$ is not a submanifold for $\ell \geq 2$, $m \geq 2$. For, assume that there is a smooth map $F: \mathbb{R}^{\ell m} \to \mathbb{R}^n$, $n \leq \ell m$, defined on a neighbourhood $\mathcal{O}$ of $0 \in \mathbb{R}^{\ell m}$ such that $F(\lambda_1 p, \dots, \lambda_m p) = 0$ for all $p \in S^{\ell-1}$ and λ in a neighbourhood of $0 \in \mathbb{R}^m$ and rank $DF(x) = n$ on $\mathcal{O}$ (see 2.1.7). Then, differentiating the coordinates $F^j]_{j=1}^n$ with respect to $\lambda_i]_{i=1}^m$ we obtain

$$\frac{\partial F^j}{\partial x_i} (\lambda_1 p, \dots, \lambda_m p) \cdot p = 0 .$$

In $x \in \Gamma$ the vector $p \in S^{\ell-1}$ is, up to its sign, uniquely determined, whereas each $p \in S^{\ell-1}$ satisfies $(\lambda_1 p, \dots, \lambda_m p) = 0$ for $(\lambda_1, \dots, \lambda_m) = 0$.
Hence it follows that $\frac{\partial F^j}{\partial x_i} (0) \cdot p = 0$ for all $p \in S^{\ell-1}$ and all i,j. So rank $DF(0) = 0$ and consequently F is constant on a neighbourhood of 0, in contradiction with 2.6.2.

CHAPTER 3
THE SET T

Introduction

The equations

$$Du_i(x_i) - \lambda_i p = 0]_{i=1}^m , \quad p \in S^{\ell-1}$$

in the unknowns (λ,x,p) may have several kinds of solution sets, depending on the m-tuple $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. In general one expects that $m\ell$ equations in $m + m\ell + \ell - 1$ unknowns have an $m + \ell - 1$ dimensional solution set, according to the implicit function theorem. In this chapter we define a subset T of $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$ having the property that the equations above establish an $m + \ell - 1$ dimensional set, for each $u \in T$.
Section 3.1 contains the definition of T and a manageable second order criterion for tuples u to be an element of T.
In Section 3.2 the number of unknowns is reduced in replacing the equation $Du_i(x_i) = \lambda_i p$ by $Du_i(x_i) = (Du_i(x_i)\cdot p)p$. This leads to a redefinition of T.
Section 3.3 is devoted to the proof that our results generalize those of Smale.
In Section 3.4 we justify the use of utility functions instead of preference relations by proving that two utility tuples, representing the same preferences, are either both in T or both not in T.
Section 3.5 contains the proof that T is dense, and in Section 3.6 we discuss the question whether T is open or not.

3.1. A first definition of T

Let $\ell \geq 2$, $m \geq 2$ and $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. Then u induces a smooth map

$$\tilde{g}_u\colon \mathbb{R}^m \times \mathbb{R}^{m\ell} \times \mathbb{R}^\ell \to \mathbb{R}^{\ell m} \times \mathbb{R}$$

as follows:

$$\tilde{g}_u(\lambda,x,p) := (Du_i(x_i) - \lambda_i p]_{i=1}^m , p \cdot p - 1) \ .$$

3.1.1. DEFINITION

$$T := \{u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m \mid \tilde{g}_u \pitchfork 0\} \ .$$

In other words: T is the set of m-tuples u for which $\tilde{g}_u$ has $0 \in \mathbb{R}^{\ell m+1}$ as a regular value.
Given $(\delta\lambda,\delta x,\delta p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times \mathbb{R}^\ell$, the derivative $D\tilde{g}_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p)$ equals

$$(D^2u_i(x_i)\delta x_i - \delta\lambda_i p - \lambda_i \delta p]_{i=1}^m , 2p \cdot \delta p) \ .$$

Rearrangement of coordinates in $\mathbb{R}^{\ell m} \times \mathbb{R}$ gives us the following matrix representation of $D\tilde{g}_u(\lambda,x,p)$:

$$D\tilde{g}_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) =$$

$$= \begin{pmatrix} D^2u_1(x_1) & \mathcal{O} & \cdots & \mathcal{O} & -p^T & \mathcal{O}_1 & \cdots & \mathcal{O}_1 & -\lambda_1 I \\ \mathcal{O} & & & \vdots & \mathcal{O}_1 & & & \vdots & \vdots \\ \vdots & & & & \vdots & & & & \\ \mathcal{O} & \cdots & & D^2u_m(x_m) & \mathcal{O}_1 & \cdots & & -p^T & -\lambda_m I \\ \mathcal{O}_1^T & \cdots & & \mathcal{O}_1^T & 0 & \cdots & & 0 & 2p \end{pmatrix} \begin{pmatrix} \delta x_1 \\ \vdots \\ \delta x_m \\ \delta\lambda_1 \\ \vdots \\ \delta\lambda_m \\ \delta p \end{pmatrix}$$

$\mathcal{O}$ is the $\ell \times \ell$ matrix with all entries zero.
$\mathcal{O}_1$ is the $\ell \times 1$ matrix with all entries zero.
I is the $\ell \times \ell$ identity matrix.

From the definition of T it follows that $u \in T$ if and only if $\tilde{g}_u(\lambda,x,p) = 0$ implies rank $D\tilde{g}_u(\lambda,x,p) = \ell m + 1$.
Let $(\alpha_1,\ldots,\alpha_m) \in \mathbb{R}^{\ell m}$ and $\beta \in \mathbb{R}$. Then

$$(\alpha_1,\dots,\alpha_m,\beta)\begin{pmatrix} D^2u_1(x_1) & 0 & \dots & 0 & -p^T & 0_1 & \dots & 0_1 & -\lambda_1 I \\ \vdots & & \ddots & \vdots & \vdots & & & & \\ 0 & \dots & & D^2u_m(x_m) & 0_1 & \dots & & -p^T & -\lambda_m I \\ 0_1^T & \dots & & 0_1^T & 0 & \dots & & 0 & 2p \end{pmatrix} =$$

$$= \left(D^2u_i(x_i)\alpha_i]_{i=1}^m , -p\cdot\alpha_i]_{i=1}^m , -\sum_{i=1}^m \lambda_i\alpha_i + 2\beta p\right) .$$

So, the $(m\ell+1) \times (m+m\ell+\ell)$ matrix, representing $D\tilde{g}_u(\lambda,x,p)$ has not full rank if and only if there is some $(\alpha_1,\dots,\alpha_m,\beta) \neq 0$ such that

$$D^2u_i(x_i)\alpha_i = 0 \text{ for all } i, \quad p\cdot\alpha_i = 0 \text{ for all } i, \quad \sum_{i=1}^m \lambda_i\alpha_i = 2\beta p .$$

3.1.2. DEFINITION.

$$S := \{(p,\alpha_1,\dots,\alpha_m) \in S^{\ell-1} \times S^{m\ell-1} \mid p\cdot\alpha_i = 0]_{i=1}^m\} .$$

(S is a compact submanifold of dimension $\ell m + \ell - m - 2$.)

Given $p \in S^{\ell-1}$ we denote by $\Pi_p\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ the orthogonal projection on the orthoplement $p^\perp$ of p.

3.1.3. DEFINITION. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. *Then*

$$G_u \in C^\infty(\mathbb{R}^{\ell m} \times S, \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times \mathbb{R}^\ell)$$

is defined as follows:

$$G_u(x,p,\alpha) := \left(\Pi_p\, Du_i(x_i)]_{i=1}^m , D^2u_i(x_i)\alpha_i]_{i=1}^m , \sum_{i=1}^m (Du_i(x_i)\cdot p)\alpha_i\right) .$$

Using the map G_u we give the following characterization of the set T.

3.1.4. THEOREM. $u \in T$ *if and only if* $G_u(x,p,\alpha) \neq 0$ *for all* $(x,p,\alpha) \in \mathbb{R}^{\ell m} \times S$.

PROOF. Let $G_u(x,p,\alpha) = 0$. Since $\Pi_p\, Du_i(x_i) = 0$ for all i, and $p \in S^{\ell-1}$ there are reals λ_i such that $Du_i(x_i) = \lambda_i p$, $i = 1,\dots,m$. So $\tilde{g}_u(\lambda,x,p) = 0$. Since $\lambda_i = Du_i(x_i)\cdot p$, it follows that the nonzero tuple $(\alpha_1,\dots,\alpha_m,0)$ belongs to the kernel of the transposed of the matrix $D\tilde{g}_u(\lambda,x,p)$. Hence $u \notin T$. On the other hand, we assume $u \notin T$. Then there is a point $(\lambda,x,p) \in \tilde{g}_u^{\leftarrow}(0)$ where $D\tilde{g}_u(\lambda,x,p)$ is not of full rank. Hence there is a nonzero pair $(\alpha,\beta) \in \mathbb{R}^{\ell m} \times \mathbb{R}$ satisfying:

$$D^2u_i(x_i)\alpha_i = 0]_{i=1}^m \,, p\cdot\alpha_i = 0]_{i=1}^m \,, \sum_{i=1}^m \lambda_i\alpha_i = 2\beta p \;.$$

It follows that

$$2\beta p\cdot p = \sum_{i=1}^m \lambda_i\alpha_i\cdot p = 0 \;.$$

So $\beta = 0$ since $p\cdot p = 1$ and consequently $\alpha \neq 0$. Since

$$Du_i(x_i) = \lambda_i p = (Du_i(x_i)\cdot p)p$$

we have

$$G_u(x,p,\frac{\alpha}{\|\alpha\|}) = 0 \;. \qquad \Box$$

For maps between Euclidean spaces the matrix representation provides a straightforward tool in determination of the rank of the derivative, whereas in general for maps between manifolds rank determination can be complicated (see definition 2.1.9). This was the reason why the map $\tilde{g}_u$ was introduced, making possible the straightforward proof of 3.1.4.
Now we return to the convention $p \in S^{\ell-1}$.

3.1.5. DEFINITION. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. *Then the smooth map*

$$g_u\colon \mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1} \to \mathbb{R}^{\ell m}$$

is defined as follows:

$$g_u(\lambda,x,p) := (Du_i(x_i) - \lambda_i p]_{i=1}^m) \;.$$

3.1.6. LEMMA.

$$T = \{u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m \mid g_u \pitchfork 0\} \;.$$

In other words: $u \in T$ *if and only if* g_u *has* $0 \in \mathbb{R}^{\ell m}$ *as a regular value.*

PROOF. Given $(\lambda,x,p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1}$ and $(\delta\lambda,\delta x,\delta p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times p^\perp$, the derivative $Dg_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p)$ equals $(D^2u_i(x_i)\delta x_i - \delta\lambda_i p - \lambda_i \delta p]_{i=1}^m)$.
Let $u \in T$ and $g_u(\lambda,x,p) = 0$. Then $\tilde{g}_u(\lambda,x,p) = 0$ and for each $(w_1,\ldots,w_m,\gamma) \in \mathbb{R}^{\ell m} \times \mathbb{R}$ there is a $(\delta\lambda,\delta x,\delta p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times \mathbb{R}^\ell$ solving the equation $D\tilde{g}_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) = (w,\gamma)$ as a consequence of the definition of T. So, choosing $\gamma = 0$, for each $w \in \mathbb{R}^{\ell m}$ there is a $(\delta\lambda,\delta x,\delta p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times p^\perp$ satisfying $Dg_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) = w$. This means that g_u has 0 as a regular value.
On the other hand we assume that g_u has 0 as a regular value. Let $\tilde{g}_u(\lambda,x,p) = 0$. Then $g_u(\lambda,x,p) = 0$ and the equation $Dg_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) = w$ has a solution with $\delta\lambda \in \mathbb{R}^m$, $\delta x \in \mathbb{R}^{\ell m}$, $\delta p \in p^\perp$, for each $w \in \mathbb{R}^{\ell m}$.
We choose $(\tilde{w},\gamma) \in \mathbb{R}^{\ell m} \times \mathbb{R}$ and consider the equations $D\tilde{g}_u(\lambda,x,p)(\tilde{\delta\lambda},\tilde{\delta x},\tilde{\delta p}) = (\tilde{w},\gamma)$. Let $w_i := \tilde{w}_i + \frac{1}{2}\lambda_i\gamma p$, $i = 1,\ldots,m$. Then it is easily seen that $Dg_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) = w$ implies

$$D\tilde{g}_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p + \tfrac{1}{2}\gamma p) = (\tilde{w},\gamma) \quad .$$

So $\tilde{g}_u$ has 0 as a regular value. □

Now we give a small refinement of 3.1.6.

3.1.7. LEMMA. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. *Then* $u \in T$ *if and only if for all points* $(\lambda,x,p) \in g_u^{\leftarrow}(0)$ *and for all* $v = (v_1,\ldots,v_m) \in (p^\perp)^m$ *the system* $\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i\delta p = v_i]_{i=1}^m$ *has a solution* $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^\perp$.

PROOF. Let $u \in T$ and $g_u(\lambda,x,p) = 0$. We choose $v \in (p^\perp)^m$ and solve the equation

$$D^2u_i(x_i)\delta x_i - \delta\lambda_i p - \lambda_i\delta p = v_i]_{i=1}^m \quad .$$

Then $\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i\delta p = v_i]_{i=1}^m$, since $\delta p \in p^\perp$.
On the other hand, let $g_u(\lambda,x,p) = 0$ and $w \in \mathbb{R}^{\ell m}$. Let $(\delta x,\delta p)$ satisfy $\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i\delta p = \Pi_p\, w_i]_{i=1}^m$. Then $(\delta\lambda,\delta x,\delta p)$ satisfies $Dg_u(\lambda,x,p)(\delta\lambda,\delta x,\delta p) = w$, if we take $\delta\lambda_i = p \cdot (D^2u_i(x_i)\delta x_i - w_i)]_{i=1}^m$. □

3.2. *An alternative definition of* T

The parameters λ, introduced in 1.2.3, play an important role in the definition of g_u, whereas they have vanished from the scene in 3.1.4.
In this section we give a characterization of T in which we do not make use explicitly of λ. As will be seen, this description of T is useful in Chapter 4, whereas 3.1.1 deserves its place in Chapters 3 and 5.

3.2.1. LEMMA. *Let* $f \in C^\infty(\mathbb{R}^{\ell m} \times S^{\ell-1}, \mathbb{R}^m)$ *be the map which sends* $(z_1,\ldots,z_m,p)$ *to* $(z_1 \cdot p,\ldots,z_m \cdot p)$. *Then* rank $Df(z,p) = m$ *for all* (z,p).

PROOF. Given $(\delta z,\delta p) \in \mathbb{R}^{\ell m} \times p^\perp$ the derivative

$$Df(z,p)(\delta z,\delta p) = (\delta z_i \cdot p + z_i \cdot \delta p]_{i=1}^m) .$$

Let $\gamma = (\gamma_1,\ldots,\gamma_m) \in \mathbb{R}^m$, then $Df(z,p)(\gamma p,0) = (\gamma_1,\ldots,\gamma_m)$, where $\gamma p := (\gamma_1 p,\ldots,\gamma_m p)$. □

From 3.2.1 it follows that each point in $\mathbb{R}^m$ is a regular value of f, and consequently that $f^{\leftarrow}(c)$ is a submanifold of dimension $\ell m + \ell - 1 - m$, for each $c \in \mathbb{R}^m$. We turn our attention to $f^{\leftarrow}(0)$.

3.2.2. DEFINITION. $V := f^{\leftarrow}(0)$.

In the next lemma we sum up the most important properties of V.

3.2.3. LEMMA. V *is a submanifold of dimension* $\ell m + \ell - 1 - m$ *and*

$$T_{(z,p)}V = \{(\delta z,\delta p) \in \mathbb{R}^{\ell m} \times p^\perp \mid p \cdot \delta z_i + z_i \cdot \delta p = 0]_{i=1}^m\}$$

for each $(z,p) \in V$.

PROOF. This follows directly from 2.4.4 by substituting $\mathbb{R}^{\ell m} \times S^{\ell-1}$ for X, $\mathbb{R}^m$ for Y and $\{0\}$ for Z. □

3.2.4. DEFINITION. $V_0 := \{(0,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1}\}$.

Since $\{0\} \subset \mathbb{R}^{\ell m}$ is a submanifold of dimension 0 and $\dim S^{\ell-1} = \ell - 1$, V_0 is a submanifold and $\dim V_0 = \ell - 1$ (see 2.1.10). Furthermore, $V_0 \subset V$.

3.2.5. DEFINITION. *Let* $u \in C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$. *Then the smooth map* $\varphi_u \colon \mathbb{R}^{\ell m} \times S^{\ell-1} \to V$ *is defined by*

$$\varphi_u(x,p) := (\Pi_p\, Du_i(x_i)]_{i=1}^m , p) \ .$$

So

$$\varphi_u(x,p) = (Du_i(x_i) - (Du_i(x_i) \cdot p)p]_{i=1}^m , p) \ ,$$

since $p \in S^{\ell-1}$ and Π_p is the projection $\mathbb{R}^{\ell} \to p^{\perp}$.
Given $(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1}$ and $(\delta x, \delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$, the derivative $D\varphi_u(x,p)(\delta x,\delta p)$ equals

3.2.6. $$(\Pi_p\, D^2u_i(x_i)\delta x_i - (Du_i(x_i) \cdot \delta p)p - (Du_i(x_i) \cdot p)\delta p]_{i=1}^m , \delta p) \ .$$

The map φ_u enables us to give the following characterization of T.

3.2.7. LEMMA. $T = \{u \in C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m \mid \varphi_u \pitchfork V_0\}$.

PROOF. Let $(x,p) \in \varphi_u^{\leftarrow}(V_0)$. Then $Du_i(x_i) = \lambda_i p$, where $\lambda_i = Du_i(x_i)\cdot p]_{i=1}^m$, since $p \in S^{\ell-1}$. Hence it follows from 3.2.6 that

$$D\varphi_u(x,p)(\delta x,\delta p) = (\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i(p\cdot\delta p)p - \lambda_i\delta p]_{i=1}^m , \delta p) =$$

$$= (\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i\delta p]_{i=1}^m , \delta p) \ .$$

From 3.2.3 we deduce

$$T_{(0,p)}V = \{(\delta z,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp} \mid p\cdot\delta z_i + 0\cdot\delta p = 0]_{i=1}^m\} =$$

$$= (p^{\perp})^m \times p^{\perp} \ .$$

From 3.2.4 and 2.1.10 it follows that

$$T_{(0,p)}V_0 = \{0\} \times p^{\perp} \ .$$

According to the definition of transversality (see 2.4.3) we have:
$\varphi_u \pitchfork V_0$ if and only if

$$D\varphi_u(x,p)(\mathbb{R}^{\ell m} \times p^{\perp}) + \{0\} \times p^{\perp} = (p^{\perp})^{m+1}$$

for each $(x,p) \in \varphi_u^{\leftarrow}(V_0)$.
Working this out we find:
$\varphi_u \pitchfork V_0$ if and only if for each point $(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1}$ for which there are reals λ_i satisfying $Du_i(x_i) = \lambda_i p]_{i=1}^{m}$, and for each $(m+1)$-tuple $v = (v_1,\ldots,v_m,v_{m+1}) \in (p^{\perp})^{m+1}$ there is a $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$ and a $q \in p^{\perp}$, satisfying:

$$\begin{cases} \Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i \delta p = v_i]_{i=1}^{m} \\ \delta p + q = v_{m+1} . \end{cases}$$

The equation $\delta p + q = v_{m+1}$ is solved by $q = v_{m+1} - \delta p$, where δp is part of the solution $(\delta x,\delta p)$ of the first equations.
Hence it follows that $\varphi_u \pitchfork V_0$ if and only if for each point $(\lambda,x,p) \in g_u^{\leftarrow}(0)$ and for each m-tuple $v = (v_1,\ldots,v_m) \in (p^{\perp})^m$ the system

$$\Pi_p\, D^2u_i(x_i)\delta x_i - \lambda_i \delta p = v_i]_{i=1}^{m}$$

has a solution $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$.
Application of 3.1.7 completes the proof. □

3.2.8. LEMMA. *If* $u \in T$ *then* $\varphi_u^{\leftarrow}(V_0)$ *is a submanifold and* $\dim \varphi_u^{\leftarrow}(V_0) = \ell + m - 1$.

PROOF. We apply 2.4.4, with

$$X = \mathbb{R}^{\ell m} \times S^{\ell-1} , \quad Y = V \quad \text{and} \quad Z = V_0 .$$

Since $\varphi_u \in C^{\infty}(\mathbb{R}^{\ell m} \times S^{\ell-1}, V)$ intersects V_0 transversally, the inverse image $\varphi_u^{\leftarrow}(V_0)$ is a submanifold and

$$\dim \varphi_u^{\leftarrow}(V_0) = \dim X - \dim Y + \dim Z = \ell m + \ell - 1 - (\ell m + \ell - 1 - m) + \ell - 1 = \ell + m - 1 .$$

□

For $u \in T$, application of 2.4.4 to the smooth map $g_u: \mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1} \to \mathbb{R}^{\ell m}$, having $0 \in \mathbb{R}^{\ell m}$ as a regular value, leads to the conclusion that also $g_u^{\leftarrow}(0)$ is a submanifold of dimension $\ell + m - 1$.
In other words, the set

$$g_u^{\leftarrow}(0) = \{(\lambda,x,p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1} \mid Du_i(x_i) = \lambda_i p]_{i=1}^m\}$$

is a submanifold of dimension $\ell + m - 1$, for $u \in T$.
Lemma 3.2.8 states that the set

$$\varphi_u^{\leftarrow}(V_0) = \{(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1} \mid \Pi_p\, Du_i(x_i) = 0]_{i=1}^m\}$$

is a submanifold of dimension $\ell + m - 1$ for $u \in T$.
The map $\varphi_u^{\leftarrow}(V_0) \to g_u^{\leftarrow}(0)$, which sends (x,p) to (λ,x,p) where $\lambda_i := Du_i(x_i) \cdot p]_{i=1}^m$, is smooth, as well as its inverse. In other words, these manifolds are diffeomorphic to each other.

3.3. *Comparison with the results of Smale*

In our set-up it is possible that utility functions have stationary points. In [15] Smale considers only utility functions without critical points. In that case it is sufficient, in search of extended equilibria, to study the set of points $(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1}$ where

$$\frac{Du_i(x_i)}{\| Du_i(x_i) \|} = p]_{i=1}^m .$$

This leads naturally to the following:
Let U be the open set of functions $f \in C^\infty(\mathbb{R}^\ell,\mathbb{R})$ for which $Df(\xi) \neq 0$ for all $\xi \in \mathbb{R}^\ell$. For $u \in U^m$, Smale defines the smooth map

$$\psi_u: \mathbb{R}^{\ell m} \times S^{\ell-1} \to (S^{\ell-1})^{m+1}$$

by

$$\psi_u(x,p) := \left(\left.\frac{Du_i(x_i)}{\| Du_i(x_i) \|}\right]_{i=1}^m , p\right) .$$

Now we consider the diagonal Δ in $(S^{\ell-1})^{m+1}$ given by

$$\Delta := \{(p_1,\dots,p_{m+1}) \in (S^{\ell-1})^{m+1} \mid p_1 = \dots = p_{m+1}\} ,$$

and prove the following:

3.3.1. LEMMA. Δ *is a submanifold and* $\dim \Delta = \ell - 1$.

PROOF. $\Delta = f^{\leftarrow}(0)$, where f is the smooth map which sends $z = (z_1,\dots,z_{m+1}) \in \in \mathbb{R}^{\ell(m+1)}$ to $(z_1 - z_{m+1}, \dots, z_m - z_{m+1}, z_{m+1} \cdot z_{m+1} - 1) \in \mathbb{R}^{m\ell} \times \mathbb{R}$. Then

$$Df(z)(\delta z) = (\delta z_1 - \delta z_{m+1}, \dots, \delta z_m - \delta z_{m+1}, 2z_{m+1} \cdot \delta z_{m+1})$$

for $\delta z \in \mathbb{R}^{\ell(m+1)}$. If $f(z) = 0$ and $(w,\gamma) = (w_1,\dots,w_m,\gamma) \in \mathbb{R}^{\ell m} \times \mathbb{R}$, the equation $Df(z)(\delta z) = (w,\gamma)$ is solved by

$$\delta z_i = \delta z_{m+1} + w_i]_{i=1}^{m} \quad \text{where} \quad \delta z_{m+1} = \tfrac{1}{2}\gamma z_{m+1} .$$

So $0 \in \mathbb{R}^{\ell m} \times \mathbb{R}$ is a regular value of the map f, or in other words: $f \pitchfork \{0\}$. Hence it follows (see 2.4.4) that Δ is a submanifold of $\mathbb{R}^{\ell(m+1)}$ and

$$\dim \Delta = \ell(m+1) - (m\ell + 1) = \ell - 1 . \qquad \square$$

3.3.2. DEFINITION (Smale).

$$Y := \{u \in U^m \mid \psi_u \pitchfork \Delta\} .$$

For $u \in U^m$ it is not longer necessary to define the map g_u (see 3.1.5) on the whole of $\mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1}$, if we are only interested in extended equilibria (x,p) where $Du_i(x_i) = \lambda_i p,\ \lambda_i > 0]_{i=1}^{m}$. So we can confine ourselves, with regard to λ, to the positive orthant $\mathbb{R}_+^m$ of $\mathbb{R}^m$. Then $u \in U^m$ induces the smooth map g_u^+ being the restriction of g_u to $\mathbb{R}_+^m \times \mathbb{R}^{\ell m} \times S^{\ell-1}$.

3.3.3. DEFINITION. $T^+ := \{u \in U^m \mid g_u^+ \pitchfork \{0\}\}$.

The following illustrates the relation of our results and those of Smale for $u \in U^m$:

3.3.4. LEMMA. $Y = T^+$.

PROOF. Let $\sigma: \mathbb{R}^\ell \setminus \{0\} \to S^{\ell-1}$ be the projection with $\sigma(v) := \frac{v}{\|v\|}$. Then for $v \neq 0$ and $\delta v \in \mathbb{R}^\ell$ the derivative $D\sigma(v)(\delta v)$ equals

$$\frac{\delta v}{\|v\|} - \left(\sigma(v) \cdot \frac{\delta v}{\|v\|}\right)\sigma(v) \ .$$

Since

$$\psi_u(x,p) = (\sigma(Du_i(x_i))]_{i=1}^m, p)$$

we derive, using the chain rule:

$$D\psi_u(x,p)(\delta x,\delta p) = (D\sigma(Du_i(x_i)) \circ D^2u_i(x_i)(\delta x_i)]_{i=1}^m, \delta p) =$$

$$= \left(\frac{D^2u_i(x_i)\delta x_i}{\|Du_i(x_i)\|} - \left(\frac{Du_i(x_i)}{\|Du_i(x_i)\|} \cdot \frac{D^2u_i(x_i)\delta x_i}{\|Du_i(x_i)\|}\right) \frac{Du_i(x_i)}{\|Du_i(x_i)\|}\Bigg|_{i=1}^m, \delta p\right)$$

for all $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$.

If $(x,p) \in \psi_u^{\leftarrow}(\Delta)$ there are positive reals λ_i such that $Du_i(x_i) = \lambda_i p]_{i=1}^m$. As is easily seen: $\lambda_i = Du_i(x_i) \cdot p$, since $p \in S^{\ell-1}$. Hence, for $(x,p) \in \psi_u^{\leftarrow}(\Delta)$ we have, introducing this λ_i in the formula:

$$D\psi_u(x,p)(\delta x,\delta p) = \left(\frac{1}{\lambda_i}\Pi_p D^2u_i(x_i)\delta x_i]_{i=1}^m, \delta p\right) .$$

The tangent space to $(S^{\ell-1})^{m+1}$ in a point $(p,\ldots,p) \in \Delta$ consists of all $(m+1)$-tuples $v = (v_1,\ldots,v_{m+1})$ satisfying $p \cdot v_i = 0$, $i = 1,\ldots,m+1$. The tangent space to the diagonal Δ in a point $(p,\ldots,p)$, being the kernel of $Df(p,\ldots,p)$ (see 3.3.1), consists of all $(q_1,\ldots,q_{m+1}) \in (p^{\perp})^{m+1}$ satisfying $q_1 = \ldots = q_{m+1}$.

We rewrite the transversality condition $\psi_u \pitchfork \Delta$ as follows:

$\psi_u \pitchfork \Delta$ if and only if for each $(x,p) \in \psi_u^{\leftarrow}(\Delta)$ and each $v \in (p^{\perp})^{m+1}$ there are $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$ and $q \in p^{\perp}$ such that

$$\begin{cases} \frac{1}{\lambda_i}\Pi_p D^2u_i(x_i)\delta x_i + q = v_i]_{i=1}^m , \\ \delta p + q = v_{m+1} . \end{cases}$$

Once we have solved $(\delta x,q)$ from the first equations, we have $\delta p = v_{m+1} - q$ as a solution for the last one.

Multiplying by λ_i and replacing $\lambda_i v_i$ by v_i we find:

$\psi_u \pitchfork \Delta$ if and only if for each $(\lambda,x,p) \in g_u^{+\leftarrow}(0)$ and each $v \in (p^{\perp})^m$ there are $(\delta x,q) \in \mathbb{R}^{\ell m} \times p^{\perp}$ such that

$$\Pi_p\ D^2u_i(x_i)\delta x_i + \lambda_i q = v_i]_{i=1}^m \ .$$

Comparison with the definition of T^+ and application of an adapted version of 3.1.7 completes the proof. □

In order to make clear that our methods apply to a more general class of utility functions we give the following simple example:
Let $u_i(x_i) = \frac{1}{2}A_i(x_i,x_i)$ where A_i is a nonsingular quadratic form on $\mathbb{R}^\ell \times \mathbb{R}^\ell$ for $i = 1,\ldots,m$. Then $D^2u_i(x_i)\delta x_i = A_i\ \delta x_i$, so $u \in T$ as is easily seen by inspection of the definition of T, and $u \notin Y$. Then for each $(\lambda,p) \in \mathbb{R}^m \times S^{\ell-1}$ the point $(\lambda,x,p) \in g_u^{\leftarrow}(0)$ when we take $x_i = \lambda_i A_i^{-1}\ p]_{i=1}^m$. As one sees the $m+\ell-1$ parameters (λ,p) give a full description of $g_u^{\leftarrow}(0)$. Here even disastrous allocations, corresponding with $\lambda = 0$, are allowed.

3.4. T *is not dependent on scale transformations*

As stated in 1.1 we are obliged to use utility functions when describing the preference relations in the economy, in order to use the calculus of manifolds and maps. We must bear in mind that two smooth functions u_i and v_i represent the same preference relation if there is a smooth function $f_i\colon \mathbb{R} \to \mathbb{R}$ with a positive derivative on the whole of $\mathbb{R}$ such that $v_i = f_i \circ u_i$. Such a function f_i is called a *scale function*, and if such a scale function exists the utility functions u_i and v_i are called *equivalent* (denoted by $u_i \sim v_i$). Hence it seems sane if the definition of T is invariant with respect to such scale transformations.

3.4.1. LEMMA. *If* $u \in T$ *and* $v_i \sim u_i]_{i=1}^m$, *then* $v \in T$.

PROOF. Let $u \in T$ and $v_i = f_i \circ u_i]_{i=1}^m$, where each f_i is a scale function. Given $x_i \in \mathbb{R}^\ell$ we denote $f_i'(u_i(x_i))$ by τ_i and $f_i''(u_i(x_i))$ by σ_i. Then straightforward calculations show:

$$Dv_i(x_i)\cdot\delta x_i = \tau_i(Du_i(x_i)\cdot\delta x_i)]_{i=1}^m$$

and

$$D^2v_i(x_i)\delta x_i = \tau_i\ D^2u_i(x_i)\delta x_i + \sigma_i(Du_i(x_i)\cdot\delta x_i)Du_i(x_i)]_{i=1}^m \ .$$

Or:

$$Dv_i(x_i) = \tau_i \; Du_i(x_i)]_{i=1}^{m}$$

and

$$D^2v_i(x_i) = \tau_i \; D^2u_i(x_i) + \sigma_i \; Du_i(x_i)^T \; Du_i(x_i)]_{i=1}^{m} \; .$$

It is sufficient to prove: $v \notin T$ implies $u \notin T$. We use 3.1.4 and assume that $G_v(x,p,\alpha) = 0$ for some $(x,p,\alpha) \in \mathbb{R}^{\ell m} \times S$ (see 3.1.3). Then we have

$$\begin{cases} \Pi_p \; Dv_i(x_i) = 0 & \text{for all } i, \\ D^2v_i(x_i)\alpha_i = 0 & \text{for all } i, \\ p \cdot \alpha_i = 0 & \text{for all } i, \\ \sum_{i=1}^{m} (Dv_i(x_i) \cdot p)\alpha_i = 0 \; . \end{cases}$$

Substituting the formulas for $Dv_i(x_i)$ and $D^2v_i(x_i)$ above and observing that $\tau_i > 0$ we find:

$$\begin{cases} \Pi_p \; Du_i(x_i) = 0 \; , \\ D^2u_i(x_i)\alpha_i = 0 \; , \\ p \cdot \alpha_i = 0 \; , \\ \sum_{i=1}^{m} (Du_i(x_i) \cdot p)\tau_i\alpha_i = 0 \; . \end{cases}$$

Since $\alpha = (\alpha_1, \ldots, \alpha_m) \in S^{m\ell-1}$ and $\tau_i > 0$ for all i, we obtain

$$\alpha^* = (\tau_1\alpha_1, \ldots, \tau_m\alpha_m) \neq 0 \quad \text{and} \quad \left(p \, , \frac{\alpha^*}{\|\alpha^*\|}\right) \in S \; .$$

Then

$$G_u\left(x, p \, , \frac{\alpha^*}{\|\alpha^*\|}\right) = 0 \; ,$$

so $u \notin T$. □

3.5. T *is a dense subset of* $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$

The proof that T is dense requires some more lemmas and definitions.

3.5.1. LEMMA. *Let* $D \subset C^\infty(\mathbb{R}^n,\mathbb{R})$ *have the following property:*
For each $f \in C^\infty(\mathbb{R}^n,\mathbb{R})$ *and for each nonnegative integer* k *there is a compact subset* C *of* $\mathbb{R}^n$ *and a sequence of functions* $f_j \in D$ *such that*

(1) $f_j(x) = f(x)$ *for all* $x \notin C$, $j = 1,2,\ldots,$

(2) $j^k f_j \to j^k f$ *uniformly on* C.

Then D *is dense in* $C^\infty(\mathbb{R}^n,\mathbb{R})$.

PROOF. Let $k \geq 0$ and let $\mathcal{O}$ be an open neighbourhood of f in the C^k Whitney topology. Then there is a continuous positive function δ on $\mathbb{R}^n$ such that

$$B_n^k(f;\delta) \subset \mathcal{O} ,$$

where

$$B_n^k(f;\delta) = \{g \in C^\infty(\mathbb{R}^n,\mathbb{R}) \mid d_n^k(j^k g(x), j^k f(x)) < \delta(x) \text{ for all } x \in \mathbb{R}^n\} .$$

(See 2.3.1.)
Hence it follows from (1) and (2) that for j sufficiently large $f_j \in B_n^k(f;\delta)$, since $j^k f_j = j^k f$ outside C and δ has a positive minimum on C. So each open neighbourhood $\mathcal{O}$ of f contains an element of D. In other words: D is dense in the C^k-topology for each $k \geq 0$. □

The next lemma is standard and we omit the proof.

3.5.2. LEMMA. *Let* $A \subset \mathbb{R}^n$ *be compact and* $f \in C^0(\mathbb{R}^m,\mathbb{R}^n)$. *Then the map* $x \to \min_{a\in A} \|f(x) - a\|$ *is continuous on* $\mathbb{R}^m$.

We recall the definition of Γ (see 2.6.1 and 2.6.2):

$$\Gamma = \{(\lambda_1 p,\ldots,\lambda_m p) \mid (\lambda_1,\ldots,\lambda_m) \neq 0,\ p \in S^{\ell-1}\} .$$

3.5.3. DEFINITION. $W := \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \Gamma$.

The set W, being a product of submanifolds, is a submanifold of dimension $\ell m + m + \ell + m - 1$ (see 2.1.10).

3.5.4. DEFINITION. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. *Then* $j^1u\colon \mathbb{R}^{\ell m} \to \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \mathbb{R}^{\ell m}$ *is the smooth map given by*

$$j^1u(x) = j^1u(x_1,\ldots,x_m) :=$$

$$= (x_1,\ldots,x_m\, ,\, u_1(x_1),\ldots,u_m(x_m)\, ,\, Du_1(x_1),\ldots,Du_m(x_m)) \; .$$

As one sees, $j^1u(x)$ is obtained by rearrangement of coordinates in

$$(j^1u_1(x_1),\ldots,j^1u_m(x_m)) = (x_1,u_1(x_1),Du_1(x_1),\ldots,x_m,u_m(x_m),Du_m(x_m)) \; .$$

Let $x \in \mathbb{R}^{\ell m}$ and $\delta x \in \mathbb{R}^{\ell m}$. Then

3.5.5. $$Dj^1u(x)(\delta x) = (\delta x_i]_{i=1}^m \, , \, Du_i(x_i)\cdot\delta x_i]_{i=1}^m \, , \, D^2u_i(x_i)\delta x_i]_{i=1}^m) \, .$$

3.5.6. DEFINITION.

(1) M *is the set of all* m*-tuples* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$, *each* u_i *being a Morse function (see* 2.4.6*).*

(2) T_0 *is the set of all* m*-tuples* u *for which* $j^1u \pitchfork W$.

The relationship between M, T_0 and T is expressed by:

3.5.7. LEMMA. $T_0 \cap M$ *is a subset of* T.

PROOF. Let $u \in T_0 \cap M$ and $(\lambda,x,p) \in g_u^{\leftarrow}(0)$, so $Du_i(x_i) = \lambda_i p]_{i=1}^m$. We choose $w = (w_1,\ldots,w_m) \in \mathbb{R}^{\ell m}$ and we consider the following system of equations in the unknowns $(\delta\lambda,\delta x,\delta p) \in \mathbb{R}^m \times \mathbb{R}^{\ell m} \times p^\perp$:

3.5.7.1. $$D^2u_i(x_i)\delta x_i - \delta\lambda_i p - \lambda_i \delta p = w_i]_{i=1}^m \; .$$

(1) If $(\lambda_1,\ldots,\lambda_m) = 0$, then each $D^2u_i(x_i)$ is nonsingular, since each u_i is a Morse function. Hence 3.5.7.1 has a solution.

(2) If $\lambda \neq 0$, then $j^1u(x) \in W$. Since $u \in T_0$ we have

$$Dj^1u(x)(\mathbb{R}^{\ell m}) + T_{j^1u(x)}W = \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \mathbb{R}^{\ell m} \; .$$

In other words: for each $(v,\gamma) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{m}$ there are $\delta x \in \mathbb{R}^{\ell m}$ and $(\delta z,\delta c,\delta y) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{m} \times T_{\lambda p}\Gamma$, where $\lambda p = (\lambda_i p]_{i=1}^{m})$, such that

3.5.7.2. $Dj^1u(x)(\delta x) + (\delta z,\delta c,\delta y) = (v,\gamma,w)$.

We use 2.6.2 and 3.5.5 to rewrite 3.5.7.2:

$$\begin{cases} \delta x_i + \delta z_i = v_i & , \quad i = 1,\dots,m , \\ Du_i(x_i) \cdot \delta x_i + \delta c_i = \gamma_i & , \quad i = 1,\dots,m , \\ D^2u_i(x_i)\delta x_i + \delta\lambda_i p + \lambda_i \delta p = w_i & , \quad i = 1,\dots,m . \end{cases}$$

The last equations tell us that 3.5.7.1 has a solution. Hence g_u has 0 as a regular value, so $u \in T$. □

In order to prove that T is dense it is sufficient to prove that $T_0 \cap M$ is dense. Since the product of open and dense sets is open and dense in the product topology it follows from 2.4.7 that M is open and dense in $C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$. So, if T_0 is dense, then $T_0 \cap M$ is dense as a consequence of the next

3.5.8. **LEMMA.** *Let* X *be a topological space,* $\mathcal{O}$ *an open and dense subset of* X, *and* A *a dense subset of* X. *Then* $A \cap \mathcal{O}$ *is dense in* X.

PROOF. Let $\mathcal{U} \subset X$ be open, $\mathcal{U} \neq \emptyset$. Then $\mathcal{U} \cap \mathcal{O} \neq \emptyset$ and $\mathcal{U} \cap \mathcal{O}$ is open. Hence $(\mathcal{U} \cap \mathcal{O}) \cap A \neq \emptyset$, in other words: $\mathcal{U} \cap (\mathcal{O} \cap A) \neq \emptyset$. □

Remains the proof that T_0 is dense. We use the Baire property of $C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$ (see 2.3.5), by defining a countable set $\{T_n\}_{n\in\mathbb{N}}$, each T_n being open and dense, such that

$$T_0 = \bigcap_{n=1}^{\infty} T_n .$$

To come to the definition of the sets T_n we choose an open countable covering $\{W_n\}_{n\in\mathbb{N}}$ of W, where each W_n is the Cartesian product

$$\mathcal{O}_n(1) \times \dots \times \mathcal{O}_n(m) \times I_n(1) \times \dots \times I_n(m) \times \Gamma_n$$

of open subsets of $\mathbb{R}^{\ell}$, $\mathbb{R}$ and Γ, respectively, such that $\bar{W}_n \subset W$ and $\bar{W}_n$ are compact. We take

$$\Gamma_n = \{(\lambda_1 p,\dots,\lambda_m p) \mid \lambda \in \Lambda_n,\ p \in S^{\ell-1}\}\ ,$$

Λ_n being an open subset of $\mathbb{R}^m \setminus \{0\}$ with $\bar{\Lambda}_n \subset \mathbb{R}^m \setminus \{0\}$ (see 2.6.1). In short notation: $W_n = \mathcal{O}_n \times I_n \times \Gamma_n$.
We define T_n to be the set of m-tuples u for which j^1u intersects W transversally on $\bar{W}_n$ (see 2.4.2). Since $W = \bigcup_{n=1}^{\infty} \bar{W}_n$ it follows that $T_0 = \bigcap_{n=1}^{\infty} T_n$.
We claim that each T_n is open and dense.

3.5.9. T_n *is open for all* $n \in \mathbb{N}$.

PROOF. Let $n \in \mathbb{N}$ and $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. We recall the definition of G_u (see 3.1.3):

$$G_u(x,p,\alpha) = \left(\Pi_p\, Du_i(x_i)]_{i=1}^m\,,\, D^2u_i(x_i)\alpha_i]_{i=1}^m\,,\ \sum_{i=1}^m (Du_i(x_i)\cdot p)\alpha_i\right).$$

Let $\delta\colon \mathbb{R}^\ell \to \mathbb{R}_+$ be a positive continuous function. Then δ defines an open neighbourhood $B_\delta(u)$ of u consisting of all v satisfying:

$$\left.\begin{array}{l} |v_i(x_i) - u_i(x_i)| < \delta(x_i) \\ \|Dv_i(x_i) - Du_i(x_i)\| < \delta(x_i) \\ \|D^2v_i(x_i) - D^2u_i(x_i)\| < \delta(x_i) \end{array}\right\} \text{ for all } x \in \mathbb{R}^{\ell m}\ .$$

From the definition of G_u and $B_\delta(u)$ it follows that there is some $\mu > 0$ such that

$$\|G_v(x,p,\alpha) - G_u(x,p,\alpha)\| < \mu\delta(x) \quad\text{and}\quad d(j^1v(x),j^1u(x)) < \mu\delta(x)$$

for all $x \in \mathbb{R}^{\ell m}$, $v \in B_\delta(u)$ and $(p,\alpha) \in S$, where $\delta(x) := \max_{1\le i\le m} \{\delta(x_i)\}$.
Since S is compact and G_u is smooth for $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$, the function $f_u\colon \mathbb{R}^{\ell m} \to \mathbb{R}$ which sends x to

$$\min\,\{\|G_u(x,p,\alpha)\| \mid (p,\alpha) \in S\}$$

is continuous and obviously nonnegative. (See 3.5.2.)
It follows from 3.1.4 that $f_u(x) > 0$ for all x implies $j^1u \pitchfork W$.

For each smooth m-tuple u we consider the continuous nonnegative function σ_u on $\mathbb{R}^{\ell m}$ given by:

$$\sigma_u(x) := d(j^1u(x),\bar{W}_n) + f_u(x) .$$

We assume $u \in T_n$. Then $\sigma_u(x) > 0$ for all x, for otherwise there is a point x for which $j^1u(x) \in \bar{W}_n$ and $f_u(x) = 0$, in contradiction with the definition of T_n. So

$$\sigma := \frac{1}{3\mu} \min \{\sigma_u(x) \mid x \in \bar{\mathcal{O}}_n\}$$

is positive.

We consider σ as a positive continuous constant function on $\mathbb{R}^{\ell}$ and claim that $B_\sigma(u) \subset T_n$. If so, then T_n is C^2-open and consequently T_n is C^∞-open.

To prove our claim, we assume that there is some $v \in B_\delta(u)$ being not an element of T_n. Then there is some (z,p,α) satisfying $j^1v(z) \in \bar{W}_n$ and $G_v(z,p,\alpha) = 0$. From this we deduce $z \in \bar{\mathcal{O}}_n$ and $\|G_u(z,p,\alpha)\| < \mu\sigma$. The last inequality implies $f_u(z) < \mu\sigma$. Since $j^1v(z) \in \bar{W}_n$ and $d(j^1u(z),j^1v(z)) < \mu\sigma$ we also have $d(j^1u(z),\bar{W}_n) < \mu\sigma$. Hence it follows that

$$f_u(z) + d(j^1u(z),\bar{W}_n) < 2\mu\sigma$$

or

$$\sigma_u(z) < 2\mu \frac{1}{3\mu} \min \{\sigma_u(x) \mid x \in \bar{\mathcal{O}}_n\} .$$

This is impossible. □

3.5.10. T_n *is dense for all* $n \in \mathbb{N}$.

PROOF. Let $n \in \mathbb{N}$, and $V_n = V_n(1) \times \dots \times V_n(m) \subset \mathbb{R}^{\ell m}$, $J_n = J_n(1) \times \dots \times J_n(m) \subset \mathbb{R}^m$ be open such that $\bar{V}_n(i)$, $\overline{J_n(i)}$ are compact and

$$\mathcal{O}_n(i) \subset \bar{\mathcal{O}}_n(i) \subset V_n(i) \subset \overline{V_n(i)} ,$$

$$I_n(i) \subset \bar{I}_n(i) \subset J_n(i) \subset \bar{J}_n(i) ,$$

for all i.

Furthermore, we consider for all i smooth functions

$$\rho_i\colon \mathbb{R}^{\ell} \to [0,1] \quad\text{and}\quad \rho_i'\colon \mathbb{R} \to [0,1]$$

satisfying

$$\operatorname{supp} \rho_i \subset V_n(i) \ , \quad \bar{O}_n(i) \subset \operatorname{int} \rho_i^{\leftarrow}(\{1\}) \ ,$$

$$\operatorname{supp} \rho_i' \subset J_n(i) \ , \quad \bar{I}_n(i) \subset \operatorname{int} \rho_i'^{\leftarrow}(\{1\}) \ .$$

(see 2.1.14).

The following picture illustrates the definition of ρ_i':

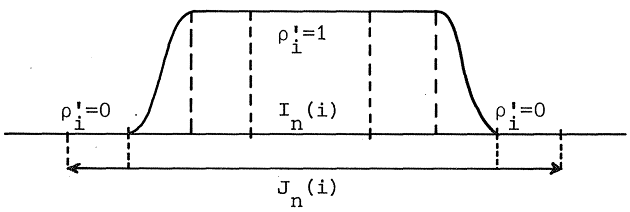


Figure 6.
Graph of ρ_i'.

Let

$$\varepsilon_i := \tfrac{1}{2} \min \{ d(\operatorname{supp} \rho_i', \mathbb{R} \setminus J_n(i)) \ , d(\bar{I}_n(i), \rho_i'^{\leftarrow}([0,1))) \} \ .$$

It follows (see Figure 6) that $\varepsilon_i > 0$ for all i, and consequently that $\varepsilon := \min \{ \varepsilon_i \mid i = 1,\dots,m \}$ is positive.

For $i = 1,\dots,m$ the open subsets B_i of $\mathbb{R}^{\ell} \times \mathbb{R}$ are given by

$$B_i := \{ (z_i, c_i) \in \mathbb{R}^{\ell} \times \mathbb{R} \mid |c_i + z_i \cdot x_i| < \varepsilon \text{ for all } x_i \in \operatorname{supp} \rho_i \} \ .$$

Let

$$B := B_1 \times \dots \times B_m \subset (\mathbb{R}^{\ell} \times \mathbb{R})^m \ .$$

Obviously, B is an open neighbourhood of 0 in $(\mathbb{R}^{\ell} \times \mathbb{R})^m$.

For $b = (b_1,\dots,b_m) \in B$, $x = (x_1,\dots,x_m) \in \mathbb{R}^{\ell m}$, $u = (u_1,\dots,u_m) \in C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$ we define

$$v_b(x) := (v_{b_1}(x_1),\dots,v_{b_m}(x_m)) \in \mathbb{R}^m$$

by

$$v_{b_i}(x_i) := u_i(x_i) + \rho_i(x_i)\rho_i'(u_i(x_i))b_i(x_i) \ , \quad i = 1,\dots,m \ ,$$

where

$$b_i(x_i) := c_i + z_i \cdot x_i \ , \quad i = 1,\dots,m \ .$$

It follows from the definition of v_b that

(1) for each integer $k \geq 0$ there is some $M_k > 0$ such that

$$d(j^k v_{b_i}(x_i), j^k u_i(x_i)) \leq M_k \|b_i\| \quad \text{for all } i \text{ and all } x_i \in \bar{V}_n(i) \ ;$$

(2) $v_{b_i}(x_i) = u_i(x_i)$ for all i and all $x_i \notin \bar{V}_n(i)$.

We claim the existence of a sequence $b_j = (b_{1,j},\dots,b_{m,j}) \in B$, converging to 0, such that $v_{b_j} \in T_n$ for all j. Then it follows from (1) and (2) that $v_{b_{i,j}} \to u_i]_{j=1}^m$ in the C^k-topology for all $k \geq 0$. Application of 3.5.1 leads to the conclusion that T_n is a dense subset of $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$.
To prove our claim we define the smooth map $\Phi\colon \mathbb{R}^{\ell m} \times B \to \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \mathbb{R}^{\ell m}$ by

$$\Phi(x,b) := j^1 v_b(x) \ .$$

So

$$\Phi(x,b) = (x_1,\dots,x_m, v_{b_1}(x_1),\dots,v_{b_m}(x_m), Dv_{b_1}(x_1),\dots,Dv_{b_m}(x_m))$$

(see 3.5.4).
We assume $(x,b) \in \Phi^{\leftarrow}(\bar{W}_n)$. Then

$$x_i \in \bar{O}_n(i) \quad \text{and} \quad v_{b_i}(x_i) \in \bar{I}_n(i) \quad \text{for all } i \ .$$

Since

$$|v_{b_i}(x_i) - u_i(x_i)| \leq |b_i(x_i)| < \varepsilon$$

we find

$$u_i(x_i) \in \operatorname{int} \rho_i'^{\leftarrow}(\{1\})$$

and hence

$$v_{b_i}(x_i) = u_i(x_i) + b_i(x_i) \ , \quad i = 1,\dots,m \ .$$

It follows from continuity of v_b that there is a neighbourhood A of (x,b) such that the same holds for $(x',b') \in A$. So

$$v_{b'_i}(x'_i) = u_i(x'_i) + b'_i(x'_i) \ , \quad i = 1,\dots,m \ ,$$

for $(x',b') \in A$.
The set A is the Cartesian product of an open neighbourhood of x in $\mathbb{R}^{\ell m}$ and an open neighbourhood B' of 0 in B.
We consider the map Φ on A, and its derivative $D\Phi(x,b)$. Since

$$\Phi(x,b) = (x_i]_{i=1}^m \ , u_i(x_i) + c_i + z_i \cdot x_i]_{i=1}^m \ , Du_i(x_i) + z_i]_{i=1}^m)$$

it follows that

$D\Phi(x,b)(\delta x,\delta b) =$

$$= (\delta x_i]_{i=1}^m \ , Du_i(x_i) \cdot \delta x_i + \delta c_i + \delta z_i \cdot x_i + z_i \cdot \delta x_i]_{i=1}^m \ , D^2u_i(x_i)\delta x_i + \delta z_i]_{i=1}^m) \ .$$

So, $D\Phi(x,b)(\delta x,\delta b) = 0$ if and only if

$$\left.\begin{aligned} &\delta x_i = 0 \\ &Du_i(x_i) \cdot \delta x_i + \delta c_i + \delta z_i \cdot x_i + z_i \cdot \delta x_i = 0 \\ &D^2u_i(x_i)\delta x_i + \delta z_i = 0 \end{aligned}\right\} \text{ for all } i \ .$$

As is easily seen this implies that $D\Phi(x,b)(\delta x,\delta b) = 0$ if and only if $(\delta x,\delta b) = 0$. Taking into account that range and domain of Φ have the same dimension $m\ell + m + m\ell$, we conclude that Φ is locally a diffeomorphism (see 2.1.5). Since $\Phi(x,b) \in \bar{W}_n$ implies that Φ is a diffeomorphism on a neighbourhood of (x,b) we conclude that there is an open neighbourhood $U_n \subset W$ of $\bar{W}_n$ such that $\Phi \pitchfork U_n$, the set U_n being a submanifold.
We apply 2.4.5 and conclude that the set $\{b \in B' \mid \Phi_b \pitchfork U_n\}$ is dense in B'. Since B' is a metric space and $0 \in B'$ there is a sequence $b_j \in B'$ converging to 0 such that $\Phi_{b_j} \pitchfork U_n$ and consequently $\Phi_{b_j} \pitchfork W$ on $\bar{W}_n$. In other words, $j^1 v_{b_j} \pitchfork W$ on $\bar{W}_n$ and by definition $v_{b_j} \in T_n$ for all j. This proves our claim and the assertion that T_n is dense. □

As a combination of 3.5.7, 3.5.8, 3.5.9 and 3.5.10 we present the following

3.5.11. THEOREM. T *is a dense subset of* $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$.

REMARK. There is a theorem in transversality theory, known as Thom's Transversality Theorem (see [6], page 54). In a version adapted to our setting it has the following form:
Let W be a submanifold of $\mathbb{R}^{\ell m} \times \mathbb{R}^m \times \mathbb{R}^{\ell m^2}$. Then the set

$$T_W := \{f \in C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m) \mid j^1 f \pitchfork W\}$$

is residual in $C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$. In other words, T_W is a countable intersection of open and dense sets.
However, we were not entitled to use this theorem as a proof for the statement that T_0 is dense, since it is not true that each $u \in C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$ is an element of $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. So we had to modify Thom's Theorem and its proof in a way which fits to each submanifold W.

3.6. *Is* T *open in* $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$ *?*

As pointed out in 1.2, the question whether T is dense is more interesting then the one whether T is open. However, Smale [15] and Van Geldrop [4] assert without proof that Y and T, respectively, are open in the C^2-topology. In this section we discuss the problems which arise, when one tries to give a proof of openness. Our conclusion is that openness of Y or T is still an open problem.
Intuitively one could think that T is open since surjective maps form an open subset. But the problem lies in the fact that the Whitney C^∞-topology on $C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$ induces on the subset $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$ a topology different from the product topology on $C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$.
To make our reasoning more concrete we define for $v \in C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$ the partial derivatives

$$\frac{\partial v_i}{\partial x_j} := \left(\frac{\partial v_i}{\partial x_j^1}, \ldots, \frac{\partial v_i}{\partial x_j^\ell}\right), \quad i = 1,\ldots,m, \; j = 1,\ldots,m,$$

where $v(x) = (v_1(x),\ldots,v_m(x))$.
We extend the notion of g_v to such maps v and define
$g_v \in C^\infty(\mathbb{R}^m \times \mathbb{R}^{\ell m} \times S^{\ell-1}, \mathbb{R}^{\ell m})$ by

$$g_v(\lambda,x,p) := \left(\frac{\partial v_i}{\partial x_i}(x) - \lambda_i p]_{i=1}^m\right).$$

We extend also G_v and we define

$$G_v(x,p,\alpha) := \left(\Pi_p \frac{\partial v_i}{\partial x_i}(x)]_{i=1}^m, \sum_{j=1}^m \frac{\partial^2 v_j}{\partial x_j\, \partial x_i}\alpha_j]_{i=1}^m, \sum_{i=1}^m \left(\frac{\partial v_i}{\partial x_i}\cdot p\right)\alpha_i\right),$$

where

$$\frac{\partial^2 v_j}{\partial x_j\, \partial x_i} := \begin{pmatrix} \frac{\partial^2 v_j}{\partial x_j^1\, \partial x_i^1} & \cdots & \frac{\partial^2 v_j}{\partial x_j^\ell\, \partial x_i^1} \\ \frac{\partial^2 v_j}{\partial x_j^1\, \partial x_i^\ell} & \cdots & \frac{\partial^2 v_j}{\partial x_j^\ell\, \partial x_i^\ell} \end{pmatrix}.$$

3.6.1. DEFINITION. T' *is the set of* $v \in C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$ *for which* g_v *has* $0 \in \mathbb{R}^{\ell m}$ *as a regular value.*

3.6.2. LEMMA. $v \in T'$ *if and only if* $G_v(x,p,\alpha) \neq 0$ *for all* $(x,p,\alpha) \in \mathbb{R}^{\ell m} \times S$.

PROOF. The proof is basically the same as the one of 3.1.4. □

We use 3.6.2 in the proof of

3.6.3. LEMMA. T' *is open in* $C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$.

PROOF. Let $v \in T'$. Then the continuous map $f_v: \mathbb{R}^{\ell m} \to \mathbb{R}$, given by

$$f_v(x) := \min\{\|G_v(x,p,\alpha)\| \mid (p,\alpha) \in S\}$$

satisfies $f_v(x) > 0$ for all x, due to 3.6.2.
We assume $\delta: \mathbb{R}^{\ell m} \to \mathbb{R}$ to be continuous and $\delta(x) > 0$ for all x. There is a positive constant μ such that for all $v' \in C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$ whose partial derivatives up to order 2 in $x \in \mathbb{R}^{\ell m}$ do not differ more than $\delta(x)$ from the corresponding derivatives of v, the following holds:

$$\|G_{v'}(x,p,\alpha) - G_v(x,p,\alpha)\| < \mu\delta(x) \quad \text{for all } (p,\alpha) \in S.$$

These functions v' constitute an open neighbourhood $B_\delta(v)$ of v in the C^2-

topology. Hence it follows that

$$|f_{v'}(x) - f_v(x)| < \mu\delta(x) \quad \text{for all } v' \in B_\delta(v) .$$

We choose

$$\delta(x) < \frac{1}{\mu} f_v(x) .$$

Then $B_\delta(v) \subset T'$ as is easily seen. So T' is C^2-open and consequently T' is C^∞-open. □

Now we define the map

$$\eta : C^\infty(\mathbb{R}^\ell,\mathbb{R})^m \to C^\infty(\mathbb{R}^{\ell m},\mathbb{R}^m)$$

by

$$\eta(u)(x) := (u_1(x_1),\dots,u_m(x_m)) \quad \text{for } u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m .$$

The map η is injective and $T = \overleftarrow{\eta}(T')$. However, η is not continuous and it is this fact that causes the problem in the proof of openness of T, and by restriction to the open subset of functions without critical points one meets the same obstacle in trying to prove that Y is open.

CHAPTER 4

THE SET OF EQUILIBRIA IN A PURE EXCHANGE ECONOMY

Introduction

In this chapter we rewrite the equations 1.2.3 which define points $(x,p) \in E_{cr}(r,u)$ by:

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i \, , \quad p \cdot x_i = p \cdot r_i]_{i=1}^{m-1} \, , \quad \Pi_p \, Du_i(x_i) = 0]_{i=1}^{m} \, .$$

We prove that for $u \in T$ the set of r for which $E_{cr}(r,u)$ is discrete, is dense in $\mathbb{R}^{\ell m}$. Since T is dense this implies that the set of pairs (r,u) for which $E_{cr}(r,u)$ and consequently $E_{ex}(r,u)$ is discrete, is dense in the set of economies (r,u).

Section 4.1 is devoted to the concept of regular economies. This is a set of pairs (r,u) for which we show that $E_{cr}(r,u)$ is discrete.

Section 4.2 contains an elaborate example with utility functions having stationary points.

In Section 4.3 we introduce an open subset E of $C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})$ such that $E^m \subset T$. For utility tuples $u \in E^m$ the manifold structure of $\overset{\leftarrow}{\varphi_u}(V_0)$ (see 3.2.8) as a set, parametrized by $\ell + m - 1$ parameters, is made explicit.

4.1. Regular economies

As before, let $\ell \geq 2$ be the number of commodities and $m \geq 2$ the number of agents in the economy. Agent i is characterized by a pair $(r_i,u_i) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})$, where r_i is his initial bundle and u_i the utility function representing his preference relation on $\mathbb{R}^{\ell}$.

An economy is a pair $(r,u) = (r_1,\dots,r_m, u_1,\dots,u_m)$ and the set of economies is topologized by the product topology induced by the metric topology on $\mathbb{R}^{\ell m}$ and the product Whitney C^{∞}-topology on $C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$.

We recall the definition of $E_{cr}(r,u)$ (see 1.2.3 and the definition of Π_p):

4.1.1. $$E_{cr}(r,u) = \left\{(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1} \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i \,, \right.$$
$$\left. p \cdot x_i = p \cdot r_i]_{i=1}^{m-1} \,, \Pi_p\, Du_i(x_i) = 0]_{i=1}^{m} \right\} .$$

4.1.2. DEFINITION. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$. *Then*

$$F_u\colon \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1} \to \mathbb{R}^\ell \times \mathbb{R}^{m-1} \times V$$

is the smooth map:

$$F_u(r,x,p) := \left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} r_i \,, p \cdot x_i - p \cdot r_i]_{i=1}^{m-1} \,, \varphi_u(x,p)\right) .$$

For the definition of V, V_0 and φ_u see Section 3.2.
As one sees we can write $F_u(r,x,p)$ as

$$F_u(r,x,p) = (F_1(r,x,p) \,, \varphi_u(x,p))$$

where

$$F_1 \in C^\infty(\mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1} \,, \mathbb{R}^\ell \times \mathbb{R}^{m-1}) .$$

4.1.3. LEMMA. Rank $DF_1(r,x,p) = \ell + m - 1$ *for all* (r,x,p).

PROOF. Since by definition

$$F_1(r,x,p) = \left(\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} r_i \,, p \cdot x_i - p \cdot r_i]_{i=1}^{m-1}\right) ,$$

the derivative $DF_1(r,x,p)(\delta r,\delta x,\delta p)$ equals

$$\left(\sum_{i=1}^{m} \delta x_i - \sum_{i=1}^{m} \delta r_i \,, p \cdot \delta x_i - p \cdot \delta r_i + \delta p \cdot x_i - \delta p \cdot r_i]_{i=1}^{m-1}\right)$$

for all $(\delta r,\delta x,\delta p) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times p^\perp$.
We choose $(w,\gamma) = (w,\gamma_1,\dots,\gamma_{m-1}) \in \mathbb{R}^\ell \times \mathbb{R}^{m-1}$. Then for each $(\delta x,\delta p)$ the equation $DF_1(r,x,p)(\delta r,\delta x,\delta p) = (w,\gamma)$ is solved when taking

$$\delta r_i = (p \cdot \delta x_i + \delta p \cdot x_i - \delta p \cdot r_i - \gamma_i)p]_{i=1}^{m-1} ,$$

$$\delta r_m = \sum_{i=1}^{m} \delta x_i - w - \sum_{i=1}^{m-1} \delta r_i . \qquad \square$$

The proof of 4.1.3 enables us to give the following characterization of T.

4.1.4. LEMMA. *Let* $u \in C^\infty(\mathbb{R}^\ell, \mathbb{R})^m$. *Then* $u \in T$ *if and only if* $F_u \pitchfork \{0\} \times V_0$, *where* $\{0\} \in \mathbb{R}^\ell \times \mathbb{R}^{m-1}$.

PROOF. Let $u \in T$ and $(r,x,p) \in F_u^{\leftarrow}(\{0\} \times V_0)$. It must be shown that

$$DF_u(r,x,p)(\mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times p^\perp) + T_{F_u(r,x,p)}(\{0\} \times V_0) =$$

$$= T_{F_u(r,x,p)}(\mathbb{R}^\ell \times \mathbb{R}^{m-1} \times V) .$$

We choose $v \in T_{\varphi_u(x,p)}V$ and $(w,\gamma) \in \mathbb{R}^\ell \times \mathbb{R}^{m-1}$, and consider the following system of equations in $(\delta r, \delta x, \delta p, v_0) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times p^\perp \times T_{\varphi_u(x,p)}V_0$:

$$(DF_1(r,x,p)(\delta r,\delta x,\delta p) , D\varphi_u(x,p)(\delta x,\delta p)) + (0,v_0) = (w,\gamma,v) .$$

In components:

$$\begin{cases} DF_1(r,x,p)(\delta r,\delta x,\delta p) = (w,\gamma) , \\ D\varphi_u(x,p)(\delta x,\delta p) + v_0 = v . \end{cases}$$

Since $u \in T$, the second equation has a solution $(\delta x, \delta p, v_0)$ and it follows from the proof of 4.1.3 that the first equation has a solution. Hence $F_u \pitchfork \{0\} \times V_0$.

Now let $F_u \pitchfork \{0\} \times V_0$ and $(x,p) \in \varphi_u^{\leftarrow}(V_0)$. Then, as is easily seen, $F_u(x,x,p) \in \{0\} \times V_0$. We choose $v \in T_{\varphi_u(x,p)}V$ and consider the equation

$$(DF_1(x,x,p)(\delta r,\delta x,\delta p) , D\varphi_u(x,p)(\delta x,\delta p) + (0,v_0) = (0,0,v) ,$$

in the unknowns $(\delta r, \delta x, \delta p, v_0)$.

Since $F_u \pitchfork \{0\} \times V_0$, this system has a solution and hence $D\varphi_u(x,p)(\delta x,\delta p) + v_0 = v$ has a solution. So $\varphi_u \pitchfork V_0$ and, by 3.2.7, $u \in T$. $\square$

4.1.5. DEFINITION. *Let* $u \in C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$. *Then*

$$Z_u := \left\{ (r,x,p) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1} \mid \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i , \right.$$

$$\left. p \cdot x_i = p \cdot r_i]_{i=1}^{m-1} , \Pi_p \, Du_i(x_i) = 0]_{i=1}^{m} \right\} .$$

4.1.6. LEMMA. *For* $u \in T$ *the set* Z_u *is a submanifold of dimension* $m\ell$, *and the tangent space*

$$T_{(r,x,p)} Z_u = \left\{ (\delta r, \delta x, \delta p) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times p^{\perp} \mid \sum_{i=1}^{m} \delta x_i = \sum_{i=1}^{m} \delta r_i , \right.$$

$$\delta p \cdot x_i + p \cdot \delta x_i = \delta p \cdot r_i + p \cdot \delta r_i]_{i=1}^{m-1} ,$$

$$\left. \Pi_p \, D^2 u_i(x_i) \delta x_i = (Du_i(x_i) \cdot p) \delta p]_{i=1}^{m} \right\}$$

for $(r,x,p) \in Z_u$.

PROOF. Let $u \in T$, so $F_u \pitchfork \{0\} \times V_0$. Hence application of 2.4.4, where

$$X = \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1} , \quad Y = \mathbb{R}^{\ell} \times \mathbb{R}^{m-1} \times V , \quad Z = \{0\} \times V_0 ,$$

leads to the following:
$F_u^{\leftarrow}(\{0\} \times V_0)$ is a submanifold and

$$\dim F_u^{\leftarrow}(\{0\} \times V_0) = \dim X - \dim Y + \dim Z =$$

$$= \ell m + \ell m + \ell - 1 - (\ell + m - 1 + \ell m + \ell - 1 - m) + \ell - 1 = \ell m .$$

Since $Z_u = F_u^{\leftarrow}(\{0\} \times V_0)$, the first part has been proved. The second part is an easy consequence of the definition of Z_u and 2.4.4. □

As one sees the dimension of the manifold Z_u is the same as the dimension of the space in which the first coordinate r moves. Se we can draw Z_u in the (r,x,p) space as in Figure 7:

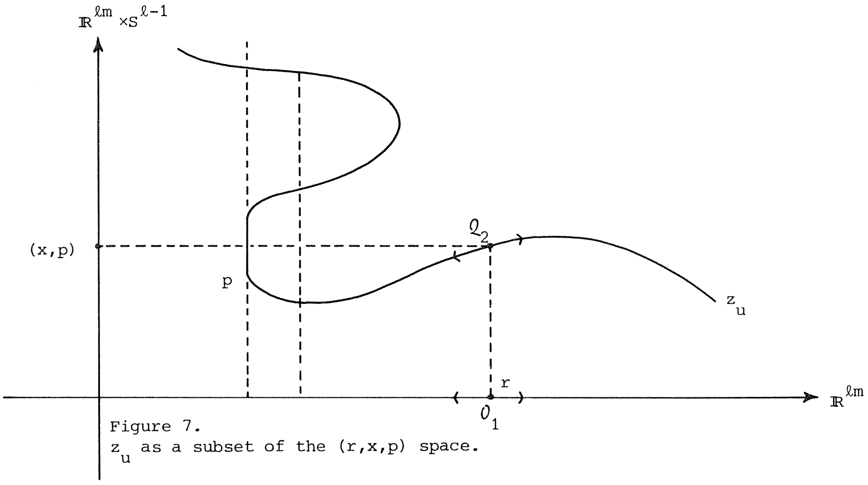


Figure 7.
Z_u as a subset of the (r,x,p) space.

It follows from the definition of Z_u that $(r,x,p) \in Z_u$ implies $(x,p) \in E_{cr}(r,u)$. A glance at Figure 7 tells us that in general, for $u \in T$, the set $E_{cr}(r,u)$ is discrete.
To make things more precise, we consider for $u \in T$ the smooth map

$$P\colon Z_u \to \mathbb{R}^{\ell m} \quad \text{where} \quad P(r,x,p) := r\ .$$

4.1.7. DEFINITION. *An economy* (r,u) *is said to be regular provided*

(1) $u \in T$;

(2) r *is a regular value of* P.

4.1.8. LEMMA. *Let* (r,u) *be a regular economy. Then* $E_{cr}(r,u)$ *is a discrete set.*

PROOF. Since $r \in \mathbb{R}^{\ell m}$ is a regular value of P and $\dim Z_u = \ell m$, the inverse image $P^{\leftarrow}(r)$ of r is a discrete subset of Z_u (see 2.4.4 with $X = Z_u$, $Y = \mathbb{R}^{\ell m}$, $Z = \{r\}$). Obviously, $P^{\leftarrow}(r) = \{r\} \times E_{cr}(r,u)$. This completes the proof. □

The next Theorem shows that $E_{cr}(r,u)$ is in general discrete.

4.1.9. THEOREM. *The set of pairs* (r,u) *for which* $E_{cr}(r,u)$ *is discrete, is dense in* $\mathbb{R}^{\ell m} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$.

PROOF. It is sufficient to show that the set of regular economies is dense. Let $O \neq \emptyset$ be open in $\mathbb{R}^{\ell m} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$. Then there are open subsets

$\mathcal{O}_1 \subset \mathbb{R}^{\ell m}$, $\mathcal{O}_2 \subset C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$ such that $\mathcal{O}_1 \times \mathcal{O}_2 \subset \mathcal{O}$. Since T is dense there is some $u \in \mathcal{O}_2 \cap T$. We consider the manifold Z_u. Since the set of regular values of $\mathcal{P}$ on Z_u is dense (see 2.2.6 or 2.2.7), there is some $r \in \mathbb{R}^{\ell m}$ being a regular value and $r \in \mathcal{O}_1$. Hence it follows that $(r,u) \in \mathcal{O}$ is a regular economy. □

In the next lemma we work out the definition of a regular economy.

4.1.10. LEMMA. *A pair* (r,u) *is a regular economy if and only if*

(1) $u \in T$;

(2) *for all* $(r,x,p) \in Z_u$ *the system of equations*

4.1.10.1
$$\begin{cases} \sum_{i=1}^{m} \delta x_i = 0 , \\ \delta p \cdot x_i + p \cdot \delta x_i = \delta p \cdot r_i]_{i=1}^{m-1} , \\ \Pi_p D^2 u_i(x_i)\delta x_i = (Du_i(x_i) \cdot p)\delta p]_{i=1}^{m} , \end{cases}$$

in the unknowns $(\delta x,\delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$ *has only the trivial solution* $\delta x = 0$, $\delta p = 0$.

PROOF. Let $u \in T$ and $(r,x,p) \in Z_u$. Then

$$D\mathcal{P}(r,x,p)(\delta r,\delta x,\delta p) = \delta r \quad \text{for} \quad (\delta r,\delta x,\delta p) \in T_{(r,x,p)}Z_u .$$

It follows from the definition 4.1.7 of regular economies that (r,u) is regular if and only if $D\mathcal{P}(r,x,p)(\delta r,\delta x,\delta p) = 0$ implies $(\delta r,\delta x,\delta p) = (0,0,0)$. In other words: $(0,\delta x,\delta p) \in T_{(r,x,p)}Z_u$ implies $\delta x = 0$, $\delta p = 0$. Application of the second part of 4.1.6 completes the proof. □

Now we discuss the position of disastrous allocations in the whole and prove the following theorem (see 1.4).

4.1.11. THEOREM (GENERAL RESULT I). *Let* $u \in T$. *Then there is a dense subset* R_u *of* $\mathbb{R}^{\ell m}$ *such that*

(1) (r,u) *is a regular economy for all* $r \in R_u$.

(2) *If* $r \in R_u$ *the set* $E_{cr}(r,u)$ *does not contain points* (x,p) *where* x *is a disastrous allocation.*

The proof of 4.1.11 requires one more lemma on the set T.

4.1.12. LEMMA. *Let* $u \in T$ *and* $x \in \mathbb{R}^{\ell m}$ *be such that* $Du_i(x_i) = 0]_{i=1}^m$. *Then* $D^2u_i(x_i)$ *is nonsingular for all* i.

PROOF. We assume there is some $\alpha_j \in \mathbb{R}^\ell \setminus \{0\}$ such that $D^2u_j(x_j)\alpha_j = 0$. Then we may also assume $\alpha_j \in S^{\ell-1}$. Choosing $p \in S^{\ell-1} \cap \alpha_j^\perp$, we consider $(p,\alpha) \in S$, where $\alpha_i = 0$ for $i \neq j$ and α_j as given above. Then $G_u(x,p,\alpha) = 0$ (see 3.1.4) which contradicts $u \in T$. □

Now we come to the proof of 4.1.11. Consider the smooth map

$$\xi_u\colon \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1} \to \mathbb{R}^\ell \times \mathbb{R}^{m-1} \times \mathbb{R}^{\ell m}$$

where

$$\xi_u(r,x,p) := \left(\sum_{i=1}^m x_i - \sum_{i=1}^m r_i \,, p \cdot x_i - p \cdot r_i]_{i=1}^{m-1} \,, Du_i(x_i)]_{i=1}^m\right) .$$

Then

$$D\xi_u(r,x,p)(\delta r,\delta x,\delta p) = \left(\sum_{i=1}^m \delta x_i - \sum_{i=1}^m \delta r_i \,,\right.$$

$$\left. p \cdot \delta x_i - p \cdot \delta r_i + \delta p \cdot x_i - \delta p \cdot r_i]_{i=1}^{m-1} \,, D^2u_i(x_i)\delta x_i]_{i=1}^m\right) ,$$

for $(\delta r,\delta x,\delta p) \in \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times p^\perp$.
If $(r,x,p) \in \xi_u^{\leftarrow}(0)$, the equation

$$D\xi_u(r,x,p)(\delta r,\delta x,\delta p) = (v,\gamma,w)$$

has a solution for each $v \in \mathbb{R}^\ell$, $\gamma \in \mathbb{R}^{m-1}$, $w \in \mathbb{R}^{\ell m}$, as a consequence of 4.1.12 and 4.1.3. So ξ_u has 0 as a regular value and consequently $\xi_u^{\leftarrow}(0)$ is a submanifold of dimension $\ell m - m$. This follows from 2.4.4 where $X = \mathbb{R}^{\ell m} \times \mathbb{R}^{\ell m} \times S^{\ell-1}$, $Y = \mathbb{R}^\ell \times \mathbb{R}^{m-1} \times \mathbb{R}^{\ell m}$, $Z = 0$. The restriction to $\xi_u^{\leftarrow}(0)$ of the smooth map $(r,x,p) \to r$ maps the $(\ell m - m)$ dimensional submanifold $\xi_u^{\leftarrow}(0)$ into $\mathbb{R}^{\ell m}$, and consequently the set of $r \in \mathbb{R}^{\ell m}$ for which there is some (x,p) satisfying $\xi_u(r,x,p) = 0$ is thin in $\mathbb{R}^{\ell m}$ (see 2.2.3).

Since the set of critical values of the map $\mathcal{P}: Z_u \to \mathbb{R}^{\ell m}$ is thin in $\mathbb{R}^{\ell m}$ (see 2.2.6) and the union of two thin sets is a thin set, we obtain the following:

If $u \in T$ the set of $r \in \mathbb{R}^{\ell m}$ such that either

(1) r is a critical value of $\mathcal{P}$, or

(2) there is a disastrous allocation x and a price system p such that

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i \ , \quad p \cdot x_i = p \cdot r_i]_{i=1}^{m-1} \ ,$$

is thin in $\mathbb{R}^{\ell m}$. Hence its complement is dense in $\mathbb{R}^{\ell m}$, and we denote this complement by R_u. Then it follows that for all $r \in R_u$ the economy (r,u) is regular, and $E_{cr}(r,u)$ does not contain points (x,p) with $Du_i(x_i) = 0]_{i=1}^{m}$. □

4.2. An example

In this section we reconsider the last example of 1.6.

Let

$$u_i(x_i) := -\tfrac{1}{2}(x_i - a_i)\cdot(x_i - a_i)]_{i=1}^{m} \ ,$$

where the bundle $a_i \in \mathbb{R}^{\ell}$ represents the ultimate desire of agent i. Then

$$Du_i(x_i) = a_i - x_i \quad \text{and} \quad D^2u_i(x_i) = -I \ .$$

Since each $D^2u_i(x_i)$ is nonsingular we have $u \in T$.

Let $r \in \mathbb{R}^{\ell m}$. We shall prove:

(1) (r,u) is not a regular economy if

$$\sum_{i=1}^{m} r_i = \sum_{i=1}^{m} a_i \ ;$$

(2) (r,u) is a regular economy if

$$\sum_{i=1}^{m} r_i \neq \sum_{i=1}^{m} a_i \ .$$

As one sees, all initial bundles r for which the disastrous allocation $a = (a_1,\dots,a_m)$ is admissible, give rise to an economy which is not a regular one. The set of r satisfying $\Sigma_{i=1}^m r_i = \Sigma_{i=1}^m a_i$ is a $(m-1)\ell$ dimensional submanifold of $\mathbb{R}^{\ell m}$ and hence a thin set.

Firstly we consider Z_u. It is easily seen that $(r,x,p) \in Z_u$ if and only if

$$4.2.1. \quad \begin{cases} x_i = \Pi_p a_i + (p \cdot r_i)p]_{i=1}^m , \\ \Pi_p\left(\sum_{i=1}^m r_i - \sum_{i=1}^m a_i\right) = 0 . \end{cases}$$

We apply 4.1.10. Let $(r,x,p) \in Z_u$. The system 4.1.10.1 for this case has the form

$$4.2.2. \quad \begin{cases} \sum_{i=1}^m \delta x_i = 0 , \\ \delta p \cdot \Pi_p a_i + p \cdot \delta x_i = \delta p \cdot r_i]_{i=1}^{m-1} , \\ \Pi_p \delta x_i = ((r_i - a_i) \cdot p)\delta p]_{i=1}^m . \end{cases}$$

The second and the third line in 4.2.2 yield

$$\delta x_i = (\delta p \cdot (r_i - \Pi_p a_i))p + ((r_i - a_i) \cdot p)\delta p]_{i=1}^{m-1} ,$$

$$\Pi_p \delta x_m = ((r_m - a_m) \cdot p)\delta p .$$

Combining this with $\Sigma_{i=1}^m \delta x_i = 0$ and 4.2.1, we find

$$4.2.3. \quad \begin{cases} \left(\left(\sum_{i=1}^m a_i - \sum_{i=1}^m r_i\right) \cdot p\right)\delta p = 0 , \\ \Pi_p\left(\sum_{i=1}^m r_i - \sum_{i=1}^m a_i\right) = 0 . \end{cases}$$

If $\Sigma_{i=1}^m r_i = \Sigma_{i=1}^m a_i$, then $(r,x,p) \in Z_u$ for each $p \in S^{\ell-1}$ where $x_i = \Pi_p a_i + (p \cdot r_i)p]_{i=1}^m$. Moreover, the first line in 4.2.3 is solved by each $\delta p \in p^{\perp}$. So r is not a regular value of P and $P^{\leftarrow}(r) \subset Z_u$ is not a discrete set.

If $\Sigma_{i=1}^m r_i \neq \Sigma_{i=1}^m a_i$, then only

$$\pm \frac{\sum_{i=1}^{m} r_i - \sum_{i=1}^{m} a_i}{\| \sum_{i=1}^{m} r_i - \sum_{i=1}^{m} a_i \|}$$

is a solution for p and 4.2.3 is only solved by $\delta p = 0$ and consequently $\delta x = 0$, so (r,u) is a regular economy. For instance: $r_i = a_i]_{i=1}^{m-1}$ and $r_m \neq a_m$ leads to a regular economy and

$$(x,p) \in E_{cr}(r,u) \quad \text{if} \quad x_i = a_i]_{i=1}^{m-1}, \; x_m = r_m, \; p = \frac{r_m - a_m}{\| r_m - a_m \|} .$$

It seems a little unsatisfactory that for this utility tuple u each initial allocation r which allows the optimal allocation $(a_1, \ldots, a_m)$ is not regular. However, we must bear in mind that in general regularity of the economy (r,u) has to do with a kind of stability. Precisely: if (r,u) is a regular economy and $(x,p) \in E_{cr}(r,u)$, then there is a neighbourhood $\mathcal{O}_1$ of r in $\mathbb{R}^{\ell m}$ and a neighbourhood $\mathcal{O}_2$ of (r,x,p) in Z_u such that the restriction of $\mathcal{P}$ to $\mathcal{O}_2$ is bijective $\mathcal{O}_2 \to \mathcal{O}_1$ with a smooth inverse, see Figure 7 and 2.1.5. This means that there are smooth functions x and p, defined on $\mathcal{O}_1$ such that $(x(r),p(r)) \in E_{cr}(r,u)$ and $(r,x(r),p(r)) \in \mathcal{O}_2$.
Returning to our example we see that for r satisfying $\Sigma_{i=1}^m r_i = \Sigma_{i=1}^m a_i$ each $p \in S^{\ell-1}$ together with $x_i = \Pi_p a_i + (p \cdot r_i)p]_{i=1}^m$ belongs to $E_{cr}(r,u)$. But every neighbourhood of r contains points r' not satisfying $\Sigma_{i=1}^m r_i' = \Sigma_{i=1}^m a_i$, for which $E_{cr}(r',u)$ consists of two points. It is not difficult to understand that for each $(x,p) \in E_{cr}(r,u)$ there is a sequence of points r'(n) such that the corresponding $E_{cr}(r'(n),u)$ do not converge to (x,p), which implies that the equilibrium (x,p) is not stable in the sense we described above.

4.3. The set E

It seems interesting to indicate a subset of $C^\infty(\mathbb{R}^\ell, \mathbb{R})$ such that for all m each m-tuple u is an element of T if each u_i belongs to that subset.
In this section we define an open subset E of $C^\infty(\mathbb{R}^\ell, \mathbb{R})$ satisfying $E^m \subset T$.

4.3.1. DEFINITION. E *is the set of all smooth functions* $f: \mathbb{R}^\ell \to \mathbb{R}$ *satisfying*

$$Df(x)^\perp \cap \text{Kernel } D^2f(x) = \{0\} \quad \textit{for all } x \in \mathbb{R}^\ell .$$

4.3.1.1. It follows from 4.3.1 that each $f \in C^\infty(\mathbb{R}^\ell,\mathbb{R})$ for which $D^2f(x)$ is nonsingular for all x, is an element of E and that each $f \in E$ is a Morse function.

4.3.1.2. Moreover, if f is locally strict convex or concave, then $f \in E$. If, on the contrary, there are some $x \in \mathbb{R}^\ell$, $v \in Df(x)^\perp \cap \text{kernel } D^2f(x)$, $v \neq 0$, then $D^2f(x)(v,v) = 0$ in contradiction with strict convexity or strict concavity.

4.3.2. LEMMA. $E^m \subset T$.

PROOF. Let $u = (u_1,\ldots,u_m) \in E^m$. If $G_u(x,p,\alpha) = 0$ for some (x,p,α) (see 3.1.3), then

$$\begin{cases} Du_i(x_i) = \lambda_i p]_{i=1}^m , \\ p \cdot \alpha_i = 0]_{i=1}^m , \\ D^2u_i(x_i)\alpha_i = 0]_{i=1}^m , \\ \sum_{i=1}^m \lambda_i\alpha_i = 0 , \\ \alpha \neq 0 . \end{cases}$$

Hence

$$\alpha_i \in Du_i(x_i)^\perp \cap \text{kernel } D^2u_i(x_i) \quad \text{for all } i ,$$

in contradiction with $\alpha \neq 0$.
Application of 3.1.4 completes the proof. □

4.3.3. LEMMA. E *is open in* $C^\infty(\mathbb{R}^\ell,\mathbb{R})$.

PROOF. For $f \in C^\infty(\mathbb{R}^\ell,\mathbb{R})$ we consider the smooth map σ_f which sends $(x,p) \in \mathbb{R}^\ell \times S^{\ell-1}$ to $(Df(x)\cdot p , D^2f(x)p) \in \mathbb{R} \times \mathbb{R}^\ell$. Inspection of the definition yields: $f \in E$ if and only if $\sigma_f(x,p) \neq 0$ for all (x,p). Moreover, there is some $\mu > 0$ such that for all positive continuous functions δ on $\mathbb{R}^\ell$ and for any pair of smooth functions f and g, the following holds for all $x \in \mathbb{R}^\ell$:

$$d(j^2g(x), j^2f(x)) < \delta(x)$$

implies

$$\| \sigma_g(x,p) - \sigma_f(x,p) \| < \mu\delta(x) \ .$$

We assume $f \in E$ and consider the C^2 neighbourhood of f consisting of all $g \in C^\infty(\mathbb{R}^\ell, \mathbb{R})$ satisfying

$$d(j^2g(x), j^2f(x)) < \tfrac{1}{2}\mu \min\{ \| \sigma_f(x,p) \| \mid p \in S^{\ell-1}\} \ .$$

It is not difficult to understand that this neighbourhood is contained in E. So E is C^2-open and consequently C^∞-open. □

<u>REMARK</u>. For $\ell \geq 2$ the set E is not dense in $C^\infty(\mathbb{R}^\ell, \mathbb{R})$.

Now we have an open subset E^m of $C^\infty(\mathbb{R}^\ell, \mathbb{R})^m$ with $E^m \subset T$. The next lemma provides a nice description of the set $E_{cr}(r,u)$ for $u \in E^m$.

4.3.4. LEMMA. *Let* $f \in E$ *and* $p \in S^{\ell-1}$. *Then the set*

$$\{x \in \mathbb{R}^\ell \mid \Pi_p Df(x) = 0\}$$

is a submanifold of dimension 1.

PROOF. We assume $p^\ell \neq 0$, where $p = (p^1, \ldots, p^\ell)$, and consider the smooth map $\psi: \mathbb{R}^\ell \to \mathbb{R}^{\ell-1}$, which sends $x \in \mathbb{R}^\ell$ to

$$p^\ell\, \overline{Df(x)} - \frac{\partial f}{\partial x^\ell}(x)\bar{p} \ .$$

As before, the bar denotes that the ℓ-th coordinate is skipped. Then

$$\psi^{\leftarrow}(0) = \{x \in \mathbb{R}^\ell \mid \Pi_p Df(x) = 0\} \ .$$

We claim: ψ has $0 \in \mathbb{R}^{\ell-1}$ as a regular value. If so, we have proved the lemma by application of 2.4.4.

To prove our claim we derive from

$$\psi^j(x) = p^\ell \frac{\partial f}{\partial x^j}(x) - p^j \frac{\partial f}{\partial x^\ell}(x)$$

that

$$\frac{\partial \psi^j}{\partial x^k} = p^\ell \frac{\partial^2 f}{\partial x^k \, \partial x^j} - p^j \frac{\partial^2 f}{\partial x^k \, \partial x^\ell} \quad \text{for } j = 1,\dots,\ell-1 \, , \ k = 1,\dots,\ell \, .$$

So $D\psi(x)$ has the matrix representation:

$$D\psi(x) = p^\ell \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \, \partial x^1} & \cdots & \frac{\partial^2 f}{\partial x^\ell \, \partial x^1} \\ \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x^1 \, \partial x^{\ell-1}} & \cdots & \frac{\partial^2 f}{\partial x^\ell \, \partial x^{\ell-1}} \end{pmatrix} - \begin{pmatrix} p^1 \\ \vdots \\ p^{\ell-1} \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial x^1 \, \partial x^\ell} & \cdots & \frac{\partial^2 f}{\partial x^\ell \, \partial x^\ell} \end{pmatrix} .$$

Let $\alpha \in \mathbb{R}^{\ell-1}$ and consequently

$$\beta := (p^\ell \alpha \, , - \bar{p} \cdot \alpha) \in p^\perp \, .$$

Then

$$\alpha \, D\psi(x) = D^2 f(x) \beta \in \mathbb{R}^\ell$$

as can be easily derived from the matrix $D\psi(x)$. Hence $\alpha \, D\psi(x) = 0$ implies $\beta \in p^\perp \cap \text{kernel } D^2 f(x)$. So, if $\psi(x) = 0$ and $\alpha \, D\psi(x) = 0$, then $\alpha = 0$. This proves our claim. □

Roughly speaking, the set $(\Pi_p \circ Df)^{\leftarrow}(0)$ is a curve for each $p \in S^{\ell-1}$ and $f \in E$.

Now we assume $u = (u_1,\dots,u_m)$ to be in E^m. For each $p \in S^{\ell-1}$ we consider the curves

$$\{x_i \in \mathbb{R}^\ell \mid \Pi_p \, Du_i(x_i) = 0\} \, , \quad i = 1,\dots,m \, .$$

Each of these curves is locally parametrized by a parameter t_i. So the $(\ell + m - 1)$ dimensional submanifold

$$\{(x,p) \in \mathbb{R}^{\ell m} \times S^{\ell-1} \mid \Pi_p \, Du_i(x_i) = 0]_{i=1}^m\}$$

is parametrized by $(p,t_1,\dots,t_m)$, being $\ell - 1 + m$ parameters.

The additional $\ell + m - 1$ equations

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} r_i \ , \quad p \cdot x_i = p \cdot r_i]_{i=1}^{m-1} \ ,$$

which make $(x,p) \in E_{cr}(r,u)$ have locally at most one solution for a regular economy (r,u). This implies that $E_{cr}(r,u)$ consists of isolated points.

CHAPTER 5

THE SET OF LOCAL STRICT PARETO OPTIMA IN A PURE EXCHANGE ECONOMY

Introduction

In this chapter we deal with a pure exchange economy with ℓ goods and m agents, in which the total resources are fixed. We consider pairs $(w,u) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^{m}$, where $w \in \mathbb{R}^{\ell}$ represents the total resources and u the m-tuple of utility functions for the agents in the economy.
Given a pair (w,u) an allocation $x \in \mathbb{R}^{\ell m}$ is called *admissible* if $\Sigma_{i=1}^{m} x_i = w$.
Let x be admissible. Then one considers the set of admissible allocations y satisfying $u_i(y_i) \geq u_i(x_i)]_{i=1}^{m}$. If these inequalities are satisfied only by x itself the allocation x is called a *global strict Pareto optimum*. If there is a neighbourhood of x, on which these inequalities are satisfied only by x itself, the allocation x is called a *local strict Pareto optimum*. Our primary interest is fixed upon the set $\theta(w,u)$, shortly denoted by θ, of the local strict Pareto optima.
It is not possible to give necessary and sufficient first or second order conditions for a point x to be in θ, since each kind of degeneracy may occur.
So we describe in this chapter subsets of the set of admissible allocations which either are contained in θ or contain θ.
Section 5.1 is devoted to the properties of the sets $\theta_{ex}(w,u)$ and $\theta_{cr}(w,u)$ (see 1.3), whose definitions are based upon first order necessary conditions for points x to be in θ. The main theorem of this section is the general result II.
In Section 5.2 we extend the notion of the set N_z and the map H_z (see 2.5.1 and 2.5.2) to points z where some $Du_i(z_i)$ may be 0, and we prove that $H_z(v) \neq 0$ for all $v \in N_z \setminus \{0\}$ implies that $\theta_{cr}(\Sigma_{i=1}^{m} z_i, u)$ is locally an $(m-1)$-dimensional submanifold for $u \in T$, and $j^1u(z) \in W$ (see 3.5.3).
Section 5.3 contains the definition and properties of a subset $\theta \subset \theta$, being open in θ_{ex}. We restrict ourselves there to the dense subset $T_0 \cap M$ of T

(see 3.5.6).
Section 5.4 is devoted to some examples.

5.1. *Regular pairs*

Let $w \in \mathbb{R}^{\ell}$. We denote by A_w the set of admissible allocations. Then A_w is an $(m-1)\ell$ dimensional submanifold of $\mathbb{R}^{\ell m}$ and for each $x \in A_w$ the tangent space $T_x A_w$ consists of all $\delta x \in \mathbb{R}^{\ell m}$ satisfying $\Sigma_{i=1}^{m} \delta x_i = 0$.

We recall the definition of a local strict Pareto optimum for a pair (w,u): a point $z \in A_w$ is a local strict Pareto optimum if there is a neighbourhood $\mathcal{O}$ of z in $\mathbb{R}^{\ell m}$ such that for all admissible allocations $x \in \mathcal{O}\setminus\{z\}$ there is some i such that $u_i(x_i) < u_i(z_i)$ (see 1.3).

We denote the set of local strict Pareto optima by $\theta(w,u)$ or by θ.

5.1.1. DEFINITION. *Let* $(w,u) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$. *Then* $\theta_{cr}(w,u)$ *is the set of admissible allocations* x *such that* $\Pi_p\, Du_i(x_i) = 0]_{i=1}^{m}$ *for some* $p \in S^{\ell-1}$.

5.1.2. DEFINITION. *Let* $(w,u) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$. *Then* $\theta_{ex}(w,u)$ *is the set of allocations* $x \in A_w$ *such that* $\Pi_p\, Du_i(x_i) = 0]_{i=1}^{m}$ *for some* $p \in S^{\ell-1}$ *and* $Du_i(x_i) \cdot Du_j(x_j) \geq 0$ *for all* i, j.

The definitions of θ, θ_{ex} and θ_{cr} are parallel to those of equilibria, extended equilibria and critical equilibria.

5.1.3. LEMMA.

(1) $\theta \subset \theta_{ex} \subset \theta_{cr}$;

(2) θ_{ex} *and* θ_{cr} *are closed sets.*

PROOF. (1) Only $\theta \subset \theta_{ex}$ requires a proof. Let $z \in A_w$ and $z \notin \theta_{ex}$. Then z is not disastrous, since otherwise $Du_i(z_i) = \lambda_i p$, with $\lambda_i = 0$ for all i and all $p \in S^{\ell-1}$. Moreover, there are at least two indices i_1, i_2 such that

$$Du_{i_1}(z_{i_1}) \neq 0\ , \quad Du_{i_2}(z_{i_2}) \neq 0 \quad \text{and} \quad \frac{Du_{i_1}(z_{i_1})}{\|Du_{i_1}(z_{i_1})\|} \neq \frac{Du_{i_2}(z_{i_2})}{\|Du_{i_2}(z_{i_2})\|}\ .$$

There is some $v \in \mathbb{R}^{\ell}$ such that $Du_{i_1}(z_{i_1})\cdot v > 0$ and $Du_{i_2}(z_{i_2})\cdot v < 0$.
We consider the curve $x(t) \in A_w$:

$$\begin{cases} x_{i_1}(t) := z_{i_1} + tv\,, \\ x_{i_2}(t) := z_{i_2} - tv\,, \\ x_i(t) := z_i \quad \text{for } i \neq i_1,\ i \neq i_2\,. \end{cases}$$

Since

$$\lim_{t\to 0} \frac{u_{i_1}(z_{i_1} + tv) - u_{i_1}(z_{i_1})}{t} = Du_{i_1}(z_{i_1})\cdot v > 0$$

and

$$\lim_{t\to 0} \frac{u_{i_2}(z_{i_2} - tv) - u_{i_2}(z_{i_2})}{t} = -\,Du_{i_2}(z_{i_2})\cdot v > 0$$

it follows that there is some $t_0 > 0$ such that $u_i(x_i(t)) \geq u_i(z_i)]_{i=1}^{m}$ for all $t \in (0,t_0)$. So we have $z \notin \theta$.

(2) Since θ_{ex} is the intersection of θ_{cr} with the closed set of points x, satisfying $Du_i(x_i) \cdot Du_j(x_j) \geq 0$ for all i, j, it is sufficient to prove that θ_{cr} is a closed set.
Let $z \notin \theta_{cr}$. Then $Du_{i_1}(z_{i_1})$ and $Du_{i_2}(z_{i_2})$ are independent for at least one pair of indices. There is a neighbourhood of z such that $Du_{i_1}(z_{i_1})$ and $Du_{i_2}(x_{i_2})$ are independent for all x in this neighbourhood. □

Our first concern is now the structure of the set $\theta_{cr}(w,u)$.

5.1.4. LEMMA. *Let* $u \in T$. *The smooth map* $h_u\colon \mathbb{R}^{\ell} \times \mathbb{R}^{\ell m} \to \mathbb{R}^{\ell} \times \mathbb{R}^{\ell m}$ *which sends* (w,x) *to*

$$\left(\sum_{i=1}^{m} x_i - w\,,\ Du_i(x_i)]_{i=1}^{m}\right)$$

has 0 *as a regular value.*

PROOF. The derivative $Dh_u(w,x)(\delta w,\delta x)$ equals

$$\left(\sum_{i=1}^{m} \delta x_i - \delta w\,,\ D^2u_i(x_i)\,\delta x_i]_{i=1}^{m}\right) \quad \text{for } (\delta w,\delta x) \in \mathbb{R}^{\ell} \times \mathbb{R}^{\ell m}\,.$$

We assume $(w,x) \in h_u^{\leftarrow}(0)$. Then, by 4.1.12, each $D^2u_i(x_i)$ is nonsingular. Hence the equation $Dh_u(w,x)(\delta w,\delta x) = v$ has a solution for all $v \in \mathbb{R}^\ell \times \mathbb{R}^{\ell m}$. □

5.1.5. COROLLARY. *Let* $u \in T$. *Then the set of* $w \in \mathbb{R}^\ell$ *for which there are admissible disastrous allocations, is thin in* $\mathbb{R}^\ell$.

PROOF. As an immediate result of 5.1.4 we have: $h_u^{\leftarrow}(0)$ is a submanifold of dimension 0, in other words, a discrete set. Hence it is at most countable. But this means that the set of points $w \in \mathbb{R}^\ell$ for which there are $x \in \mathbb{R}^{\ell m}$ such that $h_u(w,x) = 0$ is at most countable. The fact that a countable subset of $\mathbb{R}^\ell$ is thin in $\mathbb{R}^\ell$ completes the proof. □

5.1.6. LEMMA. *Let* $u \in T$. *Then* $j^1u \pitchfork W$, *where* $j^1u(x) = (x, u_i(x_i)]_{i=1}^m, Du_i(x_i)]_{i=1}^m)$ *(see* 3.5.4*) and* $W = \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \Gamma$ *(see* 3.5.3*).*

PROOF. Let $x \in \mathbb{R}^{\ell m}$ and $j^1u(x) \in W$. The proof follows immediately from the definition of T and the proof of 3.5.7. □

5.1.7. LEMMA. *Let* $u \in T$. *Then the set* $\{x \in \mathbb{R}^{\ell m} \mid (Du_1(x_1),\dots,Du_m(x_m)) \in \Gamma\}$ *is a submanifold of dimension* $\ell + m - 1$.

PROOF. This set is the inverse image of W with respect to the map j^1u. We apply 2.4.4 with $X = \mathbb{R}^{\ell m}$, $Y = \mathbb{R}^{\ell m} \times \mathbb{R}^m \times \mathbb{R}^{\ell m}$, $Z = W$, and 5.1.6 to find that the dimension of $j^1u^{\leftarrow}(W)$ equals $\ell m - (\ell m + m + \ell m) + \ell m + m + m + \ell - 1 = m + \ell - 1$. □

5.1.8. DEFINTION. *Let* $u \in T$. *The smooth map*

$$\Sigma\colon \{x \in \mathbb{R}^{\ell m} \mid (Du_1(x_1),\dots,Du_m(x_m)) \in \Gamma\} \to \mathbb{R}^\ell$$

sends x *to* $\Sigma_{i=1}^m x_i \in \mathbb{R}^\ell$.

Now we come to the main theorems of this section.

5.1.9. THEOREM. *Let* $u \in T$. *Then the set of* $w \in \mathbb{R}^\ell$ *for which*

(1) $\theta_{cr}(w,u)$ *is a submanifold of dimension* $m - 1$,

(2) A_w, *and consequently* $\theta_{cr}(w,u)$, *contains no disastrous allocations,*

is dense in $\mathbb{R}^\ell$.

PROOF. The set of critical values of the map Σ is thin in $\mathbb{R}^{\ell}$ (see 2.2.6). The set of $w \in \mathbb{R}^{\ell}$ for which A_w contains disastrous allocations is also thin in $\mathbb{R}^{\ell}$ (see 5.1.5). So the union of these two sets is thin in $\mathbb{R}^{\ell}$. Hence the set consisting of those regular values of Σ for which A_w contains no disastrous allocations is dense in $\mathbb{R}^{\ell}$, its complement in $\mathbb{R}^{\ell}$ being thin. For each w in this set the inverse image

$$\Sigma^{\leftarrow}(w) = \left\{x \in \mathbb{R}^{\ell m} \mid \sum_{i=1}^{m} x_i = w\ ,\ (Du_1(x_1),\ldots,Du_m(x_m)) \in \Gamma\right\} = \theta_{cr}(w,u)$$

is a submanifold of dimension $\ell + m - 1 - \ell$. □

5.1.10. DEFINITION. *A pair* $(w,u) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$ *is said to be regular provided*

(1) $u \in T$;

(2) A_w *contains no disastrous allocations;*

(3) w *is a regular value of* Σ.

It follows from 5.1.9 that the set of regular pairs is dense in the set of all pairs (w,u) (see also the proof of 4.1.9).

5.1.11. THEOREM (GENERAL RESULT II). *The set of pairs* (w,u) *for which* A_w *contains no disastrous allocations and* $\theta_{cr}(w,u)$ *is an* $(m-1)$-*dimensional submanifold, is dense in* $\mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$.

PROOF. Follows immediately from the fact that this set contains the set of regular pairs. □

The next lemma provides a criterion for (w,u) to be a regular pair.

5.1.12. LEMMA. *Let* $(w,u) \in \mathbb{R}^{\ell} \times C^{\infty}(\mathbb{R}^{\ell},\mathbb{R})^m$. *Then* (w,u) *is a regular pair if and only if*

(1) $u \in T$;

(2) A_w *contains no disastrous allocations;*

(3) *the system of equations*

5.1.12.1. $$\begin{cases} \sum_{i=1}^{m} \delta x_i = v\ , \\ [\Pi_p\ D^2u_i(x_i)\,\delta x_i = \lambda_i \delta p]_{i=1}^{m} \end{cases}$$

in the unknowns $(\delta x, \delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$ *has a solution for each* $v \in \mathbb{R}^{\ell}$ *in all points* $x \in \mathbb{R}^{\ell m}$*, satisfying*

$$\sum_{i=1}^{m} x_i = w \, , \quad Du_i(x_i) = \lambda_i p]_{i=1}^{m} \, , \quad \textit{where } \lambda \neq 0 \, , \quad p \in S^{\ell-1} \, .$$

PROOF. (1) and (2) are part of the definition (5.1.10) of regular pairs. As for (3), we consider the tangent space to the manifold $j^1 u^{\leftarrow}(W)$ at x. This tangent space consists of all $\delta x \in \mathbb{R}^{\ell m}$ for which there is some $\delta p \in p^{\perp}$ such that $\Pi_p \, D^2 u_i(x_i)\delta x_i = \lambda_i \delta p]_{i=1}^{m}$. Then (3) is nothing else than the property that the derivative of Σ has rank ℓ at x. □

5.1.13. LEMMA. *Let* (w,u) *be a regular pair and* $x \in \theta_{cr}(w,u)$. *We assume* $Du_i(x_i) = \lambda_i p]_{i=1}^{m}$, *where* $\lambda \neq 0$ *and* $p \in S^{\ell-1}$. *Then* $\delta x \in T_x \, \theta_{cr}(w,u)$ *if and only if*

(1) $\sum_{i=1}^{m} \delta x_i = 0$;

(2) *there is some* $\delta p \in p^{\perp}$ *such that* $\Pi_p \, D^2 u_i(x_i)\delta x_i = \lambda_i \delta p]_{i=1}^{m}$.

PROOF. This is an immediate result of the definition of θ_{cr} and of 2.4.4. □

5.2. *Local structure of* $\theta_{cr}(\Sigma_{i=1}^{m} z_i \, , u)$

In 5.1 we started by choosing pairs (w,u) and the definitions of θ, θ_{ex} and θ_{cr}. The total resources w are supposed to come from some initial allocation in the economy. The structure of θ_{cr} for regular pairs (w,u) as an (m - 1)-dimensional submanifold is a global result, based on transversality theorems. One could take a more local point of view, and ask whether there is some criterion which tells that, given some allocation z and an m-tuple u, the set $\theta_{cr}(\Sigma_{i=1}^{m} z_i \, , u)$ is an (m - 1)-dimensional submanifold only in a neighbourhood of z. This section is devoted to this question.

Let $u \in T$ *and* $z \in \mathbb{R}^{\ell m}$. *We assume* $(Du_1(z_1),\dots,Du_m(z_m)) \in \Gamma$ *and we denote by* I(z) *the set of indices* i *for which* $Du_i(z_i) \neq 0$, *by* $I^*(z)$ *the set of indices* i, *for which* $Du_i(z_i) = 0$. *Obviously,* $I(z) \neq \emptyset$.

5.2.1. DEFINITION. *Let* $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$, $z \in \mathbb{R}^{\ell m}$ *and* $(Du_1(z_1),\ldots,Du_m(z_m)) = (\lambda_1 p,\ldots,\lambda_m p) \in \Gamma$.

(1) $N_z := \left\{(v_1,\ldots,v_m) \in \mathbb{R}^{\ell m} \mid \sum_{i=1}^m v_i = 0,\ p\cdot v_i = 0]_{i=1}^m,\ v_i = 0 \text{ for all } i \in I^*(z)\right\}$;

(2) $H_z(v) := \sum_{i\in I(z)} \frac{1}{\lambda_i} D^2u_i(z_i)(v_i,v_i)\ ,\quad v \in N_z$.

The set N_z and the map $H_z\colon N_z \to \mathbb{R}$ are extensions of those defined in 2.5.1 and 2.5.2. If $I^*(z) = \emptyset$ both definitions coincide.

5.2.2. THEOREM. *Let* $u \in T$ *and* $Du_i(z_i) = \lambda_i p]_{i=1}^m$, *and* $\lambda \neq 0$. *If* $H_z(v) \neq 0$ *for all* $v \in N_z\setminus\{0\}$, *there is an open neighbourhood* $O \subset \mathbb{R}^{\ell m}$ *of* z *such that* $\theta_{cr}(\Sigma_{i=1}^m z_i, u) \cap O$ *is a submanifold of dimension* $m-1$.

PROOF. We assume

$$\frac{\partial u_m}{\partial x_m^\ell}(z_m) \neq 0\ ,$$

and define for $i = 1,\ldots,m-1$ the smooth maps $F_i\colon \mathbb{R}^\ell \times \mathbb{R}^\ell \to \mathbb{R}^{\ell-1}$, which send (x_i,x_m) to

$$\frac{\partial u_m}{\partial x_m^\ell}\overline{Du_i(x_i)} - \frac{\partial u_i}{\partial x_i^\ell}\overline{Du_m(x_m)}$$

in the usual convention (see 2.5).
Furthermore, we denote by $\overline{D^2u_i(x_i)}$ the $(\ell-1)\times\ell$ matrix, obtained by skipping the last row $\partial^2u_i/\partial x_i\,\partial x_i^\ell$ of the matrix $D^2u_i(x_i)$.
Let $F\colon \mathbb{R}^{\ell m} \to \mathbb{R}^{(m-1)(\ell-1)} \times \mathbb{R}^\ell$ be the smooth map:

$$F(x_1,\ldots,x_m) := \left(F_1(x_1,x_m),\ldots,F_{m-1}(x_{m-1},x_m)\ ,\ \sum_{i=1}^m x_i - \sum_{i=1}^m z_i\right).$$

So $F(z) = 0$. Since $\frac{\partial u_m}{\partial x_m^\ell}(x_m) \neq 0$ on a neighbourhood of z, it is easily seen that there is a neighbourhood O_1 of z in $\mathbb{R}^{\ell m}$ such that

$$F^{\leftarrow}(0) \cap \mathcal{O}_1 = \theta_{cr}\left(\sum_{i=1}^{m} z_i \,, u\right) \cap \mathcal{O}_1 \;.$$

We claim that rank $DF(z) = (m-1)(\ell-1)+\ell$. If so, then rank $DF(x) = (m-1)(\ell-1)+\ell$ on a neighbourhood $\mathcal{O}$ of z in $\mathbb{R}^{\ell m}$. Hence it follows from 2.4.4 that $F^{\leftarrow}(0) \cap \mathcal{O}$ is a submanifold of $\mathbb{R}^{\ell m}$, passing through z, of dimension $\ell m - (m-1)(\ell-1) - \ell = m-1$.
To prove the claim we consider the derivative $DF(z)$ as a matrix with $(\ell-1)(m-1)+\ell$ rows and ℓm columns.
Let $\delta z = (\delta z_1,\ldots,\delta z_m) \in \mathbb{R}^{\ell m}$. Then

$$DF(z)(\delta z) = \left(DF_i(z_i,z_m)(\delta z_i,\delta z_m)]_{i=1}^{m-1} \,, \sum_{i=1}^{m} \delta z_i\right) =$$

$$= \left(\frac{\partial F_i}{\partial x_i}\,\delta z_i + \frac{\partial F_i}{\partial x_m}\,\delta z_m\Big|_{i=1}^{m-1} \,, \sum_{i=1}^{m} \delta z_i\right) ,$$

where

$$\frac{\partial F_i}{\partial x_i} = \begin{bmatrix} \dfrac{\partial F_i^1}{\partial x_i^1} & \cdots & \dfrac{\partial F_i^1}{\partial x_i^\ell} \\ \vdots & & \vdots \\ \dfrac{\partial F_i^{\ell-1}}{\partial x_i^1} & \cdots & \dfrac{\partial F_i^{\ell-1}}{\partial x_i^\ell} \end{bmatrix} , \quad i = 1,\ldots,m-1 \;;$$

$$\frac{\partial F_i}{\partial x_m} = \begin{bmatrix} \dfrac{\partial F_i^1}{\partial x_m^1} & \cdots & \dfrac{\partial F_i^1}{\partial x_m^\ell} \\ \vdots & & \vdots \\ \dfrac{\partial F_i^{\ell-1}}{\partial x_m^1} & \cdots & \dfrac{\partial F_i^{\ell-1}}{\partial x_m^\ell} \end{bmatrix} , \quad i = 1,\ldots,m-1 \;.$$

Since

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial u_m}{\partial x_m^\ell}\,\overline{D^2 u_i(x_i)} - \overline{Du_m(x_m)}^T\,\frac{\partial^2 u_i}{\partial x_i\,\partial x_i^\ell}$$

and

$$\frac{\partial F_i}{\partial x_m} = \overline{Du_i(x_i)}^T \frac{\partial^2 u_m}{\partial x_m \, \partial x_m^\ell} - \frac{\partial u_i}{\partial x_i^\ell} \overline{D^2 u_m(x_m)}$$

we have in x = z:

$$5.2.2.1. \begin{cases} \dfrac{\partial F_i}{\partial x_i} = \lambda_m\left(p^\ell \, \overline{D^2 u_i(z_i)} - \bar{p}^T \dfrac{\partial^2 u_i}{\partial x_i \, \partial x_i^\ell}(z_i)\right) , \\[2ex] \dfrac{\partial F_i}{\partial x_m} = \lambda_i\left(\bar{p}^T \dfrac{\partial^2 u_m}{\partial x_m \, \partial x_m^\ell} - p^\ell \, \overline{D^2 u_m(x_m)}\right) . \end{cases}$$

The derivative DF(z) has the matrix

$$DF(z) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(z_1,z_m) & 0 & \cdots & 0 & \frac{\partial F_1}{\partial x_m}(z_1,z_m) \\ 0 & & & & \vdots \\ \vdots & & & & \\ 0 & \cdots & & \frac{\partial F_{m-1}}{\partial x_{m-1}}(z_{m-1},z_m) & \frac{\partial F_{m-1}}{\partial x_m}(z_{m-1},z_m) \\ I & \cdots & \cdots & \cdots & I \end{pmatrix}$$

Here 0 denotes the $(\ell - 1) \times \ell$ matrix with all entries zero, and I is the $\ell \times \ell$ identity matrix.

Let $\bar{\alpha}_i \in \mathbb{R}^{\ell-1}]_{i=1}^{m-1}$, and $\beta \in \mathbb{R}^\ell$. For $i = 1,\ldots,m-1$ we denote by $\alpha_i \in \mathbb{R}^\ell$ the vector $(p^\ell \bar{\alpha}_i \, , -\bar{p}\cdot\bar{\alpha}_i)$. Hence $\alpha_i \in p^\perp$ and $\alpha_i = 0$ if and only if $\bar{\alpha}_i = \bar{0}$. (Note that the definition of α_i is not consistent with the bar convention.)

Now assume that $(\bar{\alpha}_1,\ldots,\bar{\alpha}_{m-1},\beta)DF(z) = 0 \in \mathbb{R}^{\ell m}$. Then

$$5.2.2.2. \begin{cases} \bar{\alpha}_i \dfrac{\partial F_i}{\partial x_i}(z_i,z_m) + \beta = 0]_{i=1}^{m-1} , \\[2ex] \sum\limits_{i=1}^{m-1} \bar{\alpha}_i \dfrac{\partial F_i}{\partial x_m}(z_i,z_m) + \beta = 0 . \end{cases}$$

Combining 5.2.2.1 and 5.2.2.2 with the definition of α_i, we find

$$5.2.2.3. \begin{cases} \lambda_m\, D^2u_i(z_i)\alpha_i + \beta = 0]_{i=1}^{m-1}\,, \\ -\, D^2u_m(x_m)\left(\sum_{i=1}^{m-1} \lambda_i\alpha_i\right) + \beta = 0\,. \end{cases}$$

Let

$$\alpha_m := -\frac{1}{\lambda_m}\sum_{i=1}^{m-1} \lambda_i\alpha_i \quad \text{and} \quad v_i := \lambda_i\alpha_i]_{i=1}^{m}\,.$$

Then $v = (v_1,\ldots,v_m) \in N_z$ and 5.2.2.3 yields

$$5.2.2.4. \quad \lambda_m\, D^2u_i(z_i)\alpha_i + \beta = 0]_{i=1}^{m}\,.$$

Hence

$$\sum_{i\in I(z)} \frac{1}{\lambda_i} D^2u_i(z_i)(v_i,v_i) = \sum_{i\in I(z)} \frac{1}{\lambda_i} D^2u_i(z_i)(\lambda_i\alpha_i,\lambda_i\alpha_i) =$$

$$= \sum_{i\in I(z)} \lambda_i\, D^2u_i(z_i)(\alpha_i,\alpha_i) = \sum_{i\in I(z)} \lambda_i\,(D^2u_i(z_i)(\alpha_i)\cdot\alpha_i) =$$

$$= \sum_{i\in I(z)} \lambda_i\left(-\frac{1}{\lambda_m}\,\beta\cdot\alpha_i\right) = -\frac{1}{\lambda_m}\,\beta\cdot\left(\sum_{i\in I(z)} \lambda_i\alpha_i\right) = -\frac{1}{\lambda_m}\,\beta\cdot\sum_{i=1}^{m} v_i = 0\,.$$

So, $v \in N_z$ and $H_z(v) = 0$ and consequently $v = 0$. Hence $\alpha_i = 0$ for all $i \in I(z)$, and $\beta = 0$ (see 5.2.2.4). So we have

$$\begin{cases} D^2u_i(z_i)\alpha_i = 0]_{i=1}^{m}\,, \\ p\cdot\alpha_i = 0]_{i=1}^{m}\,, \\ \sum_{i=1}^{m} \lambda_i\alpha_i = 0\,, \quad \text{where} \quad \lambda_i = Du_i(z_i)\cdot p]_{i=1}^{m}\,, \\ \Pi_p\, Du_i(z_i) = 0]_{i=1}^{m}\,. \end{cases}$$

If $(\alpha_1,\ldots,\alpha_m) \neq (0,\ldots,0)$, then there is some $\alpha^* \in S^{\ell m-1}$ such that $G_u(z,p,\alpha^*) = 0$, in contradiction with 3.1.4.
Hence $(\alpha_1,\ldots,\alpha_m) = 0$ and consequently $(\bar{\alpha}_1,\ldots,\bar{\alpha}_m,\beta) = 0$. It follows that rank $DF(z) = (m-1)(\ell-1) + \ell$. □

For completeness' sake we compute $T_z \theta_{cr}$ as the kernel of DF(z). Let P be the $\ell \times \ell$ matrix

$$\begin{pmatrix} p^\ell & 0 & \cdots & 0 & -p^1 \\ \vdots & & & & \\ 0 & \cdots & & p^\ell & -p^{\ell-1} \\ 0 & \cdots & & 0 & 0 \end{pmatrix}$$

The kernel of P is spanned by p. We do not go into detail, but we find:

$$T_z \theta_{cr} = \left\{ \delta z \in \mathbb{R}^{\ell m} \mid \sum_{i=1}^{m} \delta z_i = 0 \,,\; \lambda_m PD^2 u_i(z_i)\delta z_i = \lambda_i PD^2 u_m(z_m)\delta z_m]_{i=1}^{m} \right\} =$$

$$= \left\{ \delta z \in \mathbb{R}^{\ell m} \mid \sum_{i=1}^{m} \delta z_i = 0 \,,\; \Pi_p D^2 u_i(z_i)\delta z_i = \lambda_i \delta p]_{i=1}^{m} \text{ for some } \delta p \right\}$$

in accordance with 5.1.13.

As a consequence of 5.2.2 and 4.3.2 we have the next result, which, for utility functions without critical points, has already been mentioned in the literature ([16]).

5.2.3. LEMMA. *Let* $(w,u) \in \mathbb{R}^\ell \times C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$ *be such that each* u_i *is strictly convex and* A_w *contains no disastrous allocations. Then* $\theta_{cr}(w,u)$ *is a submanifold of dimension* $m-1$.

PROOF. Since each u_i is strictly convex we have: $u \in E^m$ (see 4.3.1.2) and $H_z(v) < 0$ for all $z \in A_w$, $v \in N_z \setminus \{0\}$. □

5.3. *A local optimal part of* θ_{cr}

The definitions of θ_{cr} and θ_{ex} are based on first order necessary conditions for θ. So, if one wants to know whether a point $z \in \theta_{cr}$ belongs to θ, a higher order criterion is useful. Up to now we did not exceed second order derivatives in the definitions of T, H_z, etcetera. In order to avoid degeneracies for utility functions beyond the first derivative we assume in this section that each utility function is a Morse function (see 2.4.6).

In the following we consider a pair (w,u) *such that*

(1) $u \in T_0 \cap M$ *(see* 3.5.6*);*

(2) (w,u) *is a regular pair (see* 5.1.10*).*

It should be noticed that the set of the pairs (w,u) satisfying these conditions is dense, since $T_0 \cap M$ is dense and because of 5.1.9.

5.3.1. LEMMA. *Let* $x \in \theta$. *Then* $D^2u_i(x_i)$ *is definite negative for all* $i \in I^*(x)$.

PROOF. Since x is not disastrous we may assume $Du_1(x_1) \neq 0$. Let $j \in I^*(x)$, $j \neq 1$, and $D^2u_j(x_j)$ not definite negative. Then, since u_j is a Morse function, there is some $v \in \mathbb{R}^\ell$ satisfying $Du_1(x_1)\cdot v > 0$ and $D^2u_j(x_j)(v,v) > 0$. Let $x(t) = (x_1(t),\dots,x_m(t))$ be the curve in A_w, through x:

$$\begin{cases} x_1(t) := x_1 + tv , \\ x_j(t) := x_j - tv , \\ x_i(t) := x_i , \quad i \neq 1,\ i \neq j . \end{cases}$$

We have:

$$\lim_{t\to 0} \frac{u_1(x_1(t)) - u_1(x_1)}{t} = Du_1(x_1)\cdot v > 0 ,$$

$$\lim_{t\to 0} \frac{u_j(x_j(t)) - u_j(x_j)}{t^2} = \tfrac{1}{2} D^2 u_j(x_j)(v,v) > 0 ,$$

$$u_i(x_i(t)) = u_i(x_i) , \quad i \neq 1,\ i \neq j .$$

So there is some $t_0 > 0$ such that $u_i(x_i(t)) \geq u_i(x_i)$ for all $t \in (0,t_0)$, in contradiction with the assumption $x \in \Theta$. □

5.3.2. DEFINITION. θ *is the subset of* A_w *consisting of all* $z \in A_w$ *satisfying*

(1) $z \in \theta_{ex}$;

(2) $D^2u_i(z_i)$ *is definite negative for all* $i \in I^*(z)$;

(3) $H_z(v) < 0$ *for all* $v \in N_z \setminus \{0\}$.

The next lemma gives a sufficient condition for points in A_w to be in Θ.

5.3.3. LEMMA. *If* $x \in \theta$ *then* $x \in \Theta$.

PROOF. Let $z \in \theta$ and $k := \# I(z)$. Then $k \geq 1$. Since $D^2u_i(z_i)$ is definite negative for $i \in I^*(z)$ there is a neighbourhood $\mathcal{O}$ of z such that:

$$\{x \in \mathcal{O} \cap A_w \mid u_i(x_i) \geq u_i(z_i)]_{i=1}^m\} =$$

$$= \{x \in \mathcal{O} \cap A_w \mid x_i = z_i \text{ for } i \in I^*(z),\ u_i(x_i) \geq u_i(z_i) \text{ for } i \in I(z)\} =$$

$$= \{x \in \mathcal{O} \mid x_i = z_i \text{ for } i \in I^*(z)\ ,\ \sum_{i\in I(z)} x_i = \sum_{i\in I(z)} z_i\ ,$$

$$u_i(x_i) \geq u_i(z_i) \text{ for } i \in I(z)\}\ .$$

If $k = 1$, and $u_i(x_i) \geq u_i(z_i)]_{i=1}^m$, $x \in \mathcal{O} \cap A_w$, then $x = z$, as is easily seen. If $k \geq 2$, we apply 2.5.3 with $\sum_{i\in I(z)} z_i$ instead of w, k instead of m. □

CONCLUSION. It follows from $\theta \subset \Theta \subset \theta_{ex} \subset \theta_{cr}$, 2.5.3 and 5.3.1 that the only points of θ_{cr} where properties of first and second order derivatives are not sufficient to conclude whether they belong to Θ are those points $z \in \theta_{ex}$, where

(1) $D^2u_i(z_i)$ is definite negative for all $i \in I^*(z)$;

(2) $H_z(v) \leq 0$ for all $v \in N_z$ and $H_z(v) = 0$ for some $v \in N_z \setminus \{0\}$.

The definition of θ gives rise to the conjecture that θ is somehow an open set. Hence the next

5.3.4. THEOREM. θ *is an open subset of* θ_{ex} *in the relative topology.*

PROOF. Since θ_{ex} is closed in A_w and $\mathbb{R}^{\ell m}$ (5.1.3) it is sufficient to prove that for a sequence $z(n) \in \theta_{ex} \setminus \theta$ converging to $z \in \theta_{ex}$ the limit z is not an element of θ.
Let $z(n) = (z_1(n),\dots,z_m(n))$ be a sequence in $\theta_{ex} \setminus \theta$ and

$$\lim_{n\to\infty} z(n) = z = (z_1,\dots,z_m) \in \theta .$$

Then

$$Du_i(z_i(n)) = \lambda_i(n)p(n)]_{i=1}^m , \quad \lambda_i(n) \geq 0 , \quad p(n) \in S^{\ell-1} ,$$

and

$$Du_i(z_i) = \lambda_i p]_{i=1}^m , \quad \lambda_i \geq 0 , \quad p \in S^{\ell-1} .$$

The pair (w,u) is a regular one, so there is at least one j such that

$$p = \frac{Du_j(z_j)}{\| Du_j(z_j)\|} .$$

Hence, for this index j, we have

$$\lim_{n\to\infty} \frac{Du_j(z_j(n))}{\| Du_j(z_j(n))\|} = \frac{Du_j(z_j)}{\| Du_j(z_j)\|} ,$$

so

$$\lim_{n\to\infty} p(n) = p ,$$

and consequently

$$\lim_{n\to\infty} \lambda_i(n) = \lim_{n\to\infty} Du_i(z_i(n)) \cdot p(n) = Du_i(z_i) \cdot p = \lambda_i$$

for $i = 1,\dots,m$.
Each u_i is a Morse function which implies that the critical points of each u_i form a discrete set. Moreover, by a continuity argument we can show that for all $i \in I^*(z)$ there is a neighbourhood of z_i such that $D^2u_i(x_i)$ is definite negative for all x_i in that neighbourhood.
These two arguments boil down to: If n is large enough, then:

(1) $I(z) \subset I(z(n))$;
(2) $D^2u_i(z_i(n))$ is definite negative for all $i \in I^*(z)$.

We assume that (1) and (2) hold for all n. It follows from (1) and (2) that for all n, $D^2u_i(z_i(n))$ is definite negative for all $i \in I^*(z(n)) \subset I^*(z)$. Since $z(n) \notin \theta$ there is for all n some $v(n) \in N_{z(n)} \setminus \{0\}$ such that

$$H_{z(n)}(v(n)) = \sum_{i\in I(z(n))} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i(n),v_i(n)) \geq 0 .$$

The sequence $\frac{v(n)}{\|v(n)\|} \in S^{m\ell-1}$ has a limit point on $S^{m\ell-1}$.
We assume that the sequence $v'(n) := \frac{v(n)}{\|v(n)\|}$ converges to $v' \in S^{m\ell-1}$.
Since

$$\sum_{i=1}^{m} v_i(n) = 0 , \quad p(n) \cdot v_i(n) = 0]_{i=1}^{m} \quad \text{for all } n ,$$

we have

$$\sum_{i=1}^{m} v_i' = 0 , \quad p \cdot v_i' = 0]_{i=1}^{m} .$$

Furthermore:

$$\text{(3)} \qquad H_{z(n)}(v'(n)) = \sum_{i\in I(z(n))} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i'(n),v_i'(n)) =$$

$$= \sum_{i\in I(z)} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i'(n),v_i'(n)) +$$

$$+ \sum_{i\in I(z(n))\setminus I(z)} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i'(n),v_i'(n)) \geq 0 .$$

Since $i \notin I(z)$ implies $D^2u_i(z_i(n))$ is definite negative we have

$$\text{(4)} \qquad \sum_{i\in I(z)} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i'(n),v_i'(n)) \geq$$

$$\geq - \sum_{i\in I(z(n))\setminus I(z)} \frac{1}{\lambda_i(n)} D^2u_i(z_i(n))(v_i'(n),v_i'(n)) \geq 0 .$$

Hence, letting $n \to \infty$,

$$\text{(5)} \qquad \sum_{i\in I(z)} \frac{1}{\lambda_i} D^2u_i(z_i)(v_i',v_i') \geq 0 .$$

Furthermore, if $v_j' \neq 0$ for some $j \in I^*(z)$, then $j \in I(z(n))$ for infinitely many indices n. For otherwise, $j \in I^*(z(n))$ for n large enough and then

$v_j(n) = 0$ by definition and consequently $\lim_{n\to\infty} v_j'(n) = 0$. So there is a subsequence $n_k \to \infty$ with $j \in I(z(n_k)) \setminus I(z)$. But this means that

$$\lim_{k\to\infty} \frac{-1}{\lambda_j(n_k)} D^2u_j(z_j(n_k))(v_j'(n_k),v_j'(n_k)) = \infty ,$$

since

$$\lambda_j(n_k) \to \lambda_j = 0$$

and

$$D^2u_j(z_j(n_k))(v_j'(n_k),v_j'(n_k)) \to D^2u_j(z_j)(v_j',v_j') \neq 0 .$$

This is impossible because (4) yields

$$\sum_{i\in I(z)} \frac{1}{\lambda_i(n_k)} D^2u_i(z_i(n_k))(v_i'(n_k),v_i'(n_k)) \geq$$

$$\geq -\frac{1}{\lambda_j(n_k)} D^2u_j(z_j(n_k))(v_j'(n_k),v_j'(n_k))$$

and the left-hand term converges for $k \to \infty$. Hence $v_j' = 0$ for all $j \in I^*(z)$. So $v' \in N_z$, $v' \neq 0$ and (5) yields $H_z(v') \geq 0$. This contradicts the assumption that $z \in \theta$. □

5.4. *Some examples*

In this section we discuss some examples of pairs (w,u) where $u \in T$. We shall see that, dependent on the choice of (w,u), θ can coincide with θ_{ex}, be empty or be a proper subset of θ_{ex}.

5.4.1. Our first example deals with the utility tuple of 4.2

$$u_i(x_i) = -\tfrac{1}{2}(x_i - a_i)\cdot(x_i - a_i)]_{i=1}^m .$$

For $w \in \mathbb{R}^\ell$ we consider

$$\theta_{cr}(w,u) = \left\{x \in \mathbb{R}^{m} \mid \sum_{i=1}^m x_i = w , \Pi_p Du_i(x_i) = 0]_{i=1}^m \text{ for some } p \in S^{\ell-1}\right\} .$$

Then $x \in \theta_{cr}(w,u)$ if and only if there are $\lambda_i]_{i=1}^m$, $p \in S^{\ell-1}$ such that:

$$\begin{cases} x_i = a_i - \lambda_i p]_{i=1}^m , \\ w - \sum_{i=1}^m a_i = \left(- \sum_{i=1}^m \lambda_i\right) p . \end{cases}$$

(1) If $w = \Sigma_{i=1}^m a_i$, then each $p \in S^{\ell-1}$ and $\lambda \in \mathbb{R}^m$ with $\Sigma_{i=1}^m \lambda_i = 0$ satisfies

$$w - \sum_{i=1}^m a_i = \left(- \sum_{i=1}^m \lambda_i\right) p .$$

Hence $\theta_{cr}(w,u)$ is parametrized by $\ell - 1 + m - 1$ parameters and so it is not a $(m-1)$-dimensional submanifold. Not unexpected since w allows disastrous allocations in this case, so (w,u) is not a regular pair.

(2) Let $w \neq \Sigma_{i=1}^m a_i$. We claim that (w,u) is a regular pair, the parametrization of $\theta_{cr}(w,u)$ being obviously:

$$p = \pm \frac{w - \sum_{i=1}^m a_i}{\| w - \sum_{i=1}^m a_i \|} , \quad \sum_{i=1}^m \lambda_i = \mp \| w - \sum_{i=1}^m a_i \| .$$

Since $u \in T$ and w does not allow disastrous allocations we have by 5.1.12 to show that for all $x \in \theta_{cr}(w,u)$ the equations in the unknowns $(\delta x, \delta p) \in \mathbb{R}^{\ell m} \times p^{\perp}$, where $Du_i(x_i) = \lambda_i p]_{i=1}^m$,

$$\begin{cases} \sum_{i=1}^m \delta x_i = v , \\ \Pi_p D^2 u_i(x_i) \delta x_i = \lambda_i \delta p]_{i=1}^m \end{cases}$$

have a solution for all $v \in \mathbb{R}^{\ell}$.
We choose one of the two solutions for p. Since $(\Sigma_{i=1}^m \lambda_i) p \neq 0$ and $D^2 u_i(x_i) = -I$ we solve the equations (5.1.12.1) for this case by

$$\delta p = - \frac{\Pi_p v}{\sum_{i=1}^m \lambda_i} , \quad \delta x_i = - \lambda_i dp + \delta \lambda_i p]_{i=1}^m ,$$

where

$$\sum_{i=1}^{m} \delta\lambda_i = p \cdot v \ .$$

Hence (w,u) is a regular pair if and only if $w \neq \Sigma_{i=1}^{m} a_i$.

Since each D^2u_i is definite negative and A_w contains no disastrous allocations we apply 5.3.3 to conclude that each point of θ_{ex} belongs to θ, so $\theta = \theta_{ex} = \theta$.

We can prove more, due to the convexity of each u_i: each point $z \in \theta$ is a global strict Pareto optimum.

PROOF. Let $z \in \theta_{ex}$ and $x \in A_w$ such that $u_i(x_i) \geq u_i(z_i)$ for all i. It must be shown that x = z. Assume $Du_i(z_i) = \lambda_i p$ where $\lambda_i \geq 0$ for all i,

$$p = \frac{\sum_{i=1}^{m} a_i - w}{\left\| \sum_{i=1}^{m} a_i - w \right\|} \quad \text{and} \quad \sum_{i=1}^{m} \lambda_i = \left\| \sum_{i=1}^{m} a_i - w \right\| .$$

There are reals $\mu_i]_{i=1}^{m}$ and vectors $q_i \in p^{\perp}]_{i=1}^{m}$ such that

$$x_i = a_i + \mu_i p + q_i]_{i=1}^{m} \ .$$

Since

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{m} a_i + \left(\sum_{i=1}^{m} \mu_i\right) p + \sum_{i=1}^{m} q_i = w$$

it follows from the expression for p that

$$\sum_{i=1}^{m} \mu_i = - \sum_{i=1}^{m} \lambda_i \quad \text{and} \quad \sum_{i=1}^{m} q_i = 0 \ .$$

Furthermore,

$$u_i(x_i) = -\tfrac{1}{2}(\mu_i p + q_i) \cdot (\mu_i p + q_i) = -\tfrac{1}{2}\mu_i^2 - \tfrac{1}{2} q_i \cdot q_i]_{i=1}^{m}$$

and

$$u_i(z_i) = -\tfrac{1}{2}(-\lambda_i p) \cdot (-\lambda_i p) = -\tfrac{1}{2}\lambda_i^2]_{i=1}^{m} \ .$$

Hence by assumption

$$\lambda_i^2 \geq \mu_i^2 + q_i \cdot q_i]_{i=1}^m .$$

So

$$|\lambda_i| = \lambda_i \geq |\mu_i| \geq -\mu_i]_{i=1}^m$$

and consequently

$$\lambda_i + \mu_i \geq 0]_{i=1}^m .$$

Then by

$$\sum_{i=1}^m (\lambda_i + \mu_i) = 0$$

we have

$$\lambda_i + \mu_i = 0]_{i=1}^m .$$

This means $q_i \cdot q_i \leq 0]_{i=1}^m$, so $q_i = 0]_{i=1}^m$, and $x_i = a_i - \lambda_i p$ for all i. Hence $x = z$. □

5.4.2. Let $m = 2$, $\ell \geq 2$ and $q \in S^{\ell-1}$. We define u_1 and u_2:

$$\begin{cases} u_1(x_1) := q \cdot x_1 , \\ u_2(x_2) := q \cdot x_2 + \tfrac{1}{2} (\Pi_q x_2) \cdot x_2 , \end{cases}$$

and we prove: (w,u) is a regular pair for all $w \in \mathbb{R}^\ell$, and $\theta(w,u) = \emptyset$ for all w. So there are no local strict Pareto optima.

Since $Du_1(x_1) = q$, $D^2u_1(x_1) = 0$, $Du_2(x_2) = q + \Pi_q x_2$, $D^2u_2(x_2) = \Pi_q$, we have for

$$(x,p,\alpha) = (x_1,x_2,p,\alpha_1,\alpha_2) \in \mathbb{R}^{2\ell} \times S :$$

$$G_u(x,p,\alpha) = (\Pi_p q , \Pi_p q + \Pi_p \Pi_q x_2 , 0 , \Pi_q \alpha_2 , (q\cdot p)\alpha_1 + ((q + \Pi_q x_2)\cdot p)\alpha_2)$$

(see 3.1.3).

Hence $G_u(x,p,\alpha) = 0$ yields

$$\Pi_p q = 0 , \ \Pi_p x_2 = 0 , \ \Pi_q \alpha_2 = 0 , \ (q\cdot p)\alpha_1 + (q\cdot p)\alpha_2 = 0 , \ p\cdot \alpha_1 = 0 , \ p\cdot\alpha_2 = 0 .$$

This implies $q = \pm p$ and from $\Pi_q \alpha_2 = 0$, $p \cdot \alpha_2 = 0$, $\alpha_1 + \alpha_2 = 0$, it follows that $\alpha_2 = 0$, $\alpha_1 = 0$, in contradiction with the definition of S. So $G_u(x,p,\alpha) \neq 0$ for all (x,p,α) and consequently $u \in T$ (see 3.1.4). Furthermore, since $Du_1(x_1) = q \in S^{\ell-1}$, there are no disastrous allocations at all.

Now let $w \in \mathbb{R}^{\ell}$ and assume $(x_1,x_2) \in \theta_{cr}(w,u)$. So

$$\left.\begin{aligned} x_1 + x_2 &= w \\ Du_1(x_1) &= \lambda_1 p \\ Du_2(x_2) &= \lambda_2 p \end{aligned}\right\} \quad \text{where } p \in S^{\ell-1} .$$

Then

$$x_1 + x_2 = w \;,\quad p = q \;,\quad \Pi_q x_2 = 0 \;,\quad \lambda_1 = \lambda_2 = 1$$

or

$$x_1 + x_2 = w \;,\quad p = -q \;,\quad \Pi_q x_2 = 0 \;,\quad \lambda_1 = \lambda_2 = -1 \;.$$

In both cases the equations 5.1.12.1

$$\begin{cases} \delta x_1 + \delta x_2 = v \;, \\ \Pi_p \, 0 \; \delta x_1 = \lambda_1 \delta p \;, \\ \Pi_p \Pi_q \delta x_2 = \lambda_2 \delta p \;, \end{cases}$$

are solved by $\delta x_1 = v_1$, $\delta x_2 = 0$, $\delta p = 0$.

Hence (w,u) is a regular pair for all w. Furthermore,

$$\theta_{ex}(w,u) = \{(x_1,x_2) \mid x_1 + x_2 = w \;,\; x_2 = \gamma q\} \;.$$

So $\theta_{ex}(w,u)$ is parametrized by the scalar γ and $\theta_{cr} = \theta_{ex}$ for all w. Let $(x_1,x_2) \in \theta_{cr}(w,u)$. Then $Du_1(x_1) = q \neq 0$ and $Du_2(x_2) = q + \Pi_q x_2 = q \neq 0$.

$$N_x = \{(v_1,v_2) \in \mathbb{R}^{2\ell} \mid v_1 + v_2 = 0 \,,\, q \cdot v_1 = q \cdot v_2 = 0\}$$

(see 2.5.1). For $v \in N_x$ we have:

$$H_x(v) = \frac{1}{\lambda_1} D^2u_1(x_1)(v_1,v_1) + \frac{1}{\lambda_2} D^2u_2(x_2)(v_2,v_2) =$$

$$= \Pi_q(v_2,v_2) = (\Pi_q v_2)\cdot v_2 = v_2 \cdot v_2$$

(see 2.5.2). Hence there are $v \in N_x$ such that $H_x(v) > 0$ and consequently x is not a local strict Pareto optimum (see 2.5.3). So $\theta(w,u) = \emptyset$ for all $w \in \mathbb{R}^\ell$.

5.4.3. For $m = \ell = 2$ we can draw instructive (however deceptive) pictures of $\theta_{cr}(w,u)$ by means of the Edgeworth box (see 1.6).
Let $w \in \mathbb{R}^2$ and $(x_1,x_2) \in A_w$. Then $x_2 = w - x_1$, and, representing x_1 by (x,y), utility functions $\tilde{u}_1(x_1)$ and $\tilde{u}_2(x_2)$, restricted to points $(x_1,x_2) \in A_w$, give rise to functions $\tilde{u}_1(x,y)$ and $\tilde{u}_2(w-(x,y))$ on $\mathbb{R}^2$.
We write $u_1(x,y)$ and $u_2(x,y)$ for $\tilde{u}_1(x,y)$ and $\tilde{u}_2(w-(x,y))$, respectively.
Then we have the following characterization of θ_{cr} and θ_{ex}:

$$\theta_{cr} = \{(x,y) \in \mathbb{R}^2 \mid \det(Du_1(x,y), Du_2(x,y)) = 0\} ,$$

$$\theta_{ex} = \{(x,y) \in \theta_{cr} \mid Du_1(x,y)\cdot Du_2(x,y) \le 0\} .$$

Furthermore, we define for $(x,y) \in \theta_{cr}$ the set $N_{(x,y)}$ by:

(1) $$N_{(x,y)} = Du_1(x,y)^\perp \cap Du_2(x,y)^\perp \quad \text{if} \quad Du_i(x,y) \ne 0]_{i=1}^2 ,$$

(2) $$N_{(x,y)} = \{0\} \quad \text{otherwise}$$

(see 5.2.1), and the function $H_{(x,y)}$ on $N_{(x,y)}$ is defined as

$$H_{(x,y)}(v) = \frac{1}{\lambda_1} D^2u_1(x,y)(v,v) + \frac{1}{\lambda_2} D^2u_2(x,y)(v,v) ,$$

only in points $(x,y) \in \theta_{cr}$ where

$$Du_i(x,y) \ne 0]_{i=1}^2 , \quad Du_1(x,y) = \lambda_1 p , \quad Du_2(x,y) = -\lambda_2 p .$$

(see 5.2.1). Then the set θ consists of those points (x,y) where, either

(1) $Du_1(x,y) = 0$ or $Du_2(x,y) = 0$, and $D^2u_i(x,y)$ is definite negative if $Du_i(x,y) = 0$,

or

(2) $Du_1(x,y) \neq 0$, $Du_2(x,y) \neq 0$, $Du_1(x,y) = \lambda_1 p$, $Du_2(x,y) = -\lambda_2 p$, with $\lambda_1 > 0$, $\lambda_2 > 0$ and $p \in S^1$, and $H_{(x,y)}(v) < 0$ for all $v \in p^{\perp} \setminus \{0\}$.

Since we consider regular economies and only Morse functions in the following examples, we conclude by 5.3.1 and 5.3.3 that a point $(x,y) \in \theta_{ex}$ is not a local strict Pareto optimum for (u_1,u_2) whenever either

(1) $Du_1(x,y) = 0$ or $Du_2(x,y) = 0$ and $D^2u_i(x,y)$ is not definite negative for $i \in I^*(x,y)$,

or

(2) $Du_1(x,y) \neq 0$ and $Du_2(x,y) \neq 0$ and $H_{(x,y)}(v) > 0$ for some $v \in N_{(x,y)}$.

Now we give some concrete examples.

5.4.3.1.

$$\begin{cases} u_1(x,y) := -\tfrac{1}{2}x^2 + \tfrac{1}{2}y^2 , \\ u_2(x,y) := (x-2)y . \end{cases}$$

This pair is originated by the regular pair $(w,\tilde{u})$ with

$$\begin{cases} \tilde{u}_1(x_1) = -\tfrac{1}{2}(x_1^1)^2 + \tfrac{1}{2}(x_1^2)^2 , \\ \tilde{u}_2(x_2) = x_2^1\, x_2^2 , \\ w = (2,0) . \end{cases}$$

We have:

$$\begin{cases} Du_1(x,y) = (-x,y) , \\ Du_2(x,y) = (y,x-2) , \\ D^2u_1(x,y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \\ D^2u_2(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} . \end{cases}$$

So $D^2u_1(x,y)$ and $D^2u_2(x,y)$ are not definite negative.
As is easily seen:

$$\theta_{cr} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x + y^2 = 0\} ,$$

$$\theta_{ex} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x + y^2 = 0, \ y \geq 0\} ,$$

$$\theta = \{(x,y) \in \theta_{ex} \mid y > 0 \text{ and } H_{(x,y)} \text{ definite negative}\} .$$

Let $(x,y) \in \theta_{ex}$ and $y > 0$. Then $N_{(x,y)}$ is spanned by $(y,x) = (\sqrt{2x-x^2}, x)$

$$\left.\begin{aligned} \lambda_1 &= \sqrt{x^2+y^2} = \sqrt{2x} \\ \lambda_2 &= \sqrt{y^2 + (x-2)^2} = \sqrt{4-2x} \end{aligned}\right\} \quad 0 < x < 2 .$$

We only need to consider $H_{(x,y)}(v)$ with

$$v = (\sqrt{2x-x^2}, x) , \qquad 0 < x < 2 .$$

Substitution yields

$$H_{(x,y)}(v) = \frac{1}{\sqrt{2x}}(-(2x-x^2) + x^2) + \frac{2}{\sqrt{4-2x}} x\sqrt{2x-x^2} = (2x-1)\sqrt{2x} .$$

Hence it follows that

$$\theta = \{(x,y) \in \theta_{ex} \mid 0 < x < \tfrac{1}{2}\}$$

and that points in θ_{ex}, where $x = 0$, $x = 2$ or $x > \frac{1}{2}$ are not optimal.

It remains to check whether the point $(\frac{1}{2}, \frac{1}{2}\sqrt{3}) \in \theta_{ex}$ is in θ or not. There is a neighbourhood $\mathcal{O}$ of this point such that $(x,y) \in \mathcal{O}$ and $u_i(x,y) \geq u_i(\frac{1}{2}, \frac{1}{2}\sqrt{3})]_{i=1}^2$ imply:

$$\sqrt{x^2 + \tfrac{1}{2}} \leq y \leq \frac{3\sqrt{3}}{4(2-x)} .$$

We consider the function

$$g(x) = \frac{3\sqrt{3}}{4(2-x)} - \sqrt{x^2 + \tfrac{1}{2}}$$

in a neighbourhood of $x = \frac{1}{2}$. Then

$$g(\tfrac{1}{2}) = 0 , \quad g'(\tfrac{1}{2}) = 0 , \quad g''(\tfrac{1}{2}) = 0 , \quad g'''(\tfrac{1}{2}) > 0 .$$

So each neighbourhood of $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ contains points (x,y) with

$$x > \tfrac{1}{2} \quad \text{and} \quad \sqrt{x^2 + \tfrac{1}{2}} < y < \frac{3\sqrt{3}}{4(2-x)}$$

and consequently

$$u_i(x,y) > u_i(\tfrac{1}{2}, \tfrac{1}{2}\sqrt{3})]_{i=1}^{2} .$$

Hence $(\frac{1}{2}, \frac{1}{2}\sqrt{3}) \notin \theta$.

The foregoing results are laid down in Figure 8.

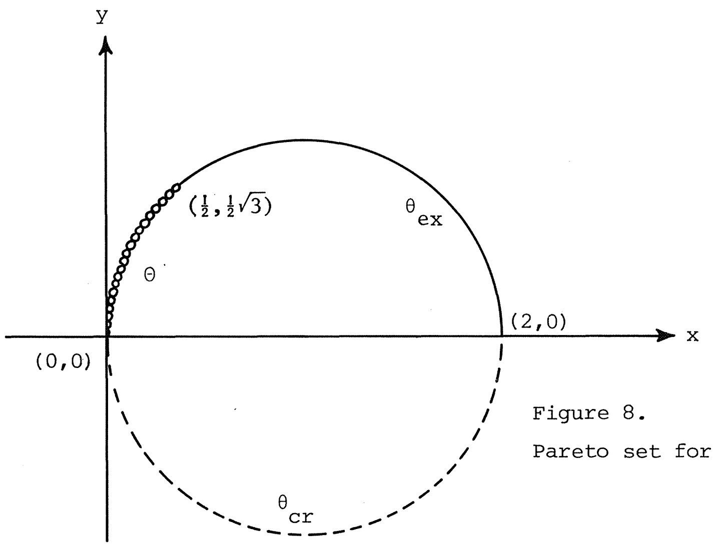


Figure 8.
Pareto set for example 5.4.3.1.

5.4.3.2. We consider the pair $u = (u_1, u_2)$ where

$$\begin{cases} u_1(x,y) := -\tfrac{1}{2}x^2 - \tfrac{1}{2}y^2 , \\ u_2(x,y) := (x-2)y . \end{cases}$$

Application of the same methods as in 5.4.3.1 combined with the fact that $D^2u_1(x,y)$ is definite negative, yields:

$$\left.\begin{array}{l} \theta_{cr} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x - y^2 = 0\} \\ \theta_{ex} = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x - y^2 = 0,\ y(x-1) \geq 0\} \\ \theta = \theta_{ex} \setminus \{(2,0)\} \\ \boldsymbol{\theta} = \theta \end{array}\right\} \text{see Figure 9.}$$

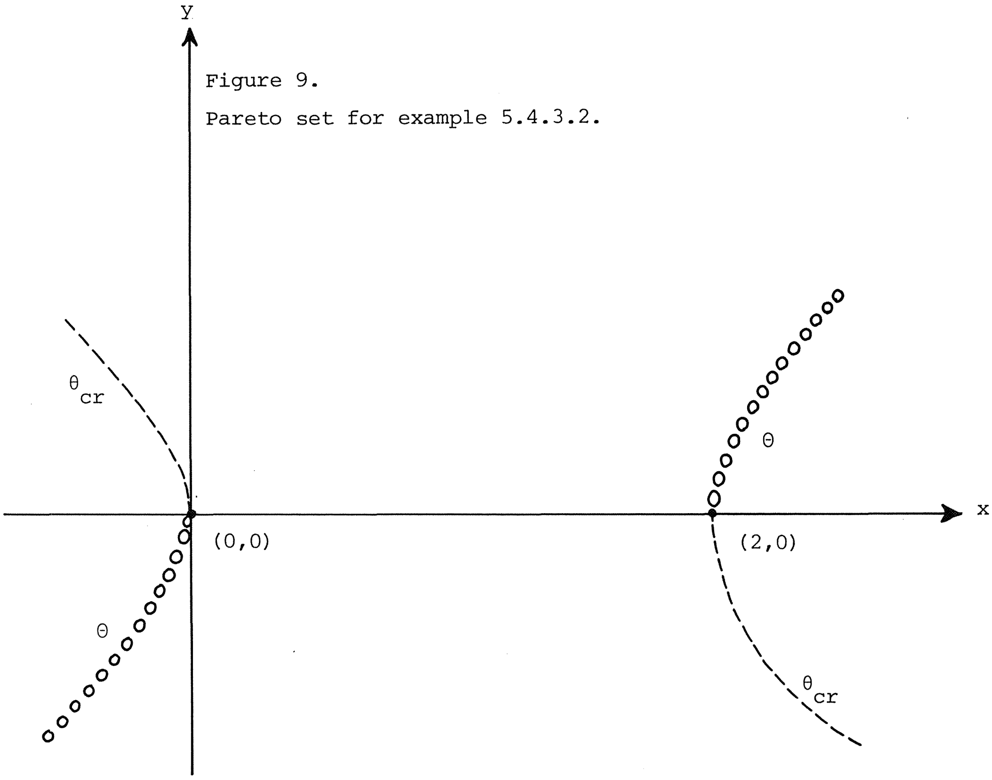


Figure 9.

Pareto set for example 5.4.3.2.

CHAPTER 6

TRADE CURVES

Introduction

Given a pure exchange economy with fixed total resources $w \in \mathbb{R}^\ell$ and a utility tuple $u \in C^\infty(\mathbb{R}^\ell,\mathbb{R})^m$, we consider trade curves, being C^1-curves $x(t) \in A_w$, defined on an interval $a \le t < b$ satisfying:

(1) $u_i(x_i(t))$ is non-decreasing in time t, $i = 1,\dots,m$, so

$$\frac{d}{dt} u_i(x_i(t)) \ge 0 \quad \text{for all } i \text{ and all } t;$$

2) $\Sigma_{i=1}^m u_i(x_i(t))$ is strictly increasing in time t, so

$$\frac{d}{dt} \sum_{i=1}^m u_i(x_i(t)) > 0 \quad \text{for all } t.$$

Obviously no trade curve starts at $z \in \theta$.
There are several ways to generate trade curves, two of which are mentioned here:

(i) Starting from some point $x \notin \theta$ one makes at any time t a choice for the direction $\dot{x}(t)$ from a set, defined by equalities and inequalities, based upon (1) and (2).

(ii) One defines globally a vector field on A_w, the integral curves being trade curves.

In both cases there are two stability problems in discussion:

(A) Given a point $x \notin \theta$, is there a trade curve which starts from x and converges to some point of θ ?

(B) Do trade curves stay near a point of θ if they start from points sufficiently close to that point?

(A) is called the accessibility problem, (B) is the Lyapunov stability problem.

Smale [19], Friedmann [3] and Schecter [11], considered mainly trade curves of type (i), the so-called non-deterministic trade curves. Under classical convexity and monotonicity hypotheses (together with boundary conditions, since they dealt with the open or closed positive orthant as consumption sets for economic agents) they proved that from each initial state $\notin \theta$ a trade curve converges to a point in θ. A slight modification of their methods leads to a Lyapunov stability theorem. See Wan [24].
Recently, Wan [24] considered deterministic trade curves (of type (ii)) dropping the monotonicity and convexity conditions, instead of which he imposed a condition, denoted by (θ), on the pair (w,u).

By definition, (w,u) *satisfies* (θ) *if and only if for all* $x \in \theta$ *there is a neighbourhood* O *of* x *such that for all* $y \in \theta$, $y \in O$, $y \neq x$, *there are indices* i *and* j *such that* $u_i(y_i) < u_i(x_i)$ *and* $u_j(y_j) > u_j(x_j)$.

Making use of (θ) he proves:
There is an open and dense set of pairs (w,u), such that for each (w,u) in this set the following holds:

(I) For each given initial state $\notin \theta$ there exists a trade curve converging to a point in θ.

(II) For each neighbourhood of $x \in \Theta$ trade curves starting sufficiently close to x remain in that neighbourhood. This holds for all $x \in \Theta$.

As indicated by Wan the convexity and monotonicity assumptions on the utility functions imply that (w,u) satisfies (θ) for all w. The results of Wan are rather exhaustive, so we can only try to add some details.
Section 6.1 is devoted to the explicit construction of a vectorfield, generating trade curves. The existence of such vectorfields is generally assumed in literature. Indeed, if utility functions do not have critical points, it is easily seen that

$$v_i(x) = \left[\frac{Du_i(x_i)}{\|Du_i(x_i)\|} - \frac{1}{m}\sum_{j=1}^{m}\frac{Du_j(x_j)}{\|Du_j(x_j)\|}\right]_{i=1}^{m}$$

satisfies $Du_i(x_i) \cdot v_i(x) > 0]_{i=1}^{m}$ for $x \notin \theta_{ex}$ and $v = 0$ on θ_{ex}.
The construction turns out to be more complicated for utility functions with critical points.

Section 6.2 contains an example of a pair (w,u) satisfying Wan's condition (θ) where Lyapunov stability is proved by straightforward calculations.

Once again we assume throughout this chapter that $u \in T_0 \cap M$ *and* (w,u) *is a regular pair.*

6.1. Construction of a vector field, generating trade curves

Let $x(t)$, $a \le t < b$, be a trade curve in A_w. Then by definition:

$$6.1.1.\quad \begin{cases} Du_i(x_i(t)) \cdot \dot{x}_i(t) \ge 0 & \text{for all } i \text{ and all } t, \\ \sum_{i=1}^{m} Du_i(x_i(t)) \cdot \dot{x}_i(t) > 0 & \text{for all } t, \\ \sum_{i=1}^{m} \dot{x}_i(t) = 0 & \text{for all } t. \end{cases}$$

In order to get some insight in the set of $\dot{x}(t)$ defined by 6.1.1 we give the next lemma.

6.1.2. LEMMA. *Let* $v = (v_1,\ldots,v_m): A_w \to \mathbb{R}^{\ell m}$ *be a* C^1*-map such that*

$$\sum_{i=1}^{m} v_i(x) = 0\ ,\quad Du_i(x_i) \cdot v_i(x) \ge 0]_{i=1}^{m} \quad \textit{for all } x \in A_w\ .$$

Then, given $z \in A_w$, $v_i(z) = 0$ *for all* $i \in I^*(z)$.

PROOF. Let $z \in A_w$ and $j \in I^*(z)$. Consider the map $g: A_w \to \mathbb{R}$, where

$$g(x) := Du_j(x_j) \cdot v_j(x)\ .$$

Then g is C^1, $g(x) \ge 0$ for all x, and $g(z) = 0$. Hence $Dg(z)(\delta z) = 0$ for all $\delta z \in T_z A_w$.
Since $Du_j(z_j) = 0$, the derivative $Dg(z)(\delta z)$ equals $(D^2u_j(z_j)\delta z_j) \cdot v_j(z)$ for all $\delta z \in T_z A_w$.
One may choose δz_j arbitrarily. Hence the assumption that u_j is a Morse function, yields $v_j(z) = 0$. □

The result of 6.1.2 justifies the next

6.1.3. DEFINITION. *For* $x \in A_w$ *the set* $C_0(x)$ *is the set of* $v = (v_1,\dots,v_m) \in \mathbb{R}^{\ell m}$ *satisfying*

(1) $\sum_{i=1}^{m} v_i = 0$ *;*

(2) $v_i = 0$ *for all* $i \in I^*(x)$ *;*

(3) $Du_i(x_i) \cdot v_i \geq 0$ *for all* i *;*

(4) $\sum_{i=1}^{m} Du_i(x_i) \cdot v_i > 0$.

Since $C_0(x)$ may be empty we define a set $C(x) \neq \emptyset$.

6.1.4. DEFINITION. *Let* $x \in A_w$. *Then* $C(x) := C_0(x) \cup \{0\}$, *where* $\{0\} \subset \mathbb{R}^{\ell m}$.

The relation between the set $C_0(x)$ and possible Pareto optima is subject of

6.1.5. LEMMA. $C_0(x) = \emptyset$ *if and only if* $x \in \theta_{ex}$.

PROOF. Let $x \in \theta_{ex}$ and $v \in C_0(x)$. There are nonnegative reals $\lambda_i]_{i=1}^m$ and some $p \in S^{\ell-1}$ such that $Du_i(x_i) = \lambda_i p$ for all i. Hence:

$$\sum_{i=1}^{m} v_i = 0\ ,\quad v_i = 0 \text{ if } \lambda_i = 0\ ,\quad p \cdot v_i \geq 0 \text{ if } \lambda_i > 0$$

and

$$\sum_{i=1}^{m} \lambda_i p \cdot v_i = \sum_{i \in I(x)} \lambda_i p \cdot v_i > 0\ .$$

But

$$\sum_{i=1}^{m} v_i = \sum_{i \in I(x)} v_i = 0$$

implies

$$\sum_{i \in I(x)} p \cdot v_i = 0\ .$$

Together with $p \cdot v_i \geq 0$ for all i, this yields $p \cdot v_i = 0$ for all i, in contradiction with $\Sigma_{i=1}^m \lambda_i p \cdot v_i > 0$. So $x \in \theta_{ex}$ implies $C_0(x) = \emptyset$.
Now let $x \notin \theta_{ex}$. Let $k := \# I(x)$, so $k \geq 2$. We choose $v = (v_1,\dots,v_m)$ by

$$\begin{cases} v_i := 0 \quad \text{for } i \in I^*(x) \ , \\ v_i := k \dfrac{Du_i(x_i)}{\| Du_i(x_i)\|} - \displaystyle\sum_{j \in I(x)} \dfrac{Du_j(x_j)}{\| Du_j(x_j)\|} \quad \text{for } i \in I(x) \ . \end{cases}$$

Claim: $v(x) \in C_0(x)$. To prove this claim we denote $\dfrac{Du_i(x_i)}{\| Du_i(x_i)\|}$ by g_i for $i \in I(x)$.
For each $i \in I(x)$ there is at least one j such that $g_i \cdot g_j < 1$ by the assumption $x \notin \theta_{ex}$. Furthermore, $g_i \cdot g_j \leq 1$ for all $i,j \in I(x)$. Then we have, for $i \in I(x)$:

$$g_i \cdot v_i = g_i \cdot \ kg_i - \sum_{j \in I(x)} g_i \cdot g_j = k - \sum_{j \in I(x)} g_i \cdot g_j > 0 \ .$$

This proves the claim. So $x \notin \theta_{ex}$ implies $C_0(x) \neq \emptyset$. □

The sets $C(x)$ form the basic tool in the construction of vector fields generating trade curves. One tries to make a vector field v such that $v(x) \in C(x)$ for all $x \in A_w$. Then necessarily $v = 0$ on θ_{ex}.

6.1.6. *It is easily seen that the set* $C_0(x)$ *is a cone in the following sense: If* $v \in C_0(x)$, $v' \in C_0(x)$, $\alpha \geq 0$, $\alpha' \geq 0$ $(\alpha,\alpha') \neq (0,0)$, *then* $\alpha v + \alpha' v' \in C_0(x)$.

The next lemma is the first step in the construction of a vector field v.

6.1.7. LEMMA. *Let* $z \in A_w$ *and* $z \notin \theta_{ex}$. *Then there is an open neighbourhood* $O_z \subset A_w$ *of* z *and a smooth vector field* v_z *defined on* O_z *such that* $v_z(x) \in C_0(x)$ *and* $\|v_z(x)\| \leq 1$ *for all* $x \in O_z$.

PROOF. We assume

$$Du_1(z_1) \neq 0 \ , \quad Du_2(z_2) \neq 0 \quad \text{and} \quad \frac{Du_1(z_1)}{\| Du_1(z_1)\|} \neq \frac{Du_2(z_2)}{\| Du_2(z_2)\|} \ .$$

Define $g_i(x_i)$ to be the normed gradient $\frac{Du_i(x_i)}{\| Du_i(x_i)\|}$, for i = 1,2 on a neighbourhood $\mathcal{U}_1$ of z.
Since $g_1(z_1) \cdot g_2(z_2) < 1$, it is assumed that

$$g_1(x_1) \cdot g_2(x_2) < 1 \quad \text{for } x \in \mathcal{U}_1 .$$

There are two alternatives:

(1) $g_1(z_1) \cdot g_2(z_2) > -1$. Then $-1 < g_1(x_1) \cdot g_2(x_2) < 1$ on a neighbourhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of z. Then we take on $\mathcal{U}_2$:

$$\tilde{v}_i(x) := Du_i(x_i) \quad \text{for } i > 2 \quad \text{and} \quad S(x) := -\sum_{i>2} \tilde{v}_i(x) .$$

(Note: if m = 2 then $S(x) := 0$.)
For i = 1,2 there are smooth functions $\alpha_i\colon \mathcal{U}_2 \to \mathbb{R}$, and there is a smooth vector field $q\colon \mathcal{U}_2 \to \mathbb{R}^\ell$ such that

$$S(x) = \alpha_1(x)g_1(x_1) + \alpha_2(x)g_2(x_2) + q(x) ,$$

where

$$q(x) \cdot g_1(x_1) = q(x) \cdot g_2(x_2) = 0 , \quad x \in \mathcal{U}_2 .$$

This is not difficult to prove, since $g_1(x_1)$ and $g_2(x_2)$ are linearly independent on $\mathcal{U}_2$, because $-1 < g_1(x_1) \cdot g_2(x_2) < 1$. The functions α_1, α_2 and the field q correspond with the decomposition of the smooth field S(x).
Next we take $\tilde{v}_1(x)$, $\tilde{v}_2(x)$ as follows:

$$\left.\begin{aligned} \tilde{v}_1(x) &:= y_1(x)g_1(x_1) + y_2(x)g_2(x_2) + q(x) \\ \tilde{v}_2(x) &:= y_3(x)g_1(x_1) + y_4(x)g_2(x_2) \end{aligned}\right\} \quad x \in \mathcal{U}_2$$

where (y_1,y_2,y_3,y_4) is solution of the equations:

$$\left.\begin{aligned} y_1(x) \qquad\qquad + y_3(x) \qquad\qquad &= \alpha_1(x) \\ y_2(x) \qquad\qquad + y_4(x) &= \alpha_2(x) \\ y_1(x) + \quad \varepsilon(x)\, y_3(x) \qquad\qquad &= 1 \\ \varepsilon(x)\, y_3(x) + y_4(x) &= 1 \end{aligned}\right\} \quad x \in \mathcal{U}_2$$

where

$$\varepsilon(x) := g_1(x_1) \cdot g_2(x_2) .$$

This system has a smooth solution (y_1, y_2, y_3, y_4) defined on $\mathcal{U}_2$ since the determinant

$$\begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & \varepsilon(x) & 0 & 0 \\ 0 & 0 & \varepsilon(x) & 1 \end{vmatrix} = \varepsilon^2(x) - 1 \neq 0 \quad \text{on } \mathcal{U}_2 \ .$$

Hence

$$g_1(x_1) \cdot \tilde{v}_1(x) = y_1 + \varepsilon(x) y_2 = 1$$

and

$$g_2(x_2) \cdot \tilde{v}_2(x) = \varepsilon(x) y_3 + y_4 = 1 \ .$$

Consider the smooth field $\tilde{v}(x) := (\tilde{v}_1(x), \dots, \tilde{v}_m(x))$ on $\mathcal{U}_2$. It follows from the construction of $v(\tilde{x})$ that for all $x \in \mathcal{U}_2$:

$$\begin{cases} Du_i(x_i) \cdot \tilde{v}_i(x) > 0 & \text{for all } i \in I(x) \ , \\ \tilde{v}_i(x) = 0 & \text{for all } i \in I^*(x) \ , \\ \sum_{i=1}^{m} \tilde{v}_i(x) = 0 \ . \end{cases}$$

Let $\mathcal{O}_z$ be an open neighbourhood of z such that $\mathcal{O}_z \subset \overline{\mathcal{O}}_z \subset \mathcal{U}_2$. There is some $M > 0$ satisfying $\|\tilde{v}(x)\| < M$ for all $x \in \overline{\mathcal{O}}_z$. The smooth vector field $\tilde{v}_z(x) := \frac{\tilde{v}(x)}{M}$ on $\mathcal{O}_z$ satisfies $v_z(z) \in C_0(x)$ and $\|v_z(x)\| \leq 1$ on $\mathcal{O}_z$.

Now we come at the second alternative:

(2) $g_1(z_1) \cdot g_2(z_2) = -1$. As in the first case, $\tilde{v}_i(x)$, $i > 2$, and $S(x)$ are defined on $\mathcal{U}_1$.
There is a neighbourhood $\mathcal{U}_2 \subset \mathcal{U}_1$ of z and a real $\alpha > 0$ such that $g_1(x_1) \cdot g_2(x_2) < 0$ and $S(x) \cdot g_1(x_1) < \alpha$ on $\mathcal{U}_2$.
Furthermore, let $S(x) = \alpha_1(x) g_1(x_1) + q(x)$ be the smooth decomposition of $S(x)$, with $q(x) \cdot g_1(x_1) = 0$ on $\mathcal{U}_2$. Hence $\alpha_1(x) < \alpha$ on $\mathcal{U}_2$.
Let $p_1 > \alpha$. We take

$$\begin{cases} \tilde{v}_1(x) := p_1 g_1(x_1) + q(x) \ , \\ \tilde{v}_2(x) := (\alpha_1(x) - p_1) g_1(x_1) \ . \end{cases}$$

Then

$$g_1(x_1)\cdot\tilde{v}_1(x) = p_1 > 0$$

and

$$g_2(x_2)\cdot\tilde{v}_2(x) = (\alpha_1(x)-p_1)g_1(x_1)\cdot g_2(x_2) > 0 .$$

The smooth field $\tilde{v}(x) = (\tilde{v}_1(x),\ldots,\tilde{v}_m(x))$ satisfies the same properties as the one in alternative (1). The remainder of the proof is the same. □

What we have to do now is gluing together the vector fields v_z in order to get a vector field on $A_w \setminus \theta_{ex}$. Partitions of unity (see 2.1.13) are the main tool here.

6.1.8. LEMMA. *There is a smooth vector field* v^* *on* $A_w \setminus \theta_{ex}$ *such that* $v^*(x) \in C_0(x)$ *and* $\|v^*(x)\| \le 1$ *for all* x.

PROOF. We denote by X the open subset $A_w \setminus \theta_{ex}$ of the submanifold A_w.
The sets $\{\mathcal{O}_z\}_{z\in X}$ as described in 6.1.7 form an open covering of X. There is a subset $J \subset X$ such that $\{\mathcal{O}_z\}_{z\in J}$ form a locally finite open covering of X (see [9], page 43).
There is a family $\{\rho_z\}_{z\in J}$ of smooth functions defined on X such that:

(1) $\rho_z(x) \in [0,1]$ for all $z \in J$, $x \in X$;

(2) $\operatorname{supp} \rho_z \subset \mathcal{O}_z$ for all $z \in J$;

(3) $\{\operatorname{supp} \rho_z\}_{z\in J}$ is a locally finite cover of X;

(4) $\sum_{z\in J} \rho_z(x) = 1$ for all $x \in X$.

(See 2.1.13.)
Let $x \in X$. Then

$$J_x := \{z \in J \mid x \in \mathcal{O}_z\} .$$

Since $\{\mathcal{O}_z\}_{z\in J}$ is a locally finite cover of X, it is clear that J_x is finite and nonempty.
Let v^* be the vector field, on X defined by

$$v^*(x) := \sum_{z\in J_x} \rho_z(x)v_z(x) ,$$

where v_z comes from 6.1.7.

We claim that $v^*(x) \in C_0(x)$, $\|v^*(x)\| \le 1$ for all $x \in X$ and that v^* is smooth on X.

Since all $\rho_z(x) \ge 0$ and $\rho_z(x) > 0$ for at least one $z \in J_x$, it follows from the cone-property 6.1.6 that $v^*(x) \in C_0(x)$ for all x. Furthermore,

$$\|v^*(x)\| \le \sum_{z\in J_x} |\rho_z(x)|\,\|v_z(x)\| \le \sum_{z\in J_x} \rho_z(x) \le 1 .$$

Remains the proof that v^* is smooth.

Let $x \in X$. Since $\{\mathcal{O}_z\}_{z\in J}$ is locally finite, there is an open neighbourhood $\mathcal{U}_x$ of x and a finite subset $K_x \subset J$ such that

$$\mathcal{U}_x \subset \bigcup_{z\in K_x} \mathcal{O}_z \quad \text{and} \quad \mathcal{U}_x \cap \mathcal{O}_z = \emptyset \quad \text{for } z \notin K_x .$$

Obviously, $J_x \subset K_x$. There is an open neighbourhood V_x of x in X satisfying:

(i) $\quad V_x \subset \bigcap_{z\in J_x} \mathcal{O}_z$;

(ii) $\quad V_x \cap \operatorname{supp} \rho_z = \emptyset$ for all $z \in K_x \setminus J_x$.

Then $J_x \subset J_y \subset K_x$ for all $y \in V_x$, and consequently we have for $y \in V_x$:

$$v^*(y) = \sum_{z\in J_y} \rho_z(y)v_z(y) = \sum_{z\in J_x} \rho_z(y)v_z(y) + \sum_{z\in J_y\setminus J_x} \rho_z(y)v_z(y) =$$

$$= \sum_{z\in J_x} \rho_z(y)v_z(y) , \quad \text{since } \rho_z(y) = 0 \text{ for } z \in J_y \setminus J_x .$$

Hence it follows that v^* is smooth on X. □

6.1.9. LEMMA. *There is a continuous vector field* v *on* A_w *such that* v *is smooth on* $A_w \setminus \theta_{ex}$, $v(x) \in C_0(x)$ *for all* $x \notin \theta_{ex}$, $v(x) = 0$ *on* θ_{ex}.

PROOF. By 2.1.15 there is a smooth function α defined on $\mathbb{R}^{\ell m}$ such that $\alpha(x) \ge 0$ for all x and $\theta_{ex} = \overset{\leftarrow}{\alpha}(\{0\})$. Then the restriction of α to A_w is smooth. We define v on A_w by:

$$\begin{cases} v(x) := \alpha(x)v^*(x) & \text{if } x \notin \theta_{ex} , \\ v(x) := 0 & \text{if } x \in \theta_{ex} , \end{cases}$$

where v^* comes from 6.1.8.
Since $\|v^*(x)\| \leq 1$ for all x, it is clear that v is continuous on A_w; v^* is smooth on $A_w \setminus \theta_{ex}$. Furthermore, since $\alpha(x) > 0$ for $x \notin \theta_{ex}$, it is clear that $v(x) \in C_0(x)$ for all $x \notin \theta_{ex}$. Hence v has the desired properties. □

REMARK. The foregoing construction holds for all pairs (w,u).

6.2. *An example*

This example deals with Lyapunov stability for points $z \in \theta$, where $I(z) \neq \emptyset$ and $\# I(z) > 1$. The pair (w,u) satisfies Wan's conditions (θ), so Lyapunov stability is guaranteed. For convention's sake we restrict ourselves to the positive orthant as the consumption space.
Let $m = 3$, $\ell = 2$, and (w,u) be the regular pair:

$$u_1(x_1) := x_1^1 x_1^2 , \quad u_2(x_2) := x_2^1 x_2^2 , \quad u_3(x_3) = -\tfrac{1}{2}(x_3 - e)\cdot(x_3 - e) \quad , \quad w = 3e \; ,$$

where $e = (1,1)$.
Since on the positive orthant each u_i satisfies local convexity, we have $\theta = \theta_{ex} = \Theta$. Some calculations show:

$$\Theta = \{(\alpha_1 e, \alpha_2 e, \alpha_3 e) \mid \alpha_i > 0]_{i=1}^3, \; \alpha_1 + \alpha_2 + \alpha_3 = 3, \; \alpha_3 \leq 1\} \; .$$

As a two-dimensional subset of $\mathbb{R}^3$, parametrized by $(\alpha_1, \alpha_2, \alpha_3)$, the set Θ is given in Figure 10. Θ is the shaded region.

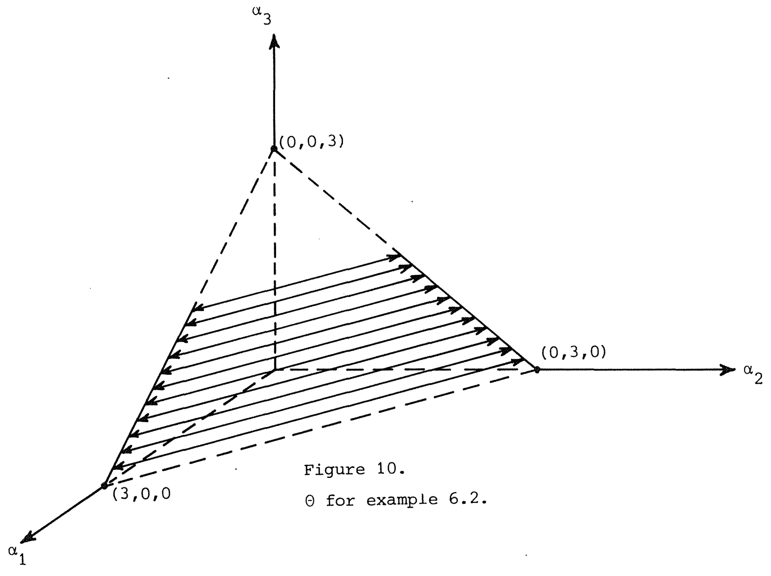


Figure 10.
Θ for example 6.2.

Obviously θ is not a submanifold, but it is a disjoint union of submanifolds of dimensions 2, 1 and 0.
Let $x \in \theta$ be $(\alpha_1 e, \alpha_2 e, \alpha_3 e)$. Then

$$u_1(x_1) = \alpha_1^2 \ , \quad u_2(x_2) = \alpha_2^2 \ , \quad u_3(x_3) = -(1-\alpha_3)^2 \ .$$

Hence $u_i(x_i) \le u_i(y_i)]_{i=1}^3$ where $y = (\beta_1 e, \beta_2 e, \beta_3 e) \in \theta$ leads to the inequalities:

$$\alpha_1 \le \beta_1 \ , \quad \alpha_2 \le \beta_2 \ , \quad \alpha_3 \le \beta_3 \ .$$

Since $\alpha_1 + \alpha_2 + \alpha_3 = \beta_1 + \beta_2 + \beta_3 = 3$, we have $x = y$. So (w,u) satisfies condition (θ).
Let us consider the point $z := (e,e,e) \in \theta$, where $I^* \ne \emptyset$ and $\# I = 2$. Let $x \in A_w$ be a point near to z, and $(\alpha_1 e, \alpha_2 e, \alpha_3 e) \in \theta$ such that

$$u_i(\alpha_i e) \ge u_i(x_i)]_{i=1}^3 \ .$$

Then

6.2.1. $$\alpha_1 \ge \sqrt{u_1(x_1)} \ , \quad \alpha_2 \ge \sqrt{u_2(x_2)} \ , \quad \alpha_1 + \alpha_2 \le 2 + \sqrt{-u_3(x_3)} \ .$$

So the point x defines a subset $B_x \subset \theta$, consisting of all points $(\alpha_1 e, \alpha_2 e, \alpha_3 e) \in \theta$ satisfying 6.2.1.
Each trade curve starting from x and converging to θ converges to a point in B_x. As a result of some calculations we have: $y \in B_x$ implies

$$\|y - z\| \le d(x) \ ,$$

where

$$d(x) := 3 \max \{ |1 - \sqrt{u_1(x_1)}| \ , \ |1 - \sqrt{u_2(x_2)}| \ , \ \sqrt{-u_3(x_3)} \} \ .$$

So $d(z) = 0$, and $d(x)$ is continuous. Hence for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all x with $\|x - z\| < \delta$ the set B_x is contained in the ball with radius ε and centre z. This means that z is Lyapunov stable, in accordance with Wan's theorem.

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