SIMPLICIAL FIXED POINT ALGORITHMS

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ACKNOWLEDGEMENTS

CONTENTS

CHAPTER 1

INTRODUCTION

Many problems in economics, nonlinear programming and other fields can be solved by using fixed point algorithms, which should be regarded as one of the major breakthroughs in computational methods for nonlinear problems. Already in 1910 the Dutch mathematician L.E.J. Brouwer proved that any continuous function f from a compact convex set into itself has at least one fixed point, i.e. there exists at least one point x^* in the set such that $f(x^*) = x^*$. The generalization of this theorem to upper semi-continuous point to set mappings is due to Von Neumann [1937] and Kakutani [1941]. However, the originial proofs of these theorems were not concerned with computational methods. In spite of the simplification of the proofs during the subsequent years and the widespread use of the theorems as a mathematical tool in several fields it is only in the last fifteen years that fixed point theorems have been embedded in a computational setting.

The first method to compute a fixed point of a mapping from the unit simplex $s^n = \{x \in R_+^n | \Sigma_{i=1}^{n+1} x_i = 1\}$ into itself was presented in the pioneering work of Scarf [1967b], see also Scarf [1973]. He utilized an argument of Lemke and Howson [1964] and Lemke [1965] and the relationship of Sperner's lemma [1928] to the Brouwer fixed point theorem. Lemke and Howson provided a finite method for the computation of a

Nash equilibrium point in a two-person nonzero-sum game using ideas from the pivot theory introduced by Dantzig [1951,1963] for the solution of linear programming problems. To prove the convergence of the method, Lemke did not use a monotonicity property but a combinatorial argument. To solve problems with a nonlinear character, Scarf replaced the sequence of linear programming pivot steps by the alternative construction of primitive sets, introduced by Scarf in an earlier paper [1967a]. Scarf proved that the algorithm generates a path of adjacent primitive sets, and terminates within a finite number of iterations with a primitive set which

yields a good approximation of a fixed point. Unfortunately, the procedure
needs a high computer storage. A method to overcome this difficulty was
discovered by Hansen [1968]. He chose the points of the primitive sets in
a systematic way. However, then the pivot steps become identical to the
pivot steps using a regular triangulation discovered by Freudenthal [1942]
and made operational by Kuhn [1960]. Most of the recently developed fixed
point algorithms are based on the concept of a triangulation. The first
of them were already developed in 1968 and 1969 by Kuhn.

The algorithms mentioned above suffer from the disadvantage that the mesh
of the triangulation is fixed and that they must start outside the region
of interest. This means that when on a certain mesh an approximate fixed
point is found the information obtained is of no use to find an
approximation on a finer mesh. Fortunately, several algorithms
followed, which were able to handle this drawback. A first method is due
to Merrill [1971,1972] and was later rediscovered by Kuhn and MacKinnon
[1975]. Closely related methods can be found in Lüthi [1975] and Fisher,
Gould and Tolle [1977]. This algorithm, called the Sandwich method, can
start at an arbitrary point. This opens the opportunity to use the
approximation obtained by applying the algorithm on a certain mesh as the
starting point in a subsequent application with a finer mesh. A second
method was independently and simultaneously discovered by Eaves [1972]
for mappings on S^n and generalized by Eaves and Saigal [1972] for
mappings on R^n. In this method the mesh of the triangulation is
automatically refined during the course of the algorithm. In both methods
an extra dimension is used . They can be considered as methods which trace
a path from the solution of an artificial mapping to the approximation of
a solution of the original one. Algorithms without an extra dimension were
developed by Tuy [1979] and by Garcia [1975] and Garcia and Gould
[1976,1979]. However, as argued by Todd [1978b], the method of Tuy needs a
lot of computation time whereas the algorithm of Garcia and Gould may
fail to find an approximate fixed point.

Beside the development of more sophisticated algorithms many related
issues were studied. Computational experiences showed that the computation
time depends highly upon the underlying triangulation (see e.g. Saigal
[1977a] and Todd [1978a]). Measures for the efficiency of a triangulation
of R^n were developed by Saigal, Solow and Wolsey [1975] and Todd
[1976b,1978a]. However, for these measures it is not known how the optimal
triangulation looks like. Van der Laan and Talman [1980a] suggested a new

measure which enables the calculation of the optimal triangulation within
a reasonable class. They also proposed a triangulation of the affine hull
of the unit simplex whose measure is the same as that of the optimal
triangulation of R^n. An other important subject is the convergence of
fixed point algorithms on unbounded regions. Conditions that guarantee
convergence were studied by Merrill [1971,1972], Gould and Tolle [1975],
Saigal [1977c], Todd [1976a,1978b,1980], Saigal and Todd [1978], Reiser
[1978a,b], Van der Laan and Talman [1980c] and Talman [1980].
To apply fixed point algorithms the grid points of a triangulation are
labelled with an integer or vector label. The relation between convergence
and labelling rules was studied by Fisher, Gould and Tolle [1977],
MacKinnon [1977], Garcia and Gould [1976,1979] and Reiser [1978a].
Recently, Kojima [1978a] and Todd [1978c,d] developed pivot methods which
take advantage of the separability respectively the linearity of a function.
Under certain conditions the convergence of a simplicial algorithm can be
improved near a solution of a smooth mapping. Acceleration techniques which
result in superlinear convergence were given by Saigal [1977c] and Saigal
and Todd [1978] for vector labelling (see also Wolsey [1974]). In these
studies the approximate Jacobian (see Todd [1978e] and also Saigal [1979b])
plays an important role. An accelerated version of an algorithm using
integer labelling was given by Reiser [1978a].
A development resulting in more insight into the simplicial fixed point
algorithms is the theory of orientations and index. This theory was
introduced by Shapley [1974] for bimatrix games and generalized by Lemke
and Grotzinger [1976], Eaves and Scarf [1976] and Todd [1976c].
The path of simplices generated by fixed point algorithms was studied by
Saigal [1976], Garcia and Gould [1978] and Saari and Saigal [1979].
Survey papers about fixed point algorithms and the related topics are those
of Gould and Tolle [1974], Eaves [1976], Saigal [1977b] and Allgower and
Georg [1980]. We refer the reader also to the monographs of Todd [1976a] and
Lüthi [1976] and to the conference proceedings edited by Karamardian [1977]
and Peitgen and Walther [1979].

In this monograph new algorithms to compute a fixed point are presented.
Also conditions to guarantee the convergence of the algorithms on
unbounded regions are stated. Most of the work is based on the papers of
Van der Laan and Talman [1979a,b,1980b,c].
The monograph is organized as follows. In chapter 2 we give the most

important fixed point theorems and some applications as the equilibrium price vector in economic models and the solution of an unconstrained optimization problem. Also the concepts of triangulation and piecewise linear approximation to a mapping with respect to a triangulation are given.

In chapter 3 triangulations of R^n and S^n are discussed together with measures for the efficiency of a triangulation.

Chapter 4 presents a survey of the algorithms of Scarf, Kuhn, Merrill and Eaves and Saigal. We show that the two algorithms of Kuhn are the two extreme cases of a class of algorithms. It will appear that the algorithms of Merrill and Eaves and Saigal supercede those of Scarf and Kuhn.

In chapter 5 a new algorithm to compute fixed points on S^n and its affine hull T^n is introduced. The method is a restart algorithm and generates a path of simplices of variable dimension. In contrast to the algorithms of Merrill and Eaves and Saigal no extra dimension is needed.

In chapter 6 the application of the algorithm on the unit cube and on R^n is discussed. Conditions to guarantee the convergence of the algorithm on R^n are given.

In chapter 7 we define an orientation of a simplex. In view of the algorithm introduced in the chapters 5 and 6 we generalize this concept. Some preliminary results about the number of completely labelled simplices are also given. Also a method which gives an odd number of completely labelled simplices is discussed.

Chapter 8 presents another new algorithm. It is a method with an automatic refinement of the grid size. It differs from the algorithm of Eaves and Saigal in that it allows for any factor of refinement of the grid size.

CHAPTER 2

PRELIMINARIES

2.1. INTRODUCTION .

Fixed point properties of functions or mappings are of fundamental importance in many problems e.g. the existence of a solution to a system of nonlinear equations or the existence of an equilibrium in an economy can be proved by using a fixed-point theorem. The most famous fixed point theorem is due to Brouwer [1912]. We give a proof of this theorem using a combinatorial lemma of Sperner [1928]. We follow this approach since this lemma is closely related to arguments which play an important role in algorithms for computing fixed points. In particular the concept of a triangulation of a set is of importance.

In section 2 Brouwer's theorem is stated. Moreover, a lemma of Knaster, Kuratowski and Mazurkiewicz [1929] is given and it is proved that this lemma implies Brouwer's fixed point theorem. In section 3 we introduce the concept of a triangulation and some other definitions. In section 4 Sperner's lemma is given and Brouwer's theorem is proved whereas in section 5 the generalization of this theorem by Kakutani [1941] is discussed. Also the crucial concept of a piecewise linear approximation of a mapping is introduced here. In section 6 some applications of fixed point theorems are given.

2.2. BROUWER'S THEOREM .

Let C be a nonempty set in the n-dimensional Euclidian space R^n, endowed with the standard norm $\|x\| = (\Sigma_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. It is an important problem whether for any continuous function f from C into itself there exists at least one point x^* in C such that $f(x^*) = x^*$, i.e. whether such a function has at least one fixed point. Under the intuitively plausible conditions of convexity and compactness of C this statement was first proved by Brouwer.

THEOREM 2.2.1. (BROUWER). Let C be a compact and convex subset of R^n. Then any continuous function f: C → C has at least one fixed point.

It is easy to see that the theorem is true for n = 1. If n > 1 the proof of the theorem is much more complicated. Without loss of generality we can restrict ourselves to continuous functions on the unit simplex.

DEFINITION 2.2.2. The n-dimensional unit simplex S^n is defined by

$$S^n = \{x \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \geq 0, \ i = 1,\ldots,n+1\};$$

for i = 1,...,n+1 we define the i-th boundary S_i^n of S^n by

$$S_i^n = \{x \in S^n \mid x_i = 0\}.$$

Observe that the vertices of S^n are the (n+1)-dimensional unit vectors to be denoted by e(i), i=1,...,n+1. In particular, S^2 is the equilateral triangle with the vertices e(1), e(2) and e(3).

Following Todd [1976a] we prove Brouwer's theorem by using Sperner's lemma and the lemma of Knaster, Kuratowski and Mazurkiewicz [1929] (the K-K-M lemma).

LEMMA 2.2.3. (K-K-M). Let $\{C_1,\ldots,C_{n+1}\}$ be a collection of closed subsets of S^n such that

a) $S^n = \bigcup_{i=1}^{n+1} C_i$

b) $\bigcap_{i \in I} S_i^n \subseteq \bigcup_{j \notin I} C_j$ for any $I \subset \{1,\ldots,n+1\}$.

Then $\bigcap_{i=1}^{n+1} C_i$ is nonempty.

The K-K-M lemma is illustrated for n = 2 in figure 2.2.1.
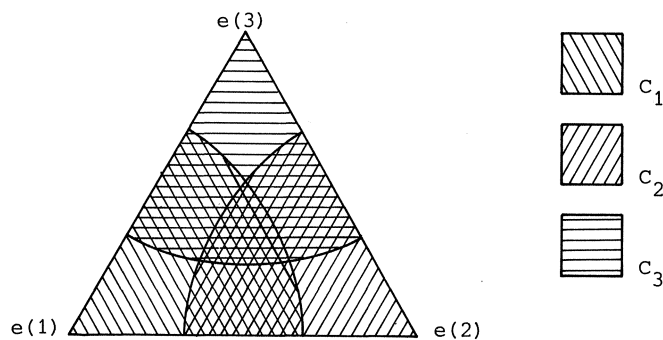
Figure 2.2.1. The K-K-M lemma.

We prove the lemma in section 4. Here we show that the theorem of Brouwer is implied by the lemma of K-K-M.

LEMMA 2.2.4. The K-K-M lemma implies Brouwer's theorem.

PROOF. Define C_i to be the set $\{x \in S^n \mid f_i(x) \leq x_i\}$ for all $i=1,\ldots,n+1$. Then it can easily be verified that the sets $C_1,\ldots,C_{n+1}$ satisfy the two conditions of the K-K-M lemma. Therefore $\bigcap_{i=1}^{n+1} C_i \neq \emptyset$. Choose $x^* \in \bigcap_{i=1}^{n+1} C_i$. From the definition of C_i we obtain that

$$f_i(x^*) \leq x_i^* \qquad i=1,\ldots,n+1.$$

Since $\sum_{i=1}^{n+1} f_i(x^*) = \sum_{i=1}^{n+1} x_i^* = 1$ it follows that $f_i(x^*) = x_i^*$, $i=1,\ldots,n+1$. Hence x^* is a fixed point of f. $\square$

To prove the K-K-M lemma, we need the fundamental concept of a triangulation. This concept will be discussed in the next section.

2.3. TECHNIQUE OF TRIANGULATION.

Before defining a triangulation of a set we introduce some basic concepts. Let $w^i \in R^n$ for $i=1,\ldots,t+1$. The $t+1$ points $w^1,\ldots,w^{t+1}$ are called affinely independent if $\Sigma_{i=1}^{t+1} \lambda_i w^i = 0$ and $\Sigma_{i=1}^{t+1} \lambda_i = 0$ imply $\lambda_i = 0$ for $i=1,\ldots,t+1$. The convex hull of $t+1$ points $w^1,\ldots,w^{t+1}$ is the set

$$\{x \in R^n | \; x = \sum_{i=1}^{t+1} \alpha_i w^i \text{ with } \alpha \in S^t\}.$$

The convex hull of a set D, to be denoted by conv(D), is defined as the intersection of all convex sets containing D. The point $x = \Sigma_{i=1}^{t+1} \alpha_i w^i$ is called an affine combination of the points $w^1,\ldots,w^{t+1}$ if $\Sigma_{i=1}^{t+1} \alpha_i = 1$. The affine hull of a set D, to be denoted by aff(D), is the set of all finite affine combinations of points of D.

Now we give the following definitions.

DEFINITION 2.3.1. A t-dimensional simplex or t-simplex, to be denoted by σ, is the convex hull of $t+1$ affinely independent points $w^1,\ldots,w^{t+1}$ of R^n. We write $\sigma = \sigma(w^1,\ldots,w^{t+1})$. The points $w^1,\ldots,w^{t+1}$ are called the vertices of σ.

DEFINITION 2.3.2. Let σ be a t-simplex in R^n. Then a k-simplex τ $(k \leq t)$ is a face of σ if all the vertices of τ are vertices of σ. A $(t-1)$-face of σ is called a facet of σ. The facet τ of σ is said to be opposite to the vertex w^i if w^i is a vertex of σ but is not a vertex of τ, $i=1,\ldots,t+1$.

Note that a t-simplex has $t+1$ facets and that the vertex opposite to a facet of σ is uniquely determined.

DEFINITION 2.3.3. Two different simplices σ_1 and σ_2 are adjacent either if one of them is a facet of the other or if σ_1 and σ_2 share a common facet.

Let C be an m-dimensional convex subset of R^n.

<u>DEFINITION 2.3.4</u>. A collection G of m-simplices is a triangulation of C
if

      a)   C is the union of all simplices in G,

      b)   the intersection of two simplices in G is either empty
          or a common face.

This definition excludes the cases drawn in figure 2.3.1. In the first
case e.g. the intersection of σ_1 and σ_2 is neither empty nor a common
face and in the second case σ is not a simplex. Examples of a triangulation
are shown in figure 2.3.2.
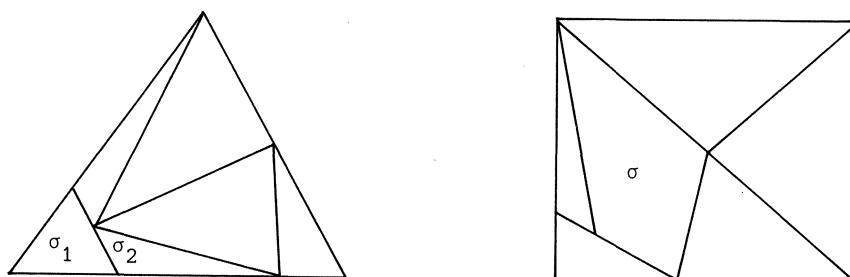
Figure 2.3.1. Examples which are not a triangulation.
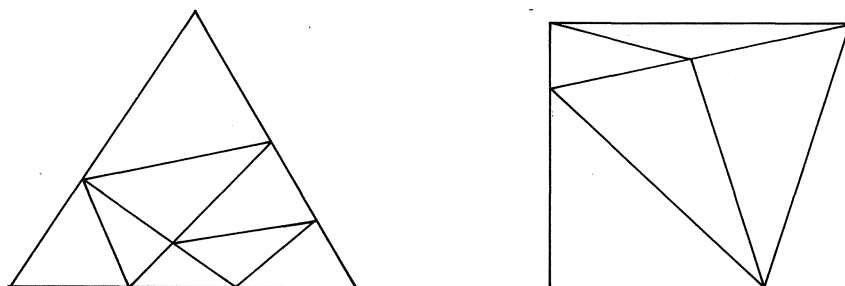
Figure 2.3.2. Examples of a triangulation.

Some important properties of a triangulation are given in the following
two theorems. Their proofs can be found in Spanier [1966]. For any set
$C \subset R^n$, denote by bd C and int C the boundary of C respectively the
interior of C, relatively to the affine hull of C, cf. Berge [1966].

THEOREM 2.3.5. If τ is a facet of a simplex of a locally finite triangula-
tion G of the m-dimensional convex subset C of R^n then either τ belongs to
bd C and is a facet of just one simplex of G, or τ does not belong to bd C
and is a facet of exactly two simplices of G.

THEOREM 2.3.6 Let D be an (m-1)-dimensional subset in bd C such that
C $\cap$ aff(D) is contained in D. Then D is triangulated by the set of (m-1)-
simplices which are contained in D and which are facets of simplices of
the triangulation G of C.

A triangulation G of C is locally finite is each $x \in C$ has a neighborhood
meeting only a finite number of simplices of G. Although in chapter 6 a
nonlocally finite triangulation will be described, in the remaining of
this monograph we only deal with triangulations having the property of
theorem 2.3.5.
Both theorems are illustrated by figure 2.3.3, where C is the 2-dimen-
sional unit simplex. In this figure τ_1 does not belong to bd C and is a
facet of the two simplices σ_1 and σ_2, whereas τ_2 belongs to bd C and is a
facet of just one simplex. This simplex is σ_3. The triangulation of the
convex hull of e(2) and e(3) consists of the three one-dimensional
simplices τ_2, τ_3 and τ_4.
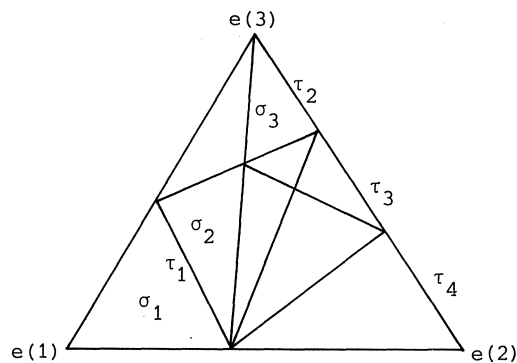
Figure 2.3.3. Illustration of the theorems 2.3.5 and 2.3.6.

We now define the mesh of a triangulation.

DEFINITION 2.3.7. Let C be an m-dimensional convex subset of R^n and let G be a triangulation of C. The diameter of a simplex σ of G, to be denoted by diam σ, is defined by

$$\text{diam } \sigma = \max_{x,y \in \sigma} \{||x-y||\}.$$

The mesh of a triangulation G, to be denoted by mesh G, is defined by

$$\text{mesh } G = \sup_{\sigma \in G} \{\text{diam } \sigma\}.$$

Observe that the diameter of a simplex σ with vertices $w^1,\ldots,w^{m+1}$ is equal to the maximum of the distance between two vertices, i.e.

$$\text{diam } \sigma(w^1,\ldots,w^{m+1}) = \max_{1 \leq i,j \leq m+1} \{||w^i-w^j||\}.$$

The following lemma is obvious.

LEMMA 2.3.8. Let G_1 be the set of all one-faces of the simplices of G. Then

$$\text{mesh } G = \sup_{\tau \in G_1} \{\text{diam } \tau\}.$$

After this introduction of the concept of a triangulation of a set C, we are able to prove the K-K-M lemma by using a lemma of Sperner.

2.4. SPERNER'S LEMMA .

Let G be a triangulation of the m-dimensional convex subset C of R^n and assume that each vertex of the triangulation is labelled with one of the integers 1,...,m+1, i.e. there is a labelling function ℓ from the set of vertices to the set {1,...,m+1}. In the following the set {1,...,n} of

integers will be denoted by I_n.

DEFINITION 2.4.1. An m-simplex of the triangulation G is completely labelled if all its vertices are differently labelled, i.e. $\sigma(w^1,\ldots,w^{m+1})$ is completely labelled if $\ell(w^i) \neq \ell(w^j)$ for all $i \neq j$, $j=1,\ldots,m+1$. Otherwise stated we have that if for any $j \in I_{m+1}$ there exists an index $i_j \in I_{m+1}$ such that $\ell(w^{i_j}) = j$, then $\sigma(w^1, \ldots, w^{m+1})$ is completely labelled.

The following lemma due to Sperner [1928] gives a condition for the existence of a completely labelled simplex of a triangulation of the unit simplex S^n.

LEMMA 2.4.2.(SPERNER). Let G be some triangulation of S^n. Assume that each vertex of its n-simplices is labelled with one of the integers of the set I_{n+1}, such that no vertex in S^n_i has label i for all i = 1,...,n+1. Then there exists at least one completely labelled n-simplex.

The lemma is obvious for n = 1 as is illustrated in figure 2.4.1. The one-simplices σ_1, σ_2 and σ_3 are completely labelled. For n = 2 the lemma is illustrated in figure 2.4.2.
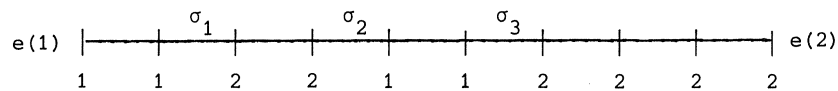
Figure 2.4.1. Illustration of Sperner's lemma for n = 1.

A proof of Sperner's lemma by induction on n can be found in Todd [1976a]. He proves a strong form of the lemma, which says that there is an odd number of completely labelled simplices. We do not give the proof, since the lemma follows as a corollary in chapter 4.
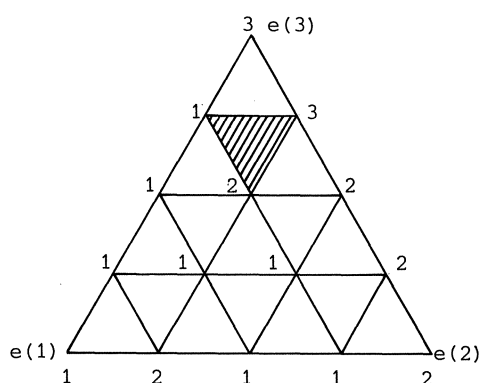
Figuur 2.4.2. Illustration of Sperner's lemma for n = 2.

The following lemma completes the proof of Brouwer's theorem.

LEMMA 2.4.3. Sperner's lemma implies the lemma of K-K-M.

PROOF. Let the collection $\{C_1,\ldots,C_{n+1}\}$ of closed subsets of S^n be defined as in lemma 2.2.4. Furthermore, let $\{G_k, k=1,2,..\}$ be some sequence of triangulations of S^n such that ε_k = mesh $G_k \to 0$ if $k \to \infty$. Each point y in S^n receives an integer label $\ell(y)$ defined by

$$\ell(y) = i, \text{ where } i = \min_{j} \{j \mid y \in C_j \text{ and } y_j > 0\}.$$

Note that, by the definition of C_j, there exists always such an index i. Since this labelling satisfies the condition of Sperner's lemma, there is at least one completely labelled simplex σ_k in G_k. Let $y^i(k)$ be the vertex of σ_k with label i, i = 1,...,n+1. Then for some subsequence $\{k_j, j=1,2,\ldots\}$ of integers with $k_j \to \infty$ if $j \to \infty$ we have that the sequence $\{y^1(k_j), j=1,2,\ldots\}$ converges to a point x^* in S^n. Since $\varepsilon_k \to 0$ if $k \to \infty$ it follows that $\lim_{j \to \infty} y^i(k_j) = x^*$ for all i=1,...,n+1. Hence $x^* \in C_i$, i=1,...,n+1 which proves that $\bigcap_{i=1}^{n+1} C_i \neq \emptyset$. $\square$

In the context of Brouwer's theorem this lemma means that, roughly speaking, any sequence of "shrinking" completely labelled simplices has a subsequence converging to a fixed point.

2.5. KAKUTANI'S THEOREM.

In many practical problems we need a fixed point theorem more general than the theorem of Brouwer. The theorem of Kakutani generalizes Brouwer's theorem to point-to-set mappings (cf. Berge [1966]). For any $\alpha > 0$ let $B^n(x,\alpha)$ be defined as the set $\{y \in R^n| \ ||y-x|| \le \alpha\}$.

DEFINITION 2.5.1. Let C be a subset of R^n and let ϕ be a point-to-set mapping from C to the set of all subsets of R^m. Then ϕ is upper-semi-continuous if

      a)   for all $x \in C$, $\phi(x)$ is compact,

      b)   for all $x \in C$ and for all $\varepsilon > 0$ there is a $\delta > 0$ such that for any $z \in B^n(x,\delta) \cap C$ holds $\phi(z) \in B^m(w,\varepsilon)$ for some $w \in \phi(x)$.

An interesting property of upper-semicontinuous (u.s.c.) mappings is stated in the following corollary (cf. Todd [1976a]).

COROLLARY 2.5.2. Let ϕ be an u.s.c. mapping from C to the set of subsets of R^m. If $\{x^k\}$ is a sequence of points in C with limit x^* and if $\{y^k\}$ is a sequence of points in R^m with $y^k \in \phi(x^k)$ for all k and with limitpoint y^*, then $y^* \in \phi(x^*)$.

The following theorem gives a sufficient condition which guarantees that an u.s.c. mapping has a fixed point, i.e. a point x^* such that $x^* \in \phi(x^*)$.

THEOREM 2.5.3. (KAKUTANI). Let C be a compact, convex, nonempty m-dimensional subset of R^n and let ϕ be an u.s.c. mapping from C to the set of nonempty convex subsets of C. Then there is a point $x^* \in C$ such that $x^* \in \phi(x^*)$.

To prove this theorem we use the concept of a piecewise linear approximation to a mapping.

DEFINITION 2.5.4. Let G be a triangulation of a nonempty, convex m-dimensional subset C of R^n and ϕ a nonempty mapping from C to the set of subsets of R^p. For any vertex w of G let f(w) be some arbitrarily chosen value of $\phi(w)$. Furthermore, let for a point $x \in C$, $w^1,\ldots,w^{t+1}$ be the vertices of the t-simplex such that x is contained in the interior of the simplex. Then there is a unique positive vector $\lambda = (\lambda_1,\lambda_2,\ldots\lambda_{t+1})$ such that $\Sigma_{i=1}^{t+1} \lambda_i = 1$ and $x = \Sigma_{i=1}^{t+1} \lambda_i w^i$. Then a piecewise linear approximation to the mapping ϕ with respect to G is the function f: $C \to R^p$ defined by

$$f(x) = \sum_{i=1}^{t+1} \lambda_i f(w^i) .$$

COROLLARY 2.5.5. The piecewise linear approximation f is continuous.

PROOF OF THEOREM 2.5.3. We follow the proof of Eaves [1971]. Let C' be an m-simplex containing C and let c be an interior point of C. Define the mapping $\overline{\phi}$ from C' to the set of nonempty, convex, compact subsets of C' by

$$\overline{\phi}(x) = \phi(x) \qquad \text{if } x \in \text{int C}$$

$$= \text{conv } (\phi(x) \cup \{c\}) \qquad \text{if } x \in \text{bd C}$$

$$= \{c\} \qquad \text{if } x \in C'\backslash C.$$

It is easy to see that $\overline{\phi}$ is upper-semicontinuous on C'. Moreover, it is clear that $\overline{\phi}(x)$ is nonempty and convex for all $x \in C'$. Suppose that x^* is a fixed point of $\overline{\phi}$. It is obvious that $x^* \in C$. If $x^* \in \text{int C}$, then $x^* \in \phi(x^*)$ and x^* is a fixed point of ϕ. If $x^* \in \text{bd C}$, then for some $y \in \phi(x^*)$, $x^* = \lambda c+(1-\lambda)y$ for some λ, $0 \leq \lambda \leq 1$. Since $c \in \text{int C}$, we have that $\lambda > 0$ implies that $x^* \in \text{int C}$ which contradicts $x^* \in \text{bd C}$. Hence,

$\lambda = 0$ and $x^* = y \in \phi(x^*)$. Therefore x^* is a fixed point of ϕ if x^* is a fixed point of $\bar{\phi}$. To prove that $\bar{\phi}$ has indeed a fixed point on C', let $\{G_k, k = 1,2,\ldots\}$ be a sequence of triangulations of C' with $\epsilon_k = \text{mesh } G_k \to 0$ if $k \to \infty$, and let f^k be a piecewise linear approximation to $\bar{\phi}$ with respect to the triangulation G_k. Bij Brouwer's theorem f^k has a fixed point, say x^k. Moreover, by definition 2.5.4, there is a simplex of G_k with vertices $w^1(k), \ldots, w^{m+1}(k)$ containing x^k such that for unique nonnegative numbers $\lambda_1^k, \ldots, \lambda_{m+1}^k$ with $\lambda^k = (\lambda_1^k, \ldots, \lambda_{m+1}^k)^T \in S^m$ holds

$$x^k = f^k(x^k) = \sum_{i=1}^{m+1} \lambda_i^k f^k(w^i(k)).$$

Since C' is compact, there is a subsequence k_j, $j=1,2,\ldots$ such that $x^{k_j} \to x^*$, $\lambda^{k_j} \to \lambda^*$ and $f^{k_j}(w^i(k_j)) \to \tilde{f}^i$, $i = 1,\ldots,m+1$, if $j \to \infty$. Since $\epsilon_k \to 0$, $w^i(k_j)$ converges to x^* for all i. Because of the upper-semi-continuity of $\bar{\phi}$, $\tilde{f}^i \in \bar{\phi}(x^*)$, for all i. Taking limits in the equation above, we obtain that $x^* = \sum_{i=1}^{m+1} \lambda_i^* \tilde{f}^i$ with $\lambda^* \in S^m$, while $\tilde{f}^i \in \bar{\phi}(x^*)$ for all i. Therefore, since $\bar{\phi}(x^*)$ is convex, $x^* \in \bar{\phi}(x^*)$, which completes the proof. □

Note that the property of corollary 2.5.2 is sufficient to prove the theorem. Algorithms to compute "Kakutani fixed points" will be based on the technique of piecewise linear approximation to the mapping with respect to a sequence of triangulations with mesh going to zero.

2.6. APPLICATIONS.

In this section we give some examples of Brouwer's theorem and the theorem of Kakutani. With respect to Brouwer's theorem we discuss the problem of the existence of an equilibrium in an exchange economy. As an application of Kakutani's theorem we give the unconstrained optimization problem.

EXAMPLE 2.6.1. (Exchange economy).

Let us consider an economy without production in which n+1 commodities are exchanged. We assume that there are m consumers each having a utility function $u^i(x)$, i=1,...,m, where $x = (x_1,\ldots,x_{n+1})$ denotes a vector of commodities. This function is defined from R_+^{n+1} to R and reveals the utility of the i-th consumer. Furthermore, we assume that each consumer has a

vector of initial endowments $e^i = (e^i_1, \ldots, e^i_{n+1})^T$. Given a price vector $p = (p_1, \ldots, p_{n+1})^T \in R^{n+1}_+ \setminus \{0\}$ each consumer i maximizes his utility under the budget constraint $p^T x \leq p^T e^i$. Let $x^i(p)$ be the demand vector for consumer i which maximizes his utility given the price vector p. Clearly, $x^i(p)$ is homogeneous of degree zero in p. Therefore we can normalize the price vector p such that $p \in S^n$. So $x^i(p)$ is a function from S^n to R^{n+1}_+ assuming that $x^i(p)$ exists and that $x^i(p)$ is unique for all p. Furthermore, we assume that $x^i(p)$ is continuous in p, for all i. The economic consequences of these assumptions are discussed by many authors, e.g. Arrow and Hahn [1971]. Finally, if $u^i(x)$ is a monotonic increasing function of x, each consumer spends all his income, i.e. $p^T x^i(p) = p^T e^i$, $i=1,\ldots,m$. Then we have

$$p^T \sum_{i=1}^m (x^i(p) - e^i) = 0$$

which formula is known as Walras' law. Since $x^i(p)$ is continuous for all i, the excess demand function g from S^n to R^{n+1} defined by $g(p) = \sum_{i=1}^m (x^i(p) - e^i)$ is also continuous in p. The economy is said to be in equilibrium if there exists a price vector p^* in S^n such that $g(p^*) \leq 0$. By Walras' law it is clear that in equilibrium $p_j^* = 0$ if $g_j(p^*) < 0$, $j=1,\ldots,m+1$.

Using Brouwer's fixed point theorem we prove that an equilibrium exists under the assumptions stated above. We define the function f from S^n into itself by

$$f_j(p) = [p_j + \max\{0, g_j(p)\}]/c(p) \qquad j = 1, \ldots, n+1$$

where $c(p) = 1 + \sum_{j=1}^{n+1} \max\{0, g_j(p)\}$. Clearly, f satisfies the conditions of Brouwer's theorem. Therefore, there exists a price p^* such that $p^* = f(p^*)$. We show that p is a fixed point of the function f if and only if p is an equilibrium price vector. If p^* is a fixed point of f then

$$p_j^* = [p_j^* + \max\{0, g_j(p^*)\}]/c(p^*) \qquad j = 1, \ldots, n+1.$$

Assume $g_j(p^*) > 0$ for some j. Then $c(p^*) > 1$ and hence $g_k(p^*) > 0$ if $p_k^* > 0$. This implies that $\sum_{j=1}^{n+1} p_j^* g_j(p^*) > 0$ which contradicts Walras' law. Consequently, $g_j(p^*) \leq 0$ $j = 1, \ldots, n+1$, which proves that p^* is an equilibrium price vector. Conversely, if p^* is such a vector then

$\max\{0, g_j(p^*)\} = 0$, $j = 1, \ldots, n+1$ and hence $f(p^*) = p^*$.   □

The function f defined above is not particularly suitable for use in algorithms for computing fixed points, since by considering the maximum over zero and the excess demand in stead of the excess demand, information is lost about the structure of the function. In a later chapter the computation of an equilibrium price vector will be based on the excess demand.

For many economic models the existence of an equilibrium was proved, e.g. Debreu [1959], Drèze [1975], Greenberg [1977], Van der Laan [1980]. In general Brouwer's theorem is not sufficient. For instance in an economy with linear production activities the excess demand is an upper-semicontinuous point-to-set mapping and the existence proof and computation of an equilibrium will be based on Kakutani's theorem (see e.g. Scarf [1973] and Todd [1976a, 1977b]). About computational results for economic models obtained by using fixed point algorithms we refer the reader also to Shoven and Whalley [1973], Whalley [1975], MacKinnon [1976], Richter [1978] and Reif [1978].

Before we discuss the example of unconstrained optimization we give a useful extension of Kakutani's theorem. The next lemma is due to Merrill [1971,1972] (see also Todd [1976a] and Allgower and Georg [1980]).

LEMMA 2.6.2. (Merrill's condition).

Let ϕ be an upper-semicontinuous mapping from R^n to the collection of nonempty, convex subsets of R^n. Assume that there exist $w \in R^n$, $\mu > 0$ and $\delta > 0$ such that for all $x \notin B^n(w, \mu)$, $f(x) \in \phi(x)$ and $z \in B^n(x, \delta)$ holds

$$(f(x) - x)^T(w-z) > 0.$$

Then the mapping ϕ has a fixed point in $B^n(w, \mu)$.

PROOF. Let C be the compact convex set $B^n(w, 2\mu)$. Note that $w \in \text{int } C$. Define the mapping $\phi'(x)$ by

$$\phi'(x) = \phi(x) \qquad \text{if } x \in \text{int } C$$
$$= \text{conv}(\phi(x) \cup \{w\}) \qquad \text{if } x \in \text{bd } C.$$

Then $\phi'(x)$ is an u.s.c. mapping from C to the collection of nonempty convex subsets of R^n such that $w \in \phi'(x)$ for all $x \in$ bd C. Following Eaves [1971] we show that ϕ' has a fixed point. Clearly, since ϕ' is u.s.c. $\phi'(C)$ is compact and hence the set C' defined by

$$C' = \text{conv}(C \cup \phi'(C))$$

is compact. Define the u.s.c. mapping $\bar{\phi}'$ from C' into itself by

$$\bar{\phi}'(x) = \phi'(x) \qquad \text{if } x \in C$$
$$= \{w\} \qquad \text{if } x \in C'\backslash C.$$

Since $\bar{\phi}'$ satisfies the conditions of Kakutani's theorem there exists a fixed point $x^* \in C'$ of $\bar{\phi}'$ and analogous as in the proof of theorem 2.5.3, it follows that $x^* \in C$ and that x^* is a fixed point of $\phi'(x^*)$. If $x^* \in B^n(w,\mu)$, then $x^* \in$ int C and hence $x^* \in \phi(x^*)$.
Therefore we prove that $x^* \in B^n(w,\mu)$. Suppose to the contrary that $x^* \notin B^n(w,\mu)$. Since $x^* \in \phi'(x^*) \subset \text{conv}(\phi(x^*) \cup \{w\})$, for some λ, $\lambda \in \ulcorner 0,1 \urcorner$, and some $f(x^*) \in \phi(x^*)$ we have that

$$x^* = \lambda f(x^*) + (1-\lambda)w$$

so that

$$\lambda(f(x^*) - x^*) + (1-\lambda)(w-x^*) = 0.$$

However, $(w-x^*)^T(w-x^*) > 0$ while by the condition of the theorem $(f(x^*)-x^*)^T(w-x^*) > 0$. So a contradiction is obtained which proves that $x^* \in B^n(w,\mu)$. $\square$

Lemma 2.6.2. is used in the following example.

EXAMPLE 2.6.3. (Unconstrained optimization problem).
Let $f: R^n \to R$ be a convex and finite function. We want to compute a point x such that $f(x) \le f(z)$ for all $z \in R^n$. Let $df(x)$ be the set of all subgradients of f at x, i.e.

$$df(x) = \{y \in R^n | f(z) \ge f(x) + y^T(z-x) \text{ for all } z \in R^n\}.$$

Since f is convex, the mapping df from R^n to the set of nonempty, convex subsets of R^n is u.s.c. (Rockafellar [1970]). Define the u.s.c. mapping ϕ by $\phi(x) = \{x\} - df(x)$, $x \in R^n$.

Assume there exists an α such that the set $F(\alpha) = \{y \mid f(y) \leq \alpha\}$ is nonempty and bounded. Then $F(\beta)$ is bounded for all real β (Rockafeller [1970]). For arbitrarily chosen $w \in R^n$ and $\delta > 0$, let $\beta = \max\{f(x) \mid x \in B^n(w,\delta)\}$. Then $F(\beta)$ is bounded so we can choose an $\mu > 0$ such that $F(\beta) \subset B^n(w,\mu)$. Furthermore, for any $x \notin B^n(w,\mu)$, $g(x) \in \phi(x)$ and $z \in B^n(x,\delta)$ we have that $x - g(x) \in df(x)$ and hence

$$(g(x)-x)^T(w-z) = (g(x)-x)^T\{(w-z+x)-x\} \geq -f(w-z+x)+f(x).$$

By $x \notin B^n(w,\mu)$ we have $f(x) > \beta$ and by $w-z+x \in B^n(w,\delta)$ we have $-f(w-z+x) > -\beta$. Therefore $(g(x)-x)^T(w-z) > 0$, which implies that $\phi(x)$ satisfies Merrill's condition. So there exists a point x^* with $x^* \in \phi(x^*)$. Clearly $0 \in df(x^*)$ and hence $f(x^*) \leq f(z)$ for all $z \in R^n$. $\quad\square$

As an illustration of this example, we want to minimize $f(x) = x_1^2 + 2x_2^2 - x_1x_2 + x_1 + x_2$. Then $df(x) = (2x_1 - x_2 + 1, 4x_2 - x_1 + 1)^T$ and the function $\phi(x) = (-x_1 + x_2 - 1, x_1 - 3x_2 - 1)^T$ has $(-5/7, -3/7)^T$ as a fixed point. Clearly this point minimizes $f(x)$.

For other applications of fixed point theorems see e.g. Todd [1976a], Saigal [1977b,1979a], Allgower and Georg [1980] and Talman [1980].

CHAPTER 3

TRIANGULATIONS AND MEASURES OF EFFICIENCY

3.1. INTRODUCTION.

In this chapter we discuss special triangulations and measures for the
efficiency of triangulations. Computational experience has shown that the
efficiency of fixed point algorithms is very sensitive to the underlying
triangulation (see e.g. Saigal [1977a] and Todd [1978a]). Basic
triangulations of R^n used in fixed point algorithms are the so called K and
H triangulations which are defined in section 2. Measures for the efficiency
of a triangulation of R^n were given by Saigal, Solow and Wolsey [1975],
Todd [1976b, 1978a], Van der Laan and Talman [1980a] and Talman [1980].
Whereas earlier measures do not allow the calculation of the optimal
triangulation, for their measure Van der Laan and Talman succeeded in
calculating an optimal triangulation within a reasonable subclass of
triangulations. The measures and the optimal triangulation for the measure
of Van der Laan and Talman [1980a] are given in section 3. Section 4 gives
triangulations of the unit simplex and its affine hull including one which
is comparable with the optimal triangulation of R^n. Finally in section 5 the
replacement step between two full-dimensional adjacent simplices is given.

3.2. BASIC TRIANGULATIONS OF R^n.

Since the accuracy of an approximation of a fixed point will depend on
the mesh of the triangulation we need triangulations without long, skinny
simplices. In other words "regular" triangulations are preferred in fixed
point algorithms (see Saigal [1977a , 1979b]). The first regular
triangulation was constructed by Freudenthal [1942] and rediscovered by
Tucker in 1945 (see Lefschetz [1949, page 140]).This triangulation is
called the K triangulation and is based on the "standard" subdivision of
the unit cube (Kuhn [1960]).

DEFINITION 3.2.1. Let z be an arbitrarily chosen point in R^n. Then the K triangulation with grid size $\delta > 0$ is the collection of all simplices $\sigma(y^1, \pi)$ with vertices $y^1, \ldots, y^{n+1}$ such that

    a) each component of $y^1 - z$ is a multiple of δ

    b) $\pi = (\pi_1, \ldots, \pi_n)$ is a permutation of the elements of the set I_n

    c) $y^{i+1} = y^i + \delta e(\pi_i)$ $i = 1, \ldots, n$.

In general z will be chosen as the zero vector. Observe that for $\delta = 1$ the collection of simplices $\{\sigma(\underline{0}, \pi) \mid \pi$ is a permutation of the elements of $I_n\}$ triangulates the unit cube $C = \{x \in R^n \mid 0 \le x_i \le 1 \quad i = 1, \ldots, n\}$. From the definition it follows that the diameter of each simplex is equal to $\delta \left\| \sum_{i=1}^n e(i) \right\| = \delta \sqrt{n}$. So, the mesh of the K triangulation with grid size δ is equal to mesh $K_\delta = \delta \sqrt{n}$. We denote the K triangulation with grid size δ by K_δ and write $K_\delta = K$ when $\delta = 1$.

A triangulation closely related to the K triangulation was used by Merrill [1972] and Eaves and Saigal [1972]. It is called the H triangulation. For $z = 0$ this triangulation is built up by simplices $\sigma(y^1, \pi)$ such that the components of y^1 are again multiples of δ and $y^{i+1} = y^i + \delta \tilde{q}(\pi_i)$, $i = 1, \ldots, n$, where $\tilde{q}(j)$ is the j-th column of the $n \times n$ matrix $\tilde{Q}$ defined by

$$\tilde{Q} = \begin{bmatrix} -1 & 0 & . & . & . & 0 \\ 1 & -1 & & & & . \\ 0 & 1 & . & . & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 0 \\ 0 & . & . & . 0 & 1 & -1 \end{bmatrix} .$$

Of course, any arbitrarily chosen point z can be made a grid point by a shift of the grid, such that the components of $y^1 - z$ become integer multiples of δ. Clearly for the H triangulation with grid size δ, denoted by H_δ (H if $\delta = 1$), the mesh is equal to $\delta \max_{S \subset I_n} \left\| \sum_{j \in S} \tilde{q}(j) \right\| = \delta \sqrt{n}$.

More general, for any nonsingular matrix A, we can define the triangulation AK_δ with grid size δ as follows.

DEFINITION 3.2.2. Let A be a nonsingular n×n matrix. Then the triangulation AK_δ is the collection of all simplices $\sigma(Ay^1,\ldots,Ay^{n+1})$ such that $\sigma(y^1,\ldots,y^{n+1})$ is a simplex of K_δ.

Observe that for z=0 the triangulation AK is the collection of simplices $\sigma(y^1,\pi)$ such that all components of $A^{-1}y^1$ are integers and $y^{i+1} = y^i + a(\pi_i)$ i=1,...,n, where a(j) is the j-th column of the matrix A. If we take A equal to $\tilde{Q}$ we obtain the H triangulation.

To compute the mesh of an AK triangulation, observe that diam $\sigma(y^1,\pi)$ is independent of y^1. So, by setting $y^1 = \underline{0}$ the next corollary follows immediately.

COROLLARY 3.2.3. mesh AK = $\max\limits_{S \subset I_n} \left\| \sum\limits_{j \in S} a(j) \right\|$.

Clearly, mesh $AK_\delta = \delta$ mesh AK. So, the mesh can be made arbitrarily small by taking the grid size δ small enough.
Another triangulation well known from the literature but not of the form AK is the so-called "Union Jack" triangulation. It is a centrally symmetric triangulation due to Tucker (see Lefschetz [1949, page 140], cf. also Whitney [1957, page 358], Todd [1977a] and Kojima [1978b]).

3.3. MEASURES FOR THE EFFICIENCY OF TRIANGULATIONS.

A first measure used for the efficiency of a triangulation of the unit cube is the number of simplices in which the unit cube is triangulated. The K and H triangulations both subdivide the n-dimensional unit cube in n! simplices. However, Saigal [1977a] experienced that in particular for large n the K triangulation performs much better than the H triangulation. This fact is not unexpected since the H triangulation yields many shapes of simplices, whereas the K triangulation results in one single shape, i.e. all simplices of this triangulation are congruent. So, this measure is too rough and must be rejected. We mention however, that Mara [1972] succeeded in finding a triangulation which divides the unit cube in 5, 16 and 68 simplices for respectively n = 3, 4 and 5. A second measure was given by Saigal, Solow and Wolsey [1975]. They defined the measure of a triangulation of the unit cube as its diameter

which is the maximum of the minimal length path between any pair of
boundary facets of the triangulation. The length of a path between two
facets is the number of simplices met by the path. Saigal [1977a] stated
that the diameter of the H triangulation was of the order or at least n^3,
whereas the diameter of the K triangulation is equal to $\frac{1}{2}n(n-1)+1$.

A more sophisticated measure which can be calculated for triangulations of
R^n was introduced by Todd [1976b, 1978a]. This measure is called the
average directional density and is roughly the average number of simplices
met by straight lines per unit length. Formally, for some triangulation
G of R^n denote, for $x,d \in R^n$ and $t > 0$, the ratio between the number of
simplices of G intersecting the line segment $[x,x+td]$ and t by
$N(G,x,d,t)$. Denoting by $N(G,d,t)$ the limit for $r \to \infty$ of the average of
$N(G,x,d,t)$ for x uniformly distributed in $\{y\in R^n | \|y\| < r\}$, the directional
density of G with direction d, to be denoted by $N(G,d)$, is the limit of
$N(G,d,t)$ for $t \to \infty$ provided that both limits exist. Finally, the average
directional density $N(G)$ of G is the average of $N(G,d)$ for d uniformly
distributed on the unit ball.

In the following, A is a nonsingular n×n matrix and $b(j)$ is the j-th
row of A^{-1}. Todd proved the following theorem. ·

THEOREM 3.3.1. The average directional density of the triangulation AK_δ is

$$N(AK_\delta) = \delta^{-1}\{ \sum_{j=1}^{n} \|b(j)\| + \sum_{j=1}^{n} \sum_{i=1}^{j-1} \|b(i)-b(j)\|\} g_n,$$

where $g_n = 2\Gamma(\frac{1}{2}n)/\{(n-1)\sqrt{\pi}\Gamma(\frac{1}{2}n-\frac{1}{2})\}$.

From this theorem the next corollary follows immediately.

COROLLARY 3.3.2.

$$N(K_\delta) = \delta^{-1}\{n + n(n-1)/\sqrt{2}\}g_n,$$

$$N(H_\delta) = \delta^{-1}\{ \sum_{j=1}^{n} (n-j+1)\sqrt{j}\}g_n.$$

An optimal triangulation according to the average directional density is
unknown. However, Todd [1978a] gave a lower bound for the average
directional density of a triangulation of the form AK. Since the average
directional density depends on the mesh, we will   restrict ourselves

to triangulations with mesh equal to or less than $\sqrt{n}$.

LEMMA 3.3.3. For any triangulation AK with mesh AK $\leq \sqrt{n}$ holds

$$N(AK) > (4n^2/25\sqrt{5})g_n.$$

This result was strengthened by Van der Laan and Talman [1980a] for triangulations within a subclass A of AK triangulations.

DEFINITION 3.3.4. An AK triangulation of R^n belongs to the class A if for some $\alpha > n-1$, $a_{ii} = \alpha$, $i=1,\ldots,n$ and $a_{ij} = -1$, $i \neq j$.

In chapter 6 it will be motivated that elements of this subclass are very suitable for use in fixed point algorithms, see also Van der Laan and Talman [1980a] and Talman [1980]. An element of this class with grid size δ will be denoted by $A(\alpha)K_\delta$.

LEMMA 3.3.5. The mesh of an $A(\alpha)K$ triangulation is equal to

$$\text{mesh } A(\alpha)K = [k\{\alpha-(k-1)\}^2 + (n-k)k^2]^{\frac{1}{2}}$$

where

$$
\begin{aligned}
k &= n && \text{if } b \geq n \\
&= b && \text{if } b \leq n-1 \text{ and } \|y^{b+1}\| \geq \|y^{b+2}\| \\
&= b+1 && \text{if } b \leq n-1 \text{ and } \|y^{b+1}\| < \|y^{b+2}\|
\end{aligned}
$$

with b the entier of $\{(\alpha+1)^2/(4+4\alpha-2n)\}$ and where $y^1,\ldots,y^{n+1}$ are the vertices of the simplex $\sigma(\underline{0},\pi)$ with $\pi_i = i$ $i=1,\ldots,n$.

The proof can be found in Talman [1980]. Note that the $A(\alpha)K_\delta$ triangulation with grid size $\delta = \alpha^{-1}$ converges to the K triangulation for $\alpha \to \infty$. The proof of the following lemma can be found in Van der Laan and Talman [1980a] and Talman [1980].

LEMMA 3.3.6. For any $A(\alpha)K_\delta$ triangulation with mesh $A(\alpha)K_\delta \leq \sqrt{n}$ holds

$$N(A(\alpha)K_\delta) > \{\tfrac{1}{2}(n-1)+(n-1)^2/\sqrt{8}\}g_n.$$

It is an open problem whether this lemma holds for all triangulations AK_δ with mesh $AK_\delta \leq \sqrt{n}$.

Van der Laan and Talman [1980a] introduced a new measure for the efficiency of triangulations. This measure makes it possible to calculate the optimal triangulation within the class A. The measure called the SV-measure is based on both the total squared length of the one-faces of the simplices and the volume of these simplices.

DEFINITION 3.3.7. The SV-measure of a triangulation G, to be denoted by $SV(G)$, is defined by

$$SV(G) = \sup_{\sigma \in G} \left[\left\{ \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} \|y^j - y^i\|^2 \right\}^{1/2} / \{\Delta\sigma\}^{1/n} \right]$$

where $\Delta\sigma$ denotes the volume of the simplex $\sigma = \sigma(y^1, \ldots, y^{n+1})$.

The numerator reflects the total squared length of all one-faces of the simplex σ. By raising this term to the power a half and the term in the denominator to the power $1/n$, we obtain a measure which is homogeneous of degree zero in the mesh. If the triangulation is of the form AK, each simplex σ has the same volume $|\det A|/n!$.

Roughly speaking the SV-measure of a triangulation is a measure for the number of grid points per unit cube with respect to the total squared length of all one-faces of the "worst" simplex σ. Clearly, for $n=2$, the SV-measure is minimized for a triangulation generating equilateral triangles. In Van der Laan and Talman [1980a] and Talman [1980] the following lemma's are proved.

LEMMA 3.3.8. The optimal triangulation within A according to the SV-measure is the $A(\alpha^*)K$ triangulation with $\alpha^* = n + \sqrt{n+1}$.

LEMMA 3.3.9. The SV-measures of the K, H and $A(\alpha^*)K$ triangulations are

a) $SV(K) = r_n \sqrt{2}$

b) $SV(H) = r_n \{3n/(n+1)\}^{1/2}$ if n is even

         $= r_n \{3(n^3 + 2n^2 - n + 2)/n(n+1)(n+2)\}^{1/2}$ if n is odd

c) $SV(A(\alpha^*)K) = r_n (n+1)^{1/2n}$,

where $r_n = \{n(n+1)(n+2)/12\}^{1/2}(n!)^{1/n}$.

LEMMA 3.3.10. The mesh of the $A(\alpha^*)K_\delta$ triangulation is equal to

$$\text{mesh } A(\alpha^*)K_\delta = \tfrac{1}{2}\delta(1+\sqrt{n+1})(n+1) \qquad \text{if } n \text{ is odd}$$
$$= \tfrac{1}{2}\delta(1+\sqrt{n+1}) \qquad\qquad \text{if } n \text{ is even.}$$

LEMMA 3.3.11. The average directional density of the $A(\alpha^*)K_\delta$ triangulation is equal to

$$N(A(\alpha^*)K_{\delta*}) = g_n\{n(n+1)/8\}^{\tfrac{1}{2}}(n+1) \qquad \text{if } n \text{ is odd}$$
$$= g_n\{n(n+1)/8\}^{\tfrac{1}{2}}\{n(n+2)\}^{\tfrac{1}{2}} \qquad \text{if } n \text{ is even.}$$

where δ^* is taken such that mesh $A(\alpha^*)K_{\sigma*} = \sqrt{n}$.

Note that $N(A(\alpha^*)K_{\delta*})$ converges to the lower bound $g_n n^2/\sqrt{8}$ of lemma 3.3.6 as $n \to \infty$. Furthermore, the $A(\alpha^*)K$ triangulation has the nice properties that the barycenter of any simplex has the same distance $\{n(n+2)\}^{\tfrac{1}{2}}(1+\sqrt{n+1})/2\sqrt{3}$ to each vertex of the simplex and that for any simplex $\sigma(y^1,\pi)$, $\|y^j - y^i\| = \|y^{j-k} - y^{i-k}\|$ for all $1 \le k < i < j \le n+1$. These properties suggest to consider this triangulation of R^n as the generalization of the equilateral triangulation of R^2 in the sense that the triangulation yields simplices which are as "round" as possible.

3.4. TRIANGULATIONS OF S^n AND T^n.

A familiar triangulation of S^n is given in the following definition.

DEFINITION 3.4.1. (Standard triangulation of S^n).
The standard triangulation of S^n with grid size δ is the collection of all simplices $\sigma(y^1,\pi)$ with vertices $y^1,\ldots,y^{n+1}$ such that

    a) δ^{-1} is a positive integer m, and each component of y^1 is a multiple of δ

    b) $\pi = (\pi_1,\ldots,\pi_n)$ is a permutation of the elements of I_n

    c) $y^{i+1} = y^i + \delta q(\pi_i)$ $i = 1,\ldots,n$,

where $q(j)$ is the j-th column of the $(n+1)\times n$ matrix Q

$$Q = \begin{bmatrix} -1 & 0 & . & . & . & 0 \\ 1 & -1 & . & & & . \\ 0 & . & & . & & . \\ . & & . & & & 0 \\ . & & & . & & -1 \\ . & & & . & & . \\ 0 & . & . & . & 0 & 1 \end{bmatrix}$$

As proved by Todd [1976a] the mesh of the standard triangulation Q with grid size δ is equal to mesh $Q_\delta = \delta\sqrt{n+1}$ if n is odd and mesh $Q_\delta = \delta\sqrt{n}$ if n is even. The following lemma has been proved in Van der Laan and Talman [1980a] and Talman [1980].

LEMMA 3.4.2. For the standard triangulation of S^n holds

$$SV(Q) = r_n\sqrt{3}\{(n^3+3n^2-n+5)/(n^3+3n^2+2n)\}^{\frac{1}{2}}(n+1)^{-1/2n} \text{ if n is odd}$$
$$= r_n\sqrt{3}(n+1)^{-1/2n} \qquad\qquad \text{if n is even.}$$

Other triangulations of S^n include a triangulation related to the Union Jack triangulation of R^n and the iterated barycentric subdivision (see Todd [1976a]).
We now discuss some triangulations of T^n, the affine hull of S^n.

DEFINITION 3.4.3. Let z be an arbitrarily chosen point of T^n. Then the standard triangulation Q of T^n with respect to z with grid size δ is the collection of all simplices $\sigma(y^1,\pi)$ with vertices $y^1,\ldots,y^{n+1}$ such that

a) $y^1 = z + \sum_{i=1}^n \alpha_i q(i)$ where α_i is a multiple of δ for all i

b) $\pi = (\pi_1,\ldots,\pi_n)$ is a permutation of the elements of I_n

c) $y^{i+1} = y^i + \delta q(\pi_i)$ \qquad $i = 1,\ldots,n.$

Of course the mesh of this triangulation is equal to the mesh of the standard triangulation of S^n. Also lemma 3.4.2 holds for this triangulation of T^n. Observe that $SV(Q)/r_n$ converges to $\sqrt{3}$ for $n \to \infty$ as is also the case for the H triangulation of R^n. Since $SV(H)$ is much higher than the SV-measure of the $A(\alpha^*)K$ triangulation of R^n, it seems to be worthwhile

to look for a triangulation of T^n whose SV-measure is comparable with $SV(A(\alpha^*)K)$.

A class of triangulations of T^n can be defined in the following way. Let A be an $(n+1)\times n$ matrix of rank n such that $\sum_{i=1}^{n+1} a_{ij} = 0$ for all $j=1,\ldots,n$. Again, let z be an element of T^n. The A triangulation of T^n with grid size δ is defined as the set of all simplices $\sigma(y^1,\pi)$ such that

$y^1 = z + \sum_{i=1}^{n} \alpha_i a(i)$ where α_i is a multiple of δ for all i,

$\pi = (\pi_1,\ldots,\pi_n)$ is a permutation of the elements of I_n and

$y^{i+1} = y^i + \delta a(\pi_i)$, $i = 1,\ldots,n$. Clearly mesh $A = \max_{S \subset I_n} \left\| \sum_{j \in S} a(j) \right\|$.

In Van der Laan and Talman [1980a] and Talman [1980] the following triangulation is proposed for use in fixed point algorithms. This triangulation is defined by the matrix

$$T = \begin{bmatrix} -n & 1 & . & . & . & 1 \\ 1 & -n & . & & & . \\ . & & . & . & & . \\ . & & & . & & 1 \\ . & & & & . & -n \\ 1 & . & . & . & & 1 \end{bmatrix}, $$

The simplices of this triangulation have the same nice properties as the simplices of the $A(\alpha^*)K$ triangulation of R^n. In particular these two triangulations have both $r_n(n+1)^{1/2n}$ as SV-measure. We return to this triangulation of T^n in chapter 5.

To use a triangulation of T^n in algorithms to compute a fixed point on S^n it may be necessary to extend the function or mapping from S^n into itself to a mapping from T^n to S^n.

3.5. REPLACEMENT STEP.

In this section we describe the replacement step between two full-dimensional adjacent simplices. Let σ and $\bar{\sigma}$ be two of such simplices. Then $\bar{\sigma}$ is obtained from σ by replacing a vertex of σ by a new vertex. The rule to obtain this new vertex is quite simple when a regular triangulation is used.

Let P be a triangulation matrix e.g. P is an $n \times n$ matrix to triangulate R^n or an $(n+1) \times n$ matrix to triangulate T^n. Let $\sigma(y^1, \pi)$ be a simplex of the triangulation, so y^1 is a grid point of the triangulation and $y^{i+1} = y^i + \delta p(\pi_i)$, $i = 1, \ldots, n$, where δ is the grid size and $p(j)$ the j-th column of P. Then the rule to obtain the simplex $\bar{\sigma}(\bar{y}^1, \bar{\pi})$ by replacing the vertex y^i of $\sigma(y^1, \pi)$ is given in table 3.5.1.

TABLE 3.5.1. i is the index of the vertex to be replaced.

	$\bar{y}^1$ becomes	$\bar{\pi}$ becomes
i = 1	$y^1 + \delta p(\pi_1)$	$(\pi_2, \ldots, \pi_n, \pi_1)$
$2 \le i \le n$	y^1	$(\pi_1, \ldots, \pi_{i-2}, \pi_i, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n)$
i = n + 1	$y^1 - \delta p(\pi_n)$	$(\pi_n, \pi_1, \ldots, \pi_{n-1})$

It is easy to see that the facet $\tau(y^1, \ldots, y^{i-1}, y^{i+1}, \ldots, y^{n+1})$ is indeed a common facet of $\sigma(y^1, \pi)$ and $\bar{\sigma}(\bar{y}^1, \bar{\pi})$. For instance if $i = 1$ $\bar{y}^i = y^{i+1}$, $i = 1, \ldots, n$, whereas $\bar{y}^{n+1} = y^{n+1} + \delta p(\pi_1)$.

Although some modifications can be necessary, table 3.5.1 will be used in the algorithms to be treated in this monograph. Triangulations in which the replacement step can be described by table 3.5.1 are called triangulations by reflecting since the new vertex is obtained by reflecting the old vertex with respect to the common facet. A full description of triangulations by reflecting can be found in Allgower and Georg [1980].

CHAPTER 4

ALGORITHMS FOR APPROXIMATING FIXED POINTS

4.1. INTRODUCTION.

In this chapter we consider some algorithms for computing a fixed point of a function or mapping from a set into itself where both integer and vector labelling will be treated. A first algorithm applied on the unit simplex was introduced in 1967 by H.E. Scarf. This algorithm is based on the concept of primitive sets rather than using a simplicial subdivision. In 1973 Scarf described a method for using the standard triangulation of the unit simplex. Earlier in 1968 and 1969 Kuhn developed already two methods to compute fixed points on the unit simplex which both were based on the standard triangulation. The first method adds a layer of artificially labelled points to S^n and starts with an $(n-1)$-simplex on the boundary, whereas the other one uses no extra points, but starts with one of the vertices of the unit simplex. Analogous methods for computing fixed points of a continuous function from R^n_+ to itself can be easily derived. Whereas all these methods were introduced at first for integer labelling, an algorithm with vector labelling to compute a fixed point of an upper-semicontinuous mapping from a convex compact set into itself was developed by Hansen and Scarf in 1969. This method was based on primitive sets. However, Eaves [1971] developed an algorithm using a triangulation of a simplex containing the convex compact set.

The algorithms mentioned above all suffer from computational inefficiency because the start is made outside the region of interest while the grid size is kept fixed throughout the algorithm. This means that if for a given grid size an approximation is found and one wants to restart the method with a finer grid to obtain a better approximation, the method must be started again outside the region of interest. So all information about the fixed point is lost.

A number of algorithms to avoid this disadvantage were developed during the last decennium. One method is due to Merrill [1971,1972] and was

independently found by Kuhn and MacKinnon [1975] (see also Lüthi [1975] and Fisher, Gould and Tolle [1977]). This algorithm can be applied both on S^n and R^n and is called the Sandwich method. It can start anywhere. So, if this method is applied for a decreasing sequence of grid sizes one can choose in each stage the starting point in such a way that it is the most appropriate one by considering the information obtained from the approximation in the previous stage. An other method which takes advantage of the information already obtained in the course of the algorithm was introduced by Eaves [1972] for application on S^n and by Eaves and Saigal [1972] for R^n. In this method, called the homotopy algorithm, the grid size is automatically decreased with a factor of at most two. Both methods were introduced with vector labelling to compute a fixed point of an u.s.c. mapping.

This chapter is organized as follows. In section 2 we discuss the concept of integer labelling and we present some theorems about the accuracy of an approximation. In the sections 3-7 we treat the algorithms mentioned above for integer labelling. The algorithm of Scarf is given in section 3, the methods of Kuhn including a unified approach of his methods are given in section 4, the analogous methods on R_+^n in section 5, the Sandwich method in section 6 and the homotopy algorithm in section 7. In section 8 we present the concept of vector labelling. The basic algorithm of Eaves is presented in section 9 while the Sandwich method and the homotopy algorithm using vector labelling are described in section 10 for mappings on R^n. The computation of a fixed point of a mapping on S^n is discussed briefly in section 11. In section 12 we give some concluding remarks.

4.2. INTEGER LABELLING AND APPROXIMATION.

In chapter 2 it was proved that a continuous function f from S^n into itself has at least one fixed point by taking a sequence of triangulations $\{G_k, k=1,2,\ldots\}$ with mesh $G_k \to 0$ if $k \to \infty$ and by labelling each grid point x by an integer label $\ell(x)$. Now we give a labelling rule which is more appropriate to be used in algorithms for computing a fixed point.

DEFINITION 4.2.1. (Standard integer labelling rule on S^n).
A point x in S^n receives the label $\ell(x) = i$, if

$$i = \min\{j \in I_{n+1} \mid f_j(x) - x_j = \min_{m \in I_{n+1}} f_m(x) - x_m \text{ and } x_j > 0\}.$$

Clearly there exists always such an index. Note that $x_j > 0$ is not really a restriction since if $f_j(x) - x_j = \min_{m \in I_{n+1}} f_m(x) - x_m$ and $x_j = 0$ for some j, then x must be a fixed point. We call a labelling rule which satisfies the condition $\ell(x) \neq i$ if $x_i = 0$ proper. By Sperner's lemma, for any triangulation G_k, $k=1,2,\ldots$ there exists a completely labelled simplex σ_k of G_k if the labelling rule is proper. Before introducing algorithms which generate a completely labelled simplex, we show how close an arbitrarily chosen point x in a completely labelled simplex is to its image $f(x)$.

LEMMA 4.2.2. Let G be a triangulation of S^n and let $\varepsilon, \delta > 0$ be such that for every simplex σ of G

$$\max_{i \in I_{n+1}} |f_i(x) - f_i(y)| < \varepsilon \qquad \text{for all } x,y \in \sigma$$

and

$$\max_{i \in I_{n+1}} |x_i - y_i| < \delta \qquad \text{for all } x,y \in \sigma.$$

Then, for any completely labelled simplex σ^* of G and $x^* \in \sigma^*$,

$$\max_{i \in I_{n+1}} |f_i(x^*) - x_i^*| < n(\varepsilon + \delta).$$

PROOF. Let w^i be the vertex of σ^* with label i, $1 \leq i \leq n+1$. Clearly $f_i(w^i) - w_i^i$ is nonpositive for all i. Hence, for all i,

$$f_i(x^*) - x_i^* = (f_i(x^*) - f_i(w^i)) + (f_i(w^i) - w_i^i) + (w_i^i - x_i^*) < \varepsilon + \delta.$$

On the other hand, since $\sum_{j=1}^{n+1} (f_j(x^*) - x_j^*) = 0$, we have for all i,

$$f_i(x^*) - x_i^* = -\sum_{j \neq i} (f_j(x^*) - x_j^*) > -n(\varepsilon + \delta).$$

Combining these two inequalities we get the desired result. $\qquad \Box$

The lemma guarantees that a completely labelled simplex in S^n is indeed an approximation of a fixed point if the standard labelling rule is used. Consider now a continuous function f from R^n to R^n and assume that this function has a fixed point (c.f. lemma 2.6.2). The standard integer labelling on R^n is defined as follows.

DEFINITION 4.2.3. (Standard integer labelling on R^n).
For some $x \in R^n$, let $i = \min\{j \in I_n \mid f_j(x) - x_j = \max_{m \in I_n} f_m(x) - x_m\}$. Then x receives the label $\ell(x)$ with

$$\ell(x) = i \quad \text{if } f_i(x) - x_i \geq 0$$
$$n+1 \quad \text{otherwise.}$$

The next lemma gives the accuracy of the approximation of a fixed point when a completely labelled simplex σ^* is found.

LEMMA 4.2.4. Let the triangulation G of R^n be such that
$$\max_{i \in I_n} |x_i - y_i| < \delta \text{ for all } x,y \in \sigma^* \text{ and let } \epsilon > 0 \text{ be such that}$$

$$\max_{i \in I_n} |f_i(x) - f_i(y)| < \epsilon \text{ for all } x,y \in \sigma^*. \text{ Then for all } x^* \text{ in } \sigma^*$$

$$\max_{i \in I_n} |f_i(x^*) - x_i^*| < \epsilon + \delta.$$

PROOF. Again, let w^i be the vertex of σ^* with $\ell(w^i) = i$, $i=1,\ldots,n+1$.
Clearly, $f_i(w^i) - w_i^i \geq 0$ for all $1 \leq i \leq n$ and $f_j(w^{n+1}) - w_j^{n+1} < 0$, $j \in I_n$.
Hence, for $i = 1,\ldots,n$

$$f_i(x^*) - x_i^* = (f_i(x^*) - f_i(w^i)) + (f_i(w^i) - w_i^i) + (w_i^i - x_i^*) > -(\epsilon + \delta)$$

and

$$f_i(x^*) - x_i^* = (f_i(x^*) - f_i(w^{n+1})) + (f_i(w^{n+1}) - w_i^{n+1}) + (w_i^{n+1} - x_i^*) < \epsilon + \delta.$$

□

So, if the standard labelling rule is used, a completely labelled simplex in R^n yields an approximation of a fixed point.

4.3. SCARF's ALGORITHM.

The original algorithm of Scarf $\lceil$1967b$\rceil$is based on the concept of primitive sets in stead of a triangulation of a simplex. Moreover, a labelling rule different from that of definition 4.2.1 was used. However, in this section we will follow Todd [1976a] using the standard labelling. To describe the algorithm we define the set $\tilde{S}^n$ by

$$\tilde{S}^n = \{x \in R^{n+1} \mid \sum_{i=1}^{n+1} x_i = 1 \text{ and } x_i \leq 2, \ i \in I_{n+1}\}.$$

$\tilde{S}^n$ is the n-simplex with vertices y^i defined by $y^i = 2e-(2n+1)e(i)$, $i=1,\ldots,n+1$, where e is the vector with $e_i = 1$ for all $1 \leq i \leq n+1$. Recall that $e(i)$ is the i-th unit vector of R^{n+1}.

In the next definition, let for some fixed k with k > n+1 the points $y^{n+2},\ldots,y^k$ be k-n-1 points in $\tilde{S}^n$ such that $y_i^{m_1} \neq y_i^{m_2}$ for all $m_1,m_2 \geq n+1$, $m_1 \neq m_2$ and for all $i \in I_{n+1}$.

DEFINITION 4.3.1. (Primitive sets).

For some $J \subset I_k$, $|J| = n+1$, the set of n+1 points $\{y^j \mid j \in J\}$ forms a primitive set if there is no m with $n+2 \leq m \leq k$ such that

$y_i^m > \min_{j \in J} y_i^j$ for all i.

Observe that the set $\{y^j \mid j \in J = I_{n+1}\}$ of vertices of $\tilde{S}^n$ does not form a primitive set. Scarf's algorithms is based on the following lemma (see Scarf [1967b]).

LEMMA 4.3.2.

a) For any $i \in I_{n+1}$, there exists a unique index $j_i > n+1$, such that the set of points $\{y^j \mid j \in (I_{n+1} \cup \{j_i\})\setminus\{i\}\}$ forms a primitive set.

b) If $\{y^j \mid j \in J\}$ is a primitive set and $m_1 \in J$ is such that $\{y^j \mid j \in J\setminus\{m_1\}\}$ is not a subset of $\{y^j \mid j \in I_{n+1}\}$, then there is a unique $m_2 \notin J$ such that $\{y^j \mid j \in (J \cup \{m_2\})\setminus\{m_1\}\}$ forms a primitive set.

Scarf's algorithm searches for a completely labelled primitive set, i.e. a primitive set $\{y^j \mid j \in J\}$ such that all elements are differently labelled. Observe that a completely labelled primitive set indeed provides an approximation of a fixed point.
To start the algorithm the points outside $\tilde{S}^n$, viz. $y^1,\ldots,y^{n+1}$ are

labelled by $\ell(y^1) = n+1$ and $\ell(y^i) = i-1$, $i=2,\dots,n+1$. From lemma 4.3.2.a there is a unique primitive set $\{y^2,\dots,y^{n+1},y^{j_1}\}$. Observe that $I_n = \{\ell(y^j)$, $j=2,\dots,n+1\}$. Starting with this primitive set the algorithm makes a search for label n+1. Therefore $\ell(y^{j_1})$ is calculated. If $\ell(y^{j_1}) = n+1$, the set is completely labelled and the algorithm terminates. Otherwise the point y^{m_1} where m_1 is the index such that $\ell(y^{m_1}) = \ell(y^{j_1})$ is replaced by y^{m_2} such that $\{y^2,\dots,y^{m_1-1},y^{m_2},y^{m_1+1},\dots,y^{n+1},y^{j_1}\}$ is again a a primitive set. Continuing by calculating the label of the new point and replacing the point having this label the algorithm generates a path of "adjacent" primitive sets until label n+1 is found. Scarf proved that the method will terminate within a finite number of replacement steps since all steps are unique and feasible whereas the number of primitive sets is finite.

For the replacement step a search through all points $y^1,\dots,y^k$ is required, unless these points are chosen in a systematic way. This is indeed possible but then there is a direct correspondence between primitive sets and simplices of the standard triangulation of S^n (see Scarf [1973, chapter 7]). The algorithm becomes then very close to the variable dimension algorithm of Kuhn to be discussed in the next section.

4.4. KUHN'S ALGORITHMS.

Besides the pioneering work of Scarf, two algorithms based on the standard triangulation of S^n and using the standard labelling rule were developed by Kuhn [1968] and [1969], (see also Todd [1976a]), namely the artificial start algorithm and the variable dimension algorithm.

Before we present these algorithms we give some definitions. Recall that S^n is the n-dimensional unit simplex, T^n the affine hull of S^n and S^n_i the i-th boundary of S^n, $i=1,\dots,n+1$. Let δ be the grid size of the standard triangulation of T^n as defined in definition 3.4.3 such that e(1) is a grid point and $\delta^{-1} = m$ is a positive integer.

DEFINITION 4.4.1.
a) $\bar{S}^n = \{x \in T^n \mid x_i \geq 0$, $i \in I_n$ and $x_{n+1} \geq -m^{-1}\}$.
b) $\bar{S}^n_i$ is the i-th boundary of $\bar{S}^n$ i.e.

$$\bar{S}^n_i = \{x \in \bar{S}^n \mid x_i = 0\}, \ i \in I_n$$

and

$$\bar{S}^n_{n+1} = \{x\epsilon\bar{S}^n \,|\, x_{n+1} = -m^{-1}\}.$$

c) $S^n(k) = \{x\epsilon S^n \,|\, x_i = 0, \, i = k+1,\ldots,n+1\}, \, k = 1,\ldots,n+1.$

Obviously, since e(1) is a grid point of the triangulation of T^n, the collection of simplices σ such that $\sigma n \bar{S}^n = \sigma$ triangulates $\bar{S}^n$. Moreover, $\bar{S}^n_{n+1}$ is triangulated (see theorem 2.3.6) and all grid points of $\bar{S}^n$ outside S^n are lying in $\bar{S}^n_{n+1}$.

In the following a permutation of the elements of the set I_k is denoted by $\pi(I_k) = (\pi_1,\ldots,\pi_k)$. So, a simplex σ of the triangulations given in 3 is denoted by $\sigma = \sigma(y^1,\pi(I_n))$. A k-simplex σ with vertices $y^1,\ldots,y^{k+1}$ is also denoted by $\sigma(y^1,\ldots,\ldots,y^{k+1})$.

The artificial start algorithm.

To apply this algorithm the points of $\bar{S}^n_{n+1}$ are artificially labelled by

$$\ell(x) = \min\{j \,|\, x_j = \max_{k\epsilon I_{n+1}} x_k\} \text{ if } x \epsilon \bar{S}^n_{n+1}.$$

The points of S^n are labelled according to definition 4.2.1. Clearly, $\ell(x) \neq n+1$ if $x \epsilon \bar{S}^n_{n+1}$ and $\ell(x) \neq i$ if $x \epsilon \bar{S}^n_i$, $i = 1,\ldots,n$. Kuhn assumed that m is a multiple of n, say m = pn. Let $\tau^0(y^1,\pi(I_{n-1}))$ be the (n-1)-simplex in $\bar{S}^n_{n+1}$ defined by (see figure 4.4.1)

a) $y^1_1 = (p+1)/m, \, y^1_i = p/m, \, i=2,\ldots,n, \, y^1_{n+1} = -m^{-1}$

b) $\pi_i = i \quad i=1,\ldots,n-1.$

<u>DEFINITION 4.4.2.</u> For $k \epsilon I_{n+1}$, a (k-1)-simplex σ is I_k-complete if the k vertices of σ carry all the labels $1,\ldots,k.$

Note that an I_{n+1}-complete n-simplex is a completely labelled simplex. The next lemma follows from the labelling rule on $\bar{S}^n_{n+1}$, defined just above. It is a special case of lemma 4.4.8 to be proved hereafter.

<u>LEMMA 4.4.3.</u> The (n-1)-simplex $\tau^0(y^1,\pi(I_{n-1}))$ is the only I_n-complete simplex in $\bar{S}^n_{n+1}.$

By theorem 2.3.5 there exists a unique n-simplex $\sigma^0(y^1,\ldots,y^{n+1})$ of $\bar{S}^n$ having τ^0 as a facet. Clearly $\sigma^0(y^1,\ldots,y^{n+1}) = \sigma^0(y^1,\pi(I_n))$ with y^1 as defined just above and $\pi_i = i$, $i \in I_n$. Observe that y^{n+1} is the only vertex of σ^0 in S^n. Moreover, y^{n+1} is the barycenter of S^n_{n+1}.

The simplex σ^0 will be the starting simplex of the algorithm. From this simplex on a path of adjacent n-simplices $\sigma^0,\sigma^1,\sigma^2,\ldots$ is generated such that the common $(n-1)$-dimensional facets are I_n-complete, until a completely labelled simplex is found, i.e. the algorithm terminates if a grid point having label n+1 is generated. Remember that the replacement step between two adjacent simplices is described in section 3.5. The algorithm can be described as follows.

Step 0. Set σ equal to $\sigma^0(y^1,\pi(I_n))$ and $\bar{y}$ equal to y^{n+1}.

Step 1. Calculate $\ell(\bar{y})$. If $\ell(\bar{y}) = n+1$, σ is completely labelled and the algorithm terminates. Otherwise, there is exactly one vertex $y^s \neq \bar{y}$ for which $\ell(y^s) = \ell(\bar{y})$.

Step 2. The simplex $\sigma(y^1,\pi(I_n))$ is adapted according to table 3.5.1 by replacing y^s. Return to step 1 with $\bar{y}$ equal to the new vertex of σ.

We now prove that the algorithm indeed terminates. The arguments used will also play a crucial role in the further simplicial algorithms. Firstly, from lemma 4.4.3 and the fact that $\ell(x) \neq i$ if $x \in \bar{S}^n_i$, it follows that τ^0 is the only I_n-complete $(n-1)$-simplex on the boundary of $\bar{S}^n$. Therefore, all the replacement steps are feasible , i.e. the new simplex is always a simplex of $\bar{S}^n$, unless the algorithm returns in σ^0 while y^{n+1} has to be replaced. However, the algorithm can not return in an earlier generated simplex. Suppose the contrary and let j be the index such that for some $i < j$, $\sigma^i = \sigma^j$ and for all $k,h < j$, $\sigma^k \neq \sigma^h$, i.e. σ^j is the first simplex which was generated earlier. Then σ^{j-1} and $\sigma^j = \sigma^i$ have a common I_n-complete facet. Clearly, any generated simplex has only two I_n-complete facets. Hence σ^{j-1} must be equal to either σ^{i-1} or σ^{i+1}. If $i \geq 1$ then $\sigma^{j-1} \neq \sigma^{i-1}$ since $\sigma^k \neq \sigma^h$ for all $k,h < j$. If $i = 0$, σ^0 has only σ^1 as adjacent simplex such that the common facet is I_n-complete. So, in both cases $\sigma^{j-1} = \sigma^{i+1}$. Moreover $j-1 = i+1$, again by that fact that $\sigma^k \neq \sigma^h$ for all $k,h < j$. Hence $j = i+2$ and $\sigma^i = \sigma^{i+2}$. Because of step 2 this is impossible.

Consequently, a path of differently adjacent simplices in $\bar{S}^n$ is generated. Since the number of simplices is finite, the algorithm must terminate within a finite number of replacement steps with a completely labelled simplex in $\bar{S}^n$. Since $\ell(x) \neq n+1$ if $x \in \bar{S}^n_{n+1} \cup S^n_{n+1}$, this simplex must be a simplex in S^n. The algorithm is illustrated for the example of figure 2.4.2 in figure 4.4.1.

Figure 4.4.1. Kuhn's artificial start algorithm.

In other algorithms to be discussed in this section the arguments to prove non-cycling are very similar to the arguments above. Therefore, they are not repeated each time.

Kuhn's variable dimension algorithm.

This algorithm does not start with a full-dimensional simplex but with the single point $e(1)$. Recall the set $S^n(k)$ of definition 4.4.1.c and note that $S^n(1)=e(1)$ and that $S^n(n+1)=S^n$. Furthermore $S^n(n)=S^n_{n+1}$. Clearly points x in $S^n(k)$ carry only labels of the set I_k since $x_i=0$ for $i \geq k+1$. By theorem 2.3.6, $S^n(k)$ is triangulated in $(k-1)$-simplices. It is obvious that $S^n(k)$ corresponds with S^{k-1}, for any $1 \leq k \leq n+1$, whereas a proper labelling on S^n induces a proper labelling on $S^n(k)$. So, by Sperner's lemma $S^n(k)$ has at least one I_k-complete simplex, $k=1,\ldots,n+1$. The algorithm starts with the $\{1\}$-complete zero-dimensional simplex $\{e(1)\}$ of $S^n(1)$. As soon as for some k, $k=1,\ldots,n$, an I_k-complete simplex in $S^n(k)$ is found, the algorithm continues with a path of k-simplices in $S^n(k+1)$ having I_k-complete common facets, until either an I_{k+1}-complete simplex in $S^n(k+1)$ is found or a

simplex having an I_k-complete facet in $S^n(k)$ is generated. In the first case the dimension is increased, whereas in the latter case the dimension is decreased. The algorithm terminates if a completely labelled simplex is found. Formally, the steps are as follows.

Step 0. Set $k=1$, $y^1=e(1)$, $\sigma=\sigma^0(y^1,\pi(\emptyset))$ and $\bar{y}=y^1$.

Step 1. Calculate $\ell(\bar{y})$. If $\ell(\bar{y})=k$, an I_k-complete simplex in $S^n(k)$ is found and go to step 3. Otherwise, there is exactly one vertex $y^s \neq \bar{y}$ such that $\ell(y^s) = \ell(\bar{y})$.

Step 2. If $s=k$ and $y_k^{k-1}=0$, the facet $\tau(y^1,\ldots,y^{k-1})$ of σ is an I_{k-1}-complete simplex in $S^n(k-1)$ and go to step 4. Otherwise $\sigma(y^1,\pi(I_{k-1}))$ is adapted according to table 4.4.2 by replacing y^s. Return to step 1 with $\bar{y}$ equal to the new vertex of σ.

Step 3. If $k=n+1$, a completely labelled simplex is found and the algorithm terminates. If $k<n+1$, $\pi(I_{k-1})$ becomes $\pi(I_k) = (\pi(I_{k-1}),k)$, σ becomes $\sigma(y^1,\pi(I_k))$. Then k becomes $k+1$ and return to step 1 with $\bar{y} = y^k = y^{k-1} + q(k-1)/m$.

Step 4. Set σ equal to τ and set $\pi(I_{k-1})$ equal to $\pi(I_{k-2}) = (\pi_1,\ldots,\pi_{k-2})$. Then k becomes $k-1$ and return to step 2 with y^s equal to the vertex of σ having label k.

Table 4.4.2. s is the index of the vertex to be replaced.

	y^1 becomes	$\pi(I_{k-1})$ becomes
$s = 1$	$y^1+q(\pi_1)/m$	$(\pi_2,\ldots,\pi_{k-1},\pi_1)$
$2 \leq s \leq k-1$	y^1	$(\pi_1,\ldots,\pi_{s-2},\pi_s,\pi_{s-1},\pi_{s+1},\ldots,\pi_{k-1})$
$s = k$	$y^1-q(\pi_{k-1})/m$	$(\pi_{k-1},\pi_1,\ldots,\pi_{k-2})$

Note that table 4.4.2 is the analogon of table 3.5.1 adapted for the fact that the variable dimension k-1 is used in stead of fixed dimension n. By the proper labelling all replacement steps are feasible, whereas also the increasing of the dimension in step 3 is feasible. Using analogous arguments as above the algorithm can never return in an earlier generated simplex. Therefore, within a finite number of iterations a completely labelled simplex is found. A more detailed proof will be given in chapter 5. In figure 4.4.2 the algorithm is illustrated for the example above.
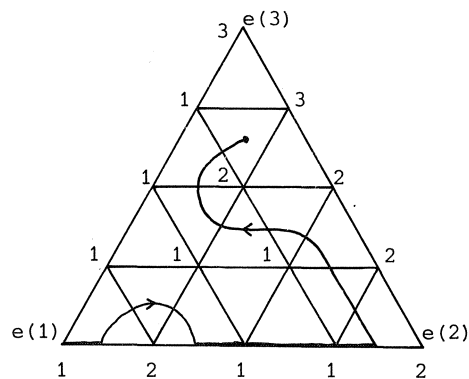
Figure 4.4.2. Kuhn's variable dimension algorithm.

It is clear that from a theoretical point of view the variable dimension algorithm can be implemented for any (not necessarily regular) triangulation of S^n. Since always a completely labelled n-simplex will be generated we have the following corollary, known as Sperner's lemma.

COROLLARY 4.4.4. If the vertices of a triangulation of S^n are properly labelled, then there exists at least one completely labelled n-simplex.

We will now show that the number of completely labelled simplices is odd. Suppose σ_1 is the completely labelled simplex generated by the variable dimension algorithm and let $\sigma_2(\bar{y}^{-1}, \bar{\pi}(I_n))$ be another completely labelled simplex (if any). Then we can apply the algorithm by a start in step 2 with $k = n+1$, $\sigma = \sigma_2$, $y^1 = \bar{y}^1$ and $\pi(I_n) = \bar{\pi}(I_n)$ and s equal to i if i is the index such that $\ell(y^i) = n+1$. Since the path between $\{e(1)\}$ and σ_1 is unique, by performing the algorithm the path of generated simplices does not cross the path between $\{e(1)\}$ and σ_1 and hence a new completely labelled simplex is found, say σ_3. The same holds if we pick up a completely labelled simplex σ_4 etc. So we have the following corollary, which is known as a stronger form of Sperner's lemma.

COROLLARY 4.4.5. If the vertices of a triangulation of S^n are properly labelled, the number of completely labelled simplices is odd.

We will return to this matter in chapter 7.

The algorithms of Kuhn have the disadvantage that they must start on the boundary respectively in one of the corners of the unit simplex. So, information obtained by applying the algorithm with a coarse grid cannot be used for an application with a finer grid. The Sandwich method to be discussed in section 6 is related to the artificially start algorithm but can start anywhere. In chapter 5 we will introduce an algorithm related to the variable dimension algorithm and which can also start anywhere.
It will appear that compared with the Sandwich method the latter algorithm has the advantage that there are no artificially labelled points and that it generates a path of simplices of variable dimension, starting with only one point.

We conclude this section with a class of algorithms of which the two algorithms of Kuhn are the extreme cases.

A class of algorithms

Before presenting the algorithms we give some definitions. Recall that T^n is triangulated with grid size m^{-1} such that $e(1)$ is a grid point and with m some positive integer.

DEFINITION 4.4.6.

a) $\bar{S}^n(k) = \{x \in T^n \mid x_i \geq 0,\ i \in I_k,\ x_{k+1} \geq -m^{-1},\ x_i = 0,\ i = k+2, \ldots, n+1\}$ $k = 1, \ldots, n$.

b) For $k = 1, \ldots, n$, $\bar{S}^n_i(k)$ is the i-th boundary of $\bar{S}^n(k)$, i.e.

$$\bar{S}^n_i(k) = \{x \in \bar{S}^n(k) \mid x_i = 0\}   i = 1, \ldots, k$$

$$\bar{S}^n_{k+1}(k) = \{x \in \bar{S}^n(k) \mid x_{k+1} = -m^{-1}\}.$$

Note that $\bar{S}^n(n) = \bar{S}^n$. It is obvious that $\bar{S}^n(k)$ is triangulated in k-simplices and that grid points of $\bar{S}^n(k)$ outside S^n are points of $\bar{S}^n_{k+1}(k)$. For given k, the algorithm starts with a k-simplex of $\bar{S}^n(k)$ having an I_k-complete (k-1)-facet in $\bar{S}^n_{k+1}(k)$. Therefore, let y be some grid point in $S^n(k)$ (see definition 4.4.1.c). Clearly $y \in \bar{S}^n(k)$, since $S^n(k) \subset \bar{S}^n(k)$. Now the points of $\bar{S}^n_{k+1}(k)$ are artificially labelled such that $\ell(z) = i$ if

$$i = \min\{j \mid y_j - z_j = \min_{h \in I_{n+1}} y_h - z_h\}.$$

Of course, the points of S^n are labelled according to the standard rule. Let $\tau(w^1,\ldots,w^k)$ be the $(k-1)$-simplex of $\bar{S}^n_{k+1}(k)$ defined by

$$w^k = y - q(k)/m \text{ and } w^i = w^{i+1} - q(i)/m \quad i=k-1,\ldots,1.$$

Then we have the following lemma.

LEMMA 4.4.7. The $(k-1)$-simplex τ is the only I_k-complete simplex in $\bar{S}^n_{k+1}(k)$.

PROOF. We first prove that τ is I_k-complete. Clearly, for $i = 1,\ldots,k$, $w^i_j = y_j$, $j \neq i,k+1$, $w^i_{k+1} = y_{k+1} - m^{-1}$ and $w^i_i = y_i + m^{-1}$. This implies $\ell(w^i) = i$, so τ is I_k-complete. It remains to prove the second part of the lemma. Suppose that the simplex $\rho(u^1,\ldots,u^k)$ is an I_k-complete simplex in $\bar{S}^n_{k+1}(k)$. If for some i,j, $1 \leq i,j \leq k$, $u^i_j = y_j - m^{-1}$, then $u^h_j \leq y_j$ for $h = 1,\ldots,k$, contradicting that $\ell(u^h) = j$ for some h. Hence, $u^i_j \geq y_j$ for all i,j, $1 \leq i,j \leq k$. Moreover, by the definition of $S^n(k)$ and $\bar{S}^n_{k+1}(k)$ we have that $y_j = u^i_j = 0$ $(j = k+2,\ldots,n+1;\ i = 1,\ldots,k)$, $y_{k+1} = 0$ and $u^i_{k+1} = -m^{-1}$, $(i = 1,\ldots,k)$. Hence $\sum_{j=1}^k u^i_j = \sum_{j=1}^k y_j + m^{-1}$ for $i = 1,\ldots,k$. It follows that for $i = 1,\ldots,k$

$$u^i_{\phi(i)} = y_{\phi(i)} + m^{-1} \text{ for some index } \phi(i) \text{ with } 1 \leq \phi(i) \leq k$$

and

$$u^i_j = y_j, \qquad \text{for } j \neq \phi(i),\ k+1.$$

Recall that the triangulation of $\bar{S}^n_{k+1}(k)$ is induced by the standard triangulation of T^n. Therefore, since ρ is a simplex of $\bar{S}^n_{k+1}(k)$, we must have that $u^{i+1} = u^i + q(\pi_i)/m$, $i = 1,\ldots,k-1$, for some permutation $\pi(I_{k-1})$. Let h be the index such that $\pi_h = 1$. Now assume $u^1_1 = y_1$. Then $u^{h+1}_1 = y_1 - m^{-1}$ which contradicts $u^{h+1}_1 \geq y_1$. Consequently, $u^1_1 = y_1 + m^{-1}$ and hence $\phi(1) = 1$ and $u^1_i = y_i$, $i=2,\ldots,k$. So $u^1 = w^1$. Moreover π_1 must be equal to one, implying that $u^2_2 = y_2 + m^{-1}$, i.e. $\phi(2) = 2$. Consequently $\phi(i) = \pi_i = i$, $i = 1,\ldots,k-1$ which proves that $\rho = \tau$. $\quad\square$

It is obvious that lemma 4.4.7 implies lemma 4.4.3 by taking k=n and
$y_j = p/m$ j=1,...,n, $y_{n+1} = 0$.
We are now prepared to sketch the algorithm associated with a given k^* and
a given point y being a grid point in $S^n(k^*)$. The method starts with the
k-simplex $\sigma^0(y^1,...,y^{k^*+1})$, where $y^{k^*+1} = y$ and where $y^1,...,y^{k^*}$ are the
vertices of the simplex τ of lemma 4.4.7, i.e. $\sigma^0 = \sigma^0(y^1,\pi(I_{k^*}))$ with
$\pi_i = i$, i=1,...,k^*. By applying Kuhn's artificial start algorithm, in which
we take k^* in stead of n, an I_{k^*+1}-complete simplex $\rho = \rho(v^1,\bar{\pi}(I_{k^*}))$ of
$S^n(k^*+1)$ must be found within a finite number of steps. Then, we continue
with the variable dimension algorithm by setting in step 0, $k = k^*+2$,
$\pi(I_{k^*+1}) = (\bar{\pi}_1, ..,\bar{\pi}_{k^*}, k^*+1)$, $\sigma(y^1,\pi(I_{k^*+1})) = \sigma(v^1, \pi(I_{k^*+1}))$ and
$\bar{y} = v^{k^*+2}$. Within a finite number of iterations, the algorithm either
terminates with a completely labelled simplex or generates a simplex of
$S^n(k^*+2)$ having an (I_{k^*+1})-complete facet in $S^n(k^*+1)$. In the latter case
the method proceeds again according to the artificial start method and
generates simplices in $\bar{S}^n(k^*)$ having I_{k^*}-complete common facets, until
again an I_{k^*+1}-complete simplex of $S^n(k^*+1)$ is found etc. Again using the
fact that all steps are feasible and that cycling can not occur, a
completely labelled simplex must be found within a finite number of steps.
Observe that for $k^* = n$ we have Kuhn's artificial start algorithm whereas
for $k^* = 1$ the method is just like the variable dimension algorithm.

4.5. ALGORITHMS TO COMPUTE FIXED POINTS ON R^n_+.

In this section a class of algorithms for computing a fixed point (if
any) of a continuous function f from R^n_+ to R^n_+ is presented. This class is
analogous to the class of algorithms described in the previous section.
An application of these algorithms is the nonlinear complementarity
problem.
For all members of the class of algorithms, R^n_+ is triangulated by the
K_δ-triangulation. For ease of notation we take $\delta=1$ in the discussion below.
Analogous to the class of algorithms on S^n we want to have a proper
labelling, i.e. a labelling such that $\ell(x) \neq i$ if $x_i = 0$. Therefore the
standard labelling rule on R^n is not appropriate and we define the
following labelling rule.

DEFINITION 4.5.1. (Labelling rule on R_+^n).

For $x \in R_+^n$, let i be the index such that $i=\min\{j \mid f_j(x)-x_j=\min_{h \in I_n} f_h(x)-x_h\}$.
Then

$$\ell(x) = i \qquad \text{if } f_i(x)-x_i < 0$$
$$= 0 \qquad \text{otherwise.}$$

Note that the labelling rule is proper. In particular we have that $\ell(\underline{0}) = 0$. Furthermore it is easy to see that a completely labelled simplex yields an approximation of a fixed point. We call a k-simplex $(I_k \cup \{0\})$-complete if the k+1 vertices carry all the labels $0,1,\ldots,k$.
We give now some definitions which are analogous to definition 4.4.6 and definition 4.4.1.cand which are helpful to describe the starting procedure.

DEFINITION 4.5.2.

a) $\bar{R}^n(k) = \{x \in R^n \mid x_i \geq 0,\ i \in I_k,\ x_{k+1} \geq -1 \text{ and } x_i=0,\ i=k+2,\ldots,n\}$ $k=0,\ldots,n-1$.

b) For $k=0,\ldots,n-1$, $\bar{R}_i^n(k)$ is the i-th boundary of $\bar{R}^n(k)$, i.e.

$$\bar{R}_i^n(k) = \{x \in \bar{R}^n(k) \mid x_i=0\} \qquad i=1,\ldots,k;$$

$$\bar{R}_{k+1}^n(k) = \{x \in \bar{R}^n(k) \mid x_{k+1}=-1\}$$

c) $R^n(k) = \{x \in R^n \mid x_i \geq 0, i \in I_k \text{ and } x_i=0,\ i=k+1,\ldots,n\}$ $k=0,\ldots,n$.

The K-triangulation of R^n implies that for given $k=k^*$ with $0 \leq k^* \leq n-1$, $\bar{R}^n(k^*)$ is triangulated in such a way that the grid points of $\bar{R}^n(k^*)$ not in R_+^n are points of $\bar{R}_{k^*+1}^n(k^*)$.
Now, let y be an arbitrarily chosen grid point of $R^n(k^*)$, i.e. $y_i \geq 0$ for $i \in I_{k^*}$. The algorithm associated with a given k^* starts with the simplex $\sigma(y^1, \pi(I_{k^*+1}))$ where $y^{k^*+2} = y$ and $y^i = y^{i+1} - e(i)$, $i=1,\ldots,k^*+1$, i.e. $y^1 = y - \sum_{i=1}^{k^*+1} e(i)$ and $\pi_i = i$, $i=1,\ldots,k^*+1$. To guarantee that the facet $\tau(y^1,\ldots,y^{k^*+1})$ is the only $(I_{k^*} \cup \{0\})$-complete simplex of $\bar{R}_{k^*+1}^n(k^*)$ the points of $\bar{R}_{k^*+1}^n(k^*)$ are artificially labelled by the following rule. Let i be the index such that

$$i = \min\{j \mid y_j-x_j-j/(k^*+1) = \min_{h \in I_{k^*}} y_h-x_h-h/(k^*+1)\}.$$

Then $\ell(x) = i$ if $y_i - x_i - i/k^*+1 < 0$ and $\ell(x) = 0$ otherwise.

Following the proof of lemma 4.4.7 it follows immediately that τ is indeed the only $(I_{k^*} \cup \{0\})$-complete simplex of $\bar{R}^n_{k^*+1}(k^*)$. In particular holds: $\ell(y^i) = i-1$, $i=1,\ldots,k^*+1$.

The algorithm proceeds as follows.

Step 0. Set $k = k^*+1$, $\sigma = \sigma(y^1, \pi(I_{k^*+1}))$ and $\bar{y} = y^{k+1} (= y^{k^*+2} = y)$.

Step 1. Calculate $\ell(\bar{y})$. If $\ell(\bar{y}) = k$, an $(I_k \cup \{0\})$-complete simplex of $R^n(k)$ is found and go to step 3. Otherwise there is exactly one vertex $y^s \neq \bar{y}$ such that $\ell(y^s) = \ell(\bar{y})$.

Step 2. If $s=k+1$, $y^k_k = 0$ and $k > k^*+1$, the facet $\tau(y^1,\ldots,y^k)$ of σ is an $(I_{k-1} \cup \{0\})$-complete simplex of $R^n(k-1)$ and go to step 4. Otherwise σ is adapted according to table 4.5.1 by replacing y^s. Return to step 1 with $\bar{y}$ equal to the new vertex of σ.

Step 3. If $k=n$, a completely labelled simplex is found and the algorithm terminates. If $k<n$, $\pi(I_k)$ becomes $\pi(I_{k+1}) = (\pi(I_k),k+1)$, σ becomes $\sigma(y^1, \pi(I_{k+1}))$. Then k becomes $k+1$ and return to step 1 with $\bar{y} = y^{k+1} = y^k + e(k)$.

Step 4. Set σ equal to τ and set $\pi(I_k)$ equal to $\pi(I_{k-1}) = (\pi_1,\ldots,\pi_{k-1})$. Then k becomes $k-1$ and return to step 2 with y^s equal to the vertex having label k.

Observe that table 4.5.1 is analogous to table 4.4.2 and that it represents the replacement step between two adjacent $(k+1)$-simplices. Recall that $R^n(k)$ and $\bar{R}^n(k-1)$ are k dimensional subspaces of R^n.

Table 4.5.1. s is the index of the vertex to be replaced.

	y^1 becomes	$\pi(I_k)$ becomes
s=1	$y^1 + e(\pi_1)$	$(\pi_2,\ldots,\pi_k,\pi_1)$
$2 \leq s \leq k$	y^1	$(\pi_1,\ldots,\pi_{s-2},\pi_s,\pi_{s-1},\pi_{s+1},\ldots,\pi_k)$
$s = k+1$	$y^1 - e(\pi_k)$	$(\pi_k,\pi_1,\ldots,\pi_{k-1})$.

The class of algorithms corresponds to the class of algorithms on S^n, described in the previous section. In particular, the case $k^* = n-1$ corresponds to Kuhn's artificial start algorithm on S^n, whereas the case $k^* = 0$ is the analogon of the variable dimension method on S^n. In the latter case the algorithm starts with the point $y = \underline{0}$ of $R^n(0)$.

By the proper labelling, it is easy to see that all replacement steps are feasible. Moreover, using the same arguments as for the artificial start algorithm of section 4, cycling cannot occur. Therefore the algorithm either terminates with a completely labelled simplex in R_+^n or an infinite path of simplices going to infinity is generated. The algorithm is illustrated in figure 4.5.1 for n=2 and $k^*=0$ respectively $k^*=1$.
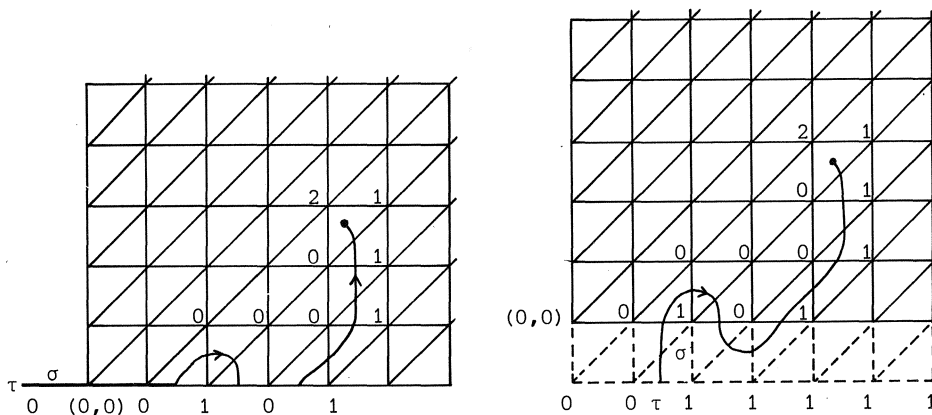
Figure 4.5.1.a n=2, $k^*=0$ $y=(0,0)^T$. Figure 4.5.1.b n=2, $k^*=1$, $y=(2,0)^T$.

Lüthi [1976] proposed an algorithm which is closely related to the case $k^*=0$. This algorithm is based on the standard labelling rule on R^n. It can be viewed as a generalization of the case $k^*=0$ in the sense that the sequence of found labels is not predetermined. On the other hand it is a special case of the variable dimension restart algorithm to be presented in the chapters 5 and 6.

4.6. SANDWICH METHOD.

All the methods described in the previous sections suffer from inefficiency, since they are characterized by a start outside the region of interest and by a fixed grid throughout the algorithm. If we have a small grid size the number of iterations becomes very high whereas the approximation is bad if the grid size is large.

A method which avoids this disadvantage is the Sandwich method introduced for functions on S^n by Kuhn and MacKinnon [1975]. The method was discovered earlier by Merrill [1971,1972] for mappings on unbounded regions.

The Sandwich method is a straightforward application of Kuhn's artificial start algorithm. Suppose we want to compute a fixed point of a continuous function f from S^n into itself. Therefore S^n is embedded in the (n+1)-dimensional simplex S^{n+1}, which is triangulated according to definition 3.4.1. Clearly S_{n+2}^{n+1} corresponds to S^n i.e. $x=(x_1,\ldots,x_{n+1},0)^T$ is an element of S_{n+2}^{n+1} iff $(x_1,\ldots,x_{n+1})^T$ is an element of S^n. Let ℓ denote the standard labelling rule on S^n as defined in 4.2.1. Now the points of S^{n+1} are labelled by $\bar{\ell}(x)$ defined as follows

$$\bar{\ell}(x) = n+2 \qquad \text{if } x_{n+2} > 0$$

$$\bar{\ell}(x) = \ell(x_1,\ldots,x_{n+1}) \qquad \text{if } x_{n+2} = 0$$

So all points with $x_{n+2} > 0$ are labelled by n+2.
Let y be an arbitrarily chosen point of S_{n+2}^{n+1}. Note that in Kuhn's algorithm y must be the barycenter of S_{n+2}^{n+1} but that the extreme case $k^*=n$ of the class of algorithms discussed in section 4 allows for an arbitrarily chosen point y. Then starting with an (n+1)-simplex such that y is one of its vertices and the convex hull of the other vertices is the unique I_{n+1}-complete simplex (see definition 4.4.2) of $\bar{S}_{n+2}^{n+1}$, the algorithm generates a path of adjacent simplices until a completely labelled (n+1)-simplex of S^{n+1} is found. Clearly this is the case as soon as a grid point is generated such that $x_{n+2} > 0$, i.e. if a simplex $\sigma(y^1,\pi(I_{n+1}))$ is generated with $y_{n+2}^1 = 0$ and $\pi_{n+1} = n+1$. But then $y_{n+2}^i = 0$ for $i=1,\ldots,n+1$ and $\tau(y^1,\ldots,y^{n+1})$ is an I_{n+1}-complete facet of σ lying in S_{n+2}^{n+1} or analogously τ is a completely labelled simplex of S^n and yields an approximation of a fixed point of f. In figure 4.6.1 the Sandwich method is illustrated for the example of figure 4.4.1 by drawing the intersection of the path of simplices in $\bar{S}^{n+1}$ with the (n+2)-th boundary S_{n+2}^{n+1} of S^{n+1}. Observe that seven function evaluations are made, whereas also seven points of the artificial labelled boundary $\bar{S}_{n+2}^{n+1}$ are generated. The generalization to R_+^n is immediately clear from the generalization of Kuhn's artificial start algorithm to R_+^n as described in section 4.5. The extension to R^n can easily be made and will be discussed in section 10 for

vector labelling in stead of integer labelling.

The advantage of this method is that y can be chosen to be any grid point of S_{n+2}^{n+1}, or equivalently, of S^n. Therefore we may apply the algorithm for a
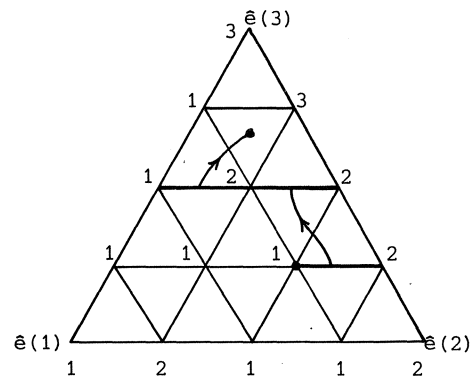
Figure 4.6.1. The intersection of the path of simplices, generated by the Sandwich method, with S_{n+2}^{n+1}, $y=(¼,½,¼)^T$, n=2, $ê(i)=(e^T(i),0)^T$.

decreasing sequence of grid sizes. In any stage the approximate solution of the previous stage can be used as the starting point. So, a sequence of approximations is generated, until the desired accuracy is achieved.

4.7. HOMOTOPY ALGORITHM.

A second algorithm which avoids the disadvantages of the methods discussed in the sections 3 until 5 was developed by Eaves [1972] for mappings on S^n and by Eaves and Saigal [1972] for mappings on R^n. To handle mappings, vector labelling has to be used. Then the method traces a path of fixed points of a homotopy function from a given linear function to the function of interest. Therefore the method was called the homotopy algorithm. We return to the matter of vector labelling in the following sections. In this section we discuss the method using integer labelling. The algorithm is characterized by an automatically decreasing grid size. We treat at first the simplest applicationwith the decreasing sequence 1,1/2,1/3,... to compute a fixed point of a continuous function f on S^n. Therefore T^{n+1} is triangulated by the standard triangulation with grid size δ=1 and with e(1) a grid point of the triangulation. Now, let

U^{n+1} be defined by

$$U^{n+1} = \{x \epsilon T^{n+1} | x_1, \ldots, x_{n+1} \geq 0 \text{ and } x_{n+2} \leq 0\}$$

and let $U^{n+1}(m)$ be defined by

$$U^{n+1}(m) = \{x \epsilon U^{n+1} | x_{n+2} = -m+1\} \quad m=1,2,\ldots .$$

Clearly the triangulation of T^{n+1} induces a triangulation of U^{n+1}. Moreover, the induced triangulation of $U^{n+1}(m)$ corresponds with the standard triangulation of S^n with grid size m^{-1}, i.e. the point $(x_1, \ldots, x_{n+1}, -m+1)^T$ of $U^{n+1}(m)$ corresponds with the point $m^{-1}(x_1, \ldots, x_{n+1})^T$ of S^n in the sense that the latter point is the intersection of the line from $(x_1, \ldots, x_{n+1}, -m+1)^T$ to $(0, \ldots, 0, 1)^T$ with $U^{n+1}(1)$. Consequently, the point $(x_1, \ldots, x_{n+1}, x_{n+2})$ of U^{n+1} is labelled by the standard labelling rule $\ell(y)$ on S^n where $y_i = x_i/(1-x_{n+2})$, $i=1, \ldots, n+1$. Obviously $\ell(x) \neq i$ if $x_i = 0$ for any $x \epsilon U^{n+1}$. Furthermore an I_{n+1}-complete simplex of $U^{n+1}(m)$ corresponds with a completely labelled simplex of the standard triangulation of S^n with grid size m^{-1}.

Since $U^{n+1}(1)$ corresponds with S^n it is I_{n+1}-complete. The algorithm starts with the simplex $\sigma^0(y^1, \pi(I_{n+1}))$ of U^{n+1} such that $y^1 = (1, 0, \ldots, 0, 1, -1)^T$, $\pi(1) = n+1$ and $\pi_i = i-1$, $i=2, \ldots, n+1$, i.e. $\sigma^0(y^1, \pi(I_{n+1}))$ is the unique simplex of the triangulation of U^{n+1} having $U^{n+1}(1)$ as a facet. According to the replacement steps of table 3.5.1 the algorithm generates a path of adjacent $(n+1)$-simplices of U^{n+1} having I_{n+1}-complete common facets. Clearly $U^{n+1}(1)$ is the only I_n-complete facet on the boundary of U^{n+1}. Therefore, since cycling cannot occur all steps are feasible and an I_{n+1}-complete simplex τ of $U^{n+1}(m)$ $m=2,3,\ldots$ is generated within a finite number of steps. The method is illustrated in figure 4.7.1.

Observe that it is possible that the algorithm returns to a previous level. The algorithm can be broken down if either an a priori chosen value of m is attained or the approximation is good enough. Observe that in the example the vertices of the facet τ between the levels 2 and 3 represent the same points as the vertices of facet ρ on level 6. Therefore triangulations which allow for a faster decreasing of the grid size are interesting.
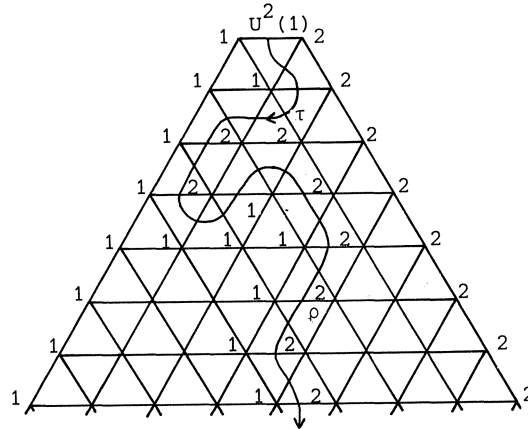
Figure 4.7.1. Illustration of the homotopy algorithm for n=1 with the
decreasing sequence of grid sizes 1, 1/2, 1/3,... .

In Eaves [1972] a triangulation for a decreasing sequence of grid sizes
with a factor of two was given. Let G be the standard triangulation of the
convex hull of $U^{n+1}(1)$ and $U^{n+1}(2)$ and let G(k) be the standard
triangulation of $U^{n+1}(m)$, where $m=2^k$, for k=1,2,... . The grid size of G and
G(k) k=1,2,... is one. Clearly, an (n+1)-simplex σ of G is characterized
by $\sigma=\sigma(y^1,\pi(I_{n+1}))$, whereas an n-simplex τ of G(k) is characterized by
$\tau=\tau(y^1,\pi(I_n))$. We are now prepared to present a triangulation of U^{n+1} with
a factor 2 between two subsequent levels. The (n+1)-simplices of U^{n+1} are
of the form $T_\tau(\sigma)$ with for some n-simplex τ of G(k), k=1,2,... and for
some (n+1)-simplex σ of G, $T_\tau(\sigma)$ defined by

$$T_\tau(\sigma) = \bar{\sigma}(T_\tau(y^1),\dots,T_\tau(y^{n+2}))$$

where $y^1,\dots,y^{n+2}$ are the vertices of σ and where for j=1,...,n+2 the h-th
component $t_h(y^j)$ of the vector $T_\tau(y^j)$ is equal to

$$t_h(y^j) = \sum_{i=1}^{n+1} u_h^i y_i^j \qquad h=1,\dots,n+1$$

and

$$t_{n+2}(y^j) = 1 - \sum_{h=1}^{n+1} t_h(y^j)$$

if $u^1,\dots,u^{n+1}$ are the vertices of τ.

For this triangulation the algorithm is illustrated in figure 4.7.2.
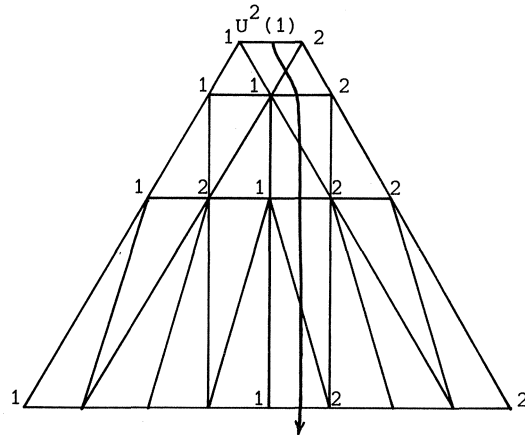
Figure 4.7.2. Illustration of the homotopy algorithm for n=1 with a
factor of two between two subsequent levels.

In Eaves and Saigal [1972] the homotopy algorithm was generalized for the
computation of a fixed point of a function on R^n (if any). Let G(0) be the
standard triangulation of $R^n \times \{1\}$ with grid size δ. Let τ be a simplex of
G(0). Then the triangulation of $\tau \times [1, \infty)$ is a scaled copy of the
triangulation of U^{n+1}. In their paper Eaves and Saigal showed that these
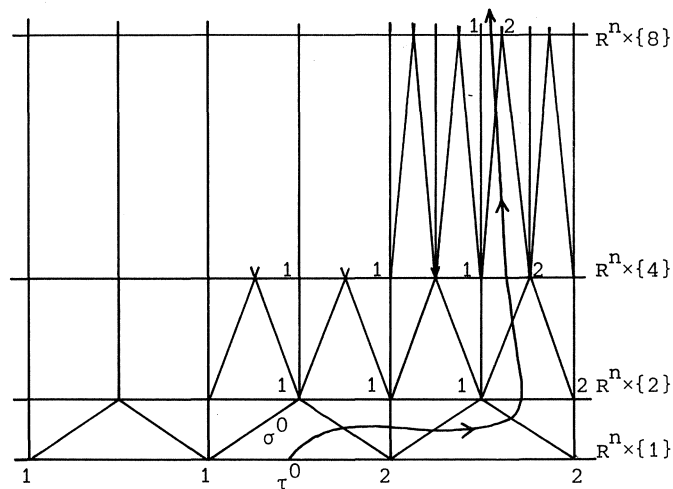
Figure 4.7.3. Illustration of the homotopy algorithm on R^n for n=1.

triangulations of $\tau \times [1,\infty)$ can be combined to a triangulation of $R^n \times [1,\infty)$. Then the algorithm is started with a simplex σ^0 having some grid point y in $R^n \times \{2\}$ as a vertex while the facet τ^0 opposite y is a facet of $R^n \times \{1\}$. To guarantee feasibility of the replacement steps the points of $R^n \times \{1\}$ are artificially labelled in such a way that τ^0 is the only completely labelled n-simplex of $R^n \times \{1\}$. Starting with σ^0 a path of adjacent (n+1)-simplices having common completely labelled facets is generated. An illustration for n=1 is given in figure 4.7.3.

4.8. VECTOR LABELLING AND APPROXIMATION.

Assume we want to compute a fixed point of an upper-semicontinuous mapping from R^n into itself. Therefore we introduce the concept of vector labelling.

Assume that f is a piecewise linear approximation to ϕ with respect to a triangulation G of R^n (see definition 2.5.4). Then we define the following labelling rule.

DEFINITION 4.8.1. (Vector labelling on R^n).
Let b be a nonnegative vector in R^{n+1} such that $b_{n+1} > 0$. Then $x \in R^n$ receives the (n+1)-dimensional vector label $\ell(x)$ with

$$\ell_i(x) = f_i(x) - x_i + b_i \qquad i=1,\ldots,n,$$

$$\ell_{n+1}(x) = b_{n+1}.$$

For vector labelling we define a completely labelled simplex as follows.

DEFINITION 4.8.2. A simplex $\sigma(w^1,\ldots,w^{n+1})$ of G is completely labelled if the set of linear equations

$$\sum_{i=1}^{n+1} \lambda_i \ell(w^i) = b$$

has a nonnegative solution $\lambda^* = (\lambda_1^*,\ldots,\lambda_{n+1}^*)$.

The following lemma states that a completely labelled simplex yields a fixed point of f.

LEMMA 4.8.3. The piecewise linear approximation f to ϕ has a fixed point in the simplex $\sigma(w^1,\ldots,w^{n+1})$ if and only if σ is completely labelled. If λ^* is the solution of the system of linear equations, then

$$x^* = \sum_{i=1}^{n+1} \lambda_i^* w^i$$

is the fixed point of f in $\sigma(w^1,\ldots,w^{n+1})$.

PROOF. Clearly, since $\ell_{n+1}(w^i) = b_{n+1}$ for all i, we have that $\sum_{i=1}^{n+1} \lambda_i^* = 1$. Moreover, by the definition of f and x^* we have that

$$b_j = \sum_{i=1}^{n+1} \lambda_i^* \ell_j(w^i) = \sum_{i=1}^{n+1} \lambda_i^* f_j(w^i) - \sum_{i=1}^{n+1} \lambda_i^* w_j^i + b_j \sum_{i=1}^{n+1} \lambda_i^* =$$

$$= f_j(x^*) - x_j^* + b_j \qquad j=1,\ldots,n,$$

which proves that x^* is a fixed point of f.

Conversely, if x is a point of $\sigma(w^1,\ldots,w^{n+1})$, then there is a $\lambda \in S^n$, such that $x = \sum_{i=1}^{n+1} \lambda_i w^i$. So $\ell(x) = \ell(\sum_{i=1}^{n+1} \lambda_i w^i) = \sum_{i=1}^{n+1} \lambda_i \ell(w^i)$. If x^* is a fixed point then $\ell_j(x^*) = f_j(x^*) - x_j^* + b_j = b_j$ for j=1,...,n, which implies that

$$\sum_{i=1}^{n+1} \lambda_i^* \ell(w^i) = b,$$

when $x^* = \sum_{i=1}^{n+1} \lambda_i^* w^i$. Hence $\sigma(w^1,\ldots,w^{n+1})$ is completely labelled. ☐

Let σ^* be a completely labelled simplex and let x^* be the fixed point of f in σ^*. The next theorem shows the accuracy of x^* as approximate fixed point of the mapping ϕ.

THEOREM 4.8.4. Let $\epsilon > 0$ be such that for all $g(y) \in \phi(y)$ there exists an element $g(x) \in \phi(x)$ such that $\max_{i \in I_n} |g_i(y) - g_i(x)| < \epsilon$ for all $x,y \in \sigma^*$.

Then there is an element $g(x^*) \in \phi(x^*)$ such that $\max_{i \in I_n} |g_i(x^*) - x_i^*| < \epsilon$.

PROOF. We have that $x^* = \sum_{i=1}^{n+1} \lambda_i^* w^i = \sum_{i=1}^{n+1} \lambda_i^* f(w^i)$. Since $w^j \in \sigma^*$ there exists an element $g(x^*) \in \phi(x^*)$ such that

$$\max_{i\in I_n} |f_i(w^j) - g_i(x^*)| < \varepsilon, \quad j=1,\ldots,n+1.$$

Hence

$$\max_{i\in I_n} |g_i(x^*) - x_i^*| = \max_{i\in I_n} |\sum_{j=1}^{n+1} \lambda_j^* \{g_i(x^*) - f_i(w^j)\}|$$

$$\leq \sum_{j=1}^{n+1} \lambda_j^* \max_{i\in I_n} |g_i(x^*) - f_i(w^j)| < \varepsilon. \qquad \square$$

COROLLARY 4.8.5. Let ϕ be a continuous function from R^n into itself and let x^* be the fixed point of the piecewise linear approximation f to ϕ in the completely labelled simplex σ^*. Then

$$\max_{i\in I_n} |\phi_i(x^*) - x_i^*| < \varepsilon.$$

Observe that for a continuous function ϕ from R^n into itself $\max_i |\phi_i(x^*) - x_i^*|$ is bounded by ε when vector labelling is used whereas this amount is bounded by $\varepsilon+\delta$ when using integer labelling (see lemma 4.2.4). Using vector labelling to compute a fixed point of a continuous function on S^n the upper bound becomes ε instead of $n(\varepsilon+\delta)$ in case of integer labelling (see lemma 4.2.2). The result of theorem 4.8.4 while true, may not be at all useful. Of course, in case of a continuous function diam σ^* close to zero implies that ε becomes very small. However, it can easily be seen that there exist u.s.c. mappings such that ε may be rather large, (e.g. as large as 1), even if diam σ^* is arbitrarily small. Nevertheless, computing a fixed point of ϕ, we have that a sequence of triangulations with mesh going to zero, the sequence of approximate fixed points contains a subsequence converging to a fixed point of ϕ (if any, see theorem 2.5.3).

4.9. A BASIC ALGORITHM TO COMPUTE A FIXED POINT OF A MAPPING.

One of the first methods to find a completely labelled simplex using vector labelling was developed by Eaves [1971]. Let ϕ be an upper-semicontinuous mapping from an n-dimensional compact, convex subset C into

itself and let c be an element in the interior of C. The mapping ϕ is replaced by the u.s.c. mapping $\bar{\phi}$ from an n-simplex C' containing C in its interior to C as in the proof of theorem 2.5.3, i.e.

$$\begin{aligned}
\bar{\phi}(x) &= \phi(x) &&\text{if } x \in C \\
&= \text{conv}(\phi(x) \cup \{c\}) &&\text{if } x \in \text{bd } C \\
&= \{c\} &&\text{if } x \in C' \backslash C
\end{aligned}$$

Recall that a fixed point of $\bar{\phi}$ is a fixed point of ϕ.

Let G be a triangulation of C' and let the starting point x^0 be an arbitrarily chosen point on bd C', such that x^0 is in the interior of a facet τ^0 of the triangulation G. Let σ^0 be the unique simplex of G having τ^0 as a facet (see theorem 2.3.5) and let d be the (n+1)-dimensional vector $(x^0-c)^T,0)^T + b$.

Before describing the algorithm we introduce the concept of an almost complete simplex.

DEFINITION 4.9.1. For k=n-1 or k=n, a k-simplex $\sigma(w^1,\ldots,w^{k+1})$ of G is almost complete if the system of linear equations

$$\sum_{i=1}^{k+1} \lambda_i \ell(w^i) + \mu d = b$$

has a nonnegative solution $(\lambda_1^*,\ldots,\lambda_{k+1}^*, \mu^*)$.

Although degeneracy can occur in fixed point algorithms we assume in the following that it does not happen. A method to resolve the problem of degeneracy can be found in e.g. Todd [1976a]. So assuming non degeneracy the system of linear equations has a unique solution if σ is an almost complete (n-1)-simplex. However, if σ is an almost complete n-simplex the above system of linear equations has two solutions having each just one variable equal to zero. Clearly, σ is completely labelled if for one of these two solutions $\mu^*=0$. The algorithm searches for a simplex having such a solution.

Since τ^0 is a boundary facet its vertices y^i $i \in I_n$ are labelled by $\ell(y^i) = (c-y^i)^T,0)^T+b$. Clearly, $x^0 = \sum_{i=1}^n \alpha_i y^i$ for some $\alpha \in S^{n-1}$. Since x^0 is in the interior of τ^0, $\alpha_i > 0$ $i \in I_n$. So, the (n-1)simplex τ^0 is almost complete with solution $(\lambda_1^*,\ldots,\lambda_n^*,\mu^*)$ where $\lambda_i^* = \frac{1}{2}\alpha_i$ $i \in I_n$ and $\mu^*=\frac{1}{2}$. Moreover, τ^0 is the only almost complete (n-1)-simplex on the boundary of C'

(see Eaves [1971] or Todd [1976a]). Now the algorithm is described by the following steps, where $\sigma^0 = \sigma(y^1,\ldots,y^{n+1})$ is the unique simplex containing τ^0.

Step 0.   Set σ equal to $\sigma(y^1,\ldots,y^{n+1})$ and let $\bar{y} = y^t$ be the vertex of σ^0 not in τ^0. Let $\sum_{i \neq t} \lambda_i \ell(y^i) + \mu d = b$ be the set of linear equations induced by τ^0. This system has the above mentioned solution $(\lambda_1^*,\ldots,\lambda_n^*,\mu^*)$.

Step 1.   Calculate $\ell(\bar{y})$ and make a standard linear programming pivot step by bringing $\ell(\bar{y})$ in the system of linear equations to obtain a new solution with one of variables equal to zero. If μ becomes zero, i.e. if d is eliminated, σ is a completely labelled simplex and the algorithm terminates. Otherwise, for some s, $\ell(y^s)$ is eliminated by the pivot step.

Step 2.   Adapt σ by replacing y^s. Let $\bar{y}$ be the new vertex of σ and return to step 1.

Since non degeneracy is assumed, all the replacement steps are unique, so cycling cannot occur (see section 4). Moreover, all replacement steps are feasible since τ^0 is the only almost complete facet on the boundary of C'. Hence the algorithm terminates within a finite number of iterations with a completely labelled simplex. Note that the algorithm has the same disadvantages of those discussed in the sections 3 until 5. It must start outside C and the grid size is kept fixed.

4.10. SANDWICH AND HOMOTOPY METHODS ON R^n.

Let ϕ be an upper semi-continuous mapping from R^n into itself and suppose we want to compute a fixed point (if any). As remarked in section 6 the Sandwich method is the application of Kuhn's artificial start algorithm on $R^n \times R_+$ by adding an extra layer of points with $x_{n+1} = -1$. From the discussion in 6 it is clear that the algorithm terminates as soon as a point is generated with $x_{n+1} = 1$. This means that in practice only simplices $R^n \times [-1,0]$ are generated, i.e. the algorithm starts with the $(n+1)$-simplex having the unique completely labelled n-simplex in $R^n \times \{-1\}$ of artificially labelled points as a facet and terminates with an $(n+1)$-simplex having a completely labelled facet in $R^n \times \{0\}$, whose vertices are labelled according to the standard labelling rule induced by the mapping ϕ. Therefore we can restrict ourselves to the set $R^n \times [-1,0]$ or

equivalently to $R^n \times [0,1]$, where $R^n \times \{0\}$ will be the level of artificially

labelled points.

To apply the Sandwich method we define the mapping $\bar{\phi}$ from $R^n \times [0,1]$ to R^n by

$$\bar{\phi}(x,t) = \{ty + (1-t)v \mid y \in \phi(x)\},$$

where $v \in R^n$ is an arbitrarily chosen point, called the starting point.

Now, a point (x^*, t^*) is called a fixed point of $\bar{\phi}$ if $x^* \in \bar{\phi}(x^*, t^*)$.

Clearly, if $(x^*, 1)$ is a fixed point of $\bar{\phi}$, then x^* is a fixed point of ϕ.

Furthermore, it is obvious that $(v,0)$ is the only fixed point of $\bar{\phi}$ in

$R^n \times \{0\}$. Let G be the triangulation of $R^n \times [0,1]$ induced by the AK triangulation

of R^{n+1}, where A is the diagonal matrix such that $a_{ii} = \delta$, $i = 1, \ldots, n$ and

$a_{ii} = 1$ for $i = n+1$, i.e. the grid points G are either in $R^n \times \{0\}$ or

$R^n \times \{1\}$ and the induced triangulations of $R^n \times \{0\}$ and $R^n \times \{1\}$ have grid size

δ. We assume that $(v,0)$ is in the interior of an n-simplex τ^0 of $R^n \times \{0\}$.

A point (x,t) receives the vector label $\ell(x)$ defined in 4.8.1, i.e.

$\ell_i(x,t) = f_i(x,t) - x_i + b_i$, $i = 1, \ldots, n$ and $\ell_{n+1}(x,t) = b_{n+1}$ with $f(x,t)$ a

piecewise linear approximation to $\bar{\phi}$ with respect to G. Since $(v,0)$ is the

only fixed point with $t=0$ it follows from lemma 4.8.3 that τ^0 is the only

completely labelled n-simplex in $R^n \times \{0\}$.

The algorithm proceeds as follows.

Step 0. Set σ equal to the unique (n+1)-simplex having τ^0 as a facet and
let $\bar{y}$ be the vertex of σ not in τ^0. Let $\Sigma_{i=1}^{n+1} \lambda_i \ell(y^i) = b$ be the
system of linear equations induced by τ^0 where $y^1, \ldots, y^{n+1}$ are the
vertices of τ^0. The solution of this system is $\lambda^* = (\lambda_1^*, \ldots, \lambda_{n+1}^*)$
if $v = \Sigma_{i=1}^{n+1} \lambda_i^* y^i$.

Step 1. Calculate $\ell(\bar{y})$. Make a linear programming step with $\ell(\bar{y})$ in the
system of linear equations. Let s be the index such that $\ell(y^s)$ is
eliminated by $\ell(\bar{y})$.

Step 2. If the facet of σ opposite y^s lies in $R^n \times \{1\}$, this facet is a
completely labelled n-simplex of $R^n \times \{1\}$ and the algorithm
terminates. Otherwise, adapt σ by replacing y^s and return to
step 1 with $\bar{y}$ equal to the new vertex.

Following these steps, the algorithm generates a path of adjacent

simplices such that the common facets are completely labelled. It means,

that the algorithm can be seen as a method which traces a path of fixed

points of the piecewise linear approximation f(x,t) to $\bar{\phi}$(x,t). Using the standard arguments the algorithm either terminates within a finite number of steps or generates a path of simplices going to infinity. It can be proved that the algorithm terminates if Merrill's condition (see lemma 2.6.2) is satisfied (c.f. Merrill ⌈1972⌉ and also Todd ⌈1976a⌉). The approximation of the fixed point can be chosen as the starting point v in a new application of the method with a smaller grid size.

To apply the homotopy algorithm we define a function $\tilde{\phi}$ from $R^n \times \lceil 1, \infty)$ to R^n such that

$$\tilde{\phi}(x, 1) = v \text{ and } \tilde{\phi}(x, 2^k) = f^k(x)$$

where v is the starting point and $f^k(x)$ a piecewise linear approximation to ϕ with respect to the triangulation of $R^n \times \{2^k\}$. $R^n \times [1, \infty)$ is triangulated as in section 7 such that v is in the interior of an n-simplex τ^0 in $R^n \times \{1\}$. The points (x,t) are labelled as above with f(x,t) a piecewise linear approximation to $\tilde{\phi}$(x,t). Hence τ^0 is the only completely labelled n-simplex in $R^n \times \{1\}$. Starting with the (n+1)-simplex having τ^0 as a facet the algorithm generates a sequence of adjacent simplices such that the common facets are completely labelled. That means that a path of fixed points of the piecewise linear approximation f(x,t) is traced. It can easily be seen that this path converges to a fixed point of ϕ(x) if t goes to infinity. The algorithm can be terminated as soon as a completely labelled simplex on a predetermined level t is generated. This occurs in a finite number of steps if Merrill's condition is satisfied.

4.11. MAPPINGS FROM S^n INTO S^n.

To compute a fixed point of an u.s.c. mapping ϕ on S^n we define the following labelling rule given a triangulation G of S^n.

DEFINITION 4.11.1. (Vector labelling on S^n).
A point x in S^n receives the (n+1)-dimensional vector label ℓ(x) defined by

$$\ell(x) = -f(x) + x + b$$

where f(x) is a piecewise linear approximation to ϕ with respect to G and where b is a nonnegative vector with at least one component positive. An n-simplex $\sigma(w^1,\ldots,w^{n+1})$ of G is called completely labelled if the system of linear equations

$$\sum_{i=1}^{n+1} \lambda_i \ell(w^i) = b$$

has a nonnegative solution $\lambda^* = (\lambda_1^*,\ldots,\lambda_{n+1}^*)$.

LEMMA 4.11.2. The point $x^* = \sum_{i=1}^{n+1} \lambda_i^* w^i$ is a fixed point of f if and only if $\sigma(w^1,\ldots,w^{n+1})$ is completely labelled with $\lambda^* = (\lambda_1^*,\ldots,\lambda_{n+1}^*)$ the solution of the system of linear equations.

PROOF. The proof of the lemma is analogous to the proof of lemma 4.8.3. Therefore we only show that $\sum_{i=1}^{n+1} \lambda_i^* = 1$ if λ^* is the solution of the system of linear equations. By summing up over all equations we have that

$$\sum_{i=1}^{n+1} \lambda_i^* \sum_{j=1}^{n+1} (-f_j(w^i) + w_j^i + b_j) = \sum_{j=1}^{n+1} b_j.$$

Hence $\sum_{i=1}^{n+1} \lambda_i^* = 1$ since $\sum_{j=1}^{n+1} f_j(w^i) = \sum_{j=1}^{n+1} w_j^i = 1$ for all i. □

Analogous to the proof of theorem 4.8.4 the next corollary follows immediately.

COROLLARY 4.11.3. Let mesh G<δ and let the mapping ϕ be such that $\phi(y) \in \phi(x) + B^n(0,\varepsilon)$ if $y \in B^n(x,\delta)$. Then $x^* \in \phi(x^*) + B^n(0,\varepsilon)$.

To approximate a fixed point of a mapping on S^n we can use the Sandwich method with a decreasing sequence of grid sizes and restarting in any stage with the approximation of the previous stage or the homotopy method involving an automatically decreasing grid size.
The application of the Sandwich method can be done as follows. As discussed in section 6, $\bar{S}^{n+1}$ is triangulated in the standard way with grid size m^{-1}. Clearly, only simplices are generated in the convex hull of $\bar{S}_{n+2}^{n+1}$ and S_{n+2}^{n+1}. Therefore we consider the corresponding set $S^n \times \lceil 0,1 \rceil$, where $S^n \times \{0\}$ is triangulated with grid size $(m+1)^{-1}$, whereas $S^n \times \{1\}$ is triangulated with grid size m^{-1}. Now we define the mapping $\bar{\phi}(x,t)$ by

$$\bar{\phi}(x,t) = t\phi(x) + (1-t)\{v\}$$

where v is an arbitrarily chosen starting point such that (v,0) is in the interior of an n-simplex of the triangulation of $S^n \times \{0\}$. Clearly (v,0) is a fixed point of $\bar{\phi}(x,t)$. Again a path of fixed points of the piecewise linear approximation $f(x,t)$ to $\bar{\phi}(x,t)$ is traced, starting with (v,0) and terminating with a point on level one.

To apply the homotopy method we triangulate the set $S^n \times [1,\infty)$ in stead of U^{n+1}. This set is triangulated by (n+1)-simplices such that $S^n \times \{2^m\}$ is triangulated by the standard triangulation with grid size 2^{-m}. A path of fixed points of the piecewise linear approximation to $\tilde{\phi}(x,t)$ is traced with $\tilde{\phi}(x,t)$ a correspondence from $S^n \times \{t\}$ to S^n such that $\tilde{\phi}(x,2^m) = f^m(x)$ where $f^m(x)$ is a piecewise linear approximation to $\phi(x)$ with respect to the triangulation of $S^n \times \{2^m\}$ $m=0,1,\ldots$.

4.12. CONCLUDING REMARKS.

All algorithms discussed in this chapter are based on the principle of generating a sequence of adjacent simplices (or primitive sets) until a completely labelled simplex yielding an approximation of a fixed point is found. An algorithm based upon the approximation of a curve by means of numerical solution of an initial value problem was introduced by Kellogg, Li and Yorke [1976] and has since been superceded by the so called continuation methods (see Allgower and Georg [1980]). The most sophisticated simplicial algorithms are the "Sandwich" method and the homotopy algorithm. In both methods the n-dimensional problem is embedded in an (n+1)-dimensional one. The first one has the disadvantage that a layer of artificially labelled points is needed. So with vector labelling a path of fixed points is traced starting with a fixed point of a constant function at each restart. This can be improved by using the approximate Jacobian (see e.g. Todd [1978a]). In the homotopy method the mapping is continuously deformed. Moreover if corrections have to be made, e.g. if on a previous level a completely labelled simplex has been found which corresponds to a point which is nearly but not actually fixed, the simplices generated can move from fine triangulations back to coarser ones. On the other hand the grid size is reduced by a factor of at most two, whereas for the Sandwich method this factor can be chosen arbitrarily. So, in the latter method it

is possible to accelerate the convergence by taking an increasing sequence of factors with which the grid size is reduced. In chapter 8 we present a homotopy algorithm which allows for an arbitrary factor between two subsequent levels.

CHAPTER 5

VARIABLE DIMENSION RESTART ALGORITHM

5.1. INTRODUCTION.

In the previous chapter several algorithms to compute a fixed point
of a function or mapping from S^n into itself were discussed. Two efficient
methods are the restart algorithm of Merrill and Kuhn and MacKinnon, which
is referred to as the "Sandwich" method, and the continuous deformation
algorithm of Eaves (and Saigal), called the homotopy method. In both methods
a path of (n+1)-dimensional simplices is generated.
In this chapter a restart algorithm is presented which does not use a
layer of artificially labelled points (see also Van der Laan and Talman
[1979a,1980c]. Moreover it differs from the algorithms mentioned above by the
fact that it does not generate a path of (n+1)-dimensional simplices but
a path of t-simplices, where t varies between 0 and n. More precisely,
the method starts with a zero-dimensional simplex, called the starting
point, generates a path of adjacent simplices of variable dimension and
terminates with an n-dimensional simplex yielding an approximation of
a fixed point. This algorithm will be referred to as the "variable
dimension restart" algorithm, since the starting point can be arbitrarily
chosen. Therefore the method can be applied for a decreasing sequence of
grid sizes using at any stage the approximation obtained in the previous
stage as the starting point.
In section 2 a full description of the steps of the algorithm for integer
labelling is given. A geometric interpretation of the method is discussed
in section 3 and its convergence is proved in section 4. In section 5 the
algorithm is put in the framework of the above mentioned methods by adding
n+1 differently labelled points on an extra level. The application of the
algorithm for vector labelling is discussed in section 6 and the

generalization for computing a fixed point of a mapping on T^n is
treated in section 7. In particular we give some convergence conditions
in this section. Numerical results are given in section 8.

5.2. DESCRIPTION OF THE STEPS OF THE ALGORITHM.

Assume that we want to approximate a fixed point of a continuous
function f from S^n into itself. Every point of S^n receives an integer
label based on the standard labelling rule on S^n as given in definition
4.2.1. Let S^n be triangulated in the standard way with grid size m^{-1}
(see definition 3.4.1). The amendments required for the T triangulation
are straightforward. In definition 3.4.1 the $(n+1) \times n$ matrix Q is
introduced to describe the simplices of the standard triangulation of S^n,
whereas the simplices generated by the algorithms of chapter 4 were of the
form $\sigma(y^1, \pi(I_k))$. It will appear that the $(n+1) \times n$ matrix Q is not
sufficient to describe the simplices generated by the variable dimension
restart algorithm. Therefore the matrix Q is redefined as the $(n+1) \times (n+1)$
matrix

$$
\Omega = \begin{bmatrix}
-1 & 0 & \cdot & \cdot & \cdot & 0 & 1 \\
1 & \cdot & & & & & 0 \\
0 & \cdot & & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot & & \cdot \\
\cdot & & & \cdot & & & \cdot \\
\cdot & & & & \cdot & -1 & \cdot \\
0 & \cdot & \cdot & \cdot & 0 & 1 & -1
\end{bmatrix} \quad \cdot
$$

So the matrix is extended with the column $q(n+1) = -\sum_{j=1}^{n} q(j) = (1,0,..,0,-1)^T$.
Now each column $q(i)$ of the matrix Q can be interpreted as corresponding
with the i-th direction, $i=1,...,n+1$. This means that a search in the i-th
direction involves a decrease of the i-th component and an increase of the
$(i+1)$-th component with $i+1 = 1$ when $i = n+1$. The algorithm to be described
below builds up the set of labels consecutively. If $T \subset I_{n+1}$ is the set of
labels already found a search is made in the directions i, $i \in T$ to find the
other labels. Recall that by the proper labelling $\ell(x) \neq i$ if $x_i = 0$.
We define now a t-dimensional simplex $\sigma(y^1, \pi(T))$.

DEFINITION 5.2.1. For $t \leq n$, the t-dimensional simplex $\sigma(y^1, \pi(T))$ of the standard triangulation of S^n is the simplex with vertices $y^1, \ldots, y^{t+1}$ such that

      a) y^1 is a grid point of the triangulation

      b) $\pi(T) = (\pi_1, \ldots, \pi_t)$ is a permutation of the elements of T, ($|T|=t$)

      c) $y^{i+1} = y^i + q(\pi_i)/m$ $i=1, \ldots, t$.

Observe that any t-dimensional simplex $\sigma(y^1, \pi(T))$ has a unique representation when $t \leq n-1$, whereas any n-simplex has n+1 representations. Let $\pi(T)$ be a permutation of n elements of the set I_{n+1}. Then $\sigma(y^1, \pi(T)) = \sigma(y^1, \pi^i(T^i))$, where $T^i = I_{n+1} \setminus \{\pi_{i-1}\}$ and $\pi^i(T^i) =$
$= (\pi_i, \ldots, \pi_n, h, \pi_1, \ldots, \pi_{i-2})$, $i=2, \ldots, n+1$, with h the unique element of I_{n+1} not in T.

We give now a generalization of definition 4.4.2.

DEFINITION 5.2.2. A $(t-1)$-simplex σ, $1 \leq t \leq n+1$, is called T-complete if the t vertices of σ carry all the labels of the set T ($|T|=t$).

Note that $T = I_{n+1}$ if $t = n+1$ and that any zero-dimensional simplex $\{w\}$ is $\{\ell(w)\}$-complete. We are now ready to describe the steps of the algorithm. In this description the (n+1)-dimensional vector R represents the "distance" between the starting point v and the vertex y^1 of the last generated simplex $\sigma(y^1, \ldots, y^{t+1})$, i.e. $y^1 = v + \sum_{j=1}^{n+1} R_j q(j)/m$. The algorithm starts with an arbitrary chosen grid point v and proceeds as follows.

Step 0.   Set $t=0$, $y^1=v$, $T=\emptyset$, $\sigma=\sigma^0(y^1, \pi(\emptyset))$, $\bar{y}=y^1$ and $R_j=0$, $j=1, \ldots, n+1$.

Step 1.   Calculate $\ell(\bar{y})$. If $\ell(\bar{y})$ is not an element of T go to step 3. Otherwise, there is exactly one vertex $y^s \neq \bar{y}$ such that $\ell(y^s) = \ell(\bar{y})$.

Step 2.   If $s = t+1$ and $R_{\pi_t} = 0$ go to step 4. Otherwise $\sigma(y^1, \pi(T))$ and R are adapted according to table 5.2.1 by replacing y^s. Return to step 1 with $\bar{y}$ equal to the new vertex of σ.

Step 3. If $t=n$, a completely labelled simplex is found and the algorithm terminates. If $t<n$, a $(Tu\{\ell(\bar{y})\})$-complete simplex is found and the dimension is increased. T becomes $Tu\{\ell(\bar{y})\}$, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_t,\ell(\bar{y}))$, σ becomes $\sigma(y^1,\pi(T))$. Then t becomes $t+1$ and return to step 1 with $\bar{y} = y^{t+1}$.

Step 4. The dimension is decreased by deleting y^s. For exactly one index s', $s'\leq t$ we have that $y^{s'}$ is the vertex of σ with label π_t. Now T becomes $T\setminus\{\pi_t\}$, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{t-1})$, σ becomes $\sigma(y^1,\pi(T))$. Then t becomes $t-1$ and return to step 2 with $s=s'$.

Table 5.2.1. s is the index of the vertex to be replaced.

	y^1 becomes	$\pi(T)$ becomes	R becomes
$s=1$	$y^1+q(\pi_1)/m$	$(\pi_2,\ldots,\pi_t,\pi_1)$	$R+e(\pi_1)$
$2\leq s\leq t$	y^1	$(\pi_1,\ldots,\pi_{s-2},\pi_s,\pi_{s-1},\pi_{s+1},\ldots,\pi_t)$	R
$s=t+1$	$y^1-q(\pi_t)/m$	$(\pi_t,\pi_1,\ldots,\pi_{t-1})$	$R-e(\pi_t)$

The algorithm generates a path of adjacent simplices starting with the zero-dimensional simplex $\{v\}$. First a search is made in the $\ell(v)$-th direction to find the missing labels, i.e. the $\ell(v)$-th component of v is decreased whereas the $(\ell(v)+1)$-th component is increased where $\ell(v)+1=1$ if $\ell(v) = n+1$. So one-dimensional simplices $\sigma(y^1,y^2)$ are generated with $y^1 = v +R_{\ell(v)}q(\ell(v))/m$ and $y^2 = y^1+q(\ell(v))/m$, such that the common zero-dimensional facet is $\{\ell(v)\}$-complete. The algorithm proceeds with two-dimensional simplices as soon as a second label is found, which must occur since $\ell(x)\neq i$ if $x_i=0$. In general if in step 1 a new label is found, say label k, we have for certain t,subset T of I_{n+1}, permutation $\pi(T)$, simplex $\sigma(y^1,\ldots,y^{t+1})$ and vector R generated by the algorithm that

a) $y^1 = v + \Sigma_{j=1}^{n+1} R_j q(j)/m$

b) $R_j = 0$ for $j \notin T$ and $R_j \geq 0$ for $j \in T$

c) $y^{i+1} = y^i + q(\pi_i)/m$, $\quad i=1,\ldots,t$

d) σ is $(Tu\{k\})$-complete.

Then the algorithm terminates with the completely labelled simplex $\sigma(y^1,\ldots,y^{n+1})$ if $t=n$. Otherwise, the dimension is increased by setting $T=Tu\{k\}$ and $t=t+1$, and a search is made with the labels i, $i\in T$ in all the

directions i, i$\in$T, to find the other labels by generating a path of adjacent simplices $\sigma(y^1,\pi(T))$ such that the common facets are T-complete. Doing so we must have that within a finite number of steps either again a new label is found or in step 2, s = t+1 and R_{π_t}=0. In the latter case, by performing a replacement step according to table 5.2.1, R_{π_t} becomes negative indicating that the method searches in the direction $-(\pi_t)$ whereas label π_t already has been found. To avoid this situation the dimension is decreased by setting $T=T\setminus\{\pi_t\}$ and t=t-1. The last vertex is deleted and the vertex with label π_t is removed. In the first case the algorithm terminates or the dimension is again increased. So, the algorithm generates a path of adjacent simplices $\sigma^0,\sigma^1,\dots$ of variable dimension. In section 4 we prove that a completely labelled simplex is found within a finite number of steps. Clearly, using this approximation the algorithm can be restarted with a finer grid, until a given accuracy is obtained. In figure 5.2.1 the method is illustrated for the example of figure 2.4.2. The results of the computation of the labels are indicated in the figure.
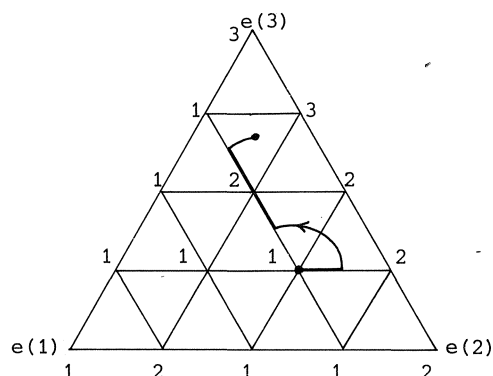
Figure 5.2.1. Illustration of the variable dimension restart method; the starting point v = $(\frac{1}{4},\frac{1}{2},\frac{1}{4})^T$, n=2, m=4.

In this example the algorithm performs the following steps, where $\sigma^0,\sigma^1,\dots,\sigma^6$ is the sequence of generated simplices.

1. Step 0: $t=0$, $u^1=v=(\frac{1}{4},\frac{1}{2},\frac{1}{4})^T$, $T=\emptyset$, $\sigma(y^1,\pi(T))=\sigma^0=\sigma(u^1)$ and $R=(0,0,0)^T$.

2. Step 1: $\ell(u^1)=1\notin T$.

3. Step 3: $T=\{1\}$, $\pi(T)=(1)$, $\sigma(y^1,\pi(T))=\sigma^1=\sigma(u^1,u^2)$ with the new vertex $u^2=u^1+q(1)/4=(0,\frac{3}{4},\frac{1}{4})^T$, $t=1$.

4. Step 1: $\ell(u^2)=2\notin T$.

5. Step 3: $T=\{1,2\}$, $\pi(T)=(1,2)$, $\sigma(y^1,\pi(T))=\sigma^2=\sigma(u^1,u^2,u^3)$ with the new vertex $u^3=u^2+q(2)/4=(0,\frac{1}{2},\frac{1}{2})^T$, $t=2$.

6. Step 1: $\ell(u^3)=2\in T$, $\ell(u^3)=\ell(u^2)$, $u^2=y^2$, i.e. $s=2$.

7. Step 2: the vertex u^2 is replaced, $\pi(T)=(2,1)$, $\sigma(y^1,\pi(T))=\sigma^3=$ $=\sigma(u^1,u^4,u^3)$ with the new vertex $u^4=u^1+q(2)/4=(\frac{1}{4},\frac{1}{4},\frac{1}{2})^T$, $R=(0,0,0)^T$.

8. Step 1: $\ell(u^4)=2\in T$, $\ell(u^4)=\ell(u^3)$, $u^3=y^3$, i.e. $s=3$.

9. Step 2: $s=3=t+1$ and $R_{\pi_t}=R_1=0$.

10. Step 4: The vertex $y^3=u^3$ is deleted, $\ell(u^1)=\pi_2=1$, i.e. $s'=1$; $T=\{2\}$, $\pi(T)=2$, $\sigma(y^1,\pi(T))=\sigma^4=\sigma(u^1,u^4)$, $t=1$, $s=s'=1$.

11. Step 2: the vertex u^1 is replaced, $\pi(T)=2$, $\sigma(y^1,\pi(T))=\sigma^5=\sigma(u^4,u^5)$ with the new vertex $u^5=u^4+q(2)/4=(\frac{1}{4},0,\frac{3}{4})^T$, $R=(0,1,0)^T$.

12. Step 1: $\ell(u^5)=1\notin T$.

13. Step 3: $T=\{1,2\}$, $\pi(T)=(2,1)$, $\sigma(y^1,\pi(T))=\sigma^6=\sigma(u^4,u^5,u^6)$ with the new vertex $u^6=u^5+q(1)/4=(0,\frac{1}{4},\frac{3}{4})^T$, $t=2$.

14. Step 1: $\ell(u^6)=3\notin T$.

15. Step 3: $t=2=n$, a completely labelled simplex is found and the algorithm terminates.

The steps 1-14 are summarized in table 5.2.2.

5.3. GEOMETRIC INTERPRETATION.

In this section a geometric interpretation of the variable dimension restart algorithm is given. Observe again that throughout the algorithm $R_j=0$ if $j\notin T$ and $R_j\geq0$ if $j\in T$. Therefore it is natural to define the regions $A(T)$ for $T\subset I_{n+1}$, $|T|\leq n$, by

$$A(T) = \{x\in S^n \mid x=v+ \sum_{j\in T} \lambda_j q(j) \text{ for nonnegative } \lambda_j, \ j\in T\}.$$

Number	t	T	$\pi(T)$	R^T	σ	$\bar{y}^T$	$\ell(\bar{y})$	s	s'
1,2	0	$\emptyset$		(0,0,0)	$\sigma(u^1)$	$(\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{4})$	1		
3,4	1	{1}	(1)	(0,0,0)	$\sigma(u^1,u^2)$	$(0,\tfrac{3}{4},\tfrac{1}{4})$	2		
5,6	2	{1,2}	(1,2)	(0,0,0)	$\sigma(u^1,u^2,u^3)$	$(0,\tfrac{1}{2},\tfrac{1}{2})$	2	2	
7,8,9	2	{1,2}	(2,1)	(0,0,0)	$\sigma(u^1,u^4,u^3)$	$(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2})$	2	3	
10	1	{2}	(2)	(0,0,0)	$\sigma(u^1,u^4)$			1	1
11,12	1	{2}	(2)	(0,1,0)	$\sigma(u^4,u^5)$	$(\tfrac{1}{4},0,\tfrac{3}{4})$	1		
13,14	2	{1,2}	(2,1)	(0,1,0)	$\sigma(u^4,u^5,u^6)$	$(0,\tfrac{1}{4},\tfrac{3}{4})$	3		

Table 5.2.2. The steps of the variable dimension restart algorithm for the example of figure 5.2.1.

Note that $A(\emptyset) = \{v\}$ and that any point x of S^n is a proper point of just one region $A(T)$ where $x \in A(T)$ is called proper if $x \in \overset{\circ}{A}(T)$ with

$$\overset{\circ}{A}(T) = \{x\epsilon A(T) \mid x=v+ \underset{j\epsilon T}{\Sigma} \lambda_j q(j) \text{ for positive } \lambda_j, \ j\epsilon T\}.$$

So the regions $\overset{\circ}{A}(T)$ partition S^n.

As described in section 2, if a set T, a permutation $\pi(T)$ and a vector R are generated, the current simplex $\sigma(y^1,\ldots,y^{t+1})$ is given by
$y^1 = v+\Sigma_{j=1}^{n+1} R_j q(j)/m$ and $y^{i+1} = y^1+\Sigma_{j=1}^{i} q(\pi_j)/m$, i=1,\ldots,t. Since $R_j=0$ if
$j\notin T$ and $R_j\geq 0$ if $j\epsilon T$, the vertices y^i, i=1,\ldots,t+1, are elements of $A(T)$,
whereas y^{t+1} must be a proper point of $A(T)$. Moreover, if y^i is a proper
point of $A(T)$ then y^j is proper for all j=i+1,\ldots,t+1. More precisely, let
$T(i)$ be the subset of T, such that $y^i\epsilon\overset{\circ}{A}(T(i))$, i=1,\ldots,t+1. Then
$T(1) \subseteq T(2) \subseteq \ldots \subseteq T(t+1) = T$ with $\mid T(i+1)\backslash T(i) \mid$ equal to zero or one.
The algorithm generates t-simplices in $A(T)$ if T is the current set of labels.
So we have the following corollaries.

COROLLARY 5.3.1. Let σ and τ be two adjacent t-simplices in $A(T)$ generated
by the variable dimension restart algorithm. Then the common facet of σ and
τ is T-complete.

COROLLARY 5.3.2. Let T be the current set of labels, $\mid T \mid \leq n$. Then the
variable dimension restart algorithm generates t-simplices in a
t-dimensional linear manifold, being the affine hull of $A(T)$.

The latter corollary emphasizes that the algorithm operates actually with
full-dimensional simplices in a t-dimensional linear manifold.
The dimension is increased if a new label is found and decreased if a
point outside $A(T)$ should be generated and T is the current set of labels.

COROLLARY 5.3.3. Let T be the current set of labels, $\mid T \mid <n$, $\sigma(y^1,\pi(T))$ a
t-simplex in $A(T)$ generated by the algorithm and $j\notin T$ a label just found.
Then $\sigma(y^1,\pi(T))$ is a $(T\cup\{j\})$-complete simplex in $A(T)$ and the dimension is
increased.

COROLLARY 5.3.4. Let T be the current set of labels, $\mid T \mid \leq n$, and $\sigma(y^1,\pi(T))$ a
t-simplex in $A(T)$ generated by the algorithm. If y^{t+1} is the only vertex of σ in
$\overset{\circ}{A}(T)$ and y^{t+1} has to be removed then the facet $\tau(y^1,\ldots,y^t)$ is a T-complete $(t-1)$-
simplex in $A(T\backslash\{k\})$ for some $k\epsilon T$ and the dimension is decreased.
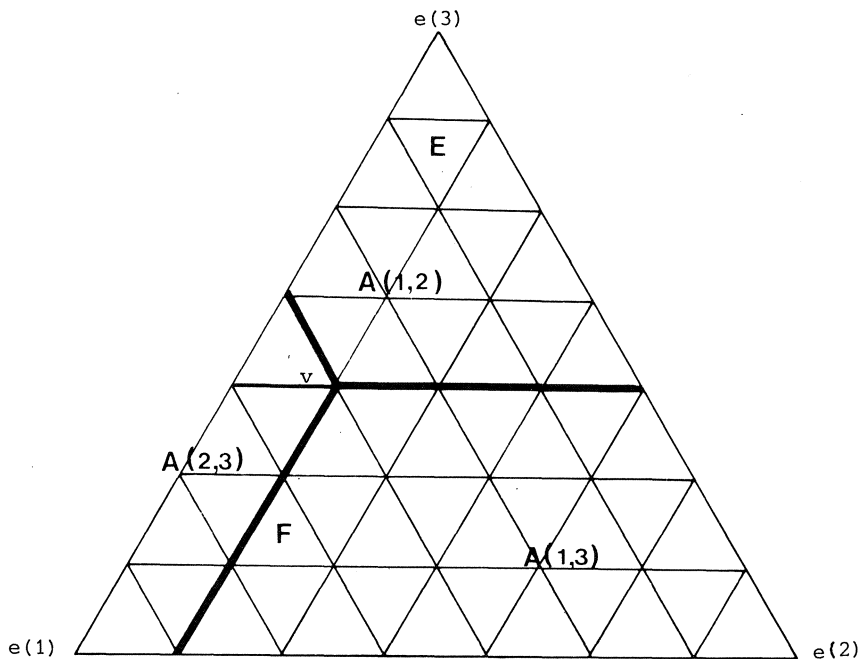
Figure 5.3.1. Geometric interpretation of the variable dimension algorithm.

We illustrate the above mentioned features and corollaries by some examples (see figure 5.3.1). $A(T)$ is denoted by $A(i_1,\ldots,i_t)$ if $T = \{i_1,\ldots,i_t\}$. If the simplex E is generated we have that $R_1=R_2=2$ and $R_3=0$, i.e. $y^1=v+\{2q(1)+2q(2)\}/7$ is a proper point of $A(1,2)$. Consequently $y^2=y^1+q(2)/7$ and $y^3=y^2+q(1)/7$ are also proper in $A(1,2)$. Moreover either two vertices have the same label, belonging to $T=\{1,2\}$, whereas the label of the third vertex is equal to the other element of T, or E is completely labelled. In the first case E has two adjacent simplices such that the common facets are T-complete.

The vertices of the simplex F are $y^1=v+q(3)/7$, $y^2=y^1+q(3)/7$ and $y^3= =y^2+q(1)/7$. If the 1-simplex $\sigma(y^1,y^2)$ in $A(3)$ is generated and σ is $\{1,3\}$-complete then the dimension is increased and the simplex F of $A(1,3)$ is generated. Conversely if the 2-simplex F in $A(1,3)$ is generated and y^3 has to be removed the dimension is decreased since y^3 is the only vertex of F in $\mathring{A}(1,3)$ or equivalently in step 2 we have that $s=3$ and $R_1=R_{\pi_t}=0$. Then the algorithm continues with the one-simplex $\sigma(y^1,y^2)$ of $A(3)$ by replacing the vertex having label 1. Observe that replacing the vertex y^3 of F according to table 5.2.1 implies that a simplex in

$A((T\cup\{2\})\setminus\{1\})=A(2,3)$ should be generated with $T=\{1,3\}$ the current set of labels. To avoid this the dimension is decreased. In general the dimension is decreased if a replacement step according to table 5.2.1 results in generating a simplex not in $A(T)$ but in an "adjacent" region $A((T\cup\{j\})\setminus\{k\})$ for some $j\notin T$ and $k\in T$. Then the algorithm continues with simplices in $A(T\setminus\{k\})$.

We conclude this section with some remarks about the path of simplices generated by the algorithm. Let $C(i)$ be defined by

$$C(i) = \{x\in S^n \mid \ell(x) = i\} \quad i=1,\ldots,n+1.$$

Let $i\in I_{n+1}$ be the index such that the starting point $v\in C(i)$. Then $\ell(v)=i$ and the algorithm generates one-simplices in $A(i)$ until a grid point y is generated such that $y\in C(j)$ for some $j\neq i$. Then $\ell(y)=j$ and the algorithm continues with 2-simplices in $A(i,j)$. Suppose that $\tau(w^1,w^2)$ is a common facet of two adjacent simplices. Then, for some permutation (h_i,h_j) of the set $\{1,2\}, w^{h_i}\in C(i)$ and $w^{h_j}\in C(j)$, i.e. $\ell(w^{h_i})=i$ and $\ell(w^{h_j})=j$. So by definition 4.2.1 we have that

$$f_i(w^{h_i})-w_i^{h_i} \leq f_k(w^{h_i})-w_k^{h_i} \quad k\neq i$$

and

$$f_j(w^{h_j})-w_j^{h_j} \leq f_k(w^{h_j})-w_k^{h_j} \quad k\neq j.$$

So, roughly speaking, the simplex $\tau(w^1,w^2)$ yields an approximation of a point x such that

$$f_i(x) - x_i = f_j(x) - x_j \leq f_k(x) - x_k \quad k\neq i,j.$$

A sequence of approximations of such points is generated until either a point in $C(h)$, $h\neq i,j$, is generated i.e. a new label is found, or a point outside $A(i,j)$ should be generated by replacing a vertex according to table 5.2.1. In general, if T is the current set of labels and $\tau(w^1,\ldots,w^t)$ is a common facet of two adjacent t-simplices generated by the algorithm, for some permutation $(h_j,j\in T)$ of the elements of the set $\{1,\ldots,t\}$ we have

that $w^{h_j} \in C(j)$, $j \in T$. Hence $\tau(w^1, \ldots, w^t)$ yields an approximation of a point x such that

$$f_i(x) - x_i = f_j(x) - x_j \quad \text{for all } i, j \in T$$

and

$$f_i(x) - x_i \leq f_k(x) - x_k \quad \text{for all } i \in T, \ k \notin T.$$

Note that x is such a point if $x \in C(T)$, where $C(T) = \bigcap_{j \in T} \bar{C}(j)$ with $\bar{C}(j)$ the closure of $C(j)$. Observe that it is possible that $C(T) \cap \tau = \emptyset$. Again a sequence of approximations of such points is generated until either a point in $C(h)$, $h \notin T$, is generated or a simplex having a T-complete facet in $A(T \setminus \{k\})$, for some $k \in T$, is generated. In the first case the algorithm continues with simplices in $A(T \cup \{h\})$ and generates a sequence of approximations of points $x \in C(T \cup \{h\})$, whereas in the latter case the algorithm proceeds with simplices in $A(T \setminus \{k\})$, generating a sequence of approximations of points $x \in C(T \setminus \{k\})$. In figure 5.3.2, $\bar{C}(i)$, $i = 1, 2, 3$, and the regions $A(T)$, $T \subset I_{n+1}$, are given. The set S^n is triangulated with grid size 1/9. Only the grid points and the completely labelled simplex generated by the algorithm are given.
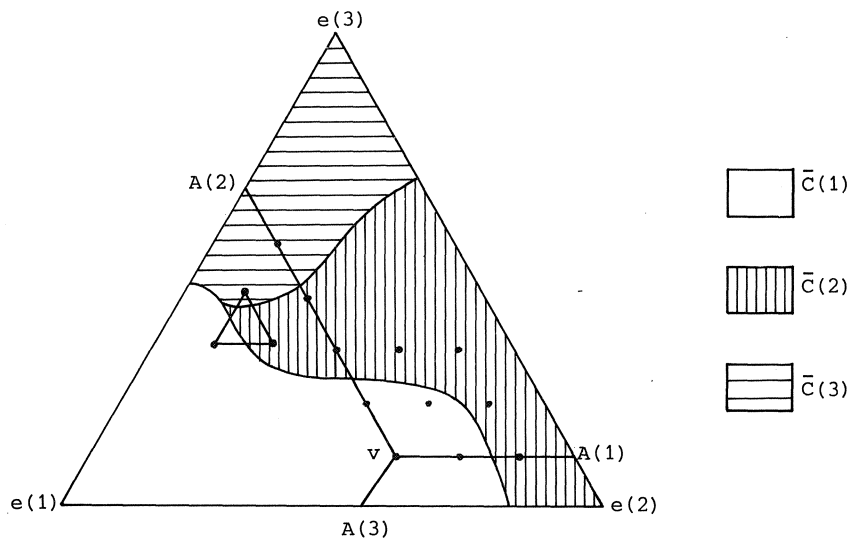

Figure 5.3.2. $n = 2$, $v = (1/3, 5/9, 1/9)^T$, $m = 9$. The dots are the grid points generated by the algorithm.

5.4. CONVERGENCE OF THE ALGORITHM.

In this section it is proved that the variable dimension restart algorithm always succeeds in finding a completely labelled simplex by establishing that all steps are feasible and the algorithm never returns in a simplex visited already before.

First, it is shown that a replacement step according to table 5.2.1 is always feasible. Clearly, if a vertex of the t-dimensional simplex $\sigma(y^1,\ldots,y^{t+1})$ has to be replaced according to table 5.2.1, the replacement step produces again a t-simplex of the triangulation of S^n except when all other vertices are points on some boundary S_j^n. Suppose for some s, $s \leq t+1$, y^s is the vertex to be replaced and assume that, for some j, y^s is the only vertex not on the j-th boundary. Then $y_j^i = 0$ for all $i \neq s$. If the starting point is not on the j-th boundary, j must be an element of T, otherwise the j-th component could not have been decreased. Since the common facet of two adjacent t-simplices is T-complete, one of the vertices y^i must have label j. Because of the proper labelling $\ell(y^i) \neq j$, $i \neq s$. Hence y^s is the only vertex with label j and cannot be replaced. This proves that the replacement step is feasible if the starting point is not on the j-th boundary.

If v is an element of S_j^n, either j is an element of T or it is not. The first case is identical to the case just mentioned. In the other case, s must be equal to t+1, since otherwise $0 = y_j^{s+1} = y_j^s - m^{-1}$ which is only possible if $j \in T$. Moreover we have that $R_{\pi_t} = 0$ since $\pi_t = j-1$ and $y_j^1 = v_j + R_{j-1}/m$ =0. So step 4 should be performed contradicting the fact that y^{t+1} must be replaced according to table 5.2.1. These facts together prove that every replacement step is feasible. Moreover, a vertex obtained in step 3 by extending a t-dimensional simplex to a (t+1)-dimensional simplex ($t \leq n-1$) is a point in S^n. To prove this, let $\sigma(y^1,\ldots,y^{t+1})$ be a t-dimensional $(T \cup \{j\})$-complete simplex in $A(T)$ generated by the algorithm, where $j \notin T$ is the new label. So $\ell(y^s) = j$ for some $1 \leq s \leq t+1$. Because of the proper labelling rule $y_j^s > 0$. Since j is not an element of T, $\pi_h \neq j$ for $h=1,\ldots,t$ implying that y_j^{t+1} is also positive. Hence the j-th component of the new vertex $y^{t+2} = y^{t+1} + q(j)/m$ is nonnegative and therefore y^{t+2} must be a point in S^n. Consequently, all steps in the algorithm are feasible.

Next we prove that the algorithm never generates a simplex visited already before. Let $\sigma^0, \sigma^1, \sigma^2, \ldots$ be the sequence of simplices generated by the algorithm, so $\sigma^0 = \{v\}$. We prove that each generated simplex σ^i, $i \geq 1$, can

be entered from just two adjacent simplices, while σ^0 can be entered from just one adjacent simplex. Then by the standard arguments of section 4.2 the algorithm can never revisite a simplex. For given $i \geq 1$, let σ^i be a t-dimensional simplex in a region $A(T)$. Now two cases can occur.

a) σ^i has just two (t-1)-dimensional T-complete facets
b) σ^i is $(T \cup \{k\})$-complete for some $k \notin T$, and σ^i has exactly one (t-1)-dimensional T-complete facet.

In the first case σ^i has exactly two different adjacent simplices from which it can be entered by the algorithm. Both simplices are simplices in $A(T)$ or one is a simplex in $A(T \setminus \{j\})$ for some $j \in T$ and the other is a simplex in $A(T)$. In the second case σ^i has also exactly two adjacent simplices from which it can be entered, one (t+1)-dimensional simplex in $A(T \cup \{k\})$ whereas the other one is either a simplex in $A(T)$ or in $A(T \setminus \{j\})$ for some $j \in T$. So in all cases, σ^i is entered from the simplex σ^{i-1} and there is just one other simplex σ^{i+1} which can be entered from σ^i. Moreover σ^0 is a zero-dimensional simplex which is $(T \cup \{k\})$-complete with $T = \emptyset$ and $k = \ell(v)$. Since $T = \emptyset$, σ^0 has no T-complete facet. Therefore σ^0 can be entered from just one other simplex being the simplex $\sigma^1 = \sigma(y^1, y^2)$ with $y^1 = v$ and $y^2 = v + q(\ell(v))/m$.

Since the number of t-simplices, $0 \leq t \leq n$, is finite the algorithm must terminate within a finite number of steps with a completely labelled n-simplex.

5.5. INTERPRETATION OF THE ALGORITHM IN $S^n \times [0,1]$.

The variable dimension restart algorithm can be easily put in the framework of other restart algorithms by adding an extra level of points. This interpretation was independently found by Barany [1979a], Van der Laan and Talman [1979a] and Todd [1978b]. In Van der Laan and Talman [1978a] the variable dimension restart algorithm is applied to compute a fixed point of a function in the product space of unit simplices. Interpretations with an extra level of points of this more general case can be found in Van der Laan and Talman [1979c]. An interpretation of a closely related algorithm on R^n (c.f. Van der Laan and Talman [1978b]) was given by Todd and Wright [1979] and Barany [1979b] (see also Talman [1980]).

Using an extra level of points we triangulate $S^n \times [0,1]$, where the extra level $S^n \times \{0\}$ is not triangulated (or triangulated with grid size one) and the natural level $S^n \times \{1\}$ is triangulated with grid size m^{-1}. So, the only grid points on the zero level are the vertices $\hat{e}(i) = (e(i)^T, 0)^T$, $i=1,\ldots,n+1$. Now we give the triangulation.

THEOREM 5.5.1. Let G be the collection of $(n+1)$-simplices obtained by connecting each grid point $x \in \overset{\circ}{A}(T)$ on level one with the vertices $e(i)$ $i \notin T$ on level zero. Then G is a triangulation of $S^n \times [0,1]$.

PROOF. In this proof we use the following notation. A point of $S^n \times [0,1]$ is denoted by (x,t) where $x \in S^n$ and $t \in [0,1]$. The set $E(T)$ denotes the set of vertices $\hat{e}(i)$, $i \notin T$.

We prove now that each point (x,t) lies in a simplex of G and that the intersection of two simplices of G is a face of each. Since the collection of regions $\overset{\circ}{A}(T)$ partition S^n, there is for given (x,t) a unique subset T of I_{n+1}, such that $x \in \overset{\circ}{A}(T)$. Moreover, for any $j \in T$ there are unique numbers α_j and $\lambda_j(k)$, $k \in T \setminus \{j\}$, and nonnegative numbers $\beta_j(i)$, $i \notin T$, such that

$$x = \alpha_j v + \sum_{k \in T \setminus \{j\}} \lambda_j(k) q(k) + \sum_{i \notin T} \beta_j(i) e(i).$$

Since $x \in \overset{\circ}{A}(T)$ we have that $\alpha_j \leq 1$. Let $\bar{T}$ be a subset of T such that

(i) $j \in \bar{T}$ if $\alpha_j < t$

(ii) $j \notin \bar{T}$ if $\alpha_j > t$.

Clearly, the intersection of the affine hull of the elements of $E(\bar{T})$ and (x,t) with the region $\overset{\circ}{A}(\bar{T})$ on the one level is a point, say $(\bar{x},1)$. Let $\sigma(y^1, \pi(\bar{T}))$ be a $\bar{t}$-simplex containing $\bar{x}$. Then (x,t) lies in the $(n+1)$-simplex $\Sigma(\sigma(y^1, \pi(\bar{T})))$ defined as the simplex with vertices the elements of $E(\bar{T})$ and the vertices of σ on the one level. This proves the first part of the lemma.

Next, suppose Σ_1 and Σ_2 are two simplices of G both containing (x,t). Let σ_1 and σ_2 be the corresponding simplices on the one level, i.e. $\sigma_i \times \{1\} = \Sigma_i \cap (S^n \times \{1\})$, $i=1,2$. Then (x,t) is contained in the face with vertices the elements of $E(T^*)$ and the vertices of $\sigma_1 \cap \sigma_2$ on the one-level,

where T^* is the largest subset of T such that (i) and (ii) are satisfied. Clearly, this face is a face of both Σ_1 and Σ_2. $\square$

The triangulation of $S^n \times [0,1]$ is illustrated for $n=1$, $m=8$, $v=(3/8,5/8)^T$ in figure 5.5.1 and for $n=2$, $m=3$ and $v=(1/3,1/3,1/3)^T$ in figure 5.5.2.
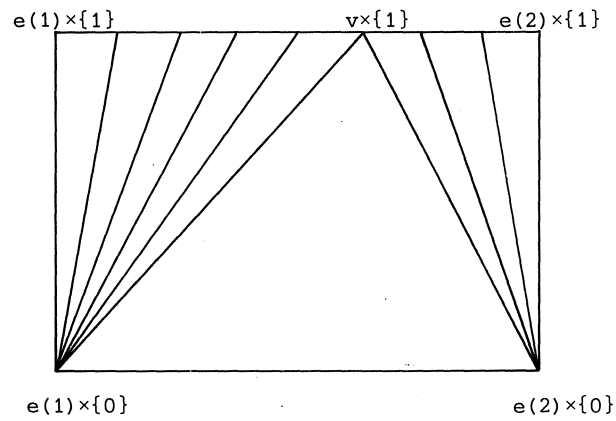

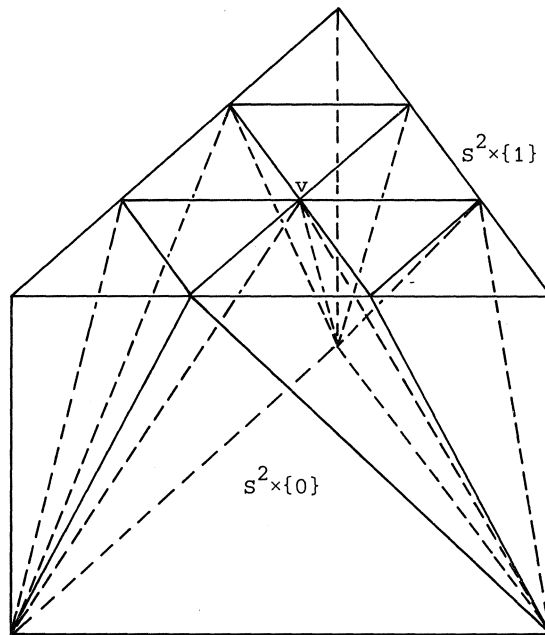Figure 5.5.1. Triangulation of $S^1 \times [0,1]$, $m=8$, $v=(3/8,5/8)^T$.

Figure 5.5.2. Triangulation of $S^2 \times [0,1]$, $m=3$, $v=(1/3,1/3,1/3)^T$.

The grid points of $S^n \times [0,1]$ are labelled according to the standard labelling on S^n, i.e. $\ell(\hat{e}(i))=i$, $i=1,\ldots,n+1$ and $\ell((x,1))=\ell(x)$ for a point $(x,1)$ in $S^n \times \{1\}$. Clearly, $S^n \times \{0\}$ is a completely labelled n-simplex and is a facet of just one (n+1)-simplex of the triangulation of $S^n \times \lceil 0,1 \rceil$. This simplex is the convex hull of $v \times \{1\}$ and $S^n \times \{0\}$. Starting with this simplex a path of adjacent (n+1)-simplices is generated by replacing in every generated simplex the vertex having the same label as the new vertex. Using standard arguments the algorithm must terminate with a simplex having a completely labelled facet on the boundary of $S^n \times \lceil 0,1 \rceil$ and this facet cannot be the facet $S^n \times \{0\}$. Because of the proper labelling there are no completely labelled n-simplices in (bd S^n) $\times \lceil 0,1 \rceil$, i.e. there is no completely labelled n-simplex σ such that, for some i, $x_i=0$ for all $x \in \sigma$. Hence, the algorithm terminates with an (n+1)-simplex having a completely labelled facet in $S^n \times \{1\}$. This facet yields a good approximation of a fixed point. The intersection of the path of (n+1)-simplices with $S^n \times \{1\}$ gives the sequence of simplices of the variable dimension restart algorithm. Observe that when in the variable dimension restart method a new label k is found, the current simplex $\sigma(y^1,\ldots,y^{t+1})$ is extended to the simplex $\sigma(y^1,\ldots,y^{t+2})$ with $y^{t+2}=y^{t+1}+q(k)/m$. In the interpretation with an artificial level the vertex of $S^n \times \{0\}$ having label k, that is the vertex $\hat{e}(k)$, is replaced by y^{t+2}. Also, if in the variable dimension restart algorithm the vertex y^{t+1} of a simplex $\sigma(y^1,\ldots,y^{t+1})$ should be removed while $R_{\pi_t}=0$, the vertex y^{t+1} is deleted and the vertex y^i, $i \leq t$, having label π_t is removed. In the method with the artificial level y^{t+1} on the one-level is replaced by $\hat{e}(\pi_t)$ and in the next step the vertex y^i having label π_t is replaced since $\ell(\hat{e}(\pi_t))=\pi_t$.

Although the variable dimension restart algorithm method can be interpreted as a method with an extra level as just described, this level does not have an artificial influence on the path and is not needed to prove the convergence. Moreover there are only n+1 extra points, whereas in the Sandwich method the number of extra points, which are artificially labelled, is of the order of the number of grid points on the natural level. This means that even if the variable dimension restart method is applied with an extra level, the number of grid points on this level, generated by the algorithm, is very small (in most examples zero). Observe that decreasing the dimension corresponds to generating a grid point on the extra level. In the Sandwich method the number of grid points on the extra

level generated by the algorithm is of the same order as the number of grid points that is generated on the natural level. In particular this is a disadvantage if vector labelling is used since then each grid point generated by the algorithm implies that a linear programming step should be made.

5.6. VECTOR LABELLING.

In this section we discuss the application of the variable dimension restart algorithm with vector labelling. Each point x of S^n receives the standard vector label of definition 4.11.1 with b=e, i.e.

$$\ell(x) = -f(x)+x+e$$

where $f(x)$ is a piecewise linear approximation to the mapping ϕ with respect to the standard triangulation of S^n with grid size m^{-1}.
Let T be a subset of I_{n+1} with $|T|=t$. Then we have the following definition.

DEFINITION 5.6.1. A $(t-1)$-simplex σ with vertices $w^1,\ldots,w^t$ is called T-complete if the system of linear equations

$$\sum_{i=1}^{t} \lambda_i \ell(w^i) + \sum_{h \notin T} \mu_h e(h) = e$$

has a nonnegative solution λ_i^*, $i=1,\ldots,t$ and μ_h^*, $h \notin T$.

Observe that any zero-dimensional simplex $\sigma(w)$ is i_0-complete with i_0 the index such that $\ell_{i_0}(w) = \max_j \ell_j(w)$. The solution of the system is $\lambda_1^* = \{\ell_{i_0}(w)\}^{-1}$ and $\mu_h^* = 1 - \{\ell_h(w)/\ell_{i_0}(w)\}$, $h \neq i_0$. The index i_0 is unique if $\ell_{i_0}(w) > \ell_j(x)$, $j \neq i_0$. If i_0 is not unique there are more solutions and the system of linear equations is degenerated. In the following we assume that degeneracy does not occur (see also Todd [1976a]).

LEMMA 5.6.2. For any T-complete simplex $\sigma(w^1,\ldots,w^t)$ holds

$$(n+1) \sum_{i=1}^{t} \lambda_i^* + \sum_{h \notin T} \mu_h^* = n+1$$

with λ_i^*, $i=1,\ldots,t$; μ_h^*, $h \notin T$, the solution of the system of linear equations.

PROOF. From definition 5.6.1 and the labelling rule we have that

$$\sum_{i=1}^{t} \lambda_i^*(-f(w^i)+w^i+e) + \sum_{h \not\in T} \mu_h^* e(h) = e.$$

Summing up over all components we obtain

$$\sum_{i=1}^{t} \lambda_i^*[\sum_{j=1}^{n+1} \{-f_j(w^i)+w_j^i+1\} + \sum_{h \not\in T} \mu_h^* = n+1.$$

Since, for all i, $\sum_{j=1}^{n+1} f_j(w^i) = \sum_{j=1}^{n+1} w_j^i = 1$, the lemma now follows immediately. $\square$

It is obvious that if $T = I_{n+1}$, a T-complete simplex $\sigma(w^1,\ldots,w^{n+1})$ is completely labelled. Then $\sum_{i=1}^{n+1} \lambda_i^* = 1$ en $\sum_{i=1}^{n+1} \lambda_i^* w^i$ is an approximation of a fixed point of the mapping ϕ (see lemma 4.11.2).

We are now ready to describe the algorithm for vector labelling. Again the algorithm starts with an arbitrarily chosen grid point v of the triangulation of S^n. Moreover the system of linear equations is set equal to $I\mu=e$ with I the n+1 identity matrix. By alternating linear programming and replacement steps a path of adjacent simplices of variable dimension is generated until a completely labelled simplex is found. Formally, the algorithm is described by the following steps.

Step 0. Set t=0, $y^1=v$, $T=\emptyset$, $\sigma=\sigma^0(y^1,\pi(\emptyset))$ and $R_j=0$, j=1,...,n+1.
         Set p=1 and $\mu_h=1$, $h\epsilon I_{n+1}$.
Step 1. Calculate $\ell(y^p)$. Make a linear programming step by bringing $\ell(y^p)$ in the system of linear equations.

$$\sum_{\substack{i=1 \\ i \neq p}}^{t+1} \lambda_i \ell(y^i) + \sum_{h \not\in T} \mu_h e(h) = e.$$

         If e(j) is eliminated for some $j\not\in T$ go to step 3. Otherwise $\ell(y^s)$ is eliminated for just one vertex $y^s \neq y^p$.
Step 2. If s=t+1 and $R_{\pi_t}=0$ go to step 4. Otherwise $\sigma(y^1,\pi(T))$ and R are adapted according to table 5.2.1 by replacing y^s. Return to step 1 with p the index of the new vertex of σ.

Step 3. If $t=n$, a completely labelled simplex is found and the algorithm terminates. If $t<n$, a $(T\cup\{j\})$-complete simplex is found and the dimension is increased. T becomes $T\cup\{j\}$, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_t,j)$, σ becomes $\sigma(y^1,\pi(T))$. Then t becomes $t+1$ and return to step 1 with $p=t+1$.

Step 4. The dimension is decreased. A linear programming step is made by bringing $e(\pi_t)$ in the set of linear equations

$$\sum_{i=1}^{t} \lambda_i \ell(y^i) + \sum_{h\notin T} \mu_h e(h) = e.$$

Either for some $j\notin T$, $e(j)$ is eliminated or for some $s'\leq t$, $\ell(y^{s'})$ is eliminated. Now T becomes $(T\setminus\{\pi_t\})$, $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{t-1})$ and σ becomes $\sigma(y^1,\pi(T))$. Then t becomes $t-1$. Return to step 3 if for some j, $e(j)$ has been eliminated. Otherwise return to step 2 with $s=s'$.

Analogously to the integer labelling case, a search is made in the directions i if the vectors $e(i)$ are eliminated. More precisely if for a certain subset T of I_{n+1} a unit vector $e(j),j\notin T$, is eliminated in step 1, we have that for the permutation $\pi(T)$ and the simplex $\sigma(y^1,\pi(T))$ generated by the algorithm the conditions a-d of section 2 are satisfied. Then, if $t<n$, T is set equal to $(T\cup\{j\})$ and a search is made in the directions i, $i\in T$, by generating a path of simplices in $A(T)$ with T-complete common facets until either again a unit vector is eliminated or a simplex having a T-complete facet in $A(T\setminus\{k\})$ is generated for some $k\in T$. In the latter case a linear programming step is made with $e(k)$. Note that a unit vector can be eliminated by another one, implying that the algorithm continues with simplices in an "adjacent" equal-dimensional region, which behaviour is not possible in case of integer labelling.

We prove now that the algorithm must find a completely labelled simplex. First we show that the algorithm never returns to a simplex generated already before. Let σ_i, $i\geq1$ be a t-dimensional simplex of the sequence σ_1, $\sigma_2,\ldots$ of generated simplices in the region $A(T)$, $T\neq\emptyset$. Then either

a) σ_i has just two (t-1)-dimensional T-complete facets

or

b) σ_i has just one (t-1)-dimensional T-complete facet and σ_i is $(T\cup\{k\})$-complete for some k not in T.

As in the case of integer labelling, in both cases σ_i can be entered from only two adjacent simplices. Moreover the starting simplex $\sigma(v)$ can be entered from only one adjacent simplex being $\sigma(y^1, y^2)$ where $y^1 = v$ and $y^2 = v + q(i_0)/m$ if $e(i_0)$ is eliminated by $\ell(v)$ in step 1. Together these observations prove that the algorithm can never return to a previously generated simplex.

Next we prove that all steps are feasible. Clearly, all linear programming steps are feasible. Next we show that a replacement step according to table 5.2.1 is feasible. Let $\sigma(y^1, \ldots, y^{t+1})$ be a simplex generated by the algorithm and let y^s, $1 \leq s \leq t+1$, be the vertex to be replaced. This step is feasible unless for some $j \in I_{n+1}$, $y^1, \ldots, y^{s-1}, y^{s+1}, \ldots, y^{t+1}$ are elements of S_j^n and $y_j^s > 0$.

THEOREM 5.6.3. Let $\sigma(y^1, \ldots, y^{t+1})$ be a t-dimensional simplex such that for some j $y_j^i = 0$ for $i \neq s$. If $v_j \neq 0$ the system of linear equations

$$\sum_{\substack{i=1 \\ i \neq s}}^{t+1} \lambda_i \ell(y^i) + \sum_{h \notin T} \mu_h e(h) = e$$

has no feasible solution.

PROOF. If the system of linear equations has a feasible (nondegenerated) solution λ_i^*, $i \neq s$, and μ_h^*, $h \notin T$, we have from the proof of lemma 5.6.2 that

$$(n+1) \sum_{\substack{i=1 \\ i \neq s}}^{t+1} \lambda_i^* + \sum_{h \notin T} \mu_h^* = n+1 \ ,$$

implying that $\sum_{\substack{i \neq s}} \lambda_i^* < 1$. Since the starting point v is not on the j-th boundary of S^n, j must be an element of T, i.e. $e(j)$ is eliminated already. So, the j-th equation of the system yields

$$\sum_{\substack{i=1 \\ i \neq s}}^{t+1} \lambda_i^* \ell_j(y^i) = \sum_{\substack{i=1 \\ i \neq s}}^{t+1} \lambda_i^* (-f_j(y^i) + y_j^i + 1) = 1.$$

Since $y_j^i = 0$ and $f_j(y^i) \geq 0$, this implies that $\sum_{\substack{i \neq s}} \lambda_i^* \geq 1$ which gives a contradiction. $\square$

The theorem proves that a vertex has not to be replaced if all other vertices are   on a same boundary given that the starting point is not

on this boundary. If, however, $v_j=0$ either $j\epsilon T$ or $j\notin T$. The first case is identical to the case just discussed. As shown for integer labelling the latter case implies that $s=t+1$ and $R_{\pi_t}=0$, so that y^{t+1} has to be deleted and a linear programming step has to be made with $e(\pi_t)$. Hence all replacement steps according to table 5.2.1 are feasible. Finally we prove that the extension to a higher dimensional simplex is always feasible. Let for some $j\notin T$, $\sigma(y^1,\ldots,y^{t+1})$ be a t-dimensional $(T\cup\{j\})$-complete simplex of $A(T)$. Since $j\notin T$, $\pi_h\neq j$, $h=1,\ldots,t$. Hence if $y_j^{t+1}=0$, $y_j^i=0$ for all i. Then analogous to the proof of theorem 5.6.3, $e(j)$ could not have been eliminated. Hence, $y_j^{t+1}>0$ and the new vertex $y^{t+2}=y^{t+1}+q(j)/m$ is a grid point of S^n. Consequently, all steps are feasible.

By the fact that all steps are feasible and that the algorithm can never return to a previously generated simplex, a completely labelled simplex must be found, since the number of simplices is finite.

The variable dimension algorithm applied with vector labelling generates a path of simplices of variable dimension such that if a simplex of $A(T)$ is generated we have that the unit vectors $e(j)$, $j\epsilon T$, have been eliminated. This means that the common facet of two adjacent simplices in $A(T)$, generated by the algorithm, is T-complete. Let $\tau(w^1,\ldots,w^t)$ be such a common facet of two adjacent simplices in $A(T)$. Since τ is T-complete we have that the system of linear equations

$$(5.6.1) \qquad \sum_{i=1}^{t} \lambda_i \ell(w^i) + \sum_{h\notin T} \mu_h e(h) = e$$

has a nonnegative solution λ_i^*, $i=1,\ldots,t$, μ_h^*, $h\notin T$. From lemma 5.6.2 we have that $\sum_{i=1}^{t} \lambda_i^* <1$. Define $\alpha=(1-\sum_{i=1}^{t}\lambda_i^*)/\sum_{i=1}^{t}\lambda_i^*$ and define $\bar{\lambda}_i=\lambda_i^*/\sum_{i=1}^{t}\lambda_i^*$. Clearly $\alpha>0$ and $\sum_{i=1}^{t}\bar{\lambda}_i=1$.

From (5.6.1) it follows that for $j\epsilon T$ holds

$$\sum_{i=1}^{t} \lambda_i^* \ell_j(w^i) = \sum_{i=1}^{t}\lambda_i^*\{-f_j(w^i)+w_j^i+1\} = 1$$

or equivalently

$$\sum_{i=1}^{t} \bar{\lambda}_i \{-f_j(w^i)+w_j^i\} = \alpha.$$

Since f is a piecewise linear approximation to the mapping ϕ we have that

(5.6.2) $f_j(\bar{x}) - \bar{x}_j = -\alpha$ $j \in T$

with $\bar{x} = \Sigma_{i=1}^t \bar{\lambda}_i w^i$. For $h \notin T$ the h-th equation of (5.6.1) can be rewritten as

$$\sum_{i=1}^t \bar{\lambda}_i \{-f_h(w^i) + w_h^i\} = \alpha - \mu_h^* / \Sigma_{i=1}^t \lambda_i^*.$$

Since $\mu_h > 0$ it follows that

(5.6.3) $f_h(\bar{x}) - \bar{x}_h > -\alpha$ $h \notin T$.

So by generating a sequence $\sigma_1, \sigma_2, \ldots$ of simplices in $A(T)$ the variable dimension restart algorithm traces a path of points $x \in A(T)$ such that for some $\alpha(x)$ (5.6.2) and (5.6.3) hold, until $f_h(x) - x_h$ becomes equal to $\alpha(x)$ for some $h \notin T$, i.e. $e(h)$ is eliminated, or the path crosses a region $A(T \backslash \{k\})$ for some $k \in T$ in which case a linear programming step is made with $e(k)$. Recall from the end of section 3 that there is an analogous behaviour in case of integer labelling.

The interpretation with an extra level with n+1 grid points is straight-forward if the vertex $\hat{e}(i)$ of $S^n \times \{0\}$ is labelled with the vector $e(i)$ $i = 1, \ldots, n+1$. Then the method starts with the (n+1)-simplex being the convex hull of $S^n \times \{0\}$ and $v \times \{1\}$. The facet $S^n \times \{0\}$ of this simplex is completely labelled with $\mu_j^* = 1$, $j = 1, \ldots, n+1$, as the solution of the system of linear equations. By alternating linear programming and replacement steps a sequence of (n+1)-simplices is generated such that the common facets are completely labelled. This sequence terminates with an (n+1)-simplex having a completely labelled facet in $S^n \times \{1\}$. In Van der Laan and Talman [1979c] and Talman [1980] it is proved that the method can be seen as a method which traces a path of zeroes of a piecewise linear approximation $f(x,t)$ to a mapping $\bar{\phi}(x,t)$ with respect to the triangulation of $S^n \times [0,1]$, where $\bar{\phi}(x,t)$ is such that

$$\bar{\phi}(x,0) = \{(n+1)x - e\} \text{ and } \bar{\phi}(x,1) = \{x\} - \phi(x).$$

It is not necessary to initiate the variable dimension restart algorithm with the system of linear equations $I\mu = e$. Another possibility is to start the algorithm with the system of linear equations $A\mu = e$ where $a(i) = \ell(e(i)) = e(i) - \tilde{f}(e(i)) + e$ and $\tilde{f}$ is a linear approximation to the

mapping ϕ with respect to the standard triangulation with grid size 1. Clearly this system has a unique solution since $\tilde{f}$ has a fixed point on S^n (see lemma 4.11.2).

5.7. CONVERGENCE CONDITIONS FOR MAPPINGS ON T^n.

In the previous sections we discussed the variable dimension restart algorithm to approximate a fixed point of a function or mapping from the compact convex unit simplex into itself. The generalization of the method to compute a fixed point of a function or mapping from T^n to itself can easily be done. Of course T^n can be triangulated by the standard triangulation as defined in definition 3.4.3. However, here we use the T triangulation given in section 3.4 which seems to be much more appropriate to be used in fixed point algorithms. So T^n is triangulated according to the T triangulation with grid size δ for some positive number δ and with respect to some point $z \in T^n$. In the following z will be taken as the starting point v, which can be arbitrarily chosen. Analogously to section 2 we redefine the matrix T as the $(n+1) \times (n+1)$ matrix

$$T = \begin{bmatrix} -n & 1 & . & . & . & 1 \\ 1 & -n & & & & . \\ . & & . & & & . \\ . & & & . & & . \\ . & & & & . & 1 \\ 1 & . & . & . & 1 & -n \end{bmatrix}.$$

The algorithm starts in v and operates exactly in the same way as the variable dimension restart method on S^n does, except that q(i) is changed into t(i), $i=1,\ldots,n+1$. So, a path of adjacent simplices of variable dimension is generated such that the common facet of two adjacent t-simplices σ_1 and σ_2 generated by the algorithm is T-complete if σ_1 and σ_2 are two simplices in A(T). The regions A(T), $T \subset I_{n+1}$, are redefined by

$$A(T) = \{x \in T^n | x = v + \sum_{j \in T} \lambda_j t(j) \text{ for nonnegative } \lambda_j, j \in T\}.$$

Observe that the regions A(T) depend on the underlying triangulation. Again the dimension is increased if a $(T \cup \{j\})$-complete simplex in A(T) is generated and the dimension is decreased if a simplex in A(T) is generated

having a T-complete facet in $A(T\setminus\{k\})$ for some $k\epsilon T$. Of course
all steps are feasible. Moreover a simplex can only be visited once. Hence
the sequence of simplices either terminates with a completely labelled
simplex or goes to infinity. First we give sufficient conditions to
guarantee that the algorithm terminates in case of integer labelling.
Define for $\mu < 1/(n+1)$, $\bar{B}^n(\mu)$ by

$$\bar{B}^n(\mu) = \{x\epsilon T^n \mid \min_{i\epsilon I_{n+1}} x_i = \mu\}.$$

<u>THEOREM 5.7.1</u>. Let f be a continuous function on T^n and let v be the
starting point. Assume there exist numbers $\mu < \min_i v_i$ and $\epsilon > 0$ such that for
all $x\epsilon\bar{B}^n(\mu)$ there exists an index $i\epsilon I_{n+1}$ such that

(5.7.1) $x_i < v_i$ and $f_j(x) - x_j < f_i(x) - x_i - \epsilon$

for at least one index $j\epsilon I_{n+1}$, $j\neq i$. Then the variable dimension restart
algorithm applied with integer labelling and using the T-triangulation
terminates with a completely labelled simplex if the grid size is small
enough.

<u>PROOF</u>. Let the grid size be so small that

(5.7.2) $\sup_{x,y\epsilon\sigma} \max_{i\epsilon I_{n+1}} \left| f_i(x) - x_i - f_i(y) + y_i \right| < \tfrac{1}{2}\epsilon$

for every simplex σ such that $\sigma\cap\bar{B}^n(\mu)\neq\emptyset$. For some T, let $\sigma(w^1,\ldots,w^{t+1})$ be a
t-dimensional simplex in $A(T)$ having points in common with $\bar{B}^n(\mu)$. Then for
any point $x\epsilon\bar{B}^n(\mu)\cap\sigma(w^1,\ldots,w^{t+1})$ there is an index i such that (5.7.1) is
satisfied for some index $j\neq i$. Hence with (5.7.2) it follows that

$$f_j(w^k) - w_j^k < f_i(w^k) - w_i^k \qquad k=1,\ldots,t+1.$$

So $\ell(w^k)\neq i$ for all k. Moreover, since $x_i < v_i$ we must have that $i\epsilon T$.
Consequently the simplex σ cannot be generated by the algorithm since if so,
$i\epsilon T$ implies that at least one vertex w^k must have label i. Hence the
algorithm can only generate simplices in the interior of $\text{conv}(\bar{B}^n(\mu))$. Since
this set is compact, the number of simplices meeting this set is finite,
which proves the theorem.   □

Observe that since the T triangulation is used, $x_i < v_i$ implies $i \in T$.
In general this holds for any triangulation such that the diagonal
elements of the triangulation matrix are negative and the off-diagonal
elements are nonnegative.

The condition of theorem 5.7.1 depends on the starting point v. Hence it is
not sure that the condition is again satisfied if the algorithm is
restarted with the last found approximation as the new starting point.
Conditions to guarantee that at any stage a completely labelled simplex
is found are given in the next theorem.

THEOREM 5.7.2. Let f be a continuous function on T^n. Assume there exist
numbers $\mu < 1/(n+1)$ and $\varepsilon > 0$ such that for all $x \in \bar{B}^n(\mu)$ we have that for at
least one index i with $x_i = \mu$,

$$f_j(x) - x_j < f_i(x) - x_i - \varepsilon$$

for at least one index $j \neq i$. Then, for any starting point v in the interior of
$\text{conv}(\bar{B}^n(\mu))$, the variable dimension restart algorithm applied with integer
labelling and using the T triangulation terminates with a completely
labelled simplex if the grid size is small enough.

PROOF. Let the grid size be so small that

$$\sup_{x,y \in \sigma} \max_{i \in I_{n+1}} \left| f_i(x) - x_i - f_i(y) + y_i \right| < \tfrac{1}{2}\varepsilon$$

for every simplex σ such that $\sigma \cap \bar{B}^n(\mu)$ is nonempty. Then the proof of
theorem 5.7.1 holds by noting that the condition of theorem 5.7.1. is
implied by the condition of theorem 5.7.2.   □

Clearly if theorem 5.7.2 holds the algorithm terminates at any stage if

    a) the grid size in the first stage is small enough and the
       sequence of grid sizes is decreasing

    b) the starting point in the first stage is chosen in the
       interior of the convex hull of $\bar{B}^n(\mu)$

    c) the starting point in any other stage is chosen within the
       completely labelled simplex found in the previous stage.

In the next theorem a condition is given such that the algorithm
converges for every grid size and for any starting point.

THEOREM 5.7.3. Let f be a continuous function on T^n. Assume that there
exists a number $\mu < 1/(n+1)$ such that for all x with $x_i < \mu$ for at least one
index $i \in I_{n+1}$, there exists an index $j \notin K$ such that

$$f_j(x) - x_j < f_h(x) - x_h \text{ for all } h \in K,$$

where K is the set of indices i with $x_i < \mu$. Then the variable dimension
restart algorithm applied with integer labelling and using the T
triangulation terminates with a completely labelled simplex for every grid
size and for any starting point.

PROOF. Let β be the mesh of the triangulation. Define μ' by

$$\mu' = \min(\mu, \min_{i \in I_{n+1}} v_i) - 2\beta$$

Let $\sigma(w^1, \ldots, w^{t+1})$ be a t-simplex of $A(T)$ meeting $\bar{B}^n(\mu')$. Then there is an
index h such that $x_h \leq \mu' + \beta$ for all $x \in \sigma$. Hence $w_h^i \leq \mu' + \beta < \mu$ for all $i = 1, \ldots, t+1$.
Consequently, using the assumptions of the theorem, $\ell(w^i) \neq h$ for all i.
Moreover we have that $x_h \leq \mu' + \beta < v_h$ for all $x \in \sigma$ which implies that $h \in T$.
Therefore the simplex σ cannot be generated by the algorithm, which proves
that only simplices in the interior of $\text{conv}(\bar{B}^n(\mu'))$ are generated. $\square$

Analogous theorems can be given if vector labelling is used. We only give a
theorem analogously to theorem 5.7.1. Theorems similar to the theorems
5.7.2 and 5.7.3 can be easily formulated.

THEOREM 5.7.4. Let f be a continuous function on T^n and let v be the
starting point. Then the algorithm applied with vector labelling terminates
under the same conditions as formulated in theorem 5.7.1.

PROOF. Let the grid size be so small that

$$(5.7.3) \qquad \sup_{x,y \in \sigma} \max_{i \in I_{n+1}} |f_i(x) - x_i - f_i(y) + y_i| < \tfrac{1}{2}\varepsilon$$

for every simplex σ such that $\sigma \cap \bar{B}^n(\mu) \neq \emptyset$.

For some T, let $\sigma(w^1,\ldots,w^{t+1})$ be a simplex in $A(T)$ such that $\sigma \cap \bar{B}^n(\mu) \neq \emptyset$. Let $x \in \bar{B}^n(\mu) \cap \sigma$. Then from (5.7.1) and (5.7.3) it follows that there exists an index i and an index $j \neq i$ such that $x_i < v_i$ and

$$\bar{f}_j(w^k) - w^k_j < \bar{f}_i(w^k) - w^k_i \quad k=1,\ldots,t+1$$

Hence $\bar{f}_i(y) - y_i > \bar{f}_j(y) - y_j$ for all $y \in \sigma(w^1,\ldots,w^{t+1})$, where $\bar{f}$ is the piecewise linear approximation to f with respect to the triangulation. As argued in the previous section the algorithm traces a path of points y such that for some $\alpha(y)$

$$\bar{f}_h(y) - y_h = -\alpha(y) \quad \text{if } h \in T$$

and

$$\bar{f}_h(y) - y_h > -\alpha(y) \quad \text{if } h \notin T.$$

So if $\sigma(w^1,\ldots,w^{t+1})$ is generated by the algorithm we must have that $i \notin T$ since $\bar{f}_i(y) - y_i > \bar{f}_j(y) - y_j$ for all $y \in \sigma$. On the other hand we have $x_i < v_i$ implying that $i \in T$. Hence a contradiction is obtained and consequently σ cannot be generated, which proves the theorem. $\quad\square$

As argued in section 4.8, condition (5.7.3) cannot be guaranteed for any grid size if we deal with an u.s.c. mapping ϕ instead of a function. Then we need e.g. that, for some $\mu < \min_i v_i$ and some $0 < \delta < \min_i v_i - \mu$, (5.7.1) holds for any $y \in T^n \cap B^{n+1}(x,\delta)$ and for any $f(y) \in \phi(y)$. Then the algorithm converges if the mesh of the T triangulation is less than δ.

In the next theorem a condition to guarantee the convergence of the algorithm in case of an u.s.c. mapping is stated, which is comparable with Merrill's condition given in lemma 2.6.2. Recall that in the theorems 5.7.1-5.7.4 there was a weak dependency of the proof on the underlying triangulation. In the proof of the next theorem this dependency is much stronger. In particular it will appear that the proof is not valid if the standard triangulation is used.

THEOREM 5.7.5. Let ϕ be an u.s.c. mapping on T^n. For some $\delta>0$, let there exist $w \in T^n$ and $\mu<1/(n+1)$ such that for all $x \in T^n$ with $\min\limits_{i \in I_{n+1}} x_i<\mu$ and all $z \in T^n \cap B^{n+1}(x,\delta)$ holds

$$(x-h(x))^T(w-z) < 0$$

for all $h(x) \in \phi(x)$. Then the variable dimension restart algorithm applied with vector labelling and using the T triangulation terminates with a completely labelled simplex if the mesh of the triangulation is less then δ.

PROOF. Let v be an arbitrarily chosen starting point. Define μ' by

$$\mu'=\min\{\mu-\delta, \min_{i \in I_{n+1}} v_i + \min_{i \in I_{n+1}} n(v_i-w_i)\}.$$

For some T, let $\sigma(w^1,\ldots,w^{t+1})$ be a t-simplex in $A(T)$ such that for some $y \in \sigma$, $\min\limits_{i \in I_{n+1}} y_i<\mu'$. We prove that the system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \ell(w^i) + \sum_{h \notin T} \mu_h e(h) = e$$

does not have a nonnegative solution λ_i^*, $i=1,\ldots,t+1$ and μ_h^*, $h \notin T$. Suppose there exists such a solution. Then

$$(5.7.4) \qquad (n+1) \sum_{i=1}^{t+1} \lambda_i^* + \sum_{h \notin T} \mu_h^* = n+1.$$

Furthermore, we have that

$$(5.7.5) \qquad [\sum_{i=1}^{t+1} \lambda_i^* \ell(w^i) + \sum_{h \notin T} \mu_h^* e(h)]^T (w-y) = e^T(w-y) = 0$$

since $w,y \in T^n$. We obtain a contradiction to (5.7.5) by proving that $\ell(w^i)^T(w-y)<0$ for all i and $e(h)^T(w-y)<0$ for all $h \notin T$ since by (5.7.4) at least one λ_i^* or μ_h^* is positive. Let f be the piecewise linear approximation to ϕ with respect to the triangulation. Clearly $f(w^i) \in \phi(w^i)$, $i=1,\ldots,t+1$. Let k be the index such that $y_k = \min\limits_{i \in I_{n+1}} y_i$. Then $w_k^i \le y_k+\delta<\mu'+\delta\le\mu$, which implies that

$$\ell(w^i)^\top(w-y) = (w^i - f(w^i) + e)^\top(w-y) = (w^i - f(w^i))^\top(w-y) < 0 \quad i=1,\dots,t+1.$$

It remains to prove that $w_h - y_h < 0$ for all $h \notin T$. First note that

$\min_{i \in I_{n+1}} (v_i - w_i) \le 0$. Hence $y_k = \min_{i \in I_{n+1}} y_i < \mu' \le \min_{i \in I_{n+1}} v_i + \min_{i \in I_{n+1}} n(v_i - w_i) \le v_k$ implying that

$k \in T$ by the structure of the T triangulation. Moreover it follows that

$$y_k < v_k + n(v_h - w_h) \quad \text{for all } h \notin T$$

whereas, again by the structure of the T triangulation

$$(5.7.6) \qquad y_h \ge v_h + n^{-1}(v_k - y_k) \quad \text{for all } h \notin T.$$

Both inequalities together give

$$y_h > v_h - (v_h - w_h) = w_h \quad \text{for all } h \notin T,$$

and a contradiction to (5.7.5) is obtained. Hence the algorithm can only generate simplices in $\mathrm{conv}(\bar{B}^n(\mu'))$. $\square$

Observe that (5.7.6) does not hold if the standard triangulation is used. Only in the case that $v=w$ we obtain immediately $y_h \ge w_h$, $h \notin T$, which is sufficient to obtain a contradiction to (5.7.5). On the other hand the theorem can be proved for any triangulation such that the diagonal elements of the matrix are negative and the off diagonal elements positive, by taking μ' large enough.

Clearly the condition "for some δ" can be strengthened to "for all δ". Then the algorithm converges for every grid size.

5.8. COMPUTATIONAL RESULTS.

The variable dimension restart algorithm is applied for vector labelling to three examples in which the equilibrium price vector in a pure exchange economy has to be computed. The data of these problems can be found in Scarf [1967b]. The algorithm is applied with the standard triangulation as well as the T triangulation. However, in stead of the

matrix Ω (T) we use the matrix $-\Omega(-T)$. In addition, a point x outside the interior of S^n receives the label e(i) with i the smallest index such that $x_i \leq 0$, whereas a point x$\in$int S^n receives the label $\ell(x)$ defined by

$$\ell_j(x) = g_j(x)/a_j \qquad j=1,\ldots,n+1,$$

where $g_j(x)$ is the total demand of good j, given that x is the price vector, and a_j is the total supply of good j. It can easily be seen that a completely labelled simplex yields a good approximation of a fixed point. Let $w^*(m)$ be the approximate fixed point in stage m, i.e.

$$w^*(m) = \sum_{i=1}^{n+1} \lambda_i^m w^i(m)$$

where $w^1(m),\ldots,w^{n+1}(m)$ are the vertices of the completely labelled simplex generated by the algorithm in stage m and $\lambda_1^m,\ldots,\lambda_{n+1}^m$ the corresponding solution to the system of linear equations. Then the starting point in stage m+1 is $w^*(m)$, whereas the starting point in the first stage is $w^*(0)$ defined by $w_i^*(0)=(n+1)^{-1}$, $i\in I_{n+1}$. Finally we have to make a remark about the sequence of grid sizes. Let h_m be defined by

$$h_m = \max_{i\in I_{n+1}} (w_i^*(m)-w_i^*(m-1)) \qquad m=1,2,\ldots$$

and let δ_m and G_m be the grid size and the triangulation in stage m. Then

$$\delta_{m+1} = h_m \delta_m \sqrt{n+1}/\text{mesh } G_m \quad \text{if } h_m<(\text{mesh } G_m)/2\sqrt{n+1}$$

$$\delta_{m+1} = \tfrac{1}{2}\delta_m \qquad\qquad \text{if } h_m\geq(\text{mesh } G_m)/2\sqrt{n+1}.$$

The results are given in the tables 5.8.1-5.8.3. In all examples the number of function evaluations is equal to the number of linear programming steps, i.e. the dimension is never decreased.

Table 5.8.1. Pure exchange economy, $n=4$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max_{i \in I_{n+1}} |(g_i-a_i)/a_i|$.

Stage	T-triangulation			Standard triangulation		
	M	N	E	M	N	E
1	.051	7	1.02	.093	8	1)
2	.026	13	.18	.047	16	.21
3	.013	18	.09	.023	26	.15
4	.006	24	.04	.012	31	.05
5	$1.2 \ 10^{-3}$	30	$2 \ 10^{-3}$	$5.4 \ 10^{-3}$	36	10^{-2}
6	$6.1 \ 10^{-4}$	36	$8 \ 10^{-5}$	$1.0 \ 10^{-3}$	47	$4 \ 10^{-4}$
7	$3.8 \ 10^{-5}$	43	$2 \ 10^{-7}$	$2.2 \ 10^{-4}$	54	$3 \ 10^{-6}$
8	$1.5 \ 10^{-6}$	50	$3 \ 10^{-11}$	$8.4 \ 10^{-6}$	63	10^{-9}

1) $w^*(1) \notin \text{int } S^n$.

Table 5.8.2. Pure exchange economy, $n=7$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max_{i \in I_{n+1}} |(g_i-a_i)/a_i|$.

Stage	T-triangulation			Standard triangulation		
	M	N	E	M	N	E
1	.032	11	1.32	.055	9	.89
2	.016	20	.22	.019	38	.26
3	.008	28	.12	.009	49	.09
4	$3.2 \ 10^{-3}$	42	.03	$4.2 \ 10^{-3}$	66	.02
5	$1.6 \ 10^{-3}$	59	$5 \ 10^{-3}$	$1.2 \ 10^{-3}$	79	10^{-3}
6	$4.1 \ 10^{-4}$	79	$2 \ 10^{-4}$	$2.7 \ 10^{-4}$	95	$2 \ 10^{-5}$
7	$7.2 \ 10^{-5}$	100	$2 \ 10^{-6}$	$1.8 \ 10^{-5}$	111	$3 \ 10^{-8}$
8	$3.2 \ 10^{-6}$	121	$8 \ 10^{-10}$	$3.2 \ 10^{-7}$	127	$< 10^{-13}$

Table 5.8.3. Pure exchange economy, $n=9$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max\limits_{i\in I_{n+1}} \left|(g_i-a_i)/a_i\right|$.

Stage	T triangulation			Standard triangulation		
	M	N	E	M	N	E
1	.025	12	.30	.045	10	.59
2	.013	26	.18	.011	65	.04
3	.006	42	.06	.006	84	.01
4	.003	58	.01	$8.7 \cdot 10^{-4}$	110	$3 \cdot 10^{-4}$
5	$8.7 \cdot 10^{-4}$	75	$8 \cdot 10^{-4}$	$2.1 \cdot 10^{-4}$	136	$4 \cdot 10^{-6}$
6	$2.1 \cdot 10^{-4}$	91	10^{-5}	$6.3 \cdot 10^{-6}$	169	10^{-9}
7	$1.4 \cdot 10^{-5}$	108	$2 \cdot 10^{-8}$	$7.0 \cdot 10^{-8}$	205	$< 10^{-13}$
8	$2.5 \cdot 10^{-7}$	124	$< 10^{-13}$			

CHAPTER 6

VARIABLE DIMENSION RESTART ALGORITHM ON R^n

6.1. INTRODUCTION.

In this chapter we discuss the application of the variable dimension restart algorithm to compute a fixed point of a function or mapping from R^n into itself. This chapter is based on Van der Laan and Talman [1979b,1980c]. Again the algorithm starts with a zero-dimensional simplex and generates a path of adjacent t-dimensional simplices, $0 \leq t \leq n$. This path either terminates with an n-dimensional simplex yielding a good approximation of a fixed point or goes to infinity. Before we treat the application on the unbounded region R^n, we first describe in section 2 the application of the algorithm on the unit cube.
Conditions for the convergence of the method on R^n are given in section 3. Further research about the variable dimension algorithm is summarized in section 4. Computational results are given in section 5.

6.2. THE UNIT CUBE.

To approximate a fixed point of a function or mapping on the n-dimensional unit cube $C^n = \{x \in R^n, \ 0 \leq x_i \leq 1, \ i=1,\ldots,n\}$, this set is triangulated by the K triangulation with grid size m^{-1}, where m is a positive integer, and z=0 (see definition 3.2.1). We first describe the algorithm for integer labelling. We call a labelling rule proper if $\ell(x) \neq i$ if $x_i=1$ and $\ell(x) \neq n+1$ if $x_i=0$ for some index $i \in I_{n+1}$. The standard integer labelling on R^n defined in definition 4.2.3 induces a labelling on C^n. We now take this labelling rule and assume it is a proper labelling rule as is the case when f is a function from C^n into the interior of C^n. The triangulation matrix corresponding to the K triangulation is the n×n identity matrix. To apply the variable dimension restart algorithm we define the n×(n+1) matrix U by $U=(I,-e)$. So $u(i)=e(i)$, $i=1,\ldots,n$ and

$u(n+1)=-e=-\sum_{i=1}^{n}u(i)$, where $u(i)$ is the i-th column of U. As in section 5.2 $u(i)$ corresponds to the i-th direction in which a search is to be made if label i has been found. For a given subset T of I_{n+1}, $|T|\leq n$, the t-simplex $\sigma(y^1,\pi(T))$ is the simplex with vertices $y^1,\ldots,y^{t+1}$ such that y^1 is a grid point, $\pi(T)$ a permutation of the elements of T and $y^{i+1}=y^i+u(i)/m$, $i=1,\ldots,t$. Observe that such a simplex is indeed a face of a simplex of the K triangulation. Again a (t-1)-simplex σ is called T-complete if the t vertices of σ carry all the labels of the set T.

Let the grid point v be an arbitrarily chosen starting point. Then the variable dimension restart algorithm starts with the zero dimensional simplex $\{v\}$ and proceeds according to the steps 0-4 of section 5.2 except that the columns of the matrix Q in table 5.2.1 must be changed into the columns of U. Again we can give an interpretation of the method by defining the region $A(T)$ as

$$A(T) = \{x\in C^n | x=v+ \sum_{j\in T} \lambda_j u(j) \text{ for nonnegative } \lambda_j, \ j\in T\},$$

where T is a proper subset of I_{n+1}. Then we have that a path of simplices of variable dimension is generated such that the common facet of two adjacent t-simplices in $A(T)$ is T-complete. Moreover, the dimension is increased if, for some $j\notin T$, a $(T\cup\{j\})$-complete simplex in $A(T)$ is generated whereas the dimension is decreased if, for some $k\in T$, a simplex in $A(T)$ is generated having a T-complete facet in $A(T\setminus\{k\})$. In the first case the algorithm continues with a simplex in $A(T\cup\{j\})$, whereas in the latter case the method proceeds with a simplex in $A(T\setminus\{k\})$. The algorithm is illustrated in figure 6.2.1 for n=2 and m=7. In this example we have that τ_1 is a $\{1,2\}$-complete one-simplex of $A(1)$. So the dimension is increased and the label of the new vertex u is calculated. The one-simplex τ_2 is an $\{1,2\}$-complete facet of σ_1 in $A(\{1,2\}\setminus\{1\})$. The dimension is decreased and the vertex w having label 1 is replaced. The simplex τ_3 is a $\{2,3\}$-complete facet in $A(2)$ and the dimension is again increased. Then the algorithm continues with 2-simplices in $A(2,3)$ until the completely labelled simplex σ_2 is found.
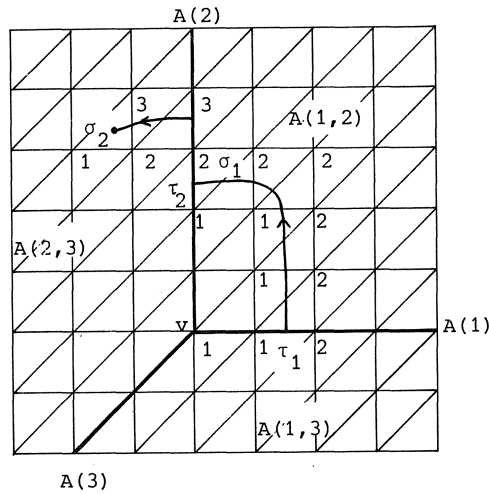
Figure 6.2.1. Illustration of the variable dimension restart algorithm on C^n, $n=2$, $m=7$, $v=(3/7,2/7)^T$.

The proof that the algorithm can never return in a simplex is identical to the proof given in section 5.4. Therefore the algorithm finds always a completely labelled simplex if all steps are feasible. First we show that a replacement step is always feasible. Let $\sigma(y^1,\ldots,y^{t+1})$ be a t-simplex in $A(T)$ generated by the algorithm and let y^s be the vertex of σ to be replaced. Clearly this replacement step is feasible except when for some j ($1\leq j\leq n$) either $y^s_j=m^{-1}$ and $y^i_j=0$, $i\neq s$, or $y^s_j=1-m^{-1}$ and $y^i_j=1$, $i\neq s$. Suppose the first case is true. If $v_j>0$, we must have that n+1 is an element of T otherwise the j-th component could not have been decreased. However by the proper labelling $\ell(y^i)\neq n+1$ for all $i\neq s$, whereas y^s has to be replaced iff the facet with vertices y^i, $i\neq s$, is T-complete. So a contradiction is obtained. This proves that if y^s has to be replaced there exists an index i, $i\neq s$, such that $y^i_j>0$. Hence the replacement step is feasible.

If $v_j=0$ then either $n+1\epsilon T$ which is identical to the case just mentioned, or $n+1\not\epsilon T$. Then, however, $s=t+1$ and $R_{\pi_t}=0$ implying that the dimension should be decreased and contradicting the fact that y^s must be replaced according to table 5.2.1.

Suppose now that $y^i_j=1$ for all $i\neq s$. If $v_j<1$ we must have $j\epsilon T$. Again because of the proper labelling $\ell(y^i)\neq j$ for all $i\neq s$. Hence, if y^s has to be

replaced there exists an index i such that $y_j^i < 1$ implying that the replacement step is feasible. If $v_j = 1$, either $j \in T$ and then there is an index i with $y_j^i < 1$, or $j \notin T$ again implying that the dimension should be decreased. So all replacement steps are feasible.

Next we prove that the extension from a t to a (t+1)-simplex, occurring when a new label is found, is also feasible. So, let $\sigma(y^1, \pi(T))$ be a $(T \cup \{j\})$-complete simplex in $A(T)$, generated by the algorithm. Since $j \notin T$ we have that $\pi_i \neq j$, $i = 1, \ldots, t$. Moreover we have that for some vertex y^s, $\ell(y^s) = j$. If $j = n+1$ then because of the proper labelling $y_h^s \geq m^{-1}$ for all h. Hence for all $i \geq s$, $y_h^i \geq m^{-1}$ for all h, since $\pi_i \neq n+1$ for all i. Consequently, $0 \leq y_h^{t+2} = y_h^{t+1} - m^{-1} \leq 1$ for all h, i.e. y^{t+2} is a point in C^n. If $j \neq n+1$ then again because of the proper labelling $y_j^s \leq 1 - m^{-1}$. This implies that $y_j^{t+1} \leq 1 - m^{-1}$, since $\pi_i \neq j$, $i \geq s$. Hence $1 \geq y_j^{t+2} = y_j^{t+1} + m^{-1} \geq 0$. Moreover $y_h^{t+2} = y_h^{t+1}$ for all $h \neq j$, so that y^{t+2} is a point in C^n. Consequently all steps of the algorithm are feasible.

We now discuss for vector labelling the application of the algorithm to approximate a fixed point of an u.s.c. mapping ϕ. Each point x of C^n receives a vector label $\ell(x)$ based on the rule given in definition 4.8.1. Taking $b = e$, ℓ is a function from C^n to R^{n+1} with

$$\ell_i(x) = f_i(x) - x_i + 1 \qquad i = 1, \ldots, n$$

and

$$\ell_{n+1}(x) = 1,$$

where $f(x)$ is a piecewise linear approximation to the mapping ϕ with respect to the triangulation of C^n. Analogously to theorem 5.6.1 a $(t-1)$-simplex $\sigma(w^1, \ldots, w^t)$ is called T-complete if the system of $n+1$ linear equations

$$\sum_{i=1}^{t} \lambda_i \ell(w^i) + \sum_{h \notin T} \mu_h e(h) = e$$

has a nonnegative solution λ_i^*, $i = 1, \ldots, t$ and μ_h^*, $h \notin T$. In the following we assume that degeneracy does not occur. The next corollary follows immediately by considering the (n+1)-th equation of the system of linear equations.

COROLLARY 6.2.1. For some T, let $\sigma(w^1, \ldots, w^t)$ be a T-complete simplex with λ_i^*, $i = 1, \ldots, t$ and μ_h^*, $h \notin T$ the solution of the system of linear equations. If $n+1 \notin T$, then

$$\sum_{i=1}^{t} \lambda_i^* + \mu_{n+1}^* = 1.$$

If $n+1 \in T$, then

$$\sum_{i=1}^{t} \lambda_i^* = 1.$$

From this corollary it follows immediately that $\sum_{i=1}^{n+1} \lambda_i^* = 1$ if $\sigma(w^1, \ldots, w^{n+1})$ is completely labelled, i.e. if σ is an I_{n+1}-complete simplex. Then $x^* = \sum_{i=1}^{n+1} \lambda_i^* w^i$ is a good approximation of a fixed point (see lemma 4.8.3).

The steps of the algorithm are identical to those of section 5.6 except that the matrix U must be used in stead of the matrix Q. So, starting with an arbitrarily chosen grid point $v \in C^n$, the algorithm generates a path of adjacent simplices. As in section 5.6 a simplex can never be revisited, so a completely labelled n-simplex must be found provided that the steps are feasible. Since C is bounded, all linear programming steps are feasible. Concerning the other steps we first prove that the extension to a higher dimensional simplex is always feasible. For some T, let $\sigma(y^1, \pi(T)$ be a $(T \cup \{h\})$-simplex in $A(T)$ generated by the algorithm. First consider the case where $h=n+1$. Then $e(n+1)$ is eliminated by the last linear programming step and it follows from corollary 6.2.1 that $\sum_{i=1}^{t+1} \lambda_i^* = 1$. We shall now verify that $y_j^{t+1} > 0$ for all j. To do this, suppose to the contrary that $y_j^{t+1} = 0$ for some j. Since $n+1 \notin T$, we have that $\pi_i \neq n+1$, $i=1, \ldots, t$. Hence $y_j^i = 0$ for all i, implying that $\pi_i \neq j$, $1 \leq i \leq t$. So $j \notin T$ and the j-th equation of the system of linear equations is

$$\sum_{i=1}^{t+1} \lambda_i^* \ell_j(y^i) + \mu_j^* = 1$$

with $\mu_j^* > 0$. Since $y_j^i = 0$, $\ell_j(y^i) \geq 1$ for all i. Consequently $\sum_{i=1}^{t+1} \lambda_i^* < 1$ which contradicts that $\sum_{i=1}^{t+1} \lambda_i^* = 1$. Hence, if $h=n+1$, $y_j^{t+1} > 0$ for all j and so $y^{t+2} = y^{t+1} + u(n+1)/m = y^{t+1} - e/m \geq 0$.

Next we prove that the extension is feasible if $h \neq n+1$. Now we verify that $y_h^{t+1} < 1$. Again we suppose to the contrary that $y_h^{t+1} = 1$. Since $h \notin T$, $\pi_i \neq h$ for $i=1, \ldots, t$ implying that $y_h^i = 1$ for all i and therefore $n+1 \notin T$. So from corollary 6.2.1 we obtain that $\sum_{i=1}^{t+1} \lambda_i^* = 1 - \mu_{n+1}^* < 1$. However, since σ is $(T \cup \{h\})$-complete, $e(h)$ is eliminated and the h-th equation is

$$\sum_{i=1}^{t+1} \lambda_i^* \ell_h(y^i) = 1.$$

Since $f(y^i) \epsilon c^n$, $\ell_h(y^i) \leq 1$, $i=1,\ldots,t+1$. Hence $\sum_{i=1}^{t+1} \lambda_i^* \geq 1$, contradicting the assumption that $y_h^{t+1}=1$. Consequently $y_h^{t+1}<1$ and so $y^{t+2}=y^{t+1}+e(h)/m$ is a point in c^n.

To prove that all replacement steps are feasible we assume that piecewise linear approximation f to ϕ is a continuous function from c^n into the interior of c^n. For some T, let $\sigma(y^1,\ldots,y^{t+1})$ be a t-simplex in $A(T)$ generated by the algorithm and let y^s be the vertex to be replaced according to table 5.2.1. This step is feasible unless either

or

a) for some j, $y_j^i=0$ for all $i \neq s$ and $y_j^s=m^{-1}$

b) for some j, $y_j^i=1$ for all $i \neq s$ and $y_j^s=1-m^{-1}$.

We prove that in both cases the system of linear equations

$$(6.2.1) \qquad \sum_{i \neq s} \lambda_i \ell(y^i) + \sum_{h \notin T} \mu_h e(h) = e$$

has no feasible solution. Suppose to the contrary that the system has a feasible solution λ_i^*, $i \neq s$ and μ_h^*, $h \notin T$. Consider case a) with $v_j>0$. Then $n+1 \epsilon T$ and so by corollary 6.2.1 $\sum_{i \neq s} \lambda_i^*=1$. The j-th equation of the system (6.2.1) is given by

$$(6.2.2) \qquad \sum_{i \neq s} \lambda_i^* \ell_j(y^i) + \bar{\mu}_j = 1$$

with $\bar{\mu}_j=\mu_j^*$ if $j \notin T$ and $\bar{\mu}_j=0$ if $j \epsilon T$. Since f is a function from c^n into the interior of c^n we have that $\ell_j(y^i)=f_j(y^i)-y_j^i+1>1$. Hence from (6.2.2) it follows that $\sum_{i \neq s} \lambda_i^*<1$ which contradicts $\sum_{i \neq s} \lambda_i^*=1$. So there is no feasible solution if $v_j>0$. Consider now the case with $v_j=0$. If $n+1 \epsilon T$, we can repeat the above arguments. If $n+1 \notin T$ it follows that $s=t+1$ and $\pi_t=j$. Hence $R_{\pi_t}=R_j=y_j^1-v_j=0$, implying that step 4 should be performed.
In case b) we distinguish between $v_j<1$ and $v_j=1$. If $v_j<1$ we have that $j \epsilon T$. Hence the j-th equation of (6.2.1) is

$$\sum_{i \neq s} \lambda_i \ell_j(y^i) = 1$$

with $\ell_j(y^i) < 1$ for all $i \neq s$ implying that $\sum_{i \neq s} \lambda_i^* > 1$.

However from corollary 6.2.1 it follows that $\sum_{i \neq s} \lambda_i^* \leq 1$, which again gives a

contradiction. Finally if $v_j = 1$ we have that, either $j \epsilon T$ and we can

repeat the arguments as for $v_j < 1$, or $s = t+1$ and $R_{\pi_t} = 0$, implying that step 4

should be performed. Hence, if f is a function from C^n into the interior of

C^n, all replacement steps are feasible and the algorithm terminates with

a completely labelled simplex within a finite number of iterations.

6.3. CONVERGENCE CONDITIONS ON R^n.

In this section the variable dimension restart algorithm is applied

to compute a fixed point of a function or mapping on R^n. Conditions are

given to guarantee that the algorithm converges. For some positive δ, R^n is

triangulated by AK_δ where A is a triangulation matrix (c.f. section 3.3),

such that the arbitrarily or by priory information chosen starting point v

is a grid point of the triangulation. It seems to be adequate to use an

element of the subclass A of AK triangulations in fixed point algorithms.

Adding the i-th column of a matrix of the form $A(\alpha)$ to a point y of R^n

means that only the i-th component of y is increased, whereas all other

components are affected by the same amount. So in going from y^1 to y^{n+1}

where y^i, $i = 1, \ldots, n+1$, are the vertices of a simplex $\sigma(y^1, \pi(I_n))$ of the

triangulation, all components have been increased by the same amount.

Note that in all the algorithms discussed in chapter 4 only the columns

of the triangulation matrix (if AK is the underlying triangulation) appear

in the rules for the replacement steps. However, the variable dimension

restart algorithm uses the extra column $a(n+1) = -\sum_{i=1}^{n} a(i)$. So using an

$A(\alpha)K_\delta$ triangulation a search in the i-th direction, $i \leq n$, means that the

i-th component is increased and all other components are decreased

with the same amount, whereas a search in the (n+1)-th direction means that

all components are decreased with the same amount. In section 3.3 we

summarized some nice properties of the $A(n+\sqrt{n+1})K_\delta$ triangulation. These

properties suggested that it is preferable to use this element of the

subclass A.   The application of the variable dimension restart algorithm

provides an extra argument. Recall that the regions $A(T)$ depend on the

underlying triangulation. So, using the $A(n+\sqrt{n+1})K_\delta$ triangulation the

n-dimensional real space R^n is subdivided in subsets $A(T)$, $T \subset I_{n+1}$ and $|T| \leq n$, defined by

$$A(T) = \{x \in R^n \mid x = v + \sum_{j \in T} \lambda_j a(j) \text{ for nonnegative } \lambda_j, \ j \in T\},$$

where $a(i)$ is the i-th column of the $n \times (n+1)$ matrix $A(n+\sqrt{n+1})$ defined by

$$A(n+\sqrt{n+1}) = \begin{bmatrix} n+\sqrt{n+1} & -1 & . & . & . & -1 & -1-\sqrt{n+1} \\ -1 & . & & & & . & . \\ . & & . & & & . & \\ . & & & . & & . & . \\ . & & & & -1 & & . \\ -1 & . & . & . & -1 & n+\sqrt{n+1} & -1-\sqrt{n+1} \end{bmatrix}.$$

Consider the n-dimensional subsets $A(T_k)$ with $T_k = I_{n+1} \setminus \{k\}$, $k=1,\ldots,n+1$. Then the $A(n+\sqrt{n+1})K_\delta$ triangulation has the property that it divides R^n in $n+1$ n-dimensional subsets $A(T_k)$ of "equal size". With subsets of equal size we mean the following. Recall that $B^n(v,\mu)$ is a ball with v as centre point and μ as length of the radius. Then for any (positive) μ the volume of the intersection of $B^n(v,\mu)$ and $A(T_k)$ is equal for all $k \in I_{n+1}$. This property follows immediately from the fact that, as pointed out by Todd [1978f], the angle between two half lines $v+\lambda a(k)$, $\lambda \geq 0$, is equal for any two indices k_1, k_2. So within a ball with the starting point v as centre point the number of n simplices in $A(T_k)$ is roughly the same for all $k \in I_{n+1}$. This is illustrated in figure 6.3.1 for $n=2$. In general we have that for all μ the number of t-simplices in $B^n(v,\mu) \cap A(T)$ is the same for every T with $|T|=t$.

To apply the variable dimension algorithm, in case of integer labelling any point x receives the standard label of definition 4.2.3, whereas in case of vector labelling each point receives a label as defined in definition 4.8.1 with b equal to e. The algorithm is started in v and proceeds in the same way as on T^n with in the replacement steps the n+1 columns of the matrix defined above in stead of the columns of T. A path of adjacent simplices is generated such that the common facet of two adjacent simplices in a region $A(T)$ is T-complete. Using standard arguments we have that the path of simplices either terminates with a completely labelled simplex or goes to infinity. Before we give some convergence theorems we consider the behaviour of the path. We restrict ourselves to vector labelling.
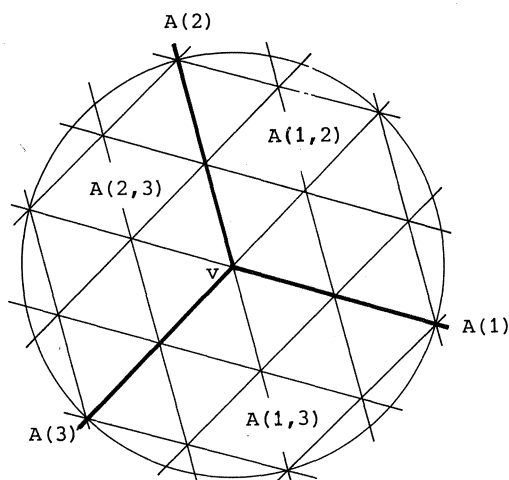
Figure 6.3.1. Illustration of the properties of the $A(n+\overline{\sqrt{n+1}})K_\delta$ triangulation, $n=2$, $\mu=2\delta$.

For some T, let $\tau(w^1,\ldots,w^t)$ be the common facet of two adjacent simplices generated by the algorithm. Then we have that τ is T-complete and so the system of linear equations

$$\sum_{i=1}^{t} \lambda_i \ell(w^i) + \sum_{h \notin T} \mu_h e(h) = e$$

has a nonnegative solution λ_i^*, $i=1,\ldots,t$ and μ_h^*, $h\notin T$. First suppose that $n+1\notin T$. Then it follows from corollary 6.2.1 that $\sum_{i=1}^{t}\lambda_i^*<1$. Following the arguments of section 5.6 we have that

(6.3.1) $f_j(\bar{x})-\bar{x}_j=\alpha$ $j\epsilon T$

and

(6.3.2) $f_h(\bar{x})-\bar{x}_h<\alpha$ $h\notin T$

where $\alpha=(1-\sum_{i=1}^{t}\lambda_i^*)/\sum_{i=1}^{t}\lambda_i^*$ and $\bar{x}=\sum_{i=1}^{t}\bar{\lambda}_i w^i$ with $\bar{\lambda}_i=\lambda_i^*/\sum_{i=1}^{t}\lambda_i^*$. Hence if $n+1\notin T$ the variable dimension restart algorithm traces a path of points $x\epsilon A(T)$ such that for some $\alpha(x)$ (6.3.1) and (6.3.2) hold. Observe that the number of equations of (6.3.1) is equal to $|T|=t$, whereas α depends on x. Suppose now that $n+1\epsilon T$. Then we have that $\sum_{i=1}^{t}\lambda_i^*=1$. Hence for $j\epsilon T$, $j\neq n+1$ we have that

$$\sum_{i=1}^{t} \lambda_i^* \{f_j(w^i) - w_j^i\} = 0$$

or

$$(6.3.3) \qquad f_j(x^*) - x_j^* = 0 \qquad j \in T \setminus \{n+1\}$$

with $x^* = \sum_{i=1}^{t} \lambda_i^* w^i$.   Moreover for $h \notin T$ it follows that

$$\sum_{i=1}^{t} \lambda_i^* \{f_h(w^i) - w_h^i\} + \mu_h^* = 0$$

or

$$(6.3.4) \qquad f_h(x^*) - x_h^* < 0 \qquad h \notin T.$$

So a path of points $x \in A(T)$ is traced such that (6.3.3) and (6.3.4) hold. Observe that for each x the right hand side of (6.3.3) is equal to zero, but that the number of equations in (6.3.3) is equal to $|T| - 1 = t - 1$.

We give now some theorems concerning the convergence of the algorithm on R^n. First we consider integer labelling. Define $\tilde{B}^n(\mu)$ by

$$\tilde{B}^n(\mu) = \{x \in R^n \mid \max_{i \in I_n} |y_i| = \mu\} \quad .$$

THEOREM 6.3.1. Let f be a continuous function on R^n and let v be the starting point. Assume there exist numbers $\mu > \max_i |v_i|$ and $\varepsilon > 0$ such that for all $x \in \tilde{B}^n(\mu)$ holds

    a) $e^T x < e^T v$ and there is at least one index j such that

        $f_j(x) - x_j > \varepsilon$

or

    b) there exists an index $i \in I_n$ such that $x_i > v_i$, and we have that

        (i) $f_j(x) - x_j > f_i(x) - x_i + \varepsilon$ for at least one index $j \in I_n$, $j \neq i$

    or (ii) $f_j(x) - x_j < -\varepsilon$ for all $j \in I_n$.

Then the variable dimension restart algorithm applied with integer labelling terminates with a completely labelled simplex for any triangulation $A(\alpha)K_\delta$ if δ is small enough.

PROOF. Let the grid size be so small that

(6.3.5) $\sup\limits_{x,y\in\sigma} \max\limits_{i\in I_n} |f_i(x) - x_i - f_i(y) + y_i| < \tfrac{1}{2}\varepsilon$

for every simplex σ such that $\sigma\cap\tilde{B}^n(\mu)\neq\emptyset$. For some T, let $\sigma(w^1,\ldots,w^{t+1})$ be a simplex in $A(T)$ such that $\sigma\cap\tilde{B}^n(\mu)\neq\emptyset$. Then for any point $x\in\tilde{B}^n(\mu)\cap\sigma$ holds at least one of the conditions a) and b). If a) holds we must have that $n+1\in T$ since $e^T x < e^T v$. So if σ is generated by the algorithm $\ell(w^i)=n+1$ for at least one index $i=1,\ldots,t+1$. However, there is at least one index j such that $f_j(x) - x_j > \varepsilon$. So with (6.3.5) it follows that

$$f_j(w^k) - w^k_j > 0 \quad k=1,\ldots,t+1,$$

implying that $\ell(w^k)\neq n+1$ for all k. So a contradiction is obtained. If b) holds we must have that $i\in T$ since $x_i > v_i$. If there exists an index $j\neq i$ such that $f_j(x) - x_j > f_i(x) - x_i + \varepsilon$ it follows from (6.3.5) that

$$f_j(w^k) - w^k_j > f_i(w^k) - w^k_i \quad k=1,\ldots,t+1,$$

implying that $\ell(w^k)\neq i$ for all k. If for all $j\in I_n$, $f_j(x) - x_j < -\varepsilon$ then we have that

$$f_j(w^k) - w^k_j < 0 \quad k=1,\ldots,t+1,$$

implying that $\ell(w^k)=n+1$ for all k. In both cases $\sigma(w^1,\ldots,w^{t+1})$ cannot be generated by the algorithm. Therefore the algorithm generates only simplices in the interior of $\mathrm{conv}(\tilde{B}^n(\mu))$. This proves the theorem. $\square$

Theorem 6.3.1 is the analogon of theorem 5.7.1. Theorems analogous to the theorems 5.7.2 and 5.7.3 can easily be given for any element of the subclass A. Here we give only the analogon of theorem 5.7.2. Moreover we restrict ourselves to the K_δ triangulation.

THEOREM 6.3.2. Let f be a continuous function on R^n. Assume there exist numbers $\mu>0$ and $\varepsilon>0$ such that for all $x\in\tilde{B}^n(\mu)$ holds that

a) For some i, $x_i = -\mu$ and there exists at least one index $j \in I_n$
such that $f_j(x) - x_j > \epsilon$.

or

b) For some i, $x_i = \mu$ and $f_j(x) - x_j > f_i(x) - x_i + \epsilon$ for at least one
index $j \in I_n$, $j \neq i$, or $f_j(x) - x_j < -\epsilon$ for all $j \in I_n$.

Then the variable dimension restart algorithm applied with integer
labelling and using the K_δ triangulation terminates with a completely
labelled simplex for any starting point v in the interior of $\text{conv}(\tilde{B}^n(\mu))$ if
δ is small enough.

PROOF. For some T, let $\sigma(w^1, \ldots, w^{t+1})$ be a simplex in $A(T)$ and $\sigma \cap \tilde{B}^n(\mu) \neq \emptyset$. It
can easily be seen that, if the grid size is small enough, there exists an index
$i \in T$ such that $\ell(w^k) \neq i$, $k = 1, \ldots, t+1$. Hence only simplices in the interior of
$\text{conv}(\tilde{B}^n(\mu))$ can be generated. $\square$

We conclude this chapter with two convergence theorems on R^n when vector
labelling is used. They are analogous to the theorems 5.7.4 and 5.7.5
concerning the convergence on T^n.

THEOREM 6.3.3. Let f be a continuous function on R^n and let v be the
starting point. Then the algorithm applied with vector labelling terminates
under the same conditions as formulated in theorem 6.3.1.

PROOF. Let the grid size be so small that

(6.3.6) $$\sup_{(x,y) \in \sigma} \max_{i \in I_{n+1}} |f_i(x) - x_i - f_i(y) + y_i| < \tfrac{1}{2}\epsilon$$

for every simplex σ such that $\sigma \cap \tilde{B}^n(\mu) \neq \emptyset$. For some T, let $\sigma(w^1, \ldots, w^{t+1})$ be
a simplex in $A(T)$ such that $\sigma \cap \tilde{B}^n(\mu) \neq \emptyset$. Let x be a point such that
$x \in \tilde{B}^n(\mu) \cap \sigma$. If a) holds we have that $n+1 \in T$. Moreover with (6.3.6) it
follows that $f_j(w^k) - w_j^k > 0$, $k = 1, \ldots, t+1$. Hence $\overline{f}_j(y) - y_j > 0$ for all $y \in \sigma$, where
$\overline{f}$ is the piecewise linear approximation to f with respect to the triangu-
lation. Therefore, by (6.3.3) or (6.3.4) σ cannot be generated by the

algorithm. If b) holds we have that $i \in T$. If bi) holds it follows that $\bar{f}_j(y) - y_j > \bar{f}_i(y) - y_i$ for all $y \in \sigma$ and by (6.3.1) or (6.3.2), σ cannot be generated. If bii) holds it follows that for all $y \in \sigma$, $\bar{f}_j(y) - y_j < 0$ for all $j \in I_n$. In particular we have that $\bar{f}_i(y) - y_i < 0$ and by (6.3.1) (if $n+1 \notin T$) or (6.3.3) (if $n+1 \in T$) it follows again that σ cannot be generated. Hence only simplices in the interior of $\text{conv}(\tilde{B}^n(\mu))$ are generated. $\square$

As in the previous chapter we have again that in case of an upper-semi-continuous mapping (5.7.5) possibly does not hold for any $\delta > 0$. Therefore we need a stronger condition to prove convergence, e.g. in case a) $f_j(x) - x_j > \varepsilon$ has to be replaced by $f_j(z) - z_j > 0$ for all $z \in B^n(x, \delta)$ and $f(z) \in \phi(z)$ for some $0 < \delta < \mu - \max_i |v_i|$. Then for a triangulation with grid size less than δ it can easily be seen that if a) holds for some $x \in \tilde{B}^n(\mu)$, $\bar{f}_j(w^k) - w_j^k > 0$ for all vertices $w^1, \ldots, w^{t+1}$ of the simplex σ containing x, implying that $n+1 \notin T$. The same adaptions have to be made for the condition b).

Another sufficient condition to prove convergence in case of a mapping is Merrill's condition, as is stated in the next theorem.

THEOREM 6.3.4. Let ϕ be an u.s.c. mapping on R^n. For some $\delta > 0$, let there exist $w \in R^n$ and $\bar{\mu} > 0$ such that for all $x, z \in R^n$ with $x \notin B^n(w, \bar{\mu})$ and $z \in B^n(x, \delta)$ holds

$$(h(x) - x)^T (w - z) > 0$$

for all $h(x) \in \phi(x)$. Then the variable dimension restart algorithm applied with vector labelling and using the $A(\alpha)K$ triangulation with $A(\alpha) \in \mathring{A}$ terminates with a completely labelled simplex if the mesh of the triangulation is less than δ.

PROOF. Let $\mu > 0$ be such that for all $x \in B^n(w, \bar{\mu})$ holds $\max_i |x_i| < \mu$. So for all x with $\max_i |x_i| \geq \mu$ we have that $x \notin B^n(w, \bar{\mu})$. Let v be the starting point and define μ' by

$$\mu' = \max[\mu + \delta, \max_{i \in I_n} |v_i| + \max\{\max_{i \in I_n} \alpha |v_i - w_i|, \alpha(\alpha - n + 1)^{-1} |\Sigma_i (w_i - v_i)|\}].$$

For some T, let $\sigma(y^1, \pi(T))$ be a t-simplex in $A(T)$ such that for some $y \in \sigma$, $\max_i |y_i| > \mu'$. We prove that the system of linear equations

$$\sum_{i=1}^{t+1} \lambda_i \ell(w^i) + \sum_{h \notin T} \mu_h e(h) = e$$

does not have a feasible solution λ_i^*, $i=1,\ldots,t+1$ and μ_h^*, $h \notin T$, implying that σ cannot be generated by the algorithm. Suppose there exists such a solution. Then we have that

$$[\sum_{i=1}^{t+1} \lambda_i^* \ell(w^i) + \sum_{h \notin T} \mu_h^* e(h)]^T [(w-y)^T, 0]^T = e^T [(w-y)^T, 0]^T$$

or by defining $\mu_{n+1}^* = 0$ if $n+1 \in T$ and letting f be the piecewise linear approximation to ϕ

$$(6.3.7) \qquad \sum_{i=1}^{t+1} \lambda_i^* \{f(w^i) - w^i\}^T (w-y) + \sum_{h \notin T} \mu_h^* (w_h - y_h) - \mu_{n+1}^* \sum_{i=1}^{n} (w_i - y_i) = 0$$

using the fact that $\mu_{n+1}^* = 1 - \sum_{i=1}^{t+1} \lambda_i^*$ (c.f. corollary 6.2.1). Since $\max_i |y_i| > \mu'$ we have that $\max_i |w_i^k| > \mu' - \delta \geq \mu$. Hence

$$(6.3.8) \qquad \{f(w^k) - w^k\}^T (w-y) > 0 \qquad k=1,\ldots,t+1.$$

Now we prove that $w_h - y_h > 0$ for all $h \notin T$. We first consider the case where, for some k, $y_k = \max_i |y_i|$. Hence

$$y_k > \mu' > v_k + \alpha(v_h - w_h) \qquad h \notin T.$$

Moreover, by the structure of the $A(\alpha)K$ triangulation we have that

$$y_h \leq v_h - \alpha^{-1}(y_k - v_k) \qquad h \notin T.$$

Combining both inequalities we obtain that

$$(6.3.9a) \qquad y_h < v_h - (v_h - w_h) = w_h \qquad h \notin T.$$

In the case where, for some k, $-y_k = \max_i |y_i|$ we again find by the structure of $A(\alpha)$

$$y_h = v_h + (y_h - v_h) \le v_h + (y_k - v_k) \qquad h \notin T.$$

Together with $-y_k > \mu' \ge -v_k + \alpha |v_h - w_h| \ge -v_k + v_h - w_h$ this again implies that

(6.3.9b) $\qquad y_h < w_h \qquad h \notin T.$

If $\mu^*_{n+1} = 0$, i.e. $n+1 \in T$, then (6.3.8) and (6.3.9a,b) contradict (6.3.7), implying that there is no feasible solution to the system of linear equations. It remains to prove that $\sum_i (w_i - y_i) < 0$ if $(n+1) \notin T$. In the case where for some k, $y_k = \max_i |y_i|$, it follows again by $A(\alpha)$ that

$$\sum_{i \ne k} y_i \ge \sum_{i \ne k} v_i - (n-1)\alpha^{-1}(y_k - v_k).$$

Moreover $y_k > v_k$ and

(6.3.10) $\qquad y_k > \mu' \ge v_k + \alpha(\alpha - n + 1)^{-1} \sum_{i=1}^{n} (w_i - v_i).$

Hence

$$\sum_{i=1}^{n} (w_i - y_i) \le \sum_{i=1}^{n} (w_i - v_i) + (n-1)\alpha^{-1}(y_k - v_k)$$

and with (6.3.10)

$$\sum_{i=1}^{n} (w_i - y_i) < 0.$$

Finally, in the case where for some k, $-y_k = \max_i |y_i|$ we have that

$$-y_k > \mu' \ge -v_k + \alpha(\alpha - n + 1)^{-1} \left| \sum_{i=1}^{n} (w_i - v_i) \right|$$

$$\ge -v_k + (\alpha - n + 1)^{-1} \sum_{i=1}^{n} (w_i - v_i).$$

Again from the structure of $A(\alpha)$ it follows that

$$\sum_{i \ne k} (y_i - v_i) \ge (\alpha - n + 2)(v_k - y_k).$$

Hence

$$\sum_{i=1}^{n} (w_i - y_i) < 0.$$

So there does not exist a feasible solution to the system of linear equations. Hence only simplices can be generated in the bounded set $\text{conv}(\tilde{B}^n(\mu'))$. $\square$

Observe that in the proof of the theorem $\mu' \geq \max_{i \in I_{n+1}} \alpha |v_i - w_i|$. Hence, if $\alpha \rightarrow \infty$, μ' is unbounded unless $v=w$. Therefore the theorem is not true if the K triangulation is used, except in the case where $v=w$.

Finally, we note that the condition "for some $\delta>0$" can be changed into the strong condition of Merrill "for all $\delta>0$". In this case the algorithm converges for every grid size.

6.4. RECENT DEVELOPMENTS.

As on S^n the variable dimension restart algorithm can be interpreted with $n+1$ points on an extra level (c.f. Todd [1978b], Barany [1979a], Van der Laan and Talman [1979c,1980c]). Therefore, let $R^n \times \{1\}$ be the natural level and $R^n \times \{0\}$ the artificial level. Now we choose $n+1$ points $u(i) \in R^n$ such that

$$u(i) \in \overset{\circ}{A}(I_{n+1} \setminus \{i\}) \qquad i=1,\ldots,n+1.$$

Let the n-simplex σ be the convex hull of the points $u(i)$ and let G be the collection of $(n+1)$-simplices obtained by connecting each grid point $x \in \overset{\circ}{A}(T)$ on level one with the vertices $u(i)$, $i \notin T$, on level zero. Then we have that G triangulates the set $(\sigma \times \{0\}) \cup (R^n \times (0,1])$. The proof of this proposition can be found in Todd [1978b]. Observe that this triangulation is not locally finite, but still satisfies the property of theorem 2.3.5. We present now briefly the research of Todd [1978b] about the variable dimension restart algorithm of Van der Laan and Talman [1979a,b] .
Let f be a continuous function on R^n and let the points $u(i)$ be defined by

$$u(i) = \delta(e-(n+1)e(i)+w) \quad i=1,\ldots,n$$
$$u(n+1) = \delta(e+w),$$

where δ is the grid size of the K (or J_1) triangulation of $R^n \times \{1\}$ and w is the n-dimensional vector such that $w_j=(2n-2j+1)/2n$, $j=1,\ldots,n$. Using vector labelling the system of linear equations is initiated with $B\mu=b$ where $b=(0,\ldots,0,1)^T$ and B is the matrix with $b(j)=((\delta w-u(j))^T,1)^T$. Todd proves that the algorithm traces a path of zeroes of the piecewise linear approximation to the function

$$h(x,t)=(1-t)(\delta w-x)+t(f(x)-x).$$

Using this fact it is proved that Merrill's condition is sufficient to guarantee that the algorithm converges. So the condition is sufficient with K the underlying triangulation. However, observe that the system of linear equations is not initiated with $I\mu=e$ as in the previous section. Starting with $I\mu=e$ has the advantage that when the dimension of the simplex is small, the basis matrix is sparse, which yields that efficient linear programming steps can be made.

Todd gives also a technique to accelerate the algorithm. Therefore he uses the set-up of the accelerated algorithms in Saigal [1977] and Saigal and Todd [1978]. He proves that in the limit an approximation in a certain stage can be obtained in n+1 iterations and that the algorithm converges Q-quadratically, when the function is continuously differentiable with a Lipschitz continuous derivative.

A note about the path of the variable dimension restart algorithm was written by Kojima [1980]. He showed that the same path can be obtained by utilizing the differential equations approach when the function is continuously differentiable.

In Van der Laan and Talman [1980b] the triangulation of $s^n \times [0,1]$ as given in section 5.5 is used to construct a triangulation of $s^n \times [1,\infty)$ which allows for any factor of grid refinement between two successive levels. We treat this matter in chapter 8.

The variable dimension restart algorithm is generalized in Van der Laan and Talman [1978a,b]. In the first paper an algorithm is given which can be applied to compute a fixed point of a function on the product space of unit simplices. The second paper is closely related and presents a class of

algorithms on R^n. One of the extreme cases of this class is the algorithm presented in this chapter. The other extreme case is an algorithm with 2n labels when integer labelling is used and a system of 2n linear equations when vector labelling is used. In addition, starting with a point v, a search in 2n directions can be made. The class of algorithms is treated in full detail by Talman [1980]. An interpretation of the "2n-case" with points on an extra level can be found in Todd and Wright [1979] and Todd [1980]. Interpretations with extra points for any element of the class are given by Van der Laan and Talman [1979c] (see also Talman [1980]). In Todd and Wright [1979] it is showed that the 2n-case is very appropriate to be used for antipodal fixed point algorithms. Finally we mention a paper of Wright [1979]. He generalized the variable dimension restart algorithm to an algorithm with 2^n directions. As argued by Wright the algorithm can be considered to be dual to the 2n-case.

6.5. COMPUTATIONAL RESULTS.

To test the algorithm the three pure exchange economy problems of Scarf [1967b] were again solved. The problems were converted to problems in R^n_+ by setting the price of one of the commodities equal to one. The algorithm was applied with vector labelling and the K triangulation. A point x outside int R^n_+ received the label e(i) with i the smallest index with $x_i \leq 0$, whereas a point x∈int R^n_+ received the same label as in section 5.8. In the first stage the starting point was the point x with $x_i = 1$, i=1,...,n and the mesh was $(n+1)(n+2)^{-1}\sqrt{n}$. The approximate fixed point $w^*(m)$ on stage m was chosen to be the starting point at stage m+1, m=1,2,... . The grid size δ_{m+1} at stage m+1 is defined by

$$\delta_{m+1} = \tfrac{1}{2}\delta_m \text{ if } h_m \geq \tfrac{1}{2}\delta_m \text{ and } \delta_{m+1} = h_m \text{ if } h_m < \tfrac{1}{2}\delta_m$$

with h_m as defined in section 5.8. The results are given in table 6.5.1.

Table 6.5.1. Three pure exchange economy models, n=4, 7 and 9, M=mesh $G_m/\sqrt{n}$, N is the total number of iterations, E= $\max_{i \in I_{n+1}} |(g_i - a_i)/a_i|$.

Dimension	m	M	N	E
4	5	6.10^{-3}	47	$2 \cdot 10^{-5}$
	7	10^{-5}	69	$< 10^{-12}$
7	5	7.10^{-3}	72	$6 \cdot 10^{-6}$
	6	$9 \cdot 10^{-5}$	80	$2 \cdot 10^{-9}$
9	5	$3 \cdot 10^{-3}$	102	10^{-6}
	6	10^{-4}	123	$2 \cdot 10^{-10}$

In the three problems above the number of function evaluations is equal to the number of linear programming steps. The algorithm was also applied on the first production economy problem of Scarf [1973].

To achieve an accuracy of less than 5.10^{-3} the algorithm needed 176 function evaluations and 184 linear programming steps, i.e. 8 times step 4 was performed, bringing a unit vector in the system of linear equations. A comparison of the number of iterations with the results as reported by Todd [1978a] shows that the variable dimension restart algorithm takes significantly fewer iterations than the algorithm of Merrill and the method of Eaves and Saigal. Even in the case that quasi-Newton steps were used our results are better or comparable. Of course, taking quasi-Newton steps can also be implemented in our algorithm.

The algorithm was also applied on the problems 1 and 2 of Saigal [1977c]. Problem 1 is a 4-dimensional unconstrained optimization problem, originally due to Colville [1968]. Problem 2 is a 20-dimensional version of a problem considered by Kellogg, Li and Yorke [1976]. For both problems the grid size in stage 1 is set equal to 2, whereas $\delta_{m+1} = \frac{1}{2}\delta_m$ for all m≥1. The starting point was $(-3,-1,-3,-1)^T$ for problem 1, and the zero point for problem 2. The results are given in table 6.5.2.

Table 6.5.2. Problem 1, unconstrained optimization, $E = \max_{i \in I_n} \left| \frac{\partial f(x)}{\partial x_i} \right|_{x = w^*(m)}$;

Problem 2, 20-dimensional fixed point problem,

$$E = \max_{i \in I_n} \left| f_i(w^*(m)) - w_i^*(m) \right| ,$$

N is the total number of iterations.

Problem	m	N	E
1	5	51	2.6
	7	89	.037
	9	107	$2 \cdot 10^{-4}$
2	1	37	.38
	3	90	10^{-3}
	5	136	10^{-7}

Compared with the results of Saigal [1977c] it follows that the variable dimension algorithm again takes much fewer iterations than the homotopy algorithm. Especially, the number of iterations is very small in the first stage. Finally we remark that for both problems the K triangulation was used. For the impact of the $A(\alpha^*)K$ triangulation we refer the reader to Talman [1980].

CHAPTER 7

THE ORIENTATION OF SIMPLICES

7.1. INTRODUCTION.

In this chapter we consider the concept of the orientation of
simplices. Each completely labelled simplex receives an index being an
element of the set $\{+1,-1\}$. It is discussed by several authors that this
index is closely related to the Brouwer degree of a mapping (c.f. Charnes,
Garcia and Lemke [1977], Prüfer [1977], Garcia and Zangwill [1978],
Peitgen and Prüfer [1979] and Prüfer and Siegberg [1979]). The history
of the Brouwer degree can be found in Siegberg [1980].
An orientation or index theory for bimatrix games is introduced by
Shapley [1974] and is generalized by Lemke and Grotzinger [1976]. Deeper
and more abstract discussions can be found in Eaves and Scarf [1976] and
Todd [1976c]. In this chapter we discuss the relationship between oriented
simplices and fixed point algorithms. We restrict ourselves to integer
labelling. The generalization to vector labelling can easily be made.
In section 2 we give some definitions and lemmas and we prove the well-
known result that the sum of the indices of the completely labelled
simplices in S^n is equal to one if the labelling rule satisfies the
conditions of Sperner's lemma. We will also simplify the definition of the
orientation of a completely labelled simplex. In section 3 this
simplification is used to define the orientation of a $(T \cup \{j\})$-complete
simplex of a region $A(T)$. We give a lower bound on the number of completely
labelled simplices by considering all $(T \cup \{j\})$-complete simplices in $A(T)$
with $|T| = n-1$. In section 4 we prove that the variable dimension algorithm
always generates a positive oriented completely labelled simplex.
Conditions for the existence of at least k completely labelled simplices
for some odd number k are given. Using the concept of orientation it is
shown that under these conditions the variable dimension restart algorithm
can be used to generate at least k completely labelled simplices.

7.2. ORIENTATION.

Let S^n be triangulated in the standard way with grid size m^{-1} and let $\sigma = \sigma(y^1, \pi(I_n))$ be a completely labelled simplex. Then we define the orientation of σ, to be denoted by or σ, as follows.

DEFINITION 7.2.1. The orientation of a completely labelled simplex $\sigma(y^1, \pi(I_n))$ is defined by

$$\text{or } \sigma = \text{sign det } [y^{j_1} \; y^{j_2} \ldots y^{j_{n+1}}]$$

where j_i is the index such that $\ell(y^{j_i}) = i$, $i = 1, \ldots, n+1$.

We also define the orientation of an I_n-complete $(n-1)$-facet τ of a simplex $\sigma(y^1, \pi(I_n))$ where either σ is completely labelled or σ has two I_n-complete facets.

DEFINITION 7.2.2. Let $\tau(y^{j_1}, \ldots, y^{j_n})$ be an I_n-complete facet of a simplex $\sigma(y^1, \pi(I_n))$ and let the vertices be ordered in such a way that y^{j_i} has label i, $i = 1, \ldots, n$. Let $y^{j_{n+1}}$ be the vertex of σ oppposite to τ. Then the orientation of τ with respect to σ, to be denoted by or$_\sigma \tau$, is defined by

$$\text{or}_\sigma \tau = \text{sign det } [y^{j_1} \; y^{j_2} \ldots y^{j_n} \; y^{j_{n+1}}].$$

COROLLARY 7.2.3. Let $\sigma(y^1, \pi(I_n))$ be a completely labelled simplex and τ be the I_n-complete facet of σ. Then

$$\text{or } \sigma = \text{or}_\sigma \tau.$$

Both the definitions 7.2.1 and 7.2.2 hold for any triangulation of S^n. Utilizing the structure of the standard triangulation of S^n we rewrite the definitions in another notation. It will appear that using this notation the definitions can easily be generalized to $(T \cup \{j\})$-complete t-simplices and T-complete $(t-1)$-simplices in $A(T)$, $T \subset I_{n+1}$ and $|T| \leq n$. Recall that $q(j)$ is the j-th column of the $(n+1) \times (n+1)$ triangulation matrix Q as defined in chapter 5. Then for each vertex w of the given triangulation of S^n there exist nonnegative integers $\alpha_1(w), \ldots, \alpha_n(w)$

such that

$$w = e(1) + \sum_{j=1}^{n} \delta\alpha_j(w)q(j)$$

with $\delta = m^{-1}$.

In the following, for a simplex $\sigma(y^1, \pi(I_n))$ of the triangulation, let α^i be the n-dimensional vector $(\alpha_1^i, \ldots, \alpha_n^i)^T$, $i=1, \ldots, n+1$, with $\alpha_j^i = \alpha_j(y^i)$, $j=1, \ldots, n$.

LEMMA 7.2.4. For any simplex $\sigma(y^1, \pi(I_n))$ holds

$$\det[y^1 \ldots y^{n+1}] = \delta^n \det \begin{bmatrix} 1 & \cdots & 1 \\ \alpha^1 & \cdots & \alpha^{n+1} \end{bmatrix}.$$

PROOF. Since $y^i = e(1) + \sum_{j=1}^{n} \delta\alpha_j^i q(j)$ we have that $y^i = (1-\delta\alpha_1^i, \delta\alpha_1^i - \delta\alpha_2^i, \ldots$ $\ldots, \delta\alpha_{n-1}^i - \delta\alpha_n^i, \delta\alpha_n^i)^T$. So adding successively the i-th row to the (i-1)-th row, $i=n+1, n, \ldots, 2$, we obtain

$$\det[y^1 \ldots y^{n+1}] = \det \begin{bmatrix} 1 & \cdots & 1 \\ \delta\alpha^1 & \cdots & \delta\alpha^{n+1} \end{bmatrix} = \delta^n \det \begin{bmatrix} 1 & \cdots & 1 \\ \alpha^1 & \cdots & \alpha^{n+1} \end{bmatrix}.$$

$\square$

LEMMA 7.2.5. Let $\sigma(y^1, \pi(I_n))$ be a simplex of the triangulation of S^n. Then

$$\det \begin{bmatrix} 1 & \cdots & 1 \\ \alpha^1 & \cdots & \alpha^{n+1} \end{bmatrix} = \pm 1.$$

PROOF. Since $\sigma = \sigma(y^1, \pi(I_n))$ we have that $y^{i+1} = y^i + \delta q(\pi_i)$, $i=1, \ldots, n$. Hence $\alpha^{i+1} - \alpha^i = e(\pi_i)$, $i=1, \ldots, n$ and therefore

$$\det \begin{bmatrix} 1 & \cdots & 1 \\ \alpha^1 & \cdots & \alpha^{n+1} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha^1 & \alpha^2-\alpha^1 & \cdots & \alpha^{n+1}-\alpha^n \end{bmatrix} =$$

$$= \det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \alpha^1 & e(\pi_1) & \cdots & e(\pi_n) \end{bmatrix} = \pm 1. \qquad \square$$

The lemmas 7.2.4 and 7.2.5 allow for the alternative definitions of or σ and $or_\sigma\tau$, given in the next corollary.

COROLLARY 7.2.6. a) Let $\sigma(y^1, \pi(I_n))$ be a completely labelled simplex in S^n and let j_i be the index such that $\ell(y^{j_i}) = i$, $i=1, \ldots, n+1$. Then

$$\text{or } \sigma = \det \begin{bmatrix} 1 & \ldots & 1 \\ j_1 & & j_{n+1} \\ \alpha & \ldots & \alpha \end{bmatrix}.$$

b) Let $\tau(y^{j_1}, \ldots, y^{j_n})$ be an I_n-complete facet of a simplex $\sigma(y^1, \pi(I_n))$ such that $\ell(y^{j_i}) = i$, $i=1, \ldots, n$ and let $y^{j_{n+1}}$ be the vertex of σ opposite to τ. Then

$$\text{or}_\sigma \tau = \det \begin{bmatrix} 1 & \ldots & 1 \\ j_1 & & j_{n+1} \\ \alpha & \ldots & \alpha \end{bmatrix}.$$

In the next sections we combine the orientation theory with the variable dimension restart algorithm treated in the chapters 5 and 6. Here we use the variable dimension algorithm of Kuhn, discussed in chapter 4, to prove that the sum of the orientations of the completely labelled simplices in S^n is equal to one if the labelling rule is proper. Remember that Kuhn's algorithm is a special case of the algorithm presented in chapter 5. Analogously it will appear that the definition of the orientation of lower-dimensional simplices to be given below is a special case of a more general definition to be given in section 3.

Consider the sets $S^n(k)$ as defined in definition 4.4.1.c, i.e. $S^n(k) = \{x \in S^n \mid x_i = 0, \ i=k+1, \ldots, n+1\}$, $k=1, \ldots, n+1$. Recall that $S^n(n+1) = S^n$ and $S^n(n) = S^n_{n+1}$. Let w be a vertex of the triangulation of S^n in $S^n(k)$. Since $w_i = 0$, $i=k+1, \ldots, n+1$, it follows that there exist nonnegative integers α_j, $j=1, \ldots, k-1$, such that

$$w = e(1) + \sum_{j=1}^{k-1} \delta \alpha_j q(j).$$

In the following, for a $(k-1)$-simplex $\sigma(y^1, \pi(I_{k-1}))$ of the triangulation of $S^n(k)$, let α^i be the $(k-1)$-dimensional vector of nonnegative integers $\alpha^i_1, \ldots, \alpha^i_{k-1}$ corresponding to y^i, $i=1, \ldots, k$.

DEFINITION 7.2.7. Let $\sigma(y^1, \pi(I_{k-1}))$ be an I_k-complete $(k-1)$-simplex in $S^n(k)$ and let j_i be the index such that $\ell(y^{j_i}) = i$, $i=1, \ldots, k$. Then

$$\text{or } \sigma = \det \begin{bmatrix} 1 & \ldots & 1 \\ j_1 & & j_k \\ \alpha & \ldots & \alpha \end{bmatrix}.$$

DEFINITION 7.2.8. Let $\tau(y^{j_1},\ldots,y^{j_{k-1}})$ be an I_{k-1}-complete facet of a (k-1)-simplex $\sigma(y^1,\pi(I_{k-1}))$ in $S^n(k)$ and let j_i be the index such that $\ell(y^{j_i}) = i$, $i=1,\ldots,k-1$. Let y^{j_k} be the vertex of σ opposite to τ. Then

$$\text{or}_\sigma\tau = \det \begin{bmatrix} 1 & \ldots & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_k} \end{bmatrix}.$$

Of course, it follows again that, if σ is I_k-complete and τ is the I_{k-1}-complete facet of σ, or $\sigma = \text{or}_\sigma\tau$. Moreover we have that the determinant is equal to ± 1.

We prove now some lemmas (see also Allgower and Georg [1980]).

LEMMA 7.2.9. Let $\sigma(y^1,\pi(I_{k-1}))$ be a (k-1)-simplex in $S^n(k)$ having two I_{k-1}-complete facets τ^1 and τ^2. Then

$$\text{or}_\sigma\tau^1 = -\text{or}_\sigma\tau^2.$$

PROOF. Let y^{t_1} and y^{t_2} be the two vertices having the same label and let τ^i be the facet with vertices y^j, $j=1,\ldots,t_h-1,t_h+1,\ldots,k$, $h \neq i$, $i=1,2$. Let j_i, $i=1,\ldots,k$ be the permutation of the elements of I_k such that $y^{j_k} = y^{t_2}$ and $\ell(y^{j_i}) = i$, $i=1,\ldots,k-1$. Let $s\in I_{k-1}$ be the index such that $y^{t_1}=y^{j_s}$. Then

$$\text{or}_\sigma\tau^1 = \det \begin{bmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_s-1} & \alpha^{t_1} & \alpha^{j_s+1} & \ldots & \alpha^{j_{k-1}} & \alpha^{t_2} \end{bmatrix}$$

and

$$\text{or}_\sigma\tau^2 = \det \begin{bmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_s-1} & \alpha^{t_2} & \alpha^{j_s+1} & \ldots & \alpha^{j_{k-1}} & \alpha^{t_1} \end{bmatrix}.$$

Clearly $\text{or}_\sigma\tau^1 = -\text{or}_\sigma\tau^2$. □

LEMMA 7.2.10. Let τ be an I_{k-1}-complete common facet of two (k-1)-simplices $\sigma_1(y^1,\pi(I_{k-1}))$ and $\sigma_2(\bar{y}^1,\bar{\pi}(I_{k-1}))$ in $S^n(k)$. Then

$$\text{or}_{\sigma_1}\tau = -\text{or}_{\sigma_2}\tau.$$

PROOF. Let τ be the facet of σ_1 with vertices $y^1,\ldots,y^{i-1},y^{i+1},\ldots,y^k$. Then σ_2 is obtained from σ_1 by replacing y^i. It can easily be verified that the vertex of σ_2 opposite to τ is equal to $\bar{y}^i = y^{i-1}+y^{i+1}-y^i$ where $i-1 = k$ if $i=1$ and $i+1 = 1$ if $i=k$. Let $j_1,\ldots,j_k$ be the permutation of the elements of I_k such that $y^{j_k} = y^i$ and $\ell(y^{j_h}) = h$, $h \in I_{k-1}$. Then

$$\text{or}_{\sigma_1}\tau = \det\begin{bmatrix} 1 & \ldots & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_k} \end{bmatrix} = -\det\begin{bmatrix} 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_{k-1}} & \alpha^{j_k-1}+\alpha^{j_k+1}-\alpha^{j_k} \end{bmatrix}$$

$$= -\det\begin{bmatrix} 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_{k-1}} & \bar{\alpha} \end{bmatrix} = -\text{or}_{\sigma_2}\tau$$

where $\bar{\alpha} = (\bar{\alpha}_1,\ldots,\bar{\alpha}_{k-1})^T$ is such that $\bar{y}^i = e(1) + \sum_{j=1}^{k-1} \delta\bar{\alpha}_j q(j)$. $\quad\square$

LEMMA 7.2.11. For $k \le n$, let $\tau(y^1,\pi(I_{k-1}))$ be an I_k-complete simplex in $S^n(k)$. Let $\sigma(y^1,\pi(I_k))$ be the k-simplex of $S^n(k+1)$ having τ as a facet. Then

$$\text{or }\tau = \text{or}_\sigma\tau.$$

PROOF. In $S^n(k)$ we have that $y^i = e(1) + \sum_{j=1}^{k-1} \delta\alpha_j^i q(j)$, $i=1,\ldots,k$. In $S^n(k+1)$ we have that $y^i = e(1) + \sum_{j=1}^{k} \delta\bar{\alpha}_j^i q(j)$, $i=1,\ldots,k$, with

$$\bar{\alpha}_j^i = \alpha_j^i \qquad j=1,\ldots,k-1$$

$$\bar{\alpha}_j^i = 0 \qquad j=k.$$

Moreover, the vertex of σ opposite to τ is equal to
$y^{k+1} = e(1) + \sum_{j=1}^{k} \delta\bar{\alpha}_j^{k+1} q(j)$ with $\bar{\alpha}_j^{k+1} = \alpha_j^k$, $j=1,\ldots,k-1$ and $\bar{\alpha}_k^{k+1} = 1$. Let j_i be the index such that $\ell(y^{j_i})=i$, $i=1,\ldots,k$ and $y^{j_{k+1}}=y^{k+1}$. Hence

$$\text{or }\tau = \det\begin{bmatrix} 1 & \ldots & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_k} \end{bmatrix} = \det\begin{bmatrix} 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_k} & \alpha^k \\ 0 & \ldots & 0 & 1 \end{bmatrix} =$$

$$= \det\begin{bmatrix} 1 & \ldots & 1 & 1 \\ \bar{\alpha}^{j_1} & \ldots & \bar{\alpha}^{j_k} & \bar{\alpha}^{j_{k+1}} \end{bmatrix} = \text{or}_\sigma\tau. \quad\square$$

The following theorem states that the sum of the orientations of the I_k-complete simplices in $S^n(k)$ is equal to 1.

THEOREM 7.2.12. Let $O(k)$ be the set of I_k-complete simplices in $S^n(k)$ and let the labelling rule be such that the conditions of Sperner's lemma are satisfied. Then

$$\sum_{\sigma \in O(k)} \text{or } \sigma = 1 \qquad k=1,\ldots,n+1.$$

PROOF. We prove this theorem by induction on k. Clearly, for $k=1$ we have that $e(1)$ is the only I_1-complete zero-simplex of $S^n(1)$ and $\text{or}\{e(1)\} = \det 1 = 1$. Let the theorem be true for $k=j$. By applying the steps of Kuhn's variable dimension algorithm, paths of j-simplices in $S^n(j+1)$ will be generated such that the common facets of two adjacent simplices are I_j-complete. Three different types of paths can occur.

a) Starting with a j-simplex in $S^n(j+1)$ having an I_j-complete $(j-1)$-simplex τ in $S^n(j)$ as a facet, the algorithm generates an I_{j+1}-complete simplex σ in $S^n(j+1)$, or conversely. By applying the lemmas 7.2.9, 7.2.10, 7.2.11 and corollary 7.2.3 we have that or τ = or σ.

b) Starting with a j-simplex in $S^n(j+1)$ having an I_j-complete $(j-1)$-simplex τ_1 in $S^n(j)$ as a facet, the algorithm generates a j-simplex in $S^n(j+1)$ having an I_j-complete $(j-1)$-simplex τ_2 in $S^n(j)$ as a facet. From the lemmas 7.2.9, 7.2.10 and 7.2.11 it follows that or τ_1 = -or τ_2.

c) Starting with an I_{j+1}-complete j-simplex σ_1 in $S^n(j+1)$ an I_{j+1}-complete j-simplex σ_2 in $S^n(j+1)$ is generated. From corollary 7.2.3 and the lemmas 7.2.9 and 7.2.10 it follows that or σ_1 = - or σ_2.

By the same arguments as in section 4.4 we have that two different paths are disjunct. So, combining these three cases we conclude $\sum_{\sigma \in O(j+1)}$ or σ = $\sum_{\tau \in O(j)}$ or τ, which proves that $\sum_{\sigma \in O(k)}$ or σ = 1, $k=1,\ldots,n+1$. □

The proof of the theorem is illustrated in figure 7.2.1. The path of τ_1 to σ_1 is of type 1, or σ_1 = or τ_1 = 1; the path between τ_2 and τ_3 is of type 2,

or τ_2 = -or τ_3 = -1; and the paths between σ_2 and σ_3 respectively σ_4 and σ_5 are of type 3 with or σ_2 = -or σ_3 = -1 and or σ_4 = -or σ_5 = -1. Observe that it is impossible to generate a path between two simplices with the same orientation.
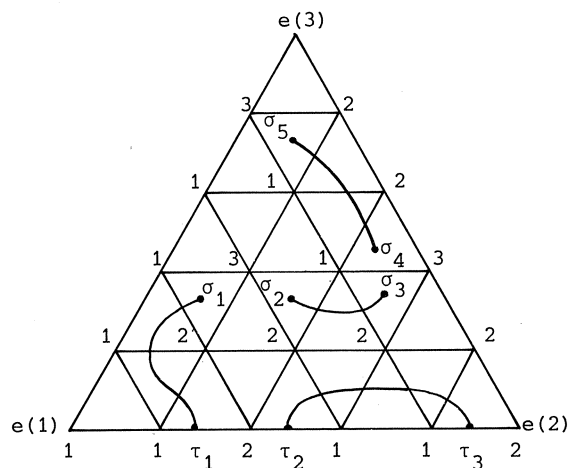
Figure 7.2.1. Illustration of theorem 7.2.12; or$\{e(1)\}$= 1;

or τ_1 = or τ_3 = 1, or τ_2 = -1; or σ_1 = or σ_3 = or σ_5 = 1,

or σ_2 = or σ_4 = -1.

In particular theorem 7.2.12 holds if k=n and k=n+1. Since we have a proper labelling there are no I_n-complete simplices in S_i^n, i≠n+1. So we have that the sum of the orientations of the completely labelled simplices of S^n is equal to the sum of the orientations of the I_n-complete simplices in bd S^n. A more general result can be found in Prüfer and Siegberg [1979].

7.3. THE VARIABLE DIMENSION RESTART ALGORITHM.

In this section we apply the concept of orientation to the variable dimension algorithm treated in chapter 5.

Let the arbitrarily chosen grid point v be the starting point of the variable dimension algorithm. Recall that for any $T\epsilon I_{n+1}$, $|T|\le n$, the regions A(T) are defined by

$$A(T) = \{x \in S^n | x = v + \sum_{j \in T} \lambda_j q(j) \text{ for nonnegative numbers } \lambda_j, \ j \in T\}.$$

For any grid point $w \in A(T)$, $|T| = t$, there is a unique t-dimensional vector α of nonnegative integers α_j, $j \in T$, such that $w = v + \sum_{j \in T} \delta \alpha_j q(j)$. In the following, for a simplex $\sigma(y^1, \pi(T))$ in $A(T)$, let α^i be the t-dimensional vector $(\alpha^i_{s_1}, \ldots, \alpha^i_{s_t})^T$ such that

$$y^i = v + \sum_{s_j \in T} \delta \alpha^i_{s_j} q(s_j),$$

where $(s_1, \ldots, s_t)$ is the permutation of the elements of T such that $s_i < s_{i+1}$, $i = 1, \ldots, t-1$.

DEFINITION 7.3.1. Let $\sigma(y^1, \pi(T))$ be a $(T \cup \{j\})$-complete t-simplex in $A(T)$, $j \notin T$ and let $j_1, \ldots, j_{t+1}$ be a permutation of the elements of I_{t+1}, such that $\ell(y^{j_{t+1}}) = j$ and $\ell(y^{j_i}) = s_i$, $i = 1, \ldots, t$. Then the orientation of $\sigma(y^1, \pi(T))$ is

$$\text{or } \sigma = \det \begin{bmatrix} 1 & \ldots & 1 & 1 \\ \alpha^{j_1} & \ldots & \alpha^{j_t} & \alpha^{j_{t+1}} \end{bmatrix}.$$

Observe that this definition is a straightforward generalization of definition 7.2.7. In particular if $v = e(1)$ we have that $S^n(t+1) = A(I_t)$, whereas an $(I_t \cup \{j\})$-complete simplex of $S^n(t+1)$ must be I_{t+1}-complete. Of course, or $\sigma = \pm 1$. Definition 7.3.1 is illustrated in figure 7.3.1.

Remark that the orientation as given in definition 7.3.1 depends on the starting point v. In general we have that if the orientation of a $(T \cup \{j\})$-complete simplex is defined for some v, it is not defined for a starting point $w \neq v$. For example in the figure, τ_1 is a $(\{1,2\})$-complete simplex in $A(1)$. However if the point $w = (\frac{3}{7}, \frac{2}{7}, \frac{2}{7})^T$ is chosen as the starting point, τ_1 is a facet of a simplex in $A(1,3)$ and or τ_1 is not defined. Of course the orientation of a completely labelled simplex is always defined by definition 7.3.1 since for any v there exist a j such that σ is a completely labelled simplex of $A(I_{n+1} \setminus \{j\})$. We prove that the orientation of a completely labelled simplex as defined in 7.3.1. is independent of the starting point v.
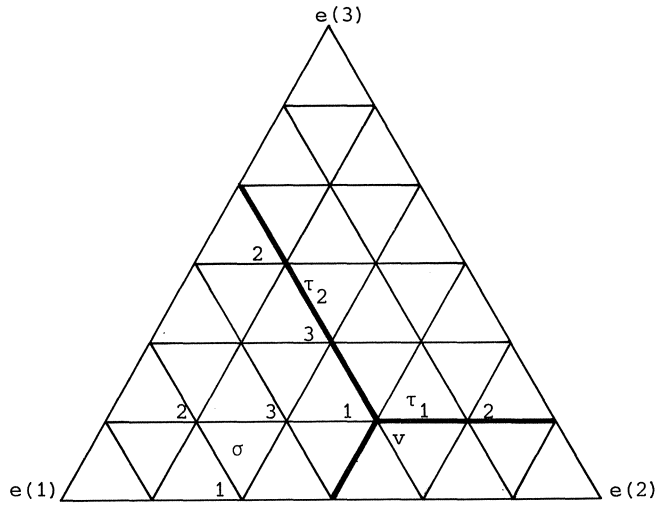
Figure 7.3.1. {v} is a {1}-complete simplex in A(∅),or {v} = 1;

τ_1 is a {1,2}-complete simplex in A(1), or τ_1 = 1;

τ_2 is a {2,3}-complete simplex in A(2), or τ_2 = -1;

σ is a {1,2,3}-complete simplex in A(2,3), or σ = -1.

THEOREM 7.3.2. The orientation of a completely labelled simplex as defined in 7.3.1 is consistent with the orientation as defined in 7.2.1.

PROOF. Let $\sigma(y^1, \pi(T))$ be a completely labelled simplex in A(T) with $T = I_{n+1} \setminus \{k\}$ for some k ∈ I_{n+1}. Let j_i be the index such that $\ell(y^{j_i}) = i$, i=1,...,n+1. Let $\alpha^i = (\alpha^i_1, \ldots, \alpha^i_{k-1}, \alpha^i_{k+1}, \ldots, \alpha^i_{n+1})^T$ be the n-vector such that

(7.3.1) $\quad y^i = v + \sum_{j \neq k} \delta\alpha^i_j q(j).$

Then by definition 7.3.1,

(7.3.2) $\quad$ or $\sigma = \det \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ \alpha^{j_1} & \cdots & \alpha^{j_{k-1}} & \alpha^{j_{k+1}} & \cdots & \alpha^{j_{n+1}} & \alpha^{j_k} \end{bmatrix}.$

We prove that definition 7.2.1 is equivalent with (7.3.2). From (7.3.1) it follows that for i = 1,....,n+1,

$$y_h^i = v_h - \delta\alpha_h^i + \delta\alpha_{h-1}^i \quad h \neq k, \; k+1$$

$$y_h^i = v_h + \delta\alpha_{h-1}^i \quad\quad\quad h = k$$

$$y_h^i = v_h - \delta\alpha_h^i \quad\quad\quad\quad h = k+1$$

with the convention that h-1 = n+1 if h=1 and k+1 = 1 if k=n+1. So the h-th element of the i-th column of the matrix $[y^{j_1} \ldots y^{j_{n+1}}]$ becomes

$$v_1 - \delta\alpha_1^{j_i} + \delta\alpha_{h-1}^{j_i} \quad\quad h = 1$$

$$v_h - \delta\alpha_h^{j_i} + \delta\alpha_{h-1}^{j_i} \quad\quad h = 2,\ldots,k-1$$

$$v_k + \delta\alpha_{k-1}^{j_i} \quad\quad\quad\quad h = k$$

$$v_{k+1} - \delta\alpha_{k+1}^{j_i} \quad\quad\quad h = k+1$$

$$v_h - \delta\alpha_h^{j_i} + \delta\alpha_{h-1}^{j_i} \quad\quad h = k+2,\ldots,n+1.$$

By adding successively the h-th row to the (h-1)-th row, h = k,k-1,...,2, the first row to the last row and then successively the i-th row to the (i-1)-th row, i = n+1,n,...,k+2, we obtain that or σ is equal to the sign of the determinant of the matrix with the h-th element of the i-th row equal to

$$\Sigma_{j=h}^k v_j + \delta\alpha_{h-1}^{j_i} \quad\quad\quad h = 1,\ldots,k$$

$$\Sigma_{j=1}^{n+1} v_j = 1 \quad\quad\quad\quad h = k+1$$

$$\Sigma_{j=1}^k v_j + \Sigma_{j=h}^{n+1} v_j + \delta\alpha_{h-1}^{j_i} \quad h = k+2,\ldots,n+1.$$

It follows now immediately that the orientation of σ as defined in definition 7.2.1 becomes

$$
\text{or } \sigma = \text{sign } \delta^n \det
\begin{bmatrix}
{}^{j_1}\alpha_{n+1} & \cdot & \cdot & \cdot & \cdot & {}^{j_{n+1}}\alpha_{n+1} \\
{}^{j_1}\alpha_1 & \cdot & \cdot & \cdot & \cdot & {}^{j_{n+1}}\alpha_1 \\
\vdots & & & & & \vdots \\
{}^{j_1}\alpha_{k-1} & \cdot & \cdot & \cdot & \cdot & {}^{j_{n+1}}\alpha_{k-1} \\
1 & \cdot & \cdot & \cdot & \cdot & 1 \\
{}^{j_1}\alpha_{k+1} & \cdot & \cdot & \cdot & \cdot & {}^{j_{n+1}}\alpha_{k+1} \\
\vdots & & & & & \vdots \\
{}^{j_1}\alpha_n & \cdot & \cdot & \cdot & \cdot & {}^{j_{n+1}}\alpha_n
\end{bmatrix}
=
$$

$$
= \det
\begin{bmatrix}
1 & \ldots & 1 & 1 & \ldots & 1 & 1 \\
\alpha^{j_1} & \ldots & \alpha^{j_{k-1}} & \alpha^{j_{k+1}} & \ldots & \alpha^{j_{n+1}} & \alpha^{j_k}
\end{bmatrix}. \qquad \square
$$

Besides the orientation of a $(T \cup \{j\})$-complete simplex in $A(T)$ we also define the orientation of a T-complete facet of a simplex in $A(T)$.

DEFINITION 7.3.3. Let $\tau(y^{j_1}, \ldots, y^{j_t})$ be a T-complete facet of a simplex $\sigma(y^1, \pi(T))$ in $A(T)$ and let the vertices be ordered in such a way that $\ell(y^{j_i}) = s_i$, $i = 1, \ldots, t$ and $y^{j_{t+1}}$ is the vertex of σ opposite to τ. Then the orientation of τ with respect to σ is defined by

$$
\text{or}_\sigma \tau = \det
\begin{bmatrix}
1 & \cdot & \cdot & \cdot & 1 \\
{}^{j_1}\alpha & \cdot & \cdot & \cdot & {}^{j_{t+1}}\alpha
\end{bmatrix}.
$$

COROLLARY 7.3.4. If τ is the T-complete facet of a $(T \cup \{j\})$-complete simplex σ in $A(T)$ then $\text{or}_\sigma \tau = \text{or } \sigma$.

The following two lemmas are a straightforward generalization of the lemmas 7.2.9 and 7.2.10. Therefore we omit the proofs of the lemmas.

LEMMA 7.3.5. Let $\sigma(y^1, \pi(T))$ be a simplex in $A(T)$ having two T-complete facets τ^1 and τ^2. Then

$$
\text{or}_\sigma \tau^1 = - \text{or}_\sigma \tau^2.
$$

LEMMA 7.3.6. Let τ be a T-complete common facet of two simplices σ_1 and σ_2 in $A(T)$. Then

$$or_{\sigma_1} \tau = - or_{\sigma_2} \tau.$$

The next lemma is a straightforward generalization of lemma 7.2.11. The proof however has to be adapted since the vertices have to be reordered, if a $(T \cup \{j\})$-complete simplex in $A(T)$ is extended to a simplex in $A(T \cup \{j\})$.

LEMMA 7.3.7. For $|T| \leq n-1$, let $\tau(y^1, \pi(T))$ be a $(T \cup \{j\})$-complete simplex in $A(T)$. Let $\sigma(y^1, \pi(T \cup \{j\}))$ be the simplex in $A(T \cup \{j\})$ having τ as a facet. Then

$$or \ \tau = or_\sigma \tau.$$

PROOF. We have that $y^i = v + \Sigma_{s_h \in T} \delta\alpha^i_{s_h} q(s_h)$, $i=1,\ldots,t+1$, whereas the vertex of σ opposite to τ is equal to $y^{t+2} = y^{t+1} + \delta q(j)$. Let j_i be the index such that $\ell(y^{j_i}) = s_i$, $i=1,\ldots,t$ and $\ell(y^{j_{t+1}}) = j$. Then

$$or \ \tau = \det \begin{bmatrix} 1 & \cdots & 1 & 1 \\ j_1 & & j_t & j_{t+1} \\ \alpha_1 & \cdots & \alpha_t & \alpha_{t+1} \end{bmatrix} = \det \begin{bmatrix} 1 & \cdots & 1 & 1 \\ \alpha^{j_1} & \cdots & \alpha^{j_{t+1}} & \alpha^{t+2} \\ 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Let i, $0 \leq i \leq t$, be the index such that $s_i < j < s_{i+1}$ with the convention that $s_i < j$ if $i=t$ and $j < s_1$ if $i=0$. Then, by definition 7.3.3, $or_\sigma \tau =$

$$\det \begin{bmatrix} 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\ \alpha^{j_1}_{s_1} & \cdots & \alpha^{j_{i-1}}_{s_1} & \alpha^{j_{t+1}}_{s_1} & \alpha^{j_{i+1}}_{s_1} & \cdots & \alpha^{j_t}_{s_1} & \alpha^{t+2}_{s_1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha^{j_1}_{s_i} & \cdots & \alpha^{j_{i-1}}_{s_i} & \alpha^{j_{t+1}}_{s_i} & \alpha^{j_{i+1}}_{s_i} & \cdots & \alpha^{j_t}_{s_i} & \alpha^{t+2}_{s_i} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ \alpha^{j_1}_{s_{i+1}} & \cdots & \alpha^{j_{i-1}}_{s_{i+1}} & \alpha^{j_{t+1}}_{s_{i+1}} & \alpha^{j_{i+1}}_{s_{i+1}} & \cdots & \alpha^{j_t}_{s_{i+1}} & \alpha^{t+2}_{s_{i+1}} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \alpha^{j_1}_{s_t} & \cdots & \alpha^{j_{i-1}}_{s_t} & \alpha^{j_{t+1}}_{s_t} & \alpha^{j_{i+1}}_{s_t} & \cdots & \alpha^{j_t}_{s_t} & \alpha^{t+2}_{s_t} \end{bmatrix}.$$

The lemma follows now immediately. □

We give now some results, which can be viewed as a generaliation of theorem 7.2.12. The next theorem states that the sum of the orientations of the $(T \cup \{j\})$-complete simplices in $A(T)$ is equal to the sum of the orientations of the T-complete simplices in $\cup_{k \in T} A(T \setminus \{k\})$. For a fixed v, let $H(T)$ be the set of $(T \cup \{j\})$-complete simplices in $A(T)$ $(j \notin T)$, and let $\bar{H}(T)$ be the set of T-complete simplices in $\cup_{k \in T} A(T \setminus \{k\})$.

THEOREM 7.3.8. Let the labelling rule be such that the conditions of Sperner's lemma are satisfied. Then

$$\sum_{\sigma \in H(T)} \text{or } \sigma = \sum_{\tau \in \bar{H}(T)} \text{or } \tau.$$

PROOF. The proof of this theorem runs along the lines of the proof of theorem 7.2.12. By applying the variable dimension restart algorithm paths of simplices in $A(T)$ are generated having T-complete common facets. Again three cases can happen.

a) Starting with a t-simplex in $A(T)$ having a T-complete facet τ in $A(T \setminus \{k\})$ for some $k \in T$, a $(T \cup \{j\})$-complete simplex σ in $A(T)$ is generated,or conversely. By corollary 7.3.4 and the lemmas 7.3.5, 7.3.6 and 7.3.7 we have that or τ = or σ.

b) Starting with a simplex in $A(T)$ having a T-complete facet τ_1 in $A(T \setminus \{k_1\})$ for some $k_1 \in T$, a simplex in $A(T)$ is generated having a T-complete facet τ_2 in $A(T \setminus \{k_2\})$ for some $k_2 \in T$. By the lemmas 7.3.5 -7.3.7 we have that or τ_1 = -or τ_2.

c) Starting with a $(T \cup \{j_1\})$-complete simplex σ_1 in $A(T)$ with $j_1 \notin T$, a $(T \cup \{j_2\})$-complete simplex σ_2 in $A(T)$ is generated $(j_2 \notin T)$ and from corollary 7.3.4 and the lemmas 7.3.5 and 7.3.6 it follows that or σ_1 = -or σ_2.

The theorem follows now immediately.   □

Observe that bd $A(T) = (bd(S^n) \cap A(T)) \cup (\cup_{k \in T} A(T \setminus \{k\}))$. Since the labelling rule is proper there are no T-complete $(t-1)$-simplices in bd $(S^n) \cap A(T)$. So theorem 7.3.8 states that the sum of the orientations of the T-complete simplices in bd $A(T)$ is equal to the sum of the orientations of the $(T \cup \{j\})$-complete simplices in $A(T)$ if the labelling rule is proper. Assume now that the labelling rule is not proper and let $\tilde{H}(T)$ be the set of the T-complete simplices in bd $A(T)$. Let τ be an element of $\tilde{H}(T)$. If, for some $k \in T$, τ is in $A(T \setminus \{k\})$, or τ is defined by definition 7.3.1. If τ is in bd$(S^n) \cap A(T)$, let σ be the simplex in $A(T)$ such that τ is a facet of σ and define or τ by or $\tau = \text{or}_\sigma \tau$ (see definition 7.3.3). Then we have the following corollary, whose proof is analogous to the proof of theorem 7.3.8.

COROLLARY 7.3.9. $\sum\limits_{\sigma \in H(T)}$ or $\sigma = \sum\limits_{\tau \in \tilde{H}(T)}$ or τ.

THEOREM 7.3.10. Let $T(k)$ be the collection of sets $T \subset I_{n+1}$ such that $|T| = k$, $k \leq n$. If the labelling rule satisfies the conditions of Sperner's lemma, then

$$\sum\limits_{T \in T(k)} \sum\limits_{\sigma \in H(T)} \text{or } \sigma = 1 \quad \text{for } k=1,\ldots,n.$$

PROOF. Again the proof follows by induction on k. For k=0 we have that $\{v\}$ is the only $\{\mathcal{U}(v)\}$-complete simplex of $A(\emptyset)$ and $\emptyset$ is the only element of $T(0)$. Moreover, or $\{v\}$ = det $[1]$ = 1. Hence

$$\sum\limits_{T \in T(0)} \sum\limits_{\sigma \in H(T)} \text{or } \sigma = \sum\limits_{\sigma \in H(\emptyset)} \text{or } \sigma = \text{or } \{v\} = 1.$$

Let the statement be true for k=j. We prove that it also holds for k = j+1. From theorem 7.3.8 it follows that

$$\sum\limits_{T \in T(j+1)} \sum\limits_{\sigma \in H(T)} \text{or } \sigma = \sum\limits_{T \in T(j+1)} \sum\limits_{\tau \in \tilde{H}(T)} \text{or } \tau.$$

Moreover we have that

$$\sum\limits_{\tau \in \tilde{H}(T)} \text{or } \tau = \sum\limits_{h \in T} \sum\limits_{\substack{\sigma \in A(T \setminus \{h\}) \\ \sigma \text{ is } T\text{-complete}}} \text{or } \sigma .$$

Hence

$$\sum_{T\in T(j+1)} \sum_{\sigma\in H(T)} \text{or } \sigma = \sum_{T\in T(j+1)} \sum_{h\in T} \sum_{\sigma\in A(T\setminus\{h\})} \text{or } \sigma =$$
$$\sigma \text{ is } T\text{-complete}$$

$$= \sum_{T\in T(j)} \sum_{i\not\in T} \sum_{\sigma\in A(T)} \text{or } \sigma =$$
$$\sigma \text{ is } (T\cup\{i\})\text{-complete}$$

$$= \sum_{T\in T(j)} \sum_{\sigma\in H(T)} \text{or } \sigma = 1. \qquad \Box$$

By taking $v = e(1)$ we have that $T = I_k$ is the only feasible element of $T(k)$, i.e. $\overset{o}{A}(T) = \emptyset$ for all $T \in T(k)$, $T \neq I_k$. Hence theorem 7.2.12 is a special case of theorem 7.3.10.

THEOREM 7.3.11. Let $N(S^n)$ be the number of completely labelled simplices in S^n. Then

$$N(S^n) \geq \sum_{j=1}^{n+1} |d_j|$$

where $d_j = \sum_{\sigma\in\tilde{H}(I_{n+1}\setminus\{j\})} \text{or } \sigma.$

PROOF. With corollary 7.3.9 it follows that $\sum_{\sigma\in H(I_{n+1}\setminus\{j\})} \text{or } \sigma =$

$= \sum_{\tau\in\tilde{H}(I_{n+1}\setminus\{j\})} \text{or } \tau = d_j$. Moreover, if $\sigma\in H(I_{n+1}\setminus\{j\})$, then σ is

completely labelled. Hence the number of completely labelled simplices in $A(I_{n+1}\setminus\{j\})$ is at least equal to $|d_j|$. The theorem follows by adding over all $j \in I_{n+1}$. $\qquad \Box$

7.4. A SEARCH TO AN ODD NUMBER OF COMPLETELY LABELLED SIMPLICES.

In the previous section we defined the orientation of a $(T\cup\{j\})$-complete simplex in $A(T)$. However we restricted ourselves to the standard triangulation of S^n. In this section we generalize the definitions 7.3.1 and 7.3.3 for simplices of a PK_δ triangulation of R^n. P is an $n\times(n+1)$ triangulation matrix, i.e. $p(n+1) = -\sum_{j=1}^{n} p(j)$ and the $n\times n$ matrix consisting

of the first n columns of P is nonsingular. A possible choice is P = U (c.f. section 6.2). Recall that the regions A(T) depend on P and are defined by

$$A(T) = \{x \in R^n \mid x = v + \sum_{j \in T} \lambda_j p(j) \text{ for nonnegative numbers } \lambda_j, \ j \in T\}$$

where the grid point v is the starting point of the variable dimension restart algorithm.

Let T be a subset of I_{n+1}, $|T| \leq n$, and $(s_1, \ldots, s_t)$ the permutation of the elements of T such that $s_j < s_{j+1}$, $j = 1, \ldots, t-1$. Then the orientation of a $(T \cup \{j\})$-complete simplex $\sigma(y^1, \pi(T))$ in A(T) and the orientation of a T-complete simplex τ with respect to σ where τ is a facet of the simplex $\sigma(y^1, \pi(T))$ in A(T), are defined as in the definitions 7.3.1 and 7.3.3 respectively, with $\alpha^i = (\alpha^i_{s_1}, \ldots, \alpha^i_{s_t})^T$ the t-vector such that

$$y^i = v + \sum_{s_j \in T} \delta \alpha^i_{s_j} p(s_j) \qquad i = 1, \ldots, t+1.$$

THEOREM 7.4.1. For some $T^* \subset I_{n+1}$, $|T^*| = n$, let $\sigma(y^1, \pi(T^*))$ be a completely labelled simplex in $A(T^*)$ generated by the variable dimension restart algorithm. Then or $\sigma = 1$.

PROOF. The proof is in fact constructive. Observe that the algorithm starts with the zero-dimensional simplex $\{v\}$ and that $or\{v\} = 1$. For some h, let $T^1, \ldots, T^h$ with $T^1 = \{\ell(v)\}$ and $T^h = T^*$ be the sequence of subsets of I_{n+1} such that the algorithm generates t-simplices $(t \geq 1)$ in successively $A(T^1), A(T^2), \ldots, A(T^h)$. Define $T^0 = \emptyset$ and $T^{h+1} = I_{n+1}$. For any i, $1 \leq i \leq h$, we have one of the following cases

a) $T^{i-1} \subset T^i \subset T^{i+1}$. Then the dimension is increased by changing from T^{i-1} to T^i and from T^i to T^{i+1}. Starting with a simplex in $A(T^i)$ having a T^i-complete facet τ in $A(T^{i-1})$, a T^{i+1}-complete simplex σ of $A(T^i)$ is generated. By corollary 7.3.4 and the lemmas 7.3.5, 7.3.6 and 7.3.7, we have that or $\tau =$ or σ (If i=h, the dimension is not increased).

b) $T^{i-1} \subset T^i$ and $T^{i+1} \subset T^i$. Then the dimension is increased by changing from T^{i-1} to T^i and decreased by changing from T^i to T^{i+1}. Starting with a simplex in $A(T^i)$ having a T^i-complete facet τ_1 in $A(T^{i-1})$, a simplex in $A(T^i)$ is generated having a T^i-complete facet τ_2 in $A(T^{i+1})$.

By the lemmas 7.3.5.-7.3.7, we have that or τ_2 = -or τ_1.

c) $T^{i+1} \subset T^i \subset T^{i-1}$. The dimension is decreased by changing from T^{i-1} to T^i and from T^i to T^{i+1}. This case is the reverse of case a. Starting with a T^{i-1}-complete simplex σ in $A(T^i)$ a simplex in $A(T^i)$ is generated having a T^i-complete facet τ in $A(T^{i+1})$. Again we have that or σ = or τ.

d) $T^i \subset T^{i-1}$ and $T^i \subset T^{i+1}$. The dimension is decreased by changing from T^{i-1} to T^i and increased by changing from T^i to T^{i+1}. Starting with a T^{i-1}-complete simplex σ_1 in $A(T^i)$, a T^{i+1}-complete simplex σ_2 in $A(T^i)$ is generated. From corollary 7.3.4 and the lemmas 7.3.5 and 7.3.6, it follows that or σ_2 = -or σ_1.

Observe that the orientation changes in the cases b and d. Let $i_1, \ldots, i_r$ with $i_j < i_{j+1}$, $j=1, \ldots, r-1$, be the sequence of indices for which case b occurs. Then we must have a sequence of indices $h_1, \ldots, h_r$ with $i_j < h_j < i_{j+1}$, $j=1, \ldots, r-1$ and $h_r > i_r$ for which case d occurs. By these arguments it follows that or σ = or$\{v\}$ = 1. $\square$

Let P^1 and P^2 be two $n \times (n+1)$ triangulation matrices such that for some i_1 and i_2 holds

$$(7.4.1) \quad \begin{cases} p^1(j) = p^2(j) & j \neq i_1, i_2 \\ p^1(i_1) = p^2(i_2) \text{ and } p^1(i_2) = p^2(i_1) \end{cases}$$

where $p^h(j)$ is the j-th column of P^h, h=1,2.

So P^2 is obtained from P^1 by interchanging the columns i_1 and i_2. Clearly, each simplex of the P^1K_δ triangulation is a simplex of the P^2K_δ triangulation and conversely. The next theorem states that the sign of the orientation of a completely labelled simplex changes if two columns of the triangulation matrix are interchanged.

THEOREM 7.4.2. Let P^1 and P^2 be two $n \times (n+1)$ triangulation matrices such that (7.4.1) holds. Let σ be a completely labelled simplex and let or$^h\sigma$ be the orientation of σ if P^h is the triangulation matrix, h=1,2. Then

$$or^1\sigma = -or^2\sigma.$$

PROOF. In theorem 7.3.2 it was proved that the orientation of a completely labelled simplex, as defined in definition 7.3.1, of the standard triangulation of S^n is independent of v. It can be easily verified that this theorem also holds for a completely labelled simplex of a PK_δ triangulation of R^n. Therefore we can choose v such that σ is in some region $A(T)$ ($|T| = n$) with i_1 and i_2 both an element of T. For some vertex y^i of σ let $\alpha^{hi} = (\alpha^{hi}_{s_1}, \ldots, \alpha^{hi}_{s_n})^T$ be the vector such that

$$y^i = v + \sum_{s_j \in T} \delta\alpha^{hi}_{s_j} p^h(s_j) \qquad h=1,2.$$

Clearly, for all i, $i=1,\ldots,n+1$, $\alpha^{1i}_j = \alpha^{2i}_j$ for all $j \neq i_1, i_2$ whereas $\alpha^{1i}_{i_1} = \alpha^{2i}_{i_2}$ and $\alpha^{1i}_{i_2} = \alpha^{2i}_{i_1}$. Hence α^{2i} is obtained from α^{1i} by interchanging two elements. So, with $j_1, \ldots, j_{n+1}$ such that $\ell(y^{j_i})=s_i$, $1 \le i \le n$ and $\ell(y^{j_{n+1}})=h$, $h \notin T$,

$$or^1\sigma = \det\begin{bmatrix} 1 & \cdots & 1 \\ \alpha^{1j_1} & \cdots & \alpha^{1j_{n+1}} \end{bmatrix} = -\det\begin{bmatrix} 1 & \cdots & 1 \\ \alpha^{2j_1} & \cdots & \alpha^{2j_{n+1}} \end{bmatrix} = -or^2\sigma.$$

$\square$

Now we have a tool to generate more completely labelled simplices (if exist) by using two matrices P^1 and P^2 such that (7.4.1) holds. Assume that the variable dimension restart algorithm generates a completely labelled simplex σ_1 with P^1K_δ the underlying triangulation. Clearly by theorem 7.4.1 we have that $or^1\sigma_1 = 1$ and by theorem 7.4.2 $or^2\sigma_1 = -1$. Let the variable dimension restart algorithm now be applied with P^2K_δ as underlying triangulation. Then either a path of simplices going to infinity is generated or a path which terminates with a completely labelled simplex σ_2. Clearly $or^2\sigma_2 = 1$ and hence $\sigma_2 \neq \sigma_1$. Moreover, a search for a third completely labelled simplex can be made. Therefore, let T^h_j ($|T^h_j| = n$) be the set of indices such that $\sigma_j \in A(T^h_j)$ if P^hK_δ is the underlying triangulation, $h=1,2$. So

$$T^2_j = T^1_j \text{ if } i_1, i_2 \in T^1_j$$
$$T^2_j = (T^1_j \cup \{i_2\}) \setminus \{i_1\} \text{ if } i_2 \notin T^1_j$$

$$T_j^2 = (T_j^1 \cup \{i_1\}) \setminus \{i_2\} \text{ if } i_1 \notin T_j^1.$$

Now using the $P^2 K_\delta$ triangulation and starting with the completely labelled simplex σ_1 by replacing the vertex with the label k, $k \notin T_1^2$, the algorithm generates a sequence of simplices of variable dimension such that the common facet of two adjacent simplices in $A(T)$ is T-complete. Clearly, by theorem 7.4.1 this sequence of simplices cannot terminate with the zero-dimensional simplex $\{v\}$ since $or^2 \sigma_1 = -1$. Hence either a sequence of simplices going to infinity is generated or the algorithm terminates with a completely labelled simplex $\sigma_3 \neq \sigma_1$. Using analogous arguments as in the proof of theorem 7.4.1, it follows that $or^2 \sigma_3 = -or^2 \sigma_1 = 1$. However using the standard arguments, $\sigma_3 \neq \sigma_2$, since there exists a path of simplices which starts with $\{v\}$ and terminates with σ_2. Therefore three completely labelled simplices are generated. The method is illustrated in the figures 7.4.1 and 7.4.2. We have that

$$P^1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \text{ and } P^2 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}.$$

For $i=1,2$, we define $A^i(T)$ by

$$A^i(T) = \{x \in R^n \mid x = v + \sum_{j \in T} \lambda_j p^i(j) \text{ for nonnegative numbers } \lambda_j, \ j \in T\}.$$

In figure 7.4.1 the regions $A^1(T)$ are given. Starting in v and using the $P^1 K$ triangulation the completely labelled simplex σ_1 is generated. Observe that $T_1^1 = \{1,3\}$. Hence $T_1^2 = \{1,3\} \cup \{2\}/\{1\} = \{2,3\}$. Using the $P^2 K$ triangulation and starting with σ_1 by replacing the vertex having label 1, the algorithm generates the completely labelled simplex σ_3, as is illustrated in figure 7.4.2. Moreover, starting from v the completely labelled simplex σ_2 is generated.

We give now a condition for the existence of at least k completely labelled simplices, $k=1,3,5,\ldots$.

THEOREM 7.4.3. Let f be a continuous function on R^n. Assume that for some integer r there exists an increasing sequence of positive numbers

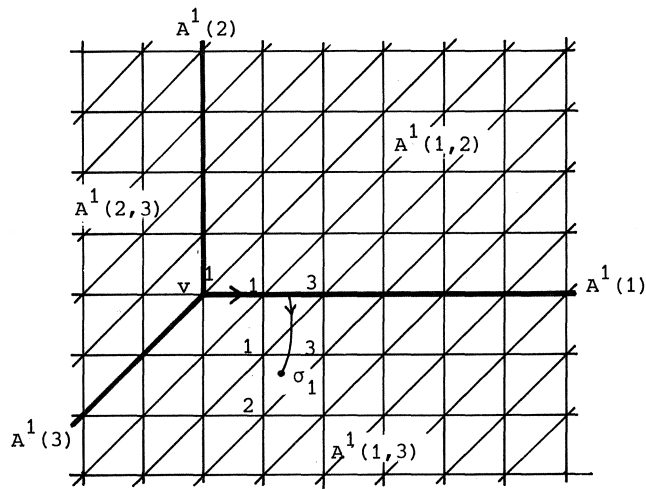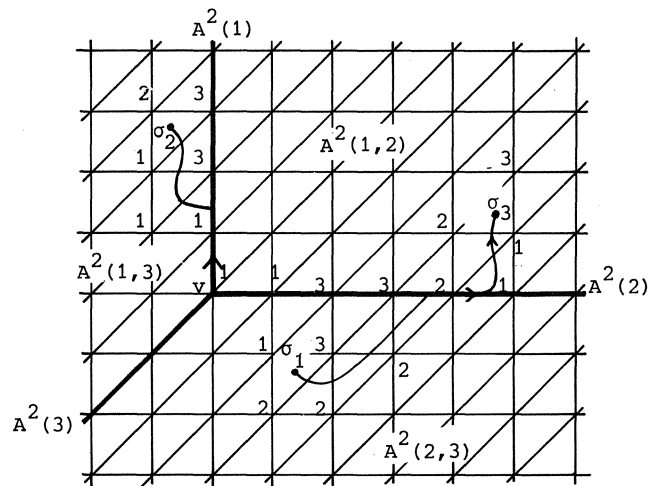Figure 7.4.1. Using the P^1K triangulation σ_1 is generated.

Figure 7.4.2. Using the P^2K triangulation σ_2 and σ_3 are generated.

$\mu_1, \mu_2, \ldots, \mu_r$ and that there exist a number $\varepsilon > 0$ and two indices i_1, i_2, such that for all $x \in \tilde{B}^n(\mu_h)$, $h=1,2,\ldots,r$, holds that

a) For some i, $x_i = -\mu_h$ and $f_j(x) - x_j > \varepsilon$ for at least one index $j \in I_n$

or

b) For some i, $x_i = \mu_h$ and we have that

    (i) $f_j(x) - x_j < -\varepsilon$ for all $j \in I_n$

or

    (ii) there exists at least one index j, $j \neq s$, with

        $f_j(x) - x_j > f_s(x) - x_s + \varepsilon$, where $s=i$ in the case that h is odd and, if h is even, $s=i$ when $i \neq i_1, i_2$, $s=i_2$ when $i=i_1$ and $s=i_1$ when $i=i_2$.

Let R^n be triangulated by K_δ. Then, if δ is small enough, at least $2h-1$ completely labelled simplices in $\text{conv}(\tilde{B}^n(\mu_h))$, $h=1,2,\ldots,r$, can be generated by applying the variable dimension restart algorithm.

PROOF. Let P^1 be the $n \times (n+1)$ matrix with $p^1(i) = e(i)$, $i \in I_n$ and $p^1(n+1) = -e$ and P^2 the $n \times (n+1)$ matrix with $p^2(i) = p^1(i)$ for $i \neq i_1, i_2$, $p^2(i_1) = p^1(i_2)$ and $p^2(i_2) = p^1(i_1)$. Clearly, both matrices induce the K triangulation of R^n. Let the grid size δ be so small that

$$\sup_{(x,y) \in \sigma} \max_{i \in I_n} \left| f_i(x) - x_i - f_i(y) + y_i \right| < \tfrac{1}{2}\varepsilon$$

for every simplex σ such that $\sigma \cap \tilde{B}^n(\mu_h) \neq \emptyset$ for some $h \in I_r$. Clearly, if h is odd, the conditions of theorem 6.3.2 are satisfied. So with $P^1 K_\delta$ as underlying triangulation and starting in $\text{conv}(\tilde{B}^n(\mu_h))$ the algorithm generates a completely labelled simplex in $\text{conv}(\tilde{B}^n(\mu_h))$, $h=1,3,5,\ldots,r$ (or $r-1$). However recall, that the convergence proof is based on the fact that if $\sigma(w^1,\ldots,w^{t+1})$ is a simplex in $A(T)$ and $\sigma \cap \tilde{B}^n(\mu_h) \neq \emptyset$, there exists an index $i \in T$ such that $\ell(w^k) \neq i$, $k=1,\ldots,t+1$. It can be easily verified that this argument also holds if h is even with $P^2 K_\delta$ the underlying triangulation. Hence, starting in $\text{conv}(\tilde{B}^n(\mu_h))$ and with $P^i K_\delta$ the underlying triangulation, where $i = 2-h \pmod 2$, the algorithm generates a

completely labelling simplex in $\text{conv}(\tilde{B}^n(\mu_h))$. Now the theorem follows by induction on h. Starting in a grid point v in $\text{conv}(\tilde{B}^n(\mu_1))$ and using the P^1K_δ triangulation the algorithm generates a completely labelled simplex σ_1 in $\text{conv}(\tilde{B}^n(\mu_1))$. Clearly v and σ_1 are in $\text{conv}(\tilde{B}^n(\mu_2))$. Hence using the P^2K_δ triangulation and starting with v respectively σ_1 the algorithm generates completely labelled simplices σ_2 and σ_3 in $\text{conv}(\tilde{B}^n(\mu_2))$ with, as argued above, $\sigma_i \neq \sigma_j$ if $i \neq j$.

Now, let σ_{2k-2} and σ_{2k-1} be the completely labelled simplices in $\text{conv}(\tilde{B}^n(\mu_k))$ found by the algorithm. Clearly σ_{2k-2} and σ_{2k-1} are in $\text{conv}(\tilde{B}^1(\mu_{k+1}))$. Now using the P^iK_δ triangulation where $i=k\,(\text{mod }2)+1$, and starting with σ_{2k-2} respectively σ_{2k-1} as described above, the algorithm generates two completely labelled simplices σ_{2k} and σ_{2k+1} in $\text{conv}(\tilde{B}^n(\mu_{k+1}))$. If k+1 is odd we have that there exist paths of simplices between $\{v\}$ and σ_1, σ_{4h-2} and σ_{4h}, $h=1,\ldots,\frac{1}{2}k-1$, and σ_{4h-1} and σ_{4h+1}, $h=1,\ldots,\frac{1}{2}k-1$. Hence $\sigma_{2k} \neq \sigma_i$, $i < 2k$, and $\sigma_{2k+1} \neq \sigma_i$, $i < 2k+1$. If k+1 is even we have that there exist paths of simplices between $\{v\}$ and σ_2, σ_{4h} and σ_{4h+2}, $h=1,\ldots,\frac{1}{2}(k-3)$ and σ_{4h-3} and σ_{4h-1}, $h=1,\ldots,\frac{1}{2}(k-1)$, which again implies that $\sigma_{2k} \neq \sigma_i$, $i < 2k$, and $\sigma_{2k+1} \neq \sigma_i$, $i < 2k+1$. Hence, for $h=1,\ldots,r$, at least 2h-1 completely labelled simplices can be generated in $\text{conv}(\tilde{B}^n(\mu_h))$. □

COROLLARY 7.4.4. If the conditions of theorem 7.4.3 are satisfied there exist at least 2h-1 completely labelled simplices in $\text{conv}(\tilde{B}^n(\mu_h))$, $h=1,..,r$.

Note that the conditions of theorem 7.4.3 can be easily modified. For example the theorem is also true if in b(ii) the condition for h is even and h is odd interchanged. Approximating fixed points, it is worthwile to investigate if the function satisfies conditions similar to those of theorem 7.4.3. Finally, we remark that a condition for the existence of two fixed points is given by Allgower and Georg [1980]. They present also a simplicial algorithm to approximate both points. The algorithm is based on an idea of Jeppson [1972].

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CHAPTER 8

A CONTINUOUS DEFORMATION ALGORITHM WITH ARBITRARY
REFINEMENT FACTORS

8.1. INTRODUCTION.

In chapter 4 we discussed the continuous deformation algorithm developed by Eaves [1972] and Eaves and Saigal [1972], to compute a fixed point of a function or mapping. These authors constructed a triangulation of $S^n \times [1,\infty)$ respectively $R^n \times [1,\infty)$. Restricting ourselves to the first set, $S^n \times \{d_k\}$ was triangulated with grid size d_k^{-1} by the standard triangulation, where $d_0, d_1, \ldots$ is an increasing sequence of integers with $d_0=1$. Then for each i, $i=0,1,\ldots,S^n \times [d_i, d_{i+1}]$ was triangulated by $(n+1)$-simplices such that the vertices of these simplices are grid points in $S^n \times \{d_i\}$ or $S^n \times \{d_{i+1}\}$. Until recently, only triangulations were known such that $d_{i+1}/d_i \leq 2$ for all i (see Todd [1976a]). This restriction is a serious drawback in comparison with restart methods.

In this chapter, which is based on Van der Laan and Talman [1980b] we present a continuous deformation algorithm which allows for an arbitrarily chosen factor of incrementation $k_i=d_{i+1}/d_i$, $i \geq 0$. In chapter 5 we showed how the variable dimension restart algorithm can be interpreted as an algorithm with $n+1$ points on an extra level. In this interpretation $S^n \times \{0\}$ is triangulated with grid size one, $S^n \times \{1\}$ with an arbitrarily chosen grid size m^{-1} and $S^n \times [0,1]$ is triangulated by $n+1$ simplices such that all vertices are grid points in $S^n \times \{0\}$ or $S^n \times \{1\}$. Van der Laan and Talman used this triangulation of $S^n \times [0,1]$ to construct a triangulation of $S^n \times [1,\infty)$ such that the factors d_{i+1}/d_i, $i=0,1,\ldots$, are arbitrary integers larger than one. Using this triangulation in a fixed point algorithm the levels d_i can be chosen throughout the algorithm, i.e. d_{m+1} is chosen as soon as a completely labelled simplex σ_m on level d_m is found. Moreover we have that the grid point on level d_{m+1} connected with the vertices of σ_m on level d_m is an arbitrarily chosen grid point $v(\sigma_m)$ of

$\sigma_m \times \{d_{m+1}\}$. Of course, the choices of d_{m+1} and $v(\sigma_m)$ are done on ground of the information obtained from the completely labelled simplex $\sigma_m \times \{d_m\}$. As soon as they are chosen, the triangulation of $S^n \times [d_m, d_{m+1}]$ is fixed throughout the remaining of the algorithm.

In section 2 the triangulation of $S^n \times [1, \infty)$ is described. In section 3 we give a concise description of the algorithm, whereas the replacement steps are given in section 4. In section 5 we discuss the application on R^n. In section 6 some remarks are made and some numerical results are presented.

8.2. TRIANGULATION OF $S^n \times [1, \infty)$.

Let $d_0, d_1, \ldots$ be a sequence of increasing integers such that $d_0 = 1$ and $d_{i+1} = k_i d_i$, $i = 0, 1, \ldots$, with k_i an arbitrary integer larger than 1. In the following we triangulate, for some m, the set $S^n \times [d_m, d_{m+1}]$ such that all grid points are points of $S^n \times \{d_m\}$ or $S^n \times \{d_{m+1}\}$. Combining the triangulations of $S^n \times [d_i, d_{i+1}]$ over all pairs (d_i, d_{i+1}) we obtain a triangulation of $S^n \times [1, \infty)$. For $i = 0, 1, \ldots$ let G_i be the standard triangulation of S^n with grid size d_i^{-1}. So, by the definitions 3.4.1 and 5.2.1, G_m is the collection of n-simplices $\sigma(y^1, \pi(T))$ $(T \subset I_{n+1}, |T| = n)$ with vertices $y^1, \ldots, y^{n+1}$ such that

a) the components of y^1 are a multiple of d_m^{-1}

b) $\pi(T) = (\pi_1, \ldots, \pi_n)$ is a permutation of the elements of T

c) $y^{i+1} = y^i + q(\pi_i)/d_m$ $i = 1, \ldots, n$,

where $q(j)$ is the j-th column of the $(n+1) \times (n+1)$ matrix defined in section 5.2. Observe that $y^1 = y^{n+1} + q(\pi_{n+1})/d_m$ where π_{n+1} is the element of I_{n+1} not in T. Therefore, in the following an n-simplex $\sigma(y^1, \pi(T))$ is also denoted by $\sigma(y^1, \pi(I_{n+1}))$ where $\pi(I_{n+1}) = (\pi(T), \pi_{n+1})$ with $\{\pi_{n+1}\} = I_{n+1} \setminus T$. As pointed out in section 5.2 we have that every simplex has n+1 representations, since each vertex can be chosen to be y^1. However, it will appear that it is more appropriate to represent an n-simplex in a unique way. This can be done as follows (see also Eaves and Saigal [1972]). For a given grid point w of G_m, let $\alpha(w) = (\alpha_1(w), \ldots, \alpha_n(w))^T$ be the n-vector with

$$\alpha_h(w) = (1 - \sum_{j=1}^{h} w_j) d_m, \quad h = 1, \ldots, n.$$

Clearly, every $\alpha_h(w)$ is an integer. Now we define a "labelling function" s from the set of grid points of G_m to the set I_{n+1} by

$$s(w) = 1 + (\sum_{h=1}^{n} \alpha_h(w)) \bmod (n+1).$$

__LEMMA 8.2.1.__ Let $\sigma(y^1, \pi(I_{n+1}))$ be an n-simplex of G_m. Then, for all $i \in I_{n+1}$, there exists an index j_i such that $s(y^{j_i}) = i$.

__PROOF.__ We prove that for each j, $j = 1, \ldots, n$,

$$(8.2.1) \qquad s(y^{j+1}) = s(y^j) \bmod (n+1) + 1.$$

Then the lemma follows immediately. For some j, let $s(y^j) = i$. First suppose that $\pi_j \neq n+1$. Then $y_h^{j+1} = y_h^j$, $h \neq \pi_j$, $\pi_j + 1$, $y_h^{j+1} = y_h^j - d_m^{-1}$, $h = \pi_j$ and $y_h^{j+1} = y_h^j + d_m^{-1}$, $h = \pi_j + 1$. Hence $\alpha_h(y^{j+1}) = \alpha_h(y^j)$ for $h \neq \pi_j$ and $\alpha_h(y^{j+1}) = \alpha_h(y^j) + 1$ for $h = \pi_j$. Therefore $\sum_{h=1}^{n} \alpha_h(y^{j+1}) = 1 + \sum_{h=1}^{n} \alpha_h(y^j)$, which proves that (8.2.1) holds. If $\pi_j = n+1$ we have that $y_h^{j+1} = y_h^j$, $h \neq 1, n+1$, $y_1^{j+1} = y_1^j + d_m^{-1}$ and $y_{n+1}^{j+1} = y_{n+1}^j - d_m^{-1}$.

Hence $\alpha_h(y^{j+1}) = \alpha_h(y^j) - 1$, $h = 1, \ldots, n$, and therefore $\sum_{h=1}^{n} \alpha_h(y^{j+1}) = \sum_{h=1}^{n} \alpha_h(y^j) + 1 - (n+1)$, which again proves that (8.2.1) holds. $\square$

The lemma means that each simplex is "completely labelled" with respect to the function $s(w)$. Moreover if $s(y^1) = 1$ we have that $s(y^j) = j$, $j = 1, \ldots, n+1$. In the following we assume that every n-simplex $\sigma(y^1, \pi(I_{n+1}))$ of G_m is represented in such a way that $s(y^1) = 1$. This representation is called the s-representation of a simplex σ. Let $\sigma_1(y^1, \pi(I_{n+1}))$ have the s-representation and let $\sigma_2(\bar{y}^1, \bar{\pi}(I_{n+1}))$ be the adjacent simplex, obtained from σ_1 by replacing y^i. Then σ_2 has also the s-representation if y^i is replaced according to table 8.2.1. This can easily be seen by observing that $\bar{y}^h = y^h$ $h \neq i$. Hence $s(\bar{y}^h) = s(y^h)$, $h \neq i$ and therefore also $s(\bar{y}^i) = s(y^i)$.

Table 8.2.1. s is the index of the vertex to be replaced.

	y^1 becomes	$\pi(I_{n+1})$ becomes
s=1	$y^{n+1} + q(\pi_1)/d_m$	$(\pi_{n+1}, \pi_2, \ldots, \pi_n, \pi_1)$
$2 \leq s \leq n+1$	y^1	$(\pi_1, \ldots, \pi_{s-2}, \pi_s, \pi_{s-1}, \pi_{s+1}, \ldots, \pi_{n+1})$

We are now ready to triangulate $s^n \times [d_m, d_{m+1}]$ for some given m. First we take a particular simplex $\sigma_m(u^1, \beta(I_{n+1}))$ of the triangulation G_m, called the starting simplex on level d_m, with u^1 and $\beta(I_{n+1})$ chosen in such a way that we have the s-representation of σ_m. Then we choose nonnegative integers $\lambda_1^m, \ldots, \lambda_{n+1}^m$ with sum equal to $k_m = d_{m+1}/d_m$ and we call the point

$$v(\sigma_m) = \sum_{i=1}^{n+1} k_m^{-1} \lambda_i^m u^i$$

the centrepoint of σ_m. Observe that $v(\sigma_m)$ is a grid point of G_{m+1}. Applying the continuous deformation algorithm to approximate a fixed point of a function or mapping, σ_m is the first completely labelled n-simplex generated by the algorithm on level d_m and λ_i^m, $0 \le i \le n+1$, are chosen such that $v(\sigma_m)$ is the grid point of G_{m+1} nearest to the approximate fixed point x^* in σ_m. Moreover, in practice we have that k_m is not predetermined but that k_m is chosen as soon as σ_m is generated. It will appear that the triangulation of $s^n \times [d_m, d_{m+1}]$ is determined by the numbers $\lambda_1^m, \ldots, \lambda_{n+1}^m$. Let $\tau(y^1, \pi(I_{n+1}))$ be an arbitrarily chosen simplex of G_m with $s(y^1) = 1$, i.e. τ has the s-representation. To triangulate $s^n \times [d_m, d_{m+1}]$ we first triangulate $\tau(y^1, \pi(I_{n+1})) \times [d_m, d_{m+1}]$ and then we prove that the union of the triangulations of $\tau \times [d_m, d_{m+1}]$ over all n-simplices τ of G_m is a triangulation of $s^n \times [d_m, d_{m+1}]$. In the sequel we call the grid point $v(\tau) = \sum_{i=1}^{n+1} k_m^{-1} \lambda_i^m y^i$ the centrepoint of the simplex $\tau(y^1, \pi(I_{n+1}))$. The triangulation of $s^n \times [d_m, d_{m+1}]$ will be such that, for all τ of G_m, the (n+1)-simplex which is the convex hull of $\tau \times \{d_m\}$ and $v(\tau) \times \{d_{m+1}\}$ is an (n+1)-simplex of this triangulation. So, in particular we have that the (n+1)-simplex ψ_m, being the convex hull of $\sigma_m \times \{d_m\}$ and $v(\sigma_m) \times \{d_{m+1}\}$ is a simplex of the triangulation.

To triangulate the set $\tau(y^1, \pi(I_{n+1})) \times [d_m, d_{m+1}]$ we define for any proper subset T of I_{n+1} the regions $\overset{\circ}{A}{}^\tau(T)$ by

$$\overset{\circ}{A}{}^\tau(T) = \{x \in \tau \mid x = v(\tau) + \sum_{j \in T} \mu_j q(\pi_j) \text{ for positive numbers } \mu_j, \ j \in T\}$$

Now a triangulation of $\tau \times [d_m, d_{m+1}]$ is obtained by connecting all the grid points x of G_{m+1} in $\overset{\circ}{A}{}^\tau(T)$ on level d_{m+1} with the vertices y^i, $i \notin T$, on level d_m. The proof that $\tau \times [d_m, d_{m+1}]$ is indeed triangulated is analogous to the proof of theorem 5.5.1. In fact, the triangulation of $\tau \times [d_m, d_{m+1}]$ is analogous to the triangulation of $s^n \times [0,1]$, as described in section 5.5. In comparison with the definition of the regions $\overset{\circ}{A}(T)$ in section 5.3, the regions

$\overset{\circ}{A}^\tau(T)$ are adapted for the fact that for the simplex $\tau(y^1,\pi(I_{n+1}))$, $\pi(I_{n+1})$ is an arbitrary permutation of the elements of I_{n+1}, whereas S^n can be considered as the simplex $\sigma(e(1),\bar\pi(I_{n+1}))$ of G_0 with $\bar\pi_i=i$, $i=1,\ldots,n+1$. For a simplex $\sigma(y^1,\pi(I_{n+1}))$ with $\pi(I_{n+1})=(1,3,2)$ the regions $A^\tau(T)$ (the closure of $\overset{\circ}{A}^\tau(T)$ in τ) are given in figure 8.2.1.
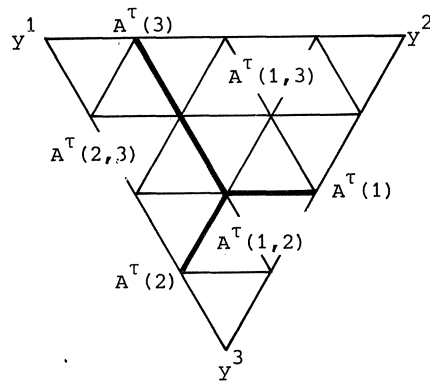
Figure 8.2.1. The regions $A^\tau(T)$ where $\tau(y^1,\pi(I_{n+1}))$ is a simplex with $\pi(I_{n+1})=(1,3,2)$.

Observe that $v(\tau)\times\{d_{m+1}\}$ is connected with all the vertices y^i, $i\in I_{n+1}$. Hence, $\mathrm{conv}(v(\tau)\times\{d_{m+1}\}\cup\tau\times\{d_m\})$ is indeed a simplex of the triangulation. Moreover, we have that for $m=0$ G_0 consists of only one simplex, being S^n itself. It can easily be seen that $s(e(i))=i$, $i\in I_{n+1}$. So it follows immediately that the triangulation of $S^n\times[d_0,d_1]$ is identical to the triangulation of $S^n\times[0,1]$ (section 5.5).

THEOREM 8.2.2. The union of the triangulations of $\tau(y^1,\pi(I_{n+1}))\times[d_m,d_{m+1}]$ over all n-simplices $\tau(y^1,\pi(I_{n+1}))$ of G_m triangulates $S^n\times[d_m,d_{m+1}]$.

PROOF. The triangulation of $\tau\times[d_m,d_{m+1}]$ is well-defined for any simplex τ of G_m. Let τ_1 and τ_2 be two adjacent n-simplices of G_m and let x be a grid point of G_{m+1} in the common facet. Then it is sufficient to prove that, if in the triangulation of $\tau_1\times[d_m,d_{m+1}]$, x is connected with a vertex y of $\tau_1\cap\tau_2$, x is also connected with y in the triangulation of $\tau_2\times[d_m,d_{m+1}]$.

Let τ_1 be $\tau_1(v^1, \gamma(I_{n+1}))$ and let τ_2 be $\tau_2(w^1, \delta(I_{n+1}))$. We have for both simplices the s-representation. Since τ_1 and τ_2 are adjacent, there exists a unique index j such that $v^i = w^i$, $i \neq j$ and $\gamma_i = \delta_i$ $i \neq j-1, j$ with the convention that $j-1 = n+1$ if $j=1$. Moreover $\delta_{j-1} = \gamma_j$ and $\delta_j = \gamma_{j-1}$. So all the vertices y in $\tau_1 \cap \tau_2$ have the same index, i.e. if for some index i $y = v^i$ we have that $y = w^i$ for the same index i. Therefore we only have to prove that for any grid point x of G_{m+1} in $\tau_1 \cap \tau_2$, $x \in \overset{\circ}{A}^{\tau_1}(T)$ implies that $x \in \overset{\circ}{A}^{\tau_2}(T)$. Since x is a grid point of G_{m+1} in the facet of τ_1 opposite to v^j, there exist unique integers Θ_i, $i \neq j-1, j$, such that

(8.2.1)
$$x = v^{j+1} + \sum_{i \neq j-1, j} \Theta_i q(\gamma_i)/d_{m+1}$$

with the convention that $v^{j+1} = v^1$ if $j = n+1$. Since $x \in \overset{\circ}{A}^{\tau_1}(T)$ for just one T, there exist also unique integers μ_h with $\mu_h > 0$, $h \in T$ and $\mu_h = 0$, $h \notin T$, such that

(8.2.2)
$$x = v(\tau_1) + \sum_{h=1}^{n+1} \mu_h q(\gamma_h)/d_{m+1}$$

or

$$x = \sum_{h=1}^{n+1} \lambda_h^m v^h / k_m + \mu_{j-1} q(\gamma_{j-1})/d_{m+1} + \mu_j q(\gamma_j)/d_{m+1} +$$

$$+ \sum_{h \neq j-1, j} \mu_h q(\gamma_h)/d_{m+1}$$

$$= \sum_{h=1}^{j} \lambda_h^m \{ v^{j+1} + \sum_{i=j+1}^{n+1} q(\gamma_i)/d_m + \sum_{i=1}^{h-1} q(\gamma_i)/d_m \}/k_m +$$

$$+ \sum_{h=j+1}^{n+1} \lambda_h^m \{ v^{j+1} + \sum_{i=j+1}^{h-1} q(\gamma_i)/d_m \}/k_m + \mu_{j-1} q(\gamma_{j-1})/d_{m+1} +$$

$$+ \mu_j q(\gamma_j)/d_{m+1} + \sum_{h \neq j-1, j} \mu_h q(\gamma_h)/d_{m+1}$$

$$= v^{j+1} + (\sum_{h=1}^{j} \lambda_h^m)(\sum_{i=j+1}^{n+1} q(\gamma_i)/d_{m+1}) + \sum_{h=1}^{j-1} \sum_{i=1}^{h-1} \lambda_h^m q(\gamma_i)/d_{m+1} +$$

$$+ \sum_{i=1}^{j-2} \lambda_j^m q(\gamma_i)/d_{m+1} + \lambda_j^m q(\gamma_{j-1})/d_{m+1} + \sum_{h=j+1}^{n+1} \sum_{i=j+1}^{h-1} \lambda_h^m q(\gamma_i)/d_{m+1} +$$

$$+ \mu_{j-1} q(\gamma_{j-1})/d_{m+1} + \mu_j(- \sum_{i \neq j} q(\gamma_i))/d_{m+1} + \sum_{h \neq j-1, j} \mu_h q(\gamma_h)/d_{m+1}.$$

Defining $\Theta_i^!$ to be the coefficient of $q(\gamma_i)$, $i \neq j-1,j$, we obtain that

(8.2.3) $\qquad x = v^{j+1} + \sum_{i \neq j-1,j} \Theta_i^! q(\gamma_i)/d_{m+1} + (\lambda_j^m + \mu_{j-1} - \mu_j) q(\gamma_{j-1})/d_{m+1}.$

Comparing (8.2.3) with (8.2.1) we obtain that $\Theta_i^! = \Theta_i$, $i \neq j-1,j$ and

(8.2.4) $\qquad \lambda_j^m + \mu_{j-1} - \mu_j = 0.$

Moreover from (8.2.2) it follows that

$$x = \sum_{h \neq j} \lambda_h^m v^h / k_m + \lambda_j^m (v^{j-1} + q(\gamma_{j-1})/d_m)/k_m + \mu_{j-1} q(\gamma_{j-1})/d_{m+1} +$$

$$+ \mu_j q(\gamma_j)/d_{m+1} + \sum_{h \neq j-1,j} \mu_h q(\gamma_h)/d_{m+1}.$$

Hence, since $v^i = w^i$, $i \neq j$, $\gamma_i = \delta_i$, $i \neq j-1,j$, $\gamma_{j-1} = \delta_j$ and $\gamma_j = \delta_{j-1}$, we obtain that

$$x = \sum_{h \neq j} \lambda_h^m w^h / k_m + \lambda_j^m (w^{j-1} + q(\delta_j)/d_m)/k_m + \mu_{j-1} q(\delta_j)/d_{m+1} +$$

$$+ \mu_j q(\delta_{j-1})/d_{m+1} + \sum_{h \neq j-1,j} \mu_h q(\delta_h)/d_{m+1} + \lambda_j^m q(\delta_{j-1})/d_{m+1} -$$

$$- \lambda_j^m q(\delta_{j-1})/d_{m+1}.$$

From (8.2.4) it follows that $\mu_j - \lambda_j^m = \mu_{j-1}$ and $\mu_{j-1} + \lambda_j^m = \mu_j$. Consequently,

$$x = v(\tau_2) + \sum_{h=1}^{n+1} \mu_h q(\delta_h)/d_{m+1}$$

with μ_h the same coefficients as in (8.2.2). Hence $x \in \overset{\circ}{A}^{\tau_2}(T)$ which proves the theorem. $\qquad \square$

From the proof of theorem 8.2.2 we obtain the following corollary.

COROLLARY 8.2.3. If a grid point x of G_{m+1} belongs to two adjacent simplices $\tau_1(v^1, \gamma(I_{n+1}))$ and $\tau_2(w^1, \delta(I_{n+1}))$ of G_m and if, for some T,

$$x = v(\tau_1) + \sum_{h \in T} \mu_h q(\gamma_h)/d_{m+1},$$

then, for the same coefficients μ_h,

$$x = v(\tau_2) + \sum_{h \in T} \mu_h q(\delta_h)/d_{m+1} .$$

This important fact will be frequently used in the replacement steps of the algorithm.

Theorem 8.2.2 is illustrated in figure 8.2.2 for $n=2$, $d_1=4$, $d_2=16$ and $\lambda^1=(2,1,1)$. So the grid size of the triangulation on level 1 is $1/4$ and the grid size of the triangulation on level d_2 is $1/16$. Let $\sigma_1(u^1, \beta(I_{n+1}))$ with $u^1=(\frac{1}{2},\frac{1}{4},\frac{1}{4})^T$ and $\beta(I_{n+1})=(1,3,2)$ be the starting simplex on level d_1. Then the centrepoint u of σ_1 is $v(\sigma_1)=\sum_{i=1}^{n+1} \lambda_i^1 u^i = (7/16, 3/8, 3/16)^T$. For the arbitrarily chosen simplex $\tau(y^1, \pi(I_{n+1}))$ of G_1 with $y^1 = (0, \frac{1}{2}, \frac{1}{2})^T$ and $\pi(I_{n+1}) = (2,3,1)$ we have that the centrepoint v is $v(\tau)=(1/16, 3/8, 9/16)^T$. In the figure the triangulation G_2 on level d_2 is pictured. Furthermore, the vertices of G_1 are drawn heavily. For any simplex τ of G_1 the regions $A^\tau(T)$ are indicated by the heavy lines. Observe that for any two simplices τ_1 and τ_2 of G_1 we have that

$$(\tau_1 \cap \tau_2) \cap A^{\tau_1}(T) = (\tau_1 \cap \tau_2) \cap A^{\tau_2}(T) .$$

As is illustrated for the simplices τ and ρ with τ as the simplex defined above and $\rho = \rho(\bar{y}^1, \bar{\pi}(I_{n+1}))$ with $\bar{y}^1=(\frac{1}{4},0,\frac{3}{4})$ and $\bar{\pi}(I_{n+1})=(1,3,2)$, the region $A^\tau(1,2)$ adjoins in a consistent way to the region $A^\rho(1,2)$. The grid points of $A^\tau(1,2)$ and $A^\rho(1,2)$ on level d_2 are connected with the heavily drawn vertex $(\frac{1}{4},\frac{1}{4},\frac{1}{2})^T$ on level d_1.
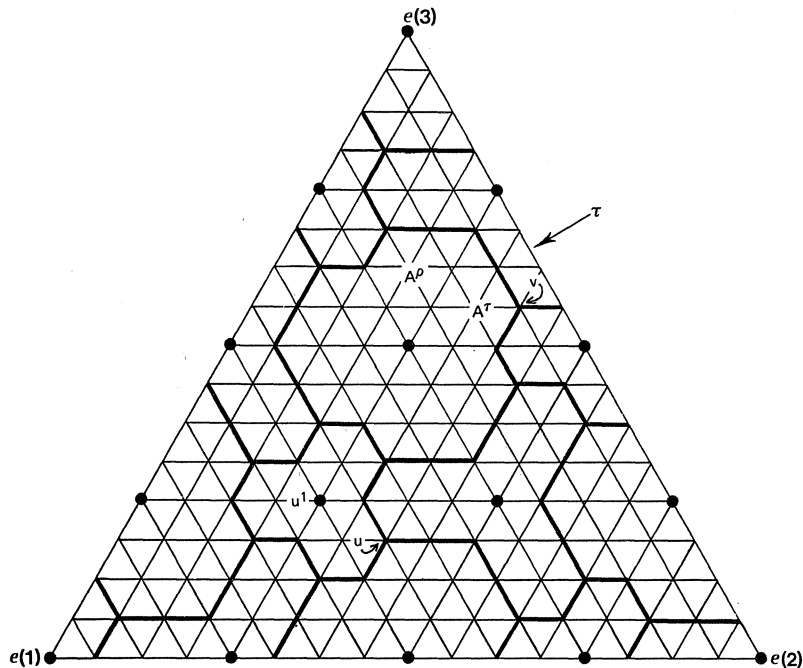
Figure 8.2.2. $n=2$, $d_2=16$, $\lambda^1=(2,1,1)$, $\sigma=\sigma_1(u^1,\beta(I_{n+1}))$ with $u^1=(\frac{1}{2},\frac{1}{4},\frac{1}{4})^T$ and $\beta(I_{n+1})=(1,3,2)$, $u=v(\sigma_1)$, $v=v(\tau)$ where $\tau=\tau(y^1,\pi(I_{n+1}))$ with $y^1=(0,\frac{1}{2},\frac{1}{2})^T$ and $\pi(I_{n+1})=(2,3,1)$; $A^\tau=A^\tau(1,2)$ and $A^\rho=A^\rho(1,2)$. The vertices of G_1 are drawn heavily. The grid points of G_2 on level d_2 within or on the boundary of a region surrounded by heavy lines are connected with the grid point of G_1 in the middle of it on level d_1.

8.3. THE ALGORITHM.

Suppose that we want to compute a fixed point of a continuous function f from S^n into itself by applying the continuous deformation algorithm with a triangulation of $S^n\times[1,\infty)$ as described in the previous section. Using integer labelling, each point $(x^T,d_i)^T$ of $S^n\times\{d_i\}$, $i=0,1,\ldots$, receives the standard integer label $\ell(x)$ as defined in definition 4.1.1.

For some arbitrarily chosen grid point v of G_1, e.g. v is the grid point nearest to the barycenter $(1/n+1,\ldots,1/n+1)^T$, $s^n \times [d_0, d_1]$ is triangulated as described in the previous section with $\sigma_0 = s^n$ and $v = v(\sigma_0) = \sum_{i=1}^{n+1} \lambda_i^0 e(i)$ the centrepoint of σ_0. The algorithm starts with the simplex ψ_0, being the convex hull of the vertices of s^n on level d_0 and the point v on level d_1, and proceeds along a path of adjacent $(n+1)$-simplices in $s^n \times [d_0, d_1]$ such that the common facets are completely labelled. Note that $s^n \times \{d_0\}$ is a completely labelled facet of ψ_0. Clearly, since the labelling rule is proper, there are no completely labelled facets in (bd $s^n) \times [d_0, d_1]$. Therefore all replacement steps are feasible and the algorithm generates within a finite number of steps a simplex in $s^n \times [d_0, d_1]$ having a completely labelled facet in $s^n \times \{d_1\}$, say $\sigma_1(u^1, \beta(I_{n+1}))$. Note that the intersection of the path of adjacent simplices with $s^n \times \{d_1\}$ is the path of adjacent t-simplices $(0 \le t \le n)$ of G_1 generated by the variable dimension restart algorithm as described in chapter 5. Now an integer $k_1 \ge 2$ is chosen and $s^n \times [d_1, d_2]$ is triangulated as described in the previous section with σ_1 as the starting simplex and an arbitrary grid point $u = \sum_{i=1}^{n+1} \lambda_i^1 u^i / k_1$ of G_2 in σ_1 as the centre point $v(\sigma_1)$ of σ_1. Then the algorithm continues the path of adjacent simplices with completely labelled common facets by replacing the vertex of $s^n \times \{d_0\}$ connected with the vertices of $\sigma_1 \times \{d_1\}$ by $v(\sigma_1) \times \{d_2\}$ and computing $\ell(v(\sigma_1))$. Simplices in $s^n \times [d_1, d_2]$ are generated until a simplex is found having either a completely labelled facet, say τ_1, in $s^n \times \{d_1\}$ or a completely labelled facet in $s^n \times \{d_2\}$. In the latter case k_2 and $\lambda_1^2, \ldots, \lambda_{n+1}^2$ are chosen and the algorithm continues with simplices in $s^n \times [d_2, d_3]$. In the first case the vertex $v(\tau_1) \times \{d_2\}$, which is connected with the vertices of τ_1, is replaced by the vertex of s^n on level d_0 connected with the vertices of τ_1, and the algorithm proceeds with a path of adjacent simplices of the triangulation of $s^n \times [d_0, d_1]$, until again a simplex having a completely labelled facet, say $\tau_2(y^1, \pi(I_{n+1}))$, in $s^n \times \{d_1\}$ is generated. Then the algorithm continues with simplices in $s^n \times [d_1, d_2]$ by replacing the vertex in $s^n \times \{d_0\}$ by $v(\tau_2) \times \{d_2\}$, where $v(\tau_2) = \sum_{i=1}^{n+1} \lambda_i^1 y^i / k_1$ is the centrepoint of τ_2.

Within a finite number of steps the algorithm generates a simplex having a completely labelled facet in $s^n \times \{d_2\}$, since the replacement steps are unique and feasible and the number of simplices of the triangulation of $s^n \times [d_0, d_2]$ is finite. Then k_2 and λ_i^2, $i \in I_{n+1}$, are chosen and the algorithm proceeds along a path of simplices in $s^n \times [d_0, d_3]$ etc. The algorithm can be

terminated if it generates a simplex of G_m with grid size d_m^{-1} small enough. Clearly, within a finite number of steps the algorithm finds a completely labelled simplex of G_m.

Using vector labelling a vertex $e(h)$ of S^n on level d^0 is labelled by $e(h)$, $h=1,\ldots,n+1$, and a point x on level d_i, $i \geq 1$, is labelled by $\ell(x) = x - f(x) + e$. The algorithm starts with the same simplex ψ_0 by introducing the vector $\ell(v(\sigma_0))$ in the system of $n+1$ linear equations $I\mu = e$. To compute a fixed point of a mapping ϕ, a point x on level d_i, $i \geq 1$, is labelled by $\ell(x) = x - f^i(x) + e$, where $f^i(x)$ is a piecewise linear approximation to ϕ with respect to G_i. The algorithm can be seen as tracing a path of zeroes of the piecewise linear function $f(x,t)$ obtained by taking the functions $\bar{\phi}(x,d_i)$, $i \geq 1$, defined by

$$\bar{\phi}(x,1) = (n+1)x - e \quad \text{and} \quad \bar{\phi}(x,d_i) = x - f^i(x) \quad i \geq 1$$

and extending them linearly on each simplex of the triangulation of $S^n \times [1,\infty)$.

8.4. THE REPLACEMENT STEPS.

To give the replacement steps it is more appropriate to describe the triangulation of the set $\tau(y^1, \pi(I_{n+1})) \times [d_m, d_{m+1}]$ as follows. For any subset T of I_{n+1}, let the regions $\overset{o}{A}{}^\tau(T)$ be redefined by

$$\overset{o}{A}{}^\tau(T) = \{x \epsilon \tau \mid x = v(\tau) + \sum_{j \epsilon T} \mu_j q(j) \text{ for positive numbers } \mu_j, \ j \epsilon T\}.$$

Then $\tau \times [d_m, d_{m+1}]$ is triangulated by connecting all the grid points x of G_{m+1} in $\overset{o}{A}{}^\tau(T)$ on level d_{m+1} with every vertex y^i of τ on level d_m, such that $\pi_i \notin T$. It can easily be verified that this triangulation of $\tau \times [d_m, d_{m+1}]$ is identical to the triangulation given in section 2. However we remark the following. In section 2 a grid point x in $\overset{o}{A}{}^\tau(T)$ on level d_{m+1} is connected with the vertex y^i of τ on level d_m if $i \notin T$. By this definition $x \epsilon \tau_1 \cap \tau_2$ and $x \epsilon \overset{o}{A}{}^{\tau_1}(T)$ implies $x \epsilon \overset{o}{A}{}^{\tau_2}(T)$ as proved in theorem 8.2.2. Now we have that x in $\overset{o}{A}{}^\tau(T)$ is connected with y^i if $\pi_i \notin T$. So, for some s, let y^j, $j \neq s$, be the common vertices of τ_1 and τ_2 and let T^1 and T^2 be the subsets of I_{n+1} such that $x \epsilon \overset{o}{A}{}^{\tau_i}(T^i)$, $i=1,2$. Then, with the convention that $s-1=n+1$ if $s=1$,

$$T^2 = T^1 \text{ if } \pi_{s-1}, \pi_s \notin T^1 \text{ or } \pi_{s-1}, \pi_s \in T^1$$

$$T^2 = (T^1 \cup \{\pi_s\}) \setminus \{\pi_{s-1}\} \text{ if } \pi_{s-1} \in T^1 \text{ and } \pi_s \notin T^1$$

$$T^2 = (T^1 \cup \{\pi_{s-1}\}) \setminus \{\pi_s\} \text{ if } \pi_{s-1} \notin T^1 \text{ and } \pi_s \in T^1.$$

Moreover, let $\mu_1, \ldots, \mu_{n+1}$ with $\mu_h > 0$ if $h \in T^1$ and $\mu_h = 0$ if $h \notin T^1$ be the numbers such that

$$x = v(\tau_1) + \sum_{h=1}^{n+1} \mu_h q(h)/d_{m+1}.$$

Then, by corollary 8.2.3,

$$x = v(\tau_2) + \sum_{h=1}^{n+1} \mu_h' q(h)/d_{m+1}$$

with $\mu_h' = \mu_h$, $h \neq s-1, s$, $\mu_{s-1}' = \mu_s$ and $\mu_s' = \mu_{s-1}$.

Now for some m, let ψ be a simplex in $S^n \times [d_m, d_{m+1}]$ generated by the algorithm. Since ψ is a simplex of the triangulation of $S^n \times [d_m, d_{m+1}]$, there exist a simplex $\tau(v^1, \gamma(I_{n+1}))$ of $S^n \times \{d_m\}$ with $v(\tau) = \sum_{i=1}^{n+1} \lambda_i v^i$ as its centrepoint, a subset T of t elements of I_{n+1}, a permutation $\pi(T)$ and a nonnegative vector $(R_1, \ldots, R_{n+1})^T$ of integers such that the following conditions hold:

1. The intersection of ψ and $S^n \times \{d_m\}$ is the convex hull of the n+1-t vertices v^i of τ such that $\gamma_i \notin T$. This set of vertices is called the set of active vertices of τ, whereas the other vertices are called inactive,

2. The intersection of ψ and $S^n \times \{d_{m+1}\}$ is the t-simplex $\sigma(y^1, \pi(T))$ with vertices $y^1, \ldots, y^{t+1}$ in $\tau \times \{d_{m+1}\}$ such that

    a) $y^1 = v(\tau) + \sum_{j=1}^{n+1} R_j q(j)/d_{m+1}$

    b) $y^{i+1} = y^i + q(\pi_i)/d_{m+1}$ $i = 1, \ldots, t$,

3. $R_j = 0$ for $j \notin T$ and $R_j \geq 0$ for $j \in T$.

Observe that the conditions 2a,b and 3 are analogous to respectively a,c and b on page 66. In particular the conditions 1-3 are satisfied for the starting simplex of the algorithm given by the convex hull of $S^n \times \{d_0\}$ and $v(\sigma_0) \times \{d_1\}$. So the algorithm is initiated with $\tau(v^1, \gamma(I_{n+1})) = \sigma_0(e(i), \tilde{\gamma}(I_{n+1}))$

with $\tilde{\gamma}_i = i$, $i \in I_{n+1}$, $T = \emptyset$, $\pi(T) = (\emptyset)$ and $R_i = 0$, $i \in I_{n+1}$.

Now any replacement step can be described by adapting $\tau(v^1, \gamma(I_{n+1}))$, the subset T, the permutation $\pi(T)$ and the vector R. Recall that τ has always the s-representation.

As described in the previous section two facets of ψ are completely labelled. So, only two vertices of ψ have the same label and one of them is the last vertex generated by the algorithm. Then the other one must be replaced. Two cases can occur,

> A) the vertex to be replaced is an active vertex of $\tau(v^1, \gamma(I_{n+1}))$, say v^{i_0},
>
> B) the vertex to be replaced is a vertex of $\sigma(y^1, \pi(T))$, say y^{j_0}.

We consider now these two cases.

Case A.

1^e. Suppose that v^{i_0} is the only active vertex of $\tau(v^1, \gamma(I_{n+1}))$. Then

$|T| = n$ and $\sigma(y^1, \pi(T))$ is a completely labelled n-simplex of S^n on level d_{m+1}. Now the simplex σ is written in the s-representation and τ is set equal to σ, R is set equal to zero and T becomes the empty set. The algorithm continues by computing the label of the centrepoint $v(\tau) = \sum_{i=1}^{n+1} \lambda_i^{m+1} v^i$ of the new simplex τ. If it is the first time that a completely labelled simplex is found on level d_{m+1}, this simplex is the starting simplex σ_{m+1} on level d_{m+1} and the integers λ_i^{m+1}, $i = 1, \ldots, n+1$, can be chosen arbitrarily.

2^e. The vertex v^{i_0} is not the only active vertex of $\tau(v^1, \gamma(I_{n+1}))$.

First suppose that

$$\lambda_{i_0}^m \neq 0 \text{ or } \gamma_{i_0 - 1} \in T \text{ (with } i_0 - 1 = n+1 \text{ if } i_0 = 1).$$

Since v^{i_0} is active we have that $\gamma_{i_0} \notin T$ and therefore y^{t+1} is not on the facet of τ opposite to v^{i_0}. Hence the point $y^{t+2} = y^{t+1} + q(\gamma_{i_0})/d_{m+1}$ is a grid point of G_{m+1} in τ. Now v^{i_0} becomes an inactive vertex of τ and is replaced by y^{t+2}, i.e. T is set equal to $T \cup \{\gamma_{i_0}\}$ and $\pi(T)$ becomes $(\pi_1, \ldots, \pi_t, \gamma_{i_0})$, whereas the simplex τ and the vector R do not change. If both

$$\lambda^m_{i_0} = 0 \text{ and } \gamma_{i_0-1} \notin T,$$

then $y^1, \ldots, y^{t+1}$ are points on the facet of τ opposite to v^{i_0} and the point $y^{t+1} + q(\gamma_{i_0})/d_{m+1}$ is a grid point of G_{m+1} not in τ. Now τ is adapted according to table 8.2.1 by replacing v^{i_0}. Let τ_1 be the new simplex on level d_m and let $\bar{v}^{-i_0}$ be the new vertex of τ_1. Since $y^1, \ldots, y^{t+1}$ are on the common facet of τ and τ_1, it follows from the proof of theorem 8.2.2 that again a simplex of the triangulation is obtained if $v^{i_0} \times \{d_m\}$ is replaced by $\bar{v}^{-i_0} \times \{d_m\}$. By corollary 8.2.3 and the fact that γ_{i_0} and γ_{i_0-1} are both not in T it follows that T, $\pi(T)$ and R do not change. Hence τ becomes τ_1 and $\bar{v}^{-i_0} \times \{d_m\}$ is the new vertex.

Case B.

Suppose that $t > 0$. Then we consider the following cases,

(i) $\quad j_0 = 1$

(ii) $\quad 2 \leq j_0 \leq t$

(iii) $\quad j_0 = t+1$.

In case (i) we first suppose that

$$\gamma_{r-1} \in T \text{ (with } r-1=n+1 \text{ if } r=1 \text{) or } R_{\gamma_r} + 1 < \lambda^m_r,$$

where r is the index such that $\pi_1 = \gamma_r$. Then y^{t+1} is not on the facet of τ opposite v^r and therefore $y^{t+1} + q(\pi_1)/d_{m+1}$ is a grid point of G_{m+1} in $A^\tau(T)$. Now $\pi(T)$ and R are adapted according to table 5.2.1, i.e. y^1 is replaced by $y^{t+1} + q(\pi_1)/d_{m+1}$.

If both

$$\gamma_{r-1} \notin T \text{ and } R_{\gamma_r} + 1 = \lambda^m_r,$$

then $y^2, \ldots, y^{t+1}$ are on the facet of τ opposite v^r and $y^{t+1} + q(\pi_1)/d_m$ is not in τ. Let $\tau(\bar{v}^{-1}, \bar{\gamma}(I_{n+1}))$ be the simplex obtained from $\tau(v^1, \gamma(I_{n+1}))$ by replacing the inactive vertex v^r ($\gamma_r = \pi_1 \in T$) according to table 8.2.1. Now T becomes $(T \cup \{\gamma_{r-1}\}) \setminus \{\gamma_r\}$, R_{γ_r} and $R_{\gamma_{r-1}}$ are interchanged, π_1 becomes γ_{r-1} and τ is set equal to $\tau(\bar{v}^{-1}, \bar{\gamma}(I_{n+1}))$, implying that y^1 is replaced by the new

vertex y^1, defined by $v(\tau)+\Sigma_{j=1}^{n+1}R_j q(j)$ for the new τ and R.
In case (ii) suppose that

$$\pi_{j_0-1}\neq\gamma_{r-1} \quad \text{(with } r-1=n+1 \text{ if } r=1) \text{ or } R_{\gamma_r}-R_{\gamma_{r-1}}<\lambda_r^m,$$

where r is the index such that $\pi_{j_0}=\gamma_r$. Then $\pi(T)$ and R are adapted according to table 5.2.1, i.e. y^{j_0} is replaced by $y^{j_0-1}+q(\pi_{j_0})/d_{m+1}$ which is a grid point in $A^\tau(T)$.
If both

$$\pi_{j_0-1}=\gamma_{r-1} \quad \text{and } R_{\gamma_r}-R_{\gamma_{r-1}}=\lambda_r^m$$

then y^i, $i\neq j_0$, is on the facet of τ opposite v^r and $y^{j_0-1}+q(\pi_{j_0})/d_{m+1}$ is not in τ. Now $\pi(T)$ is adapted according to table 5.2.1, R_{γ_r} and $R_{\gamma_{r-1}}$ are interchanged and $\tau(v^1,\gamma(I_{n+1}))$ is adapted according to table 8.2.1 by replacing the inactive vertex v^r, i.e. y^{j_0} is replaced by the new vertex $\bar{y}^{j_0}$ defined by the new τ, R and $\pi(T)$. Observe that T does not change since γ_r and γ_{r-1} are both an element of T.
In case (iii) suppose that $R_{\pi_t}\geq 1$. Then y^{t+1} is replaced by the grid point $y^1-q(\pi_t)/d_{m+1}$ in $A^\tau(T)$, i.e. $\pi(T)$ and R are adapted according to table 5.2.1. If $R_{\pi_t}=0$, $y^1-q(\pi_t)/d_{m+1}$ is not in $A^\tau(T)$. Now y^{t+1} must be replaced by the inactive vertex v^r of $\tau(v^1,\gamma(I_{n+1}))$, where r is the index with $\gamma_r=\pi_t$. So $\pi(T)$ becomes $(\pi_1,\ldots,\pi_{t-1})$ and T becomes $T\backslash\{\pi_t\}$.

Finally we consider the case t=0. Then $T=\emptyset$ and the only vertex y^1 of the zero-dimensional simplex $\sigma(y^1,\pi(T))$ has to be replaced. Since $T=\emptyset$, $R_j=0$ $j\in I_{n+1}$ and y^1 is the centrepoint $v(\tau)$ of $\tau(v^1,\gamma(I_{n+1}))$, which is connected with all vertices of τ. So, all these vertices are active and τ is a completely labelled simplex of G_m. Hence y^1 has to be replaced by a grid point of G_{m-1} on level d_{m-1}. Therefore we have to compute both the simplex $\bar{\rho}(\bar{z}^1,\bar{\beta}(I_{n+1}))$ of $s^n\times\{d_{m-1}\}$ such that all vertices of τ are grid points of G_m in $\bar{\rho}$, and the vertex of $\bar{\rho}$, say $\bar{z}^{i_0}$, which is connected with all the vertices of τ. To do so, we choose an arbitrary interior point of τ, say x, and we calculate the vector $\alpha(x)$ as described in section 2 with $d_i=d_{m-1}$. Since x is not a grid point of G_{m-1}, $\alpha(x)$ is not a vector of integers. Let $\bar{\alpha}_i$ be the entier of $\alpha_i(x)$, $i=1,\ldots,n$, and let z^1 be the grid

point of G_{m-1} such that $z^1=e(1)+\Sigma_{i=1}^n\bar{\alpha}_iq(i)/d_{m-1}$. Since x is an interior point of τ, $\alpha_i(x)-\bar{\alpha}_i\neq\alpha_j(x)-\bar{\alpha}_j$ for all $i\neq j$, $j=1,\ldots,n$. Let $\beta(I_{n+1})$ be the permutation of the elements of I_{n+1} such that

$$\alpha_{\beta_1}(x)-\bar{\alpha}_{\beta_1}>\alpha_{\beta_2}(x)-\bar{\alpha}_{\beta_2}>\ldots>\alpha_{\beta_n}(x)-\bar{\alpha}_{\beta_n}$$

and $\beta_{n+1}=n+1$. Then x is an interior point of the simplex $\rho(z^1,\beta(I_{n+1}))$ of G_{m-1}. Therefore all vertices of $\tau(v^1,\gamma(I_{n+1}))$ are points of ρ and we set $\bar{\rho}(\bar{z}^1,\bar{\beta}(I_{n+1}))$ equal to $\rho(z^1,\beta(I_{n+1}))$ in such a way that $\bar{\rho}$ has the s-representation. It remains to find $\bar{z}^i0$. Let $v(\bar{\rho})=\Sigma_{i=1}^{n+1}\lambda_i^{m-1-i}\bar{z}^i/d_m$ be the centrepoint of $\bar{\rho}$. It can be easily seen that

$$v^1=v(\bar{\rho})+\sum_{j=1}^n\Theta_jq(j)/d_m$$

with $\Theta_j=\Sigma_{i=1}^j(v_i(\bar{\rho})-v_i^1)$, $j=1,\ldots,n$. Set Θ_{n+1} equal to zero and define

(8.4.1) $\bar{\Theta}_j=\Theta_j-\min_{i\in I_{n+1}}\Theta_i$ $\quad j=1,\ldots,n+1$.

Since $\Sigma_{i=1}^{n+1}q(i)=0$, we obtain that $v^1=v(\bar{\rho})+\Sigma_{j=1}^{n+1}\bar{\Theta}_jq(j)/d_m$. Define for $i=1,\ldots,n+1$, $\Theta^i=(\Theta_1^i,\ldots,\Theta_{n+1}^i)$ by

(8.4.2) $\begin{cases}\Theta_j^i=\bar{\Theta}_j+1 & \text{if there exists an index } h\in\{1,\ldots,i-1\} \text{ with } \gamma_h=j \\ \\ =\bar{\Theta}_j & \text{otherwise.}\end{cases}$

Since $v^i=v^1+\Sigma_{j=1}^{i-1}q(\gamma_j)/d_m$ it follows that

$$v^i=v(\bar{\rho})+\sum_{j=1}^{n+1}\Theta_j^iq(j)/d_m \quad i=1,\ldots,n+1.$$

By (8.4.1) and (8.4.2) we have that $\Theta_j^i\geq0$ for all i,j. Let $H(v^i)$ be the set of indices $\{j|\Theta_j^i=0\}$, $i=1,\ldots,n+1$. By definition, $|H(v^1)|\geq1$, $|H(v^{n+1})|\leq1$ and $|H(v^i)|-|H(v^{i+1})|$ is equal to zero or one. So there exists at least one index h such that

$$|H(v^h)| = 1.$$

Let s be the index such that $\{s\}=H(v^h)$ for some h with $|H(v^h)|=1$. Since $H(v^{i+1})\subseteq H(v^i)$, s is unique. Let i_0 be the index such that $\gamma_{i_0}=s$. Define

$$\bar{T}=I_{n+1}\backslash\{\gamma_{i_0}\}$$

$$\bar{\pi}(\bar{T})=\{\gamma_{i_0+1},\ldots,\gamma_{n+1},\gamma_1,\ldots,\gamma_{i_0-1}\}$$

$$\bar{R}_j=\Theta_j^{i_0+1}-\min_{i\in I_{n+1}}\Theta_i^{i_0+1}\qquad j=1,\ldots,n+1.$$

It can easily be seen that $\bar{R}_j=0$ for $j=\gamma_{i_0+1}$, i.e. $\bar{R}_j=0$ for $j\notin\bar{T}$. Moreover $\bar{R}_j\geq0$, $j\in\bar{T}$. Define y^1 to be v^{i_0+1}. Then $\tau(v^1,\gamma(I_{n+1}))$ can be written as $\bar{\sigma}(y^1,\bar{\pi}(\bar{T}))$. Since

$$y^1=v^{i_0+1}=v(\bar{\rho})+\sum_{j=1}^{n+1}\bar{R}_jq(j)/d_m$$

it follows that $\bar{\sigma}$ is in $A^{\bar{\rho}}(\bar{T})$. So, finally we have that τ becomes $\bar{\rho}(z^1,\bar{\beta}(I_{n+1}))$, T becomes $\bar{T}$, $\pi(T)$ becomes $\bar{\pi}(\bar{T})$ and R becomes $\bar{R}$, i.e. the only vertex y^1 of σ on level d_{m+1} is replaced by the vertex z^{j_0} of $\bar{\rho}$ on level d_{m-1}, where j_0 is the index such that $\bar{\beta}_{j_0}=s$. The zero-dimensional simplex σ on level d_{m+1} becomes the n-dimensional simplex $\bar{\sigma}$ on level d_m.

8.5. THE APPLICATION ON R^n.

Let $d_0,d_1,\ldots$ be a sequence of increasing positive numbers such that $d_{i+1}=k_id_i$, $i=0,1,\ldots$ with k_i an arbitrary integer larger than 1. For some $n\times n$-triangulation matrix A, $R^n\times\{d_m\}$ is triangulated by the AK triangulation with grid size d_m^{-1}. On each level we choose $z=\underline{0}$ to be a grid point. Doing so, the grid points of the triangulation on level d_m are also a grid point on level d_i, $i>m$. With $a(n+1)=-\sum_{j=1}^na(j)$ we obtain again that each n-simplex σ has n+1 representations $\sigma(y^1,\pi(I_{n+1}))$. As in section 2 each simplex is given in the s-representation. Therefore we have to define the function s(w). So, let w be a grid point on level d_m. Then we define $\alpha(w)$ by

$$\alpha(w)=d_mA^{-1}w$$

and s(w) by $s(w)=1+(\sum_{h=1}^n\alpha_h(w))\bmod(n+1)$.

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Now we triangulate $R^n \times [d_m, d_{m+1}]$ for some $m \geq 0$. Combining these triangulations for all pairs $[d_i, d_{i+1}]$, $i=0,1,\ldots$ we obtain the triangulation of $R^n \times [d_0, \infty)$. The triangulation of $R^n \times [d_m, d_{m+1}]$ is done in the same way as described in section 2 for $S^n \times [d_m, d_{m+1}]$. So, let $\sigma_m(u^1, \beta(I_{n+1}))$ be the starting simplex on level d_m, and let

$$v(\sigma_m) = \sum_{i=1}^{n+1} k_m^{-1} \lambda_i^m u^i$$

be its centrepoint for arbitrarily chosen λ_i^m, $i \in I_{n+1}$, such that $\sum_{i=1}^{n+1} \lambda_i^m = k_m$. To triangulate $\tau(y^1, \pi(I_{n+1})) \times [d_m, d_{m+1}]$, with τ given in the s-representation we define again for any proper subset T of I_{n+1}, the regions $\mathring{A}^\tau(T)$ as in section 2, and connect all grid points x on level d_{m+1} in region $\mathring{A}^\tau(T)$ with the vertices y^i of τ on level d_m with $i \notin T$. Combining the triangulations $\tau \times [d_m, d_{m+1}]$ over all simplices τ, we get a consistent triangulation of $R^n \times [d_m, d_{m+1}]$.

We now discuss the use of this triangulation of $R^n \times [d_0, \infty)$ in a fixed point algorithm. Assume we want to compute a fixed point of a mapping ϕ. Let f^i be a piecewise linear approximation to ϕ with respect to the triangulation of R^n on level d_i. Then a point x of $R^n \times \{d_i\}$ receives the vector label as defined in definition 4.8.1 with b=e and f=f^i, i=0,1,... . Assume that Merrill's condition is satisfied and that $R^n \times \{d_i\}$ is triangulated by $A(\alpha)K_{\delta_i}$ with $\delta_i = d_i^{-1}$, i=0,1,... . Let σ^0 be a completely labelled simplex on level d_0. Then, starting with the simplex having σ^0 as a facet, a sequence of simplices of the triangulation of $R^n \times [d_0, \infty)$ is generated, such that the common facets are completely labelled. By Merrill's condition this sequence cannot diverge if δ_0 is small enough. If, for some m large enough, a completely labelled simplex of $R^n \times \{d_m\}$ is generated, a good approximation of a fixed point is found and the algorithm terminates. It remains to consider the following questions:

1^e. How to find a completely labelled simplex σ^0 in $R^n \times \{d_0\}$?

2^e. How to continue the algorithm if, starting from the simplex having σ^0 as a facet, a simplex of $R^n \times [d_0, d_1]$ is generated having a completely labelled facet $\sigma(y^1, \pi(I_{n+1}))$ in $R^n \times \{d_0\}$?

Both problems can be solved by applying the variable dimension restart algorithm. Starting in an arbitrarily chosen grid point $(v^T, d_0)^T$ in $R^n \times \{d_0\}$, this algorithm generates a sequence of simplices of variable

dimension on level d_0. By theorem 6.3.4 this sequence terminates with a completely labelled simplex σ^0 in $R^n \times \{d_0\}$. To solve the second question, observe that, for some $i \in I_{n+1}$, $\sigma(y^1, \pi(I_{n+1}))$ is in the region

$$A(T) = \{x \in R^n \mid x = v + \Sigma \lambda_j a(j) \text{ for nonnegative numbers } \lambda_j, \; j \in T\}$$

with $T = I_{n+1} / \{i\}$. Then the variable dimension restart algorithm can again be applied. Starting with σ and reintroducing the i-th unit column, a sequence of simplices of variable dimension in level d_0 is generated until again a completely labelled simplex is found. Then the algorithm continues again with simplices of the triangulation of $R^n \times [d_0, \infty)$. In fact we can use the interpretation of the variable dimension restart algorithm with n+1 points on an extra level, as discussed in section 6.4, to triangulate for some σ $(\sigma \times \{0\}) \cup (R^n \times (0, d_0])$. Then, starting with the simplex being the convex hull of the vertices of σ on level 0 and $(v^T, d_0)^T$, a sequence of n+1-simplices having completely labelled common facets is generated.

8.6. SOME REMARKS AND NUMERICAL RESULTS.

In the algorithm of Eaves [1972] and Eaves and Saigal [1972] the factor of grid refinement is (at most) equal to two. So, their algorithm cannot be used to obtain quadratic convergence. Using a restart method in a continuous deformation algorithm, Saigal [1977c] and Saigal and Todd [1978] developed an acceleration technique to achieve quadratic convergence when the underlying function is continuously differentiable and the derivative is Lipschitz continuous. Using the triangulation presented in this chapter, quadratic convergence can be obtained without the necessity of making restarts. Saigal and Todd also proved that, using the information obtained from the last found approximation, a completely labelled simplex on the new level can be generated in n+1 pivots. It is worthwile to investigate whether the same result can be proved for the continuous deformation algorithm with an arbitrary factor of grid refinement.

In the triangulations of $S^n \times [1, \infty)$ and $R^n \times [d_0, \infty)$, described in section 2 respectively 5, the starting point on a new level can be chosen on basis of information obtained earlier. So, for vector labelling one could choose the grid point nearest to the approximation obtained from the last found completely labelled simplex. Therefore, even when

the factor of incrementation is two, the triangulation is more general
than that of Eaves and Saigal. This is illustrated in the figures 8.6.1.a
and 8.6.1.b for n=1. In 8.6.1.a, $\tau(y^1,y^2) \times \lceil d_m, 2d_m \rceil$ is triangulated according
to the triangulation of Eaves and Saigal. Observe that $y^i \times \{d_m\}$, i=1,2, must
be connected with $v \times \{2d_m\}$. In 8.6.1.b, $\tau(y^1,y^2) \times \lceil d_m, 2d_m \rceil$ is triangulated by
connecting $y^i \times \{d_m\}$, i=1,2 with $y^1 \times \{2d_m\}$, and $y^2 \times \{d_m\}$ with the other two
grid points of $\tau \times \{2d_m\}$. Observe however that the vertices $y^i \times \{d_m\}$, i=1,2,
can be connected with any grid point of $\tau \times \{2d_m\}$.
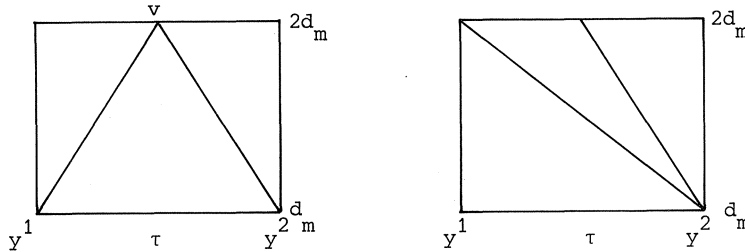
Figure 8.6.1.a Eaves-Saigal's
triangulation of $\tau \times \lceil d_m, 2d_m \rceil$

Figure 8.6.1.b. Example of a
triangulation of $\tau \times \lceil d_m, 2d_m \rceil$ as given
in section 2.

Tuy [1979] presented a restart method to approximate a fixed point of a
function, using primitive sets (see also Todd [1978b]). Although his
approach is different, Van der Heyden [1979b] presented a similar method.
He introduced also a geometric interpretation of the algorithm. Using this
interpretation he constructed a triangulation of $S^n \times \lceil 1, \infty)$ permitting an
arbitrary factor of grid refinement between two successive levels. However,
this is achieved at the expense of an asymmetry in the roles played by the
different coordinates. More precisely, $\tau(y^1, \ldots, y^{n+1}) \times \lceil d_m, d_{m+1} \rceil$ is triangulated
as in section 2, but with the restriction that, for some i, $v=y^1$. Although a
partial remedy is given, this feature makes the triangulation less attractive.
The difference between Van der Heyden's triangulation and the triangulation
introduced in section 2 is illustrated in the figures 8.6.2.a,b. Finally we
mention a paper of Shamir [1979]. Independently of Van der Laan and Talman
[1980b] he constructed the same triangulation of $S^n \times \lceil 1, \infty)$ using also the
interpretation of the variable dimension restart algorithm with an extra level.
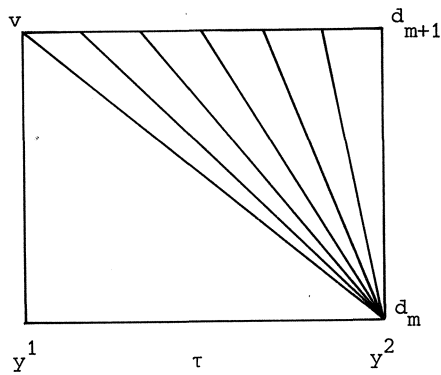
Figure 8.6.2.a. Van der Heyden's triangulation of $\tau \times [d_m, d_{m+1}]$, n=1, $k_m=6$, $v=y^1$ or y^2.
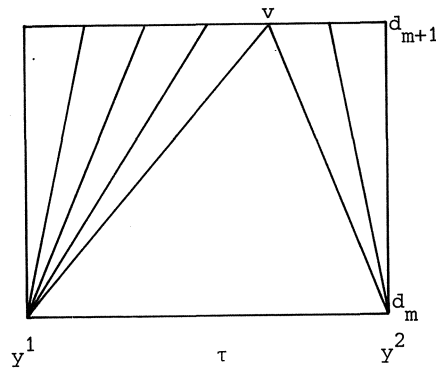
Figure 8.6.2.b. Example of a triangulation of $\tau \times [d_m, d_{m+1}]$ as given in section 2, n=1, $k_m=6$, v is an arbitrary grid point on level d_{m+1}.

We conclude this section with some computational results. We applied the algorithm for vector labelling to the three examples of section 5.8. In all runs the T triangulation is used with again -T in stead of T. The labelling rule is the same as in section 5.8. The starting point on level d_{m+1} is chosen as the grid point nearest to $w^*(m)$ with $w^*(m)$ the approximate fixed point on level d_m. The algorithm is applied with $d_{i+1}/d_i=2$ for all i, and with a variable factor of grid refinement. In the latter case $\delta_{m+1}=d_{m+1}^{-1}$ is chosen analogously as in section 5.8. However to guarantee that d_{m+1} is a multiple of d_m, δ_{m+1} is defined by $\delta_{m+1}=\delta_m/k_m$ with

$$k_m = 2, \qquad \text{if } h_m \geq (\text{mesh } G_m)/2\sqrt{n+1}$$

$$k_m = \text{Entier } (h_m\sqrt{n+1}/\text{mesh } G_m)^{-1} \text{ if } h_m < (\text{mesh } G_m)/2\sqrt{n+1}$$

The results are given in the tables 8.6.1-8.6.3. Comparing the columns 3 and 7 it turns out that the algorithm is considerably improved by using a variable factor of grid refinements. Moreover a comparison with the tables 5.8.1-5.8.3 learns that for $d_{i+1}/d_i=k_i$ the number of iterations of the continuous deformation algorithm is of the same order as the number of iterations of the variable dimension restart algorithm. More

computational experience is needed to decide which algorithm is superior. We remark the following:

1. The variable dimension restart algorithm is easier to implement on the computer.

2. In each stage the restart algorithm is initiated with $I\mu=e$. This is a drawback, since the deformation algorithm continues with the current system of linear equations. However in the acceleration technique of Todd [1978b] the system of linear equations corresponding to the completely labelled simplex is used to initiate the system in the next stage. This technique should be used when the restart algorithm is compared with the deformation algorithm.

3. The examples are very nice. In particular, the deformation algorithm never returns to a previous level. Both algorithms should be tested on more complicated problems.

Table 8.6.1. Pure exchange economy, $n=4$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max_{i \in I_{n+1}} |(g_i - a_i)/a_i|$.

$d_{i+1}/d_i=2$				$d_{i+1}/d_i=k_i$			
m	M	N	E	m	M	N	E
1	.051	7	1.02	1	.051	7	1.02
3	.013	20	.05	2	.026	13	.29
5	.003	33	$8 \ 10^{-3}$	3	.013	20	.05
7	$8 \ 10^{-4}$	43	$2 \ 10^{-4}$	4	.006	26	.03
9	$2 \ 10^{-4}$	55	$2.5 \ 10^{-5}$	5	$2 \ 10^{-3}$	31	$3 \ 10^{-3}$
11	$5 \ 10^{-5}$	69	$2 \ 10^{-6}$	6	10^{-3}	38	10^{-3}
13	$1.2 \ 10^{-5}$	81	$3 \ 10^{-8}$	7	$8 \ 10^{-5}$	45	$4 \ 10^{-6}$
15	$3 \ 10^{-6}$	91	$5 \ 10^{-9}$	8	$4 \ 10^{-5}$	52	$2 \ 10^{-6}$
17	$8 \ 10^{-7}$	102	$4 \ 10^{-10}$	9	$6 \ 10^{-8}$	59	$< 10^{-10}$

Table 8.6.2. Pure exchange economy, $n=7$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max\limits_{i\in I_{n+1}} |(g_i-a_i)/a_i|$.

$d_{i+1}/d_i=2$				$d_{i+1}/d_i=k_i$			
m	M	N	E	m	M	N	E
1	.032	11	1.32	1	.032	11	1.32
3	.008	36	.05	2	.016	27	.23
5	.002	59	$5\ 10^{-3}$	3	.008	36	.05
7	$5\ 10^{-4}$	83	$2\ 10^{-4}$	4	$3.8\ 10^{-3}$	46	.02
9	$1.2\ 10^{-4}$	102	10^{-5}	5	$5.6\ 10^{-4}$	62	$2\ 10^{-4}$
11	$3\ 10^{-5}$	118	10^{-6}	6	$2.8\ 10^{-4}$	70	$5\ 10^{-5}$
13	$8\ 10^{-6}$	142	$8\ 10^{-8}$	7	$3.2\ 10^{-6}$	80	$2\ 10^{-8}$
15	$2\ 10^{-6}$	166	$4\ 10^{-9}$	8	$1.6\ 10^{-6}$	93	$4\ 10^{-9}$

Table 8.6.3. Pure exchange economy, $n=9$, $M=(\text{mesh } G_m)/2\sqrt{n+1}$, N is the cumulative number of iterations, $E= \max\limits_{i\in I_{n+1}} |(g_i-a_i)/a_i|$.

$d_{i+1}/d_i=2$				$d_{i+1}/d_i=k_i$			
m	M	N	E	m	M	N	E
1	.025	12	.30	1	.025	12	.30
3	.006	43	.03	2	.013	32	.08
5	$1.5\ 10^{-3}$	83	$2\ 10^{-3}$	3	.006	43	.03
7	$4\ 10^{-4}$	112	10^{-4}	4	.002	67	$3\ 10^{-3}$
9	10^{-4}	142	$6\ 10^{-6}$	5	$7\ 10^{-4}$	78	$3\ 10^{-4}$
11	$2.5\ 10^{-5}$	166	$4\ 10^{-7}$	6	$5\ 10^{-5}$	102	10^{-6}
13	$6\ 10^{-6}$	193	$3\ 10^{-8}$	7	$6\ 10^{-6}$	120	$3\ 10^{-8}$

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