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**THE INTERVAL FUNCTION
OF A GRAPH**

H.M. MULDER

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INTRODUCTION

When trying to prove a theorem, many mathematicians proceed as follows: they make some more or less vague images in their mind of the concepts involved in the theorem, and they try to "prove", or "disprove", the conjectured theorem by operating with these images. Once they have convinced themselves of its truth, they construct a "formal" proof of the theorem. It is often helpful to visualize the arguments in the process of "proving" by drawing pictures on the blackboard or on paper, preferably on the back of paper already printed on the other side. For example, Venn-diagrams are useful when we deal with sets.

One of the attractive features of graph theory is its inherent pictorial character -that is, the images and pictures we make of the objects we deal with can be rather precise. Sometimes we can even use figures in the written proof of a theorem.

In this monograph I have included many figures to provide examples and to illustrate the arguments in proofs. Often the argument does not lie so much in the figure itself as in the process of drawing the figure. The reader is therefore invited to draw his own figures where necessary.

We deal with intervals in graphs. An interval in a graph can be considered as an analogue of the notion of interval on the real line in the following way: the interval $[a,b]$ on the real line consists of all real numbers "between" a and b ; a vertex w in a graph G is said to lie between two vertices u and v if w lies on a shortest path from u to v in G ; the interval between u and v in G is the set of all vertices between u and v .

We use this concept to study several classes of graphs. Some of these classes are related to algebraic structures, and some are related to other discrete mathematical structures (symmetric block designs, Helly hyper-graphs). In Section 1.3 we discuss the unoriented Hasse diagrams of finite

modular and distributive lattices. In Chapter 2 we introduce $(0, \lambda)$ -graphs. These are connected graphs in which any two vertices have λ common neighbours or none at all. Such graphs are regular. Examples of $(0, \lambda)$ -graphs are the n -cube and the incidence graph of a symmetric block design. When the diameter is at least 4, we can easily obtain a lower bound for the degree of a $(0, \lambda)$ -graph involving the diameter. The "extremal" $(0, \lambda)$ -graphs are the hypercubes and the Hadamard graphs (Sections 2.2, 2.3 and 2.4).

In the next two chapters we consider median graphs. A median graph is a connected graph such that, for any three vertices u , v and w , there exists exactly one vertex $x = x(u, v, w)$ lying simultaneously on a shortest (u, v) -path, a shortest (v, w) -path and a shortest (w, u) -path. This vertex x is called the median of u , v and w . There is a close relationship between median graphs and some algebraic structures (e.g. median semilattices and median algebras; see Section 3.3). One of the main theorems of this monograph is the "median graph theorem", which states that any median graph can be obtained from a median graph with fewer vertices by an expansion procedure (Section 3.2). A consequence of this theorem is that a median graph admits a proper colouring of its edges (which is uniquely determined) such that any colour class is a cutset. The components with respect to these cutsets can be taken as the edges of a hypergraph, which turns out to be a maximal self-complementary Helly hypergraph (Section 4.1). Another consequence is that a median graph can be embedded in an n -cube as a distance-preserving "median-closed" subgraph (Section 3.4).

In Chapter 5 we give a characterization of Hamming graphs (graph products of complete graphs), a natural generalization of the n -cube. In the last two chapters we study quasi-median graphs, which generalize median graphs. We prove suitable analogues of many of the results in Chapter 3. For example, we prove a "quasi-median theorem" (Section 6.2), and we establish the relationship between quasi-median graphs and quasi-median algebras (Chapter 7).

To make this thesis self-contained, we add Chapter 0 in which we list the relevant definitions and results from the literature.

The n -cube plays an important role in most of the chapters. To use a metaphor, the n -cube runs through the story like a thread. From this point of view the contents of this tract can be divided conveniently into five parts. In the first part (Section 1.1) we introduce the interval function of a graph as the main tool in our analysis. In the second part (Sections

1.2 and 1.3 and Chapter 2) we discuss several classes of graphs, which all generalize some aspect of the n -cube: interval-regular graphs, unoriented Hasse diagrams of finite lattices, $(0, \lambda)$ -graphs and Hadamard graphs. The third part (Chapters 3 and 4) deals with median-closed subgraphs of the n -cube and their relationships with other finite mathematical structures. In the fourth part (Chapter 5) we single out the most obvious of all possible generalizations of the n -cube, that of the Hamming graphs, and in the last part (Chapters 6 and 7) we study the "quasi-median closed" subgraphs of these generalized hypercubes.

As is usual in mathematics the material of this monograph was not developed in the way it is presented here. I have included the problem from which this research "originated" as a digression in Section 4.2. Some years ago a colleague at the Vrije Universiteit, Jan van Mill (a topologist) posed to me the problem of determining an upper bound for the number of edges in k -Helly hypergraphs. This problem arose in the context of the theory of supercompact spaces in topology. We settled it for $k \leq 2$ fairly easily, but the case $k \geq 3$ resisted our attempts. An alternative approach of the case $k = 2$ led to the so-called maximal Helly copair hypergraphs (see Definition 4.1.5 and Corollaries 4.1.16 and 4.1.17).

A suggestion of my friend and colleague at the Mathematical Centre Lex Schrijver to use his notion of interval structure led to a fruitful collaboration (see MULDER & SCHRIJVER [MS]), the results of which form the contents of Section 4.1. And so the stone began to roll, gathering moss on its way. I still can point out the exact place in Keszthely where, on a walk during the Fifth Hungarian Combinatorial Colloquium, I was "struck" by the idea ultimately leading to the median graph theorem (Theorem 3.2.4).

In one of the final stages of this research, Ladislav Nebeský, who obtained some of the results in Chapter 3 independently (see [N2]), visited the Vrije Universiteit. During our discussions, we came across the ideas which led to the results in [N3] and in Chapter 7.

With hindsight, the original problem was not difficult, and its solution need not involve the graphs that grew out of it (see Corollary 4.2.5).

CHAPTER 0

BASIC CONCEPTS

In this chapter we list those definitions and results from the literature that are needed in the sequel. It is intended for those readers who are not familiar with the basic concepts of graph theory, poset and lattice theory, and design theory.

0.1. GRAPHS

With some minor adaptations we adopt the terminology of BONDY & MURTY [BM] and WILSON [Wi]. Other general references for graph theory are BERGE [Be] and BIGGS, LLOYD & WILSON [BLW].

A *graph* G consists of a finite set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of distinct vertices called *edges* (so we consider only simple undirected finite graphs). If we use the letter G to indicate a graph, then we always assume that V is its vertex-set and E its edge-set. We give examples of graphs in Figures 0.1 and 0.2. In these figures vertices are depicted as small circles, and any edge $\{u,v\}$ (usually denoted by uv) is depicted as a line *joining* the *ends* u and v . If uv is an edge in G , then u is *incident* with uv , and we say that u and v are *adjacent*, or that u is a *neighbour* of v . Two edges are *adjacent* if they have a vertex (end) in common.

The *complementary graph* \bar{G} of G has V as vertex-set and two distinct vertices are adjacent in \bar{G} whenever they are not adjacent in G .

A *subgraph* of G is a graph G' whose vertex-set V' is a subset of V , and whose edge-set E' is a subset of E such that any edge in E' joins two vertices in V' . If all edges of G joining two vertices in V' are in E' , then G' is *induced* by V' , and is denoted by $G[V']$. If V' consists of all ends of edges in E' , then G' is an (*edge-*)*induced* subgraph of G , and

is denoted by $G[E']$. A *spanning subgraph* of G is a subgraph which has V as its vertex-set.

A *clique* in G is a set of vertices in which any two distinct vertices are adjacent. If V is a clique, then G is the *complete graph* K_n , where $n = |V|$ is the *size* of the set V . The graph K_3 is called *triangle*.

The *degree* $d(v) = d_G(v)$ of a vertex v of G is the number of neighbours of v . If all vertices of G have the same degree $d(G)$, then G is *regular* of degree $d(G)$.

A sequence of distinct vertices $P = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_n$ in G such that any two consecutive vertices are adjacent is called a *path* from u_0 to u_n (a (u_0, u_n) -path, for short) of *length* n . If $n \geq 2$, then u_1, \dots, u_{n-1} are the *internal vertices* of P , and P *passes through* these vertices. A graph G is *connected* if for any two vertices u and v of G there exists a (u, v) -path. The *distance* $d(u, v) = d_G(u, v)$ between two vertices u and v is the length of any shortest (u, v) -path in G . The *diameter* $\text{diam}(G)$ of G is the largest distance in G . A *disconnected* graph consists of the *disjoint union* of connected graphs called the *components* of the graph. A *disconnecting set* in a connected graph G is a set $F \subseteq E$ such that the graph with vertex-set V and edge-set $E \setminus F$ is disconnected. A *cutset* in a connected graph is a disconnecting set, no proper subset of which is a disconnecting set. Note that the deletion of a cutset from a connected graph splits the graph into exactly two components.

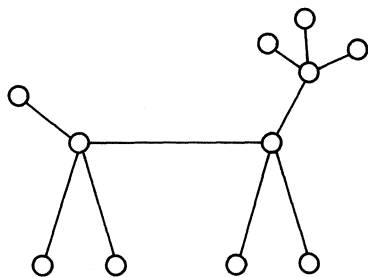


FIGURE 0.1.

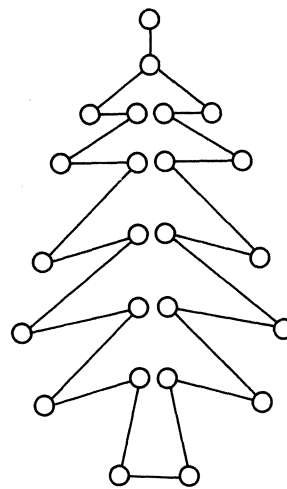


FIGURE 0.2.

A *circuit* of length n in G is a sequence of vertices $C = u_1 \rightarrow \dots \rightarrow u_n \rightarrow u_1$, in which u_1, \dots, u_n are distinct vertices and any two consecutive vertices are adjacent. A *tree* is a connected graph without circuits. The graph in Figure 0.1 is a tree, whereas the graph in Figure 0.2 is not. A *Hamiltonian graph* is a graph containing a spanning circuit.

A *bipartite graph* contains no *odd circuits* (circuits of odd length). In such a graph G the vertex-set can be partitioned into two sets V_1 and V_2 such that any edge of G joins a vertex in V_1 to a vertex in V_2 . By *counting the edges twice*, we get the formula

$$\sum_{v \in V_1} d(v) = |E| = \sum_{v \in V_2} d(v),$$

or the inequality

$$p|V_1| \leq q|V_2|,$$

where p is a lower bound for the degrees in V_1 and q is an upper bound for the degrees in V_2 . If all possible edges between V_1 and V_2 are in G , then G is the *complete bipartite graph* $K_{n,m}$, where $n = |V_1|$ and $m = |V_2|$.

The graph $K_{1,1,m}$ consists of an edge uv together with m vertices adjacent to both u and v .

A *matching* in G is a set of non-adjacent edges. If $|V|$ is even, then a *perfect matching* in G is a matching of size $\frac{1}{2}|V|$, so that any vertex of G is "matched" to another vertex. An *edge colouring* of G with k colours is an assignment of k colours $\alpha_1, \alpha_2, \dots, \alpha_k$ to the edges of G such that edges of the same colour form a matching. We use the term *colour* α_i to indicate the colour α_i that is assigned to an edge, as well as to indicate the set of all edges assigned colour α_i (that is, the *colour class* α_i).

A graph G is *homomorphic* to a graph H if there exists a *homomorphism* from G onto H -that is, a mapping f from $V(G)$ onto $V(H)$ such that if $W \subseteq V(H)$ induces a connected subgraph of H , then the set $f^{-1}[W]$ induces a connected subgraph of G (or, informally, if we can obtain H from G by *contracting edges*). The graphs G and H are *isomorphic* if there exists an *isomorphism* between G and H -that is, a bijection $f: V(G) \rightarrow V(H)$ such that both f and f^{-1} are homomorphisms. Usually we do not distinguish between isomorphic graphs. An *automorphism* of G is an isomorphism of G onto itself. The graph G is *transitive* if, for any two vertices u and v

of G , there exists an automorphism f of G such that $f(u) = v$. We call G *distance-transitive* if, for any u, v, x and y in V with $d(u,v) = d(x,y)$, there exists an automorphism f such that $f(u) = x$ and $f(v) = y$. A graph G is *homeomorphic* to a graph H if there exists a homomorphism f from G onto H such that $f^{-1}(uv)$ induces a path in G , for any edge uv of H . A graph is *outerplanar* if it contains no subgraph homeomorphic to K_4 or $K_{2,3}$.

The *adjacency matrix* of a graph G with vertices v_1, \dots, v_n is the $n \times n$ matrix $A = (a_{ij})$ with

$$a_{ij} = \begin{cases} 1 & \text{if } v_i v_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

The *product* of the graphs G and H is the graph $G \times H$ whose vertex-set is the Cartesian product $V(G) \times V(H)$, in which (u,v) is adjacent to (u',v') if and only if either $u = u'$ and $vv' \in E(H)$ or $v = v'$ and $uu' \in E(G)$.

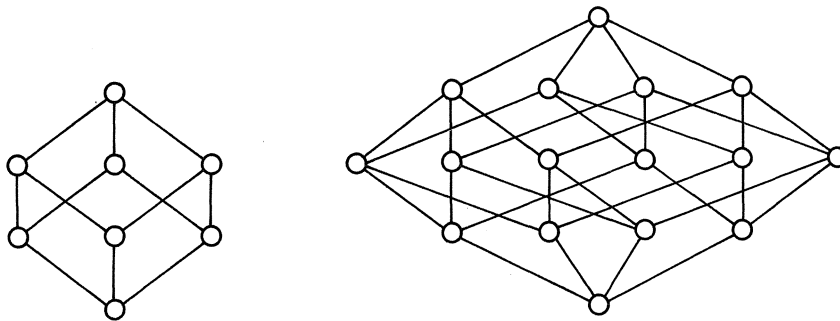


FIGURE 0.3.

We conclude this section with an example that plays an important role in this monograph. The *hypercube* Q_n of dimension n (the *n-cube*, for short) has the $(0,1)$ -vectors of length n as vertices, two vertices being joined if they differ in exactly one coordinate (this is the *vector representation* of Q_n). The n -cube has 2^n vertices and $n2^{n-1}$ edges. It is regular of degree n , bipartite and distance-transitive. An alternative definition of Q_n is the following: let V be a set of size n (an *n-set*, for short), and let $\mathcal{P}(V)$ be the *power-set* of V ; then Q_n has $\mathcal{P}(V)$ as its vertex-set, and two

vertices A and B are adjacent whenever $|A \Delta B| = 1$, where $A \Delta B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference of the sets A and B (this is the *subset representation* of Q_n). The last definition can be translated into the first one by labelling any vertex $A \subseteq V$ with its characteristic function. Note that we use capitals to indicate vertices represented by subsets of a set. The small hypercubes are the 0-cube K_1 , the 1-cube K_2 , and the 2-cube $K_{2,2}$. In Figure 0.3 we give Q_3 and Q_4 . Note that for non-negative integers n and m we have $Q_{n+m} = Q_n \times Q_m$.

0.2. POSETS AND LATTICES

A general reference for the theory of lattices and partially ordered sets is BIRKHOFF [Bi].

A finite *partially ordered set* (poset) $P = (V, \leq)$ consists of a finite set V and a reflexive, transitive, antisymmetric relation \leq on V . If $u \leq v$ and $u \neq v$, then we write $u < v$. If u and v are in V , then v *covers* u in P if $u < v$ and there is no w in V with $u < w < v$. Using the covering relation we can obtain a graphical representation of P , called the *Hasse diagram* of P , as follows: draw a small circle to represent each element of P , placing v higher than u whenever $u < v$; draw a straight line joining u and v whenever v covers u . The Figures 0.3, 0.4, 0.5 and 0.6 are Hasse diagrams of posets. Note that the Hasse diagram of a poset is "*oriented*" downwards.

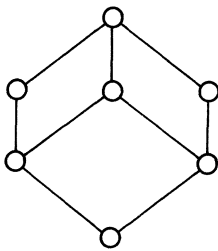


FIGURE 0.4.

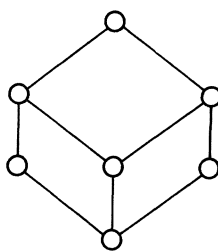


FIGURE 0.5.

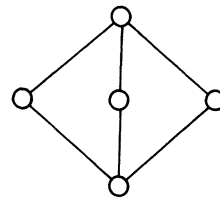


FIGURE 0.6.

The *greatest lower bound* of two elements u and v in P is an element x such that $x \leq u$ and $x \leq v$ and for any y in V with $y \leq u$ and $y \leq v$, we have $y \leq x$. If such a greatest lower bound of u and v exists, then it is unique,

by the antisymmetry of \leq , and we denote it by $u \wedge v$. Similarly the *least upper bound* of u and v is an element x in V such that $u \leq x$ and $v \leq x$ and for any y in V with $u \leq y$ and $v \leq y$, we have $x \leq y$. If such a least upper bound exists, then it is unique, and we denote it by $u \vee v$. A *universal lower bound* of P is an element 0 in V with $0 \leq u$, for all u in V . Similarly a *universal upper bound* of P is an element 1 in V such that $u \leq 1$, for all u in V .

A finite *lattice* is a finite poset P in which any two elements have a greatest lower bound and a least upper bound. It follows that a finite lattice has a universal lower bound and a universal upper bound. Note that a finite poset with universal upper bound in which any two elements have a greatest lower bound is a lattice.

A finite *modular lattice* (V, \leq) is a finite lattice satisfying the following two *covering conditions*, for any two elements u and v in V :

- (i) if u and v cover $u \wedge v$, then $u \vee v$ covers both u and v ;
- (ii) if $u \vee v$ covers both u and v , then u and v cover $u \wedge v$.

Figures 0.4 and 0.5 are Hasse diagrams of non-modular lattices. Figure 0.4 satisfies covering condition (i), and Figure 0.5 satisfies covering condition (ii). Figure 0.6 is the Hasse diagram of a modular lattice.

A finite *distributive lattice* P is a modular lattice, which does not contain Figure 0.6.

Finally, the finite *Boolean lattice* on 2^n elements is the poset $(\mathcal{P}(V), \subseteq)$, where V is an n -set. The Hasse diagrams of the Boolean lattices on 8 and 16 elements are depicted in Figure 0.3. Note that a Boolean lattice is a distributive lattice in which each element has a unique "complement".

0.3. BLOCK DESIGNS

General references for the theory of block designs are CAMERON & VAN LINT [CL], HALL [H] and RYSER [R1].

A *block design* $D = (X, \mathcal{B})$ with parameters (b, v, r, k, λ) consists of a finite v -set X of *points* together with a family \mathcal{B} of k -subsets of X (called *blocks*) such that each pair of distinct points is contained in

exactly λ blocks in \mathcal{B} . It can be verified that any point of the design is contained in a fixed number of blocks (we denote this number by r). The number of blocks in the design is denoted by b . The following equalities are easily verified by simple counting arguments:

$$bk = vr,$$

and

$$r(k-1) = (v-1)\lambda.$$

A block design is *symmetric* if $b = v$. Note that in this case $r = k$. Hence the *parameters* of a symmetric block design are (v, k, λ) . It can be verified that any two distinct blocks have λ points in common (see [H] or [R1]).

The *incidence matrix* of a block design $D = (X, \mathcal{B})$ with $X = \{p_1, \dots, p_v\}$ and $\mathcal{B} = \{B_1, \dots, B_b\}$ is the $b \times v$ matrix $M = (m_{ij})$ such that

$$m_{ij} = \begin{cases} 1 & \text{if } p_j \in B_i \\ 0 & \text{if } p_j \notin B_i. \end{cases}$$

Note that a symmetric block design need not have a symmetric incidence matrix.

The *complementary block design* \bar{D} of a block design $D = (X, \mathcal{B})$ with parameters (b, v, r, k, λ) has point-set X and set of blocks $\bar{\mathcal{B}} := \{X \setminus B \mid B \in \mathcal{B}\}$. The parameters of \bar{D} are $(b, v, b-r, v-k, b-2r+\lambda)$.

A *Hadamard matrix* of order n is an $n \times n$ matrix H with entries ± 1 satisfying

$$HH^T = H^T H = nI,$$

where I is the identity matrix of order n . Changing the signs of rows and columns does not affect the defining equation, and using this property we can easily verify that $n = 2$ or $n \equiv 0 \pmod{4}$. There exist several constructions of block designs from Hadamard matrices, one of which is described in Chapter 2 (see [CL] for this one and for other constructions).

CHAPTER 1

INTERVALS

In this chapter we introduce the interval function of a graph. After proving some elementary results on intervals in graphs, we study interval-regular graphs, and use the interval function of a graph to characterize the unoriented Hasse diagrams of finite modular and distributive lattices.

1.1. THE CONCEPT OF INTERVAL

First a preliminary remark: if we use the letter G in this monograph to denote a graph, then we always assume that V is its vertex-set and E its edge-set. We start by defining the concept that plays a central role in the theorems and proofs of this monograph.

1.1.1. DEFINITION. Let G be a graph, and let $\mathcal{P}(V)$ be the power-set of V . The mapping $I_G: V \times V \rightarrow \mathcal{P}(V)$ defined by

$$I_G(u,v) := \{w \in V \mid w \text{ lies on a shortest } (u,v)\text{-path in } G\}$$

is the *interval function* of G .

Note that w lies in $I_G(u,v)$ if and only if $d(u,v) = d(u,w) + d(w,v)$. Each set $I_G(u,v)$ is called an *interval* in G . When no confusion can arise we write I instead of I_G .

EXAMPLE 1. In a tree the interval $I(u,v)$ consists of the vertices on the unique path from u to v .

EXAMPLE 2. Consider the n -cube Q_n with its vector representation, and let

$u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be vertices of Q_n . The interval $I(u, v)$ consists of those vertices of Q_n that have u_i as i -th coordinate, for those values of i for which $u_i = v_i$.

Figures are often used to elucidate the arguments in proofs or to give examples, and sometimes a figure forms an essential step in a proof. A shortest path from u to v is usually depicted as in Figure 1.1, and the interval $I(u, v)$ as in Figure 1.2.

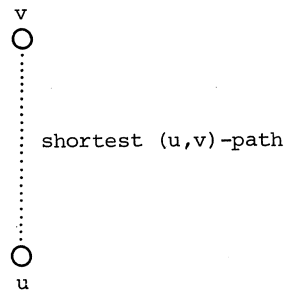


FIGURE 1.1.

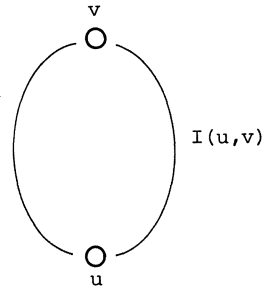


FIGURE 1.2.

Note that a graph G with interval function I is connected if and only if there are no empty intervals in G . The following properties of the interval function can easily be verified.

1.1.2. PROPOSITION. *Let G be a connected graph with interval function I . Then, for any u and v in V ,*

- (i) $u, v \in I(u, v)$;
- (ii) $I(u, v) = I(v, u)$;
- (iii) if $x \in I(u, v)$, then $I(u, x) \subseteq I(u, v)$;
- (iv) if $x \in I(u, v)$, then $I(u, x) \cap I(x, v) = \{x\}$;
- (v) if $x \in I(u, v)$ and $y \in I(u, x)$, then $x \in I(y, v)$.

1.1.3. PROPOSITION. *Let G be a connected graph with interval function I . For any three vertices u , v and w of G there exists a vertex z in $I(u, v) \cap I(u, w)$ such that*

$$I(z, v) \cap I(z, w) = \{z\}.$$

PROOF. Let z be a vertex in $I(u,v) \cap I(u,w)$ with $d(u,z)$ as large as possible. Then we have

$$I(z,v) \cap I(z,w) = \{z\}.$$

For, if not, then let x be a vertex in $I(z,v) \cap I(z,w)$ distinct from z . It follows from Proposition 1.1.2 (v) that z lies in $I(u,x)$. Hence we have

$$d(u,x) = d(u,z) + d(z,x) > d(u,z).$$

By Proposition 1.1.2 (i) and (iii), we have

$$x \in I(x,v) \cap I(x,w) \subseteq I(z,v) \cap I(z,w) \subseteq I(u,v) \cap I(u,w),$$

which contradicts the maximality of $d(u,z)$. \square

The vertex z in this proposition need not be unique as the graph in Figure 1.3 shows. Note that in this graph $d(u,z) \neq d(u,z')$. In Figure 1.4 we give an example where the vertex z is unique. These two figures are examples by which the reader can check the arguments in the next theorem.

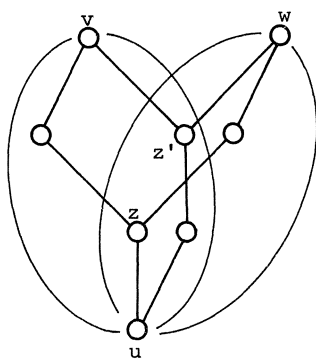


FIGURE 1.3.

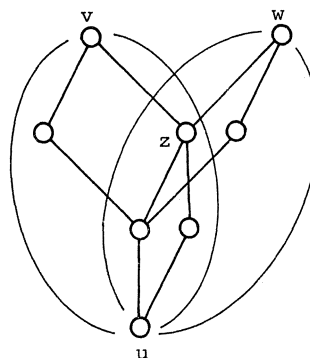


FIGURE 1.4.

1.1.4. THEOREM. Let G be a connected graph with interval function I , and let u, v, w and z be vertices of G . Then z is the only vertex in

$I(u,v) \cap I(u,w)$ such that

$$I(z,v) \cap I(z,w) = \{z\}$$

if and only if

$$I(u,v) \cap I(u,w) = I(u,z).$$

PROOF. First we prove the "if" part (see Figure 1.4). Let z be a vertex of G with $I(u,v) \cap I(u,w) = I(u,z)$. Then it follows that

$$z \in I(u,v) \cap I(u,w),$$

and so, by Proposition 1.1.2 (iv), we have

$$I(u,z) \cap I(z,v) = \{z\},$$

and

$$I(u,z) \cap I(z,w) = \{z\}.$$

Hence $I(z,v) \cap I(z,w) = \{z\}$.

Any shortest (u,z) -path forms the first part of a shortest (u,v) -path as well as the first part of a shortest (u,w) -path. So, for any x in $I(u,z)$ distinct from z , we have

$$z \in I(x,v) \cap I(x,w).$$

This shows the uniqueness of z .

Conversely (see Figure 1.3), assume that $I(z,v) \cap I(z,w) = \{z\}$, for some z in $I(u,v) \cap I(u,w)$, but

$$I(u,z) \subsetneq I(u,v) \cap I(u,w).$$

Choose a vertex x in $I(u,v) \cap I(u,w) \setminus I(u,z)$. Then any shortest (u,v) -path passing through x does not contain z . So

$$z \notin I(x,v) \cap I(x,w) \subset I(u,v) \cap I(u,w).$$

By Proposition 1.1.3, there exists a vertex y in $I(x,v) \cap I(x,w)$ with

$$I(y,v) \cap I(y,w) = \{y\}.$$

Then $y \neq z$, and so z is not unique. \square

1.1.5. DEFINITION. Let G be a graph with interval function I . A subset W of V is *convex* if $I(u,v) \subseteq W$, for any u and v in W .

A subgraph of a graph G induced by a convex subset of V is called a *convex subgraph* of G . Since the intersection of any two convex sets in G is convex, it makes sense to talk about the smallest convex set containing a subset U of V . This smallest convex set containing U is called the *convex closure* of U in G . The *convex closure* of a subgraph H of G is the subgraph of G induced by the convex closure of the vertex-set of H .

Note that in any graph, the empty set, any single vertex, any two adjacent vertices, and the whole vertex-set are always convex sets. Graphs in which these subsets are the only convex sets have been studied by BHASKARA RAO & RAO HEBBARE [BR], and RAO HEBBARE [Ra1], [Ra2].

As the graph in Figure 1.5 shows, an interval need not be convex. In this graph x and z are vertices in the interval $I(u,v)$, but $I(x,z) \not\subseteq I(u,v)$.

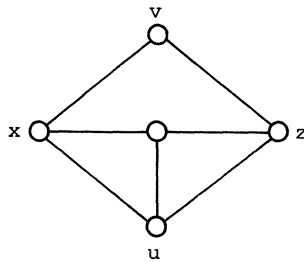


FIGURE 1.5.

1.1.6. DEFINITION. A graph G is *interval monotone* if each interval in G is convex.

Note that Q_n is interval monotone, as is any tree. In this monograph

we shall discuss several classes of interval monotone graphs.

An interesting problem is the characterization of interval monotone graphs. In particular we can look for characterizations that involve forbidden subgraphs. Two results of this kind are given in the next propositions.

1.1.7. PROPOSITION. *If G contains no induced subgraph homomorphic to $K_{2,3}$ or to the graph in Figure 1.5, then G is interval monotone.*

PROOF. Let u and v be vertices of G for which the interval $I(u,v)$ is not convex. Choose x and y in $I(u,v)$ with $I(x,y) \not\subseteq I(u,v)$, and such that $d(x,y)$ is as small as possible. Note that $d(x,y) \geq 2$, and that $x \neq u \neq y \neq v \neq x$.

Since $d(x,y)$ is as small as possible, there exists a shortest (x,y) -path $P = x \rightarrow p \rightarrow \dots \rightarrow y$ in G , which has only the vertices x and y in common with $I(u,v)$.

Let q be a neighbour of x in $I(x,u)$, and let $Q = q \rightarrow \dots \rightarrow y$ be a path from q to y in the subgraph of G induced by $I(q,u) \cup I(u,y)$ (see Figure 1.6; note that, if we require Q to be a shortest (q,y) -path in this subgraph, then u does not necessarily lie on Q). Similarly, let r be a neighbour of x in $I(x,v)$, and let $R = r \rightarrow \dots \rightarrow y$ be a path from r to y in the subgraph of G induced by $I(r,v) \cup I(v,y)$.

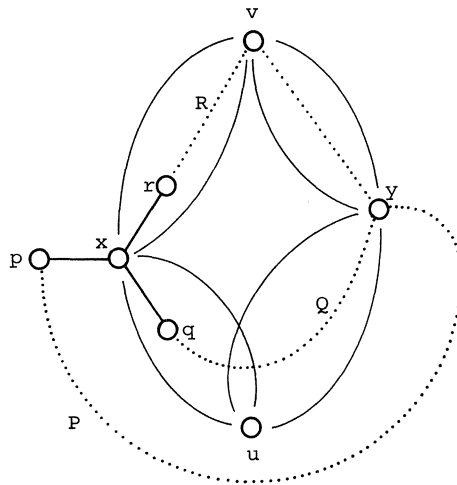


FIGURE 1.6.

Since p is not in $I(u,v)$, the vertex p is adjacent to at most one of the two vertices q and r .

Let H be the subgraph of G induced by the vertices of P , Q and R . In H we contract the edges in the subgraph H' of H induced by the set $V(H) \setminus \{x,p,q,r\}$, thus getting a graph \bar{H} with vertices x, p, q, r and s , where s is the homomorphic image of H' . In \bar{H} the vertex s is adjacent to p, q and r , but not to x . Hence it follows from the choice of x, p, q and r that \bar{H} is $K_{2,3}$, or the graph in Figure 1.5. \square

1.1.8. PROPOSITION. *If G contains no subgraph homeomorphic to $K_{2,3}$, then G is interval monotone.*

PROOF. Let the vertices u, v, x and y and the paths P, Q and R be as in the proof of the previous proposition.

If there existed a vertex w in $I(u,x) \cap I(y,v)$, then by Proposition 1.1.2 (v) we would have $x \in I(w,v) \subseteq I(y,v)$, and hence

$$I(x,y) \subseteq I(y,v) \subseteq I(u,v).$$

This contradicts the choice of x and y . So

$$I(u,x) \cap I(y,v) = \emptyset.$$

Similarly we have

$$I(u,y) \cap I(x,v) = \emptyset.$$

It follows that

$$[I(u,x) \cup I(u,y)] \cap [I(v,x) \cup I(v,y)] = \{x,y\},$$

so that the paths Q and R have only the vertex y in common. This implies that P, Q and R form a subgraph of G homeomorphic to $K_{2,3}$, which completes the proof. \square

Although the graph in Figure 1.5 contains $K_{2,3}$ as a subgraph, it does not contain an induced subgraph homeomorphic to $K_{2,3}$. Note that the

converse of this last proposition is not true. For example, Q_4 is interval monotone, but contains many induced subgraphs homeomorphic to $K_{2,3}$.

A consequence of Proposition 1.1.8 is that outerplanar graphs are interval monotone.

1.2. INTERVAL-REGULAR GRAPHS

In this monograph we discuss several classes of graphs of which Q_n is a member. One of these classes is that of interval-regular graphs, which can be considered as a broad generalization of the hypercubes. First we introduce some terminology that is useful in studying these graphs.

Let G be a graph, and let u and v be vertices of G . For $i = 0, \dots, d(u,v)$, we define

$$N_i(u,v) := \{w \in I(u,v) \mid d(u,w) = i\}.$$

The set $N_i(u,v)$ is called the i -th level in the interval $I(u,v)$. It follows immediately from the definition of $I(u,v)$ that

$$N_i(u,v) = N_{d(u,v)-i}(v,u).$$

Note that any edge in the subgraph of G induced by the interval $I(u,v)$ joins either two vertices in consecutive levels in the interval or two vertices within the same level. See Figure 1.8 for an example of how levels in an interval are indicated in a figure.

For $i \geq 0$, we define

$$N_i(u) := \{w \in V \mid d(u,w) = i\}.$$

The set $N_i(u)$ is called the i -th level of u in G . For $i = 1$, we write $N(u)$ instead of $N_1(u)$. Note that $N(u)$ is the set of neighbours of u .

1.2.1. DEFINITION. A connected graph G , with interval function I and distance function d , is *interval-regular* if

$$|I(u,v) \cap N(u)| = d(u,v) \text{ or } |I(u,v) \cap N(v)| = d(u,v),$$

for any two vertices u and v of G .

From the definition of Q_n it follows easily that Q_n is interval-regular. Note that an interval-regular graph need not be regular. For example, the graphs in Figure 1.7 are interval-regular but not regular. A connected regular graph is generally not interval-regular.

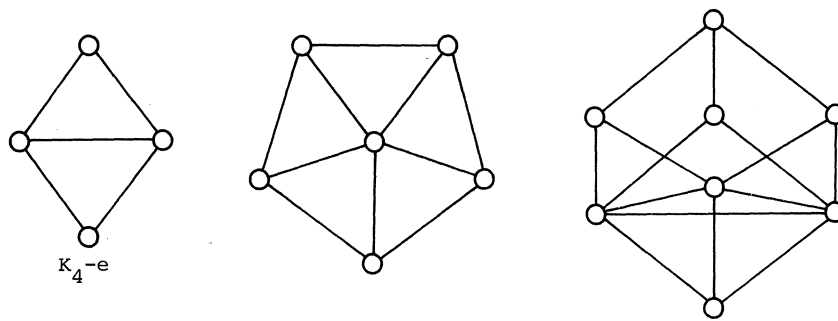


FIGURE 1.7.

The next proposition provides a useful tool for studying interval-regular graphs.

1.2.2. PROPOSITION. *Let G be an interval-regular graph with interval function I . Then*

$$|I(u,v) \cap N(u)| = d(u,v),$$

for any two vertices u and v of G .

PROOF. The proof is by induction on $d(u,v)$.

If $d(u,v) \leq 2$, the assertion is true by definition. So let $n = d(u,v) \geq 3$, and assume that

$$|I(u,v) \cap N(u)| = n.$$

We have to prove that $|I(u,v) \cap N(v)| = n$.

Let i be an integer such that $0 < i < d(u,v)$, and choose x in $N_i(u,v)$. Then by the induction hypothesis we have

$$|I(u,x) \cap N(x)| = i,$$

and

$$|I(x,v) \cap N(x)| = n-i.$$

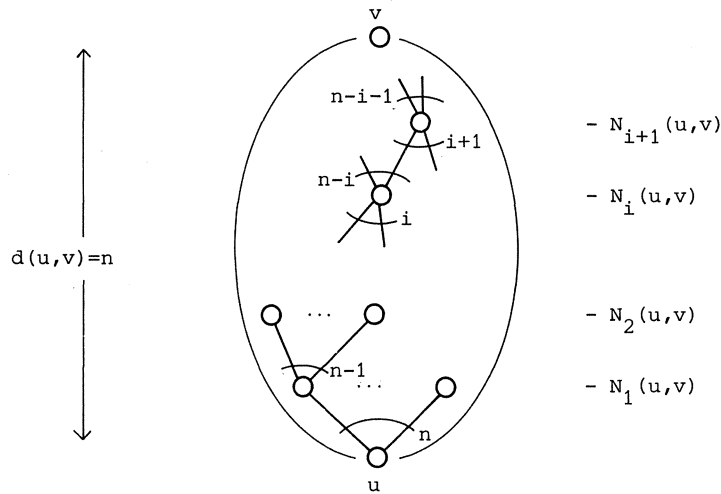


FIGURE 1.8.

Hence, counting the number of edges between $N_{i-1}(u,v)$ and $N_i(u,v)$ twice, we get

$$(n-i+1)|N_{i-1}(u,v)| = i|N_i(u,v)| \quad \text{for } i = 1, \dots, n-1.$$

Since $|N_0(u,v)| = 1$, it follows by induction on i that

$$|N_i(u,v)| = \binom{n}{i} \quad \text{for } i = 0, 1, \dots, n-1,$$

and thus $n = |N_{n-1}(u,v)| = |I(u,v) \cap N(v)|$. \square

We now show the connection between interval-regular graphs and hypercubes. Note that in Q_n with its vector representation the set F_i of edges joining vertices that differ in their i -th coordinate form a matching as

well as a cutset. This matching matches two $(n-1)$ -cubes in Q_n in such a way that adjacent vertices in the one $(n-1)$ -cube are matched to adjacent vertices in the other. By the "set of edges between levels in an interval" we mean the set of edges joining vertices in consecutive levels in the interval.

1.2.3. THEOREM. *Let G be a connected graph with interval function I . Then G is interval-regular if and only if, for any two vertices u and v of G , the subgraph induced by the set of edges between levels in the interval $I(u,v)$ is a hypercube of dimension $d(u,v)$.*

PROOF. If the edges between levels in an interval $I(u,v)$ induce a hypercube of dimension $d(u,v)$, then

$$|N(u) \cap I(u,v)| = d(u,v) = |I(u,v) \cap N(v)|,$$

and so G is interval-regular.

Conversely, let G be interval-regular. In the proof of Proposition 1.2.2, we showed that, for any two vertices u and v of G ,

$$|N_i(u,v)| = \binom{d(u,v)}{i} \quad \text{for } i = 0, 1, \dots, d(u,v),$$

so that $|I(u,v)| = 2^{d(u,v)}$.

The proof is by induction on $d(u,v)$. For $d(u,v) \leq 2$ the assertion is true by definition.

Let $n = d(u,v) \geq 3$, and let x be a neighbour of u in $I(u,v)$, so that $d(x,v) = n-1$. Then we have

$$|I(x,v) \cap N_{n-1}(u,v)| = |N_{n-2}(x,v)| = n-1.$$

Hence there is exactly one vertex y in $N_{n-1}(u,v) \setminus I(x,v)$.

If there were a vertex z in $I(u,y) \cap I(x,v)$, then, by Proposition 1.1.2 (v) and (iii), we would have $y \in I(z,v) \subseteq I(x,v)$. This contradicts the fact that y is not in $I(x,v)$, and so we have

$$I(u,y) \cap I(x,v) = \emptyset.$$

Since $d(u,y) = d(x,v) = n-1 = d(u,v)-1$, it follows that

$$|I(u,y)| + |I(x,v)| = 2^{n-1} + 2^{n-1} = 2^n = |I(u,v)|,$$

and so we have

$$I(u,v) = I(u,y) \cup I(x,v).$$

Let H' be the subgraph of G induced by the set of edges between levels in $I(u,y)$, and let H be the subgraph of G induced by the set of edges between levels in $I(x,v)$. Then by the induction hypothesis, both H and H' are $(n-1)$ -cubes. Let F be the set of edges between H and H' joining vertices in consecutive levels in $I(u,v)$. Then $E(H) \cup E(H') \cup F$ is the set of edges between levels in $I(u,v)$.

First we prove that F is a matching between H and H' . Let i be a positive integer with $i \leq n$, and let w be a vertex in $N_{i-1}(x,v) \subset N_i(u,v)$ (see Figure 1.9).

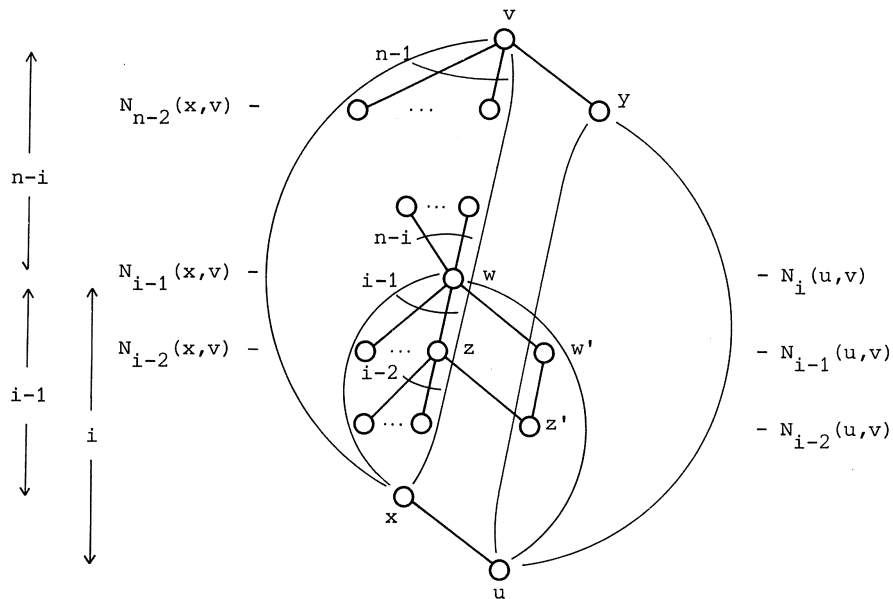


FIGURE 1.9.

Since y is not in $I(x,v)$, it follows that w cannot be adjacent to a vertex in $N_{i+1}(u,y) \subset N_{i+1}(u,v)$. Since $d(x,w) = i-1 = d(u,w)-1$, we have

$$|N_{i-2}(x,w)| = i-2 = |N_{i-1}(u,w)| - 1.$$

This implies that w has exactly one neighbour in $N_{i-1}(u,w) \setminus I(x,v)$, and so w is incident with exactly one edge of F . Similarly any vertex of H' is incident with exactly one edge of F , and so F is a matching between H and H' .

Finally we prove that F matches H and H' in the appropriate way. Let wz be an edge of H with w in $N_{i-1}(x,v)$ and z in $N_{i-2}(x,v)$, and let z' be the neighbour of z in H' with zz' in F . Then z' lies in $N_{i-2}(u,y) \subset N_{i-2}(u,v)$ (see Figure 1.9), and so

$$d(w,z') = 2.$$

Since G is interval-regular, it follows that w and z' have exactly one common neighbour w' distinct from z . Note that w' lies in $N_{i-1}(u,v)$. Since z' is a vertex of H' and F is a matching, it follows that w' is also a vertex of H' , so that ww' is the edge of F incident with w . This implies that adjacent vertices of H are matched by F to adjacent vertices of H' . By the observation concerning Q_n preceding this theorem, the proof is complete. \square

The idea of studying interval-regular graphs stems from a paper by FOLDES [F1], and the following two corollaries of Theorem 1.2.3 are due essentially to him. In [F1] he only considered the property

$$|I(u,v) \cap N(u)| = d(u,v) \quad \text{for all } u,v \in V,$$

which is equivalent to the notion of interval-regularity, as we have proved in Proposition 1.2.2.

1.2.4. COROLLARY. *Let G be a connected graph. Then the following assertions are equivalent:*

- (i) G is interval-regular;
- (ii) $|N_i(u,v)| = \binom{d(u,v)}{i}$, for $u,v \in V$ and $i = 0, \dots, d(u,v)$;

(iii) there are exactly $d(u,v)!$ shortest (u,v) -paths, for $u,v \in V$.

PROOF. It follows directly from Theorem 1.2.3 that (i) and (ii) are equivalent, and that (i) implies (iii). The proof that (iii) implies (i) is easily done by induction on $d(u,v)$ and is left to the reader. \square

1.2.5. COROLLARY. *Let G be a connected graph. Then G is a hypercube if and only if G is bipartite and interval-regular.*

PROOF. A hypercube is bipartite and interval-regular.

Conversely, let G be bipartite and interval-regular. First we prove that G is regular. Let u and v be two adjacent vertices in G . Since G is triangle-free, we have

$$N(u) \cap N(v) = \emptyset.$$

Let w be a neighbour of u distinct from v . Then it follows that $d(v,w) = 2$, and so, by the interval-regularity of G , w is adjacent to exactly one neighbour of v distinct from u . Similarly any neighbour z of v distinct from u is adjacent to exactly one vertex in $N(u) \setminus \{v\}$. This implies that there exists a matching between $N(u)$ and $N(v)$, so that $d(u) = d(v)$. Since G is connected, it follows that G is regular.

Let u and v be vertices of G with $d(u,v) = \text{diam}(G)$. Since G is bipartite, we have $N(u) \subset I(u,v)$, and it follows from Theorem 1.2.3 that the edges between levels in $I(u,v)$ induce a hypercube of dimension $|N(u)| = d(G)$. It follows from the regularity of G that G is this hypercube. \square

In Chapter 5 we discuss two classes of interval-regular graphs, in which each interval induces a bipartite subgraph. Here we give another result and an example, involving the product of two graphs.

1.2.6. PROPOSITION. *If G and H are interval-regular graphs, then the product graph $G \times H$ is interval-regular.*

PROOF. Let (u,v) and (u',v') be vertices of $G \times H$, where u and u' are vertices of G , and v and v' are vertices of H . Then

$$d_{G \times H}((u,v), (u',v')) = d_G(u,u') + d_H(v,v'),$$

where $d_{G \times H}$, d_G and d_H are the respective distance functions of $G \times H$, G and H . It is easily verified by induction on the distance between (u,v) and (u',v') that

$$I_{G \times H}((u,v), (u',v')) = I_G(u,u') \times I_H(v,v'),$$

where the right-hand side of the equality is the Cartesian product of the sets $I_G(u,u')$ and $I_H(v,v')$. From the definition of $G \times H$ it follows that

$$|I_{G \times H}((u,v), (u',v')) \cap N((u,v))| = d_G(u,u') + d_H(v,v'),$$

and so $G \times H$ is interval-regular. \square

Let n and m be integers with $n, m \geq 2$. Since $K_{1,1,m}$ is interval-regular, as the reader can easily check, it follows from Proposition 1.2.6 that the graph $G = K_{1,1,m} \times K_{n-2}$ is interval-regular. Note that G is non-bipartite and has diameter n (see Figure 1.10).

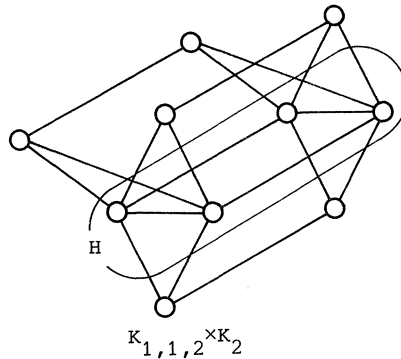


FIGURE 1.10.

The graph G can serve as a counterexample to many conjectures concerning interval-regular graphs. For example, G contains a subgraph H isomorphic to Q_{n-1} . This subgraph H has 2^{n-1} vertices and has the property that $G-H$ consists of m disjoint $(n-2)$ -cubes. For $m > 2^{n-1}$, it follows that

G is not a Hamiltonian graph.

We conclude this section by stating an unsolved problem on interval-regular graphs. The following conjecture is suggested by the fact that all interval-regular graphs known up to now are interval monotone. If the conjecture were true, it would make the handling of interval-regular graphs much easier.

1.2.7. CONJECTURE. *An interval-regular graph is interval monotone.*

1.3. THE DIAGRAPH OF A FINITE LATTICE

For definitions and properties of partially ordered sets and lattices, the reader is referred to Chapter 0 and to BIRKHOFF [Bi].

Let $P = (V, \leq)$ be a finite partially ordered set (poset, for short). The *diagraph* $G(P)$ of P is the graph with V as vertex-set, in which two vertices are joined by an edge whenever one of the two vertices covers the other in the poset. So the diagraph of a poset is the unoriented Hasse diagram of the poset. Various other terms are used in the literature - for instance, covering graph in [DR], and undirected Hasse diagram in [F1].

Note that Q_n is the diagraph of the Boolean lattice on 2^n elements. Figure 1.11 gives the (oriented) Hasse diagrams of the two posets with the circuit of length 4 as diagraph, and Figure 1.12 those with the circuit of length 5 as diagraph.

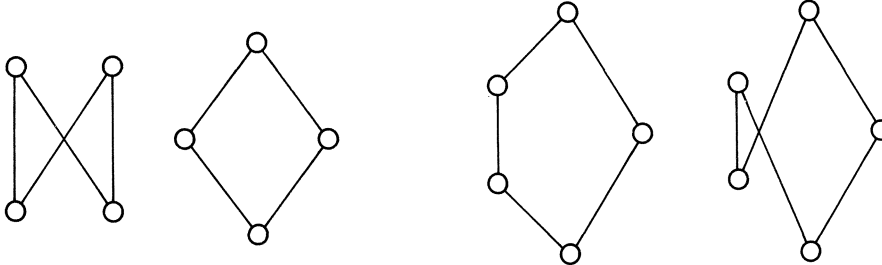


FIGURE 1.11.

FIGURE 1.12.

The characterization of those graphs that are diagraphs of some poset is still an open problem (cf. [DR]). A diagraph by definition must be

triangle-free, but this condition is not sufficient. The smallest triangle-free graph that is not a diagraph is depicted in Figure 1.13 (it was mentioned to me by RIVAL [Ri]).

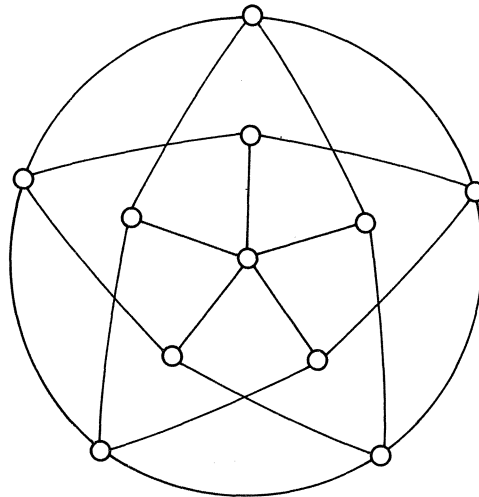


FIGURE 1.13.

A *graded poset* $P = (V, \leq)$ is a poset with *height function* $h: V \rightarrow \mathbb{Z}$ such that

- (H1) if $u < v$, then $h(u) < h(v)$,
- (H2) if v covers u , then $h(v) = h(u) + 1$.

The integer $h(v)$ is called the *height* of v .

1.3.1. PROPOSITION. *A graph G is connected and bipartite if and only if G is the diagraph of a finite graded poset with universal lower bound.*

PROOF. If G is the diagraph of a finite graded poset with universal lower bound x , then G admits a bipartition, where one set consists of those vertices with even height and the other set consists of those vertices

with odd height. Furthermore, there exists a path from x to any other vertex of G , and so G is connected.

Conversely, let G be bipartite and connected, and let u be a vertex of G . For $i = 0, 1, \dots$, we direct the edges between $N_i(u)$ and $N_{i+1}(u)$ from $N_{i+1}(u)$ to $N_i(u)$. Defining $v \leq_u w$ whenever there exists a directed path from w to v , gives a poset (V, \leq_u) , of which G is the diagram. This poset is graded with the function $h_u: V \rightarrow \mathbb{N} \cup \{0\}$ defined by

$$h_u(v) = d(u, v) \quad \text{for } v \in V;$$

- that is, $h_u(v) = i$, for any vertex v in $N_i(u)$. Since G is connected, we have

$$u \leq_u v \quad \text{for } v \in V,$$

and so u is the universal lower bound. \square

The ordering \leq_u on V constructed in the above proof is called the *canonical ordering* of G with respect to u . In Figure 1.14 we give an example of a graph and one of its canonical orderings.

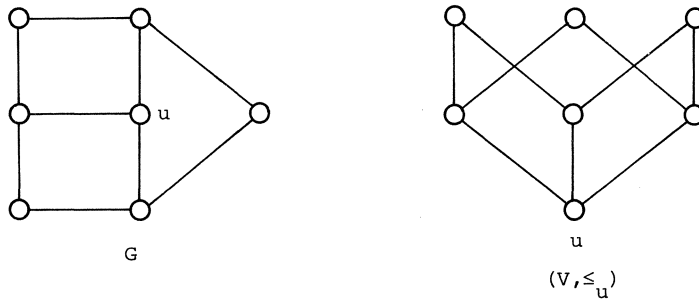


FIGURE 1.14.

In the rest of this section we use the interval function to characterize diagrams of classes of finite lattices. The interval function seems to be useful only when we consider lattices which are graded.

1.3.2. DEFINITION. Let G be a connected graph. Two vertices u and v of G are *diametrical* if $d(u,v) = \text{diam}(G)$.

If u and v are diametrical vertices in a graph, then u is said to be a *diametrical vertex* of v and vice versa.

1.3.3. PROPOSITION. Let G be a connected graph with interval function I . Then G is the diagraph of a finite graded lattice if and only if G is bipartite and contains two diametrical vertices 0 and 1 such that

- (i) $I(0,1) = V$,
- (ii) for any $u, v \in V$ there exists a vertex $z = z(u,v)$ of G such that $I(0,u) \cap I(0,v) = I(0,z)$.

PROOF. Let $P = (V, \leq)$ be a finite graded lattice with 0 as universal lower bound, 1 as universal upper bound, and G as its diagraph. Since P is graded, it follows that $V = I(0,1)$. Furthermore, the interval $I(0,w)$ is the set of elements u in V such that $u \leq w$. So for any u and v , we have

$$I(0,u) \cap I(0,v) = I(0,z),$$

where z is the greatest lower bound of u and v .

Conversely, let G be a bipartite graph satisfying (i) and (ii). Define

$$u \leq_0 v \quad \text{if} \quad u \in I(0,v).$$

Note that this ordering is the canonical ordering of G with respect to 0 . Hence G is the diagraph of a graded poset P with height function h_0 defined by

$$h_0(u) = d(0,u) = \text{diam}(G) - d(u,1).$$

In this poset, 0 is the universal lower bound and 1 is the universal upper bound. Furthermore, it follows from Theorem 1.1.4 that the greatest lower bound of u and v in P is the unique vertex z in G with

$I(0,u) \cap I(0,v) = I(0,z)$. Since V is finite, it follows that P is a lattice graded with height function h_0 . \square

In [A1] ALVAREZ has given characterizations of the diagraphs of finite modular and distributive lattices. In these characterizations the following conditions occur.

- (α) G contains two diametrical vertices x_0 and x_1 such that if two distinct vertices u and v in $N_i(x_j)$ have a common neighbour in $N_{i-1}(x_j)$, then u and v have a unique common neighbour in $N_{i+1}(x_j)$, for $j = 0,1$, and $i = 1, \dots, \text{diam}(G)$.
- (β) If the graph in Figure 1.15 is a subgraph of G , then there exists a vertex u_8 in G such that the graph in Figure 1.16 is a subgraph of G .

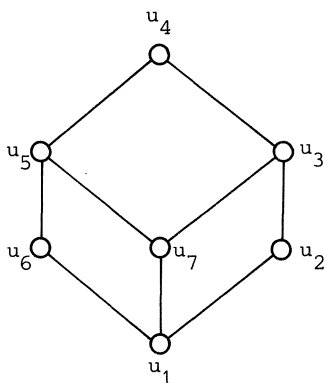


FIGURE 1.15.

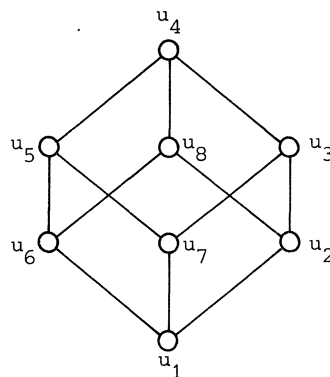


FIGURE 1.16.

In Figure 1.17 we exhibit a graph that satisfies (α), but not (β). Any even circuit of length at least six satisfies (β), but not (α).

1.3.4. THEOREM. (ALVAREZ, 1965). *Let G be a connected graph. Then G is the diagraph of a finite modular lattice if and only if G is bipartite and satisfies conditions (α) and (β).*

1.3.5. THEOREM. (ALVAREZ, 1965). *Let G be a connected graph. Then G is the diagraph of a finite distributive lattice if and only if G is*

bipartite, does not contain $K_{2,3}$ as a subgraph, and satisfies conditions (α) and (β) .

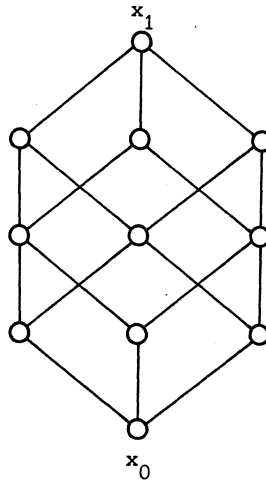


FIGURE 1.17.

In the next two theorems we replace condition (β) by the following weaker condition (γ) .

- (γ) If the graph in Figure 1.15 is a subgraph of G and if u_1, u_2, u_3 and u_4 are in four consecutive levels of $I(x_0, x_1)$, then the graph in Figure 1.16 is a subgraph of G .

It is easy to prove that (α) and (γ) imply (β) for connected bipartite graphs. The advantage gained is not so much that the theorems are slightly stronger, but that we can use the interval function, by which we can simplify the proofs.

1.3.6. THEOREM. *Let G be a connected graph. Then G is the diagraph of a finite modular lattice if and only if G is bipartite and satisfies conditions (α) and (γ) .*

PROOF. Let G be a diagraph of a finite modular lattice with x_0 as universal lower bound and x_1 as universal upper bound. Since a modular lattice is graded, it follows from Proposition 1.3.1 that G is bipartite. Moreover, condition (α) is the graph-theoretic

translation of the two covering conditions characterizing finite modular lattices (see Section 0.2). Condition (γ) follows from the modularity of the lattice and from condition (α) .

Conversely, let G be a connected bipartite graph satisfying conditions (α) and (γ) . Denote x_0 by 0 and x_1 by 1.

First let us prove that $V = I(0,1)$. Assume the contrary. A vertex v in $N_i(0) \setminus I(0,1)$ cannot be adjacent to a vertex w in $N_{i+1}(0,1)$, since otherwise we would have $v \in I(0,w) \subseteq I(0,1)$. Since G is bipartite, v cannot be adjacent to a vertex in $N_i(0)$. Hence if v is joined to a vertex w in $I(0,1)$, then w lies in $N_{i-1}(0,1)$.

Choose a vertex v in $N_i(0) \setminus I(0,1)$ joined to a vertex w in $N_{i-1}(0,1)$, such that i is as large as possible. Note that the existence of v and w is guaranteed by the connectedness of G . Let u be a neighbour of w in $I(w,1)$. Then u lies in $N_i(0,1)$. It follows from condition (α) that there is a unique vertex x in $N_{i+1}(0)$ that is adjacent to both u and v . Since u lies in $N_i(0,1)$ and x lies in $N_{i+1}(0)$, it follows from the maximality of i that x lies in $N_{i+1}(0,1) \subseteq I(0,1)$. Hence v lies in $I(w,x) \subseteq I(0,1)$, which gives the required contradiction. So we have proved that

$$V = I(0,1).$$

Let \leq_0 be the canonical ordering of G with respect to 0. Then (V, \leq_0) is a poset with 0 as universal lower bound and 1 as universal upper bound. From (α) we deduce the two covering conditions that characterize modularity in finite lattices. The proof that (V, \leq_0) is a lattice consists of two steps and involves Proposition 1.3.3.

STEP 1. We shall prove that, for any neighbour u of 0, the subgraph H of G induced by the interval $I(u,1)$ satisfies conditions (α) and (γ) . That H satisfies (γ) is easily verified and is left to the reader.

For any vertex v in $I(u,1)$ we have $I(v,1) \subseteq I(u,1)$. Hence, if two distinct vertices w and w' in $N_i(u,1)$ have a common neighbour v in $N_{i-1}(u,1)$, then the unique common neighbour of w and w' in $N_{i+2}(0) = N_{i+2}(0,1)$ also lies in $I(u,1)$.

Assume that H does not satisfy condition (α) . Let d and e be two distinct vertices in $N_i(u,1)$ with a common neighbour f in $N_{i+1}(u,1)$, and such that their (unique) common neighbour b_1 in $N_i(0,1)$ is not a vertex

of H . We may choose d and e such that i is as small as possible. The situation is now as in Figure 1.18, where the shortest paths P and Q possibly have internal vertices in common. Let P be the path $c_1 \rightarrow c_2 \rightarrow \dots \rightarrow c_i = u$. Note that $i \geq 2$.

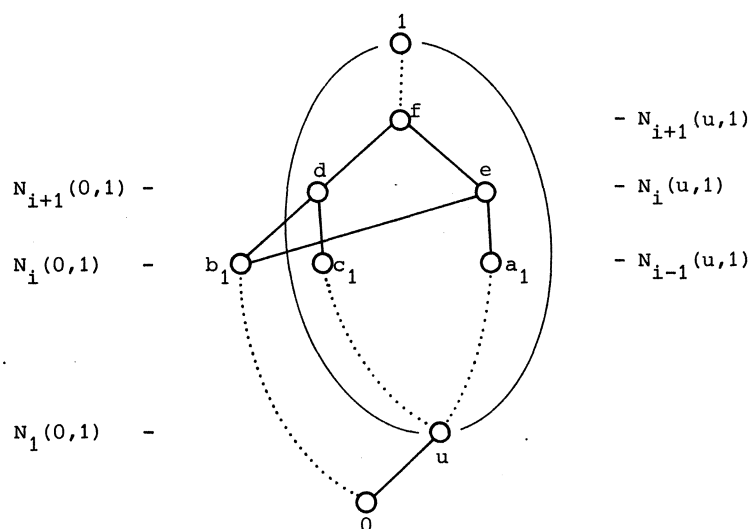


FIGURE 1.18.

Since G satisfies condition (α), the vertices a_1 and b_1 must have a common neighbour a_2 in $N_{i-1}(0,1)$. Note that a_2 cannot lie in $I(u,1)$, since b_1 is not in $I(u,1)$. Similarly b_1 and c_1 have a common neighbour b_2 in $N_{i-1}(0,1) \setminus I(u,1)$.

If $a_2 = b_2$, we get the situation in Figure 1.19. From condition (γ) we deduce the existence of a vertex c_0 , which is adjacent to a_1 , c_1 and f . Hence a_1 and c_1 have a common neighbour c_0 in $N_i(u,1)$ and a common neighbour b_2 in $N_{i-1}(0,1) \setminus I(u,1)$. This contradicts the minimality of i . The situation is therefore as depicted in Figure 1.20, where the paths R and R' possibly have internal vertices in common.

Let a_3 be the common neighbour of a_2 and b_2 in $N_{i-2}(0,1)$, and let b_3 be the common neighbour of b_2 and c_2 in $N_{i-2}(0,1)$. Because of the minimality of i , it follows that a_3 and b_3 are not in $I(u,1)$. If $a_3 = b_3$, then condition (γ) implies the existence of a vertex d_1 joined to a_2 , c_2 and d . So d_1

must be in $I(u,1)$. Again this contradicts the minimality of i . Hence $a_3 \neq b_3$.

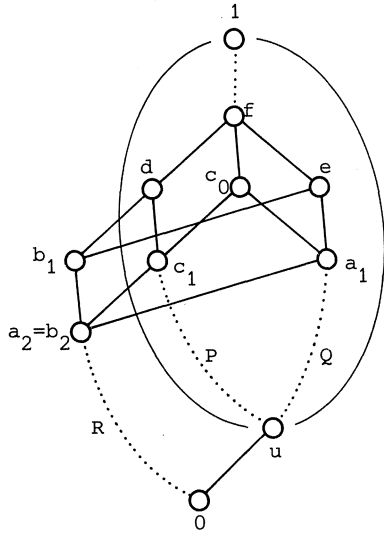


FIGURE 1.19.

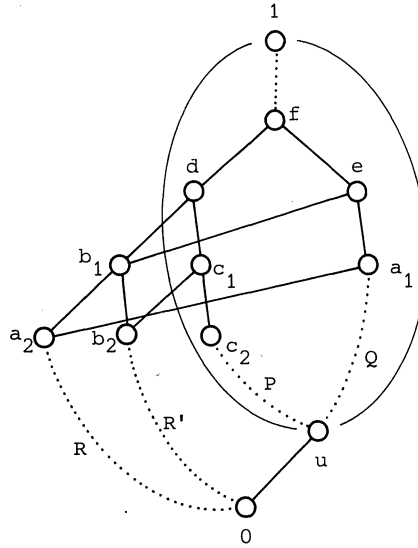


FIGURE 1.20.

Repeating this procedure, we find vertices a_{i-1} , a_i , b_{i-1} and b_i as in Figure 1.21, where a_i and b_i are in $N(0)$. Condition (γ) implies the existence of a vertex d_{i-1} adjacent to a_i , c_i and c_{i-2} , so that d_{i-1} lies in $I(u, c_{i-2}) \subseteq I(u,1)$. Since $i \geq 2$, this again contradicts the minimality of i . (See next page for Figure 1.21.)

We conclude that H also satisfies condition (α) . By using the same argument, we can also deduce that, for any two vertices v and w on the same shortest $(0,1)$ -path, the subgraph of G induced by the interval $I(v,w)$ satisfies conditions (α) and (γ) .

STEP 2. We shall show that, for any two vertices u and v of G , there exists a vertex z such that

$$I(0,u) \cap I(0,v) = I(0,z).$$

We prove this assertion by induction on $d(0,1)$. Let u and v be two vertices of G . If $I(0,u) \cap I(0,v) = \{0\}$, we take $z = 0$. If u lies in

$I(0,v)$, we take $z = u$, and if v lies in $I(0,u)$, we take $z = v$.

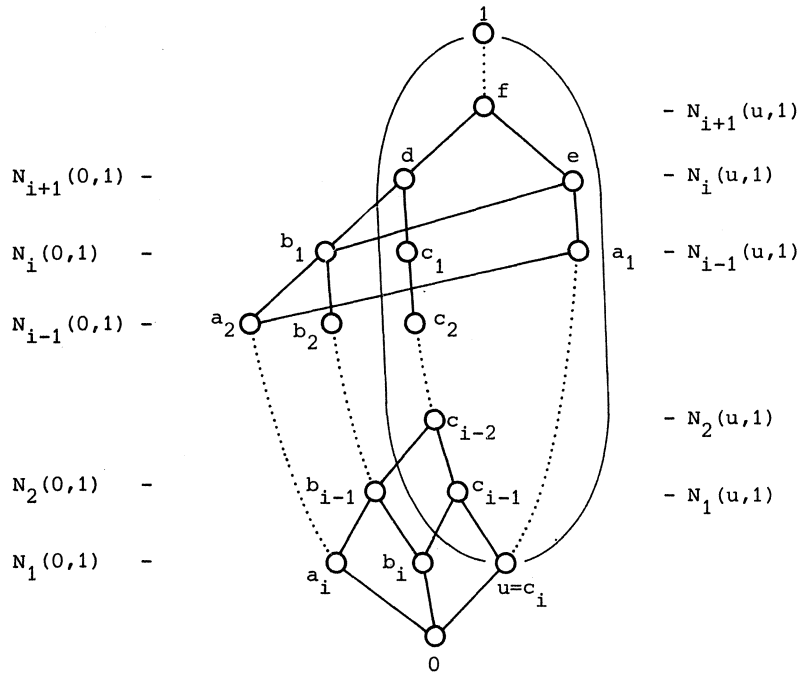


FIGURE 1.21.

In the remaining case let x be a neighbour of 0 in $I(0,u) \cap I(0,v)$. Note that u and v are in $I(x,1)$. By the induction hypothesis, it follows from the previous step of the proof that $(I(x,1), \leq'_0)$ is a lattice, where \leq'_0 is the restriction of the ordering \leq_0 on the interval $I(x,1)$. Hence there exists a vertex z_x in $I(x,1)$ such that

$$I(x,u) \cap I(x,v) = I(x,z_x).$$

If x is the only neighbour of 0 in $I(0,u) \cap I(0,v)$, then we have

$$I(0,u) \cap I(0,v) = \{0\} \cup I(x,z_x) = I(0,z_x),$$

and we are done.

Otherwise, let y be a neighbour of 0 in $I(0,u) \cap I(0,v)$ distinct

from x . Using a similar argument as above, we deduce that there exists a vertex z_y in $I(y,1)$ such that

$$I(y,u) \cap I(y,v) = I(y,z_y).$$

Let p be the common neighbour of x and y in $N_2(0,1)$. It follows from the previous step of the proof that p is in $I(0,u)$ as well as in $I(0,v)$. By the induction hypothesis we have

$$I(p,u) \cap I(p,v) = I(p,z_p) \subset I(0,u) \cap I(0,v),$$

for some vertex z_p of G . Since p lies in $I(x,1)$, it follows from the fact that $(I(x,1), \leq'_0)$ is a lattice that $z_p = z_x$ (see Figure 1.22). Similarly, we have $z_p = z_y$.

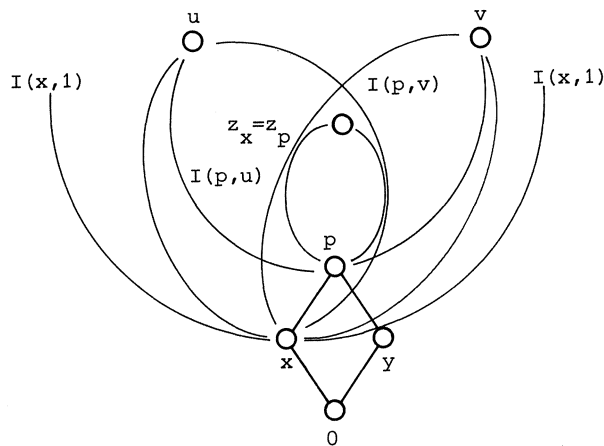


FIGURE 1.22.

We have therefore proved that there is a vertex z in G such that, for any neighbour x of 0 in $I(0,u) \cap I(0,v)$,

$$I(x,u) \cap I(x,v) = I(x,z).$$

It follows from the definition of an interval that, for this vertex z , we have

$$\begin{aligned}
I(0,u) \cap I(0,v) &= \{0\} \cup \bigcup_{x \in N_1(0,u) \cap N_1(0,v)} I(x,u) \cap I(x,v) \\
&= \{0\} \cup \bigcup_{x \in N_1(0,u) \cap N_1(0,v)} I(x,z) \\
&= I(0,z).
\end{aligned}$$

This establishes Step 2. Hence it follows from Proposition 1.3.3 that (V, \leq_0) is a lattice, by which the proof is complete. \square

The next theorem is the analogue of Theorem 1.3.5. Its proof follows easily from Theorem 1.3.6.

1.3.7. THEOREM. *Let G be a connected graph. Then G is the diagraph of a finite distributive lattice if and only if G is bipartite, does not contain $K_{2,3}$ as a subgraph, and satisfies conditions (α) and (γ) .*

1.3.8. DEFINITION. A connected graph G is *diametrical* if each vertex of G has a unique diametrical vertex.

A diametrical graph need not be regular, or even bipartite, as the graphs in Figure 1.23 show.

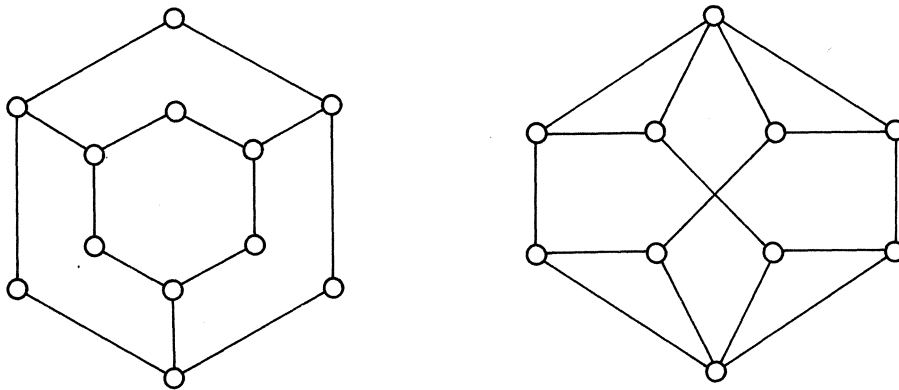


FIGURE 1.23.

We conclude this chapter by stating a theorem which we can prove easily by using two results from Chapter 3 (Theorem 3.3.3 and Corollary 3.4.3). The proof of the theorem is therefore postponed to the appropriate place in Chapter 3.

1.3.9. THEOREM. *Let G be a connected graph with $\text{diam}(G) = n$. Then G is the n -cube if and only if G is diametrical and bipartite, does not contain $K_{2,3}$ as a subgraph, and satisfies conditions (α) and (γ) .*

CHAPTER 2

 $(0, \lambda)$ -GRAPHS

In this chapter we generalize the property of a hypercube that any two vertices have exactly two common neighbours or none at all. This leads to the more general class of $(0, \lambda)$ -graphs. These graphs can be considered as a generalization of symmetric block designs. It turns out that the extremal $(0, \lambda)$ -graphs are the hypercubes, the Clebsch graph, and the newly-constructed Hadamard graphs.

2.1. $(0, \lambda)$ -GRAPHS

In the n -cube Q_n any two distinct vertices have exactly two common neighbours or none at all.

Another class of graphs having this property (with one exception) consists of the halfcubes. The *halfcube* $\frac{1}{2}Q_{2n}$ is constructed from Q_{2n} , as follows: give Q_{2n} the subset representation; take the subgraph of Q_{2n} induced by $\{A \subset \{1, 2, \dots, 2n\} \mid |A| \leq n\}$, and identify complementary vertices in the n -th level of \emptyset .

The halfcube $\frac{1}{2}Q_2$ is the complete graph K_2 , and $\frac{1}{2}Q_4$ is the complete bipartite graph $K_{4,4}$. With the exception of $\frac{1}{2}Q_4$, the halfcubes have the above property that any two distinct vertices have exactly two common neighbours or none at all. For $n \geq 2$, the halfcube $\frac{1}{2}Q_{2n}$ has 2^{2n-1} vertices and $n2^{2n-1}$ edges. It has degree $2n$ and diameter n and is distance-transitive. For any two vertices u and v with $d(u, v) = n$, the interval $I(u, v)$ induces a hypercube of dimension $d(u, v)$. For any two diametrical vertices u and \bar{u} , there are exactly $(2n)!$ shortest (u, \bar{u}) -paths (cf. Corollary 1.2.4). Furthermore, $\frac{1}{2}Q_{2n}$ has Q_{2n-2} as an induced subgraph and Q_{2n-1} as a spanning subgraph.

Let us study a more general class, to which both of these graphs

belong.

2.1.1. DEFINITION. Let λ be an integer with $\lambda \geq 2$. A connected graph G is a $(0, \lambda)$ -graph if any two distinct vertices in G have exactly λ common neighbours or none at all.

Examples of $(0, \lambda)$ -graphs are the complete graphs K_1 , K_2 and $K_{\lambda+2}$, the complete bipartite graph $K_{\lambda, \lambda}$, and the graph $K_{\lambda+2, \lambda+2}$ minus a perfect matching. Another example is depicted in Figure 2.1. We give more examples later in this section and in the last two sections of this Chapter.

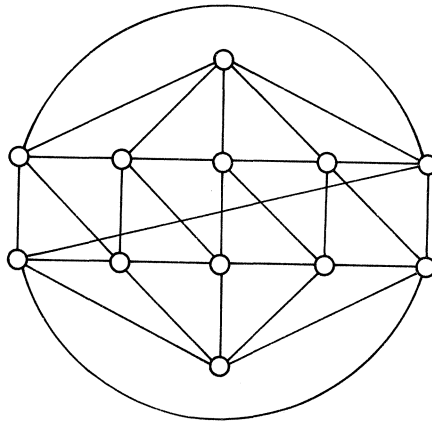


FIGURE 2.1.

In the above definition we exclude the "degenerate" case $\lambda = 1$, for in this case, the corresponding condition is that G does not contain a circuit of length four. If $\lambda \geq 2$, then much more can be proved.

2.1.2. PROPOSITION. Let G be a $(0, \lambda)$ -graph. Then G is regular.

PROOF. Let u and v be two neighbours in G . Colour all neighbours of u blue, except v . Colour all neighbours of v red, except u . Note that the common neighbours of u and v are coloured red as well as blue.

Choose an arbitrary blue vertex, w say. Then w and v have u as common neighbour. So w is adjacent to exactly $\lambda - 1$ red vertices. Similarly

any red vertex is adjacent to exactly $\lambda-1$ blue vertices. This implies that there are as many red vertices as blue vertices, and so $d(u) = d(v)$. Since G is connected, it follows that G is regular. \square

We denote the degree of each vertex of G by $d(G)$ (see Section 0.1).

2.1.3. PROPOSITION. *Let G be a $(0, \lambda)$ -graph with interval function I . Then*

$$|I(u, v) \cap N(u)| \geq d(u, v) + \lambda - 2,$$

for any two vertices u and v of G with $d(u, v) \geq 2$.

PROOF. The proof is by induction on $d(u, v)$.

If $d(u, v) = 2$, then

$$I(u, v) \cap N(u) = N(u) \cap N(v) \neq \emptyset.$$

So $|I(u, v) \cap N(u)| = \lambda$.

Let $d(u, v) \geq 3$, and let w be a neighbour of u in $I(u, v)$. Then $d(w, v) = d(u, v) - 1$, and so

$$|I(w, v) \cap N(w)| \geq d(u, v) + \lambda - 3.$$

Colour the vertices of $I(w, v) \cap N(w)$ red, and colour the vertices of $I(u, v) \cap N(u) \setminus \{w\}$ blue. Any red vertex and u have w as common neighbour, so red vertices are adjacent to exactly $\lambda-1$ blue vertices. Any blue vertex and w have u as common neighbour, so blue vertices are joined to at most $\lambda-1$ red vertices (and possibly to some neighbours of w outside $I(u, v)$). Hence there are at least as many blue vertices as red vertices, and so

$$|I(u, v) \cap N(u)| \geq 1 + |I(w, v) \cap N(w)| \geq d(u, v) + \lambda - 2.$$

\square

2.1.4. PROPOSITION. *Let G be a $(0, \lambda)$ -graph. If $\text{diam}(G) \geq 4$, then*

$$d(G) \geq \text{diam}(G) + 2\lambda - 4.$$

PROOF. Let u and v be two diametrical vertices of G , and let w be a vertex in the interval $I(u,v)$ such that

$$d(u,w) \geq 2, \text{ and } d(w,v) \geq 2.$$

Then, since $I(u,w) \cap I(w,v) = \{w\}$, it follows from Proposition 2.1.3 that

$$\begin{aligned} d(G) &= |N(w)| \geq |I(u,w) \cap N(w)| + |I(w,v) \cap N(w)| \\ &\geq (d(u,w) + \lambda - 2) + (d(w,v) + \lambda - 2) = d(u,v) + 2\lambda - 4 \\ &= \text{diam}(G) + 2\lambda - 4. \end{aligned}$$

□

In the next sections we discuss $(0,\lambda)$ -graphs G with $d(G) = \text{diam}(G) + 2\lambda - 4$. The condition " $\text{diam}(G) \geq 4$ " in the last proposition is necessary as we shall see below. First we deduce some bounds for the number of vertices of a $(0,\lambda)$ -graph. In the proof of the next theorem we use the "counting twice principle" (see Section 0.1).

2.1.5. THEOREM. *Let G be a $(0,\lambda)$ -graph of degree d . Then*

$$1 + \frac{d(d-1)}{\lambda} \leq |V| \leq 1 + d + \frac{(\lambda-1)!(d-\lambda)!}{(d-2)!} \sum_{i \geq 0} \binom{d}{\lambda+i}.$$

PROOF. Let u be a vertex of G . By definition $|N_1(u)| = d$. The degree of any vertex in the subgraph induced by $N_1(u)$ is 0 or λ . So any vertex in $N_1(u)$ has $d-1$ or $d-\lambda-1$ neighbours in $N_2(u)$. Hence, by counting the edges between $N_1(u)$ and $N_2(u)$, we get

$$(d-\lambda-1)|N_1(u)| \leq \lambda|N_2(u)| \leq (d-1)|N_1(u)|;$$

-that is,

$$(1) \quad \frac{d(d-\lambda-1)}{\lambda} \leq |N_2(u)| \leq \frac{d(d-1)}{\lambda}.$$

So

$$\begin{aligned} |V| &\geq |N_2(u)| + |N_1(u)| + |N_0(u)| \geq \frac{d(d-\lambda-1)}{\lambda} + d + 1 \\ &= 1 + \frac{d(d-1)}{\lambda}. \end{aligned}$$

By counting the edges between $N_i(u)$ and $N_{i-1}(u)$, we get from Proposition 2.1.3

$$(d-\lambda-i+3) |N_{i-1}(u)| \geq (\lambda+i-2) |N_i(u)|, \quad \text{for } i \geq 3.$$

It follows from (1) and the last inequality that, if $i \geq 3$, then

$$\begin{aligned} |N_i(u)| &\leq \frac{d(d-1)}{\lambda} \cdot \frac{d-\lambda}{\lambda+1} \cdot \dots \cdot \frac{d-\lambda-i+3}{\lambda+i-2} \\ &= \frac{(\lambda-1)!}{(d-\lambda+1)\dots(d-2)} \binom{d}{\lambda+i-2}. \end{aligned}$$

Since $|V| = \sum_{i \geq 0} |N_i(u)|$, the right-hand inequality of the theorem follows after some computation. \square

2.1.6. COROLLARY. *Let G be a $(0,2)$ -graph of degree d . Then*

$$1 + \binom{d}{2} \leq |V| \leq 2^d.$$

PROOF. Put $\lambda = 2$ in Theorem 2.1.5. \square

The next two corollaries can be deduced from the proof of the last theorem. But they can also be proved directly, as the reader can easily check.

2.1.7. COROLLARY. *Let G be a $(0,\lambda)$ -graph. Then $d(G) = \lambda$ if and only if G is $K_{\lambda,\lambda}$.*

2.1.8. COROLLARY. *Let G be a $(0,\lambda)$ -graph. Then $d(G) = \lambda+1$ if and only if G is $K_{\lambda+2,\lambda+2}$ minus a perfect matching, or $K_{\lambda+2}$.*

For bipartite graphs the lower bound can be sharpened as follows; the proof is left to the reader.

2.1.9. PROPOSITION. *Let G be a bipartite $(0, \lambda)$ -graph of degree d . Then*

$$2 + 2 \frac{d(d-1)}{\lambda} \leq |V|.$$

In the next section, we study the $(0, \lambda)$ -graphs that attain the upper bound. Before doing this, let us consider $(0, \lambda)$ -graphs with the minimum number of vertices.

For the definition and the properties of block designs used in the sequel, the reader is referred to Section 0.3 and to [CL] and [R1].

Let $\lambda \geq 2$. Let G be a graph in which any two distinct vertices have exactly λ common neighbours. Then it follows from the definition of a symmetric block design that the adjacency matrix of G is the incidence matrix of a symmetric block design with parameters $(|V|, d(G), \lambda)$. Conversely, let D be a symmetric block design with parameters (v, d, λ) and with symmetric incidence matrix with zero diagonal. This incidence matrix is the adjacency matrix of a graph, in which any two distinct vertices have exactly λ common neighbours. This fact also follows immediately from the definitions, as the reader can easily check.

2.1.10. PROPOSITION. *Let G be a $(0, \lambda)$ -graph of degree d . Then*

$$|V| = 1 + \frac{d(d-1)}{\lambda}$$

if and only if G corresponds to a symmetric block design with parameters $(|V|, d, \lambda)$, which has a symmetric incidence matrix with zero diagonal.

PROOF. If G is a $(0, \lambda)$ -graph of degree d with $|V| = 1 + \frac{d(d-1)}{\lambda}$, then it follows from the proof of Theorem 2.1.5 that any two distinct vertices in G have exactly λ common neighbours.

The assertion now follows from the remarks preceding this proposition. □

REMARK. Graphs satisfying the conditions in the above proposition are strongly regular graphs with parameters $(|V|, d, \lambda, \lambda)$ (cf. [CL] and [Se]).

The *incidence graph* $G(D)$ of a block design D , with point-set X and set of blocks \mathcal{B} , is defined as follows: $X \cup \mathcal{B}$ is the vertex-set of $G(D)$, and there is an edge between p in X and B in \mathcal{B} whenever p is an element of B .

2.1.11. PROPOSITION. *Let G be a regular graph of degree d . Then the following assertions are equivalent:*

- (i) G is a bipartite $(0, \lambda)$ -graph with $\text{diam}(G) \leq 3$;
- (ii) G is a bipartite $(0, \lambda)$ -graph with $|V| = 2 + 2 \frac{d(d-1)}{\lambda}$;
- (iii) G is the incidence graph of a symmetric block design with parameters $(\frac{1}{2}|V|, d, \lambda)$.

PROOF. The proof is straightforward and is left to the reader. \square

Let G be a $(0, \lambda)$ -graph of degree $d \geq \lambda$ with

$$|V| = 2 + \frac{d(d-1)}{\lambda}.$$

If $\lambda = 2$, then for any vertex u of G , the set $N(u)$ induces a subgraph of G consisting of the disjoint union of circuits, none of which is a circuit of length 4.

Since the adjacency matrix A of G is a symmetric $(0, 1)$ -matrix, it follows that

$$A^2 = A^T A = (d - \lambda)I + \lambda J,$$

where I is the identity matrix of order $|V|$ and J is the all-one matrix of order $|V|$. Using eigenvalue techniques, RYSER [R2] has given a nice one-page proof that the above matrix equation implies the existence of a positive integer k such that

$$d = k^2 + \lambda, \text{ and } k \mid \lambda.$$

Hence for given λ , there exist only finitely many $(0, \lambda)$ -graphs with a minimum number of vertices.

For $\lambda = 2$, the only possible values of d are 3 and 6. A symmetric

block design with $\lambda = 2$ is called a *biplane* (see [C1]). There are finitely many biplanes known. These are described in [As], [AMS] and [C1]. There exists exactly one biplane when d equals 3, 4 or 5. There exist exactly three biplanes when d equals 6, and exactly four biplanes when d equals 9. There are four biplanes known when $d = 11$, and two when $d = 13$. For other values of d with $d \leq 15$, there are no biplanes.

The biplane with $d = 3$ has Q_3 as incidence graph, and one of the three biplanes with $d = 6$ has $\frac{1}{2}Q_6$ as incidence graph. It is easy to see that the biplane with $d = 3$ corresponds to the $(0,2)$ -graph on four vertices, namely K_4 . The biplane with $\frac{1}{2}Q_6$ as incidence graph gives rise to exactly two non-isomorphic $(0,2)$ -graphs on 16 vertices. One of these is the product graph $K_4 \times K_4$, and the other is the *Shrikhande graph* (see [Sh] or [B1]). This latter graph is depicted in Figure 2.2, where vertices with the same label should be identified. In this graph for each vertex u , the subgraph induced by $N(u)$ is a circuit of length 6.

Other biplanes do not have a symmetric incidence matrix with zero diagonal.

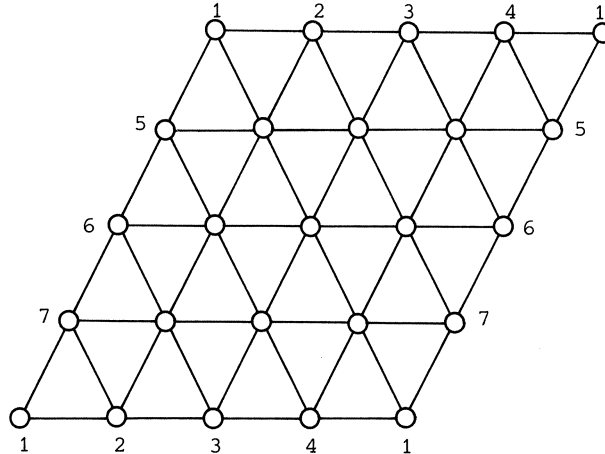


FIGURE 2.2.

We saw in Proposition 2.1.4 that if G is a $(0,\lambda)$ -graph with $\text{diam}(G) \geq 4$, then

$$(*) \quad d(G) \geq \text{diam}(G) + 2\lambda - 4.$$

We now give examples to show that this need not be satisfied for $(0, \lambda)$ -graphs G with $\text{diam}(G) \leq 3$.

2.1.12. PROPOSITION. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 1$. Then G is K_2 , or $K_{\lambda+2}$.*

Clearly G does not satisfy inequality (*) when G is K_2 ($\lambda \geq 3$), or when G is $K_{\lambda+2}$ ($\lambda \geq 5$).

2.1.13. PROPOSITION. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 2$. If $d(G) = \lambda$, or if G is bipartite, then G is $K_{\lambda, \lambda}$.*

Clearly $K_{\lambda, \lambda}$ does not satisfy inequality (*) for $\lambda \geq 3$.

2.1.14. THEOREM. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 2$. If $\lambda < d(G) \leq 2\lambda - 1$, then any two distinct vertices of G have exactly λ common neighbours.*

PROOF. Let u be a vertex of G . Then

$$V = \{u\} \cup N(u) \cup N_2(u).$$

Let W be the set of vertices distinct from u that have no common neighbour with u . Since each vertex in $N_2(u)$ has common neighbours with u , we have

$$W \subseteq N(u).$$

Assume that $W \neq \emptyset$, and let w be a vertex in W . Then

$$N(w) \setminus \{u\} \subseteq N_2(u).$$

Let v be a vertex in $N_2(u)$, and let $d = d(G)$. Since

$$\begin{aligned} |N(v) \cap N_2(u)| &= |N(v) \setminus N(u)| = |N(v)| - |N(v) \cap N(u)| \\ &= d - \lambda \leq \lambda - 1, \end{aligned}$$

it follows that v has no common neighbour with w . Hence

$$N(w) \setminus \{u\} \subseteq N_2(u) \setminus N(v),$$

and so

$$|N_2(u) \setminus N(v)| \geq d-1.$$

Furthermore, it follows that w lies in $N(v)$. Since w is an arbitrary vertex of W , we have

$$W \subseteq N(v) \cap N(u).$$

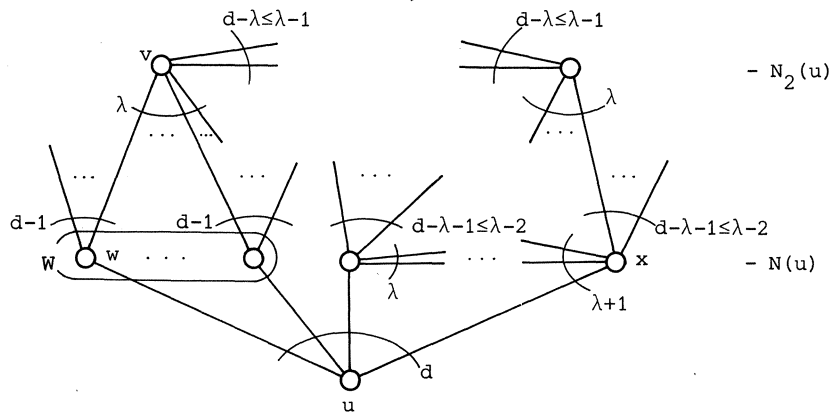


FIGURE 2.3.

Since $d > \lambda$, there exists a vertex x in $N(u) \setminus N(v)$. Then x is not in W , so x has λ common neighbours with u . This implies that

$$N(x) \cap W = N(x) \cap N(u) \cap W = \emptyset,$$

and so

$$|W| \leq |N(u) \setminus (N(x) \cup \{x\})| = d - \lambda - 1.$$

Counting the number of edges between $N(u)$ and $N_2(u)$ in two ways, we get

$$\begin{aligned} (d-1)|W| + (d-\lambda-1)(d-|W|) &= \lambda|N_2(u)| \\ &= \lambda|N_2(u) \setminus N(v)| + \lambda|N(v) \setminus N(u)|. \end{aligned}$$

From this we get

$$\begin{aligned} |W| &= |N_2(u) \setminus N(v)| + 1 - \frac{(d-\lambda)(d-\lambda-1)}{\lambda} \\ &\geq d - \frac{(d-\lambda)(d-\lambda-1)}{\lambda} > d - \frac{\lambda(\lambda-1)}{\lambda} \\ &= d - \lambda + 1 \geq |W| + 2, \end{aligned}$$

which is a contradiction. So $W = \emptyset$, and u has exactly λ common neighbours with any other vertex of G . Since u is an arbitrary vertex of G , the theorem follows. \square

2.1.15. THEOREM. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 3$. If $d(G) \leq 2\lambda - 1$, then G is bipartite.*

PROOF. By Proposition 2.1.3, we have

$$d(G) \geq \text{diam}(G) + \lambda - 2 = \lambda + 1.$$

Since $\text{diam}(G) = 3$, it follows that if G has odd circuits then a smallest odd circuit in G must be of length 3, 5 or 7.

Let 0 and 1 be two diametrical vertices of G .

First assume that there exists a vertex y in $N(0) \setminus N_2(1)$. Then y lies in $N_3(1)$, so y and 1 have no common neighbours (see Figure 2.4). For any vertex x in $N_2(1) \cap N(0)$, we have

$$|N(x) \cap N(1)| = \lambda.$$

Hence

$$|N(x) \cap N(y)| \leq d(G) - |N(x) \cap N(1)| \leq 2\lambda - 1 - \lambda = \lambda - 1.$$

This implies that x and y cannot have any common neighbour, which contradicts the fact that x and y both are neighbours of 0. So we have proved that

- (a) if $u, v \in V$ with $d(u, v) = 3$, then $N(u) \subseteq N_2(v)$.

It follows from (a) that a smallest odd circuit in G cannot have length 7.

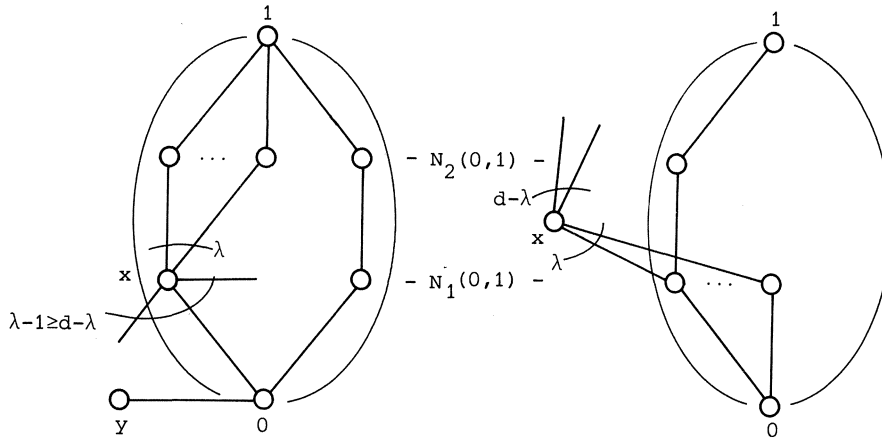


FIGURE 2.4.

FIGURE 2.5.

Let x be a neighbour of 0 . By (a) we have $x \in N_2(1)$, so x and 1 have exactly λ common neighbours, and

$$|N(x) \cap N(0)| \leq d(G) - |N(x) \cap N(1)| = d(G) - \lambda \leq \lambda - 1.$$

Thus we have proved that

- (b) if $u, v \in V$ with $d(u, v) = 3$, then any neighbour of u has no common neighbours with u .

Let x be a vertex in $N_2(0) \setminus N(1)$ (see Figure 2.5). Then x and 0 have λ common neighbours, so

$$|N(x) \cap N(1)| \leq d(G) - |N(x) \cap N(0)| = d(G) - \lambda \leq \lambda - 1.$$

Hence x has no common neighbours with 1 . Since x is not adjacent to 1 , it follows that $d(x, 1) = 3$. So we have proved that

- (c) if $u, v \in V$ with $d(u, v) = 3$, then $N_2(u) \cap N_2(v) = \emptyset$.

Let x be a neighbour of 0 (see Figure 2.6). By (a), we have $d(x,1) = 2$, so x and 1 have exactly λ common neighbours. Since $d(G) \geq \lambda+1$, there exists a neighbour y of 1 , which is not adjacent to x . By (b), a common neighbour of x and y has distance 2 from 0 and 1 . But this contradicts (c). So $d(x,y) = 3$. Since G is connected, this implies that

(d) any vertex of G has a diametrical vertex.

Combining (b) and (d), we conclude that G is triangle-free.

Finally, assume that G contains a circuit of length 5. By (d), we may assume that 0 is on a circuit of length 5, say $0 \rightarrow u_1 \rightarrow u_2 \rightarrow u_3 \rightarrow u_4 \rightarrow 0$. Then u_1 and u_4 are adjacent to 0 . Furthermore, since G is triangle-free, it follows that u_2 and u_3 lie in $N_2(0)$. Since u_2 and u_3 are adjacent, it follows from (b) and (c) that u_2 and u_3 lie in $N_3(1)$ (see Figure 2.7).

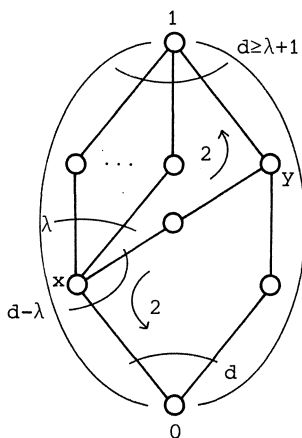


FIGURE 2.6.

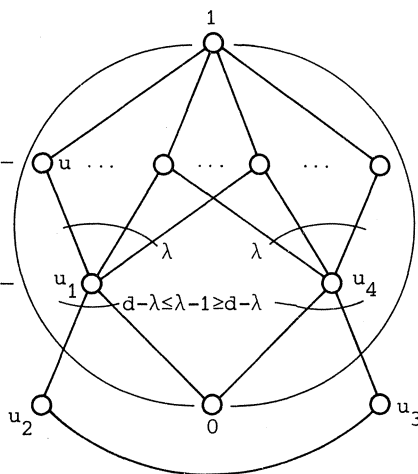


FIGURE 2.7.

By (a), we have

$$|N(u_1) \cap N(1)| = \lambda = |N(u_4) \cap N(1)|.$$

We also have

$$|N(u_1) \cap N(u_4) \cap N(1)| \leq |N(u_1) \cap N(u_4) \setminus \{0\}| = \lambda - 1.$$

So there exists a common neighbour u of u_1 and 1 , which is not adjacent to u_4 . As in the proof of (d), it follows that u and u_4 are diametrical. As can be checked in Figure 2.7, we have $d(u_2, u) = 2 = d(u_2, u_4)$. This establishes a contradiction with (c). So G does not contain a circuit of length 5, which completes the proof. \square

We conclude this section by exhibiting an infinite sequence of $(0, \lambda)$ -graphs with diameter 3, which do not satisfy inequality (*).

Let n be a prime power. A finite projective plane of order n is a symmetric block design D with parameters $(n^2+n+1, n+1, 1)$, see [R1]. The complementary block design \bar{D} is a symmetric block design with parameters (n^2+n+1, n^2, n^2-n) . The incidence graph $G(\bar{D})$ of the block design \bar{D} is a $(0, \lambda)$ -graph with diameter 3, such that $\lambda = n^2-n$ and $d(G(\bar{D})) = n^2$. For $n \geq 3$, we have $\lambda < d(G(\bar{D})) < 2\lambda-1$.

2.2. THE n -CUBE AS A $(0, 2)$ -GRAPH

In Section 1.2 we characterized a hypercube as a bipartite interval-regular graph; this was slightly stronger than the characterization of FOLDES [F1]. We can now weaken the condition of interval-regularity and obtain an even stronger characterization.

2.2.1. THEOREM. *Let G be a bipartite $(0, 2)$ -graph with $\text{diam}(G) = n$, and let 0 and $\hat{0}$ be two diametrical vertices of G . If*

$$|I(0, u) \cap N(u)| = d(0, u), \quad \text{and} \quad |I(u, \hat{0}) \cap N(u)| = d(u, \hat{0})$$

for any vertex u of G , then G is Q_n .

PROOF. Since G is bipartite, it follows that $N(0) \subseteq I(0, \hat{0})$. Hence we have

$$d(G) = |N(0)| = |I(0, \hat{0}) \cap N(0)| = \text{diam}(G) = n.$$

Furthermore, it follows for any vertex u of G that

$$\begin{aligned} d(G) &\geq |I(0, u) \cap N(u)| + |I(u, \hat{0}) \cap N(u)| = d(0, u) + d(u, \hat{0}) \\ &\geq \text{diam}(G) = d(G), \end{aligned}$$

and so

$$N(u) = (I(0,u) \cap N(u)) \cup (I(u,\hat{0}) \cap N(u)).$$

This implies that $V = I(0,\hat{0})$.

Counting the edges between $N_{i-1}(0,\hat{0})$ and $N_i(0,\hat{0})$ in two ways, we get

$$i|N_i(0,\hat{0})| = (n-i+1)|N_{i-1}(0,\hat{0})| \quad \text{for } i = 1, \dots, n.$$

Since $|N_0(0,\hat{0})| = 1$, it follows by induction on i that

$$|N_i(0,\hat{0})| = \binom{n}{i} \quad \text{for } i = 0, 1, \dots, n.$$

Let G_j be the subgraph of G induced by $\bigcup_{i=0}^j N_i(0,\hat{0})$. By induction on j we prove that G_j is isomorphic to the subgraph of Q_n induced by $\{A \subseteq N(0) \mid |A| \leq j\}$, where any vertex u of G_j corresponds to the set $I(0,u) \cap N(0)$.

For $j = 0$ or 1 the assertion is clearly true, so let $j > 1$. By the induction hypothesis the vertices of $N_{j-1}(0,\hat{0})$ are represented by the $(j-1)$ -subsets of $N_1(0)$. Let A and B represent two distinct vertices in $N_{j-1}(0,\hat{0})$. It follows that if $|A \cap B| \neq j-2$, then A and B have no common neighbour in $N_{j-2}(0,\hat{0})$. Furthermore, if $|A \cap B| = j-2$, then A and B have exactly one common neighbour (represented by $A \cap B$) in $N_{j-2}(0,\hat{0})$. In the latter case the other common neighbour of A and B must lie in $N_j(0,\hat{0})$.

The number of pairs of distinct vertices in $N_{j-1}(0,\hat{0})$ having a common neighbour in $N_{j-2}(0,\hat{0})$ equals the number of pairs of edges between $N_{j-2}(0,\hat{0})$ and $N_{j-1}(0,\hat{0})$ having a vertex in $N_{j-2}(0,\hat{0})$ in common. This equals

$$\binom{n}{j-2} \binom{n+j-2}{2} = \binom{n}{j} \binom{j}{2}.$$

The right-hand side of this equality equals the number of pairs of distinct vertices in $N_{j-1}(0,\hat{0})$ having a common neighbour in $N_j(0,\hat{0})$. Hence two distinct vertices in $N_{j-1}(0,\hat{0})$ have a (unique) common neighbour in $N_j(0,\hat{0})$ if and only if they have a (unique) common neighbour in $N_{j-2}(0,\hat{0})$. This implies that two distinct vertices in $N_j(0,\hat{0})$ have at most one common neighbour in $N_{j-1}(0,\hat{0})$.

Finally we prove that the vertices in $N_j(0, \hat{0})$ can be represented by the j -subsets of $N(0)$ in the required way. Let w be a vertex in $N_j(0, \hat{0})$, and let A_1, \dots, A_j be the neighbours of w in $N_{j-1}(0, \hat{0})$. Then $|A_1| = \dots = |A_j| = j-1$, and A_s and A_t have a common neighbour in $N_{j-2}(0, \hat{0})$, for any two integers s and t with $1 \leq s < t \leq j$. Hence

$$|A_s \cap A_t| = j-2 \quad \text{for } s, t \text{ with } 1 \leq s < t \leq j.$$

This implies that $A = \bigcup_s A_s$ is a j -subset of $N(0)$, and that A_1, \dots, A_j are the $(j-1)$ -subsets of A . Furthermore, it follows that

$$A = N(0) \cap I(0, w).$$

So w can be represented by the set A .

Distinct vertices in $N_j(0, \hat{0})$ are thus represented by distinct j -subsets of $N(0)$. Since $|N_j(0, \hat{0})| = \binom{n}{j}$, it follows that all j -subsets of $N(0)$ are used as representatives. So G_j is isomorphic to the "bottom $j+1$ levels" of \emptyset in Q_n . This completes the proof. \square

In Figure 2.8 we give an example that shows the necessity of the condition that G is a $(0, 2)$ -graph in the above theorem. (For a result resembling the above result see Theorem 5.10 in [C2].)

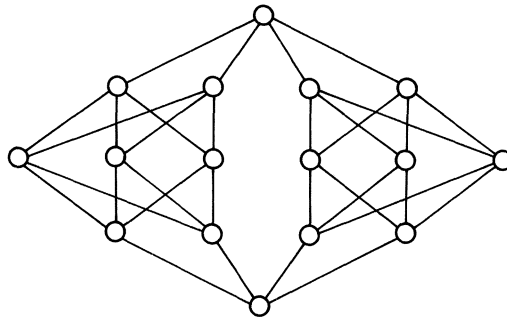


FIGURE 2.8.

We are now in a position to study those $(0, \lambda)$ -graphs that attain the upper bound for the number of vertices derived in Propositions 2.1.5 and 2.1.6.

2.2.2. PROPOSITION. *Let G be a $(0,2)$ -graph of degree d . Then $|V| = 2^d$ if and only if G is Q_d .*

PROOF. If G is Q_d , then G is a $(0,2)$ -graph of degree d with $|V| = 2^d$.

Let G be a $(0,2)$ -graph of degree d with $|V| = 2^d$, and let u be a vertex of G . It follows from the proof of Proposition 2.1.5 that each vertex in $N_i(u)$ has i neighbours in $N_{i-1}(u)$, and $d-i$ neighbours in $N_{i+1}(u)$, for $i = 1, \dots, d$. Hence G is bipartite and there exists a vertex v in G such that $\{v\} = N_d(u)$. Furthermore, $N_i(u) = N_{d-i}(v)$ for $i = 0, 1, \dots, d$. So G satisfies the conditions of Theorem 2.2.1 with u as 0 and v as $\bar{0}$, and G is the d -cube Q_d . \square

REMARK. LABORDE [La] has given a characterization of the d -cube, which is a corollary of the last proposition. It turns out that the conditions in Proposition 2.2.2 are included in the conditions involved in the characterization in [La]. For other characterizations of Q_d see McFALL [Mf1], [Mf2]. Some of these characterizations can be deduced from the above results.

2.2.3. THEOREM. *Let G be a $(0,\lambda)$ -graph of degree $d \geq \lambda+2$. If*

$$|V| = 1 + d + \frac{(\lambda-1)!(d-\lambda)!}{(d-2)!} \sum_{i \geq 0} \binom{d}{\lambda+i},$$

then $\lambda = 2$ and G is Q_d .

PROOF. Let u be a vertex of G . It follows from Proposition 2.1.5 that any vertex in $N(u)$ has $d-1$ neighbours in $N_2(u)$. Furthermore, it follows for $i = 2, \dots, d-\lambda+2$, that each vertex in $N_i(u)$ has $\lambda + i - 2$ neighbours in $N_{i-1}(u)$, and $d - \lambda - i + 2$ neighbours in $N_{i+1}(u)$. Hence G has diameter $d - \lambda + 2$. Moreover, there are no edges within the levels of u , so G is bipartite.

Since $d \geq \lambda + 2$, we have

$$\text{diam}(G) = d - \lambda + 2 \geq 4,$$

and it follows from Proposition 2.1.4 that

$$d \geq \text{diam}(G) + 2\lambda - 4 \geq 2\lambda.$$

If $d = \lambda + 2$, it follows that $d = 4$ and $\lambda = 2$. Let $d \geq \lambda + 3$, and let v be an arbitrary vertex in $N_3(u)$. Then $I(u,v) \setminus \{u,v\}$ induces a regular bipartite subgraph of G of degree λ . So

$$|I(u,v) \cap N(u)| = |I(u,v) \cap N_2(u)| = \lambda + 1;$$

-that is, $I(u,v) \setminus \{u,v\}$ induces a $K_{\lambda+1, \lambda+1}$ minus a perfect matching in G .

Represent each vertex w in $N_2(u) \cup N_3(u)$ by the set $I(u,w) \cap N(u)$. Let the set A represent a vertex in $N_3(u)$. Then A is a $(\lambda+1)$ -subset of $N(u)$, and the $\lambda+1$ neighbours of A in $N_2(u)$ are represented by the λ -sets $A \setminus \{a\}$, a in A . Since $\lambda + 1 \geq 3$, it follows that distinct vertices of $N_3(u)$ are represented by distinct $(\lambda+1)$ -subsets of $N(u)$.

Let x and y be two distinct vertices in $N(u)$. Then x and y have $\lambda-1$ common neighbours in $N_2(u)$. Let $P \subset N(u)$ represent a common neighbour of x and y in $N_2(u)$. Note that P is a λ -set.

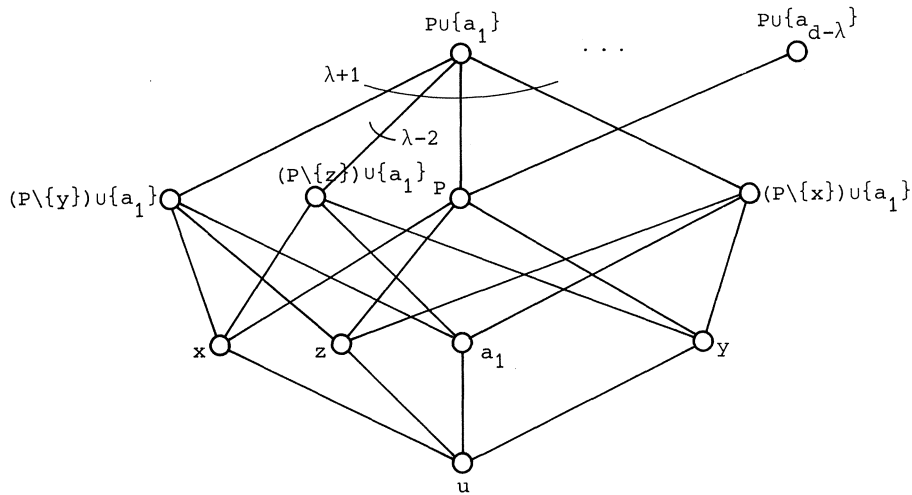


FIGURE 2.9.

Let $P \cup \{a_s\}$, $s = 1, \dots, d-\lambda$, be the neighbours of P in $N_3(u)$ (see Figure 2.9). Then, for $1 \leq s < t \leq d - \lambda$, we have $a_s \neq a_t$. Furthermore, $P \cup \{a_s\}$ and $P \cup \{a_t\}$ have only P as common neighbour in $N_2(u)$. Since x and y are in P , it follows that, apart from P , $\lambda - 2$ of the neighbours of $P \cup \{a_s\}$

in $N_2(u)$ are common neighbours of x and y .

Hence, since $d \geq 2\lambda$, we have

$$\begin{aligned} \lambda &\geq (d-\lambda)(\lambda-2) + 2 \geq \lambda(\lambda-2) + 2 = (\lambda-1)^2 + 1 \\ &\geq (\lambda-1) + 1 = \lambda. \end{aligned}$$

This implies that $\lambda = 2$.

Using the previous proposition, we conclude that G is Q_d . \square

We conclude this section with an observation inspired by the last proof. Let G be a $(0, \lambda)$ -graph, and let u be a vertex of G such that there are no edges within the first and the second levels of u . It then follows from the proof of the last theorem that if any vertex in $N_3(u)$ has exactly $\lambda + 1$ neighbours in $N_2(u)$, then $\lambda = 2$.

2.3. HADAMARD GRAPHS

In this section we construct a class of $(0, \lambda)$ -graphs with diameter 4 from Hadamard matrices.

Let H be a Hadamard matrix of order $4t$ (see Chapter 0 or [R1]). We shall assume that H is "normalized", so that each entry in the first row is $+1$. Any other row has $2t$ entries $+1$ and $2t$ entries -1 . Such a row determines two sets of $2t$ columns, one corresponding to the $+1$ entries and the other corresponding to the -1 entries. The $8t - 2$ sets of $2t$ columns so obtained form the blocks of a self-complementary block design D_H with parameters $(8t-2, 4t, 4t-1, 2t, 2t-1)$. Any two non-complementary blocks in D_H intersect in t points. Since any 3-set of columns is contained in exactly $t-1$ blocks, such a block design is called a Hadamard 3-design of order $4t$. This construction is reversible: from a self-complementary block design with parameters $(8t-2, 4t, 4t-1, 2t, 2t-1)$, in which any two non-complementary blocks intersect in t points, we get a Hadamard matrix of order $4t$.

2.3.1. DEFINITION. Let D_H be a Hadamard 3-design of order 2λ , and let $\{1, 2, \dots, 2\lambda\}$ be the set of points of D_H . The Hadamard graph G_H corresponding to the Hadamard matrix H is constructed from D_H as follows:

- (i) take as vertices of G_H the blocks of D_H together with the sets $\emptyset, \{1\}, \dots, \{2\lambda\}, \{1\}^c, \dots, \{2\lambda\}^c, \{1, \dots, 2\lambda\}$, where $\{i\}^c = \{1, \dots, 2\lambda\} \setminus \{i\}$;
- (ii) join two distinct vertices A and B by an edge whenever $A \subset B$ and $|B \setminus A| \leq \lambda - 1$.

The smallest Hadamard graph is Q_4 , which corresponds to the following Hadamard matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

It follows easily from the properties of D_H that G_H is a bipartite $(0, \lambda)$ -graph of degree 2λ with diameter 4, such that

$$d(G_H) = 2\lambda = \text{diam}(G_H) + 2\lambda - 4.$$

Furthermore, G_H is diametrical -that is, any vertex in G_H has a unique diametrical vertex (see Definition 1.3.8).

REMARK. A Hadamard graph is also the incidence graph of a transversal design with group size 2 and block size 2λ (see HANANI [Ha]).

2.3.2. THEOREM. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 4$. If $d(G) = 2\lambda$, then G is a Hadamard graph.*

PROOF. Let 0 and $\hat{0}$ be two diametrical vertices of G . To avoid unnecessary repetition we make use of a step in the proof of Theorem 2.4.2. There we shall deduce that under the above conditions G is bipartite and $V = I(0, \hat{0})$. Then it follows that $\hat{0}$ is the unique diametrical vertex of 0 . Furthermore,

$$|I(u, \hat{0}) \cap N(u)| = 2\lambda - 1 \quad \text{for } u \in N_1(0, \hat{0}),$$

and

$$|I(0, u) \cap N(u)| = \lambda = |I(u, \hat{0}) \cap N(u)| \quad \text{for } u \in N_2(0, \hat{0}),$$

and

$$|I(0,u) \cap N(u)| = 2\lambda - 1 \quad \text{for } u \in N_3(0,\hat{0}).$$

Let u be a vertex in $N_3(0,\hat{0})$. Then $I(0,u) \setminus \{0,u\}$ induces a regular bipartite subgraph of degree λ , and so

$$|I(0,u) \cap N(0)| = |I(0,u) \cap N(u)| = 2\lambda - 1 \quad \text{for } u \in N_3(0,\hat{0}).$$

Similarly, we have

$$|I(v,\hat{0}) \cap N(\hat{0})| = |I(v,\hat{0}) \cap N(v)| = 2\lambda - 1 \quad \text{for } v \in N_1(0,\hat{0}).$$

From these two equations we deduce that if two distinct vertices in $N_2(0,\hat{0})$ have common neighbours, then they have at least one common neighbour in $N_1(0,\hat{0})$ as well as in $N_3(0,\hat{0})$, and so they have at most $\lambda-1$ common neighbours in $N_1(0,\hat{0})$.

Label the neighbours of 0 by $1, 2, \dots, 2\lambda$, respectively. Represent each vertex u of G by the set $I(0,u) \cap N(0)$. Then any neighbour j of 0 has as unique diametrical vertex the vertex in $N_3(0,\hat{0})$ represented by $\{1, \dots, 2\lambda\} \setminus \{j\}$ (that is, the vertex of $N_3(0,\hat{0})$, which is not in $I(j,\hat{0}) \cap N_3(0,\hat{0})$). Since G is connected, it follows that any vertex in G has a unique diametrical vertex.

The above observations imply that distinct vertices in $N_2(0,\hat{0})$ are represented by distinct λ -subsets of $N(0)$, and that if $A \subset N(0)$ represents a vertex u in $N_2(0,\hat{0})$, then $N(0) \setminus A$ represents the diametrical vertex of u (which lies in $N_2(0,\hat{0})$). So the λ -sets representing the vertices of $N_2(0,\hat{0})$ form the blocks of a self-complementary block design D with parameters $(4\lambda-2, 2\lambda, 2\lambda-1, \lambda, \lambda-1)$.

Finally, let A and B represent two distinct non-diametrical vertices in $N_2(0,\hat{0})$. Then A and B are two non-disjoint λ -subsets of $N(0)$. Furthermore, the λ common neighbours of A and B are the vertices represented by $\{j\}$, for $j \in A \cap B$, and by $\{j\}^c$, for $j \in \{1, \dots, 2\lambda\} \setminus (A \cup B)$. Hence

$$\lambda = |A \cap B| + |\{1, \dots, 2\lambda\} \setminus (A \cup B)| = 2|A \cap B|.$$

So λ is even, and A and B intersect in $\frac{1}{2}\lambda$ points; that is, D is a Hadamard 3-design, and so G is a Hadamard graph. \square

2.4. EXTREMAL $(0, \lambda)$ -GRAPHS

In this section we discuss $(0, \lambda)$ -graphs G satisfying

$$d(G) = \text{diam}(G) + 2\lambda - 4.$$

The only complete graphs that satisfy this equation are K_1 and K_2 , with $\lambda = 2$, and K_6 , with $\lambda = 4$.

Another graph satisfying this equation is the *Clebsch graph*, introduced by SEIDEL [Se]. The Clebsch graph is a $(0, 6)$ -graph of degree 10, and has 16 vertices. Its vertex-set is $\{A \subset \{1, \dots, 5\} \mid |A| \leq 2\}$, and two vertices A and B are adjacent whenever $2 \leq |A \Delta B| \leq 3$. This graph has very nice properties. For instance, it is distance-transitive, and for any vertex u , the subgraph induced by $N(u)$ is the complement of the Petersen graph. The complementary graph of the Clebsch graph is the Greenwood-Gleason graph E_3 described in Chapter 5 (for a picture see Figure 5.7).

2.4.1. PROPOSITION. *Let G be a $(0, \lambda)$ -graph with $\text{diam}(G) = 2$. If $d(G) = 2\lambda - 2$, then G is Q_2 or the Clebsch graph.*

PROOF. If $\lambda = 2$, then $d(G) = 2 = \lambda$, and so G is $K_{2,2}$, which equals Q_2 (see Corollary 2.1.7).

Let $\lambda \geq 3$. Then $2\lambda - 1 > d(G) = 2\lambda - 2 > \lambda$, and so any two distinct vertices in G have exactly λ common neighbours (see Theorem 2.1.14). It follows from the proof of Theorem 2.1.5 that

$$|V| = 1 + \frac{d(d-1)}{\lambda},$$

and so

$$\lambda(|V|-1) = (2\lambda-2)(2\lambda-3).$$

Hence λ divides 6 -that is, λ equals 3 or 6.

If $\lambda = 3$, then $|V| = 5$, which would imply that G is K_5 , contradicting the fact that $\text{diam}(G) = 2$. So $\lambda = 6$, and hence $d(G) = 10$ and $|V| = 16$. It is left to the reader to verify that G is the Clebsch graph. \square

We are now ready to prove the main theorem of this chapter, which contains another characterization of Q_n .

2.4.2. THEOREM. *Let G be a $(0, \lambda)$ -graph. If $d(G) = \text{diam}(G) + 2\lambda - 4$, then*

$\lambda = 4$ and G is K_6 ,
or $\lambda = 6$ and G is the Clebsch graph,
or G is a Hadamard graph of degree 2λ ,
or G is Q_n , where $n = \text{diam}(G)$.

PROOF. If $\text{diam}(G) \leq 2$, then it follows from Proposition 2.4.1 and from the observations preceding this proposition that G is one of the graphs Q_0, Q_1, Q_2, K_6 , or the Clebsch graph.

If $\text{diam}(G) = 3$, then by Theorem 2.1.15 it follows that G is bipartite. By Proposition 2.1.11 we have $|V|$ is even and

$$|V| = 2 + \frac{2(2\lambda-1)(2\lambda-2)}{\lambda}.$$

So $\lambda = 2$ and $d(G) = 3$, which implies that G is Q_3 .

Let $n = \text{diam}(G) \geq 4$, and let 0 and 1 be two diametrical vertices of G . The main part of the proof is to show that $V = I(0,1)$, and that G is bipartite.

Let i be an integer with $1 < i < n-1$, and let v be a vertex in $N_i(0,1)$. Then by Proposition 2.1.3 we have

$$\begin{aligned} n + 2\lambda - 4 &= |N(v)| \geq |N_{i-1}(0,1) \cap N(v)| + |N_{i+1}(0,1) \cap N(v)| \\ &= |I(0,v) \cap N(v)| + |I(v,1) \cap N(v)| \\ &\geq d(0,v) + \lambda - 2 + d(v,1) + \lambda - 2 \\ &= n + 2\lambda - 4. \end{aligned}$$

Hence $N(v) \subseteq N_{i-1}(0,1) \cup N_{i+1}(0,1)$, and

$$|I(0,v) \cap N(v)| = i + \lambda - 2,$$

and

$$|I(v,1) \cap N(v)| = n - i + \lambda - 2.$$

It follows that there are no edges within $N_i(0,1)$, for $1 < i < n-1$.

Assume that there is a vertex u in $N(0) \setminus N_1(0,1)$. Then

$$n \geq d(u,1) \geq d(0,1) = n,$$

so $d(u,1) = n$. Let x be a neighbour of 0 in $I(0,1)$. By Proposition 2.1.3, we have

$$\begin{aligned} |N(x) \setminus I(x,1)| &= n + 2\lambda - 4 - |N(x) \cap I(x,1)| \\ &\leq n + 2\lambda - 4 - (n + \lambda - 3) = \lambda - 1. \end{aligned}$$

Hence $|N(u) \cap N(x)| \leq \lambda - 1$ -that is, u and x have no common neighbours, contradicting the fact that 0 is a common neighbour of u and x . So we have proved that $N(0) \subset I(0,1)$. Similarly, $N(1) \subset I(0,1)$.

Let u be a neighbour of 0 . By Proposition 2.1.3, we have

$$\begin{aligned} |N(u) \cap N(0)| &\leq n + 2\lambda - 4 - |N(u) \cap I(u,1)| \\ &\leq n + 2\lambda - 4 - (n + \lambda - 3) \\ &= \lambda - 1, \end{aligned}$$

and so u has no common neighbour with 0 -that is, there are no edges within $N(0)$.

Assume that u has a neighbour v outside $I(0,1)$ (see Figure 2.10). Then

$$d(v,1) \geq d(u,1) = n - 1.$$

Since u is a common neighbour of v and 0 , it follows that v and 0 have exactly λ common neighbours. Let w be an arbitrary neighbour of u in $N_2(0,1)$. Then v and w have u as common neighbour, and so v and w have λ common neighbours. Since $N(w) = (N(w) \cap N(0)) \cup (N(w) \cap N_3(0,1))$, it follows that

$$N(w) \cap N(v) = N(w) \cap N(0) = N(v) \cap N(0).$$

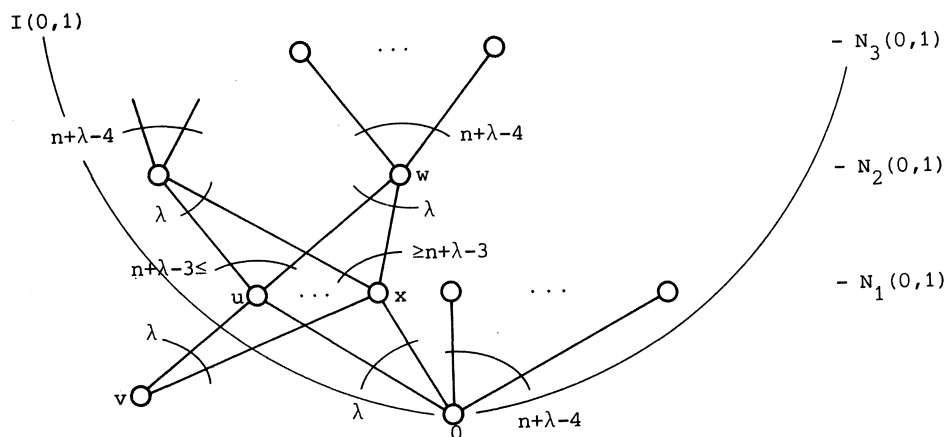


FIGURE 2.10.

Let x be a common neighbour of 0 and v distinct from u . Then we have

$$\{0, v\} \cup (N(u) \cap N_2(0, 1)) \subseteq N(x) \cap N(u),$$

and so

$$|N(x) \cap N(u)| \geq 2 + n + \lambda - 3 \geq \lambda + 3,$$

which is impossible. Hence we have proved that

$$N(u) \subseteq \{0\} \cup N_2(0, 1) \quad \text{for } u \in N(0),$$

and similarly,

$$N(u) \subseteq \{1\} \cup N_{n-2}(0, 1) \quad \text{for } u \in N(1).$$

Summarizing, we have proved that $V = I(0, 1)$, and, since there are no edges within the levels in $I(0, 1)$, that G is bipartite. So if $\text{diam}(G) = 4$, then by Theorem 2.3.2, G is a Hadamard graph of degree 2λ . If $\text{diam}(G) \geq 5$, then it follows from the observation following Theorem 2.2.3 that $\lambda = 2$. By Theorem 2.2.1, G is the n -cube. This concludes the proof. \square

CHAPTER 3

MEDIAN GRAPHS

In this chapter we consider median graphs. These are graphs in which $|I(u,v) \cap I(v,w) \cap I(w,u)| = 1$, for any three vertices u , v and w . All trees and all hypercubes satisfy this condition. We give several characterizations of median graphs involving the interval function. Every median graph can be obtained from a median graph with fewer vertices by an expansion procedure, and this characterization provides a tool for embedding a median graph in a hypercube, and for the study of the relation of median graphs with "median-like" algebraic structures.

3.0. INTRODUCTION

The central notion of this chapter is that of a median graph. This notion has been introduced independently by AVANN [Av] (who used the term unique ternary distance graph) and by NEBESKÝ [N2]. They studied the relation of median graphs with certain algebraic structures. (A. Schrijver and the author studied the relationship between median graphs and Helly hypergraphs, which is the theme of Chapter 4.)

The algebraic structures related to median graphs are those of median semilattices, median segments, and median betweenness introduced by SHOLANDER [S1], [S2], [S3]. A median semilattice can be defined in terms of a binary operation, with matching partial order (see Definition 3.3.2), or in terms of a ternary operation. The last variant has been introduced independently by NEBESKÝ [N1] in the setting of graphic algebras, under the name of simple graphic algebras. It follows from results in [S2], and also from results in Chapter 7, that the notion of a normal graphic algebra (also introduced by NEBESKÝ [N1], see Definition 3.3.1) coincides with that of a simple graphic algebra.

We explore the relationship between median graphs and median semi-lattices in Section 3.3. There we give new, graph-theoretical proofs of the results first obtained by AVANN [Av] and NEBESKÝ [N2] using the structural characterization of median graphs obtained in Section 3.2. We study the relationship between median graphs and median segments (in our terminology median interval structures) in Theorems 3.1.4 and 3.1.5.

3.1. MEDIANS AND INTERVALS

We start with some notation. Let I be a mapping of the Cartesian product $X \times X$ into the power-set $\mathcal{P}(X)$ of X . Then, for any three elements x, y and z of X , we write

$$I(x, y, z) := I(x, y) \cap I(y, z) \cap I(z, x).$$

For three vertices u, v and w of a graph G with interval function I , the set $I(u, v, w)$ can be of any size. This is illustrated by the graphs in Figure 3.1.

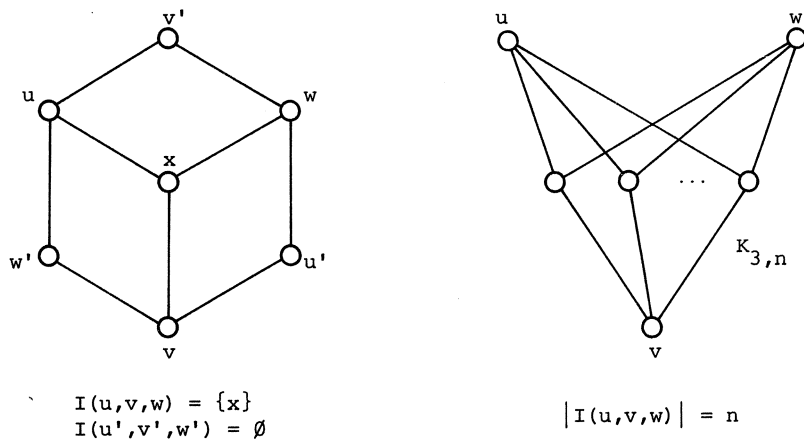


FIGURE 3.1.

3.1.1. PROPOSITION. *Let G be a graph with interval function I such that $I(u, v, w) \neq \emptyset$, for any three vertices u, v and w of G . Then G is connected and bipartite.*

PROOF. Since any interval in G must be non-empty, G is connected.

Assume that G is not bipartite, and let $u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_{2k} \rightarrow u_0$ be an odd circuit of smallest length. Then

$$d(u_0, u_k) = d(u_0, u_{k+1}) = k.$$

Hence u_{k+1} is not in $I(u_0, u_k)$, and u_k is not in $I(u_0, u_{k+1})$, so that $I(u_0, u_k, u_{k+1}) = \emptyset$, which is impossible -that is, G is bipartite. \square

In the sequel we focus on graphs in which the set $I(u, v, w)$ is a singleton for any three vertices u, v and w .

3.1.2. DEFINITION. A graph G with interval function I is a *median graph* if $|I(u, v, w)| = 1$ for any three vertices u, v and w of G .

An alternative formulation of this definition is the following:

let G be a connected graph with distance function d ; then G is a median graph if, for any three vertices u, v and w of G , there exists a unique vertex $x = x(u, v, w)$ such that

$$d(u, x) + d(x, v) = d(u, v),$$

$$d(v, x) + d(x, w) = d(v, w),$$

and

$$d(w, x) + d(x, u) = d(w, u).$$

The vertex x in the above definition, denoted by $x = \langle u, v, w \rangle$, is called the *median* of u, v and w .

We give an example of a median graph in Figure 3.2. All trees and all hypercubes are median graphs. In the n -cube with its vector representation we can determine the median $x = (x_1, \dots, x_n)$ of the vertices $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ as follows: to fix x_i the vertices u, v and w have elections and cast their i -th coordinate as their vote; of course, the majority wins. For example, in Q_4 the median of $(0, 0, 1, 1)$, $(0, 1, 0, 1)$ and $(1, 1, 1, 0)$ is $(0, 1, 1, 1)$. Note that a convex subgraph of a median graph is a median graph.

In this section we give a number of characterizations of median graphs using the interval function.

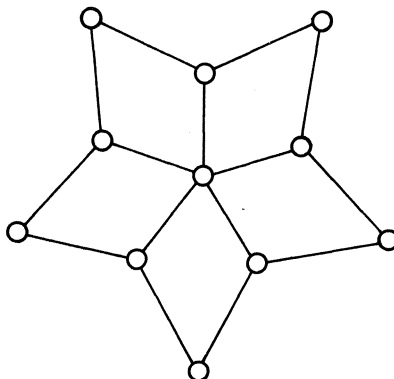


FIGURE 3.2.

IN [BS] SCHRIJVER introduced the following concept.

3.1.3. DEFINITION. Let X be a finite set. A mapping $I: X \times X \rightarrow \mathcal{P}(X)$ is an *interval structure* on X if I satisfies

- (I1) $x, y \in I(u, v)$ if and only if $I(x, y) \subseteq I(u, v)$, for $x, y, u, v \in X$,
- (I2) $I(u, v, w) \neq \emptyset$ for all $u, v, w \in X$.

It follows from (I1) that $I(u, v) = I(v, u)$ for any two elements u and v of X .

Let I be an interval structure on the set X . Each set $I(u, v)$ is called an *interval* in X . A subset Y of X is *I-convex* if, for any two elements u and v of Y , the interval $I(u, v)$ is contained in Y . It follows from (I1) that each interval in X is *I-convex*.

If (X, \leq) is a lattice, then we can obtain an example of an interval structure on X by taking

$$I(u, v) := \{w \in X \mid u \wedge v \leq w \leq u \vee v\}.$$

If the mapping $I: X \times X \rightarrow \mathcal{P}(X)$ satisfies condition (I1) and also

$$(I2') \quad |I(u,v,w)| = 1 \quad \text{for all } u,v,w \in X,$$

then I is called a *median interval structure* on X (see MULDER & SCHRIJVER [MS]; SHOLANDER [S2] used the term median segments). An interval structure obtained from a lattice in the above way is a median interval structure if and only if the lattice is distributive. Further examples of median interval structures can be obtained from median graphs, as we show now.

3.1.4. THEOREM. *Let G be a median graph with interval function I . Then I is a median interval structure on V .*

PROOF. We have only to verify the interval monotonicity of I .

Assume the contrary, and let $I(u,v)$ be a non-convex interval in G . Let x and y be two vertices in $I(u,v)$ such that $I(x,y) \not\subseteq I(u,v)$ with $d(x,y)$ as small as possible. Note that $d(x,y) \geq 2$. Then there exists a shortest (x,y) -path P such that all internal vertices of P lie outside $I(u,v)$. Let z be an internal vertex of P .

It follows from the minimality of $d(x,y)$ that $I(u,v) \cap I(x,z) = \{x\}$.

Hence

$$I(u,x) \cap I(x,z) = \{x\}.$$

Since $I(u,x,z) \neq \emptyset$, it follows that x lies in $I(u,z)$. Similarly, x lies in $I(v,z)$, and so

$$x \in I(u,v,z),$$

-that is, x is the median of u , v and z . In the same way it follows that y is also the median of u , v and z . So $x = y$, contradicting the fact that $d(x,y) \geq 2$. \square

A converse of this theorem also holds.

3.1.5. THEOREM. *Let I be a median interval structure on the finite set V . Let G_I be the graph with vertex-set V , in which two distinct vertices u and v are adjacent whenever $I(u,v) = \{u,v\}$. Then G_I is a median graph with I as*

its interval function.

PROOF. We prove that G_I is connected and that the interval function I_{G_I} of G_I is identical with I . It then follows that G_I is a median graph.

First observe that for u, v and w in V , we have

$$w \in I(u,v) \text{ if and only if } I(u,w) \cap I(w,v) = \{w\}.$$

We use this continually in the sequel without mention. For example, for w in $I(u,v) \setminus \{u,v\}$, we have

$$u \notin I(w,v) \subset I(u,v),$$

and

$$v \notin I(u,w) \subset I(u,v).$$

Using this, we can easily verify by induction on $|I(u,v)|$ that $I(u,v)$ induces a connected subgraph of G_I , for all u and v in V . Hence G_I is connected.

To prove that $I(u,v) = I_{G_I}(u,v)$ for all u and v in V , we use induction on $d(u,v)$, where d is the distance function of G_I .

By definition, we have $I(u,v) = I_{G_I}(u,v)$ for any two vertices u and v of G_I with $d(u,v) \leq 1$. So let u and v be vertices with $d(u,v) > 1$.

Let w be a vertex in $I_{G_I}(u,v) \setminus \{u,v\}$. It follows that $d(u,w) < d(u,v)$ and $d(w,v) < d(u,v)$, and so by the induction hypothesis, we have

$$I_{G_I}(u,w) = I(u,w),$$

and

$$I_{G_I}(w,v) = I(w,v).$$

Since $I_{G_I}(u,w) \cap I_{G_I}(w,v) = \{w\}$, it follows that w lies in $I(u,v)$, and so

$$I_{G_I}(u,v) \subseteq I(u,v).$$

Assume that there exists a vertex w in $I(u,v) \setminus I_{G_I}(u,v)$. Then

$$I(u,w) \cap I(w,v) = \{w\}.$$

For any vertex w' in $I(u,w)$, we have

$$w \in I(w',w) \cap I(w,v) \subseteq I(u,w) \cap I(w,v) = \{w\}.$$

Hence $I(w',w) \cap I(w,v) = \{w\}$ -that is, w lies in $I(w',v)$. So, if there exists a vertex w' in $I(u,w) \cap I_{G_I}(u,v)$ distinct from u , then it follows from the preceding observation and the induction hypothesis that

$$w \in I(w',v) = I_{G_I}(w',v) \subset I_{G_I}(u,v).$$

This contradicts the choice of w . So

$$I(u,w) \cap I_{G_I}(u,v) = \{u\},$$

and

$$I(w,v) \cap I_{G_I}(w,v) = \{v\}.$$

Since $I(u,w)$ and $I(w,v)$ induce connected subgraphs of G_I , there exists a path P from u to v passing through w , all of whose internal vertices lie in $I(u,v) \setminus I_{G_I}(u,v)$. Then P is not a shortest (u,v) -path, and so the length of P exceeds $d(u,v)$. Since $d(u,v) \geq 2$, it follows that we can find an internal vertex x of P distinct from w , and that we can find a vertex y in $I_{G_I}(u,v) \setminus \{u,v\}$.

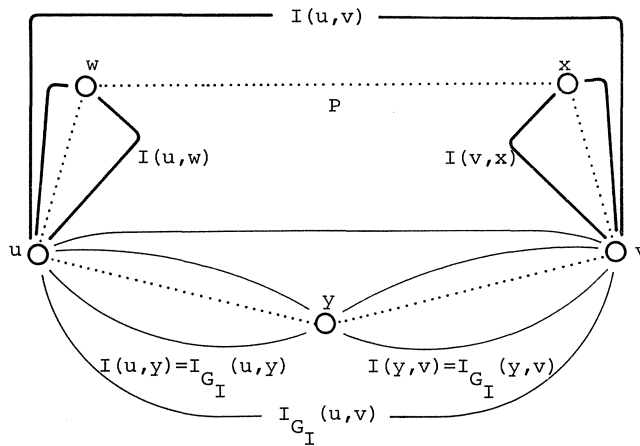


FIGURE 3.3.

By the induction hypothesis we have

$$I(u,y) = I_{G_I}(u,y),$$

and

$$I(y,v) = I_{G_I}(y,v).$$

It follows that

$$\begin{aligned} u \in I(u,y) \cap I(u,w) &= I_{G_I}(u,y) \cap I(u,w) \\ &\subseteq I_{G_I}(u,v) \cap I(u,w) = \{u\}. \end{aligned}$$

This implies that u lies in $I(y,w)$. Similarly v lies in $I(y,w)$, and so by (I1),

$$I(u,v) \subseteq I(y,w) \subseteq I(u,v).$$

Similarly, $I(u,v) = I(y,x)$. This implies that

$$\begin{aligned} w,x \in I(w,x) &= I(u,v) \cap I(w,x) \\ &= I(y,w) \cap I(w,x) \cap I(x,y), \end{aligned}$$

contradicting the fact that I is a median interval structure. So we have proved that $I(u,v) = I_{G_I}(u,v)$, which completes the proof. \square

In this proof we saw that $I_{G_I} = I$ for a median interval structure I . Furthermore, it follows from Theorems 3.1.4 and 3.1.5 that, if G is a median graph with interval function I , then $G_{I_G} = G$. This establishes a one-to-one correspondence between the median interval structures on the (finite) set V and the median graphs with V as vertex-set (see also Theorem 4.1.13).

3.1.6. THEOREM. *Let G be a graph with interval function I . If I is an interval structure on V , then G is a median graph.*

PROOF. Assume the contrary, and let u, v and w be vertices of G such that

$|I(u,v,w)| \geq 2$. Then, since G is interval monotone, we can choose two adjacent vertices x and y in $I(u,v,w)$.

By Proposition 3.1.1 we know that G is bipartite. Hence $d(u,x)$ and $d(u,y)$ differ by exactly 1, say

$$d(u,y) = d(u,x) + 1.$$

Since x and y lie in $I(u,v) \cap I(u,w)$, it follows that

$$d(y,v) = d(x,v) - 1,$$

and

$$d(y,w) = d(x,w) - 1.$$

Hence, since x and y lie in $I(v,w)$, we have

$$\begin{aligned} d(v,w) &= d(v,y) + d(y,w) = d(v,x) + d(x,w) - 2 \\ &= d(v,w) - 2, \end{aligned}$$

which is impossible. \square

Note that the interval monotonicity in Theorem 3.1.6 is necessary, as the graph in Figure 3.4 shows. In this graph we have $I(u,v,w) = \{x,y\}$.

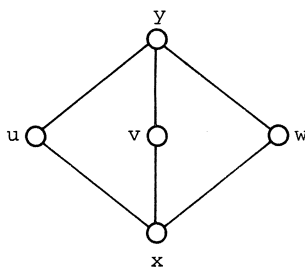


FIGURE 3.4.

This example shows also that the interval monotonicity of G is necessary in the next theorem.

3.1.7. THEOREM. *Let G be a connected graph with interval function I . Then G is a median graph if and only if G is interval monotone and I satisfies the following condition:*

$$\text{if } I(u,v) \cap I(v,w) = \{v\}, \text{ then } d(u,w) = d(u,v) + d(v,w) \\ \text{for } u,v,w \in V.$$

PROOF. The "only if" part of the theorem follows from Theorem 3.1.4 and the fact that, if $I(u,v) \cap I(v,w) = \{v\}$, then v is the median of u , v and w .

Conversely, in view of Theorem 3.1.6 it suffices to prove that $I(u,v,w) \neq \emptyset$ for any three vertices u , v and w of G .

Let u , v and w be arbitrary vertices of G , and let z in $I(u,v) \cap I(v,w)$ be such that

$$I(u,z) \cap I(z,w) = \{z\}.$$

Then $d(u,w) = d(u,z) + d(z,w)$ -that is, z lies in $I(u,w)$, and so z is in $I(u,v,w)$. \square

In Propositions 5.1.3, 5.1.4 and 5.1.5 we discuss variations of the condition on I in the above theorem.

The aim at the minimality of conditions in definitions and theorems is often contrary to the aim of elegance of formulation and arguments. My position in this is that I prefer elegance. Then, if possible, I give a theorem in which the minimality of conditions is explored. The next theorem serves this purpose with respect to median graphs.

3.1.8. THEOREM. *Let G be a connected triangle-free graph. If $|I(u,v,w)| = 1$ for any three vertices u , v and w of G such that $d(u,v) = 2$, then G is a median graph.*

PROOF. Note that $K_{2,3}$ is not a subgraph of G .

First we prove that G is bipartite. Assume the contrary, and let $u = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_{2k} \rightarrow u$ ($k \geq 2$) be an odd circuit in G of smallest length. Then $d(u, u_k) = k$, $d(u_k, u_{k+2}) = 2$ and $d(u, u_{k+2}) = k-1$. So u , u_k and

u_{k+2} have a median x in G . It follows that x equals neither u_k , nor u_{k+2} . Hence x is a common neighbour of u_k and u_{k+2} . This implies that

$$k-1 = d(u, u_{k+2}) = d(u, x) + 1 = d(u, u_k) = k,$$

which is impossible. So G is bipartite.

We deduce from this that if u and v are adjacent vertices, and w is any other vertex in G , then either u or v is the median of u , v and w .

The proof that any three vertices in G have a median consists of two steps.

STEP 1: $I(u, v, w) \neq \emptyset$ for any three vertices u , v and w of G .

Assume the contrary, and let u , v and w be vertices such that $I(u, v, w) = \emptyset$ and $d(u, v) + d(v, w) + d(w, u)$ is as small as possible. Furthermore, let $d(u, v)$ be as small as possible under these conditions. We may assume without loss of generality that $d(v, w) \geq d(u, w)$. So we have the following situation:

$$d(v, w) \geq d(u, w) \geq d(u, v) \geq 3.$$

It follows from the minimality of $d(u, v) + d(v, w) + d(w, u)$ that

$$I(u, v) \cap I(u, w) = \{u\},$$

and

$$I(u, v) \cap I(v, w) = \{v\}.$$

Let u' be a neighbour of u in $I(u, v)$. Since G is bipartite and u' is not in $I(u, w)$, we have

$$d(u', w) = 1 + d(u, w).$$

Then u' , v and w are vertices with

$$d(u', v) + d(v, w) + d(w, u') = d(u, v) + d(v, w) + d(w, u).$$

Since $d(u', v) = d(u, v) - 1$, it follows from the minimality of $d(u, v)$ that $I(u', v, w) \neq \emptyset$. Since

$$v \in I(u',v) \cap I(v,w) \subseteq I(u,v) \cap I(v,w) = \{v\},$$

it follows that v lies in $I(u',w)$. Hence

$$\begin{aligned} 1 + d(u,w) &= d(u',w) = d(u',v) + d(v,w) \\ &= d(u,v) - 1 + d(v,w) \geq d(v,w) + 2, \end{aligned}$$

which contradicts the fact that $d(v,w) \geq d(u,w)$. So for any three vertices u , v and w of G , we have $I(u,v,w) \neq \emptyset$.

STEP 2: $|I(u,v,w)| \leq 1$ for any three vertices u , v and w of G .

Again assume the contrary, and let u , v and w be such that $|I(u,v,w)| \geq 2$ and $d(u,v) + d(v,w) + d(w,u)$ is as small as possible.

Let x and y be two distinct vertices in $I(u,v,w)$. It follows from the minimality of $d(u,v) + d(v,w) + d(w,u)$ that

$$I(u,x) \cap I(u,y) = \{u\}.$$

Choose a neighbour u_x of u in $I(u,x)$ and a neighbour u_y of u in $I(u,y)$. Then, since G is triangle-free, we have $d(u_x, u_y) = 2$ (see Figure 3.5).

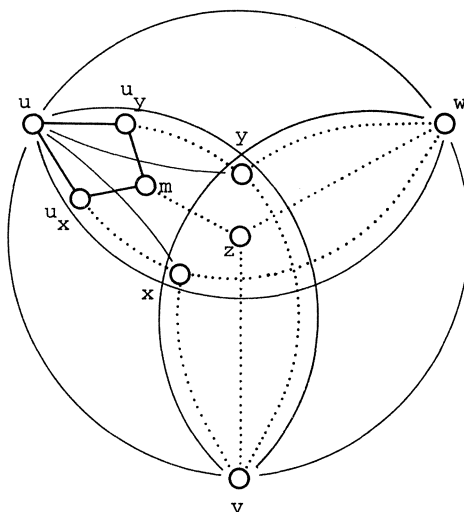


FIGURE 3.5.

Let m be the median of u_x , u_y and v , and let m' be the median of u_x , u_y and w . If $m \neq m'$, then the vertices u , m , m' , u_x and u_y would induce a $K_{2,3}$ in G , which is forbidden. So $m = m'$.

It follows from Step 1 that $I(m, v, w) \neq \emptyset$. Choose a vertex z from $I(m, v, w)$. Then z cannot equal both x and y , so suppose $z \neq x$. It follows that z and x lie in $I(u_x, v, w)$ (see Figure 3.5). Furthermore,

$$d(u_x, v) + d(v, w) + d(w, u_x) = d(u, v) + d(v, w) + d(w, u) - 2,$$

contradicting the choice of u , v and w . This concludes the proof. \square

3.2. THE STRUCTURE OF MEDIAN GRAPHS

Let us now try to get a deeper insight into the structure of median graphs. Unfortunately we have to prepare the ground a little bit before undertaking our enquiry. Some of the possible surprise about its outcome is thus taken away.

3.2.1. DEFINITION. A *cutset colouring* of a connected graph is an edge colouring of the graph such that, for any colour i , the set of edges assigned colour i is a cutset.

It can be shown that a connected graph G has a cutset colouring only if G is a simple bipartite graph that does not contain $K_{2,3}$ as a subgraph. In Figure 3.6 we give a graph with two distinct cutset colourings.

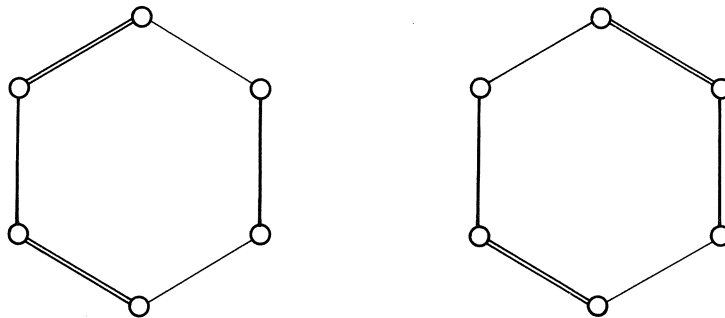


FIGURE 3.6.

3.2.2. DEFINITION. A connected graph is *uniquely cutset colourable* if it admits exactly one cutset colouring up to the labelling of the colours.

In a tree any cutset consists of a single edge, and so all trees are uniquely cutset colourable. Note that if we want to establish a cutset colouring of a connected graph, we are forced to assign the same colours to non-adjacent edges in any circuit of length four. Hence the graphs in Figure 3.7 are all uniquely cutset colourable. Likewise all hypercubes are uniquely cutset colourable. To each coordinate in the vector representation of Q_n there corresponds a cutset of the cutset colouring: the i -th cutset joins the vertices of the $(n-1)$ -cube induced by the vertices with 0 as the i -th coordinate to the vertices of the $(n-1)$ -cube induced by the vertices with 1 as the i -th coordinate.

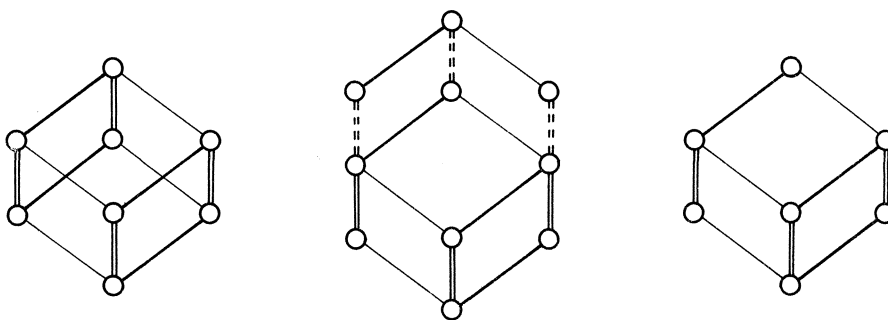


FIGURE 3.7.

For subsets S and T of the vertex-set of a graph, $[S, T]$ denotes the set of edges with one end in S and the other in T .

3.2.3. DEFINITION. Let G be a connected graph, and let W and W' be two subsets of V such that $W \cup W' = V$ and $W \cap W' \neq \emptyset$ and $[W \setminus W', W' \setminus W] = \emptyset$. The *expansion* of G with respect to W and W' is the graph G' constructed as follows from G :

- (i) replace each vertex $v \in W \cap W'$ by two vertices u_v and u'_v , which are joined by an edge;
- (ii) join u_v to all neighbours of v in $W \setminus W'$ and join u'_v to all

neighbours of v in $W' \setminus W$;

(iii) if $v, w \in W \cap W'$ are adjacent in G , then join u_v to u_w and u'_v to u'_w .

If W and W' are convex sets in G , then G' is called a *convex expansion* of G . We illustrate this construction in Figure 3.8.

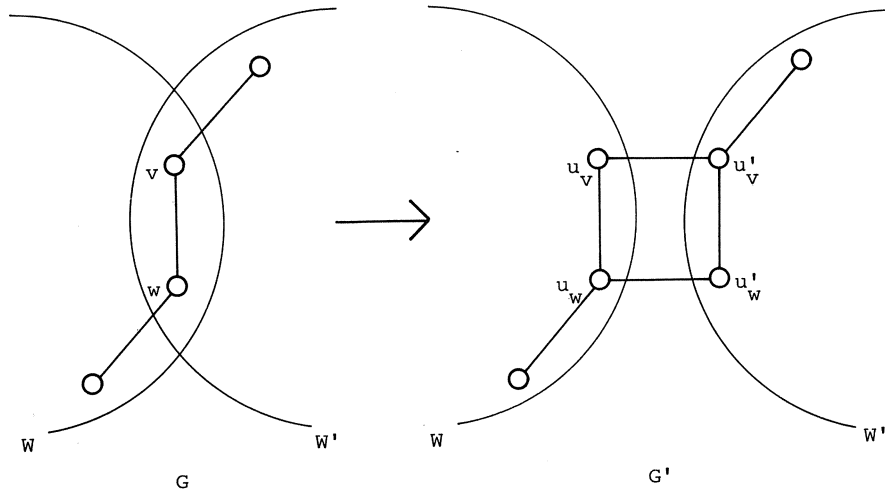


FIGURE 3.8.

In the next few pages we study median graphs in further detail. We reverse the order of theorem and proof -that is, we first give the "proof", and then we state the outcome of our enquiry in Theorem 3.2.4. The line of reasoning below is split into a number of steps.

Let G be a median graph. Recall that G is bipartite, and that we denote the median of u , v and w by $\langle u, v, w \rangle$. For any two adjacent vertices u and v of G , we let

$$W_u := \{w \in V \mid d(u, w) + 1 = d(v, w)\},$$

$$W_v := \{w \in V \mid d(u, w) = d(v, w) + 1\},$$

$$F_{uv} := [W_u, W_v],$$

$$U_u := \{w \in W_u \mid w \text{ is an end of an edge in } F_{uv}\},$$

$$U_v := \{w \in W_v \mid w \text{ is an end of an edge in } F_{uv}\}.$$

Fix an edge $e = ab$, and write $F = F_{ab}$. Note that W_a is the set of all vertices nearer to a than to b , and W_b is the set of all vertices nearer to b than to a (see Figure 3.9).

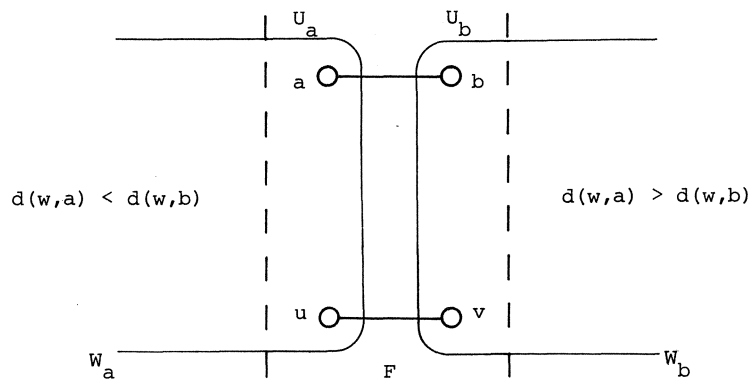


FIGURE 3.9.

- (1) *If two vertices u and v of G are adjacent, then $\langle u, v, w \rangle$ equals either u or v for any vertex w of G .*

PROOF. The assertion follows immediately from the definition of median. \square

- (2) *Let u and v be two adjacent vertices of G . Then*

$$W_u = \{w \in V \mid \langle u, w, v \rangle = u\},$$

$$\text{and } W_v = \{w \in V \mid \langle u, w, v \rangle = v\}.$$

PROOF. Use (1). \square

- (3) *Let u and v be two adjacent vertices of G . Then $W_v = V \setminus W_u$.*

PROOF. Use (1) and (2). \square

- (4) *$I(v, a) \subseteq W_a$ for any vertex v in W_a ,*

and $I(v,b) \subseteq W_b$ for any vertex v in W_b .

PROOF. The assertion follows immediately from the definition of W_a and W_b . \square

(5) F is a cutset.

PROOF. It follows from (3) that F is a disconnecting set. It follows from (4) that W_a and W_b induce connected subgraphs of G , and so F is a cutset. \square

(6) If uv is an edge in F with u in U_a and v in U_b , then $d(u,a) = d(v,b)$.

PROOF. Since uv is in F , we have

$$d(u,a) = d(u,b) - 1 \leq d(v,b) = d(v,a) - 1 \leq d(u,a).$$

\square

(7) $I(u,a) \subseteq U_a$ for any vertex u in U_a ,
and $I(u,b) \subseteq U_b$ for any vertex u in U_b .

PROOF. We prove only the first assertion by induction on $d(a,u)$.

Let v be a neighbour of u in U_b , and let w be a neighbour of u in $I(u,a) \subseteq W_a$. Then, since G is bipartite, we have $d(w,v) = 2$. It follows from (6) that

$$d(v,b) = d(u,a) = d(w,a) + 1 = d(w,b),$$

and so $\langle v,w,b \rangle$ is a common neighbour of v and w lying in $I(b,v) \subseteq W_b$ -that is w has a neighbour in W_b , and so w lies in U_a . By the induction hypothesis we have $I(a,w) \subseteq U_a$, and so it follows that $I(a,u) \subseteq U_a$. \square

(8) $W_a = W_u$ and $W_b = W_v$ for any edge uv in F with u in U_a and v in U_b .

PROOF. First let u be a neighbour of a , so that, by (6), v is a neighbour

of b . By (3) it suffices to prove that $W_a \subseteq W_u$ and $W_b \subseteq W_v$.

Choose a vertex w in W_a , and let $d(w,a) = k$. Then $d(w,b) = k + 1$.

CASE 1: $\langle a,u,w \rangle = u$.

Note that $d(u,w) = d(a,w) - 1 = k - 1$. Then we have

$$d(u,w) + 1 \geq d(v,w) \geq d(b,w) - 1 = k + 1 - 1 = k > d(u,w),$$

and so $d(v,w) = d(u,w) + 1$ -that is, w lies in W_u .

CASE 2: $\langle a,u,w \rangle = a$.

Note that $d(u,w) = d(a,w) + 1 = d(b,w) = k + 1$. Hence $\langle u,w,b \rangle = a$. Since G is bipartite, it follows that $d(v,w)$ equals either k or $k + 2$. If $d(v,w) = k$, then we would have $v = \langle u,w,b \rangle = a$, which is impossible. Hence

$$d(v,w) = k + 2 = d(u,w) + 1$$

-that is, w lies in W_u .

Similarly it follows that $W_b \subseteq W_v$.

Using (7), we deduce the general case $d(u,a) \geq 1$ by induction on $d(u,a)$. \square

(9) U_a and U_b are convex sets in G .

PROOF. Let u and u' be two vertices in U_a , and let v' be a neighbour of u' in U_b . By (8) we have $W_a = W_{u'}$, and $W_b = W_{v'}$, and so it follows that

$$U_a = U_{u'}, \quad \text{and} \quad U_b = U_{v'}.$$

Hence, if we replace a by u' , it follows from (7) that $I(u,u') \subseteq U_{u'} = U_a$. So U_a is convex. Likewise, U_b is convex. \square

(10) W_a and W_b are convex sets in G .

PROOF. The convexity of W_a and W_b follows from the convexity of U_a and U_b and the fact that

$$[W_a \setminus U_a, W_b \setminus U_b] = \emptyset.$$

□

(11) F is a matching between U_a and U_b .

PROOF. Assume the contrary, and assume that u in U_a has two distinct neighbours v and v' in U_b . Since G is bipartite, v and v' are not adjacent, so that u is in $I(v, v')$. It follows from the convexity of U_b that u lies in U_b , contradicting the choice of u . □

(12) The mapping $f: U_a \rightarrow U_b$, defined by $f(u) = v$ whenever $uv \in F$, induces an isomorphism between $G[U_a]$ and $G[U_b]$.

PROOF. Since F is a matching between U_a and U_b , the mapping f is bijective.

Let u and u' be two vertices in U_a . By (8), the result of (6) still holds, when we replace a by u' and b by $f(u')$, and so

$$d(u, u') = d(f(u), f(u')).$$

It follows that u and u' are adjacent if and only if $f(u)$ and $f(u')$ are adjacent. □

(13) G is uniquely cutset colourable.

PROOF. Let uv be an edge in F with u in U_a and v in U_b . By (8) we have $F = [W_u, W_v] = F_{uv}$. So, if we define the relation R on the edge-set E of G by

$$wx R yz \quad \text{if} \quad wx \in F_{yz} \quad \text{for } wx, yz \in E,$$

then this relation R is an equivalence relation on E . The equivalence classes of R are of the form $[W_x, W_y]$, where xy is an edge in G . Hence the equivalence classes are matchings as well as cutsets, and so they form a cutset colouring of G .

Let $a = u_0 \rightarrow u_1 \rightarrow \dots \rightarrow u_p = u$ be a path from a to u in $G[U_a]$. Such a path exists, since U_a is convex. Then it follows from (12) that

$b = f(u_0) \rightarrow \dots \rightarrow f(u_p) = v$ is a path from b to v in $G[U_b]$. As observed above, non-adjacent edges in a circuit of length four in G must be assigned the same colours in any cutset colouring of G . So the edges $ab = u_0f(u_0), u_1f(u_1), \dots, u_pf(u_p) = uv$ are to be assigned the same colour in any cutset colouring. This implies that F is the set of edges with the same colour as ab in any cutset colouring of G , and so the cutset colouring defined above is unique. \square

(14) *The mapping f , defined in (12), is colour-preserving.*

PROOF. Let uu' be an edge in $G[U_a]$. Then $u \rightarrow f(u) \rightarrow f(u') \rightarrow u' \rightarrow u$ is a circuit of length four in G , and so uu' and $f(u)f(u')$ have the same colour in the cutset colouring of G . \square

(15) *G can be obtained as a convex expansion from a median graph with fewer vertices, unless G is K_1 .*

PROOF. Construct the graph G' from $G[V \setminus U_b]$ by joining each vertex u in U_a to all the neighbours of $f(u)$ in $W_b \setminus U_b$. So G' is obtained from G by "contracting" F .

$$\text{Let } W'_b := (W_b \setminus U_b) \cup U_a.$$

Then we have (if G and H are isomorphic graphs, we write $G \cong H$)

$$G'[W_a] = G[W_a],$$

$$G'[W'_b] \cong G[W_b],$$

and

$$G'[U_a] = G[U_a] \cong G[U_b].$$

Using (9) and (10), we deduce from these equalities and isomorphisms that U_a, W_a and W'_b are convex sets in G' , the verification of which is left to the reader. It follows from the construction of G and Definition 3.2.3 that G is the convex expansion of G' with respect to W_a and W'_b .

It remains to prove that G' is a median graph. Let u, v and w be three vertices in W_a . Since W_a is convex in G , it follows that u, v and w have a median in G . Since $G[W_a] = G'[W_a]$ and W_a is convex in G' , this median of u, v and w in G is also the median of u, v and w in G' . Similarly, any

three vertices in W'_b have a median in G' .

Let u and v be vertices in W_a , and let w be a vertex in $W'_b \setminus W_a$. Then u , v and w have a median $x = \langle u, v, w \rangle$ in G . Note that by (10) x lies in W_a . Let P be a shortest (w, u) -path in G . It follows from the convexity of U_a and U_b that P passes through exactly one edge $f(u_p)u_p$ in F .

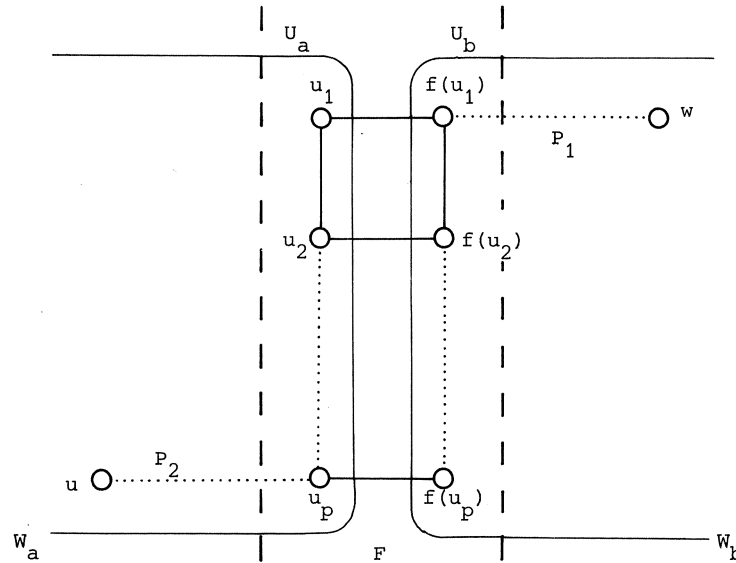


FIGURE 3.10.

Let $f(u_1)$ be the first vertex of P from w that lies in U_b . Since U_b is convex in G , it follows that P is of the form $P_1 \rightarrow f(u_1) \rightarrow f(u_2) \rightarrow \dots \rightarrow f(u_p) \rightarrow u_p \rightarrow P_2$, where $f(u_1) \rightarrow \dots \rightarrow f(u_p)$ is a shortest path from $f(u_1)$ to $f(u_p)$ in U_b (see Figure 3.10).

Let Q be the path $P_1 \rightarrow f(u_1) \rightarrow u_1 \rightarrow \dots \rightarrow u_p \rightarrow P_2$. Then Q is also a shortest (w, u) -path in G . Furthermore, Q contains exactly one vertex in U_b . It is clear that in determining the median of u , v and w in G , we can confine ourselves to paths of the same form as Q (that is, paths containing exactly one vertex in U_b). When we contract F , we obtain from Q a shortest (w, u) -path Q' in G' , where $Q' = P_1 \rightarrow u_1 \rightarrow \dots \rightarrow u_p \rightarrow P_2$. Any shortest (w, u) -path in G' can be obtained in this way from a shortest (w, u) -path in G of "type Q ". It follows from these observations that x is the unique median of u , v and w in G' .

Similarly, we prove that any two vertices u and v in W'_b and any vertex w in W_a have a unique median in G' . So G' is a median graph, and the proof is complete. \square

Before stating the outcome of our enquiry in Theorem 3.2.4, we prove a converse of (15).

(16) *Let G be a median graph, and let G' be a convex expansion of G with respect to the convex sets W and W' in G . Then G' is a median graph.*

PROOF. Write

$$U := \{u_v \mid v \in W \cap W'\},$$

and

$$U' := \{u'_v \mid v \in W \cap W'\},$$

where u_v and u'_v are as in Definition 3.2.3, and let

$$Z := (W \setminus W') \cup U,$$

and

$$Z' := (W' \setminus W) \cup U'.$$

Then $Z \cup Z'$ is the vertex-set of G' . It follows from the definition of expansion that

$$G'[Z] \cong G[W],$$

$$G'[Z'] \cong G[W'],$$

and

$$G'[U] \cong G'[U'].$$

Furthermore, it follows that Z and Z' are convex sets in G' . Since $G[W]$ and $G[W']$ are median graphs, as convex subgraphs of the median graph G , it follows that any three vertices in Z (or in Z') have a unique median in Z (or in Z').

Let u , v and w be vertices in G' not all in Z or all in Z' . Assume

that u and v lie in Z and w lies in Z' . Let x be the median of the vertices of G corresponding to u , v and w . Any shortest path P' in G' between a vertex in Z and a vertex in Z' can be obtained from a shortest path P in G between the corresponding vertices by "adding" an edge between U and U' to the path P . Since Z is convex in G' , the interval $I_{G'}(u,v)$ is contained in Z . Hence the vertex of G' in Z corresponding to x is the (uniquely determined) median of u , v and w in G' .

The case that u and v lie in Z' and w lies in Z is treated similarly. Hence G' is a median graph. \square

Using the above results, we deduce the following characterization of median graphs.

3.2.4. THEOREM. *A graph G is a median graph if and only if G can be obtained from K_1 by a sequence of convex expansions.*

In proving this theorem we have obtained several properties of median graphs that are interesting in their own right. We can also deduce these properties directly from the theorem. For convenience we state them as corollaries.

3.2.5. COROLLARY. *A median graph is uniquely cutset colourable.*

In Chapter 4 we discuss the family of sets W such that $[W, V \setminus W]$ is a colour class in the cutset colouring of a median graph G .

3.2.6. COROLLARY. *Let G be a median graph, and let $F = [W, V \setminus W]$ be a colour class in the cutset colouring of G . Let $U = \{u \in W \mid u \text{ is an end of an edge in } F\}$ and $U' = \{u \in V \setminus W \mid u \text{ is an end of an edge in } F\}$.*

Then

(i) $G[U]$, $G[U']$, $G[W]$ and $G[V \setminus W]$ are convex subgraphs of G , and hence median graphs;

(ii) the mapping $f: U \rightarrow U'$, defined by

$$f(u) = u' \quad \text{whenever } uu' \in F \text{ and } u \in U,$$

induces a colour-preserving isomorphism between $G[U]$ and $G[U']$.

Note that not all uniquely cutset colourable graphs are median graphs. For example, two of the three uniquely cutset colourable graphs in Figure 3.7 are not median graphs. Furthermore, when we delete a k -cube from Q_n , with $0 \leq k \leq n-3$, we get a uniquely cutset colourable graph, which is not a median graph.

From the theorem we can deduce that we can embed every median graph in a hypercube -that is, the hypercube contains an induced subgraph G' , which is isomorphic to G .

A *distance-preserving subgraph* G' of a graph G is a subgraph of G such that for any two vertices u and v of G' we have $d_{G'}(u,v) = d_G(u,v)$. Note that a distance-preserving subgraph is an induced subgraph.

3.2.7. THEOREM. *A graph G is a median graph if and only if G is a distance-preserving subgraph of a hypercube Q such that the median in Q of any three vertices of G is also a vertex of G .*

PROOF. If G is a subgraph of a hypercube Q as described in the theorem, then it follows that G is connected and for any three vertices of G their median in Q is also their (unique) median in G . So G is a median graph.

Conversely, let G be a median graph that is not K_1 . Then by Theorem 3.2.4, G is the convex expansion of a median graph H with fewer vertices, with respect to (say) W_1 and W_2 .

The proof is by induction on the number of vertices -that is, we may assume that H is a distance-preserving "median-closed" subgraph of an n -cube Q with vertex-set X .

We give a sketch of the proof. We double Q by a convex expansion with respect to X and X' ($= X$). Thereby we obtain an $(n+1)$ -cube \bar{Q} , of which Q is a "half" with respect to the newly-introduced colour class $[X, X']$ in the cutset colouring of \bar{Q} . The other "half" is the n -cube Q' induced by X' .

In doubling Q we have doubled also H , say, to \bar{H} with "halves" H in Q and H' in Q' . Let $W'_1 \cup W'_2$ be the vertex-set of H' , where W'_i corresponds to W_i in H ($i = 1, 2$). Then G is the subgraph of \bar{H} induced by $W_1 \cup W'_2$. It follows easily from the properties of the expansion procedure that G is a distance-preserving median-closed subgraph of \bar{H} , and so is of \bar{Q} . \square

We can sharpen this characterization of median graphs.

3.2.8. THEOREM. *A graph G is a median graph if and only if G is a connected induced subgraph of a hypercube Q such that the median in Q of any three vertices of G is also a vertex of G .*

PROOF. By the previous theorem it suffices to prove that a connected induced "median-closed" subgraph G of a hypercube Q is distance-preserving.

Assume the contrary, and let u and v be vertices of G with $d_G(u,v) \neq d_Q(u,v)$ such that $k = d_G(u,v)$ is as small as possible. Since G is an induced subgraph of Q , it follows that

$$d_Q(u,v) < d_G(u,v) = k.$$

Hence, since u and v are distinct, we have $k \geq 2$.

If $k = 2$, then we would have $1 \leq d_Q(u,v) < 2$, and so u and v would be adjacent in Q . This would imply that u and v are also adjacent in G , which contradicts $k \geq 2$. So we have $k \geq 3$.

Let x be a neighbour of v in G with $d_G(u,x) = k - 1$. Then it follows from the minimality of k that

$$d_G(u,x) = k - 1 = d_Q(u,x).$$

Hence $d_Q(u,v) \geq k - 2$. Since Q is bipartite, it follows that $d_Q(u,v) = k - 2$.

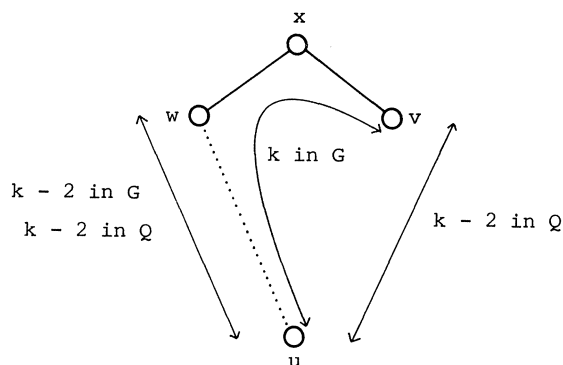


FIGURE 3.11.

Let w be a neighbour of x in G with $d_G(u,w) = k - 2$. Then $d_Q(u,w) = k - 2$, and $d_G(w,v) = d_Q(w,v) = 2$. Let z be the median of u, v and w in Q . Then z is a common neighbour of w and v in Q , and

$$d_Q(u,z) = k - 3 = d_Q(u,w) - 1 = d_Q(u,v) - 1.$$

Since G is median-closed, it follows that z is a vertex in G . Since G is an induced subgraph, it follows that z is a common neighbour of v and w in G . Hence

$$d_G(u,z) \leq d_G(u,w) + 1 = k - 1.$$

By the minimality of k we have

$$d_G(u,z) = d_Q(u,z) = d_Q(u,w) - 1 = k - 3,$$

and so

$$k = d_G(u,v) \leq d_G(u,z) + 1 = k - 2,$$

which is absurd. This gives the required contradiction. \square

Let G be a median graph with n colours in its cutset colouring. The smallest hypercube in which G can be embedded in the above sense is Q_n . If G is embedded in Q_m with $m \geq n$, then G is a subgraph of the n -dimensional subcube of Q_m induced by the colour classes of Q_m corresponding to the colours of G .

3.3. MEDIAN GRAPHS AND MEDIAN SEMILATTICES

One of the algebraic structures introduced by SHOLANDER ([S1], [S3]) is that of a median semilattice. M. Sholander has given two definitions of a median semilattice, one involving a binary operation with matching partial ordering (see Definition 3.3.2), and the other involving a ternary operation. In the latter case we use the term median algebra instead of median semilattice (see Definition 3.3.1). The axioms of the ternary

operation in Definition 3.3.1 are that of a normal graphic algebra, introduced by NEBESKÝ [N1].

3.3.1. DEFINITION. A *median algebra* (V, m) consists of a finite set V and a ternary operation $m: V \times V \times V \rightarrow V$ satisfying the following conditions:

- (m1) $m(u, v, u) = u$ for $u, v \in V,$
- (m2) $m(u, v, w) = m(w, v, u) = m(v, u, w)$ for $u, v, w \in V,$
- (m3) $m(m(u, v, w), w, x) = m(u, m(v, w, x), w)$ for $u, v, w, x \in V.$

The relationship between median algebras and median graphs has been established independently by AVANN [Av] and NEBESKÝ [N2], and follows from results in Chapter 7.

Here we study the relationship between median graphs and median semilattices in the sense of Definition 3.3.2.

A *semilattice* (V, \leq) is a finite poset in which any two elements u and v have a unique greatest lower bound, denoted by $u \wedge v$. Since V is finite, it follows that a semilattice contains a universal lower bound, usually denoted by 0 .

For u and v in V , let us denote

$$[u, v] := \{w \in V \mid u \leq w \leq v\}.$$

Such a set $[u, v]$ is called an *order interval* in the semilattice. Since V is finite, any non-empty order interval $[u, v]$ is a lattice with respect to the ordering \leq .

A semilattice is called *distributive* if each order interval $[0, u]$ is a distributive lattice. In Figure 3.12 we give the Hasse diagrams of three distributive semilattices.

If two elements u and v of a semilattice have a least upper bound, then this least upper bound is unique (and is denoted by $u \vee v$). For, assume that u and v have two distinct least upper bounds x and y . Note that in this case we have $x \neq x \wedge y \neq y$. Then it follows that $u, v \leq x \wedge y < x, y$ -that is, neither x nor y is a least upper bound.

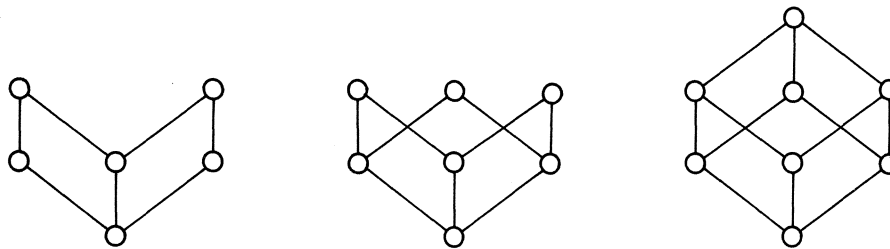


FIGURE 3.12.

A semilattice is said to have the *coronation property* if for any three elements u, v and w of the semilattice such that the three least upper bounds $u \vee v, v \vee w$ and $w \vee u$ exist, there exists a common least upper bound $u \vee v \vee w$. Of the three semilattices in Figure 3.12 the middle one does not have the coronation property.

3.3.2. DEFINITION. A *median semilattice* is a finite distributive semilattice with the coronation property.

On a median semilattice (V, \leq) we can define a ternary operation $\langle u, v, w \rangle = (u \wedge v) \vee (v \wedge w) \vee (w \wedge u)$, as is shown below in the proof of Theorem 3.3.3. It follows from results of SHOLANDER [S3], and also from results in Chapter 7, that this ternary operation defines a median algebra on V .

3.3.3. THEOREM. A graph G is a *median graph* if and only if G is the *diagraph* of a median semilattice.

PROOF. Let G be a median graph with interval function I , and embed G in a hypercube Q as in Theorem 3.2.7. Fix a vertex 0 of G , and let Q be the Hasse diagram of the Boolean lattice B on 2^n elements with 0 as universal lower bound. The orientation of Q induces an orientation of G , by which G is the Hasse diagram of a poset (V, \leq) with 0 as universal lower bound. Note that this ordering of V can also be defined as follows:

$$u \leq v \text{ whenever } u \in I(0,v) \qquad \text{for } u,v \in V.$$

Let u and v be two vertices of G . Since the greatest lower bound of u and v in B is the median of u , v and 0 in Q , it follows that u and v have this median as their unique greatest lower bound in the poset (V, \leq) . So (V, \leq) is a semilattice.

It follows from Theorem 1.3.6 and the properties of median graphs that each order interval $[0, u]$ in (V, \leq) is a distributive lattice. The coronation property of (V, \leq) follows from the fact that, for any three vertices u , v and w in Q , the median of $u \vee v$, $v \vee w$ and $w \vee u$ is precisely $u \vee v \vee w$. So (V, \leq) is a median semilattice with G as its diagraph.

Conversely, let G be the diagraph of a median semilattice (V, \leq) , and let d be the (graph-theoretical) distance function of G . Note that a distributive semilattice is a graded poset. Hence the ordering \leq is the canonical ordering of G with respect to the universal lower bound 0 of the semilattice (see Proposition 1.3.1).

First we prove by induction on $d(u, v)$ that

$$d(u, v) = d(u, u \wedge v) + d(u \wedge v, v) \qquad \text{for } u, v \in V.$$

If $u \leq v$ or $v \leq u$, then the assertion is clear. So let u and v be such that $u \wedge v < u$ and $u \wedge v < v$, and let $d(u, v) = k$. Note that $k \geq 2$.

Let w be a neighbour of v with $d(u, w) = k - 1$.

CASE 1: $w < v$.

Assume that $u \wedge w \neq u \wedge v$. Then, since $[0, v]$ is a distributive lattice and v covers w , it follows that $u \wedge v$ covers $u \wedge v \wedge w = u \wedge (v \wedge w) = u \wedge w$ (see Figure 3.13).

By the induction hypothesis we have

$$\begin{aligned} d(u, v) &= d(u, w) + 1 = d(u, u \wedge w) + d(u \wedge w, w) + 1 \\ &= d(u, u \wedge v) + 1 + d(u \wedge v, v) + 1 \\ &\geq d(u, v) + 2, \end{aligned}$$

which is a contradiction. Hence, it follows that $u \wedge w = u \wedge v$.

By the induction hypothesis we have

$$\begin{aligned} d(u,v) &= d(u,w) + 1 = d(u, u \wedge w) + d(u \wedge w, w) + 1 \\ &= d(u, u \wedge v) + d(u \wedge v, w) + 1 = d(u, u \wedge v) + d(u \wedge v, v). \end{aligned}$$

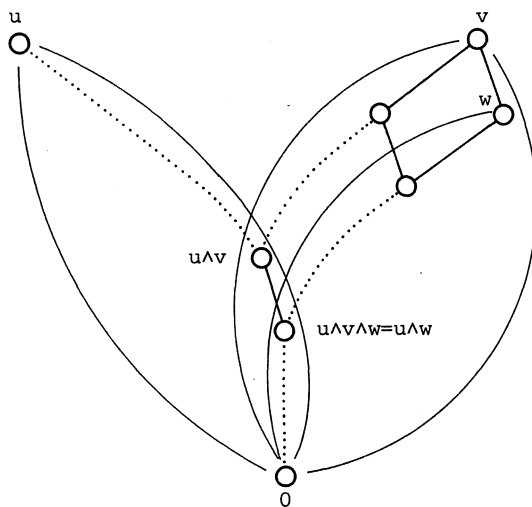


FIGURE 3.13.

CASE 2: $w > v$.

By the induction hypothesis we can find a neighbour w' of w with

$$d(u, w') = d(u, w) - 1 = k - 2,$$

such that $w' < w$. Let $v' = w' \wedge w$. Since w covers both v and w' in the distributive lattice $[0, w]$, it follows that w' and v both cover v' . Hence we have

$$d(u, v) - 1 \leq d(u, v') \leq d(u, w') + 1 = d(u, w) = d(u, v) - 1,$$

and so $d(u, v') = k - 1$. With this we have reduced Case 2 to Case 1.

By analogous reasoning we can prove that if u and v have a least

upper bound $u \vee v$, then

$$d(u,v) = d(u, u \vee v) + d(u \vee v, v).$$

In the proof we have to use the fact that $[0, u \vee v]$ is a distributive lattice containing u and v .

From these results we deduce that, if two vertices u and v have a least upper bound $u \vee v$, then

$$I(u,v) = \{w \mid u \wedge v \leq w \leq u \vee v\} = [u \wedge v, u \vee v],$$

where I is the interval function of G . Then, for any two vertices u and v of G , we have

$$I(u,v) = \bigcup [x \wedge y, x \vee y],$$

where the union is taken over all pairs of vertices x and y such that x lies in $[u \wedge v, u]$, y lies in $[u \wedge v, v]$ and $x \vee y$ exists. Hence

$$I(u,v) = \bigcup_{\substack{x,y \in I(u,v) \\ x \vee y \text{ exists}}} [x \wedge y, x \vee y].$$

This implies that G is interval monotone.

Finally, let u , v and w be vertices of G , and let $x = u \wedge v$, $y = u \wedge w$ and $z = v \wedge w$. Then x and y lie in $[0, u]$, and so $x \vee y$ exists. Similarly, it follows that $x \vee z$ and $y \vee z$ exist. Hence it follows from the coronation property of (V, \leq) that $x \vee y \vee z$ exists in (V, \leq) . Since

$$x \vee y \vee z = (u \wedge v) \vee (u \wedge w) \vee (v \wedge w),$$

it follows from the above derived properties of I that

$$x \vee y \vee z \in I(u,v,w).$$

Hence I is an interval structure on V , and so, by Theorem 3.1.6, G is a median graph. \square

The next two theorems follow directly from the proof of the last theorem. Theorem 3.3.5 is due to SHOLANDER [S3].

3.3.4. THEOREM. *Let V be a finite set. There exists a one-to-one correspondence between the median semilattices with V as set of elements and the ordered pairs $(G,0)$, where G is a median graph with vertex-set V and 0 is a vertex of G .*

3.3.5. THEOREM. *Let (V,\leq) be a median semilattice, and let a be an element in V . Define the ordering \leq_a of V by $u \leq_a v$ whenever $u \geq a \wedge v$ and $u = (a \wedge u) \vee (u \wedge v)$. Then (V,\leq_a) is a median semilattice with a as universal lower bound.*

3.4. THE n -CUBE AS A MEDIAN GRAPH

We conclude this chapter by deducing some new characterizations of Q_n from Theorem 3.2.7 (see [M3]).

3.4.1. THEOREM. *A graph G is (isomorphic to) Q_n if and only if G is a median graph with maximum degree n such that G contains two diametrical vertices, at least one of which has degree n .*

PROOF. The "only if" part of the proof follows immediately from the properties of Q_n .

Let G be a median graph, and let 0 and $\hat{0}$ be two diametrical vertices with $d(0) = n$. Embed G in a hypercube Q with its subset representation (as in Theorem 3.2.7) so that the vertex \emptyset of Q corresponds to 0 . Let $\{1\}, \{2\}, \dots, \{n\}$ represent the neighbours of 0 in G . Note that the distance functions of G and Q coincide on the vertex-set of G .

We prove that all subsets of $\{1,2,\dots,n\}$ are precisely the vertices of G , and so G is Q_n . This is accomplished by induction on the cardinality of the subsets of $\{1,2,\dots,n\}$.

Since 0 and $\hat{0}$ are diametrical, it follows that $N(0) \subset I(0,\hat{0})$, where I is the interval function of G .

Let $i \geq 1$, and assume that all sets $A \subseteq \{1,\dots,n\}$ with $|A| \leq i$ are vertices of G and that

$$d(A, \hat{0}) = \text{diam}(G) - |A|,$$

for all such sets A .

Choose a set $A \subseteq \{1, \dots, n\}$ with $|A| = i + 1$, and let B and C be two distinct i -subsets of A . Then

$$B \cup C = A,$$

and

$$|B \cap C| = i - 1.$$

Furthermore, the common neighbours of B and C in Q are A and $B \cap C$. By the induction hypothesis, B , C and $B \cap C$ are vertices of G . Moreover, we have

$$d(B \cap C, \hat{0}) = d(B, \hat{0}) + 1 = d(C, \hat{0}) + 1.$$

Since $d(B, \hat{0}) = d(C, \hat{0})$ and $d(B, C) = 2$, it follows that the median of B , C and $\hat{0}$ in Q is a common neighbour of B and C that has smaller distance from $\hat{0}$ than B or C . So A has to be the median of B , C and $\hat{0}$ in Q . This implies that A is in G and

$$d(A, \hat{0}) = d(B, \hat{0}) - 1 = \text{diam}(G) - |B| - 1 = \text{diam}(G) - |A|.$$

This completes the proof. \square

3.4.2. COROLLARY. *A graph G is Q_n if and only if G is a regular median graph of degree n .*

3.4.3. COROLLARY. *A graph G is Q_n if and only if G is a diametrical median graph of diameter n .*

Since by Theorem 3.3.3 the diagraph of a distributive lattice is a median graph, Theorem 1.3.9 follows from this last corollary. It is of course not necessary to use these heavy results for the proof of Theorem 1.3.9. But since we have them, why not use them?

CHAPTER 4

HELLY HYPERGRAPHS

In this chapter we discuss the relationship between median graphs and yet another mathematical structure: Helly hypergraphs. As a digression we derive an upper bound for the number of edges in a k -Helly hypergraph.

4.1. MEDIAN GRAPHS AND HELLY HYPERGRAPHS

The results in this section have been obtained in collaboration with A. Schrijver (see MULDER & SCHRIJVER [MS]).

Besides the algebraic structures mentioned in Chapter 3, there is another mathematical structure related to median graphs. This is a special class of Helly hypergraphs.

First we introduce some terminology. A *hypergraph* $H = (V, E)$ consists of a finite *vertex-set* V and a family $E \subset P(V)$ of non-empty subsets of V , the members of which are called *edges*. Note that in BERGE's terminology this is a simple hypergraph (see [Be]). Occasionally we write \bar{E} instead of (V, E) . For any two elements u and v in V , we write

$$I_{\bar{E}}(u, v) := \{B \in E \mid u \text{ and } v \text{ lie in } B\}.$$

4.1.1. DEFINITION. The *underlying graph* $G_{\bar{E}}$ of a hypergraph (V, E) has V as vertex-set, and two distinct vertices u and v are joined by an edge in $G_{\bar{E}}$ whenever $I_{\bar{E}}(u, v) \neq \emptyset$.

A hypergraph (V, E) is a *Helly hypergraph* if it has the *Helly property* -that is, each subfamily of E , any two members of which meet, has a non-

empty intersection. The underlying graph of a Helly hypergraph is triangle-free. Helly hypergraphs have been studied in particular by BERGE and DUCHET (see e.g. [Be], [BD] and [Du]). A characterization of Helly hypergraphs by GILMORE (see [Gi], or [Be, p. 396]) can be formulated as follows.

4.1.2. THEOREM. *A hypergraph (V, E) has the Helly property if and only if I_E is an interval structure on V .*

4.1.3. COROLLARY. *Let I be an interval structure on V . Then any family E of non-empty I -convex subsets of V has the Helly property.*

For any subset B of V , the family $\{B, V \setminus B\}$ is called a *copair* of V ; the copair $\{\emptyset, V\}$ is the *trivial copair* of V .

4.1.4. DEFINITION. A hypergraph (V, E) is a *copair hypergraph* if $V \setminus B$ is an edge for any edge B of E . A Helly copair hypergraph is a copair hypergraph with the Helly property.

4.1.5. DEFINITION. A *maximal Helly copair hypergraph* (V, E) is a Helly copair hypergraph such that, if $\{A, V \setminus A\}$ is a non-trivial copair of V and $E \cup \{A, V \setminus A\}$ has the Helly property, then A and $V \setminus A$ are edges of E .

In Figure 4.1 we give two examples of maximal Helly copair hypergraphs on six vertices (if a copair contains a singleton, we draw only the singleton in the figure). The corresponding underlying graphs are drawn in Figure 4.2.

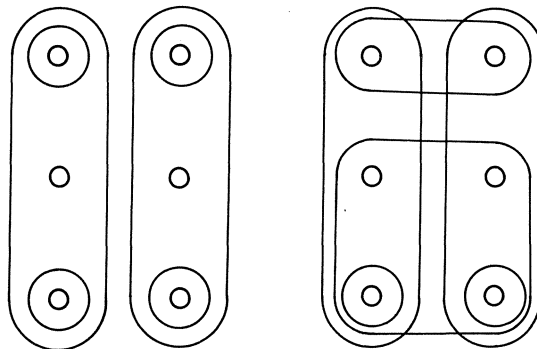


FIGURE 4.1.

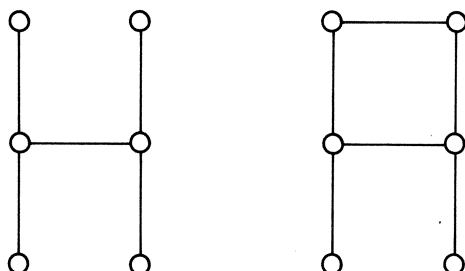


FIGURE 4.2.

4.1.6. DEFINITION. A hypergraph (V, \bar{E}) *separates vertices* if for any two distinct vertices u and v in V , there exists an edge A in \bar{E} such that u is in A and v is not.

Note that a hypergraph (V, \bar{E}) separates vertices if and only if $I_{\bar{E}}(v, v) = \{v\}$ for any vertex v in V .

4.1.7. THEOREM. Let (V, \bar{E}) be a Helly hypergraph. Then \bar{E} is maximal if and only if \bar{E} separates vertices.

PROOF. Assume that \bar{E} does not separate vertices, and let v be a vertex such that $|I_{\bar{E}}(v, v)| \geq 2$. Then the copair hypergraph $\bar{E} \cup \{\{v\}, V \setminus \{v\}\}$ has the Helly property, the verification of which is left to the reader. So \bar{E} is not maximal.

Let \bar{E} separate vertices, and let $\{A, V \setminus A\}$ be a non-trivial copair of V not in \bar{E} . Choose a vertex u in A and a vertex v in $V \setminus A$ such that $|I_{\bar{E}}(u, v)|$ is as small as possible.

First we prove that

$$I_{\bar{E}}(u, v) \cap A = \{u\}.$$

Assume the contrary, and let w be a vertex in $I_{\bar{E}}(u, v) \cap A$ distinct from u . Since \bar{E} separates vertices, there exists a copair $\{C, V \setminus C\}$ in \bar{E} with w in C and u in $V \setminus C$. Since w lies in $I_{\bar{E}}(u, v)$, it follows that v lies in C . This implies that u is not in $I_{\bar{E}}(w, v) \subset I_{\bar{E}}(u, v)$, contradicting the minimality of $I_{\bar{E}}(u, v)$.

Likewise we have

$$I_{\bar{E}}(u,v) \setminus A = \{v\}.$$

Hence we have proved that

$$I_{\bar{E}}(u,v) = \{u,v\}.$$

Let $\{B, V \setminus B\}$ be a copair in \bar{E} with v in B and u in $V \setminus B$. Since A is not in \bar{E} , and so A is not in $\{B, V \setminus B\}$, it follows that $A \cap B \neq \emptyset$ or $(V \setminus A) \cap (V \setminus B) \neq \emptyset$. Without loss of generality, let $A \cap B \neq \emptyset$. Then A and B , together with the set of edges containing both u and v , form a family of subsets of V , any two members of which meet. The intersection of this family is

$$I_{\bar{E}}(u,v) \cap A \cap B = \{u,v\} \cap A \cap B = \emptyset.$$

Hence $\bar{E} \cup \{A, V \setminus A\}$ is not a Helly hypergraph, from which the maximality of \bar{E} follows. \square

We can obtain from this theorem a lower bound for the number of edges in a maximal Helly copair hypergraph. It follows from Theorem 4.1.12 that this lower bound is best possible.

4.1.8. COROLLARY. *Let (V, \bar{E}) be a maximal Helly copair hypergraph. Then*

$$|\bar{E}| \geq 2 \lceil \log |V| \rceil,$$

where the logarithm is taken to base 2.

PROOF. Fix a vertex u in V . For any vertex v we define

$$E_v = \{B \in \bar{E} \mid u, v \in B\}.$$

Clearly $E_v \subseteq E_u$ for each vertex v . Since \bar{E} separates vertices, it follows that $E_v \neq E_w$ for any two distinct vertices v and w in V . Hence

$$|V| \leq 2^{|E_u|} = 2^{\frac{1}{2}|E|}.$$

The required inequality follows by taking logarithms. \square

In the next theorems we discuss the relationship between maximal Helly copair hypergraphs and median interval structures.

4.1.9. THEOREM. *Let (V, E) be a maximal Helly copair hypergraph. Then I_E is a median interval structure on V .*

PROOF. It follows from the definition of a Helly hypergraph that I_E is an interval structure.

Assume that there exist vertices u, v and w such that

$$x, y \in I_E(u, v, w),$$

for two distinct vertices x and y . By Theorem 4.1.7 there is a copair $\{B, V \setminus B\}$ in E with x in B and y in $V \setminus B$. It follows that one of the edges B and $V \setminus B$ must contain at least two of the three vertices u, v and w . Without loss of generality, let u and v be in B . Then y is not in $I_E(u, v)$, contradicting the choice of u, v and w . \square

4.1.10. THEOREM. *Let I be a median interval structure on a finite set V , and let*

$$E_I := \{B \subset V \mid \emptyset \neq B \neq V, \text{ and } B \text{ and } V \setminus B \text{ are } I\text{-convex}\}.$$

Then (V, E_I) is a maximal Helly copair hypergraph.

PROOF. It follows from Corollary 4.1.3 that E_I is a Helly copair hypergraph. Hence by Theorem 4.1.7 it suffices to show that E_I separates vertices.

Assume the contrary, and let u and v be two distinct vertices, for which there is no separating copair in E_I , and such that $|I(u, v)|$ is as small as possible.

First we prove that $I(u, v) = \{u, v\}$. Assume that there exists a vertex w in $I(u, v)$ distinct from u and v . Since v is not in $I(u, w) \subset I(u, v)$, it

follows from the minimality of $I(u,v)$ that there exists an edge A with u in A and w not in A . Since u and v cannot be separated, we have v in A . By definition A is an I -convex set in V , and so w lies in $I(u,v) \subset A$, contradicting the fact that w is not in A . Hence

$$I(u,v) = \{u,v\}.$$

Let

$$B := \{z \in V \mid v \notin I(u,z)\}.$$

Since I is a median interval structure, it follows that $I(u,z) \cap I(z,v) \cap I(u,v)$ is a singleton. This implies that

$$V \setminus B = \{z \in V \mid u \notin I(z,v)\}.$$

Then $\{B, V \setminus B\}$ is a copair which separates u and v . So if B and $V \setminus B$ are I -convex, then the required contradiction is established.

We now prove that B is I -convex. Since each interval in V is I -convex (condition (I1) in Definition 3.1.3), it follows from the definition of B that

$$I(u,z) \subseteq B \quad \text{for } z \in B.$$

Let x and y be two vertices in B . It follows from the definition of B that u lies in $I(v,x)$ as well as in $I(v,y)$, and so

$$I(u,x) \subseteq I(v,x),$$

and

$$I(u,y) \subseteq I(v,y).$$

Hence there exists a vertex z in B such that

$$\{z\} = I(u,x) \cap I(x,y) \cap I(y,u) = I(v,x) \cap I(x,y) \cap I(y,v).$$

Assume that $I(x,y) \not\subseteq B$, and let w be a vertex in $I(x,y) \setminus B$. Then it follows from the fact that z lies in $I(v,x) \cap I(y,v)$ that

$$\{z\} \subseteq I(z,w) \cap I(z,v) \subseteq I(x,y) \cap I(z,v)$$

$$\subseteq I(x,y) \cap I(x,v) \cap I(y,v) = \{z\},$$

and so $I(w,z) \cap I(z,v) = \{z\}$. Since I is a median interval structure, it follows that z lies in $I(w,v)$, and so

$$I(z,v) \subseteq I(w,v).$$

Since w is not in B , it follows that $I(w,v) \subseteq V \setminus B$, and so z is in $V \setminus B$, contradicting the fact that z lies in B . So we have proved that B is I -convex. The I -convexity of $V \setminus B$ is treated similarly. This completes the proof. \square

From these two theorems we deduce the following result.

4.1.11. COROLLARY. *Let I be a median interval structure on the set V .*

Then

$$I_{E_I} = I.$$

Let (V,E) be a maximal Helly copair hypergraph. Then

$$E_{I_E} = E.$$

By these theorems we have established a relationship between median graphs and maximal Helly copair hypergraphs. The direct relationship between these mathematical structures is given in the next theorem.

Let G be a median graph. Any cutset from the cutset colouring of G induces a copair of V : the deletion of the cutset from G splits the graph into two components, the vertex-sets of which form the complementary subsets of a copair. Let us call the copairs of V induced by the cutset colouring of G the *canonical copairs* of G .

4.1.12. THEOREM. *A hypergraph (V,E) is a maximal Helly copair hypergraph*

if and only if \mathcal{E} consists of the canonical copairs of a median graph with vertex-set V .

PROOF. Let G be a median graph with vertex-set V and interval function I , and let $\mathcal{E}_G \subset \mathcal{P}(V)$ be the family consisting of the canonical copairs of G . Then \mathcal{E}_G consists of I -convex subsets of V . Furthermore, \mathcal{E}_G separates vertices, and so \mathcal{E}_G is a maximal Helly copair hypergraph. Note that two distinct vertices of G are adjacent if and only if

$$n \{B \in \mathcal{E}_G \mid u, v \in B\} = \{u, v\},$$

-that is, $G_{I_{\mathcal{E}_G}} = G$.

Conversely, let (V, \mathcal{E}) be a maximal Helly copair hypergraph, and let $G_{\mathcal{E}}$ be its underlying graph. By Theorems 4.1.9 and 3.1.5 it follows that $G_{\mathcal{E}}$ is a median graph. It follows from Corollary 4.1.11 that each edge B in \mathcal{E} is $I_{G_{\mathcal{E}}}$ -convex. Hence, for any copair $\{B, V \setminus B\}$ in \mathcal{E} , the set $[B, V \setminus B]$ must be a cutset in the cutset colouring of $G_{\mathcal{E}}$. So \mathcal{E} consists of canonical copairs of $G_{\mathcal{E}}$. From the first part of the proof we conclude that \mathcal{E} consists of all the canonical copairs of $G_{\mathcal{E}}$. \square

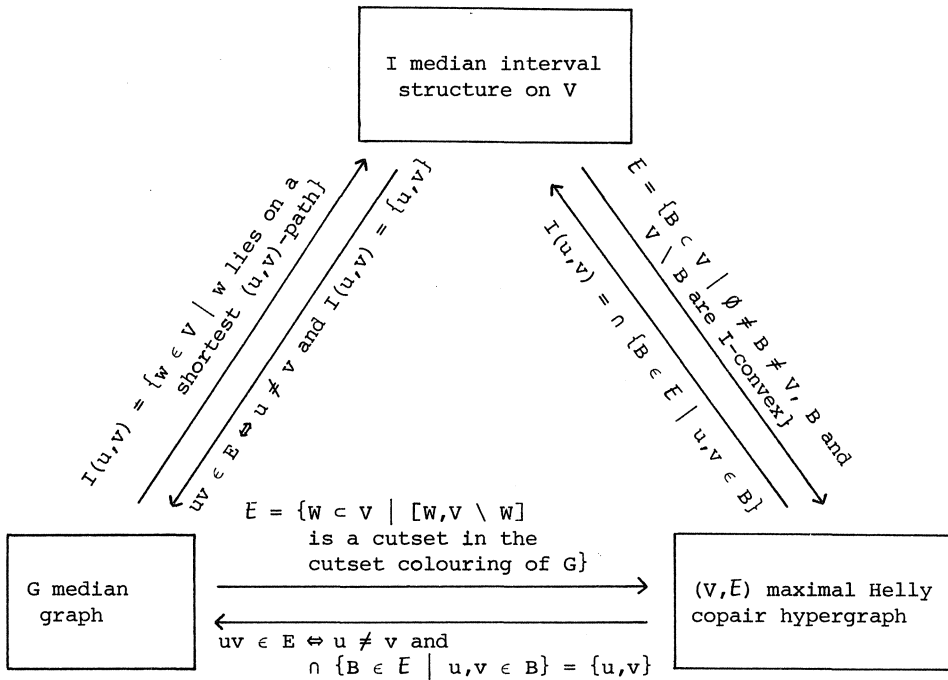
We recapitulate the above results and Theorems 3.1.4 and 3.1.5 in the next theorem, which is the main result of MULDER & SCHRIJVER [MS].

4.1.13. THEOREM. *Let V be a finite set. Then there exist one-to-one correspondences between the median interval structures on V , the maximal Helly copair hypergraphs with vertex-set V , and the median graphs with vertex-set V as follows:*

- (i) *let I be a median interval structure on V ; then*
 - (V, \mathcal{E}) is a maximal Helly copair hypergraph, where \mathcal{E} consists of the I -convex copairs of V ;
 - G is a median graph, where two distinct vertices u and v are adjacent whenever $I(u, v) = \{u, v\}$;
- (ii) *let (V, \mathcal{E}) be a maximal Helly copair hypergraph; then*
 - I is a median interval structure, where
 - $I(u, v) = n\{B \in \mathcal{E} \mid u, v \in B\}$, for $u, v \in V$;

- G is a median graph, where G is the underlying graph of \bar{E} ;
- (iii) let G be a median graph; then
 - I is a median interval structure, where I is the interval function of G ;
 - (V, \bar{E}) is a maximal Helly copair hypergraph, where \bar{E} consists of the canonical copairs of G .

This theorem can be summarized in the following diagram.



Since the number of edges in a maximal Helly copair hypergraph (V, \bar{E}) equals twice the number of colours in the cutset colouring of its underlying graph, it follows from Theorem 3.2.8 that the lower bound for the number of edges derived in Corollary 4.1.8 is best possible.

In the rest of this section we derive an upper bound for the number of edges in a maximal Helly copair hypergraph.

Let G be a connected graph with n vertices admitting a cutset colouring. Since each cutset contains edges of a spanning tree, the number of colours in the cutset colouring of G is at most $n - 1$.

4.1.14. PROPOSITION. *Let G be a connected graph with n vertices admitting a cutset colouring. Then the number of colours in the cutset colouring of G is $n - 1$ if and only if G is a tree.*

PROOF. If G is a tree, then any cutset in G consists of a single edge, and so G is uniquely cutset colourable with $n - 1$ colours.

Conversely, let T be a spanning tree of G . Then T has $n - 1$ edges, so the edges of T all have different colours. Hence every edge of T determines exactly one cutset of the cutset colouring -that is, if e is an edge of T and $T[W]$ and $T[V \setminus W]$ are the components of $T - e$, then $[W, V \setminus W]$ is the cutset in the cutset colouring of G containing e .

Assume that there is an edge joining u and v in G that is not in T . Then the (u, v) -path in T contains at least two distinct edges f_1 and f_2 . So the edge uv is in the colour class determined by f_1 as well as in the colour class determined by f_2 . This is impossible, and so G equals T . \square

We use the term *maximum* to mean: "with a maximal number of edges". The next theorem follows immediately from Theorem 4.1.12 and the last proposition.

4.1.15. THEOREM. *A hypergraph (V, E) is a maximum Helly copair hypergraph if and only if E consists of the canonical copairs of a tree with vertex-set V .*

4.1.16. COROLLARY. *Let (V, E) be a Helly copair hypergraph. Then*

$$|E| \leq 2(|V| - 1).$$

An upper bound for the number of edges in a Helly hypergraph follows easily from this corollary and is due to MILNER (see [Er]).

4.1.17. COROLLARY. *Let (V, E) be a Helly hypergraph. Then*

$$|E| \leq 2^{|V|-1} + |V| - 1.$$

PROOF. In $\mathcal{P}(V)$ there are $2^{|V|-1}$ copairs. Of at most $|V| - 1$ of these

copairs both complementary sets are in \bar{E} . \square

This upper bound is attained by those hypergraphs (V, \bar{E}) in which there exists a vertex v such that \bar{E} consists of all subsets of V containing v . In other words, \bar{E} consists of the canonical copairs of the star $K_{1, n-1}$, with v as vertex of degree $n - 1$, together with all possible non-empty intersections of members of these copairs. This observation can be established by direct verification. It follows also from Theorem 4.2.6.

4.2. DIGRESSION: THE NUMBER OF EDGES IN A k -HELLY HYPERGRAPH

In this section we give extensions of the last inequality of the previous section, and some related results.

4.2.1. DEFINITION. A hypergraph \bar{E} is k -linked if any k members, not necessarily distinct, have a non-empty intersection.

Note that every hypergraph is both 0-linked and 1-linked.

4.2.2. DEFINITION. A hypergraph \bar{E} is a k -Helly hypergraph if any k -linked subhypergraph $\bar{E}' \subseteq \bar{E}$ has a non-empty intersection.

A 2-Helly hypergraph is just a Helly hypergraph in the usual sense. A hypergraph (V, \bar{E}) is a 1-Helly hypergraph (and simultaneously a 0-Helly hypergraph) if and only if $\cap \bar{E} \neq \emptyset$. It is easily verified that, for $k \geq 2$, any subset of V of size less than k can be added to a k -Helly hypergraph without destroying the " k -Helly" property.

A k -uniform hypergraph has all its edges of size k . By $\mathcal{P}_k(V)$ we denote the family of all k -subsets of V . For v in V , the family of subsets of V containing v is denoted by $\{v\}^\uparrow$.

Before deriving an upper bound for the number of edges in a k -Helly hypergraph we prove another result.

4.2.3. DEFINITION. Let \bar{E} be a hypergraph. A *pointer* of an edge B in \bar{E} is a subset of B of size $|B| - 1$, which is not contained in any other edge of \bar{E} .

4.2.4. THEOREM. Let p be an integer with $p \geq 2$, and let (V, E) be a p -uniform hypergraph with $|V| = n$. If each edge of E has a pointer, then

$$|E| \leq \binom{n-1}{p-1}.$$

Furthermore, equality holds if and only if there exists a vertex v in V such that

$$E = \{v\}^\uparrow \cap \mathcal{P}_p(V).$$

PROOF. For each edge B of E we fix a preferred pointer B' in $\mathcal{P}_{p-1}(V)$. Let \mathcal{B} be the family of preferred pointers. We define a bipartite graph G as follows: $E \cup (\mathcal{P}_{p-1}(V) \setminus \mathcal{B})$ is the vertex-set of G and there is an edge between B in E and A in $\mathcal{P}_{p-1}(V) \setminus \mathcal{B}$ whenever A is contained in B . Then each vertex B in E has degree $p - 1$ in G , and each vertex A in $\mathcal{P}_{p-1}(V) \setminus \mathcal{B}$ has degree at most $n - p + 1$ in G .

By counting the edges in G between E and $\mathcal{P}_{p-1}(V) \setminus \mathcal{B}$ twice, we get

$$\begin{aligned} (p-1)|E| &\leq (n-p+1)|\mathcal{P}_{p-1}(V) \setminus \mathcal{B}| \\ &= (n-p+1)\left(\binom{n}{p-1} - |E|\right), \end{aligned}$$

from which the upper bound follows.

If equality holds, then each vertex A in $\mathcal{P}_{p-1}(V) \setminus \mathcal{B}$ must have degree exactly $n - p + 1$ in G , and so $A \cup \{u\}$ is an edge in E for each u in $V \setminus A$. Furthermore, any edge B in E has exactly one pointer: its preferred pointer.

Let B be an edge in E , and let $B' = B \setminus \{v\}$ be its pointer. Then we have to prove that any k -subset of V containing v is in E . Let $A = B \setminus \{w\}$ be a $(p-1)$ -subset of B containing v . Then A is not a pointer and has degree $n - p + 1$ in G . Let u be an arbitrary vertex in $V \setminus B$. Then $C = A \cup \{u\}$ is in E .

If $C \setminus \{v\}$ is not the pointer of C , then $(C \setminus \{v\}) \cup \{w\}$ would be an edge in E . Since

$$B' = B \cap ((C \setminus \{v\}) \cup \{w\}),$$

this contradicts the fact that B' is the pointer of B . Hence $C \setminus \{v\}$ is the pointer of C .

So we have proved that any p -subset C of V containing v with $|B \Delta C| = 2$ is an edge of E .

Applying the preceding argument on any edge containing v , we get the required result. \square

4.2.5. COROLLARY. *Let (V, E) be a p -uniform k -Helly hypergraph with $|V| = n$. If $p > k$, then*

$$|E| \leq \binom{n-1}{p-1}.$$

Furthermore, equality holds if and only if there exists a vertex v in V such that

$$E = \{v\}^\uparrow \cap P_p(V).$$

PROOF. If $k \leq 1$, then $nE \neq \emptyset$, and the assertion follows immediately.

If $k \geq 2$, then it follows from the fact that $p > k$ that each edge in E must have a pointer. Hence the result follows from the previous theorem. \square

4.2.6. THEOREM. *Let (V, E) be a k -Helly hypergraph with $|V| = n$. Then*

$$|E| \leq 2^{n-1} + \sum_{i=1}^{k-1} \binom{n-1}{i}.$$

Furthermore, equality holds if and only if there exists a vertex v in V such that

$$E = \{v\}^\uparrow \cup \bigcup_{i=1}^{k-1} P_i(V).$$

PROOF. Write $E_i = E \cap P_i(V)$. Then

$$|E_i| \leq \binom{n}{i} \quad \text{for } i = 1, \dots, k-1.$$

Furthermore, by Corollary 4.2.5 we have

$$|E_i| \leq \binom{n-1}{i-1} \quad \text{for } i = k+2, \dots, n.$$

Consider $E_k \cup E_{k+1}$. Each edge in E_{k+1} must have a pointer that is not an edge in E_k . Let A be the family of non-edge pointers of edges in E_{k+1} . Then we have

$$\begin{aligned} |E_k \cup E_{k+1}| &= |E_k| + |E_{k+1}| \leq |E_k| + |A| \\ &= |E_k \cup A| \leq \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}. \end{aligned}$$

Combining the three inequalities we get the upper bound for $|E|$.

To get a maximum k -Helly hypergraph we must have equality in all of the above inequalities (if possible). So

$$E_i = P_i(v) \quad \text{for } i = 1, \dots, k-1.$$

Moreover, there must exist a vertex v_i in V such that

$$E_i = \{v_i\}^\uparrow \cap P_i(v) \quad \text{for } i = k+2, \dots, n.$$

Finally, we have

$$|E_k \cup E_{k+1}| = \binom{n}{k}.$$

If $k \geq n - 1$, then the assertion is easily verified, and is left to the reader. If $k \leq n - 2$, then it follows that $v = v_{k+2} = \dots = v_n$, for some vertex v in V . Furthermore, every edge in $E_k \cup E_{k+1}$ must contain v , and so we have

$$E_i = \{v\}^\uparrow \cap P_i(v) \quad \text{for } i = k, k+1.$$

□

A maximum 1-uniform 1-Helly hypergraph \bar{E} consists of a unique singleton. A maximum 2-uniform 2-Helly hypergraph is just a triangle-free graph with the maximum number of edges -that is, a complete bipartite graph $K_{\lfloor \frac{1}{2}n \rfloor, \lfloor \frac{1}{2}n \rfloor}$. (This follows from the famous theorem of P. Turán concerning K_p -free graphs with a maximum number of edges -see, for

example, [Be]; the special case of K_3 -free graphs used here has already been proved in 1907 by WYTHOFF [Wy].)

We give a preliminary result concerning the number of edges in maximum k -uniform k -Helly hypergraphs.

4.2.7. PROPOSITION. *Let $f_k(n)$ be the number of edges in a maximum k -uniform k -Helly hypergraph with n vertices. Then*

$$f_k(n+1) \leq \left\lfloor \frac{n+1}{n-k+1} f_k(n) \right\rfloor \quad \text{for } n \geq k.$$

PROOF. Let (V, E) be a maximum k -uniform k -Helly hypergraph with $|V| = n+1$. Then, for each vertex v in V , we have

$$|E \cap \mathcal{P}(V \setminus \{v\})| \leq f_k(n).$$

Hence

$$\sum_{v \in V} |E \cap \mathcal{P}(V \setminus \{v\})| \leq (n+1) f_k(n).$$

On the left-hand side of this inequality the edges of E are each counted with multiplicity $(n-k+1)$, and so the left-hand side equals $(n-k+1)f_k(n+1)$. □

The above mentioned theorem of Wythoff follows from this proposition by direct verification. We conclude this section by stating a conjecture on 3-uniform 3-Helly hypergraphs.

4.2.8. CONJECTURE. *A 3-uniform 3-Helly hypergraph (V, E) is maximum if and only if there exists a partition A_0, A_1, A_2 of V with*

$$||A_i| - |A_j|| \leq 1 \quad \text{for } 0 \leq i < j \leq 2$$

such that E consists of the following 3-subsets of V :

- (i) $B \in \mathcal{P}_3(V)$ such that $|B \cap A_i| = 1$ for $i = 0, 1, 2$;
- (ii) $B \in \mathcal{P}_3(V)$ such that $|B \cap A_i| = 2$ and $|B \cap A_{i+1}| = 1$, where $i + 1$ is taken modulo 3, for $i = 0, 1, 2$;

With the use of Proposition 4.2.7, it can be verified that the conjecture is true for $|v| \leq 12$.

CHAPTER 5

HAMMING GRAPHS

In this chapter we study another generalization of hypercubes: the Hamming graphs (graph products of complete graphs). This generalization is more restricted than those in Chapters 1 and 2, and paves the way for the study of quasi-median graphs in the next chapter. We characterize Hamming graphs using the interval function, and introduce the extended odd graphs as a class of graphs that shows the scope of some properties involved in characterizations derived in this Chapter and Chapter 1.

5.1. A CHARACTERIZATION OF HAMMING GRAPHS

Let a_1, \dots, a_n be positive integers, and let V be the Cartesian product $\prod_{i=1}^n \{0, 1, \dots, a_i - 1\}$. In coding theory the following terms are used: the *Hamming distance* of two vectors in V is the number of coordinates in which they differ, and the *weight* of a vector is the number of its non-zero coordinates (see [CL]). The following definition gives a natural generalization of hypercubes with their vector representation.

5.1.1. DEFINITION. Let a_1, \dots, a_n be positive integers. The *Hamming graph* H_{a_1, \dots, a_n} is the graph with vertex-set $\prod_{i=1}^n \{0, 1, \dots, a_i - 1\}$, in which two vertices are joined by an edge if and only if the corresponding vectors differ in exactly one coordinate.

For example, Q_n is the Hamming graph H_{a_1, \dots, a_n} with $a_1 = \dots = a_n = 2$. The Hamming graphs $H_{4,2}$ and $H_{3,3}$ are given in Figure 5.1.

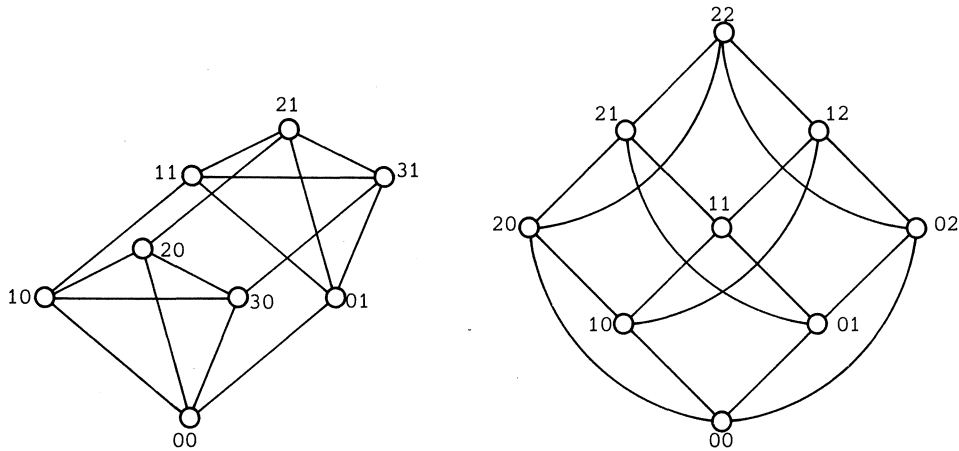


FIGURE 5.1.

Note that a Hamming graph H_{a_1, \dots, a_n} is the graph with vertex-set $V = \prod_{i=1}^n \{0, 1, \dots, a_i - 1\}$ such that the Hamming distance and the distance function of the graph coincide. It follows from Definition 5.1.1 and the definition of the product of graphs that

$$H_{a_1, \dots, a_n} = K_{a_1} \times \dots \times K_{a_n}.$$

This graph is regular of degree $\sum_i a_i - 1$. Furthermore, for any two vertices u and v in V , the interval $I(u, v)$ induces a hypercube of dimension $d(u, v)$. This follows easily from Definition 5.1.1 and immediately from Proposition 1.2.6. So a Hamming graph is interval-regular. Moreover, K_4 -e is not an induced subgraph of a Hamming graph (see Figure 1.7). For any vertex u of H_{a_1, \dots, a_n} , the subgraph induced by $N(u)$ consists of the disjoint union of complete graphs of sizes $a_1 - 1, \dots, a_n - 1$.

The automorphism group of H_{a_1, \dots, a_n} is generated by those automorphisms of the following types:

- (i) renumbering the elements of a coordinate-set;
- (ii) interchanging two coordinate-sets of the same size.

Hence a Hamming graph is vertex-transitive -that is, any vertex can be taken as the zero vector. Furthermore, H_{a_1, \dots, a_n} is distance-transitive if and only if $a_1 = \dots = a_n$. Coordinate-sets of size 1 do not influence the structure of a Hamming graph and are therefore not considered in the sequel. The problem of characterizing Hamming graphs was posed by FOLDES [F2]. One other term before we study Hamming graphs in more detail: the j -th level of a Hamming graph consists of all vertices of weight j .

5.1.2. THEOREM. *Let G be a connected graph with interval function I . Then G is a Hamming graph if and only if G is interval-regular and does not contain K_4 -e as an induced subgraph and I satisfies the following condition:*

$$I(u,v) \cap I(v,w) = \{v\} \Rightarrow d(u,w) \geq \max\{d(u,v), d(v,w)\},$$

for any three vertices u, v and w of G .

PROOF. To prove the "only if" part, let u, v and w be vertices of a Hamming graph with $I(u,v) \cap I(v,w) = \{v\}$. Without loss of generality, we may assume that v corresponds to the zero vector. Then u has exactly $d(u,v)$ non-zero coordinates, and w has exactly $d(v,w)$ non-zero coordinates. Furthermore, u and w have no equal non-zero coordinates. Hence u and w differ in at least $\max\{d(u,v), d(v,w)\}$ coordinates. The other conditions on G have already been checked in the remarks preceding the theorem.

The "if" part of the proof requires two preliminary steps.

First assume that G contains an induced circuit of length 5, $u \rightarrow v \rightarrow w \rightarrow x \rightarrow y \rightarrow u$ say. Then we have $d(u,w) = d(u,x) = 2$, and $d(w,x) = 1$. Hence $I(u,w) \cap I(u,x)$ contains a vertex p , which is adjacent to u, w and x (see Figure 5.2). Since K_4 -e does not occur in G , it follows that p is not adjacent to y or to v . Similarly, there is a vertex q adjacent to x, u and v , but not to y or w . So y, p and q are three distinct common neighbours of u and x , contradicting the interval-regularity of G . Hence each circuit of length 5 in G contains a diagonal.

The next step is to prove that, for any two vertices u and v of G , the interval $I(u,v)$ induces a hypercube of dimension $d(u,v)$. By Theorem

1.2.3, it suffices to prove that there are no edges joining two vertices within the same level in $I(u,v)$.

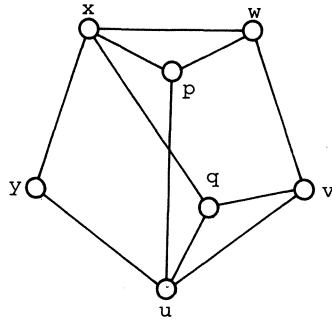


FIGURE 5.2.

Assume the contrary, and let x and y be two neighbours in the i -th level in $I(u,v)$. Choose a vertex z in $I(u,x) \cap I(u,y)$ such that $d(z,x) = d(z,y)$ is minimal. Then we have

$$I(z,x) \cap I(z,y) = \{z\},$$

and so

$$1 = d(x,y) \geq \max\{d(z,x), d(z,y)\} \geq 1.$$

Hence z is adjacent to x and y , and z lies in $N_{i-1}(u,v)$. Similarly there is a vertex w in $N_{i+1}(u,v)$, which is adjacent to x and y . Thus x, y, z and w induce a K_4 -e in G , which is forbidden. So each interval in G induces a hypercube.

Let us now prove that G is a Hamming graph. Fix a vertex u of G . Since K_4 -e does not occur in G , it follows that the subgraph induced by $N(u)$ is the disjoint union of complete graphs. Let A_1, \dots, A_n be the maximal cliques in $N(u)$. Since $I(u,v)$ induces a hypercube for any vertex v of G , we have

$$|I(u,v) \cap A_i| \leq 1 \quad \text{for } i = 1, \dots, n, \text{ and for } v \in V.$$

Let $|A_i| = a_i$, for $i = 1, \dots, n$, and label the vertices of clique A_i

with $1, \dots, a_i$. By induction on j we prove that $\bigcup_{i=0}^j N_i(u)$ is isomorphic to the subgraph of H_{a_1+1, \dots, a_n+1} induced by the bottom $j+1$ levels. We do this by representing each vertex v of G by a vector (v_1, \dots, v_n) in $\prod_{i=1}^n \{0, 1, \dots, a_i\}$, where

$$v_i = 0 \quad \text{if } I(u, v) \cap A_i = \emptyset,$$

and

$$\{v_i\} = I(u, v) \cap A_i \quad \text{if } I(u, v) \cap A_i \neq \emptyset.$$

Note that u is represented by the zero vector, and any vertex in the i -th level is represented by a vector of weight i .

For $j \leq 1$ the assertion is clear, so let $j > 1$, and assume that the set $\bigcup_{i=0}^{j-1} N_i(u)$ induces the bottom j levels of H_{a_1+1, \dots, a_n+1} .

Choose a vertex v in $N_j(u)$. Since the interval $I(u, v)$ induces a j -cube Q_j in G , it follows that v is adjacent to a vertex x in $N_{j-1}(u)$ if and only if the Hamming distance between v and x is 1. Let x and y be two distinct neighbours of v in $N_{j-1}(u)$. Then it follows that $d(x, y) = 2$, and that x and y have a unique common neighbour in $N_{j-2}(u)$. Since G is interval-regular, it follows that x and y have no common neighbour other than v in $N_j(u)$. This implies that two distinct vertices in $N_j(u)$ have at most one common neighbour in $N_{j-1}(u)$. Hence it follows from the induction hypothesis that distinct vertices in $N_j(u)$ are represented by distinct vectors of weight j .

Choose a vector of weight j , $(v_1, \dots, v_j, 0, \dots, 0)$, say. By the induction hypothesis the vertices in $N_{j-1}(u)$ represented by $(v_1, \dots, v_{j-1}, 0, \dots, 0)$ and $(0, v_2, \dots, v_j, 0, \dots, 0)$ have a unique common neighbour in $N_{j-2}(u)$ and no common neighbour in $N_{j-1}(u)$. So their second common neighbour lies in $N_j(u)$, and this common neighbour must be represented by $(v_1, \dots, v_j, 0, \dots, 0)$. Hence all vectors of weight j are used as representatives of vertices in $N_j(u)$.

Finally, the edges joining two vertices in $N_j(u)$ have to be checked.

First let v and v' be two distinct vertices in $N_j(u)$ with Hamming distance 1 - say, $v = (v_1, v_2, \dots, v_j, 0, \dots, 0)$ and $v' = (v'_1, v_2, \dots, v_j, 0, \dots, 0)$ with $v_1 \neq v'_1$. In Figure 5.3 we depict a circuit of length 5 containing v and v' . The only diagonal possible in this circuit is vv' . So by the first step in the proof v and v' are adjacent.

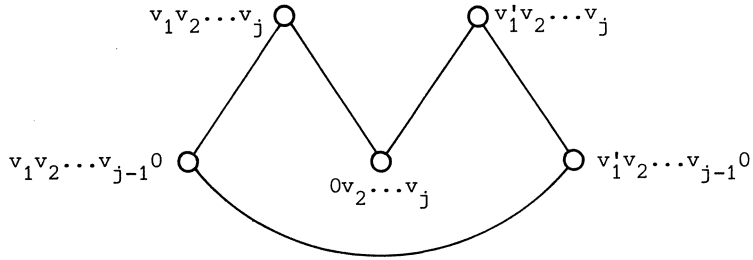


FIGURE 5.3.

Let v and w be two vertices in $N_j(u)$ with Hamming distance at least 2. Let z be a vertex in $I(u,v) \cap I(u,w)$ with $I(z,v) \cap I(z,w) = \{z\}$. If $d(v,z) \geq 2$, or if $d(z,w) \geq 2$, then v and w are not adjacent. So assume that $d(v,z) = d(z,w) = 1$ (see Figure 5.4).

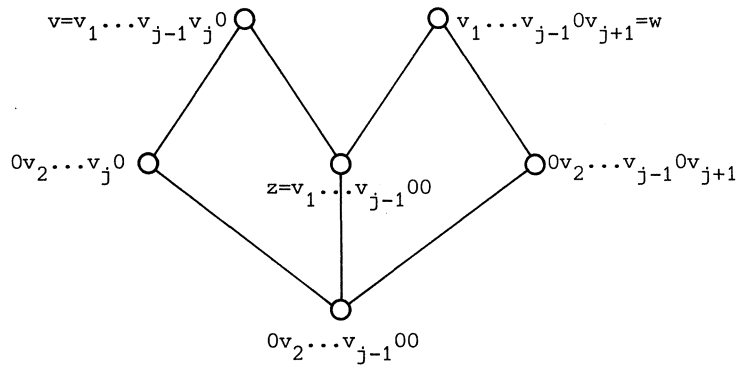


FIGURE 5.4.

If v and w were joined by an edge, then a circuit of length 5 without diagonals would be formed. Hence v and w are not adjacent.

So we have proved that $\bigcup_{i=0}^j N_i(u)$ induces a subgraph of G , on which the Hamming distance and the graph distance coincide. This completes the proof. \square

We conclude this section with some easy results concerning graphs G

with interval function I satisfying

$$I(u,v) \cap I(v,w) = \{v\} \Rightarrow d(u,w) \geq \max\{d(u,v), d(v,w)\},$$

for any three vertices u , v and w of G . In the next section we give an example of a graph showing that this condition on I is necessary in the foregoing theorem.

5.1.3. PROPOSITION. *If G is a triangle-free graph with interval function I such that*

$$I(u,v) \cap I(v,w) = \{v\} \Rightarrow d(u,w) \geq \max\{d(u,v), d(v,w)\},$$

for any three vertices u , v and w of G , then G is bipartite.

PROOF. The argument is the same as in the step of the proof of Theorem 5.1.2, where we verified that each interval induces a hypercube. \square

5.1.4. PROPOSITION. *Let G be a connected graph with interval function I . Then G is complete if and only if*

$$I(u,v) \cap I(v,w) = \{v\} \Rightarrow d(u,w) = \max\{d(u,v), d(v,w)\},$$

for any three distinct vertices u , v and w of G .

PROOF. If G is complete, then for any three distinct vertices u , v and w of G ,

$$I(u,v) \cap I(v,w) = \{v\},$$

and

$$d(u,v) = d(v,w) = d(w,u) = 1.$$

Conversely, assume that G is not complete. Let u and w be vertices of G with $d(u,w) = 2$, and let v be a common neighbour of u and w . Then we have

$$I(u,v) \cap I(v,w) = \{v\},$$

and

$$d(u,w) = 2 > 1 = d(u,v) = d(v,w).$$

□

5.1.5. PROPOSITION. *Let G be a connected graph with interval function I . Then $I(u,v,w) \neq \emptyset$ for any three vertices u, v and w of G if and only if*

$$I(u,v) \cap I(v,w) = \{v\} \Rightarrow d(u,w) > \max\{d(u,v), d(v,w)\},$$

for any three distinct vertices u, v and w of G .

PROOF. First we prove the "if" part. Assume the contrary, and let u, v and w be three distinct vertices of G such that $I(u,v,w) = \emptyset$ and $d(u,v) + d(v,w) + d(w,u)$ is as small as possible. Then we have

$$I(u,v) \cap I(v,w) = \{v\},$$

and

$$I(v,w) \cap I(w,u) = \{w\}.$$

It follows that

$$d(u,w) > d(v,w) > d(u,w),$$

which is impossible.

Conversely, let u, v and w be distinct vertices of G with $I(u,v) \cap I(v,w) = \{v\}$. Since $I(u,v,w) \neq \emptyset$, it follows that v lies in $I(u,w)$. So v lies on a shortest (u,w) -path, and v is distinct from u and w . This implies that $d(u,w) > \max\{d(u,v), d(v,w)\}$. □

5.2. EXTENDED ODD GRAPHS

We might wonder whether, for a connected graph G , the condition " $I(u,v)$ induces a $d(u,v)$ -dimensional hypercube for any two vertices u and v of G " implies that G is a Hamming graph. That this is not the case is shown by the example below.

Let k be an integer with $k \geq 2$. The *odd graph* O_k has the $(k-1)$ -subsets of $\{1, \dots, 2k-1\}$ as vertices and two vertices are adjacent if their

corresponding subsets are disjoint (see [Ko], [BFB], and [B2]). The graph O_k ($k \geq 2$) is a distance-transitive graph which is regular of degree k . The small odd graphs are the triangle K_3 , and the *Petersen graph* (see Figure 5.5). For $k \geq 4$, the smallest circuit in O_k has length 6.

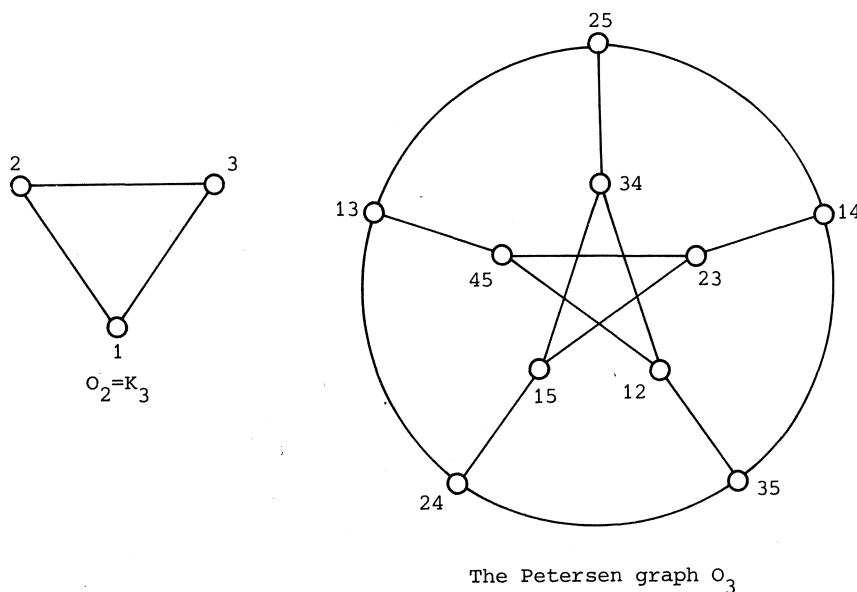


FIGURE 5.5.

5.2.1. DEFINITION. Let k be an integer with $k \geq 2$. The *extended odd graph* E_k has $\{A \subset \{1, \dots, 2k-1\} \mid |A| \leq k-1\}$ as vertex-set, and two vertices A and B are joined by an edge whenever

$$|A \Delta B| = 1, \quad \text{or} \quad |A \Delta B| = 2k-2.$$

The small extended odd graphs are the complete graph K_4 , and the *Greenwood-Gleason graph* (see Figure 5.6). This graph was introduced by GREENWOOD & GLEASON [GG] in order to construct a colouring of the edges of K_{16} with three colours and without monochromatic triangles. KALBFLEISCH & STANTON [KS] have proved that each colour class in such a colouring of K_{16} induces a graph isomorphic to the Greenwood-Gleason graph. This graph is the complement of the Clebsch graph (see Section 2.4), and it contains the graph in Figure 1.13, so that it is not the diagraph of a poset.

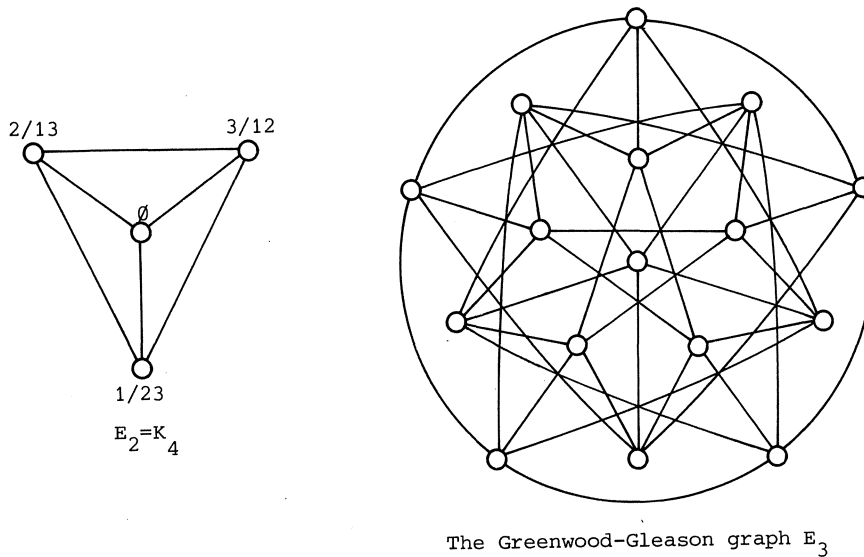


FIGURE 5.6.

The graph E_k is regular of degree $2k-1$. It consists of the "lower half" of Q_{2^k-1} with the odd graph O_k on the $(k-1)$ th level. Let I be the interval function of E_k . Then for any vertex A of E_k , the interval $I(\emptyset, A)$ is just the power set $\mathcal{P}(A)$ of A . Hence if A and B are two disjoint $(k-1)$ -subsets of $\{1, \dots, 2k-1\}$, then

$$I(\emptyset, A) \cap I(\emptyset, B) = \{\emptyset\},$$

and

$$d(\emptyset, A) = d(\emptyset, B) = k-1 \geq 1 = d(A, B).$$

So, for $k \geq 3$, the graph E_k does not satisfy the condition on I in Theorem 5.1.2. Note that E_k is not a Hamming graph, for $k \geq 3$.

The extended odd graphs and the halfcubes introduced in Section 2.1 can be brought under a "common" definition involving the concept of copair (see Section 4.1). Let n be a positive integer, and let V be a set of size n . The graph G_n has the copairs of V as vertices, and two vertices $\{A, V \setminus A\}$ and $\{B, V \setminus B\}$ are joined by an edge whenever

$$|A \Delta B| = 1 \quad \text{or} \quad |A \Delta V \setminus B| = 1.$$

For $n = 2k$, the graph G_n is the halfcube $\frac{1}{2}Q_{2k}$, whereas for $n = 2k-1$, the graph G_n is the extended odd graph E_k ($k \geq 2$).

5.2.2. PROPOSITION. *The graph E_k is distance-transitive.*

PROOF. If A and B are vertices of E_k with $d(\emptyset, A) = d(\emptyset, B)$, then $|A| = |B|$. Let f be a permutation of the set $\{1, \dots, 2k-1\}$ which maps A onto B . Then f induces an automorphism of E_k which maps \emptyset onto \emptyset and A onto B .

Let $\{j\}$ be a neighbour of \emptyset . Then

$$\begin{aligned} A \rightarrow A \cup \{j\} & \quad \text{if } j \notin A \text{ and } |A| < k-1, \\ A \rightarrow A \setminus \{j\} & \quad \text{if } j \in A, \\ A \rightarrow \{1, \dots, 2k-1\} \setminus (A \cup \{j\}) & \quad \text{if } j \notin A \text{ and } |A| = k-1, \end{aligned}$$

defines an automorphism of E_k .

The automorphisms of E_k of this type generate the automorphism group of E_k . Hence E_k is distance-transitive. \square

5.2.3. COROLLARY. *For any two vertices A and B of E_k , the subgraph induced by $I(A, B)$ is a hypercube of dimension $d(A, B)$.*

PROOF. It follows from the definition of E_k that $I(\emptyset, A)$ induces a hypercube of dimension $|A|$, for any vertex A of E_k . Since E_k is distance-transitive, the assertion follows for any two vertices of E_k . \square

5.2.4. COROLLARY. *The smallest odd circuit in E_k has length $2k-1$.*

PROOF. Let C be an odd circuit in E_k . Since E_k is distance-transitive, we may assume that \emptyset is a vertex of C . Hence it follows from the definition of E_k that C must contain an edge joining two vertices within the $(k-1)$ th level. \square

The extended odd graph E_k shows that in Corollary 1.2.5 the condition that G is bipartite cannot be weakened to " G does not contain odd circuits

of length less than $2k-1$ ". This answers a question of FOLDES [F1] in the negative. Using Seidel's eigenvalue techniques (see [Se], or [CL]), J.A. Bondy had already exhibited a sequence of graphs, the first of which is E_3 , showing that the condition "G is triangle-free" is not sufficient ([F2]).

CHAPTER 6

QUASI-MEDIAN GRAPHS

In this chapter we deal with quasi-median graphs, which are a generalization of median graphs analogous to the generalization of the hypercubes to the Hamming graphs. We characterize quasi-median graphs using the interval function, and also by an expansion procedure.

6.1. PSEUDO-MEDIANS AND QUASI-MEDIANS

The n -cube Q_n is a median graph, whereas its generalization the Hamming graph H_{a_1, \dots, a_n} is usually not a median graph. A natural generalization of a median of three vertices in a graph can be given analogous to the generalization of Q_n to H_{a_1, \dots, a_n} .

6.1.1. DEFINITION. Let G be a graph, and let (u, v, w) be an ordered triple of vertices of G . An ordered triple (x, y, z) of vertices of G is a *pseudo-median* of the triple (u, v, w) if it satisfies the following conditions:

- (P1) there is a shortest (u, v) -path in G on which both x and y lie,
 there is a shortest (v, w) -path in G on which both y and z lie,
 and there is a shortest (w, u) -path in G on which both z and x lie;
- (P2) $d(x, y) = d(y, z) = d(z, x)$;
- (P3) $d(x, y)$ is minimal under the conditions (P1) and (P2).

The distance $d(x, y)$ is the *size* of the pseudo-median (x, y, z) .

An alternative formulation of condition (P1) is the following:

$$d(u,x) + d(x,y) + d(y,v) = d(u,v),$$

$$d(v,y) + d(y,z) + d(z,w) = d(v,w),$$

and

$$d(w,z) + d(z,x) + d(x,u) = d(w,u).$$

The triple (u,v,w) in the graph of Figure 6.1 has two pseudo-medians of size 1, and in Figure 6.2, the triple (u,v,w) has a unique pseudo-median.

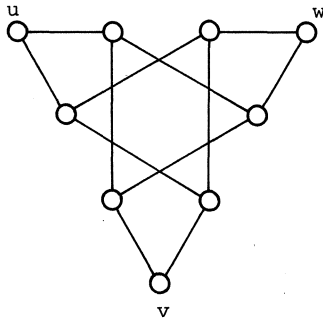


FIGURE 6.1.

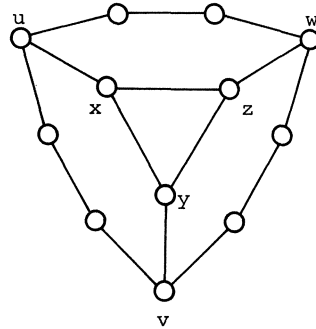


FIGURE 6.2.

6.1.2. DEFINITION. Let G be a graph, and let (u,v,w) be an ordered triple of vertices of G . An ordered triple (x,y,z) of vertices of G is the *quasi-median* of the triple (u,v,w) if it is a pseudo-median of (u,v,w) , and if (u,v,w) has no pseudo-median distinct from (x,y,z) .

The triple (u,v,w) in Figure 6.2 has a quasi-median. In accordance with the terminology developed in Chapter 3, a quasi-median of size 0 is called a *median*.

Let $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots, w_n)$ be vertices in the Hamming graph H_{a_1, \dots, a_n} . Then the triple (u,v,w) has a quasi-median (x,y,z) . This quasi-median can be determined as follows: if u_i , v_i and w_i are all distinct, then x has u_i as i -th coordinate, y has v_i as i -th coordinate, and z has w_i as i -th coordinate; if u_i , v_i and w_i are not all distinct, then x , y and z all have p_i as i -th coordinate, where p_i is the integer occurring at least twice among u_i , v_i and w_i . The size of the quasi-median is the number of coordinates in which u , v and w are all distinct.

Let G be a graph, and let u, v and w be vertices of G . Let (x,y,z) be a pseudo-median of (u,v,w) . Then (x,y,z) is the quasi-median of the triple (x,y,z) , or (more informally) the triple (x,y,z) is its own quasi-median. The vertices x, y and z satisfy the following condition:

$$(P1') \quad \begin{aligned} x &\in I(u,v) \cap I(u,w), \\ y &\in I(v,u) \cap I(v,w), \\ \text{and } z &\in I(w,u) \cap I(w,v). \end{aligned}$$

Let $(P3')$ be condition $(P3)$, where $(P1)$ is replaced by $(P1')$. The conditions $(P1')$, $(P2)$ and $(P3')$ do not imply that (x,y,z) is a pseudo-median of (u,v,w) -see, for example, Figure 6.3, where x, y and z satisfy $(P1')$, $(P2)$ and $(P3')$. In this graph x, y and z have p as median, whereas p is not in $I(u,v,w)$.

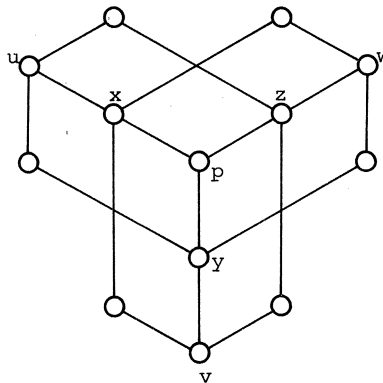


FIGURE 6.3.

It follows from the definition of pseudo-median that if the vertices u, v, w, x, y and z of a graph G satisfy condition $(P1)$, then a pseudo-median of (x,y,z) is also a pseudo-median of (u,v,w) . The next result follows easily from this observation.

6.1.3. PROPOSITION. *Let G be a graph in which each ordered triple of vertices has a pseudo-median. Then*

$$I(x,y) \cap I(x,z) = \{x\},$$

$$I(y,x) \cap I(y,z) = \{y\},$$

and

$$I(z,x) \cap I(z,y) = \{z\},$$

for any pseudo-median (x,y,z) in G .

The graph in Figure 6.4 shows the necessity of the condition "each ordered triple has a pseudo-median" in the above proposition.

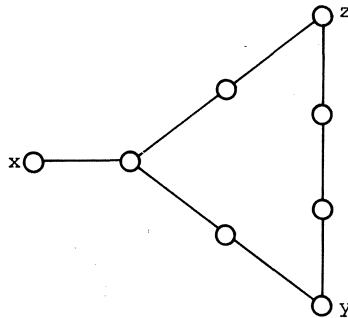


FIGURE 6.4.

6.2. THE STRUCTURE OF QUASI-MEDIAN GRAPHS

The following definition gives the right generalization of the concept of median graph. Although at first sight it seems rather odd, the reason for choosing this definition will become clear as this section progresses. We denote a circuit of length n by C_n .

6.2.1. DEFINITION. A connected graph G is a *quasi-median graph* if it satisfies the following conditions:

- (Q1) each ordered triple of vertices of G has a quasi-median;
- (Q2) $K_4 - e$ is not an induced subgraph of G ;
- (Q3) each induced C_6 in G has Q_3 or $H_{3,3}$ as convex closure.

In Figure 6.5 we give the graphs occurring in this definition (for an alternative picture of $H_{3,3}$ see Figure 5.1). All median graphs and all

Hamming graphs are quasi-median graphs (see the observations following Definition 6.1.2). In Figure 6.6 we give another example of a quasi-median graph.

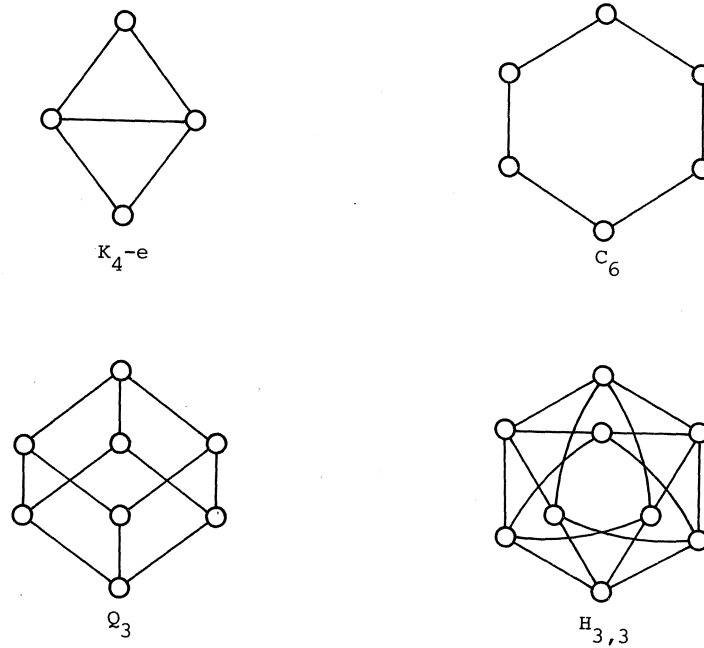


FIGURE 6.5.

In the above definition we can replace condition (Q3) by the weaker but less elegant condition

(Q3') if $u_1 \rightarrow \dots \rightarrow u_6 \rightarrow u_1$ is an induced C_6 in G with $d(u_1, u_4) = 3$, then there exist vertices u_7 and u_8 in G such that $\{u_1, \dots, u_8\}$ induces a Q_3 in G .

Essential steps in the proof that (Q1), (Q2) and (Q3') imply (Q3) are

- (i) $K_{2,3}$ is not an induced subgraph of G ,
- (ii) C_5 is not an induced subgraph of G .

It is left to the reader to complete the proof of this assertion.

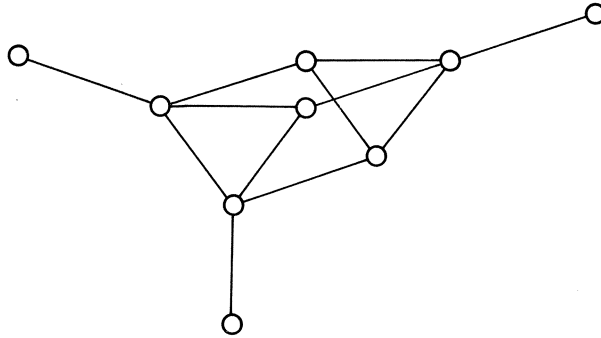


FIGURE 6.6.

The three conditions (Q1), (Q2) and (Q3) in Definition 6.2.1 are independent. For example, C_6 satisfies the first two conditions, and the same holds for the graph in Figure 6.7. The graphs C_5 and $K_{2,3}$ satisfy the last two conditions. The wheel on six vertices (see Figure 6.8) and K_4-e satisfy conditions (Q1) and (Q3).

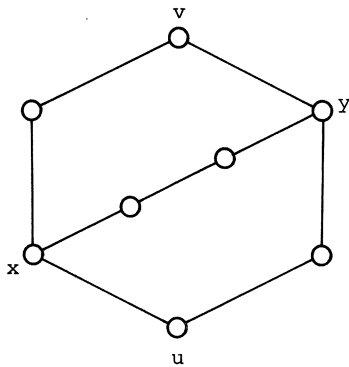


FIGURE 6.7.

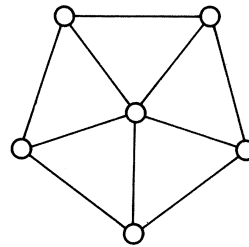


FIGURE 6.8.

6.2.2. THEOREM. *If G is a quasi-median graph with interval function I , then G is interval monotone, and I satisfies the following condition:*

- (Q4) *if $I(u,v) \cap I(v,w) = \{v\}$, then $d(u,w) \geq \max\{d(u,v), d(v,w)\}$, for any three vertices u, v and w of G .*

PROOF. In order to prove that I satisfies condition (Q4), let u, v and w be vertices of G with $I(u,v) \cap I(v,w) = \{v\}$. Then it follows that there are vertices x and z in G such that (x,v,z) is the quasi-median of (u,v,w) . It follows from (P1) that

$$\begin{aligned} d(u,w) &= d(u,x) + d(x,z) + d(z,w) \\ &= d(u,v) + d(z,w) \\ &= d(u,x) + d(v,w). \end{aligned}$$

Hence I satisfies condition (Q4).

The proof that G is interval monotone is more tedious. First we prove that each interval in G induces a bipartite subgraph.

Let u and v be two vertices, and assume that there exist two adjacent vertices x and y within the i -th level in $I(u,v)$. Then there is a vertex w such that (w,x,y) is the quasi-median of (u,x,y) . Since x and y are adjacent, w is joined to both x and y . Moreover, we have $d(u,w) = d(u,x) - 1$, and so w lies in the $(i-1)$ th level in $I(u,v)$. Similarly, there is a vertex z in the $(i+1)$ th level in $I(u,v)$ adjacent to x and y . Then w, x, y and z induce a K_4 -e in G , which is not allowed. Hence $I(u,v)$ induces a bipartite graph.

Assume that G is not interval monotone. Choose vertices u, v, x and y in G such that

$$x, y \in I(u,v), \text{ and } I(x,y) \not\subseteq I(u,v)$$

with $d(u,x)$ as small as possible. Note that $x \neq u$, and $d(x,y) \geq 2$. We write $k = d(x,y)$.

Since $I(x,y) \not\subseteq I(u,v)$, there exists a shortest (x,y) -path $x = x_0 \rightarrow x_1 \rightarrow \dots \rightarrow x_{k-1} \rightarrow x_k = y$, of which at least one internal vertex does not lie in $I(u,v)$. Let y_1 be a neighbour of x in $I(u,x)$. Then we have

$$d(u,y_1) = d(u,x) - 1.$$

Hence it follows from the minimality of $d(u,x)$ that

$$I(y_1, y) \subseteq I(u, v).$$

Since x and y_1 are adjacent, we have

$$1 + d(x, y) \geq d(y_1, y) \geq d(x, y) - 1.$$

If $d(y_1, y) = 1 + d(x, y)$, then x lies on a shortest (y_1, y) -path, and so $I(x, y) \subset I(y_1, y) \subseteq I(u, v)$. This contradicts the choice of u, v, x and y . If $d(y_1, y) = d(x, y)$, then the quasi-median of (x, y_1, y) is of size 1, and so a triangle would be introduced in $I(x, y_1) \cup I(y_1, y) \subseteq I(u, v)$, contradicting the fact that $I(u, v)$ induces a bipartite subgraph in G . So

$$d(y_1, y) = d(x, y) - 1 = k - 1 = d(x_1, y),$$

from which it follows that y_1 lies in $I(x, y) \cap I(u, x)$ (see Figure 6.9).

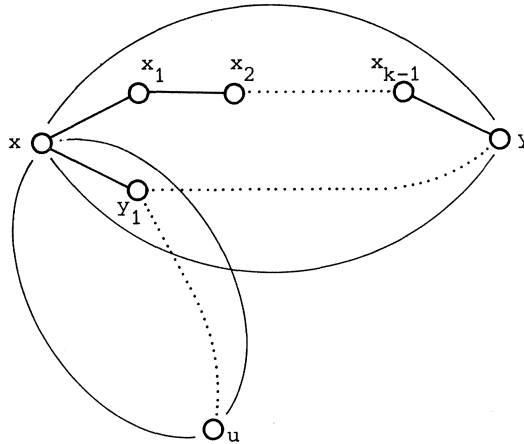


FIGURE 6.9.

Since $I(x, y)$ induces a bipartite subgraph, it follows that x_1 and y_1 are not adjacent, and so $d(x_1, y_1) = 2$. Consider the quasi-median of (x_1, y_1, y) . If it has size 1, then a triangle would be induced in $I(x_1, y) \cup I(y_1, y) \subset I(x, y)$, which is forbidden.

Assume that it has size 2, and let (x_1, y_1, z) be the quasi-median of (x_1, y_1, y) . Then x, x_1, y_1 and z must be contained in a C_6 , which is

induced in G . Since

$$d(z,y) = d(x_1,y) - 2 = d(x,y) - 3,$$

it follows that $d(z,x) = 3$. By (Q3) the convex closure of this C_6 in G is a 3-cube. This implies that x_1 , y_1 and z have a median which is also a median of x_1 , y_1 and y , contradicting the assumption that the quasi-median of (x_1, y_1, y) has size 2.

So we have proved that x_1 , y_1 and y have a median y_2 , which is adjacent to x_1 and y_1 , and which lies in $I(y_1, y) \subseteq I(u, v)$. Note that $x \rightarrow x_1 \rightarrow y_2 \rightarrow y_1 \rightarrow x$ is an induced C_4 in G (see Figure 6.10).

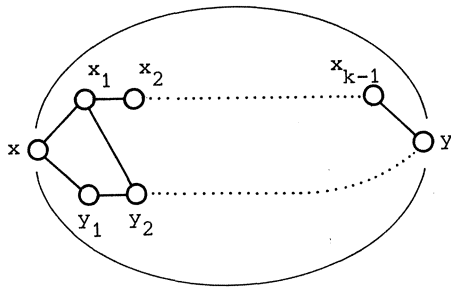


FIGURE 6.10

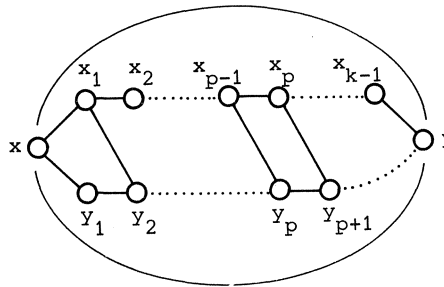


FIGURE 6.11.

Repeating the preceding argument, if necessary, we can find a number p , with $1 \leq p \leq k-1$, and vertices y_p and y_{p+1} , such that

$$x_{p-1}, y_p, y_{p+1} \in I(u, v) \cap I(x, y),$$

and

$$x_p \in I(x, y) \setminus I(u, v),$$

as in Figure 6.11, where $x_{p-1} \rightarrow x_p \rightarrow y_{p+1} \rightarrow y_p \rightarrow x_{p-1}$ is an induced C_4 in G . This leads to a contradiction as the following argument shows.

Since y_p and y_{p+1} are adjacent vertices in $I(u, v)$, we have $d(u, y_p) \neq d(u, y_{p+1})$. Assume that

$$d(u, y_{p+1}) > d(u, y_p).$$

The case $d(u, y_{p+1}) < d(u, y_p)$ can be treated similarly.

Let y_{p+1} be in the $(j+1)$ th level in $I(u, v)$, and let y_p be in the j -th level in $I(u, v)$. Then it follows that x_{p-1} is either in the $(j+1)$ th level in $I(u, v)$, or in the $(j-1)$ th level in $I(u, v)$. In the latter case we have

$$x_p \in I(x_{p-1}, y_{p+1}) \subseteq I(u, y_{p+1}) \subseteq I(u, v),$$

contradicting the choice of x_p . So x_{p-1} lies in $N_{j+1}(u, v)$.

Using the above argument concerning the existence of y_2 , we deduce that x_{p-1} , y_{p+1} and v have a median z in G . If $z \neq x_p$, then x_{p-1} , x_p , y_p , y_{p+1} and z would induce a $K_{2,3}$ in G , which contradicts the fact that G satisfies (Q1). Hence $z = x_p$, contradicting the choice of x_p , which was such that x_p lies in $I(x, y) \setminus I(u, v)$. This completes the proof. \square

The condition (Q3) is necessary in the above theorem as the graph in Figure 6.7 shows. This graph satisfies conditions (Q1) and (Q2), but is not interval monotone.

In the next pages we study quasi-median graphs in further detail. Our enquiry is split into quite a number of steps, some of which are natural analogues of those in the proof of the median graph theorem (Theorem 3.2.4). Unfortunately the technical details are more involved than in the case of median graphs, but the payoff of the enquiry below is worth the trouble. We state the outcome of the enquiry in Theorem 6.2.4.

In the following pages G is a connected graph with interval function I satisfying the following conditions:

- (Q2) $K_4 - e$ is not an induced subgraph of G ;
- (Q3) the convex closure of an induced C_6 in G is Q_3 or $H_{3,3}$;
- (Q4) if $I(u, v) \cap I(v, w) = \{v\}$, then $d(u, w) \geq \max\{d(u, v), d(v, w)\}$, for any three vertices u , v and w of G ;
- (Q5) G is interval monotone.

Note that, as in the case of Definition 6.2.1, the condition (Q3) can be replaced by condition (Q3').

We note first that these four conditions are independent. For example, the graph in Figure 6.12 satisfies the last three conditions only. Note that in this graph the triple (u,v,w) has two pseudo-medians of size 1. The circuit C_5 fails to satisfy (Q4), and C_6 fails to satisfy (Q3). Finally, $K_{2,3}$ satisfies all conditions except the last.

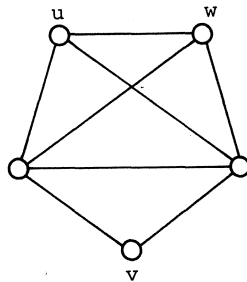


FIGURE 6.12.

We now introduce some terminology, which corresponds to that on page 78. For any two adjacent vertices u and v of G , we denote

$$A_{uv} := \{w \in V \mid d(w,u) = d(w,v)\},$$

$$W_u^{uv} := \{w \in V \mid d(w,u) < d(w,v)\},$$

$$W_v^{uv} := \{w \in V \mid d(w,u) > d(w,v)\},$$

$$U_u^{uv} := \{w \in W_u^{uv} \mid w \text{ has a neighbour in } W_v^{uv}\},$$

$$U_v^{uv} := \{w \in W_v^{uv} \mid w \text{ has a neighbour in } W_u^{uv}\},$$

$$F_{uv} := [W_u^{uv}, W_v^{uv}].$$

Fix an edge $e = ab$ of G , and write $A = A_{ab}$, $W_a = W_a^{ab}$, $W_b = W_b^{ab}$, $U_a = U_a^{ab}$, $U_b = U_b^{ab}$, $F = F_{ab}$. Note that A , W_a and W_b partition V . In Figure 6.13 the terminology is visualized.

The reader is invited to draw pictures when reading the proofs below. It is helpful in understanding what is going on. In some cases, when the

arguments are rather technical, we have included the appropriate figures.

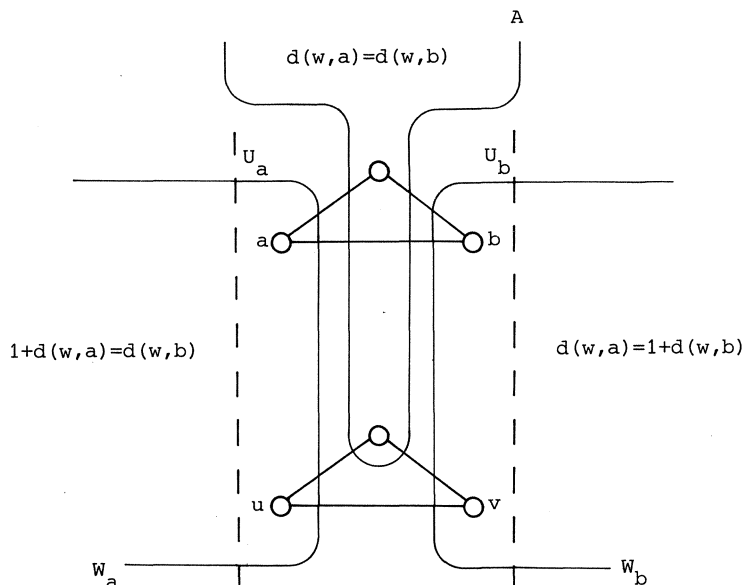


FIGURE 6.13.

(1) Any ordered triple (u,v,w) of vertices in G has a pseudo-median.

PROOF. Let x be a vertex in $I(u,v) \cap I(u,w)$ with

$$I(x,v) \cap I(x,w) = \{x\},$$

and let y be a vertex in $I(v,x) \cap I(v,w)$ with

$$I(y,x) \cap I(y,w) = \{y\},$$

and lastly, let z be a vertex in $I(w,x) \cap I(w,y)$ with

$$I(z,x) \cap I(z,y) = \{z\}.$$

By (Q4) it follows that $d(x,y) = d(y,z) = d(z,x)$. Hence (x,y,z) is an ordered triple in G satisfying (P1) and (P2), and so (u,v,w) has a pseudo-median. \square

(2) Each interval $I(u,v)$ in G induces a bipartite subgraph.

PROOF. The proof is similar to that of the biparticity of $G[I(u,v)]$ in the proof of Theorem 6.2.2. \square

(3) $|I(u,v,w)| \leq 1$ for any three vertices u, v and w of G .

PROOF. The proof is similar to that of Theorem 3.1.6. \square

(4) G does not contain C_5 as an induced subgraph.

PROOF. Assume the contrary, and let $u_1 \rightarrow \dots \rightarrow u_5 \rightarrow u_1$ be an induced C_5 in G . Then we have

$$d(u_1, u_3) = d(u_1, u_4) = 2.$$

Since u_3 and u_4 are adjacent, it follows that there exists a vertex x in G , adjacent to u_1, u_3 and u_4 , such that (x, u_3, u_4) is a pseudo-median of (u_1, u_3, u_4) . Since K_4 -e does not occur in G , x cannot be adjacent to u_1 or u_4 .

Similarly there is a vertex y adjacent to u_1, u_2 and u_4 , but not to u_3 or u_5 . Note that x and y cannot be adjacent, for otherwise u_1, x, y and u_4 would induce a K_4 -e in G . Hence u_1, u_4, u_5, x and y induce a $K_{2,3}$ in G , which contradicts the fact that $|I(u_5, x, y)| \leq 1$. \square

(5) Let u and v be vertices of G , and let x and y be two distinct neighbours of u in $I(u,v)$. Then x, y and v have a median.

PROOF. By (3), we have only to prove that (x,y,v) has a pseudo-median of size 0.

Assume the contrary. Since $I(u,v)$ induces a bipartite subgraph, x and y are not adjacent, and so $d(x,y) = 2$. If (x,y,v) has a pseudo-median of size 1, then this pseudo-median would introduce a triangle in $I(v,x) \cup I(v,y) \subset I(u,v)$. Hence a pseudo-median of (x,y,v) must have size 2. Let (x,y,z) be a pseudo-median of (x,y,v) . Then u, x, y and z are contained in an induced C_6 in G (see Figure 6.14).

It follows that $d(u,z) = 3$. Hence by (Q3), the convex closure of the circuit C is a Q_3 , and this implies that x, y and z have a median. This

median is also a pseudo-median of size 0 for (x,y,v) , contrary to the assumption that (x,y,v) has no median. This completes the proof.

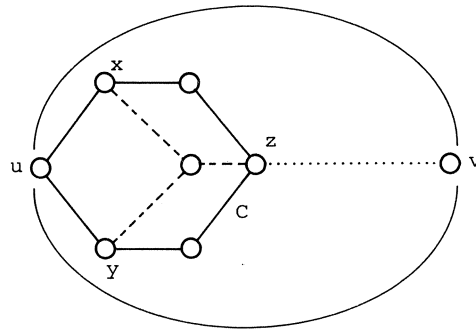


FIGURE 6.14.

□

- (6) $I(w,a) \subseteq W_a$ for any vertex w in W_a ,
and $I(w,b) \subseteq W_b$ for any vertex w in W_b .

PROOF. The inclusions follow immediately from the definition of W_a, W_b and A . □

- (7) If uv is an edge in F with u in U_a and v in U_b , then
 $d(u,a) = d(v,b)$.

PROOF. Since u and v are adjacent, we have

$$d(v,a) \leq d(u,a) + 1 = d(u,b) \leq d(v,b) + 1 = d(v,a).$$

□

- (8) $I(u,a) \subseteq U_a$ for any vertex u in U_a ,
and $I(v,b) \subseteq U_b$ for any vertex v in U_b .

PROOF. We prove the first inclusion by induction on $d(u,a)$.

Let v be a neighbour of u in U_b , and let u' be a neighbour of u in W_a such that

$$d(a,u') = d(a,u) - 1.$$

Then (7) implies that

$$d(b,v) = d(a,u) = d(a,u') + 1 = d(b,u') = d(b,u) - 1.$$

Hence v and u' are neighbours of u in $I(u,b)$, and so by (5), the vertices b , v and u' have a median v' in $I(b,v) \subseteq W_b$. Furthermore, it follows that v' is adjacent to u' -that is, u' lies in U_a .

The second inclusion follows in the same way. \square

(9) If uv is an edge in F with u in U_a and v in U_b , then $W_u^{uv} = W_a$, $W_v^{uv} = W_b$, $U_u^{uv} = U_a$, $U_v^{uv} = U_b$, $A_{uv} = A$, and $F_{uv} = F$.

PROOF. If we prove that $W_u^{uv} = W_a$ and $W_v^{uv} = W_b$, then the other equalities follow immediately.

First let u be adjacent to a , so that by (7), v is adjacent to b . It follows from the definition of U_a and U_b that a , b , v and u induce a C_4 in G . We prove that $W_a \subseteq W_u^{uv}$.

Let w be a vertex in W_a . Note that

$$d(w,u) + 1 \geq d(w,v) \geq d(w,b) - 1 = d(w,a).$$

To show that w lies in W_u^{uv} , it suffices to prove that $d(w,v) \geq d(w,u) + 1$.

CASE 1: $d(w,u) = d(w,a) - 1$.

Then we have $d(w,v) \geq d(w,a) = d(w,u) + 1$.

CASE 2: $d(w,u) = d(w,a)$.

In this case a pseudo-median of (w,u,a) must have size 1. Let (x,u,a) be such a pseudo-median (see Figure 6.15).

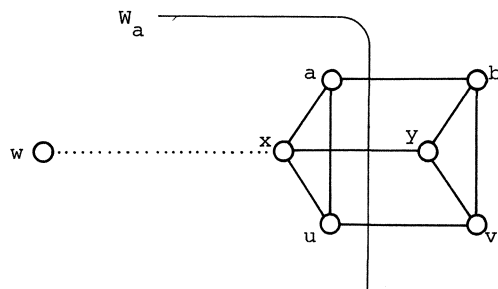


FIGURE 6.15.

Since K_4 -e does not occur in G , it follows that x is not adjacent to b or v , and so we have

$$d(x,b) = 2 = d(x,v).$$

Hence a pseudo-median of (x,b,v) has size 1. Let (y,b,v) be such a pseudo-median. Then y is adjacent to x , and so

$$d(w,y) \leq d(w,x) + 1 = d(w,a) = d(w,b) - 1 \leq d(w,y).$$

If $d(w,v) = d(w,y) = d(w,b) - 1$, then a triangle would occur in $I(b,w)$. So $d(w,v) = d(w,b) = d(w,a) + 1 = d(w,u) + 1$.

CASE 3: $d(w,u) = d(w,a) + 1 = d(w,b)$.

In this case the vertex a is the median of w, u and b . If $d(w,v) = d(w,a)$, then v would be also a median of w, u and b , and so we have $d(w,v) \geq d(w,a) + 1 = d(w,u)$.

Assume that $k = d(w,v) = d(w,u)$. Let (x,b,v) be a pseudo-median of (w,b,v) (see Figure 6.16).

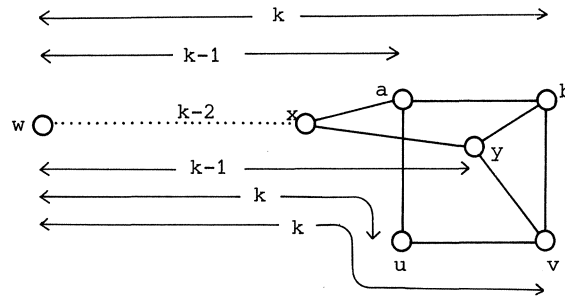


FIGURE 6.16.

Then a and x are two distinct neighbours of b in $I(b,w)$, and so by (5), the vertices x, a and w have a median y in $I(w,a) \subseteq W_a$.

By (4), the circuit $y \rightarrow a \rightarrow u \rightarrow v \rightarrow x \rightarrow y$ must have a diagonal. The only possible diagonal is that between u and y , but, as indicated in Figure 6.16, we have $d(w,y) = d(w,u) - 2$. This establishes a contradiction, and so our assumption that $d(w,v) = d(w,u)$ is incorrect -that is,

$d(w,v) \geq d(w,u) + 1$. This settles Case 3.

So we have proved that $W_a \subseteq W_u^{uv}$. The same line of thought shows that $W_u^{uv} \subseteq W_a$, and so we have $W_a = W_u^{uv}$. Using a similar method we prove that $W_b = W_v^{uv}$, completing the proof in the case that u is adjacent to a . The general case follows by induction on $d(a,u)$, using (8). \square

(10) U_a and U_b are convex sets in G .

PROOF. Combining (8) and (9), we deduce the convexity of U_a and U_b . \square

(11) F is a matching between U_a and U_b .

PROOF. Assume the contrary, and assume that u in U_a has two distinct neighbours v and v' in U_b . By (7), we have

$$d(v,b) = d(u,a) = d(u,b) - 1 = d(v',b),$$

and so v and v' are two distinct neighbours of u in $I(u,b)$. Hence $d(v,v') = 2$, so that by the convexity of U_b we have

$$u \in I(v,v') \subseteq U_b,$$

contradicting the fact that u is in U_a . \square

(12) The mapping $f: U_a \rightarrow U_b$, defined by $f(u)$ is v whenever uv is an edge in F with u in U_a , induces an isomorphism between $G[U_a]$ and $G[U_b]$.

PROOF. Since F is a matching, it follows that f is bijective.

Let u and u' be vertices in U_a . By (9), the result of (7) still holds, when we replace a by u' and b by $f(u')$ -that is,

$$d(u,u') = d(f(u),f(u')).$$

Hence u and u' are adjacent if and only if $f(u)$ and $f(u')$ are adjacent. \square

- (13) *If two vertices of a triangle are in U_a , then the third vertex of the triangle is also in U_a . The same holds for U_b .*

PROOF. Let $u \rightarrow u' \rightarrow u'' \rightarrow u$ be a triangle in G with u and u' in U_a , and let $v = f(u)$ and $v' = f(u')$. Then $u \rightarrow v \rightarrow v' \rightarrow u' \rightarrow u$ is an induced C_4 in G . Since K_4 -e does not occur in G , neither v nor v' is adjacent to u'' -that is,

$$d(u'', v) = d(u'', v') = 2.$$

It follows that u'' lies in $W_u^{uv} = W_a$. Let (v, v', v'') be a pseudo-median of (v, v', u'') . Then v'' is adjacent to u'' , v and v' , but not to u (otherwise K_4 -e would be induced in G). This implies that v'' lies in $W_v^{uv} = W_b$, and so u'' lies in U_a and v'' lies in U_b .

The assertion for U_b follows in the same way. \square

- (14) *If a vertex x in A has a neighbour in $U_a \cup U_b$, then there is an edge uv in F with u in U_a such that*

$$N(x) \cap U_a = \{u\}, \text{ and } N(x) \cap U_b = \{v\}.$$

PROOF. Assume that x is adjacent to a vertex u in U_a , and let v be the neighbour of u in U_b . By (9) the vertex x lies in $A = A_{uv}$, and so x is adjacent to v .

Since U_a is convex and x is not in U_a , it follows that every neighbour of x in U_a is adjacent to u . If x is adjacent to a neighbour u' of u in U_a , then u', x, u and v would induce K_4 -e in G (since F is a matching, u' and v are not adjacent).

So x has no neighbours in U_a besides u , and similarly, x has no neighbours in U_b besides v . \square

- (15) *Let $B := \{w \in A \mid w \text{ has a neighbour in } U_a \text{ and in } U_b\}$, and let $|N(a) \cap N(b)| = k - 2$. Then there exists an isomorphism ϕ between $G[U_a \cup U_b \cup B]$ and $G[U_a] \times K_k$, and there exists a vertex p of K_k such that $\phi(u) = (u, p)$ for any vertex u in U_a .*

PROOF. The first step of the proof is visualized in Figure 6.17: if the graph H consisting of a complete graph K_n that has an edge in common with

C_4 occurs in G , then the convex closure of H in G is $H_{2,n}$. The verification of this property of G is left to the reader (recall that by (3) $K_{2,3}$ does not occur in G).

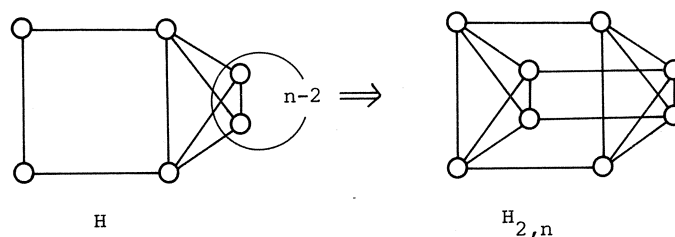


FIGURE 6.17.

Let uv and $u'v'$ be edges in F with u and u' in U_a such that u and u' are adjacent (and so v and v' are also adjacent). Then by (14) and the above observation we have

$$|N(u) \cap N(v)| = |N(u') \cap N(v')|.$$

Since $G[U_a]$ is connected (being a convex subgraph of G), it follows that for any edge uv in F ,

$$|N(u) \cap N(v)| = k - 2.$$

It follows from the above observations and (10), (12) and (13) that $G[U_a \cup U_b \cup B]$ contains a spanning subgraph G_0 isomorphic to $G[U_a] \times K_k$. In Figure 6.18 we give an example to visualize the argument.

It follows from (14) that all edges between B and $U_a \cup U_b$ in G are in G_0 , and from the first step of the proof and the convexity of U_a that all edges in G between vertices in B are in G_0 . This concludes the proof.

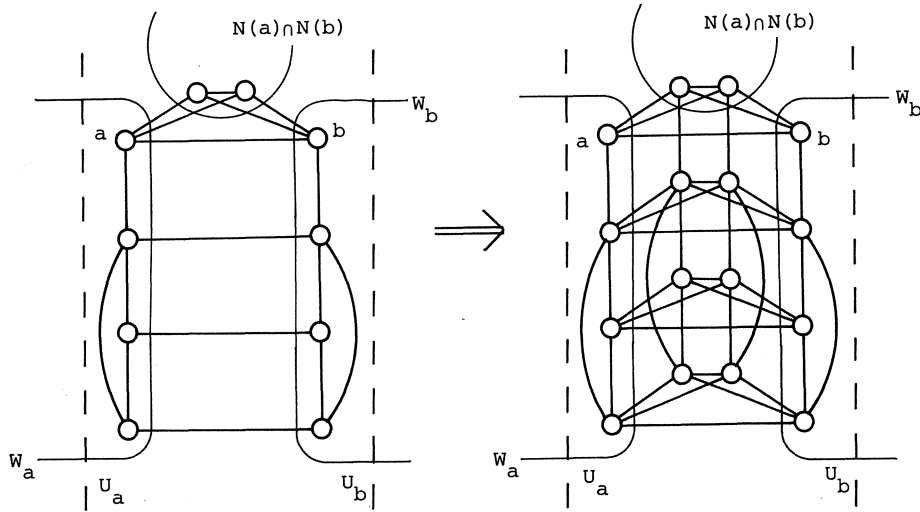


FIGURE 6.18.

□

- (16) If c is a common neighbour of a and b , then $W_a^{ac} = W_a$ and $U_a^{ac} = U_a$.

PROOF. Let c be a common neighbour of a and b , so that c lies in A . For any vertex w in W_a , we have

$$d(w,a) = d(w,b) - 1 \leq d(w,c) \leq d(w,a) + 1 = d(w,b).$$

If $d(w,c) = d(w,a)$, then a , b and c would induce a triangle in $I(w,b)$, and so we have

$$d(w,c) = d(w,b) = d(w,a) + 1.$$

Hence w lies in W_a^{ac} . So we have proved that $W_a \subseteq W_a^{ac}$. In a similar way it follows that $W_a^{ac} \subseteq W_a$ -that is,

$$W_a^{ac} = W_a.$$

It follows from (15) and the convexity of U_a that $U_a \subseteq U_a^{ac}$.

Interchanging the role of the edges ab and ac in the foregoing argument, we deduce that $U_a^{ac} \subseteq U_a$, and so

$$U_a^{ac} = U_a.$$

□

$$(17) \quad [W_a \setminus U_a, A] = \emptyset = [W_b \setminus U_b, A].$$

PROOF. Choose a vertex x in A . Since $d(x,a) = d(x,b)$, a pseudo-median of (a,b,x) must be of the form (a,b,c) , where c is a common neighbour of a and b (so that c is in A). It follows that x lies in W_c^{ac} . So x is adjacent to a vertex w in $W_a = W_a^{ac}$ only if w lies in $U_a^{ac} = U_a$. Hence there are no edges between A and $W_a \setminus U_a$.

It follows similarly that $[W_b \setminus U_b, A] = \emptyset$. □

$$(18) \quad W_a \text{ and } W_b \text{ are convex sets in } G.$$

PROOF. The convexity of W_a and W_b follows from (15) and (17) and the convexity of U_a and U_b . □

FIRST INTERMISSION

Before proceeding, let us summarize what we have proved so far. For any vertex c in $N(a) \cap N(b)$, we write $W_c = W_c^{ac}$ and $U_c = U_c^{ac}$. Let C be the maximal clique in G to which both a and b belong, and let $|C| = k$. Note that $C = (N(a) \cap N(b)) \cup \{a,b\}$.

Since the choice of the edge ab was arbitrary, it follows that what we have proved so far still holds when we replace the edge ab by any other edge in $G[C]$ and adapt the assertions accordingly. So we have established the following features of G (see Figure 6.19):

- $\{W_x\}_{x \in C}$ partitions V ;
- $[W_x \setminus U_x, W_y] = \emptyset$ for any two distinct vertices x and y in C ;
- for any subset C' of C with $|C'| = h$ and for any y in C , the graphs $G[U_{x \in C'}, U_x]$ and $G[U_y] \times K_h$ are isomorphic;

- for any x in C the sets W_x and U_x are convex in G ;
- if two vertices of a clique lie in U_x (respectively W_x), then the whole clique is in U_x (respectively W_x), for any vertex x in C ;
- for any subset C' of C the set $\bigcup_{x \in C'} W_x$ is convex in G .

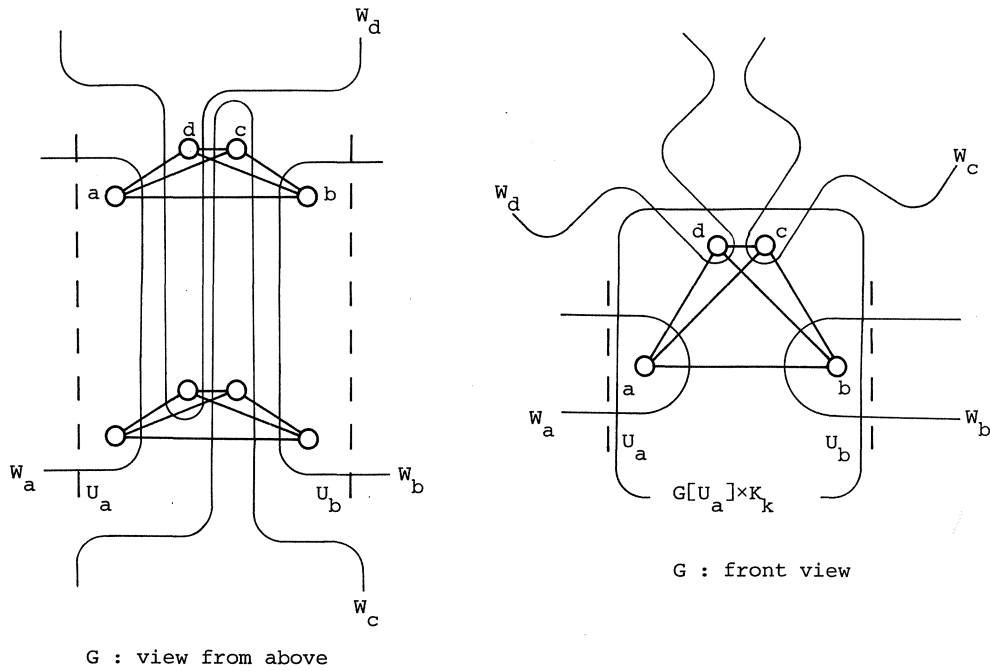


FIGURE 6.19.

It follows from (9) that for any two edges g and g' of G , we have

$$g' \in F_g \text{ if and only if } F_g = F_{g'}.$$

So we can define an equivalence relation on E by calling two edges g and g' equivalent if $F_g = F_{g'}$. Each of the equivalence classes is a matching, as was proved in (11). Hence the partition of E induced by this equivalence relation is a proper edge colouring of G . This edge colouring is called the *canonical edge colouring* of G .

In the sequel, each edge of G is assigned its canonical colour.

Each Hamming graph is also endowed with its canonical colouring. By the "colours occurring in the set $W \subseteq V$ " we mean the colours of the edges in the subgraph of G induced by W .

Let us resume our study of the properties of G .

END OF FIRST INTERMISSION

(19) *The mapping $f: U_a \rightarrow U_b$ defined in (12) induces a colour-preserving isomorphism between $G[U_a]$ and $G[U_b]$.*

PROOF. In an induced C_4 in G , non-adjacent edges are assigned the same colours. Combining this observation with (12), we get the required result. \square

(20) *For any two vertices u and v of G , the edges on a shortest (u,v) -path all have different colours.*

PROOF. This follows immediately from the convexity of the sets W_x^{xy} , where x and y are two adjacent vertices in G . \square

(21) *If u and v are vertices of G , then on two shortest (u,v) -paths the same colours occur.*

PROOF. If a shortest (u,v) -path P contains an edge xy , say $P = u \rightarrow \dots \rightarrow x \rightarrow y \rightarrow \dots \rightarrow v$, then u lies in W_x^{xy} and v lies in W_y^{xy} . Since $W_x^{xy} \cup W_y^{xy}$ is convex, it follows that any shortest (u,v) -path contains an edge in $[W_x^{xy}, W_y^{xy}] = F_{xy}$. \square

(22) *Let u and v be vertices of G . If a colour occurs in $I(u,v)$, then the edges of that colour in $G[I(u,v)]$ form a cutset in $G[I(u,v)]$.*

PROOF. Use (20) and (21). \square

(23) *Let u , v and w be vertices of G . Then $I(u,v) \cap I(u,w) = \{u\}$ if and only if there is no colour occurring in $I(u,v)$ and $I(u,w)$ simultaneously.*

PROOF. If $I(u,v) \cap I(u,w) \neq \{u\}$, then there is a colour occurring in $I(u,v) \cap I(u,w)$, proving the "if" part of the assertion.

Let $I(u,v) \cap I(u,w) = \{u\}$, and assume that the colour of the colour class F occurs in both $I(u,v)$ and $I(u,w)$. Without loss of generality we may assume that u lies in W_a . Choose vertices p and r in U_a such that

$$p, f(p) \in I(u,v),$$

and

$$r, f(r) \in I(u,w).$$

Since U_a is convex, it follows that $I(p,r) \subseteq U_a$.

Let q in $I(u,v) \cap I(p,r)$ be such that

$$I(u,v) \cap I(q,r) = \{q\},$$

and let s in $I(u,w) \cap I(q,r)$ be such that

$$I(u,w) \cap I(q,s) = \{s\}.$$

It follows from (Q4) that $d(u,q) = d(q,s) = d(s,u)$. Since G is interval monotone, it follows from the convexity of U_a and the fact that $G[U_a \cup U_b]$ is isomorphic to $G[U_a] \times K_2$ that

$$f(q) \in I(q, f(p)) \subseteq I(u,v),$$

and

$$f(s) \in I(s, f(r)) \subseteq I(u,w).$$

Note that by (12), the graphs $G[I(q,s)]$ and $G[I(f(q), f(s))]$ are isomorphic. Hence we have

$$d(u,q) = d(q,s) = d(f(q), f(s)).$$

Since $I(u,v) \cap I(q,s) = \{q\}$, it follows that

$$I(u,v) \cap I(f(q), f(s)) = \{f(q)\}.$$

Similarly we have

$$I(u,w) \cap I(f(q),f(s)) = \{f(s)\}.$$

By (Q4) this implies that

$$\begin{aligned} d(f(q),f(s)) &= d(u,f(q)) = d(u,q) + 1 = d(q,s) + 1 \\ &= d(f(q),f(s)) + 1, \end{aligned}$$

which is a contradiction. Hence the colour F does not occur in both $I(u,v)$ and $I(u,w)$. Since F is an arbitrary colour class in the canonical colouring of G , the proof is complete. \square

(24) *Let u, v and w be vertices of G , and let (x,y,z) be a pseudo-median of (u,v,w) . If u and v lie in W_p for some p in C , then x, y and z lie in W_p .*

PROOF. We deduce from the convexity of W_p that x and y lie in W_p . If z lies in W_q for some q ($\neq p$) in C , then the colour F_{pq} occurs in both $I(x,z)$ and $I(y,z)$, and so by (23) we have $|I(x,z) \cap I(y,z)| \geq 2$. This contradicts the fact that $I(x,z) \cap I(y,z) = \{z\}$, which follows from (1) and Proposition 6.1.3. \square

(25) *Let u, v and w be vertices of G such that (u,v,w) is its own quasi-median, and let $k = d(u,v)$. Then the convex closure of $\{u,v,w\}$ is isomorphic to K_3^k .*

PROOF. The proof is by induction on k . For $k \leq 1$, the assertion is evident. For $k = 2$, it follows from (Q3).

Let $k \geq 3$. For convenience we rename the vertices and start with a triple (u,v',w'') that is its own quasi-median of size k .

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be the colours in $I(u,v')$, let $\beta_1, \beta_2, \dots, \beta_k$ be the colours in $I(v',w'')$ and let $\gamma_1, \gamma_2, \dots, \gamma_k$ be those in $I(w'',u)$. Then by (23) and (24) the colours $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k, \gamma_1, \gamma_2, \dots, \gamma_k$ are all different.

Let w be a neighbour of u in $I(u,w'')$, and let γ_1 , say, be the colour of the edge uw . Note that

$$d(w,v') \leq d(u,v') + 1 = k + 1.$$

Since γ_1 does not occur in $I(u, v')$, it follows that w is not in $I(u, v')$, and so $d(w, v') \geq k$.

If $d(w, v') = k + 1$, then u lies in $I(w, v')$, and so $\gamma_1, \alpha_1, \alpha_2, \dots, \alpha_k$ are the colours in $I(w, v')$ and $\gamma_2, \dots, \gamma_k$ are those in $I(w, w'')$. Hence by (23) we have $I(w, v') \cap I(w, w'') = \{w\}$, and so $k = d(v', w'') \geq d(w, v') = k + 1$, which is impossible. So we have proved that

$$d(v', w) = d(v', u) = k.$$

Let (u, v, w) be a pseudo-median of (u, v', w) . Then v is a common neighbour of u and w . Furthermore, v lies in $I(u, v')$, which implies that the colour of the edge uv must be one of $\alpha_1, \alpha_2, \dots, \alpha_k$. Without loss of generality, let it be α_1 . The situation we have arrived at is depicted in Figure 6.20. Let δ be the colour of the edge wv , so that the colours in $I(v', w)$ are $\delta, \alpha_2, \dots, \alpha_k$.

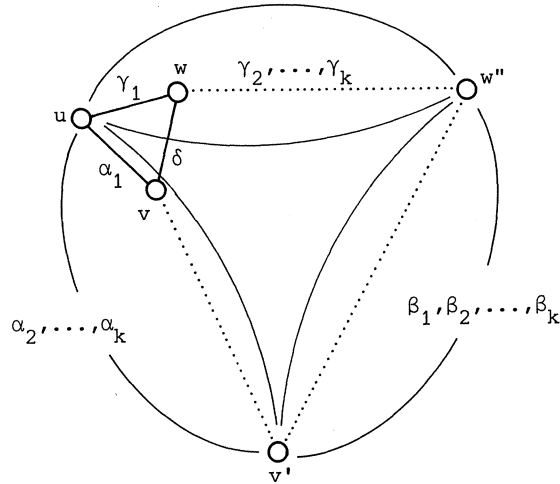


FIGURE 6.20.

Consider a pseudo-median of (w, v', w'') . Since $\gamma_2, \dots, \gamma_k$ are the colours in $I(w, w'')$, and $\beta_1, \beta_2, \dots, \beta_k$ are those in $I(v', w'')$, this pseudo-median must have size $k - 1$. So there is a neighbour w' of v' in $I(v', w'') \cap I(v', w)$ such that (w, w', w'') is this pseudo-median of size $k - 1$.

Since w' lies in $I(v', w'')$, the colour of the edge $v'w'$ must be one

of $\beta_1, \beta_2, \dots, \beta_k$. Let it be β_1 . We know that w' lies in $I(v', w)$, and so β_1 is one of the colours $\delta, \alpha_2, \dots, \alpha_k$. This implies that β_1 and δ are the same colour. Note that $\alpha_2, \dots, \alpha_k$ are the colours in $I(w, w')$.

In the same way as we have deduced that $d(v', w) = k$, it follows that

$$d(u, w') = d(u, v') = k.$$

Hence w lies in $I(u, w')$, and $\gamma_1, \alpha_2, \dots, \alpha_k$ are the colours in $I(u, w')$.

Let (u', v', w') be a pseudo-median of (u, v', w') , so that u' is a common neighbour of v' and w' in $I(u, v') \cap I(u, w')$. Then $d(u, u') = k - 1$, and so the colours in $I(u, u')$ are $\alpha_2, \dots, \alpha_k$ - that is, the common colours in $I(u, v')$ and $I(u, w')$. It follows that γ_1 is the colour of the edge $u'w'$, and α_1 is the colour of the edge $u'v'$ (see Figure 6.21).

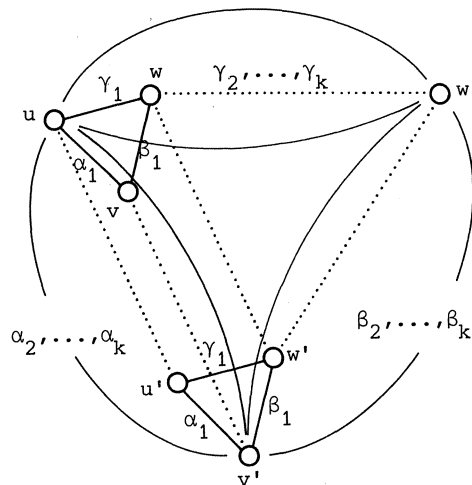


FIGURE 6.21.

In the same way as we found the vertex w' , we can find a neighbour u'' of w'' in $I(u, w'') \cap I(u', w'')$ such that (u, u', u'') is a pseudo-median of size $k - 1$ of (u, u', w'') and γ_1 is the colour of the edge $u''w''$.

Finally we can find a neighbour v'' of w'' in $I(v, w'')$ such that (u'', v'', w'') is a pseudo-median of (u'', v', w'') . As before, it follows that β_1 is the colour of the edge $v''w''$ and α_1 is the colour of the edge $u''v''$.

It follows from the above observations that the situation is as

depicted in Figure 6.22, where (u, u', u'') is its own quasi-median of size $k - 1$, as are (v, v', v'') and (w, w', w'') .

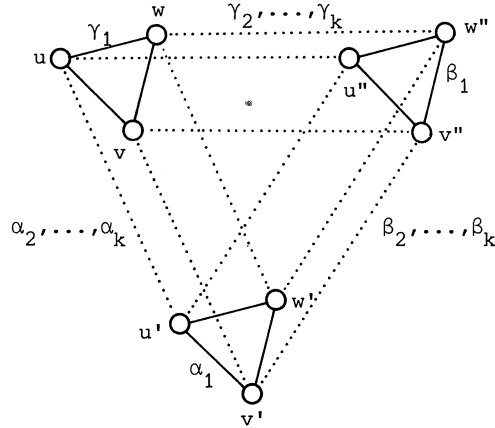


FIGURE 6.22.

It follows that

$$u, u', u'' \in U_u^{uv} = U_u^{uw},$$

$$v, v', v'' \in U_v^{uv} = U_v^{vw},$$

and

$$w, w', w'' \in U_w^{uw} = U_w^{vw}.$$

Note that F_{uv} is the colour α_1 , F_{vw} is the colour β_1 , and F_{uw} is the colour γ_1 .

Let X_u be the convex closure of $\{u, u', u''\}$. Since (u, u', u'') is its own quasi-median of size $k - 1$, it follows from the induction hypothesis that X_u induces a K_3^{k-1} in G with $\alpha_2, \dots, \alpha_k, \beta_2, \dots, \beta_k, \gamma_2, \dots, \gamma_k$ as its canonical colours. Since U_u^{uv} is convex, it follows that $X_u \subseteq U_u^{uv}$.

Similarly the convex closure X_v of $\{v, v', v''\}$ and the convex closure X_w of $\{w, w', w''\}$ both induce a K_3^{k-1} in G with the colours $\alpha_2, \dots, \alpha_k, \beta_2, \dots, \beta_k, \gamma_2, \dots, \gamma_k$ as their canonical colours. Furthermore, we have $X_v \subseteq U_v^{uv}$ and $X_w \subseteq U_w^{uw}$.

By (19) each of colours α_1, β_1 and γ_1 induces a colour-preserving isomorphism between the two graphs that are matched by that colour. So

$X = X_u \cup X_v \cup X_w$ induces a K_3^k in G with $\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k, \gamma_1, \gamma_2, \dots, \gamma_k$ as its canonical colours.

Using (20) and (21), we conclude that, for any two vertices x and y in X , all shortest (x,y) -paths lie entirely in the K_3^k . So X is the convex closure of $\{u, u', u'', v, v', v'', w, w', w''\}$, and hence of $\{u, v', w''\}$. \square

SECOND INTERMISSION

In this intermission we construct a new graph G' from G . Recall that in (12) we introduced the mapping $f: U_a \rightarrow U_b$, defined by

$$f(u) = v \text{ whenever } uv \in F \text{ and } u \in U_a.$$

Let us construct the graph G' from G by "contracting" the colour F -that is, we obtain G' from the subgraph $G[V \setminus U_b]$ of G by joining each vertex u in U_a to all neighbours of $f(u)$ in $W_b \setminus U_b$. In this construction we have "identified" the colour F_{bc} with the colour F_{ac} , for each vertex c in $N(a) \cap N(b)$.

We write

$$v' = \begin{cases} v & \text{for } v \in V \setminus U_b \\ f^{-1}(v) & \text{for } v \in U_b, \end{cases}$$

$$Z' = \{v' \mid v \in Z\} \quad \text{for } Z \subseteq V$$

In what follows, any set Z' refers to a set of vertices in G' . Note that $V' = V \setminus U_b$, and $W_b' = (W_b \setminus U_b) \cup U_a$. In Figure 6.23 we give the figures of G' analogous to those of G in Figure 6.19.

Let I' be the interval function of G' , and let d' be the distance function of G' . In the next few steps we prove that G' also satisfies the conditions (Q2), (Q3), (Q4) and (Q5).

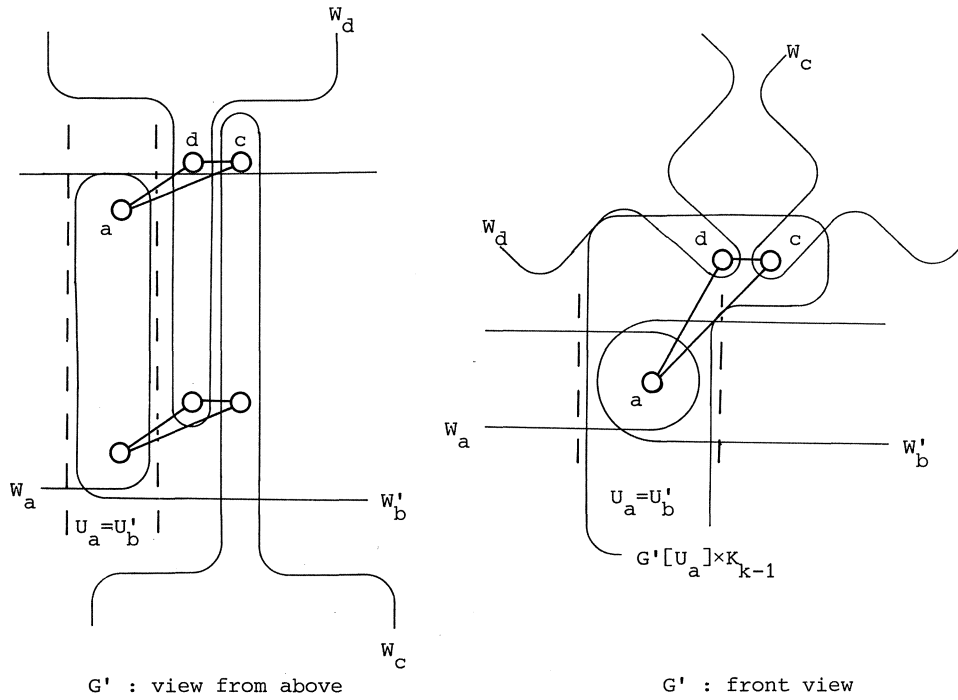


FIGURE 6.23.

END OF SECOND INTERMISSION

- (26) *The mapping $g: V \rightarrow V'$, defined by $g(v) = v'$ for any vertex v in V , induces an isomorphism between $G[W_a \cup A]$ and $G'[W_a \cup A]$, and one between $G[W_b \cup A]$ and $G'[W'_b \cup A]$.*

PROOF. The two isomorphisms follow immediately from the structure of G and the construction of G' . \square

- (27) *If u and v are vertices of G such that the colour F does not occur in $I(u,v)$, then the mapping g , defined in (26), induces an isomorphism between $G[I(u,v)]$ and $G'[I'(u',v')]$.*

PROOF. Note that, for any vertex c in $N(a) \cap N(b)$, at most one of the colours F_{ac} and F_{bc} occurs in $I(u,v)$. The assertion follows immediately

from (26). \square

(28) *If u is a vertex in W_a and v is a vertex in W_b , then*
 $I'(u,v') = I(u,v) \setminus U_b$.

PROOF. It follows from the convexity of $W_a \cup W_b$ that, for any vertex c in $N(a) \cap N(b)$, the colours F_{ac} and F_{bc} do not occur in $I(u,v)$. Hence in G' the colour F_{ac} does not occur in $I'(u,v')$, for any vertex c in $N(a) \cap N(b)$. With this in mind we deduce the assertion from the construction of G' . \square

(29) *In G' the sets $W_a, W_b, W_a \cup W_b, A, W_a \cup A$ and $W_b \cup A$ are convex.*

PROOF. Use (27) and (28). \square

(30) *For any two vertices u' and v' of G' , the interval $I'(u',v')$ induces a bipartite graph in G' .*

PROOF. By (26) and (27) it suffices to prove the assertion for u in W_a and v in W_b . From (22) we know that the edges of F in $G[I(u,v)]$ form a cutset. Hence in each circuit of $G[I(u,v)]$ there are an even number of edges from F . Since we obtain $G'[I'(u',v')]$ from $G[I(u,v)]$ by contracting F , it follows from (2) that $I'(u',v')$ induces a bipartite graph in G' . \square

(31) $[W_a' \setminus U_a', A'] = \emptyset = [W_b' \setminus U_b', A'] = [W_a' \setminus U_a', W_b' \setminus U_b']$.

PROOF. The equalities follow immediately from the structure of G and the construction of G' . \square

(32) *Let u', v' and w' be vertices of G' . Then the triple (u',v',w') has a pseudo-median (x',y',z') in G' , and the convex closure of $\{x',y',z'\}$ induces a K_3^h , where h is the size of (x',y',z') .*

PROOF. First let (u,v,w) be its own quasi-median in G , and let k be the size of (u,v,w) .

If the colour F does not occur in one of the intervals $I(u,v)$, $I(v,w)$ and $I(w,u)$, then by (26) and (29) the triple (u',v',w') must be its own quasi-median in G' , and the convex closure of $\{u',v',w'\}$ in G' induces a K_3^k .

So let F be one of the colours in $I(v,w)$, say. By (24) the convex closure X of $\{u,v,w\}$ induces a K_3^k in G with F as one of its canonical colours. Hence u lies in $B \subseteq A$, and v lies in U_a , say, and w lies in U_b . Note that X' induces in G' a subgraph isomorphic to $K_3^{k-1} \times K_2$.

Let x and $f(x)$ be the neighbours of u in X such that the edge $xf(x)$ is in F . If we contract F , then in G' we have

$$I'(u,v) \cap I'(u,w') = \{u,x\}.$$

Furthermore, (x,v,w') is the quasi-median of (u,v,w') in G' , and the convex closure of $\{x,v,w'\}$ induces a subgraph in G' isomorphic to K_3^{k-1} .

To prove the general case, let u, v and w be vertices of G , and let (x,y,z) be a pseudo-median (u,v,w) of size k . If the colour F does not occur in one of the intervals $I(u,v)$, $I(v,w)$ and $I(w,u)$, then it follows from (26) and (29) that (x',y',z') is a pseudo-median of (u',v',w') in G' with size k . Furthermore, we deduce from (25) and (26) that the convex closure of $\{x',y',z'\}$ in G' induces a K_3^k .

If F does occur in two of the three intervals, then (23), (26) and (27) imply that (x',y',z') is a pseudo-median of size k of (u',v',w') in G' . Furthermore, using (25) we deduce that the convex closure of $\{x',y',z'\}$ induces a K_3^k in G' .

If F does occur in exactly one of the three intervals, then F is one of the colours of the K_3^k induced by the convex closure of $\{x,y,z\}$ in G . Then (27) and (28) imply that a pseudo-median of (x',y',z') in G' is also a pseudo-median of (u',v',w') in G' . The assertion follows by the first step of the proof.

Since $|I(u,v,w)| \leq 1$, it follows from (23) that the colour F does not occur in all three intervals, by which the proof is complete. \square

(33) G' is interval monotone.

PROOF. This follows immediately from (27), (28) and the interval monotonicity of G . \square

(34) G' satisfies condition (Q4).

PROOF. Use (32). \square

(35) G' does not have K_4 -e as an induced subgraph.

PROOF. If K_4 -e does occur in a graph, then the vertices of this K_4 -e all lie in the interval between the two non-adjacent vertices of the K_4 -e. Hence, by (30), K_4 -e does not occur in G' . \square

(36) G' satisfies condition (Q3).

PROOF. Let $u_1 \rightarrow u_2 \rightarrow \dots \rightarrow u_6 \rightarrow u_1$ be an induced C_6 in G' .

If $d'(u_1, u_4) = 3$, then u_1, \dots, u_6 lie in $I'(u_1, u_4)$. In this case the triple (u_1, u_3, u_5) cannot have a pseudo-median of size 1 or 2, for this would introduce a triangle in $I'(u_1, u_4)$. Hence (u_1, u_3, u_5) has a pseudo-median of size 0. Similarly (u_2, u_4, u_6) has a pseudo-median of size 0. Since G' is interval monotone (by (33)), $K_{2,3}$ does not occur in G' , and so the convex closure of the circuit is Q_3 .

If there is no pair of vertices with distance 3 in the circuit, then (u_1, u_3, u_5) must be its own quasi-median. For let (v, u_3, u_4) be a pseudo-median of size 1 of the triple (u_1, u_3, u_4) . Then v is a common neighbour of u_1, u_3 and u_4 . Since $K_{2,3}$ does not occur in G' , it follows that

$$I'(u_1, u_3) = \{u_1, u_2, v, u_3\}.$$

Since K_4 -e does not occur in G' , the vertex v is not adjacent to u_5 . Hence $I'(u_1, u_3) \cap I'(u_3, u_5) = \{u_3\}$. From this we can deduce that (u_1, u_3, u_5) is its own quasi-median. By (32), the convex closure of the circuit is $K_3^2 = H_{3,3}$. \square

Finally our study of the properties of the graph G satisfying conditions (Q2), (Q3), (Q4) and (Q5) has come to an end. Before stating the outcome of it in Theorem 6.2.4, we give a definition.

6.2.3. DEFINITION. Let G be a connected graph, and let U, W, W', A and B be subsets of V such that

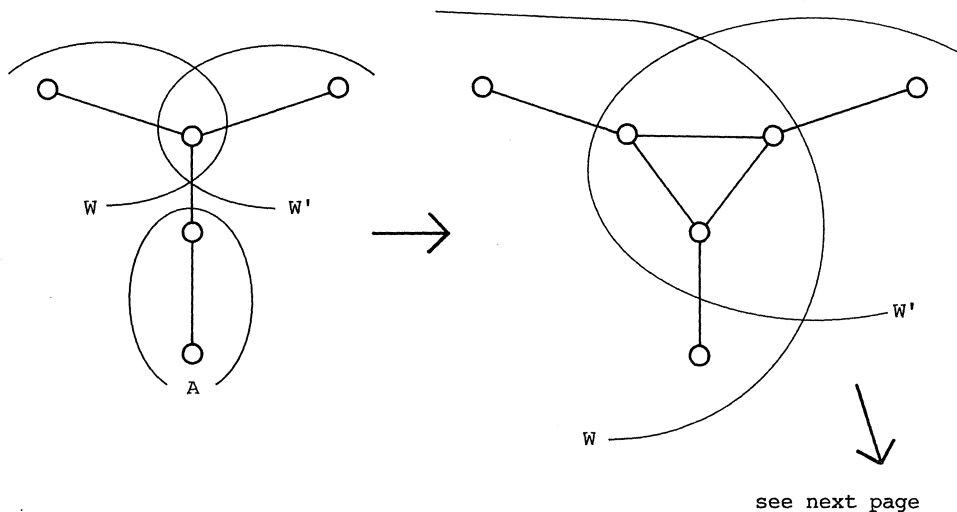
- $U = W \cap W' \neq \emptyset$,
- $A = V \setminus (W \cup W')$,
- $B = \{x \in A \mid x \text{ has a neighbour in } U\}$,
- $[W \setminus U, W' \setminus U] = [W \setminus U, A] = [W' \setminus U, A] = \emptyset$,

- W and W' are convex sets in G ,
- if two vertices of a triangle lie in U , then the third lies also in U ,
- there exists an isomorphism ϕ between $G[U \cup B]$ and $G[U] \times K_k$, for some positive integer k , and there exists a vertex p of K_k such that $\phi(u) = (u, p)$ for any vertex u in U .

The *quasi-median expansion* of G with respect to W and W' is the graph G' constructed as follows:

- (i) replace each vertex v in U by two vertices u_v and u'_v which are joined by an edge;
- (ii) join u_v to all neighbours of v in $V \setminus W'$, and join u'_v to all neighbours of v in $V \setminus W$;
- (iii) if v and w are two adjacent vertices in U , join u_v to u_w and u'_v to u'_w .

Note that in the above definition, it follows from the structure of G that U , $W \cup W'$, $W \cup A$, $W' \cup A$ and A are also convex in G . If B is empty, so that A is empty and $k = 1$, and U is triangle-free, then the quasi-median expansion is just the convex expansion introduced in Chapter 3. In Figure 6.24 we give an example of two successive quasi-median expansions leading to the graph in Figure 6.6.



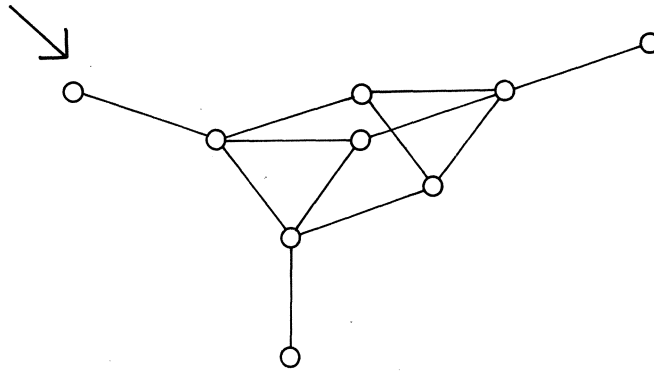


FIGURE 6.24.

The tour-de-force that has been performed in the last twenty-five pages is condensed in the following theorem.

6.2.4. THEOREM. *Let G be a connected graph with interval function I satisfying the following conditions:*

- (Q2) $K_4 - e$ is not an induced subgraph of G ;
- (Q3) the convex closure of an induced C_6 in G is Q_3 or $H_{3,3}$;
- (Q4) if $I(u,v) \cap I(v,w) = \{v\}$, then $d(u,w) \geq \max\{d(u,v), d(v,w)\}$, for any three vertices u, v and w of G ;
- (Q5) G is interval monotone.

Then G can be obtained from K_1 by a sequence of quasi-median expansions.

Recall that in the above theorem, condition (Q3) can be replaced by the weaker condition (Q3') (see the remark following Definition 6.2.1).

Before we close the circle, by which we shall obtain a number of characterizations of quasi-median graphs, one further step is required. (For the definition of a distance-preserving subgraph see page 87.)

6.2.5. THEOREM. *Let G be a connected distance-preserving subgraph of a Hamming graph H such that the quasi-median in H of any ordered triple of vertices of G consists of vertices of G . Then G is a quasi-median graph.*

PROOF. Since G is distance-preserving, the quasi-median in H of any triple of vertices in G is also the quasi-median of this triple in G . The conditions (Q2) and (Q3) follow immediately. \square

Finally we can close the circle.

6.2.6. THEOREM. *Let G be a graph obtained from K_1 by a sequence of quasi-median expansions. Then G is a connected distance-preserving subgraph of a Hamming graph H such that the quasi-median in H of any ordered triple of vertices of G consists of vertices of G .*

PROOF. The proof is by induction on the number of vertices of G . So let G be a connected distance-preserving "quasi-median closed" subgraph of a Hamming graph H , and let W and W' be convex sets in G with $U = W \cap W' \neq \emptyset$. Write $A = V \setminus (W \cup W')$ and $B = \{x \in A \mid x \text{ has a neighbour in } U\}$. Assume that U, W, W', A and B satisfy the conditions in Definition 6.2.3, and let G' be the quasi-median expansion of G with respect to W and W' . Then we have to prove that G' is embeddable in a Hamming graph in the prescribed way.

Fix a vertex p in U . Since H is transitive, we may assume that p is the zero vector.

If $A = \emptyset$, then $k = 1$. Let us assume in this case that all vertices of H have zero as first coordinate -that is, that $H = K_1 \times H'$ for some Hamming graph H' . If necessary, this can be attained by adding a dummy first coordinate.

If $A \neq \emptyset$, then we may assume that $(1,0,\dots,0), (2,0,\dots,0), \dots, (k-1,0,\dots,0)$ are the neighbours of p in B , and thus are in A . Since $[W \setminus U, A] = [W' \setminus U, A] = \emptyset$, all neighbours of p in $W \cup W'$ have 0 as first coordinate. If there is a vertex z in G with first coordinate $h > 0$, then the quasi-median of $(p, (1,0,\dots,0), z)$ in H is $(p, (1,0,\dots,0), (h,0,\dots,0))$, where $(h,0,\dots,0)$ is a vertex of G . Hence we have $0 < h \leq k-1$, and so it follows from the convexity of $W \cup W'$ that z lies in A . So again all vertices in $W \cup W'$ have first coordinate 0, and we may assume that

$$H = K_k \times H',$$

for some Hamming graph H' .

Let

$$\bar{H} = K_{k+1} \times H'.$$

For w in W' , let \bar{w} be the vertex of \bar{H} obtained from w by replacing the first coordinate of w , which is 0, by k . Write $\bar{W} = \{\bar{w} \mid w \text{ is in } W'\}$. It follows from the construction of G' that G' is (isomorphic to) the subgraph of \bar{H} induced by the set $W \cup \bar{W} \cup A$. Note that $\bar{H}[\bar{W} \cup A]$ is isomorphic to $\bar{H}[W' \cup A]$, which equals $H[W' \cup A]$. Since G is a distance-preserving subgraph of H , it follows that G' is a distance-preserving subgraph of \bar{H} .

It remains to prove that G' is "quasi-median closed" in \bar{H} . Let u, v and w be vertices of G' . If either all three vertices lie in $W \cup A$, or all three vertices lie in $\bar{W} \cup A$, then it is clear that the quasi-median of (u, v, w) in \bar{H} lies also in G' .

Choose u in W , v in A and w in W' . Then the vertices of the quasi-median (x, y, z) of (u, v, w) in H all have zero as first coordinate. It follows from the convexity of $W \cup A$ that x lies in W . Similarly z lies in W' . Since y lies in $(W \cup A) \cap (W' \cup A) \cap (W \cup W')$, it follows that y lies in U . From the structure of $G[U \cup B]$ we know that y has a unique neighbour y' in $A \cap I_H(v, y)$. So (x, y', \bar{z}) is the quasi-median of (u, v, \bar{w}) in \bar{H} , and x, y' and \bar{z} lie in $W \cup \bar{W} \cup A$.

For the final case that u, v and w lie in $W \cup W'$, we first deduce some properties of G .

By Theorem 6.2.5 we know that G is a quasi-median graph. Hence (25) in the proof of Theorem 6.2.4 holds also for G -that is, if (x, y, z) is its own quasi-median of size k in G , then the convex closure in G of $\{x, y, z\}$ induces a K_3^k . It is clear that this is also the K_3^k induced by the convex closure of $\{x, y, z\}$ in H .

Let x, y and z be vertices of G with x and y in U such that (x, y, z) is its own quasi-median. Since with any two vertices in U of a triangle the third vertex of the triangle lies also in U , it follows that the convex closure of $\{u, v, w\}$ lies entirely in U , so that w lies in U .

Let u, v and w be vertices of G with u and v in W and w in W' , and let (x, y, z) be the quasi-median of (u, v, w) . Then it follows from the convexity of W that x and y lie in W . Assume that z is in W' . Choose a

vertex x' in $I_G(x,z) \cap U$, and a vertex y' in $I_G(y,z) \cap U$. Note that such vertices exist because of the fact that $W \cup W'$ is convex and $[W \setminus U, W' \setminus U] = \emptyset$. Let (x'', y'', z) be the quasi-median of (x', y', z) . Since U is convex, x'' and y'' lie in U , and so by the previous observation z lies in U . So we have proved that x , y and z lie in W .

Similarly, if u , v and w are vertices of G with u in W and v and w in W' , and if (x, y, z) is the quasi-median of (u, v, w) , then x , y and z lie in W' .

Now we know enough of G to finish the proof that G' is "quasi-median closed" in \bar{H} . Let u , v and w be vertices with u and v in W and w in W' , and let (x, y, z) be the quasi-median of (u, v, w) in G . Then, as established above, x , y and z lie in W , and it follows that (x, y, z) is the quasi-median in \bar{H} of (u, v, \bar{w}) .

Finally, let u , v and w be vertices with u in W and v and w in W' , and let (x, y, z) be the quasi-median of (u, v, w) in G . Then, as established above, x , y and z lie in W' , and so $(\bar{x}, \bar{y}, \bar{z})$ is the quasi-median of (u, \bar{v}, \bar{w}) in \bar{H} . By definition \bar{x} , \bar{y} and \bar{z} lie in \bar{W} . This completes the proof of the theorem. \square

In the theorems proved up to now in this section, we have given three characterizations of quasi-median graphs. We close this section with one further characterization (the appropriate generalization of Theorem 3.2.8), the proof of which is straightforward and left to the reader.

6.2.7. THEOREM. *A graph G is a quasi-median graph if and only if G is a connected induced subgraph of a Hamming graph H such that the quasi-median in H of any ordered triple of vertices of G consists of vertices of G .*

6.3. HAMMING GRAPHS AS QUASI-MEDIAN GRAPHS

As might be expected, a regular quasi-median graph is a Hamming graph. We devote the last section of this chapter to the proof of this fact.

6.3.1. THEOREM. *A graph G is a Hamming graph if and only if it is a quasi-median graph containing two diametrical vertices, at least one of which has maximum degree.*

PROOF. The "only if" part of the proof is evident. So let G be a quasi-median graph, and let u and v be diametrical vertices such that u has maximum degree. Embed G in a Hamming graph H as in Theorem 6.2.6. It follows from the results in the foregoing section that if (x,y,z) is its own quasi-median of size k in G , then the convex closure of $\{x,y,z\}$ in H is the convex closure of $\{x,y,z\}$ in G , and so induces a K_3^k in G .

Without loss of generality, we may assume that u is the zero vector and v is a $(0,1)$ -vector of weight $\text{diam}(G)$. Let $(1,0,\dots,0), \dots, (a_1,0,\dots,0), (0,1,0,\dots,0), \dots, (0,a_2,0,\dots,0), \dots, (0,\dots,0,a_k,0,\dots,0)$ be the neighbours of u in G , so that $N_G(u)$ induces the disjoint union of $K_{a_1}, K_{a_2}, \dots, K_{a_k}$ in G . Since G is quasi-median closed in H , the interval $I_G(u,v)$ contains exactly one vertex of each clique in $N_G(u)$. The other vertices in $N_G(u)$ all have distance $\text{diam}(G)$ at v . Hence

$$I_G(u,v) \cap N(u) = \{(1,0,\dots,0), \dots, (0,\dots,0, \underset{\substack{\uparrow \\ k}}{1},0,\dots,0)\}.$$

As in the proof of Theorem 3.4.1, it follows that if $p_i = 0,1$, for $i = 1, \dots, k$, then $(p_1, \dots, p_k, 0, \dots, 0)$ is a vertex of G .

Without loss of generality, we may assume that

$$a_1 \geq a_2 \geq \dots \geq a_k \geq 1.$$

If $a_1 = 1$, then G is Q_k , and we are done.

So let $a_1 \geq 2$. We first prove that the vertex $(2,0,\dots,0)$ of G , which has distance $\text{diam}(G)$ at v , has $(2,1,0,\dots,0), \dots, (2,a_2,0,\dots,0), \dots, (2,0,\dots,0,a_k,0,\dots,0)$ as neighbours in G , so that the degree of $(2,0,\dots,0)$ equals $d(u)$, the maximum degree in G . Since the vertex $(2,1,0,\dots,0)$ is the median of v , $(2,0,\dots,0)$ and $(0,1,0,\dots,0)$, this vertex is a vertex of G . If, for $\alpha \geq 2$, the vertex $(0,\alpha,0,\dots,0)$ is in G , then the vertex $(2,\alpha,0,\dots,0)$ is also in G . For $(2,\alpha,0,\dots,0)$ lies in the convex closure of $\{(2,0,\dots,0), (0,\alpha,0,\dots,0), (1,1,0,\dots,0)\}$, and $((2,0,\dots,0), (0,\alpha,0,\dots,0), (1,1,0,\dots,0))$ is the quasi-median of $((2,0,\dots,0), (0,\alpha,0,\dots,0), v)$ in H , and so is in G . We can similarly determine all above-mentioned neighbours of $(2,0,\dots,0)$. From this we deduce that the vertices $(p_1, \dots, p_k, 0, \dots, 0)$ with $p_1 = 0,1,2$, and $p_i = 0,1$, for $i = 2, \dots, k$, are all vertices of G .

By a similar argument we conclude that all vertices

$(p_1, \dots, p_k, 0, \dots, 0)$ with $0 \leq p_i \leq a_i$, for $i = 1, \dots, k$,

are in G . Since the maximum degree in G equals $d_G(u)$, which is $a_1 + \dots + a_k$, it follows that G is the Hamming graph H_{a_1+1, \dots, a_k+1} . \square

6.3.2. COROLLARY. *A graph G is a Hamming graph if and only if G is a regular quasi-median graph.*

6.3.3. COROLLARY. *A graph G is a Hamming graph if and only if G is a quasi-median graph in which each vertex of G has at least one diametrical vertex.*

CHAPTER 7

QUASI-MEDIAN ALGEBRAS

In this final chapter we deal with an algebraic structure related to quasi-median graphs. As a consequence of our results we establish the relationship between median graphs and median algebras.

7.1. NEBESKÝ ALGEBRAS

The results of this section are due to NEBESKÝ [N3]. He used these results to study the algebraic properties of Husimi trees (connected graphs in which the convex closure of any circuit is a complete graph).

7.1. DEFINITION. A *Nebeský algebra* $A = (V, q)$ consists of a finite set V and a ternary operation $q: V \times V \times V \rightarrow V$ satisfying the following conditions, for any u, v, w and x in V (for convenience we write uvw instead of $q(u, v, w)$):

$$(q1) \quad uvu = u;$$

$$(q2) \quad uvw = wvu;$$

$$(q3) \quad uv(uvw) = uvw;$$

$$(q4) \quad u(uvw)w = uvw;$$

$$(q5) \quad (uvw)wx = u(vwx)w.$$

We use (q1) and (q2) in the sequel without mention.

7.1.2. PROPOSITION. *If $A = (V, q)$ is a Nebeský algebra, then, for any u, v, w and x in V :*

- (n1) $uuv = u$;
- (n2) *if $uvx = x$, then $vux = x$;*
- (n3) $vu(uvw) = uvw$;
- (n4) $u(uvw)v = uvw$;
- (n5) *if $uvw = vuw$, then $uvw = uvw$;*
- (n6) *if $uvx = uwx$, then $vuw = vxw$.*

PROOF. (n1) By (q5), we have $uuv = (uvu)uv = u(vuv)u = uvu = u$.
 (n2) If $uvx = x$, then by (q5) we have

$$\begin{aligned} vux &= xuv = (uvx)uv = (xvu)uv && \text{(q5)} \\ &= x(vuv)u = xv u = uvx = x. \end{aligned}$$

- (n3) The equality follows from (q3) and (n2).
- (n4) By (q5) and (n1), we have $u(uvw)v = (uuv)vw = uvw$.
- (n5) Let $uvw = vuw$. Then we have

$$\begin{aligned} uvw &= wvu && \text{(n3)} \\ &= vw(wvu) = (wvu)wv \\ &= (uvw)wv = (vuw)wv && \text{(q5)} \\ &= v(uvw)w && \text{(n4)} \\ &= v(vwu)w = vw u \\ &= uvw. \end{aligned}$$

- (n6) Let $uvx = uwx$. Then we have

$$\begin{aligned} vuw &= (vuw)(vuw)x && \text{(n1)} \\ &= (vu(vuw))(vuw)x && \text{(q3)} \\ &= v(u(vuw)x)(vuw) && \text{(q5)} \\ &= v(x(vuw)u)(vuw) \\ &= v((xvu)uw)(vuw) && \text{(q5)} \\ &= v((uvx)uw)(vuw) \\ &= v((uwx)uw)(vuw) = v(wu(uwx))(vuw) \end{aligned}$$

$$\begin{aligned}
& \stackrel{(n3)}{=} v(uwx)(vuw) = (vuw)(uvx)v \stackrel{(q5)}{=} ((vuw)uv)vx \\
& \stackrel{(q3)}{=} (vu(vuw))vx = (vuw)vx = (wuv)vx \\
& \stackrel{(q5)}{=} w(uvx)v.
\end{aligned}$$

Interchanging the roles of u and x and of v and w in the above computation we get

$$vxw = wxv = v(xwu)w = w(uwx)v = w(uvx)v = vuw.$$

□

If A is a Nebeský algebra, then, for u and v in V , we let

$$[u, v] := \{w \in V \mid uvw = w\},$$

and

$$[u, v]^* := \{w \in V \mid uwv = w\}.$$

7.1.3. PROPOSITION. *Let A be a Nebeský algebra. Then, for u and v in V :*

- (n7) $u, v \in [u, v]$;
- (n8) $[u, v] = [v, u]$;
- (n9) $[u, u] = \{u\}$;
- (n10) if $x \in [u, v]$, then $[u, x] \subseteq [u, v]$;
- (n11) if $x \in [u, v]$ and $y \in [u, x]$, then $x \in [y, v]$;
- (n12) $[u, v] \subseteq [u, v]^*$.

PROOF. (n7), (n8) and (n9) follow easily from (q1), (q2), (n1) and (n2).

(n10) Let x be in $[u, v]$ and y be in $[u, x]$. Then by (n2), we have

$$vux = x \quad \text{and} \quad yxu = uxy = y.$$

By (q5), we have

$$vuy = yuv = (yxu)uv = y(xuv)u = yxu = y,$$

and so by (n2), it follows that y lies in $[u, v]$.

(n11) Let x be in $[u,v]$ and y be in $[u,x]$. By (n10), we have

$$y \in [u,x] \subseteq [u,v],$$

and so $uxy = uvy$. By (n6) and (n2), we have

$$vyx = vux = x.$$

(n12) Let x be in $[u,v]$. Then by (n2), we have

$$uvx = x = vux.$$

It follows from (n5) that

$$x = uvx = uxv,$$

so that x lies in $[u,v]^*$. \square

7.1.4. PROPOSITION. *Let A be a Nebeský algebra. Then, for u, v and w in V ,*

$$[u,v] \cap [v,w] \cap [u,w]^* = \{uvw\}.$$

PROOF. Denote $X = [u,v] \cap [v,w] \cap [u,w]^*$. By (q3), (q4) and (n3), we have

$$uvw = uv(uvw) = u(uvw)w = vw(uvw),$$

and so uvw lies in X .

Choose x in X , so that $x = vux = vwx = uxw$. Then it follows from (n6) that

$$uvw = uxw = x,$$

which completes the proof. \square

7.1.5. PROPOSITION. *Let A be a Nebeský algebra. Then, for u and v in V ,*

$$[u,v]^* = \{x \in V \mid [u,x] \cap [x,v] = \{x\}\}.$$

PROOF. First let x be in $[u,v]^*$, so that $uxv = x$. It follows from (n1) that x lies in $[u,x] \cap [x,v]$. Choose a vertex y in $[u,x] \cap [x,v]$. Then by definition, $uxy = xvy = y$. By (q5) we have

$$y = uxy = ux(xvy) = (yvx)xu = y(vxu)x = yxx = x,$$

which tells us that $[u,x] \cap [x,v] = \{x\}$.

Conversely, let x in V be such that $[u,x] \cap [x,v] = \{x\}$. Then by (q3) and (n3), we have

$$ux(uxv) = xv(uxv) = uxv,$$

and so uxv lies in $[u,x]$ as well as in $[x,v]$ -that is, $uxv = x$. So by definition, x lies in $[u,v]^*$. \square

7.1.6. DEFINITION. Let $A = (V,q)$ be a Nebeský algebra. The *underlying graph* G_A of A has V as vertex-set, and two distinct vertices u and v are joined by an edge, whenever $[u,v] = \{u,v\}$.

7.1.7. PROPOSITION. Let A be a Nebeský algebra. Then the set $[u,v]$ induces a connected subgraph of G_A , for any two vertices u and v of G_A .

PROOF. The proof is by induction on $|[u,v]|$.

If $|[u,v]| \leq 2$, then the assertion is true by definition. So let $|[u,v]| \geq 3$, and choose a vertex x in $[u,v]$ distinct from u and v . By (n12) and Proposition 7.1.5, we have

$$[u,x] \cap [x,v] = \{x\},$$

and so v does not lie in $[u,x]$, and u does not lie in $[x,v]$. It follows from (n10) that

$$|[u,x]| < |[u,v]|, \quad \text{and} \quad |[x,v]| < |[u,v]|.$$

So by the induction hypothesis, $[u,x]$ and $[x,v]$ induce connected subgraphs of G_A . Hence, by (n10), there exist a (u,x) -path and an (x,v) -path in the subgraph induced by $[u,v]$. \square

Note that by this proposition the underlying graph of a Nebeský algebra is connected. An example of a Nebeský algebra A is the following: A has $K_4 - e$ as underlying graph (see Figure 7.1), and $[u,w] = \{u,v,w\}$. It is left to the reader to check that this description defines a Nebeský algebra on the set $\{u,v,w,x\}$. Note that $I_{K_4 - e}(u,w) \neq [u,w]$.

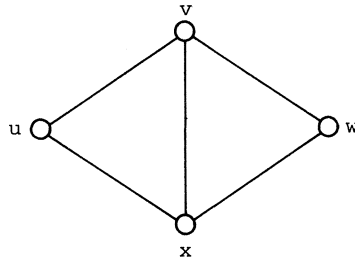


FIGURE 7.1.

7.2. QUASI-MEDIAN GRAPHS AND QUASI-MEDIAN ALGEBRAS

In this final section we discuss an algebraic structure associated with quasi-median graphs.

7.2.1. DEFINITION. A quasi-median algebra $A = (V,q)$ is a Nebeský algebra satisfying the following condition, for any u, v and w in V :

$$(q6) \quad \text{if } uvw = v \text{ and } [u,w] = \{u,w\}, \text{ then } [u,v] = \{u,v\}.$$

Note that condition (q6) can also be formulated in terms of the underlying graph G_A of A : if uw is an edge in G_A , and if $uvw = v$, for a vertex v distinct from u and w , then v is adjacent to u (and so is also adjacent to w).

In the sequel we prove that the underlying graph of a quasi-median algebra is a quasi-median graph. Once more we reverse the order of theorem and proof, and once more the proof is split into a number of steps. For the conditions (Q1), (Q2) and (Q3'), which characterize quasi-median graphs, see pages 128 and 129.

Let A be a quasi-median algebra with underlying graph G , and let I be the interval function of G .

$$(1) \quad I(u,v) = [u,v] \quad \text{for } u,v \in V.$$

PROOF. The proof is by induction on $d(u,v)$. For $d(u,v) \leq 1$, the assertion is true by definition.

Let $n = d(u,v) \geq 2$. First we prove that $I(u,v) \subseteq [u,v]$. Let w be a neighbour of u in $I(u,v)$. Then $d(w,v) = n - 1$, and so by the induction hypothesis,

$$[w,v] = I(w,v).$$

Assume that w is not in $[u,v]$, so that $uvw \neq w$. By Proposition 7.1.4 we have

$$uvw \in [v,w] = I(w,v),$$

and so $uvw \neq u$. By (q3), we have $uvw = u(uvw)w$, and so (q6) implies that uvw is a common neighbour of u and w in $[w,v] = I(w,v)$. It follows that

$$d(u,v) \leq d(uvw,v) + 1 = d(w,v) = d(u,v) - 1,$$

which is a contradiction. So we have

$$I(x,y) \subseteq [x,y],$$

for any two vertices x and y in G with $d(x,y) \leq n$.

The proof that $[u,v] \subseteq I(u,v)$ is more tedious. Assume the contrary.

We first prove that, for any vertex p in $[u,v] \setminus I(u,v)$ and any vertex q in $I(u,v) \setminus \{u,v\}$, we have

$$[u,p] \cap [u,q] = \{u\},$$

(*)

$$\text{and } [v,p] \cap [v,q] = \{v\}.$$

We prove only the first equality. Assume that there exists a vertex r in

$[u,p] \cap [u,q]$ distinct from u . Note that $r \neq v$. Then by (n11), p lies in $[r,v]$. By the induction hypothesis, we have $[u,q] = I(u,q)$, and so r lies in $I(u,q) \subset I(u,v)$. So by the induction hypothesis, we have $I(r,v) = [r,v]$, from which it follows that p lies in $I(r,v) \subset I(u,v)$. This contradicts the choice of p . This proves (*).

It follows from (*) and Proposition 7.1.7 that we may choose a vertex w in $[u,v] \setminus I(u,v)$ adjacent to v . Note that

$$d(u,w) \geq d(u,v) = n \geq 2.$$

Let z be a neighbour of u in $I(u,v)$. Then $d(z,v) = n - 1 \geq 1$, and so by the induction hypothesis $[z,v] = I(z,v) \subset I(u,v)$. Since w does not lie in $I(u,v)$, it follows that

$$n - 1 = d(z,v) \leq d(z,w) \leq d(z,v) + 1 = n.$$

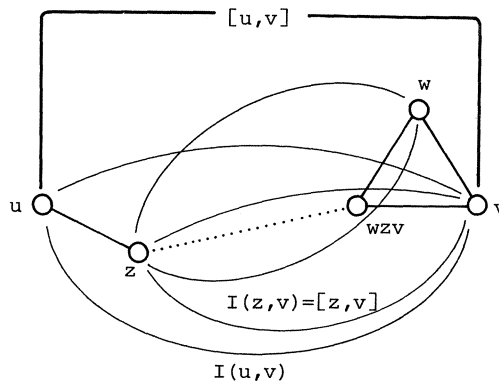


FIGURE 7.2.

CASE 1: $d(z,w) = d(z,v) = n - 1$ (see Figure 7.2).

By the induction hypothesis, $[z,w] = I(z,w)$, and so

$$v, w \notin I(z,w) \cap I(z,v) = [z,w] \cap [z,v].$$

Hence (q6) implies that wzv , which lies in $[z,w] \cap [z,v]$, is a common neighbour of v and w in $I(z,v) \subset I(u,v)$. It follows that

$$d(u,w) = n \text{ and } wzv \in N_1(w,u).$$

Using the first part of the proof, we get

$$\begin{aligned} u \neq wzv \in I(u,w) \cap I(u,wzv) &= I(u,w) \cap [u,wzv] \\ &\subseteq [u,w] \cap [u,wzv]. \end{aligned}$$

This contradicts (*), settling Case 1.

CASE 2: $d(z,w) = d(z,v) + 1 = n$.

Then v lies in $I(z,w)$, and by the first part of the proof we have $I(z,w) \subseteq [z,w]$. Hence

$$(**) \quad vwz = v.$$

Since $d(u,w) \geq d(u,v) = n \geq 2$, we may choose a vertex y in $[u,w]$ distinct from u , which is adjacent to w but not to v . (Note that it may be necessary to change our choice of w .) Note that by (*), y lies in $[u,v] \setminus I(u,v)$ (see Figure 7.3).

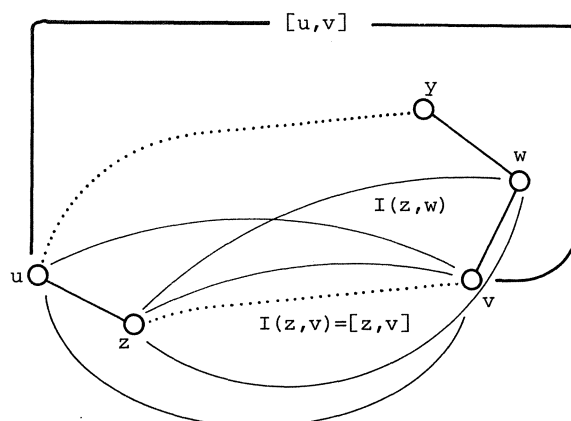


FIGURE 7.3.

Assume that u is not in $[z,w]$. Then wuz lies in $[u,z] \setminus [z,w] = \{u\}$, and so $wuz = u$. Since w and z lie in $[u,v]$, it follows from (q5) and (*)

that

$$z = wz = w(vuz)u = (wvu)uz = wuz = u,$$

which is a contradiction. So

$$u \in [z, w].$$

This implies that y lies in $[u, w] \subseteq [z, w]$, so that

$$(***) \quad wzy = y.$$

It follows from (***) and (q5) that

$$zyv = vyz = v(wzy)z = (vwz)zy = vzy = yzv.$$

Using (n5) and (*), we deduce that

$$zyv = zvy = v,$$

so that v lies in $[z, y]$. Since $d(v, y) = 2 \leq n$, it follows from the first part of the proof that

$$w \in I(v, y) \subseteq [v, y] \subseteq [z, y].$$

Hence $wzy = w$, which contradicts (***). This settles Case 2, and so $[u, v] \subseteq I(u, v)$. \square

(2) K_4 -e does not occur in G .

PROOF. Assume the contrary, and let u, v, w and x induce a K_4 -e in G such that u and w are the non-adjacent vertices. By (1) we have

$$v, x \in I(u, w) = [u, w],$$

and so by (q5),

$$w = vw = (vuw)wx = v(uwx)w = vxw = x,$$

which establishes the required contradiction. \square

(3) $I(u,v)$ induces a bipartite subgraph in G for any u and v in V .

PROOF. Assume the contrary, and let x and y be two adjacent vertices in the j -th level of the interval $I(u,v)$. Then by (1),

$$xuy \in [u,x] \cap [u,y] = I(u,x) \cap I(u,y),$$

and so xuy is distinct from x and y . By (q6), the vertex xuy is a common neighbour of x and y in the $(j-1)$ th level, and similarly, xvy is a common neighbour of x and y in the $(j+1)$ th level. Hence x , y , xuy and xvy induce a K_4 -e in G , contradicting (2). \square

(4) If x and y are two distinct neighbours of v in $I(u,v)$, then xuy is a common neighbour of x and y , for any u and v in V .

PROOF. Note that $z = xuy$ lies in $I(u,x) \cap I(u,y)$. Since $I(u,v)$ induces a bipartite subgraph, it follows that

$$d(x,y) = 2.$$

Assume that z is not a common neighbour of x and y , so that z is not in $I(x,y) = [x,y]$. It follows from (1), Proposition 7.1.5, and the fact that x and y are neighbours of v , that

$$I(z,x) \cap I(z,y) = \{z\},$$

$$I(x,z) \cap I(x,y) = \{x\},$$

and

$$I(y,x) \cap I(y,z) = \{y\};$$

- that is, $xzy = z$, $zxy = x$ and $xyz = y$.

Since x and y lie in $I(z,v)$, we have

$$xvz = x, \text{ and } vzy = y.$$

Hence by (q5), we have

$$z = xzy = (xvz)zy = x(vzy)z = xyz = y,$$

which establishes the required contradiction. \square

(5) G satisfies condition (Q3').

PROOF. Let $u_1 \rightarrow \dots \rightarrow u_6 \rightarrow u_1$ be an induced C_6 in G with $d(u_1, u_4) = 3$. Then u_1, \dots, u_6 lie in $I(u_1, u_4)$. Hence by (4), the vertex $u_3u_1u_5$ is a common neighbour of u_3 and u_5 in $I(u_1, u_3) \cap I(u_1, u_5)$, so that $u_3u_1u_5$ is also adjacent to u_1 , and similarly, the vertex $u_2u_4u_6$ is a common neighbour of u_2, u_4 and u_6 .

Since $I(u_1, u_4)$ induces a bipartite subgraph of G , the eight vertices obtained above induce a Q_3 in G . \square

(6) Let u, v and w be vertices of G with $I(u, v) \cap I(u, w) = \{u\}$,
 $I(v, u) \cap I(v, w) = \{v\}$ and $I(w, u) \cap I(w, v) = \{w\}$. Then
 $d(u, v) = d(v, w) = d(w, u)$.

PROOF. By (1) and Proposition 7.1.5, we have

$$[x, y]^* = \{z \in V \mid I(x, z) \cap I(z, y) = \{z\}\},$$

for any two vertices x and y of G .

Assume the contrary, and let u, v and w be such that

- $u \in [v, w]^*$, $v \in [u, w]^*$, and $w \in [u, v]^*$;
- $d(u, v) \geq d(v, w) \geq d(w, u)$;
- $d(u, v) > d(w, u)$;
- $d(u, v) + d(v, w) + d(w, u)$ is minimal under these three conditions;
- $d(w, u)$ is minimal under these four conditions.

Note that by (q6), we have $d(u, w) \geq 2$.

Let x be a neighbour of w in $I(w, u)$. Since $I(x, u) \subseteq I(w, u)$, it follows that

$$u \in [x, v]^*.$$

Furthermore,

$$d(v,w) \leq d(v,x) \leq d(v,w) + 1.$$

CASE 1: $d(v,x) = d(v,w) + 1$.

Then $d(u,v) + d(v,w) + d(w,u) = d(u,v) + d(v,x) + d(x,u)$. If there is a neighbour y of x in $I(x,u) \cap I(x,v)$, then by (4) the vertex wy is a common neighbour of w and y in $I(v,w) \cap I(w,y) \subseteq I(v,w) \cap I(w,u)$, which contradicts the fact that w is in $[u,v]^*$. Hence

$$x \in [u,v]^*.$$

If v lies in $[u,x]^*$, then this contradicts the minimality of $d(u,w)$ in our choice of u , v and w . Hence there exists a neighbour z of v in $I(v,x) \cap I(v,u)$. Since z is not in $I(v,w)$, it follows from (4) that the situation is as depicted in Figure 7.4, where P is a shortest path from z to x in $I(v,x) \setminus I(v,w)$, which is "matched" to a shortest (v,w) -path Q .

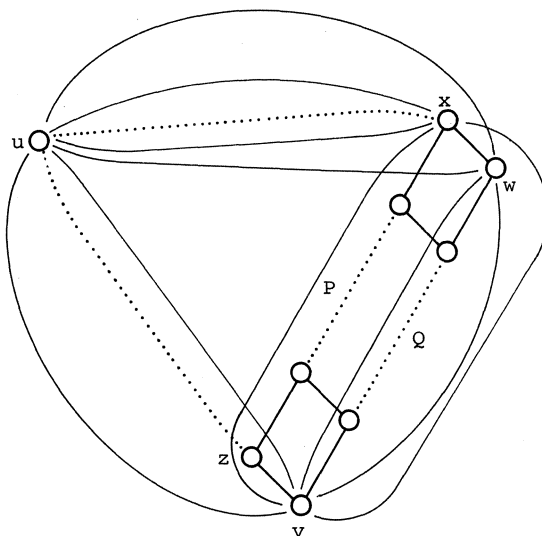


FIGURE 7.4.

If z lies in $[u,x]^*$, then it follows from the minimality of $d(u,v) + d(v,w) + d(w,u)$ that

$$d(u,v) - 1 = d(u,z) = d(u,x) = d(u,w) - 1,$$

contradicting the fact that $d(u,v) > d(u,w)$.

Hence there is a neighbour y of z in $I(z,u) \cap I(z,x)$. We deduce from (4) that the situation must be as in Figure 7.5, for some vertices p, q and r , and a shortest (y,q) -path "matched" to the first part of the (z,x) -path P .

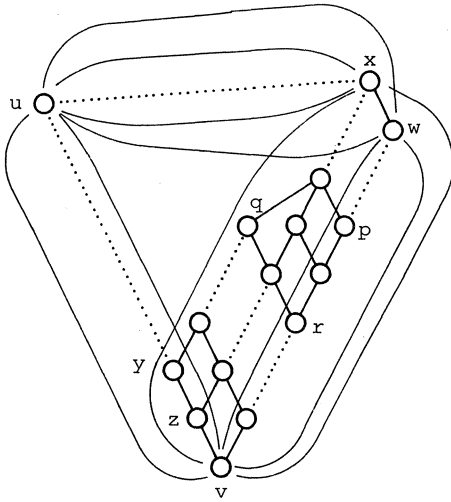


FIGURE 7.5.

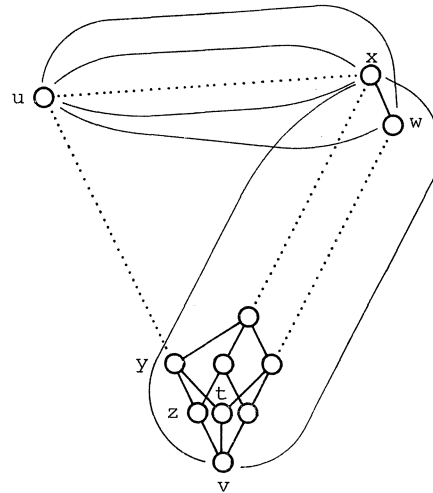


FIGURE 7.6.

Property (5) implies the existence of a common neighbour s of p, q and r in $I(v,x)$. Note that s lies in $I(p,r) \subseteq I(v,w)$. Hence q has a neighbour in $I(v,w)$. By applying (5) a number of times, we conclude that the situation is as depicted in Figure 7.6.

So t is a neighbour of v in $I(v,u) \cap I(v,w)$, contradicting the fact that v is in $[u,w]^*$. This settles Case 1.

CASE 2: $d(v,x) = d(v,w)$.

By (q6), the vertex $w' = wvx$ is a common neighbour of w and x in $I(v,w) \cap I(v,x)$. It follows from the minimality of $d(u,v) + d(v,w) + d(w,u)$ that the situation is as in Figure 7.7, for some non-negative integers k, l and m , where $d(v,x) = k + l + m$.

Then we have

$$k + l = d(u,v) \geq d(v,w) = d(v,x) = k + l + m,$$

so that $m = 0$. Furthermore, we have

$$k + 1 = d(u,v) > d(u,w) = 1 + 1,$$

so that $k \geq 2$.

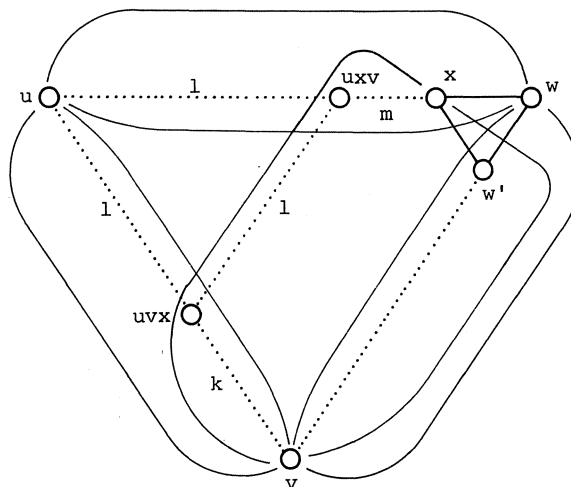


FIGURE 7.7.

Using a similar argument as in Case 1, we deduce a contradiction. \square

(7) *Each ordered triple (u,v,w) of vertices in G has a quasi-median.*

PROOF. The assertion follows immediately from (1) and (6) and Propositions 7.1.4 and 7.1.5. \square

Combining (2), (5) and (7), we have the proof of the following theorem.

7.2.2. THEOREM. *Let G be the underlying graph of a quasi-median algebra. Then G is a quasi-median graph.*

We can associate a quasi-median algebra with a quasi-median graph, by which we establish a converse of the above theorem.

7.2.3. THEOREM. *Let G be a quasi-median graph with interval function I . Define the mapping $q: V \times V \times V \rightarrow V$ by $q(u,v,w) = x$, whenever x lies in $I(u,v) \cap I(v,w)$ and $I(u,x) \cap I(x,w) = \{x\}$. Then $A = (V,q)$ is a quasi-median algebra with G as its underlying graph.*

PROOF. As above we write uvw instead of $q(u,v,w)$. Note that q is a well-defined mapping, and that for any three vertices u, v and w of G , the quasi-median of (u,v,w) is the triple (wuv,uvw,vwu) . The conditions (q1) - (q4) and (q6) follow easily from the properties of G .

To prove that q satisfies condition (q5) embed G in a Hamming graph H_{a_1, \dots, a_n} as in Theorem 6.2.5, so that each vertex u of G is labelled by a vector (u_1, \dots, u_n) in $\prod_{i=1}^n \{0, 1, \dots, a_i - 1\}$. Let u, v, w and x be vertices of G , and let i be an integer with $1 \leq i \leq n$. It follows that if w_i is one of u_i, v_i and x_i , then the i -th coordinates of $(uvw)wx$ and $u(vwx)w$ both equal w_i . If w_i is distinct from u_i, v_i and x_i , it follows that v_i is the i -th coordinate of uvw , and so the i -th coordinate of $(uvw)wx$ equals the i -th coordinate of vwx . Furthermore, it follows that the i -th coordinate of $u(vwx)w$ equals the i -th coordinate of vwx . We have therefore proved that $(uvw)wx = u(vwx)w$. \square

We conclude this section by establishing the relationship between median graphs and median algebras. Note that a median algebra (V,m) is a Nebeský algebra, so that it makes sense to speak about the underlying graph of a median algebra.

7.2.4. COROLLARY. *The underlying graph G of a median algebra (V,m) is a median graph.*

PROOF. It is easily verified that a median algebra is a quasi-median algebra such that

$$uvw = vuw = vwu,$$

for any u, v and w in V . This implies that G is a quasi-median graph in which any quasi-median has size 0 -that is, G is a median graph. \square

7.2.5. COROLLARY. *Let G be a median graph with interval function I . Define the mapping $m: V \times V \times V \rightarrow V$ by $I(u,v,w) = \{m(u,v,w)\}$, for any three vertices u, v and w of G . Then (V,m) is a median algebra.*

PROOF. By Theorem 7.2.3, (V,m) is a quasi-median algebra with $uvw = vuv =$
 $= wvu$ for any u, v and w in V , and so (V,m) is a median algebra. \square

We have come to the end of this monograph. We have occupied ourselves with studying a number of classes of graphs, all more or less remotely related to the n -cube. As the main tool of our analysis we have used the interval function of a graph. The topic of intervals in graphs is of course far from exhausted. The reader will certainly have thought of questions related to the results presented here. There are also other problems where the interval function could be used. It is the hope of the author that such questions will stimulate further research on this topic.

REFERENCES

- [Al] ALVAREZ, L., *Undirected graphs realizable as graphs of modular lattices*, *Canad. J. Math.* 17 (1965), 923-932.
- [As] ASCHBACHER, M., *On collineation groups of symmetric block designs*, *J. Combinatorial Theory Ser. B.* 11 (1971), 272-281.
- [AMS] ASSMUS, Jr. E.F., J.A. MEZZAROBÀ & J. SALWACH, *Planes and biplanes*, in: "Higher Combinatorics" (M. Aigner ed.), Reidel, Dordrecht (1977), pp. 205-212.
- [Av] AVANN, S.P., *Metric ternary distributive semi-lattices*, *Proc. Amer. Math. Soc.*, 12 (1961), 407-414.
- [BFB] BALABAN, A.T., D. FARCAȘIU & R. BANIÇA, *Graphs of multiple 1,2-shifts in carbonium ions and related systems*, *Rev. Roumaine Chim.*, 11 (1966), 1205-1227.
- [Be] BERGE, C., *Graphs and Hypergraphs*, North-Holland, Amsterdam, 1973.
- [BD] BERGE, C. & P. DUCHET, *A Generalization of Gilmore's theorem*, in: "Recent Advances in Graph Theory" (Proc. Symp. Prague June 1974; M. Fiedler, ed.), Academia, Praha, 1975.
- [BR] BHASKARA RAO, S. & S.P. RAO HEBBARE, *Characterization of planar distance convex simple graphs*, *Proc. Symposium in Graph Theory*, ISI, Calcutta, 1976.
- [B1] BIGGS, N., *Algebraic Graph Theory*, Cambridge Univ. Pr., London, 1974.
- [B2] BIGGS, N., *An edge-colouring problem*, *Amer. Math. Monthly*, 79 (1972), 1018-1020.
- [BLW] BIGGS, N., E.K. LLOYD & R.J. WILSON, *Graph Theory 1736-1936*, Clarendon Press, Oxford, 1976.
- [Bi] BIRKHOFF, G., *Lattice Theory*, A.M.S. Coll. Publ. Vol. XXV, 3rd ed., Amer. Math. Soc., Providence R.I., 1963.
- [BM] BONDY, J.A. & U.S.R. MURTY, *Graph Theory with Applications*, Macmillan, London, 1976.
- [BS] BROUWER, A.E. & A. SCHRIJVER, *A characterization of supercompactness with an application to treelike spaces*, Report Math. Centre ZW 34/74, Amsterdam, 1974.

- [C1] CAMERON, P.J., *Biplanes*, Math. Z., 131 (1973) 85-101.
- [C2] CAMERON, P.J., *Parallelisms of Complete Designs*, London Math. Soc. Lecture Notes 23, Cambridge Univ. Pr., London, 1976.
- [CL] CAMERON, P.J. & J.H. van LINT, *Graph Theory, Coding Theory and Block Designs*, London Math. Soc. Lecture Notes 19, Cambridge Univ. Pr., London, 1975.
- [Du] DUCHET, P., *Propriété de Helly et problèmes de représentation*, in: "Problèmes de Combinatoires et Théorie des Graphes" (J.-C. Bermond et al. eds), Colloques Internationaux CNRS No. 260, CNRS, Paris, 1978.
- [DR] DUFFUS, D. & I. RIVAL, *Path length in the covering graph of a lattice*, Discrete Math., 19 (1977), 139-158.
- [Er] ERDÖS, P., *Topics in combinatorial analysis*, in: Proc. 2nd Louisiana Conf. Combinatorics, Graph Theory and Computing, (R.C. Mullin et al. eds), Baton Rouge, 1971.
- [F1] FOLDES, S., *A characterization of hypercubes*, Discrete Math., 17 (1977), 155-159.
- [F2] FOLDES, S., Private communication.
- [Gi] GILMORE, P.C., *Families of sets with faithful graph representation*, Report Thomas J. Watson Research Centre, Yorktown Heights N.J., 1962.
- [GG] GREENWOOD, R.E. & A.M. GLEASON, *Combinatorial relations and chromatic graphs*, Canad. J. Math., 7 (1955), 1-7.
- [H] HALL Jr, M., *Combinatorial Theory*, Blaisdell, London, 1967.
- [Ha] HANANI, H., *On transversal designs*, in: "Combinatorics; part 1: Theory of designs, finite geometry and coding theory" (M. Hall Jr & J.H. van Lint eds), Math. Centre Tracts 55, Math. Centre, Amsterdam, 1974, pp. 42-52.
- [KS] KALBFLEISCH, J. & R. STANTON, *On the maximal triangle-free edge-chromatic graphs in three colors*, J. Combinatorial Theory, 5 (1968), 9-20.
- [Ko] KOWALEWSKI, A., *Topologische Deutung von Buntordnungsproblemen*, Sitzungsber. Akad. Wiss. Wien (Abt IIa), 126 (1917), 963-1007.

- [La] LABORDE, J.M., *Characterisation locale du graphe du n-cube*, Journées Combinatoires, Grenoble, June 1978.
- [Mf1] MCFALL, J.D., *Hypercubes and their characterizations*, Univ. Waterloo, Dept. of Combinatorics and Optimization Research Report CORR 78-26, August 1978.
- [Mf2] MCFALL, J.D., *Characterizing hypercubes*, to appear.
- [M1] MULDER, H.M., *The structure of median graphs*, *Discrete Math.*, 24 (1978), 197-204.
- [M2] MULDER, H.M., *$(0, \lambda)$ -graphs and n-cubes*, *Discrete Math.*, 28 (1979), 179-188.
- [M3] MULDER, H.M., *n-cubes and median graphs*, *J. Graph Theory*, 4 (1980), 107-110.
- [MS] MULDER, H.M. & A. SCHRIJVER, *Median graphs and Helly hypergraphs*, *Discrete Math.*, 25 (1979), 41-50.
- [N1] NEBESKÝ, L., *Graphic algebras*, *Comment. Math. Univ. Carolinae*, 11 (1970), 533-544.
- [N2] NEBESKÝ, L., *Median graphs*, *Comment. Math. Univ. Carolinae*, 12 (1971), 317-325.
- [N3] NEBESKÝ, L., *Algebraic properties of Husimi trees*, to appear.
- [Ra1] RAO HEBBARE, S.P., *Some problems on distance convex simple graphs*, Tech. Report No. 2/77, ISI, Calcutta, 1977.
- [Ra2] RAO HEBBARE, S.P., *A class of distance convex simple graphs*, *Ars Combinatoria*, 7 (1979), 19-26.
- [Ri] RIVAL, I., Private communication.
- [R1] RYSER, H.J., *Combinatorial Mathematics*, Carus Math. Monograph 14, Wiley, New York, 1963.
- [R2] RYSER, H.J., *Intersection properties of finite sets*, *J. Combinatorial Theory Ser. A.*, 14 (1973), 79-92.
- [Se] SEIDEL, J.J., *Strongly regular graphs with $(-1, 1, 0)$ adjacency matrix having eigenvalue 3*, *Linear Algebra and Appl.*, 1 (1968), 281-298.
- [S1] SHOLANDER, M., *Trees, lattices, order, and betweenness*, *Proc. Amer.*

- Math. Soc., 3 (1952), 369-381.
- [S2] SHOLANDER, M., *Medians and betweenness*, Proc. Amer. Math. Soc., 5 (1954), 801-807.
- [S3] SHOLANDER, M., *Medians, lattices, and trees*, Proc. Amer. Math. Soc., 5 (1954), 808-812.
- [Sh] SHRIKHANDE, S.S., *The uniqueness of the L_2 association scheme*, Ann. Math. Statist., 30 (1959), 781-798.
- STEENROD, N.E., P.R. HALMOS, M.M. SCHIFFER & J.A. DIEUDONNÉ, *How to write mathematics*, Amer. Math. Soc., Providence R.I., 1973.
- [Wi] WILSON, R.J., *Introduction to Graph Theory*, 2nd edition, Longman Group, Harlow, Essex, 1979.
- [Wy] WYTHOFF, W.A., *Oplossing vraagstuk 28*, Nieuw Archief voor de Wiskunde, 10 (1906-1910), 60-61.

INDEX OF SYMBOLS AND NOTATIONS

in order of their first appearances in the text.

G	graph	4
$V(G)$	vertex-set of G	4
$E(G)$	edge-set of G	4
V	vertex-set of G	4
E	edge-set of G	4
uv	edge joining u and v	4
\bar{G}	complementary graph of G	4
$G[V']$	subgraph of G induced by $V' \subseteq V$	4
$G[E']$	subgraph of G edge-induced by $E' \subseteq E$	5
K_n	complete graph on n vertices	5
$ V $	size of set V	5
$d(v) = d_G(v)$	degree of vertex v in G	5
$d(G)$	degree of regular graph G	5
$u \rightarrow \dots \rightarrow v$	path from u to v	5
$d(u,v) = d_G(u,v)$	distance between u and v in G	5
$\text{diam}(G)$	diameter of G	5
$u \rightarrow \dots \rightarrow v \rightarrow u$	circuit	6
$K_{n,m}$	complete bipartite graph	6
$K_{1,1,m}$		6
$G \times H$	product of G and H	7
Q_n	n -cube	7
$\mathcal{P}(V)$	power-set of V	7
$A \Delta B$	symmetric difference of A and B	8
$P = (V, \leq)$	poset	8
$u \leq v$		8
$u \wedge v$	greatest lower bound of u and v	9
$u \vee v$	least upper bound of u and v	9
0	universal lower bound	9
1	universal upper bound	9
$D = (X, \mathcal{B})$	block design with point-set X and block-set \mathcal{B}	9
(b, v, r, k, λ)	parameters of block design	9
(v, k, λ)	parameters of symmetric block design	10
$\bar{D} = (X, \bar{\mathcal{B}})$	complementary block design of D	10
H	Hadamard matrix	10

$I_G(u,v) = I(u,v)$	interval between u and v in G	11
$I = I_G$	interval function of G	11
$N_i(u,v)$	i -th level in $I(u,v)$	18
$N_i(u)$	i -th level of u	18
$N(u)$	set of neighbours of u	18
$K_4 - e$	K_4 minus an edge	19
$G(P)$	diagraph of poset P	26
\leq_u	canonical ordering with respect to u	28
$\frac{1}{2}Q_{2n}$	halfcube	39
$G(D)$	incidence graph of block design D	45
D_H	Hadamard 3-design	57
G_H	Hadamard graph	57
$I(u,v,w)$	$I(u,v) \cap I(v,w) \cap I(w,u)$	65
$\langle u,v,w \rangle$	median of u, v and w	66
G_I		68
$[S,T]$	set of edges between S and T	77
(V,m)	median algebra	90
$[u,v]$	order interval	90
(V,E)	hypergraph	97
E	(edge-set of) hypergraph	97
I_E		97
G_E	underlying graph of E	97
E_I		101
$P_k(V)$	family of all k -subsets of V	107
$\{v\}^\uparrow$	family of all subsets of V containing v	107
H_{a_1, \dots, a_n}	Hamming graph	113
O_k	odd graph	120
E_k	extended odd graph	121
C_n	circuit of length n	128
(V,q)	Nebesky algebra	165
uvw	$q(u,v,w)$	165
$[u,v]$		167
$[u,v]^*$		167
G_A	underlying graph of Nebesky algebra A	169

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adjacent vertices	4	degree	5
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bipartite graph	6	diameter	5
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incident	4	parameters of symmetric block design	10
induced subgraph	4	partially ordered set	8
internal vertex	5	pass through a vertex	5
interval in a graph	11	path	5
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