Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

MATHEMATICAL CENTRE TRACTS 125

2-D SYSTEMS, AN ALGEBRAIC APPROACH

R. EISING

MATHEMATISCH CENTRUM AMSTERDAM 1980

1980 Mathematics subject classification: 93B25

ISBN 90 6196 198 X

ACKNOWLEDGEMENTS

This monograph is a slightly revised version of my doctoral dissertation submitted to the Eindhoven University of Technology.

To the thesis advisors Prof.dr.ir. M.L.J. Hautus and Prof.dr.ir. J.C. Willems I owe many thanks for their constructive criticism and valuable suggestions.

I am indebted to Prof.dr. W. Peremans and Prof.dr.ir. J. de Graaf for their suggestions for improvement.

I thank the Mathematical Centre for the opportunity to publish this monograph in their series Mathematical Centre Tracts.

CONTENTS

I INTRODUCTION AND SUMMARY

In recent years the field of digital processing of two-dimensional sampled data has attracted many researchers. The reasons for the interest in this field are on one side the richness in potential application areas and on the other side the abundance of non-trivial theoretical problems.
Potential areas of application are digital image processing, seismic signal processing, gravity and magnetic field mapping. Here a digital image can be thought of as a collection of digitized (photographic) data where each pixel (picture element) represents a gray level in the case of black and white photography (for instance a newspaper photograph). In color photography in each pixel some numbers, coding the color and the color intensity, are given.
Of course it is not necessary that an image is formed using visible light. Other ways of obtaining an image are for instance radar, infra red photography (agricultural applications and reconnaissance), ultra sonic imaging and X-ray photography (medical applications) and particles such as electrons may also be an intermediate between the object and the image (electron microscopic photography).
Processing of the two-dimensional sampled data (often just called images) may consist of some of the following operations. Restoration of blurred images. Enhancement of noisy images by reduction of the noise level (noise filtering combined with other techniques). Feature extraction (detection of edges etc.).
Sources of the blur may be movement of the camera or for instance atmospheric turbulance. Noise sources corrupting the image may be inherent to the imaging system or may arise in the transmission of an image (space craft photography).
As an example of image enhancement may serve the Mariner 6 and 7 pictures of Mars processed at the Jet Propulsion Laboratory, Pasadena, California (see [59]).
For a very readable introduction see [3] and for more information concerning restoration techniques and more technical aspects of image processing see [37], [2].

The enhancement of a picture is a difficult task, partly because there are almost no theoretical foundations and on the other hand because there are very severe storage requirements. Reasons for this to be the case are because one needs a picture with say 1024 × 1024 pixels in order to obtain a reasonable resolution and in the case of color photography for each pixel one needs 24 bits to code the basic colors and the color levels. Furthermore the number of computations is enormous. This can be seen from the following convolution which may serve as a prototype of operation by which an image is processed:

$$(1.1.1) \qquad y_{kh} = \sum_{i,j} F_{k-i,h-j}\, u_{ij} \, .$$

Here u_{ij} denotes the pixel at position (i,j) of the original picture and y_{kh} denotes the pixel at position (k,h) of the processed picture. From this model it is clear that there are serious computational problems and the number of computations involved depends strongly on the number of non-zero elements F_{mn} of the double sequence $(F_{m,n})$, the so called point spread function or impulse response. A convolution as described in (1.1.1) is called a 2-D system or a 2-D filter (2-D stands for two-dimensional).
The main problem in the field of image processing is the design of 2-D filters such that the output image with pixels y_{kh} is more satisfactory according to some quality criterion than the input image with pixels u_{ij}. Many papers appeared in this field. However, most of these present some ad hoc solutions to the problem and the main reason for this is the lack of a quantitative quality criterion. See [39], [75], [38].
A very important aspect of a 2-D filter is stability which means, roughly speaking, that small disturbances in the input only have a small effect on the output. Some references are [30], [37].
A severe restriction on the possible application of filters is the computation time, especially for serial problems (on line filtering), where there is only a limited processing time available for each image. Sometimes this problem is circumvented by taking a smaller number of pixels describing the image. However, this may reduce the resolution considerably. Hence there is a need for fast processing techniques. One of the approaches to this problem is the use of transform techniques such as the Fast Fourier Transform which may reduce the computational effort considerably. See for instance [37], [69], [14]. Recently, fast algorithms, based on polynomial transforms, have been developed. See [58].

Another approach to the problem of reducing the computational effort and at the same time decreasing the amount of memory, necessary for the image processing problem, is the introduction of state space techniques. For 1-D systems which can be described by a convolution like

$$(1.1.2) \qquad y_k = \sum_i F_{k-i}\, u_i$$

state space models have proved to be very useful. The standard state space model in this case is (given some initial conditions)

$$(1.1.3) \qquad \begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ y_k &= Cx_k + Du_k \end{aligned}$$

From this model it can be seen that the convolution can be calculated very easily because the output y_k depends only on the last input u_k and the last state x_k. Of course u_k, x_k, y_k may be members of different vector spaces and A, B, C, D are matrices with appropriate dimensions. The model (1.1.3) is called a realization of (1.1.2). The only condition that has to be satisfied is:

$$F_i = 0 \qquad \text{for } i < 0 \;, \quad F_0 = D$$

$$F_i = CA^{i-1}B \;\; \text{for } i = 1,2,3,\ldots \;.$$

Furthermore it is clear that in order to calculate y_k all past inputs (u_h for $h < k$) can be forgotten. All information concerning past inputs, necessary to determine the output, is contained in the state x_k. It will be clear that, besides the possible reduction of the number of computations, also the amount of memory may be reduced considerably.
This idea of introducing an extra variable x_k (the state) playing the role of a memory device, which to some extent enables us to reduce the redundancy in the convolution defining the 1-D system, will be introduced in the context of image processing problems (2-D systems).
However, the state space that has to be introduced in this case is generally infinite dimensional. See [48], [24]. The reason for this is that in the defining model for a 2-D system the input image and the output image have infinite extent. In real image processing problems the images have finite extent, so that the state space can be taken finite dimensional (although large).

The fact that the output can be computed recursively from the input, as is the case for the state space model (1.1.3) will also hold in the 2-D case (of course under some conditions). This recursive nature of the image processing problem may be very advantageous.

State space models for image processing problems, or more generally 2-D systems, recently appeared in the literature. See for instance [4], [24], [48], [16], [74]. The proposed models look quite different at first sight but in fact they are closely related. All papers except [16] have in common that the models are only local state space models. This means that they just give the equations for the recurrent computation of the output given the input (furthermore initial conditions are given). They do not have a real state space character in the sense that there is some variable such as x_k above, which comprises the relevant information from the past, because this would presuppose an ordering of the index set.

In this monograph a state space model together with a local state space model is presented and both models can be obtained from one another in an easy and straightforward way (see also [16]). This is the reason why the other models can be considered special cases of the model presented in this approach.

The model, describing the input/output behavior of the 2-D system we will be working with, is given by

$$(1.1.4) \qquad y_{kh} = \sum_{i=0,j=0}^{k,h} F_{k-i,h-j}\, u_{ij}\,, \quad k,h = 0,1,2,\ldots\ .$$

This input/output model is used by all authors who work on (local) state space models for 2-D systems, although in some cases F_{00} is taken to be zero a priori.

A figure which supports the intuition and is useful in visualizing the image processing problem is the following.

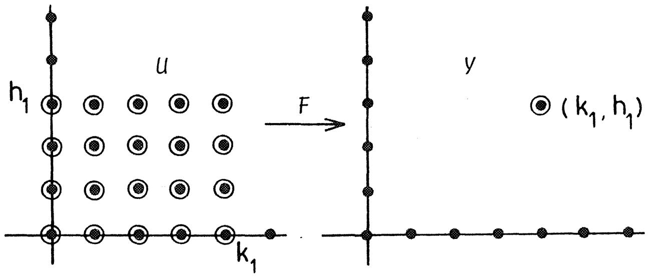

Figure 1

$\mathcal{U} = (u_{i,j})$; $i,j = 0,1,2,\ldots$ is the input image,

$\mathcal{Y} = (y_{k,h})$; $k,h = 0,1,2,\ldots$ is the output image,

$\mathcal{F} = (F_{m,n})$; $m,n = 0,1,2,\ldots$ is the point spread function or impulse response.

The input and output image and the impulse response are thought of as having infinite extent in the positive i, j, k, h, m and n direction. The equations by which y_{kh} is computed from the input are given in (1.1.4). Observe that y_{k_1,h_1} only depends on inputs u_{ij} with $i \le k_1$ and $j \le h_1$ (the encircled points in $\mathcal{U}$).

The local state space models for such an image processing system as presented in [4], [25], [28] respectively, can be described as follows

$$x_{k+1,h+1} = A_1\, x_{k+1,h} + A_2\, x_{k,h+1} - A_1 A_2 x_{kh} + Bu_{kh}$$

(1.1.5)

$$y_{kh} = Cx_{kh} ,$$

where $A_1A_2 = A_2A_1$ (see Attasi [4]), and

$$x_{k+1,h+1} = A_1\, x_{k+1,h} + A_2\, x_{k,h+1} + B_1\, u_{k+1,h} + B_2\, u_{k,h+1}$$

(1.1.6)

$$y_{kh} = Cx_{kh} .$$

This is the last model of a series of models proposed by Fornasini-Marchesini in [16], [23], [25]. It can be shown that the Attasi model is a special case of (1.1.6).

A third model, closely related to the model which we propose in Chapter IV and further on, is given by

$$(1.1.7)\qquad \begin{aligned} \begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} &= \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh} \\ y_{kh} &= [C_1 , C_2] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + Du_{kh} . \end{aligned}$$

Again it can be shown that the Fornasini-Marchesini model, and therefore the Attasi model can be written in the form of this model, due to Roesser and Givone. See [28], [61], [52]. Observe that in the Roesser model (1.1.7) y_{kh} depends on u_{kh} while in the other two models this is not the case. However, this can be incorporated in the models of Attasi and Fornasini-Marchesini as well. Furthermore observe that the Roesser model is a "first order" model while the other two are "second order".

A different approach to the realization problem can be found in [53], [52]. The realizations described there are so called circuit realizations. This means that the equations of a local state space model are written in the form of a circuit. These circuit realizations are also closely related to the Roesser model and therefore to ours.

In [48] a polynomial matrix approach to the study of 2-D systems was presented which since then attracted some attention of other researchers such as Fornasini and Marchesini. This method is also related to ours. This will be shown in Chapter V.

The main idea of this approach is that 2-D systems can be seen as 1-D systems over a ring. This means that a model like (1.1.4) can be described by a model as (1.1.2) where the matrices and vectors now have entries in a ring. The state space model which we will present has the form of (1.1.3) where the matrices A, B, C, D are matrices over a ring. The local state space model which we will derive is closely related to (1.1.7) and this model can be obtained from A, B, C, D quite easily. Also the state space model over the ring, to be defined, can be obtained from the local state space model in a straightforward way. This approach to 2-D systems was presented in [16] and simultaneously in [74]. (The results of Sontag's [74] and our [16] were closely related.) See also [73].

The objectives of this thesis are to describe the relation of 2-D systems theory with 1-D systems theory over rings and show how the theory of 1-D systems over rings can contribute to 2-D systems theory. Another goal is to present the results ultimately in algorithmic form and not to give only existence results. Of course this is very important if application oriented research is being done. Some rather abstract mathematics appears in Chapters II and III. The reason for this is that the results can be used as a tool in the construction of algorithms for the solution of some problems in 2-D systems theory.
As is often the case, if problems are described on a more abstract level, the algorithms and theory developed on that abstract level also have applications in other fields. Some examples of this will be given in Chapter III.

We now give a survey of the monograph.
In Chapter II we give an introduction and some results in the theory of systems over rings. We will indicate some differences with the theory of systems over a field and we also present some results common to both the field case and the ring case. Most parts of the chapter will be concerned with realization theory.
In Chapter III realization algorithms are developed for systems over a principal ideal domain. These algorithms are related to some of the existing algorithms for systems over a field. Also some applications to delay differential systems and systems over the integers are given.
In Chapter IV it is shown that 2-D systems can be seen as 1-D systems over a principal ideal domain. Some state space models and local state space models are developed. We also introduce some causality concepts and it is shown that these are closely related to recursiveness of the defining equations of the local state space models under consideration. Furthermore it is shown that the existing models are special cases of the models developed in this chapter.
Chapter V gives various properties of the 2-D state space models. For instance, it is shown that the problem of investigating the stability of a 2-D system can be coped with if this system is viewed as a 1-D system over a principal ideal domain. Conditions for the existence of observers, useful for filtering problems in the field of image enhancement, are given and an algorithm in order to obtain such observers is shown to be a generalization of an already existing so called pole placement algorithm. Also some results concerning invertibility of 2-D systems are obtained. This may be

used in inverse filtering of images. Concepts of reachability and observability (well-known in 1-D systems theory) are defined and relations with other approaches are shown. Furthermore, an algorithm for constructing low order local state space models, which improves on the other algorithms in the literature, is presented. Also some results concerning generic properties are presented.
In Chapter VI the realization algorithms, developed in Chapter III are modified in such a way that the specific structure of 2-D systems can be exploited.
Finally, in the Appendix, definitions of a few algebraic concepts which arise in Chapters II and III are given.

II *LINEAR SYSTEMS OVER PRINCIPAL IDEAL DOMAINS*

II.1. *General introduction*

Many dynamic discrete time phenomena can be described by means of linear equations of the form

$$(2.1.1) \qquad y_k = \sum_h F_{kh} u_h , \qquad h \in \mathbb{Z}, \; k \in \mathbb{Z},$$

where $u_h \in \mathbb{R}^p$, $y_k \in \mathbb{R}^m$ and $F_{kh} \in \mathbb{R}^{m\times p}$. The vector u_h is called the input at time h and y_k is called the output at time k. Here $\mathbb{R}^p$ ($\mathbb{R}^m$) denotes the vector space of real p-vectors (m-vectors) and $\mathbb{R}^{m\times p}$ is the space of real $m \times p$-matrices. $\mathbb{Z}$ is the ring of integers. Later on we will impose a finiteness condition on (2.1.1) to avoid convergence problems. A large part of linear systems theory is implicitly or explicitly concerned with equations like (2.1.1). In most cases a so called causality assumption is made, i.e. it is required that the output at time k be not influenced by future inputs. The words "time" and "future" stem from the fact that in many applications $\mathbb{Z}$ is interpreted as a time set designating the order, according to which the process (phenomenon), of which (2.1.1) is a model, evolves. This time set gives the possibility to use (2.1.1) as a model for phenomena, showing dynamic behavior.
The model (2.1.1) will be called an *input/output system* (also I/O system). The input/output system (2.1.1) will be called *causal* if the output at time k is not influenced by future inputs, i.e. if

$$(2.1.2) \qquad F_{kh} = 0 , \qquad h > k .$$

REMARK. Usually (2.1.1) is called a discrete time input/output system but we will almost exclusively consider discrete time input/output systems. Therefore we will omit the term "discrete time". □

In many cases the dynamic behavior of a process does not explicitly depend upon the time itself. In other words, if a sequence $(u_h)_{h\in\mathbb{Z}}$ is related to a sequence $(y_k)_{k\in\mathbb{Z}}$ through (2.1.1), then the shifted sequence $(u_{h+n})_{h\in\mathbb{Z}}$ is related to the corresponding $(y_{k+n})_{k\in\mathbb{Z}}$ for each $n \in \mathbb{Z}$. In this case, F_{kh}

only depends on the difference $k-h$. Henceforth we will make this assumption of *time invariance* (also called *shift invariance* when $\mathbb{Z}$ is not directly related to time) and we denote the input/output system (2.1.1) by

$$(2.1.3) \qquad y_k = \sum_h F_{k-h} u_h , \quad h \in \mathbb{Z} , \ k \in \mathbb{Z} .$$

We will make two further assumptions concerning the input sequence $(u_h)_{h\in\mathbb{Z}}$ and the sequence $(F_n)_{n\in\mathbb{Z}}$, namely, we assume that the input sequences have *finite past*, i.e., there exists an h_0 such that

$$(2.1.4) \qquad u_h = 0 , \quad h < h_0$$

for all input sequences.
By a reindexing of the input sequences (2.1.4) can be expressed as

$$(2.1.5) \qquad u_h = 0 , \quad h < 0 .$$

We will also assume that the input/output system is causal (see (2.1.2)). This means that

$$(2.1.6) \qquad F_n = 0 , \quad n < 0 .$$

The assumptions imply that $y_k = 0$ for $k < 0$. Assuming causality, time invariance and the finite past condition on the inputs, (2.1.1) can be written as

$$(2.1.7) \qquad y_k = \sum_{h=0}^{k} F_{k-h} u_h , \quad k = 0,1,2,\ldots .$$

This is the standard equation (see [12]) for a causal, discrete time, time invariant, linear input/output system. When no confusion can arise we simply call (2.1.7) an input/output system.

REMARK. It is not necessary for the time set (index set) to be a totally ordered set. In Chapter IV we will be concerned with systems where the index set is a partially ordered set. □

II.2. *The impulse response*

First we will introduce a generalization. In the foregoing we assumed that the inputs (outputs) were real p- (m-) vectors. This assumption will now be dropped.

From now on R will denote a commutative integral domain with identity. Henceforth we will be concerned with R-*modules*. An R-module (see Appendix) M is just a vector space where the scalars belong to a ring R. We will only consider the case where M is a finitely generated R-module. M is finitely generated if there exist elements $m_1,\dots,m_p \in M$ such that every element $m \in M$ can be written as $m = \lambda_1 m_1 + \dots + \lambda_p m_p$ for some $\lambda_1,\dots,\lambda_p \in R$. If for each $m \in M$, $\lambda_1,\dots,\lambda_p$ are unique, then $m_1,\dots,m_p$ is a *basis* for M. A *free* R-module is a finitely generated R-module which has a basis. An example of a free R-module is R^p, the set of column p-vectors with entries in R, with the usual (just like for a vector space) addition and scalar multiplication.

It can easily be seen that every free R-module M is isomorphic to a module R^p where p is the number of basis elements.

As is the case for vector spaces a linear map $A: R^p \to R^m$ is completely characterized by an $m \times p$-matrix A with entries in R. Therefore the map A can be identified with the matrix $A \in R^{m\times p}$.

We now generalize the concept of I/O systems to the case of free R-modules. Therefore we say that (2.1.7) is an input/output system over R if $u_h \in R^p$, $h = 0,1,2,\dots$; $y_h \in R^m$, $h = 0,1,2,\dots$; $F_i \in R^{m\times p}$, $i = 0,1,2,\dots$; where $R^{m\times p}$ denotes the set of $m \times p$-matrices over R.

In most parts of Chapter II it will not be necessary to be concerned with the discrete time dynamical interpretation of the input/output system (2.1.7). It will be sufficient to work with an abstract notion of I/O system. This concept will be the impulse response.

Suppose that $m = p = 1$, then (2.1.7) is called a *scalar* I/O system. Applying an *impulse* to the I/O system, i.e. an input sequence such that $u_0 = 1$ and $u_h = 0$, $h = 1,2,3,\dots$, the response, the output sequence, will be $(y_k)_{k\in\mathbb{Z}_+}$, where $y_k = F_k$, $k \in \mathbb{Z}_+$ (the set of non-negative integers). For this reason the sequence $(F_n)_{n\in\mathbb{Z}_+}$ is called the *impulse response* (also called *Markov sequence*).

Analogously to the scalar case the matrix sequence $(F_n)_{n\in\mathbb{Z}_+}$ is also called the impulse response for the general case (2.1.7). Given the impulse response and a sequence of inputs one can compute the output sequence by just computing the convolution in (2.1.7). Therefore, the impulse response completely characterizes the I/O system under consideration. The abstract notion of I/O system, mentioned above, will be the impulse response. Therefore we give the following definition.

(2.2.1) DEFINITION. *An impulse response* F *(over the ring* R*) is a sequence of* $m \times p$*-matrices over* R, $F = (F_n)_{n \in \mathbb{Z}_+}$ *for some integers* m, p. □

II.3. *The formal power series approach*

In this section another approach of I/O systems over R will be considered. For this purpose we need the concept of formal power series. Suppose we are given an R-sequence $R = (r_n)_{n \in \mathbb{Z}_+}$ $(r_n \in R)$.

(2.3.1) DEFINITION. *The formal power series* r(z) *in the variable* z^{-1} *associated with the sequence* $R = (r_n)_{n \in \mathbb{Z}_+}$ *is*

$$r(z) = \sum_{n=0}^{\infty} r_n z^{-n} .$$ □

REMARK. In fact r(z) is just another way of writing down the sequence $(r_0, r_1, r_2, \ldots)$ and z^{-1} is a position marker. (We do not require "convergence".) Introducing formal power series enables us to write down convolutions (like (2.1.7)) as products. Formal power series can also be used when we are dealing with so called realization problems. This will become clear in the sequel. □

To illustrate the use of formal power series we introduce $R[[z^{-1}]]$, the set of formal power series over R. The set $R[[z^{-1}]]$ can be given a ring structure. Suppose $r_1(z), r_2(z) \in R[[z^{-1}]]$, then the product $r_1(z)r_2(z)$ is defined by formally carrying out the multiplication of the two series and collecting the terms with the same power in z^{-1}, to obtain a member of $R[[z^{-1}]]$ again. The sum of two elements $r_1(z)$ and $r_2(z)$ is defined by addition of the corresponding coefficients (of the same powers in z^{-1}). This makes $R[[z^{-1}]]$ a ring.

REMARK. Multiplication of two elements in the ring $R[[z^{-1}]]$ in fact performs the convolution of the corresponding sequences. □

The formal power series associated with a sequence can also be seen as the (formal) Z-transform of this sequence ("formal" because no convergence is required).

Analogously to the scalar case we also have a formal power series associated with a vector sequence $(u_h)_{h\in\mathbb{Z}_+}$, $(y_k)_{k\in\mathbb{Z}_+}$ and also for a matrix sequence F. So we have

(2.3.2) $$u(z) = \sum_{h=0}^{\infty} u_h z^{-h}$$

(2.3.3) $$y(z) = \sum_{k=0}^{\infty} y_k z^{-k}$$

(2.3.4) $$F(z) = \sum_{n=0}^{\infty} F_n z^{-n}$$

where

$u(z) \in R[[z^{-1}]]^p$, the free $R[[z^{-1}]]$ module of p-vectors,

$y(z) \in R[[z^{-1}]]^m$, the free $R[[z^{-1}]]$ module of m-vectors,

$F(z) \in R[[z^{-1}]]^{m\times p}$, the set of $m \times p$-matrices over $R[[z^{-1}]]$.

A causal, discrete time, time invariant, linear input/output system (2.1.7) can equivalently be described by

(2.3.5) $$y(z) = F(z)u(z)$$

where $y(z)$, $F(z)$, $u(z)$ should not be thought of as functions of z. This is just another way of writing down (2.1.7).

II.4. *Free systems over a ring*

In this section we will explain what we mean by a free system over a ring R. It will be shown that every free system gives rise to an I/O system but not every I/O system is related to a free system in a natural way.

(2.4.1) DEFINITION. *A finite dimensional free system* Σ *over* R *is a quadruple of* R*-matrices* (A,B,C,D) *where* $A \in R^{n\times n}$, $B \in R^{n\times p}$, $C \in R^{m\times n}$, $D \in R^{m\times p}$ *for* *some integers* m*,* n*,* p*.* n *is called the dimension of* Σ*. If* $m = p = 1$ *then* Σ *is called scalar.* □

Because we will only deal with finite dimensional free systems, Σ will be called a free system.
Parallel to the dynamic interpretation of an impulse response over R we can proceed here in the following way. Again we will use an interpretation in terms of discrete time dynamics.

$$x_{k+1} = Ax_k + Bu_k \,, \quad x_0 = 0 \,, \qquad (2.4.2)$$
$$y_k = Cx_k + Du_k \,, \quad k = 0,1,2,\ldots \,.$$

Usually $x_k \in R^n$ is called the *state*, R^n is called the *state space*, again $u_k \in R^p$ is the input and $y_k \in R^m$ is the output.
We can now eliminate the states x_k, $k \in \mathbb{Z}_+$ and obtain an I/O system

$$(2.4.3) \qquad y_k = \sum_{h=0}^{k} F_{k-h}\, u_h \,, \quad k = 0,1,2,\ldots$$

where

$$(2.4.4) \qquad F_0 = D \,, \quad F_i = CA^{i-1}B \,, \quad i = 1,2,3,\ldots \,.$$

Thus with every free system $\Sigma = (A,B,C,D)$ we can associate an impulse response $F_\Sigma = (D,CB,CAB,CA^2B,\ldots)$. F_Σ will also be called the impulse response of Σ.
Later on we will also deal with non-free systems. Then the name "free system" will be justified. Until then we will omit the word "free" and simply call Σ a system.
Now let us be given an impulse response $F = (F_0,F_1,F_2,\ldots)$. We say that the system $\Sigma = (A,B,C,D)$ *realizes* F if $F = F_\Sigma$, i.e. if (2.4.4) holds. Σ is also called a *realization* of F.
A system $\Sigma = (A,B,C,D)$ with the dynamical interpretation (2.4.2) is usually called a (discrete time) *state space system*.
One of the reasons that state space systems are important is that an important class of I/O systems can be realized, thus providing more structure in an input/output system which in turn is very important for the construction of regulators and observers (see [49], [71]).

Another important feature of state space systems is the following. Suppose we have an I/O system with impulse response F. Let $F = F_\Sigma$ for some $\Sigma = (A,B,C,D)$ (Σ realizes F). Now consider the I/O system

$$y_k = \sum_{h=0}^{k} F_{k-h} u_h , \quad k = 0,1,2,\dots .$$

If we want to compute y_k we have to store $(u_0,u_1,\dots,u_k)$ and $(F_0,F_1,\dots,F_k)$ and y_k is the convolution of the two sequences. No matter how large k is, we can never "forget" some of the inputs and some part of the given impulse response. Furthermore the evaluation of the convolution represents an ever increasing amount of computations. On the other hand, if y_k is given by

$$(2.4.5) \qquad \begin{aligned} x_{k+1} &= Ax_k + Bu_k , \quad x_0 = 0 , \\ y_k &= Cx_k + Du_k , \quad k = 0,1,2,\dots , \end{aligned}$$

then in order to compute y_k we just have to know the fixed matrices A, B, C, D, the last input u_k and the last state x_k. We may forget all the previous inputs and the previous states. If the dimension of Σ is not too large, the amount of computations will be reduced considerably, primarily because the state and the output can be evaluated recursively. Furthermore the memory requirements may be reduced considerably. We say that the state contains all the relevant information from the past, that is, the state may be considered some kind of memory device (see also [45], [71]).
Having motivated the study of state space systems a little bit, we will now be concerned with the conditions that have to be imposed on an impulse response F such that F can be realized by a system Σ.
If $F = (F_0,F_1,F_2,\dots)$ is the impulse response of a system $\Sigma = (A,B,C,D)$, then

$$(2.4.6) \qquad F_0 = D , \quad F_i = CA^{i-1} B , \quad i = 1,2,3,\dots .$$

However, by the Cayley-Hamilton theorem we have

$$(2.4.7) \qquad A^n = \alpha_0 I + \alpha_1 A + \dots + \alpha_{n-1} A^{n-1}$$

for some $\alpha_i \in R$, $i = 0,\dots,n-1$. (The Cayley-Hamilton theorem holds for every matrix over a commutative ring, see [31]. This is where commutativity becomes important.) Therefore we have

$$(2.4.8) \qquad F_{k+n} = \sum_{i=0}^{n-1} \alpha_i F_{i+k} , \qquad k = 1,2,3,\ldots .$$

F is called *recurrent*. $\alpha_0,\ldots,\alpha_{n-1}$ are called *recurrence parameters* of (2.4.8).

This recurrency condition is also sufficient for F to be realizable.

(2.4.9) THEOREM. *If* $F = (F_0,F_1,F_2,\ldots)$ *is recurrent with recurrence parameters* $\alpha_0,\ldots,\alpha_{n-1}$, *then* $\Sigma = (A,B,C,D)$ *realizes* F *where* $D = F_0$,

$$A = \begin{bmatrix} 0 & & \cdots & & 0 & \alpha_0 I \\ I & 0 & & & \vdots & \\ 0 & I & \ddots & & & \vdots \\ \vdots & \cdot & \ddots & \cdot & 0 & \\ & & \ddots & \cdot & & \\ 0 & & \cdots & 0 & I & \alpha_{n-1} I \end{bmatrix}, \quad B = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad C' = \begin{bmatrix} F_1' \\ F_2' \\ \vdots \\ F_n' \end{bmatrix}$$

where all matrices I *and* 0 *are* $p \times p$-*matrices*.

PROOF. For a proof see [42] (straightforward computation). □

Usually, the dimension of this realization can be reduced considerably, unless in the case of a scalar impulse response when we have a minimal number of recurrence parameters.

The fact that realizability of an impulse response F is equivalent to recurrency of F can also be stated in terms of the formal power series associated with F. We therefore introduce the following notation.

$R[z]$ denotes the ring of polynomials in the variable z with coefficients in R.

$R(z)$ denotes the field of "rational functions" in z, i.e.

$$R(z) = \{r_1(z)/r_2(z) \mid r_1(z) \in R[z],\ r_2(z) \in R[z]\} .$$

Although $r_1(z)/r_2(z)$ is not a rational function (mapping) we will use the phrase "rational function" for a member of $R(z)$ because the meaning is nowhere ambiguous. In fact, $R(z)$ is a set of equivalence classes.

A polynomial $r(z)$ is called *monic* if the leading coefficient is the identity i.e. $r(z) = r_0 + r_1 z + \ldots + r_{n-1}z^{n-1} + z^n$. A rational function $r_1(z)/r_2(z)$ is called *proper* if $r_2(z)$ is monic and $\deg(r_2(z)) \geq \deg(r_1(z))$ where $\deg(r_i(z))$ denotes the degree (in z) of $r_i(z)$.

The ring of proper rational functions will be called $R_c(z)$. The "c" stands for causal. The ring $R_c(z)$ can be embedded in $R[[z^{-1}]]$ because an element of $R_c(z)$ can be expanded in a formal power series associated with an impulse response of a causal I/O system. A formal power series $r(z)$ is called rational if it is the expansion of a rational function $r_1(z)/r_2(z) \in R_c(z)$. Rationality of $F(z)$ means that every entry of $F(z)$ is rational. Observe that the rational power series $F(z)$ is a member of $R_c(z)^{m\times p}$.
We can now state the following theorem.

(2.4.10) THEOREM. *Let* $F = (F_0, F_1, F_2, \ldots)$ *be an impulse response where* $F_i \in R^{m\times p}$. *Let* $F(z)$ *be its associated formal power series. Then* F *is recurrent iff* $F(z)$ *is rational.*

PROOF. For a proof see [15]. □

(2.4.11) DEFINITION. *Let* $F = (F_0, F_1, F_2, \ldots)$ *be a realizable impulse response and let* $F(z)$ *be the associated formal power series. Then* $F(z)$ *(which is a proper rational matrix) is called the transfer matrix of the I/O system with impulse response* F. □

Now we can say that every transfer matrix $F(z)$ has at least one realization namely a realization of the impulse response of which $F(z)$ is the associated formal power series.
We can also say that every system $\Sigma = (A,B,C,D)$ has a transfer matrix $F(z)$ if we define $F(z)$ to be the formal power series associated with the impulse response $F = (D, CB, CAB, \ldots)$. Then we have

(2.4.12) THEOREM. *For the system* $\Sigma = (A,B,C,D)$ *the transfer matrix* $F(z)$ *is given by*

$$F(z) = C[zI - A]^{-1} B + D .$$

PROOF. Expanding $C[zI - A]^{-1} B + D$ in a formal power series immediately gives the result. □

Up to now we have obtained the following result: Every recurrent impulse response (proper rational matrix) has a state space realization. Generally there is a lot of redundancy in the realization (2.4.9). Next we will try to find a realization (given an impulse response or a transfer matrix) of minimal dimension, a so called *minimal realization*. This will be the subject of the next section.

II.5. *Realizations of I/O systems over rings*

The main tool in this section and also in the next chapter will be the *Hankel matrix* associated with an impulse response.

(2.5.1) DEFINITION. *Let* $F = (F_0, F_1, F_2, \ldots)$ *be an impulse response. Then the Hankel matrix* $H(F)$ *associated with* F *is defined by the following infinite block matrix. The* (i,j)*-th "element" is* F_{i+j-1} *for* $i,j = 1,2,3,\ldots$.

$$H(F) = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & \cdots \\ F_2 & F_3 & F_4 & & \\ F_3 & F_4 & & & \\ F_4 & & & & \\ \vdots & & & & \end{bmatrix}$$

and $H(F)_{\ell,k}$ *is the following Hankel block*

$$H(F)_{\ell,k} = \begin{bmatrix} F_1 & F_2 & F_3 & F_4 & \cdots & F_k \\ F_2 & F_3 & F_4 & & & \cdot \\ F_3 & F_4 & & & & \cdot \\ F_4 & & & & & \cdot \\ \vdots & & & & & \vdots \\ F_\ell & \cdot & \cdot & \cdot & \cdots & F_{\ell+k-1} \end{bmatrix} .$$

□

Many properties of a system over a ring can be derived if the system is considered to be a system over the *quotient field* (see appendix), if it exists. A commutative integral domain R can be embedded in its quotient field $Q(R)$ (see [8]). Therefore a matrix over R is, a fortiori, a matrix over $Q(R)$, so the rank of a matrix over R can be defined to be the rank over $Q(R)$. Therefore we are able to define the rank of the Hankel matrix $H(F)$ in (2.5.1) by

(2.5.2) $$\text{rank } H(F) = \sup_{\ell,k} \text{rank } H(F)_{\ell,k}, \quad \ell,k = 1,2,3,\ldots$$

where $H(F)_{\ell,k}$ are considered matrices over $Q(R)$.
We can now state the following theorem.

(2.5.3) THEOREM. *Suppose that rank* $H(F) = n$, *then there exists a realization* $\Sigma_q = (A_q, B_q, C_q, D_q)$ *over* $Q(R)$ *with dimension* n. *This realization is* *minimal (minimality of a realization means that there are no realizations having smaller dimension). Every other minimal realization* $\Sigma_q = (\bar{A}_q, \bar{B}_q, \bar{C}_q, \bar{D}_q)$ *is isomorphic to* Σ*, i.e., there exists an invertible matrix* T_q *over* $\mathcal{Q}(R)$ *such that* $\bar{A}_q = T_q A_q T_q^{-1}$, $\bar{B}_q = T_q B_q$, $\bar{C}_q = C_q T_q^{-1}$, $\bar{D}_q = D_q$.

PROOF. See [45]. □

Observe that the condition rank $H(F) < \infty$ means that F is recurrent over $\mathcal{Q}(R)$.

We will now impose some extra conditions on R such that recurrence over $\mathcal{Q}(R)$ implies recurrence over R. In that case we can say that F is realizable over R iff rank $H(F) < \infty$. We therefore state the following theorem.

(2.5.4) THEOREM. *Let* R *be a Noetherian, integrally closed domain (see appendix). Let* F *be an impulse response over* R. *Suppose that* F *is realizable over* $\mathcal{Q}(R)$. *Let* $\alpha_0, \dots, \alpha_{n-1}$ *be the recurrence parameters of a minimal recursion for* F *over* $\mathcal{Q}(R)$. *Then* $\alpha_i \in R$, $i = 0, \dots, n-1$.

PROOF. For a proof see [64]. □

(2.5.5) REMARK. Using theorem (2.5.4), theorem (2.4.9) gives us a realization which, generally, may not be expected to be minimal. However, in the case of a scalar input/output system the realization in (2.4.9), using the minimal recurrence in theorem (2.5.4), is minimal because it is minimal over $\mathcal{Q}(R)$. □

The following theorem may now be stated.

(2.5.6) THEOREM. *Let* R *be a principal ideal domain (see appendix). Suppose that rank* $H(F) = n$. *Then there exists a realization over* R *of dimension* n. □

A proof will be given in Chapter III.

The above theorem implies that, in the case where R is a principal ideal domain, there exists a minimal realization over R if rank $H(F) < \infty$.

We already mentioned the role of the state of a system as some kind of memory. Therefore it is important that the state space is small. This is the reason that we are interested in minimal realizations. On the other hand we are interested in state space systems which contain no more information than is already available in the I/O description. This idea is closely related to the concept of Nerode equivalence classes (see [45]). From that point of view the state space is just the set of Nerode equivalence classes provided with some structure. In formalizing these ideas the notions reachability and observability enter. We formulate the following definitions for the state space system Σ

$$(2.5.7)\qquad \begin{aligned} x_{k+1} &= Ax_k + Bu_k\ , \quad x_0 = 0\ , \\ y_k &= Cx_k + Du_k\ , \quad k = 0,1,2,\ldots \end{aligned}$$

where $u_k \in R^p$, $y_k \in R^m$, $x_k \in R^n$ and the matrices have appropriate dimensions.

(2.5.8) DEFINITION. Σ *is reachable over* R *if the columns of the block matrix* $[B,AB,\ldots,A^{n-1}B]$ *generate* R^n *over* R. □

(2.5.9) DEFINITION. Σ *is observable over* R *if* $Cx = CAx = \ldots = CA^{n-1}x = 0$ *implies* $x = 0$. □

(2.5.10) DEFINITION. Σ *is canonical over* R *if* Σ *is reachable and observable.* □

These are only formal definitions. The intuitive notions of reachability and observability imply and are implied by these conditions (see [45]). When no confusion can arise we leave out "over R".

Observe that reachability (observability) of the system Σ is only concerned with the pair (A,B) ((C,A)). Therefore we will also be working with the reachable pair (A,B) and the observable pair (C,A) by which we mean that the conditions in (2.5.8) and (2.5.9) are satisfied respectively.
If we have a system $\Sigma = (A,B,C,D)$, then the triple (A,B,C) will be called canonical if (A,B) is a reachable pair and (C,A) is an observable pair. (This is the same as: Σ is canonical.)

Given a realization $\Sigma = (A,B,C,D)$ of an I/O system one might try, by some reduction method, similar to the one as is used for systems over a field, to reduce the state space until reachability and observability are obtained. However, in general, canonical (free) realizations do not exist.
This idea motivates the introduction of a generalized notion of a realization.

(2.5.11) DEFINITION. *A non-free system* Σ *is a quintuple* $\Sigma = (X,A,B,C,D)$ *where* X *is a finitely generated* R*-module and* A, B, C, D *are* R*-linear maps:* $A: X \to X$, $B: R^p \to X$, $C: X \to R^m$, $D: R^p \to R^m$ *for some integers* m *and* p. □

REMARK. Considering the state space to be the set of Nerode equivalence classes it can be provided with an R[z]-module structure and furthermore it can be shown (see [71]) that the state space is in fact isomorphic to the R-module X_F generated by the columns of the Hankel matrix associated with the I/O system. □

(2.5.12) DEFINITION. *A non-free system* $\Sigma = (X,A,B,C,D)$ *realizes an impulse response* $F = (F_0,F_1,F_2,\ldots)$ *if* $F_0 = D$ *and* F_n *is the following composition of maps:* $F_n = C \circ A^{n-1} \circ B$, $n = 1,2,3,\ldots$. □

Observe that if an impulse response can be realized by a free system $\Sigma = (A,B,C,D)$, then it can, a fortiori, be realized by a non-free system (by taking $X = R^n$). The converse is also true. For if we fix a set of generators for X then the R-linear maps can be represented (non-uniquely) by matrices.
Now we introduce reachability and observability for a non-free system $\Sigma = (X,A,B,C,D)$.

(2.5.13) DEFINITION. Σ *is reachable if the set* $(Be_i, A\circ Be_i, \ldots, A^{n-1}\circ Be_i)$, $i = 1,\ldots,p$, *generates* X. *Here* e_i *denotes the i-th basis vector in* R^p. □

(2.5.14) DEFINITION. Σ *is observable if* $Cx = C\circ Ax = \ldots = C\circ A^{n-1}x = 0$ *implies* $x = 0$. □

(2.5.15) DEFINITION. Σ *is canonical if* Σ *is reachable and observable.* □

As is the case for systems over a field, canonical non-free realizations of an impulse response are only unique up to isomorphism (see (2.5.3)). An analogous result is formulated in the *realization isomorphism theorem.*

(2.5.16) THEOREM. *Suppose that* $F = (F_0,F_1,F_2,\ldots)$ *is a realizable impulse response and suppose that we have two canonical non-free realizations* $\Sigma = (X,A,B,C,D)$ *and* $\bar{\Sigma} = (\bar{X},\bar{A},\bar{B},\bar{C},\bar{D})$, *then there exists an invertible* R*-homomorphism* $T: \bar{X} \to X$ *such that* $\bar{A} = T^{-1}\circ A\circ T$, $\bar{B} = T^{-1}\circ B$, $\bar{C} = C\circ T$, $\bar{D} = D$.

PROOF. For a proof see [15]. □

Σ and $\bar{\Sigma}$ are called *isomorphic*.

Now suppose that $F = (F_0, F_1, F_2, \ldots)$ is a realizable impulse response. Then there exists a canonical non-free realization where the state space is the above mentioned X_F (for a proof see [71]) and if X_F is a free R-module then this realization is also minimal for in this case we take a basis in X_F and then the maps A, B, C, D can be represented by matrices, thus constituting a free realization. This realization is, a fortiori, a realization over $\mathcal{Q}(R)$ which is canonical and therefore is minimal (see [45]).

In the following we will mainly be concerned with systems over a principal ideal domain R. In this case a canonical (non-free) realization is always free, for we can take X_F as the state space. X_F is torsion free (see appendix) as a module over R and is finitely generated. Therefore X_F is free (see [31]). In Chapter III we give an algorithmic proof of this result. So, in the case of a principal ideal domain we can always work with matrices, when dealing with canonical realizations. Furthermore these canonical realizations have minimal dimension. It is, however, not true that minimal realizations are canonical as is the case for systems over a field (see [45]).

From now on we will again omit the words "free" and "non-free" when there cannot be any ambiguity. Unless otherwise stated we will assume R to be a principal ideal domain.

Let $\Sigma = (A,B,C,D)$ be a system of dimension n over R with state space interpretation

$$\text{(2.5.17)} \qquad \begin{aligned} x_{k+1} &= Ax_k + Bu_k, \quad x_0 = 0, \\ y_k &= Cx_k + Du_k, \quad k = 0,1,2,\ldots . \end{aligned}$$

Sometimes one is interested in modifying the characteristic polynomial of A. In section V.5 some aspects of this are studied. In some occasions the stability properties of a system have to be improved by means of a regulator. One of the main problems concerning regulators or observers (see [49], [71]) is: How can the characteristic polynomial of A ($\det(zI - A)$) be modified by using *feedback* $u_k = Kx_k$. The next theorem is concerned with the question of *pole assignability*. (If R is not a principal ideal domain then this theorem does not necessarily hold. For a counterexample see [11].)

(2.5.18) THEOREM. *Suppose that* $A \in R^{n\times n}$ *and* $B \in R^{n\times p}$. *Then a necessary and sufficient condition for* (A,B) *to be a reachable pair is: For every set*

$(p_1,\dots,p_n) \in R$ *there exists a matrix* $K \in R^{p\times n}$ *such that* $\det(zI - A + BK) =$ $= (z - p_1) \dots (z - p_n)$.

PROOF. For a proof see [55]. □

In the scalar case we can say more.

(2.5.19) THEOREM. *Let* (A,B) *be a reachable pair over* R, *let* $A \in R^{n\times n}$, $B \in R^{n\times 1}$ *(where* R *is not necessarily a principal ideal domain). Then for every polynomial* $p(z)$ *with degree* n *there exists a row vector* K *such that* $\det(zI - A + BK) = p(z)$.

PROOF. The proof can be given using the so called standard controllable form (see also Chapter IV). The construction of K can also be achieved along the lines of a stabilization algorithm due to Ackermann [1]. □

Observe that in (2.5.19) every n-th degree polynomial p(z) can be the characteristic polynomial of the matrix $A - BK$, whereas in (2.5.18) only polynomials of a special form can be obtained. The fact that every n-th degree polynomial can be obtained by means of feedback is called *coefficient assignability*. Coefficient assignability can also be obtained in the case of a system over a local ring (see appendix) or even a semi local ring [71] (see appendix). This can be done using a generalization of Heymann's Lemma [34]. We will state this result only for a local ring, for this is the only case we will be needing.

(2.5.20) THEOREM. *Let* (A,B) *be a reachable pair over a local ring* R, *then there exists a matrix* K *and a vector* u *such that* $(A + BK, Bu)$ *is a reachable pair.*

PROOF. The proof can be given along the lines of [71] where the problem is reduced to a similar problem over a field in which case Heymann's Lemma can be applied [34]. □

The above methods for pole assignability and coefficient assignability cannot immediately be used for the observer case. For this case one would need that the *dual system* $\Sigma' = (A',C',B',D')$ be reachable. However, reachability and observability are not dual properties. It is not even true that a minimal realization Σ satisfies: Σ or Σ' is canonical.

For example take $F = (6,6,6,\ldots)$, a scalar impulse response over $\mathbb{Z}$. A canonical realization is $\Sigma_1 = (1,1,6,6)$. A realization Σ_2 such that Σ_2' is canonical is $\Sigma_2' = (1,6,1,6)$. A minimal realization Σ_3, such that Σ_3 nor Σ_3' is canonical, is $\Sigma_3 = (1,2,3,6)$.
In the case of a principal ideal domain we can also construct a realization $\Sigma = (A,B,C,D)$, for an impulse response F, such that $\Sigma' = (A',C',B',D')$, is canonical. This can be achieved by constructing a canonical realization for the transposed impulse response $F' = (F_0',F_1',F_2',\ldots)$. Hence we can in fact use (2.5.18), (2.5.19) and (2.5.29) to construct observers. For more information on observers see [49], [79].

III *ALGORITHMS*

III.1. *Matrices over a principal ideal domain*

In this chapter we are going to construct canonical realizations for an impulse response F over a principal ideal domain R. Also a recursive algorithm, including some results concerning the partial realization problem, is presented. Furthermore it will be shown that Ho's algorithm (see [45]) and an algorithm due to Zeiger (see [44]) can be generalized to the ring case.

We will also present an algorithm that constructs a realization given the transfer matrix of a system over R. This will be done by constructing first a realization over $Q(R)$ and afterwards reducing this realization to a canonical realization over R. For this algorithm a realization method presented by Kalman in [41] and a realization algorithm described in [33] by Heymann will be very useful.

In all the algorithms to be presented in this chapter the existence of a Hermite form or a Smith form is crucial. We will need a somewhat modified Hermite form and also a modified Smith form will do because the usual divisibility properties of the Smith form are irrelevant for our purposes.

First of all we will introduce the Hermite form, the modified (in a certain sense) Hermite form and the Smith form of a matrix over a principal ideal domain. We start by observing that in a principal ideal domain R the Bezout identity holds. This means that for $r_1, r_2 \in R$ a greatest common divisor d can be defined such that d is a linear combination of r_1 and r_2, i.e., there exist $c_1, c_2 \in R$ such that $d = c_1r_1 + c_2r_2$. This can be generalized to the case of n elements $r_1, \ldots, r_n \in R$. Again a greatest common divisor d can be defined such that $d = c_1r_1 + \ldots + c_nr_n$. Furthermore it can easily be shown that d is a "greatest" common divisor of $r_1, \ldots, r_n$, i.e., a divisor such that every other divisor of $r_1, \ldots, r_n$ divides d (a divisor q of $r_1, \ldots, r_n$ is an element of R such that $r_i = qd_i$ for some elements d_i, $i = 1, \ldots, n$). In general a greatest common divisor of $r_1, \ldots, r_n$ is not unique for if d is a greatest common divisor, then du is a greatest common divisor whenever u is a unit.

We will use the following notation. If $r_1,\ldots,r_n \in R$, then $(r_1,\ldots,r_n)$ denotes a greatest common divisor of $r_1,\ldots,r_n$.
In the sequel we will frequently use unimodular matrices over R.

(3.1.1) DEFINITION. *A unimodular matrix over* R *is a square matrix which has an inverse over* R. □

Let A be a matrix over R (not necessarily square) then the following operations are called *elementary row (column) operations:*

i) Interchanging two rows (columns);

ii) multiplication of a row (column) by a unit of R;

iii) Addition of r times a row (column) to another row (column), where $r \in R$.

It can easily be seen that each of these row (column) operations corresponds to the left (right) multiplication of A by a unimodular matrix.

(3.1.2) DEFINITION. *Let* A, B *be two matrices over* R, *then* A *is called left equivalent to* B *if* A = UB. A *is called right equivalent to* B *if* A = BV. A *is called equivalent to* B *if* A = UBV. *Here* U, V *are unimodular matrices.* □

(3.1.3) THEOREM. *Let* A *be an* $n \times m$*-matrix over* R. *Then* A *is right equivalent to a lower triangular matrix* B *(*$b_{ij} = 0$ *if* $j > i$*) where* b_{ii} *is unique up to a unit if* A *is regular and* $b_{i\ell}$ *is an element of the residue class modulo* b_{ii} *where, in case* A *is regular,* $b_{i\ell}$ *is also unique up to a unit. Here* $\ell < i$ *and* $i = 1,\ldots,min(m,n)$.

PROOF. For a proof see [57]. □

In order to obtain the matrix B one only has to perform elementary column operations and a right multiplication with a unimodular matrix based on the Bezout identity, while in the case where R is a *Euclidean domain* (see appendix) only elementary column operations are sufficient. The matrix B in the above theorem is called the *Hermite form* of A (also called Hermite normal form).
What we need is not precisely the Hermite form of A but a lower triangular matrix $B = [\bar{B},0]$ such that $\bar{B}$ has full column rank over $\mathcal{Q}(R)$. In general this cannot be obtained by just right multiplying A with a unimodular matrix V. If we also allow multiplying with a permutation matrix Π on the left this special form, equivalent to A, can be obtained. We will not need the special properties of the diagonal elements of B, nor will we need the special relation of row elements and the corresponding diagonal elements.

Every lower triangular matrix $B = [\bar{B},0]$ such that $\bar{B}$ has full column rank over $\mathcal{Q}(R)$ where

(3.1.4) $\quad B = [\bar{B},0] = \Pi AV$

will be called a *modified Hermite form* of A. We now have the following theorem.

(3.1.5) THEOREM. *Let* A *be a matrix over* R. *Then there exists a permutation matrix* Π *and a unimodular matrix* V *such that* $B = \Pi AV$ *is a modified Hermite form.*

PROOF. Although the theorem is valid for a principal ideal domain, the proof will be given for a Euclidean domain R because this is the only case we will be needing. Suppose that A is a $m \times p$ matrix over a Euclidean domain R. If $A = 0$ then A is a modified Hermite form and we are finished. If $A \neq 0$, then there is some element $a_{ij} \neq 0$. This element can be moved to the leading position (1,1) by just applying row and column permutations. Hence we may assume that $a_{11} \neq 0$. We may also assume that a_{11} has the smallest φ value among the elements of the first row. Hence we may write

$$a_{1j} = q_{1j}\, a_{11} + r_{1j} \qquad \text{for } j = 2,\dots,p$$

and $\varphi(r_{1j}) < \varphi(a_{11})$ where φ is a Euclidean function for R (see appendix). Hence by adding appropriate multiples of the first column to the second up to the p-th column we can achieve that

$$AV_1 = \begin{bmatrix} a_{11} & r_{12} & \cdots & r_{1p} \\ a_{21} & & & \\ \vdots & & A_1 & \\ a_{m1} & & & \end{bmatrix}$$

where V_1 is a unimodular matrix. By applying a column permutation we can obtain that the element in the (1,1) position has smallest φ value among the elements in the first row. Again we can add appropriate multiples of the first column to the other columns and we obtain that all elements (up to the (1,1) element) in the first row have φ value smaller than the φ value of the element in position (1,1). Eventually we end up with a matrix

$$A\bar{V} = \begin{bmatrix} \bar{a}_{11} & 0 & \dots & 0 \\ \bar{a}_{21} & & & \\ \vdots & & \bar{A} & \\ \bar{a}_{m1} & & & \end{bmatrix}.$$

The same procedure can be applied to $\bar{A}$ and ultimately we obtain

$$\Pi AV = B = [\bar{B},0]$$

where Π is a permutation matrix which comprises all row permutations which occur in the described process and V is the product of all unimodular matrices representing the applied elementary column operations. Furthermore $\bar{B}$ is a lower triangular matrix having full column rank. □

REMARK. In this way not only a modified Hermite form but also the properties concerning the offdiagonal elements, mentioned in theorem (3.1.3), can be obtained. In addition, also the uniqueness (up to a unit) result concerning the diagonal elements also holds for theorem (3.1.5). □

(3.1.6) THEOREM. *Let* A *be an* $n \times m$*-matrix over* R. *Then* A *is equivalent to a matrix* $\bar{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$ *where* $D = \text{diag}(d_1,\dots,d_r)$ *and* d_i *divides* d_{i+1} *for* $i = 1,\dots,r-1$. *Here* r *denotes the rank of* A *over* Q(R) *and some of the zero matrices are possibly empty. Furthermore,* d_i *is unique up to a unit for* $i = 1,\dots,r$.

PROOF. For a proof see [57]. □

The matrix $\bar{D}$ is called the *Smith form* of A (also called Smith normal form or Smith canonical form). Again we will not exactly need the Smith form. We do not need the divisibility result "d_i divides d_{i+1}, $i = 1,\dots,r-1$". We only need the diagonal character of $\bar{D}$. This often simplificates the algorithm to obtain $\bar{D}$ considerably.
Every diagonal matrix $\bar{D} = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$, where D has full rank, equivalent to A will be called a *modified Smith form* for A.
The modified Hermite form and the modified Smith form for a certain matrix will be fundamental for the realization algorithms to follow.

III.2. *Realization algorithms for an impulse response*

In this section we will derive algorithms to construct a canonical realization (2.4.7), (2.5.10) of an impulse response F.
In [63] Silverman's algorithm [68] is used to compute a realization of an impulse response over a principal ideal domain. The realization is obtained by first computing a realization over the quotient field and then applying a suitable state space transformation. In this section a more direct realization algorithm is proposed, which is related to an algorithm due to Zeiger (cf. [44]). It is also shown that the original Zeiger algorithm and the Ho algorithm [45] can be extended to systems over a principal ideal domain, but the algorithm described in this section seems to be more appealing. Furthermore a recursive algorithm, similar to Rissanen's algorithm [60], is described, which to some extent can also be used for obtaining partial realizations (see [43]).
The principle objective of this thesis will be the application of the theory of systems over a principal ideal domain to the case of 2-D systems, but in this chapter we will also present some applications to the case of systems over the integers [63] and the case of delay differential systems [55], [46], [47].
In Chapter II it was shown that for systems over a principal ideal domain it is not necessary to consider non-free systems and therefore we can work with matrices. For this reason we will introduce some matrix language. A matrix $A \in R^{m\times n}$ will be called *right regular* if there does not exist a non-zero vector $x \in R^n$ satisfying $Ax = 0$. Equivalently A is right regular if rank A = n. The matrix A is called *right invertible* if there exists a matrix $A^+ \in R^{n\times m}$ such that $AA^+ = I$. *Left regularity* and *left invertibility* are defined similarly.
Given a system $\Sigma = (A,B,C,D)$ we define for $k = 1,2,3,\ldots$

(3.2.1) $$Q(\Sigma,k) = [B,AB,\ldots,A^{k-1}B] \;,$$

(3.2.2) $$P(\Sigma,k) = [C',A'C',\ldots,(A')^{k-1}C']' \;.$$

Observe that Σ is reachable (see (2.5.8)) if $Q(\Sigma,n)$ is right invertible and observable (see (2.5.9)) if $P(\Sigma,n)$ is right regular.
In order to construct a canonical realization of a given impulse response $F = (F_0,F_1,F_2,\ldots)$ we consider the Hankel matrix $H = H(F)$ (see (2.5.1)) and Hankel blocks $H_{\ell k} = H(F)_{\ell,k}$ (see (2.5.1)) which we write down again for

convenience

$$(3.2.3)\qquad H_{\ell k} = \begin{bmatrix} F_1 & F_2 & F_3 & \cdots & F_k \\ F_2 & F_3 & & & \cdot \\ F_3 & & & & \cdot \\ \vdots & & & & \vdots \\ F_\ell & \cdot & \cdot & \cdots & F_{\ell+k-1} \end{bmatrix}.$$

Remember that rank H is defined as (see (2.5.2)) rank $H = \sup_{\ell,k} \operatorname{rank} H_{\ell k}$.

The following result is instrumental and constitutes the main theorem of this chapter.

(3.2.4) THEOREM. *Suppose that for a certain pair of integers* ℓ, k *we have* rank $H_{\ell k}$ = rank $H = n$. *If matrices* $P \in R^{\ell m\times n}$, $Q \in R^{n\times kp}$, $Q_k \in R^{n\times p}$ *satisfy*

1° $H_{\ell,k+1} = P[Q,Q_k]$,
2° Q *is right invertible,*
3° P *is right regular,*

then there exists a unique canonical realization $\Sigma = (A,B,C,D)$ *of* F *such that* $P = P(\Sigma,\ell)$, $[Q,Q_k] = Q(\Sigma,k+1)$, *viz.*

$$(3.2.5)\qquad A = [Q_1,\ldots,Q_{k-1},Q_k]Q^+ \qquad B = Q_0\ ,\quad C = P_0\ ,\quad D = F_0$$

where P_0 *is the matrix consisting of the first* m *rows of* P, $Q_i \in R^{n\times p}$ *is defined by the block decomposition* $Q = [Q_0,Q_1,\ldots,Q_{k-1}]$ *and* Q^+ *is a right inverse of* Q.

PROOF. Considering F as an impulse response over $\mathcal{Q}(R)$, we find a canonical $\mathcal{Q}(R)$-realization $\bar\Sigma = (\bar A,\bar B,\bar C,\bar D)$ of F of dimension n (see [45]). Then we have

$$(3.2.6)\qquad PQ = H_{\ell k} = \bar P\bar Q$$

where $\bar P := P(\bar\Sigma,\ell)$, $\bar Q := Q(\bar\Sigma,k)$. Let $\bar P^+$ be a left inverse (over $\mathcal{Q}(R)$) of $\bar P$ and $\bar Q^+$ a right inverse of $\bar Q$. Then we have

$$\bar P^+ PQ\bar Q^+ = I\ .$$

Thus if we define $S := Q\bar Q^+ \in \mathcal{Q}(R)^{n\times n}$, then S is invertible and $S^{-1} = \bar P^+P$. The system $\Sigma = (A,B,C,D)$ defined by $A := S\bar AS^{-1}$, $B := S\bar B$, $C := \bar CS^{-1}$, $D = \bar D$

is also a realization of F over $\mathcal{Q}(R)$. Equation (3.2.6) implies

$$Q = S\bar{Q} \ , \quad P = \bar{P}S^{-1} \ .$$

Hence $P = P(\Sigma,\ell)$, $Q = Q(\Sigma,k)$. But then we must have $C = P_0 \in R^{m\times n}$, $B = Q_0 \in R^{n\times p}$. In addition,

$$H_{\ell,k+1} = P[Q,Q_k] = PQ(\Sigma,k+1)$$

and consequently $Q_k = A^kB$. It follows that

$$(3.2.7) \qquad [Q_1,\ldots,Q_k] = [AQ_0,\ldots,AQ_{k-1}] = AQ$$

and therefore we have (3.2.6), which implies $A \in R^{n\times n}$. Consequently Σ is a realization over R. Σ is also a canonical system over R, for if $Q = Q(\Sigma,k)$ is right invertible and $k \le n$, then $Q(\Sigma,n)$ is right invertible,

$$Q(\Sigma,n)\begin{bmatrix} Q(\Sigma,k)^+ \\ 0 \end{bmatrix} = I$$

where $Q(\Sigma,k)^+$ is a right inverse of $Q(\Sigma,k)$ and 0 is a $(n-k)p \times n$ matrix consisting only of zeroes. If $Q = Q(\Sigma,k)$ is right invertible and $k > n$ then we have, by the Cayley-Hamilton theorem,

$$Q_{n+i} = \alpha_0 Q_i + \alpha_1 Q_{i+1} + \ldots + \alpha_{n-1} Q_{n-1+i} \ , \quad i = 0,\ldots,k-n-1$$

and therefore

$$(3.2.8) \qquad [Q_0,Q_1,\ldots,Q_{n-1},Q_n,\ldots,Q_{k-1}]V = [Q_0,Q_1,\ldots,Q_{n-1},0,\ldots,0]$$

for some unimodular matrix V. Now suppose that Q^+ is a right inverse of Q, Then $V^{-1}Q^+$ is a right inverse of $[Q_0,Q_1,\ldots,Q_{n-1},0,\ldots,0]$ and therefore $Q(\Sigma,n)$ is right invertible. In the same way right regularity of $P(\Sigma,n)$ can be proved. Therefore Σ is a canonical realization over R. □

(3.2.9) REMARK. In theorem (3.2.4) we did not use that R is a principal ideal domain. Obviously theorem (3.2.4) remains valid if R is any commutative integral domain. □

Some problems that remain are: How to determine n and how to choose ℓ and k. Furthermore, given that F is realizable, is it possible to obtain a factorization of $H_{\ell,k+1}$ as is required in 1° of theorem (3.2.4)?

Whether or not H has finite rank cannot be decided in general but, if the impulse response F stems from a transfer matrix, then we are sure that rank H is finite. But if the I/O system is given by its transfer matrix the realization method using the Hankel matrix seems to be a long way around. For this case we will give a more direct realization algorithm in Section III.3.

Let us now suppose that we know the rank of the Hankel matrix, then in order to find a Hankel block $H_{\ell k}$ such that rank $H_{\ell k}$ = rank H, it suffices to take $\ell = k = n$. This follows from the Cayley-Hamilton theorem. Again we are left with the problem whether or not a factorization of $H_{\ell,k+1}$ as in 1° of theorem (3.2.4) is possible. The following result states that for sufficiently large k a factorization of the form

$$H_{\ell,k+1} = P[Q,Q_k]$$

is always possible, once the factorization

$$H_{\ell k} = PQ$$

is given.

(3.2.10) THEOREM. *Let* $P \in R^{\ell m\times n}$, $Q \in R^{n\times kp}$ *satisfy the conditions* 2° *and* 3° *of theorem* (3.2.4) *and assume that* rank $H_{\ell k}$ = rank $H \le k$. *If*

$$H_{\ell k} = PQ$$

then there exists a unique $Q_k \in R^{m\times n}$ *such that*

$$H_{\ell,k+1} = P[Q,Q_k] \;.$$

PROOF. There exists a realization of rank $\le$ k (see [64]). By the Cayley-Hamilton theorem the impulse response satisfies a recurrence relation of the form

$$F_{k+j} = \alpha_0 F_j + \dots + \alpha_{k-1} F_{k-1+j} \qquad (j = 1,2,3,\dots)$$

where $\alpha_i \in R$. If we write $W := [\alpha_0 I,\dots,\alpha_{k-1} I]' \in R^{kp\times p}$, then it follows that

$$(3.2.11)\quad H_{\ell,k+1} = [H_{\ell k}, H_{\ell k}W] = P[Q,QW] \;.$$

Hence we may choose $Q_k = QW$. Because P is right regular, Q_k is unique. □

(3.2.12) REMARK. Also this result is valid for more general rings than principal ideal domains. Obviously, it suffices that F satisfies a recurrence relation of order $\leq k$. This is for example the case for Noetherian integral domains which are integrally closed (see [64]). □

Now the question arises of how to compute a factorization of $H_{\ell,k+1}$ such that the conditions of theorem (3.2.4) are satisfied. One way of doing this depends on the Smith form (see (3.1.6)). We start by factorizing $H_{\ell k}$ as follows. There exist unimodular matrices U and V and an $n \times n$ diagonal matrix D such that

$$(3.2.13) \quad H_{\ell k} = U\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}V .$$

Some of the zero matrices in (3.2.13) are possibly empty. The matrix D is regular (i.e. right regular and left regular). We do not require that the diagonal elements of D satisfy the divisibility properties in theorem (3.1.6). If we define

$$(3.2.14) \quad P := U\begin{bmatrix} D \\ 0 \end{bmatrix} , \quad Q = [I,0]V , \quad Q^{+} := V^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix} ,$$

we see that P is right regular and $QQ^{+} = I$, so that Q is right invertible. In addition $H_{\ell k} = PQ$. Now if we decompose $H_{\ell,k+1}$ as

$$(3.2.15) \quad H_{\ell,k+1} = [H_{\ell k},S]$$

it follows from theorem (3.2.10) that there exists a matrix Q_k such that $S = PQ_k$, hence

$$U^{-1}S = \begin{bmatrix} DQ_k \\ 0 \end{bmatrix} ,$$

i.e., the first n rows of $U^{-1}S$ are divisible by the corresponding diagonal element of D, and the remaining rows are zero. Thus we are able to determine Q_k.

(3.2.16) REMARK. If these divisibility conditions on $U^{-1}S$ are not satisfied, this implies that F does not have a realization of rank $\leq k$. This indicates how a recursive realization algorithm could be constructed, losely speaking by increasing ℓ or k in a sensible way until the conditions on S are satisfied (see (3.4.5)). □

The computation of a Smith form might be rather elaborate even if we do not require the diagonal elements of D to satisfy the divisibility properties. Therefore we will now describe a realization algorithm based on the modified Hermite form of $H_{\ell k}$.
There exist a permutation matrix Π, a unimodular matrix V and a lower triangular matrix G such that G has full rank and (see (3.1.5))

(3.2.17) $H_{\ell k} = \Pi[G,0]V$

where the zero matrix is possibly empty. Then we define $P := \Pi G$, $Q := [I,0]V$ and we have the desired factorization. The matrix Q_k in $H_{\ell,k+1} = P[Q,Q_k]$ has to be determined from the equation $PQ_k = S$ where S is the same matrix as in (3.2.1). Therefore $GQ_k = \Pi^{-1}S$. However, since $[I_n,0]G$ is a regular matrix, Q_k is uniquely determined by the $n \times n$ equation

$$[I_n,0]GQ_k = [I_n,0]\Pi^{-1}S$$

where I_n is the $n \times n$ identity matrix and n is the rank of $H_{\ell k}$. This equation is easy to solve because of the triangular character of $[I_n,0]G$. It follows from theorem (3.2.10) that a solution exists and satisfies the equation $GQ_k = \Pi^{-1}S$ provided that rank $H \le k$.

(3.2.18) REMARK. The algorithm given is closely related to Zeiger's algorithm (cf [44]). In this algorithm for systems over a field the factorization $H_{\ell k} = PQ$ with Q right invertible and P left invertible yields the realization $A = P^+(\sigma H)_{\ell k}Q^+$, $B = Q_0$, $C = P_0$, $D = F_0$, where P^+ is a left inverse of P and $(\sigma H)_{\ell k}$ is the shifted Hankel block,

$$(3.2.19)\qquad (\sigma H)_{\ell k} = \begin{bmatrix} F_2 & F_3 & F_4 & \cdots & F_{k+1} \\ F_3 & F_4 & & & \cdot \\ F_4 & & & & \cdot \\ \vdots & & & & \vdots \\ F_{\ell+1} & \cdot & \cdot & \cdots & F_{\ell+k} \end{bmatrix}.$$

In the case of a system over a principal ideal domain this algorithm is not directly applicable since it is usually not possible to factorize $H_{\ell k}$ in such a way that P is left invertible and Q is right invertible (see also remark (3.2.21)). However, in our algorithm P is right regular and therefore left invertible over $\mathcal{Q}(R)$. If one is willing to perform calculations in $\mathcal{Q}(R)$ then one can use Zeiger's algorithm since it follows from theorems (3.2.4) and (3.2.10) that the resulting realization is a realization over R because, given the factorization over R, the resulting realization is unique. □

(3.2.20) REMARK. The method of computing a factorization using the Smith form (3.1.6) is obviously related to Ho's algorithm (see [45]). The proper generalization of Ho's algorithm to the ring case is the following: starting from the factorization

$$UH_{\ell k}V = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

where U and V are unimodular matrices and D is a regular diagonal matrix, we construct $\Sigma = (A,B,C,D)$ from

$$DA = [I_n,0]U(\sigma H)_{\ell k} V \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

$$DB = [I_n,0]U\, H_{\ell k} \begin{bmatrix} I_p \\ 0 \end{bmatrix}$$

$$C = [I_m,0]H_{\ell k}\, V \begin{bmatrix} I_n \\ 0 \end{bmatrix}$$

$$D = F_0$$

where $(\sigma H)_{\ell k}$ is given by (3.2.19). Then $\Sigma = (A,B,C,D)$ is the realization of F corresponding to the factorization $H_{\ell k} = PQ$ where

$$P = U^{-1} \begin{bmatrix} D \\ 0 \end{bmatrix}, \quad Q = [I,0]V^{-1} .$$

The solvability of the equations for A and B again follows from theorems (3.1.4) and (3.2.10). Again this shows that one can work with the Ho algorithm if one is willing to perform calculations over $\mathcal{Q}(R)$. □

Observe that one cannot apply Zeiger's algorithm or Ho's algorithm starting with a factorization of $H_{\ell k}$ over $\mathcal{Q}(R)$. One has to perform the factorization over R and from that point one can work with these algorithms.
The algorithm proposed in the foregoing, in particular if the modified Hermite form is used, is simpler than the algorithms which are modifications of Zeiger's of Ho's algorithm. For Zeiger's algorithm it is necessary to do calculations in $\mathcal{Q}(R)$ and inverses of both P and Q have to be calculated. For Ho's algorithm it is necessary to compute the Smith form (without the divisibility properties) which is more elaborate than the modified Hermite form.

(3.2.21) REMARK. A realization $\Sigma = (A,B,C,D)$ is called *split* if both (A,B) and (A',C') are reachable pairs (see [71]), for "reachable pair" see (2.5.8). If an impulse response F admits a split realization Σ, then every canonical realization $\bar{\Sigma}$ of F is split, since it follows from the realization isomorphism theorem for the case of a principal ideal domain that $P(\bar{\Sigma},n)T = P(\Sigma,n)$ for some invertible matrix T. Obviously the realization given in theorem (3.1.6) is split iff P is left invertible. Therefore, if we construct P and Q using (3.2.10), the realization is split if the invariant factors of $H_{\ell k}$, i.e. the diagonal elements of the matrix D in the Smith form of $H_{\ell k}$, are invertible in R. Thus we recover a result of Sontag [71]. □

Working with matrices (over R) the realization isomorphism theorem can be derived quite easily from the corresponding theorem in the field case. Suppose that F is an impulse response with canonical realizations $\Sigma = (A,B,C,D)$ and $\bar{\Sigma} = (\bar{A},\bar{B},\bar{C},\bar{D})$, then Σ and $\bar{\Sigma}$ are a fortiori realizations of F over $\mathcal{Q}(R)$. Therefore there exists a matrix T over $\mathcal{Q}(R)$ such that $\bar{A} = TAT^{-1}$, $\bar{B} = TB$, $\bar{C} = CT^{-1}$, $\bar{D} = D$. Then $Q(\bar{\Sigma},n) = TQ(\Sigma,n)$ and thus $T = Q(\bar{\Sigma},n)Q(\Sigma,n)^{+} \in R^{n\times n}$.
When we are interested in a realization $\Sigma = (A,B,C,D)$ such that $\Sigma' = (A',C',B',D')$ is canonical, then we can also use the realization algorithm based on the factorization (3.2.13). In this case we factorize

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} [D,0] .$$

Hence $P := U\begin{bmatrix} I \\ 0 \end{bmatrix}$ is left invertible and $Q = [D,0]V$ is left regular and the construction of Σ' is straightforward.

The realizations based on the factorizations

$$\begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D \\ 0 \end{bmatrix}[I,0] \quad \text{and} \quad \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}[D,0]$$

respectively, are both minimal. These two realizations are not necessarily isomorphic for this would imply that they were split, which generally is not the case. The example on the last page of Chapter II may serve to show this.

(3.2.22) EXAMPLE. In [63] an example of an impulse response over $R = \mathbb{Z}$ is given

$$F_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 2 & -2 \\ 2 & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \quad F_3 = F_4 = \ldots = 0 .$$

We will compute a realization for this sequence. It is easily seen that rank H_{22} = rank $\mathcal{H} = 2$. We compute a modified Hermite form of H_{22}

$$H_{22} = \begin{bmatrix} 2 & -2 & 2 & 2 \\ 2 & 0 & 1 & 1 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} .$$

Hence, we obtain

$$Q = [Q_0,Q_1] = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & -1 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 0 \\ 2 & 1 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}$$

The matrix Q_2 is determined from the equation $PQ_2 = S := [F_3', F_4']' = 0$. Hence $Q_2 = 0$. Consequently we find the following canonical realization

$$A = [Q_1,Q_2]Q^+ = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = Q_0 = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad C = P_0 = \begin{bmatrix} 2 & 0 \\ 2 & 1 \end{bmatrix}, \quad D = 0 .$$

This realization is a split realization since P is left invertible. Indeed, in fact

$$\begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 2 \end{bmatrix} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

III.3. *Realization algorithms for an I/O system given by a transfer matrix*

In many practical situation the impulse response $F = (F_0, F_1, F_2, \ldots)$ is not directly available. However, often one is given the transfer matrix of an I/O system (see (2.4.11)). Let $F(z)$ be a $m \times p$-transfer matrix ($F(z)$ is a proper rational matrix), then $F(z)$ can be written as $F(z) = M(z)/m(z)$ where $M(z) \in R[z]^{m \times p}$ and $m(z)$ is a monic polynomial in $R[z]$. Indeed, $F(z) \in R_c(z)^{m \times p}$ means that all entries of $F(z)$ are proper rational functions with monic denominators. Then we can take $m(z)$ to be a common multiple of the denominators ($R[z]$ is a unique factorization domain). The formal power series expansion of $F(z)$ can be obtained by long division and then we could use the associated impulse response to construct a realization (existence is guaranteed because $F(z)$ is proper rational) by means of the algorithm described above.

(3.3.1) EXAMPLE. As has been pointed out in [55], [46], delay-differential systems can be modeled as systems over the ring $R = \mathbb{R}[d]$. For instance, if we introduce the delay operator d by $dy(t) = y(t-1)$ in the system of equations (see [46])

$$\begin{aligned} \ddot{y}_1(t) + \dot{y}_1(t-1) &= 2\dot{u}_1(t-2) - 6u_2(t) \\ \ddot{y}_2(t) + \dot{y}_2(t-1) &= -2\dot{u}_1(t-3) - 2\dot{u}_2(t) + 4u_2(t-1) \end{aligned} \tag{3.3.2}$$

we obtain $y = Wu$ where

$$W = \frac{1}{s^2 + ds} \begin{bmatrix} 2d^2 s & -6 \\ -2d^3 s & -2s+4d \end{bmatrix}$$

and s denotes the differentiation operator $sy = \dot{y}$. (We assume zero initial conditions.)

We want to obtain a representation of the equation (3.3.2) in the form

$$\begin{aligned} \dot{x}(t) &= A(d)x(t) + B(d)u(t) \\ y(t) &= C(d)x(t) \quad . \end{aligned} \tag{3.3.3}$$

To this end, we consider W a rational matrix over $\mathbb{R}[d]$ and we expand in powers of s^{-1}

$$(3.3.4) \qquad W = F_1(d)s^{-1} + F_2(d)s^{-2} + \dots .$$

Then the matrices A(d), B(d), C(d) in (3.3.3) have to satisfy $CA^kB = F_{k+1}$, $k = 0,1,2,\dots$, or equivalently $W = C[sI - A]^{-1}B$ which is more appropriate for this example; i.e., (A,B,C,0) has to be a realization of the impulse response $(0,F_1,F_2,\dots)$.

We now have (by long division for example)

$$F_1(d) = \begin{bmatrix} 2d^2 & 0 \\ -2d^3 & -2 \end{bmatrix}, \quad F_2(d) = \begin{bmatrix} -2d^3 & -6 \\ 2d^4 & 6d \end{bmatrix}, \quad F_3(d) = \begin{bmatrix} 2d^4 & 6d \\ -2d^5 & -6d^2 \end{bmatrix}.$$

It is seen that $[s^2 + ds]W = L_1 s + L_2$ and hence $F_{k+1} + dF_k = 0 \quad (k = 2,3,\dots)$. Consequently rank H = rank $H_{22} = 2$. We compute a modified Hermite form of H_{22}:

$$(3.3.5) \qquad H_{22} = \begin{bmatrix} 2d^2 & 0 & -2d^3 & -6 \\ -2d^3 & -2 & 2d^4 & 6d \\ -2d^3 & -6 & 2d^4 & 6d \\ 2d^4 & 6d & -6d^5 & -6d^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ d & -2 & 0 & 0 \\ d & -6 & 0 & 0 \\ -d^2 & 6d & 0 & 0 \end{bmatrix} \begin{bmatrix} -2d^2 & 0 & 2d^3 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

It follows that

$$Q = \begin{bmatrix} -2d^2 & 0 & 2d^3 & 6 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad Q^+ = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ \frac{1}{6} & 0 \end{bmatrix}, \quad P = \begin{bmatrix} -1 & 0 \\ d & -2 \\ d & -6 \\ -d^2 & 6d \end{bmatrix},$$

The matrix Q_2 is easily obtained from $PQ_2 = S := [F_3', F_4']'$ which yields

$$Q_2 = \begin{bmatrix} -2d^4 & -6d \\ 0 & 0 \end{bmatrix}.$$

Notice, that it is not necessary to know F_4 explicitly, since Q_2 is uniquely determined by the equation $P_0 Q_2 = F_3$.

Thus we find the following realization

$$A = \begin{bmatrix} -d & 6 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -2d^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ d & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

For the equation (3.3.3) we obtain

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t-1) + 6x_2(t) - 2u_1(t-2) \\ \dot{x}_2(t) &= u_2(t) \\ y_1(t) &= -x_1(t) \\ y_2(t) &= x_1(t-1) - 2x_2(t) . \end{aligned} \tag{3.3.6}$$

Notice that P is actually left invertible, because its diagonal elements are invertible. It follows that we have a split realization (see (3.2.21)).

The above way of realizing a transfer matrix may be a long detour and we will therefore construct a canonical realization directly from the transfer matrix. This will be done by first constructing a minimal realization over $\mathcal{Q}(R)$ and afterwards reducing this realization to a canonical realization over R by means of a state space isomorphism $T \in \mathcal{Q}(R)^{n\times n}$ where n is the dimension of the minimal realization (the *McMillan degree* of F(z) as a transfer matrix over $\mathcal{Q}(R)$, see [41], [33]).
Suppose $\bar{F}(z)$ is a transfer matrix over R. Then $\bar{F}(z)$ is a fortiori a transfer matrix over $\mathcal{Q}(R)$. Therefore $\bar{F}(z)$ has a minimal $\mathcal{Q}(R)$ realization $\Sigma_q = (F,G,H,K)$, i.e.,

$$\bar{F}(z) = H[zI - F]^{-1} G + K . \tag{3.3.7}$$

Suppose that n is the rank of Σ. Then we know that rank $\mathcal{H}$ = rank H_{nn} where H is the associated Hankel matrix and H_{nn} is the Hankel block

$$H_{nn} = \begin{bmatrix} HG & HFG & HF^2G & \dots & HF^{n-1}G \\ HFG & HF^2G & & & \cdot \\ HF^2G & & & & \cdot \\ \vdots & & & & \vdots \\ HF^{n-1}G & \cdot & \cdot & \dots & HF^{2n-2}G \end{bmatrix} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} [G, FG, \dots, F^{n-1}G] . \tag{3.3.8}$$

H_{nn} is an R-matrix because $\bar{F}(z)$ is a transfer matrix over R. By minimality $Q(\Sigma_q,n)$ has full rank. Therefore there exists a unimodular R-matrix U such that

$$(3.3.9) \quad [G,FG,\ldots,F^{n-1}G]U = [T,0]$$

where $T \in \mathcal{Q}(R)^{n\times n}$ and T is regular.
This can be achieved in the following way. Let ℓ be a common multiple of the denominators of the entries of $Q(\Sigma_q,n)$. This element $\ell \in R$ is well defined because R is a principal ideal domain and therefore a unique factorization domain. Now the matrix $\bar{Q}(\Sigma_q,n) := \ell Q(\Sigma_q,n)$ is a matrix over R. For $\bar{Q}(\Sigma_q,n)$ we can construct a modified Hermite form

$$(3.3.10) \quad \Pi\bar{Q}(\Sigma_q,n)\bar{U} = [\bar{T},0]$$

where we can even take $\Pi = I$.
Hence in (3.3.9) we can take $T = \bar{T}/\ell$ and $U = \bar{U}$.
That T is regular follows from the minimality of Σ_q. Now we have

$$(3.3.11) \quad T^{-1}[G,FG,\ldots,F^{n-1}G]U = [I,0] \ .$$

We will now prove that $\Sigma = (A,B,C,D) = (T^{-1}FT,T^{-1}G,HT,K)$ is a canonical realization over R. Observe that

$$(3.3.12) \quad T^{-1}[G,FG,\ldots,F^{n-1}G] = [I,0]U^{-1} \ ,$$

therefore $T^{-1}G$ is an R-matrix, furthermore $T^{-1}[G,FG,\ldots,F^{n-1}G]$ is a right invertible R-matrix. Hence

$$\begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix} T = H_{nn}\left[T^{-1}[G,FG,\ldots,F^{n-1}G]\right]^{+}$$

Therefore $[H',F'H',\ldots,(F^{n-1})'H']'\,T$ is a right regular R-matrix. Hence we have obtained a factorization of H_{nn} in such a way that theorem (3.2.4) can be applied. We will also show how the (unique once the factorization is given) ring realization is related to the quotient field realization. HT is an R-matrix, $T^{-1}FT$ is also an R-matrix, for

$$T^{-1}[G,FG,\dots,F^{n-1}G,F^nG]\begin{bmatrix} U & W \\ 0 & I \end{bmatrix} = [I,0]$$

where $W = [-\alpha_0 I,\dots,-\alpha_{n-1}I]'$ and α_i ($\in R$), $i = 0,\dots,n-1$, are the coefficients of the characteristic polynomial of F. Therefore $T^{-1}[FG,\dots,F^nG]$ is an R-matrix and we have

(3.3.13) $\quad T^{-1}FT = T^{-1}[FG,\dots,F^nG][G,FG,\dots,F^{n-1}G]^+ T$.

Hence Σ is a canonical R-realization of $\bar{F}(z)$.

REMARK. The construction of a ring realization from a quotient field realization as described in (3.3.11) is based on the same technique as the realization algorithm which Rouchaleau describes in [63]. □

We can now construct a canonical R-realization if we have a realization algorithm to construct a minimal $\mathcal{Q}(R)$ realization starting from the transfer matrix.
In [41] Kalman gave a realization algorithm for real transfer matrices. This algorithm can be generalized to the case of transfer matrices over an arbitrary field. The algorithm is based on the McMillan form of a proper rational matrix (see [41]) and gives a minimal $\mathcal{Q}(R)$-realization.

In [33] Heymann gave a realization algorithm for transfer matrices over an arbitrary field. The algorithm is based on a diagonal rational matrix, equivalent to the transfer matrix in a certain sense and this algorithm also gives a minimal $\mathcal{Q}(R)$-realization.
Because this algorithm is easier to apply than the algorithm due to Kalman we will state this algorithm in the form of a theorem. Before we state the appropriate realization theorem of Heymann we introduce the following definition.

(3.3.14) DEFINITION. *A rational function* $p(z)/q(z) \in R(z)$ *is called strictly proper if* $q(z)$ *is monic and* $\deg(p(z)) < \deg(q(z))$. *A rational matrix is called strictly proper if every entry is strictly proper.* □

The set of strictly proper rational functions will be denoted by $R_{sc}(z)$.
Suppose that $\bar{F}(z)$ is a strictly proper rational $m \times p$-matrix ($\bar{F}(z) \in R_{sc}(z)^{m\times p}$), then $\bar{F}(z)$ can be written as $\bar{F}(z) = M(z)/m(z)$ where $m(z)$ is a monic polynomial and $\deg M(z) < \deg m(z)$ where the degree of a matrix

is defined as the maximum of the degrees of the entries. Again the polynomial $m(z)$ is a common multiple of the denominators of the entries of $\bar{F}(z)$.

Every matrix $F(z) \in R_c(z)^{m\times p}$ can be written as $F(z) = \bar{F}(z) + F_0$ where $\bar{F}(z) \in R_{sc}(z)^{m\times p}$ and $F_0 \in R^{m\times p}$. A realization $\Sigma = (A,B,C,D)$ of a strictly proper rational matrix necessarily has the matrix D equal to zero. (The realization exists because $R_{sc}(z) \subset R_c(z)$.)

Now, let $\bar{F}(z) = M(z)/m(z)$ be a strictly proper rational $m \times p$-matrix. Then $\bar{F}(z)$ is a fortiori a strictly proper rational matrix with coefficients in $\mathcal{Q}(R)$, i.e., $\bar{F}(z) \in \mathcal{Q}(R)_{sc}(z)^{m\times p}$. Furthermore, we will suppose $M(z)$ and $m(z)$ to be relatively prime which means that there is no non-trivial common factor of $m(z)$ and all entries of $M(z)$.

The matrix $M(z)$ can be considered a matrix over the principal ideal domain $\mathcal{Q}(R)[z]$ and is therefore equivalent (see [31]) to a diagonal matrix over $\mathcal{Q}(R)[z]$, for instance one might take the modified Smith form.

We can now state the "realization theorem" of [33].

(3.3.15) THEOREM. *Let* $F(z) = F_0 + \bar{F}(z) = F_0 + M(z)/m(z)$ *be a transfer matrix with coefficients in* R, *where* $\bar{F}(z)$ *is a strictly proper rational matrix over* $R_{sc}(z)$ *and therefore over* $\mathcal{Q}(R)_{sc}(z)$. *Let* $U(z)$ *and* $V(z)$ *be unimodular matrices over* $\mathcal{Q}(R)[z]$ *and* $D(z) = \mathrm{diag}[\varepsilon_1/\psi_1,\ldots,\varepsilon_r/\psi_r,0,\ldots,0]$ *be a strictly proper diagonal matrix where, for each* i, ε_i *and* ψ_i *are relatively prime polynomials and* ψ_i *is monic. Suppose that*

$$M(z) = U(z)(m(z)D(z))V(z) \bmod (m(z))$$

where $m(z)$ *is the least common multiple of the denominators of the entries of* $\bar{F}(z)$. *Construct a system* $\Sigma = (F,G,H,K)$ *as follows*

1° *For each* $i = 1,\ldots,r$ *let* $(F_i,G_i,H_i,0)$ *be the system given by*

$$F_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdot & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \cdot & 0 \\ 0 & & \cdots & 0 & 1 \\ -a_{i,0} & & \cdots & & -a_{i,n_i-1} \end{bmatrix}, \quad G_i = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$H_i = \left[b_{i,0},\ldots,b_{i,m_i},0,\ldots,0\right],$$

where the a_{ij} *and the* b_{ij} *are the coefficients of* ψ_i *and* ε_i *respectively.*

Here $\psi_i(z) = \sum_{j=0}^{n_i} a_{ij} z^j$ *where* $a_{i,n_i} = 1$, *and* $\varepsilon_i(z) = \sum_{j=0}^{m_i} b_{ij} z^j$. *(These systems will be minimal realizations of the transfer functions* $\varepsilon_i(z)/\psi_i(z)$.*)*

2° *Define* $(\tilde{F},\tilde{G},\tilde{H},\tilde{K})$ *to be the system* $\tilde{F} = \mathrm{diag}[F_1,\dots,F_r]$, $\tilde{G} = \mathrm{diag}[G_1,\dots,G_r]$, $\tilde{H} = \mathrm{diag}[H_1,\dots,H_r]$, $\tilde{K} = 0$, *and then augmenting, if necessary, by adding columns of zeroes to* $\tilde{G}$ *and/or rows of zeroes to* $\tilde{H}$ *in order to make* $\tilde{H}[zI - \tilde{F}]^{-1}\tilde{G}$ *of the same size as* $\bar{F}(z)$. *This system will be a minimal realization of* D(z)

3° *Let* $U(z) = \tilde{U}(z) \bmod (m(z))$, $V(z) = \tilde{V}(z) \bmod (m(z))$ *and let* $\tilde{U}(z) = \sum_{i=0}^{\mu-1} \tilde{U}_i z^i$, $\tilde{V}(z) = \sum_{i=0}^{\mu-1} \tilde{V}_i z^i$ *where* $\mu = \deg(m(z))$. *Furthermore, let* $F = \tilde{F}$, $G = \sum_{j=0}^{\mu-1} \tilde{F}^j \tilde{G} \tilde{V}_j$, $H = \sum_{i=}^{\mu-1} \tilde{U}_i \tilde{H} \tilde{F}^i$, $K = F_0$.

Then (F,G,H,K) *is a minimal realization of* F(z) *over* $\mathcal{Q}(R)$.

PROOF. For a proof see [33]. □

(3.3.16) REMARK. In theorem (3.3.15) one may also take as a minimal realization $(\bar{F},\bar{G},\bar{H},\bar{K})$ where

$$\bar{F} = \tilde{F}\ ,\quad \bar{G} = \sum_j \tilde{F}^j \tilde{G} V_j\ ,\quad \bar{H} = \sum_i U_i \tilde{H}\tilde{F}^i\ ,\quad \bar{K} = F_0\ ,$$

where $U(z) = \sum_i U_i z^i$, $V(z) = \sum_j V_j z^j$ (see [35]). □

Observe that (F,G,H,K) and $(\bar{F},\bar{G},\bar{H},\bar{K})$ are isomorphic because they both are minimal realizations of the same transfer matrix F(z).
Concerning the degree in z of the matrices U(z) and V(z) in theorem (3.3.15) almost nothing can be said except in the case where we are dealing with two equivalent matrices which both are regular (over the quotient field) and have degree at most one. In this case the unimodular transformations can be chosen z-independent (see [27]). This is the case in the following example (see (3.3.18)).
In the next example we will construct a ring realization for the transfer matrix given in example (3.3.1) by using theorem (3.3.15) and the method described in (3.3.9) through (3.3.13). Again z is replaced by s because this transfer matrix stems from a continuous time system.

(3.3.17) **EXAMPLE.** Let $\bar{F}(s) = M(s)/m(s)$ be a strictly proper rational matrix given by

$$M(s) = \begin{bmatrix} 2d^2s & -6 \\ -2d^3s & -2s+4d \end{bmatrix}, \quad m(s) = s^2 + ds$$

where $R = \mathbb{R}[d]$ (see also (3.3.1)). Now we have

$$(3.3.18) \quad \begin{bmatrix} -2 & -3/d \\ d & 1 \end{bmatrix} \begin{bmatrix} 2d^2s & -6 \\ -2d^3s & -2s+4d \end{bmatrix} \begin{bmatrix} 1 & -3/d^3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2d^2s & 0 \\ 0 & -2s-2d \end{bmatrix}.$$

Using the notation as in theorem (3.3.15) we have $\varepsilon_1(s)/\psi_1(s) = 2d^2/(s+d)$, $\varepsilon_2(s)/\psi_2(s) = -2/s$. A minimal realization over $\mathcal{Q}(R)$ given by theorem (3.3.15) is $\Sigma = (F,G,H,K)$ where

$$F = \begin{bmatrix} -d & 0 \\ 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 3/d^3 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 2d^2 & -6/d \\ -2d^3 & 4 \end{bmatrix}, \quad K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now we apply the method described in (3.3.9) - (3.3.13). In this case we have (see (3.3.9))

$$T = \begin{bmatrix} 3/d^3 & 0 \\ 1 & d \end{bmatrix}.$$

The canonical $\mathbb{R}[d]$- realization is

$$(3.3.19) \quad A = T^{-1}FT = \begin{bmatrix} -d & 0 \\ 1 & 0 \end{bmatrix}, \quad B = T^{-1}G = \begin{bmatrix} d^3/3 & 1 \\ -d^2/3 & 0 \end{bmatrix}$$

$$C = HT = \begin{bmatrix} 0 & -6 \\ -2 & 4d \end{bmatrix}, \quad D = K = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

(3.3.20) **REMARK.** The realizations in (3.3.17) and (3.3.1) are not the same. However, they are both canonical and therefore they are isomorphic by the realization isomorphism theorem (2.5.16). The state space isomorphism is:

$$T = \begin{bmatrix} 0 & 6 \\ 1 & d \end{bmatrix}.$$

□

III.4. *Partial realizations*

In many situations, the total impulse response is not always immediately available. For this reason it is useful to have so called *partial realization* algorithms, where finite impulse response sequences are processed and where the computational results are updated as soon as new data are available.

(3.4.1) DEFINITION. *A system* $\Sigma = (A,B,C,D)$ *is an* N*-partial realization of a finite sequence* $F_N = (F_0,F_1,\ldots,F_N)$ *if* $D = F_0$ *and* $F_i = CA^{i-1}B$, $i = 1,\ldots,N$.□

For systems over a field such partial realization algorithms are known (see [45], [43], [60]) and a nice result on partial realizations is:

(3.4.2) THEOREM. *Let* $F_N = (F_0,F_1,\ldots,F_N)$ *be a finite sequence of* $m \times p$-*matrices over a field. Let* ℓ *and* k *be positive integers such that* $\ell + k = N$. *Then* F_N *has one and only one extension to an infinite impulse response sequence* F *such that*

$$\text{rank } H = \text{rank } H_{\ell k}$$

iff the following two conditions are satisfied:

(3.4.3) $\quad \text{rank } H_{\ell k} = \text{rank } H_{\ell,k+1}$

(3.4.4) $\quad \text{rank } H_{\ell k} = \text{rank } H_{\ell+1,k}$.

PROOF. For a proof see [43]. □

Observe that a realization can then be found by applying theorem (3.2.4) to $H_{\ell,k+1}$.

For systems over rings the problem of finding canonical (minimal) partial realizations is still unsolved even for the scalar case.
To some extent, the following theorem gives a result on partial realization over a principal ideal domain.

(3.4.5) THEOREM. *Let* $F_N = (F_0,F_1,\ldots,F_N)$ *be a finite sequence of* $m \times p$-*matrices over a principal ideal domain* R. *Let* k *and* ℓ *be positive integers such that* $\ell + k = N$. *Suppose that we have the factorization*

(3.4.6) $\quad H_{\ell,k+1} = P[Q,Q_k]$

where P *is right regular and* Q *is right invertible with right inverse* Q^+. *If*

$$\text{(3.4.7)} \qquad \operatorname{rank} H_{\ell+1,k} = \operatorname{rank} H_{\ell k} =: n$$

and $k \geq n$, *then there exists a unique partial realization* $\Sigma = (A,B,C,D)$ *satisfying* $[Q,Q_k] = Q(\Sigma,k+1)$, $P = P(\Sigma,\ell)$, *viz.*

$$A = [Q_1,\dots,Q_k]Q^+ \;, \quad B = Q_0 \;, \quad C = P_0 \;, \quad D = F_0$$

where $P_0 \in R^{m\times n}$ *consists of the first* m *rows of* P *and* $Q_i \in R^{n\times p}$ *is defined by the block decomposition* $Q = [Q_0,Q_1,\dots,Q_{k-1}]$.

PROOF. Defining $S \in R^{\ell m\times p}$ by the decomposition $H_{\ell,k+1} = [H_{\ell k},S]$ we conclude from (3.2.10) that $S = H_{\ell k}W$ where $W := Q^+Q_k$. If we decompose W by $W = [W_1',\dots,W_k']'$ where $W_i \in R^{p\times p}$, then the sequence F_N satisfies the following recurrence relation

$$\text{(3.4.8)} \qquad F_{k+j} = F_j W_1 + F_{j+1} W_2 + \dots + F_{k+j-1} W_k$$

for $j = 1,\dots,\ell$. Now let us define F_i for $i > N$ by this recurrence relation. Then the result will follow from theorem (3.2.6) if we know that rank H = = rank $H_{\ell k}$ =: n. According to [68] it suffices to show that

$$\text{(3.4.9)} \qquad \operatorname{rank} H_{\ell+1,k+j} = n$$

for $j = 1,2,\dots$. For $j = 0$ this equality follows from (3.4.7). For $j \geq 0$ we have by (3.4.8)

$$H_{\ell+1,k+j+1} = [H_{\ell+1,k+j}\,, H_{\ell+1,k+j}\tilde{W}_j]$$

where $\tilde{W}_j := [0,0,\dots,0,W_1',\dots,W_k']' \in R^{(k+j)p\times p}$.
This equation implies (3.4.9). □

Let us suppose that we are given an infinite sequence $F = (F_0,F_1,F_2,\dots)$ and we want to compute a partial realization of $(F_0,F_1,\dots,F_N)$ where N is a given positive integer. The algorithm is based on recursive construction of a modified Hermite form, $\Pi_{\ell k}$, $V_{\ell k}$, $T_{\ell k}$, $G_{\ell k}$ of $H_{\ell k}$, that is,

$$\Pi_{\ell k} H_{\ell k} V_{\ell k} = T_{\ell k} = [G_{\ell k},0]$$

whose rank $G_{\ell k} = n$. We start constructing a modified Hermite form of

$H_{11} = F_1$ ($\ell = 1$, $k = 1$) (see (3.1.4)). Thus we obtain matrices Π_{11}, V_{11}, T_{11}, G_{11} such that

$$\Pi_{11} H_{11} V_{11} = T_{11} = [G_{11}, 0]$$

and G_{11} is right regular and lower triangular. If $F_1 = 0$, then G_{11} is the empty matrix.
We proceed recursively as in case α or case β depending upon the following properties (for general ℓ, k)

$$\mathcal{P}: \; n \leq k \, , \; n+p \leq km \, , \; V_{\ell k} = \begin{bmatrix} U_{\ell k} & W_{\ell k} \\ 0 & I_p \end{bmatrix}$$

for suitable matrices $U_{\ell k}$, $W_{\ell k}$.

Case α. Property $\mathcal{P}$ is satisfied: we add a block row to $H_{\ell k}$ and write

$$\begin{bmatrix} \Pi_{\ell k} & 0 \\ 0 & I_m \end{bmatrix} H_{\ell+1,k} V_{\ell k} = \begin{bmatrix} G_{\ell k} & 0 \\ S_1 & S_2 \end{bmatrix} ,$$

then, if $S_2 = 0$ we obtain a partial realization of $(F_1, \ldots, F_{k+\ell})$ as follows: define

$$P := \Pi_{\ell k}^{-1} G_{\ell k} \, , \quad Q := [I_n, 0] U_{\ell k}^{-1} \, , \quad Q_{k-1} := Q W_{\ell k} \, ,$$

then we write $H_{\ell k} = [H_{\ell,k-1}, S]$ and we have

$$[H_{\ell,k-1}, S] \begin{bmatrix} U_{\ell k} & W_{\ell k} \\ 0 & I_p \end{bmatrix} = P[I_n, 0, 0]$$

where $[I_n, 0, 0] \in R^{n \times (n+(km-n-p)+p)}$. It follows that

$$H_{\ell,k-1} \, U_{\ell k} = P[I_n, 0]$$

and hence $H_{\ell,k-1} = PQ$ and

$$H_{\ell,k-1} \, W_{\ell k} + S = 0$$

and hence $S = PQ_{k-1}$. Consequently, we have the relation (3.4.6) with k replaced by k - 1. Also, it is clear that P is right regular and Q is right invertible.

By P we have $k \geq n$ and (3.4.7) follows from the equation $S_2 = 0$. Thus we may apply theorem (3.4.5).
If $\ell+k \geq N$, the algorithm has terminated. If not, we notice that property P is still satisfied with ℓ replaced by $\ell+1$ and we proceed with case α. If $S_2 \neq 0$ we determine a modified Hermite form of S_2 and therewith a modified Hermite form of $H_{\ell+1,k}$. Then we check again whether P is satisfied (with ℓ replaced by $\ell+1$).

Case β. Property P is not satisfied. We add a block column to $H_{\ell k}$ and write

$$\Pi_{\ell k}\ H_{\ell,k+1}\begin{bmatrix} V_{\ell k} & 0 \\ 0 & I_p \end{bmatrix} = [G_{\ell k},0,S] \ .$$

We try to find a matrix W such that

$$\Pi_{\ell k}\ H_{\ell,k+1}\begin{bmatrix} V_{\ell k} & W \\ 0 & I_p \end{bmatrix} = [G_{\ell k},0,0] \ .$$

The existence of such a W can be investigated by performing elementary column operations on the matrix $[G_{\ell k},0,S]$. Due to the special form of $G_{\ell k}$, this investigation is very simple and explicit conditions for the existence of W can be given:

1° The i-th row of S is divisible by $(G_{\ell k})_{ii}$.

2° If the appropriate multiple of the i-th column is substracted from the columns of S (so as to make the i-th row zero) for $i = 1,\dots,n$, then the resulting columns have to be zero.

If we are able to construct W, then we check whether $k \geq n$. If so, we are in case α. If not, or if W does not exist, we are again in case β. In the latter case, we of course have to update the modified Hermite form.
We will now show that the procedure terminates provided H has finite rank. First we note that for a fixed value of ℓ, we cannot have infinitely often that case β holds. For k increases at every step and we must have $k \geq n$ after a number of steps, because $n \leq \ell m$. Also, condition 1° of case β cannot be violated infinitely often, since at every step the ideal in R generated by $(G_{\ell k})_{ii}$ will strictly increase unless condition 1° is satisfied. Furthermore, condition 2° will certainly be satisfied if $n = \text{rank } H$ and every time 2° is not satisfied, n will increase. Similarly in case α, $S_2 = 0$ will hold if $n = \text{rank } H$ and otherwise n will increase. This shows

the finiteness of the algorithm. The algorithm given here is not a true algorithm for partial realization, since one needs an infinite impulse response in order to complete the algorithm. Of course, one can always extend a finite sequence such that the resulting sequence has a Hankel matrix of finite rank. However, it is not at all obvious how to extend a finite sequence such that the corresponding Hankel matrix has minimal rank (cf. [63]).

If we apply this algorithm to a scalar impulse response, the algorithm gives a canonical realization where the recurrence parameters of a minimal recurrence are in the matrix (now a vector) $W_{\ell k}$. But in this case we do not have to carry out the complete algorithm, for in this case the realization given in theorem (2.4.9) is a canonical realization. Observe that we only need the recurrence parameters in the case of a scalar system. Therefore we can also apply the algorithm due to Rissanen [60] and perform calculations over $\mathcal{Q}(R)$. This will give the required recurrence parameters and they are elements of R (see [64]).

A large part of this chapter can also be found in [21].

IV 2-D SYSTEMS

IV.1. *Introduction*

As already mentioned in Chapter II the time set of an input/output system may also be a partially ordered set. In this chapter we will be concerned with input/output systems where the time set is $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ with a partial order $\preceq$ defined by (using row vector notation)

(4.1.1) $(k,h) \preceq (m,n)$ iff $k \le m$ and $h \le n$.

In many cases when we are dealing with input/output systems in which the inputs and outputs depend on two "time" parameters (which may be actually space parameters), the dependence of the outputs on the inputs can be characterized by equations of the form

(4.1.2) $$y_{kh} = \sum_{i,j} F_{k,h,i,j}\, u_{ij} \qquad (k,h) \in \mathbb{Z}^2,\ (i,j) \in \mathbb{Z}^2$$

where $u_{ij} \in \mathbb{R}^p$, $y_{kh} \in \mathbb{R}^m$ and $F_{k,h,i,j} \in \mathbb{R}^{m\times p}$ for some integers m and p. Again (as in Chapter II) we will impose some finiteness conditions on the index set that insure that (4.1.2) denotes a finite sum for all k, h. The partial order defined in (4.1.1) enables us to define causality for a system described by (4.1.2). In Section IV.4 we will allow a more general partial order on $\mathbb{Z}^2$. In fact, it can be shown that every partial order on $\mathbb{Z}^2$ gives rise to some causality notion. For the input/output system (4.1.2) causality is defined in an analogous way as in (2.1.2).
The input/output system (4.1.2) is called causal if the output at (k,h) is only dependent on past inputs (u_{ij} with $i \le k$ and $j \le h$). This means that

(4.1.3) $$F_{k,h,i,j} = 0\ , \quad i < k \text{ or } j < h\ .$$

In many cases the input/output system does not explicitly depend on (k,h). In other words: If a double sequence $(u_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is related to a double sequence $(y_{k,h})_{(k,h)\in\mathbb{Z}^2}$ then the shifted input sequence $(u_{i+m,j+n})_{(i,j)\in\mathbb{Z}^2}$ gives rise to the output $(y_{k+m,h+n})_{(k,h)\in\mathbb{Z}^2}$ for all $(m,n) \in \mathbb{Z}^2$. In this case $F_{k,h,i,j}$ only depends on the difference (k-i,h-j).

This assumption of shift invariance will be made and we replace (4.1.2) by

(4.1.4) $$y_{kh} = \sum_{i,j} F_{k-i,h-j}\, u_{ij}\ , \quad (k,h) \in \mathbb{Z}^2\ .$$

Now the causality condition reduces to

(4.1.5) $$F_{nm} = 0\ , \quad n < 0 \text{ or } m < 0\ .$$

We will make again (as in (2.1.4)) a *finite past* assumption on $(u_{i,j})_{(i,j)\in\mathbb{Z}^2}$ i.e.,

$$u_{ij} = 0 \quad \text{if } i < 0 \text{ or } j < 0\ .$$

Then the causality condition implies that $y_{kh} = 0$ for $k < 0$ or $h < 0$. It follows that, assuming causality and time invariance, we may write the input/output system as

(4.1.6) $$y_{kh} = \sum_{i=0,j=0}^{k,h} F_{k-i,h-j}\, u_{ij}\ , \quad k = 0,1,2,\dots\ , \ h = 0,1,2,\dots\ .$$

This is the standard equation (see [16]) for a 2-D causal, discrete time shift invariant, linear input/output system.
As in II.2 we call the double sequence $(F_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$ the *impulse response* of the input/output system (4.1.6)
As in the 1-D case (that is, the case of one time parameter) the 2-D input/output system (4.1.6) can also be described via formal power series but now in two variables s^{-1} and z^{-1}.

(4.1.7) DEFINITION. *The formal power series* $r(s,z)$ *in the variables* s^{-1} *and* z^{-1}, *associated with the double sequence* $(r_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$ *is*

$$r(s,z) = \sum_{m=0,n=0}^{\infty,\infty} r_{mn}\, s^{-n} z^{-m}\ . \qquad \square$$

A formal power series $r(s,z)$ is also called the formal 2-D Z-*transform* of the double sequence $(r_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$. Again the word "formal" is used because one does not actually want to calculate the sum, but one uses it as just another notation for $(r_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$. As in the 1-D case s^{-1} and z^{-1} are only position markers.
The set of formal power series in two variables with real coefficients is denoted by $\mathbb{R}[[s^{-1},z^{-1}]]$. The set $\mathbb{R}[[s^{-1},z^{-1}]]$ is a ring with the usual

definitions for the sum and the product of two elements.
Observe that $\mathbb{R}[[s^{-1},z^{-1}]]$ can also be written as $\mathbb{R}[[s^{-1}]][[z^{-1}]]$, that is the ring of formal power series in z^{-1} where the coefficients are formal power series in s^{-1}.
Analogously to the scalar case we also have a formal power series associated with a vector sequence $(u_{i,j})_{(i,j)\in\mathbb{Z}_+^2}$, $(y_{k,h})_{(k,h)\in\mathbb{Z}_+^2}$ and also for a matrix sequence $(F_{m,n})_{(m,n)\in\mathbb{Z}_+^2}$. Thus we have

$$u(s,z) = \sum_{i=0,j=0}^{\infty,\infty} u_{ij}\, s^{-j}\, z^{-i}$$

$$(4.1.8) \qquad y(s,z) = \sum_{k=0,h=0}^{\infty,\infty} y_{kh}\, s^{-h}\, z^{-k}$$

$$F(s,z) = \sum_{m=0,n=0}^{\infty,\infty} F_{mn}\, s^{-n}\, z^{-m}$$

where

$u(s,z) \in \mathbb{R}[[s^{-1},z^{-1}]]^p$, the set of p-vectors over $\mathbb{R}[[s^{-1},z^{-1}]]$,

$y(s,z) \in \mathbb{R}[[s^{-1},z^{-1}]]^m$, the set of m-vectors over $\mathbb{R}[[s^{-1},z^{-1}]]$,

$F(s,z) \in \mathbb{R}[[s^{-1},z^{-1}]]^{m\times p}$, the set of $m \times p$-matrices over $\mathbb{R}[[s^{-1},z^{-1}]]$.

Using formal power series, the 2-D input/output system (4.1.6) can equivalently be described by

$$(4.1.9) \qquad y(s,z) = F(s,z)u(s,z) \ .$$

In this chapter our main concern will be the construction of a state space realization for a 2-D input/output system. We will give an intuitive reasoning why a state space will generally be infinite dimensional. Suppose we have a state space X such that y_{kh} depends only on x_{kh} and u_{kh} for all (k,h) and $x_{kh} \in X$. Suppose further that x_{kh} depends only on $u_{k',h'}$ for some $(k',h') \preceq (k,h)$ and on former states (for $\preceq$ see (4.1.1)).
Now consider figure 2:

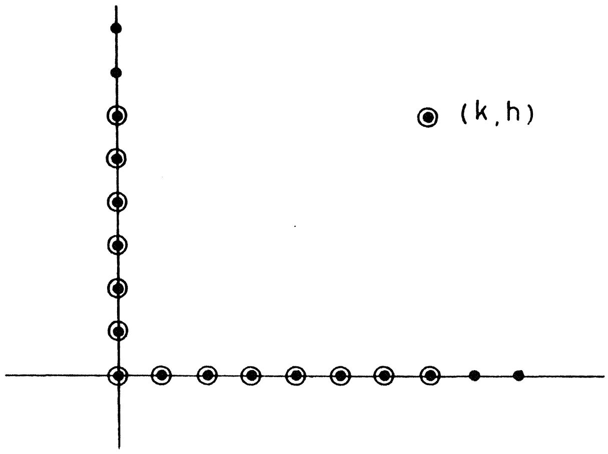

Figure 2

Because the output at (k,h) depends on u_{mn} for $m \leq k$, $n \leq h$ (see (1.4.6)) the state x_{kh} will depend on former states and at least will depend on $x_{0,0}, x_{0,1}, x_{1,0}, x_{1,1}, \ldots, x_{k-1,h}, x_{k,h-1}$. Because (k,h) is arbitrary it will be intuitively clear that the state space generally is infinite dimensional because for each (k,h) the state x_{kh} at least contains the information concerning the initial conditions $x_{0,0}, x_{0,1}, \ldots, x_{0,h}$ and $x_{1,0}, x_{2,0}, \ldots, x_{k,0}$.

REMARK. The fact that the state space is generally infinite dimensional follows also from considerations using Nerode equivalence classes. See [24], [48]. □

Because we are interested in recursive state space models and, again intuitively speaking, recursiveness of state space models is closely connected with rationality of transfer functions, henceforth we will mainly be concerned with the case where $F(s,z)$ in (4.1.9) is a rational function in s and z. Therefore we introduce some notation. The field of real rational functions in the variables s and z will be denoted by $\mathbb{R}(s,z)$. The ring of real polynomials in s and z will be denoted by $\mathbb{R}[s,z]$.

REMARK. $\mathbb{R}[s,z]$ can also be written as $\mathbb{R}[s][z]$, i.e., every polynomial in two variables s and z can also be written as a polynomial in z where the coefficients are polynomials in s (also $\mathbb{R}[s,z] = \mathbb{R}[z][s]$).

$\mathbb{R}(s,z)$ can be written as $\mathbb{R}[s](z)$, i.e., a rational function in two variables can be written as a rational function in one of the variables with, as coefficients, polynomials in the other variable.
Because $\mathbb{R}(s,z) = \mathbb{R}[s](z)$ we can divide all coefficients of an element of $\mathbb{R}(s,z)$ by the coefficients of the leading term of the denominator of this element. Thereby $\mathbb{R}(s,z)$ can be seen as $\mathbb{R}(s)(z)$ where the denominator polynomial is monic. □

As in the 1-D case (see [45]) we will work with rational functions which can be expanded in a formal power series in the variables s^{-1} and z^{-1}. In the 1-D case the necessary and sufficient condition for this to be possible is that the rational function is proper. This will now be generalized.
Using the identifications in the above remark we will use the following notation. A polynomial $q \in \mathbb{R}[s,z]$, viewed as an element of $\mathbb{R}[s][z]$, will be written as $\bar{q}$. An analogous notation will be used for P and $\bar{P}$ where $P \in \mathbb{R}[s,z]^{m\times p}$ and $\bar{P} \in \mathbb{R}[s][z]^{m\times p}$, where $\mathbb{R}[s,z]^{m\times p}$ and $\mathbb{R}[s][z]^{m\times p}$ are the sets of $m \times p$-matrices over $\mathbb{R}[s,z]$ and $\mathbb{R}[s][z]$, respectively.
Let $F(s,z)$ be an $m \times p$-matrix over $\mathbb{R}(s,z)$, then $F(s,z)$ can be written as $F(s,z) = P/q = \bar{P}/\bar{q}$ where P, $\bar{P}$, q, $\bar{q}$ are as above.

(4.1.10) DEFINITION. *A rational matrix* $F \in \mathbb{R}(s,z)^{m\times p}$ *is called proper if for some representation* $F = \bar{P}/\bar{q}$

1° *The degree in* z *of* $\bar{q}(z)$ *is not less than the degree in* z *of* $\bar{P}(z)$.
2° *The degree in* s *of the coefficient (the so called leading coefficient) of the highest power in* z *of* $\bar{q}(z)$ *is not less than the degree of each other coefficient of* $\bar{q}(z)$ *and the entries of* $\bar{P}(z)$.

F *is called strictly proper if "not less" is replaced by "larger" in* 1° *and* 2°. □

It can easily be seen that a representation $F = \tilde{P}/\tilde{q}$ for a proper F, where $\tilde{P}$ and $\tilde{q}$ are coprime, satisfies 1° and 2°.
Let $q(s,z) \in \mathbb{R}[s,z]$ and suppose the degree in s of $q(s,z)$ is m and the degree in z of $q(s,z)$ is n. Then, for q to be the denominator of a proper $F = P/q$, it is necessary and sufficient that, besides $\deg_s(P) \le m$ and $\deg_z(P) \le n$, the coefficient of the monomial $s^m z^n$ be non-zero. In this case this coefficient can be assumed to be one. This could be used as the definition of properness and in fact would constitute a symmetric (in s and z) definition. However, for our purposes definition (4.1.10) is more appropriate.

EXAMPLES. $\frac{1}{s+z}$ and $\frac{sz+s^2+1}{z^2s+s}$ are not proper.

$\frac{z+3s}{z^2s+zs+1}$ is proper.

(4.1.11) DEFINITION. *A proper rational matrix in two variables* s *and* z *is called a 2-D transfer matrix.* □

A transfer matrix in irreducible form and a representation where common factors occur in the numerators and denominators of the entries will not be distinguished, unless this is mentioned explicitly.

The set of 2-D transfer matrices $(m \times p)$ will be denoted by $\mathbb{R}_c(s,z)^{m\times p}$. If $F(s,z) \in \mathbb{R}_c(s,z)^{m\times p}$, then F can be written as $\bar{P}/\bar{q}$, where $\bar{P} \in \mathbb{R}[s][z]^{m\times p}$ and $\bar{q}(s,z) = q_0(s) + q_1(s)z + \ldots + q_n(s)z^n \in \mathbb{R}[s][z]$.
We divide all coefficients of $\bar{q}$ and $\bar{P}$ by $q_n(s)$ (the leading coefficient of $\bar{q}(s,z)$). By the properness of F all coefficients become proper rational functions in s and $\bar{q}$ becomes a monic polynomial.
Now F(s,z) can be seen as an element of $R_c(z)^{m\times p}$ where $R = \mathbb{R}_c(s)$. This is the key idea in the present approach to 2-D systems and it gives the possibility to interpret 2-D systems as 1-D systems over rings.
The following theorem shows that $\mathbb{R}_c(s,z)^{m\times p}$ can be embedded in $\mathbb{R}[[s^{-1},z^{-1}]]^{m\times p}$.

(4.1.12) THEOREM. *Let* F(s,z) *be a formal power series in* $\mathbb{R}[[s^{-1},z^{-1}]]^{m\times p}$. *If* $F(s,z) \in \mathbb{R}(s,z)^{m\times p}$ *then* $F(s,z) \in \mathbb{R}_c(s,z)^{m\times p}$. *Furthermore, every* $F(s,z) \in \mathbb{R}_c(s,z)^{m\times p}$ *can be expanded in a formal power series in* $\mathbb{R}[[s^{-1},z^{-1}]]^{m\times p}$.

PROOF. The proof of the first part is immediate. For the second part we observe that $F(s,z) \in \mathbb{R}_c(s)_c(z)^{m\times p}$. Therefore F(s,z) can be expanded in a formal power series in $\mathbb{R}_c(s)[[z^{-1}]]$. Then we expand all the coefficients in $\mathbb{R}_c(s)$ in a formal power series in $\mathbb{R}[[s^{-1}]]$. Therefore we have that $F(s,z) \in \mathbb{R}[[s^{-1}]][[z^{-1}]] = \mathbb{R}[[s^{-1},z^{-1}]]$. □

The proof can also be given by considering F(s,z) a complex function in two variables which is analytic at infinity. Then the associated formal power series is the series expansion (in s^{-1} and z^{-1}) at infinity.

The fact that $\mathbb{R}_c(s,z)^{m\times p}$ can be identified with $\mathbb{R}_c(s)_c(z)^{m\times p}$ enables us to consider a 2-D transfer matrix as a 1-D transfer matrix over the ring $\mathbb{R}_c(s)$. Also, a 2-D impulse response $F = (F_{k,h})_{(k,h)\in\mathbb{Z}_+^2}$ can be considered as a 1-D impulse response $(T(s)_k)_{k\in\mathbb{Z}_+}$ where $T(s)_k$ is a formal power series in s^{-1}. Again we are primarily interested in the case where $T(s)_k$ is a proper rational matrix for each $k \in \mathbb{Z}_+$ (that is, $T(s)_k$ is a 1-D transfer matrix for each $k \in \mathbb{Z}_+$).
Therefore, the ring which will be of central importance here is the ring $\mathbb{R}_c(s)$ of proper rational functions in one variable s (the ring of 1-D transfer functions).

In order to apply the realization results of Chapter II and the algorithms in Chapter III it is necessary that $\mathbb{R}_c(s)$ be a principal ideal domain. In the next section we will prove that this is indeed the case and also that $\mathbb{R}_c(s)$ shows some additional structure from which we may benefit when actually performing the computations necessary for the realization algorithms.

IV.2. *The ring of 1-D transfer functions:* $\mathbb{R}_c(s)$

In this section we will prove that $\mathbb{R}_c(s)$ is a principal ideal domain. First of all we will always assume that for $r_1(s)/r_2(s) \in \mathbb{R}_c(s)$ we have that $r_1(s)$ and $r_2(s)$ are coprime, i.e., $r_1(s)$ and $r_2(s)$ have no common factor other than unity. Constant common factors other than unity are ruled out because $r_2(s)$ is supposed to be monic. In this way we obtain in a certain sense the simplest representation for an element of $\mathbb{R}_c(s)$. A ring which will be very useful is the following

(4.2.1) $\quad \bar{\mathbb{R}}_c(s) = \{\bar{r}_1(s)/\bar{r}_2(s) \in \mathbb{R}(s) \mid \bar{r}_2(0) \neq 0\}$.

Observe that in the representation $\bar{r}_1(s)/\bar{r}_2(s) \in \bar{\mathbb{R}}_c(s)$ we may assume that $\bar{r}_2(0) = 1$.
One of the main tools will be the following ring isomorphism

(4.2.2) $\quad S: \mathbb{R}_c(s) \to \bar{\mathbb{R}}_c(s)$

defined by

$$S(r_1(s)/r_2(s)) = r_1(1/s)/r_2(1/s) =: \bar{r}_1(s)/\bar{r}_2(s)$$

where we suppose $r_2(s)$ to be monic and $\bar{r}_2(0) = 1$.

The fact that S is an isomorphism follows immediately from the following observations.

Let

$$r_1(s)/r_2(s) = (r_{10}+r_{11}s+\ldots+r_{1m}s^m)/(r_{20}+r_{21}s+\ldots+r_{2,n-1}s^{n-1}+s^n) ,$$

then

$$\bar{r}_1(s)/\bar{r}_2(s) = (r_{10}s^n+r_{11}s^{n-1}+\ldots+r_{1m}s^{n-m})/(r_{20}s^n+r_{21}s^{n-1}+\ldots+r_{2,n-1}s+1) ,$$

where $n \geq m$.

Observe that $\mathbb{R}[s] \subset \bar{\mathbb{R}}_c(s)$.

$\bar{\mathbb{R}}_c(s)$ is a *ring of fractions of* $\mathbb{R}[z]$ *with respect to* D_c (see [77] or the appendix). Notation: $\bar{\mathbb{R}}_c(s) = \mathbb{R}[s]_{D_c}$ where the multiplicative set D_c is

$$(4.2.3) \qquad D_c = \{d(s) \mid d(s) \in \mathbb{R}[s], d(0) \neq 0\} .$$

Since $\mathbb{R}[s]$ is a principal ideal domain (see [77]), it follows that $\bar{\mathbb{R}}_c(s) = \mathbb{R}[s]_{D_c}$ is also a principal ideal domain (see [8]). Therefore $\mathbb{R}_c(s)$ is a principal ideal domain. This enables us to apply the realization algorithms described in Chapter III. However, in actually performing the required computations one can benefit from the finer structure of $\mathbb{R}_c(s)$. It is not difficult to see that $\mathbb{R}_c(s)$ is in fact a Euclidean domain. In fact, the required Euclidean function φ can be taken to be the degree difference of denominator and numerator, that is, for $n(s)/d(s) \in \mathbb{R}_c(s)$ we may define $\varphi(n(s)/d(s)) = \deg(d(s)) - \deg(n(s))$. Now suppose that $n_2(s)/d_2(s)$ and $n_1(s)/d_1(s)$ be such that $\varphi(n_2(s)/d_2(s)) \geq \varphi(n_1(s)/d_1(s))$. Then there exist $p(s)/q(s) \in \mathbb{R}_c(s)$ and $r_1(s)/r_2(s) \in \mathbb{R}_c(s)$ such that $\varphi(r_1(s)/r_2(s)) < \varphi(n_1(s)/d_1(s))$ and $n_2(s)/d_2(s) = (p(s)/q(s))n_1(s)/d_1(s) + r_1(s)/r_2(s)$, for we can take $p(s)/q(s) = (n_2(s)d_1(s))/(n_1(s)d_2(s))$ and $r_1(s)/r_2(s) = 0$, where $p(s)/q(s) \in \mathbb{R}_c(s)$ because $\varphi(n_2(s)/d_2(s)) \geq \varphi(n_1(s)/d_1(s))$. This shows that in the ring $\mathbb{R}_c(s)$ we even have for two elements a and b that $a \mid b$ or $b \mid a$ ($a \mid b$ denotes a divides b). $\mathbb{R}_c(s)$ is a local ring whose maximal ideal is generated by $1/s$. This can be exploited in the realization algorithms.

IV.3. *2-D systems as 1-D systems over a principal ideal domain*

In this section, we will introduce state space realizations for a 2-D system. In general, the state space associated with a 2-D input/output system will be infinite dimensional (see figure 2). We will give updating equations for this state space realization.

Suppose we are given a 2-D impulse response $\mathcal{F} = (F_{k,h})_{(k,h)\in\mathbb{Z}_+^2}$ where $F_{kh} \in \mathbb{R}^{m\times p}$. This impulse response can be identified with the 1-D impulse response $\mathcal{T} = (T_k(s))_{k\in\mathbb{Z}_+}$ over $\mathbb{R}[[s^{-1}]]$ where

$$T_k(s) = \sum_{h=0}^{\infty} F_{kh}\, s^{-h} .$$

Therefore we can, assuming realizability, construct a realization of $\mathcal{F}$ over $\mathbb{R}[[s^{-1}]]$. However, we will primarily be interested in the case where $T_k(s) \in \mathbb{R}_c(s)^{m\times p}$ for $k = 0,1,2,\ldots$. This occurs when we have obtained the impulse response $\mathcal{T}$ from a 2-D transfer matrix. The reason for us to require $T_k(s) \in \mathbb{R}_c(s)^{m\times p}$ is that we will be able to construct a finite dimensional local state space model (this concept will be introduced later on). This local state space model is an effective tool in computing the output given the input, because the computations can be done recursively. Although the underlying state space model is infinite dimensional over $\mathbb{R}$, our state space model is finite dimensional over $\mathbb{R}_c(s)$. In [24] the authors state that it is not clear what the state space equations should look like. In the following it is shown that the equations of the state space realization over $\mathbb{R}_c(s)$ can in fact serve as updating equations.

In the above, the roles played by s and z can be interchanged. This can be seen as follows. Suppose we have a realizable impulse response $\mathcal{T} = (T_k(s))_{k\in\mathbb{Z}_+}$ over $\mathbb{R}_c(s)$. This means that the associated input/output system has a transfer matrix $T(z) \in \mathbb{R}_c(s)_c(z)^{m\times p}$ for some integers m and p. However, $\mathbb{R}_c(s)_c(z)$ and $\mathbb{R}_c(z)_c(s)$ can be identified and therefore $T(z)$ can be seen as a transfer matrix over $\mathbb{R}_c(z)$. From this transfer matrix an impulse response $\overline{\mathcal{T}}$ over $\mathbb{R}_c(z)$ can be constructed and we may say that $\mathcal{T}$ and $\overline{\mathcal{T}}$ both represent the same 2-D impulse response.

REMARK. Although the variables s and z play a completely comparable role in the transfer matrix description of a 2-D input/output system, they loose this symmetric property as soon as the 2-D transfer matrix is considered a

1-D transfer matrix over the ring $\mathbb{R}_c(s)$ or $\mathbb{R}_c(z)$. These two possible interpretations will only be mentioned a few times in this text. □

In the next we will introduce state space models for a 2-D system given by an impulse response or a transfer matrix. If the system is given by a transfer matrix, then a state space model can either be obtained via the impulse response or directly from the transfer matrix, depending on which realization algorithm is preferable (cf. section III.3).
Let $F(s,z) \in \mathbb{R}_c(s,z)^{m\times p}$ (that is, $F(s,z)$ is a 2-D transfer matrix) and identify $F(s,z)$ with a transfer matrix $T(z)$ over $\mathbb{R}_c(s)$. $T(z)$ can be expanded in a formal power series

$$(4.3.1) \qquad T(z) = \sum_{k=0}^{\infty} T_k z^{-k}$$

where $T_k \in \mathbb{R}_c(s)^{m\times p}$ for $k = 0,1,2,\ldots$.
To obtain a canonical realization over $\mathbb{R}_c(s)$ (which will also be minimal because $\mathbb{R}_c(s)$ is a principal ideal domain) we apply theorem (2.5.6) or one of the realization algorithms in Chapter III and obtain a canonical realization $\Sigma = (A(s),B(s),C(s),D(s))$ over $\mathbb{R}_c(s)$, where

$$(4.3.2) \qquad \begin{aligned} &A(s) \in \mathbb{R}_c(s)^{n\times n}, \quad B(s) \in \mathbb{R}_c(s)^{n\times p}, \\ &C(s) \in \mathbb{R}_c(s)^{m\times n}, \quad D(s) \in \mathbb{R}_c(s)^{m\times p}. \end{aligned}$$

Here we have supposed that the dimension of the realization is n. Thus we have

$$(4.3.3) \qquad F(s,z) = T(z) = C(s)[zI - A(s)]^{-1} B(s) + D(s) .$$

The dynamical interpretation is given by the equations

$$(4.3.4) \qquad \begin{aligned} \bar{x}_{k+1}(s) &= A(s)\bar{x}_k(s) + B(s)\bar{u}_k(s) , \quad \bar{x}_0(s) = 0 , \\ \bar{y}_k(s) &= C(s)\bar{x}_k(s) + D(s)\bar{u}_k(s) , \quad k = 0,1,2,\ldots . \end{aligned}$$

Here

$$\bar{u}_k(s) = \sum_{h=0}^{\infty} u_{kh} s^{-h} , \qquad k = 0,1,2,\ldots ,$$

$$\text{(4.3.5)} \quad \bar{y}_k(s) = \sum_{h=0}^{\infty} y_{kh}\, s^{-h} , \quad k = 0,1,2,\ldots ,$$

$$\bar{x}_k(s) = \sum_{h=0}^{\infty} x_{kh}\, s^{-h} , \quad k = 0,1,2,\ldots ,$$

where u_{kh} and y_{kh} are the inputs and outputs to the 2-D system (4.1.6) and $\bar{x}_k(s) \in \mathbb{R}[[s^{-1}]]^n$ is the state of the system.
In this way we have obtained infinite dimensional state space equations (over $\mathbb{R}$) for the system (4.1.6), although they are finite dimensional over $\mathbb{R}_c(s)$.
A realization $\Sigma = (A(s),B(s),C(s),D(s))$ of $F(s,z)$ of the form (4.3.4) will be called a *first level realization* (see also [16]).
Because $\mathbb{R}_c(s) \subset \mathbb{R}[[s^{-1}]]$ the multiplications in (4.3.4) are well defined. More details concerning the realization procedures for 2-D systems will be given in Chapter VI.
As was stated in theorem (2.5.16) each pair of canonical realizations of $F(s,z)$ is related by a state space isomorphism $S(s)$ which is also a proper rational matrix.
The matrices $A(s)$, $B(s)$, $C(s)$, $D(s)$ can themselves be viewed as 1-D transfer matrices. Realizing each of them we obtain minimal realizations

$$\text{(4.3.6)} \quad \begin{array}{ll} (AA,AB,AC,AD) & \text{for } A(s) \\ (BA,BB,BC,BD) & \text{for } B(s) \\ (CA,CB,CC,CD) & \text{for } C(s) \\ (DA,DB,DC,DD) & \text{for } D(s) \ . \end{array}$$

(Here all of them are single matrices, not products.) So we have

$$A(s) = AC[sI - AA]^{-1} AB + AD$$

and analogous formulas for $B(s)$, $C(s)$ and $D(s)$. A state space isomorphism can of course be given an analogous dynamical interpretation. The sequence of matrices $AA,AB,\ldots,DD$ will be called the *second level realization* of $F(s,z)$.
The dynamical interpretation of the second level realization is the following. Introduce vectors b_{kh}, a_{kh}, c_{kh}, d_{kh} satisfying the equations

$$b_{k,h+1} = BA\ b_{kh} + BB\ u_{kh}$$

(4.3.7) $$\begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} BC \\ 0 \end{bmatrix} b_{kh} + \begin{bmatrix} BD \\ 0 \end{bmatrix} u_{kh}$$

$$c_{k,h+1} = CA\ c_{kh} + CB\ x_{kh}$$

$$d_{k,h+1} = DA\ d_{kh} + DB\ u_{kh}$$

$$y_{kh} = CD\ x_{kh} + CC\ c_{kh} + DC\ d_{kh} + DD\ u_{kh}$$

where the vectors have suitable dimensions and all initial conditions are equal to zero (see also [16]).
Furthermore, we have (see (4.3.5))

$$\bar{x}_k(s) = \sum_{h=0}^{\infty} x_{kh}\ s^{-h}$$

$$\bar{u}_k(s) = \sum_{h=0}^{\infty} u_{kh}\ s^{-h}$$

$$\bar{y}_k(s) = \sum_{h=0}^{\infty} y_{kh}\ s^{-h} .$$

REMARK. In (4.3.7) x_{kh}, a_{kh}, b_{kh}, c_{kh}, d_{kh} are *local states* (cf. [48]) because we have state space systems for each k. Although the matrices are the same for each k, the states will generally not be the same. □

From (4.3.7) it is clear that the second level realization enables us to compute the output recursively for a given input. Note that, also in this respect, 2-D systems and 1-D systems differ very much, because in the 1-D case the state space equations are also the equations used for the recursive computations of the outputs, whereas in the 2-D case the equations for the recursive computation of the outputs (4.3.7) are finite dimensional and the state space equations (4.3.4) are infinite dimensional.
A flow diagram revealing the hierarchic nature of the first and second level realization is shown in figure 3.

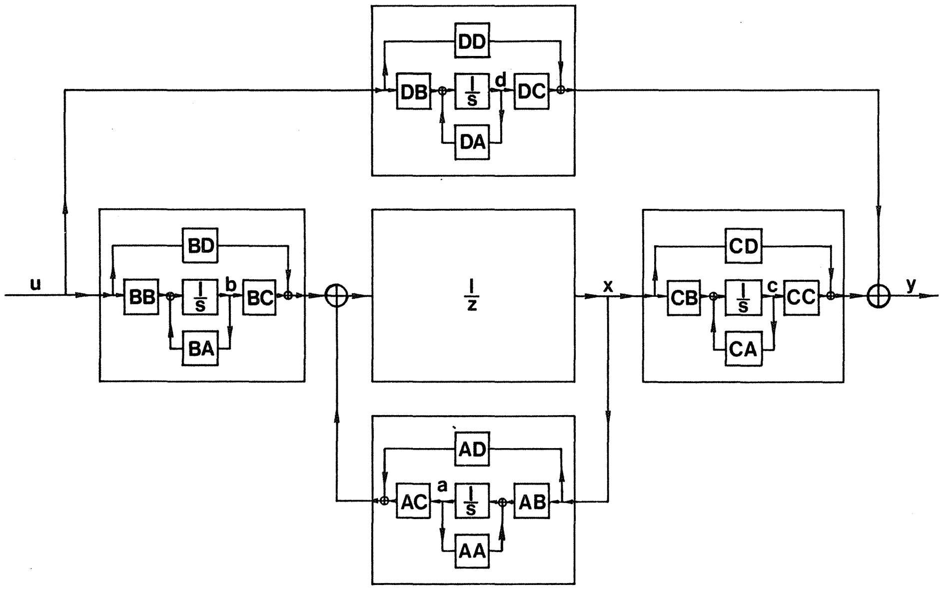

Figure 3. Flow diagram of a 2-D system.

REMARK. In 1-D systems theory the blocks containing A(s), B(s), C(s) and D(s) only contain constant matrices. In fact, this flow diagram specializes to the 1-D case if we leave out the parts concerning the s-dynamics. It will also be clear what a flow diagram for an n-D system could look like. For instance, in the 3-D case, the blocks containing constant matrices in figure 3 will contain 1-D systems again. □

We will show that the models of [61] and [24] are equivalent to our second level realization in the sense that a model as presented in [61] and [24] can be rewritten in the form of our model and vice versa. Furthermore, given the input sequences, the generated output sequences are the same, independently of the model chosen for the description of the underlying 2-D system. It suffices to show that Roesser's model [61] is equivalent to ours in the above sense since the Fornasini-Marchesini model is known to be equivalent to Roesser's model (see [48], [25]). A local state space model due to Attasi (see [43]) is a special case of ours since it is a special case of [24]. In [24], [48], [25] only local state space models are given; these papers do not contain equations for the (infinite dimensional) state space realization. Because our model combines a local state space model (the second level realization) and a state space model (the first level realization) which are related in a simple way, the other models can be considered special cases of our model.

Fornasini and Marchesini work with various local state space models which are strongly related (cf. [23], [24], [25]). For completeness we give here the definition of their last local state space model which they presented in [25].

$$x_{k+1,h+1} = A_1\, x_{k+1,h} + A_2\, x_{k,h+1} + B_1\, u_{k+1,h} + B_2\, u_{k,h+1}$$

$$y_{kh} = C\, x_{kh}\,.$$

Here u_{kh} and y_{kh} denote the input and output and x_{kh} is a local state. A D-matrix such that $y_{kh} = C\,x_{kh} + D\,u_{kh}$ can be incorporated in their theory without causing any problems.

In [61] the following local state space model is considered (with notation as in [61])

$$\begin{bmatrix} R_{k+1,h} \\ S_{k,h+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

(4.3.8)

$$y_{kh} = [C_1\,,\,C_2] \begin{bmatrix} R_{kh} \\ S_{kh} \end{bmatrix} + D\,u_{kh}\,.$$

In fact, the matrix D is zero in [61] but it is easy to extend the model to the case where D is present.

(4.3.9) THEOREM. *The local state space model* (4.3.8) *can be written in the form* (4.3.4) *and* (4.3.7). *The corresponding matrices are*

$$A(s) = A_2[sI - A_4]^{-1} A_3 + A_1 , \quad B(s) = A_2[sI - A_4]^{-1} B_2 + B_1 ,$$

$$C(s) = C_2[sI - A_4]^{-1} A_3 + C_1 , \quad D(s) = C_2[sI - A_4]^{-1} B_2 + D$$

and

$$AA = A_4 , \quad AB = A_3 , \quad AC = A_2 , \quad AD = A_1 ,$$

$$BA = A_4 , \quad BB = B_2 , \quad BC = A_2 , \quad BD = B_1 ,$$

$$CA = A_4 , \quad CB = A_3 , \quad CC = C_2 , \quad CD = C_1 ,$$

$$DA = A_4 , \quad DB = B_2 , \quad DC = C_2 , \quad DD = D .$$

Vice versa, the second level realization (4.3.7) *can be written in the form* (4.3.8). *Then the corresponding matrices and vectors are*

$$R_{kh} = x_{kh} , \quad A_1 = AD , \quad A_2 = [AC,BC,0,0] , \quad B_1 = BD ,$$

$$S_{kh} = \begin{bmatrix} a_{kh} \\ b_{kh} \\ c_{kh} \\ d_{kh} \end{bmatrix} , \quad A_3 = \begin{bmatrix} AB \\ 0 \\ CB \\ 0 \end{bmatrix} , \quad A_4 = \begin{bmatrix} AA & 0 & 0 & 0 \\ 0 & BA & 0 & 0 \\ 0 & 0 & CA & 0 \\ 0 & 0 & 0 & DA \end{bmatrix} , \quad B_2 = \begin{bmatrix} 0 \\ BB \\ 0 \\ DB \end{bmatrix} \tag{4.3.10}$$

$$C_1 = CD , \quad C_2 = [0,0,CC,DC] , \quad D = DD .$$

PROOF. Suppose that $u(s,z)$ and $y(s,z)$ are the formal power series associated with the input and output of the system defined by (4.3.8). Then we have

$$y(s,z) = \left\{[C_1,C_2]\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D\right\} u(s,z) .$$

It follows from the equality

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1}\begin{bmatrix} zI-A_1-A_2[sI-A_4]^{-1}A_3 & -A_2[sI-A_4]^{-1} \\ 0 & I \end{bmatrix}^{-1}$$

by calculating both inverses in the right-hand side that

$$y(s,z) = F(s,z)u(s,z)$$

where

$$F(s,z) = \{C_1 + C_2[sI-A_4]^{-1}A_3\}\{zI - A_1 - A_2[sI-A_4]^{-1}A_3\}^{-1}\{B_1 + A_2[sI-A_4]^{-1}B_2\} +$$

$$+ C_2[sI-A_4]^{-1}B_2 + D \ .$$

The second part of the theorem is proved by just reorganizing (4.3.7). □

If we insist on a model in the form (4.3.8), the matrices tend to be "large". This can be seen if, for instance, the method described in [48] (only for scalar transfer functions) is used. It can also be seen from (4.3.10). Our second level realization gives more matrices but they are "smaller".

In [74] a method to find a local state space model of the form (4.3.8) is described, starting with a first level realization. This is done using

$$(4.3.11) \quad W(s) = \begin{bmatrix} A(s) & B(s) \\ C(s) & D(s) \end{bmatrix}$$

as a single 1-D transfer matrix. In [74] a model of the form (4.3.8) is obtained by partitioning a realization (F,G,H,K) of W(s) as

$$(4.3.12) \quad F = A_4 \ , \quad G = [A_3,B_2] \ , \quad H = \begin{bmatrix} A_2 \\ C_2 \end{bmatrix} \ , \quad K = \begin{bmatrix} A_1 & B_1 \\ C_1 & D \end{bmatrix}$$

where the matrices A_1, A_2, A_3, A_4, B_1, B_2, C_1, C_2, D are the same as in (4.3.8).

(4.3.13) REMARK. Sontag's paper [74] appeared at the same time as [16] in which most of this chapter is described. For more comments concerning the relations with other papers and the history of the 2-D realization problem see Chapter I. □

(4.3.14) EXAMPLE. Consider a scalar proper rational function (a 2-D transfer function)

$$F(s,z) = \frac{\sum_{i=0}^{n} a_i(s) z^i}{\sum_{j=0}^{n} b_j(s) z^j} = \frac{\sum_{i=0}^{n} \alpha_i(s) z^i}{\sum_{j=0}^{n} \beta_j(s) z^j} \ ,$$

where (because of properness)

$$\alpha_i(s) = \frac{a_i(s)}{b_n(s)} \in \mathbb{R}_c(s) \quad \text{and} \quad \beta_j(s) = \frac{b_j(s)}{b_n(s)} \in \mathbb{R}_c(s) \ .$$

In order to simplify the example we assume that $a_n(s) = 0$. The first level realization can very easily be found for scalar transfer functions because one can use the so called standard controllable form for a realization, see [9].

A first level realization is $\Sigma = (A(s), B(s), C(s), D(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \\ -\beta_0(s) & & \cdots & & -\beta_{n-1}(s) \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C(s) = [\alpha_0(s), \ldots, \alpha_{n-1}(s)] \ , \quad D(s) = 0 \ .$$

The second level realization gives CD, CC, CA, CB, AD, AC, AA, AB, BD; BC = BA = BB = 0.

The resulting local state space equations are

$$\begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} BD \\ 0 \end{bmatrix} u_{kh} \ ,$$

$$c_{k,h+1} = CA\ c_{kh} + CB\ x_{kh} \ ,$$

$$y_{kh} = CD\ x_{kh} + CC\ c_{kh} \ ,$$

where $AD \in \mathbb{R}^{n\times n}$, $AA \in \mathbb{R}^{n\times m}$, $CA \in \mathbb{R}^{m\times m}$ and m is the degree of $b_n(s)$. We can even take AA = CA.

Two kinds of system matrices have been obtained:

$$\begin{bmatrix} AD & AC \\ AB & AA \end{bmatrix} \in \mathbb{R}^{(n+m)\times(n+m)}$$

representing dynamics in two directions and

CA

representing dynamics in one direction

In [48] an $(n+2m) \times (n+2m)$ system matrix is obtained for this transfer function because the authors wanted system equations in Roesser's form.

By a system matrix of a system $\Sigma = (A,B,C,D)$ or a local state space system (4.3.8) is meant the matrix A or in the latter case the matrix

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} .$$

In theorem (4.3.9) we have seen that the local state space system (4.3.8) is theoretically equivalent to the second level realization (4.3.7). The structure of the equations (4.3.8) is also present in (4.3.7). Although in (4.3.7) one usually deals with smaller matrices, which can be advantageous on a computational level, Roesser's equations (4.3.8) seem to be more attractive theoretically because the model contains less matrices. Therefore we will often work with (4.3.8) when we are dealing with local state space models. In the next, local state space models will be called state space models because it will always be clear whether a first level realization or a second level realization is under consideration.

IV.4. *Weakly causal 2-D systems*

In this section the results concerning state space realization of a causal 2-D system, as described in the first part of Chapter IV, will be generalized to a larger class of 2-D systems. This will give a generalized notion of state space realization, for which the state, and therefore the output, can still be evaluated in a recursive way. These 2-D systems will be called weakly causal. The results also include a realization method for a class of Non Symmetric Half Plane filters (NSHP filters), see [22].

Weakly causal 2-D systems arise in a natural way when one studies inverse 2-D systems. Generally, a causal 2-D system does not have a causal inverse, even not a causal inverse with inherent delay, see [20]. In Chapter V these new notions will become clear when inverse systems will be discussed.

The variables z and s can be interpreted as shift operators in the following way

$$z(x)_{kh} = x_{k+1,h}\ ,\quad s(a)_{kh} = a_{k,h+1}\ .$$

Thus the local state space model (4.3.8) can be written as

$$\begin{bmatrix} z(x)_{kh} \\ s(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}\ ,$$

(4.4.1)

$$y_{kh} = [C_1\ , C_2] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D\,u_{kh}\ ,\qquad k,h = 0,1,2,\ldots\ ,$$

where, instead of R and S, we have taken x and a respectively, and the initial conditions are $x_{0h} = 0$, $h = 0,1,2,\ldots$; $a_{k0} = 0$, $k = 0,1,2,\ldots$.
In the generalized state space model for weakly causal 2-D systems, as defined below, the shift operators z and s will be replaced by more general shift operators.
Consider the 2-D input/output system

(4.4.2) $$y_{kh} = \sum_{(i,j)\in J} F_{k-i,h-j}\, u_{ij}\ ,\qquad (k,h) \in J \subset \mathbb{Z}^2\ .$$

The index set J will be specified later on. $F_{ij} \in \mathbb{R}^{m\times p}$ for $(i,j) \in \mathbb{Z}^2$ and some integers m and p.
The *support* of the impulse response $F = (F_{m,n})_{(m,n)\in\mathbb{Z}^2}$ is the set

(4.4.3) $$S_F = \{(m,n) \mid (m,n) \in \mathbb{Z}^2,\ F_{mn} \neq 0\}\ .$$

A *cone* C is a subset of $\mathbb{R}^2$ (with row vector notation) such that if $(x,y) \in C$, then $(\lambda x, \lambda y) \in C$ for all $\lambda \geq 0$. The closed first quadrant of $\mathbb{R}^2$ will be denoted by $\mathbb{R}^2_+$.

(4.4.4) DEFINITION. *The input/output system* (4.4.2) *will be called weakly causal if*

$$S_F \subset C\ ,\quad J \subset C$$

for some closed convex cone C *satisfying*

(4.4.5)
$$1^\circ:\ C \cap (-C) = \{0\}\ ,$$
$$2^\circ:\ \mathbb{R}^2_+ \subset C\ .$$

□

From now on C will always denote a closed convex cone satisfying 1° and 2°. In the following we will be interested in invertible mappings φ

$$\varphi: C \cap \mathbb{Z}^2 \to \mathbb{Z}_+^2$$

such that the origin is a fixed point ($\varphi(0,0) = (0,0)$). Therefore we introduce first the notion of causality cone.

(4.4.6) DEFINITION. *A causality cone* C_c *is the intersection of two half-planes* $H_{p,r}$ *and* $H_{q,t}$ *where*

$$H_{p,r} = \{(x,y) \mid (x,y) \in \mathbb{R}^2,\ px + ry \geq 0\}\ ,$$

$$H_{q,t} = \{(x,y) \mid (x,y) \in \mathbb{R}^2,\ qx + ty \geq 0\}\ ,$$

where p, r, q, t *are non-negative integers satisfying*

$$qr - pt = -1\ .$$ □

The next figure shows the causality cone based upon $H_{1,0}$ and $H_{2,1}$.

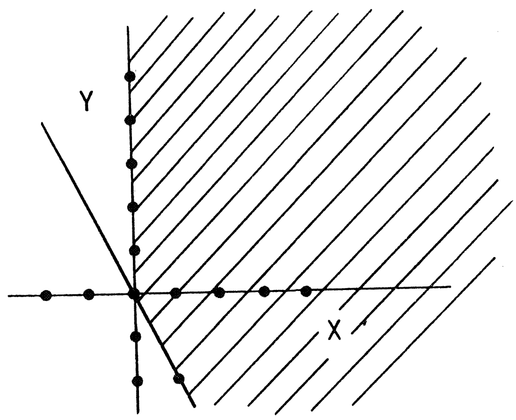

Figure 4

(4.4.7) LEMMA. *Every causality cone has the properties* 1° *and* 2° *in* (4.4.5).

PROOF. The proof is straightforward and will be omitted. □

REMARK. Every causality cone induces a partial order on $\mathbb{Z}^2$ in the same way as $\mathbb{Z}_+^2$ (the causality cone for a causal system) does, see [66]. This partial order enables us to introduce double sequences with finite past as is done in the 1-D case (see (2.1.4)); see also [56]. □

(4.4.8) LEMMA. *Suppose that* C *is a closed convex cone satisfying* 1° *and* 2° *in* (4.4.5). *Then there exists a causality cone* C_c *such that* $C \subset C_c$.

PROOF. $C \subset C'$ where C' is the intersection of two halfplanes $H_{p',r'}$ and $H_{q',t'}$, such that $q'r' - p't' < 0$ and p' and r' are coprime. Then there exist integers q_1 and t_1 such that $q_1r' - p't_1 = -1$ and thus $(q_1 + np')r' - p'(t_1 + nr') = -1$ for all $n \in \mathbb{Z}$. Because $q'/t' < p'/r'$ we have for sufficiently large n_0 that $q'/t' < (q_1 + n_0p')/(t_1 + n_0r')$. Now take $p = p'$, $r = r'$, $q = q_1 + n_0p'$, $t = t_1 + n_0r'$ and $C_c = H_{p,r} \cap H_{q,t}$ is a causality cone satisfying $C \subset C_c$. □

REMARK. Lemma (4.4.8) gives a result on existence of C_c. In fact, C_c is not unique at all. For instance, a causality cone containing C_c suffices also in lemma (4.4.8). □

(4.4.9) THEOREM. *If* C_c *is a causality cone, then there exists a map* φ, *one-one and onto,*

$$\varphi\colon C_c \cap \mathbb{Z}^2 \to \mathbb{Z}_+^2$$

such that

$$\varphi(k_1+k_2,h_1+h_2) = \varphi(k_1,h_1) + \varphi(k_2,h_2) .$$

PROOF. Suppose $C_c = H_{p,r} \cap H_{q,t}$, then the map φ defined by

$$\varphi(k,h) = (pk+rh,qk+ht)$$

is a possible one. □

This map φ will be used to transform a weakly causal input/output system into a causal input/output system, which, in turn, will be used to construct a state space realization for the weakly causal input/output system. We will take a formal power series point of view for (4.4.2) (or apply the 2-D Z-transform to (4.4.2)). Then we obtain

$$(4.4.10)\quad y(s,z) = F(s,z)u(s,z) .$$

Observe that $F(s,z) = \sum_{(k,h)\in J} F_{kh}\, s^{-h} z^{-k}$ cannot be seen as a matrix over the ring of formal power series as is defined in (4.1.7), because positive powers of s or z may arise. Nevertheless we may use the word "formal power series" when we are dealing with the matrix F(s,z) or the vectors y(s,z) and u(s,z) because we have

(4.4.11) THEOREM. *For any* $C_c = H_{p,r} \cap H_{q,t}$ *the set*

$$\hat{S}_{p,r,q,t} = \{F(s,z) \mid \text{there exists } F = (F_{k,h})_{(k,h)\in\mathbb{Z}^2} \text{ such that}$$

$$F(s,z) = \sum_{(k,h)\in\mathbb{Z}^2} F_{kh}\, s^{-h} z^{-k} \text{ and } S_F \subset C_c\}$$

is a ring with the usual addition and multiplication. Furthermore $\hat{S}_{p,r,q,t}$ *is isomorphic to* $\hat{S}_{1,0,0,1}$.

PROOF. Define the ring homomorphism

$$\Phi\colon \hat{S}_{p,r,q,t} \to \hat{S}_{1,0,0,1}$$

by

$$\Phi(F)(\alpha,\beta) = \sum_{m=0,n=0}^{\infty,\infty} F_{\varphi^{-1}(n,m)}\, \alpha^{-m} \beta^{-n}$$

where φ is the same as in (4.4.9). Now the proof is just a matter of verification that Φ is indeed a ring isomorphism. □

For a formal power series, in the enlarged sense, rationality is, as usually, defined as being the quotient of polynomials.

(4.4.12) THEOREM. *Let* Φ *be the ring isomorphism defined in theorem* (4.4.11). *Then* $\Phi(F)(\alpha,\beta)$ *is rational iff* $F(s,z) \in \mathbb{R}(s,z)$.

PROOF. The equality

$$F_{\varphi^{-1}(n,m)}\, \alpha^{-m}\beta^{-n} = F_{k,h}\, \alpha^{-qk-ht}\, \beta^{-pk-rh} = F_{kh}(\alpha^t\beta^r)^{-h}(\alpha^q\beta^p)^{-k}$$

implies

$$\Phi(F)(\alpha,\beta) = F(\alpha^t\beta^r, \alpha^q\beta^p)$$

from which the result follows. □

The ring isomorphism can also be defined for $\hat{S}^{m\times\ell}_{p,r,q,t}$, the set of $m\times\ell$ matrices over $\hat{S}_{p,r,q,t}$, because Φ can be applied to every entry of a matrix over $\hat{S}_{p,r,q,t}$. We will make a little abuse of notation and also write $\Phi(F)(\alpha,\beta)$ where $F(s,z)$ is a matrix over $\hat{S}_{p,r,q,t}$. By this we will mean that Φ is applied to every entry of $F(s,z)$ separately. Furthermore, by a rational matrix over $\hat{S}_{p,r,q,t}$ we will mean a matrix where every entry is rational (in $\hat{S}_{p,r,q,t}$). Again, by a formal power series expansion (representation) of a matrix $F(s,z)$ over $\hat{S}_{p,r,q,t}$ we mean a matrix whose entries are formal power series expansions (representations) of the corresponding entries of $F(s,z)$.

(4.4.13) DEFINITION. *A rational* $F(s,z)$*, corresponding to a weakly causal input/output system, will be called a weakly causal transfer matrix.* □

The following theorem is concerned with the uniqueness of a formal power series expansion, in the enlarged sense, for a weakly causal transfer matrix.

(4.4.14) THEOREM. *Suppose* $F(s,z)$ *is a weakly causal transfer matrix and that*

$$F(s,z) = \sum_{(k,h)\in\mathbb{Z}^2} F_{kh}\, s^{-h} z^{-k} = \sum_{(k,h)\in\mathbb{Z}^2} G_{kh}\, s^{-h} z^{-k} .$$

Furthermore, suppose that $S_F \subset C_c$, $S_G \subset C_c$, *where* C_c *is a causality cone and* $F = (F_{k,h})_{(k,h)\in\mathbb{Z}^2}$, $G = (G_{k,h})_{(k,h)\in\mathbb{Z}^2}$. *Then* $F_{kh} = G_{kh}$ *for all* k, h.

PROOF. Let Φ be defined as in (4.4.11). Now $\Phi(F)(\alpha,\beta)$ is a causal transfer matrix. The formal power series expansion for $\Phi(F)(\alpha,\beta)$ is unique (see [26]). From this the proof follows. □

Notice that a weakly causal transfer matrix may have more than one formal power series representation with support in a (different by (4.4.14)) causality cone.

EXAMPLE.

$$\frac{1}{s-z} = \frac{1}{s}\sum_{k=0}^{\infty}\left(\frac{z}{s}\right)^k \quad \text{with} \quad C_c = \{(x,y) \mid y \geq 0,\ x \geq -y\} ;$$

$$\frac{1}{s-z} = -\frac{1}{z}\sum_{k=0}^{\infty}\left(\frac{s}{z}\right)^k \quad \text{with} \quad \bar{C}_c = \{(x,y) \mid x \geq 0,\ y \geq -x\} .$$

Starting with a weakly causal transfer matrix it is of course a long way around to transform it into a causal transfer matrix via the associated formal power series. Therefore we state the following lemma which enables us to construct a causal transfer matrix directly from a weakly causal one.

(4.4.15) LEMMA. *The isomorphism* Φ, *as defined in* (4.4.11), *can also be described by the substitution*

$$s = \alpha^{t}\beta^{r}, \quad z = \alpha^{q}\beta^{p}$$

with inverse

$$\alpha = s^{p}z^{-r}, \quad \beta = s^{-q}z^{t}.$$

PROOF. The proof follows immediately from the proof of theorem (4.4.12). The fact that the inverse of the substitution $s = \alpha^{t}\beta^{r}$, $z = \alpha^{q}\beta^{p}$ also has integer exponents is due to $qr - pt \doteq -1$. □

Next we will derive a local state space realization for a weakly causal transfer matrix. The equations for this realization will be generalizations of (4.4.1). For this purpose we transform this weakly causal transfer matrix into a causal one. Then we construct a local state space model for this causal transfer matrix as was done in IV.3 which will be written in Roesser's form (4.4.1). The ring isomorphism Φ can also be defined for the obtained state space realization. This is done by means of lemma (4.4.15). The obtained state space realization is not a first order model anymore but the equations can still be used for recursive computation of the state and the output. We will now describe the procedure in more detail.

Suppose $F(s,z)$ is a weakly causal transfer matrix over $\hat{S}_{p,r,q,t}$. Suppose $T(\alpha,\beta) = F(\alpha^{t}\beta^{r},\alpha^{q}\beta^{p})$. Then $T(\alpha,\beta)$ is a causal transfer matrix. Now $T(\alpha,\beta)$ has a second level realization (see (4.3.7)) which can be written in the form (4.4.1)

(4.4.16a) $$\begin{bmatrix}\beta(x)_{kh}\\ \alpha(a)_{kh}\end{bmatrix} = \begin{bmatrix}A_1 & A_2\\ A_3 & A_4\end{bmatrix}\begin{bmatrix}x_{kh}\\ a_{kh}\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u_{kh},$$

(4.4.16b) $$y_{kh} = [C_1, C_2]\begin{bmatrix}x_{kh}\\ a_{kh}\end{bmatrix} + D\,u_{kh}$$

and

$$T(\alpha,\beta) = [C_1,C_2]\begin{bmatrix} \beta I-A_1 & -A_2 \\ -A_3 & \alpha I-A_4 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + D \ .$$

Since $\beta = s^{-q} z^{t}$, $\alpha = s^{p} z^{-r}$, equation (4.4.16a) can be written as

$$\begin{bmatrix} s^{-q}z^{t}(x)_{kh} \\ s^{p}z^{-r}(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh}$$

or

$$\begin{bmatrix} z^{t}(x)_{kh} \\ s^{p}(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 s^{q} & A_2 s^{q} \\ A_3 z^{r} & A_4 z^{r} \end{bmatrix}\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 s^{q} \\ B_2 z^{r} \end{bmatrix} u_{kh} \ , \tag{4.4.17}$$

i.e.

$$\begin{aligned} x_{k+t,h} &= A_1\, x_{k,h+q} + A_2\, a_{k,h+q} + B_1\, u_{k,h+q} \ , \quad th + qk \geq 0 \ , \\ a_{k,h+p} &= A_3\, x_{k+r,h} + A_4\, a_{k+r,h} + B_2\, u_{k+r,h} \ , \quad rh + pk \geq 0 \ . \end{aligned} \tag{4.4.18}$$

Instead of the initial conditions for (4.4.1) we now have the following

$$\begin{aligned} x_{-rm,pm} &= 0 \ , \quad m = 0,1,2,\ldots \ , \\ a_{tn,-qn} &= 0 \ , \quad n = 0,1,2,\ldots \ , \end{aligned} \tag{4.4.19}$$

see also figure 5.

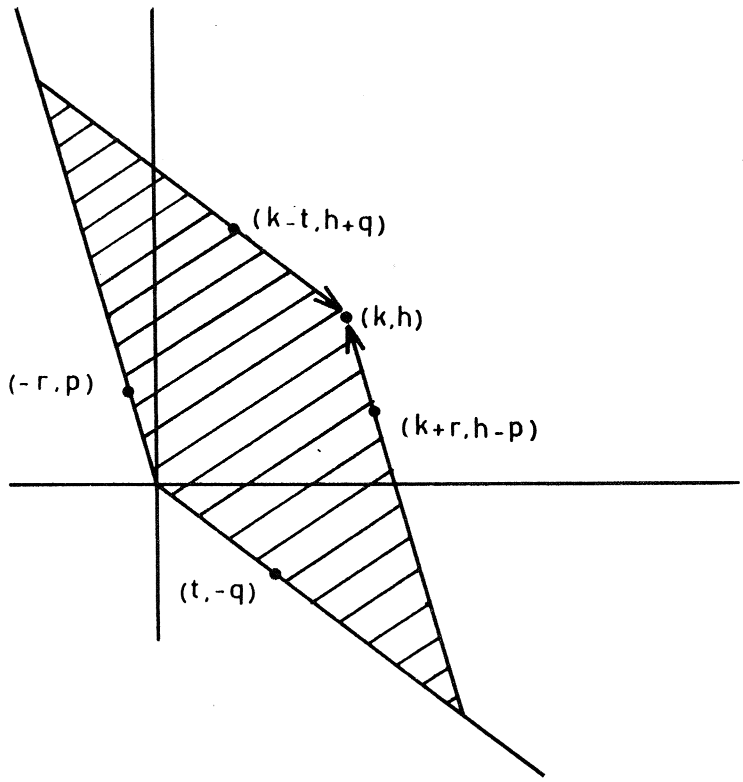

Figure 5

Observe that, in order to compute the (local) state at (k,h), only the states in the shaded area have to be known. From this we can also see that the state can be evaluated in a recursive way and for every state a finite number of steps is necessary.

REMARK. Because there are many ring isomorphisms transforming a weakly causal transfer matrix into a causal one, we may not expect uniqueness results. Even for equations as (4.4.1) no uniqueness results can be stated up to now because it is not clear how minimality or non-minimiality of first level realizations influences minimality or non-minimality of the second level realization. Furthermore, up to now, a good definition of minimality of equations like (4.4.1) has not appeared in the literature. There is still another complication, for it is not clear what the effects are if we interchange s and z. □

The derived realization technique can also be applied to a class of NSHP filters (Non Symmetric Half Plane filters), see [22].

(4.4.20) DEFINITION. *An* NSHP *filter is an input/output system with support in an* NSHP, *i.e., a subset of* $\mathbb{Z}^2$ *of the following kind*

$$\{(k,h) \mid (k,h) \in \mathbb{Z}^2,\ k > 0 \textit{ or } (k = 0 \textit{ and } h \geq 0)\}\ . \qquad \square$$

For more details on NSHP filters we refer to [22]. Now consider an NSHP filter with support in a set H_q where q is positive integer and

$$(4.4.21) \quad H_q = \{(k,h) \mid (k,h) \in \mathbb{Z}^2,\ k \geq 0,\ h \geq -qk\}\ .$$

It is clear that H_q is in fact a causality cone so that the above method can be applied.

(4.4.22) REMARK. By allowing transformations like $\alpha = s^{-1}$, $\beta = z^{-1}$ one can realize transfer matrices having their support in a closed convex cone C containing another quadrant. C still has to satisfy $C \cap (-C) = \{0\}$. Also by interchanging s and z all the results in this chapter remain valid and possibly other state space models are obtained. □

(4.4.23) EXAMPLE. The transfer function

$$F(s,z) = \frac{-s + (s-1)z}{sz + s^2}$$

is weakly causal, for if we substitute (or apply an isomorphism Φ which is equivalent to this substitution, see (4.4.15))

$$s = \alpha\ , \quad z = \alpha\beta$$

then

$$\Phi(F)(\alpha,\beta) = \frac{-1 + (\alpha-1)\beta}{\alpha + \alpha\beta}$$

is causal.

$\Phi(F)$ can be realized as follows:

$$\begin{bmatrix} \beta(x)_{kh} \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_{kh} ,$$

$$y_{kh} = [-1 , -1] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + u_{kh}$$

and we obtain a local state space realization of $F(s,z)$

$$x_{k+1,h} = - x_{k,h+1} + u_{k,h+1} ,$$

$$a_{k,h+1} = u_{kh} ,$$

$$y_{kh} = - x_{kh} - a_{kh} + u_{kh} , \quad k = 0,1,2,\dots; \ h = -k,-k+1,-k+2,\dots .$$

The initial conditions are

$$x_{0,h} = 0 , \quad a_{k,-k} = 0 , \quad h = 0,1,2,\dots; \ k = 0,1,2,\dots .$$

V REFINEMENT OF THE STATE SPACE MODELS AND PROPERTIES

V.1. *Stability*

If a 2-D system is not only an object of theoretical study and one has practical applications in mind, then stability is one of the first properties the system must have. Stability of an input/output system reflects the idea that if a bounded (in some sense) input is applied to the system, then the output has to be bounded.
Suppose we have the input/output system

$$(5.1.1) \qquad y_{kh} = \sum_{i=0,j=0}^{k,h} F_{k-i,h-j}\, u_{ij} \qquad (k,h) \in \mathbb{Z}_+^2$$

where again $u_{ij} \in \mathbb{R}^p$, $y_{kh} \in \mathbb{R}^m$ for some integers m, p.
Let $\| \ \|$ be some norm on $\mathbb{R}^p$, then the norm on $\mathbb{R}^m$ will also be denoted by $\| \ \|$. The operator norm on $\mathbb{R}^{m\times p}$ induced by $\| \ \|$ will also be denoted by $\| \ \|$.
Next we give the definition of BIBO stability (Bounded Input - Bounded Output stability) for the input/output system (5.1.1).

(5.1.2) DEFINITION. *The input/output system* (5.1.1) *is* BIBO *stable if for all* $M > 0$ *there exists an* $N > 0$ *such that, if* $\|u_{ij}\| \le M$ *for all* (i,j), *then* $\|y_{kh}\| \le N$ *for all* (k,h). □

(5.1.3) THEOREM. *The input/output system* (5.1.1) *is* BIBO *stable iff*

$$\sum_{k=0,h=0}^{\infty,\infty} \|F_{kh}\| < \infty .$$

PROOF. See [76]. □

The property of stability can also be checked if we have the transfer matrix of an input/output system. Suppose that F(s,z) is a 2-D transfer matrix and let q(s,z) be the least common multiple of the denominators of the entries of F(s,z). Then F(s,z) = P(s,z)/q(s,z) where P(s,z) is a polynomial matrix. We will assume that P(s,z) and q(s,z) are relatively prime, i.e., a common divisor of q(s,z) and all entries of P(s,z) is necessarily a constant. A sufficient condition for an input/output system with transfer matrix $F(s,z)$ to be BIBO stable can be expressed as a condition on the polynomial $q(s,z)$.

(5.1.4) THEOREM. *The input/output system with transfer matrix* $F(s,z) = P(s,z)/q(s,z)$ *where* $P(s,z)$ *and* $q(s,z)$ *are relatively prime is* BIBO *stable if* $q(s,z) \neq 0$ *for* $|z| \geq 1$, $|s| \geq 1$.

PROOF. The proof, given in [30] for the scalar case, can easily be extended to the multivariable case. □

The condition $q(s,z) \neq 0$ for $|z| \geq 1$, $|s| \geq 1$ is not a necessary condition for BIBO stability. This is because of the possible occurrence of non-essential singularities of the second kind on $|s| = 1$, $|z| = 1$ for the two variable complex function $P(s,z)/q(s,z)$, see [30]. Thus, for a stable 2-D transfer matrix $F(s,z) = P(s,z)/q(s,z)$ we have that $q(s,z) \neq 0$ for $|z| \geq 1$, $|s| \geq 1$, except possibly on $|s| = 1$; $|z| = 1$.
For weakly causal input/output systems BIBO stability is defined in an analogous way. For instance we have that the weakly causal input/output system is stable iff

$$\sum_{(k,h)\in J} \|F_{kh}\| < \infty .$$

Furthermore we have that the weakly causal input/output system with transfer matrix $F(s,z) \in \hat{S}^{m\times p}_{p,q,r,t}$ is BIBO stable iff the transformed causal input/output system with transfer matrix $\Phi(F)(\alpha,\beta)$ is BIBO stable. See (4.4.2), (4.4.11).

Because a different form of the above theorem is more convenient for us we state a theorem of Huang [36].

(5.1.5) THEOREM. $q(s,z) \neq 0$ *for* $|z| \leq 1$, $|s| \leq 1$ *iff*

1° $q(s,0) \neq 0$ *for* $|s| \leq 1$,

2° $q(s,z) \neq 0$ *for* $|z| \leq 1$, $|s| = 1$.

PROOF. See [36]. □

Because we are interested in the region $|z| \geq 1$, $|s| \geq 1$, we state

(5.1.6) THEOREM. $q(s,z) = \sum_{j=0}^{n} q_j(s) z^j \neq 0$ *for* $|s| \geq 1$, $|z| \geq 1$ *(where* $q_n(s)$ *is not the zero polynomial) iff*

1° $q_n(s) \neq 0$ *for* $|s| \geq 1$,
2° $q(s,z) \neq 0$ *for* $|s| = 1$, $|z| \geq 1$.

PROOF. Consider $q(\frac{1}{s}, \frac{1}{z})$. Multiply with appropriate powers of s and z (in order to obtain a polynomial in s and z again) and then use Huang's theorem. □

Various stability tests have appeared in the literature. A good entrance to the "2-D stability tests" literature is [37].

The possible occurrence of non-essential singularities of the second kind can only violate condition 2° of (5.1.6). Therefore BIBO stability of F(s,z) implies that $q_n(s) \neq 0$ for $|s| \geq 1$.
This motivates us to introduce a subring $\mathbb{R}_\sigma(s)$ of $\mathbb{R}_c(s)$:

$$(5.1.7) \quad \mathbb{R}_\sigma(s) = \left\{ \frac{r_1(s)}{r_2(s)} \;\middle|\; \frac{r_1(s)}{r_2(s)} \in \mathbb{R}_c(s),\ r_2(s) \neq 0 \text{ for } |s| \geq 1 \right\}.$$

V.2. *The ring of stable 1-D transfer functions*

Analogously to Section IV.2 we will prove in this section that $\mathbb{R}_\sigma(s)$ is a principal ideal domain. Because we have $\mathbb{R}_\sigma(s) \subset \mathbb{R}_c(s)$ we implicitly assume for $r_1(s)/r_2(s) \in \mathbb{R}_\sigma(s)$ that $r_1(s)$ and $r_2(s)$ are coprime and that $r_2(s)$ is monic.
We will also consider the following ring

$$(5.2.1) \quad \bar{\mathbb{R}}_\sigma(s) = \left\{ \frac{\bar{r}_1(s)}{\bar{r}_2(s)} \in \mathbb{R}(s) \;\middle|\; \bar{r}_2(s) \neq 0 \text{ for } |s| \leq 1 \right\}.$$

$\mathbb{R}_\sigma(s)$ is isomorphic to $\bar{\mathbb{R}}_\sigma(s)$ where the ring isomorphism is the same as in (4.2.2). Again $\mathbb{R}[s] \subset \bar{\mathbb{R}}_\sigma(s)$. The ring $\bar{\mathbb{R}}_\sigma(s)$ is a ring of fractions (see [77] or the appendix) where the multiplicative set D_σ is

$$(5.2.2) \quad D_\sigma = \{d(s) \mid d(s) \in \mathbb{R}[s],\ d(s) \neq 0 \text{ for } |s| \leq 1\}.$$

So $\bar{\mathbb{R}}_\sigma(s)$ is the ring $\mathbb{R}[s]_{D_\sigma}$ (see [77]). Because $\mathbb{R}[s]$ is a principal ideal domain, $\bar{\mathbb{R}}_\sigma(s)$ is also a principal ideal domain. Therefore $\mathbb{R}_\sigma(s)$ is also a principal ideal domain. Actually $\mathbb{R}_\sigma(s)$ is a Euclidean domain because $\bar{\mathbb{R}}_\sigma(s)$ is. This can be seen as follows. Suppose $n(s)/d(s) \in \bar{\mathbb{R}}_\sigma(s)$ and $n(s) = n_\sigma(s)n_*(s)$ where $n_\sigma(s) \in D_\sigma$ and $n_*(s)$ has only zeroes for $|s| > 1$.

Then a possible Euclidean function φ can be defined by

(5.2.3) $\quad \varphi(n(s)/d(s)) = \deg_s(n_*(s))$.

This can be shown as follows.
Let $r_1, r_2 \in \bar{\mathbb{R}}_\sigma(s)$. Then $r_1 = r_{1*} u_1$, $r_2 = r_{2*} u_2$ where r_{i*} is a polynomial for $i = 1,2$ and u_i is a unit in $\bar{\mathbb{R}}_\sigma(s)$ for $i = 1,2$. Suppose that $\varphi(r_1) \geq \varphi(r_2)$. This means that $\deg r_{1*} \geq \deg r_{2*}$. Therefore $r_{1*} = q\, r_{2*} + p$ and $\deg p < \deg r_{2*}$. Hence $r_1 = u_1 u_2^{-1} q\, r_2 + p\, u_1$ and we have proved that $r_1 = \tilde{q}\, r_2 + \tilde{p}$ where $\varphi(\tilde{p}) < \varphi(r_2)$.

V.3. *First level realizations of stable input/output systems and stabilization of 2-D systems*

In Section V.1 we saw that the transfer matrix of a stable 2-D system can be seen as a transfer matrix over $\mathbb{R}_\sigma(s)$ which is a principal ideal domain. Therefore the realization algorithms in Chapter III allow us to obtain a first level realization $\Sigma_\sigma = (A(s),B(s),C(s),D(s))$ over $\mathbb{R}_\sigma(s)$. This first level realization can be considered to be built out of stable 1-D transfer functions, which is very important for practical reasons. Suppose that we have a, not necessarily stable, 2-D transfer matrix $F(s,z)$ which can be considered a 1-D transfer matrix over $\mathbb{R}_\sigma(s)$. Observe that this is only a condition on the polynomial $q_n(s)$ where $F(s,z) = P(s,z)/q(s,z)$ and $q(s,z) = q_0(s) + q_1(s)z + \ldots + q_n(s)z^n$ ($P(s,z)$ is a polynomial matrix and $q(s,z)$ is the least common multiple of the denominators of the entries of $F(s,z)$. Let $\Sigma_\sigma = (A(s),B(s),C(s),D(s))$ be a canonical first level realization of $F(s,z)$ over $\mathbb{R}_\sigma(s)$. Then

$$(5.3.1) \qquad \begin{aligned} \bar{x}_{k+1}(s) &= A(s)\bar{x}_k(s) + B(s)\bar{u}_k(s) \ , \quad \bar{x}_0(s) = 0 \ , \\ \bar{y}_k(s) &= C(s)\bar{x}_k(s) + D(s)\bar{u}_k(s) \ , \quad k = 0,1,2,\ldots \ . \end{aligned}$$

Suppose that the state $\bar{x}_k(s)$ is the quantity that we want to know, but only the output can be measured. Then a possible approach to this problem is to construct an observer for this system (see [49] for the 1-D case). An approach to this 2-D observer problem can be found in [5].
In the present context this comes down to the construction of an $\mathbb{R}_\sigma(s)$ matrix $W(s)$ such that

$$[zI - A(s) + W(s)C(s)]^{-1}$$

is a stable 2-D transfer matrix, or equivalently

$$[zI - A(s)' + C(s)'W(s)']^{-1}$$

is a stable 2-D transfer matrix, Now it will be clear that the construction of an observer for $(A(s),B(s),C(s))$ is equivalent to the construction of a stabilizing regulator for the dual system $(A(s)',C(s)',B(s)')$. Because this last problem can be formulated more easily we will first consider this regulator problem.

If $F(s,z)$ is unstable, then we are interested in stabilizing (5.3.1) by means of feedback. This means that we try to find a matrix $K(s)$ over $\mathbb{R}_\sigma(s)$ such that if we choose $\bar{u}_k(s) = -K(s)\bar{x}_k(s)$ the new transfer matrix

$$C(s)[zI - A(s) + B(s)K(s)]^{-1} B(s) + D(s) \tag{5.3.2}$$

is a stable 2-D transfer matrix. We can achieve this in the following way. Suppose that $A(s) \in \mathbb{R}_\sigma(s)^{n\times n}$ and choose $p_1,\dots,p_n \in \mathbb{R}$ such that $(z-p_1)(z-p_2)\dots(z-p_n)$ is a stable polynomial, i.e.,

$$|p_i| < 1 \quad \text{for } i = 1,2,\dots,n . \tag{5.3.3}$$

Now $\mathbb{R}_\sigma(s)$ is a principal ideal domain by theorem (2.5.18). Hence there exists a matrix $K(s)$ such that $\det[zI - A(s) + B(s)K(s)] = (z-p_1)\dots(z-p_n)$. Therefore the system (5.3.1) can be stabilized by means of a feedback $\bar{u}_k(s) = -K(s)\bar{x}_k(s)$.

The matrix $K(s)$ can be given a dynamical interpretation by canonically realizing the 1-D transfer matrix $K(s)$ as follows:

$$\ell_{k,h+1} = K_A\,\ell_{kh} - K_B\,x_{kh}$$

$$u_{kh} = K_C\,\ell_{kh} - K_D\,x_{kh}$$

with appropriate dimensions and zero initial conditions.

The matrix K_A is stable (has only eigenvalues λ with $|\lambda| < 1$) because $K(s)$ is a stable transfer matrix. Observe that the feedback is composed of stable "building blocks".

Of course there is an observer counterpart to this stabilization technique. Here one starts with a first level realization $\Sigma_\sigma = (A(s),B(s),C(s),D(s))$

such that the dual system $\Sigma'_\sigma = (A(s)', C(s)', B(s)', D(s)')$ is canonical and one constructs a matrix $K(s)$ such that $\det[zI - A(s) + K(s)C(s))]$ is a stable polynomial, in the sense of (5.3.3). See [49] for more background information, [80] for applications and [55] for a construction method. Although the construction method in [55] for a feedback matrix $K(s)$ is for the case of systems over $\mathbb{R}[s]$ it can be generalized to the case of systems over $\mathbb{R}_\sigma(s)$ in a straightforward way.
In the next section we will consider more closely the rather restrictive reachability condition required in the above procedure. (Theorem (2.5.18) requires reachability of the pair $(A(s),B(s))$ over $\mathbb{R}_\sigma(s)$.)

V.4. *Canonical first level realizations over* $\mathbb{R}_c(s)$ *and* $\mathbb{R}_\sigma(s)$

In this section we give rather explicit conditions for the reachability and observability of systems over $\mathbb{R}_c(s)$ and $\mathbb{R}_\sigma(s)$. Reachability of a pair $(A(s),B(s))$ where $A(s) \in \mathbb{R}_c(s)^{n\times n}$ $(\mathbb{R}_\sigma(s)^{n\times n})$ and $B(s) \in \mathbb{R}_c(s)^{n\times p}$ $(\mathbb{R}_\sigma(s)^{n\times p})$ is equivalent to the existence of a matrix $L(s) \in \mathbb{R}_c(s)^{np\times n}$ $(\mathbb{R}_\sigma(s)^{np\times n})$ such that (see (2.5.8))

$$(5.4.1) \qquad [B(s), A(s)B(s), \ldots, A(s)^{n-1}B(s)]L(s) = I$$

because the columns of $[B(s), A(s)B(s), \ldots, A(s)^{n-1}B(s)]$ generate the standard basis vectors for $\mathbb{R}_c(s)^n$ $(\mathbb{R}_\sigma(s)^n)$ iff $(A(s),B(s))$ is reachable.
First we investigate the $\mathbb{R}_c(s)$ case. Suppose that $(A(s),B(s))$ is a reachable pair over $\mathbb{R}_c(s)$. Then we have the following theorem.

(5.4.2) THEOREM. $(A(s),B(s))$ *is a reachable pair over* $\mathbb{R}_c(s)$ *iff* (AD,BD) *is a reachable pair over* $\mathbb{R}$ *where* AD *and* BD *are the constant terms in the formal power series expansions of* $A(s)$ *and* $B(s)$ *respectively.*

PROOF. If $[B(s), A(s)B(s), \ldots, A(s)^{n-1}B(s)]L(s) = I$ for some $L(s)$, then $[B(s), A(s)B(s), \ldots, A(s)^{n-1}B(s)]L(s) = [BD, AD\,BD, \ldots, AD^{n-1}BD]LD = I$ where LD is the constant term in the formal power series expansion of $L(s)$. Therefore (AD,BD) is a reachable pair over $\mathbb{R}$. Now suppose that (AD,BD) is a reachable pair over $\mathbb{R}$ and let MD be such that $[BD, AD\,BD, \ldots, AD^{n-1}BD]MD = I$ then $[B(s), A(s)B(s), \ldots, A(s)^{n-1}B(s)]MD = I + M(s)$, where $M(s)$ is a strictly proper rational matrix in $\mathbb{R}_c(s)^{n\times n}$. Let $M(s)$ have a realization $M(s) = MC[sI - MA]^{-1}MB$ then $I + M(s)$ is invertible over $\mathbb{R}_c(s)$ and

$[I+M(s)]^{-1} = I - MC[sI - MA + MB\,MC]^{-1}MB$. We choose $L(s) = MD[I+M(s)]^{-1}$ and arrive at

$$[B(s), A(s)B(s), \dots, A(s)^{n-1}B(s)]L(s) = I\ .$$

Hence $(A(s),B(s))$ is a reachable pair over $\mathbb{R}_c(s)$. □

Now let $A(s)$ and $B(s)$ be matrices over $\mathbb{R}_\sigma(s)$. Then we have:

(5.4.3) THEOREM. $(A(s),B(s))$ *is a reachable pair over* $\mathbb{R}_\sigma(s)$ *iff* $(A(s),B(s))$ *is a reachable pair over* $\mathbb{R}_c(s)$ *and* $(A(s),B(s))$ *is a reachable pair (of complex matrices) for each* $|s| \geq 1$.

PROOF. Reachability over $\mathbb{R}_\sigma(s)$ is equivalent to (5.4.1) where $L(s)$ is a matrix over $\mathbb{R}_\sigma(s)$. This implies reachability over $\mathbb{R}_c(s)$ and reachability over $\mathbb{C}$ for $|s| \geq 1$. The proof of the "if" part is as follows. Suppose that $(A(s),B(s))$ is not a reachable pair over $\mathbb{R}_\sigma(s)$. Then we have that (see [72], [13]) the greatest common divisor $g(s)$ of all $n \times n$ minors is not invertible in the ring $\mathbb{R}_\sigma(s)$. This can occur only if $g(s)$ is not invertible in $\mathbb{R}_c(s)$ or if there exists s_0 such that $|s_0| \geq 1$ and $g(s_0) = 0$. In the former case $(A(s),B(s))$ is not reachable over $\mathbb{R}_c(s)$ and in the latter case all $n \times n$ minors are zero at $s = s_0$, therefore $(A(s_0),B(s_0))$ is not reachable over $\mathbb{C}$. □

The reachability of $(A(s),B(s))$ over $\mathbb{R}_\sigma(s)$ or $\mathbb{R}_c(s)$ can be characterized in another way in terms of polynomial matrices over $\mathbb{R}_\sigma(s)[z]$ or $\mathbb{R}_c(s)[z]$. The result will be stated in terms of matrices $A(s)$, $B(s)$ over $\mathbb{R}_\sigma(s)$. An analogous result can be stated for matrices over $\mathbb{R}_c(s)$.

(5.4.4) THEOREM. $(A(s),B(s))$ *is reachable over* $\mathbb{R}_\sigma(s)$ *iff* $[zI - A(s), B(s)]$ *is right invertible over* $\mathbb{R}_\sigma(s)[z]$, *i.e., there exist matrices* $P(z)$ *and* $Q(z)$ *over* $\mathbb{R}_\sigma(s)[z]$ *such that* $[zI - A(s)]P(z) + B(s)Q(z) = I$.

PROOF. Right invertibility of $[zI - A(s), B(s)]$ over $\mathbb{R}_\sigma(s)[z]$ is equivalent to right invertibility of the matrix (which contains n block rows where n is the dimension of $A(s)$)

$$(5.4.5)\qquad \begin{bmatrix} I & 0 & & & & & 0 & B(s)\\ -A(s) & I & \cdot & & & & & 0 \\ 0 & -A(s) & \ddots & & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \\ & & \ddots & \cdot & & & \cdot^{\cdot^{\cdot}} & \\ & \ddots & \ddots & I & 0 & B(s) & & \\ & & 0 & -A(s) & B(s) & 0 & & \end{bmatrix},$$

see [62]. The proof in [62] is for matrices over a field but it can be extended to the ring case in a straightforward way. Now right invertibility of (5.4.5) is equivalent to right invertibility of

$$(5.4.6)\qquad \left[\begin{array}{ccc|cccc} I & & & 0 & & \cdots & 0 \\ & \ddots & & \vdots & & & \vdots \\ & & I & 0 & & \cdots & 0 \\ \hline 0 & \cdots & 0 & B(s), & A(s)B(s) & \cdots & A(s)^{n-1}B(s) \end{array}\right].$$

The matrix (5.4.6) can be obtained from (5.4.5) by elementary row and column operations which proves the claimed equivalence. Finally, right invertibility of (5.4.6) is equivalent to reachability of $(A(s),B(s))$ over $\mathbb{R}_\sigma(s)$. □

Another way of expressing conditions for the reachability of $(A(s),B(s))$, which gives a more compact formulation, is by substituting $\frac{1}{s}$ for s in $A(s)$ and $B(s)$ (see also [16]). Suppose $A(s)$ and $B(s)$ are matrices over $\mathbb{R}_\sigma(s)$. Substitute $\frac{1}{s}$ for s and multiply with appropriate powers of s to obtain again rational matrices $\bar{A}(s)$, $\bar{B}(s)$ such that $\bar{A}(0)$ and $\bar{B}(0)$ are well defined. This can be achieved in a straightforward way. Then theorem (5.4.3) can be stated as

(5.4.7) THEOREM. $(A(s),B(s))$ *is a reachable pair over* $\mathbb{R}_\sigma(s)$ *iff* $(\bar{A}(s),\bar{B}(s))$ *is reachable over* $\mathbb{C}$ *for* $|s| \le 1$.

PROOF. The proof follows from the above considerations and will be omitted.□

Reachability of $(A(s),B(s))$ over $\mathbb{R}_\sigma(s)$ can also be expressed as

(5.4.8) THEOREM. $(A(s),B(s))$ *is a reachable pair over* $\mathbb{R}_\sigma(s)$ *iff* $[zI - \bar{A}(s),\bar{B}(s)]$ *has full row rank for* $|s| \le 1$ *and all* $z \in \mathbb{C}$.

PROOF. Combining theorems (5.4.4) and (5.4.7) immediately gives the result.□

Observability of a pair $(C(s),A(s))$ over $\mathbb{R}_\sigma(s)$ (see (2.5.9)) means that

(5.4.9) $\quad C(s)x(s) = C(s)A(s)x(s) = \ldots = C(s)A(s)^{n-1}x(s) = 0$

implies that $x(s) = 0$ (again n is the dimension of $A(s)$). This means that

$$(5.4.10) \qquad \begin{bmatrix} C(s) \\ C(s)A(s) \\ \vdots \\ C(s)A(s)^{n-1} \end{bmatrix}$$

is right regular over $\mathbb{R}_\sigma(s)$ (see (3.2.2)). This is equivalent to right regularity of (5.4.10) over the quotient field of $\mathbb{R}_\sigma(s)$ which is in fact $\mathbb{R}(s)$.

Concerning observability of $(C(s),A(s))$ over $\mathbb{R}_\sigma(s)$ we have the following theorem.

(5.4.11) THEOREM. $(C(s),A(s))$ *is an observable pair over* $\mathbb{R}_\sigma(s)$ *iff there exists a complex number* s_0 *such that* $C(s_0)$ *and* $A(s_0)$ *are well defined and* $(C(s_0),A(s_0))$ *is an observable pair over* $\mathbb{C}$.

PROOF. Suppose that (5.4.10) is right regular over $\mathbb{R}_\sigma(s)$ and hence over $\mathbb{R}(s)$. Then (5.4.10) is left invertible over $\mathbb{R}(s)$. Thus there exists a s_0 such that $(C(s_0),A(s_0))$ is an observable pair over $\mathbb{C}$. Now suppose that (5.4.10) is not right regular over $\mathbb{R}_\sigma(s)$. Then it is not right regular over $\mathbb{R}(s)$ and so we have that all $n \times n$-minors are zero. Thus for every $s_0 \in \mathbb{C}$ for which $C(s_0)$ and $A(s_0)$ are defined $(C(s_0),A(s_0))$ is not an observable pair over $\mathbb{C}$. □

REMARK. If $(C(s),A(s))$ is observable for some s_0, then this pair is observable for almost all $s \in \mathbb{C}$, because the points $s \in \mathbb{C}$, such that $C(s)$ and $A(s)$ are well defined, where observability fails to be hold are the zeroes of the $n \times n$ minors of (5.4.10). This is related to genericity which will be discussed briefly in Section V.9. □

Because observability of a pair $(C(s),A(s))$ is equivalent to right regularity of (5.4.10) over $\mathbb{R}(s)$ the above results concerning observability of a pair $(C(s),A(s))$ over $\mathbb{R}_\sigma(s)$ are also valid for matrices over $\mathbb{R}_c(s)$. ($\mathbb{R}_\sigma(s)$ and $\mathbb{R}_c(s)$ have the same quotient field, namely $\mathbb{R}(s)$.)

Observe that observability is very easily satisfied. Reachability over $\mathbb{R}_c(s)$ is a somewhat stronger condition while reachability over $\mathbb{R}_\sigma(s)$ is a very severe condition.

(5.4.12) EXAMPLE. Consider the pair $(A(s), B(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{s} & 1 & \frac{1}{s} \end{bmatrix} ; \quad B(s) = \begin{bmatrix} 0 \\ 0 \\ \frac{s+2}{2s+1} \end{bmatrix} .$$

This pair is reachable over $\mathbb{R}(s)$ as well as over $\mathbb{R}_c(s)$ but it is not reachable over $\mathbb{R}_\sigma(s)$. This can easily be seen because

$$\det[B(s), A(s)B(s), A(s)^2 B(s)] = \left(\frac{s+2}{2s+1}\right)^3$$

which is invertible in $\mathbb{R}(s)$ and in $\mathbb{R}_c(s)$ but not in $\mathbb{R}_\sigma(s)$.

V.5. *Separability of 2-D transfer matrices*

In this section we will be concerned with transfer matrices whose denominator polynomial is separable, i.e., is the product of a polynomial in s and a polynomial in z.

(5.5.1) DEFINITION. *A transfer matrix* $F(s,z) = P(s,z)/q(s,z)$, *where* $P(s,z)$ *and* $q(s,z)$ *are coprime, will be called separable iff* $q(s,z)$ *can be written as a product* $q(s,z) = q_1(s)q_2(z)$ *with* $q_1(s)$ *and* $q_2(z)$ *polynomials.* □

(5.5.2) LEMMA. *A transfer matrix* $F(s,z)$ *is separable iff there exists a first level realization* $\Sigma_s = (A, B(s), C(s), D(s))$ *where* A *is a constant matrix.*

PROOF. Suppose that $F(s,z) = P(s,z)/q_1(s)q_2(z)$. Now $P(s,z)$ can be written as

$$P(s,z) = P_1(s)P_2(z) .$$

This fact is an immediate generalization of the scalar case, for suppose that $p(s,z)$ is a two variable polynomial. Then $p(s,z)$ can be written as

$$p(s,z) = [p_{11}(s),\dots,p_{1r}(s)]\begin{bmatrix} p_{21}(z) \\ \vdots \\ p_{2r}(z) \end{bmatrix}$$

where r is the rank of the matrix of coefficients of p(s,z). From this the proof of the lemma follows immediately. □

From lemma (5.5.2) it is clear that a separable F(s,z) can be realized as a cascade connection of two 1-D systems with dynamics in different directions because

$$F(s,z) = \frac{p_1(s)}{q_1(s)}\ \frac{p_2(z)}{q_2(z)} \ .$$

(5.5.3) DEFINITION. *Two systems* $\Sigma_1 = (A_1(s),B_1(s),C_1(s),D_1(s))$ *and* $\Sigma_2 = (A_2(s),B_2(s),C_2(s),D_2(s))$ *are feedback equivalent if there exists a state space isomorphism* T(s) *and a feedback matrix* K(s) *such that (see* [10]*)*

1° $\quad A_1(s) = T(s)^{-1}[A_2(s) - B_2(s)K(s)]T(s)$,

2° $\quad B_1(s) = T(s)^{-1}B_2(s)$,

3° $\quad C_1(s) = C_2(s)T(s)$,

4° $\quad D_1(s) = D_2(s)$. □

(5.5.4) DEFINITION. *Two transfer matrices* $F_1(s,z)$ *and* $F_2(s,z)$ *are called feedback equivalent if* $F_1(s,z)$ *and* $F_2(s,z)$ *have first level realizations* Σ_1 *and* Σ_2 *respectively, which are feedback equivalent in the sense of* (5.5.3). □

We will now derive a result on feedback equivalence.

(5.5.5) THEOREM. *Every reachable system* $\Sigma_1 = (A_1(s),B_1(s),C_1(s),D_1(s))$ *over* $\mathbb{R}_c(s)$ *is feedback equivalent to a system* $\Sigma_2 = (A_2,B_2(s),C_2(s),D_2(s))$ *where* A_2 *is a matrix over* $\mathbb{R}$.

PROOF. Because $\mathbb{R}_c(s)$ is a local ring we can apply theorem (2.5.21), a generalization of Heymann's Lemma, and obtain a matrix K(s) and a vector u(s) such that (A(s) - B(s)K(s) , B(s)u(s)) is a reachable pair. This K(s) and u(s) are very easily found if one notes that $(A_1(s),B_1(s))$ is a reachable pair over $\mathbb{R}_c(s)$ iff (see (5.4.2)) (AD_1, BD_1) is a reachable pair over $\mathbb{R}$. Therefore there exist a matrix K and a vector u over $\mathbb{R}$ such that $(AD_1 - BD_1K, BD_1u)$ is reachable. Such a K and u can be constructed using the algorithm in [34]. Now, again by theorem (5.4.2) $(A_1(s) - B_1(s)K, B_1(s)u)$ is reachable over $\mathbb{R}_c(s)$ and can therefore be transformed into control canonical form. Thus there exists a state space isomorphism T(s) such that (see [18])

$$T(s)^{-1}[A_1(s) - B_1(s)K]T(s)$$

is a companion matrix, thus has the form

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \\ -\alpha_0(s) & & \cdots & & -\alpha_{n-1}(s) \end{bmatrix}$$

where $\alpha_i(s) \in \mathbb{R}_c(s)$ for $i = 0,\ldots,n-1$, and n is the dimension of Σ_1, and

$$T(s)^{-1}B_1(s)u = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .$$

It is clear that there exists a row vector $k(s) = (k_0(s),\ldots,k_{n-1}(s)) \in$ $\in \mathbb{R}_c(s)^{1\times n}$ such that

$$(5.5.6) \qquad T(s)^{-1}[A_1(s) - B_1(s)K]T(s) - T(s)^{-1}B_1(s)u\,k(s)$$

is a matrix having entries in $\mathbb{R}$. The matrices, required for feedback equivalence, are T(s) and $K(s) = K + uk(s)T(s)^{-1}$. The matrix A_2 in Σ_2 is given by (5.5.6). This proves the theorem. □

REMARK. An algorithm to determine the state space isomorphism can be given analogous to the 1-D case, see for instance [12]. □

A transfer matrix $F_1(s,z)$ can be transformed into a separable transfer matrix $F_2(s,z)$ by just applying state feedback to a reachable realization of $F_1(s,z)$. For if $(A_1(s),B_1(s),C_1(s),D_1(s))$ is a reachable realization of

$F_1(s,z)$, then there exists a matrix $K(s)$ such that $[A_1(s) - B_1(s)K(s)]$ has a characteristic polynomial which does not depend on s. This can be achieved because a reachable system is pole assignable (see (2.5.18)). In the case of a system over $\mathbb{R}_c(s)$ this can be achieved by the method described in the proof of theorem (5.5.5) and in the case of a system over $\mathbb{R}_\sigma(s)$ the construction method in [55] gives the result. In the $\mathbb{R}_\sigma(s)$-case $F_2(s,z)$ cannot be guaranteed to have a realization $(A_2,B_2(s),C_2(s),D_2(s))$ where A_2 is a matrix over $\mathbb{R}$ and has the same dimension as $A_1(s)$, whereas in the $\mathbb{R}_c(s)$ case this can be achieved, as is obvious from theorem (5.5.5). We then have the following theorem.

(5.5.7) THEOREM. *Every* 2-D *transfer matrix is feedback equivalent to a separable* 2-D *transfer matrix.*

PROOF. This follows immediately from the foregoing considerations. □

REMARK. Because separable transfer matrices possess a somewhat simpler structure the above results may be useful in designing regulators and observers. The separability of transfer matrices may be advantageous in the construction of second level realizations (see also Section V.8). □

EXAMPLE. Consider the transfer matrix

$$F(s,z) = [s+s^2, -1+(1+s)z]/(1-s(s+1)z+(s^2-1)z^2) \; .$$

A first level canonical realization is

$$(A_1(s),B_1(s),C_1(s),D_1(s)) = \left(\begin{bmatrix} 1 & \frac{1}{s+1} \\ \frac{s}{s-1} & \frac{1}{s-1} \end{bmatrix}, \begin{bmatrix} 1 & \frac{1}{s+1} \\ 0 & \frac{1}{s-1} \end{bmatrix}, [0,1], 0\right)$$

$$AD_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad BD_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} .$$

If we take

$$K = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

then

$$(AD_1 - BD_1K\,,\, BD_1u) = \left(\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

and

$$(A_1(s) - B_1(s)K\,,\, B_1(s)u) = \left(\begin{bmatrix} 2 & \frac{2}{s+1} \\ \frac{s}{s-1} & \frac{2}{s-1} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

which is a reachable pair. Now using the same construction as for systems over a field we get

$$T(s) = \begin{bmatrix} \frac{-2}{s-1} & 1 \\ \frac{s}{s-1} & 0 \end{bmatrix}, \quad T(s)^{-1} = \begin{bmatrix} 0 & \frac{s-1}{s} \\ 1 & \frac{2}{s} \end{bmatrix}$$

and

$$(T(s)^{-1}[A_1(s) - B_1(s)K]T(s)\,,\, T(s)^{-1}B_1(s)u) = \left(\begin{bmatrix} 0 & 1 \\ \frac{-2s-4}{s^2-1} & \frac{2s}{s-1} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

If we choose

$$k(s) = \left[\frac{2s-4}{s^2-1}, \frac{2s}{s-1}\right]$$

then

$$K(s) = \begin{bmatrix} \frac{s+1}{s-1} & \frac{2s^2+2s+4}{s^2-s} \\ 0 & -1 \end{bmatrix}$$

and $F(s,z)$ is feedback equivalent to a separable transfer matrix $C(s)[zI - A]^{-1}\, B(s) + D(s)$, where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 & \frac{1}{s} \\ 1 & \frac{s^2+s+2}{s^3-s} \end{bmatrix},$$

$$C(s) = \left[\frac{s}{s-1}, 0\right], \quad D(s) = 0$$

V.6. *Invertibility of 2-D input/output systems*

In this section we will be concerned with invertibility of scalar weakly causal 2-D input/output systems (see also [20]). The concept of inverse system becomes important if one wants to know the input of a system in the case where the output is available. Applications of inverse 2-D systems can be found for instance in [70], [38], where they were used for the restoration of degraded images. Most of the theorems will be given in terms of transfer functions. The associated impulse response, or equivalently the formal power series expansion of the transfer function, will be very helpful in proving the theorems.

(5.6.1) DEFINITION. *Suppose* $F(s,z)$ *is a weakly causal transfer function. Then a weakly causal transfer function* $G(s,z)$ *is an inverse of* $F(s,z)$ *if*

1° *There exists a causality cone* C_c *such that* $S_F \subset C_c$, $S_G \subset C_c$.

2° $G(s,z)F(s,z) = 1$. □

Usually this inverse $G(s,z)$ is denoted by $F(s,z)^{-1}$.
Condition 1° is part of the definition in order to be able to define the multiplication in 2° properly.

(5.6.2) THEOREM. *Suppose that* $F(s,z) \in \hat{S}_{p,q,r,t}$. *Then*

1° $F(s,z)$ *has a weakly causal inverse iff* $F_{00} \neq 0$.

2° *If* $F(s,z)$ *has a weakly causal inverse* $F(s,z)^{-1}$ *then* $F(s,z)^{-1} \in \hat{S}_{p,q,r,t}$.

PROOF. $\hat{S}_{p,q,r,t}$ is isomorphic to $\hat{S}_{1,0,0,1}$, the isomorphism being Φ, as defined in the proof of (4.4.11). Now $\Phi(F)(\alpha,\beta)$ is invertible iff $F_{00} \neq 0$ (see [81]). From this 1° and 2° follow immediately. □

If a second level realization for an invertible transfer function $F(s,z)$ ($F_{00} \neq 0$) is given, then one can immediately write down a second level realization for $F(s,z)^{-1}$.
Suppose that a second level realization is given in the form (4.4.1), (4.4.16)

$$\begin{bmatrix} \beta(x)_{kh} \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh} ,$$

$$y_{kh} = [C_1, C_2]\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D\,u_{kh}\,.$$

Then $D = F_{00}$ and a second level realization for the inverse system is

(5.6.3)
$$\begin{bmatrix} \beta(x)_{kh} \\ \alpha(a)_{kh} \end{bmatrix} = \begin{bmatrix} A_1 - B_1D^{-1}C_1 & A_2 - B_1D^{-1}C_1 \\ A_3 - B_2D^{-1}C_1 & A_4 - B_2D^{-1}C_2 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1D^{-1} \\ B_2D^{-1} \end{bmatrix} y_{kh}\,,$$

$$u_{kh} = [-D^{-1}C_1\,,\,-D^{-1}C_2]\begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D^{-1}\,y_{kh}\,.$$

Theorem (5.6.2) gives a positive result only if $F_{00} \neq 0$. In that case a weakly causal inverse exists. If this condition is not satisfied it is possible to introduce a generalized notion of inverse system (compare [51], [67]) which still can be given a useful interpretation. The idea is the following. If one is interested in the input of a system, given the output, but the system is not invertible then one might be satisfied if one could obtain a shifted (in some direction) version of the input. If one has applications in mind in the field of image restoration (see [70]), then this may be a satisfactory solution.

By a shifted version of the input $(u_{k,h})$ we mean $(u_{k+m,h+n})$ for some integers m, n. In terms of formal power series representations this is translated into:

By a shifted version of u(s,z) we mean $z^{-m}s^{-n}u(s,z)$ for some integers m, n.

For these reasons we will now consider inverses with inherent delay (short w.i.d.). See also [51], [67].

(5.6.4) DEFINITION. *Suppose that* F(s,z) *is a weakly causal transfer function. Then a weakly causal transfer function* G(s,z) *is said to be an inverse with inherent delay* (M,N) *if*

1° *there exists a causality cone* C_c *such that* $S_F \subset C_c$, $S_G \subset C_c$, $(M,N) \in C_c$,

2° $G(s,z)F(s,z) = z^{-M}s^{-N}$. □

(5.6.5) REMARK. The condition 2° is an immediate generalization of the 1-D counterpart, while 1° is incorporated in the definition because otherwise the product $G(s,z)F(s,z)$ cannot be well defined (in such a way that it still represents the convolution of the impulse responses G and F associated with $G(s,z)$ and $F(s,z)$). For instance, take $F(s,z) = \frac{1}{s-z}$ and $G(s,z) = \frac{1}{s+z}$ as representations for the formal power series

$$F(s,z) = \frac{1}{s} + \frac{z}{s^2} + \frac{z^2}{s^3} + \ldots$$

$$G(s,z) = \frac{1}{z} - \frac{s}{z^2} + \frac{s^2}{z^3} - \ldots .$$

Here the product of the two formal power series (they do not share a common causality cone) cannot be well defined whereas $\frac{1}{s-z}\,\frac{1}{s+z} = \frac{1}{s^2-z^2}$. □

Now we have (for Φ and φ, see (4.4.11), (4.4.8) respectively)

(5.6.6) THEOREM. *Suppose that* $F(s,z)$ *is a weakly causal transfer function and that* $\Phi(F)(\alpha,\beta) = T(\alpha,\beta)$ *is a causal transfer function. If* $U(\alpha,\beta)$ *is a weakly causal inverse of* $T(\alpha,\beta)$ *w.i.d.* (M',N')*, then* $\Phi^{-1}(U)(s,z)$ *is a weakly causal inverse of* $F(s,z)$ *w.i.d.* $(M,N) = \varphi^{-1}(M',N')$.

PROOF. Suppose that

$$U(\alpha,\beta)T(\alpha,\beta) = \beta^{-M'}\alpha^{-N'} ,$$

then

$$\Phi^{-1}(UT)(s,z) = z^{-M}s^{-N}$$

or

$$\Phi^{-1}(U)(s,z)\Phi^{-1}(T)(s,z) = z^{-M}s^{-N} .$$ □

By the above theorem it is clear that, for the construction of inverses w.i.d. for weakly causal systems, we can restrict ourselves to causal systems. Now, one might expect (as in the 1-D case) every non-zero causal transfer function to have a causal inverse w.i.d. However, this is not the case. A condition for this to be true is as follows.

(5.6.7) THEOREM. *Suppose that* $F(s,z)$ *is a causal transfer function, so that*

$$F(s,z) = \frac{p_0(s) + p_1(s)z + \ldots + p_m(s)z^m}{q_0(s) + q_1(s)z + \ldots + q_n(s)z^n},$$

$p_m(s) \neq 0$, $q_n(s)$ *is monic, thus* $\neq 0$. *Furthermore by causality* $n \geq m$ *and*

$$\deg(q_n(s)) \geq \deg(q_i(s)), \quad i = 0,\ldots,n-1,$$

$$\deg(q_n(s)) \geq \deg(p_j(s)), \quad j = 0,\ldots,m.$$

Then $F(s,z)$ *has a causal inverse w.i.d. iff*

$$\deg(p_m(s)) \geq \deg(p_j(s)), \quad j = 0,\ldots,m-1.$$

PROOF. Let $M = n - m$ and $N = \deg(q_n(s)) - \deg(p_m(s))$. Then $F(s,z)z^M s^N$ is invertible and the inverse is causal. Therefore $F(s,z)$ has a causal inverse w.i.d. (M,N). Conversely, suppose that $G(s,z)$ is a causal inverse w.i.d. (M,N). Then $M \geq 0$ and $N \geq 0$. Now $G(s,z)F(s,z) = z^{-M}s^{-N}$ which implies $\deg(p_m(s)) \geq \deg(p_j(s))$, $j = 0,\ldots,m-1$. □

Observe that for every (M_1,N_1) such that $M_1 \geq M$ and $N_1 \geq N$ there exists a causal transfer function which can serve as an inverse w.i.d. (M_1,N_1). Here (M,N) is the inherent delay in the proof of theorem (5.6.7).
Furthermore the inverse w.i.d. (M,N) is invertible (without delay). By theorem (5.6.7) not every non-zero causal transfer function has a causal inverse w.i.d., but we will show in the next that every causal transfer function does have a weakly causal inverse w.i.d., which itself is invertible without delay.
Suppose that $F(s,z)$ is a causal transfer function. Let $S_F \subset \mathbb{R}_+^2$ be its support and define $\text{conv}^+ S_F$ by

(5.6.8) $$\text{conv}^+ S_F = \text{conv } S_F + \mathbb{R}_+^2$$

where conv S_F denotes the convex hull of S_F (the intersection of all convex sets containing S_F) and again we use row vector notation for $\mathbb{R}_+^2$.
Furthermore, let (M,N) be an extremal point of $\text{conv}^+ S_F$ (a point such that $\text{conv}^+ S_F \setminus (M,N)$ is still convex). Then it is clear that $(M,N) \in S_F$ (see also [6]).
Furthermore, $H(s,z) = F(s,z)z^M s^N$ is a weakly causal transfer function and $H_{00} \neq 0$. Hence, by theorem (5.6.2), $H(s,z)$ has a weakly causal inverse.

Therefore $F(s,z)$ has a weakly causal inverse w.i.d. (M,N) which itself has an inverse (without delay) namely $H(s,z)^{-1}$.
Summarizing we have

(5.6.9) THEOREM. *Suppose that* $F(s,z)$ *is a causal transfer function. Let* (M,N) *be an extremal point of* $\text{conv}^+ S_F$. *Then there exists an inverse w.i.d.* (M,N) *which itself is an invertible weakly causal transfer function.* □

Some geometrical insight is provided by figure 6.

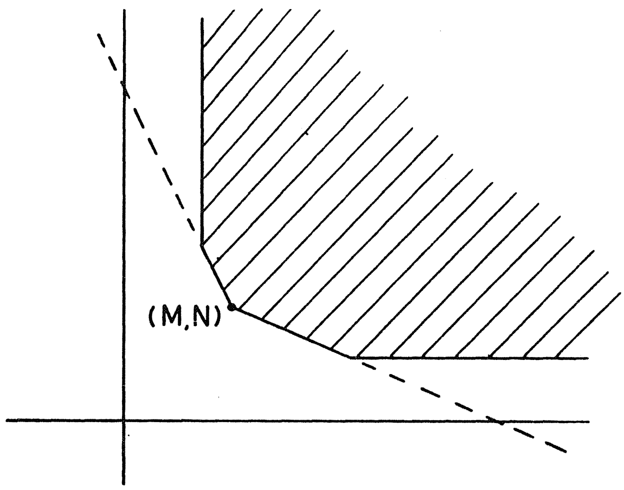

Figure 6

The shaded area denotes $\text{conv}^+ S_F$, the dotted lines correspond to a possible shifted cone giving rise to the required causality cone.
In fact the above theorem says the following. Look for a point (M,N) in S_F such that there exists a causality cone (a closed convex cone satisfying (4.4.5) will also suffice because of lemma (4.4.8)) with the property that if we shift it in a way such that the origin $(0,0)$ becomes (M,N) the support S_F is still contained in the shifted causality cone. Then (M,N) is a possible candidate for the inherent delay. Furthermore, this holds for all extremal points of $\text{conv}^+ S_F$.

(5.6.10) REMARK. The extremal points of $\text{conv}^+ S_F$ are the analogues of the minimal delay in the 1-D case, because they give rise to inverse transfer functions which are invertible without delay. □

In the 1-D case every delay larger than the minimal delay may serve as an inherent delay for some inverse (which is not necessarily invertible without delay). Because of the lack of a natural order in the 2-D case an inverse with minimal delay is not well defined. Therefore we will characterize all the possible delays corresponding to weakly causal inverses of some causal transfer function. The construction of possible inverses with inherent delay will be based on theorem (5.6.9). The following theorem enables us to construct more inverses w.i.d. whenever one inverse w.i.d. (which is itself invertible), based on an extremal point of $\text{conv}^+ S_F$, is known.

(5.6.11) THEOREM. *Suppose that* $G(s,z)$ *is a weakly causal invertible transfer function. Let* $S_G \subset C$ *where* C *is a closed convex cone satisfying* (4.4.5). *Let* $(M,N) \in \mathbb{Z}^2 \setminus (-C \cap \mathbb{Z}^2)$. *Then* $G(s,z)z^{-M}s^{-N}$ *is a weakly causal transfer function.*
If $M \le 0$ *and* $N \le 0$, $(M,N) \ne (0,0)$, *then* $G(s,z)z^{-M}s^{-N}$ *is not weakly causal.*

PROOF. If $G(s,z)$ is invertible then $G_{00} \ne 0$. This means that for every $(M,N) \in \mathbb{Z}^2 \setminus (-C \cap \mathbb{Z}^2)$ there exists a causality cone such that if we shift it to (M,N) it still contains S_G. Therefore $G(s,z)z^{-M}s^{-N}$ is weakly causal. The second assertion follows from $G_{00} \ne 0$. □

Now consider a causal transfer function $F(s,z)$.
Let $\mathcal{M}$ and $\mathcal{N}$ denote the sets

(5.6.12)
$$\mathcal{M} = \{M \mid (M,N) \in \text{conv}^+ S_F \text{ for some integer } N\} ,$$
$$\mathcal{N} = \{N \mid (M,N) \in \text{conv}^+ S_F \text{ for some integer } M\} .$$

Let $\bar{M}$ and $\bar{N}$ be defined by

(5.6.13) $\bar{M} = \min_{M \in \mathcal{M}} M , \quad \bar{N} = \min_{N \in \mathcal{N}} N .$

We can now characterize the set of possible inherent delays corresponding to some causal transfer function.

(5.6.14) THEOREM. *Suppose that* F(s,z) *is a causal transfer function. Let* $\bar{M}$, $\bar{N}$ *be defined as in* (5.6.13). *Then we have*

1° *If* $M > \bar{M}$ *or* $N > \bar{N}$ *there exists a weakly causal inverse* $G_{M,N}(s,z)$ *w.i.d.* (M,N).
If $M \leq \bar{M}$ *and* $N \leq \bar{N}$ *and* $(M,N) \neq (\bar{M},\bar{N})$ *there does not exist a weakly causal inverse w.i.d.* (M,N).

2° $G_{M,N}(s,z)$ *is invertible (without delay) iff* (M,N) *is an extremal point of* $\text{conv}^+ S_F$.

3° $G_{M,N}(s,z)$ *is causal iff* $(\bar{M},\bar{N}) \in \text{conv}^+ S_F$ *and* $M \geq \bar{M}$, $N \geq \bar{N}$.

PROOF. Applying theorem (5.6.11) to $F(s,z)z^M s^N$ for every extremal point (M,N) of $\text{conv}^+ S_F$ gives the proof of 1°. The proof of 2° follows from theorem (5.6.9). The proof of 3° follows from theorem (5.6.7). □

(5.6.15) EXAMPLE. Suppose we have a causal input/output system with transfer function $F(s,z) = \dfrac{s+z}{-sz+(s-1)z^2}$. Observe that F(s,z) does not satisfy the conditions of theorem (5.6.7) so that F(s,z) does not have a causal inverse. Now consider figure 7.

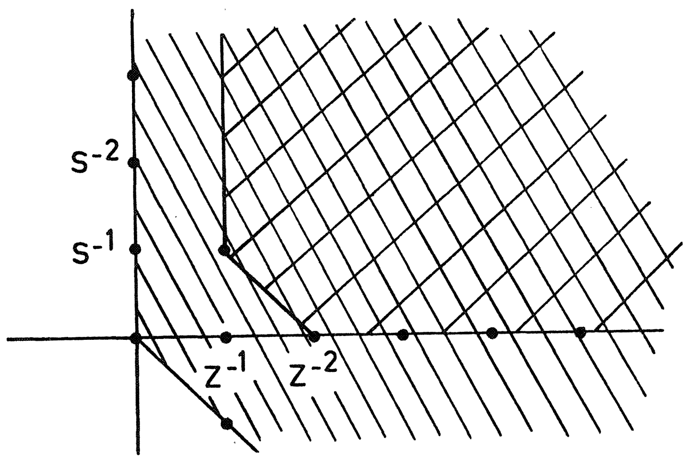

Figure 7

Conv$^+$ S_F is the double shaded area, $(\bar{M},\bar{N}) = (1,0)$. The extremal points of conv$^+$ S_F are (2,0) and (1.1). By theorem (5.6.14) there exists a weakly causal inverse w.i.d. (0,1). Indeed, if we take

$$G_{0,1}(s,z) = \frac{-sz + (s-1)z^2}{sz + s^2} ,$$

then $G_{0,1}(s,z)F(s,z) = s^{-1}$ and $G_{0,1}(s,z)$ is weakly causal, for if we substitute (see lemma (4.4.15))

$$s = \alpha\beta^2 , \quad z = \beta ,$$

then

$$G_{0,1}(\alpha\beta^2,\beta) = \frac{-\alpha\beta + \alpha\beta^2 - 1}{\alpha\beta + \alpha^2\beta^2}$$

which is a causal transfer function.

By theorem (5.6.14) there exists a weakly causal inverse w.i.d. (1,1) which is invertible itself without delay. The weakly causal inverse w.i.d. (1,1) is

$$G_{1,1}(s,z) = \frac{-s + (s-1)z}{sz + s^2} .$$

$G_{1,1}(s,z)$ is weakly causal, for if we substitute

$$s = \alpha , \quad z = \alpha\beta ,$$

then

$$G_{1,1}(\alpha,\alpha\beta) = \frac{-1 + (\alpha-1)\beta}{\alpha + \alpha\beta}$$

which is causal and invertible.

The shaded area in figure 7 denotes the causality cone of $G_{1,1}(s,z)$.

A second level realization of this transfer function can be found in (4.4.23).

Observe that $S_F \subset \mathbb{Z}_+^2 \subset S_{G_{1,1}}$. At first stage y_{kh} is defined for $(k,h) \in \mathbb{Z}_+^2$. We have to add zeroes in the sense that $y_{kh} = 0$ for $(k,h) \in S_{G_{1,1}} \setminus \mathbb{Z}_+^2$.

Again one should bear in mind that in an expression like $y(s,z) = F(s,z)u(s,z)$ the product is only defined in the case where $y(s,z)$, $F(s,z)$, $u(s,z)$ have their support in the same causality cone (belong to the same ring). See also (5.6.5).

(5.6.16) REMARK. In this section it was shown that weakly causal 2-D systems arise in a natural way if one is interested in inverse systems. This is one of the reasons to study weakly causal systems and the generalized state space equations associated with them. □

The transformation of a weakly causal input/output system, to obtain a causal one, can be seen as a unimodular transformation of an integer lattice (compare [50]). This observation may be useful if one is interested in further generalization.

The results of this section concerning inverses and inverses w.i.d. can be generalized to the multivariable case (m outputs and p inputs) if one supposes $D = F_{00}$ to be left invertible (which also requires $m \geq p$). In the case of an inverse w.i.d. the results can be generalized if F_{MN} is supposed to be left invertible, where (M,N) denotes an extremal point of $\mathrm{conv}^+ S_F$.

To the author's knowledge the more general case has not yet been solved.

V.7. *Reachability and observability of second level realizations*

In the literature some attempts have been made to give useful definitions of reachability and observability of local state space models which appear as second level realizations of 2-D systems. See for instance [61], [48], [24]. In [48] polynomial matrices in two variables are used to give definitions for these notions where the local state space model is taken to be Roesser's model (see 4.3.8).

In this section we will indicate some connections between [48] and the approach to 2-D systems as is developed in this thesis (see also [17]). The local state space model will be Roesser's which is equivalent to our second level realization (4.3.6). We will write down the equations for this model again for convenience:

(5.7.1)
$$\begin{bmatrix} x_{k+1,h} \\ a_{k,h+1} \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u_{kh} ,$$
$$y_{kh} = [C_1 , C_2] \begin{bmatrix} x_{kh} \\ a_{kh} \end{bmatrix} + D\, u_{kh} .$$

In [48] the concepts of modal controllability and modal observability are defined for (5.7.1). The concepts are also related to "minimality" of the local state space model which will be a topic in the next section. To state these definitions we need the notion of coprimeness of polynomial matrices in two variables.

(5.7.2) DEFINITION. *Two matrices* $P(s,z)$ *and* $Q(s,z)$ *over* $\mathbb{R}[s,z]$ *with the same number of rows are called left coprime with respect to* $\mathbb{C}[s,z]$ *if for every left common factor* $D(s,z)$ *such that* $P(s,z) = D(s,z)\tilde{P}(s,z)$ *and* $Q(s,z) = D(s,z)\tilde{Q}(s,z)$ *where* $D(s,z)$, $\tilde{P}(s,z)$ *and* $\tilde{Q}(s,z)$ *are matrices over* $\mathbb{C}[s,z]$ *and* $D(s,z)$ *is a square matrix, we necessarily have that* $\det(D(s,z))$ *is a complex number* $d \neq 0$. □

In this definition we have to work with polynomials with complex coefficients because the field, where the coefficients belong to, has to be algebraically closed. Right coprimeness is defined analogously (see [54]). Now we can state the definitions of modal controllability and modal observability.

(5.7.3) DEFINITION. *The system* (5.7.1) *is modally controllable if*

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

are left coprime with respect to $\mathbb{C}[s,z]$. □

(5.7.4) DEFINITION. *The system* (5.7.1) *is modally observable if*

$$[C_1, C_2] \quad \text{and} \quad \begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix}$$

are right coprime with respect to $\mathbb{C}[s,z]$. □

Furthermore we need the following definition where we suppose that K is a field.

(5.7.5) DEFINITION. *Two matrices* $P(z)$ *and* $Q(z)$ *over* $K[z]$ *with the same number of rows are called left coprime with respect to* $K[z]$ *if for every left common factor* $D(z)$ *such that* $P(z) = D(z)\tilde{P}(z)$ *and* $Q(z) = D(z)\tilde{Q}(z)$ *where*

$D(z)$, $\tilde{P}(z)$ *and* $\tilde{Q}(z)$ *are matrices over* $K[z]$ *and* $D(z)$ *is a square matrix, we necessarily have that* $\det(D(z))$ *is a unit in* $K[z]$. □

In [54] the following theorem is obtained.

(5.7.6) THEOREM. *The matrices*

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

are left coprime with respect to $\mathbb{C}[s,z]$ *iff they are left coprime with respect to* $\mathbb{C}(s)[z]$ *and also with respect to* $\mathbb{C}(z)[s]$. □

It can easily be seen that if A_1, A_2, A_3, A_4, B_1, B_2 are real matrices, which will always be presupposed whenever we are dealing with the system (5.7.1), we may take $\mathbb{R}(s)[z]$ and $\mathbb{R}(z)[s]$ instead of $\mathbb{C}(s)[z]$ and $\mathbb{C}(z)[s]$ respectively in theorem (5.7.6).
Theorem (5.7.6) gives a result which is related to modal controllability of (5.7.1). An analogous result can be stated for modal observability.
Theorem (5.7.6) will enable us to relate modal controllability and modal observability of (5.7.1) to reachability and observability respectively of a first level realization for the transfer matrix of (5.7.1). A possible first level realization over $\mathbb{R}_c(s)$ (see 4.3.9)) for the transfer matrix $F(s,z)$ of (5.7.1) is $\Sigma = (A(s),B(s),C(s),D(s))$, where

$$\begin{aligned} A(s) &= A_1 + A_2[sI-A_4]^{-1} A_3 \\ B(s) &= B_1 + A_2[sI-A_4]^{-1} B_2 \\ C(s) &= C_1 + C_2[sI-A_4]^{-1} A_3 \\ D(s) &= D + C_2[sI-A_4]^{-1} B_2 \,. \end{aligned} \tag{5.7.7}$$

Recall that this means that

$$F(s,z) = D + [C_1,C_2]\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix}^{-1}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = D(s) + C(s)[zI-A(s)]^{-1} B(s) \;.$$

By interchanging the role of s and z we can construct another first level realization. A possible first level realization over $\mathbb{R}_c(z)$ is then

$\bar{\Sigma} = (\bar{A}(z), \bar{B}(z), \bar{C}(z), \bar{D}(z))$, where

$$(5.7.8) \qquad \begin{aligned} \bar{A}(z) &= A_4 + A_3[zI - A_1]^{-1} A_2 \\ \bar{B}(z) &= B_2 + A_3[zI - A_1]^{-1} B_1 \\ \bar{C}(z) &= C_2 + C_1[zI - A_1]^{-1} A_2 \\ \bar{D}(z) &= D + C_1[zI - A_1]^{-1} B_1 \ . \end{aligned}$$

Now, using theorem (5.7.6) and the first level realizations (5.7.7), (5.7.8), we can state a theorem which links the approach to 2-D systems of [61] and ours.

(5.7.9) THEOREM. *The system* (5.7.1) *is modally controllable iff*

1° $(A_1 + A_2[sI - A_4]^{-1} A_3, B_1 + A_2[sI - A_4]^{-1} B_2)$ *is a reachable pair over* $\mathbb{R}(s)$

and

2° $(A_4 + A_3[zI - A_1]^{-1} A_2, B_2 + A_3[zI - A_1]^{-1} B_1)$ *is a reachable pair over* $\mathbb{R}(z)$.

PROOF.

$$\begin{bmatrix} zI-A_1 & -A_2 & B_1 \\ -A_3 & sI-A_4 & B_2 \end{bmatrix} = \begin{bmatrix} I & -A_2 \\ 0 & sI-A_4 \end{bmatrix} \begin{bmatrix} zI-A(s) & 0 & B(s) \\ -[sI-A_4]^{-1}A_3 & I & [sI-A_4]^{-1}B_2 \end{bmatrix}$$

where $A(s)$ and $B(s)$ are defined as in (5.7.7). If $(A(s),B(s))$ is not a reachable pair over $\mathbb{R}(s)$ then $[zI - A(s), B(s)]$ is not right invertible over $\mathbb{R}(s)[z]$ (see theorem (5.4.4), this theorem is stated for systems over $\mathbb{R}_\sigma(s)$ but is also true for systems over $\mathbb{R}(s)$). Because $\mathbb{R}(s)[z]$ is a principal ideal domain (or, equivalently, because $\mathbb{R}(s)$ is a field) this means that $zI - A(s)$ and $B(s)$ have a non-unimodular left common factor $D(s,z)$, see [78]. This means that $D(s,z)$ is a square matrix over $\mathbb{R}(s)[z]$ and furthermore $\deg_z[\det(D(s,z))] \geq 1$. By the above factorization it is clear that

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

also have a non-unimodular left common factor, namely

$$\begin{bmatrix} D(s,z) & -A_2 \\ 0 & sI-A_4 \end{bmatrix}$$

which implies that they are not left coprime with respect to $\mathbb{C}(s)[z]$. By theorem (5.7.6) the system (5.7.1) is not modally controllable. Failure of 2° gives the same result in an analogous way and therefore necessity has been proven.

The condition 1° together with 2° is also sufficient. Suppose that 1° is true. Then $[zI - A(s), B(s)]$ is right invertible. Therefore there exist matrices L and Q over $\mathbb{R}(s)[z]$ such that

$$[zI - A(s)]L + B(s)Q = I .$$

Now it is straightforward to verify that

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix} \begin{bmatrix} L & LA_2[sI-A_4]^{-1} \\ [sI-A_4]^{-1}[A_3L-B_2Q] & [sI-A_4]^{-1}[A_3L-B_2Q]A_2[sI-A_4]^{-1}+[sI-A_4]^{-1} \end{bmatrix} +$$

$$+ \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} Q , & QA_2[sI-A_4]^{-1} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} .$$

This implies left coprimeness of

$$\begin{bmatrix} zI-A_1 & -A_2 \\ -A_3 & sI-A_4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

with respect to $\mathbb{C}(s)[z]$. In the same way 2° implies left coprimeness with respect to $\mathbb{C}(z)[s]$ and therefore modal controllability of (5.7.1) has been proven. □

By duality an analogous result can be proven for modal observability.

(5.7.10) THEOREM. *The system* (5.7.1) *is modally observable iff*

1° $(C_1 + C_2[sI - A_4]^{-1} A_3, A_1 + A_2[sI - A_4]^{-1} A_3)$ *is an observable pair over* $\mathbb{R}(s)$

and

2° $(C_2 + C_1[zI - A_1]^{-1} A_2, A_4 + A_3[zI - A_1]^{-1} A_2)$ *is an observable pair over* $\mathbb{R}(s)$.

□

V.8. *Low order second level realizations*

While minimality of a first level realization of a 2-D input/output system is a well defined concept, for second level realizations it is not clear how this notion should be defined. For first level realizations we have that a system is minimal if the dimension of the state space is minimal. This property is satisfied if one deals with a canonical realization. Therefore one can test a first level realization for minimality by checking whether or not it is reachable and observable. This idea was used in [24], [48], [61] to obtain minimal second level realizations, in the sense that the authors defined reachability and observability for the models they used but these notions did not imply the intuitive property of minimality. Of course one would expect the notion of minimality of a second level realization to agree with "smallest possible number of equations that define the model". However, up to now, conditions on a second level realization which imply this kind of minimality have not appeared in the literature.
In [48] the authors defined minimality of a local state space system by requiring the model to have the properties of modal controllability and modal observability. However, this definition does not agree with the intuitive notion of minimality. In an important special case, the case of a transfer function with separable numerator or separable denominator, the intuitive notion of minimality and the concept of minimality as defined in [48] coincide. The reason for this is that if a local state space model of the form (5.7.1) realizing a transfer function has the properties of modal controllability and modal observability, then we must have that A_1 is an $n \times n$ matrix and A_4 is an $m \times m$ matrix where n is the degree in z of the denominator polynomial and m is the degree in s of the denominator polynomial of the transfer function. In the case where the numerator or the denominator of the transfer function is separable it is indeed possible to construct a local state space model where A_1 is an $n \times n$ matrix and A_4 is an $m \times m$ matrix (n and m have the same meaning as above). For the more general case (not requiring the numerator or the denominator to be separable) it is not known whether or not there exists a realization (second level) with $n+m$ states. For this reason there has been a search for low order second level realizations. In [48] and [74] it was shown that a transfer function $F(s,z) = p(s,z)/q(s,z)$ with $\deg_z(q(s,z)) = n$ and $\deg_s(q(s,z)) = m$ could be (second level) realized in the form of a Roesser model (5.7.1), where

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

is a matrix with dimension := min(n+2m,m+2n). In [19] an improvement on this result was published, which we will explain here in more detail. Consider therefore a 2-D causal transfer function F(s,z)

$$(5.8.1) \qquad F(s,z) = \frac{p(s,z)}{q(s,z)} = \frac{p_0(s) + p_1(s)z + \ldots + p_n(s)z^n}{q_0(s) + q_1(s)z + \ldots + q_n(s)z^n} .$$

Here p(s,z) and q(s,z) are coprime two variable real polynomials, $p_i(s)$, $q_i(s)$ are polynomials in s for i = 0,...,n.
Recall that causality means that $q_n(s)$ is a monic polynomial (thus $\neq 0$) and that the following degree conditions are satisfied

$$\deg(q_n(s)) \geq \deg(q_i(s)) , \quad i = 0,\ldots,n-1 ,$$

$$\deg(q_n(s)) \geq \deg(p_i(s)) , \quad i = 0,\ldots,n .$$

Let us assume that

$$\deg(q_n(s)) = m .$$

We will now explicitly describe the construction of a local state space model for F(s,z) in the form of the model as in (5.7.1). As in Chapter IV section 3 a first level realization will be an intermediate step in this construction.
Let $\varphi(s)$ be a greatest common divisor of $p_0(s),\ldots,p_n(s)$. Let $\deg(\varphi(s)) := \ell$. This factor is called the *content* of p(s,z) and $\bar{p}(s,z)$, defined by $p(s,z) = \varphi(s)\bar{p}(s,z)$, is called the *primitive part* of p(s,z). See also [77].
We factorize $q_n(s)$ as follows

$$(5.8.2) \qquad q_n(s) = \psi_1(s)\psi_2(s)$$

such that $\psi_1(s)$ and $\psi_2(s)$ are monic polynomials satisfying

$$(5.8.3) \qquad \deg(\psi_2(s)) \geq \deg(\varphi(s)) , \quad \deg(\psi_1(s)) \geq \max_i \deg(p_i(s)) - \ell .$$

If a factorization as in (5.8.2) satisfying (5.8.3) does not exist such that $\psi_2(s)$ is a real polynomial, we proceed as follows (observe that this can happen only in the case where $\varphi(s)$ is a polynomial with odd degree).

Let $\bar{\varphi}(s)$ be a common factor of $p_0(s),\dots,p_n(s)$ such that $\deg(\bar{\varphi}(s)) = \ell - 1$. Now a factorization as in (5.8.2) in real polynomials such that $\deg(\psi_2(s)) = \deg(\bar{\varphi}(s))$ does exist and we can use $\bar{\varphi}(s)$ instead of $\varphi(s)$ in the following.

Let $p_i(s) = \varphi(s)\bar{p}_i(s)$, $i = 0,\dots,n$. It is clear that $\Sigma = (A(s),B(s),C(s),D(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \\ \dfrac{-q_0(s)}{q_n(s)} & & \cdots & & \dfrac{-q_{n-1}(s)}{q_n(s)} \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \dfrac{\varphi(s)}{\psi_2(s)} \end{bmatrix}, \tag{5.8.4}$$

$$C(s) = \left[\frac{\bar{p}_0(s)}{\psi_1(s)},\dots,\frac{\bar{p}_{n-1}(s)}{\psi_1(s)}\right] + \frac{\bar{p}_n(s)}{\psi_1(s)}\left[\frac{-q_0(s)}{q_n(s)},\dots,\frac{-q_{n-1}(s)}{q_n(s)}\right],$$

$$D(s) = \frac{p_n(s)}{q_n(s)}$$

is a first level realization of $F(s,z)$.

We will use the notation

$$\bar{A}(s) = \left[\frac{-q_0(s)}{q_n(s)},\dots,\frac{-q_{n-1}(s)}{q_n(s)}\right], \quad \bar{B}(s) = \frac{\varphi(s)}{\psi_2(s)},$$

$$\bar{C}(s) = \left[\frac{\bar{p}_0(s)}{\psi_1(s)},\dots,\frac{\bar{p}_{n-1}(s)}{\psi_1(s)}\right], \quad \bar{D}(s) = \frac{\bar{p}_n(s)}{\psi_1(s)}.$$

We can realize these 1-D transfer matrices and obtain the following realizations

$(\overline{AA},\overline{AB},\overline{AC},\overline{AD})$ for $\bar{A}(s)$

$(\overline{BA},\overline{BB},\overline{BC},\overline{BD})$ for $\bar{B}(s)$

$(\overline{CA},\overline{CB},\overline{CC},\overline{CD})$ for $\bar{C}(s)$

$(\overline{DA},\overline{DB},\overline{DC},\overline{DD})$ for $\bar{D}(s)$

(DA,DB,DC,DD) for $D(s)$.

It is clear that we can do this in a way such that

$$(5.8.5) \qquad \begin{aligned} &\overline{AA} = \overline{BA} = DA\ , \quad \overline{CA} = \overline{DA}\ , \\ &\overline{AC} = \overline{BC} = DC\ , \quad \overline{CC} = \overline{DC}\ . \end{aligned}$$

This can be seen as follows.
A 1-D transfer function

$$T(s) = \frac{b_0 + b_1 s + \ldots + b_{n-1}s^{n-1}}{a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n}$$

has a realization $\Sigma = (A,B,C,D)$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & & \ldots & 0 & 1 \\ -a_0 & & \ldots & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C = [\, b_0 \ , \ \ldots \ , \ b_{n-1}] \ , \quad D = 0\ .$$

Hence $\Sigma' = (A',C',B',D')$ is also a realization. This shows that the matrix B' can be chosen independent from the coefficients of the numerator polynomial and A' depends only on the coefficients of the denominator polynomial. This can be generalized to single output systems and henceforth (5.8.5) is possible.

To obtain (5.8.5) we can also use the so called *standard observable realization* (see [9]).

These realizations can be "tied together" to form a realization of the type (5.7.1) in the following way

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ & 0 & & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ & & \overline{AD} & & \end{bmatrix}_{(n \times n)}, \quad A_2 = \begin{bmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \ldots & 0 & & & \\ & \overline{AC} & & 0 & \ldots & 0 \end{bmatrix}_{(n \times (2m-\ell))}$$

$$(5.8.6) \qquad A_3 = \begin{bmatrix} & \overline{AB} & \\ \overline{DB} & \overline{AD} + \overline{CB} \end{bmatrix}_{((2m-\ell) \times n)}, \quad A_4 = \begin{bmatrix} \overline{AA} & 0 \\ \overline{DB}\ \overline{AC} & \overline{CA} \end{bmatrix}_{((2m-\ell) \times (2m-\ell))}$$

$$C_1 = [\overline{CD} + \overline{DD}\ \overline{AD}]_{(1\times n)} \ , \quad C_2 = [\overline{DD}\ \overline{AC}\ , \overline{CC}]_{(1\times(2m-\ell))}$$

$$B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \overline{BD} \end{bmatrix}_{(n\times 1)} \ , \quad B_2 = \begin{bmatrix} \overline{BB} \\ \overline{DB}\ \overline{BD} \end{bmatrix}_{((2m-\ell)\times 1)} \ , \quad D = DD \ .$$

Again $\overline{AA}$, $\overline{AB}$, DD etc. are single matrices, not products of matrices. (In particular D = DD should *not* be read as $D = D^2$.)
Observe that if p(s,z) is a primitive polynomial, then the construction gives a realization with dimension $n+2m$ (see also [48], [74]) and in the case $p(s,z) = \varphi(s)\varphi_1(z)$ (separable numerator) we obtain a realization of dimension $n+m$.
It is a matter of straightforward verification that

$$A(s) = A_1 + A_2[sI - A_4]^{-1} A_3 \ , \quad B(s) = B_1 + A_2[sI - A_4]^{-1} B_2 \ ,$$

$$C(s) = C_1 + C_2[sI - A_4]^{-1} A_3 \ , \quad D(s) = D + C_2[sI - A_4]^{-1} B_2 \ ,$$

as is required. See (4.3.9).
Summarizing we have

(5.8.7) THEOREM. *Let* $F(s,z) = p(s,z)/q(s,z)$ *be a causal* 2-D *transfer function. Suppose that* $\varphi(s)$ *is the content of* p(s,z) *and* $\deg(\varphi(s)) = \ell$. *Then there exists a realization of the form* (5.7.1) *with dimension* $n+2m-\ell$. *This realization is possibly in complex numbers (depending upon the factorization* (5.8.2)*) but there exists always a real realization with dimension* $n+2m-\ell+1$ *where* $\ell \geq 1$. □

(5.8.8) REMARK. By interchanging s and z the same kind of result can be obtained and one can take the minimum of the two for the dimension of a local state space realization. □

We will now derive an analogous result for the denominator case. Suppose that $\psi(s)$ is the content of q(s,z) and let the degree of $\psi(s)$ be r. Let $q_i(s) = \psi(s)\bar{q}_i(s)$ for $i = 0,\dots,n$. A first level realization of F(s,z) is then $\Sigma = (A(s),B(s),C(s),D(s))$ where

$$A(s) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & & \cdots & 0 & 1 \\ \frac{-\bar{q}_0(s)}{\bar{q}_n(s)} & & \cdots & & \frac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)} \end{bmatrix}, \quad B(s) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C(s) = \left[\frac{p_0(s)}{q_n(s)}, \ldots, \frac{p_{n-1}(s)}{q_n(s)}\right] + \frac{p_n(s)}{q_n(s)} \left[\frac{-\bar{q}_0(s)}{\bar{q}_n(s)}, \ldots, \frac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)}\right],$$

$$D(s) = \frac{p_n(s)}{q_n(s)} .$$

Let

$$\tilde{A}(s) = \left[\frac{-\bar{q}_0(s)}{\bar{q}_n(s)}, \ldots, \frac{-\bar{q}_{n-1}(s)}{\bar{q}_n(s)}\right], \quad \tilde{B}(s) = 1 ,$$

$$\tilde{C}(s) = \left[\frac{p_0(s)}{q_n(s)}, \ldots, \frac{p_n(s)}{q_n(s)}\right], \quad \tilde{D}(s) = \frac{p_n(s)}{q_n(s)} .$$

We can now proceed in completely the same way as in (5.8.6) and obtain an analogous result.
Observe that the realization we obtain in this way is always real.
Hence we have

(5.8.9) THEOREM. *Let* $F(s,z) = p(s,z)/q(s,z)$ *be a causal* 2-D *transfer function. Suppose that* $\psi(s)$ *is the content of* $q(s,z)$ *and* $\deg(\psi(s)) = r$. *Then there exists a real realization of the form* (5.7.1) *with dimension* $n + 2m - r$. □

This is a generalization of the separable denominator result.

(5.8.10) REMARK. Again, by interchanging s and z, the same kind of result can be obtained and the minimum of the two can be taken as the dimension of a local state space realization. □

Summarizing, we can state that the dimension of a local state space model which is a second level realization of a causal 2-D transfer function can be taken the minimum of the numbers given by theorems (5.8.7) and (5.8.9) and the remarks (5.8.8) and (5.8.10).

V.9. *Generic properties*

In many cases one is interested in questions like: Is some property E easily satisfied? This is especially the case if E is one of the conditions for a theorem to be true or a condition such that an algorithm can be executed. We will be concerned with properties which can be viewed as a mapping $E_a\colon \mathbb{R}^N \to \{\text{"true"},\text{"false"}\}$ where N is the number of parameters which are used to describe the situation on which the property E is to be tested. Then one looks for the set $K(E_a) = \{x \in \mathbb{R}^N \mid E_a(x) = \text{"false"}\}$. If $K(E_a)$ is small in some sense, then the property E is said to be generically satisfied. This ideas will now be presented in a rigorous way.

(5.9.1) DEFINITION. *A subset* $K \subset \mathbb{R}^N$ *is said to be Zariski closed if there exists a real polynomial* p *in* N *variables such that* $(x_1,\ldots,x_N)' \in K$ *iff* $p(x_1,\ldots,x_N) = 0$. *The complement of* K *is called a Zariski open subset of* $\mathbb{R}^N$. □

By using this definition of Zariski open and Zariski closed sets we can define a topology for $\mathbb{R}^N$, the so called Zariski topology (see [32]), which is a powerful tool in algebraic geometry.
It can be proved that Zariski closed sets $\neq \mathbb{R}^N$ have Lebesgue measure zero and may therefore be considered "small".
Consider $\mathbb{R}^N$ for some positive number N. Let E be a property that a point in $\mathbb{R}^N$ may have. In other words, consider the mapping

$$E_a\colon \mathbb{R}^N \to \{\text{"true"},\text{"false"}\}$$

defined by

$$E_a((x_1,\ldots,x_N)') = \text{"true"} \quad \text{if } (x_1,\ldots,x_N)' \text{ has property E,}$$
$$E_a((x_1,\ldots,x_N)') = \text{"false" otherwise.}$$

We will now define the concept of generic property.

(5.9.2) DEFINITION. *A property* E *is called a generic property if* $K(E_a)$ *is contained in a Zariski closed set* $\neq \mathbb{R}^N$, *where* $K(E_a)$ *is defined as above.* □

Intuitively this means that if E is a generic property, then E will be satisfied for almost every point $(x_1,\ldots,x_N)' \in \mathbb{R}^N$. Furthermore, for almost every point $(x_1,\ldots,x_N)' \in \mathbb{R}^N$ there exists an open set U (in the standard

topology on $\mathbb{R}^N$) containing $(x_1,\dots,x_N)'$ such that E is satisfied for all points in U.
We will now prove that reachability of a pair $(A(s),B(s))$ over $\mathbb{R}_c(s)$ is a generic property. We will suppose that $A(s) = \bar{A}(s)/a(s)$ where $\deg(a(s)) \le d_a$, $\deg(\bar{A}(s)) \le d_a$, $d_a > 0$ ($\bar{A}(s)$ is a polynomial matrix and $a(s)$ is a monic polynomial) and $B(s) = \bar{B}(s)/b(s)$ where $\deg(b(s)) \le d_b$, $\deg(\bar{B}(s)) \le d_b$, $d_b > 0$ ($\bar{B}(s)$ is a polynomial matrix and $b(s)$ is a monic polynomial). Therefore $(A(s),B(s))$ is specified by a point in $\mathbb{R}^{N_1}$ where

$$N_1 = n^2(d_a+1) + np(d_b+1) + d_a + d_b \ .$$

(5.9.3) REMARK. If we do not suppose $(A(s),B(s))$ to be given in the form $(\bar{A}(s)/a(s),\bar{B}(s)/b(s))$, we can also deal with the case where we suppose that for every entry $a_{ij}(s)/\alpha_{ij}(s)$ we have that $\deg(a_{ij}(s)) \le \delta_a$, $\deg(\alpha_{ij}(s)) \le \delta_a$, $\delta_a > 0$ for all i and j, and for every entry $b_{ij}(s)/\beta_{ij}(s)$ we suppose that $\deg(b_{ij}(s)) \le \delta_b$, $\deg(\beta_{ij}(s)) \le \delta_b$, $\delta_b > 0$ for all i and j. In this case the number N_2, the dimension of the parameter space, has to be chosen as

$$N_2 = n^2(2\delta_a+1) + np(2\delta_b+1) \ . \qquad \square$$

(5.9.4) THEOREM. *Let* $A(s) \in \mathbb{R}_c(s)^{n\times n}$ *and* $B(s) \in \mathbb{R}_c(s)^{n\times p}$ *be given. Then reachability over* $\mathbb{R}_c(s)$ *of the pair* $(A(s),B(s))$ *is a generic property if* $p \ge 1$.

PROOF. By theorem (5.4.2) reachability of $(A(s),B(s))$ over $\mathbb{R}_c(s)$ is equivalent to reachability of (AD,BD) over $\mathbb{R}$ where $AD = \lim_{s\to\infty} A(s)$, $BD = \lim_{s\to\infty} B(s)$. Reachability of (AD,BD) over $\mathbb{R}$ only fails if all $n\times n$ minors of $[BD, AD\,BD,\dots,AD^{n-1}BD]$ are zero. This corresponds to a Zariski closed set K_0 in $\mathbb{R}^{N_0}$ where $N_0 = n(n+m)$. The set K_0 is also a Zariski closed set in $\mathbb{R}^{N_1}$ and also in $\mathbb{R}^{N_2}$ where N_1 and N_2 are defined above. Furthermore, $K_0 \ne \mathbb{R}^{N_0}$ because there exist reachable pairs. This proves the theorem. $\square$

(5.9.5) THEOREM. *Let* $A(s) \in \mathbb{R}_c(s)^{n\times n}$, $B(s) \in \mathbb{R}_c(s)^{n\times p}$ *be given. Then reachability over* $\mathbb{R}(s)$ *of the pair* $(A(s),B(s))$ *is a generic property if* $p \ge 1$.

PROOF. By (5.9.4) we have that reachability over $\mathbb{R}_c(s)$ is a generic property. Hence reachability over $\mathbb{R}(s)$ is a generic property because $\mathbb{R}_c(s) \subset \mathbb{R}(s)$. $\square$

Observe that theorem (5.9.5) can also be proved by using the fact that reachability over $\mathbb{R}_c(s)$ is a stronger property than reachability over $\mathbb{R}(s)$.

By duality it is clear that observability is also a generic property (again $m \geq 1$).

We will now show that modal controllability and modal observability for a local state space model as given in (5.7.1) are generic properties.

(5.9.6) THEOREM. *Consider a local state space model given by the equations* (5.7.1). *Then modal controllability of the pair*

$$\left(\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right)$$

and modal observability of the pair

$$\left([C_1, C_2], \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\right)$$

are generic properties if $p \geq 1$ *and* $m \geq 1$, *respectively.*

PROOF. Combining theorem (5.9.5) and theorem (5.7.9) immediately gives the result. By duality the result concerning modal observability is also clear. □

Summarizing we may say that the properties of reachability over $\mathbb{R}_c(s)$, reachability over $\mathbb{R}(s)$, observability over $\mathbb{R}_c(s)$ (which is the same as observability over $\mathbb{R}(s)$), observability over $\mathbb{R}(s)$, modal controllability over $\mathbb{R}$, modal observability over $\mathbb{R}$ are easily satisfied.

VI *REALIZATION ALGORITHMS FOR 2-D SYSTEMS*

VI.1. *Introduction*

In Chapter III we developed some realization algorithms for systems over a principal ideal domain. These algorithms can be applied to 2-D systems which can be viewed as systems over a principal ideal domain. This was shown in Chapter IV. One of the most important steps in the algorithms was the factorization of a matrix over a principal ideal domain to obtain a modified Smith form or a modified Hermite form for such a matrix. In section IV.2 we showed that $\mathbb{R}_c(s)$ is in fact a Euclidean domain and in section V.2 we showed that $\mathbb{R}_\sigma(s)$ is also a Euclidean domain. Because $\mathbb{R}_c(s)$ and $\mathbb{R}_\sigma(s)$ occur as coefficient rings for 2-D systems the factorization of matrices which occur in realization problems for 2-D systems can be obtained by just applying elementary column operations and elementary row operations (see (3.1.3)). However, the factorization of a matrix by means of elementary column operations and elementary row operations is a rather complicated process. Therefore we will make use of some special properties of $\mathbb{R}_c(s)$ and $\mathbb{R}_\sigma(s)$ which simplify the factorization problem considerably.

VI.2. *Factorization algorithms for matrices over* $\mathbb{R}_c(s)$ *or* $\mathbb{R}_\sigma(s)$

First we will be concerned with the factorization of matrices over $\mathbb{R}_c(s)$. The ring $\mathbb{R}_c(s)$ is a local ring (see [8]) whose maximal ideal is generated by $\frac{1}{s}$. Therefore we have that, for two elements a and b in $\mathbb{R}_c(s)$, $a \mid b$ or $b \mid a$ where $a \mid b$ means that a divides b.
Let $\varphi\colon \mathbb{R}_c(s) \to \mathbb{Z}_+$ be the Euclidean function defined by

$$\varphi(n(s)/d(s)) = \deg(d(s)) - \deg(n(s)) \ .$$

This is the same function as is used in section IV.2.

Consider an $n \times p$ matrix $M(s)$ over $\mathbb{R}_c(s)$

$$M(s) = \begin{bmatrix} m_{11}(s) & \cdots & m_{1p}(s) \\ \vdots & & \vdots \\ m_{n1}(s) & \cdots & m_{np}(s) \end{bmatrix} .$$

We will describe a series of elementary column operations and elementary row operations that reduces M(s) to a modified Smith form (see (3.1.6)). Suppose that $M(s) \neq 0$ (otherwise M(s) is already a modified Smith form). Then we may assume $m_{11}(s) \neq 0$ for a non-zero entry of M(s) can be brought to the left upper most position by row and column permutations. Furthermore it can be assumed that $m_{11}(s)$ has minimal φ value among the entries of the first row and first column. Therefore we have that $m_{11}(s) \mid m_{1j}(s)$ for $j = 2,\ldots,p$ and $m_{11}(s) \mid m_{i1}(s)$ for $i = 2,\ldots,n$. By adding suitable multiples of the first column to the other columns and by adding suitable multiples of the first row to the other rows one can obtain a matrix $M_1(s)$, equivalent to M(s), which has the form

$$(6.2.1) \qquad M_1(s) = \begin{bmatrix} m_{11}(s) & 0 & \cdots & 0 \\ 0 & \bar{m}_{22}(s) & \cdots & \bar{m}_{2p}(s) \\ \vdots & \vdots & & \vdots \\ 0 & \bar{m}_{n2}(s) & \cdots & \bar{m}_{np}(s) \end{bmatrix} = \begin{bmatrix} m_{11}(s) & 0 \\ 0 & \bar{M}(s) \end{bmatrix} .$$

The described procedure can be repeated by performing elementary row and elementary column operations on $\bar{M}(s)$. Eventually this gives a matrix $\bar{D}(s)$ (two sided) equivalent to M(s), which can be written as

$$\bar{D}(s) = \begin{bmatrix} D(s) & 0 \\ 0 & 0 \end{bmatrix}$$

where D(s) is a regular diagonal matrix and some of the zero matrices are possibly empty. $\bar{D}(s)$ is a modified Smith form for M(s).

It is clear that we can obtain a modified Hermite form in an analogous way (see (3.1.5)).

The above shows that the determination of a modified Smith form or a modified Hermite form, necessary for applying the realization algorithms of Chapter III, can be done rather easily for a matrix over $\mathbb{R}_c(s)$ (at least compared to the general case of a matrix over a Euclidean domain) by exploiting the fact that $\mathbb{R}_c(s)$ is a local ring. The matrix $\bar{D}(s)$, (two sided) equivalent to M(s), can be obtained almost as easily as if $\mathbb{R}_c(s)$ were a field. The difference is that in this case the element with the lowest φ value in the first column and the first row of M(s) has to be transferred to the (1,1) position. Afterwards one can proceed as in the field case in order to obtain (6.2.1) etc.

We will now describe how to obtain a modified Smith form for a polynomial matrix $P(s) \in \mathbb{R}[z]^{n \times m}$. This will be needed in order to construct a modified Smith form for a matrix over $\mathbb{R}_\sigma(s)$. Consider

$$P(s) = \begin{bmatrix} p_{11}(s) & \cdots & p_{1m}(s) \\ \vdots & & \vdots \\ p_{n1}(s) & \cdots & p_{nm}(s) \end{bmatrix} .$$

Because $\mathbb{R}[s]$ is a Euclidean domain we can obtain a modified Smith form by just applying elementary column operations and elementary row operations. Of course the Euclidean function is the degree function. We proceed as follows.

If $P(s) = 0$, then we already have a modified Smith form. If $P(s) \neq 0$, we may assume $p_{11}(s) \neq 0$ for a non-zero entry of P(s) can be brought to the left upper most position by applying row and column permutations. Furthermore we may assume that $p_{11}(s)$ has lowest degree among the entries of the first row and the first column, for this can also be obtained by means of row and column permutations. Then we have

$$p_{1j}(s) = p_{11}(s)q_{1j}(s) + r_{1j}(s) \;,\; p_{i1}(s) = p_{11}(s)q_{i1}(s) + r_{i1}(s)$$

where $\deg(r_{1j}(s)) < \deg(p_{11}(s))$ and $\deg(r_{i1}(s)) < \deg(p_{11}(s))$.

Now, by adding suitable multiples of the first column to the other columns and by adding suitable multiples of the first row to the other rows one can obtain a matrix $\tilde{P}(s)$, equivalent to P(s),

$$\tilde{P}(s) = \begin{bmatrix} p_{11}(s) & r_{12}(s) & \cdots & r_{1m}(s) \\ r_{21}(s) & \tilde{p}_{22}(s) & \cdots & \tilde{p}_{2m}(s) \\ \vdots & \vdots & & \vdots \\ r_{n1}(s) & \tilde{p}_{n2}(s) & \cdots & \tilde{p}_{nm}(s) \end{bmatrix} .$$

Again we can obtain, by applying suitable row and column permutations, that in the left upper most position we have an entry, unequal to zero, which has lowest degree among the entries in the first row and the first column. Now we can apply elementary row and elementary column operations just like we did for P(s) and this process can be repeated. Furthermore, this process will end for eventually we end up with a situation where all entries in the first column and the first row are zero except for the entry in the (1,1) place, because every cycle of this process will give entries in the first row and the first column that have a degree less than the degree of the entry in position (1,1). Hence we end up with a matrix $P_1(s)$ where

$$(6.2.2) \qquad P_1(s) = \begin{bmatrix} \tilde{p}_{11}(s) & 0 & \cdots & 0 \\ 0 & \bar{p}_{22}(s) & \cdots & \bar{p}_{2m}(s) \\ \vdots & \vdots & & \vdots \\ 0 & \bar{p}_{n2}(s) & \cdots & \bar{p}_{nm}(s) \end{bmatrix} = \begin{bmatrix} \tilde{p}_{11}(s) & 0 \\ 0 & \bar{P}(s) \end{bmatrix} .$$

We can continue the algorithm by replacing P(s) by $\bar{P}(s)$ and eventually we end up with a matrix $\bar{Q}(s)$, equivalent to P(s), which can be written as

$$\bar{Q}(s) = \begin{bmatrix} Q(s) & 0 \\ 0 & 0 \end{bmatrix}$$

where Q(s) is a regular diagonal matrix and some of the zero matrices are possibly empty. $\bar{Q}(s)$ is a modified Smith form for P(s).
Again, it is clear that we can obtain a modified Hermite form in an analogous way (see (3.1.5)).
This diagonalization algorithm can be executed somewhat easier than an algorithm to obtain the Smith form of a matrix because we are not interested in the usual divisibility properties of the diagonal elements of Q(s) (see [57]).
Henceforth we will be concerned with matrices over $\mathbb{R}_\sigma(s)$ and we will describe a factorization algorithm that constructs a modified Smith form for this kind of matrices.
Of course we can obtain a modified Smith form for a matrix over $\mathbb{R}_\sigma(s)$ by applying elementary row operations and elementary column operations analogous to the case of a polynomial matrix as is described above. In this case the Euclidean function, defined in section V.2, plays the role of the degree function in the factorization algorithm above. However, this approach to the factorization of a matrix over $\mathbb{R}_\sigma(s)$ is very involved and

computationally unattractive. Therefore we will reduce the factorization problem for a matrix over $\mathbb{R}_\sigma(s)$ to the case of a polynomial matrix. Let $M(s)$ be a $n \times p$ matrix over $\mathbb{R}_\sigma(s)$. As in IV.2 let the ring isomorphism $S: \mathbb{R}_\sigma(s) \to \bar{\mathbb{R}}_\sigma(s)$ be defined by

$$\bar{r}(s) = S(r(s)) = r(1/s) \ .$$

See (4.2.1) for the definition of $\bar{\mathbb{R}}_\sigma(s)$.
Let $\bar{M}(s)$ be the matrix over $\bar{\mathbb{R}}_\sigma(s)$ obtained by applying S to each entry of $M(s)$. Suppose that $\bar{m}(s)$ is the least common multiple of the denominators of the entries of $\bar{M}(s)$. Let $P(s) := \bar{m}(s)\bar{M}(s)$, then $P(s)$ is a polynomial matrix. Using the method described above for computing a modified Smith form for a polynomial matrix we obtain unimodular matrices (over $\mathbb{R}[s]$) $U(s)$ and $V(s)$, stemming from the elementary row operations and elementary column operations, and a modified Smith form $\tilde{Q}(s)$ such that

$$(6.2.3) \qquad P(s) = \bar{U}(s)\,\tilde{Q}(s)\,\bar{V}(s) \ .$$

If we define $\bar{Q}(s) := \tilde{Q}(s)/\bar{m}(s)$, then we have

$$(6.2.4) \qquad \bar{M}(s) = \bar{U}(s)\,\bar{Q}(s)\,\bar{V}(s) \ .$$

$\bar{U}(s)$ and $\bar{V}(s)$ are unimodular matrices over $\mathbb{R}[z]$. Therefore they also are unimodular over $\bar{\mathbb{R}}_\sigma(s)$ because $\mathbb{R}[s] \subset \bar{\mathbb{R}}_\sigma(s)$ (see V.2). Hence $\bar{Q}(s)$ is a modified Smith form of $\bar{M}(s)$. To obtain a modified Smith form of $M(s)$ we apply S^{-1} to all entries of $\bar{U}(s)$, $\bar{Q}(s)$ and $\bar{V}(s)$. This gives $U(s) = \bar{U}(1/s)$, $V(s) = \bar{V}(1/s)$, $Q(s) = \bar{Q}(1/s)$ and

$$(6.2.5) \qquad M(s) = U(s)\,Q(s)\,V(s) \ .$$

Hence we have obtained a modified Smith form by just performing calculations over $\mathbb{R}[s]$, which seems to be more attractive than doing calculations over $\mathbb{R}_\sigma(s)$.
Completely similarly, we can reduce the computation of a modified Hermite form over $\mathbb{R}_\sigma(s)$ to the $\mathbb{R}[s]$-case.

We will not consider numerical aspects of these factorization algorithms and problems of storage of intermediate results. This will be a topic for further investigation. The use of residue class computations is another topic which seems to be promising.

With the aid of the above described factorization algorithms we can compute a first level realization for a 2-D input/output system given by its impulse response. If the 2-D system is given by its transfer matrix, then we can compute the impulse response and apply a realization algorithm as if the impulse response were given. Alternatively we can first compute a realization over the quotient field and afterwards reduce it to a ring realization. This procedure also involves the determination of a modified Hermite form or a modified Smith form (see (3.3.10)). Here we can also benefit from the algorithms described in this chapter. If we want to compute a ring realization, starting with the transfer matrix, via the Hankel matrix approach, then it is desirable to know a bound on the dimensions of the Hankel block on which the realization will be based. Such a bound can indeed be given. Suppose that $F(s,z) \in \mathbb{R}_c(s,z)^{m\times p}$. Then $F(s,z)$ can be written as $F(s,z) = M(s,z)/m(s,z)$, where $M(s,z)$ and $m(s,z)$ are coprime (the greatest common divisor of $m(s,z)$ and all entries of $M(s,z)$ is 1).
We can also view $F(s,z)$ as a matrix over $\mathbb{R}_c(s)_c(z)$ by dividing all coefficients of $M(s,z)$ and $m(s,z)$ by the coefficient (a polynomial in s) of the highest power in z which occurs in $m(s,z)$. In this way we obtain $F(s,z) = \bar{M}(s,z)/\bar{m}(s,z)$ where $\bar{m}(s,z)$ is a monic polynomial in $\mathbb{R}_c(s)[z]$ and $\bar{M}(s,z)$ is a matrix over $\mathbb{R}_c(s)[z]$. It can be shown (cf. [63]) that $\bar{m}(s,z)$ is the minimal polynomial of the matrix $A(s)$ in a canonical realization $\Sigma = (A(s),B(s),C(s),D(s))$ of $F(s,z)$ over $\mathbb{R}_c(s)$. Hence, for the Hankel block $H_{\ell k}$ (see (3.2.3)) on which the realization algorithm is based, we can take H_{nn} where $n = \deg_z \bar{m}(s,z)$. If $\mathbb{R}_c(s)$ is replaced by $\mathbb{R}_\sigma(s)$, an analogous bound on $H_{\ell k}$ can be obtained.

REMARK. The fact that $\bar{m}(s,z)$ is the minimal polynomial of $A(s)$ in a canonical realization can also be exploited in investigating internal stability. Results can be found in the author's paper in the Proc. European Signal Processing Conf.; Lausanne, 1980. See also [25]. □

We can even compute the dimension of a canonical realization beforehand. This can be done because the dimension of a canonical realization equals the dimension of a minimal realization over the quotient field. The dimension of a minimal realization over $\mathbb{R}(s)$ of $F(s,z)$ (the quotient field), which is by definition equal to the McMillan degree of $F(s,z)$, as a matrix over $\mathbb{R}(s)_c(z)$ can be found using theorem (3.3.15). The McMillan degree equals the sum of the degrees of the polynomials $\psi_1,\dots,\psi_r$ (see (3.3.15)). To avoid this rather complicated way of determining the McMillan degree of

$F(s,z)$ as a matrix over $\mathbb{R}(s)_c(z)$ we will now prove that the McMillan degree of $F(s,z)$ as a matrix over $\mathbb{R}(s)_c(z)$ is generically equal to the McMillan degree of $F(s_0,z)$ as a matrix over $\mathbb{R}_c(z)$. Here s_0 is a real number such that $F(s_0,z)$ is well defined. This genericity result should be understood as follows.

For a fixed transfer matrix $F(s,z)$ the set of real numbers s_0 such that $F(s_0,z)$ is well defined and the McMillan degree of $F(s,z)$ (as a matrix over $\mathbb{R}(s)_c(z)$) is equal to the McMillan degree of $F(s_0,z)$ (as a matrix over $\mathbb{R}_c(z)$) is a Zariski open set in $\mathbb{R}$.

Consider the matrix $D(z)$ in theorem (3.3.15). In this case $\mathcal{Q}(R) = \mathbb{R}(s)$.

$$(6.2.6) \qquad D(z) = \begin{bmatrix} \varepsilon_1/\psi_1 & & & & \\ & \varepsilon_2/\psi_2 & & & \\ & & \ddots & & 0 \\ & & & \varepsilon_r/\psi_r & \\ & 0 & & & 0 \end{bmatrix}$$

where some of the zero matrices are possibly empty.

$\varepsilon_1,\ldots,\varepsilon_r$ are polynomials over $\mathbb{R}(s)[z]$; $\psi_1,\ldots,\psi_r$ are monic polynomials over $\mathbb{R}(s)[z]$. Furthermore, ε_i and ψ_i are relatively prime polynomials for $i = 1,\ldots,r$.

Now choose s_0 such that $\varepsilon_1,\ldots,\varepsilon_r$, $\psi_1,\ldots,\psi_r$ are well defined. First we observe that the McMillan degree of $F(s_0,z)$ is less than or equal to the McMillan degree of $F(s,z)$ as a matrix over $\mathbb{R}(s)_c(z)$. The McMillan degree of $F(s_0,z)$ will be smaller than the McMillan degree of $F(s,z)$ if for at least one of the pairs (ε_i,ψ_i), $i = 1,\ldots,r$, say (ε_j,ψ_j), we have that $\varepsilon_j(s_0,z)$ and $\psi_j(s_0,z)$ are not relatively prime as polynomials over $\mathbb{R}[z]$. Using the resultant (see [77]) of $\varepsilon_j(s_0,z)$ and $\psi_j(s_0,z)$ the failure of the coprimeness implies that a polynomial in the variable s, where the coefficients are (polynomial) functions of the defining parameters of $F(s,z)$, will have a zero at s_0. Hence we may conclude that failure of coprimeness of the corresponding numerator and denominator polynomials in $D(z)$ can occur only at isolated points s_0. The number of points s_0 where $F(s_0,z)$ is not well defined is also finite. Therefore the genericity result concerning $F(s,z)$ and $F(s_0,z)$ is proved. (Zariski closed sets in $\mathbb{R}$ are just isolated point sets.)

Summarizing we may say (see section V.9) that by picking s_0 such that $F(s_0,z)$ is well defined the property

"McMillan degree of $F(s,z)$ = McMillan degree of $F(s_0,z)$"

is easily satisfied.

VI.3. *A realization algorithm for 2-D transfer matrices*

Finally we will put together the elements for a possible realization algorithm and outline in a rather detailed way the steps which have to be made in order to construct a realization algorithm for a 2-D transfer matrix which can be considered a 1-D transfer matrix over $\mathbb{R}_\sigma(s)$.

Let $F(s,z)$ be a 1-D transfer matrix over $\mathbb{R}_\sigma(s)$ (a transfer matrix of a 2-D system) with m outputs and p inputs.

Step 1. Determine $M(s,z)$ and $m(s,z)$ such that $F(s,z) = M(s,z)/m(s,z)$ and $M(s,z)$ and $m(s,z)$ are coprime. $M(s,z)$ ($m(s,z)$) is a polynomial matrix (polynomial) over $\mathbb{R}_c(s)[z]$.
This can be obtained by applying the algorithm described in [7] and, afterwards, dividing each coefficient by the coefficient of the leading term of the denominator. Recall that $m(s,z)$ is the minimal polynomial of the matrix $A(s)$ of a first level canonical realization. The matrix $M(s,z)$ need not be computed explicitly. Its occurrence here is for notational convenience. The important part in step 1 is the determination of $m(s,z)$.

Step 2. Suppose that the degree in z of $m(s,z)$ is ρ.
Determine the first $\rho+1$ matrix coefficients (matrices over $\mathbb{R}_\sigma(s)$) $M_0(s), M_1(s), \ldots, M_\rho(s)$ of the formal power series expansion

$$F(s,z) = M_0(s) + M_1(s)z^{-1} + \ldots + M_\rho(s)z^{-\rho} + \ldots .$$

COMMENT. The coprimeness condition of $M(s,z)$ and $m(s,z)$ is not necessary because the rest of the algorithm works also for every common multiple of the denominators of the entries of $F(s,z)$. If $M(s,z)$ and $m(s,z)$ are not taken to be coprime, then a larger number of matrix coefficients of the formal power series expansion of $F(s,z)$ has to be determined in order to make the algorithm work. However, generically $m(s,z)$ (such that $M(s,z)$ and $m(s,z)$ are coprime) is just the product of the denominators of the entries of $F(s,z)$. Therefore, if $F(s,z)$ is given by practical data, the determination of $m(s,z)$ can generically be done rather easily without doing superfluous work. □

This can be obtained by long division.
Let $m(s,z) = m_0(s) + m_1(s)z + \ldots + m_{\rho-1}(s)z^{\rho-1} + z^{\rho}$. Construct the matrices $M_{\rho+1}(s),\ldots,M_{2\rho-1}(s)$ by means of the recurrence

$$M_{\rho+j}(s) = -m_{\rho-1}(s)M_{\rho+j-1}(s) - m_{\rho-2}(s)M_{\rho+j-2}(s) - \ldots - m_0(s)M_j(s)$$

for $j = 1,\ldots,\rho-1$.
(These matrices (over $\mathbb{R}_\sigma(s)$) are the next $\rho - 1$ matrix coefficients of the formal power series expansion of $F(s,z)$).

Step 3. Build the Hankel block

$$H_{\rho\rho} = \begin{bmatrix} M_1(s) & M_2(s) & \ldots & M_\rho(s) \\ M_2(s) & & & \vdots \\ \vdots & & & \\ M_\rho(s) & & \ldots & M_{2\rho-1}(s) \end{bmatrix}$$

Construct a factorization $H_{\rho\rho} = P(s)Q(s)$ where $P(s) \in \mathbb{R}_\sigma(s)^{\rho m\times n}$, $Q(s) \in \mathbb{R}_\sigma(s)^{n\times\rho p}$. Here n is the rank of $H_{\rho\rho}$. This can be obtained by constructing a modified Hermite form for $H_{\rho\rho}$

$$H_{\rho\rho} = \Pi[\bar{D}(s)\ ,\ 0]V(s)\ .$$

Here Π is a $\rho m \times \rho m$ permutation matrix, $\bar{D}(s)$ is $\rho m \times n$ lower triangular matrix with full rank and $V(s)$ is a $\rho p \times \rho p$ unimodular matrix over $\mathbb{R}_\sigma(s)$. Such a modified Hermite form can be constructed as described in (3.1.5) by exploiting the fact that $\mathbb{R}_\sigma(s)$ is a Euclidean domain. Another way of obtaining a modified Hermite form for $H_{\rho\rho}$ is described in this chapter (substitution of s by $1/s$ and application of an algorithm to construct a modified Hermite form for a polynomial matrix).
We can now take

$$P(s) = \Pi\bar{D}(s)\ ,\quad Q(s) = [I\ ,\ 0]V(s)\ .$$

Here I is the $n \times n$ identity matrix.

Step 4. $Q(s) = [Q_1(s),\ldots,Q_\rho(s)]$ where $Q_i(s) \in \mathbb{R}_\sigma(s)^{n\times p}$.

Compute $Q_{\rho+1}(s) = -(m_0(s)Q_1(s) + \ldots + m_{\rho-1}(s)Q_\rho(s))$.

Compute a right inverse $Q(s)^+$ of $Q(s)$. This can be done easily for if $Q(s) = [I \quad 0]V(s)$, then $Q(s)^+ = V(s)^{-1}\begin{bmatrix} I \\ 0 \end{bmatrix}$. Because $V(s)$ is a product of elementary matrices, which have been built in the factorization process, the matrix $V(s)^{-1}$ can easily be computed as the product (in reversed order) of the inverse elementary matrices (which can be written down immediately).

Step 5. Define

$$D(s) = M_0(s) \quad , \quad C(s) = P_1(s)$$

where $P_1(s)$ is the matrix consisting of the first m rows of $P(s)$.
Define

$$B(s) = Q_1(s) \quad \text{and} \quad A(s) = [Q_2(s),\ldots,Q_{\rho+1}(s)]Q(s)^+ \ .$$

Then $\Sigma = (A(s),B(s),C(s),D(s))$ is a canonical (first level) realization of $F(s,z)$.

Step 6. $A(s)$, $B(s)$, $C(s)$, $D(s)$ are proper rational matrices which can be realized by means of some realization algorithm for 1-D transfer matrices (see for instance [40]) and a second level realization of $F(s,z)$ can be constructed via the method (which will not be repeated here) described in Chapter IV (see (4.3.7)). Furthermore, a Roesser realization can be composed as in (4.3.9). □

In this algorithm one can take short-cuts but the present form has been chosen to facilitate the readability.

EXAMPLE.

$$F(s,z) = \begin{bmatrix} \dfrac{2s}{(1+2s)(2sz-1)} & \dfrac{s}{(1-2s)z} \\ \dfrac{4s}{1+4s} & \dfrac{1}{z} \end{bmatrix} .$$

Step 1. In this case the polynomial $m(s,z) = -\frac{1}{2s}z + z^2$.

Step 2. Thus $\rho = 2$ and $m_0(s) = 0$ and $m_1(s) = -\frac{1}{2s}$.

$$M_0(s) = \begin{bmatrix} 0 & 0 \\ \frac{4}{1+4s} & 0 \end{bmatrix}, \quad M_1(s) = \begin{bmatrix} \frac{1}{1+2s} & \frac{s}{1-2s} \\ 0 & 1 \end{bmatrix},$$

$$M_2(s) = \begin{bmatrix} \frac{1}{(1+2s)2s} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_3(s) = \frac{1}{2s} M_2(s) = \begin{bmatrix} \frac{1}{(1+2s)4s^2} & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 3.

$$H_{22} = \begin{bmatrix} \frac{1}{1+2s} & \frac{s}{1-2s} & \frac{1}{(1+2s)2s} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{(1+2s)2s} & 0 & \frac{1}{(1+2s)4s^2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrices Π, D(s), V(s) for a modified Hermite form are (obtained as described in (3.1.5))

$$\Pi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \bar{D}(s) = \begin{bmatrix} 1 & 0 \\ \frac{s}{1-2s} & \frac{1}{1+2s} \\ 0 & \frac{1}{(1+2s)2s} \\ 0 & 0 \end{bmatrix},$$

$$V(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2s} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and P(s) and Q(s) can be chosen as

$$P(s) = \begin{bmatrix} \frac{s}{1-2s} & \frac{1}{1+2s} \\ 1 & 0 \\ 0 & \frac{1}{(1+2s)2s} \\ 0 & 0 \end{bmatrix}, \quad Q(s) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2s} & 0 \end{bmatrix}.$$

(Frequently it occurs that a factorization $H_{\rho\rho} = P(s)Q(s)$ can be written down before a modified Hermite form has been obtained in the factorization process and, of course, in this case one takes this short-cut. Furthermore it is clear that only the first two rows of P(s) have to be computed. However, in the present example we have chosen to follow the lines of the algorithm described above.)

Step 4.

$$Q_3(s) = -m_1(s)Q_2(s) = \begin{bmatrix} 0 & 0 \\ \frac{1}{4s} & 0 \end{bmatrix},$$

$$Q(s)^+ = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Step 5.

$$D(s) = \begin{bmatrix} 0 & 0 \\ \frac{4s}{1+4s} & 0 \end{bmatrix}, \quad C(s) = \begin{bmatrix} \frac{s}{1-2s} & \frac{1}{1+2s} \\ 1 & 0 \end{bmatrix},$$

$$B(s) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A(s) = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2s} \end{bmatrix}.$$

Step 6. Realizing A(s), B(s), C(s), D(s) one can obtain a second level realization in Roesser's form via (4.3.9) where

$$\left[\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right] = \left[\begin{array}{c|c|c|c} AD & AC & 0 & 0 \\ \hline AB & AA & 0 & 0 \\ \hline CB & 0 & CA & 0 \\ \hline 0 & 0 & 0 & DA \end{array}\right] = \left[\begin{array}{cc|c|cc|c} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} \end{array}\right]$$

$$\begin{bmatrix} B_1 \\ \hdashline B_2 \end{bmatrix} = \begin{bmatrix} BD \\ \hdashline 0 \\ \hdashline 0 \\ \hdashline DB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ \hdashline 0 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \\ \hdashline 1 & 0 \end{bmatrix}, \quad D = DD = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$[C_1 \mid C_2] = [CD \mid 0 \mid CC \mid DC] = \begin{bmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 & 0 & -\frac{1}{4} \end{bmatrix}.$$

Observe that BA, BB and BC do not occur because

$$B(s) = BD = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

REMARK. The above realization could have been obtained faster because rank H_{11} = rank $\mathcal{H} = 2$. □

APPENDIX

Ring of fractions

Let R be a commutative ring with identity (unit element). Let S be a subset of R which is closed under multiplication and contains the identity. Such a set S is called a multiplicative set (also multiplicatively closed set). Consider the equivalence relation $\sim$ on $R \times S$, defined as follows: $(r,s) \sim (r_1,s_1)$ iff there exists an element $s_2 \in S$ such that $s_2(rs_1 - r_1s) = 0$. Here $r, r_1 \in R$ and $s, s_1 \in S$. Let R_S denote the set of equivalence classes. The equivalence class determined by (r,s) will be denoted by r/s.
R_S can be given a ring structure by defining

$$r_1/s_1 + r_2/s_2 = (r_1s_2 + r_2s_1)/s_1s_2$$
$$r_1/s_1 \cdot r_2/s_2 = r_1r_2/s_1s_2 \ .$$

R_S is called the *ring of fractions of* R *with respect to* S.

Quotient field

Let R be an integral domain (which is non-zero, commutative and has an identity). Let $S = R \setminus 0$ (which is a multiplicative set). Then R_S is called the *field of fractions* of R. This field is also called the *quotient field* of R and will be denoted by $Q(R)$.

Noetherian domain

An integral domain such that every ideal is finitely generated is called a *Noetherian domain*.

Integrally closed domain

An integral domain R is called an *integrally closed domain* if for any $q \in \mathcal{Q}(R)$ (the quotient field of R) such that

$$q^n + a_{n-1}q^{n-1} + \dots + a_0 = 0$$

for some non-negative integer n and elements $a_0, \dots, a_{n-1} \in R$ we have that $q \in R$.

Principal ideal domain

An integral domain such that every ideal is generated by one element is called a *principal ideal domain.*

Euclidean domain

A *Euclidean domain* is an integral domain R together with a map $\varphi: \{R \setminus 0\} \to \mathbb{Z}_+$ ($\mathbb{Z}_+$ denotes the set of non-negative integers) such that

1° If $a \mid b$ (a divides b) then $\varphi(a) \le \varphi(b)$ for $a,b \in R$.
2° For $a \in R$ and $b \in \{R \setminus 0\}$ there exist $q,r \in R$ such that $a = bq + r$ and either $r = 0$ or $\varphi(r) < \varphi(b)$.

The map φ is called the *Euclidean function.*

Semi local ring

A commutative ring with identity is called a *semi local ring* if it has only finitely many maximal ideals.

Local ring

A semi local ring is called a *local ring* if it has a unique maximal ideal.

Module

Let R be a commutative ring with identity 1. A *module over the ring* R, also called R-*module*, is an Abelian group M (almost invariably written additively) together with a map $(r,m) \mapsto rm$ from $R \times M$ to M satisfying the following conditions

$$r(m_1 + m_2) = rm_1 + rm_2$$

$$(r_1 + r_2)m = r_1m + r_2m$$

$$(r_1r_2)m = r_1(r_2m)$$

$$1m = m \ .$$

Sometimes the R-module, just defined, is called a left R-module. There is a similar definition of a right R-module in which the elements of R are written on the right. We will consider only left R-modules and simply call them R-modules.

Torsion free module

Let R be an integral domain. An R-module M is called a *torsion free module* if $rm = 0$ ($r \in R$, $m \in M$) implies that $r = 0$ or $m = 0$.

REFERENCES

[1] ACKERMANN, J.; Entwurf durch Polvorgabe. Regelungstechnik, 25. Jahrgang, Heft 6, pp. 173-179, 1977.

[2] ANDREWS, H.C., ed.; Tutorial and Selected Papers in Digital Image Processing. IEEE Press, New York, 1978.

[3] ANDREWS, H.C.; Digital Image Processing. IEEE Spectrum, Vol. 16, no. 4, pp. 38-49, 1979.

[4] ATTASI, S.; Systèmes linéaires homogènes à deux indices. IRIA Rapport de Recherche, no. 31, 1973.

[5] BARRY, P.E., R. Gran, C.R. Waters; Two-Dimensional Filtering - a State Estimator Approach. Proc. 1976 IEEE CDC, pp. 613-618.

[6] BERGE, C.; Topological Spaces. Oliver and Boyd, 1963.

[7] BOSE, N.K.; An Algorithm for GCF Extraction from Two Multivariable Polynomials. Proc. IEEE, Vol. 64, no. 1, pp. 185-186, 1976.

[8] BOURBAKI, N.; Commutative Algebra. Addison-Wesley, 1972.

[9] BROCKETT, R.W.; Finite Dimensional Linear Systems. Wiley, 1970.

[10] BRUNOVSKI, P.; A Classification of Linear Controllable Systems. Kybernetika, Vol. 3, pp. 173-187, 1970.

[11] BUMBY, R.T., E.D. Sontag; Reachability does not imply coefficient assignability. Notices AMS, 1978.

[12] CHEN, C.T.; Introduction to Linear System Theory. Holt, Rinehart and Winston inc., 1970.

[13] CHING, W.S.; Linear Equations over Commutative Rings. Linear Algebra Appl., Vol. 18, pp. 257-266, 1977.

[14] COOLEY, J.W., J.W. Tukey; An Algorithm for the Machine Calculation of Complex Fourier Series. Math. Comp., Vol. 19, pp. 297-301, 1966.

[15] EILENBERG, S.; Automata, Languages and Machines. Vol. A, New York Academic, 1974.

[16] EISING, R.; Realization and Stabilization of 2-D Systems. IEEE Trans. Automat. Control, Vol. AC-23, no. 5, pp. 793-799, 1978.

[17] EISING, R.; Controllability and Observability of 2-D Systems. IEEE Trans. Automat. Control, Vol. AC-24, no. 1, pp. 132-133, 1979.

[18] EISING, R.; Separability of 2-D Transfer Matrices. IEEE Trans. Automat. Control, Vol. AC-24, no. 3, pp. 508-510, 1979.

[19] EISING, R.; Low Order Realizations for 2-D Transfer Functions. Proc. IEEE, Vol. 67, no. 5, pp. 866-868, 1979.

[20] EISING, R.; State Space Realization and Inversion of 2-D Systems. To appear in IEEE Trans. Circuits and Systems.

[21] EISING, R., M.L.J. Hautus; Realization Algorithms for Systems over a Principal Ideal Domain. Memorandum COSOR 78-25, Eindhoven University of Technology, 1978.

[22] EKSTROM, M., J.W. Woods; Two-Dimensional Spectral Factorization with Applications in Recursive Digital Filtering. IEEE Trans. Acoust. Speech Signal Process., Vol. ASSP-24, no. 2, pp. 115-128, 1976.

[23] FORMASINI, E., G. Marchesini; Two-Dimensional Filters: New Aspects of the Realization Theory. Proc. of the third int. Joint Conf. on Pattern Recogn., Coronado CA, 1976.

[24] FORMASINI, E., G. Marchesini; State Space Realization Theory of Two-Dimensional Filters. IEEE Trans. Automat. Control, Vol. AC-21, no. 4, pp. 484-492, 1976.

[25] FORMASINI, E., G. Marchesini; Doubly Indexed Dynamical Systems: State Space Models and Structural Properties. Math. Systems Theory, Vol. 12, pp. 59-72, 1978.

[26] FUKS, B.A.; Analytic Functions of Several Complex Variables. Translations of Math. Monographs, A.M.S., 1963.

[27] GANTMACHER, F.R.; The Theory of Matrices. Vol I ch. VI, Chelsea, 1960.

[28] GIVONE, D.D., R.P. Roesser; Minimization of Multidimensional Linear Iterative Circuits. IEEE Trans. Comput., Vol. C-22, no. 7, pp. 673-678, 1973.

[29] GOODMAN, D.; An Alternate Proof of Huang's Stability Theorem. IEEE Trans. Acoust. Speech Signal Process., Vol. ASSP-24, no. 5, pp. 426-427, 1976.

[30] GOODMAN, D.; Some Stability Properties of Two-Dimensional Linear Shift Invariant Digital Filters. IEEE Trans. Circuits and Systems, Vol. CAS-24, no. 4, pp. 201-208, 1977.

[31] HARTLEY, B., T.O. Hawkes; Rings, Modules and Linear Algebra. Chapman and Hall, 1976.

[32] HERMANN, R.; Algebro-Geometric and Lie-Theoretic Techniques in Systems Theory, part A. Math. Sc. Press, 1977.

[33] HEYMANN, M.; Structure and Realization Problems in the Theory of Dynamical Systems. Springer, 1975.

[34] HEYMANN, M.; Comments on "Pole Assignment in Multi-Input Controllable Linear Systems". IEEE Trans. Automat. Control, Vol. AC-13, pp. 748-749, 1968.

[35] HEYMANN, M., J.A. Thorpe; Transfer Equivalence of Linear Dynamical Systems. SIAM J. Control Optim., Vol. 8, pp. 19-40, 1970.

[36] HUANG, T.S.; Stability of Two-Dimensional Recursive Filters. IEEE Trans. Audio and Electroacoust., Vol. AU-20, no. 2, pp. 158-163, 1972.

[37] HUANG, T.S., ed.; Picture Processing and Digital Filtering. Topics in Applied Physics, Vol. 6, Springer, second ed., 1979.

[38] HUNT, B.R.; Digital Image Processing. Proc. IEEE, Vol. 63, no. 4, pp. 693-708, 1975.

[39] HUNT, B.R., G.F. Sera; Power-Law Stimulus-Response Models for Measures of Image Quality in Non Performance Environments. IEEE Trans. Systems Man Cybernet., Vol. SMC-8, no. 11, pp. 781-791, 1978.

[40] JONG, L.S. de; Numerical Aspects of Realization Algorithms in Linear Systems Theory. Thesis, Eindhoven, 1975

[41] KALMAN, R.E.; Irreducible Realizations and the Degree of a Rational Matrix. SIAM J. Control Optim., Vol. 13, pp. 520-544, 1965.

[42] KALMAN, R.E.; Lectures on Controllability and Observability. CIME Summer Course 1968, Cremonese, Roma, 1969.

[43] KALMAN, R.E.; On Minimal Partial Realizations of a Linear Input/Output Map, Aspects of Network and System Theory. R.E. Kalman and N. DeClaris, eds., New York, Holt, Rinehart and Winston, pp. 385-407, 1971.

[44] KALMAN, R.E.; Realization Theory of Linear Dynamical Systems. In: Control Theory and Topics in Functional Analysis Vol. II. International Atomic Energy Agency, Vienna, pp. 235-256, 1976.

[45] KALMAN, R.E., P.L. Falb, M.A. Arbib; Topics in Mathematical System Theory. McGraw-Hill, 1969.

[46] KAMEN, E.W.; On an Algebraic Theory of Systems defined by Convolution Operators. Math. Systems Theory, Vol. 9, no. 1, pp. 57-74, 1975.

[47] KAMEN, E.W.; Lectures on Algebraic System Theory: Linear Systems over Rings. NASA Contractor Report 3016.

[48] KUNG, S.Y., B.C. Lévy, M. Morf, T. Kailath; New Results in 2-D Systems Theory, Part II, 2-D State Space Models-Realization and the Notions of Controllability, Observability and Minimality. Proc. IEEE, Vol. 65, no. 6, pp. 945-961, 1977.

[49] KWAKERNAAK, H., R. Sivan; Linear Optimal Control Systems. Wiley Interscience, 1972.

[50] LEKKERKERKER, C.G.; Geometry of Numbers. Wolters-Noordhoff and North-Holland, 1969

[51] MASSEY, J.L., M.K. Sain; Inverses of Linear Sequential Circuits. IEEE Trans. Comput., Vol. C-13, no. 2, pp. 330-337, 1968.

[52] MITRA, S.K., S. Chakrabarti; A New Realization Method for 2-D Digital Transfer Functions. IEEE Trans. Acoust. Speech Signal Process., Vol. ASSP-26, no. 6, pp. 544-550, 1978.

[53] MITRA, S.K., A.D. Sagar, N.A. Pendergrass; Realizations of Two-Dimensional Recursive Digital Filters. IEEE Trans. Circuits and Systems, Vol. CAS-22, no. 3, pp. 177-184, 1975.

[54] MORF, M., B.C. Lévy, S.Y. Kung; New Results in 2-D Systems Theory, Part I: 2-D Polynomial Matrices, Factorization and Coprimeness. Proc. IEEE, Vol. 65, no. 6, pp. 861-873, 1977.

[55] MORSE, A.S.; Ring Models for Delay Differential Systems. Automatica, Vol. 12, pp. 529-531, 1976.

[56] MULLANS, R.E., D.L. Elliott; Linear Systems on Partially Ordered Time Sets. Proc. 1973 IEEE CDC, pp. 334-337.

[57] NEWMAN, M.; Integral Matrices. Academic Press, 1972.

[58] MESSBAUMER, H.J., P. Quandalle; Fast Computation of Discrete Fourier Transforms Using Polynomial Transforms. IEEE Trans. Acoust. Speech Signal Process., Vol. ASSP-27, no. 2, pp. 169-181, 1979.

[59] RINDFLEISCH, T.C., J.A. Dunne, H.J. Frieden, W.D. Stromberg, R.M. Ruiz; Digital Processing of the Mariner 6 and 7 Pictures. Journal of Geophysical Research, Vol. 76, no. 2, pp. 394-417, 1971.

[60] RISSANEN, J.; Recursive Identification of Linear Systems. SIAM J. Control Optim., Vol. 9, no. 3, pp. 420-430, 1971.

[61] ROESSER, R.P.; A Discrete State Space Model for Linear Image Processing. IEEE Trans. Automat. Control, Vol. AC-20, no. 1, pp. 1-10, 1975.

[62] ROSENBROEK, H.H.; State Space and Multivariable Theory. Wiley, 1970.

[63] ROUCHALEAU, Y.; Linear, Discrete Time, Finite Dimensional, Dynamical Systems over Some Classes of Commutative Rings. Ph.D. Thesis, Stanford, 1972.

[64] ROUCHALEAU, Y., B.F. Wyman, R.E. Kalman; Algebraic Structure of Linear Dynamical Systems III, Realization Theory over a Commutative Ring. Proc. Nat. Acad. Sci. USA, Vol. 69, pp. 3404-3406, 1972.

[65] SAMUEL, P.; About Euclidean Rings. J. Algebra, Vol. 19, pp. 282-301, 1971.

[66] SEVIORA, R.E.; Causality and Stability in Two-Dimensional Digital Filtering. Proc. 1973 Asilomar Conf. Monterrey CA, pp. 285-289.

[67] SILVERMAN, L.M.; Inversion of Multivariable Linear Systems. IEEE Trans. Automat. Control, Vol. AC-14, no. 3, pp. 270-276, 1969.

[68] SILVERMAN, L.M.; Realization of Linear Dynamical Systems. IEEE Trans. Automat. Control, Vol. AC-16, no. 6, pp. 554-568, 1971.

[69] SINGLETON, R.C.; An Algorithm for Computing the Mixed Radix Fast Fourier Transform. IEEE Trans. Audio Electroacoust., Vol. AU-17, no. 2, pp. 93-106, 1969.

[70] SONDHI, M.M.; Image Restoration: The Removal of Spatial Invariant Degradations. Proc. IEEE, Vol. 60, no. 7, 842-853, 1972.

[71] SONTAG, E.D.; Linear Systems over Commutative Rings: a Survey. Ricerche di Automatica, Vol. 7, no. 1, pp. 1-34, 1976.

[72] SONTAG, E.D.; On Split Realizations of Response Maps over Rings. Inform. and Control, Vol. 37, pp. 23-33, 1978.

[73] SONTAG, E.D.; On Linear Systems and Noncommutative Rings. Math. Systems Theory, Vol. 9, no. 4, pp. 327-344, 1976.

[74] SONTAG, E.D.; On First-Order Equations for Multi-Dimensional Filters. IEEE Trans. Acoust. Speech Signal Process., Vol. ASSP-26, pp. 480-482, 1978.

[75] STOCKHAM,Jr., T.G.; Image Processing in the Context of a Visual Model. Proc. IEEE, Vol. 60, no. 7, pp. 828-842, 1972.

[76] VIDYASAGAR, N., N.K. Bose; Input-Output Stability of Linear Systems defined over Measure Spaces. Proc. 1975 Mid.West Symp. on Circuits and Systems.

[77] WAERDEN, B.L. v.d.; Algebra I. Springer, 1971. English translation by F. Blum and J.R. Schulenberger. Ungar, 1970.

[78] WOLOVICH, W.A.; Linear Multivariable Systems. Springer, 1974.

[79] WONHAM, W.M.; Linear Multivariable Control, A Geometric Approach. Springer, 1974.

[80] WOODS, J.W.; Markov Image Modeling. Proc. 1976 IEEE CDC, pp. 596-600.

[81] ZARISKI, O., P. Samuel; Commutative Algebra. Vol. II. v. Nostrand, 1960.

NOTATIONS

$\mathbb{R}$	the set of real numbers.
$\mathbb{R}_+$	the set of non-negative real numbers.
$\mathbb{C}$	the set of complex numbers.
$\mathbb{Z}$	the set of integers.
$\mathbb{Z}_+$	the set of non-negative integers.
$R[z]$	the set of polynomials in the variable z with coefficients in R.
$R(z)$	the set of rational functions in the variable z with coefficients in R.
$R_c(z)$	the set of proper rational functions in the variable z with coefficients in R.
$R_{sc}(z)$	the set of strictly proper rational functions in the variable z with coefficients in R.
$R[[z^{-1}]]$	the set of formal power series in the variable z^{-1} with coefficients in R.
$R[s,z]$	the set of polynomials in the variables s and z with coefficients in R.
$\mathbb{R}_\sigma(s)$	the set of stable real transfer functions.
$\mathbb{R}(s,z)$	the set of real rational functions in the variables s and z.
$\mathbb{R}_c(s,z)$	the set of proper real rational functions in the variables s and z.
$\mathbb{R}[[s^{-1},z^{-1}]]$	the set of real formal power series in the variables s^{-1} and z^{-1}.
$(x_j)_{j\in J}$	sequence with index set J (elements are denoted by x_j).
$(x_{i,j})_{(i,j)\in J}$	double sequence with index set J (elements are denoted by x_{ij} and also by $x_{i,j}$).
R^n	the set of column n-vectors over R (row vector notation is used for $\mathbb{Z}^2$ and $\mathbb{Z}_+^2$ and in IV.4 and V.6 also for $\mathbb{R}^2$ and $\mathbb{R}_+^2$).

$(x_1,\ldots,x_n)'$	column n-vector.
$R^{m\times p}$	the set of $m\times p$-matrices over R.
A'	the transpose of a matrix A (in the scalar case or in the case where A is not a matrix the "prime" is used just to discern A and A').
$(A)_{ij}$	the (i,j)-th entry of the matrix A.
det(A)	the determinant of the matrix A.
$\mathrm{diag}(d_1,\ldots,d_n)$	diagonal $n\times n$-matrix.
$\deg_s(p)$	the degree in s of the polynomial p (in the case of more than one variable).
deg(p)	the degree of the polynomial p (in the case of one variable).
$\mathcal{H}(F)$	the Hankel matrix associated with the 1-D impulse response F.
S_F	the support of the 2-D impulse response F.
conv S_F	the convex hull of S_F.
$\mathrm{conv}^+ S_F$	the sum of conv S_F and $\mathbb{R}_+^2$.
C_c	causality cone.
$\hat{S}_{p,r,q,t}$	the set of 2-D transfer functions with support in a causality cone described by the parameters p, r, q and t.
$\mathcal{Q}(R)$	the quotient field of the integral domain R.
$\lvert x\rvert$	the absolute value of x.

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