MATHEMATICAL CENTRE TRACTS 14

CALCULUS OF VARIATIONS IN MATHEMATICAL PHYSICS

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THIRD PRINTING

MATHEMATISCH CENTRUM AMSTERDAM 1980

AMS (MOS) subject classification scheme (1970): 49-01, 49H05

First printing 1966
Second printing 1973
Third printing 1980

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PREFACE

This tract represents worked-out lecture notes of a course in the calculus of variations delivered by the author to students in mathematical physics at the University of Amsterdam. In this course the calculus of variations is treated in a slightly modernized way by making full use of the language of vector spaces. Although the reader is supposed to be familiar with the fundamental notions of a Banach space and a Hilbert space, two sections are included in which these spaces are treated systematically in a condensed fashion. Much attention is paid to problems of theoretical mechanics including Noether's theorem. Some elementary knowledge of boundary value problems, e.g. vibrating string and membrane, will enable the reader to appreciate more fully those parts of the text, in which applications of Hilbert space theory are made. Much material for this course is derived from the books by Gelfand and Fomin and by Michlin. In particular, the first book represents an easily readable modern introduction to the calculus of variations and its applications.

Selected further reading is given in the bibliography at the end of this tract.

Until recently, the calculus of variations was known as a rather dull part of analysis for which there was hardly a place in the university curriculum. Its treatment was left to the physicists who needed it in their courses on theoretical mechanics. However, the situation was radically changed with the advent of the modern electronic computer. It was soon found that e.g. the reformulation of a boundary value problem as a variational problem made it more amenable to numerical computation. Moreover, the physicists found a further use of the variational formulation in problems of field theory. The success of functional-analysis as an important tool in mathematical physics also contributed to the renaissance of the calculus of variations as an independent mathematical discipline.

The recent introduction of generalized functions and derivatives by Sobolev and Schwartz gave a conclusive treatment of the famous variational principle of Dirichlet - in the past a notorious stumbling-block for many mathematicians. A treatment of these topics would go much beyond the scope of this modest introduction. The interested reader may consult the bibliography at the end of this tract for further reading.

CHAPTER I

MAXIMA AND MINIMA OF FUNCTIONS

1. Functions of a single variable

We consider real functions $y(x)$ of a real variable x. For simplicity it is assumed that $y(x)$ is defined for all values of x.

Definition 1.1

A function $y(x)$ is said to have a local maximum at $x = c$ if for some positive ε

$$y(x) - y(c) \leqq 0 \text{ for all } x \in (c-\varepsilon, c+\varepsilon). \tag{1.1}$$

A function $y(x)$ is said to have a local minimum at $x = c$ if for some positive ε

$$y(x) - y(c) \geqq 0 \text{ for all } x \in (c-\varepsilon, c+\varepsilon). \tag{1.2}$$

A common name for a maximum or a minimum is an extremum. If for an extremum the inequality (1.1) or (1.2) holds for all values of x, we say that $y(c)$ is a global extremum.

If x is restricted to a compact interval, say $a \leqq x \leqq b$, we may have boundary extrema at the ends of the interval.

For a continuous function defined on a compact interval we have the following theorem due to Weierstrass.

Theorem 1.1

A continuous function defined on a compact interval possesses a global maximum and a global minimum.

Proof

We show the existence of a point c such that $y(x) \geqq y(c)$ for all x. Obviously $y(x)$ has a lower bound μ. A minimizing sequence $\{x_n\}$, $n = 1,2,\ldots$ can be constructed with $y(x_n) > \mu - 1/n$. The compactness property guarantees the existence of a subsequence $\{x_\nu\}$, $\nu(n) \to \infty$, which converges to a certain point c. We know that c belongs to the given interval and that $y(x)$ is continuous at c. Then by taking the limit it follows that $y(c) = \mu$.

For differentiable functions we have the following necessary condition for an extremum.

Theorem 1.2

If the differentiable function $y(x)$ has an extremum at $x = c$ then $y'(c) = 0$.

Proof

Taking the case of a minimum we note that the difference quotient $\{y(c+h) - y(c)\}/h$ for sufficiently small $|h|$ has the sign of h. Hence, for $h \to 0$ with $h > 0$ it follows that $y'(c) \geqq 0$. On the other hand for $h \to 0$ with $h < 0$ it follows that $y'(c) \leqq 0$. The only possible combination is $y'(c) = 0$.

Example 1.1

a. $y(x) = x^2$ has a minimum at $x = 0$, $y'(0) = 0$.
b. for $y(x) = x^3$ we also have $y'(0) = 0$ but there is no extremum at $x = 0$.
c. $y(x) = x^2 \sin^2 1/x$ has a minimum at $x = 0$ and $y'(0) = 0$. We note, however, that $y'(x)$ is discontinuous at $x = 0$.

The points at which the derivative of a differentiable function $y(x)$ vanishes are called stationary points and we say that $y(x)$ is stationary at c when $y'(c) = 0$. For functions which are at least twice differentiable the second derivative may give supplementary information about the nature of the stationary points. More precisely we have the following statement.

Theorem 1.3

Let c be a stationary point of $y(x)$ with a continuous second derivative. Then $y(c)$ is a maximum for $y''(c) < 0$ and a minimum for $y''(c) > 0$.

Proof

According to Taylor's theorem we have for sufficiently small $|h|$ the expansion

$$y(c+h) = y(c) + \tfrac{1}{2}h^2 y''(c+\theta h), \quad 0 \leq \theta \leq 1.$$

Taking the case of a positive second derivative at c we note that the continuity of $y''(x)$ at c implies the existence of an interval $(c-\varepsilon, c+\varepsilon)$ where everywhere $y''(x) > 0$. In this neighbourhood of c we always have $y(c+h) - y(c) \geq 0$ so that $y(x)$ has a minimum at c.

2. Functions of two and more variables

Real functions $y(x_1,x_2)$ of real variables x_1 and x_2 can be visualized as a landscape in cartesian coordinates x_1,x_2 and y. Local extrema are defined as for functions of a single variable. The maxima are the peaks of the mountains and the minima the lowest points of the valleys. For a continuous function within a compact x-region the Weierstrass property holds in a similar fashion.

We shall now suppose that $y(x_1,x_2)$ has partial derivatives. Then it follows from theorem 1.2 that the necessary conditions for an extremum are the vanishing of the partial derivatives. Again all points for which

$$\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = 0 \tag{2.1}$$

are called the stationary points of y.
If we make the further assumption that y has continuous second derivatives, then sufficient conditions for the existence of extrema can be obtained.

Theorem 2.1

Let (c_1,c_2) be a stationary point of $y(x_1,x_2)$ with continuous second derivatives. Then $y(c_1,c_2)$ is a maximum if

$$\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} > \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2 \quad \text{and} \quad \frac{\partial^2 y}{\partial x_1^2} < 0 \tag{2.1}$$

and a minimum if

$$\frac{\partial^2 y}{\partial x_1^2}\frac{\partial^2 y}{\partial x_2^2} > \left(\frac{\partial^2 y}{\partial x_1 \partial x_2}\right)^2 \quad \text{and} \quad \frac{\partial^2 y}{\partial x_1^2} > 0. \tag{2.2}$$

Proof

According to Taylor's expansion we have in a sufficiently small neighbourhood of (c_1,c_2)

$$(2.3)\qquad y(c_1+h_1,c_2+h_2) = y(c_1,c_2) + \tfrac{1}{2}h_1^2 y_{11}(c_1,c_2) + $$
$$+ h_1h_2y_{12}(c_1,c_2) + \tfrac{1}{2}h_2^2 y_{22}(c_1,c_2) + \varepsilon(h_1^2+h_2^2),$$

where $\varepsilon \to 0$ for $h_1^2+h_2^2 \to 0$.
Hence the nature of the stationary point is determined by the behaviour of the quadratic form

$$(2.4)\qquad h_1^2y_{11} + 2h_1h_2y_{12} + h_2^2y_{22}.$$

A definitely positive form gives a minimum, a definitely negative form gives a maximum at (c_1,c_2).

The stationary points for which $y_{11}y_{22} < y_{12}^2$ are called saddle-points. They are the passes in the landscape of $y(x_1,x_2)$.

Example 2.1

a. $y = x_1^2 + x_2^2 + 1$ has a minimum (0,0,1). The landscape is a paraboloid of revolution. The condition (2.2) applies.
b. $y = x_1^4 + x_2^4 + 1$ has a minimum (0,0,1). However, (2.2) is of no use here.
c. $y = x_1^2 - x_2^2 + 1$ has a saddle-point (0,0,1). The landscape is a hyperbolic paraboloid.

If the (x_1,x_2) values are restricted to a compact region, then there may be boundary extrema. A typical case is discussed in the following example.

Example 2.2

We consider the function $y = x_1^2 + 2x_1x_2 + 2x_2^2 - x_1 - 2x_2$ within the circle $x_1^2 + x_2^2 \leqq 1$. Without difficulty we find the inner minimum $(0,\frac{1}{2},-\frac{1}{2})$. Possible boundary extrema are obtained from y for $x_1^2 + x_2^2 = 1$. Substitution of $x_1 = \cos\theta$, $x_2 = \sin\theta$ gives the function $y(\cos\theta, \sin\theta) = \sin^2\theta + 2\sin\theta\cos\theta - \cos\theta - 2\sin\theta + 1$ which is stationary for $\theta = 0$, $\arctan\frac{4}{3}$, $\pm\frac{2}{3}\pi$. Thus we find a minimum at (1,0), a maximum at $(\frac{3}{5},\frac{4}{5})$, a minimum at $(-\frac{1}{2},\frac{1}{2}\sqrt{3})$ and a maximum at $(-\frac{1}{2},-\frac{1}{2}\sqrt{3})$. A careful investigation is now needed to determine the behaviour of $y(x_1,x_2)$ in a neighbourhood of each of these points. Thus at (1,0) we have

$$y = (x_1-1) + \text{quadratic terms.}$$

It follows that (1,0) does not yield a boundary extremum. At $(\frac{3}{5},\frac{4}{5})$ we have

$$25y = 10 + 9(5x_1-3) + 12(5x_2-4) + \text{quadratic terms.}$$

It appears that $(\frac{3}{5},\frac{4}{5},\frac{2}{5})$ is a maximum not only at the boundary but also for the interior of the (x_1,x_2) region. A further investigation shows that this is the only boundary extremum. According to the theorem of Weierstrass, both extrema found above are also global extrema.

The generalization to functions $y(\vec{x})$ of n variables $x_1,x_2,\ldots,x_n$ or $\vec{x}$ in vector notation can easily be made. Again the Weierstrass property holds for a continuous function defined on a compact region. If $y(\vec{x})$ has continuous second derivatives its stationary points are obtained from

(2.5) $$\frac{\partial y}{\partial x_1} = \frac{\partial y}{\partial x_2} = \ldots = \frac{\partial y}{\partial x_n} = 0.$$

If for a stationary point the quadratic form

(2.6) $$\sum h_i h_j \frac{\partial^2 f}{\partial x_i \partial y_j}$$

is definitely positive (negative) we have a minimum (maximum).

3. Subsidiary conditions

One often encounters extremum problems with subsidiary conditions. Then an extremum of a function

$$y(x_1, x_2, \ldots, x_n) \tag{3.1}$$

has to be determined where the variables satisfy m ($m<n$) conditions of the type

$$z_1(\vec{x}) = C_1, z_2(\vec{x}) = C_2, \ldots, z_m(\vec{x}) = C_m, \tag{3.2}$$

where the C_i ($i = 1,2,\ldots,m$) are constants. In principle, it is possible to eliminate m of the n variables and to consider y as a function of the remaining n-m variables. In practice, however, the often tedious process of elimination can be avoided by introducing the so-called Lagrange multipliers. This method works formally as follows. We consider the function

$$F = y + \lambda_1 z_1 + \lambda_2 z_2 + \ldots + \lambda_m z_m, \tag{3.3}$$

where the variables $x_1, x_2, \ldots, x_n$ are considered independent and where the $\lambda_1, \lambda_2, \ldots, \lambda_m$ are considered as parameters. The stationary points of F are determined by the n relations

$$\frac{\partial y}{\partial x_j} + \sum_i \lambda_i \frac{\partial z_i}{\partial x_j} = 0, \quad j = 1,2,\ldots,n. \tag{3.4}$$

Thus the values of the n variables x_j and of the m parameters λ_i follow from the set (3.4) and the set (3.2) comprising m+n relations.

The proof of this elegant method is quite simple. We note that (3.4) is equivalent to noting that the rectangular matrix of m+1 columns and n rows

$$\left(\frac{\partial y}{\partial x_j}\ \frac{\partial z_1}{\partial x_j}\ \frac{\partial z_2}{\partial x_j} \cdots \frac{\partial z_m}{\partial x_j}\right) \tag{3.5}$$

has a rank $\leq$ m.
The usual process of eliminating m variables by means of the m subsidiary conditions may be carried out as follows. It is possible to introduce n-m independent variables $t_1, t_2, \ldots, t_{n-m}$ in such a way that the n variables x_j are certain functions of the n-m variables t_k ($k = 1,2,\ldots,n-m$) satisfying the conditions (3.2). The stationary points of y now follow from the n-m relations

$$\frac{\partial}{\partial t_k}\, y\{\vec{x}(t_1, t_2, \ldots, t_{n-m})\} = 0.$$

Written in full we obtain

$$\sum_j \frac{\partial y}{\partial x_j} \frac{\partial x_j}{\partial t_k} = 0. \tag{3.6}$$

To this we may add the relations

$$\left\{\begin{array}{l} \sum_j \frac{\partial z_1}{\partial x_j} \frac{\partial x_j}{\partial t_k} = 0 \\ \sum_j \frac{\partial z_2}{\partial x_j} \frac{\partial x_j}{\partial t_k} = 0 \\ - - - - - - - \\ \sum_j \frac{\partial z_m}{\partial x_j} \frac{\partial x_j}{\partial t_k} = 0, \end{array}\right. \tag{3.7}$$

which are obtained from (3.2) by partial differentiation. For each k we have in (3.6) and (3.7) m+1 homogeneous linear equations of the variables $\partial x_j/\partial t_k$. Elimination of these variables gives the result (3.5) and hence the equivalent relations (3.4).

The method of the Lagrangian multipliers can only be used for functions with continuous partial derivatives. Further it yields only stationary points and a closer investigation about the nature of these points is always needed. In some cases, however, it may be obvious by other arguments that we have to deal with an extremum. The method will be amply illustrated by the following examples.

Example 3.1

Consider $y = x_1^2 + x_2^2 - 2x_1 + 2$ with $x_1 + x_2 = -1$.
Geometrically, this is the parabolic intersection of a paraboloid and a vertical plane. Hence we may expect a minimum. Introducing a Lagrangian multiplier λ we consider

$$F = x_1^2 + x_2^2 - 2x_1 + 2 + \lambda(x_1+x_2).$$

The necessary conditions for an extremum are

$$\frac{\partial F}{\partial x_1} = 2x_1 - 2 + \lambda = 0, \quad \frac{\partial F}{\partial x_2} = 2x_2 + \lambda = 0.$$

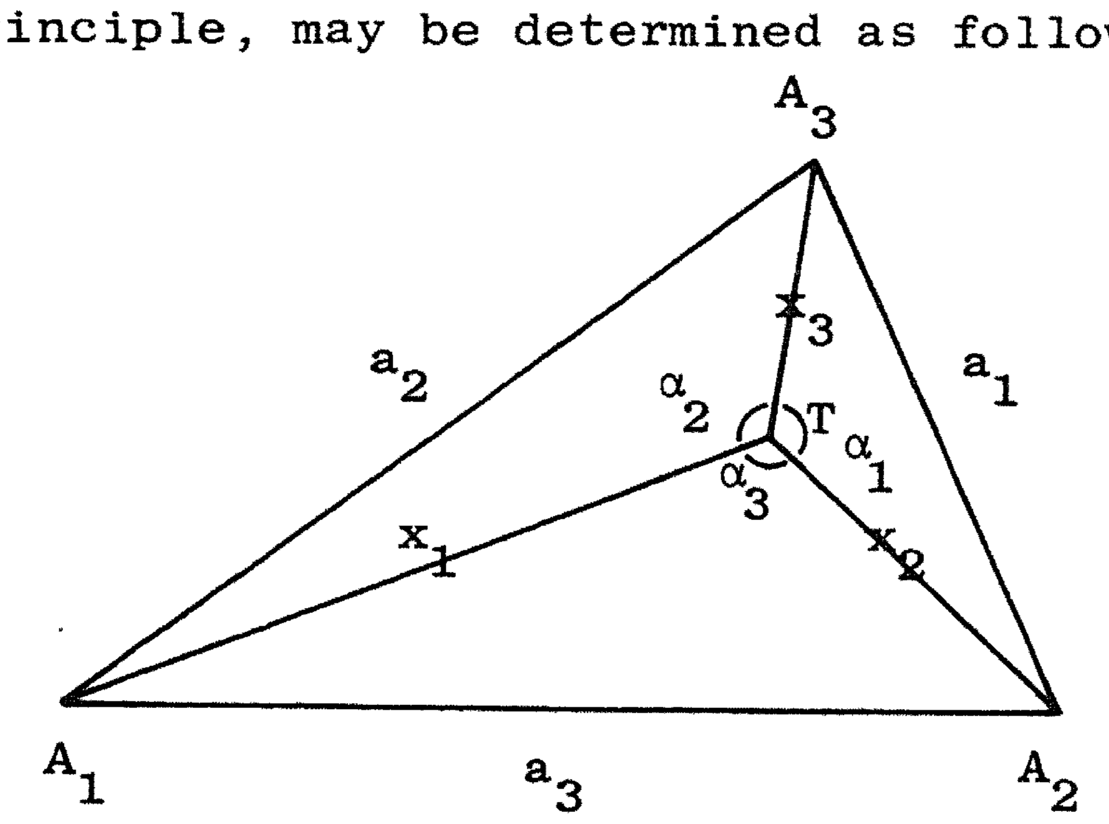
We obtain the parametric solution $x_1 = 1 - \frac{1}{2}\lambda$, $x_2 = -\frac{1}{2}\lambda$. For $\lambda = 2$ the subsidiary condition is satisfied so that the minimum (0,-1,3) is obtained.

Example 3.2

In a triangle $A_1A_2A_3$ the so-called point of Torricelli T is determined by the condition

$$TA_1 + TA_2 + TA_3 = \text{minimum}.$$

The position of this point the existence of which follows at once from the Weierstrass principle, may be determined as follows.



Using the cyclic notation $A_2A_3 = a_1$, $TA_1 = x_1$, $\angle A_2TA_3 = \alpha_1$ etc. we note that the angles α_1 etc. are simple functions of the independent variables x_1 etc. In fact, the cosine rule in the triangle A_2TA_3 gives

$$2x_2x_3\cos\alpha_1 = x_2^2 + x_3^2 - a_1^2, \text{ etc.}$$

The problem may now be formulated as that of determining the minimum of

$$y = x_1 + x_2 + x_3$$

with the subsidiary condition

$$z \equiv \alpha_1(x_2,x_3) + \alpha_2(x_3,x_1) + \alpha_3(x_1,x_2) = 2\pi.$$

The conditions (3.4) give

$$\frac{\partial z}{\partial x_1} = \frac{\partial z}{\partial x_2} = \frac{\partial z}{\partial x_3}$$

or

$$\frac{\partial \alpha_2}{\partial x_1} + \frac{\partial \alpha_3}{\partial x_1} = \frac{\partial \alpha_3}{\partial x_2} + \frac{\partial \alpha_1}{\partial x_2} = \frac{\partial \alpha_1}{\partial x_3} + \frac{\partial \alpha_2}{\partial x_3} .$$

A simple calculation shows that

$$\frac{\partial \alpha_2}{\partial x_1} + \frac{\partial \alpha_3}{\partial x_1} = \frac{S}{\sin\alpha_1} ,$$

where S is some symmetric expression of the elements of triangle $A_1A_2A_3$. Thus it follows that

$$\sin\alpha_1 = \sin\alpha_2 = \sin\alpha_3$$

so that

$$\alpha_1 = \alpha_2 = \alpha_3 = 2\pi/3.$$

CHAPTER II

A CONCRETE VARIATIONAL PROBLEM

4. Fundamental lemma's

In this section we have collected together a number of lemma's which are of fundamental importance in the calculus of variations.

Lemma 4.1

If $\phi(x)$ is continuous in (a,b) and if

$$\int_a^b \phi(x)h(x)dx = 0$$

for all $h(x) \in C(a,b)$ with $h(a) = h(b) = 0$, then $\phi(x) = 0$ for all x.

Proof

If $\phi(x) \neq 0$ for some x, say positive, there exists a subinterval (α,β) where $\phi(x) > 0$. By taking $h(x) = (x-\alpha)(\beta-x)$ for x in (α,β) and $h(x) = 0$ elsewhere we obtain

$$\int_a^b \phi(x)h(x)dx = \int_\alpha^\beta \phi(x)(x-\alpha)(\beta-x)dx > 0.$$

Since this contradicts the assumption of the theorem we must have $\phi(x) = 0$ for all x.

Lemma 4.2

If $\phi(x)$ is continuous in (a,b) and if

$$\int_a^b \phi(x)h'(x)dx = 0$$

for all $h(x) \in C_1(a,b)$ with $h(a) = h(b) = 0$, then $\phi(x) = c$ for all x, where c is a constant.

Proof

We may take the special choice

$$h(x) = \int_a^x \{\phi(\zeta) - c\} d\zeta$$

where the constant c is determined by the condition $h(b) = 0$. Then we have

$$\int_a^b \{\phi(x) - c\}^2 dx = \int_a^b \{\phi(x) - c\} h'(x) dx =$$

$$= \int_a^b \phi(x) h'(x) dx - c\{h(b) - h(a)\} = 0.$$

This obviously implies $\phi(x) = c$ for all x.

Lemma 4.3

If $\phi(x)$ and $\psi(x)$ are continuous in (a,b) and if

$$\int_a^b \{\phi(x) h(x) + \psi(x) h'(x)\} dx = 0$$

for all $h(x) \in C_1(a,b)$ with $h(a) = h(b) = 0$, then $\phi(x) = \psi'(x)$ for all x.

Remark

The latter relation implies the differentiability of $\psi(x)$, which was not assumed in advance.

Proof

Putting $\Phi(x) = \int_a^x \phi(\zeta) d\zeta$

we find after integrating by parts that

$$\int_a^b \{- \Phi(x) + \psi(x)\} h'(x) dx = 0.$$

According to the previous lemma this implies

$$\Phi(x) = \psi(x) + c,$$

where c is a constant. Since $\Phi(x)$ is differentiable the same is true for $\psi(x)$. Finally differentiation of this relation gives the required result.

<u>Lemma 4.4</u>

If $\phi(x)$ is continuous in (a,b) and if

$$\int_a^b \phi(x)h'(x)dx = 0$$

for all $h(x) \in C_1(a,b)$ with $h(a) = h(b) = 0$ which also satisfy the subsidiary equation

$$\int_a^b \psi(x)h'(x)dx = 0,$$

where $\psi(x)$ is continuous but not a constant function, then for all x

$$\phi(x) = \lambda\psi(x) + \mu,$$

where λ and μ are constants.

<u>Proof</u>

We proceed along lines similar to those in the proof of lemma 4.2. We take the special choice

$$h(x) = \int_a^x \{\phi(\zeta) - \lambda\psi(\zeta) - \mu\}d\zeta$$

where the constants λ and μ are determined by $h(b) = 0$ and $\int_a^b \psi h'dx = 0$. This gives the two linear equations

$$\lambda \int_a^b \psi d\zeta + \mu\int_a^b d\zeta = \int_a^b \phi d\zeta,$$

$$\lambda \int_a^b \psi^2 d\zeta + \mu\int_a^b \psi d\zeta = \int_a^b \phi\psi d\zeta.$$

Since, according to the Schwarz inequality,

$$\left\{\int_a^b \psi d\zeta\right\}^2 < \int_a^b d\zeta . \int_a^b \psi^2 d\zeta ,$$

unless $\psi(x)$ is a constant, these equations have a unique solution. The proof now consists in showing that with the values of λ and μ found above,

$$\int_a^b \{\phi(x) - \lambda\psi(x) - \mu\}^2 dx = 0$$

which implies the required result. In fact, the latter integral may be written as

$$\int_a^b \{\phi(x) - \lambda\psi(x) - \mu\} h'(x) dx = \int_a^b \phi(x) h'(x) dx -$$

$$- \lambda \int_a^b \psi(x) h'(x) dx - \mu\{h(b) - h(a)\} = 0.$$

<u>Remark</u>

This lemma lies at the bottom of the method of the Lagrangian multipliers in the calculus of variations.

5. The simplest variational problem

The simplest problem of the calculus of variations may be formulated as follows. Let $F(x,y,p)$ be a function with continuous partial derivatives up to the second order with respect to x, y and p. Then the problem is that of determining a function $y(x) \in C_1(a,b)$ satisfying the boundary conditions

$$y(a) = A, \quad y(b) = B \tag{5.1}$$

for which the functional

$$f[y] = \int_a^b F(x,y,p)dx \tag{5.2}$$

is an extremum.

Let $y(x)$ be the required curve for which $f[y]$ assumes, say, a minimum value. This means that there exists a neighbourhood of $y(x)$ consisting of neighbouring curves $y(x) + h(x)$, where $h(x)$ is a small function, such that

$$f[y+h] - f[y] \geqq 0. \tag{5.3}$$

The notion of a neighbourhood of a function can be made precise by introducing for functions $\phi(x)$ of the class $C_1(a,b)$ the following norm

$$\|\phi\| = \max |\phi(x)| + \max |\phi'(x)|. \tag{5.4}$$

With this norm the function space $C_1(a,b)$ becomes a normed space. By an ε-neighbourhood of a function $y(x)$ of this space is meant the set of functions $\phi(x)$ with $\|\phi-y\| < \varepsilon$. A small function $h(x)$ is a function with $\|h\| < \varepsilon$ where ε (as always) is a small positive quantity. In the variational problem under consideration we consider the subspace of functions satisfying the boundary conditions (5.1). A convenient way of describing a neighbourhood of $y(x)$ in this subspace is obtained below.

We take an arbitrary element h(x) of $C_1(a,b)$ which satisfies the boundary conditions

(5.5) $$h(a) = 0, \quad h(b) = 0.$$

Then a neighbourhood of y(x) is formed by the one-parameter families

(5.6) $$y(x) + th(x),$$

where t is a small parameter. In this way the neighbourhood can be visualized as a sphere consisting of rays (5.6). The radius of the sphere, i.e. the smallness of the neighbourhood is given by $|t| \; \|h\|$.

On each ray (5.6) the functional (5.2) becomes an ordinary function of the single variable t which of course is a minimum for t = 0. For this we have the necessary condition

(5.7) $$\frac{d}{dt} f\left[y(x) + th(x)\right] = 0 \text{ for } t = 0.$$

Substitution of (5.2) gives explicitly

(5.8) $$\int_a^b \left\{\frac{\partial F}{\partial y} h(x) + \frac{\partial F}{\partial p} h'(x)\right\} dx = 0.$$

This condition holds on each ray and hence is valid for all $h(x) \in C_1(a,b)$ satisfying (5.5). According to the fundamental lemma 4.3, this leads to the following differential equation

(5.9) $$\frac{d}{dx}\frac{\partial F}{\partial p} = \frac{\partial F}{\partial y}$$

which is usually called the Euler equation.

The solutions of (5.9) are called extremals. Since the Euler equation is of the second order, the family of extremals has two degrees of freedom. One may expect that an extremal is determined by the two boundary conditions (5.1) at the endpoints. There are, however, various complications. It may happen that more than one extremal is determined by the boundary conditions (5.1) or that no extremal of (5.9) passes through the endpoints (a,A) and (b,B). On the other hand, since (5.9) is only a necessary condition a solution of (5.9) with (5.1) does not necessarily correspond to an extremum of the given functional. These features will be amply illustrated in the examples below.

In some special cases the integration of the Euler equation becomes particularly simple.

Case 1

F does not depend on x.
In this case a first integral of (5.9) may be written down at once

$$p \frac{\partial F}{\partial p} - F = C, \tag{5.10}$$

where C is an arbitrary constant.
Indeed, differentiation of the left-hand side of (5.10) gives in view of (5.9)

$$\frac{d}{dx}(pF_p - F) = p\frac{d}{dx}F_p + F_p\frac{dp}{dx} - F_y\frac{dy}{dx} - F_p\frac{dp}{dx} =$$

$$= p\left(\frac{d}{dx}F_p - F_y\right) = 0.$$

Integration of the first order equation (5.10) involves, of course, a second constant of integration.

Case 2

F does not depend on y.
Obviously the Euler equation may be replaced by the first order one

$$\frac{\partial F}{\partial p} = C. \tag{5.11}$$

Case 3

F does not depend on p.

The Euler equation reduces in this case to an ordinary equation

$$F_y(x,y) = 0. \tag{5.12}$$

The Euler equation (5.9) is in general a differential equation of the second order. Written out in full we have

$$\frac{d}{dx}\frac{\partial F}{\partial p} \equiv F_{px} + F_{py}\frac{dy}{dx} + F_{pp}\frac{d^2y}{dx^2} = F_y. \tag{5.13}$$

One may object that the existence of the second derivative of y was not assumed in advance although the existence of the left-hand side of (5.13) is guaranteed by lemma 4.3. It can easily be shown, however, that y" exists at least at those points where $F_{pp} \neq 0$. It is known that

$$\lim_{\varepsilon\to 0}\{F_p(x+\varepsilon, y(x+\varepsilon), y'(x+\varepsilon)) - F_p(x,y,y')\}/\varepsilon = \frac{d}{dx} F_p.$$

The existence of this limit implies that of

$$\lim_{\varepsilon\to 0}\{F_p(x,y,y'(x+\varepsilon)) - F_p(x,y,y')\}/\varepsilon.$$

Since F_p has a continuous derivative with respect to its third argument the latter expression can be written as

$$\lim_{\varepsilon\to 0} \frac{y'(x+\varepsilon)-y'(x)}{\varepsilon} F_{pp}(x,y,y^*)$$

where $y^* \to y'$ for $\varepsilon \to 0$. In view of the continuity of F_{pp} the existence of y" now follows unless $F_{pp} = 0$.

Example 5.1

Consider the functional

$$f[y] = \int_{-1}^{1} y^2(p-2x)^2 dx$$

for $y \in C_1(-1,1)$ with the boundary conditions

$$y(-1) = 0, \quad y(1) = 1.$$

The minimum value of $f[y]$, obviously zero, is reached for

$$\begin{cases} y(x) = 0 & \text{for } -1 \leq x \leq 0, \\ y(x) = x^2 & \text{for } 0 \leq x \leq 1. \end{cases}$$

The Euler equation is satisfied but y'' does not exist at $x = 0$.

6. The Brachistochrone problem

In 1696 John Bernoulli proposed the following problem as a challenge to the mathematicians of his time.

A particle slides under gravity from rest along a smooth vertical curve $y = y(x)$ joining two points A and B. The problem is to find the curve for which the time to go from A to B is a minimum.

This problem was subsequently solved by James (Jacob) Bernoulli, his elder brother, by Newton, Leibniz, de l'Hôpital and of course John himself. This problem proved to be the fruitful start of the new discipline of the calculus of variations.
This so-called problem of the brachistochrone will be studied here by the method of the previous section, a method which goes back essentially to Euler.

Taking the upper point A as the origin and measuring y vertically downwards the velocity v at a depth y is $(2gy)^{\frac{1}{2}}$ and the time from A to B is

(6.1) $$\int_A^B v^{-1} ds = (2g)^{-\frac{1}{2}} \int_A^B y^{-\frac{1}{2}} (1+p^2)^{\frac{1}{2}} dx.$$

So we have to minimize the functional

(6.2) $$f[y] = \int_0^a y^{-\frac{1}{2}} (1+p^2)^{\frac{1}{2}} dx,$$

where $y(x) \in C_1(0,a)$ with the boundary conditions

(6.3) $$y(0) = 0, \quad y(a) = b,$$

where (a,b) are the coordinates of the end point B.

In this case, the integrand F does not depend on x. Therefore, instead of the Euler equation, the simpler equation (5.10) can be used. This gives

.4) $$y(1+p^2) = \text{constant}.$$

In order to find the solution in an easily recognizable form we put

(6.5) $\qquad x = x(\theta), \quad y = y(\theta)$

where the parameter θ is determined by $p = \text{cotan}\, \frac{1}{2}\theta$, $0 < \theta < 2\pi$. Then equation (6.4) gives $y = c(1-\cos\theta)$ where c is a constant of integration. The function $x(\theta)$ is obtained by integration of $dx = \tan \frac{1}{2}\theta\, dy = c(1-\cos\theta)d\theta$. Thus the parameter equation (6.5) becomes explicitly

(6.6) $\qquad x - x_0 = c(\theta-\sin\theta), \quad y = c(1-\cos\theta).$

Thus the extremals of the Euler equation are cycloids obtained by rolling a circle of radius c along a straight line (see fig. 6.1). The extremals through A are determined by $x_0 = 0$. Obviously they can be obtained from each other by a similarity transformation. In order to determine the extremal which also passes through B it suffices to start from a certain cycloid, say with $c = 1$, and to determine its inter-

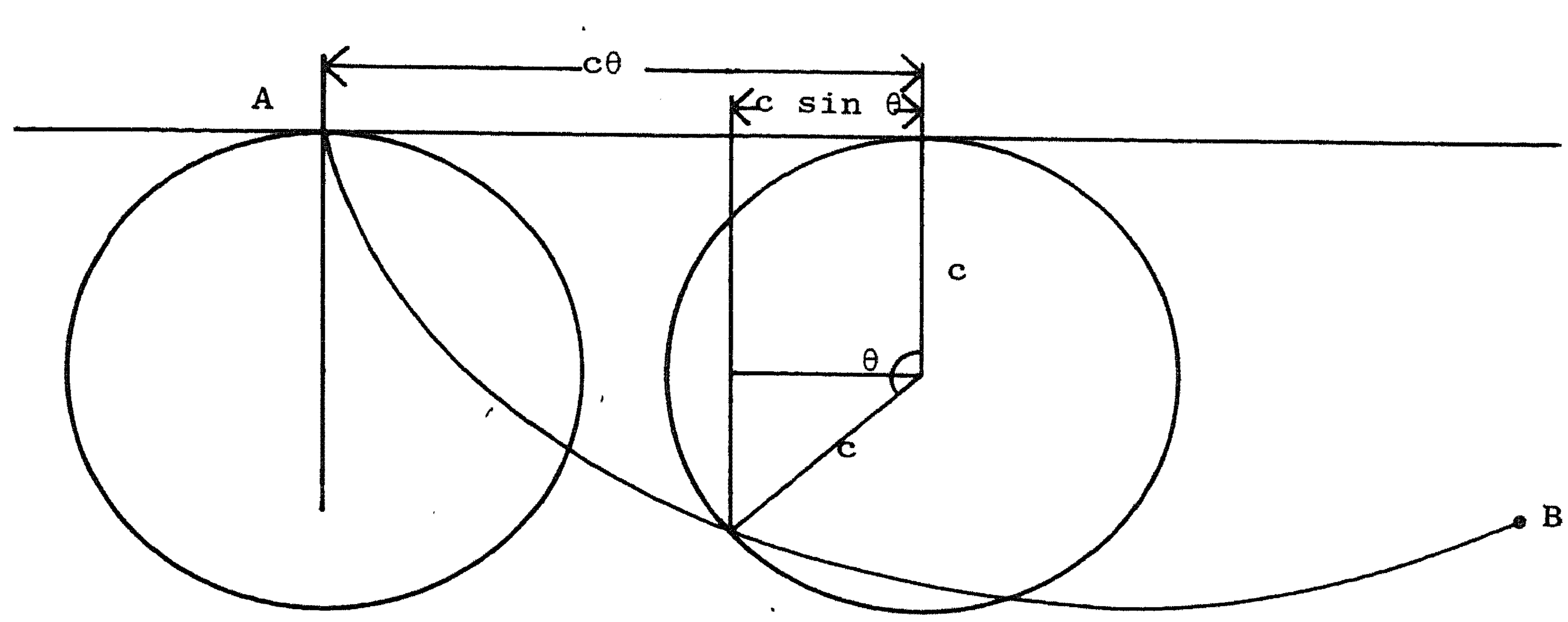


fig. 6.1

sections with the line AB. There is always an intersection B_1 with $0 < \theta' < 2\pi$. The similarity transformation by which B_1 passes into B gives the required extremal (see fig. 6.2). There may be more intersections $B_2, B_3, \ldots$ of the line AB with the given cycloid.

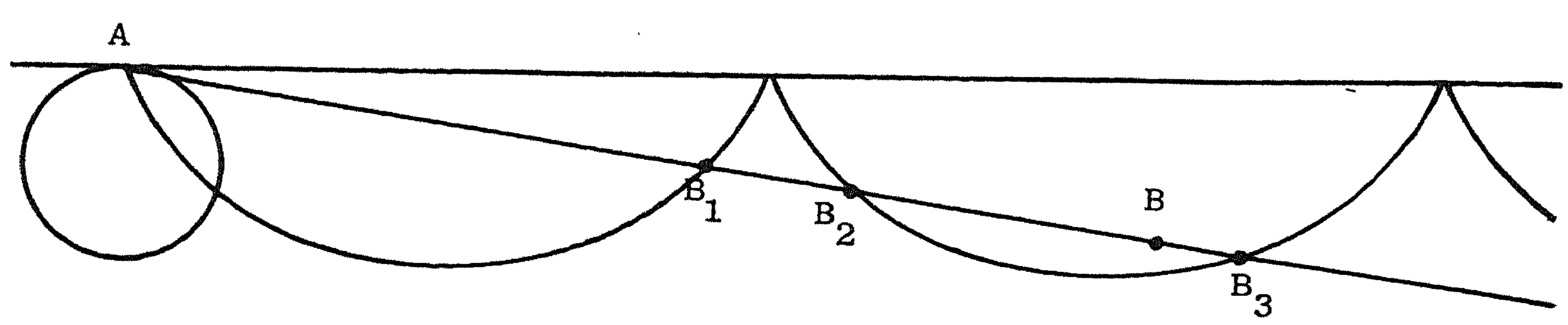


fig. 6.2

These correspond to further extremals starting at A and passing through B. Since the latter extremals contain one or more cusps they must be excluded for not being members of the class $C_1(0,a)$. Thus in all cases we are left with a single solution. That the brachistochrone obtained above really yields a minimum value of the functional (6.2) may be clear intuitively. Mathematically this follows only after a careful investigation of the functional in a neighbourhood of the found solution. In practice, however, this phase is often neglected.

The following problem gives an idea of the difficulties involved in a variational problem in general.
We consider the surfaces of revolution which are obtained by rotating a curve y(x) joining the points A(-1,a) and B(1,a) about the x-axis. The problem is to determine the function for which the surface has a minimum area.

The problem amounts to minimizing the functional

(6.7) $$f[y] = \int_{-1}^{1} y(1+p^2)^{\frac{1}{2}} dx.$$

The admissible functions are those of $C_1(-1,1)$ for which

(6.8) $$y(-1) = y(1) = a \quad (a > 0).$$

Again the Euler equation may be replaced by the simpler equation

(5.10). This gives

(6.9) $$y(1+p^2)^{-\frac{1}{2}} = C$$

which can be brought into the form

(6.10) $$\frac{dy}{(y^2-C^2)^{\frac{1}{2}}} = \frac{dx}{C}.$$

Integration gives the two-parameter set of extremals

(6.11) $$y = C \cosh(\frac{x}{C} + C'),$$

where C and C' are constants of integration. In view of the symmetry of the end point conditions we are interested only in those extremals for which C' = 0. The constant C is determined by the transcendental equation

(6.12) $$a = C \cosh C^{-1}.$$

This equation may have two, one or no solutions. One arrives at the same conclusion by observing that all symmetric extremals $y = C \cosh x/C$ are similar so that as in the brachistochrone problem the required extremal is determined by intersecting a particular extremal, say $y = \cosh x$, with the line $y = ax$. There are two, one or no intersections as the case may be. When $m = 1.51$ represents the minimum value of $C \cosh C^{-1}$, so that the lines $y = \pm mx$ are tangent to all symmetric extremals, for $a > m$, then two possible solutions of the Euler equation are obtained, whereas for $a < m$, there are none (see fig. 6.3).

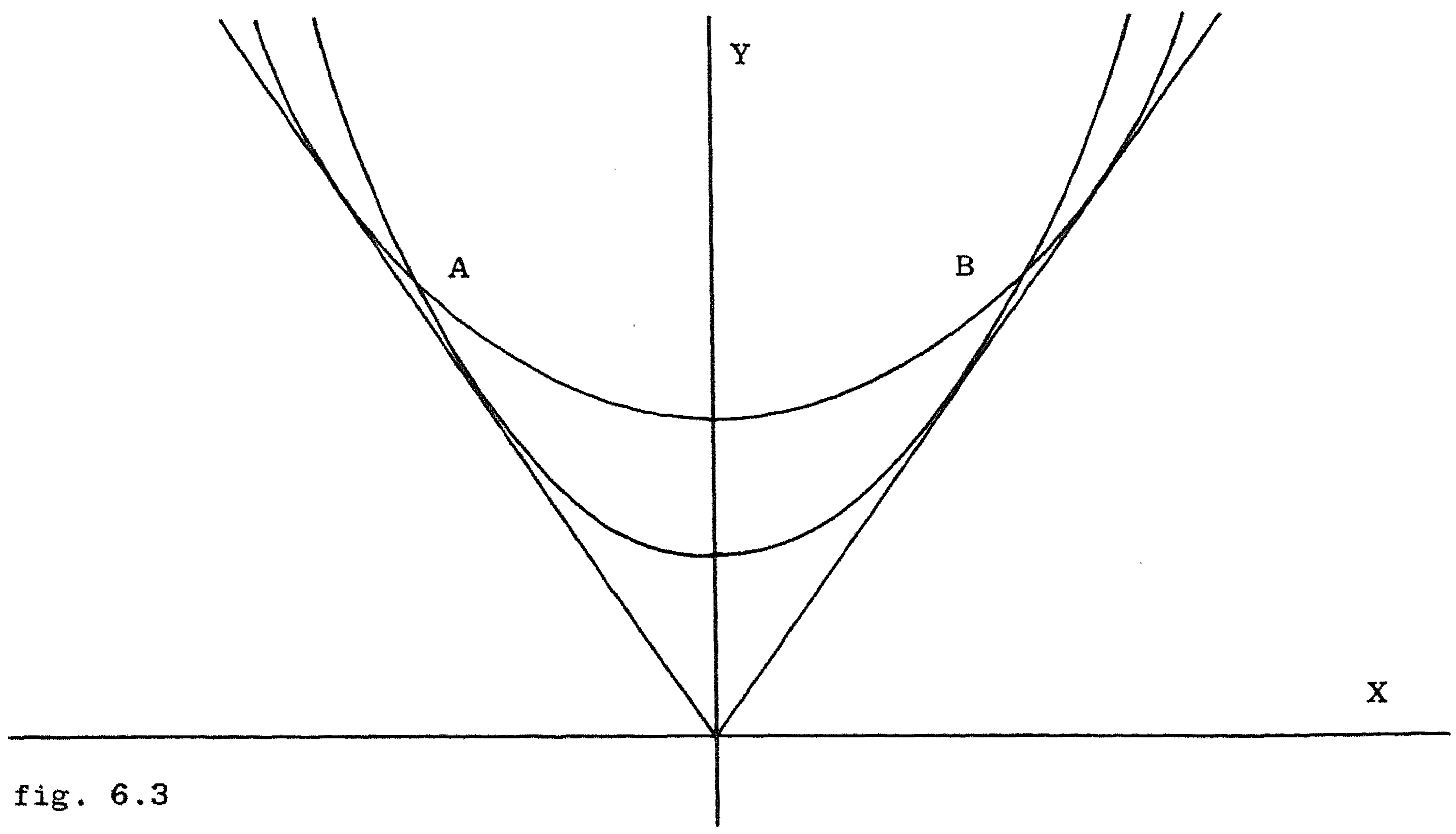


fig. 6.3

A more detailed study shows that in the case of two extremals passing through A and B only one of the curves corresponds to a surface of revolution of minimum area. In the case $a < m$, the problem has no solution, at least not in the given class $C_1(-1,1)$.
Physically a solution may be obtained experimentally by spanning a soapfilm upon two circular rings mounted on an axis. In the latter case the minimal surface appears to be degenerated into two circular disks connected by their symmetry axis. In other words the required surface is obtained by rotating the "broken extremal" which is shown in fig. 6.4.

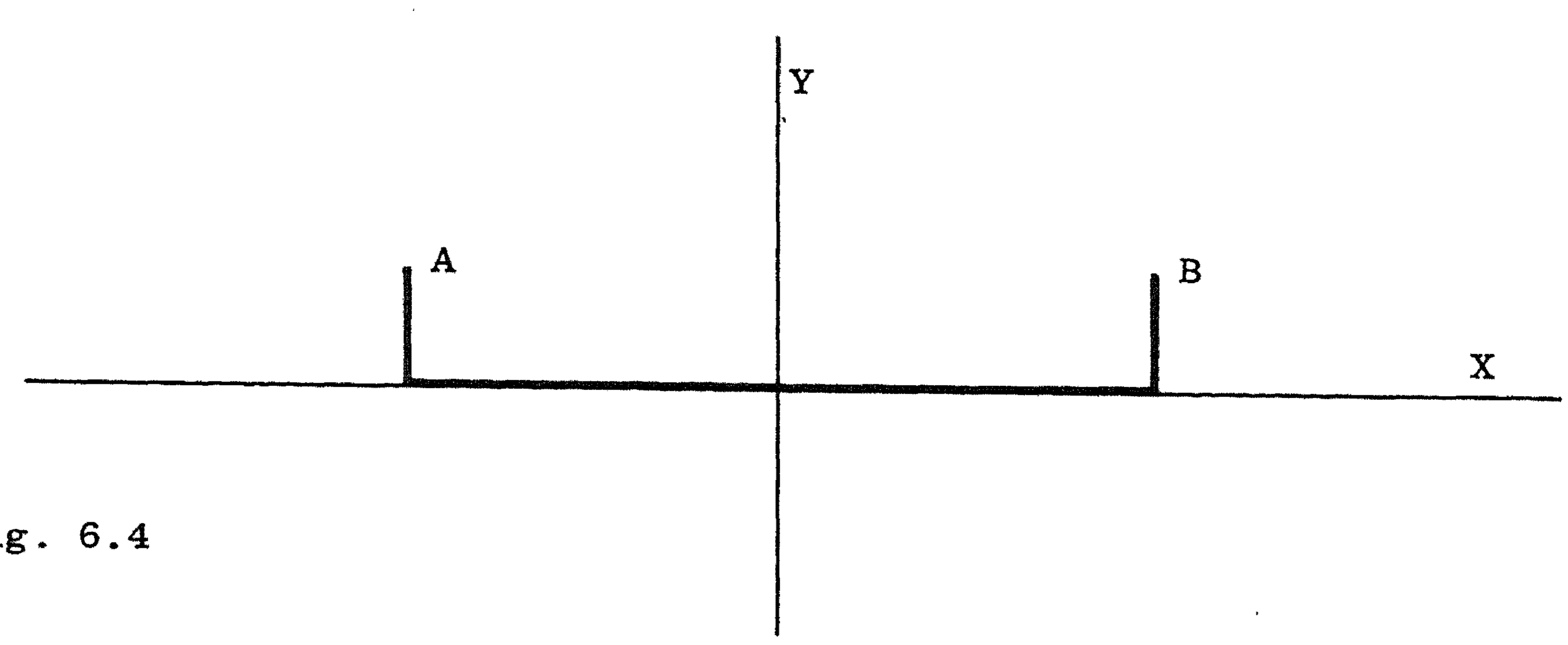


fig. 6.4

7. The catenary problem

In this section we consider the simplest variational problem of section 5, where the admissible functions satisfy not only the boundary conditions (5.1) but also the subsidiary condition

$$g[y] = \int_a^b G(x,y,p)dx = C, \tag{7.1}$$

where C is a constant and where G(x,y,p) has continuous partial derivatives up to the second order.

Let y(x) + h(x) be an admissible function in an ε-neighbourhood of an extremal y(x). This means that $||h(x)|| < \varepsilon$ with the norm (5.4), that $h(a) = h(b) = 0$, and that $g[y+h] - g[y] = 0$. Taylor expansion of F and G gives

$$f[y+h] - f[y] = \int_a^b \{F_y h + F_p h'\}dx + 0(\varepsilon^2), \tag{7.2}$$

and

$$0 = \int_a^b \{G_y h + G_p h'\}dx + 0(\varepsilon^2). \tag{7.3}$$

Thus, arguing as in section 5, the necessary condition for $f[y]$ to be an extremum is

$$\int_a^b \{F_y h + F_p h'\}dx = 0 \tag{7.4}$$

for all $h(x) \in C_1(a,b)$ with $h(a) = h(b) = 0$ and which also satisfy

$$\int_a^b \{G_y h + G_p h'\}dx = 0. \tag{7.5}$$

We can now apply the fundamental lemma 4.4. It suffices merely to rewrite (7.4) and (7.5) in the form $\int \phi h'dx = 0$ and $\int \psi h'dx = 0$ by putting

$$\phi = -\int_a^x F_y dx + F_p, \quad \psi = -\int_a^x G_y dx + G_p.$$

Hence constants λ and μ exist such that

$$-\int_a^x (F_y+\lambda G_y)dx + (F_p+\lambda G_p) = \mu.$$

Differentiation gives

$$\frac{d}{dx}(F_p+\lambda G_p) = F_y + \lambda G_y. \tag{7.6}$$

The necessary condition (7.6) may be interpreted as the Euler equation for the functional $f + \lambda g$.
In this form it is the variational counterpart of Lagrange's method of section 3 where a corresponding extremum problem was considered. Lagrange's method may easily be extended for a variational problem with an arbitrary number of subsidiary conditions. If e.g. an extremum is sought of the functional

$$f[y] = \int_a^b F(x,y,p)dx \tag{7.7}$$

with the m subsidiary conditions

$$g_i[y] = \int_a^b G_i(x,y,p)dx = C_i, \quad i = 1,2,\dots,m, \tag{7.8}$$

the obvious solution is to consider Euler's equation for the functional

$$f + \lambda_1 g_1 + \lambda_2 g_2 + \dots + \lambda_m g_m. \tag{7.9}$$

As an illustration of Lagrange's method in the calculus of variations we consider the so-called catenary problem. A flexible rope of uniformly distributed mass and of given length hangs in equilibrium under gravity with its ends attached to two fixed points A and B. We are required to find the form of the equilibrium curve.

It is obvious that the rope must hang in a vertical plane. According to a well-known principle of classical mechanics, the equilibrium position is characterized by a minimum potential energy. This is equivalent to the condition that the centre of gravity of the rope has the lowest position.

We shall consider here only the simpler case where A and B have equal height. Then taking the vertical plane as the x,y-plane with gravity acting downwards, these points may be given the coordinates A(-1,a) and B(1,a) with $a > 0$. The y-coordinate of the centre of gravity is proportional to the integral $\int y ds$. Hence we have to minimize the functional

$$(7.10) \qquad f[y] = \int_{-1}^{1} y(1+p^2)^{\frac{1}{2}} dx,$$

where $y(x) \in C_1(-1,1)$ with the boundary conditions $y(-1) = y(1) = a$ and where $y(x)$ also satisfies the subsidiary condition

$$(7.11) \qquad g[y] = \int_{-1}^{1} (1+p^2)^{\frac{1}{2}} dx = 2b,$$

where 2b is the length of the rope.

According to Lagrange we introduce the Lagrangian multiplier λ and consider now the single functional

$$(7.12) \qquad f[y] + \lambda g[y] = \int_{-1}^{1} (y+\lambda)(1+p^2)^{\frac{1}{2}} dx.$$

This functional is of the form (6.7) and so the equation of the extremals can be written down at once as

$$(7.13) \qquad y + \lambda = C \cosh \left(\frac{x}{C} + C'\right).$$

In view of the symmetry of the boundary conditions, $C' = 0$. The condition (7.11) gives

$$(7.14) \qquad b = C \sinh C^{-1}.$$

Noting that $b \geqq 1$ this equation always has a unique solution and so the complications of the corresponding problem in section 6 are not present here. Finally the value of λ is determined by requiring the extremal to pass through A and B, viz.

$$\lambda = C \cosh C^{-1} - a. \tag{7.15}$$

8. The second variation

In this section we give a further discussion of the simplest variational problem, namely to minimize the functional

$$(8.1)\qquad f[y] = \int_a^b F(x,y,p)dx$$

for functions $y(x) \in C_1(a,b)$ with the boundary conditions

$$(8.2)\qquad y(a) = A, \quad y(b) = B.$$

It will be found convenient to assume here that $F(x,y,p)$ has continuous partial derivatives up to the third order. We consider now the behaviour of (8.1) in a neighbourhood of a function $y(x)$ of the given class. Let $y(x) + h(x)$ be a neighbouring function of the same class so that $h(a) = h(b) = 0$. Then according to Taylor's theorem, we have

$$(8.3)\qquad F(x,y+h,\ p+h') - F(x,y,p) = (F_y h+F_p h') + \\ + \tfrac{1}{2}(F_{yy}h^2+2F_{yp}hh'+F_{pp}h'^2) + R,$$

where the first and second partial derivatives on the right-hand side are all taken at (x,y,p). From the continuity of the second partial derivatives it follows that the remainder R satisfies

$$(8.4)\qquad R/\|h\|^2 \to 0 \text{ for } \|h\| \to 0$$

where the norm is taken in the sense (5.4).
The corresponding increment of $f[y]$ may be written as (cf. 7.2)

$$(8.5)\qquad \Delta f[y] \overset{\text{def}}{=} f[y+h] - f[y] = \delta f[h] + \delta^2 f[h] + \varepsilon \|h\|^2,$$

where

(8.6) $$\delta f[h] \overset{\text{def}}{=} \int_a^b \{F_y h + F_p h'\} dx,$$

(8.7) $$\delta^2 f[h] \overset{\text{def}}{=} \int_a^b \{\tfrac{1}{2}F_{yy} h^2 + F_{yp} hh' + \tfrac{1}{2}F_{pp} h'^2\} dx$$

and where $\varepsilon \to 0$ as $||h|| \to 0$.

The first term $\delta f[h]$ is called the first variation of y. It may be considered as a continuous linear functional with respect to $h \in C_1(a,b)$. The second term $\delta^2 f[h]$ is known as the second variation of y. It represents a homogeneous quadratic functional of $h \in C_1(a,b)$ which is likewise continuous.

If $f[y]$ has an extremum for the function y(x) the first variation δf vanishes as has been shown in section 5. It can be expected that the behaviour of the second variation $\delta^2 f$ gives further information. In fact the following statement can be made.

Theorem 8.1

If $f[y]$ has a minimum for a function y(x) then for this function

(8.8) $$\delta^2 f[h] \geqq 0$$

for all admissible h.

Proof

According to (8.5) we have

(8.9) $$\Delta f[y] = \delta^2 f[h] + \varepsilon \, ||h||^2.$$

Supposing $\delta^2 f[h] < 0$, then the right-hand side of this relation would be negative for sufficiently small $||h||$. Since the left-hand side is always positive a contradiction would be obtained. Hence we must have the inequality (8.8).

We note that this theorem still gives a necessary condition. A similar statement holds of course for the maximum case. One might think that $\delta^2 f[h] > 0$ gives a sufficient condition for a minimum of f but this is not the case. We have, however, the following sufficient condition.

Theorem 8.2

If for a function y(x) the first variation vanishes and if there exists a positive constant k such that

$$\delta^2 f[h] \geqq k\,||h||^2 \tag{8.10}$$

then f[y] has a minimum for y(x).

Proof

Follows at once from (8.9) and (8.10).

In order to study the behaviour of the quadratic functional (8.7) more explicitly we shall put it in a more convenient form. After integration by parts we find

$$\delta^2 f[h] = \int_a^b (Ph'^2 + Qh^2)\,dx, \tag{8.11}$$

where

$$P = \tfrac{1}{2}F_{p'p'}, \quad Q = \tfrac{1}{2}\left(F_{yy} - \frac{d}{dx}F_{yp'}\right). \tag{8.12}$$

Thus in order to apply the preceding two theorems it suffices to study quadratic functionals of the special form (8.11). The first result is the following.

Lemma

If

$$\int_a^b (Ph'^2 + Qh^2)\,dx \geqq 0$$

for all $h(x) \in C_1(a,b)$ with $h(a) = h(b) = 0$ then

$$P(x) \geqq 0.$$

Proof

Assuming $P(x) < 0$ for $x = x_0$, there would exist an interval $I(x_0 - \frac{\pi}{n}, x_0 + \frac{\pi}{n})$ where $P(x) \leqq -\alpha < 0$ and where n is a sufficiently large integer. Take now

$$h(x) = \begin{cases} \sin^2 n(x-x_0) & \text{inside} \quad I, \\ 0 & \text{outside} \quad I, \end{cases}$$

so that $h(x) \in C_1(a,b)$ and $h(a) = h(b) = 0$. A simple calculation shows that

$$\int_a^b (Ph'^2 + Qh^2)\,dx < -2n\alpha\pi + \frac{2\pi}{n} \max|Q|.$$

Thus for sufficiently large n the right-hand side would become negative which violates the assumption of the lemma.

Combining theorem (8.1) and this lemma we arrive at the following famous statement which goes back to Legendre.

Theorem 8.3 (Legendre)

If the functional

$$f[y] = \int_a^b F(x,y,p)\,dx, \quad y(a) = A, \quad y(b) = B$$

has a minimum for the curve $y = y(x)$ then at every point of this curve the following inequality (Legendre's condition) is satisfied

$$\frac{\partial^2 F}{\partial p^2} \geqq 0.$$

In a number of problems of mathematical physics the Legendre condition is almost trivially satisfied and the problem remains to decide whether a certain extremal yields an extremum or not. Although there exists a quite extensive literature and although there are a number of fairly general theorems giving sufficient conditions it is often better to treat a specific variational problem by some ad hoc method. This will be illustrated in the following problem.

Example 8.1

Consider the functional

$$f[y] = \int_0^a (p^2 - y^2)\,dx \tag{8.13}$$

where the admissible functions are those of $C_1(0,a)$ for which

$$y(0) = 0, \quad y(a) = 1. \tag{8.14}$$

The Euler equation is $y'' + y = 0$. A simple calculation yields the following extremal

$$y(x) = \sin x / \sin a. \tag{8.15}$$

It appears that for $a = n\pi$ $(n = 1,2,\ldots)$ the problem has no solution. Excluding this case we investigate next a neighbourhood of the extremal. Putting

$$y = \sin x / \sin a + h(x) \tag{8.16}$$

we have

$$f[y+h] - f[y] = \int_0^a (h'^2 - h^2)\,dx. \tag{8.17}$$

The right-hand side represents at the same time the second variation of f. It is of the form (8.11) with $P = 1$ and $Q = -1$. Legendre's condition is clearly satisfied for the minimum case. The arbitrary function $h(x)$ may be expanded into the Fourier series

(8.18) $$h(x) = \sum_{n=1}^{\infty} h_n \sin n\pi x/a.$$

Since $h(x) \in C_1(0,a)$ with $h(0) = h(a) = 0$ it is known that $\sum_{n=1}^{\infty} n^2 h_n^2 < \infty$. Substitution of (8.18) into the right-hand side of (8.17) gives, after some elementary calculations,

(8.19) $$\delta^2 f[h] = \tfrac{1}{2} \sum_{n=1}^{\infty} h_n^2 (n^2\pi^2 - a^2)/a.$$

This shows that only for $a < \pi$ the second variation is positive for all possible choices of the coefficients h_n. Thus (8.15) yields a minimum of (8.13) only for $a < \pi$.

CHAPTER III

AN ABSTRACT VARIATIONAL PROBLEM

9. Linear normed spaces

An abstract linear space R of elements y is said to be a linear normed space if to each element y there is assigned a nonnegative number $||y||$, called the norm of y, such that

1. $||y|| > 0$ for $y \neq 0$,
2. $||\alpha y|| = |\alpha| \, ||y||$ for each scalar α,
3. $||y_1+y_2|| \leqq ||y_1|| + ||y_2||$.

From the second axiom it follows that $||0|| = 0$. the first axiom says that $||y|| = 0$ implies $y = 0$.

In a normed space the distance between two elements y_1 and y_2 is defined as $||y_1-y_2||$. This notion of a distance allows the use of geometric language. E.g. the set of elements y satisfying $||y-y_0|| < \rho$ is called a sphere with centre y_0 and radius ρ.

The sequence $\{y_n\}$ is said to converge to the limit y if $||y_n-y|| \to 0$ for $n \to \infty$.

The sequence $\{y_n\}$ is called a fundamental sequence if $||y_m-y_n|| \to 0$ for $m,n \to \infty$ simultaneously.

A normed space R in which every fundamental sequence converges to a limit which belongs to R is called a Banach space.

We shall mainly consider function spaces the elements of which are real functions y(x) of one or more real variables $x_1,x_2,\ldots,x_n$. The scalars α are then real numbers.

Example 9.1

The class C(a,b) of functions y(x) which are continuous for $a \leqq x \leqq b$ is a linear normed space for the norm

$$(9.1) \qquad ||y||_0 = \max |y(x)|.$$

Convergence in this norm of a sequence of functions $\{y_n(x)\}$ coincides with the familiar notion of uniform convergence. In this space a fundamental sequence always converges to a continuous function so that the space C(a,b) is a Banach space.
Similarly the class of continuous functions of n variables $x_1, x_2, \ldots, x_n$ belonging to a compact region Ω with the same norm forms the Banach space $C(\Omega)$.

Example 9.2

The class $C_1(a,b)$ of functions y(x) which are continuous and have a continuous derivative for $a \leqq x \leqq b$ is a linear normed space for the norm

$$(9.2) \qquad ||y||_1 = \max|y(x)| + \max|y'(x)|.$$

This space too is a Banach space. The same statement can be made for the class of functions $y(x_1, x_2, \ldots, x_n)$ which are continuous and have continuous partial derivatives in Ω. This space is indicated by $C_1(\Omega)$. A further generalization is obtained by taking higher derivatives into account. Thus the Banach space $C_p(\Omega)$ is characterized by the norm

$$(9.3) \qquad ||y||_p = \sum_{i=0}^{p} \max_{\Omega} |y^{(i)}(x)|.$$

In the linear space of the continuous functions y(x) having continuous derivatives in $a \leqq x \leqq b$ there are two norms available depending on the interpretation of this space as the normed space C(a,b) or as the normed space $C_1(a,b)$. Accordingly there are two types of convergence. The sequence $\{y_n\}$ is said to converge weakly to the limit y if $||y_n - y||_0 \to 0$ for $n \to \infty$ i.e. if $y_n(x) \to y(x)$ for all x uniformly in (a,b).
The sequence $\{y_n\}$ is said to converge strongly to the limit y if $||y_n - y||_1 \to 0$ for $n \to \infty$ i.e. if simultaneously $y_n(x) \to y(x)$ and

$y_n'(x) \to y'(x)$ for all x uniformly in (a,b). Obviously strong convergence implies weak convergence.

Example 9.3

The sequence $y_n(x) = n^{-1}\sin nx$ converges weakly to zero in any x-interval. However, the sequence is not convergent in the strong sense. The sequence $y_n(x) = n^{-2}\sin nx$ converges strongly, and hence also weakly, to the zero function.

A set S of elements of a normed space R is called compact if any sequence $\{y_n\}$ of elements of S contains a convergent sequence the limit of which also belongs to S.

For a finite-dimensional space, say the points or vectors of the three-dimensional Euclidean space, a compact set is the same as a bounded and closed set. This equivalence stems from Weierstrass' principle, which states that in this case any infinity of points which form a bounded set contains at least one accumulation point. However, this is not always true in a linear normed space with an infinite number of dimensions such as the spaces C(a,b) and $C_1(a,b)$.

Example 9.4

In the space C(a,b), the set formed by the functions sin ωx, where ω may take all real values, is not compact in spite of the fact that all functions have the norm 1 and that the set is closed. In fact the sequence sin nx does not converge for n = 1,2,... .

In general it is quite difficult to decide whether a given set of functions is compact or not. However, for functions of the class C(a,b) there exists a relatively simple criterion due to Arzelà and Ascoli. This theorem will be preceded by the following definition.

Definition

A set S of elements of C(a,b) is said to be equicontinuous if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$(9.4) \qquad |y(x_1) - y(x_2)| < \varepsilon$$

for all $y \in S$ provided that $|x_1 - x_2| < \delta$.

Theorem 9.1 (Arzelà - Ascoli)
A subset S of C(a,b) is compact if and only if S is bounded, closed and its elements are equicontinuous.

Proof
A simple proof may be found e.g. in Kolmogorov and Fomin, Elements of the theory of functions and functional analysis.

A functional f is defined as a mapping which maps the elements of a normed space R onto the real line. Thus to each element $y \in R$ there is associated a real number $f[y]$. The functional f is said to be continuous if $y_n \to y$ implies $f[y_n] \to f[y]$.

Example 9.5
The norm $\|y\|$ is a continuous functional since $\|y_n\| \leq \|y_n - y\| + \|y\|$ so that $\|y_n - y\| \to 0$ implies $\|y_n\| \to \|y\|$.

A linear functional $f[y]$ is characterized by the property

$$f[\alpha_1 y_1 + \alpha_2 y_2] = \alpha_1 f[y_1] + \alpha_2 f[y_2]. \tag{9.5}$$

A bilinear functional $f[y,z]$ depends on two elements $y,z \in R$ and is linear in y for a fixed z and linear in z for a fixed y. If we take $z = y$ we obtain a so-called quadratic functional.

A functional $f[y]$ is said to be bounded on a subset $S \subset R$ if

$$\sup_{y \in S} |f[y]| < \infty. \tag{9.6}$$

A bounded functional on S has an upper bound and a lower bound. However, an element of S for which these bounds are actually reached may or may not be present. For a compact subset we have the following generalization of theorem 1.1.

Theorem 9.2 (The principle of Weierstrass)
A functional $f[y]$ continuous on a compact subset S of the Banach space R is bounded and assumes a maximum and a minimum value at certain elements of S.

Proof
That $f[y]$ has a finite upper bound M may be shown in the following indirect way. If $M = \infty$ there would exist a sequence $\{y_n\}$ of elements of S with $f[y_n] \to \infty$. The compactness of S implies the existence of a subsequence $\{y_{\nu_n}\}$ converging to an element y_0 of S. The continuity of $f[y]$ means that $f[y_{\nu_n}] \to f[y_0]$ for $n \to \infty$. Since $f[y_0]$ is a finite quantity a contradiction is obtained. The second part of this theorem is proved as in the proof of theorem 1.1.

Example 9.6
Consider the continuous (quadratic) functional

$$f[y] = \int_0^1 y^2(x)dx$$

where $y(x) \in C(0,1)$ with the boundary conditions $y(0) = 0$ and $y(1) = 1$ and with the extra condition $|y(x)| \leqq 1$. The set S formed by the functions $y(x)$ satisfying these conditions is obviously a bounded subset of the Banach space $C(0,1)$ with the norm (9.1). The functional f is obviously bounded on S with upper bound 1 and lower bound 0. The lower bound is reached for the minimizing sequence $y_n = x^n$ $(n = 1,2,\dots)$ since $f[y_n] = (2n+1)^{-1}$. However, this sequence does not converge; nor does it contain a convergent subsequence. Thus $f[y]$ has no minimum. Since the Weierstrass principle does not hold here the set S cannot be compact. In fact it is not even closed.

Example 9.7
Let $K(x,t)$ be continuous in the square $a \leqq x,\ t \leqq b$. Then we state that the set S of continuous functions defined by

$$(9.7) \qquad y(x) = \int_a^b K(x,t)\phi(t)dt,$$

where $\phi(t)$ belongs to a closed sphere of $C(a,b)$, i.e. $||\phi|| \leqq \rho$, is compact.

Without difficulty we see that S is a closed and bounded subset of the same Banach space. Next we prove the equicontinuity of S. Since the kernel function $K(x,t)$ is uniformly continuous for any given $\varepsilon > 0$, a quantity $\delta(\varepsilon)$ can be found such that

$$|K(x_1,t) - K_2(x_2,t)| < \varepsilon \quad \text{for} \quad |x_1 - x_2| < \delta$$

uniformly for all t. Then we have

$$|y(x_1) - y(x_2)| \leqq \int_a^b K(x_1,t) - K(x_2,t)| |\phi(t)| dt <$$

$$< (b-a)\rho\varepsilon.$$

Since the right-hand side of this inequality does not depend on the choice of the element $y(x)$, the equicontinuity of S follows. All conditions of the Arzelà-Ascoli theorem are now satisfied, and thus the set S is compact.

10. Functionals on a Banach space

Let $f[y]$ be a functional on a Banach space R. Then $f[y]$ is said to be differentiable at the element y if the behaviour of f in a neighbourhood of y is of the following kind

$$f[y+h] = f[y] + \delta f[h] + \alpha[h], \tag{10.1}$$

where h belongs to a sufficiently small ε-sphere, i.e. $||h|| < \varepsilon$, where δf is a linear functional of h and where the remainder α satisfies the condition

$$\frac{\alpha[h]}{||h||} \to 0 \text{ for } ||h|| \to 0. \tag{10.2}$$

Obviously a functional which is differentiable at y is also continuous at y. The expansion (10.1) is the analogue of Taylor's expansion of a differentiable function. Since a function of n variables $x_1, x_2, \ldots, x_n$ can be interpreted as a functional over the n-dimensional Euclidian space of vectors $x(x_1, x_2, \ldots, x_n)$, which is a Banach space, the expansion (10.1) contains the Taylor expansion as a particular case. The linear functional δf is called the variation of f. For functions y the variation corresponds to the differential dy.

Example 10.1

The functional of the simplest variational problem of section 5

$$f[y] = \int_a^b F(x,y,p)dx$$

where F has continuous partial derivatives of the first order is a differentiable functional on the Banach space $C_1(a,b)$ with the norm (9.2). In fact we have

$$f[y+h] = f[y] + \int_a^b \{F_y(x,y^*,p^*)h(x) + F_p(x,y^*,p^*)h'(x)\}dx$$

where y^* and p^* are intermediate values which for $\|h\| \to 0$ tend to y and p. In view of the continuity of F_y and F_p, the expansion (10.1) holds, with δf, of course, given by (8.6).

For differentiable functionals we have the following abstract analogue of theorem 1.2.

Theorem 10.1

A necessary condition for a differentiable functional f defined on a Banach space R to have an extremum at the element y is that its variation vanish at y.

Proof

Taking the minimum case there exists a sufficiently small neighbourhood of y such that

$$f[y+h] - f[y] \geqq 0 \text{ for all } h \text{ with } \|h\| < \varepsilon.$$

Assuming $\delta f \neq 0$ at y it follows from (10.1) that $f[y+h] - f[y]$ has the sign of the linear functional δf provided ε is small enough. Since the latter sign changes if h is replaced by $-h$, a contradiction is obtained. Thus we must have $\delta f = 0$ at y.

Assuming further differentiability properties, an abstract treatment of the second variation as it has been discussed for the simplest variational problem in section 8 may be given.

The functional $f[y]$ is said to be twice differentiable at the element y if its increment may be written as

$$f[y+h] - f[y] = \delta f[h] + \delta^2 f[h] + \beta[h], \tag{10.3}$$

where the linear functional δf is the variation, where $\delta^2 f$ is a quadratic functional of h and where the remainder β satisfies the condition

$$\frac{\beta[h]}{\|h\|^2} \to 0 \text{ as } \|h\| \to 0. \tag{10.4}$$

The quadratic functional $\delta^2 f$ is called the <u>second variation</u> of f. The abstract analogue of theorem 8.1 is as follows.

Theorem 10.2

A necessary condition for a twice differentiable functional f to have a minimum at y is that its second variation be non-negative at y.

Proof

As that for theorem 8.1.

A quadratic functional $g[y]$ defined on a Banach space is said to be definitely positive if there exists a positive constant k such that

$$g[y] > k \|y\|^2. \tag{10.5}$$

In a similar way, a definitely negative functional is defined. These concepts are the natural analogues of positive and negative definiteness of homogeneous quadratic forms of a finite number of variables. The abstract analogue of theorem 8.2 can now be formulated as follows.

Theorem 10.3

A sufficient condition for a twice differentiable functional f to have a minimum at y is

1st $\delta f = 0$ at y

2nd $\delta^2 f$ definitely positive at y.

Proof

As that for theorem 8.2.

11. The Hilbert space

A linear vector space R is called euclidean if to each pair y,z of elements of R there is assigned a real or complex number (y,z) with the following properties

1. $(y,y) > 0$ for $y \neq 0$,
2. $(y,z) = \overline{(z,y)}$,
3. $(\alpha_1 y_1+\alpha_2 y_2,z) = \alpha_1(y_1,z) + \alpha_2(y_2,z)$.

The quantity (y,z) is called the inner product of y and z. The inner product satisfies the following fundamental inequality usually referred to as the Schwarz inequality although sometimes it is also ascribed to Bunjakovskiĭ and Cauchy.

Theorem 11.1

$$|(y,z)|^2 \leqq (y,y)(z,z), \tag{11.1}$$

where the equality sign only holds if y and z are linearly dependent.

Proof

For $z = 0$ (11.1) is trivial. For $z \neq 0$ take $\lambda = (y,z)/(z,z)$ in $(y-\lambda z, y-\lambda z) \geqq 0$.

The non-negative quantity $(y,y)^{\frac{1}{2}}$ is called the norm of y and is written as $||y||$. It can easily be shown that this norm satisfies the axioms of a linear normed space so that any euclidean space can be considered as a linear normed space. Thus the geometrical language of a normed space can also be used here. A novelty of the euclidian space is the notion of orthogonality. Two elements are said to be orthogonal if their inner product vanishes. Two subsets S_1 and S_2 of R are called orthogonal if each element of S_1 is orthogonal to each element of S_2. It can easily be shown that the totality Z of elements z which are orthogonal to a given set Y forms a subspace of R. Z is then called the orthogonal complement of Y. A set of vectors any pair of which are orthogonal is called an orthogonal set. If the vectors are also unit vectors the set is called orthonormal. Thus an orthonormal set $e_1, e_2, \ldots$ is characterized by $(e_i, e_j) = \delta_{ij}$ where δ_{ij} is the familiar symbol of Kronecker.

We now consider the problem of approximating an arbitrary element y by a linear combination of a finite orthonormal set of vectors $e_1, e_2, \ldots, e_n$. Thus we have to minimize the expression

$$\|y - (\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n)\|$$

for a suitable choice of coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$.
A simple calculation shows that the following identity holds

$$(y - \sum \alpha_i e_i, y - \sum \alpha_i e_i) = (y,y) - \sum |(y,e_i)|^2 + \sum |\alpha_i - (y,e_i)|^2. \tag{11.2}$$

Obviously the problem is solved by taking

$$\alpha_i = (y, e_i), \quad i = 1,2,\ldots\ . \tag{11.3}$$

These coefficients are called the Fourier coefficients of y with respect to the elements e_i.
An orthonormal set is called complete when there is no further vector which is orthogonal to all elements of the orthonormal set.
An euclidean space which is complete in its norm, i.e. in which every fundamental sequence has a limit, and which possesses a countable complete orthonormal set is called a Hilbert space. A Hilbert space is usually denoted by H.

Theorem 11.2

Let $e_1, e_2, \ldots$ be a complete orthonormal set of the Hilbert space H; then each element y can be expanded in a Fourier series

$$y = \sum_{i=1}^{\infty} \alpha_i e_i \tag{11.4}$$

where the coefficients α_i are the Fourier coefficients of y with respect to the set $e_i, i = 1,2,\ldots$. Further the following equality holds

$$\|y\|^2 = \sum_{i=1}^{\infty} |\alpha_i|^2, \tag{11.5}$$

the so-called relation of Parseval.

Proof

From (11.2) there follows the following inequality, the so-called inequality of Bessel

$$\sum_{i=1}^{\infty} |\alpha_i|^2 \leq \|y\|^2. \tag{11.6}$$

This proves the convergence of the series on the left-hand side. For any element y we may form its Fourier series the convergence of which easily follows from the Cauchy principle. In fact for m,n → ∞ with n > m we have

$$\|\sum_{i=m}^{n} \alpha_i e_i\|^2 = \sum_{i=m}^{n} |\alpha_i|^2 \to 0$$

so that the Fourier series is a fundamental sequence. Since H is complete the series converges to a certain element y*. The completeness of the orthonormal set implies the identity of y* and y. In fact, since y* has the same Fourier coefficients, it follows that y*-y is orthogonal to all e_i, i=1,2,... . But then y*-y = 0. This proves the expansion (11.4). The Parseval relation follows from (11.2). For $1 \leq i \leq n$ and α_i given by (11.3), the identity (11.2) gives

$$\|y\|^2 = \sum_{i=1}^{n} |\alpha_i|^2 + \|y - \sum_{i=1}^{n} \alpha_i e_i\|^2. \tag{11.7}$$

For n → ∞ the Parseval relation is obtained.

Corollary

If $y = \sum_{i=1}^{\infty} \alpha_i e_i$ and $z = \sum_{i=1}^{\infty} \beta_i e_i$ then

$$(y,z) = \sum_{i=1}^{\infty} \alpha_i \overline{\beta_i}. \tag{11.8}$$

Example 11.1

The linear space of Lebesgue measurable functions y(x), $a \leq x \leq b$ for which

(11.9) $$\int_a^b |y(x)|^2 dx < \infty$$

becomes a Hilbert space if we define the inner product of y(x) and z(x) as

(11.10) $$(y,z) = \int_a^b y(x)\ \overline{z(x)}dx.$$

This space is usually denoted as $L^2(a,b)$. We note that this space can be formed by the process of completion applied to the euclidean vector space of continuous functions for which (11.10) is the inner product. In this way the Lebesgue measurable functions appear as the limits of fundamental sequences of continuous functions - a situation completely analogous to that of the introduction of real numbers by fundamental sequences of rationals.

A linear functional $f[y]$ on a normed space R is said to be bounded if there exists a positive constant C such that

(11.11) $$|f[y]| < C\,||y|| \quad \text{for all } y \in R.$$

A bounded linear functional is also continuous, i.e.

(11.12) $$y_n \to y \quad \text{implies} \quad f[y_n] \to f[y].$$

In fact we have

$$|f[y_n] - f[y]| = |f[y_n - y]| < C\,||y_n - y|| \to 0.$$

The converse is also true. If $f[y]$ were continuous but not bounded it would be possible to find a sequence of unit vectors y_n for which $|f[y_n]| > n$. This would, however, contradict the continuity property since $y_n/n \to 0$ whereas $f[y_n/n] > 1$.
According to a theorem due to F. Riess a continuous linear functional of a Hilbert space is of a certain standard type.

Theorem 11.3 (Riess)
Every continuous linear functional $f[y]$ in a Hilbert space H can be

represented in the form

$$f[y] = (y,z), \quad z \in H.$$

The element z is uniquely determined by the functional f.

Proof

Follows easily by expanding y in a Fourier series (11.4) with respect to some complete orthonormal system.

12. Functionals on a Hilbert space

A linear operator A acting upon the elements of a normed space R is characterized by the property

$$A(\alpha_1 y_1+\alpha_2 y_2) = \alpha_1 Ay_1 + \alpha_2 Ay_2. \tag{12.1}$$

The set of elements of R for which A is defined is called the domain D_A of A. It will be assumed that A transforms the elements of D_A into elements of the same space R. The latter subset of R is called the range R_A of A.

A linear operator A is said to be bounded if there exists a positive number C such that

$$\|Ay\| < C\,\|y\| \quad \text{for all } y\in R. \tag{12.2}$$

The lower bound of the constants C for which (12.2) holds is called the norm of the operator and is written as $\|A\|$. Thus we have

$$\|A\| = \sup \frac{\|Ay\|}{\|y\|}, \tag{12.3}$$

where the supremum extends over all vectors y, except, of course, y = 0. It can always be assumed that the domain of a bounded linear operator is all of R. As for a bounded linear functional, a bounded linear operator is also continuous, i.e.

$$y_n \to y \quad \text{implies} \quad Ay_n \to Ay, \tag{12.4}$$

and a continuous linear operator is also bounded. The proof is analogous to that of the preceding section.

From now on we shall consider more particularly linear operators on a Hilbert space H. The linear operators on a Hilbert space are either bounded or unbounded. For a bounded, and hence continuous, operator A, the domain D_A coincides with H, and R_A is a subspace of H. There exists a finite norm (12.3). For an unbounded operator A it can be assumed that D_A is a dense subset in H. Further there exists a sequence of unit vectors $\{f_n\}$ such that $f_n \in D_A$ and $\|Af_n\| \to \infty$.

A linear operator A is said to be <u>invertible</u> if there exists a 1-1 correspondence between D_A and R_A. The operator which transforms the elements of R_A back into those of D_A is called the inverse of A and is written as A^{-1}. Thus $z = Ay$ is equivalent to $y = A^{-1}z$.

A linear operator A is said to be <u>symmetric</u> if

(12.5) $$(Ay,z) = (y,Az) \text{ for all } y,z \in D_A.$$

A symmetric operator A is called <u>positive</u> if

(12.6) $$(Ay,y) > 0 \text{ for } y \neq 0.$$

A positive symmetric operator A is called <u>positive definite</u> if there exists a positive constant k such that

(12.7) $$(Ay,y) \geqq k(y,y) \text{ for } y \in D_A.$$

<u>Example 12.1</u>

Consider in the Hilbert space $L^2(0,1)$ of real functions $y(x)$ the differential operator $Ay(x) = -y''(x)$ acting upon those functions of $C_2(0,1)$ for which $y(0) = y(1) = 0$. This operator is not bounded, since, for the sequence of unit vectors e_n defined by $e_n = -2^{\frac{1}{2}} \sin n\pi x$, $n = 1,2,\ldots,$ we have $\|Ae_n\| = n\pi \to \infty$. The operator is symmetric since

$$(Ay,z) - (y,Az) = \int_0^1 (yz''-y''z)dx = (yz'-y'z)\Big|_0^1 = 0.$$

The operator is also positive for we have

$$(Ay,y) = -\int_0^1 yy''dx = \int_0^1 y'^2 dx.$$

The operator is even positive definite with

$$(Ay,y) \geqq \pi^2(y,y),$$

for, by expanding y as $y = 2^{\frac{1}{2}} \sum_{n=1}^{\infty} \alpha_n \sin n\pi x$,
we have

$$Ay = 2^{\frac{1}{2}} \sum_{n=1}^{\infty} n^2\pi^2\alpha_n \sin n\pi x,$$

so that

$$(Ay,y) = \sum_{n=1}^{\infty} n^2\pi^2\alpha_n^2 \quad \text{and} \quad (y,y) = \sum_{n=1}^{\infty} \alpha_n^2.$$

Thus

$$(Ay,y) - \pi^2(y,y) = \sum_{n=2}^{\infty} (n^2-1)\pi^2\alpha_n^2 \geqq 0.$$

The operator is invertible. The equation $Ay = z$ has the solution

$$y(x) = x\int_x^1 (1-\zeta)z(\zeta)\,d\zeta + (1-x)\int_0^x \zeta z(\zeta)\,d\zeta$$

so that A^{-1} is an integration operator of the kind

$$y(x) = \int_0^1 G(x,\zeta)z(\zeta)\,d\zeta,$$

where the kernel, the Green function $G(x,\zeta)$ is given by the following symmetric expression

$$G(x,\zeta) = \tfrac{1}{2}(x+\zeta) + \tfrac{1}{2}|x-\zeta| - x\zeta.$$

The action of a linear operator A in a Hilbert space H is completely determined by the transforms of the elements of a suitable complete orthonormal set $\{e_n\}$. The transformation $z = Ay$ may be written in the form

(12.8)
$$\beta_i = \sum_{j=1}^{\infty} A_{ij}\alpha_j,$$
$$y = \sum_{i=1}^{\infty} \alpha_i e_i, \qquad z = \sum_{i=1}^{\infty} \beta_i e_i.$$

For a symmetric operator we have, in particular, $(Ae_i,e_j) = (e_i,Ae_j)$ for all i and j. This means that

(12.9) $$A_{ji} = \overline{A_{ij}}.$$

For a given operator A the expression $(Ay,\overline{z})$ represents a bilinear functional and (Ay,y) a quadratic functional. According to the end of section 8 a positive definite symmetric operator yields a positive definite quadratic functional.

<u>Theorem 12.1</u>

The operator A is invertible if and only if $Ay = 0$ has only the zero solution.

<u>Proof</u>

1. Let A be invertible. If for some y we have $Ay = 0$ then also $A^{-1}Ay = 0$ i.e. $y = 0$.
2. Let $Ay = 0$ have only the zero solution. If for a given z both $Ay_1 = z$ and $Ay_2 = z$ then we would have $A(y_1-y_2) = 0$ but this implies $y_1 = y_2$.

<u>Theorem 12.2</u>

The inverse A^{-1} of A is bounded if and only if a positive constant k exists such that

(12.10) $$\|Ay\| \geqq k\,\|y\|, \quad y \in D_A.$$

<u>Proof</u>

1. Let $B = A^{-1}$ be bounded. This means that $\|Bz\| \leqq \|B\|\,\|z\|$ for all $z \in R_A$. It suffices to replace z by Ay.
2. Let (12.10) be valid. Then $Ay = 0$ implies $y = 0$ so that according to the previous theorem A^{-1} exists. It suffices to replace y by $A^{-1}z$.

The following theorem is of fundamental importance for an important class of variational problems. In its abstract formulation is hidden the practical conclusion that boundary value problems can be reduced to variational problems.

Theorem 12.3

Let A be a positive symmetric linear operator.

1. If the equation $Ay = g$ has a solution then this solution yields a minimum value of the functional

$$f[y] = (Ay,y) - 2\mathrm{Re}(y,g). \tag{12.11}$$

2. If the latter functional takes a minimum value for the element y then this gives a solution of $Ay = g$.

Proof

The increment of $f[y]$ can be written as

$$f[y+h] - f[y] = 2\mathrm{Re}(Ay-g,h) + (Ah,h).$$

The necessary condition for $f[y]$ to have an extremum is $(Ay-g,h) = 0$ for all $h \in D_A$. Since D_A is dense in H this is equivalent to $Ay-g = 0$. From $(Ah,h) \geq 0$ it follows that there is a minimum.

Example 12.2

Consider the operator $A = -d^2/dx^2$ acting upon functions $y(x) \in L^2(0,1)$. The domain of A consists of those functions of $C_2(0,1)$ for which $y(0) = y(1)$. We know already that A is positive symmetric. Then the above theorem says that the boundary value problem

$$\frac{d^2y}{dx^2} = -2, \quad y(0) = y(1) = 0$$

is equivalent to the extremum problem

$$f[y] = \int_0^1 (yy''+4y)dx = \text{maximum}.$$

Although in this particular case the boundary value problem can be solved explicitly without difficulty - its solution is $x(1-x)$ - a slight change may make the direct approach of the boundary value problem rather difficult whereas the variational approach has always the same order of simplicity. Later on, such variational problems will be solved by so-called direct methods. These will be treated in chapter VII.

CHAPTER IV

FURTHER VARIATIONAL PROBLEMS

13. Several unknown functions

The simplest variational problem treated in section 5 allows an almost trivial generalization if the single function $y(x)$ is replaced by a set of m independent functions $y_i(x)$, $i = 1,2,\ldots,m$. In fact the only thing we have to do is to replace the y-element in the problem of section 5 by an m-dimensional vector y, i.e. an element of the Banach space which is formed by the Cartesian product of m copies of $C_1(a,b)$. Explicitly, the problem is to determine a set of functions $y_i(x) \in C_1(a,b)$ satisfying the boundary conditions

$$(13.1) \qquad y_i(a) = A_i, \quad y_i(b) = B_i \qquad i = 1,2,\ldots,m$$

for which the functional

$$(13.2) \qquad f[y] = \int_a^b F(x, y_1, y_2, \ldots, y_m, y_1', y_2', \ldots, y_m')\,dx$$

is an extremum.

For the norm in the Banach space of y we may take the expression

$$(13.3) \qquad \|y\| = \sum_{i=1}^{m} \{\max|y_i(x)| + \max|y_i'(x)|\}.$$

Assuming, as in section 5, that F has continuous partial derivatives up to the second order we find from theorem 10.1 the necessary condition

$$(13.4) \qquad \int_a^b \{ \sum_{i=1}^{m} \frac{\partial F}{\partial y_i} h_i + \sum_{i=1}^{m} \frac{\partial F}{\partial y_i'} h_i' \}\,dx = 0.$$

By taking special vector functions h with only one component differing from zero we may apply lemma 4.3. Thus for each component y_i, an Euler equation is obtained. Hence, as a necessary condition, the functions $y_1, y_2, \ldots, y_m$ are solutions of the m equations

(13.5) $$\frac{d}{dx}\frac{\partial F}{\partial y_i'} = \frac{\partial F}{\partial y_i} \qquad i = 1,2,\ldots,m.$$

If, in particular, F does not depend explicitly on x, then the following derived equation, the analogue of (5.10), results

(13.6) $$\sum_{i=1}^{m} y_i' \frac{\partial F}{\partial y_i'} - F = C,$$

where C is an arbitrary constant.
The verification of this relation is immediate.

Example 13.1

The shortest curve lying on a given surface S which connects two points of S is called a geodesic of S. The geodesics of a plane are straight lines, and those of a sphere are the great circles. We shall now determine the geodesics of the circular cylinder $x^2+y^2 = a^2$. Let the geodesics be determined in cylindrical coordinates (r,θ,z) by the equations $\theta = \theta(t)$, $z = z(t)$ where t is a parameter. The line element ds is given by

$$ds^2 = a^2 d\theta^2 + dz^2.$$

Thus we have to minimize the functional

$$\int_{t_1}^{t_2} (a^2\dot\theta^2+\dot z^2)^{\frac{1}{2}} dt.$$

The Euler equations (13.5) are

$$\frac{d}{dt}\{(a^2\dot\theta^2+\dot z^2)^{-\frac{1}{2}}\dot\theta\} = 0, \quad \frac{d}{dt}\{(a^2\dot\theta^2+\dot z^2)^{-\frac{1}{2}}\dot z\} = 0,$$

so that

$$(a^2\dot\theta^2+\dot z^2)^{-\frac{1}{2}}\dot\theta = C_1, \quad (a^2\dot\theta^2+\dot z^2)^{-\frac{1}{2}}\dot z = C_2$$

are first integrals. Thus, we deduce $\dot z/\dot\theta$ = constant or

$$\frac{dz}{d\theta} = \text{constant}.$$

Thus the geodesics are of the form

$$z = \alpha\theta + \beta, \ r = \alpha,$$

where α and β are constants. These equations represent a two-parameter family of helical lines. Evidently a geodesic is uniquely determined by its end points in this case.

Of course, in this special case, the problem can be solved in a simpler way by assuming the equation of the geodesics in the form $z = z(\theta)$ thus reducing the number of dependent variables to one.

In the general case where the geodesics are sought on a surface given in parameter form by

$$x = x(u,v), \ y = y(u,v), \ z = z(u,v),$$

we have to minimize the functional $\int ds$ where ds is the line element, viz.

$$f[u,v] = \int_{t_1}^{t_2} (Edu^2+2Fdudv+Gdv^2)^{\frac{1}{2}}. \tag{13.7}$$

The geodesics then are determined in parameter from $u = u(t)$, $v = v(t)$.

Example 13.2

The plane motion of a particle of mass m attracted by a gravitational force located at the origin is determined by Hamilton's principle which states that the integral

$$\int_{t_1}^{t_2} (T-U)dt,$$

where T is the kinetic energy and U the potential energy is stationary for motions with a given situation at the end points (cf. section 17).

In this case, we have

$$T = \tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) \text{ and } U = -k/r,$$

where r and θ are polar coordinates. Thus we need to consider the functional

$$f[r,\theta] = \int_{t_1}^{t_2} \{\tfrac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + k/r\}dt.$$

The equations (13.5) are

$$\begin{cases} m\ddot{r} - mr\dot{\theta}^2 + kr^{-2} = 0, \\ \frac{d}{dt}(r^2\dot{\theta}) = 0. \end{cases}$$

The second equation gives Kepler's second law $r^2\dot{\theta} = c$ (constant). Writing

$$\frac{d}{dt} = \frac{c}{r^2}\frac{d}{d\theta}$$

the first Euler equation can be brought into the form

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{k}{mc^2}.$$

This has the general solution

$$\frac{1}{r} = \frac{1}{r_0} + A\cos(\theta-\theta_0),$$

where A and θ_0 are constants of integration and where $r_0 = mc^2/k$. The equation thus found is the familiar equation of a conic in polar coordination with the origin as a focus.

14. A functional depending on higher-order derivatives

We consider the functional

$$(14.1) \qquad f[y] = \int_a^b F(x,y,y',y'')dx$$

where $y(x) \in C_2(a,b)$ and where F has continuous partial derivatives up to the third order. The admissible functions are those for which

$$(14.2) \qquad y(a) = A,\ y(b) = B,\ y'(a) = A',\ y'(b) = B'.$$

Again theorem 10.1 applies and we obtain as the necessary condition that

$$(14.3) \qquad \int_a^b \{F_y h + F_{y'} h' + F_{y''} h''\} dx = 0$$

for all $h(x) \in C_2(a,b)$ with $h(a) = h(b) = h'(a) = h'(b) = 0$.
Integrating this relation formally by parts we obtain

$$(14.4) \qquad \int_a^b \{F_y - \frac{d}{dx} F_{y'} + \frac{d^2}{dx^2} F_{y''}\} h\, dx = 0$$

for all continuous $h(x)$ with $h(a) = h(b) = 0$. According to lemma 4.1, this results in the Euler type equation

$$(14.5) \qquad \frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial^2 F}{\partial y''} = 0.$$

This differential equation is of the fourth order. The required extremal is a solution of this equation which also satisfies the boundary conditions (14.2). The existence of the third and fourth derivative of y was not assumed in advance so that the validity of (14.5) at first sight seems questionable. However, using an argument similar to that in the analogous situation for the variational problem of section 5, it can be shown that (14.12) holds and that y''' and y'''' exist in general. The rigorous derivation of (14.5) from (14.4) depends essentially on the use of the following lemma which closely resembles lemma 4.2.

<u>Lemma 14.1</u>

If $\phi(x)$ is continuous in (a,b) and if

$$\int_a^b \phi(x)h''(x)dx = 0$$

for all $h(x)\in C_2(a,b)$ with $h(a) = h(b) = h'(a) = h'(b) = 0$ then

$$\phi(x) = c_1x + c_2$$

where c_1 and c_2 are constants.

<u>Proof</u>

As in the corresponding proof of lemma 4.2 we take a particular function $h(x)$ determined by

$$h''(x) = \phi(x) - c_1x - c_2$$

and by the boundary conditions at a and b. The explicit expression of this choice is irrelevant but we note that

$$h(x) = \int_a^x (x-\zeta)\{\phi(\zeta) - c_1\zeta - c_2\}d\zeta$$

where the constants c_1 and c_2 are determined by

$$0 = \int_a^b (b-\zeta)\{\phi(\zeta) - c_1\zeta - c_2\}d\zeta,$$

$$0 = \int_a^b \{\phi(\zeta) - c_1\zeta - c_2\}d\zeta.$$

Then

$$\int_a^b \{\phi(x) - c_1x - c_2\}^2 dx = \int_a^b \{\phi(x) - c_1x - c_2\}h''(x)dx =$$

$$= \int_a^b \phi(x)h''(x)dx - c_1\int_a^b xh''(x)dx - c_2\int_a^b h''(x)dx = 0$$

so that

$$\phi(x) \equiv c_1x + c_2.$$

15. Variable end point conditions

We take up the variational problem of section 5 with the following slight change. We consider the functional

$$(15.1) \qquad f[y] = \int_a^b F(x,y,p)\,dx,$$

where F satisfies the usual conditions and where y(x) has a continuous derivative in (a,b). The only change is that no conditions at a and b are prescribed. As in section 5 the necessary condition for $f[y]$ to be an extremum consists in the vanishing of the first variation. Thus we must have

$$(15.2) \qquad \int_a^b (F_y h + F_p h')\,dx = 0,$$

but now for <u>all</u> $h(x) \in C_1(a,b)$ and not only for those functions which vanish at the end points. Since (15.2) holds in any case for the subclass of functions h(x) with h(a) = h(b) = 0 we again find Euler's equation

$$(15.3) \qquad \frac{d}{dx} F_p = F_y$$

but there is more. Substitution of $F_y = F_p'$ into (15.2) gives

$$\int_a^b (F_p' h + F_p h')\,dx = F_p h \Big|_a^b = 0.$$

The values of h at a and b are arbitrary and thus we must have the extra conditions

$$(15.4) \qquad F_p = 0 \text{ for } x = a \text{ and } x = b.$$

It appears that, at least in general, the extremal is determined by the second order differential equation (15.3) and the boundary conditions (15.4). These boundary conditions which were not given in advance, unlike to the boundary conditions of the variational problem of section

5, are sometimes called natural boundary conditions.

Of course one end point may correspond to a given boundary condition and the other may lead to a natural boundary condition.

Example 15.1

We consider the following variant of the brachistochrone problem. A particle slides under gravity from rest along a smooth vertical curve $y = y(x)$ from a given point A towards a given vertical line. The problem is to find the path for which the particle reaches the line in the least possible time. In the notation of section 6, the problem is that of minimizing the functional (cf. 6.2)

$$f[y] = \int_0^a y^{-\frac{1}{2}}(1+p^2)^{\frac{1}{2}}dx$$

for functions $y(x) \in C_1(0,a)$ with $y(0) = 0$ at the fixed end point at the origin. Since Euler's equation is that of the brachistochrone problem the extremals are cycloids

$$x = c(\theta - \sin\theta), \quad y = c(1-\cos\theta).$$

The condition at the variable end point becomes $p = 0$ for $x = a$. This means that the sought curve meets the given line perpendicularly. A simple calculation shows that $c = a/\pi$.

In a completely analogous fashion the variational problem of the preceding section may be treated with variable end conditions.

16. Several independent variables

The simplest variational problem for functions $y(x_1, x_2)$ of two independent variables is to determine an extremum of the functional

$$f[y] = \iint_{\Omega} F(x_1, x_2, y, p_1, p_2)\, dx_1 dx_2, \tag{16.1}$$

where p_1 and p_2 are notations for $\frac{\partial y}{\partial x_1}$ and $\frac{\partial y}{\partial x_2}$, and where Ω is the rectangular region $a_1 \leq x_1 \leq b_1$, $a_2 \leq x_2 \leq b_2$. It is assumed that F has continuous partial derivatives up to the second order with respect to all its arguments. The admissible functions can be elements of $C_2(\Omega)$ satisfying a given boundary condition. This boundary condition is the two-dimensional analogue of (5.1) i.e. that y takes prescribed values at the boundary of Ω. The condition as regards the existence of the second partial derivatives, might be somewhat relaxed as in the discussion of the corresponding problem of section 5, but we shall not enter into those subtleties.

The functional (16.1) is again of the type of that of section 5 i.e. a differentiable functional on the space $C_1(\Omega)$ which is normed by (9.3), i.e.

$$\|y\| = \max_{\Omega}|y| + \max_{\Omega}\left|\frac{\partial y}{\partial x_1}\right| + \max_{\Omega}\left|\frac{\partial y}{\partial x_2}\right|. \tag{16.2}$$

In the usual way we find for the first variation the expression

$$\delta f = \iint_{\Omega} \left(\frac{\partial F}{\partial y} h + \frac{\partial F}{\partial p_1}\frac{\partial h}{\partial x_1} + \frac{\partial F}{\partial p_2}\frac{\partial h}{\partial x_2}\right) dx_1 dx_2. \tag{16.3}$$

The necessary condition $\delta f = 0$ becomes, after integration by parts,

$$\iint_{\Omega} \left(\frac{\partial F}{\partial y} - \frac{\partial}{\partial x_1}\frac{\partial F}{\partial p_1} - \frac{\partial}{\partial x_2}\frac{\partial F}{\partial p_2}\right) h\, dx_1 dx_2 = 0 \tag{16.4}$$

for all admissible h. In a way similar to that in lemma 4.1, this yields the Euler equation

$$\frac{\partial}{\partial x_1}\frac{\partial F}{\partial p_1} + \frac{\partial}{\partial x_2}\frac{\partial F}{\partial p_2} = \frac{\partial F}{\partial y}, \tag{16.5}$$

a partial differential equation of the second order.

Example 16.1

Consider the functional

$$f[y]=\int_0^1\int_0^1 (p_1^2+p_2^2)dx_1dx_2$$

for functions y satisfying the boundary conditions

$$y = 0 \text{ for } x_1 = 0 \text{ and for } x_2 = 0,$$

$$y = x_2 \text{ for } x_1 = 1 \text{ and } y = x_1 \text{ for } x_2 = 1.$$

The Euler equation is

$$\frac{\partial^2 y}{\partial x_1^2} + \frac{\partial^2 y}{\partial x_1^2} = 0$$

so that y is a harmonic function. The problem is solved by $y = x_1x_2$ ($= \operatorname{Im}\frac{1}{2}(x_1+ix_2)^2$). A simple argument shows that the functional assumes a minimum value for this function.

We shall now consider the more general case of a functional with respect to a function y of n independent variables $x_1,x_2,\ldots,x_n$ belonging to a compact region Ω with a sufficiently smooth surface S. The smoothness of S implies the existence, at least almost everywhere, of a unit vector $\vec{n}$ normal to S in the outward direction. For such a region we have the following n-dimensional version of Gauss' theorem

(16.6) $$\int_\Omega \operatorname{div} \vec{v}\, dx = \int_S \vec{v}.\vec{n}\, d\sigma,$$

where $\vec{v}(x_1,x_2,\ldots,x_n)$ is an n-dimensional vector function with continuous partial derivatives. The necessary condition for

(16.7) $$f[y] = \int_\Omega F(x_1,\ldots,x_n,y,p_1,\ldots,p_n)dx$$

to have an extremum is again

(16.8) $$\delta f = \int_\Omega \left(\frac{\partial F}{\partial y} h + \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} \frac{\partial h}{\partial x_i}\right) dx = 0.$$

This condition can be written in the form

(16.9) $$\int_\Omega \left\{\frac{\partial F}{\partial y} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial p_i}\right\} h \, dx + \int_\Omega \operatorname{div}(h\vec{v}) dx = 0,$$

where $\vec{v}$ has the components $\frac{\partial F}{\partial p_1}, \ldots, \frac{\partial F}{\partial p_n}$.

According to Gauss' theorem (16.6) the latter condition can be put into the form

(16.10) $$\int_\Omega \left\{\frac{\partial F}{\partial y} - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{\partial F}{\partial p_i}\right\} h \, dx + \int_S h\vec{v}.\vec{n} d\sigma = 0.$$

Nothing has been supposed so far at the boundary S of Ω. However, (16.10) must hold in any case for those functions h which vanish at S. By the same argument as before, this leads to the Euler equation

(16.11) $$\frac{\partial}{\partial x_1} \frac{\partial F}{\partial p_1} + \frac{\partial}{\partial x_2} \frac{\partial F}{\partial p_2} + \ldots + \frac{\partial}{\partial x_n} \frac{\partial F}{\partial p_n} = \frac{\partial F}{\partial y}.$$

We are left with the condition (16.10) for those h which are not everywhere zero on S. This means

(16.12) $$\int_S h\vec{v}.\vec{n} d\sigma = 0.$$

Usually S consists of two parts, S_1 and S_2, such that on S_1 the admissible functions y take prescribed values and that on S_2 nothing is supposed. Thus S_1 is a fixed boundary and S_2 a free boundary. Since $h = 0$ on S_1 the condition (16.12) refers only to the free boundary S_2. Since h is arbitrary we arrive at the necessary condition

(16.13) $$\sum_{i=1}^{n} \frac{\partial F}{\partial p_i} n_i = 0$$

which generalizes the corresponding condition (15.4) of the one-dimensional problem.

In summarizing, we have found that the given variational problem can be solved by a solution of the partial differential equation

$$(16.14) \qquad \sum \frac{\partial}{\partial x_i} \frac{\partial F}{\partial p_i} = \frac{\partial F}{\partial y},$$

and the boundary conditions

$$(16.15) \qquad \begin{cases} y = \text{given on } S_1, \\ \sum_{i=1}^{n} \frac{\partial F}{\partial p_i} n_i = 0 \text{ on } S_2. \end{cases}$$

The latter boundary condition, which originates "naturally" from the given problem, is commonly known as a natural boundary condition.

Example 16.2

Consider the functional

$$f[u] = \int_\Omega (p_1^2+p_2^2+p_3^2)\,dx$$

for functions u in the cylindrical region $x_1^2+x_2^2 \leqq a^2$, $0 \leqq x_3 \leqq b$. The Euler equation (16.14) becomes the Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} = 0.$$

However, in view of the form of Ω it is better to use cylindrical coordinates r, θ, z defined by

$$x_1 = r \cos\theta, \; x_2 = r \sin\theta, \; x_3 = z.$$

Instead of a direct transformation of the Laplace equation by substitution of the new variables, we start anew from the variational problem, but now reformulated in cylindrical coordinates

$$f[u] = \int_\Omega \left\{ \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 \right\} r\,dr\,d\theta\,dz.$$

The Euler equation is now automatically transformed into

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Since nothing has been prescribed at the boundary of Ω, we have the natural boundary conditions

$$\frac{\partial u}{\partial z} = 0 \text{ for } z = 0 \text{ and } z = b, \ \frac{\partial u}{\partial r} = 0 \text{ for } r = a.$$

CHAPTER V

APPLICATIONS IN MECHANICS

17. Hamilton's principle

We consider a mechanical system with m degrees of freedom. At any instant the configuration may be given by the m components $(q_1, q_2, \ldots, q_m)$ of an m-dimensional vector $\vec{q}$. The problem is to determine the motion of the system as the result of exterior forces and the inertia forces. The motion of the system corresponds to a certain curve $\vec{q}(t)$ in the m-dimensional vector space. In classical mechanics, the possible motions are governed by a set of ordinary differential equations such as the Lagrange equations. The actual motion may be fixed by knowing the configuration at some time t_0. This means that not only the positions but also the velocities of the components of the system must be prescribed. In geometrical language, the motion is uniquely determined by starting the motion curve at a certain point P_0 and in a certain direction.

We may also try to fix the motion curve by giving two of its points, namely the initial position P_0 and the final position P_1. In the language of differential equations, this means that the Cauchy condition is replaced by a two-point boundary condition. According to Hamilton the actual motion curve from P_0 to P_1 is determined by a variational principle which states that among all possible curves between P_0 and P_1, the actual motion is an extremal of the functional

$$f[\vec{q}] = \int_{P_0}^{P_1} (T-U)\,dt, \tag{17.1}$$

where T is the kinetic energy and U the potential energy of the system. Although in many cases the functional (17.1) takes a minimum value, this is certainly not true in general, and the only thing we can say is that for the actual motion $\vec{q}(t)$ the functional (17.1) is merely stationary. For a great class of mechanical systems the kinetic energy and the potential energy are functions of the following type

$$(17.2)\qquad \begin{cases} T = \sum_{i,j} a_{ij}(t,q_1,q_2,\ldots,q_m)\dot{q}_i\dot{q}_j, \\ U = U(t,q_1,q_2,\ldots,q_m). \end{cases}$$

Moreover, for conservative systems T and U do not depend explicitly on the time t.

The variational problem (17.1) with fixed end point conditions is exactly that of section 13. The corresponding set of Euler equations (13.5) is

$$(17.3)\qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial (T-U)}{\partial q_i} \qquad i = 1,2,\ldots,m.$$

These equations are recognized as the Lagrange equations of the system.

At first sight, Hamilton's principle seems to be only an easy way of obtaining the equations of motion (17.3). However, Hamilton's principle reveals its full power when the generalization is made to so-called continuous systems, i.e. systems with infinitely many degrees of freedom. Well-known examples of such systems are the vibrating string and a vibrating elastic plate.

If the system is conservative we have the special case discussed at the end of section 13. According to (13.6), the following first integral exists,

$$\sum_{i=1}^{m} \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} - (T-U) = H,$$

where H is a constant of the motion. Since T is a homogeneous quadratic form in the derivatives $\dot{q}_i$ the latter equation can be replaced by

$$(17.4)\qquad T + U = H$$

which expresses the well-known mechanical law that for conservative systems the sum of the kinetic and the potential energy is a constant. This constant is called the total energy.

Since for conservative systems T - U = 2T - H, Hamilton's principle can be replaced by the familiar principle of "least action", according to which the actual motion "minimizes" the functional

$$f[\vec{q}] = \int_{t_0}^{t} T\,dt. \tag{17.5}$$

Again we note that in general, the functional is only stationary, and so it would be better to speak of the principle of stationary action.

The equilibrium of a conservative system, for which $T \equiv 0$, is determined by

$$\frac{\partial U}{\partial q_i} = 0 \quad i = 1,2,\ldots,m. \tag{17.6}$$

This means that an equilibrium corresponds to a stationary value of the potential energy.

We shall next investigate the possible motion of a mechanical system in the neighbourhood of an equilibrium. By a trivial transformation of the coordinates, equilibrium can be made to correspond to the origin in the m-dimensional q-space. Furthermore, it may be assumed that the potential energy at this point vanishes. In the neighbourhood of the origin, both q_i and $\dot{q}_i$ are small for all i (the C_1 norm) so that T and U can be expanded as

$$T = \sum_{i,j} a_{ij}\dot{q}_i\dot{q}_j + \text{higher order terms}, \tag{17.7}$$

$$U = \sum_{i,j} b_{ij}q_iq_j + \text{higher order terms}. \tag{17.8}$$

In these expansions, the coefficients a_{ij} and b_{ij} are constants. In order to facilitate the further discussion we suppose that the higher order terms in (17.7) and (17.8) can be omitted so that T and U are given by non-vanishing homogeneous quadratic forms.

We note that $T \geqq 0$ so that by an appropriate transformation in the q-space, the expression for T can be transformed into a pure sum of squares. This means that, without loss of generality, we may take

$a_{ij} = \delta_{ij}$. The Lagrange equations (17.3) are now given by the set

$$\ddot{q}_i + \sum_{j=1}^{m} b_{ij} q_j = 0 \quad i = 1,2,\ldots,m. \tag{17.9}$$

The solutions of this set are linear combinations of terms of the type exp $i\omega t$. The possible values of ω are given by

$$\det(b_{ij} - \omega^2 \delta_{ij}) = 0$$

so that ω^2 is an eigenvalue of the matrix (b_{ij}). A stable equilibrium is characterized by an oscillatory solution of (17.9). Then all possible values of ω must be real, and so the eigenvalues of (b_{ij}) are all positive. A homogeneous quadratic form with positive eigenvalues is non-negative and even positive definite. Also the converse is true. Thus we have obtained the result that in the neighbourhood of a stable equilibrium, $U \geqq 0$. This proves the following statement.

Theorem 17.1

For a conservative mechanical system with a finite number of degrees of freedom, and which is in a stable equilibrium, the potential energy assumes a minimum, and conversely.

Example 17.1

Consider a point of mass m oscillating harmonically along the x-axis. The kinetic and potential energy may be given as $T = \frac{1}{2}m\dot{x}^2$, $U = \frac{1}{2}cx^2$ where c is a spring constant. Hamilton's principle states that

$$\delta \int_{t_0}^{t_1} (m\dot{x}^2 - cx^2)\,dt = 0.$$

This problem has been considered in example 8.1. The Euler equation gives the following equation of motion

$$m\ddot{x} + cx = 0$$

with the solution $x(t) = C_1 \cos t\sqrt{\frac{c}{m}} + C_2 \sin t\sqrt{\frac{c}{m}}$.

According to what has been found in example 8.1, this solution yields a minimum of (17.1) only for a time interval $\Delta t < \pi\sqrt{\frac{m}{c}}$. This illustrates the fact that the actual motion in general yields only stationary values of the Hamilton functional. The system is also conservative so that (17.4) holds. In fact for the solution found above we have

$$H = T + U = c(C_1^2+C_2^2).$$

Example 17.2

The motion of a mathematical pendulum of length a and mass m may be described by the energy expressions $T = \frac{1}{2}ma^2\dot{\theta}^2$, $U = qma(1-\cos\theta)$. The system is conservative. The equation of motion is

$$a\dot{\theta} + g \sin \theta = 0.$$

The explicit solution of this equation can be given in terms of elliptic functions. A first integral is obtained from (17.4) in the form

$$\dot{\theta}^2 = 2ga^{-1}(\cos \theta + C),$$

where C is a constant. The latter relation can easily be integrated as

$$t - t_0 = \sqrt{\frac{a}{2g}}\int_{\theta_0}^{\theta} (C+\cos \theta)^{-\frac{1}{2}}d\theta.$$

The positions of equilibrium correspond to $\theta = 0$ and $\theta = \pi$. For $\theta = 0$ the potential energy is a minimum, and so this equilibrium is stable. For $\theta = \pi$ the potential energy is a maximum so that this equilibrium is unstable. The motion in the neighbourhood of the stable equilibrium is determined by $T = \frac{1}{2}ma^2\dot{\theta}^2$ and $U = \frac{1}{2}gma\theta^2 + \dots$.
The corresponding equation of motion is

$$a\ddot{\theta} + g\theta = 0$$

the solution of which are harmonic oscillations.

18. Continuous mechanical systems

Hamilton's principle can easily be extended to mechanical systems with an infinite number of degrees of freedom. For such systems the motion can be visualized as a curve $\vec{q}(t)$ in a Hilbert space H. The problem is again to determine this curve by two of its points if the kinetic energy $T(t,\vec{q},\dot{\vec{q}})$ and the potential energy $U(t,\vec{q})$ are known. Then the motion between the fixed endpoints $P_0(t_0)$ and $P_1(t_1)$ is such as to yield a stationary value of the functional

(18.1) $$f[t,\vec{q},\dot{\vec{q}}] = \int_{t_0}^{t_1} (T-U)\,dt.$$

Writing the variations of the functionals T and U as

(18.2) $$\begin{cases} \delta U = (\vec{h},\partial U) \\ \delta T = (\vec{h},\partial T) + (\dot{\vec{h}},\partial' T) \end{cases}$$

the necessary condition $\delta f = 0$ may be written as

(18.3) $$\int_{t_0}^{t} \{(\vec{h},\partial T - \partial U) + (\dot{\vec{h}},\partial' T)\}dt = 0$$

for all $\vec{h}(t)$ H vanishing at t_0 and t. Thus we find the Euler equation

(18.4) $$(\frac{d}{dt}\partial' T) = \partial T - \partial U$$

which generalizes the Lagrange equations (17.3).

Again, for equilibrium, we obtain the necessary condition

(18.5) $$\partial U = 0.$$

In order to investigate the possible motion in the neighbourhood of a point of equilibrium we proceed as in the preceding section. It can be assumed that at the origin of H, the system is at rest. Then both T and U can be approximated by homogeneous quadratic functionals of the form

$$(18.6)\qquad \begin{cases} T = (A\vec{q},\vec{q}). \\ U = (B\vec{q},\vec{q}), \end{cases}$$

where A and B are symmetric linear operators which for simplicity are assumed to be independent of t. The operator A is moreover positive and continuous in view of its physical origin. The equation of motion (18.4) can now be written as

$$(18.7)\qquad \frac{d}{dt}(A\vec{q}) + B\vec{q} = 0.$$

Since A is symmetric and positive with $A\vec{q} = 0$, there exists an inverse according to theorem 12.1 only for $\vec{q} = 0$, so that (18.7) can be transformed into

$$(18.8)\qquad \ddot{\vec{q}} + A^{-1}B\vec{q} = 0.$$

The discussion of the latter equation is similar to that for the finite dimensional case. A new metric is introduced according to

$$(18.9)\qquad [\vec{q},\vec{r}] \overset{\text{def}}{=} (A\vec{q},\vec{r}).$$

It is easily seen that this definition satisfies the axioms of a Hilbert space, and so a Hilbert space is defined by this new inner product. The metric (18.9) is called the energetic inner product.
The operator $A^{-1}B$ occuring in (18.8) turns out to be symmetric with respect to the metric (18.9) whereas in the old metric this is not the case in general. In fact, we have

$$[A^{-1}B\vec{q},\vec{r}] = (B\vec{q},\vec{r}) = (\vec{q},B\vec{r}) = (\vec{q},AA^{-1}B\vec{r}) =$$
$$= (A\vec{q},A^{-1}B\vec{r}) = [\vec{q},A^{-1}B\vec{r}].$$

In analogy to the finite dimensional case, it is natural to presume that a minimum of the potential energy corresponds to a stable equilibrium.
Let us suppose that the equilibrium at the origin is such that the operator B is positive definite, i.e. that there exists a positive con-

stant k with

$$(B\vec{q},\vec{q}) > k(\vec{q},\vec{q})$$

for all $\vec{q}$. Then the operator $A^{-1}B$ in (18.8) is also positive definite in the energy metric (18.9). In fact we have

$$[A^{-1}B\vec{q},\vec{q}] = (B\vec{q},\vec{q}) > k(\vec{q},\vec{q}) \geqq \frac{k}{\|A\|} (A\vec{q},\vec{q}) =$$

$$= \frac{k}{\|A\|} [\vec{q},\vec{q}].$$

Thus a possible motion in the neighbourhood of the origin is determined by the equation

$$(18.10) \qquad \ddot{\vec{q}} + C\vec{q} = 0$$

where C is a positive definite operator (with respect to the energy metric).

It can easily be shown that for any motion which starts in the vicinity of the origin the motion curve remains near the origin for ever. We note that (18.10) implies the existence of the following first integral

$$(18.11) \qquad [\dot{q},\dot{q}] + [Cq,q] = c,$$

where c is a constant. Hence during the motion we always have $[Cq,q] < c$. Since C is positive definite $[q,q]$ also remains bounded, thus proving the stability.

If, on the other hand, the equilibrium is stable it follows without difficulty that the operator C is positive. It may happen, however, that C is not positive definite but we shall not enter into this question.

19. Equations of mathematical physics

a. The vibrating string

We consider the motion of a string which in its equilibrium position lies along the x-axis between the fixed endpoints $x = 0$ and $x = a$. Let $u(x,t)$ denote the (small) vertical displacement of the string at the point x and at the time t as shown in fig. 19.1. If both u and u_x are small, then simple expressions for the potential and the kinetic energy can be found.

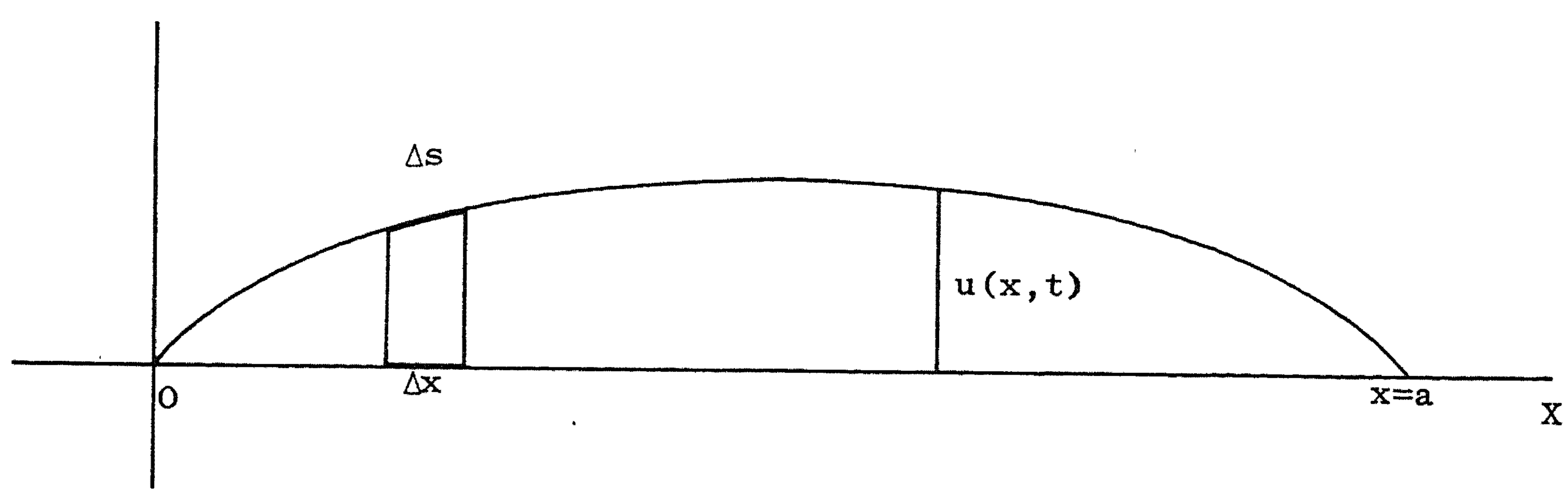


fig. 19.1

The string derives its potential energy from the tension by which the string element Δx is extended to the line element $\Delta s = (1+u_x^2)^{\frac{1}{2}}\Delta x$. Neglecting higher order terms we may write $\Delta s - \Delta x = \frac{1}{2}u_x^2\Delta x$. The total potential energy U is found by integration

$$U = \int_0^a \tau(ds-dx) = \int_0^a \tfrac{1}{2}\tau u_x^2 dx.$$

For the kinetic energy we have obviously

$$T = \int_0^a \tfrac{1}{2}\rho u_t^2 dx,$$

where ρ is the mass density of the string. Both τ and ρ may be regarded as functions of x although in most applications they are constants.

Hamilton's principle says that

$$(19.1) \qquad \delta \int_{t_1}^{t_2} \int_0^a (\rho u_t^2 - \tau u_x^2)\, dx\, dt = 0.$$

The Euler equation may be obtained as in section 16. Without difficulty, we obtain the equation of the vibrating string in the form

$$(19.2) \qquad \frac{\partial}{\partial x}\left(\tau \frac{\partial u}{\partial x}\right) = \rho \frac{\partial^2 u}{\partial t^2} .$$

To this we have to add the boundary conditions

$$(19.3) \qquad u = 0 \text{ for } x = 0 \text{ and } x = a.$$

As in the preceding section, the problem may be formulated in Hilbert space language by introducing the function space formed by $C_1(0,a)$ functions with the inner product definition

$$(19.4) \qquad (u,v) \overset{\text{def}}{=} \tfrac{1}{2}\int_0^a \rho uv\, dx.$$

Then the kinetic and the potential energy can be written in the form (18.6), where A is the identity and B the operator defined by

$$(19.5) \qquad Bu = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\tau \frac{\partial u}{\partial x}\right).$$

The equation (19.2) is then obtained from (18.8).

The operator B acts upon functions which are twice differentiable and satisfy the end point conditions (19.3). We note that B is symmetric and positive definite. The free vibrations of the string are determined by the eigenvalues, i.e. the spectrum, of B.
Let λ be an eigenvalue of B and $\phi(x)$ the corresponding eigenfunction of B. This means that

$$B\phi = \lambda\phi.$$

Since B is positive definite, the eigenvalues are positive. Then the corresponding free motion, which is a solution of (19.2) is given by

(19.6) $$u(x,t) = \phi(x) \sin\{(t-t_0)\sqrt{\lambda}\}.$$

b. A string with an external force

If to the system treated above an external force g(x,t) is added the expression for the potential energy contains an extra term as follows

$$U = \tfrac{1}{2}\int_0^a \tau u_x^2 dx - \int_0^a gu\, dx.$$

The variational condition is here

(19.7) $$\delta\int_{t_1}^{t_2}\int_0^a (\rho u_t^2 - \tau u_x^2 + 2gu)dx\, dt = 0.$$

The simplest way of deriving the Euler equation is by applying the theory of section 16. We find

(19.8) $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \rho\frac{\partial^2 u}{\partial t^2} - g(x,t).$$

If the external force is independent of the time, then there exists a stationary position determined by

(19.9) $$\frac{\partial}{\partial x}\left(\tau\frac{\partial u}{\partial x}\right) = - g(x)$$

with

(19.10) $$u(x) = 0 \text{ for } x = 0 \text{ and } x = a.$$

The latter problem, when formulated in Hilbert space language, is covered by theorem 12.3. According to this theorem the boundary value problem (19.9), (19.10) is equivalent to minimizing the functional

$$f[u] = -\int_0^a \dot{u}(x)\, \frac{\partial}{\partial x}\left(\tau\frac{\partial u}{\partial x}\right)dx - 2\int_0^a g(x)u(x)dx.$$

This functional represents twice the potential energy. In agreement with the general theory, the potential energy takes a minimum value when the system, i.e. the string, is in a stable equilibrium position.

c. A vibrating membrane

A membrane is a homogeneous flexible sheet which derives its potential energy from the tension τ by which the surface element dx dy is extended to $(1+u_x^2+u_y^2)^{\frac{1}{2}}dx\, dy$ where $u(x,y,t)$ is the vertical displacement. In a certain sense a membrane may be considered as a two-dimensional generalization of a string. Assuming that u, u_x and u_y are small, the kinetic and potential energy are given by

$$T = \iint_{\Omega} \tfrac{1}{2}\rho u_t^2 dx\, dy, \quad U = \iint_{\Omega} \tfrac{1}{2}\tau(u_x^2+u_y^2)dx\, dy.$$

Hamilton's principle says that

$$\delta\int_{t_1}^{t_2}\iint_{\Omega}(\rho u_t^2-\tau u_x^2-\tau u_y^2)dx\, dy\, dt = 0. \tag{19.11}$$

The Euler equation becomes, after (16.11),

$$\frac{\partial}{\partial x}\left(\tau\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\tau\frac{\partial u}{\partial y}\right) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{19.12}$$

which is the well-known equation of the vibrating membrane. At the part S_1 of the boundary where the membrane is kept fixed, we have the boundary condition

$$u = 0 \text{ for } (x,y)\in S_1. \tag{19.13}$$

At the part S_2 of the boundary where the membrane can move freely we have the natural boundary condition

$$\frac{\partial u}{\partial n} = 0 \text{ for } (x,y)\in S_2, \tag{19.14}$$

where $\partial/\partial n$ denotes the normal derivative.

In many cases τ and ρ are constants. Then (19.12) may be written in the simpler form

$$(19.15)\qquad c^2 \Delta u = \frac{\partial^2 u}{\partial t^2}$$

where c is a constant with the dimensions of a velocity.

In this connection, we mention the well-known problem of Dirichlet, which consists in finding a solution of the potential equation

$$(19.16)\qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

where (x,y) belongs to a simply connected domain Ω at the boundary S of which u takes prescribed values

$$(19.17)\qquad u = h(s) \text{ for } (x,y) \in S.$$

If u(x,y) is interpreted as the equilibrium position of a membrane, then we are looking, in fact, for a stable equilibrium with a minimum value of the potential energy. Thus, the equivalent variational formulation is

$$(19.18)\qquad \iint_\Omega (u_x^2+u_y^2)dx\,dy = \text{minimum},$$

where the admissible functions satisfy the condition (19.17).

d. A vibrating rod

We consider a thin elastic rod which in its equilibrium position lies along the x-axis between x = 0 and x = a. An elastic rod derives its potential energy from its resistance to bending. If again u(x,t) denotes the local vertical displacement, the potential energy due to the increase of curvature u_{xx} of a line element is given by

$$U = \int_0^a \tfrac{1}{2}\mu u_{xx}^2 dx,$$

where μ characterizes the elastic properties of the material. The kinetic energy T has its usual form, and so Hamilton's principle gives

$$\delta\int_{t_1}^{t_2}\int_0^a (\rho u_t^2-\mu u_{xx}^2)dx\,dt = 0. \tag{19.19}$$

Let us assume for simplicity that ρ and μ are constants. Then Euler's equation becomes

$$\mu\frac{\partial^4 u}{\partial x^4} + \rho\frac{\partial^2 u}{\partial t^2} = 0. \tag{19.20}$$

Various forms of boundary conditions are possible.

1. The rod may be clamped at an end point.
   This means that the following two boundary conditions are imposed from the start

$$u = 0 \text{ and } u_x = 0. \tag{19.21}$$

2. The rod may be supported at an end point.
   This means that one boundary condition is imposed

$$u = 0, \tag{19.22}$$

   and that additionally a natural boundary condition is derived as

$$u_{xx} = 0. \tag{19.23}$$

3. The rod may be free at an end point.
   In this case no conditions are imposed at first. We are led, however, to the following pair of natural boundary conditions

$$u_{xx} = 0 \text{ and } u_{xxx} = 0. \tag{19.24}$$

CHAPTER VI

ADVANCED TOPICS

20. Broken extremals

In some problems, it becomes necessary to extend the class of admissible curves to include piecewise smooth curves. We again consider the simplest variational problem of section 5. However, the admissible functions y(x) are here continuous and piecewise differentiable. We shall discuss only the simplest case, in which there is one exceptional point x_1 with $a < x_1 < b$ such that $p = y'(x)$ is continuous in the closed intervals (a, x_1) and (x_1, b). The position of such a corner is not fixed. Figure 20.1 shows two neighbouring curves y(x) and y(x) + h(x) such that p is discontinuous at x_1 and $p + h'$ at x_2.

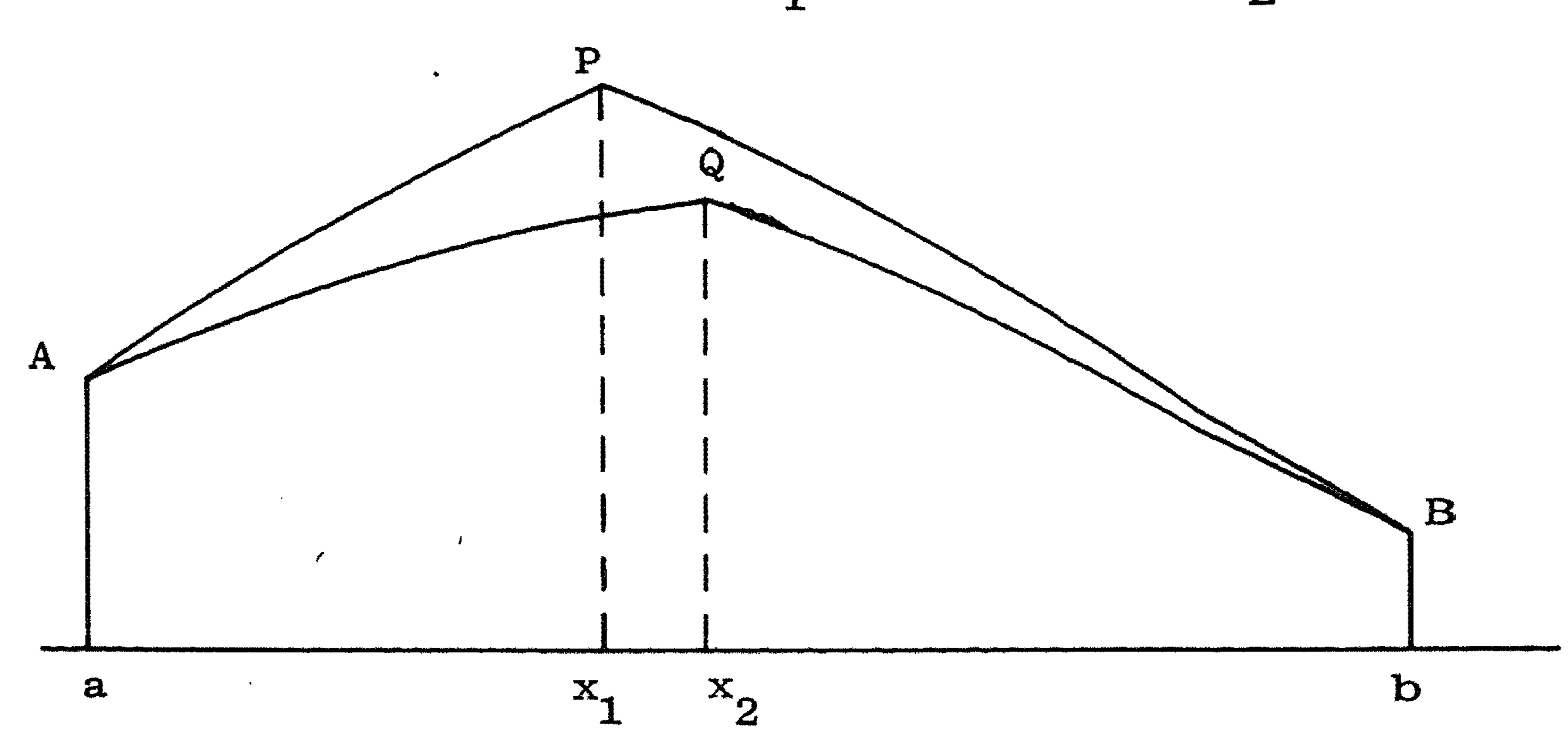


fig. 20.1

The problem is here to find a function of this extended class for which

$$(20.1) \qquad f[y] = \int_a^b F(x,y,p)\,dx$$

is stationary. Let APB be the required curve and AQB a neighbouring curve. Then we have to consider the local difference

$$(20.2) \qquad f[y+h] - f[y] = \int_a^b \{F(x,y+h,p+h') - F(x,y,p)\}\,dx.$$

In view of the discontinuities at x_1 and x_2 the interval of integration has to be divided into the three parts (a,x_1) (x_1,x_2) (x_2,b) for which separate discussions are needed.

In the well-known first order symbolism, we write

$$x_2 = x_1 + \delta x_1, \quad y_2 = y_1 + \delta y_1. \tag{20.3}$$

Then (20.2) can be written as

$$\delta f = \int_a^{x_1} + \int_{x_1+\delta x_1}^{b} (F_y h + F_p h')dx + \delta x_1 F(x,y,p)\Big|_{x_1}^{x_1+\delta x_1}.$$

After integration by parts, this passes into

$$\delta f = \int_a^b (F_y - \frac{d}{dx} F_p)h \, dx - (h(x)F_p(x,y,p))\Big|_{x_1}^{x_1+\delta x_1} + \tag{20.4}$$

$$+ \delta x_1 F(x,y,p)\Big|_{x_1}^{x_1+\delta x_1}.$$

From the small trapezium formed at the corners P and Q by the curves and the ordinates, there follows

$$\begin{cases} h(x_1) = y_1 - \delta x_1 \, p_1(x_1-0), \\ h(x_2) = y_1 - \delta x_1 \, p_1(x_1+0). \end{cases} \tag{20.5}$$

Substitution of this into (20.4) gives the following result for the first order terms

$$\delta f = \int_a^b (F_y - \frac{d}{dx} F_p)h \, dx + \delta x_1 (F - pF_p)\Big|_{x_1-0}^{x_1+0} - \delta y_1 F_p \Big|_{x_1-0}^{x_1+0}. \tag{20.6}$$

Since $\delta f = 0$ for all variations $h(x)$, δx_1, δy_1 we obtain the following two conditions.

1. The solution is a solution of the Euler equation

$$\frac{d}{dx}\frac{\partial F}{\partial p} = \frac{\partial F}{\partial y}$$

in the two intervals (a,x_1) and (x_1,b).

2. At the corner x_1 the following two functions are continuous

$$F - p\frac{\partial F}{\partial p} \quad \text{and} \quad \frac{\partial F}{\partial p}.$$

The latter two conditions are known as the Weierstrass-Erdmann conditions.

Example 20.1

Consider the functional

$$f[y] = \int_0^4 (p^2-1)^2 dx$$

for functions with just one corner and satisfying $y(0) = 0$ and $y(4) = 2$. The extremals are straight lines. We try the following broken line

$$\begin{cases} y = ax \text{ for } 0 \leqq x \leqq x_1, \\ y - 2 = b(x-4) \text{ for } x_1 \leqq x \leqq 4. \end{cases}$$

The Weierstrass-Erdmann conditions are

$$a(a^2-1) = b(b^2-1)$$
$$(3a^2+1)(a^2-1) = (3b^2+1)(b^2-1).$$

They are satisfied by either $a = 1$, $b = -1$ or $a = -1$, $b = 1$. Thus the following two extremals are obtained

$$\begin{cases} y = x \text{ for } 0 \leqq x \leqq 3 \\ y = 6 - x \text{ for } 3 \leqq x \leqq 4, \end{cases}$$

and

$$\begin{cases} y = -x \text{ for } 0 \leqq x \leqq 1 \\ y = x - 2 \text{ for } 1 \leqq x \leqq 4. \end{cases}$$

The obvious minimum value of the functional is zero.

21. Noether's theorem

So far the functional

$$(21.1)\qquad f[y] = \int_a^b F(x,y,p)\,dx$$

has been considered in a fixed interval only. If the admissible functions are allowed to vary at the end as in section 15, the first variation of f may be written as

$$(21.2)\qquad \delta f = \int_a^b \left(F_y - \frac{d}{dx} F_p\right) h(x)\,dx + F_p\,\delta y\Big]_{x=a}^{x=b} .$$

Here we shall obtain a more general expression by taking into account also a possible variation of the end points themselves.
If y(x) is a given function passing through the points $A(a,y_1)$ and $B(b,y_2)$, then a neighbouring curve may start at $A^*(a+\delta a, y_1+\delta y_1)$ and end at $B^*(b+\delta b,\ y_2+\delta y_2)$ (cf. fig. 21.1).

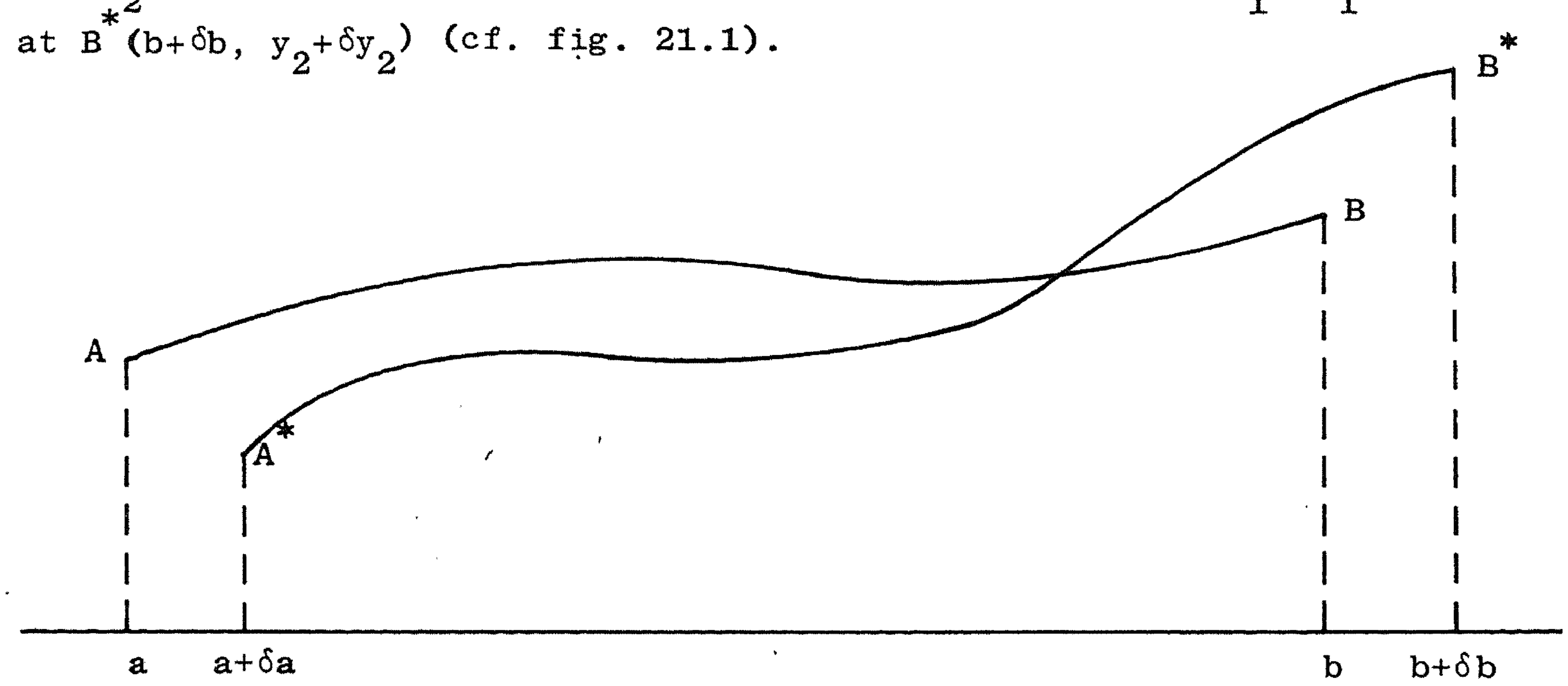


fig. 21.1

If I denotes the union of the intervals (a,b) and $(a+\delta a, b+\delta b)$, it is convenient to have a definition of y(x) and $y^*(x) = y(x) + h(x)$ for all points of I. This requires a possible continuous continuation of y(x) and $y^*(x)$ at the marginal intervals $(a,a+\delta a)$ and $(b,b+\delta b)$. This continuation can be carried out in the simplest way by extending the curve by its tangent at the end point. In this way $y(x)$ and $y^*(x)$ are defined as C_1-curves in I and it is possible to use the metric

$$\rho(y,y^*) = \max_I |y-y^*| + \max_I |p-p^*|. \tag{21.3}$$

The first variation δf of (21.1) may then be defined as a linear expression in h, h', δa, δb, δy_1, δy_2 which differs from $f[y+h] - f[y]$ by a quantity of higher order with respect to the distance $\rho(y,y+h)$.

If only the first order terms are written down, we may write

$$\delta f = \int_{a+\delta a}^{b+\delta b} F(x,y+h,p+h')dx - \int_a^b F(x,y,p)dx =$$

$$= \int_a^b (F_y h + F_p h')dx - F(x,y,p)\Big|_{x=a} \delta a + F(x,y,p)\Big|_{x=b} \delta b,$$

and so

$$\delta f = \int_a^b (F_y - \frac{d}{dx} F_p)h(x)dx + F_p h(x)\Big|_{x=a}^{x=b} + F\delta(x)\Big|_{x=a}^{x=b}. \tag{21.4}$$

At the end points, we have in the same first order approximation as in (20.5),

$$h(a) = \delta y_1 - p(a)\delta a$$

$$h(b) = \delta y_2 - p(b)\delta b.$$

Hence (21.4) may be written as

$$\delta f = \int_a^b (F_y - \frac{d}{dx} F_p)h(x)dx + F_p \delta y\Big|_{x=a}^{x=b} + (F-pF_p)dx\Big|_{x=a}^{x=b}. \tag{21.5}$$

The latter expression is known as the <u>general variation</u> of the functional $f[y]$. The obvious generalization for the general variation of a functional depending on m functions $y_1, y_2, \ldots, y_m$ is

$$\delta f = \int_a^b \sum_{i=1}^m (F_{y_i} - \frac{d}{dx} F_{p_i})h_i(x)dx + \sum_{i=1}^m F_{p_i} \delta y_i\Big|_{x=a}^{x=b} + \tag{21.6}$$

$$+ \left. (F - \sum_{i=1}^{m} p_i F_{p_i}) \delta x \right|_{x=a}^{x=b} .$$

In what follows we consider a family of transformations

$$(21.7) \qquad \begin{cases} x^* = \Phi(x,y,p,\varepsilon) \\ y^* = \Psi(x,y,p,\varepsilon) \end{cases}$$

depending on a real parameter ε. It will be assumed that Φ and Ψ are differentiable with respect to ε and that the value $\varepsilon = 0$ corresponds to the identity. Thus for small ε we may write

$$(21.8) \qquad \begin{cases} x^* = x + \varepsilon\phi(x,y,p) + o(\varepsilon) \\ y^* = y + \varepsilon\psi(x,y,p) + o(\varepsilon). \end{cases}$$

It may happen that the functional

$$(21.9) \qquad f[y] = \int_a^b F(x,y,p)dx$$

is invariant with respect to the set of transformations (21.7). This means that for all values of ε

$$(21.10) \qquad \int_{a^*}^{b^*} F(x^*,y^*,\frac{dy^*}{dx^*})dx^* = \int_a^b F(x,y,\frac{dy}{dx})dx.$$

Example 21.1

The functional

$$f[y] = \int_0^a y^{-\frac{1}{2}}(1+p^2)^{\frac{1}{2}}dx$$

of Bernoulli's Brachistochrone problem (cf. 6.2) is invariant with the following set of transformations

$$(21.11) \qquad x^* = x + \varepsilon, \quad y^* = y.$$

These transformations may be interpreted as translations in the x-direction.

According to Noether, to a set of transformations of the type (21.7), which leave a given functional (21.1) invariant, there corresponds a first integral of the Euler equation. This may be illustrated for the case of a functional where F does not depend explicitly on x. In this case, which has been considered in section 5, there exists an "energy" expression (5.10) as a first integral of the Euler equation. In fact, a functional of this special type is invariant under the set of translations (21.11). The general formulation is contained in the following statement.

Theorem 21.1 (Noether)

If the functional

$$f[y] = \int_a^b F(x,y,p)dx$$

is invariant under the family of transformations (21.7) for arbitrary end points, then along each extremal we have

(21.12) $$F_p\psi + (F-pF_p)\phi = \text{constant}.$$

Proof

Formula (21.5) for the general variation may be applied where $\delta x = \varepsilon\phi$ and $\delta y = \varepsilon\psi$ according to (21.8). This gives, for an extremal,

(21.13) $$\delta f = \varepsilon\{F_p\phi + (F-pF_p)\psi\}_a^b .$$

Since, by hypothesis, $f[y]$ is invariant under the infinitesimal transformation (21.8), therefore $\delta f = 0$, and the right-hand side of (21.13) vanishes for all choices of a and b. This fact is expressed by (21.12).

Again, Noether's theorem can easily be generalised. For a functional of the type

(21.14) $$f[y_1,y_2,\dots,y_m] = \int_a^b F(x,y_1,\dots,y_m,p_1,\dots,p_m)dx$$

which is invariant under the infinitesimal transformations

$$(21.15)\qquad \begin{cases} x^* = x + \varepsilon\phi + O(\varepsilon) \\ y_i^* = y_i + \varepsilon\psi_i + O(\varepsilon), \; i = 1,2,\dots,m \end{cases}$$

we obtain the following first integral

$$(21.16)\qquad \sum_{i=1}^{m} F_{p_i}\psi_i + (F - \sum_{i=1}^{m} p_i F_{p_i})\phi = \text{constant}.$$

Example 21.2

The plane motion of a particle of mass m attracted to the origin 0 by gravitation is determined by Hamilton's principle, where the action functional is given in polar coordinates (r,θ) by

$$f[r,\theta] = \int_{t_1}^{t_2} \{\tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{k}{r}\}dt,$$

where k is a constant. Since the functional is invariant under rotation of the axes

$$\theta^* = \theta + \varepsilon, \quad r^* = r, \quad t^* = t,$$

the first integral (21.16) therefore reduces to

$$mr^2\dot{\theta} = \text{constant},$$

which is Kepler's second law of planetary motion.

CHAPTER VII
DIRECT METHODS

22. Introduction

Problems of mathematical physics may be formulated in different ways. Sometimes a problem is given in the form of a differential equation with a boundary condition, and sometimes we have a variational problem. Consider, for instance, the equilibrium position of a membrane which has a given displacement h(s) at its boundary. The corresponding boundary value problem is, according to section 19c,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{22.1}$$

with

$$u = h(s) \text{ for points } (x,y) \text{ at the boundary.} \tag{22.2}$$

On the other hand, we have the equivalent variational formulation

$$\iint_{\Omega} (u_x^2 + u_y^2)\,dx\,dy = \text{minimum}, \tag{22.3}$$

where the admissible functions satisfy the condition (22.2).

It depends on the circumstances, e.g. the form of Ω, which version offers the best possibilities for further treatment. The minimum problem (22.3) can always be attacked in a direct way by construction of a suitable approximation. If, however, Ω has the form of e.g. a rectangle or a circle, the boundary value problem (21.1), (22.2) is more attractive, since separation of variables then becomes possible, thereby considerably simplifying the treatment.

Example 22.1

Consider the circular membrane $r \leq 1$ for which $u = h(\theta)$ at the boundary, where r and θ are polar coordinates. The potential equation (22.1) in these coordinates is of the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0,$$

and this is separable. Its solutions are $r^n \cos n\theta$ and $r^n \sin n\theta$, where $n = 0,1,2,\ldots$. The general solution is

$$u(r,\theta) = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) r^n .$$

The unknown coefficients follow from the Fourier expansion

$$h(\theta) = \sum_{n=0}^{\infty} (a_n \cos n\theta + b_n \sin n\theta).$$

Example 22.2

Consider the triangular membrane $x \geq 0$, $y \geq 0$, $x+y \leq 1$ for which, at the boundary, $u(x,y) = 0$ at $x = 0$ and at $y = 0$, and $u(x,y) = xy$ at $x+y = 1$. In this case, the variational approach is appropriate. A reasonable trial function would be a polynomial such as

$$f(x,y) \doteq xy\{a + b(x+y) + c(x+y)^2\},$$

where $a + b + c = 1$.

The condition (22.3) leads to a minimum problem for a homogeneous quadratic polynomial expression in a,b,c with the subsidiary condition $a + b + c = 1$. By choosing trial polynomials of higher degree, the required solution can be obtained to any desired precision.

A very general class of boundary value problems and equivalent variational problems is covered by theorem 12.3. The boundary value formulation is expressed by

(22.4) $\quad Au = g,$

where A is a positive symmetric linear operator acting upon a Hilbert space of functions. Usually A is a differential operator and the domain D_A is restricted by one or more boundary conditions. The variational formulation is

$$\tfrac{1}{2}(Au,u) - (u,g) = \text{minimum}. \tag{22.5}$$

<u>Example 22.3</u>

Consider the boundary value problem

$$\frac{d^2u}{dx^2} - xu = 1,\ 0 < x < 1,$$

with $u = 0$ for $x = 0$ and $x = 1$.

It is not difficult to verify that the operator $A = -\dfrac{d^2}{dx^2} + x$ is symmetric and positive in the Hilbert space $L_2(0,1)$.

Thus the problem has the variational version

$$\int_0^1 (-u_{xx}+xu+2)u\ dx = \text{minimum},$$

or, more simply

$$\int_0^1 (u_x^2+xu+2)u\ dx = \text{minimum}.$$

Although the original boundary value problem can be solved explicitly by means of Bessel functions of order 1/3, the direct attack of the minimum problem would be much easier. Approximations may be sought in the form of a polynomial such as

$$u(x) = x(1-x)(c_0+c_1x+\ldots+c_mx^m).$$

Another important field of applications is formed by eigenvalue problems where one is interested e.g. in the lowest frequency of an oscillating system. The general situation can easily be expressed again in Hilbert space terminology. Following the discussion at the end of section 18, the behaviour of the oscillating system is described by

(22.6) $$Aq + \frac{\partial^2 q}{\partial t^2} = 0$$

where A is a symmetric positive definite operator in an appropriate Hilbert space. Setting $q = ue^{i\omega t}$ we obtain the eigenvalue equation

(22.7) $$Au = \omega^2 u.$$

The lowest eigenvalue, i.e. the square of the lowest frequency, is then given by

(22.8) $$\omega_1^2 = \min \frac{(Au,u)}{(u,u)}.$$

The corresponding eigenfunction, provided the lowest eigenvalue is non-degenerate, is then given by that function for which the latter minimum is obtained.

Example 22.4

Consider the motion of a string as given by

$$-\frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial t^2} = 0, \quad 0 < x < 1,$$

with $q = 0$ at the end points $x = 0$ and $x = 1$.

The eigenvalue equation is easily solvable - at least in this very simple case. The lowest mode of vibration is given by $u = \sin \pi x$. The variational problem (22.8) is here

$$\omega_1^2 = \min \frac{\int_0^1 u_x^2 dx}{\int_0^1 u^2 dx} .$$

A first idea of the value of ω_1 may be obtained by taking the trial function $u = x(1-x)$. This gives $\omega_1^2 \leqq 10$. The true value of ω_1^2 is, of course, $\pi^2 = 9.87$, which is remarkably close.

23. Minimizing sequences

We consider the very general problem of finding the minimum of a functional $f[y]$ defined on a normed space of admissible functions y. Since a maximum of f corresponds to a minimum of -f, it is sufficient to consider the minimum case only. It will be assumed that the admissible functions for which $f[y]$ is considered form a subset S of a Banach space R. Then the existence is assumed of a lower bound

$$\mu = \inf_{y \in S} f[y]. \tag{23.1}$$

The first question to be asked is whether S contains an element $\hat{y}$ for which

$$f[\hat{y}] = \mu. \tag{23.2}$$

If the answer is affirmative, as is the case in a great variety of problems of mathematical physics, then the problem is to approximate μ and $\hat{y}$ by means of a minimizing sequence $\{y_n\}$.
By virtue of (23.1), there always exists a minimizing sequence for which $f[y_n] \to \mu$. In fact, the definition (23.1) implies the existence of elements y_n of S, $n = 1,2,\dots$, for which $f[y_n] < \mu + 1/n$.

Example 23.1

Consider the functional

$$f[y] = \int_{-1}^{1} x^2 y_x^2 \, dx$$

for functions $y(x) \in C_1(-1,1)$ satisfying the boundary conditions $y(-1) = -1$, $y(1) = 1$. The obvious minimum is $\mu = 0$. A minimising sequence is formed by

$$y_n = \frac{\arctan nx}{\arctan n}, \quad n = 1,2,\dots .$$

We note that

$$f[y_n] < \frac{8}{\pi n}.$$

However, the sequence does not converge to a C_1-function and there exists no element $\hat{y}$ for which the minimum value $\mu = 0$ is obtained.

The situation in this example is rather exceptional in practical problems. Usually, once a minimizing sequence is constructed, it converges or at least contains a converging subsequence. The art of constructing converging minimizing sequences is a well-developed branch of numerical mathematics to which the names of Rayleigh, Ritz and others are associated.

The second question to be asked concerns the validity of the relation

$$\lim_{n\to\infty} f[y_n] = f[\lim_{n\to\infty} y_n] = \mu. \tag{23.3}$$

This is always true if the functional is continuous with respect to the norm of R. However, the majority of the functionals considered in the problems of the calculus of variations do not satisfy this condition.

Example 23.2

The functional of the arc length

$$f[y] = \int_0^1 (1+y_x^2)^{\frac{1}{2}} dx$$

is not continuous in the norm of C(0,1).

However, (23.3) still holds if continuity is replaced by the weaker condition of semicontinuity.

Definition (23.1)

The functional $f[y]$ is said to be lower semicontinuous at $\hat{y}$ if, for any given positive ε, there exists a positive $\delta(\varepsilon)$ such that

$$f[y] > f[\hat{y}] - \varepsilon \text{ for } ||y-\hat{y}|| < \delta.$$

Using this definition we may state the following property.

Theorem 23.1

If $\{y_n\}$ is a minimizing sequence of $f[y]$ converging to $\hat{y}$ and if $f[y]$ is lower semicontinuous at $\hat{y}$ then

$$\lim_{n\to\infty} f[y_n] = f[\hat{y}].$$

Proof

There exists a sequence of positive ε_n, $n = 1,2,\ldots$ converging to zero such that $f[y_n] > f[\hat{y}] - \varepsilon_n$. For $n \to \infty$ this gives

$$f[\hat{y}] \leqq \lim_{n\to\infty} f[y_n].$$

On the other hand, we have, of course,

$$f[\hat{y}] \geqq \inf f[y] = \lim_{n\to\infty} f[y_n].$$

By comparison with the previous inequality, the theorem follows.

For the actual construction of a minimizing sequence, we again consider the problem (22.4). We make the further assumption that

$$Au = g \tag{23.4}$$

has a solution u_0 of finite energy, i.e. a solution for which the energy norm (Au,u) is finite.

The following theorem will show that once a minimizing sequence for (22.5) is constructed, it automatically converges in a suitable norm.

Theorem 23.2

If $Au = g$, where A is a positive symmetric linear operator in a Hilbert space, has a solution $\hat{u}$ with a finite energy norm, then each minimizing sequence of the functional

$$f[u] = (Au,u) - 2(u,g)$$

converges to $\hat{u}$ in the energy norm.

Proof

We have the relation

$$f[u_n] = (A(u_n-\hat{u}),u_n-\hat{u}) - (A\hat{u},\hat{u}).$$

Since $\{u_n\}$ is a minimizing sequence, we have

$$\lim f[u_n] = -(A\hat{u},\hat{u}).$$

Thus we must also have

$$(A(u_n-\hat{u}),u_n-\hat{u}) \to 0.$$

Using the notation

$$||u||_A^2 = (Au,u),$$

this means convergence in the energy norm

$$||u_n-\hat{u}||_A \to 0.$$

The latter theorem is the explicit statement that a boundary value problem can be solved by means of a minimizing sequence of a functional. Minimizing sequences may easily be constructed from linear combinations of a set of functions $\{\phi_n\}$ which are complete with respect to the energy norm. The n-th element of a minimizing sequence may then be written as

$$u_n = \sum_{j=1}^{n'} a_j\phi_j \tag{23.5}$$

with coefficients determined by the requirement that $F[u_n]$, now a functional of the n coefficients a_j, take a minimum value. Since

$$F[u_n] = \sum_{j,k}(A\phi_j,\phi_k)a_ja_k - 2\sum_j(\phi_j,g)a_j,$$

the conditions are

$$\frac{\partial F}{\partial a_j} = 0 \text{ for } j = 1,2,\ldots,n,$$

or, explicitly,

$$\sum_k (A\phi_j, \phi_k) a_k = (g, \phi_j). \tag{23.6}$$

This represents a set of linear equations from which the coefficients a_k can be solved.

The ideal choice of the set $\{\phi_n\}$ would be functions orthogonal with respect to the energy norm, i.e. $(A\phi_j, \phi_k) = 0$ for $j \neq k$. For practical reasons, it is desirable to approximate this situation as closely as possible without sacrificing the simplicity of the auxiliary functions.

Before giving an example, we make the final remark that for a positive definite operator A, convergence in the energy norm implies convergence in the usual norm, i.e. in the mean, since, for a positive definite operator A, there exists a positive constant c such that

$$(Au,u) \geqq c\,||u||^2.$$

Example 23.3

The boundary value problem

$$y'' + xy = -x, \quad y(0) = y(1) = 0,$$

can be converted into the variational problem

$$F[y] = \int_0^1 (y_x^2 - xy^2 - 2xy)\,dx = \text{minimum}.$$

The admissible functions are taken from $C_2(0,1)$ with $y(0) = y(1) = 0$. The required solution may be developed as a sine series. Thus, a minimizing sequence can be constructed according to

$$y_n = \sum_{j=1}^{n} a_j \sin j\pi x.$$

The coefficients a_j follow from the relations (23.6) for which here with

$$A = -\frac{d^2}{dx^2} - x \quad \text{and} \quad g = x$$

$$(A\phi_j, \phi_k) = \frac{4jk}{\pi^2(j^2-k^2)^2} \quad \text{if } j + k \text{ is odd,}$$

$$\text{"} \quad = 0 \quad \text{if } j + k \text{ is even, } j \neq k,$$

$$\text{"} \quad = \tfrac{1}{2}j^2\pi^2 - \tfrac{1}{4} \quad \text{if } j = k,$$

and

$$(g, \phi_j) = \frac{(-1)^{j-1}}{j\pi} .$$

For $n = 1$ we find $a_1 = 0.0679$.
For $n = 2$ we find a corrected value $a_1 = 0.0681$ and further $a_2 = - 0.0085$.
For $n = 3$ a slight improvement is obtained. For a_1 and a_2 the same values are obtained and next $a_3 = 0.0024$.

Recommended Reading

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