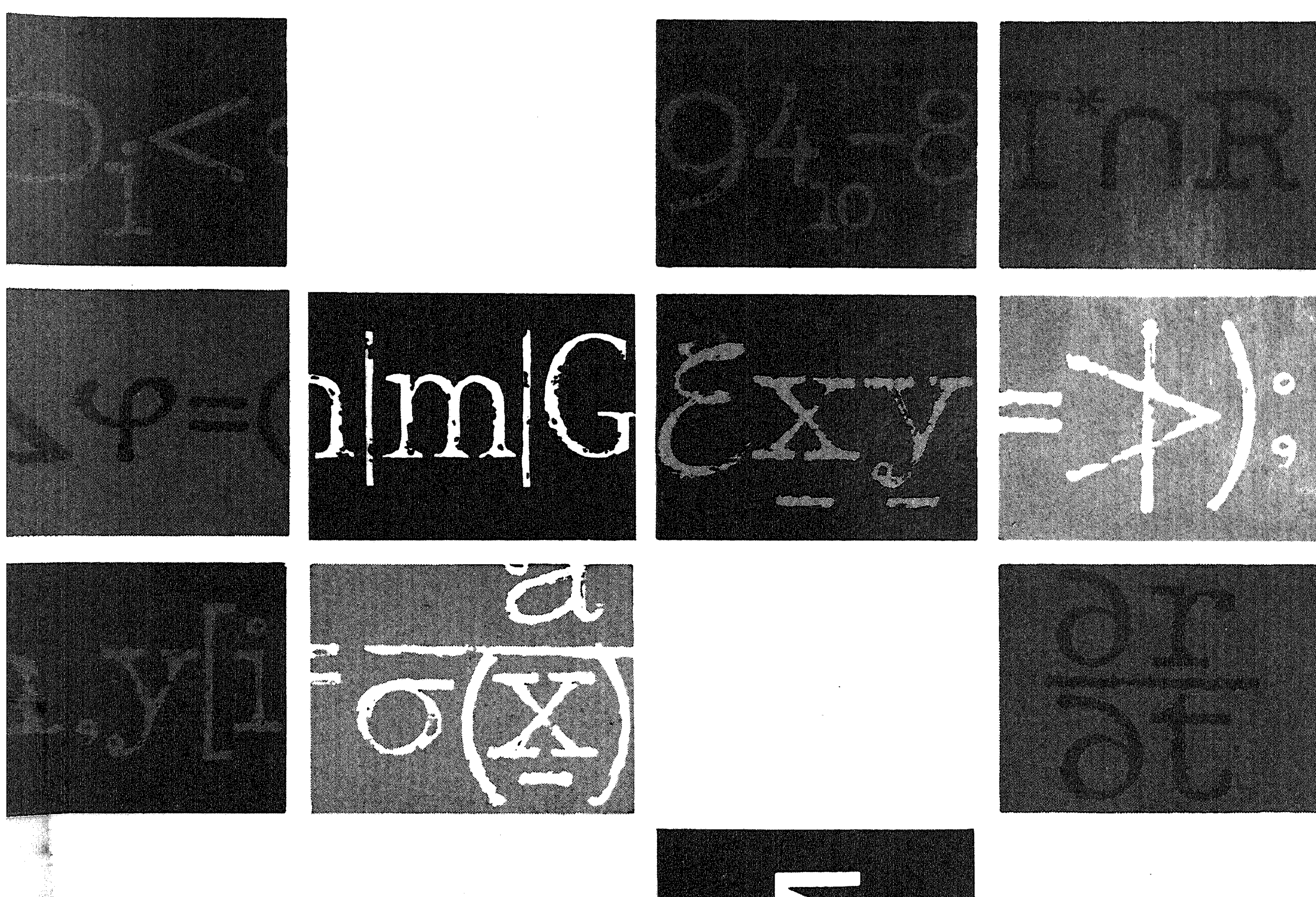
CARDINAL FUNCTIONS IN TOPOLOGY -TEN YEARS LATER

I. JUHÁSZ



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CONTENTS

PREFACE

Ten years have passed since the first edition of my book "Cardinal Functions"*) appeared, and this decade has seen a tremendous amount of activity in and development of the area. Thus when I was asked to prepare a new, updated edition of my book, I had no choice but to completely rewrite it. This new version now contains at least three times as much material as the old one. If this is not apparent at first sight it is because the new book has no appendix on combinatorial set theory. Such an appendix is no longer necessary since a number of good books and survey articles on this subject have recently appeared. In this new version I aimed at a certain kind of completeness by trying to include all the fundamental results that can be established in ZFC, i.e. ordinary set theory. This "forced" the exclusion of independence results, which, in my view at least, constitute the most significant advances of our field. Hence, in this respect, the book is certainly not complete and in fact it just cries out for a partner volume covering the basic independence results.

The material of this book has been based on a two-semester course that I gave at the University of Budapest in 1978. However, it was actually written during the second half of 1979, when I was visiting at the Mathematics Department of the Free University of Amsterdam. I would like to take this opportunity to thank this institution, in particular Professors P.C. Baayen and M.A. Maurice, for making my visit possible. I would also like to thank my former student A. Pozsonyi, whose meticulous notes of my course meant a great help for me in writing this book. I am grateful to the staff of the Mathematical Center involved in the fast and careful preparation of the manuscript, and especially to Mr. T. Jacobs, who prepared the index and the list of symbols.

Toronto, March 1980 Iṣtván Juhász

PREFACE TO THE FIRST EDITION

General topology can be considered as a natural outgrowth of set theory; the simple set theoretic nature of its fundamental notions makes it an appropriate area for the application of set theoretic methods. On the other hand, many set theoretic problems have their roots in topology and this makes the interaction between the two disciplines even more profound. The closeness of

*) MCT 34, Mathematisch Centrum, Amsterdam 1971

their relationship is perhaps most apparent in the work done by the Moscow school of topology in the early twenties.

The last decade has witnessed a very rapid development of set theoretic methods and ideas, the main sources of which were, in our opinion, the following: 1) the independence results of P. Cohen and his followers; 2) the results on "large" cardinals of A. Tarski's school, and 3) the achievements of P. Erdős, R. Rado, A. Hajnal, and others in combinatorial set theory (e.g., partition calculus). Not surprisingly, this has stirred up a renewed interest in the set theoretic aspects of general topology. A number of old problems were settled and many new ones were raised.

The aim of this tract is to present a variety of questions of this kind by centering them around the unifying concept of cardinal functions.

Since a considerable part of the means employed in our investigations are relatively recent and not easily accessible in the literature, we have found in both convenient and timely to include an appendix entirely devoted to the detailed explanation of these methods and ideas of combinatorial set theory.

This tract was written during the second half of 1969, while the author was a guest of the Department of Pure Mathematics of the Mathematical Centre in Amsterdam. The appendix is based on a series of talks given by the author during the same period at the Mathematical Centre under the title "Combinatorial Set Theory".

At this point I wish to express my gratitude towards the Mathematical Centre for their kind hospitality which gave me the opportunity to write this tract, as well as for publishing it. I am particularly grateful to Professors J. de Groot and P.C. Baayen for initiating my invitation and supporting this project.

Special thanks are also due to Albert Verbeek, who took on the difficult task of actually writing the text of the appendix, and did most of the work necessary to turn the crude manuscript into print. I would also like to thank Nelly Kroonenberg, who added A6 to the appendix.

Finally, I am greatly indebted to my friend and collegue A. Hajnal, whose help was essential in acquiring the methods used in this tract.

Budapest, December, 1970. István Juhász

CHAPTER 0

PRELIMINARIES

We shall use in this book the by now more or less standard "modern" set-theoretical notations, e.g. that of [DR 1974], [JE 1978], [K 1977] or [WI 1977]. The set of all subsets (the power set) of a set X is denoted by P(X). Functions are always sets of ordered pairs (i.e. they are identified with their graphs). The domain of a function f is denoted by D(f) and its range by R(f). Thus f: $A \to B$ means that $D(f) = A$ and $R(f) \subset B$. We shall put B^A to denote the set of all functions f: $A \to B$. If $S \subset A$ then $f \upharpoonright S \in B^S$. We shall often use the symbol H(A,B) to denote the set of all finite functions from A to B, i.e. $g \in H(A,B)$ means that D(g) is a finite subset of A and $R(g) \subset B$. If $B = 2 = \{0,1\}$, then we shall write H(A) instead of H(A,2).

Ordinals - usually denoted by greek letters - are identified with their sets of predecessors. Consequently, if α, β are ordinals then $\alpha < \beta$ means the same as $\alpha \in \beta$. A sequence s of length α is a function with $D(s) = \alpha$, hence e.g. β^{α} is the set of all sequences of length α of ordinals less than β. We shall also put $\beta^{\underline{\alpha}} = \cup\{\beta^{\nu}: \nu \in \alpha\}$, i.e. the members of $\beta^{\underline{\alpha}}$ are the sequences of length less than α.

Cardinals are the initial ordinals, κ, λ, μ will always denote infinite cardinals, ω is the smallest infinite cardinal. For any ordinal α, $cf(\alpha)$ denotes the cofinality of α that is always a regular cardinal. The cardinality of a set X is denoted by $|X|$. The successor cardinal of κ is denoted by κ^+. A non-successor cardinal is called a limit cardinal.

When indexing by all ordinals less than a given ordinal (which of course might be a cardinal) we shall usually use the symbol $\in$, while if we index by all cardinals less than a given one we always use <. Thus in $\{x_\alpha: \alpha \in \kappa\}$ the indices run through all suitable ordinals, while in $\{\rho_\lambda: \lambda < \kappa\}$ through all suitable (infinite) cardinals.

For any set X and cardinal ρ (which might be finite) $[X]^\rho$ denotes the collection of all ρ-element subsets of X; $[X]^{<\rho}$ and $[X]^{\leq\rho}$ are defined analogously.

The symbol κ^λ will have double meaning, it denotes according to our above convention the set of all maps from λ into κ, moreover it denotes the corresponding cardinal exponentiation. The λ^{th} weak power of κ, denoted by $\kappa^{\underline{\lambda}}$ is defined as $\kappa^{\underline{\lambda}} = \Sigma\{\kappa^\mu\colon \mu < \lambda\}$. The cardinal λ is called strong limit if $\kappa < \lambda$ implies $2^\kappa < \lambda$ as well. Next we collect the basic facts about cardinal exponentiation.

0.1. a) $(\kappa^+)^\lambda = 2^\lambda.\kappa^+$;

b) *if* $\lambda < \mathrm{cf}(\kappa)$ *and* κ *is limit, then*

$$\kappa^\lambda = \Sigma\{\mu^\lambda\colon \mu < \kappa\};$$

c) *if* $\mathrm{cf}(\kappa) < \lambda < \kappa$ *then*

$$\kappa^\lambda = (\Sigma\{\mu^\lambda\colon \mu < \kappa\})^{\mathrm{cf}(\kappa)};$$

d) $\mathrm{cf}(\kappa^\lambda) > \lambda$;

e) *if* $\{\kappa_\xi\colon \xi\epsilon\lambda\}$ *is increasing with* $\kappa = \Sigma\{\kappa_\xi\colon \xi \in \lambda\}$ *then*

$$\Pi\{\kappa_\xi\colon \xi \in \lambda\} = \kappa^\lambda.$$

Using these one can prove the following statement.

0.2. *Suppose that the power* κ^λ *is a jump, i.e.* $\kappa,\lambda \geq \omega$, $\mu^\lambda < \kappa^\lambda$ *if* $\mu < \kappa$ *and* $\kappa^\mu < \kappa^\lambda$ *if* $\mu < \lambda$. *Then* $\lambda = \mathrm{cf}(\kappa)$.

A set mapping F over a set X is any map of the form $F\colon [X]^{<\rho} \to P(X)$. The particular case $\rho = 2$, in which case we usually write $F\colon X \to P(x)$ instead of $F\colon [X]^1 \to P(X)$, is of particular importance for us. The basic result concerning these is Hajnal's theorem below. A set $S \subset X$ is said to be free for F if $x,y \in S$ with $x \neq y$ imply $x \notin F(y)$.

0.3. *If* $F\colon X \to [X]^{<\lambda}$ *where* $\lambda < |X|$ *then there is a free set* $S \subset X$ *for* F *with* $|S| = |X|$.

Concerning the notion of ramification systems and the basic result on them, the ramification lemma, we refer the reader to [WI 1977, Ch.2.2].

By an r-partition of X into ρ parts, where $r < \omega$, we mean a map $f\colon [X]^r \to Y$ where $|Y| = \rho$. The partition relation $\kappa \to (\lambda_\nu)^r_{\nu\epsilon\rho}$ means that whenever f is an r-partition of κ of the form $f\colon [\kappa]^r \to \rho$ there is a $\nu \in \rho$ and a set $a_\nu \in [\kappa]^{\lambda\nu}$ with $|f([a_\nu]^r)| = 1$. The following two known partition relations will be used frequently in this book.

0.4. *For any* κ *we have*

a) $\kappa \to (\kappa, \omega)^2$;

b) $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$.

An important corollary of 0.4a) is that if $<_1$ and $<_2$ are well-orderings of the same set S then there is a $D \subset S$ with $|D| = |S|$ such that $<_1$ and $<_2$ coincide on D.

If λ is a singular cardinal with $cf(\lambda) = \kappa$ and $\langle\lambda_\nu: \nu \in \kappa\rangle$ is a fixed sequence of cardinals less than λ with $\lambda = \Sigma\{\lambda_\nu: \nu \in \kappa\}$, then a family $\{S_\nu: \nu \in \kappa\}$ of subsets of λ is said to be *canonical* with respect to the r-partition $f: [\lambda]^r \to \rho$, if the following conditions are satisfied:

(i) $|S_\nu| = \lambda_\nu$;

(ii) $\nu \in \mu \in \kappa$ implies $S_\nu < S_\mu$;

(iii) if $a, a' \in [\cup\{S_\nu: \nu \in \kappa\}]^r$ are such that $|a \cap S_\nu| = |a' \cap S_\nu|$ for each $\nu \in \kappa$ then $f(a) = f(a')$.

0.5. (The canonization lemma) *If* λ *is a singular strong limit cardinal and* $f: [\lambda]^r \to \rho$ *with* $r \in \omega$ *and* $\rho \in \lambda$, *then there is a canonical family with respect to* f.

A family of sets A is called a Δ-system (or quasi disjoint) with root D if for any two $A, A' \in \mathcal{A}$ we have $A \cap A' = D$. The following is the basic result concerning Δ-systems.

0.6. *Let* $\mathcal{A}$ *be a family of sets with* $|\mathcal{A}| = \kappa$ *and* $|A| \le \lambda$ *for each* $A \in \mathcal{A}$, *where* $\kappa > \omega$ *is regular and* $\mu^\lambda < \kappa$ *for every* $\mu < \kappa$. *Then there is a subset* $\mathcal{A}' \subset \mathcal{A}$ *such that* $|\mathcal{A}'| = |\mathcal{A}|$ *and* $\mathcal{A}'$ *is a* Δ*-system.*

The following two results are easy consequences of 0.6.

0.7. (Mišcenko's lemma) *Let* $\mathcal{H} \subset P(X)$ *be such that*

$$ord(p, \mathcal{H}) = |\{H \in \mathcal{H}: p \in H\}| \le \kappa$$

for each $p \in X$. *Then there are at most* κ *finite minimal covers of* X *by members of* $\mathcal{H}$.

0.8. (Burke's lemma) *Let* $\{\mathcal{B}_\alpha: \alpha \in \kappa\}$ *and* $\{\mathcal{C}_\alpha: \alpha \in \kappa\}$ *be families of sets of size* $\le\lambda$ *such that*

(i) $\mathcal{C}_\alpha \cap \mathcal{B}_\alpha = \emptyset$ *for each* $\alpha \in \kappa$;

(ii) $C_\alpha \cap B_\beta \neq \emptyset$ *if* $\{\alpha,\beta\} \in [\kappa]^2$.
Then $\kappa \leq 2^\lambda$.

0.6 does not remain true, even for finite sets, if κ is singular. Now if λ is singular with $cf(\lambda) = \kappa$ and $\lambda = \Sigma(\lambda_\nu: \nu \in \kappa\}$ and $\lambda_\nu < \lambda$ for each $\nu \in \kappa$, then the family $\{A_\nu: \nu \in \kappa\}$ is called a double Δ-system if

(i) for each $\nu \in \kappa$, $|A_\nu| = \lambda_\nu$ and A_ν is a Δ-system with root D_ν;
(ii) $\{D_\nu: \nu \in \kappa\}$ is a Δ-system.

It is not hard to deduce now the following result from 0.6 with a suitable thinning out procedure.

0.9. *If* λ *is as above and* A *is a family of finite sets with* $|A| = \lambda$ *then there are subfamilies* $\{A_\nu: \nu \in \kappa\} \subset P(A)$ *which form a double* Δ*-system.*

Our topological notation follows in general that of [EN 1977] and is quite standard. Instead of $\langle X,\tau\rangle$, where τ is a topology (i.e. the family of all open sets) on the set X, we usually just write X to denote the corresponding topological space. Thus we sometimes write $\tau(X)$ for the topology of X.

For X a space and κ an infinite cardinal we denote by $(X)_\kappa$ the space with the same underlining set and κ-fold intersections of open set (i.e. G_κ-sets) as the base for its topology. $D(\kappa)$ denotes the discrete space on the underlying set κ.

It will be convenient for us to denote the class of all topological spaces by $\mathcal{T}$ and the class of all T_i spaces $(0 \leq i \leq 5)$ by $\mathcal{T}_i$, and the class of compact T_i spaces by $\mathcal{C}_i$. A little deviation from standard usage is that, for us, regular = T_3, i.e. always includes T_1.

CHAPTER 1

INTRODUCTION OF THE CARDINAL FUNCTIONS

In what follows, unless otherwise stated, X is an arbitrary topological space

1.1. DEFINITION.

$$o(X) = |\tau(X)|.$$

1.2. DEFINITION. $\mathcal{B} \subset \tau(X)$ is a *base* of X if every $G \in \tau(X)$ is a union of some members of $\mathcal{B}$.

$$w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } X\} + \omega.$$

w(X) is called the *weight* of X.

1.3. DEFINITION. $\mathcal{B} \subset \tau(X)$ is a *pseudo base or ψ-base* of X if for every $p \in X$ we have

$$\{p\} = \cap\{B \in \mathcal{B}: p \in B\}.$$

Clearly X has a ψ-base if and only if $X \in T_1$. Thus in the following definition $X \in T_1$. ψw(X) is called the *pseudo weight* or *ψ-weight* of X.

$$\psi w(X) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a } \psi\text{-base of } X\} + \omega.$$

1.4. DEFINITION. $\mathcal{B} \subset \tau(X)\backslash\{\emptyset\}$ is said to be a *π-base* of X is for every non-empty open set G there is a $B \in \mathcal{B}$ with $B \subset G$.

$$\pi(X) = \min\{|\mathcal{B}| : B \text{ a } \pi\text{-base of } X\} + \omega.$$

π(X) is called the *π-weight* of X.

1.5. DEFINITION. $N \subset P(X)$ is said to be a *network* in X if every open set is the union of some members of N. (Thus a base is a network consisting of open sets.)

$$nw(X) = \min\{|N|: N \text{ a network of } X\} + \omega.$$

We call nw(X) the *net weight* of X.

1.6. DEFINITION.

$$d(X) = \min\{|S|: S \subset X \text{ and } \bar{S} = X\} + \omega,$$

d(X) is called the *density* of X.

1.7. DEFINITION. $C \subset \tau(X)\setminus\{\emptyset\}$ is called a *cellular* family if the members of C are pairwise disjoint.

$$c(X) = \sup\{|C|: C \text{ cellular in } X\} + \omega.$$

c(X) is called the *cellularity* of X.

1.8. DEFINITION. X is said to be κ-*compact* (κ-*Lindelöf*) if every open cover of X has a subcover of cardinality less than κ (at most κ).

$$L(X) = \min\{\kappa: X \text{ is } \kappa\text{-Lindelöf}\} + \omega.$$

L(X) is the *Lindelöf-degree* of X. We could of course analogously have defined the compactness degree of X, but we prefer to work with L.

1.9. DEFINITION.

$$s(X) = \sup\{|D|: D \subset X, \text{ as a subspace, is discrete}\} + \omega.$$

We call s(X) the *spread* of X.

1.10. DEFINITION. A space S is called *left* (*right*) *separated* if there is a well-ordering $<$ of S such that every final (initial) segment of S under $<$ is open. Clearly, S is left (right) separated by $<$ if and only if every $p \in S$ has a neighbourhood U_p such that $q \notin U_p$ whenever $q < p$

$(q > p)$. Such U_p are called left (right) separating neighbourhoods.

$$z(X) = \sup\{|S|: S \subset X \text{ is left separated}\} + \omega;$$

$$h(X) = \sup\{|S|: S \subset X \text{ is right separated}\} + \omega.$$

We call $z(X)$ the *width* and $h(X)$ the *height* of X.

1.11. DEFINITION.

$$p(X) = \sup\{|S|: S \subset X \text{ is closed and discrete}\} + \omega.$$

1.12. DEFINITION. $S = \{p_\alpha: \alpha \in \nu\} \subset X$ is a *free sequence* of length ν in X if for each $\alpha \in \nu$ we have

$$\overline{\{p_\beta: \beta \in \alpha\}} \cap \overline{\{p_\beta: \beta \in \nu\backslash\alpha\}} = \emptyset.$$

Clearly then S is discrete.

$$F(X) = \sup\{\kappa: \exists \text{ a free sequence of length } \kappa \text{ in } X\} + \omega.$$

1.13. DEFINITION. We denote by RO(X) the family of all *regular open* sets in X, i.e. $G \in RO(X)$ if $\overline{G} = \text{Int } G$. Similarly RC(X) is the family of *regular closed* subsets of X, i.e. $F \in RC(X) \leftrightarrow X\backslash F \in RO(X)$.

$$\rho(X) = |RO(X)| = |RC(X)|.$$

1.14. DEFINITION. If $A \subset X$ a family $\mathcal{U} \subset \tau(X)$ is a *neighbourhood base* of A in X if for every open set $G \supset A$ there is a $U \in \mathcal{U}$ with $A \subset U \subset G$. We put

$$\chi(A,X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ a neighbourhood base of A in X}\},$$

and call it the *character of* A *in* X. If $p \in X$ we write $\chi(p,X)$ instead of $\chi(\{p\},X)$.

$$\chi(X) = \sup\{\chi(p,X): p \in X\}$$

is the *character* of X.

1.15. DEFINITION. If $A \subset X$ a (*local*) *ψ-base* of A in X is a family $\mathcal{V} \subset \tau(X)$ satisfying $A = \cap\mathcal{V}$.

$$\psi(A,X) = \min\{|\mathcal{V}|: \mathcal{V} \text{ a local } \psi\text{-base of } A \text{ in } X\}.$$

Again if $p \in X$ then we write $\psi(p,X) = \psi(\{p\},X)$. The *pseudo character* of X is defined for $X \in T_1$ by

$$\psi(X) = \sup\{\psi(p,X): p \in X\}.$$

The following well-known fact will play an important role: *If $X \in C_2$ and $F \subset X$ is closed then $\psi(F,X) = \chi(F,X)$. Consequently then $\psi(X) = \chi(X)$.* Variations on the same theme are the following:

$$\Psi(X) = \sup\{\psi(F,X): F \text{ closed in } X\};$$

$$\Psi_K(X) = \sup\{\psi(C,X): C \subset X \text{ compact}\}.$$

If $X \in T_2$ then every $p \in X$ is the intersection of its closed neighbourhoods, hence we can define

$$\psi_c(p,X) = \min\{|\mathcal{V}|: \mathcal{V} \subset \tau(X) \ \& \ p \in \cap\mathcal{V} \ \& \ \{p\} = \cap\{\bar{V}: V \in \mathcal{V}\}\},$$

moreover

$$\psi_c(X) = \sup\{\psi_c(p,X): p \in X\}.$$

If $X \in T_3$ then we get analogous "closed" versions of $\psi_c(F,X)$ for F closed in X and

$$\Psi_c(X) = \sup\{\psi_c(F,X): F \text{ is closed in } X\}.$$

Finally if $X \in T_1$ then we can define

$$\psi_\Delta(X) = \psi(\Delta, X \times X),$$

where $\Delta = \{\langle x,x\rangle: x \in X\}$ is the diagonal of X. $\psi_\Delta(X)$ is a kind of "symmetric pseudocharacter" as can be seen from the following (easily established) characterization: $\psi_\Delta(X) = \kappa$ is the smallest cardinal such that to every $p \in X$ one can assign a local ψ-base

$$\mathcal{V}_p = \{V_\alpha(p): \alpha \in \kappa\}$$

such that, for each $\alpha \in \kappa$,

$$p \in V_\alpha(q) \leftrightarrow q \in V_\alpha(p).$$

1.15. If $p \in X$ a *local* π*-base* of p in X is a family $\mathcal{U} \subset \tau(X)\setminus\{\emptyset\}$ such that every neighbourhood of p contains a member of $\mathcal{U}$.

$$\pi\chi(p,X) = \min\{|\mathcal{U}|: \mathcal{U} \text{ a local } \pi\text{-base of } p \text{ in } X\}$$

is the π-*character* of p in X.

$$\pi\chi(X) = \sup\{\pi\chi(p,X): p \in X\}$$

is the π-*character* of X.

1.16. Let $p \in X$, $S \subset X$ and $p \in \bar{S}$, then

$$a(p,S) = \min\{|M|: M \subset S \ \& \ p \in \bar{M}\},$$

moreover

$$t(p,X) = \sup\{a(p,S): p \in \bar{S} \subset X\}.$$

$t(p,X)$ is the *tightness* of X in p, while

$$t(X) = \sup\{t(p,X): p \in X\}$$

is the *tightness* of X.

A set $F \subset X$ is said to be κ-*closed* if $S \subset F$ and $|S| \le \kappa$ imply $\bar{S} \subset F$. It is easy to see that $t(X) \le \kappa$ holds if and only if every κ-closed set in X is closed. This characterization of tightness is useful e.g. in proving the following proposition.

1.17. *If* $f: X \to Y$ *is a continuous and closed map of* X *onto* Y *then* $t(Y) \le t(X)$.

PROOF. Let $\kappa = t(X)$ and $F \subset Y$ be κ-closed, then $f^{-1}(F)$ is also κ-closed. Indeed, if $S \in [f^{-1}(F)]^{\le\kappa}$ and $p \in \bar{S}$ then $f(S) \in [F]^{\le\kappa}$ and $f(p) \in \overline{f(S)} \subset F$, hence $p \in f^{-1}(F)$. But $t(X) \le \kappa$ implies then that $f^{-1}(F)$ is closed, hence by the closedness of f the set F is also closed. ⊣

Recall that if $\mathcal{H} \subset P(X)$ then for $p \in X$

$$\mathrm{ord}(p,\mathcal{H}) = |\{H \in \mathcal{H}: p \in H\}|$$

and

$$\mathrm{ord}(\mathcal{H}) = \sup\{\mathrm{ord}(p,\mathcal{H}): p \in X\}.$$

1.18. DEFINITION. If $X \in T_1$ we put

$$\mathrm{psw}(X) = \min\{\mathrm{ord}(\mathcal{B}): \mathcal{B} \text{ a } \psi\text{-base of } X\}.$$

1.19. DEFINITION. The cardinal κ is *a caliber* of X if whenever $\{G_\alpha: \alpha \in \kappa\} \subset \tau(X)\setminus\{\emptyset\}$ there is a subset $A \in [\kappa]^\kappa$ with $\cap\{G_\alpha: \alpha \in A\} \ne \emptyset$. Observe that we do *not* require $G_\alpha \ne G_\beta$ for $\alpha \ne \beta$. Thus it is easy to see that if κ is a caliber of X then so is $\mathrm{cf}(\kappa)$. We shall put

$$\mathrm{cal}(X) = \{\kappa: \kappa \text{ is a caliber of } X\}.$$

It is easy to see that if $\mathrm{cf}(\kappa) > d(X)$ then $\kappa \in \mathrm{cal}(X)$, hence cal(X) is not a set. Clearly, if $\kappa \in \mathrm{cal}(X)$ then there is no cellular family of size κ in X.

1.20. DEFINITION. We say that κ is *a precaliber* of X if $\{G_\alpha: \alpha \in \kappa\} \subset \tau(X)\setminus\{\emptyset\}$ implies the existence of an $A \in [\kappa]^\kappa$ such that $\{G_\alpha: \alpha \in A\}$ is centered (i.e. has the finite intersection property). We again put

$$\mathrm{precal}(X) = \{\kappa: \kappa \text{ is a precaliber of } X\}.$$

Clearly, $\kappa \in \text{precal}(X)$ implies $\text{cf}(x) \in \text{precal}(X)$. It is easy to show that

$$\text{cal}(X) \subset \text{precal}(X)$$

for any X, moreover if $X \in C_2$ then

$$\text{cal}(X) = \text{precal}(X).$$

1.21. DEFINITION. A cardinal function ϕ is said to be monotone if $Y \subset X$ implies $\phi(Y) \le \phi(X)$ (when both defined). The functions w, nw, ψw, χ, ψ, t for example are monotone, while d, π, L and $\pi\chi$ are not. For any cardinal function ϕ we put

$$\phi^*(X) = \sup\{\phi(Y): Y \subset X\},$$

ϕ^* is called the *hereditary* (*or monotone*) *version* of ϕ. Clearly ϕ is monotone if and only if $\phi = \phi^*$.

1.22. DEFINITION. Several of our cardinal functions have been defined as suprema e.g. c, s, z, h, etc. If ϕ is a cardinal function defined in this way, i.e.

$$\phi(X) = \sup\{\kappa: \kappa \text{ has property } P_\phi\},$$

then we put

$$\hat{\phi}(X) = \min\{\lambda: \lambda > \kappa \text{ if } \kappa \text{ has property } P_\phi\}.$$

We always have $\phi(X) \le \hat{\phi}(X)$ then, while $\hat{\phi}(X) = \phi(X)^+$ if $\phi(X)$ is a successor cardinal.

CHAPTER 2

INTERRELATIONS BETWEEN CARDINAL FUNCTIONS

2.1. *Trivial inequalities*

(a) $c(X) \le d(X) \le \pi(X) \le w(X) \le o(X) \le \min\{2^{|X|}, 2^{nw(X)}\}$;

(b) $\max\{d(X), L(X)\} \le n(X) \le \min\{|X|, w(X)\}$;

(c) for $X \in T_1$, $\psi w(X) \le \min\{|X|, w(X)\}$ and $|X| \le o(X)$;

(d) for every $x \in X$, $\max\{t(x,X), \pi\chi(x,X)\} \le \chi(x,X)$, moreover if $X \in T_1$ then $\psi(x,X) \le \chi(x,X)$;

(e) $\chi(X) \le w(X) \le |X| \cdot \chi(X)$, and for $X \in T_1$, $\psi(X) \le \psi w(X)$;

(f) $\pi\chi(X) \le \pi(X) \le d(X) \cdot \pi\chi(X)$ and $t(X) \le |X|$. ⊣

2.2. *If* $X \in T_0$, $(X \in T_1)$, *then* $|X| \le 2^{w(X)}$ $(|X| \le 2^{\psi w(X)})$.

PROOF. Let $\mathcal{B}$ be a base (ψ-base) for X with $|\mathcal{B}| \le w(X)$ $(|\mathcal{B}| \le \psi w(X))$. Then $x, y \in X$ and $x \ne y$ imply

$$\{B \in \mathcal{B}: x \in B\} \ne \{B \in \mathcal{B}: y \in B\}$$

since X is T_0 (T_1), i.e. we have got a 1-1 map of X into $P(\mathcal{B})$. ⊣

2.3. *Let* $X \in T_1$, *then*

(a) $|X| \le nw(X)^{\psi(X)}$;

(b) $nw(X) \le \psi w(X)^{L(X)}$, *hence* $|X| \le \psi w(X)^{L(X) \cdot \psi(X)}$.

PROOF.

(a) Suppose $\mathcal{N}$ is a network for X with $|\mathcal{N}| \le nw(X)$. For any $p \in X$ let $\mathcal{V}_p$ be a family of neighbourhoods of p with $|\mathcal{V}_p| \le \psi(X)$, and for every $V \in \mathcal{V}_p$ pick $N_V \in \mathcal{N}$ such that $p \in N_V \subset V$. Then $\mathcal{N}_p = \{N_V : V \in \mathcal{V}_p\}$ has cardinality not exceeding $|\mathcal{V}_p| \le \psi(X)$, moreover $\cap\mathcal{N}_p = \cap\mathcal{V}_p = \{p\}$, hence the map $p \to \mathcal{N}_p$ of X into $[\mathcal{N}]^{\le\psi(X)}$ is one-to-one.

(b) Let $\mathcal{P}$ be a ψ-base for X with $|\mathcal{P}| \le \psi w(X)$. For every open set U containing a given point p, its complement $X \backslash U$ can be covered by members of $\mathcal{P}$ missing p as $\mathcal{P}$ is a ψ-base. But then it can also be covered by at most $L(X\setminus U) \le L(X)$ many such members of $\mathcal{P}$. This shows that the complements of all unions formed by at most $L(X)$ members of $\mathcal{P}$ constitute a network for X, which is clearly of the required cardinality.

REMARK. Observe that in case (b) our proof actually yields the following stronger result: If $X \in \mathcal{T}_1$ and X is κ-compact then $nw(X) \le \psi w(X)^{\kappa}$. In particular, if $X \in \mathcal{C}_1$ then $nw(X) \le \psi w(X)$. ⊣

2.4. *For* $X \in \mathcal{T}_2$

$$|X| \le \exp\exp d(X).$$

PROOF. Let $S \subset X$ be dense, $|S| \le d(X)$. For any $p \in X$ we put

$$\mathcal{S}_p = \{G \cap S: p \in G \in \tau(X)\} \subset P(S).$$

Now $p \ne q$ implies $\mathcal{S}_p \ne \mathcal{S}_q$ since X is Hausdorff, hence $p \to \mathcal{S}_p$ is a one-one map of X into $P(P(S))$, which proves our assertion. ⊣

COROLLARY. *If* $X \in \mathcal{T}_2$*, then*

$$w(X) \le \exp\exp\exp d(X). \qquad ⊣$$

2.5. *For every* $X \in \mathcal{T}_2$ *we have*

$$|X| \le d(X)^{\chi(X)}.$$

PROOF. Let S be dense in X with $|S| \le d(X)$, and for each $p \in X$ let $\mathcal{U}_p$ be a neighbourhood base of p in X with $|\mathcal{U}_p| = \chi(p,X) \le \chi(X) = \kappa$. For every non-empty open set G we pick a point $q(G) \in S \cap G$, and then put $N_p = \{q(G): G \in \mathcal{U}_p\} \in [S]^{\le\kappa}$ for $p \in X$. Clearly, then $p \in \overline{U \cap N_p}$ for every neighbourhood U of p. Consequently, as X is T_2, we have

$$\{p\} = \bigcap\{\overline{U \cap N_p}: U \in \mathcal{U}_p\},$$

hence the map $p \to \{U \cap N_p: U \in \mathcal{U}_p\}$ takes X in a one-one way into $[[S]^{\le\kappa}]^{\le\kappa}$. ⊣

REMARK. We have actually established the following, somewhat stronger result: If $X \in T_2$ and κ is a cardinal, then

$$|\{p \in X: \chi(p,X) \leq \kappa\}| \leq d(X)^{\kappa}. \qquad \dashv$$

2.6. *Let* X *be arbitrary with* $S \subset X$ *dense in* X. *Then*

(a) $c(S) = c(X)$;

(b) $d(X) \leq d(S) \leq d(X)\cdot t(X)$;

(c) $\pi(S) \leq \pi(X)$ *and for any* $p \in S$, $a(p,S) \leq \pi\chi(p,S) \leq \pi\chi(p,X)$;

(d) $\rho(S) = \rho(X) \leq \min\{\pi(X)^{c(X)}, 2^{d(X)}\}$.

PROOF.

(a) Suppose first that $\mathcal{G}$ is a disjoint family of non-empty open subsets of X. Clearly, then

$$\mathcal{G}\restriction S = \{G \cap S: G \in \mathcal{G}\}$$

is a cellular family of the same cardinality in S. Now, on the other hand, let

$$\{G \cap S: G \in \mathcal{G}\}$$

be a cellular family in S with G open in X for all $G \in \mathcal{G}$. Then $G,G' \in \mathcal{G}$ with $G \neq G'$ implies $G \cap G' = \emptyset$, since otherwise $(G \cap G') \cap S = (G \cap S) \cap (G' \cap S)$ would be non-empty. $\dashv$

(b) As S is dense in X, every dense subset of S is also dense in X, hence $d(X) \leq d(S)$. Now let T be dense in X with $|T| \leq d(X)$. For each $p \in T$ we can choose a set $S_p \in [S]^{\leq t(X)}$ with $p \in \bar{S}_p$. It is easy to see then that $S' = \bigcup\{S_p: p \in T\}$ is dense in S, hence $d(S) \leq |S'| \leq d(X)\cdot t(X)$. $\dashv$

(c) Let $\mathcal{P}$ be a (local) π-base for X (for p in X). Since the members of $\mathcal{P}\restriction S = \{P \cap S: P \in \mathcal{P}\}$ are non-empty, as S in dense, $\mathcal{P}\restriction S$ is clearly a (local) π-base for S (for p in S), while $|\mathcal{P}\restriction S| \leq |\mathcal{P}|$. $\dashv$

(d) Let $\bar{G}$ (with G open) be an arbitrary regular closed subset of X and consider the map $\bar{G} \to \bar{G} \cap S$. Since G is open and S is dense we have $\bar{G} = \overline{G \cap S}$, hence

$$\bar{G} \cap S = \overline{G \cap S} \cap S = \overline{G \cap S}^{S},$$

i.e. $\bar{G} \cap S$ is regular closed in S. Moreover, every regular closed set in S has the form $\overline{G \cap S}^S$ for a G open in X, hence the above map takes RC(X) onto RC(S). Finally, if G and H are open in X with $\bar{G} \neq \bar{H}$, then either $G \setminus \bar{H} \neq \emptyset$ or $H \setminus \bar{G} \neq \emptyset$, hence as S is dense, $\bar{G} \cap S \neq \bar{H} \cap S$, i.e. our map is also one-one.

To prove the second half of (d) first note $\rho(X) \leq 2^{d(X)}$ follows immediately from what we have just proved. Next consider a π-base $\mathcal{P}$ of X with $|\mathcal{P}| \leq \pi(X)$. For any (non-empty) open set G in X let $\mathcal{C}_G$ be a maximal disjoint family of members of $\mathcal{P}$ contained in G. Clearly, then $\overline{\cup \mathcal{C}_G} = \bar{G}$, hence if $\bar{G} \neq \bar{H}$ then $\mathcal{C}_G \neq \mathcal{C}_H$. This shows that the map $G \to \mathcal{C}_G$ takes RO(X) in a one-one way into $[\mathcal{P}]^{\leq c(X)}$. ⊣

2.7. *Let* $X \in \mathcal{T}_3$ *and* $S \subset X$ *be dense in* X. *Then*

(a) $\pi(S) = \pi(X)$, *moreover* $\chi(p,S) = \chi(p,X)$ *and* $\pi\chi(p,S) = \pi\chi(p,X)$ *whenever* $p \in S$;

(b) $w(X) \leq \rho(X) \leq \pi(X)^{c(X)} \leq 2^{d(X)}$.

PROOF.

(a) The $\leq$-parts of the equalities are obvious in view of 2.6(c). For showing the $\geq$-parts let $\mathcal{B}$ be a (local) π-base (at p) or a neighbourhood base of p *in* S, respectively, and use the regularity of X to show that $\{\text{Int } \bar{B} : B \in \mathcal{B}\}$ is a corresponding family *in* X. ⊣

(b) Only $\pi(X)^{c(X)} \leq 2^{d(X)}$ needs proof. However $c(X) \leq d(X)$ is always valid, and the regularity of X, in view of 2.7(a), implies $\pi(X) \leq 2^{d(X)}$. Compare this with the Corollary of 2.4! ⊣

2.8. *Let* $X \in \mathcal{T}_2$ *and* $p \in X$. *Then*

(a) $\psi w(X) \leq \rho(X)$;

(b) $\psi w(X) \leq nw(X)$;

(c) $\psi_c(p,X) \leq \psi(p,X) \cdot L(X)$.

PROOF.

(a) In a Hausdorff space every point is equal to the intersection of its regular closed neighbourhoods. ⊣

(b) Let $\mathcal{N}$ be a network for X of minimal cardinality. Consider the set $\mathcal{M}$ of those pairs $m = \langle N_1, N_2 \rangle \in \mathcal{N} \times \mathcal{N}$, for which there are disjoint open sets G_1 and G_2 such that $N_1 \subset G_1$ and $N_2 \subset G_2$. For each $m \in \mathcal{M}$ fix such a pair $\langle G_1^{(m)}, G_2^{(m)} \rangle$. We claim that $\{G_1^{(m)} : m \in \mathcal{M}\}$ is a ψ-base for X. Indeed, let x_1 and x_2 be distinct points of X, then they have disjoint neighbourhoods U_1 and U_2, respectively. Let us choose N_1 and N_2 from $\mathcal{N}$ in such a way that $x_1 \in N_1 \subset U_1$ and $x_2 \in N_2 \subset U_2$. Then $m = \langle N_1, N_2 \rangle \in \mathcal{M}$, hence we have $x_1 \in G_1^{(m)}$ and $x_2 \in G_2^{(m)} \subset X \setminus G_1^{(m)}$. Since $|\mathcal{M}| \le nw(X)$, we are done. ⊣

<u>REMARK</u>. Observe that the ψ-base we have produced has the stronger property that for any pair of distinct points of X it has a member which contains the first but even its closure misses the second. ⊣

(c) Let U be an arbitrary open neighbourhood of p. Since $\{p\}$ is equal to the intersection of all closed neighbourhoods of p, their complements cover $X \setminus U$. But $L(X \setminus U) \le L(X)$, hence we can find a family $\mathcal{V}_U$ of closed neighbourhoods of p with $|\mathcal{V}_U| \le L(X)$ such that $\cap \mathcal{V}_U \subset U$. Now, if $\mathcal{U}$ is ψ-base at p with $|\mathcal{U}| = \psi(p,X)$, then put $\mathcal{V} = \cup\{\mathcal{V}_U : U \in \mathcal{U}\}$. Clearly $\{p\} = \cap \mathcal{V}$ and $|\mathcal{V}| \le \psi(p,X) \cdot L(X)$. ⊣

The proofs of the above inequalities can be considered elementary in that they all boiled down to more or less straightforward counting arguments. To prove our following results, however, stronger methods are needed. Another unifying feature of them is that many of them involve hereditary versions of some of the basic cardinal functions. Therefore we first prove a few easy results concerning the hereditary versions of c, L and d.

2.9. (a) $c^*(X) = s(X)$;
(b) $L^*(X) = h(X)$;
(c) $d^*(X) = z(X)$.

<u>PROOF</u>.
(a) ⊣
(b) If S is right separated in type κ, where κ is a regular cardinal then the proper initial segments of S form an open cover of S with no subcover of cardinality $<\kappa$, hence $L(S) = \kappa$. But clearly

$$h(X) = \sup\{\kappa : \kappa \text{ is regular and } \exists S \subset X \text{ right separated in type } \kappa\},$$

hence we obtain $h(X) \le L^*(X)$.
Now assume that $L(S) \ge \kappa^+$, and let $\mathcal{G}$ be an open cover of S with no subcover of size $\le\kappa$. By transfinite induction we select points $p_\xi \in S$

and open sets $G_\xi \in G$ for $\xi \in \kappa^+$ as follows. Suppose $\xi \in \kappa^+$ and for all $\eta \in \xi$ we have already picked p_η and G_η. By our assumption $\{G_\eta : \eta \in \xi\}$ does not cover S, hence we can pick

$$p_\xi \in S \setminus \cup\{G_\eta : \eta \in \xi\},$$

and then choose $G_\xi \in G$ in such a wat that $p_\xi \in G_\xi$. Clearly, $\{p_\xi : \xi \in \kappa^+\}$ is a right separated subspace of S of type κ^+, and thus we also have $L^*(X) \le h(X)$, because either $L^*(X) = \omega$ or

$$L^*(X) = \sup\{\kappa^+ : \exists Y \subset X \text{ with } L(Y) \ge \kappa^+\}.$$

(c) Now, if S is left separated in type κ, where κ is regular, then obviously $d(S) = \kappa$, hence - similarly as in (b) - we get $z(X) \le d^*(X)$.

On the other hand, if $d(S) = \kappa$ we can select a left separated subspace S' of S of type κ as follows. Suppose $\xi \in \kappa$ and for $\eta \in \xi$ we have chosen points $p_\eta \in S$. Then $\{p_\eta : \eta \in \xi\}$ is not dense in S, hence we can pick

$$p_\xi \in S \setminus \overline{\{p_\eta : \eta \in \xi\}}.$$

Obviously $S' = \{p_\xi : \xi \in \kappa\}$ is as required. Thus we get $d^*(X) \le z(X)$. ⊣

2.10. (a) *If* $X \in T_2$, *then*

$$\psi(X) \le \psi_c(X) \le h(X).$$

(b) *If* $X \in T_3$, *then*

$$h(x) = \Psi(X) \cdot L(X).$$

PROOF. (a) Let $p \in X$ with $\psi_c(p,X) = \kappa$, moreover $\{F_\xi : \xi \in \kappa\}$ be closed neighbourhoods of p such that

$$\{p\} = \cap\{F_\xi : \xi \in \kappa\}.$$

By transfinite induction we define for each $\nu \in \kappa$ a point $p_\nu \in X$ and an ordinal $\xi_\nu \in \kappa$. Suppose we have defined p_μ and ξ_μ for $\mu \in \nu \in \kappa$. Then

$$\{p\} \neq \bigcap\{F_{\xi_\mu} : \mu \in \nu\},$$

hence we can choose

$$p_\nu \in \bigcap\{F_{\xi_\mu} : \mu \in \nu\}\setminus\{p\}$$

and $\xi_\nu \in \kappa$ such that $p_\nu \notin F_{\xi_\nu}$. Clearly, then $\nu \neq \mu$ implies $p_\nu \neq p_\mu$ and $\{p_\nu : \nu \in \kappa\}$ is right separated. ⊣

(b) Since $X \in T_3$, every closed subset of X is the intersection of its closed neighbourhoods, hence the same proof as in (a) yields $\Psi(X) \le \Psi_c(X) \le h(X)$, while $L(X) \le h(X)$ is immediate from 2.9(b), $h(X) \ge \Psi(X) \cdot L(X) = \kappa$.
Thus, using 2.9(b) again, it suffices to show that X is hereditary κ-Lindelöf, which in turn follows if every open subspace of X is κ-Lindelöf. But by assumption every open set in X is the union of at most $\Psi(X) \le \kappa$ closed sets, which are all κ-Lindelöf, hence so is their union. ⊣

2.11. $$2^{\hat{s}(X)} \le o(X) \le \min\{|X|^{z(X)}, w(X)^{h(X)}\}.$$

<u>PROOF</u>. First, if $S \subset X$ is discrete, then

$$o(S) = 2^{|S|} \le o(X).$$

To prove the other two inequalities put $z(X) = \kappa$ and $h(X) = \lambda$. By 2.9(c), every closed subset F of X is of the form $F = \bar{A}$, where $A \in [X]^{\le\kappa}$, hence $o(X) \le |[X]^{\le\kappa}| = |X|^\kappa$. Next consider an open base $\mathcal{B}$ of X with $|\mathcal{B}| \le w(X)$. In view of 2.9(b) then every open subset of X is the union of at most λ members of $\mathcal{B}$, hence

$$o(X) \le |[\mathcal{B}]^{\le\lambda}| \le w(X)^\lambda. \quad ⊣$$

Discrete subspaces play a very important rôle in our investigations. The following results yield methods to deal with them.

2.12. *If* X *is both right and left separated then there is an* $S \subset X$ *with* $|S| = |X|$ *which is discrete.*

PROOF. Suppose the well-orderings $\prec_1$ and $\prec_2$ right and left separate X, respectively. By the Erdös-Dushnik-Miller theorem 0.4(a) there is $S \subset X$ with $|S| = |X|$ such that $\prec_1$ and $\prec_2$ coincide on S. But then S is discrete. $\dashv$

2.13. *Let* $\mathcal{U}$ *be an open cover of* X *and assume that* X *has no discrete subspace of cardinality* κ. *Then there are* $\mathcal{V} \in [\mathcal{U}]^{<\kappa}$ *and* $S \in [X]^{<\kappa}$ *such that*

$$X = \bigcup\mathcal{V} \cup \bar{S}.$$

PROOF. By transfinite induction we pick points $p_\xi \in X$ and sets $U_\xi \in \mathcal{U}$ as follows. Suppose we have already picked $\{p_\eta: \eta \in \xi\}$ and $\{U_\eta: \eta \in \xi\}$ and

$$X \neq \bigcup\{U_\eta: \eta \in \xi\} \cup \overline{\{p_\eta: \eta \in \xi\}}.$$

Then we choose

$$p_\xi \in X\setminus(\bigcup\{U_\eta: \eta \in \xi\} \cup \overline{\{p_\eta: \eta \in \xi\}})$$

and $U_\xi \in \mathcal{U}$ with $p_\xi \in U_\xi$. Clearly, then $\eta \neq \xi$ implies $p_\eta \neq p_\xi$, moreover the chosen points form a discrete subspace. Hence, by our assumption, this procedure must stop before step κ. $\dashv$

2.14. *If* $X \in T_2$ *contains no discrete subspace of cardinality* κ, *then for each* $p \in X$ *either* $\psi(p,X) < \kappa$ *or* $a(p,X\setminus\{p\}) < \kappa$.

PROOF. Let $\mathcal{V}$ be a family of closed neighbourhoods of p such that $\{p\} = \bigcap\mathcal{V}$. Then $\{X\setminus V: V \in \mathcal{V}\}$ is an open cover of $X\setminus\{p\}$, hence 2.13 implies the existence of $\mathcal{V}_0 \in [\mathcal{V}]^{<\kappa}$ and $A \in [X\setminus\{p\}]^{<\kappa}$ such that

$$X\setminus\{p\} \subset \bigcup\{X\setminus V: V \in \mathcal{V}_0\} \cup \bar{A}.$$

Now, if $p \in \bar{A}$, then we have $a(p,X\setminus\{p\}) < \kappa$, while if $p \notin \bar{A}$, then $\mathcal{V}_0 \cup \{X\setminus\bar{A}\}$ yields a ψ-base at p of size $<\kappa$. $\dashv$

Now we turn to the "non-elementary" results mentioned above. There have been three main methods of proof for these in the literature:

1) ramification arguments (cf. [dG 1965] or [HJ 1967])
2) partition arguments (e.g. [HJ 1969a])
3) "closure" arguments ([Š A 1972] or [PO 1974])

the second method being actually a hidden case of the others, which can both be used to prove the corresponding partition relations. The first method is perhaps the most intuitive, while the other two are in general much more elegant and simple in presentation. For each particular result I have chosen one method of proof that to my taste was the simplest and most efficient. However, the reader is advised to try proving these results also by the other methods.

2.15. (a) *If* $X \in T_1$, *then*

$$|X| \le 2^{s(X)\cdot\psi(X)};$$

(b) *if* $X \in T_2$, *then*

$$|X| \le 2^{c(X)\cdot\chi(X)}.$$

PROOF (both for (a) and (b)). Assume that, on the contrary, $|X| > 2^{\kappa}$, where $\kappa = s(X)\cdot\psi(X)$ in case (a) ($\kappa = c(X)\cdot\chi(X)$ in case (b)). Let $\prec$ be a fixed linear ordering of X. For each $p \in X$ we let $\{U_\alpha(p): \alpha \in \kappa\}$ be a ψ-base (a neighbourhood base) at p. Now, if $p,q \in X$ and $p \prec q$ we put

$$\xi(p,q) = \min\{\alpha \in \kappa: q \notin U_\alpha(p)\},$$

$$\eta(p,q) = \min\{\beta \in \kappa: p \notin U_\beta(q)\},$$

(or, in case (b), we pick $\xi(p,q) = \alpha$ and $\eta(p,q) = \beta$ in such a way that $U_\alpha(p) \cap U_\beta(q) = \emptyset$). The map $f: [X]^2 \to \kappa\times\kappa$ defined by

$$f(\{p,q\}) = \langle \xi(p,q), \eta(p,q)\rangle$$

yields a partition of $[X]^2$ into κ pieces, hence from $|X| > 2^{\kappa}$ and the partition relation 0.4(b): $(2^{\kappa})^+ \to (\kappa^+)^2_{\kappa}$, we obtain the existence of a pair $\langle\alpha,\beta\rangle \in \kappa\times\kappa$ and a set $Y \in [X]^{\kappa^+}$ such that $\xi(p,q) = \alpha$ and $\eta(p,q) = \beta$ whenever $\{p,q\} \in [Y]^2$. Let us now put for each $p \in Y$

$$G_p = U_\alpha(p) \cap U_\beta(p).$$

It is obvious then that if $p,q \in Y$ and $p \prec q$, then $p \notin G_q$ and $q \notin G_p$ (or, in case (b), that $G_p \cap G_q = \emptyset$, while each $G_p \neq \emptyset$ since $p \in G_p$); thus in case (a) Y is discrete and in case (b) $\{G_p: p \in Y\}$ is cellular with $|Y| > \kappa$, which is a contradiction. ⊣

REMARK. Our proof for case (b) actually yields the following stronger result: If $X \in T_2$, then

$$|\{p \in X: \chi(p,X) \leq \lambda\}| \leq 2^{c(X)\cdot\lambda}.$$

Moreover, even here, instead of $\chi(p,X) \leq \lambda$ it suffices to assume that there is a local π-base $\mathcal{B}_p$ at every p such that $|\mathcal{B}_p| \leq \lambda$ and $\mathcal{B}_p$ is linked, i.e. any two of its members intersect.

Another similar result with basically the same proof is this: If X is normal then every disjoint family of closed sets of character $\leq\lambda$ in X has cardinality $\leq 2^{c(X)\cdot\lambda}$.

2.16 *If* $X \in T_2$, *then* $|X| \leq 2^{h(X)}$.

PROOF. Immediate from 2.15 (a) and 2.10(a). ⊣

2.17 *If* $X \in T_2$, *then*

$$z(X) \leq 2^{s(X)}.$$

PROOF. Put $s(X) = \kappa$ and assume that $z(X) > 2^\kappa$. Then X contains a left-separated subspace Y with $|Y| > 2^\kappa$. Applying 2.16 to Y we get a right-separated subspace $S \subset Y$ with $|S| > \kappa$. But then S is both right and left separated, hence by 2.12 it contains a discrete subspace D with $|D| = |S| > \kappa$, which is a contradiction. ⊣

Next we will use a ramification argument to give a strengthening of 2.15(a) for Hausdorff spaces. For this however, it will be convenient to introduce here a new cardinal function. Let us note first that (for a T_1 space X) we can obviously define $\psi(X)$ as the smallest cardinal κ such that $(X)_\kappa$ is discrete.

2.18. DEFINITION. *Let* $X \in T_1$, *then*

$$\psi_\ell(X) = \min\{\kappa: (X)_\kappa \text{ is left separated}\}.$$

Clearly, we always have $\psi_\ell(X) \leq \psi(X)$.

2.19. *For each* $X \in T_2$ *we have*

$$|X| \leq 2^{s(X)\cdot\psi_\ell(X)}$$

PROOF. Let $\kappa \geq \psi_\ell(X)$. It suffices to show that if $|X| > 2^\kappa$ then X contains a discrete subspace of cardinality κ^+. Now, by our assumption, we can fix a well-ordering $\prec$ of X and for each point $p \in X$ a family of open neighbourhoods $\{V_\xi(p): \xi \in \kappa\}$ such that

$$\cap\{V_\xi(p): \xi \in \kappa\} \subset \{q \in X: p \preceq q\}.$$

Next we build a ramification system of height κ^+ on X. In order to have transparent notation, however, we first introduce some operations on subsets of X. Thus let $A \subset X$ with $|A| \geq 2$. We denote by $x^0(A)$ the first and by $x^1(A)$ the second member of A under $\prec$, moreover we put $F(A) = \{x^0(A), x^1(A)\}$. Since X is Hausdorff, we can choose closed sets $E^i(A)$ in X for $i \in 2$ such that

$$x^i(A) \notin E^i(A) \text{ for } i \in 2, \text{ and } E^0(A) \cup E^1(A) = X.$$

Finally, for any $\xi \in \kappa$ and $i \in 2$ we put

$$\Omega_{2\cdot\xi+i}(A) = \{y \in [A\backslash F(A)] \cap E^i(A): x^i(A) \notin V_\xi(y)\}.$$

Since $x^i(A) \prec y$ for each $y \in A\backslash F(A)$, the above definitions clearly imply that

$$A\backslash F(A) = \cup\{\Omega_\eta(A): \eta \in \kappa\}.$$

For the sake of completeness we put $F(A) = A$ and $\Omega_\eta(A) = \emptyset$ for all $\eta \in \kappa$ whenever $|A| \le 1$. Now we define the sets S_t and F_t of our ramification system for all sequences $t \in \kappa^{\underset{\smile}{\kappa^+}}$ by transfinite induction as follows. We put $S_\emptyset = X$. If S_t has been defined then we let $F_t = F(S_t)$, and for each $\eta \in \kappa$ we let

$$S_{t\frown\eta} = \Omega_\eta(S_t).$$

Finally, if $\nu \in \kappa^+$ is a limit ordinal and S_u has already been defined for each $u \in \kappa^{\underset{\smile}{\nu}}$, then we put for any $t \in \kappa^\nu$

$$S_t = \bigcap\{S_{t\upharpoonright\mu}\colon \mu \in \nu\}.$$

It is easy to see that the conditions of the ramification lemma apply, hence we can find a sequence $t \in \kappa^{\kappa^+}$ such that for each $\nu \in \kappa$ we have

$$S_{t\upharpoonright\nu} \ne \emptyset, \text{ hence } |S_{t\upharpoonright\nu}| \ge 2.$$

Now, for every $\nu \in \kappa^+$ we define $i(\nu) \in 2$ and $\xi(\nu) \in \kappa$ from the relation

$$t(\nu) = 2\cdot\xi(\nu) + i(\nu),$$

and put

$$p_\nu = x^{i(\nu)}(S_{t\upharpoonright\nu}).$$

Clearly, there is a fixed pair $\langle i,\xi\rangle \in 2\times\kappa$ such that if $a = \{\nu \in \kappa^+\colon i(\nu) = i \text{ and } \xi(\nu) = \xi\}$, then $|a| = \kappa^+$. We claim that the set $\{p_\nu\colon \nu \in a\}$ is discrete in X. Indeed, for each $\nu \in a$ we have $p_\nu = x^i(S_{t\upharpoonright\nu}) \notin E^i(S_{t\upharpoonright\nu})$, while $p_\mu \in E^i(S_{t\upharpoonright\nu})$ for each $\mu > \nu$, moreover

$$p_\mu \in S_{t\upharpoonright\nu+1} = S_{t\upharpoonright\nu\frown t(\nu)} = \Omega_{2\cdot\xi+i}(S_{t\upharpoonright\nu})$$

implies that $p_\nu \notin V_\xi(p_\mu)$ for $\nu \in a$ and $\nu < \mu$, hence $\{p_\nu\colon \nu \in a\}$ is both right and left separated in the well-ordering induced by its indices. $\dashv$

REMARK. The above proof actually yields the following somewhat stronger result: If $X \in T_2$, κ is regular, $|X| > \Sigma\{2^\lambda : \lambda < \kappa\}$ and X has a well-ordering $\prec$ such that for each $p \in X$ we can choose a system of neighbourhoods V_p with $|V_p| < \kappa$ and $q \notin \cap V_p$ for each $q \prec p$, then X has a discrete subspace of cardinality κ. I don't know whether the assumption on the regularity of κ can be omitted or not.

2.20. *For each* $X \in T_2$ *there is an* $S \subset X$ *such that*

$$|S| \le 2^{s(X)} \text{ and } X = \cup\{\bar{T}: T \in [S]^{\le s(X)}\}.$$

Consequently, we have

$$|X| \le 2^{2^{s(X)}}.$$

PROOF. Let us put $s(X) = \kappa$. By a straightforward transfinite induction we can construct a subspace $S = \{p_\xi : \xi \in \phi\} \subset X$ for some ordinal ϕ such that:

(a) no $p_\xi \in S$ is in the closure of at most κ-many previous p_η's;

(b) every point in X is in the closure of at most κ-many points from S, i.e. $X = \cup\{\bar{T}: T \in [S]^{\le\kappa}\}$. Now, it suffices to show that $|S| \le 2^\kappa$. Let us denote the initial segment $\{p_\eta : \eta \le \xi\}$ of S by S_ξ. Then S_ξ contains no discrete subspace of size κ^+, hence 2.14 applied to S_ξ and p_ξ, in view of (a), yields $\psi(p_\xi, S_\xi) \le \kappa$. In other words, we have $\psi_\ell(S) \le \kappa$, hence by 2.19 $|S| \le 2^\kappa$.

Now $|X| \le 2^{2^{s(X)}}$ follows easily from $X = \cup\{\bar{T}: T \in [S]^{\le\kappa}\}$ and 2.4. ⊣

2.21. *If* $X \in T_2$, *then* $o(X) \le \exp\exp s(X)$.

PROOF. Indeed, from 2.11, 2.20 and 2.17 we obtain

$$o(X) \le |X|^{z(X)} \le (\exp\exp s(X))^{\exp s(X)} = \exp\exp s(X). \quad ⊣$$

2.22. (a) *If* $X \in T_2$, *then*

$$\psi w(X) \le 2^{s(X)}.$$

(b) *If* $X \in T_3$, *then*

$$nw(X) \le 2^{s(X)}.$$

PROOF.

(a) We claim that the family

$$M = \{\bar{T}\colon T \in [S]^{\le s(X)}\},$$

where S is chosen as in 2.20 separates the points of X, hence $\{X\setminus\bar{T}\colon \bar{T} \in M\}$ is a ψ-base of X of cardinality at most $2^{s(X)}$. Indeed, let $p,q \in X$, $p \ne q$, moreover U be a neighbourhood of p with $q \notin \bar{U}$. Now let $\bar{T} \in M$ be such that $p \in \bar{T}$, then we also have $p \in \overline{U \cap T}$, but clearly $\overline{U \cap T} \in M$ and $q \notin \overline{U \cap T}$. ⊣

(b) In this case we claim that the above family M is a network for X. In fact, if $p \in U$ with U open in X, then by the regularity of X we can take a neighbourhood V of p such that $\bar{V} \subset U$. If we take again a $\bar{T} \in M$ with $p \in \bar{T}$, then $p \in \overline{T \cap V} \subset U$ and $\overline{T \cap V} \in M$. ⊣

Next we give two rather easy results, which do not have much to do with the above, but they still fit best here.

2.23. (a) *If* X *is hereditarily collectionwise Hausdorff, then*

$$c(X) = s(X).$$

(b) *If* X *is hereditarily paracompact, then*

$$c(X) = h(X).$$

PROOF.

(a) Suppose that $D \subset X$ is discrete. Then $D' = \bar{D}\setminus D$ is nowhere dense in X, since for any non-empty open set G if $G \cap D' \ne \emptyset$ then $G \cap D \ne \emptyset$ as well, and for any $p \in G \cap D$ there is an open neighbourhood U_p with $D \cap U_p = \{p\}$, hence either $G \cap U\setminus\{p\}$ or $\{p\}$ is a non-empty open subset of G which misses D'. Now, look at the subspace $Y = X\setminus D'$ of X, then Y is dense in X, moreover $D \subset Y$ is closed discrete in Y. But Y is collectionwise Hausdorff, hence for each $p \in D$ there is an open $V_p \ni p$ in Y such that $p \ne q$ implies $V_p \cap V_q = \emptyset$. Therefore, using 2.6(a), we have

$$c(X) = c(Y) \geq |D|. \quad \dashv$$

REMARK. We have actually shown the following somewhat stronger result: If D is a discrete subspace of X then its members can be separated by pairwise disjoint neighbourhoods in X.

(b) It suffices to show that if $h(X) > \kappa$ then $c(X) > \kappa$ as well. But if $h(X) > \kappa$, then by 2.9(b) we have a $Y \subset X$ with $L(Y) > \kappa$. Thus there is an open (in X) cover $\mathcal{G}$ of Y such that no $\mathcal{G}' \in [\mathcal{G}]^{\leq\kappa}$ covers Y. Let us put $G = \cup\mathcal{G}$. Since G is paracompact, its open cover $\mathcal{G}$ has a σ-disjoint (even σ-discrete in G) refinement

$$\mathcal{U} = \cup\{\mathcal{U}_n : n \in \omega\}.$$

By our assumption then we must have $|\mathcal{U}| > \kappa$, hence $|\mathcal{U}_n| > \kappa$ for some $n \in \omega$, but $\mathcal{U}_n$ is cellular in X. $\dashv$

Now we leave the hereditary versions and turn to another bunch of results, most of which yield upper bounds for the cardinality of spaces in terms of their Lindelöf number and some other cardinal functions. Of course the, by now well-konwn, celebrated theorem of Archangelskii is the paradigm of these results. We shall start with a set-theoretical lemma that will be crucial in what follows.

2.24. *Let* $\lambda \leq \kappa < \mu$ *be infinite cardinals such that* $\kappa^{\lambda} = \kappa$, *moreover* $G: [\mu]^{<\lambda} \to [\mu]^{\leq\kappa}$ *be a set mapping over* μ.

(a) *There exists a set* $A \in [\mu]^{\kappa}$ *which is closed with respect to* G.

(b) *If* $\mu = \kappa^+$ *and* $\lambda \leq \rho \leq \kappa$ *with* ρ *a regular cardinal, then for each* $\xi \in \kappa^+$ *there is an* $\eta \in \kappa^+ \setminus \xi$ *such that* $cf(\eta) = \rho$ *and* ρ *is closed with respect to* G.

PROOF.

(a) Since $\kappa^{cf(\kappa)} > \kappa = \kappa^{\lambda}$, we have $\lambda \leq cf(\kappa)$, hence $\lambda < \kappa$ if κ is singular. Consequently we can always choose a regular cardinal ρ with $\lambda \leq \rho \leq \kappa$. Now we define by transfinite induction sets $A_\alpha \in [\mu]^{\kappa}$ for all $\alpha \in \rho$ as follows. Let $A_0 \in [\mu]^{\kappa}$ be arbitrary and assume that $A_\beta \in [\mu]^{\kappa}$ has been defined for each $\beta \in \alpha$ with a fixed $\alpha \in \rho$.

If α is limit, put

$$A_\alpha = \bigcup\{A_\beta : \beta \in \alpha\},$$

and if $\alpha = \beta + 1$, put

$$A_\alpha = \bigcup\{G(H) : H \in [A_\beta]^{<\lambda}\} \cup A_\beta.$$

Since $\rho \leq \kappa$ and $\kappa^{\underset{\cdot}{\lambda}} = \kappa$ we clearly have $|A_\alpha| = \kappa$. Now put

$$A = \bigcup\{A_\alpha : \alpha \in \rho\}.$$

Clearly $|A| = \kappa$ as well. We claim that A is closed with respect to G. Indeed, if $H \in [A]^{<\lambda}$, then $\lambda \leq \rho = \mathrm{cf}(\rho)$ implies that $H \in [A_\alpha]^{<\lambda}$ for some $\alpha \in \rho$, hence

$$G(H) \subset A_{\alpha+1} \subset A. \quad \dashv$$

(b) The proof is quite similar to that of (a), except that now we define a strictly increasing sequence of ordinals $\xi_\alpha \in \kappa^+$ (instead of the arbitrary sets A_α) such that $\xi_0 = \max\{\xi, \kappa\}$, $\xi_\alpha = \bigcup\{\xi_\beta : \beta \in \alpha\}$ for α limit and

$$\bigcup\{G(H) : H \in [\xi_\beta]^{<\lambda}\} \subset \xi_{\beta+1}.$$

Clearly, then $\eta = \bigcup\{\xi_\alpha : \alpha \in \rho\}$ will be as required. $\dashv$

Next we prove a very general and strong result, which accordingly has a rather weird formulation. Before doing that however it will be convenient to introduce some new notation.

2.25. DEFINITION. *Let* X *be a* T_1*-space,* Y *its subspace and* $F \subset X$ *an arbitrary set. Then* $\psi(F \upharpoonright Y, X)$, *the pseudo-character of* F *relative to* Y *in* X, *is the smallest cardinality of a family* $\mathcal{U}$ *of open sets in* X *such that*

$$F \subset \bigcap\mathcal{U} \text{ and } F \cap Y = \bigcap\mathcal{U} \cap Y.$$

Note that we always have

$$\psi(F\cap Y,Y) \le \psi(F\upharpoonright Y,X) \le \psi(F,X).$$

2.26. *Let* $X \in T_1$ *and* $\lambda \le \kappa$ *be infinite cardinals such that* $\kappa^{\lambda} = \kappa$, *and* $t(X) < \kappa$; *let* $Y \subset X$ *be such that for each* $S \in [Y]^{\le\kappa}$

$$\psi(\bar{S}\upharpoonright Y,X) \le \kappa \text{ and } \bar{S} \text{ is } \lambda\text{-compact}.$$

Then

$$z(Y) \le \kappa.$$

PROOF. Assume, indirectly, that $z(Y) > \kappa$, hence Y contains a left separated subspace of cardinality κ^+. As our condition on Y is clearly inherited by its subspaces, we can actually assume that Y itself is left separated in type κ^+, i.e. $Y = \{p_\alpha : \alpha \in \kappa^+\}$ with a one-one indexing and for each $\alpha \in \kappa^+$

$$S_\alpha = \{p_\beta : \beta \in \alpha\}$$

is closed in Y. Thus if we put $F_\alpha = \bar{S}_\alpha$ (closure taken in X), then $F_\alpha \cap Y = S_\alpha$. Moreover, in view of our assumption about Y, we have

$$\psi(F_\alpha \upharpoonright Y,X) \le \kappa$$

for each $\alpha \in \kappa^+$. Consequently we can fix a family of open sets $\mathcal{U}_\alpha$ such that $|\mathcal{U}_\alpha| \le \kappa$, $F_\alpha \subset \bigcap\mathcal{U}_\alpha$ and $\bigcap\mathcal{U}_\alpha \cap Y = F_\alpha \cap Y = S_\alpha$. Put for any $\alpha \in \kappa^+$

$$\mathcal{V}_\alpha = \bigcup\{\mathcal{U}_\beta : \beta \in \alpha\},$$

clearly $|\mathcal{V}_\alpha| \le \kappa$ as well.

Next we define a set mapping $G: [Y]^{<\lambda} \to [Y]^{\le\kappa}$. Let $I \in [Y]^{<\lambda}$, then $\lambda \le \kappa < \kappa^+$ implies $I \subset S_{\alpha(I)}$ for some $\alpha(I) \in \kappa^+$. Let us put $\mathcal{W}_I = \{\mathcal{V} \in [\mathcal{V}_{\alpha(I)}]^{<\lambda} : I \subset \bigcup\mathcal{V} \;\&\; Y\setminus\bigcup\mathcal{V} \ne \emptyset\}$, and for each $\mathcal{V} \in \mathcal{W}_I$ pick a point

$$q(\mathcal{V}) \in Y\setminus\bigcup\mathcal{V}.$$

Now, finally, we can put

$$G(I) = \{q(\mathcal{V}) : \mathcal{V} \in \mathcal{W}_I\};$$

since $\kappa^{\lambda} = \kappa$, we have

$$|G(I)| \leq |\mathcal{W}_I| \leq \kappa,$$

hence G is as desired.

Next we choose a regular cardinal ρ such that $\lambda \leq \rho \leq \kappa$ and $t(X) < \rho$. Such a ρ exists because either κ is regular, and then κ itself can serve as such a ρ, or κ is singular, in which case $\lambda < \kappa$, as we have shown in the proof of 2.24(a), and then we can put

$$\rho = (\max\{\lambda, t(X)\})^+ < \kappa.$$

Now our set mapping G and the cardinals $\lambda \leq \rho \leq \kappa$ satisfy the conditions of 2.24(b), with a little, but innocent, abuse of notation, hence we can apply it to obtain a $\beta \in \kappa^+$ such that $cf(\beta) = \rho$ and S_β is closed with respect to G (the role of ξ is immaterial here).

Let us note that $t(X) < \rho = cf(\beta)$ implies

$$F_\beta = \cup\{F_\alpha : \alpha \in \beta\}.$$

Now for each $\alpha \in \beta$ we have $p_\beta \notin F_\alpha$, hence by the choice of $\mathcal{U}_\alpha$ we can pick a $V_\alpha \in \mathcal{U}_\alpha$ such that $p_\beta \notin V_\alpha$. But $F_\alpha \subset V_\alpha$, hence

$$\cup\{V_\alpha : \alpha \in \beta\} \supset F_\beta = \cup\{F_\alpha : \alpha \in \beta\},$$

moreover, $F_\beta = \bar{S}_\beta$ is λ-compact, therefore there is an $I \in [S_\beta]^{<\lambda}$ such that

$$\cup\{V_\alpha : p_\alpha \in I\} \supset F_\beta \supset I.$$

It is clear from our construction that

$$\mathcal{V} = \{V_\alpha : p_\alpha \in I\} \in [\mathcal{V}_{\alpha(I)}]^{<\lambda},$$

moreover $p_\beta \notin \cup\mathcal{V}$, hence $\mathcal{V} \in \mathcal{W}_I$. But then $q(\mathcal{V}) \in G(I)$ was chosen in such a way that

$$q(\mathcal{V}) \in Y\setminus\cup\mathcal{V} \subset Y\setminus F_\beta = Y\setminus S_\beta,$$

contradicting that S_β is closed with respect to G. ⊣

2.27. *If* $X \in T_2$, *then*

$$|X| \le 2^{L(X)\cdot\psi(X)\cdot t(X)}.$$

PROOF. Let us put $\mu = L(X)\cdot\psi(X)\cdot t(X)$, $\lambda = \mu^+$, and $\kappa = 2^\mu$; then we have $\kappa^{\lambda} = (2^\mu)^\mu = \kappa$, X is λ-compact, as being μ^+-compact is the same as being μ-Lindelöf, and $t(X) \le \mu < \kappa$. Next we are going to show that for each $S \in [X]^{\le\kappa}$

$$\psi(\bar{S},X) = \psi(\bar{S}|X,X) \le \kappa.$$

First, however, we show that if $S \in [X]^{\le\kappa}$ then $|\bar{S}| \le \kappa$ as well. Since $t(X) \le \mu$ implies

$$\bar{S} = \cup\{\bar{T}\colon T \in [S]^{\le\mu}\},$$

it suffices to show that if $T \in [X]^{\le\mu}$ then $|\bar{T}| \le \kappa$, as $|[S]^{\le\mu}| \le \kappa^\mu = \kappa$. But from 2.8(a) and 2.6(d) we have

$$\psi w(\bar{T}) \le \rho(\bar{T}) \le 2^{|T|} \le 2^\mu = \kappa,$$

moreover from 2.3 we get

$$|\bar{T}| \le \psi w(\bar{T})^{L(\bar{T})\cdot\psi(\bar{T})} \le \kappa^\mu = \kappa.$$

Now, for an $S \in [X]^{\le\kappa}$, consider for each $p \in \bar{S}$ a local ψ-base $\mathcal{V}_p$ with $|\mathcal{V}_p| \le \mu$, and put

$$\mathcal{V} = \cup\{\mathcal{V}_p\colon p \in \bar{S}\}.$$

Then $|\mathcal{V}| \le |\bar{S}|\cdot\mu \le \kappa$, hence we have

$$|[\mathcal{V}]^{\le\mu}| \le \kappa^{\mu} = \kappa$$

as well. Thus $\psi(\bar{S},X) \le \kappa$ will follow if we can show that

$$\cap\mathcal{W} = \bar{S}$$

for

$$\mathcal{W} = \{\cup\mathcal{U} : \mathcal{U} \in [\mathcal{V}]^{\le\mu} \ \& \ \bar{S} \subset \cup\mathcal{U}\}.$$

Consider any $q \in X\backslash\bar{S}$, then for each $p \in \bar{S}$ there is a $V_p \in \mathcal{V}_p$ with $q \notin V_p$. Since

$$L(\bar{S}) \le L(X) \le \mu,$$

the open cover $\{V_p : p \in \bar{S}\}$ of $\bar{S}$ has a subcover $\mathcal{U}$ with $|\mathcal{U}| \le \mu$. But then $\cup\mathcal{U} \in \mathcal{W}$, while $q \notin \cup\mathcal{U}$, hence indeed $\bar{S} = \cap\mathcal{W}$.
Thus we see that X together with λ and κ, moreover with $Y = X$, satisfy the conditions of 2.26, consequently we have

$$d(X) \le z(X) \le \kappa.$$

But then $X = \bar{S}$ for an $S \in [X]^{\le\kappa}$, hence $|X| \le \kappa$ according to what we have shown above. $\dashv$

<u>COROLLARY</u>. (Archangelskii's theorem). *If* $X \in \mathcal{T}_2$, *then*

$$|X| \le 2^{L(X)\cdot\chi(X)}. \qquad \dashv$$

A.V. Archangelskiǐ has raised the very natural problem whether $|X|$ has a similar upper bound in terms of $L(X)\cdot\psi(X)$. It is easy to see that if $X \in \mathcal{T}_1$ and $L(X)\cdot\psi(X) < \kappa$, where κ is a measurable cardinal, then $|X| < \kappa$ as well. On the other hand example 7.2 shows that if μ is the first measureable cardinal, then for each $\kappa < \mu$ there is a T_1-space X such that $L(X)\cdot\psi(X) = \omega$, but $|X| > \kappa$. Moreover, S. Shelah has recently proved the consistency of "ZFC + CH + there exists a regular space X with $L(X)\cdot\psi(X) = \omega$ and $|X| = \omega_2 > \omega_1 = 2^{\omega}$" (cf. [SH 1978], or [HJ 1980b]). This, however, leaves open an enormous gap between ω_2 (or $(2^{\omega})^{+}$, if you

like) and the first measurable cardinal.
We will now present several results which can be considered as partial solutions of Archengelskii's problem in that they yield upper bounds for $|X|$ in terms of either $L(X)\cdot\psi(X)$ plus some additional information about X or some "$L(X)\cdot\psi(X)$-like" expression.

2.28. *If* $X \in T_1$, *then*

$$|X| \le 2^{p(X)\cdot\psi_\Delta(X)} \le 2^{L(X)\cdot\psi_\Delta(X)}.$$

PROOF. Let $\kappa = p(X)\cdot\psi_\Delta(X)$, then (cf. 1.14) we can choose for each $p \in X$ a system of open neighbourhoods $\{U_\alpha(p): \alpha \in \kappa\}$ such that $\cap\{U_\alpha(p): \alpha \in \kappa\} = \{p\}$, moreover $p \in U_\alpha(q) \leftrightarrow q \in U_\alpha(p)$. Assume now that $|X| > 2^\kappa$, and for each pair $\{p,q\} \in [X]^2$ put

$$f(\{p,q\}) = \min\{\alpha \in \kappa: q \notin U_\alpha(p)\}.$$

Using $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ (cf. (0.4.b)) we obtain a homogeneous $H \in [X]^{\kappa^+}$ for the partition f of $[X]^2$, i.e. we have a fixed $\alpha \in \kappa$ such that $f(\{p,q\}) = \alpha$ whenever $\{p,q\} \in [H]^2$. Now let $S \supset H$ be maximal with the property that $f([S]^2) = \{\alpha\}$, which exists by Zorn's lemma. But then for each $q \in X$ we have a $p \in S$ with $q \in U_\alpha(p)$, where $|U_\alpha(p) \cap S| = 1$, hence q is not an accumulation point of S, i.e. S is closed discrete with

$$|S| \ge |H| = \kappa^+,$$

a contradiction to $p(X) \le \kappa$. ⊣

2.29. *Let* $X \in T_1$, *then*
(a) $d(X) \le psw(X)^{L(X)}$;
(b) $\psi w(X) \le psw(X)^{L(X)}$;
(c) $|X| \le psw(X)^{L(X)\cdot\psi(X)}$.

PROOF. (a) Let us put $L(X) = \lambda$ and $psw(X) = \kappa$. Then, by definition, we have a ψ-base $\mathcal{B}$ of X such that $ord(\mathcal{B}) = \kappa$. Next we shall define a set mapping $G: [X]^{\le\kappa^\lambda} \to [X]^{\le\kappa^\lambda}$. For any $A \in [X]^{\le\kappa^\lambda}$ put

$$\mathcal{B}_A = \{B \in \mathcal{B}: B \cap A \neq \emptyset\},$$

and

$$\mathcal{U}_A = \{\mathcal{U} \in [\mathcal{B}_A]^{\leq\lambda}: X \setminus \cup\mathcal{U} \neq \emptyset\}.$$

Since $\mathrm{ord}(\mathcal{B}) = \kappa$, we have $|\mathcal{B}_A| \leq \kappa \cdot |A| \leq \kappa^\lambda$, hence $|\mathcal{U}_A| \leq |\mathcal{B}_A|^\lambda \leq \kappa^\lambda$ as well. Now for each $\mathcal{U} \in \mathcal{U}_A$ pick a point $p(\mathcal{U}) \in X \setminus \cup\mathcal{U}$ and put

$$G(A) = A \cup \{p(\mathcal{U}): \mathcal{U} \in \mathcal{U}_A\}.$$

Then according to our above remarks $|G(A)| \leq \kappa^\lambda$.
Next we define subsets A_α of X for all $\alpha \in \lambda^+$ using the following recursive formula:

$$A_\alpha = G(\cup\{A_\beta: \beta \in \alpha\}).$$

It is shown by a straightforward transfinite induction then that $|A_\alpha| \leq \kappa^\lambda$ for each $\alpha \in \lambda^+$, moreover $A_\alpha \subset A_\beta$ for $\alpha < \beta$. Let us put

$$S = \cup\{A_\alpha: \alpha \in \lambda^+\},$$

clearly $|S| \leq \lambda^+ \cdot \kappa^\lambda = \kappa^\lambda$. We claim that S is dense in X.
Assume, on the contrary, that $X \setminus \bar{S} \neq \emptyset$, and let $q \in X \setminus \bar{S}$. Let us choose for each $p \in \bar{S}$ a set $B_p \in \mathcal{B}$ with $q \notin B_p$. Since $L(\bar{S}) \leq L(X) = \lambda$, we can find a set $T \in [\bar{S}]^{\leq\lambda}$ such that

$$\mathcal{U} = \{B_p: p \in T\}$$

covers $\bar{S}$. For each $p \in T$ we have $p \in \bar{S} \cap B_p \neq \emptyset$, hence $S \cap B_p \neq \emptyset$ as well. But then for each $p \in T$ there is an $\alpha_p \in \lambda^+$ with

$$A_{\alpha_p} \cap B_p \neq \emptyset,$$

hence because $|T| \leq \lambda$ and the sets A_α are increasing, there is a fixed $\alpha \in \lambda^+$ with

$$A_\alpha \cap B_p \neq \emptyset$$

for all $p \in T$. But then $q \notin \bigcup\mathcal{U}$ implies

$$\mathcal{U} \in \mathfrak{U}_{A_\alpha},$$

hence

$$p(\mathcal{U}) \in G(A_\alpha) = A_{\alpha+1} \subset S \subset \bar{S} \subset \bigcup\mathcal{U},$$

contradicting that $p(\mathcal{U}) \in X \setminus \bigcup\mathcal{U}$. ⊣

(b) Let us use the notations from the proof of (a). Since S is dense in X every $B \in \mathcal{B}$ intersects it, hence using $\mathrm{ord}(\mathcal{B}) \leq \kappa$,

$$\psi w(X) \leq |\mathcal{B}| \leq \kappa \cdot |S| \leq \kappa^\lambda. \quad ⊣$$

(c) From 2.3 and the above result we get

$$|X| \leq \psi w(X)^{L(X)\cdot\psi(X)} \leq psw(X)^{L(X)\cdot\psi(X)}. \quad ⊣$$

In the following two results the cardinal function $p(X)$ will take the place of $L(X)$. Therefore we first present a result concerning $p(X)$ that is of independent interest.

2.30. *If* $X \in T_1$, *then*

$$s(X) \leq p(X) \cdot \Psi(X).$$

PROOF. Let $D \subset X$ be a discrete subspace of X. We can pick for each $p \in D$ an open neighbourhood U_p such that $U_p \cap D = \{p\}$. Let us put

$$U = \bigcup\{U_p : p \in D\}.$$

Then U is open, hence it can be written as

$$U = \bigcup\{F_\alpha : \alpha \in \Psi(X)\},$$

where each F_α is closed. We claim that the set $D \cap F_\alpha$ is also closed for any $\alpha \in \Psi(X)$. Indeed, if $p \notin D \cap F_\alpha$, then either $p \in X \setminus F_\alpha$ or $p \in F_\alpha \setminus D \subset U$, hence $p \in U_q \setminus \{q\}$ for some $q \in D$, thus in either case p has a neighbourhood that misses $D \cap F_\alpha$. But then $D \cap F_\alpha$ is closed discrete, therefore

$$|D| \le \Sigma\{|D \cap F_\alpha| : \alpha \in \Psi(X)\} \le p(X)\cdot\Psi(X). \quad \dashv$$

2.31. *If* $X \in T_1$, *then*

$$|X| \le 2^{p(X)\cdot\Psi(X)}.$$

PROOF. Applying 2.15(a) and 2.30 we get

$$|X| \le 2^{s(X)\cdot\psi(X)} \le (2^{p(X)\cdot\Psi(X)})^{\psi(X)} = 2^{p(X)\cdot\Psi(X)}. \quad \dashv$$

In order to prove our next analogous result we again need a lemma of independent interest, which therefore we formulate and prove separately.

2.32. *Let* $X \in T_1$ *and* $\mathcal{G}$ *be an open cover of* X *such that* $p(X) \le \kappa$ *and* $\mathrm{ord}(\mathcal{G}) \le \kappa$. *Then* $\mathcal{G}$ *has a subcover* $\mathcal{V}$ *of cardinality at most* κ.

PROOF. Let us consider the family

$$\mathcal{S} = \{S \subset X : (\forall G \in \mathcal{G})(|S \cap G| \le 1)\}.$$

Clearly $\mathcal{S}$ is closed with respect to increasing unions of its members, hence using Zorn's lemma we can find a set $S \in \mathcal{S}$ which is maximal in $\mathcal{S}$. Now, for arbitrary $p \in X$ we have a $G \in \mathcal{G}$ with $p \in G$, and $|G \cap S| \le 1$, hence S is closed discrete in X, consequently $|S| \le \kappa$. Let us put

$$\mathcal{V} = \{G \in \mathcal{G} : G \cap S \ne \emptyset\}.$$

Since $\mathrm{ord}(\mathcal{G}) \le \kappa$, we have then

$$|\mathcal{V}| \le \kappa\cdot|S| = \kappa.$$

We claim that $\mathcal{V}$ covers X. Indeed, for any $p \in X \setminus S$ by the maximality of S we have a $G \in \mathcal{G}$ with $p \in G$ and $G \cap S \neq \emptyset$, and then $G \in \mathcal{V}$ as well. ⊣

2.33 *If* $X \in \mathcal{T}_1$, *then*

$$|X| \leq 2^{p(X) \cdot psw(X)}.$$

PROOF. Let us put $\kappa = p(X) \cdot psw(X)$ and $\mathcal{B}$ be a ψ-base of X with $ord(\mathcal{B}) \leq \kappa$. For any $p \in X$ we put

$$\mathcal{B}_p = \{B \in \mathcal{B}: p \in B\},$$

then $|\mathcal{B}_p| \leq \kappa$. Since $\psi(p,X) \leq \kappa$, the complement of $\{p\}$, i.e. $X\setminus\{p\}$ is the union of at most κ closed sets in X, consequently we also have

$$p(X\setminus\{p\}) \leq \kappa.$$

Since $\mathcal{B}$ is a ψ-base we have for each $p \in X$

$$X\setminus\{p\} = \bigcup(\mathcal{B}\setminus\mathcal{B}_p);$$

but then $ord(\mathcal{B}\setminus\mathcal{B}_p) \leq ord(\mathcal{B}) \leq \kappa$ and $p(X\setminus\{p\}) \leq \kappa$ imply, in view of 2.32, the existence of a

$$\mathcal{C}_p \in [\mathcal{B}\setminus\mathcal{B}_p]^{\leq\kappa} \text{ with } X\setminus\{p\} = \bigcup\mathcal{C}_p.$$

Now we have $|\mathcal{B}_p|, |\mathcal{C}_p| \leq \kappa$ and $\mathcal{B}_p \cap \mathcal{C}_p = \emptyset$ for each $p \in X$, moreover if $p \neq q$ then $q \in B$ for some $B \in \mathcal{C}_p$, hence

$$B \in \mathcal{B}_q \cap \mathcal{C}_p \neq \emptyset.$$

But then, applying Burke's lemma, 0.8, for the family of pairs $\{\langle\mathcal{B}_p, \mathcal{C}_p\rangle: p \in X\}$ we obtain that $|X| \leq 2^\kappa$. ⊣

Now we turn to proving a result which yields a common generalization of the inequality 2.15(b), $|X| \leq \exp(c(X)\cdot\chi(X))$, and of Archangelskii's inequality $|X| \leq \exp(L(X)\cdot\chi(X))$, but unfortunately only for $X \in \mathcal{T}_4$. It is necessary to introduce some definitions for this.

2.34 DEFINITION. *A family $\mathcal{S}$ of subsets of S is said to be a weak cover of X if $X = \overline{\cup \mathcal{S}}$, i.e. $\cup \mathcal{S}$ is dense in X. Similarly, if $A \subset X$ the family $\mathcal{S}$ is said to be a weak cover of A in X if $A \subset \overline{\cup \mathcal{S}}$. We say that X is weakly κ-Lindelöf, if every open cover of X has a weak subcover of cardinality at most κ. The weak Lindelöf number wL(X) of X is then defined as* as

$$wL(X) = \min\{\kappa: X \text{ is weakly } \kappa\text{-Lindelöf}\}.$$

Now $wL(X) \le L(X)$ is trivial, however we also have

$$wL(X) \le c(X).$$

Indeed, let $\mathcal{G}$ be any open cover of X, and $\mathcal{C}$ be a maximal cellular family refining $\mathcal{G}$. It follows immediately from the maximality of $\mathcal{C}$ that $\cup\mathcal{C}$ is dense in X. Thus if we choose for each $C \in \mathcal{C}$ a $G_C \in \mathcal{G}$ with $C \subset G_C$, then clearly

$$\{G_C: C \in \mathcal{C}\}$$

is a weak subcover of $\mathcal{G}$ of cardinality at most $c(X)$. $\dashv$

The following result is again a lemma of independent interest to be used in the proof of our above mentioned general theorem.

2.35 *Let $X \in T_4$ and put $wL(X) = \kappa$. Then every X-open cover $\mathcal{G}$ of a closed set $F \subset X$ has a weak subcover of F of size at most κ.*

PROOF. Let us put $G = \cup\mathcal{G}$, then $F \subset G$, hence by the normality of X we can find an open set U such that

$$F \subset U \subset \bar{U} \subset G.$$

Therefore $\mathcal{G} \cup \{X\setminus\bar{U}\}$ is an open cover of X, and thus has a weak subcover of cardinality at most κ, which we can assume has the form $\mathcal{G}' \cup \{X\setminus\bar{U}\}$ for some $\mathcal{G}' \in [\mathcal{G}]^{\le\kappa}$. But then, as $\cup\mathcal{G}' \cup (X\setminus\bar{U})$ is dense in X, $\cup\mathcal{G}'$ must be dense in U, i.e.

$$F \subset U \subset \overline{\cup\mathcal{G}'}. \dashv$$

2.36 *If* $X \in T_4$, *then*

$$|X| \leq 2^{wL(X)\cdot\chi(X)}.$$

PROOF. Let us put $\mu = wL(X)\cdot\chi(X)$, $\lambda = \mu^+$ and $\kappa = 2^\mu$. Then $\kappa^{\underset{\smile}{\lambda}} = \kappa$. Next choose for each $p \in X$ a neighbourhood base $\mathcal{U}_p$ with $|\mathcal{U}_p| \leq \chi(X)$, moreover write for any $A \subset X$

$$\mathcal{U}_A = \cup\{\mathcal{U}_p : p \in A\}.$$

We are going to define a set mapping

$$G: [X]^{<\lambda} \to [X]^{\leq\kappa}$$

as follows. For any $A \in [X]^{<\lambda}$ put

$$\mathcal{V}_A = \{\mathcal{U} \subset \mathcal{U}_A : X\setminus\overline{\cup\mathcal{U}} \neq \emptyset\},$$

then for each $\mathcal{U} \in \mathcal{V}_A$ choose a point

$$p(\mathcal{U}) \in X\setminus\overline{\cup\mathcal{U}}.$$

Then $|\mathcal{U}_A| \leq |A|\cdot\chi(X) \leq \mu$ implies $|\mathcal{V}_A| \leq 2^\mu = \kappa$, hence if we put

$$G(A) = \overline{\{p(\mathcal{U}) : \mathcal{U} \in \mathcal{V}_A\}} \cup A,$$

then, using $\chi(X) \leq \mu$ and 2.5, G will be as required. Now 2.24(a) can be applied to obtain a set $A \in [X]^{\leq\kappa}$ that is closed with respect to G. We claim that A is equal to X. Assume, on the contrary, that $p \in X\setminus A \neq \emptyset$. Let us note furthermore that since

$$\bar{B} \subset G(B) \subset A$$

holds for each $B \in [A]^{\leq\mu}$ and $t(X) \leq \chi(X) \leq \mu$, the set A is closed in X. Consequently, as X is regular, we can find open sets U and V such that

$$p \in U,\ A \subset V, \text{ and } U \cap V = \emptyset.$$

Let us now choose for every point $q \in A$ a basic neighbourhood $V_q \in \mathcal{U}_q$ with $V_q \subset V$. Then

$$\{V_q : q \in A\}$$

forms an open cover of A, hence by 2.35 and the closedness of A, it has a weak subcover of A of size at most $wL(X) \le \mu$. Thus we have a $B \in [A]^{\le\mu}$ with

$$A \subset \overline{\cup\{V_q : q \in B\}} \subset \bar{V} \subset X\setminus U.$$

This shows that $\mathcal{U} = \{V_q : q \in B\} \in \mathcal{V}_B$, hence

$$p(\mathcal{U}) \in G(B) \subset A,$$

contradicting that

$$p(\mathcal{U}) \in X\setminus\cup\mathcal{U} \subset X\setminus A. \quad \dashv$$

Our next result due to Šapirovskii is another application of the closure method we have just used. It will play a very important role in the next chapter where the cardinal functions on special classes of spaces will be investigated.

2.37 *If* $X \in \mathcal{T}_3$ *is non-discrete, then*

$$\rho(X) \le \pi\chi(X)^{c(X)}.$$

<u>PROOF</u>. Let us put $c(X) = \mu$, $\pi\chi(X)^{\mu} = \kappa$ and $\lambda = \mu^+$. Then $\kappa^{\lambda} = \kappa^{\mu} = \kappa$. Note that since X is not discrete $\pi\chi(X) \ge \omega$, hence κ is infinite. Next, for each $p \in X$, fix a local π-base $\mathcal{B}_p$, and for any set $A \subset X$ put

$$\mathcal{B}_A = \cup\{\mathcal{B}_p : p \in A\}.$$

We shall now define a set mapping

$$G: [X]^{<\lambda} \to [X]^{\le\kappa}.$$

Let $A \in [X]^{<\lambda}$ and put

$$\mathcal{C}_A = \{\mathcal{U} \in [\mathcal{B}_A]^{\leq \mu}: X\setminus\overline{\cup\mathcal{U}} \neq \emptyset\}.$$

Clearly then $|\mathcal{C}_A| \leq |\mathcal{B}_A|^{\mu} \leq (|A|\cdot\pi\chi(X))^{\mu} = \kappa$, hence we can put

$$G(A) = \{p(\mathcal{U}): \mathcal{U} \in \mathcal{C}_A\},$$

where, as usual, each $p(\mathcal{U})$ is chosen from $X\setminus\overline{\cup\mathcal{U}}$. We can apply 2.24(a) to obtain a set $A \in [X]^{\leq\kappa}$ that is closed with respect to G. We claim that A is dense in X. Assume, on the contrary, that $X\setminus\bar{A} \neq \emptyset$. Since X is regular we can find a non-empty open set U such that

$$U \subset \bar{U} \subset X\setminus\bar{A}.$$

Now let $\mathcal{U}$ be a maximal disjoint family of members of $\mathcal{B}_A$ disjoint from $\bar{U}$. Then

$$A \subset \overline{\cup\mathcal{U}},$$

because otherwise we could find a point

$$p \in A\setminus\overline{\cup\mathcal{U}},$$

hence also a set $V \in \mathcal{B}_p$ with

$$V \subset X\setminus(\overline{\cup\mathcal{U}} \cup \bar{U}),$$

contradicting the maximality of $\mathcal{U}$. But $|\mathcal{U}| \leq \mu$, hence we can find a set $H \in [A]^{\leq\mu}$ such that $\mathcal{U} \in [\mathcal{B}_H]^{\leq\mu}$, and thus $\mathcal{U} \in \mathcal{C}_H$. Consequently we have

$$p(\mathcal{U}) \in G(H) \subset A,$$

contradicting that

$$p(\mathcal{U}) \in X\setminus\overline{\cup\mathcal{U}} \subset X\setminus A.$$

Therefore we have

$$d(X) \le \pi\chi(X)^{c(X)},$$

hence from 2.1(f)

$$\pi(X) \le d(X)\cdot\pi\chi(X) \le \pi\chi(X)^{c(X)}$$

as well, and then using 2.7(b) we obtain

$$\rho(X) \le \pi(X)^{c(X)} = \pi\chi(X)^{c(X)}. \quad \dashv$$

COROLLARY. *If* $X \in T_3$ *and the set*

$$Y = \{p \in X: \pi\chi(p,X) \le \kappa\}$$

is dense in X, *then* $\rho(X) \le \kappa^{c(X)}$.

PROOF. Indeed, by 2.6(a), (b) and (d) and 2.37 we have

$$\rho(X) = \rho(Y) \le \pi\chi(Y)^{c(X)} \le \kappa^{c(X)}. \quad \dashv$$

We shall end this chapter with a somewhat isolated but nonetheless very interesting result of E. van Douwen, that could be best fitted here.

2.38 *If* $X \in T_2$ *and* Aut(X) *denotes the set of all autohomeomorphisms of* X, *then*

$$|\mathrm{Aut}(X)| \le 2^{\pi(X)}.$$

In particular, if X *is homogeneous, then*

$$|X| \le 2^{\pi(X)}.$$

PROOF. Let $\mathcal{B}$ be a π-base of X with $|\mathcal{B}| \le \pi(X)$. For any $h \in \mathrm{Aut}(X)$ we define a map

$$h^*: \mathcal{B} \to P(\mathcal{B})$$

as follows. For all $B \in \mathcal{B}$

$$h^*(B) = \{C \in \mathcal{B}: C \subset h(B)\}.$$

It suffices to show then that the map $h \to h^*$ of $Aut(X)$ into $P(\mathcal{B})^{\mathcal{B}}$ is one-one, because there are only

$$(2^{|\mathcal{B}|})^{|\mathcal{B}|} = 2^{|\mathcal{B}|} \le 2^{\pi(X)}$$

maps from $\mathcal{B}$ into $P(\mathcal{B})$.
Thus assume $h_0, h_1 \in Aut(X)$ and $h_0 \neq h_1$. Then there is a point $p \in X$ with

$$h_0(p) = q_0 \neq h_1(p) = q_1.$$

Let U_0 and U_1 be disjoint neighbourhoods of q_0 and q_1 respectively, and choose an open neighbourhood V of p such that $h_0(V) \subset U_0$ and $h_1(V) \subset U_1$. Now, if $B \in \mathcal{B}$ is such that $B \subset V$, and such a B exists, then $h_0(B) \cap h_1(B) = \emptyset$. But if $C \in h_i^*(B)$ then $C \subset h_i(B)$, which shows that the members of $h_0^*(B)$ are disjoint from those of $h_1^*(B)$. Consequently we have $h_0^*(B) \neq h_1^*(B)$.
The second statement now follows easily because fixing a point p of a homogeneous space X, for each $q \in X$ there is an $h_q \in Aut(X)$ such that

$$h_q(p) = q,$$

and therefore $|X| \le |Aut(X)|$. ⊣

CHAPTER 3

CARDINAL FUNCTIONS ON SPECIAL CLASSES OF SPACES

In this chapter we carry on our investigation of the interrelationships between cardinal functions on more restricted classes than in chapter 2. Of course it is rather arbitrary to draw a line in the hierarchy of spaces and say those below are general, those above are special. However in our case the results themselves help in establishing this line by their special character on the classes $\mathcal{T}_5$ and $\mathcal{C}_2$ of hereditarily normal and compact Hausdorff spaces, respectively.

3.1. a) *If* $X \in \mathcal{T}_4$ *and* $D \subset X$ *is closed discrete, then*

$$2^{|D|} \leq \rho(X).$$

b) *If* $X \in \mathcal{T}_5$ *and* $D \subset X$ *is discrete, then*

$$2^{|D|} \leq \rho(X).$$

<u>PROOF</u>. a) For each $A \subset D$, using the normality of X we can find an open set U_A with

$$A \subset U_A \quad \text{and} \quad \bar{U}_A \cap (D \setminus A) = \emptyset.$$

But then the map $A \to \bar{U}_A$ from P(D) into RC(X) is clearly one-one. ⊣

b) As was shown in the proof of 2.23a) the set $D' = \bar{D} \setminus D$ is nowhere dense in X, hence $Y = X \setminus D'$ is dense in X while D is closed discrete in Y. But now $Y \in \mathcal{T}_4$, hence we can apply a) to obtain, also using 2.6d), that

$$\rho(X) = \rho(Y) \geq 2^{|D|}.$$

Now we introduce three new local cardinal functions that will play a crucial role in the proof of our main results concerning T_5 spaces. They have been first studied by Šapirovskii, who proved all these results, though our treatment is simpler than his.

3.2. DEFINITION. *Let* $X \in \mathcal{T}_3$, *then for each closed set* $H \subset X$ *we put*

$$\psi_\rho(H,X) = \min\{|\mathcal{F}| : \mathcal{F} \subset RC(X) \ \& \ H = \cap\mathcal{F}\}.$$

(Warning: the members of $\mathcal{F}$ do *not* have to be neighbourhoods of H!) In particular, if $H = \{p\}$, then we write $\psi_\rho(p,X)$ instead of $\psi_\rho(\{p\},X)$. Obviously since $X \in \mathcal{T}_3$ we always have

$$\psi_\rho(p,X) \leq \psi(p,X),$$

moreover

$$\psi_\rho(H,X) \leq \psi(H,X) \quad \text{if } X \in \mathcal{T}_4.$$

3.3. DEFINITION. *For any* $X \in \mathcal{T}$ *and* $p \in X$ *put*

$$t_c(p,X) = \sup\{a(p,K\setminus\{p\}) : p \in K' \subset K \subset X\}.$$

Observe that $K' \subset K$ implies that K is closed.

3.4. *Let* $X \in \mathcal{T}_5$ *and* $p \in K' \subset K \subset X$. *There is a closed set* H *with*

$$p \in H \subset K \quad \text{and} \quad \psi_\rho(H,X) \leq t_c(p,X).$$

PROOF. Put $t_c(p,X) = \kappa$, then by definition there is a set $A \in [K\setminus\{p\}]^{\leq\kappa}$ with $p \in \bar{A}$. For each $x \in A$ we can choose a regular closed neighbourhood F_x of p such that $x \notin F_x$. Now $M = \cap\{F_x : x \in A\}$ is not quite the set H we want because M does not have to be contained in K. The next trick, that makes very essential use of $X \in \mathcal{T}_5$ will take care of this. Consider the subspace $Y = X\setminus(K\cap M)$, then $F_1 = M\cap Y$ and $F_2 = K\cap Y$ are disjoint closed sets in Y, hence we can find disjoint open (in Y and therefore also in X) sets G_1 and G_2 with $F_1 \subset G_1$ and $F_2 \subset G_2$. Now let us put $H = M\cap\bar{G}_2$, then clearly

$$\psi_\rho(H,X) \leq \kappa.$$

For every $x \in A$ we have $x \notin F_x \supset M$, hence $A \subset K\backslash M$, and therefore

$$p \in \bar{A} \subset \overline{K\backslash M} = \overline{K\backslash K\cap M} = \overline{F_2} \subset \overline{G_2},$$

consequently $p \in H$. On the other hand we have

$$H = M\cap\bar{G}_2 \subset M\backslash G_1 \subset M\backslash F_1 = M\backslash(X\backslash K\cap M) = K\cap M\subset K,$$

hence H is as required. $\dashv$

3.5. DEFINITION. *Let* p *be a non-isolated point in* X. *We say that the sequence of closed sets*

$$\mathcal{R} = \{K_\alpha : \alpha \in \kappa\}$$

is a well at p *if*

(i) $\alpha \in \beta \in \kappa$ *implies* $K_\beta \subset K_\alpha$;

(ii) $p \in K'_\alpha$ *for each* $\alpha \in \kappa$;

(iii) $\cap\{K_\alpha : \alpha \in \kappa\} = \{p\}$.

Next we put

$$k(p,X) = \min\{|\mathcal{R}| : \mathcal{R} \text{ is a well at } p\}.$$

Note that $k(p,X)$, when defined, is a regular cardinal as any cofinal subsystem of a well at p is again a well at p.
Our next result shows that $k(p,X)$ is defined if $X \in \mathcal{T}_3$.

3.6. *If* $X \in \mathcal{T}_3$ *and* $p \in X$ *is non-isolated, then*

$$k(p,X) \leq \psi(p,X).$$

PROOF. Put $\psi(p,X) = \kappa$ and consider a system $\{U_\alpha : \alpha \in \kappa\}$ of open neighbourhoods of p such that

$$\{p\} = \cap\{\bar{U}_\alpha : \alpha \in \kappa\}.$$

Next define for each $\alpha \in \kappa$

$$K_\alpha = \cap\{\bar{U}_\beta : \beta \le \alpha\}.$$

We claim that $\{K_\alpha : \alpha \in \kappa\}$ is a well at p. In fact, (i) and (iii) of 3.5 are obviously true. If for some $\alpha \in \kappa$ we had $p \notin K'_\alpha$ then we could choose a neighbourhood U of p with

$$U \cap K_\alpha = \{p\},$$

hence we had

$$\{p\} = \cap\{U_\beta : \beta \le \alpha\} \cap U,$$

i.e. $\psi(p,X) \le |\alpha| < \kappa$, a contradiction. ⊣

3.7. *If* $X \in T_5$ *and* $p \in X$ *is non-isolated, then*

$$\psi_\rho(p,X) \le k(p,X).\min\{\psi(p,X), t_c(p,X)\}.$$

PROOF. If $\psi(p,X) \le t_c(p,X)$, then the right-hand side of our inequality is, in view of 3.6, equal to $\psi(p,X)$, hence it is valid by our remark in 3.2. Thus we can assume that $t_c(p,X) < \psi(p,X)$, and what we have to prove is

$$\psi_\rho(p,X) \le k(p,X).t_c(p,X).$$

Let $\{K_\alpha : \alpha \in \kappa\}$ be a well at p of minimal cardinality. Then for each $\alpha \in \kappa$ we can apply 3.4 to obtain a closed set H_α with $p \in H_\alpha \subset K_\alpha$ and $\psi_\rho(H_\alpha,X) \le t_c(p,X)$. But then $\{p\} = \cap\{K_\alpha : \alpha \in \kappa\}$ implies $\{p\} = \cap\{H_\alpha : \alpha \in \kappa\}$, hence

$$\psi_\rho(p,X) \le \Sigma\{\psi_\rho(H_\alpha,X) : \alpha \in \kappa\} \le k(p,X).t_c(p,X). \quad ⊣$$

3.8. *Assume* $X \in T_5$, ρ *is a singular but not strong limit cardinal (i.e. there is a* $\lambda < \rho$ *with* $2^\lambda > \rho$) and $\rho(X) \le \rho$. Then for each $p \in X$ we have

$$\psi_\rho(p,X) < \rho.$$

PROOF. Let $p \in X$ be arbitrary. If $\psi(p,X) < \rho$, then by 3.2 we are done.

Thus in what follows we assume $\psi(p,X) = \rho$. Next we distinguish two cases a) and b).

a) For each closed set K with $p \in K'$ we have

$$\psi(p,K) = \psi(p,X) = \rho.$$

Let us put $k(p,X) = \kappa$, then by 3.6 we have $\kappa \le \rho$, moreover κ is always regular, hence actually $\kappa < \rho$.

Since $2^{\lambda} > \rho$, by 3.1b) we have no discrete subspace of X of cardinality λ. Thus if we take any closed set K with $p \in K'$, then using 2.14 we get that either $\psi(p,K) < \lambda$ or $a(p,K\setminus\{p\}) < \lambda$. Now the first case cannot happen, therefore we have $a(p,K\setminus\{p\}) < \lambda$ for every closed set K with $p \in K'$, hence $t_c(p,X) \le \lambda$. But then we get from 3.7 that

$$\psi_\rho(p,X) \le k(p,X).t_c(p,X) \le \kappa.\lambda < \rho.$$

b) There exists a closed set K with $p \in K'$ and $\psi(p,K) < \rho$. Using the regularity of X we can then find regular closed neighbourhoods $\{F_\alpha: \alpha \in \psi\}$ of p in X, where $\psi = \psi(p,K)$, such that

$$\cap\{F_\alpha: \alpha \in \psi\} \cap K = \{p\}.$$

Now put $Y = X\setminus\{p\}$ and $H_1 = K\cap Y$, $H_2 = \cap\{F_\alpha: \alpha \in \psi\} \cap Y$. Then Y is normal, moreover H_1 and H_2 are disjoint closed sets in Y, hence they have disjoint open (in X) neighbourhoods G_1 and G_2, respectively. Now we have $p \in \overline{K\setminus\{p\}} \subset \overline{G_1}$ on one hand and

$$\cap\{F_\alpha: \alpha \in \psi\} \cap \overline{G_1} \subset \cap\{F_\alpha: \alpha \in \psi\}\setminus G_2 \subset \cap\{F_\alpha: \alpha \in\psi\}\setminus H_2 = \{p\}$$

on the other hand, hence

$$\{p\} = \cap\{F_\alpha: \alpha \in \psi\} \cap \overline{G_1},$$

consequently

$$\psi_\rho(p,X) \le \psi < \rho.$$

3.9. *Let* $X \in T_5$ *and* $\rho(X) \le \rho$, *a singular but not strong limit cardinal. Then*

$$|X| \le \rho^{\underline{\rho}} = \Sigma\{\rho^{\kappa}: \kappa < \rho\}.$$

PROOF. Indeed using 3.8 we can find for each $p \in X$ a family $F_p \in [RC(X)]^{<\rho}$ with

$$\{p\} = \cap F_p. \quad \dashv$$

In order to see the strength of this result we have to consider special assumptions about the behaviour of the exponentiation function 2^{λ}. Let us denote by $S(\kappa)$ the following statement: $2^{\kappa} = \rho$ is singular and 2^{λ} is strictly increasing for cofinally many $\lambda < \rho$. It is well-known that $S(\kappa)$ is consistent with ZFC.
$S(\kappa)$ implies that ρ is not strong limit and $\rho^{\underline{\rho}} < 2^{\rho}$. Indeed, by our assumption there is a λ with $\kappa < \lambda < \rho$ and $\rho = 2^{\kappa} < 2^{\lambda}$, moreover

$$\rho^{\underline{\rho}} = \Sigma\{\rho^{\lambda}: \lambda < \rho\} = \Sigma\{(2^{\kappa})^{\lambda}: \lambda < \rho\} = \Sigma\{2^{\lambda}: \lambda < \rho\} \le 2^{\rho}.$$

But by our assumption then $cf(\rho^{\underline{\rho}}) = cf(\rho)$, hence actually

$$\rho^{\underline{\rho}} < 2^{\rho}.$$

3.10. *Assume* $X \in T_5$, $\rho = 2^{\kappa}$ *and* $S(\kappa)$. *Then*

a) $\rho(X) \le \rho = 2^{\kappa}$ *implies* $|X| \le \rho^{\underline{\rho}} < 2^{\rho}$;
b) $d(X) \le \kappa$ *implies* $|X| \le \rho^{\underline{\rho}} < 2^{2^{\kappa}}$;
c) $s(X) \le \kappa$ *implies* $|X| \le \rho^{\underline{\rho}} < 2^{2^{\kappa}}$.

PROOF.

a) is immediate from 3.9. $\dashv$
b) follows from $\rho(X) \le 2^{d(X)} \le \rho$ and a). $\dashv$
c) According to 2.20 we have a set $S \subset X$ with $|S| \le 2^{\kappa} = \rho$ such that $X = \cup\{\bar{T}: T \in [S]^{\le\kappa}\}$. Now by b) we have $|\bar{T}| \le \rho^{\underline{\rho}}$ for each $T \in [S]^{\le\kappa}$, hence

$$|X| \le \rho.\rho^{\underline{\rho}} = \rho^{\underline{\rho}} < 2^{2^{\kappa}}. \quad \dashv$$

One should compare these results with the corresponding very sharp inequalities from 2.6d), 2.4 and 2.20 respectively.
Now we leave the study of T_5 spaces and turn to the class $\mathcal{C}_2$ of compact Hausdorff spaces. The importance of this class in general topology cannot be overemphasized. According to one's expectations, as it turns out from the following results the class $\mathcal{C}_2$ behaves in a particularly nice way with respect to cardinal functions as well. I venture to speculate that the study of this special class (and perhaps of others) will become central in the investigation of cardinality problems in topology.

3.11. a) *If* $X \in \mathcal{C}_1$, *then*

$$psw(X) = \psi w(X).$$

b) *If* $X \in \mathcal{C}_2$, *then*

$$psw(X) = w(X).$$

PROOF.

a) We actually prove a little more: *whenever* $\mathcal{B}$ *is a pseudobase of* X *we have* $\mathrm{ord}(\mathcal{B}) = |\mathcal{B}|$. Now, for each point $p \in X$ and every $B \in \mathcal{B}$ with $p \in B$

$$X = B \cup \cup\{C \in \mathcal{B}\colon p \notin C\},$$

hence by the compactness of X we can select a finite *minimal* subcover $\mathcal{U}_B$ from $\{B\} \cup \{C \in \mathcal{B}\colon p \notin C\}$. But clearly we must have $B \in \mathcal{U}_B$, hence as $\mathcal{U}_B$ is finite the map $B \to \mathcal{U}_B$ from $\mathcal{B}$ into the set $\mathcal{M}(\mathcal{B})$ of all finite minimal covers of X by members of $\mathcal{B}$ is finite-to-one. By Miščenko's lemma 0.7, however $\mathcal{M}(\mathcal{B})$ has cardinality $\le \mathrm{ord}(B)$, hence so does $\mathcal{B}$. ⊣

b) By part a) it suffices to show that $\psi w(X) = w(X)$. First observe that by our remark made after 2.3b) we have $nw(X) \le \psi w(X) = \kappa$. Thus by our remark in 2.8b) we actually have a pseudobase $\mathcal{B}$ of X of cardinality κ that separates the points of X in the strong sense described there. Now take any point $p \in X$ and open set G with $p \in G$; since the family $\mathcal{B}' = \{B \in \mathcal{B}\colon p \notin \bar{B}\}$ covers $X \setminus \{p\}$, we can find finitely many members, say $B_1,\dots,B_n$, of $\mathcal{B}'$ such that

$$\bigcup_{i=1}^{n} B_i \cup G = X.$$

Thus if we put

$$C = X\setminus \bigcup_{i=1}^{n} \overline{B_i},$$

then we have $p \in C \subset G$, hence all sets C of this form constitute a base for X, and clearly their number is at most κ. ⊣

In the following results the tightness of compact Hausdorff spaces will play a crucial role. The next result of Archangelskii throws some light on this by giving a beautiful characterization of $t(X)$ for $X \in \mathcal{C}_2$.

3.12. *If* $X \in \mathcal{C}_2$, *then*

$$t(X) = F(X).$$

PROOF. Since both $t(X)$ and $F(X)$ are defined as suprema of certain cardinals which agree with the suprema of the corresponding regular or successor ordinals, it suffices to prove the following two statements:

(i) if κ is regular and the length of a free sequence, then $\kappa \le t(X)$;

(ii) if $t(X) \ge \kappa^+$ then X contains a free sequence of length κ^+.

To see (i) let $\{p_\alpha: \alpha \in \kappa\}$ be a free sequence and put

$$S_\alpha = \{p_\beta: \beta \in \alpha\}$$

for each $\alpha \le \kappa$. Since X is compact the set S_κ has a complete accumulation point, say p. Then for every neighbourhood U of p we have

$$|U \cup S_\kappa| = \kappa,$$

hence we have

$$p \notin \overline{S_\alpha}$$

for each $\alpha \in \kappa$, using that our sequence is free. But then, by the regularity of κ, we also have $p \notin \bar{A}$ for each $A \in [S_\kappa]^{<\kappa}$, hence

$$t(p,X) \geq a(p,S_\kappa) = \kappa.$$

In order to prove (ii) we need a little lemma.

LEMMA. *For any space* X *if* $A,B \subset X$ *and* $A \cap \bar{B} \neq \emptyset$ *then there is a set* $C \subset B$ *with* $|C| \leq \chi(A,X)$ *and* $A \cap \bar{C} \neq \emptyset$.

PROOF OF THE LEMMA. Let $\mathcal{U}$ be a neighbourhood base of A in X of minimal cardinality and for each $U \in \mathcal{U}$ pick a point

$$p(U) \in U \cap B,$$

which is possible by $U \cap \bar{B} \neq \emptyset$. Now set

$$C = \{p(U) : U \in \mathcal{U}\}.$$

Then $|C| \leq \chi(A,X) = |\mathcal{U}|$, moreover A must intersect $\bar{C}$, since otherwise $X \setminus \bar{C}$ were a neighbourhood of A, hence

$$p(U) \in U \subset X \setminus \bar{C}$$

would hold for some $U \in \mathcal{U}$, contradicting that

$$p(U) \in C \subset \bar{C}.$$

Now to prove (ii) assume that $t(p,X) \geq \kappa^+$. Thus we can find a set S such that $p \in \bar{S}$ but $p \notin \bar{T}$ for each $T \in [S]^{\leq\kappa}$. Let us put

$$B = \cup\{\bar{T}: T \in [S]^{\leq\kappa}\}.$$

Now if $C \in [B]^{\leq\kappa}$, then for each $x \in C$ there is a $T_x \in [S]^{\leq\kappa}$ with $x \in \bar{T}_x$, hence if we put

$$T = \cup\{T_x : x \in C\},$$

then we have $T \in [S]^{\leq\kappa}$, hence $\bar{C} \subset \bar{T} \subset B$. Clearly we also have $p \notin B$ but $p \in \bar{B} = \bar{S}$.
Now we define a free sequence $\{p_\alpha : \alpha \in \kappa^+\}$ and a sequence of sets $\{A_\alpha : \alpha \in \kappa^+\}$ by transfinite induction as follows. Assume $\beta \in \kappa^+$ and for each $\alpha \in \beta$ we have already defined the point p_α and set A_α in such a way that the following inductive hypotheses are satisfied for all $\alpha \in \beta$:

$I(\alpha)$: A_α is closed and $\chi(A_\alpha, X) \leq |\alpha| + \omega$;

$J(\alpha)$: if $\gamma \in \alpha$ then $A_\gamma \supset A_\alpha$;

$K(\alpha)$: $p \in A_\alpha$ and $p_\alpha \in B \cap A_\alpha$.

Now the set $S_\beta = \{p_\alpha : \alpha \in \beta\} \in [B]^{\leq\kappa}$, hence we have $p \notin \overline{S_\beta} \subset B$. Since X is regular we can find then a closed G_δ set H_β containing p such that $H_\beta \cap \overline{S_\beta} = \emptyset$ and put

$$A_\beta = \cap\{A_\alpha : \alpha \in \beta\} \cap H_\beta.$$

Then we have by 1.14 that

$$\chi(A_\beta, X) = \psi(A_\beta, X) \leq |\beta| + \omega,$$

i.e. $I(\beta)$ and $J(\beta)$ hold. Clearly we also have $p \in A_\beta$. Now to choose p_β observe that $p \in A_\beta \cap \bar{B} \neq \emptyset$, hence we can apply our lemma to obtain a $C \subset B$ with $|C| \leq \chi(A_\beta, X) = |\beta| + \omega \leq \kappa$ such that $A_\beta \cap \bar{C} \neq \emptyset$. Thus if we choose p_β from $A_\beta \cap \bar{C}$ then by what we have proven above $K(\beta)$ is also satisfied. Now it is easy to check that $\{p_\alpha : \alpha \in \kappa^+\}$ is free. In fact, for any $\beta \in \kappa^+$ we have

$$\overline{\{p_\alpha : \beta \leq \alpha \in \kappa^+\}} \subset A_\beta \subset H_\beta \subset X \setminus \overline{S_\beta}. \quad \dashv$$

<u>COROLLARY</u>. *If* $X \in C_2$ *then*

$$t(X) \leq s(X). \quad \dashv$$

The next result due to B. Šapirovskii is a nice strengthening of 2.17 for $X \in C_2$.

3.13. *If* $X \in \mathcal{C}_2$ *then*

$$z(X) \le s(X).t(X)^+ \le s(X)^+.$$

<u>PROOF</u>. The second inequality follows immediately from the above corollary of 3.12. In order to prove the first we start by showing that $d(X) \le s(X).t(X)^+ = \kappa$. Let Y be a dense left separated subset of X (cf. the proof of 2.9c)). Then Y does not contain any right separated subspace of cardinality κ^+, since otherwise by 2.12 it would also contains a discrete subspace of size $\kappa^+ > s(X) \ge s(Y)$. Thus, appealing to 2.9b), we obtain that $h(Y) = L^*(Y) \le \kappa$.
Since we are heading for an application of 2.26, next we calculate $\psi(\bar{S}|Y,X)$ for $S \subset Y$. As X is regular, for each $p \in Y\setminus\bar{S}$ we can select a closed neighbourhood F_p of p in X such that $F_p \cap \bar{S} = \emptyset$. Now using that

$$L(Y\setminus\bar{S}) \le L^*(Y) \le \kappa$$

we can find a set $T \in [Y\setminus\bar{S}]^{\le\kappa}$ such that

$$Y\setminus\bar{S} \subset \cup\{F_p : p \in T\}.$$

But then the family $\mathcal{U} = \{X\setminus F_p : p \in T\}$ has the properties

$$\bar{S} \subset \cap\,\mathcal{U} \text{ and } Y \cap \bar{S} = \cap\,\mathcal{U} \cap Y,$$

i.e. $\mathcal{U}$ establishes

$$\psi(\bar{S}|Y,X) \le \kappa.$$

Now we have every ingredient to apply 2.26 to our X, Y, κ and $\lambda = \omega$, as a result of which we obtain

$$d(X) \le d(Y) \le z(Y) \le \kappa,$$

since Y is dense in X.
Now, our conditions on X are inherited by its closed subspaces,

hence for any $S \subset X$ we can conclude that

$$d(\bar{S}) \leq \kappa,$$

hence by 2.6b)

$$d(S) \leq d(\bar{S}).t(\bar{S}) \leq \kappa$$

as well, i.e.

$$z(X) = d^*(X) \leq \kappa.$$

The next results are also due to Šapirovskii, though not their proofs presented here. I have lumped them together because their proofs really use the same basic ideas.

3.14. *Let* $X \in \mathcal{C}_2$ *and* $p \in X$, *then*

a) $\pi\chi(X) \leq t(X)$;

b) $\pi\chi(p,X) \leq t(p,X)$ *if* $\pi\chi(p,X) = \kappa$ *is regular;*

c) X *contains a dense subspace* Y *left separated in type* $\pi(X)$.

PROOF.

a) In view of 3.12 it clearly suffices to show that if κ is an uncountable cardinal with

$$\pi\chi(p,X) \geq \kappa$$

for some $p \in X$, then X contains a free sequence of length κ. In order to achieve this we need a little lemma that will be used repeatedly, hence we formulate it separately.

LEMMA. *For any* $X \in \mathcal{T}$ *(and* $p \in X$*) with* $\pi(X) = \pi$ $(\pi\chi(p,X) = \pi)$ *and family* $\mathcal{S}$ *of subsets of* X *with* $|\mathcal{S}| \leq \lambda < \pi$ *such that* $\chi(S,X) \leq \lambda$ *for each* $S \in \mathcal{S}$ *we have a non-empty open set* $G \subset X$ *(a neighbourhood* G *of* p*) such that* $S \backslash G \neq \emptyset$ *for each* $S \in \mathcal{S}$.

PROOF OF THE LEMMA. Let us choose for each $S \in \mathcal{S}$ a neighbourhood base $\mathcal{U}_S$ in X with $|\mathcal{U}_S| \leq \lambda$, then put

$$\mathcal{U} = \cup\{\mathcal{U}_S : S \in \mathcal{S}\}.$$

Then $|\mathcal{U}| \le \lambda < \pi$, hence $\mathcal{U}$ is not a π-base of X (or a local π-base at p), consequently there is a non-empty open set G (a neighbourhood G of p) such that $U \setminus G \neq \emptyset$ for each $U \in \mathcal{U}$. But this clearly implies $S \setminus G \neq \emptyset$ for each $S \in \mathcal{S}$ as well. $\dashv$

Now assume that

$$\pi\chi(p,X) \ge \kappa > \omega$$

with $X \in \mathcal{C}_2$ and construct the promised free sequence. To achieve that we shall construct a "triangular" matrix of the form

$$\{F^{\mu}_{\nu} : \mu \in \kappa \ \& \ \mu \le \nu \in \kappa\}$$

of non-empty closed subsets of X satisfying the following conditions for each $\nu \in \kappa$:

$$I(\nu): \ \chi(F^{\mu}_{\nu},X) \le |\nu| + \omega \quad \text{whenever } \mu \le \nu;$$

$$J(\nu): \ F^{\mu}_{\nu} \subset F^{\mu}_{\rho} \quad \text{if } \mu \le \rho \le \nu,$$

i.e. the rows of our matrix form a decreasing chain;

$$K(\nu): \ p \in F^{\nu}_{\nu}.$$

Now assume that $\nu \in \kappa$ and we have already defined the sets $F^{\mu}_{\nu'}$, whenever $\mu \le \nu' < \nu$ (i.e. the columns of our matrix with index $\nu' < \nu$) in such a way that $I(\nu')$, $J(\nu')$ and $K(\nu')$ hold for all $\nu' < \nu$. Let us put for each $\mu < \nu$

$$H^{\mu}_{\nu} = \cap\{F^{\mu}_{\nu'} : \mu \le \nu' < \nu\},$$

then by the inductive hypotheses $H^{\mu}_{\nu} \neq \emptyset$ and $\psi(H^{\mu}_{\nu},X) = \chi(H^{\mu}_{\nu},X) \le |\nu| + \omega$. We can thus apply our lemma to the family of sets $\mathcal{S}_{\nu} = \{H^{\mu}_{\nu} : \mu \in \nu\}$ to obtain a neighbourhood G_{ν} of p from which all the $H^{\mu}_{\nu} \in \mathcal{S}_{\nu}$ "hang out". Of course G_{ν} can also be assumed to be an open F_{σ} as these form a neighbourhood basis at p. Then we put

$$F_\nu^\mu = H_\nu^\mu \setminus G_\nu$$

for each $\mu < \nu$. We still have to define F_ν^ν to complete the definition of the ν^{th} column. For this we choose a closed G_δ set Z_ν such that

$$p \in Z_\nu \subseteq G_\nu,$$

and then put

$$F_\nu^\nu = \cap\{F_{\mu'}^\mu : \mu \in \nu\} \cap Z_\nu .$$

It is easy to see that $I(\nu)$, $J(\nu)$ and $K(\nu)$ are satisfied. Having completed our construction we can pick for each $\mu \in \kappa$ a point

$$p_\mu \in \cap\{F_\nu^\mu : \mu \le \nu \in \kappa\}.$$

We claim that $\{p_\mu : \mu \in \kappa\}$ is a free sequence. Indeed by our construction we have for each $\nu \in \kappa$

$$\overline{\{p_\mu : \mu \in \nu\}} \subset X \setminus G_\nu \subset X \setminus Z_\nu ,$$

and on the other hand

$$\overline{\{p_\mu : \nu \le \mu \in \kappa\}} \subset F_\nu^\nu \subset Z_\nu . \quad \dashv$$

<u>COROLLARY</u>. *If* $X \in \mathcal{C}_2$ *then*

$$\rho(X) \le 2^{s(X)}.$$

<u>PROOF</u>. By 3.14a) and 3.12 we have $\pi\chi(X) \le t(X) \le s(X)$, and obviously $c(X) \le s(X)$, hence from 2.37 we get

$$\rho(X) \le \pi\chi(X)^{c(X)} \le 2^{s(X)}. \quad \dashv$$

b) Let $\mathcal{B}$ be a local π-base at p of cardinality κ, we can then write $\mathcal{B} = \{B_\nu : \nu \in \kappa\}$. (We can assume $\kappa > \omega$ as the case $\kappa = \omega$ is trivial).

Since X is regular, we can choose for each $\nu \in \kappa$ a non-empty closed G_δ-set $A_\nu \subset B_\nu$. Next we define a triangular matrix of non-empty closed sets of the form

$$\{F_\nu^\mu : \mu \le \nu \in \kappa\}$$

with the same properties $I(\nu)$ and $J(\nu)$ (but not $K(\nu)$!) as in the proof of a) as follows. Suppose $\nu \in \kappa$ and the sets $F_{\nu'}^\mu$ have already been defined for all $\mu \le \nu' < \nu$ satisfying $I(\nu')$ and $J(\nu')$. Let us put again for each $\mu < \nu$

$$H_\nu^\mu = \cap\{F_{\nu'}^\mu : \mu \le \nu' < \nu\},$$

then $\psi(H_\nu^\mu, X) = \chi(H_\nu^\mu, X) \le |\nu| + \omega$, hence we can apply our lemma to the point p and the family $\mathcal{S}_\nu = \{A_\nu\} \cup \{H_{\nu'}^\mu : \mu < \nu\}$ to obtain an open F_σ neighbourhood G_ν of p such that

$$F_\nu^\mu = H_\nu^\mu \setminus G_\nu \neq \emptyset$$

whenever $\mu < \nu$, and

$$F_\nu^\nu = A_\nu \setminus G_\nu \neq \emptyset.$$

Thus we can define our matrix column by column, and having completed it we can again pick points

$$p_\mu \in \cap\{F_\nu^\mu : \mu \le \nu \in \kappa\}$$

from the intersections of its respective rows. Let us put

$$S = \{p_\mu : \mu \in \kappa\}.$$

Then $p \in \bar{S}$ since by our construction

$$p_\mu \in F_\mu^\mu \subset A_\mu \subset B_\mu$$

for all $\mu \in \kappa$, but also for each $\nu \in \kappa$

$$p \notin \overline{\{p_\mu : \mu \in \nu\}} \subset X \backslash G_\nu .$$

Since κ is regular this implies that $p \notin \bar{A}$ for any $A \in [S]^{<\kappa}$, hence we get

$$a(p,S) = \kappa,$$

which yields even a little more than $t(p,X) \geq \kappa$, namely $\hat{t}(p,X) > \kappa$. ⊣

c) We can assume that $\pi(X) = \kappa > \omega$, since otherwise $d(X) = \omega$, and every countable dense subspace of X is left separated. Let $\mathcal{B}$ be a π-base of X consisting of open F_σ-sets with $|\mathcal{B}| = \kappa$, say

$$\mathcal{B} = \{B_\alpha : \alpha \in \kappa\},$$

and pick for each $\alpha \in \kappa$ a non-empty closed G_δ-set $A_\alpha \subset B_\alpha$. Again we will produce a triangular matrix

$$\{F^\mu_\nu : \mu \leq \nu < \kappa\}$$

consisting of non-empty closed sets and satisfying conditions $I(\nu)$ and $J(\nu)$ for each $\nu \in \kappa$. The construction of the ν^{th} column $\{F^\mu_\nu : \mu \leq \nu\}$ is now quite similar to that in case b):

Assuming that we have got the sets $\{F^\mu_{\nu'} : \mu \leq \nu' < \nu\}$ satisfying $I(\nu')$ and $J(\nu')$ for each $\nu' < \nu$, we define the sets H^μ_ν for $\mu < \nu$ in the same way and then choose α_ν as the smallest member of κ such that for all $\mu < \nu$

$$F^\mu_\nu = H^\mu_\nu \backslash B_{\alpha_\nu} \neq \emptyset .$$

The existence of α_ν is insured by our lemma. Then we put

$$F^\nu_\nu = A_{\alpha_\nu} .$$

The points $\{p_\mu : \mu \in \kappa\}$ are again chosen in the same way:

$$p_\mu \in \cap\{F^\mu_\nu : \mu \leq \nu < \kappa\}.$$

Then $Y = \{p_\mu : \mu \in \kappa\}$ is left separated, as for each $\nu \in \kappa$ we have

$$p_\nu \in A_{\alpha_\nu} \subset B_{\alpha_\nu} \subset X \setminus \{p_\mu : \mu \in \nu\}.$$

We claim that Y is also dense in X. Indeed for any $\nu \in \kappa$ we have $\nu \leq \alpha_\nu$ by our construction, because obviously $\mu < \nu$ implies $\alpha_\mu < \alpha_\nu$. Now if $\nu < \alpha_\nu$ then for some $\mu < \nu$ we have

$$p_\mu \in F^\mu_\nu \subset H^\mu_\nu \subset B_\nu,$$

and if $\nu = \alpha_\nu$ then we have

$$p_\nu \in A_{\alpha_\nu} = A_\nu \subset B_\nu,$$

hence in any case we have $Y \cap B_\nu \neq \emptyset$. As this is true for all $\nu \in \kappa$, Y is indeed dense in X. ⊣

<u>COROLLARY</u>. *If* $X \in C_2$ *and* $\pi(X) = \kappa$ *is regular then* X *has a dense subspace* Y *with* $d(Y) = \kappa$. ⊣

<u>REMARK</u>. It is not known whether this corollary of 3.14c) or 3.14b) remain valid for singular κ, though this can easily be shown to be the case under some set theoretic hypotheses like GCH.

3.15. *Let* $X \in C_2$, *then*

a) $\pi^*(X) = z(X)$;

b) $\pi\chi^*(X) = t(X)$.

<u>PROOF</u>.

a) Since $d(X) \leq \pi(X)$ we immediately have

$$d^*(X) = z(X) \leq \pi^*(X),$$

actually for any $X \in T$. On the other hand if $Y \subset X$, then applying 3.14c) to $\bar{Y}$ we obtain a left separated $Z \subset \bar{Y}$ with

$$|Z| = \pi(\bar{Y}) \geq \pi(Y)$$

in view of 2.6c), hence $z(X) \geq \pi^*(X)$. ⊣

b) Let us put $\kappa = \pi\chi^*(X)$. First we show $t(X) \leq \kappa$ (for arbitrary X). Thus let $p \in \overline{A\setminus\{p\}} \subset X$. We have by 2.7a) and 2.6c) that

$$a(p,A) \leq \pi\chi(p,A \cup \{p\}) \leq \kappa,$$

hence as p and A were arbitrary, $t(X) \leq \kappa$. Now for the converse consider any $Y \subset X$, then by 2.6c) we have $\pi\chi(Y) \leq \pi\chi(\bar{Y})$. But 3.14a) applied to $\bar{Y}$ gives $\pi\chi(\bar{Y}) \leq t(\bar{Y}) \leq t(X)$, hence we conclude $\kappa \leq t(X)$. ⊣

The following classical result of Čech and Pospišil is a kind of converse to Archangelskii's theorem that compact Hausdorff spaces of character at most κ have cardinality at most 2^κ.

3.16. *If* $X \in C_2$ *and* $\chi(p,X) \geq \kappa$ *holds for each* $p \in X$, *then*

$$|X| \geq 2^\kappa.$$

PROOF. We will distinguish two cases according as $\kappa = \omega$ or $\kappa > \omega$.

Case 1. Now $\kappa = \omega$ and we shall prove a little more than stated, namely that X can be mapped continuously onto the interval [0,1]. To achieve this we first define by an easy induction on $n \in \omega$ non-empty open subsets U_ε of X for each finite sequence $\varepsilon \in 2^n$ in such a way that

(i) $\overline{U_{\varepsilon\frown 0}} \cup \overline{U_{\varepsilon\frown 1}} \subset U_\varepsilon$,

and

(ii) $\overline{U_{\varepsilon\frown 0}} \cap \overline{U_{\varepsilon\frown 1}} = \emptyset$

(This is where we have to use our assumption about the characters of points in X in the form that every non-empty open set in X is infinite.) Next we put for any (infinite) sequence $s \in 2^\omega$

$$F_s = \cap\{\overline{U_{s\restriction n}} : n \in \omega\}.$$

Then $F_s \neq \emptyset$ since X is compact and (i) holds, moreover $F_s \cap F_t = \emptyset$ if $s \neq t$ using (ii), hence the map $f: F = \cup\{F_s : s \in 2^\omega\} \to D(2)^\omega$ defined by

$$f(p) = s \leftrightarrow p \in F_s$$

is well-defined, continuous and onto. But clearly we have

$$F = \cup\{F_s : s \in 2^\omega\} = \bigcap_{n\in\omega} \cup\{\bar{U}_\varepsilon : \varepsilon \in 2^n\},$$

hence F is closed in X. Now the Cantor-set can be mapped onto [0,1], hence so can F, and using the Uryson extension theorem X as well.

Case 2. $\kappa > \omega$. Now we use our assumption in the form that if $F \subset X$ is closed with $\psi(F,X) = \chi(F,X) < \kappa$ then $|F| \geq 2$. Next we define by transfinite induction on the length of sequences from 2^{κ} closed sets F_s with the following properties:

(i) if $s \subset t$, then $F_s \supset F_t$;

(ii) $F_{s\frown 0} \cap F_{s\frown 1} = \emptyset$;

(iii) $\chi(F_s) \leq |s| + \omega = |\ell h(s)| + \omega$.

Thus assume $\alpha \in \kappa$ and we have already defined the sets $F_{s'}$ for $s' \in 2^{\alpha}$. If α is limit then we put

$$F_s = \cap\{F_{s\restriction\beta} : \beta \in \alpha\},$$

for each $s \in 2^\alpha$. If $\alpha = \beta+1$, then for any $s \in 2^\beta$ we pick two distinct points p_s and q_s in F_s and disjoint closed G_δ-sets, say P_s and Q_s, containing them and then put

$$F_{s\frown 0} = F_s \cap P_s \text{ and } F_{s\frown 1} = F_s \cap Q_s.$$

It is easy to see that (i)-(iii) will be satisfied in both cases. Having completed this induction we put

$$F_t = \cap\{F_{t\restriction\alpha} : \alpha \in \kappa\}$$

for each $t \in 2^\kappa$, clearly then $F_t \neq \emptyset$ as X is compact and (i) holds, moreover $s \neq t$ implies $F_s \neq F_t$, hence indeed $|X| \geq 2^\kappa$. ⊣

Now we fit here a recent result of Malyhin which is at present the only non-trivial result concerning the (pseudo) character of compact T_1 spaces.

3.17. *If* $X \in C_1$ *and* $\psi(X) \leq \omega$ *then either* $|X| \leq \omega$ *or* $|X| \geq 2^{\omega}$.

PROOF. Let us assume that $|X| > \omega$. We shall say that a closed set $F \subset X$ is big if $|F| < \omega$. Clearly, if we can show that any big closed set in X contains two disjoint big closed subsets, then we are done, because the same procedure as in the proof of the Čech-Pospišil theorem can be applied. The pseudo character being hereditary, this of course reduces to showing that X contains two disjoint big closed subsets. To establishes this, let us note first that if $\mathcal{U}_p$ is a countable ψ-base at $p \in X$, then from $X\setminus\{p\} = \cup\{X\setminus U: U \in \mathcal{U}_p\}$ we obtain the existence of a neighbourhood $U_p \in \mathcal{U}_p$ of p such that the set $F_p = X\setminus U_p$ is big. But X is compact, hence there exist finitely many points $p_1,\ldots,p_n \in X$ such that

$$X = \cup\{U_{p_i} : i = 1,\ldots,n\},$$

i.e.

$$\cap\{F_{p_i} : i = 1,\ldots,n\} = \emptyset.$$

A little reflection now shows that since we have finitely many big closed sets in X whose intersection is small, we must also have two big closed sets F and G in X such that $F \cap G$ is not big, i.e. $|F\cap G| \leq \omega$.

Now observe that for any countable (i.e. small) closed set $K \subset X$ we have $\psi(K,X) \leq \omega$. Indeed let us put

$$\mathcal{U} = \{\cup\mathcal{V}: \mathcal{V} \in [\cup\{\mathcal{U}_p: p \in K\}]^{<\omega} \ \& \ K \subset \cup\mathcal{V}\}.$$

Then $|\mathcal{U}| \leq \omega$ and we claim that $K = \cap\mathcal{U}$. Indeed, if $q \in X\setminus K$ then we can choose a $V_p \in \mathcal{U}_p$ with $q \notin V_p$ for each $p \in K$, but K is compact hence we get a set $V = \cup\mathcal{V} \in \mathcal{U}$ with

$$q \notin V \supset K.$$

Using this observation we take a countable family $\mathcal{U}$ of open neighbourhoods of $F \cap G$ with

$$F \cap G = \cap\mathcal{U}.$$

But F and G are big, i.e. uncountable, hence we can find $U \in \mathcal{U}$ and $V \in \mathcal{U}$ with

$$|F\setminus U| > \omega \text{ and } |G\setminus V| > \omega.$$

Hence if we put $W = U \cap V$, then $F\setminus W$ and $G\setminus W$ are disjoint and big closed sets. ⊣

COROLLARY. *If* $X \in \mathcal{C}_2$ *and* $\chi(X) \leq \omega$, *then either* $|X| \leq \omega$ *or* $|X| = 2^{\omega}$. ⊣

The following result, due to Šapirovskii again, is a very deep and very elegant strengthening of the Čech-Pospišil theorem. In order to formulate this however we need a bit of terminology. If X is a space a κ-dyadic system in X is a family

$$\{\langle F_\alpha^0, F_\alpha^1\rangle : \alpha \in \kappa\}$$

of pairs of closed subsets of X such that

a) $F_\alpha^0 \cap F_\alpha^1 = \emptyset$ for each $\alpha \in \kappa$;

b) $F_\varepsilon = \cap\{F_\alpha^{\varepsilon(\alpha)} : \alpha \in D(\varepsilon)\} \neq \emptyset$ for each $\varepsilon \in H(\kappa)$.

(We recall that $H(\kappa)$ denotes the set of all finite functions from κ to 2.)

3.18. *The following conditions are equivalent for* $X \in \mathcal{C}_2$:

(i) X *can be mapped continuously onto* I^κ;

(ii) *there is a closed set* $F \subset X$ *which can be mapped continuously onto* $D(2)^\kappa$;

(iii) *there is a closed set* $F \subset X$ *with*

$$\pi\chi(p,F) \geq \kappa$$

for each $p \in F$;

(iv) *there is a* κ*-dyadic system in* X.

PROOF.

(i) → (ii) is trivial since $D(2)^\kappa \subset I^\kappa$. ⊣

(ii) → (iii). Suppose that

$$f: X \to D(2)^\kappa$$

is a continuous onto map. Let us denote by $\mathcal{F}$ the family of all closed subsets F of X such that

$$f(F) = D(2)^{\kappa}.$$

It follows easily from the compactness of X that $\mathcal{F}$ is closed under intersections of decreasing chains hence by Zorn's lemma $\mathcal{F}$ contains a minimal member F. Then $f \restriction F$ is irreducible, i.e. no proper closed subset of F is mapped onto $D(2)^{\kappa}$. We claim that then

$$\pi\chi(p,F) \geq \pi\chi(f(p),D(2)^{\kappa}),$$

and the latter by 7.9 is equal to κ. Our claim follows from the simple observation that if U is a non-empty open set in F then

$$f^{\#}(U) = D(2)^{\kappa} \setminus f(F \setminus U) \neq \emptyset$$

as well since f is irreducible. Now if $\mathcal{U}$ is a local π-base at p in F then

$$\{f^{\#}(U) : U \in \mathcal{U}\}$$

is a local π-base at f(p) in $D(2)^{\kappa}$. Indeed, for every open neigbourhood G of f(p) there is a $U \in \mathcal{U}$ with $U \subset f^{-1}(G)$, hence with $f^{\#}(U) \subset G$.
(iii) $\to$ (iv). This is the really significant part of our result. Without loss of generality we can assume that F = X in (iii), i.e. $\pi\chi(p,X) \geq \kappa$ for all $p \in X$. We can also assume that $\kappa > \omega$, since the case $\kappa = \omega$ has been taken care of in case 1 of the proof of 3.16. Indeed, it suffices to put there

$$F_n^i = \cup\{U_\varepsilon : \varepsilon \in 2^n \ \&\ \varepsilon(n-1) = i\},$$

for all $n \in \omega$ and $i \in 2$. (Note also that $\chi(p,X) \geq \omega$ if and only if $\pi\chi(p,X) \geq \omega$.)
Now we have to construct a κ-dyadic system in X for $\kappa > \omega$. This will be achieved with the help of two lemmas.

LEMMA 1. *If $\mathcal{F}$ is a family of non-empty closed G_δ-sets in X with $|\mathcal{F}| < \kappa$ then we can find two closed G_δ sets K^0 and K^1 such that*
(a) *for each $F \in \mathcal{F}$ we have $K^0 \cap F \neq \emptyset$ and $K^1 \cap F \neq \emptyset$;*
(b) *there is an $F \in \mathcal{F}$ with $F \cap K^0 \cap K^1 = \emptyset$.*

PROOF OF LEMMA 1. Since

$$\psi(F,X) = \chi(F,X) \leq \omega < \kappa$$

holds for each $F \in \mathcal{F}$, we can apply the lemma from the proof of 2.14a) and conclude that every point $p \in X$ has a neighbourhood U_p such that

$$F \setminus U_p \neq \emptyset$$

whenever $F \in \mathcal{F}$. Of course we can assume that each U_p is an open F_σ, hence $C_p = X \setminus U_p$ is a closed G_δ set. As X is compact we can find finitely many points, say $p_1,\dots,p_n$ of X such that

$$\cup\{U_{p_i} : i = 1,\dots,n\} = X,$$

i.e.

$$\cap\{C_{p_i} : i = 1,\dots,n\} = \emptyset.$$

Now the sets C_{p_i} have the property required in (a), i.e. $C_{p_i} \cap F \neq \emptyset$ for $F \in \mathcal{F}$. Let k be the smallest integer such that there are k of the C_{p_i} whose intersection is disjoint from some $F \in \mathcal{F}$. Clearly then $1 < k \leq n$. Suppose that $i_1,\dots,i_k$ are the indices of k such C_{p_i}. It is easy to see that

$$K^0 = C_{p_{i_1}} \quad \text{and} \quad K^1 = \cap\{C_{p_{i_j}} : 2 \leq j \leq k\}$$

satisfy conditions (a) and (b). ⊣

LEMMA 2. *Suppose $\kappa_0 < \kappa_1 \leq \kappa$ where κ_1 is regular, and $\mathcal{F}_0$ is a κ_0-dyadic system in X composed of closed G_δ sets. Then there exists a κ_1-dyadic system $\mathcal{F}_1$ also composed of closed G_δ sets such that $|\mathcal{F}_0 \setminus \mathcal{F}_1| < \omega$.*

<u>PROOF OF LEMMA 2</u>. Let us put

$$F_0 = \{\langle F^0_\alpha, F^1_\alpha\rangle\ \alpha \in \kappa_0\}.$$

First we shall define by transfinite induction on $\alpha \in \kappa_1$ a family of pairs of closed G_δ sets $\langle C^0_\alpha, C^1_\alpha\rangle$ as follows. Suppose that $\alpha \in \kappa_1$ and the pairs $\langle C^0_\beta, C^1_\beta\rangle$ have been defined for all $\beta \in \alpha$. Define H_α as the set of all *non-empty* finite intersections composed of the elements of $\{F^i_\alpha\colon \alpha \in \kappa_0\ \&\ i \in 2\} \cup \{C^i_\beta\colon \beta \in \alpha\ \&\ i \in 2\}$. Note that an element of H_α can be written then in the form

$$F_h \cap C_g$$

with $h \in H(\kappa_0)$ and $g \in H(\alpha)$ and is of course a closed G_δ set. Now $|H_\alpha| < \kappa_1 \le \kappa$, hence lemma 1 can be applied to H_α, then we obtain two closed G_δ sets K^0_α and K^1_α which separately meet every member of H_α, but their intersection does not, i.e. we have $h_\alpha \in H(\kappa_0)$ and $g_\alpha \in H(\alpha)$ such that $F_{h_\alpha} \cap C_{g_\alpha} \in H_\alpha$ and

$$F_{h_\alpha} \cap C_{g_\alpha} \cap K^0_\alpha \cap K^1_\alpha = \emptyset.$$

Then we put

$$C^i_\alpha = F_{h_\alpha} \cap C_{g_\alpha} \cap K^i_\alpha \quad (i \in 2);$$

clearly $C^0_\alpha \cap C^1_\alpha = \emptyset$, but neither C^0_α nor C^1_α is empty. Having completed our transfinite construction of the C^i_α we now prove a claim concerning them.

<u>Claim</u>. If $h \in H(\kappa_0)$, $g \in H(\kappa_1)$ and

$$F_h \cap C_g \neq \emptyset,$$

then there is a finite set $a \in [\kappa_0]^{<\omega}$ such that for each $h' \in H(\kappa_0 \setminus a)$ we have

$$F_{h' \cup h} \cap C_g = F_{h'} \cap F_h \cap C_g \neq \emptyset$$

as well.

We shall prove this claim by induction on the maximal element α of $D(g)$. (We can put $\max \emptyset = -1$ for the empty function.) The claim is obviously true if $g = \emptyset$ (i.e. $\alpha = -1$), since $\mathcal{F}_0$ is κ_0-dyadic. Now assume that it holds whenever $g' \in H(\kappa_1)$ with $\max D(g') < \alpha$ and let $g \in H(\kappa_1)$ with $\max D(g) = \alpha$. Writing $g' = g \setminus \{<\alpha, g(\alpha)>\}$ we then have

$$F_h \cap C_g = F_h \cap C_{g'} \cap F_{h_\alpha} \cap C_{g_\alpha} \cap K_\alpha^{g(\alpha)} .$$

Now if $F_h \cap C_g \neq \emptyset$ then

$$F_{h \cup h_\alpha} \cap C_{g' \cup g_\alpha} \neq \emptyset$$

as well, moreover $\max D(g' \cup g_\alpha) < \alpha$, hence by our inductive assumption there is a finite set $a \in [\kappa_0]^{<\omega}$ as required in the claim for the pair $<h \cup h_\alpha, g' \cup g_\alpha>$. But then this same $\underline{a}$ can serve for the pair $<h,g>$ as well. Indeed, if $h' \in H(\kappa_0 \setminus a)$, then we have

$$F_{h'} \cap F_{h \cup h_\alpha} \cap C_{g' \cup g_\alpha} \neq \emptyset,$$

i.e. this set belongs to $\mathcal{H}_\alpha$, hence it meets $K_\alpha^{g(\alpha)}$:

$$F_{h'} \cap F_{h \cup h_\alpha} \cap C_{g' \cup g_\alpha} \cap K_\alpha^{g(\alpha)} = F_{h'} \cap F_h \cap C_{g'} \cap C_\alpha^{g(\alpha)} =$$

$$= F_{h'} \cap F_h \cap C_g \neq \emptyset,$$

which was to be shown.

Now we shall "thin out" the family $\{<C_\alpha^0, C_\alpha^1>: \alpha \in \kappa_1>$ to obtain our family $\mathcal{F}_1$. Let us consider for this purpose the function $f: \kappa_1 \to \kappa_1 \cup \{-1\}$ defined by

$$f(\alpha) = \max D(g_\alpha).$$

Then $f(\alpha) < \alpha$ for each $\alpha \in \kappa$, i.e. f is regressive. Thus by Neumer's theorem (also known as the pressing down lemma) there is a subset $B \subset \kappa_1$ with $|B| = \kappa_1$ and an $\alpha_0 \in \kappa_1$ with $f(\beta) = \alpha_0$ for each $\beta \in B$. Then $g_\beta \in H(\alpha_0+1)$ for each such β, moreover $|H(\alpha_0+1)| \leq |\alpha_0| + \omega < \kappa_1$, hence using the regularity of κ_1 we can take a fixed $g \in H(\alpha_0+1)$ such that

$$|\{\beta \in B\colon g_\beta = g\}| = \kappa_1 .$$

In the same manner of course we can further thin out this set to obtain an $A \in [\kappa_1]^{\kappa_1}$ so that for all $\alpha \in A$ we have $h_\alpha = h$ and $g_\alpha = g$ with fixed $h \in H(\kappa_0)$ and $g \in H(\kappa_1)$. Since then

$$C_\alpha^i = H_h \cap C_g \cap K_\alpha^i \neq \emptyset$$

for $\alpha \in A$, we have in particular that $F_h \cap C_g \neq \emptyset$, hence by our above claim there is an $a \in [\kappa_0]^{<\omega}$ with

$$F_{h'} \cap F_h \cap C_g \neq \emptyset$$

whenever $h' \in H(\kappa_0 \setminus a)$. Now we claim that

$$\mathcal{F}_1 = \{\langle F_\nu^0, F_\nu^1\rangle\colon \nu \in \kappa_0 \setminus a\} \cup \{\langle C_\alpha^0, C_\alpha^1\rangle\colon \alpha \in A\}$$

is as required. $|\mathcal{F}_0 \setminus \mathcal{F}_1| < \omega$ is trivial. To see that $\mathcal{F}_1$ (when suitably relabeled) is κ_1-dyadic we only have to show that

$$F_{h'} \cap C_{g'} \neq \emptyset$$

whenever $h' \in H(\kappa_0 \setminus a)$ and $g' \in H(A)$. Let us put

$$D(g') = \{\alpha_1, \ldots, \alpha_k\},$$

where $\alpha_1 < \ldots < \alpha_k$, moreover $g'(\alpha_j) = i_j$. Then

$$F_{h'} \cap C_{g'} = F_{h'} \cap C_{\alpha_1}^{i_1} \cap \ldots \cap C_{\alpha_k}^{i_k} =$$

$$= F_{h'} \cap F_h \cap C_g \cap K_{\alpha_1}^{i_1} \cap \ldots \cap K_{\alpha_k}^{i_k}.$$

Now, since $h' \in H(\kappa_0 \setminus a)$, we have

$$F_{h'} \cap F_h \cap C_g \neq \emptyset,$$

hence $F_{h'} \cap F_h \cap C_g \in \mathcal{H}_{\alpha_1}$, consequently

$$F_{h'} \cap F_h \cap C_g \cap K_\alpha^{i_1} = F_{h'} \cap C_{\alpha_1}^{i_1} \neq \emptyset$$

as well. But then $F_{h'} \cap C_{\alpha_1}^{i_1} \in H_{\alpha_2}$, hence similarly as before we get

$$F_{h'} \cap C_{\alpha_1}^{i_1} \cap K_{\alpha_2}^{i_2} = F_{h'} \cap C_{\alpha_1}^{i_1} \cap C_{\alpha_2}^{i_2} \neq \emptyset.$$

Continuing in this manner we shall get in k steps the required relation

$$F_{h'} \cap C_{g'} \neq \emptyset.$$

Now we can return to proving (iii)→(iv). In fact, if κ is regular we are already home because we can put $\kappa_1 = \kappa$ in lemma 2. Thus assume from now on that κ is singular, i.e. $cf(\kappa) = \rho < \kappa$. Let us write κ in the form

$$\kappa = \Sigma\{\kappa_\nu : \nu \in \rho\},$$

where $\rho < \kappa_\nu < \kappa_\mu < \kappa$ whenever $\nu \in \mu \in \rho$ and κ_ν is regular for each $\nu \in \rho$. We shall now define by transfinite induction a κ_ν-dyadic family F_ν in X for each $\nu \in \rho$ in such a way that $|F_\nu \setminus F_{\nu'}| < \rho$ if $\nu < \nu'$. To start with, let F_0 be any κ_0-dyadic family in X, which exists by lemma 2. If $\mu \in \rho$ and F_ν has been suitably defined for every $\nu \in \mu$, consider the family

$$F'_\nu = \cap\{F_\alpha : \nu \le \alpha < \mu\}$$

for every $\nu \in \mu$. By the inductive hypotheses then $|F_\nu \setminus F'_\nu| < \rho$, hence $|F'_\nu| = |F_\nu| = \kappa_\nu$ moreover it is easy to see that $F'_\nu \subset F'_{\nu'}$ for $\nu \in \nu' \in \mu$. Let us now put

$$F^{(\mu)} = \cup\{F'_\nu : \nu \in \mu\}.$$

Then $|F^{(\mu)}| = \Sigma\{\kappa_\nu : \nu \in \mu\} = \kappa^{(\mu)} < \kappa_\mu$ as κ_μ is regular and $\mu < \rho < \kappa_\mu$, moreover $F^{(\mu)}$ is clearly $\kappa^{(\mu)}$-dyadic in X. Thus we can apply lemma 2 with $F^{(\mu)}, \kappa^{(\mu)}$ and κ_μ in place of F_0, κ_0 and κ_1 to obtain an F_μ which is κ_μ-dyadic in X and satisfies $|F^{(\mu)} \setminus F_\mu| < \omega$, consequently $|F_\nu \setminus F_\mu| < \rho$ for each $\nu \in \mu$. Having completed the construction now it is easy to deduce that for $F''_\mu = \cap\{F_\nu : \mu \leq \nu < \rho\}$ we have

$$|F''_\mu| = |F_\mu| = \kappa_\mu \quad \text{and} \quad F''_\nu \subset F''_\mu \quad \text{if } \nu < \mu,$$

hence $\cup\{F_\mu : \mu \in \rho\}$ is a κ-dyadic system in X. ⊣

(iv)→(i). Let $F = \{\langle F^0_\alpha, F^1_\alpha\rangle : \alpha \in \kappa\}$ be κ-dyadic in X. Let us put $F_\alpha = F^0_\alpha \cup F^1_\alpha$ and

$$F = \cap\{F_\alpha : \alpha \in \kappa\}.$$

Then F is closed in X, moreover

$$F = \cup\{F_s : s \in 2^\kappa\},$$

where

$$F_s = \cap\{F^{s(\alpha)}_\alpha : \alpha \in \kappa\} \neq \emptyset$$

for $s \in 2^\kappa$. We also have $F_s \cap F_t = \emptyset$ if $s,t \in 2^\kappa$ and $s \neq t$, hence the map

$$f: F \to D(2)^\kappa$$

determined by the relation

$$f(p) = s \leftrightarrow p \in F_s$$

is well-defined and onto. It is also easy to see that f is continuous. It is well-known that $D(2)^\kappa$ maps continuously onto I^κ, hence so does F, thus by Uryson's extension theorem this map extends to a continuous map of X onto I^κ. ⊣

Before we give applications of this result we formulate an auxiliary result.

3.19. *Let* $X \in C_2$, $F \subset X$ *closed and* $p \in F$. *Then*

$$\pi\chi(p,X) \le \pi\chi(p,F).\chi(F,X).$$

<u>PROOF</u>. This is trivial if F is also open in X. Thus assume now that $\chi(F,X) \ge \omega$. Let us choose a family $\mathcal{F}$ of non-empty closed G_δ-sets in F such that $|\mathcal{F}| = \pi\chi(p,F)$ and every neighbourhood of p in F contains a member of $\mathcal{F}$. Clearly we have then

$$\psi(C,X) = \chi(C,X) \le \chi(F,X)$$

whenever $C \in \mathcal{F}$. Thus choosing a neighbourhood base $\mathcal{U}_C$ of minimal cardinality for each $C \in \mathcal{F}$ we have that

$$\mathcal{U} = \cup\{\mathcal{U}_C: C \in \mathcal{F}\}$$

is a local π-base at p in X, moreover

$$|\mathcal{U}| \le \pi\chi(p,F).\chi(F,X). \quad \dashv$$

3.20. *If* $X \in \mathcal{C}_2$ *does not map continuously onto* I^κ, *then*

$$S_\kappa = \{p \in X: \pi\chi(p,X) < \kappa\}$$

is dense in X.

<u>PROOF</u>. The case $\kappa = \omega$ is easy: then every closed subset of X has an isolated point, hence X is scattered, hence the set of its isolated points is dense in X. If, on the other hand, $\kappa > \omega$, then we can apply 3.18 to conclude that every closed subset $F \subset X$ has a point $p \in F$ with $\pi\chi(p,F) < \kappa$, but since every non-empty open set contains a non-empty closed G_δ-set we obtain from 3.19 that S_κ is dense in X. $\dashv$

<u>COROLLARY</u>. *If* $X \in \mathcal{C}_2$ *does not admit a continuous map onto* I^{κ^+}, *then*

$$\rho(X) \le \kappa^{c(X)}.$$

The proof of this is immediate from 3.20 and the corollary of 2.37. The following deep result of Šapirovskii now follows easily.

3.21. *If* X $\in$ C_5 *then*

$$\rho(X) \leq 2^{c(X)}.$$

PROOF. One has to notice only that X does not admit a continuous map onto I^{ω_1}, since as is well-known the closed continuous image of a T_5 space is again T_5 and $I^{\omega_1} \notin T_5$ (because e.g. the Tychonov plank embeds into I^{ω_1}), and then to apply the above corollary. ⊣

3.22. *If* βN *does not embed into* X $\in$ C_2, *then*

$$\rho(X) \leq 2^{c(X)}.$$

PROOF. In this case we claim that X does not map continuously onto $I^{\exp\omega}$. Indeed, if

$$f: X \to I^{\exp\omega}$$

is an onto map then as βN embeds into $I^{\exp\omega}$ (since $w(\beta N) = \exp\omega$) there is a closed subset F of X such that $f(F) = \beta N$ and $f \upharpoonright F$ is irreducible. But βN is extremally disconnected, moreover it is known that an irreducible map of a compact Hausdorff space onto an extremally disconnected space is a homeomorphism, hence we get $\beta N \simeq F \subset X$, a contradiction. Thus using the corollary of 3.20 we get

$$\rho(X) \leq (2^{\omega})^{c(X)} = 2^{c(X)}. \quad ⊣$$

Next we are going to present another very interesting result of Šapirovskii shedding some new light on the rather close ties that we have already seen to exist between the tightness and "π-structure" of compact Hausdorff spaces. In order to achieve a clear presentation we have broken up the proof into three sub-results.

3.22. *Let* X $\in$ C_2 *and put* $\kappa = t(X)$. *Then there exists an ireedúcible continuous onto map*

$$f: X \to Y,$$

where Y *embeds into a* Σ_κ*-power of* $I = [0,1]$.

PROOF. We shall define by transfinite induction on α continuous maps $f_\alpha: X \to I^\alpha$ (putting $I^0 = \{0\}$ a singleton space), in such a way that if $\beta < \alpha$ then

$$I(\beta,\alpha): \quad f_\beta = \pi^\alpha_\beta \circ f_\alpha,$$

where π^α_β denotes the natural projection of I^α onto I^β. Now if α is limit and f_β has been defined for all $\beta \in \alpha$ satisfying $I(\beta,\gamma)$ whenever $\beta \in \gamma \in \alpha$, then we can (and must) define f_α by putting

$$f_\alpha(p)(\beta) = f_{\beta+1}(p)(\beta)$$

for each $p \in X$ and $\beta \in \alpha$. Clearly this will insure $I(\beta,\alpha)$ for all $\beta \in \alpha$. If however $\alpha = \beta+1$, then we first examine whether f_β is irreducible onto $f_\beta(X)$. If it is, then we stop. Now, if it is not, then we choose a non-empty open set $G_\beta \subset X$ such that

$$f_\beta(X \backslash G_\beta) = f_\beta(X),$$

and then a continuous function

$$g_\beta: X \to I$$

such that

$$g_\beta(X \backslash G_\beta) = \{0\} \quad \text{and} \quad 1 \in g_\beta(G_\beta).$$

Then we define f_α using the stipulation $I(\beta,\alpha)$ and putting for each point $p \in X$

$$f_\alpha(p)(\beta) = g_\beta(p).$$

Observe that this implies: if $\gamma > \beta$ then G_β cannot show the reducibility of f_γ. Consequently we must arrive at an ordinal α such that f_α is an irreducible map of X onto its range. Hence, to conclude, its suffices to show that in this case

$$Y = f_\alpha(X) \subset \Sigma_\kappa(I^\alpha).$$

(Recall that $\Sigma_\kappa(I^\alpha) = \{f \in I^\alpha : |\{\beta \in \alpha : f(\beta) \neq 0\}| \leq \kappa\}$.) Assume, indirectly, that $y \in f_\alpha(X) \setminus \Sigma_\kappa(I^\alpha)$, i.e. we can find a set of ordinals $\{\beta_\rho : \rho \in \kappa^+\} \subset \alpha$ such that (i) $\beta_\rho < \beta_{\rho'}$ if $\rho < \rho'$, and (ii) $y(\beta_\rho) > 0$ for each $\rho \in \kappa^+$. Let us define now for each $\rho \leq \kappa^+$ the point $y_\rho \in Y$ as follows: (we put $\beta_{\kappa^+} = \cup\{\beta_\rho : \rho \in \kappa^+\}$)

$$y_\rho(\beta) = \begin{cases} y(\beta), & \text{if } \beta \in \beta_\rho; \\ 0, & \text{if } \beta \in \alpha \setminus \beta_\rho. \end{cases}$$

It follows easily from our construction that each such point y_ρ belongs to $Y = f_\alpha(X)$ as y does. Now it is obvious that y_{κ^+} is a limit point of the set $\{y_\rho : \rho \in \kappa^+\}$, while it is not a limit point of any subset of it of size at most κ, consequently $t(Y) \geq \kappa^+$. This however is impossible because by 1.17 the closed map f_α cannot raise the tightness. ⊣

REMARK. The topologically initiated reader will readily recognize that the above argument, which by the way is Šapirovskii's original approach to all of his results in this chapter, actually yields the following stronger result: If a completely regular space of tightness $\leq \kappa$ admits a perfect map onto a subspace of a Σ_κ-power of I, then it also admits an irreducible such map. It can be mentioned here that every metrizable space embeds into a Σ_ω-power of I. In order to formulate our next result it will be convenient to use the following piece of notation:

$$\pi sw(X) = \min\{\operatorname{ord}(\mathcal{B}) : \mathcal{B} \text{ is a } \pi\text{-base of } X\}.$$

This should be compared with 1.18.

3.24. *If* Y *embeds into a* Σ_κ*-power of* I *then* $\pi sw(Y) \leq \kappa$.

PROOF. For any Y embeddable into a Σ_κ-power of I let us put

$$\lambda_Y = \min\{\lambda : Y \hookrightarrow \Sigma_\kappa(I^\lambda)\}.$$

We shall prove our claim by induction on λ_Y. It holds trivially if $\lambda_Y \le \omega$, thus we put $\lambda_Y = \lambda > \omega$ and assume that 3.24 holds for all Z with $\lambda_Z < \lambda$. We shall call Y *good* if it has the property that $\lambda_G = \lambda_Y$ for each non-empty open set $G \subset Y$. Clearly every non-empty open set H in Y contains a non-empty open good subspace G, e.g. any $G \subset H$ with λ_G minimal. Therefore if $\mathcal{G}$ is a maximal disjoint family of open good subsets of Y, then $\cup\mathcal{G}$ is dense in Y, hence if each $G \in \mathcal{G}$ has a π-base of order $\le\kappa$ then so does Y. Consequently it suffices to restrict our attention to good spaces, i.e. we can also assume that Y is good. Now let us denote by L the set of all limit ordinals in λ, and for each $\alpha \in L$ put

$$Y_\alpha = \{y\restriction\alpha+\omega : y \in Y \ \&\ \exists n \in \omega\ (y(\alpha+n) > 0)\} \subset \Sigma_\kappa(I^{\alpha+\omega}).$$

Then $\lambda_{Y_\alpha} \le |\alpha+\omega| < \lambda$, hence by our inductive hypotheses we can choose for each $\alpha \in L$ a π-base $\mathcal{B}_\alpha$ in Y_α with $\mathrm{ord}(\mathcal{B}_\alpha) \le \kappa$. We can of course assume that the members of each $\mathcal{B}_\alpha$ are traces on Y_α of elementary open sets from $I^{\alpha+\omega}$, moreover that for every $B \in \mathcal{B}_\alpha$ there is a $\xi(B) \in (\alpha+\omega)\setminus\alpha$ with

$$0 \notin \mathrm{pr}_{\xi(B)}(B) \subset I.$$

For any $\alpha \in L$ and $B \in \mathcal{B}_\alpha$ let us now put

$$B' = \{y \in Y : y\restriction\alpha+\omega \in B\} = (\pi^\lambda_{\alpha+\omega})^{-1}(B)$$

and $\mathcal{B}'_\alpha = \{B' : B \in \mathcal{B}_\alpha\}$, moreover

$$\mathcal{B}' = \cup\{\mathcal{B}'_\alpha : \alpha \in L\}.$$

First we show that $\mathcal{B}'$ is a π-base for Y. Indeed, Let U be an elementary open set in I^λ with $U\cap Y \neq \emptyset$. Since Y is good we have $\lambda_{U\cap Y} = \lambda$, hence we can find an $\alpha \in L$ such that (i) the support of U is contained in α, and (ii) there eixsts a point $y \in U\cap Y$ with $y\restriction\alpha+\omega \in Y_\alpha$. Clearly, then we can find a $B \in \mathcal{B}_\alpha$ with $B \subset \pi^\lambda_{\alpha+\omega}(U\cap Y)$, hence $B' \subset U\cap Y$. Next we show that $\mathrm{ord}(\mathcal{B}') \le \kappa$. Assume that this is false. Since $\mathrm{ord}(\mathcal{B}'_\alpha) \le \kappa$ for each $\alpha \in L$, then we can choose a point $y \in Y$, a set $A \in [L]^{\kappa^+}$ and for each $\alpha \in A$ a member $B_\alpha \in \mathcal{B}_\alpha$ such that

$$y \in \cap \{B'_\alpha : \alpha \in A\}.$$

Let us write for each $\alpha \in A$

$$\xi_\alpha = \xi(B_\alpha) \in \alpha+\omega\setminus\alpha.$$

Clearly, if $\alpha \neq \alpha'$ belong to A then $\xi_\alpha \neq \xi_{\alpha'}$. But by definition of the ξ_α's we have

$$y(\xi_\alpha) \in pr_{\xi_\alpha}(B_\alpha) \subset I\setminus\{0\}$$

whenever $\alpha \in A$, contradicting that $y \in \Sigma_\kappa(I^\lambda)$. ⊣

3.25. *If* $X \in \mathcal{C}_2$, *then* $\pi sw(X) \leq t(X)$

PROOF. By 3.23 let $f: X \to Y$ be an irreducible map of X onto Y, where Y embeds into a Σ_κ-power of I (with $\kappa = t(X)$). Using 3.24 we can choose a π-base $\mathcal{B}$ of Y with $ord(\mathcal{B}) \leq \kappa$.
Now it suffices to show that

$$\{f^{-1}(B) : B \in \mathcal{B}\}$$

forms a π-base for X. Indeed, for any non-empty open $U \subset X$ we have $f^{\#}(U) = Y\setminus f(X\setminus U) \neq \emptyset$ and open, because f is closed and irreducible, but if $B \in \mathcal{B}$ satisfies $B \subset f^{\#}(U)$ then $f^{-1}(B) \subset U$. ⊣

COROLLARY. *If* $X \in \mathcal{C}_2$ *and* $t(X)^+$ *is a caliber for* X *then* $\pi(X) \leq t(X)$. ⊣

The next application of 3.25 yields an alternative and quite elegant proof of 3.13. We first formulate an auxiliary result needed for this.

3.26. *Let* $X \in \mathcal{T}$ *and* $\mathcal{G}$ *be a family of non-empty open subsets of* X *such that* $ord(p,\mathcal{G}) < \kappa$ *holds for all* $p \in X$. *Then there is a family* $\{D_\alpha : \alpha \in \kappa\}$ *of discrete subspaces of* X, *whose union* $D = \cup\{D_\alpha : \alpha \in \kappa\}$ *is "dense" in* $\mathcal{G}$, i.e. $D \cap G \neq \emptyset$ *for all* $G \in \mathcal{G}$.

PROOF. We shall construct by transfinite induction on $\alpha \in \kappa$ subfamilies

$\mathcal{G}_\alpha \subset \mathcal{G}$ and sets D_α. Thus assume we have constructed $\mathcal{G}_\beta$ and D_β for $\beta \in \alpha$. Then put $\mathcal{H}_\alpha = \mathcal{G}\setminus\cup\{\mathcal{G}_\beta: \beta \in \alpha\}$ and define D_α as a maximal subset A of $\cup\mathcal{H}_\alpha$ with the property that $|A\cap G| \le 1$ for all $G \in \mathcal{H}_\alpha$. Then D_α is clearly discrete (it is even closed discrete in $\cup\mathcal{H}_\alpha$). Next we define $\mathcal{G}_\alpha$ by

$$\mathcal{G}_\alpha = \{G \in \mathcal{H}_\alpha : G \cap D_\alpha \neq \emptyset\}.$$

First we show that, having completed the construction, we have

$$\mathcal{G} = \cup\{\mathcal{G}_\alpha : \alpha \in \kappa\}.$$

Assume, on the contrary that $p \in G \in \mathcal{G}\setminus\cup\ \{\mathcal{G}_\alpha : \alpha \in \kappa\}$. Then by our construction we have

$$p \in \cup\ \mathcal{H}_\alpha \setminus D_\alpha$$

for every $\alpha \in \kappa$, hence by the choice of D_α we must have a $G_\alpha \in \mathcal{G}_\alpha$ with $p \in G_\alpha$. But then $\alpha \neq \beta$ implies $G_\alpha \neq G_\beta$ contradicting that $\text{ord}(p,\mathcal{G}) < \kappa$. Our result now follows immediately. $\dashv$

Now if $X \in \mathcal{C}_2$ and we put $\kappa = t(X)^+$, then 3.25 yields a π-base $\mathcal{B}$ for X with $\text{ord}(p,\mathcal{B}) < \kappa$ for each $p \in X$, hence from 3.26 we have a family $\{D_\alpha : \alpha \in \kappa\}$ of discrete subsets of X such that

$$D = \cup\{D_\alpha : \alpha \in \kappa\}$$

is "dense" in $\mathcal{B}$, consequently dense in X as well. But clearly $|D| \le s(X).t(X)^+$, from which 3.13 follows easily. $\dashv$

To conclude this chapter we shall turn to a topic that might have been studied in chapter 2 as well. This concerns the following general question: if we have an inequality that places an upper bound on the cardinality of certain spaces can this be strengthened to the same upper bound for the number of all compact subsets of these spaces? Since compact sets in many respect play similar roles as points, this is not an unreasonable question. The first systematic treatment of this question was carried out by Hodel and Burke, most of the following results are due to them.

Now, for any X, we shall denote by K(X) the number of all compact subsets of X. Let us note that for any $X \in \mathcal{T}$ we have $|X| \le K(X)$ as every singleton is compact, however if $X \in \mathcal{T}_2$ then every compact set in X is closed, hence $K(X) \le o(X)$. Thus from 2.21 we immediately obtain that

$$K(X) \le \operatorname{expexp} s(X)$$

whenever $X \in \mathcal{T}_2$, hence the required strenghtening of 2.20 is indeed valid. However, as we shall see, it is not always that easy to prove such strenghtenings, even if they are valid.

3.27. *If* $X \in \mathcal{T}_1$, *then* (cf. 2.2)

$$K(X) \le 2^{\psi w(X)}.$$

PROOF. Let $\mathcal{B}$ be a pseudobase of X with $|\mathcal{B}| \le \psi w(X)$ and such that $\mathcal{B}$ is closed under finite unions. For any compact $C \subset X$ and $p \in X \setminus C$ there is a $B \in \mathcal{B}$ with $C \subset B \subset X \setminus \{p\}$. Indeed, we can choose for every $q \in C$ a $B_q \in \mathcal{B}$ with $q \in B_q$ but $p \notin B_q$. Since C is compact we can find a finite set $A \in [C]^{<\omega}$ such that

$$B = \cup\{B_q : q \in A\} \supset C.$$

Clearly B is as required. But then for every compact $C \subset X$ if we put $\mathcal{B}_C = \{B \in \mathcal{B}: C \subset B\}$ then

$$\cap \mathcal{B}_C = C,$$

hence the map $C \to \mathcal{B}_C$ is one-one, i.e. $K(X) \le |P(\mathcal{B})| \le 2^{\psi w(X)}$. ⊣

3.28. *If* $X \in \mathcal{T}_2$, *then* (cf. 2.4)

$$K(X) \le \operatorname{expexp} d(X).$$

PROOF. Since $X \in \mathcal{T}_2$, the family RO(X) forms a pseudobase for X. Consequently by 2.6d) we have

$$\psi w(X) \le \rho(X) \le 2^{d(X)},$$

hence from 3.27

$$K(X) \le \exp \psi w(X) \le \operatorname{expexp} d(X).$$

3.29. *If* $X \in T_2$ *then* (cf. 2.16)

$$K(X) \le 2^{h(X)}.$$

PROOF. Let us fix a linear order $<$ on X and for any $\{p,q\} \in [X]^2$ with $p < q$ choose disjoint open neighbourhoods $U_{p,q}$ and $V_{p,q}$ respectively. Denote by $\mathcal{B}$ the family of all finite intersections formed by sets of the form $V_{p,q}$. Then, by 2.16,

$$|\mathcal{B}| \le |X| \le 2^{h(X)}.$$

Now if $C \subset X$ is compact and $p \notin C$ we can find a finite set $A \in [C]^{<\omega}$ with

$$C \subset \cup\{U_{xp} : x \in A\},$$

hence

$$p \in \cap\{V_{xp} : x \in A\} = B_p \subset X\backslash C.$$

In other words we have

$$X\backslash C = \cup\{B_p : p \in X\backslash C\},$$

hence using $L(X\backslash C) \le h(X)$ we can find a set $S \in [X\backslash C]^{\le h(X)}$ such that

$$X\backslash C = \cup\{B_p : p \in S\}.$$

Consequently we have

$$K(X) \le |[\mathcal{B}]^{\le h(X)}| \le (2^{h(X)})^{h(X)} = 2^{h(X)}. \qquad \dashv$$

REMARK. Since 2.16 was an immediate consequence of 2.15a) it is natural to ask whether the latter has the corresponding strenghtening. In fact this is still an open problem, even if X itself is assumed to be in C_2. However from 3.13 we obtain that if $X \in T_2$ and $C \subset X$ is compact then

$$d(C) \le s(C)^+ \le s(X)^+,$$

hence, using 2.15a)

$$K(X) \le |[X]^{\le s(X)^+}| \le (2^{\psi(X).s(X)})^{s(X)^+} = 2^{\psi(X).s(X)^+},$$

which is just slightly weaker than what one would expect.
The next result due to Burke and Hodel approaches the desired strengthening of 2.15a) from another angle in that instead of $\psi(X)$ it uses the "compact pseudocharacter" $\Psi_K(X)$ defined as follows:
If $X \in T_1$, then

$$\Psi_K(X) = \sup\{\psi(C,X) : C \subset X \text{ is compact}\}.$$

3.30. *Let* $X \in T_2$, *then*

$$K(X) \le 2^{\Psi_K(X).s(X)}.$$

PROOF. Put $\kappa = \Psi_K(X).s(X)$, then from $\psi(X) \le \Psi_K(X)$ and 2.15a) we get $|X| \le 2^\kappa$. Now, in exactly the same way as in the proof of 3.29, we can obtain a family of open sets $\mathcal{B}$ with $|\mathcal{B}| \le |X| \le 2^\kappa$ such that for every compact set $C \subset X$ and $p \in X \setminus C$ there is a $B \in \mathcal{B}$ satisfying $p \in B \subset X \setminus C$. Consider a compact set C and a closed set $F \subset X \setminus C$. For each $p \in F$ we can select a $B_p \in \mathcal{B}$ with $p \in B_p \subset X \setminus C$, hence

$$\mathcal{G} = \{B_p : p \in F\}$$

is an open cover of F. Since $s(F) \le s(X) \le \kappa$, applying 2.13 to $\mathcal{G}$ we get

$$S \in [F]^{\le\kappa} \quad \text{and} \quad \mathcal{C} \in [\mathcal{G}]^{\le\kappa}$$

such that

$$F \subset \bar{S} \cup \cup\mathcal{C} = A(S,\mathcal{C}) \subset X\backslash C.$$

Now recall that $\psi(C,X) \leq \kappa$, hence

$$X\backslash C = \cup\{F_\alpha : \alpha \in \kappa\},$$

where each F_α is closed, consequently $X\backslash C$ can be written as

$$X\backslash C = \cup\{A(S_\alpha,\mathcal{C}_\alpha) : \alpha \in \kappa\}.$$

But as both $|X|$ and $|\mathcal{B}|$ are $\leq 2^\kappa$, we have at most 2^κ sets of the form $A(S,\mathcal{C})$, and thus at most 2^κ unions formed by at most κ sets of the form $A(S,\mathcal{C})$. ⊣

<u>REMARK</u>. I do not know whether $\mathcal{T}_2$ could be replaced by $\mathcal{T}_1$ here as in 2.15a).

3.31. *If* $X \in \mathcal{T}_2$, *then* (cf. 2.31)

$$K(X) \leq 2^{p(X).\Psi(X)}.$$

<u>PROOF</u>. From 2.30 we get $s(X) \leq p(X).\Psi(X)$, moreover as X is Hausdorff $\Psi_K(X) \leq \Psi(X)$ holds as well. Consequently, by 3.30 we have

$$K(X) \leq 2^{s(X).\Psi_K(X)} \leq 2^{p(X).\Psi(X)}. \quad ⊣$$

Before giving the corresponding strengthening of 2.28 we prove an auxiliary result, which generalizes for higher cardinals the well-known fact that compact T_2 spaces with G_δ diagonals are metrizable.

3.32. *If* $X \in \mathcal{C}_2$, *then*

$$\psi_\Delta(X) = w(X).$$

<u>PROOF</u>. Of course only $w(X) \leq \psi_\Delta(X) = \kappa$ needs proof. But now $\psi_\Delta(X) = \psi(\Delta, X\times X) = \chi(\Delta, X\times X)$, hence we have a neighbourhood base

$\mathcal{U}$ for Δ in $X\times X$ with $|\mathcal{U}| = \kappa$. Using the compactness of X it is easy to find for each $U \in \mathcal{U}$ a finite open cover $\mathcal{V}_U$ of X such that

$$\Delta \subset \cup\{V\times V\colon V \in \mathcal{V}_U\} \subset U.$$

We claim that

$$\mathcal{V} = \cup\{\mathcal{V}_U\colon U \in \mathcal{U}\}$$

is a base for X. Since $|\mathcal{V}| \le \kappa$ is trivial this will give what we want.

Thus let $p \in X$ and $F \subset X$ be closed with $p \notin F$. Then $F \times \{p\}$ is closed in $X\times X$ and $(F\times\{p\}) \cap \Delta = \emptyset$, hence there is a $U \in \mathcal{U}$ with $(F\times\{p\}) \cap U = \emptyset$ as well. Now if $V \in \mathcal{V}_U$ is such that $p \in V$, then

$$V\times\{p\} \subset V\times V \subset U \subset X\times X\setminus F\times\{p\},$$

i.e. $p \in V \subset X\setminus F$ and $\mathcal{V}$ is indeed a base for X. $\dashv$

3.33. *If* $X \in T_2$, *then* (cf. 2.28)

$$K(X) \le 2^{p(X)\cdot\psi_\Delta(X)}.$$

<u>PROOF</u>. Now if $C \subset X$ is compact, then $C \in C_2$, hence 3.32 implies $d(C) \le w(C) = \psi_\Delta(C) \le \psi_\Delta(X)$. Consequently using 2.28 we get

$$K(X) \le |X|^{\psi_\Delta(X)} \le (2^{p(X)\cdot\psi_\Delta(X)})^{\psi_\Delta(X)} = 2^{p(X)\cdot\psi_\Delta(X)}. \quad \dashv$$

3.34. *If* $X \in T_1$ *then* (cf. 2.33)

$$K(X) \le 2^{p(X)\cdot psw(X)}.$$

<u>PROOF</u>. Let us put $p(X)\cdot psw(X) = \kappa$ and choose a ψ-base $\mathcal{B}$ for X with $ord(\mathcal{B}) \le \kappa$. Since 2.33 implies $|X| \le 2^\kappa$ we clearly also have $|\mathcal{B}| \le 2^\kappa$. Now by Miščenko's lemma, 0.7, we know that for any compact $C \subset X$ the collection $\mathcal{H}_C$ of all finite minimal covers of C by members of $\mathcal{B}$ has cardinality $\le \kappa$. We claim that

$$C = \cap\{\cup\mathcal{V}: \mathcal{V} \in \mathcal{H}_C\}.$$

Indeed, if $p \in X\setminus C$ we can choose for each $x \in C$ a $B_x \in \mathcal{B}$ with $x \in B_x \subset X\setminus\{p\}$, hence by the compactness of C we have a finite cover of C of the form

$$\mathcal{V} = \{B_{x_1},\ldots,B_{x_n}\},$$

where of course $\mathcal{V}$ can be assumed to be a minimal cover of C, i.e. $\mathcal{V} \in \mathcal{H}_C$. But clearly $p \notin \cup\mathcal{V}$. Thus we have

$$K(X) \leq |[\mathcal{B}]^{\leq\kappa}| \leq (2^\kappa)^\kappa = 2^\kappa. \quad \dashv$$

CHAPTER 4

THE SUP = MAX PROBLEM

The functions c,s,h,z have the common feature of having been defined as the supremum of cardinalities of certain sets. Sometimes these sets are referred to as the "*defining sets*" of the corresponding cardinal functions. It is natural to ask under what conditions is this supremum actually a maximum, or in other words using the notation introduced in 1.22, if ϕ is one of the these functions, when do we have $\phi(X) < \hat{\phi}(X)$. This is what we briefly call the sup = max problem. Obviously if $\phi(X)$ is a successor cardinal then $\hat{\phi}(X) = \phi(X)^+$, i.e. our problem is trivial. The interesting cases are therefore those in which the function values are limit cardinals.

4.1. *For any* $X \in \mathcal{T}$ *if* $c(X) = \lambda$ *is singular then*

$$\hat{c}(X) = \lambda^+,$$

i.e. X *has a cellular family of cardinality* λ.

PROOF. Let us call an open non-empty set $G \subset X$ good if $c(H) = c(G)$ whenever H is a non-empty open subset of G. Now every non-empty open set in X has a good subset, e.g. one of minimal cellularity. Therefore if $\mathcal{H}$ is a maximal disjoint family of good sets in X then $\cup\mathcal{H}$ is dense in X. If $|\mathcal{H}| = \lambda$, then we are done, hence we may assume that

$$|\mathcal{H}| = \kappa < \lambda.$$

Next we show that

$$\sup\{c(H) : H \in \mathcal{H}\} = \lambda.$$

Indeed, let $\rho < \lambda$ be any regular cardinal with $\kappa < \rho$. Then from

$c(X) = \lambda > \rho$ we have a cellular family $\mathcal{D}$ in X with $|\mathcal{D}| = \rho$. Since $\rho > \kappa$ is regular and $\cup\mathcal{H}$ is dense, we conclude that some member H of $\mathcal{H}$ intersects ρ members of $\mathcal{D}$, hence $c(H) \geq \rho$. Let us write

$$\lambda = \Sigma\{\lambda_\alpha : \alpha \in \mu\},$$

where $\mu = cf(\lambda)$ and each λ_α is regular. To end the proof it clearly suffices to find a cellular family $\{G_\alpha : \alpha \in \mu\}$ such that $c(G_\alpha) > \lambda_\alpha$ for each $\alpha \in \mu$. If there is an $H \in \mathcal{H}$ with $c(H) = \lambda$, then any cellular family of size $\mu < \lambda$ taken in H will do as H is good. If on the other hand $c(H) < \lambda$ for each $H \in \mathcal{H}$, then we can easily select from $\mathcal{H}$ itself such a family, using that $\sup\{c(H) : H \in \mathcal{H}\} = \lambda$. $\dashv$

<u>REMARK</u>. The question remains what happens if $c(X) = \lambda$ is a regular limit, i.e. weakly inaccessible cardinal. We shall see (cf. 7.6) that for such a λ already $\hat{c}(X) = \lambda$ can occur.

4.2. *If* λ *is a singular strong limit cardinal and* $X \in T_2$ *with* $|X| \geq \lambda$, *then* $\hat{s}(X) > \lambda$, *i.e.* X *contains a discrete subspace of cardinality* λ.

<u>PROOF</u>. Let $\prec$ be a well-ordering of X and for each $\{x,y\} \in [X]^2$ with $x \prec y$ choose disjoint neighbourhoods U_{xy} and V_{xy} respectively. Then we define a partition of $[X]^3$ into four parts as follows: if $\{x,y,z\} \in [X]^3$ with $x \prec y \prec z$, then put

$$f(\{x,y,z\}) = \langle \varepsilon_1, \varepsilon_2 \rangle$$

according to the following stipulations

$$\varepsilon_1 = \begin{cases} 0, & \text{if } x \in U_{yz} \\ 1, & \text{otherwise,} \end{cases} \qquad \varepsilon_2 = \begin{cases} 0, & \text{if } z \in V_{xy} \\ 1, & \text{otherwise.} \end{cases}$$

Now we can apply the canonization lemma, 0.5, to this partition f to find an $H \subset X$ with $|H| = \lambda$ and a decomposition

$$H = \cup\{H_\alpha : \alpha \in \mu = f(\lambda)\}$$

of H such that the conditions of that lemma hold. Suppose that $\alpha \in \mu$

and $y \in H_\alpha$, moreover y has an immediate $\prec$-predecessor x and an immediate $\prec$-successor z in H_α. We claim that y is isolated in H. In fact, let

$$N = V_{xy} \cap U_{yz} \cap H.$$

Evidently $x,z \notin N$. Now if $p \in H$ and $p \prec x$, then $p \in V_{xy}$ implies $p \notin U_{xy}$, hence $f(\{p,x,y\}) = \langle 1,\varepsilon_2\rangle$ by the definition of f. Since H is canonical then we also have $f(\{p,y,z\}) = \langle 1,\varepsilon_2\rangle$, consequently $p \notin U_{yz} \supset N$. But if $p \notin V_{xy}$ then $p \notin N$ again. We can quite similarly show that if $z \prec q \in H$ then $q \notin N$ as well, hence $N \cap H = \{y\}$ indeed. But obviously there are altogether λ such points y in H, hence they form a discrete subspace of size λ in X. $\dashv$

<u>COROLLARY</u>. *If $\phi \in \{s,h,z\}$ and $X \in T_2$ with $\phi(X) = \lambda$, a strong limit singular cardinal, then*

$$\hat{\phi}(X) = \lambda^+. \quad \dashv$$

In our subsequent results the class H of the so-called strongly Hausdorff spaces will play an important role. Now, by definition, $X \in H$ if and only if it is Hausdorff and has the following property: from every infinite subset $A \subset X$ we can choose a sequence of points $\{p_n: n \in \omega\}$ such that the p_n have pairwise disjoint neighbourhoods in X. It can be shown that $H \supset T_3$, in fact every Uryson space (i.e. one in which two distinct points have disjoint closed neighbourhoods) is strongly Hausdorff.

4.3. *Let λ be a singular cardinal with* $cf(\lambda) = \omega$.
a) *If $X \in T_2$ and $h(X) = \lambda$ then $\hat{h}(X) = \lambda^+$.*
b) *If $X \in H$ and $\phi \in \{s,z\}$ then $\phi(X) = \lambda$ implies $\hat{\phi}(X) = \lambda^+$.*

<u>PROOF</u>. We shall prove all these three sup = max results simultaneously. Let us put

$$\lambda = \Sigma\{\lambda_k: k \in \omega\},$$

where $\omega < \lambda_k < \lambda_{k+1}$ and λ_k is regular for each $k \in \omega$. We can choose for each k a defining set $D_k \subset X$ with $|D_k| = \lambda_k$ and then assume that

$$X = \cup\{D_k : k \in \omega\},$$

since then $\phi(X) = \lambda$ will remain valid. Note that this implies that whenever $S \subset X$ satisfies $|S| \geq \kappa > \omega$, where $\kappa < \lambda$ is a regular cardinal, then S contains a defining set of cardinality κ. We can also assume that every $p \in X$ has a neighbourhood of cardinality less than λ. Indeed let Y be the union of all open subsets G of X with $|G| < \lambda$. If $|Y| = \lambda$ then we can simply replace X by Y. If $|Y| < \lambda$ then clearly every non-empty open set in the subspace $X \setminus Y$ has cardinality λ. Now let $\{G_k : k \in \omega\}$ be an infinite cellular family in $X \setminus Y$, which exists in every infinite Hausdorff space. But then for each $k \in \omega$ we have a defining set $R_k \subset G_k$ with $|R_k| \geq \lambda_k$, and then clearly $R = \cup\{R_k : k \in \omega\}$ is a defining set with $|R| = \lambda$ and we are done.
Let us denote by X_κ the set of those points $p \in X$ which have a neighbourhood U_p of cardinality less than κ. If we have $|X_\kappa| = \lambda$ for some $\kappa < \lambda$, then we can apply Hajnal's theorem, 0.3, to the set mapping

$$F(p) = U_p \cap X_\kappa$$

over X_κ and find a set $D \subset X_\kappa$ with $|D| = \lambda$ which is free for F. But clearly then D is a discrete subspace, hence a defining set for ϕ of cardinality λ. Therefore we can assume from now on that $|X_\kappa| < \lambda$ for each $\kappa < \lambda$. But then we can define by an easy induction a sequence $\{p_k : k \in \omega\}$ of distinct points of X such that every neighbourhood of p_k has cardinality at least λ_k.

Now in case a) let us just choose for every $k \in \omega$ an open neighbourhood G_k of p_k such that

$$\lambda_k \leq |G_k| < \lambda.$$

If we pass to a suitable subsequence we can also assume that $|G_k| < \lambda_{k+1}$. For each $k \in \omega$ put

$$S_k = G_k \setminus \cup\{G_\ell : \ell < k\},$$

then clearly $|S_k| = |G_k| \geq \lambda_k$, hence we can choose a right separated set $R_k \subset S_k$ with $|R_k| \geq \lambda_k$ as well. Since

$$\cup\{R_\ell : \ell < k\} \subset \cup\{G_\ell : \ell < k\} \subset X \setminus R_k,$$

it is obvious that

$$R = \cup\{R_k : k \in \omega\}$$

is also right separated and of cardinality λ. In case b) using $X \in \mathcal{H}$ and passing to a suitable subsequence we can assume that the points p_k have pairwise disjoint neighbourhoods G_k. Then we can choose again in each G_k a defining set R_k for ϕ with $|R_k| \geq \lambda_k$, and clearly

$$R = \cup\{R_k : k \in \omega\}$$

is the required defining set of size λ. $\dashv$

<u>COROLLARY</u>. *If* $X \in \mathcal{T}_2$ *and* $s(X) = \hat{s}(X) = \lambda$ *with* $cf(\lambda) = \omega < \lambda$, *then there is a* $Y \subset X$ *with* $z(Y) = \hat{z}(Y) = \lambda$. *Hence if* sup = max *fails at a singular* λ *of countable cofinality for* s *in* $\mathcal{T}_2$, *then it fails for* z *as well.*

<u>PROOF</u>. Let us put

$$Y = \cup\{D_k : k \in \omega\},$$

where $D_k \subset X$ is discrete with $|D_k| \geq \lambda_k$. Then $\hat{z}(Y) \geq z(Y) = \lambda$ is trivial. Now if $Z \subset Y$ with $|Z| = \lambda$, then clearly $s(Z) = h(Z) = \lambda$, hence by 4.3a) there is a right separated set $R \subset Z$ with $|R| = \lambda$. But then R (and thus Z) cannot be left separated, since otherwise by 2.12 it would contain a discrete subspace of cardinality λ. $\dashv$

Of course the above corollary is of use only if sup = max does fail for s in $\mathcal{T}_2$ with a singular λ of cofinality ω. The following beautiful characterization of just when this might happen is due to K. Kunen and J. Roitman. In it we use $\mathbb{C}$ to denote the Cantor set, more precesily $\mathbb{C} = D(2)^\omega$.

4.4. *Let* λ *be a singular cardinal with* $cf(\lambda) = \omega$. *Then the following two statements* $P(\lambda)$ *and* $Q(\lambda)$ *are equivalent:*

$P(\lambda)$: *If* $X \in \mathcal{T}_2$ *with* $s(X) = \lambda$ *then* $\hat{s}(X) = \lambda^+$.
$Q(\lambda)$: *If* $Y \in [\mathbb{C}]^\lambda$ *then there is a set* $B \subset Y$

with $|B| = |Y| = \lambda$ *such that* B *is meager (i.e. of first category) in* $\mathbb{C}$.

<u>PROOF</u>. As the proof is rather lenghty and complicated we shall start with a few easy lemmas concerning nowhere dense and meager subsets of $\mathbb{C}$. First we fix some notation. For any $h \in H(\omega)$ we put

$$N_h = \{f \in \mathbb{C}: h \in f\},$$

the elementary open set in $\mathbb{C}$ defined by h. Moreover we shall write NWD instead of nowhere dense and SD instead of somewhere dense = not nowhere dense.

<u>LEMMA 1</u>. $A \subset \mathbb{C}$ *is NWD if and only for each* $n \in \omega$ *there is an* $h \in H(\omega\setminus n)$ *with* $N_h \cap A = \emptyset$.

<u>PROOF OF LEMMA 1</u>. Suppose $A \subset \mathbb{C}$ is NWD and let $n \in \omega$. We enumerate the collection of all 0-1 sequences of length n in a sequence $\{k_i : i < 2^n\}$. Using that A is NWD in $\mathbb{C}$ we can easily define a sequence $\{h_i : i < 2^n\} \subset H(\omega\setminus n)$ in such a way that $h_0 \subset h_1 \subset \ldots \subset h_i \subset \ldots$ and $A \cap N_{k_i \cup h_i} = \emptyset$ for each $i < 2^n$. Now put

$$h = h_{2^n-1} = \cup\{h_i : i < 2^n\},$$

then $h \in H(\omega\setminus n)$ and for every $f \in N_h$ we have

$$f \restriction n = k_i$$

for some $i < 2^n$, hence

$$f \in N_{k_i \cup h} \subset N_{k_i \cup h_i} \subset \mathbb{C}\setminus A.$$

On the other hand, let A satisfy the condition of the lemma and consider any $p \in H(\omega)$ with $D(p) \subset n$. If $h \in H(\omega\setminus n)$ in such that $A \cap N_h = \emptyset$, then $N_p \cap N_h = N_{p\cup h} \subset N_p$ with $A \cap N_{p\cup h} = \emptyset$, hence A is NWD in $\mathbb{C}$.

<u>LEMMA 2</u>. *If* $Q(\lambda)$ *holds then for every* $\kappa < \lambda$ *there is a* $\kappa' < \lambda$ *such that every* $A \in [\mathbb{C}]^{\kappa'}$ *contains a NWD subset* B *with* $|B| > \kappa$.

PROOF OF LEMMA 2. Let us write $\lambda = \Sigma\{\lambda_n : n \in \omega\}$ with $\lambda_n < \lambda$ for each $n \in \omega$, and assume, indirectly
find for every $n \in \omega$ a set $A_n \in [\mathbb{C}]^{\lambda_n}$ such that if $B \subset A_n$ is NWD in $\mathbb{C}$ then $|B| \le \kappa$. Let us put

$$A = \cup\{A_n : n \in \omega\}.$$

Then $A \in [\mathbb{C}]^{\lambda}$, hence by $Q(\lambda)$ we can find a $B \in [A]^{\lambda}$ which is meager, i.e.

$$B = \cup\{B_m : m \in \omega\},$$

where each B_m is NWD. Consequently we have

$$|B_m \cap A_n| \le \kappa$$

for every pair $\langle m,n\rangle \in \omega\times\omega$, which implies

$$|B| = |\cup\{B_m \cap A_n : \langle m,n\rangle \in \omega\times\omega\}| \le \kappa,$$

a contradiction.

LEMMA 3. *If* $X \in [\mathbb{C}]^{\lambda}$ *and no* $Y \in [X]^{\lambda}$ *is meager in* $\mathbb{C}$ *then there is a* $\kappa < \lambda$ *such that every* $A \in [X]^{\kappa}$ *is* SD *in* $\mathbb{C}$.

PROOF OF LEMMA 3. If no such κ existed then we could find for each $n \in \omega$ a NWD set $A_n \in [X]^{\lambda_n}$. But then

$$A = \cup\{A_n : n \in \omega\}$$

would be a meager subset of X with $|A| = \lambda$.

$P(\lambda) \to Q(\lambda)$. Assume $Q(\lambda)$ fails, i.e. there is an $X \in [\mathbb{C}]^{\lambda}$ such that no $Y \in [X]^{\lambda}$ is meager; we shall construct a Hausdorff topology τ on X such that $s(X,\tau) = \hat{s}(X,\tau) = \lambda$. Since for any $f \in \mathbb{C}$ there are only countably many $g \in \mathbb{C}$ satisfying

$$|\{n \in \omega : f(n) \neq g(n)\}| < \omega$$

we can assume that if $f,g \in X$ and $f \neq g$ then

$$|\{n \in \omega : f(n) \neq g(n)\}| = \omega .$$

Next write $\lambda = \Sigma\{\lambda_n : n \in \omega\}$, where each λ_n is less than λ, and accordingly let

$$X = \cup\{X_n : n \in \omega\}$$

be a disjoint decomposition (i.e. $X_n \cap X_m = \emptyset$ if $n \neq m$) of X with $|X_n| = \lambda_n$ for $n \in \omega$. Now if $f \in X_n$ and $k \in \omega$ put

$$U_k(f) = \{f\} \cup \{g \in \cup_{m<n} X_m : \forall j < k(g(n+j) = f(n+j))\}.$$

Clearly if $k < \ell < \omega$ then $U_k(f) \supset U_\ell(f)$, moreover if $g \in U_k(f) \cap X_m$ with $m < n$ then

$$U_{n+k}(g) \subset U_k(f),$$

hence we have determined a topology τ on X with $\{U_k(f) : k \in \omega\}$ as a τ neighbourhood base of f for any $f \in X$. To see that τ is Hausdorff take $f \in X_n$, $g \in X_m$ with $m \leq n$ and $f \neq g$. We can find then a $j \in \omega$ with $f(n+j) \neq g(n+j)$, consequently, as is easy to see,

$$U_k(f) \cap U_k(g) = \emptyset$$

if $k > n + j$.

Since, for any $f \in X_n$, we have $U_0(f) \cap X_n = \{f\}$, the set X_n is discrete in (X,τ), consequently $s(X) = \lambda$ holds. Finally we show that no $Y \in [X]^\lambda$ is discrete in (X,τ). Let us put $Y_n = Y \cap X_n$ for $n \in \omega$, clearly there is a fixed $m \in \omega$ such that $|Y_m| > \kappa$, where κ is as in lemma 3, consequently Y_m is SD in $\mathbb{C}$. Thus by lemma 1 we can find an $n \in \omega$ such that $Y_m \cap N_h \neq \emptyset$ whenever $h \in H(\omega \setminus n)$; it can of course be assumed that $n > m$ and $Y_n \neq \emptyset$. Let then $f \in Y_n$, $k \in \omega$ and put

$$h = f \restriction \{n+j : j < k\} \in H(\omega \setminus n).$$

According to the above we have

$$Y_m \cap N_h \neq \emptyset,$$

and clearly $y_m \cap N_n \subset V_k(f)$, hence

$$Y_m \cap V_k(f) \neq \emptyset$$

as well. But this shows that f is an accumulation point of $Y_m \subset Y$, hence Y is indeed not discrete.

<u>$Q(\lambda) \to P(\lambda)$</u>. Let us assume that $Q(\lambda)$ holds and $X \in T_2$ with $s(X) = \lambda$. Repeating what we have done in the proof of 4.3 (and using the same notation as there) we can assume that X satisfies the following properties (i) - (iii):

(i) if $S \in [X]^\kappa$ where $\kappa < \lambda$ is regular then S contains a discrete subset of cardinality κ;

(ii) every $p \in X$ has a neighbourhood U_p of cardinality less than λ;

(iii) if $\kappa < \lambda$ then $X_\kappa = \{p \in X$: p has a neighbourhood of cardinality at most $\kappa\}$ has cardinality $< \lambda$. Let us write for any space S and point $p \in S$

$$\phi(p,S) = \min\{|U| : p \in U \text{ and } U \text{ is open in } S\},$$

thus (ii) is equivalent to $\phi(p,X) < \lambda$ for all $p \in X$, while (iii) can be rewritten as follows: if $\kappa < \lambda$ then

$$|\{p \in X: \phi(p,X) \leq \kappa\}| < \lambda.$$

Let us put for any set $U \subset X$ and $\kappa < \lambda$

$$E_\kappa(U) = \{q \in X: \phi(q, U \cup \{q\}) < \kappa\}.$$

We shall say that U is κ-good if $|U| \geq \kappa$ and $|E_\kappa(U)| < \lambda$.

<u>CLAIM</u>. If X also satisfies condition (iv) below, then it contains a discrete subset of cardinality λ, i.e. $\hat{s}(X) = \lambda^+$.

(iv) There exists a cardinal $\theta < \lambda$ such that if $\theta < \kappa < \lambda$ is a regular cardinal and $p \in X$ satisfies $\phi(p,X) \geq \kappa$, then every open set containing p is κ-good.

In order to prove our claim we shall define by induction a strictly increasing sequence of regular cardinals $\kappa_n < \lambda$ and pairs of disjoint κ_n-good open sets U_0^n and U_1^n with $|U_0^n \cup U^n{}_1| < \lambda$ as follows. Let κ_0 be regular with $\max\{\theta,\lambda_0\} < \kappa_0 < \lambda$, p_0^0 and p_1^0 be (using (iii)) distinct points of X with $\phi(p_i^0,X) \geq \kappa_0$ for $i \in 2$ and U_0^0, U_1^0 be disjoint open neighbourhoods of p_0^0 and p_1^0 respectively, such that $|U_i^0| = \phi(p_i^0,X) < \lambda$. Clearly, by (iv), both U_i^0 are κ_0 - good. Suppose now that $n \in \omega$ and we have already suitably defined κ_m, U_0^m and U_1^m for every $m \leq n$. Then we can choose a regular $\kappa_{n+1} < \lambda$ such that $\lambda_{n+1} \leq \kappa_{n+1}$,

$$\kappa_n \leq |U_i^n| < \kappa_{n+1} \qquad (i \in 2),$$

moreover $|R_n| < \kappa_{n+1}$, where

$$R_n = \cup\{E_{\kappa_m}(U_i^m) : m \leq n \text{ and } i \in 2\}.$$

This is possible because by the inductive hypothesis every U_i^m is κ_m-good. We can also assume, using lemma 2, that every $A \in [\mathbb{C}]^{\kappa_{n+1}}$ has a NWD subset of size κ_n. Then we choose distinct points p_0^{n+1}, p_1^{n+1} $\in X$ with

$$\phi(p_i^{n+1}, X) \geq \kappa_{n+1}$$

and U_0^{n+1}, U_1^{n+1} as disjoint open neighbourhoods of them with

$$\kappa_{n+1} \leq |U_i^{n+1}| = \phi(p_i^{n+1},X) < \lambda.$$

Having completed this inductive procedure let us put for any $h \in H(\omega)$

$$U_h = \cap\{U_{h(j)}^j : j \in D(h)\}.$$

We claim that if $n = \min D(h)$ then

$$|U_h| \geq \kappa_n .$$

This can be proved by induction on $|h|$. If $|h| = 1$ this just says $|U_i^n| \geq \kappa_n$. Next assume that $|h| = |D(h)| = k+1$ and we have already established our sub-claim for $h' \in H(\omega)$ with $|h'| \leq k$. Put

$$n = \min D(h) \text{ and } h' = h \setminus \{<n,h(n)>\}$$

Then

$$|U_{h'}| \geq \kappa_{\min D(h')} \geq \kappa_{n+1} > |R_n|,$$

hence we can choose a point $q \in U_{h'} \setminus R_n$. In particular then $q \notin E_{\kappa_n}(U^n_{h(n)})$, consequently

$$|U_h| = |U_{h'} \cap U^n_{h(n)}| \geq \phi(q, U^n_{h(n)} \cup \{q\}) \geq \kappa_n.$$

Now let us assign to every point $x \in X$ an $f_x \in \mathbb{C}$ by the following stipulations:

$$f_x(n) = \begin{cases} 0, & \text{if } x \in U^n_0 \\ 1 & \text{otherwise.} \end{cases}$$

Clearly, for any $h \in H(\omega)$, $x \in U_h$ implies $f_x \in N_h$.
Next we do one more inductive procedure to define for $i \in \omega$ a finite function $h_i \in H(\omega)$ with $-1 = n_i \min D(h_i) > \max D(h_{i-1})$ for $i > 0$, and sets $W_i \subset Z_i \subset X$ such that $\{f_x : x \in W_i\}$ is NWD in $\mathbb{C}$. Let us put $h_0 = \{<1,0>\}$ (i.e. $D(h_0) = \{1\}$ and $h_0(1) = 0$), $Z_0 = U_{h_0} = U^1_0$, and $W_0 \subset Z_0$ be such that $|W_0| \geq \kappa_0$ and $\{f_x : x \in W_0\}$ is NWD in $\mathbb{C}$. Now if everything has been suitably defined for $0 \leq j \leq i$, then the set

$$S_i = \{f_x : x \in \bigcup_{j=0}^{i} W_j\}$$

is NWD in $\mathbb{C}$, hence by lemma 1 we can choose $h_{i+1} \in H(\omega \setminus (m_i+2))$, where $m_i = \max D(h_i)$, so that $N_{h_{i+1}} \cap S_i = \emptyset$. Then we put $n_{i+1} = \min D(h_{i+1})-1$ and

$$Z_{i+1} = U_{h_{i+1}} \setminus \cup \{U_{h_j} : j \leq i\}.$$

Clearly, $|Z_{i+1}| = |U_{h_{i+1}}| \geq \kappa_{n_{i+1}+1}$, because $|U_{h_j}| < \kappa_{m_j+1} \leq \kappa_{n_{i+1}}$ holds for every $j \leq i$, hence we can find a subset $W_{i+1} \subset Z_{i+1}$ with

$|W_{i+1}| \geq \kappa_{n_{i+1}}$ such that $\{f_x: x \in W_{i+1}\}$ is NWD in $\mathbb{C}$. This completes the induction. Let us now choose for every $i \in \omega$ (in view of (i)) a discrete subset $D_i \subset W_i$ with $|D_i| = \kappa_{n_i}$. Observe that if $j < i$, then $U_{h_j} \cap W_i = \emptyset$ because even $U_{h_j} \cap Z_i = \emptyset$, moreover $W_j \cap U_{h_i} = \emptyset$ holds as well because $\{f_x: x \in W_j\} \cap N_{h_i} = \emptyset$. But this implies then that $D = \cup\{D_i: i \in \omega\}$ is discrete in X while $|D| = \lambda$. Thus our claim is proven.

Consequently we can assume that, if $Y \subset X$ and $|Y| = \lambda$, Y does not satisfy (iv). Indeed, otherwise, as it inherits properties (i) and (ii) from X, Y would contain a discrete subspace of cardinality λ, either because of our claim, if it also satisfies (iii), or using the same reasoning as in the proof of 4.3, if it does not satisfy (iii). Thus it remains to show $\hat{s}(X) = \lambda^+$ under the following additional assumption:

(v) For every $Y \in [X]^\lambda$ and $\theta < \lambda$ there exist a regular $\theta < \kappa < \lambda$, a point $p \in Y$ with $\phi(p,Y) \geq \kappa$ and an open neighbourhood U of p in X such that U is not κ-good in Y, i.e. $|\{y \in Y: \phi(y,(U \cap Y)\cup\{y\})<\kappa\}| = \lambda$.

We shall now use (v) to define by induction on $n \in \omega$ sets $Y_n \in [X]^\lambda$, regular cardinals $\kappa_n < \lambda$, points $p_n \in Y_n$ and open (in X) neighbourhoods U_n of p_n such that $Y_{n+1} \subset Y_n$, $\lambda_n < \kappa_n < \kappa_{n+1}$, $\phi(p_n,Y_n) \geq \kappa_n$, $|U_n| < \lambda$ and U_n is not κ_n-good in Y_n. For $n = 0$ we simply put $Y_0 = X$, and $\kappa_0 > \lambda_0$, p_0, U_0 are chosen by using (v). If we have already suitably defined everything with indices up to n, then we first put

$$Y_{n+1} = \{y \in Y_n: \phi(y,(U_n \cap Y_n)\cup\{y\}) < \kappa_n\}\setminus U_n,$$

hence $Y_{n+1} \in [Y_n]^\lambda$. Next we use (v) again to get a regular $\kappa_{n+1} > \max\{\kappa_n,\lambda_n\}$, a point $p_{n+1} \in Y_{n+1}$ with $\phi(p_{n+1},Y_{n+1}) \geq \kappa_{n+1}$ and an open set $U_{n+1} \ni p_{n+1}$ which is not κ_{n+1}-good in Y_{n+1} and satisfies $|U_{n+1}| < \lambda$ and $|U_{n+1} \cap U_k \cap Y_k|<\kappa_k$ for every $k \leq n$, the latter being possible because $p_{n+1} \in {}_{k+1}$ for each $k \leq n$. After having completed the induction, put for each $n \in \omega$

$$Z_n = U_n \cap Y_n \setminus \cup\{U_m: n < m < \omega\}.$$

Since by our construction $n < m$ implies

$$|U_m \cap U_n \cap Y_n| < \kappa_n ,$$

and $\kappa_n > \omega$ is regular we have

$$|Z_n| = |U_n \cap Y_n| \geq \phi(p_n,Y_n) \geq \kappa_n.$$

Moreover, if $m < n$ then by our construction $U_m \cap Y_{m+1} = \emptyset$ and thus $U_m \cap Y_n = \emptyset$, hence $U_m \cap Z_n = \emptyset$ as well, consequently $Z_n \cap U_m = \emptyset$ for every $m \neq n$. But then if $D_n \subset Z_n$ is a discrete subset of Z_n with $|D_n| \geq \kappa_n$, which exists by (i), then clearly $D = \cup\{D_n : n \in \omega\}$ is discrete in X with $|D| = \lambda$, as was required. ⊣

COROLLARY 1. *If* $\lambda > 2^\omega$ *with* $cf(\lambda) = \omega$ *and* $X \in T_2$ *satisfies* $s(X) = \lambda$, *then* $\hat{s}(X) = \lambda^+$. ⊣

COROLLARY 2. *If Martin's axiom holds and* $\lambda < 2^\omega$ *with* $cf(\lambda) = \omega$, then $X \in T_2$ *with* $s(X) = \lambda$ *implies* $\hat{s}(X) = \lambda^+$.

PROOF. Indeed it is well-known that under Martin's axiom every set $Y \in [\mathbb{C}]^{<2^\omega}$ is meager in $\mathbb{C}$, hence $Q(\lambda)$ holds. ⊣

REMARKS. It is well known that the natural map of $\mathbb{C}$ onto $I = [0,1]$, which assigns to every $f \in \mathbb{C}$ the member of I with dyadic expansion f, takes (non-)meager subsets of $\mathbb{C}$ onto (non-)meager subsets of I, hence in 4.4 one could replace $\mathbb{C}$ by I. It is also known that if one adds λ Cohen reals to a model of ZFC then, in the resulting model, I (or $\mathbb{C}$) has a subset of cardinality λ no uncountable subset of which is meager. This shows that sup=max might actually fail for s and thus for z in T_2 at a singular λ of countable cofinality.
In the rest of this chapter I shall give applications of our above results to the problem about the nature of $o(X)$ for $X \in T_2$. As we shall see this problem is quite closely related to the sup=max problem for s, h and z.

4.5. *If* λ *is a singular strong limit cardinal then* $\lambda \neq o(X)$ *for every* $X \in T_2$.

PROOF. If $|X| < \lambda$, then $o(X) \leq 2^{|X|} < \lambda$ as λ is strong limit. If on the other hand $\lambda \leq |X|$, then by 4.2 there is a discrete $D \subset X$ with $|X| = \lambda$, hence by 2.11

$$\lambda < 2^\lambda \leq o(X). \qquad ⊣$$

4.6. *Let* λ *be a limit but not strong limit cardinal, and assume that* 2^κ *is strictly increasing for cofinally many* $\kappa < \lambda$. *If* $X \in T_2$ *is such that*

$$|X| = \hat{s}(X) = \hat{z}(X) = \lambda,$$

then

$$o(X) = 2^{\lambda},$$

hence in particular $o(X)$ is singular with

$$cf(o(X)) = cf(2^{\lambda}) = cf(\lambda) \le \lambda < o(X).$$

PROOF. $2^{\lambda} \le o(X)$ follows immediately from 2.11. On the other hand $\hat{z}(X) = \lambda$ implies $d(Y) < \lambda$ for every $Y \subset X$, hence clearly

$$o(X) \le |[X]^{<\lambda}| = \lambda^{\lambda}.$$

But λ is not strong limit, hence there is a $\kappa < \lambda$ with $2^{\kappa} > \lambda$, and thus $2^{\lambda} = \lambda^{\lambda}$. ⊣

REMARK. As was indicated in the remark made after 4.4 it is consistent to have an $X \in \mathcal{T}_2$ with $|X| = \hat{s}(X) = \hat{z}(X) = \lambda$, where $cf(\lambda) = \omega$ (e.g. $\lambda = \aleph_{\omega}$). It is also easy to see that it is consistent to assume that at the same time the 2^{κ} function is strictly increasing. Consequently by 4.6 we have then $cf(o(X)) = \omega$.
It is shown by our next result however that for the class $\mathcal{H}$ of strongly Hausdorff spaces the situation is quite different in that it is in some sense "almost hopeless" to find an $X \in \mathcal{H}$ with $cf(o(X)) = \omega$, or even with $o(X)^{\omega} \neq o(X)$.

4.7. *Let* κ *be a cardinal such that* $o(X) = \kappa$ *for some infinite* $X \in \mathcal{H}$ *and* $\kappa < \kappa^{\omega}$. *Then there is a cardinal* β *with the following properties* (i) - (iv):

(i) $\omega < cf(\beta) = \gamma < \beta$;
(ii) $(\forall \alpha < \beta)(\alpha^{\gamma} < \beta)$;
(iii) $\beta^{\gamma} > \beta^{(\omega)}$ (= *the* ω^{th} *successor of* β);
(iv) $\kappa \ge \beta^{(\omega)}$.

PROOF. Let λ be the smallest cardinal such that $\lambda^\omega > \kappa$. Since $\lambda \le \kappa$, the power λ^ω is clearly a jump, hence by 0.2 we have $\omega = cf(\lambda)$. Moreover $\kappa = o(X) > 2^\omega$ implies $\lambda > \omega$.
For any $p \in X$ let us put

$$\sigma(p,x) = \min\{\sigma(U): p \in U, U \text{ open in } X\}$$

and

$$\sigma = \sigma(X) = \sup\{\sigma(p,X): p \in X\}.$$

Since $X \in H$ there can only be finitely many points $p \in X$ such that $\sigma(p,X) \ge \lambda$, for otherwise X would contain a disjoint family $\{U_n: n \in \omega\}$ of open sets with $o(U_n) \ge \lambda$ for all $n \in \omega$, and thus

$$o(X) \ge \lambda^\omega > \kappa = o(X)$$

would follow. On the other hand, throwing away finitely many points from X will clearly not change $o(X)$, hence we can assume that $\sigma(p,X) < \lambda$ for each $p \in X$.
Now we claim that in fact $\sigma < \lambda$ must be valid. Assume, on the contrary, that $\sigma = \lambda$. Since λ can be written as $\lambda = \Sigma\{\lambda_n: n \in \omega\}$, where $\lambda_n < \lambda$ for $n \in \omega$, then we can pick for $n \in \omega$ distinct points $p_n \in X$ such that $\sigma(p_n,X) > \lambda_n$, moreover using $X \in H$ we can assume that each p_n has a neighbourhood U_n so that the family $\{U_n: n \in \omega\}$ is disjoint. However this implies, by 1.2 c),

$$o(X) \ge \Pi\{o(U_n): n \in \omega\} \ge \prod_{n\in\omega} \lambda_n = \lambda^\omega > \kappa,$$

a contradiction.
Next we show that $|X| \le \sigma^+$. Indeed, every $p \in X$ has an open neighbourhood $U(p)$ such that $|U(p)| \le o(U(p)) \le \sigma$. Hence if $|X| > \sigma^+$ were true then $U(p)$ would be a set-mapping which satisfies the conditions of Hajnal's theorem, 0.3, hence a free set $D \subset X$ with $|D| = |X|$ would exist for $U(p)$. However this subspace D is clearly discrete, consequently $\kappa = o(X) = 2^{|X|}$, which is of course impossible.
Now consider the above defined open cover $\mathcal{U} = \{U(p): p \in X\}$ of X, then $|\mathcal{U}| \le \sigma^+$. Let τ denote the smallest cardinal for which X *does not* contain a discrete subspace of cardinality τ, i.e. $\tau = \hat{s}(X)$. As is

shown in 2.13, then every closed subset $F \subset X$ can be obtained in the following form

$$F = (F \cap (\cup \mathcal{U}_F)) \cup \bar{S}_F,$$

where $\mathcal{U}_F \in [\mathcal{U}]^{<\tau}$ and $S_F \in [X]^{<\tau}$. An easy calculation shows then that

$$\lambda \le o(X) \le (\sigma^+)^{\tau}.$$

Since $X \in \mathcal{H}$, 4.3 implies $cf(\tau) > \omega$. From this and $cf(\lambda) = \omega$ it follows then that there is a cardinal $\rho < \tau$ with $(\sigma^+)^{\rho} > \lambda$. Let γ be the smallest cardinal with $(\sigma^+)^{\gamma} > \lambda$ and then β be smallest such that $\beta^{\gamma} > \lambda$. Then $\beta \le \sigma^+ < \lambda$, hence $\gamma > \omega$ by the choice of λ. Moreover $\gamma < \tau$, hence X contains a discrete subspace of size γ, consequently $o(X) \ge 2^{\gamma} = \gamma^{\gamma}$ and thus $\beta^{\gamma} \ge \lambda^{\omega} > o(X)$ implies $\beta > \gamma$. In particular β and γ are infinite, hence the power β^{γ} is a jump and therefore $\gamma = cf(\beta)$. Now it is obvious that β satisfies conditions (i) - (iv). $\dashv$

As an immediate corollary we obtain that if $X \in \mathcal{H}$ and $o(X) < \omega_{\omega_1+\omega}$ then $o(X)^{\omega} = o(X)$. Indeed, this is obvious since ω_{ω_1} is the smallest cardinal which satisfies (i). However our result says much more than this. Indeed, the consistency of the existence of a cardinal satisfying (i) - (iii) has only been established by M. Magidor with the help of some enormously large (so called strongly compact) cardinals. Moreover by some very recent results of Jensen & Todd the existence of such a β implies that measurable cardinals exist in some inner models of set theory. This shows that constructing a "counterexample" would require some very sophisticated new method in axiomatic set theory.

It is natural to ask now whether a more definitive result than 4.7 could be obtained for more special classes of Hausdorff spaces. Our next two results are of this form. Let $\mathcal{P}$ denote the class of all hereditarily paracompact T_3 spaces.

4.8. *If* $X \in \mathcal{P}$ *and* $|X| \ge \omega$, *then* $o(X) = o(X)^{\omega}$.

<u>PROOF</u>. Suppose, on the contrary, that $\kappa = o(X) < \kappa^{\omega}$. Similarly as in the proof of 4.7 we let λ be the smallest cardinal whose ω^{th} power exceeds κ. Then $cf(\lambda) = \omega < \lambda \le \kappa$. We can of course assume that for all $Y \subset X$ with $o(Y) = \kappa$ we have $\sigma(Y) = \sigma(X)$. Since $\mathcal{P} \subset \mathcal{H}$, and the class

$\mathcal{P}$ is hereditary, the same argument as in the proof of 4.7 yields that $\sigma = \sigma(X) < \lambda$. Put $\rho = \min \{\alpha : \sigma^{\alpha} > \lambda\}$. By the choice of λ then $\rho > \omega$. The following claim is the crux of the proof.

<u>CLAIM</u>. Let $\langle\kappa_{\xi} : \xi \in \rho\rangle$ be a sequence of cardinals such that $\kappa_{\xi} < \sigma$ for every $\xi \in \rho$. Then there is a disjoint family $\{G_{\xi} : \xi \in \rho\}$ of sets open in X such that $\kappa_{\xi} < o(G_{\xi})$ for each $\xi \in \rho$. In particular $\Pi\{\kappa_{\xi} : \xi \in \rho\} \leq o(X) = \kappa$.

<u>PROOF OF THE CLAIM</u>. Clearly we have a locally finite open cover $\mathcal{U}$ of X for which $o(\bar{U}) \leq \sigma$ for every $U \in \mathcal{U}$. Now we define by transfinite induction for $\xi \in \rho$ open sets $G_{\xi} \subset X$ and $U_{\xi} \in \mathcal{U}$ such that $G_{\xi} \subset U_{\xi}$. Suppose that $\eta \in \rho$ and G_{ξ}, U_{ξ} have been defined for $\xi \in \eta$. Then

$$o(\cup \{\bar{U}_{\xi} : \xi \in \eta\}) \leq \sigma^{|\eta|} < \lambda \leq \kappa,$$

hence for $Y = X \setminus \cup\{\bar{U}_{\xi} : \xi \in \eta\}$ we have $o(Y) = \kappa$. Since $\mathcal{U}$ is locally finite Y is open, moreover $\sigma(Y) = \sigma$ by our assumption. Thus there is $p \in Y$ for which $\sigma(p,Y) = \sigma(p,X) > \kappa_{\eta}$. Now pick $U_{\eta} \in \mathcal{U}$ such that $p \in U_{\eta}$, and put $G_{\eta} = Y \cap U_{\eta}$. Then $p \in G_{\eta}$ implies $o(G_{\eta}) \geq \sigma(p,X) > \kappa_{\eta}$, and clearly $\xi \in \eta$ implies $G_{\xi} \cap G_{\eta} = \emptyset$. The claim is thus proven.
An immediate consequence of this claim is that $\tau < \sigma$ implies $\tau^{\rho} \leq \kappa$, and thus $\tau^{\rho} < \lambda$ as well (indeed, $\tau^{\rho} \geq \lambda$ would imply $\tau^{\rho} \geq \lambda^{\omega} > \kappa$). Consequently the power σ^{ρ} is a jump, hence $\rho = cf(\sigma)$ by 0.2. Now write $\sigma = \Sigma\{\kappa_{\xi} : \xi \in \rho\}$, where $\kappa_{\xi} < \sigma$ for each $\xi \in \rho$. Applying the claim to the sequence $\langle\kappa_{\xi} : \xi \in \rho\rangle$ we get a disjoint open family $\{G_{\xi} : \xi < \rho\}$ such that $o(G_{\xi}) > \kappa_{\xi}$ for $\xi < \rho$. But then by 0.1

$$\sigma^{\rho} = \Pi\{\kappa_{\xi} : \xi \in \rho\} \leq \Pi\{o(G_{\xi}) : \xi \in \rho\} \leq o(X) = \kappa < \lambda^{\omega},$$

while clearly $\sigma^{\rho} > \lambda$ implies $\sigma^{\rho} \geq \lambda^{\omega}$, a contradiction, which completes our proof. $\dashv$

Now let $\mathcal{G}$ be the class of all T_2 topological groups.

4.9 *Let* $G \in \mathcal{G}$, $|G| \geq \omega$. *Then* $o(G) = o(G)^{\omega}$.

PROOF. Let e denote the unit element of G, $\mathcal{V}$ be the neighbourhood filter of e in G, and put $\sigma = \sigma(e,G) = \sigma(G)$. We have to distinguish two cases:

Case a. There is $V \in \mathcal{V}$ such that $o(V) = \sigma$ and finitely many left translates of V cover G, i.e. there is a finite set $A \subset G$ for which $G = \cup\{aV: a \in A\}$. Clearly then $o(G) \leq \Pi\{o(a.V): a \in A\} = \sigma$, while G contains an infinite disjoint family $\{H_n: n \in \omega\}$ of non-empty open sets, hence by $o(H_n) \geq \sigma$ we have $o(G) \geq \sigma^\omega$ and consequently $o(G) = o(G)^\omega$.

Case b. There is no $V \in \mathcal{V}$ as in case a. Let $U \in \mathcal{V}$ be arbitrary with $o(U) = \sigma$ and pick a symmetric neighbourhood $V \in \mathcal{V}$ such that $V^2 \subset U$. Consider $A \subset G$ such that $\{aV: a \in A\}$ forms a maximal disjoint family of left translates of V. We claim that $\cup\{aU: a \in A\} = G$. Indeed for any $x \in G$ there is $a \in A$ with $(xV) \cup (aV) \neq \emptyset$, hence there are $v_1, v_2 \in V$ such that $xv_1 = av_2$. Then $x = av_2v_1^{-1}$, and $v_2v_1^{-1} \in U$ implies $x \in aU$.

Thus by our assumption $|A| = \alpha \geq \omega$, and obviously

$$o(G) \leq \Pi\{o(aU): a \in A\} = \sigma^\alpha$$

on one hand and

$$o(G) \geq \Pi\{o(aV): a \in A\} = \sigma^\alpha$$

on the other. But then $o(G) = \sigma^\alpha = (\sigma^\alpha)^\omega$. ⊣

REMARK. It is a very intriguing open question whether the above results are valid for compact Hausdorff spaces, i.e. whether $X \in C_2$ and $|X| \geq \omega$ imply $o(X)^\omega = o(X)$.

CHAPTER 5

CARDINAL FUNCTIONS ON PRODUCTS

The aim of this chapter is to investigate the following problem: assume ϕ is a cardinal function and

$$R = \times\{R_i : i \in I\}; \qquad (*)$$

how can we evaluate $\phi(R)$ in terms of the values $\phi(R_i)$ and the cardinality of the index set, $|I|$?

In order to exclude some trivial difficulties we assume throughout that no R_i in (*) is indiscrete, hence it contains two points p_i and q_i such that $p_i \notin \overline{\{q_i\}}$. If we denote by F the two-point T_0-space in which exactly one of the singletons is closed, then our convention obviously implies

$$F^\kappa \hookrightarrow R \text{ or } D(2)^\kappa \hookrightarrow R \qquad (**)$$

depending on whether $|\{i \in I : q_i \notin \overline{\{p_i\}}\}| = \kappa$ or not. We shall show later in 7.9 and 7.10 that the following relations hold for F^κ and $D(2)^\kappa$.

5.1 a) *If* $\phi \in \{w, nw, s, h, z, \pi, \pi\chi, t, \chi\}$ *then*

$$\phi(F^\kappa) = \phi(D(2)^\kappa) = \kappa;$$

b)

$$d(D(2)^\kappa) = \log \kappa.$$

It will be convenient to use the following notation for a product of the form (*) and a cardinal function ϕ defined for all $i \in I$:

$$\phi_I(R) = \sup\{\phi(R_i): i \in I\}.$$

5.2 a) *For every cardinal function* ϕ *we have considered*

$$\phi(R) \geq \phi_I(R).$$

b) *If* ϕ *is as in* 5.1a) *then*

$$\phi(R) \geq |I|.$$

PROOF. a) It is routine to check that this holds either using $R_i \hookrightarrow R$ or that R_i is a continuous open image of R via the projection map pr_i. ⊣

b) Except for $\phi = \pi\chi$ or $\phi = \pi$ this follows immediately from 5.1a), our conventions, and the monotonicity of ϕ. Next, as $\pi(R) \geq \pi\chi(R)$, it suffices to show $\pi\chi(R) \geq |I|$. Let G_i be a non-empty open proper subset of R_i and

$$p \in X\{G_i : i \in I\},$$

moreover assume that $\mathcal{P}$ is a local π-base at p in R. For each $i \in I$ there exists a $P_i \in \mathcal{P}$ with $P_i \subset pr_i^{-1}(G_i)$, hence clearly for every $P \in \mathcal{P}$

$$I_P = \{i \in I: P_i = P\}$$

is finite. Consequently we have $|\mathcal{P}| \geq |I|$ (assuming of course that I is infinite, the only case we really care about here), and thus $\pi\chi(R) \geq |I|$. ⊣

5.3 a) *If* $\phi \in \{w, nw, \pi, \pi\chi, \chi\}$ *then*

$$\phi(R) = |I| \cdot \phi_I(R);$$

b) *if in addition every* $R_i \in T_1$ *then*

$$\psi(R) = |I| \cdot \psi_I(R).$$

PROOF. a) Let $\mathcal{B}_i$ be a base (network, resp. π-base) in R_i of minimal cardinality. It is obvious that all sets of the form

$$\cap \{pr_i^{-1}(B_i) : i \in J\},$$

where $J \in [I]^{<\omega}$ and $B_i \in \mathcal{B}_i$ constitute a base (network, π-base) in R, and thus

$$\phi(R) \leq |\mathcal{B}| \leq |I| \ \Phi_I(R)$$

whenever $\phi \in \{w, nw, \pi\}$. A completely analogous "localized" version of this argument works for $\pi\chi$ and χ. In view of 5.2 however we actually must have equality everywhere. ⊣

b) First observe that if each $R_i \in \mathcal{T}_1$ then we have $D(2)^{|I|} \hookrightarrow R$, moreover $\psi(D(2)^{|I|}) = \chi(D(2)^{|I|}) = |I|$, thus 5.2 is valid for ψ. The rest is as in a). ⊣

We need the following result to obtain estimates for the density function.

5.4 *If* $\kappa \geq \omega$ *then*

$$d(D(\kappa)^{\exp \kappa}) \leq \kappa.$$

PROOF. Consider the space $X = D(2)^{\kappa} \in \mathcal{C}_2$ of which we know that $w(X) = \kappa$. Let us write X in the form $X = \{p_\xi : \xi \in \exp \kappa\}$ and fix a base $\mathcal{B}$ of X with $|\mathcal{B}| = \kappa$. We shall put

$$\tilde{\mathcal{B}} = \{\mathcal{C} \in [\mathcal{B}]^{<\omega} : \mathcal{C} \text{ is disjoint}\},$$

and

$$\mathcal{D} = \{d \in H(\mathcal{B},\kappa) : D(d) \in \tilde{\mathcal{B}}\},$$

i.e. the members of $\mathcal{D}$ are finite functions whose domains are in $\tilde{\mathcal{B}}$ and values are taken from κ. Clearly $|\mathcal{D}| = |\tilde{\mathcal{B}}| = \kappa$. Now for any $d \in \mathcal{D}$ we define a point $f^d \in D(\kappa)^{\exp \kappa}$ as follows:

$$f^d(\alpha) = \begin{cases} d(B), \text{ if } \alpha \in B \in D(d) \\ 0 \text{ otherwise.} \end{cases}$$

We claim that $S = \{f^d : d \in \mathcal{D}\}$ is dense in $D(\kappa)^{\exp \kappa}$. Indeed, the elementary open sets in this space of the form

$$U_h = \{f \in D(\kappa)^{\exp \kappa} : h \subset f\},$$

where $h \in H(\exp \kappa, \kappa)$, constitute a base. Since $D(h)$ is finite we can pick for every $\xi \in D(h)$ a neighbourhood $B_\xi \in \mathcal{B}$ of p_ξ in X such that $\xi \neq \eta$ implies $B_\xi \cap B_\eta = \emptyset$. Then $\{B_\xi : \xi \in D(h)\} \in \tilde{\mathcal{B}}$ and if we put

$$d(B_\xi) = h(\xi)$$

for $\xi \in D(h)$ then $d \in \mathcal{D}$. But obviously then

$$f^d \in U_h,$$

hence S is indeed dense. ⊣

5.5. a)

$$d(R) \leq \log |I| \, . \, d_I(R);$$

b) *if each* R_i *contains two disjoint non-empty open sets then*

$$d(R) = \log |I| . \, d_I(R).$$

PROOF. a) Let us put $\log |I| . \, d_I(R) = \kappa$. Then for each $i \in I$ there is a dense set $S_i \subset R_i$ with $|S_i| \leq \kappa$, hence a (continuous) map

$$g_i : D(\kappa) \overset{\text{onto}}{\to} S_i.$$

Then the continuous map

$$g = X\{g_i : i \in I\}: D(\kappa)^I \to X\{S_i : i \in I\} = S$$

maps $D(\kappa)^I$ onto the dense subset S of R, where $|I| \le \exp \kappa$ in view of $\log |I| \le \kappa$, hence we obviously have from 5.4 that

$$d(R) \le d(S) \le d(D(\kappa)^I) \le \kappa. \quad \dashv$$

b) Let U_i^0, U_i^1 be disjoint non-empty open sets in R_i and let us put for every $h \in H(I)$

$$G_h = \cap\{pr_i^{-1}(U_i^{h(i)}) : i \in D(h)\},$$

clearly G_h is a non-empty open set in R. Let now S be an arbitrary dense subset of R, then for each $h \in H(I)$ we can pick $p_h \in S \cap G_h$. Now, for any $p \in S$ we define a point $\tilde{p} \in D(2)^I$ as follows:

$$\tilde{p}(i) = \begin{cases} 0 \text{ if } p(i) \in U_i^0 \\ 1 \quad \text{otherwise.} \end{cases}$$

We claim that $\tilde{S} = \{\tilde{p} : p \in S\}$ is dense in $D(2)^I$. Indeed, it is easy to see that for any $h \in H(I)$ we have $\tilde{p}_h \supset h$. But by 5.1b) we have $d(D(2)^I) = \log |I|$ showing that

$$d(R) \ge \log |I|.$$

But $d(R) \ge d_I(R)$ is always true according to 5.2a) and thus $d(R) \ge \log |I| . d_I(R)$. $\dashv$

Next we turn to the study of cellularity, where we find the interesting phenomenon that $c(R)$ is in a sense independent of $|I|$.

5.6 $$c_I(R) \le c(R) \le 2^{c_I(R)}$$

<u>PROOF</u>. Of course only the second inequality needs proof in view of 5.2a). Let us first consider the case in which I is finite, e.g. $R = \overset{n}{\underset{i=1}{X}} R_i$, and put $\kappa = c_I(R) = \max\{c(R_i) : i=1,\dots,n\}$. Assume,

reasoning indirectly, that $c(X) > 2^{\kappa}$, hence there is a cellular family $\mathcal{G}$ in X with $|\mathcal{G}| > 2^{\kappa}$. We can of course assume that for each $G \in \mathcal{G}$

$$G = \times\{pr_i(G): i=1,\dots,n\}.$$

Then with every pair $\{G,H\} \in [\mathcal{G}]^2$ we can associate an index $j=j(\{G,H\}) \in I$ such that

$$pr_j(G) \cap pr_j(H) = \emptyset.$$

Then $j:[\mathcal{G}]^2 \to I$ is a partition for which the Erdös-Rado theorem, 0.4 6), may be a applied to obtain a $j_0 \in I$ and a $\mathcal{G}' \in [\mathcal{G}]^{\kappa^+}$ with

$$j([\mathcal{G}']^2) = \{j_0\}.$$

But then $\{pr_{j_0}(G): G \in \mathcal{G}'\}$ is a cellular family of size κ^+ in R_{j_0}, which is impossible. The general result now follows from the following:

<u>LEMMA</u>. *Put, for any* $J \subset I$, $R_J = X\{R_j:j \in J\}$, *then*

$$c(R) = \sup\{c(R_J):J \in [I]^{<\omega}\}.$$

<u>PROOF OF THE LEMMA</u>. We can assume that $c(R) > \omega$. Let κ be an uncountable regular cardinal and $\{G_\alpha:\alpha \in \kappa\}$ be a cellular family of elementary open sets in R, i.e.

$$G_\alpha = \cap\{pr_i^{-1}(G_{\alpha,i}): i \in I_\alpha\},$$

where $I_\alpha \in [I]^{<\omega}$ and $G_{\alpha,i}$ is open in R_i. By the Δ-system lemma, 0.6, we can assume that the family $\{I_\alpha:\alpha \in \kappa\}$ is a Δ-system, i.e. there is a $J \in [I]^{<\omega}$ with

$$I_\alpha \cap I_\beta = J$$

whenever $\{\alpha,\beta\} \in [\kappa]^2$. It is obvious then that

$$\{pr_J(G_\alpha): \alpha \in \kappa\}$$

is a cellular family in R_J, hence $\kappa \le c(R_J)$. As $c(R)$ is the sup of all such κ, this means that we are done. $\dashv$

5.7 *If* $\phi \in \{h,z\}$ *then*

$$|I|.\phi_I(R) \le \phi(R) \le |I|.2^{\phi_I(R)}.$$

PROOF. Because of 5.2 it again suffices to prove the second inequality. Let us put $\phi_I(R) = \kappa$ and distinguish two cases (i) and (ii).
(i) $|I| \le \kappa$. Assume indirectly that $\phi(R) > 2^\kappa$, then R has a subset S which is left (or right) separated in type $(2^\kappa)^+$, e.g. $S = \{p_\alpha : \alpha \in (2^\kappa)^+\}$ and

$$U_\alpha = \cap \{pr_i^{-1}(U_{\alpha,i}) : i \in I_\alpha\}$$

is a left (right) separating neighbourhood of p_α for each $\alpha \in (2^\kappa)^+$, where of course $I_\alpha \in [I]^{<\omega}$ and $U_{\alpha,i}$ is open in R_i. Thus if $\beta < \alpha$ then there is an $i \in I_\alpha$ with

$$pr_i(p_\beta) \notin U_{\alpha,i} \quad (pr_i(p_\alpha) \notin U_{\beta,i}).$$

Let us choose for any $\{\beta,\alpha\} \in [(2^\kappa)^+]$ with $\beta < \alpha$ such an $i = i(\{\beta,\alpha\})$, then $|I| \le \kappa$ and $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ implies the existence of an $i_0 \in I$ and an $a \in [(2^\kappa)^+]^{\kappa^+}$ such that

$$i([a]^2) = \{i_0\}.$$

But clearly then $\{pr_{i_0}(p_\alpha) : \alpha \in a\}$ is left (right) separated in R_{i_0}, which is impossible.
(ii) $|I| > \kappa$. Assume that we have a left (right) separated set $S \subset R$, with

$$|S| > |I|.2^\kappa,$$

where $S = \{p_\alpha : \alpha \in \lambda\}$ and

$$U_\alpha = \cap \{pr_i^{-1}(U_{\alpha,i}) : i \in I_\alpha\}$$

are the left (right) separating neighbourhoods of p_α for each $\alpha \in \lambda$. Since $|S| > |I| = |[I]^{<\omega}|$, we can actually assume that $I_\alpha = J$ for every $\alpha \in \lambda$. But clearly then $\{pr_J(p_\alpha): \alpha \in \lambda\}$ is a left (right) separated subspace of R_J of type $\lambda > 2^\kappa$, which however contradicts what we have proven in (i). ⊣

REMARK. 5.7 is no longer valid in full generality (i.e. for all topological spaces) if we put $\phi = s$. However for Hausdorff spaces we can prove the following much stronger result.

5.8 *If each* $R_i \in T_2$ *then*

$$s(R) \le z(R) \le |I|.2^{s_I(R)}.$$

PROOF. Case 1: I is finite, e.g. $I = \{1,2,\ldots,n\}$. We shall prove our result in this case by induction on $|I| = n$. If $n = 1$ then 5.8 reduces to 2.17. Thus assume $n > 1$ and that 5.8 has already been established for $n-1$. Let us put $\kappa = \max\{s(R_i): i = 1,\ldots,n\}$ and suppose that $S \subset R$ is left separated by a well-ordering $\prec$ in type $\lambda = (2^\kappa)^+$. We can take for each $p \in S$ a left separating neighbourhood U_p of the form

$$U_p = X\{U_{p,i}: i \in I\},$$

hence if $q \prec p$ then $q \notin U_p$, i.e.

$$pr_i(q) = q(i) \notin U_{p,i}$$

for some $i \in I$. We claim that we can assume $p(i) \neq q(i)$ for any two distinct $p,q \in S$ and $i \in I$. Indeed, we cannot have a $T \in [S]^\lambda$ and a $j \in I$ such that

$$|pr_j(T)| = 1$$

because then

$$pr_{I\setminus\{j\}}(T) = \tilde{T}$$

would clearly be a left separated subset of $X\{R_i : i \in I \setminus \{j\}\}$, contradicting our inductive hypothesis. Hence for any $j \in I$ and $x \in R_j$ we have

$$|S \cap pr_j^{-1}(\{x\})| \le 2^{\kappa} < \lambda,$$

with the help of which we can select from S, using a straightforward transfinite induction, a subset of cardinality λ that already satisfies the above claim. For simplicity's sake we shall also assume that $R_i = pr_i(S)$ for each $i \in I$.
Finally it can be assumed that, for every $i \in I$, either $h(R_i) \le 2^{\kappa}$ or R_i is right separated, because if $h(R_i) = \lambda$ then we can just pass to a subset S' of S with $|S'| = |S| = \lambda$ and $pr_i(S')$ right separated, and do this (finitely many times as I is finite) for each $i \in I$. In particular, we may assume that $h(R_i) \le 2^{\kappa}$ if $i \le \ell$ and R_i is right separated if $\ell < i \le n$. Using 2.10 we have then

$$\psi(R_i) \le 2^{\kappa}$$

whenever $i \le \ell$, hence we can choose for every $p \in S$ and $i \le \ell$ a family

$$\mathcal{V}_{p,i} = \{V_{p,i}(\xi) : \xi \in 2^{\kappa}\}$$

of neighbourhoods of $p(i)$ in R_i with

$$\cap \mathcal{V}_{p,i} = \{p(i)\}.$$

Our aim is to define a ramification system on S, and the following operations F and Ω_α on subsets of S are introduced to facilitate that. Let $\alpha \in 2^{\kappa}$ and $A \subset S$. If $|A| \le 2^{\kappa}$ we simply put $F(A) = A$ and $\Omega_\alpha(A) = \emptyset$. Next if $|A| = |S| = \lambda$, then consider the $\prec$ - first member p of A, put $F(A) = \{p\}$ and then let the map

$$f_A : A \setminus \{p\} \to I \times (2^{\kappa})^{\ell}$$

be defined as follows: if $q \in A \setminus \{p\}$ then

$$f_A(q) = \langle i, \xi_1, \ldots, \xi_\ell \rangle = \langle i, \vec{\xi} \rangle$$

if and only if i is the first member of I with

$$p(i) \notin U_{q,i},$$

moreover, for each $j \leq \ell$, ξ_j is the first ordinal in 2^κ with

$$q(j) \notin V_{p,j}(\xi_j).$$

Let $\{< i^{(\alpha)}, \vec{\xi}^{(\alpha)} > : \alpha \in 2^\kappa\}$ be an enumeration of $I \times (2^\kappa)^\ell$ in type 2^κ and then put

$$\Omega_\alpha(A) = f_A^{-1}(\{< i^{(\alpha)}, \vec{\xi}^{(\alpha)} >\})$$

for each $\alpha \in 2^\kappa$.
Now we can define the ramification system as follows. Put $S_\emptyset = S$. If ν is limit and S_t has been defined for all $t \in (2^\kappa)^{\underline{\nu}}$ then put for any $s \in (2^\kappa)^\nu$

$$S_s = \cap \{S_{t \restriction \mu} : \mu \in \nu\}.$$

Finally if S_t has been defined for all $t \in (2^\kappa)^\mu$ then put $F_t = F(S_t)$ and

$$S_{\widehat{t\alpha}} = \Omega_\alpha(S_t)$$

for every $\alpha \in 2^\kappa$. It is obvious that the conditions of the ramification lemma are then satisfied, hence there is a sequence $t \in (2^\kappa)^{\kappa^+}$ such that

$$S_{t \restriction \nu} \neq \emptyset \text{ and therefore } |S_{t \restriction \nu}| = \lambda$$

whenever $\nu \in \kappa^+$. Let us denote by $p^{(\nu)}$ the $\prec$-first member of $S_{t\restriction\nu}$, i.e. $\{p^{(\nu)}\} = F_{t\restriction\nu}$. By our construction then we have for each $\nu \in \kappa^+$ a $j(\nu) \in I$ and $\vec{\xi}(\nu) \in (2^\kappa)^\ell$ with $S_{t\restriction\nu}(p^{(\mu)}) = < j(\nu), \vec{\xi}(\nu)>$ whenever $\mu \in \kappa^+ \setminus (\nu+1)$, hence in particular

$$p^{(\nu)}(j(\nu)) \notin U_{p^{(\mu)}, j(\nu)}$$

if $\nu \in \mu \in \kappa^+$. Clearly there is a set $a \in [\kappa^+]^{\kappa^+}$ and an index $j \in I$ with $j(\nu) = j$ for all $\nu \in a$, and then $T = \{p^{(\nu)}(j) : \nu \in a\}$ is left separated (in type λ). We claim however that this set T is also right separated. Indeed, if $j > \ell$ then this is immediate as R_j itself is right separated, while on the other hand if $j \le \ell$ then for each $\nu \in \mu \in \kappa^+$ we have

$$p^{(\mu)}(j) \notin V_{p^{(\nu)},j}(\xi_j(\nu)).$$

Thus T is both right and left separated, hence by 2.12 it contains a discrete subspace D with $|D| = |T| = \kappa^+ > s(R_j)$, a contradiction.
Case 2: $|I| \ge \omega$. Let us put

$$\kappa = \sup \{s(R_J) : J \in [I]^{<\omega})$$

and show that $z(R) \le |I|.2^{\kappa} \le |I|.2^{s_I(R)}$.

Assume on the contrary that $S \subset R$ is left separated by $<$ in type $\lambda = (|I|.2^{\kappa})^+$ and

$$U_p = \cap\{pr_i^{-1}(U_{p,i}) : i \in I_p\} \qquad (I_p \in [I]^{<\omega})$$

are the left separating neighbourhoods. It can be assumed now that $I_p = J$ for all $p \in S$ because $|[I]^{<\omega}| = |I| < \lambda$. But then $pr_J(S)$ is clearly left separated in type $\lambda > 2^{\kappa} \ge 2^{s(R_J)}$ which is impossible by what we have proven in case 1. ⊣

REMARK. It is natural to ask whether z(R) could be replaced by h(R) in 5.8. While this is known to fail for Hausdorff spaces, it is easy to see that it holds if each $R_i \in \mathcal{T}_3$. Indeed then by 2.22b) we have

$$nw(R_i) \le 2^{s(R_i)},$$

hence, using 5.3,

$$h(R) \le nw(R) = |I|.\ nw_I(R) \le |I|.2^{s_I(R)}.$$

5.9. *If each* $R_i \in \mathcal{C}_2$ *then*

$$t(R) = |I| . t_I(R).$$

PROOF. By 5.2 it suffices to show that $t(R) \le |I|.t_I(R)$. We shall first consider the case $|I| = 2$, more precisely we prove the following lemma.

LEMMA. *If* $X \in \mathcal{T}_1$ and $Y \in \mathcal{C}_2$, *then*

$$t(X \times Y) \le t(X).t(Y) = \kappa.$$

PROOF OF THE LEMMA. What we have to show then is that every κ-closed set $H \subset X \times Y$ is closed. Thus suppose $\langle p,q\rangle \in \bar{H}$ and show that $\langle p,q\rangle \in H$. Now $\{p\} \times Y \subset X \times Y$ is closed, therefore $T = H \cap (\{p\} \times Y)$ is κ-closed, but $t(\{p\} \times Y) = t(Y) \le \kappa$, hence T is actually closed, and it suffices to prove that $q \in pr_Y(T)$. Assume on the contrary that $q \notin pr_Y(T)$, then as the closedness of T in $\{p\} \times Y$ implies that the projection $pr_Y(T)$ is closed in Y, we can choose a closed neighbourhood V of q in Y such that $V \cap pr_Y(T) = \emptyset$. Then $X \times V$ is a neighbourhood of the point $\langle p,q\rangle$ in $X \times Y$, hence

$$\langle p,q\rangle \in \overline{(X \times V) \cap H}.$$

Now, just like above, the closedness of $X \times V$ and the κ-closedness of H implies that $(X \times V) \cap H$ is κ-closed in $X \times Y$. But the compactness of Y implies that $pr_X: X \times Y \to X$ is a closed map, and therefore $S = pr_X((X \times V) \cap H)$ is κ-closed in X, consequently, in view of $t(X) \le \kappa$, it is also closed. By the continuity of pr_X however we have then

$$p \in \bar{S} = S = pr_X(X \times V) \cap H),$$

hence there is a point $r \in V$ with $\langle p,r\rangle \in H$, contradicting that

$$(\{p\} \times V) \cap H = (\{p\} \times V) \cap T = \emptyset.$$

From this lemma we obtain by a simple induction that $t(R) \le t_I(R)$ whenever I is finite. Let us now turn to the case in which I is infinite. Put $\kappa = |I|. t_I(R)$, consider any κ-closed set $A \subset R$, and let $p \in \bar{A}$. According to our previous result for any $J \in [I]^{<\omega}$ we have $t(R_J) \leq \kappa$, moreover as the projection pr_J is now a closed map, $pr_J(A)$ is κ-closed and therefore closed in R_J. Consequently $pr_J(p) \in pr_J(A)$, hence there is a point $q_J \in A$ such that

$$p \restriction J = q_J \restriction J.$$

Consider the set

$$B = \{q_J : J \in [I]^{<\omega}\},$$

then $B \in [A]^{\leq\kappa}$ since $|[I]^{<\omega}| = |I| \leq \kappa$, hence we clearly have

$$p \in \bar{B} \subset A$$

by the definition of B and the κ-closedness of A. $\dashv$

Next in this chapter we shall investigate calibers of spaces and their relations to products of the form (*). A classical result of this sort is the following theorem of Šanin.

5.10 *Suppose* $\kappa > \omega$ *is a regular cardinal and* $\kappa \in cal(R_i)$ *for every* $i \in I$. *Then* $\kappa \in cal(R)$ *as well.*

PROOF. Let us first consider the case in which I is finite, say $I = \{1,\dots,n\}$. Now if $\{G_\alpha : \alpha \in \kappa\}$ is a family of non-empty elementary open sets in R of the form

$$G_\alpha = \times\{G_{\alpha,i} : i \in I\},$$

then put $a_0 = \kappa$ and if for some $j < n$ we have already defined $a_j \in [\kappa]^\kappa$ then using $\kappa \in cal(R_{j+1})$ choose $a_{j+1} \in [a_j]^\kappa$ such that

$$\cap \{G_{\alpha,j+1} : \alpha \in a_{j+1}\} \neq \emptyset.$$

Obviously then

$$\cap\{G_\alpha : \alpha \in a_n\} \neq \emptyset$$

as well, hence $\kappa \in cal(R)$.

Next we consider the case of arbitrary I. Again let us start with a family $\{G_\alpha : \alpha \in \kappa\}$ of elementary open sets in R, where

$$G_\alpha = \cap \{pr_i^{-1}(G_{\alpha,i}) : i \in I_\alpha\}$$

with $I_\alpha \in [I]^{<\omega}$ for each $\alpha \in \kappa$. Applying the Δ-system lemma, 0.6, we can pick an $a \in [\kappa]^\kappa$ such that $\{I_\alpha : \alpha \in a\}$ is a Δ-system, e.g. we have $I_\alpha = J_\alpha \cup J$ for each $\alpha \in a$ and the family $\{J_\alpha : \alpha \in a\}$ is disjoint. In view of what we have established above $\kappa \in cal(R_J)$, hence there exists a $b \in [a]^\kappa$ such that

$$\cap \{pr_J(G_\alpha) : \alpha \in b\} \neq \emptyset.$$

But obviously then we also have

$$\cap \{G_\alpha : \alpha \in b\} \neq \emptyset. \qquad \dashv$$

<u>COROLLARY</u>.

$$c(R) \leq d_I(R).$$

Indeed, $d_I(R)^+$ is clearly a caliber of each R_i. $\dashv$

Although 5.10 does not remain valid for singular cardinals, the following result makes it possible to conclude just that in certain particular cases.

5.11 *Let* $\omega < \kappa = cf(\lambda) < \lambda$ *and* $d(R_i) < \kappa$ *for each* $i \in I$. *Then* $\lambda \in cal(R)$.

<u>PROOF</u>. Let us choose for each $i \in I$ a dense set $S_i \subset R_i$ with $|S_i| < \kappa$, moreover write λ in the form $\lambda = \Sigma\{\lambda_\alpha : \alpha \in \kappa\}$, where $\kappa < \lambda_\alpha < \lambda_\beta$ if $\alpha \in \beta \in \kappa$ and each λ_α is regular. Next consider a family $\{G_\nu : \nu \in \lambda\}$ of elementary open sets in R, where

$$G_\nu = \cap \{pr_i^{-1}(G_{\nu,i}) : i \in I_\nu\} \qquad (I_\nu \in [I]^{<\omega})$$

for any $\nu \in \lambda$. Now applying 0.9 we can find a subfamily $\mathcal{J}$ of $\{I_\nu : \nu \in \lambda\}$ which forms a double Δ-system, more precisely it can be assumed to have the form

$$\mathcal{J} = \{I_{\alpha\beta}: \alpha \in \kappa \ \& \ \beta \in \lambda_\alpha\},$$

where $I_{\alpha\beta} = J \cup J_\alpha \cup J_{\alpha\beta}$, $I_{\alpha\beta} \cap I_{\alpha\beta'} = J \cup J_\alpha$ for $\beta \neq \beta'$ and $I_{\alpha\beta} \cap I_{\alpha'\gamma} = J$ for $\alpha \neq \alpha'$. (Accordingly, we shall write $G_{\alpha\beta}$ instead of G_ν.) For any fixed $\alpha \in \kappa$ we can find a point $p^{(\alpha)} \in S_{J\cup J_\alpha}$ $= \times \{S_i: i \in J \cup J_\alpha\}$ (which is dense in $R_{J\cup J_\alpha}$) and a set $a^{(\alpha)} \in [\lambda_\alpha]^{\lambda\alpha}$ such that

$$p^{(\alpha)} \in \cap \{pr_{J \cup J_\alpha} (G_{\alpha\beta}) : \beta \in a^{(\alpha)}\},$$

because $|S_{J\cup J_\alpha}| < \kappa < \lambda_\alpha$ and λ_α is regular. Then $p^{(\alpha)} \restriction J \in S_J$ with $|S_J| < \kappa$, hence we can also select a set $a \in [\kappa]^\kappa$ and a point $p \in S_J$ such that $p = p^{(\alpha)} \restriction J$ for all $\alpha \in a$. Now it is easy to see however that

$$\cap \{G_{\alpha\beta}: \alpha \in a \ \& \ \beta \in a^{(\alpha)}\} \neq \emptyset . \qquad \dashv$$

COROLLARY. *If each* R_i *is separable then every cardinal of uncountable cofinality is a caliber for* R. *In particular we have*

$$\text{cal}(D(2)^I) = \{\lambda: \text{cf}(\lambda) \neq \omega\}.$$

We shall now present a few results concerning precalibers, which of course, using 1.20, yield corresponding results about calibers of compact Hausdorff spaces. They are based on the following general combinatorial result prior to whose formulation we need some definitions. Let $\leq$ be a reflexive and transitive binary relation on the set X. We shall write $I(X) = \{y \in X: y \leq x\}$ for the set of $\leq$ - predecessors of x. Two members x and x' of X are said to be *compatible* if $I(x) \cap I(x') \neq \emptyset$ and *incompatible* otherwise. $A \subset X$ is called an *antichain* if any two members of A are incompatible, and $(X,\leq)$ is said to satisfy the *κ-antichain condition* if every *antichain* in it has cardinality less than κ. Finally, $Y \subset X$ is said to be *κ-good* if $|Y| = \kappa$ and for every $y \in Y$ and $C \in [I(y)]^{<\kappa}$ there exists a $Y' \in [Y]^\kappa$ such that c and y' are compatible whenever $c \in C$ and $y' \in Y'$.
If κ and λ are cardinals, we shall write $\kappa << \lambda$ to denote that $\kappa < \lambda$ and $\kappa'^{\lambda'} < \lambda$ holds whenever $\kappa' < \kappa$ and $\lambda' < \lambda$.

5.12 *Let* $\leq$ *be a transitive binary relation on* X *satisfying the* γ*-antichain condition, and* κ *be a regular cardinal with* $\gamma \ll \kappa$. *Then for every* $Y \in [X]^{\kappa}$ *there is a* $Z \in [Y]^{\kappa}$ *which is* κ*-good.*

PROOF. Suppose that $Y \in [X]^{\kappa}$ but no $Z \in [Y]^{\kappa}$ is κ-good. Then we can choose an $x(Z) \in Z$ and a set $C(Z) \in [I(x(Z))]^{<\kappa}$ such that if we put

$$F(Z) = \{y \in Z: y \text{ is compatible with every } c \in C(Z)\},$$

then $|F(Z)| < \kappa$. We shall put $\lambda(Z) = |C(Z)|$, write $C(Z) = \{c^Z_\alpha: \alpha \in \lambda(Z)\}$ and for each $\alpha \in \lambda(Z)$ set

$$S_\alpha(Z) = \{y \in Z: y \text{ is incompatible with } c^Z_\alpha\}.$$

Obviously we have

$$Z = F(Z) \cup \cup\{S_\alpha(Z): \alpha \in \lambda(Z)\}.$$

We shall now define a ramification system of height γ over Z. Thus let us put $S_\emptyset = Z$. If S_t has been defined for some $t \in SEQ_\sigma$ with $\sigma < \gamma$ then we put $n(t) = 0$ and $F_t = S_t$ if $|S_t| < \kappa$, moreover $n(t) = \lambda(S_t)$ and $F_t = F(S_t)$ otherwise. Then, as usual, we set for each $\alpha \in \lambda(S_t) = n(t)$

$$S_{t\frown\alpha} = S_\alpha(S_t).$$

Finally, if $\sigma \in \gamma$ is a limit ordinal $t \in SEQ_\sigma$ and $S_{t\restriction\rho}$ has been defined for each $\rho \in \sigma$, then we put $S_t = \cap\{S_{t\restriction\rho}: \rho \in \sigma\}$. First we show by induction on $\sigma \in \gamma$ that $\kappa_\sigma = |N \cap SEQ_\sigma| < \kappa$, where of course $N = \{t \in SEQ: S_t \text{ is defined}\}$. Indeed, if σ is limit and $\kappa_\rho < \kappa$ holds whenever $\rho \in \sigma$, then clearly

$$\kappa_\sigma \leq \Pi\{\kappa_\rho: \rho \in \sigma\},$$

moreover the regularity of κ implies the existence of a $\kappa' < \kappa$ with $\kappa_\rho \leq \kappa'$ for all $\rho \in \sigma$. But then

$$\kappa_\sigma \leq (\kappa')^{|\sigma|} < \kappa,$$

as $\gamma << \kappa$ and $|\sigma| < \gamma$. Next if $\sigma = \rho + 1$, then clearly

$$\kappa^{\sigma} = \Sigma\{\lambda(S_t): t \in N \cap SEQ_{\rho} \ \& \ |S_t| = \kappa\} < \kappa$$

again by the regularity of κ and the inductive hypothesis. It is also clear from here that

$$n(t\restriction\rho) \le \kappa_{\rho+1} \le \kappa_{\rho+1}^{|\rho|} < \kappa$$

holds whenever $t \in N$ and $\rho < \gamma$, hence the conditions of the ramification lemma are satisfied. Thus we can find a sequence $t \in SEQ_{\gamma}$ with $S_{t\restriction\alpha} \neq \emptyset$, hence $|S_{t\restriction\alpha}| = \kappa$ for all $\alpha \in \gamma$. Let us put

$$x^{(\alpha)} = x(S_{t\restriction\alpha}),$$

moreover

$$c^{(\alpha)} = c_{t(\alpha)}^{S_{t\restriction\alpha}}.$$

It follows from our construction then that $c^{(\alpha)} \le x^{(\alpha)}$ but $c^{(\alpha)}$ is incompatible with $x^{(\beta)}$ whenever $\alpha \in \beta \in \gamma$. But then the transitivity of $\le$ implies that any two members of the set $\{c^{(\alpha)}: \alpha \in \gamma\}$ are incompatible, contradicting that $\le$ satisfies the γ-antichain condition. ⊣

Now the following two results due to Argyros and Tsarpalias follow easily. It will be useful for us to use the following definition here: a family G of open sets in a space X is called κ-*nice* if $|G| = \kappa$ and for every $G \in G$ and centered family S of open subsets of G if $|S| < \kappa$ then there is a $G' \in [G]^{\kappa}$ such that $S \cup G'$ is centered.

5.13 *If $\hat{c}(X) \le \gamma << \kappa$ and κ is regular then every family of open sets G of cardinality κ contains a κ-nice subfamily, thus in particular* $\kappa \in \text{precal}(X)$.

<u>PROOF</u>. We can apply 5.12 to the transitive relation $\subset$ on the set $X = \{G \subset X: G \neq \emptyset$ and open$\}$ since $\hat{c}(X) \le \gamma$ just means that the γ-antichain condition is satisfied. Let now $G \in [X]^{\kappa}$ be arbitrary and $G' \in [G]^{\kappa}$ be κ-good with respect to the relation $\subset$ on X. We claim

that then G' is also κ-nice. Indeed, if $G \in G'$ and S is a centered family of open subsets of G with $|S| < \kappa$ we let $C \subset G'$ be a maximal subcollection of G' for which $C \cup S$ is centered. We show that $|C| = \kappa$. In fact, if $|C| < \kappa$ held, then the κ-goodness of G' and $\{\cap C' \cap \cap S' : C' \in [C]^{<\omega} \ \& \ S' \in [S]^{<\omega}\setminus\{\emptyset\} \in [I(G)]^{<\kappa}$ would imply

$$|\{G \in G' : \{G\} \cup C \cup S \text{ is centered}\}| = \kappa,$$

which is clearly impossible if C is maximal and $|C| < \kappa$. ⊣

5.14. *Suppose* $\kappa = cf(\lambda) < \lambda$ *and* $\mu^{\kappa} < \lambda$ *hold for all* $\mu < \lambda$, *moreover* $\kappa \in$ precal(X). *Then* $\lambda \in$ precal(X) *as well.*

PROOF. Let us write $\lambda = \Sigma\{\mu_\alpha : \alpha \in \kappa\}$ with $\kappa < \mu_\alpha < \mu_\beta$ for $\alpha \in \beta \in \kappa$, and then put

$$\lambda_\alpha = (\mu_\alpha^\kappa)^+.$$

Clearly then we have $\hat{c}(X) \leq \kappa << \lambda_\alpha < \lambda$ for every $\alpha \in \kappa$. Let us now consider a family $\{G_\nu : \nu \in \lambda\}$ of open subsets of X, write $\lambda = \cup\{a^{(\alpha)} : \alpha \in \kappa\}$ where $|a_\alpha| = \lambda_\alpha$ for every $\alpha \in \kappa$, and put

$$G^{(\alpha)} = \{G_\nu : \nu \in a^{(\alpha)}\}.$$

We can clearly assume that every $G^{(\alpha)}$ is of cardinality λ_α and then, using 5.13, that $G^{(\alpha)}$ is actually λ_α-nice. Let us pick for each $\alpha \in \kappa$ a member $G^{(\alpha)} \in G^{(\alpha)}$, since $\kappa \in$ precal(X) we might also assume that the family $\{G^{(\alpha)} : \alpha \in \kappa\}$ is centered. Let us put

$$S^{(0)} = \{\cap\{G^{(\alpha)} : \alpha \in v\} \cap G^{(0)} : v \in [\kappa]^{<\omega}\},$$

then the λ_0-niceness of $G^{(0)}$ implies the existence of a $H^{(0)} \in [G^{(0)}]^{\lambda_0}$ such that $S^{(1)} = H^{(0)} \cup S^{(0)}$ is centered.

We can continue this procedure by transfinite induction as follows. Suppose that $\alpha \in \kappa$ and for every $\beta \in \alpha$ we have defined already the centered family $S^{(\beta)}$ such that $|S^{(\beta)}| = \lambda_\beta$ and $S^{(\gamma)} \subset S^{(\beta)}$ if $\gamma \in \beta$. Let us put then

$$R^{(\alpha)} = \cup\{S^{(\beta)} : \beta \in \alpha\},$$

clearly $|R^{(\alpha)}| < \lambda_\alpha$ and, because $G^{(\alpha)} \cap G^{(0)} \in S^{(0)} \subset R^{(\alpha)}$, the family

$$\{G^{(\alpha)} \cap \bigcap v : v \in [R^{(\alpha)}]^{<\omega}\}$$

is centered. Thus by the λ_α-niceness of $G^{(\alpha)}$ there is an $H^{(\alpha)} \in [G^{(\alpha)}]\,\lambda_\alpha$ for which

$$S^{(\alpha)} = H^{(\alpha)} \cup R^{(\alpha)}$$

is centered. But it is easy to see then that

$$H = \cup\,\{H^{(\alpha)} : \alpha \in \kappa\}$$

is a centered subfamily of G with $|H| = \lambda$. $\dashv$

<u>COROLLARY</u>. *If λ if a strong limit cardinal and $X \in C_2$ then $\lambda \in \mathrm{cal}(X)$ if and only if $\mathrm{cf}(\lambda) \in \mathrm{cal}(X)$.*

Now we turn to a result, due to J. Gerlits, which concerns maps defined on products of the form (*). It is customary to say that a map

$$f : R = \times\{R_i : i \in I\} \to Y$$

depends only on $J \subset I$ if for every two points $p, q \in R$ with $p\restriction J = q\restriction J$ we have $f(p) = f(q)$. Moreover, f is said to *depend on less than* κ *coordinates* if there is a $J \in [I]^{<\kappa}$ such that f depends only on J. Before presenting the main result we need an auxiliary lemma. But first some notation: Let $p, q \in R$ and $S = \{S_\xi : \xi \in \kappa\}$ be a partition of I, i.e. $S_\xi \cap S_\mu = \emptyset$ if $\xi \neq \mu$ and $\cup\{S_\xi : \xi \in \kappa\} = I$; we shall put then

$$R(S;p,q) = \{r \in R : \forall\xi \in \kappa(r\restriction S_\xi = p\restriction S_\xi \text{ or } r\restriction S_\xi = q\restriction S_\xi)\}\ .$$

5.15. *If R is as above, $p, q \in R$ and $R_i \in T_1$ with $p(i) \neq q(i)$ for every $i \in I$, moreover $S = \{S_\xi : \xi \in \kappa\} \subset [I]^{<\omega}\setminus\{\emptyset\}$ is a partition of I, then*

$$R(\mathcal{S};p,q) \simeq D(2)^\kappa.$$

PROOF. Let us put $D_i = \{p(i),q(i)\}$, then clearly $R(\mathcal{S};p,q) \subset D = \times\{D_i : i \in I\} \simeq D(2)^I$, moreover $R(\mathcal{S};p,q)$ is also closed in D, because if $r \in D \setminus R(\mathcal{S};p,q)$, then there exist an $S_\xi \in \mathcal{S}$ and $i,j \in S_\xi$ with $r(i) = p(i)$ and $r(j) = q(j)$ and the set of points of D satisfying this is open and disjoint from $R(\mathcal{S};p,q)$. Consequently $R(\mathcal{S};p,q)$ is compact. Let us now consider the map

$$F:R(\mathcal{S};p,q) \to D(2)^\kappa$$

defined as follows:

$$F(r)(\xi) = \begin{cases} 0, & \text{if } r\restriction S_\xi = p\restriction S_\xi, \\ \\ 1, & \text{if } r\restriction S \ = q\restriction S_\xi. \end{cases}$$

It is easy to see that F is a one-one and onto map. But F is also continuous, because for any subbasic open set $C_{\xi,i} = \{\varepsilon \in D(2)^\kappa : \varepsilon(\xi) = i\}$ in $D(2)^\kappa$ we have (with $p^0 = p$ and $p^1 = q$)

$$F^{-1}(C_{\xi,i}) = \{r \in R(\mathcal{S};p^0,p^1) : r\restriction S_\xi = p^i\restriction S_\xi\}$$

that is clearly open in $R(\mathcal{S};p^0,p^1)$. But then F actually is a homeomorphism. ⊣

5.16. *Let*

$$f:R = \times\{R_i : i \in I\} \to Y$$

be a continuous map, where $R_i \in \mathcal{T}_2$ *for all* $i \in I$ *and* $Y \in \mathcal{T}_2$, *moreover assume that* $\kappa > \omega$ *is a caliber for* R, *while* $D(2)^\kappa$ *is not embeddable into* Y. *Then* f *depends on less than* κ *coordinates only.*

PROOF. Let us assume indirectly that for no $J \in [I]^{<\kappa}$ depends f on J only. We shall then define by transfinite induction elementary open sets U_ξ^0, U_ξ^1 in R of the form

$$U^i_\xi = \cap\{pr_j^{-1}(U^i_{\xi,j}) : j \in I_\xi\}$$

with $I_\xi \in [I]^{<\omega}$ as follows. If we have defined U^0_η, U^1_η for all $\eta \in \xi \in \kappa$, then put

$$J_\xi = \cup\{I_\eta : \eta \in \xi\}.$$

As $|J_\xi| < \kappa$, we have, by assumption, two points $p^0_\xi, p^1_\xi \in R$ such that $p^0_\xi \upharpoonright J_\xi = p^1_\xi \upharpoonright J_\xi$, but $y^0_\xi = f(p^0_\xi) \neq y^1_\xi = f(p^1_\xi)$. Since Y is Hausdorff, we can find disjoint open neighbourhoods V^0_ξ and V^1_ξ of y^0_ξ and y^1_ξ, respectively. We can then choose U^0_ξ and U^1_ξ of the above form as elementary open neighbourhoods of p^0_ξ and p^1_ξ such that

$$f(U^i_\xi) \subset V^i_\xi .$$

We may also assume that for every $j \in I_\xi$ either $U^0_{\xi,j} \cap U^1_{\xi,j} = \emptyset$ or $U^0_{\xi,j} = U^1_{\xi,j}$, using that each $R_j \in \mathcal{T}_2$. Let us put $S_\xi = \{j \in I_\xi : U^0_{\xi,j} \cap U^1_{\xi,j} = \emptyset\} \subset I_\xi$, then $S_\xi \neq \emptyset$ because $U^0_\xi \cap U^1_\xi = \emptyset$. Moreover $\eta < \xi$ implies $S_\eta \cap S_\xi = \emptyset$, because for any $j \in S_\eta$ we have $j \in J_\xi$, hence

$$p^0_\xi(j) = p^1_\xi(j) \in U^0_{\xi,j} \cap U^1_{\xi,j} \neq \emptyset.$$

Therefore every I_ξ intersects only finitely many S_η, hence if we put for any $\xi \in \kappa$

$$F(\xi) = \{\eta \in \kappa : I_\xi \cap S_\eta \neq \emptyset\},$$

then $F: \kappa \to [\kappa]^{<\omega}$. But then $\kappa > \omega$ implies in view of Hajnal's theorem, 0.3, that there is a free set of size κ for F, hence in what follows we may actually assume that $I_\xi \cap S_\eta = \emptyset$ if $\xi \neq \eta$. Let us now put $\mathcal{S} = \{S_\xi : \xi \in \kappa\}$ and $S = \cup\mathcal{S}$, moreover $R_S = pr_S(R) = \times\{R_j : j \in S\}$. Finally fix a point $q \in R_S$ such that $q(j) \in U^1_{\xi,j}$ for every $j \in S$. Since κ is a caliber for R we can assume that $\cap\{U^0_\xi : \xi \in \kappa\} \neq \emptyset$, and then we can choose a point $p \in \cap\{U^0_\xi : \xi \in \kappa\}$. Now, if $j \in S_\xi \subset S$, then $p(j) \neq q(j)$ since $U^0_{\xi,j}$ and $U^1_{\xi,j}$ are disjoint, hence we can apply 5.15 to conclude that

$$R^* = R_S(S;p,S,q) \simeq D(2)^\kappa.$$

Let us now put

$$R^{**} = \{r \in R: r \upharpoonright S \in R^* \ \& \ r \upharpoonright (I \setminus S) = p \upharpoonright (I \setminus S)\},$$

then clearly $R^{**} \simeq R^* \simeq D(2)^\kappa$ as well. Next we show that f is one-one on R^{**}. Indeed if $r,r' \in R^{**}$ and $r \neq r'$ then there is a $\xi \in \kappa$ with e.g. $r \upharpoonright S_\xi = p \upharpoonright S_\xi$ and $r' \upharpoonright S_\xi = q \upharpoonright S_\xi$. But $I_\xi \setminus S_\xi \subset I \setminus S$ and thus $r \upharpoonright (I_\xi \setminus S_\xi) = r' \upharpoonright (I_\xi \setminus S_\xi) = p \upharpoonright (I_\xi \setminus S_\xi)$, moreover $U^0_{\xi,j} = U^1_{\xi,j}$ if $j \in I_\xi \setminus S_\xi$, hence we have $r \in U^0_\xi$ and $r' \in U^1_\xi$, consequently

$$f(r) \in V^0_\xi \text{ and } f(r') \in V^1_\xi$$

and thus $f(r) \neq f(r')$. But then $f \upharpoonright R^{**}$ is actually a homeomorphism, which contradicts our assumption that $D(2)^\kappa$ does not embed into Y. ⊣

COROLLARY. *Suppose* $f: R \to Y$, *where all the* R_i *and* Y *are Hausdorff,* $d(R_i) < cf(\kappa)$ *for each* $i \in I$ *and* $\phi(Y) < \kappa$, *where* ϕ *is a monotone cardinal function such that* $\phi(D(2)^\kappa) = \kappa$. *Then* f *depends only on less that* κ *coordinates.* ⊣

CHAPTER 6

CARDINAL FUNCTIONS ON UNIONS OF CHAINS

In this chapter we are going to study the following problem. Given a space X as the union of an increasing chain of subspaces, i.e.

$$X = \cup\{X_\alpha : \alpha \in \kappa\} \qquad (*)$$

with $X_\alpha \subset X_\beta$ if $\alpha \in \beta \in \kappa$, and knowing the values of some cardinal functions on the X_α, what can be said about X? This problem has just recently become the object of systematic study by M.G. TKAČENKO [TK 1978] and by [HJ 1981], and therefore it might have a less final character than the previous chapters. Clearly there is no loss of generality in assuming that in (*) κ is a regular cardinal and that $\alpha \in \beta \in \kappa$ imply $X_\alpha \subsetneqq X_\beta$, hence we shall assume this throughout.

6.1. *If* $\phi \in \{c,s,h,z\}$ *and* $\phi(X_\alpha) < \lambda$ *for all* $\alpha \in \kappa$ *then* $\phi(X) \le \lambda$*; if in addition* $\kappa > \lambda$ *then* $\phi(X) < \lambda$.

PROOF. Let us first consider the case $\phi = c$. Thus consider a cellular family $\mathcal{G}$ in X and put for each $\alpha \in \kappa$

$$\mathcal{G}_\alpha = \{G \in \mathcal{G} : G \cap X_\alpha \neq \emptyset\}.$$

Clearly then $|\mathcal{G}_\alpha| \le c(X_\alpha) < \lambda$, moreover $\alpha \in \beta \in \kappa$ implies $\mathcal{G}_\alpha \subset \mathcal{G}_\beta$, and finally

$$\mathcal{G} = \cup\{\mathcal{G}_\alpha : \alpha \in \kappa\}$$

Consequently we must have $|\mathcal{G}| \le \lambda$. Now if $\lambda < \kappa$ then choose for each $G \in \mathcal{G}$ $\alpha(G) \in \kappa$ such that $G \in \mathcal{G}_{\alpha(G)}$. Since κ is regular we have an $\alpha_0 \in \kappa$ with $\alpha(G) \le \alpha_0$ for each $G \in \mathcal{G}$, consequently $\mathcal{G} = \mathcal{G}_{\alpha_0}$ and there-

$|G| \le c(X_{\alpha_0}) < \lambda$. If λ is a successor cardinal then this immediately implies $c(X) < \lambda$. If on the other hand λ is a limit cardinal and $c(X) = \lambda$ would hold, we could choose for every cardinal $\mu < \lambda$ a cellular family $G^{(\mu)}$ in X with $|G^{(\mu)}| = \mu$ and then find, as we have shown above, an $\alpha_\mu < \kappa$ with $G^{(\mu)}_{\alpha_\mu} = G$, i.e.

$$\forall G \in G^{(\mu)} \ (G \cap X_{\alpha_\mu} \neq \emptyset).$$

But then again we had an ordinal $\alpha \in \kappa$ such that $\alpha_\mu \le \alpha$ for each $\mu < \lambda$, which is impossible as this would imply $c(X_\alpha) \ge \lambda$.

Now let $\phi \in \{s,h,z\}$, we shall give a joint proof using defining sets in the sense of chapter 4. If S is a defining set for ϕ in X and $\alpha \in \kappa$, then $S \cap X_\alpha$ is a defining set in X_α, hence $|S \cap X_\alpha| < \lambda$. But again if $\alpha < \beta < \kappa$ then $S \cap X_\alpha \subset S \cap X_\beta$, hence $|S| \le \lambda$, i.e. $\phi(X) \le \lambda$. If $\lambda < \kappa$ then by the regularity of κ this implies $S \subset X_\alpha$ for some $\alpha \in \kappa$, hence $|S| \le \phi(X_\alpha) < \lambda$. This implies $\phi(X) < \lambda$ if λ is a successor. If λ is a limit cardinal then $\phi(X) = \lambda$ would imply the existence of a defining set $S^{(\mu)} \subset X$ with $|S^{(\mu)}| = \mu$ for every cardinal $\mu < \lambda$ and thus the existence of an $\alpha_\mu \in \kappa$ with $S^{(\mu)} \subset X_{\alpha_\mu}$.
But there is an $\alpha \in \kappa$ with $\alpha_\mu \le \alpha$ for each $\mu < \lambda$, contradicting $\phi(X_\alpha) < \lambda$. ⊣

6.2. (i) *If* $nw(X_\alpha) \le \lambda$ *for all* $\alpha \in \kappa$ *then* $nw(X) \le \kappa.\lambda$.
(ii) *If* $X \in T_1$ *and* $\psi w(X_\alpha) \le \kappa$ *for* $\alpha \in \kappa$, *then* $\psi w(X) \le \kappa.\lambda$.
(iii) *If* $X \in T_1$, $p \in X$ *and* $\psi(p,X_\alpha) \le \lambda$ *whenever* $p \in X_\alpha$, *then* $\psi(p,X) \le \kappa.\lambda$.

PROOF.

(i) Clearly if N_α is a network in X_α then $N = \cup\{N_\alpha : \alpha \in \kappa\}$ is a network in $X = \cup\{X_\alpha : \alpha \in \kappa\}$ with $|N| \le \kappa.\lambda$. Observe that in this case the fact that $\{X_\alpha : \alpha \in \kappa\}$ is a chain is not used. ⊣

(ii) Let B_α be a family of open subsets of X such that $\{B \cap X_\alpha : B \in B_\alpha\}$ is a ψ-base for X_α and $|B_\alpha| \le \psi w(X_\alpha) \le \lambda$. We claim that $B = \cup\{B_\alpha : \alpha \in \kappa\}$ is a ψ-base for X. Indeed, if $p,q \in X$ with $p \neq q$, then there is an $\alpha \in \kappa$ with $p,q \in X_\alpha$, but then if $B \in B_\alpha$ is such that $p \in B \cap X_\alpha$ and $q \notin B \cap X_\alpha$ then $p \in B$ and $q \notin B$ hold as well. Thus we have

$$\psi w(X) \le |\mathcal{B}| \le \kappa.\lambda. \quad \dashv$$

(iii) The proof in this case is quite similar to that of (ii). $\dashv$

6.3. *If* $X \in T_2$ *and* $h(X_\alpha) \le \lambda$ *for each* $\alpha \in \kappa$ *then*

$$|X| \le 2^\lambda.$$

PROOF. By 2.16 we have $|X_\alpha| \le 2^\lambda$ for $\alpha \in \kappa$, hence $|X| \le 2^\lambda$ follows if $\kappa \le 2^\lambda$. But if $\kappa > 2^\lambda \ge \lambda^+$ then from 6.1 we get $h(X) < \lambda^+$, i.e. $h(X) \le \lambda$, hence by 2.16 again we get $|X| \le 2^\lambda$. $\dashv$

6.4. *Let* $X \in T_2$, *moreover* $h(X_\alpha) \le \lambda$ *and* $z(X_\alpha) < \lambda$ *for every* $\alpha \in \kappa$. *Then*

$$o(X) \le 2^\lambda.$$

PROOF. By 6.3 we have $|X| \le 2^\lambda$ and by 6.1 $z(X) \le \lambda$. Therefore using 2.11 we conclude

$$o(X) \le |X|^{z(X)} \le (2^\lambda)^\lambda = 2^\lambda. \quad \dashv$$

COROLLARY. *If* $X \in T_2$ *and* $nw(X_\alpha) < \lambda$ *for all* $\alpha \in \kappa$, *then*

$$o(X) \le 2^\lambda. \quad \dashv$$

6.5. *If* $X \in T_2$ *and* $s(X_\alpha) \le \lambda$ *for all* $\alpha \in \kappa$, *then*

(i) $z(X) \le 2^\lambda$;

(ii) $o(X) \le \exp_2 \lambda$;

(iii) $\psi w(X) \le 2^{\lambda^2}$;

(iv) *if, in addition,* $X \in T_3$ *then*

$$nw(X) \le 2^\lambda.$$

PROOF. If $\kappa > \lambda^+$ then by 6.1 we have $s(X) \le \lambda$ and therefore (i)-(iv) follow immediately from 2.17, 2.21 and 2.22, respectively. Thus in what follows we assume that $\kappa \le \lambda^+$ $(\le 2^\lambda)$.

(i) Assume, indirectly, that $z(X) > 2^\lambda$, i.e. X contains a left separated subset S with $|S| = (2^\lambda)^+ > \kappa$. But then we must have

$|S \cap S_\alpha| = (2^\lambda)^+$ for some $\alpha \in \kappa$ as well, which contradicts 2.17. ⊣

(ii) By 2.20 we have $|X_\alpha| \le \exp_2\lambda$ for each $\alpha \in \kappa$, hence $\kappa \le \lambda^+$ implies $|X| \le \exp_2\lambda$ as well. But then in view of (i) and 2.11

$$o(X) \le |X|^{z(X)} \le (\exp_2\lambda)^{\exp\lambda} = \exp_2\lambda$$

follows. ⊣

(iii) Using 2.20 choose for each $\alpha \in \kappa$ an $S_\alpha \in [X_\alpha]^{\le 2^\lambda}$ such that

$$X_\alpha = \cup\{\bar{T}^{X_\alpha}: T \in [S_\alpha]^{\le\lambda}\},$$

and put $S = \cup\{S_\alpha: \alpha \in \kappa\}$. Then $|S| \le \kappa.2^\lambda = 2^\lambda$, and clearly every point $p \in X$ is in the closure of a subset of S of size at most λ. But then

$$M = \{\bar{T}: T \in [S]^{\le\lambda}\}$$

yields a ψ-base of X with the same argument as in the proof of 2.22a). ⊣

(iv) If $X \in T_3$ then the family M defined in (iii) is a network of X with the same argument as in the proof of 2.22b). ⊣

6.6. *If* $X \in T_2$ *and* $\rho(X_\alpha) < \lambda$ *for all* $\alpha \in \kappa$ *then* $\psi w(X) \le \lambda$, *and for* $\lambda < \kappa$ *even* $\rho(X) < \lambda$. *If in addition* $X \in T_3$ *then* $nw(X) \le \lambda$.

PROOF. Let us first consider the case $\lambda \ge \kappa$. Since by 2.8 $\psi w(X_\alpha) \le \rho(X_\alpha)$ (or even $w(X_\alpha) \le \rho(X_\alpha)$ if $X \in T_3$) we immediately conclude then from 6.2 that $\psi w(X) \le \kappa.\lambda = \lambda$ (or $nw(X) \le \kappa.\lambda = \lambda$). Now assume that $\lambda < \kappa$ and show that $\rho(X) < \lambda$. If, on the contrary, $\rho(X) \ge \lambda$ then choose a family $G \subset RO(X)$ with $|G| = \lambda$. If $G,H \in G$ and $G \ne H$ then $\bar{G} \ne \bar{H}$ as well, hence either $G\backslash\bar{H} \ne \emptyset$ or $H\backslash\bar{G} \ne \emptyset$. Assume, by symmetry, that $G\backslash\bar{H} \ne \emptyset$ and find an ordinal $\alpha = \alpha(G,H)$ in κ such that $(G\backslash\bar{H}) \cap X_\alpha \ne \emptyset$ and $H \cap X_\alpha \ne \emptyset$. Clearly then we have

$$\overline{G \cap X_\beta}^{X_\beta} \ne \overline{H \cap X_\beta}^{X_\beta}$$

for every ordinal $\beta \in \kappa\backslash\alpha$. By $\lambda < \kappa$ and the regularity of κ we can then find a fixed $\alpha_0 \in \kappa$ such that $\alpha(G,H) \le \alpha_0$ for every pair

$\{G,H\} \in [\mathcal{G}]^2$, hence we have then

$$\overline{G \cap X_{\alpha_0}}^{X_{\alpha_0}} \neq \overline{H \cap X_{\alpha_0}}^{X_{\alpha_0}}$$

for any $G,H \in \mathcal{G}$ with $G \neq H$. But this would mean that X_{α_0} contains λ different regular closed sets, namely the $\overline{G \cap X_{\alpha_0}}^{X_{\alpha_0}}$ for $G \in \mathcal{G}$, which contradicts $\rho(X_{\alpha_0}) < \lambda$. Thus we have $\psi w(X) \le \rho(X) < \lambda$, and if $X \in \mathcal{T}_3$ then $nw(X) \le \rho(X) < \lambda$. $\dashv$

6.7. *If* $X \in \mathcal{T}_2$ *and* $d(X_\alpha) \le \lambda$ *for each* $\alpha \in \kappa$ *then* $|X| \le \exp_2\lambda$.

PROOF. If $\kappa \le \exp_2\lambda$ then this follows immediately from $|X_\alpha| \le \exp_2\lambda$ for all $\alpha \in \kappa$ (cf. 2.4). If on the other hand $\kappa > \exp_2\lambda$ use the fact that

$$\rho(X_\alpha) \le \exp d(X_\alpha) \le 2^\lambda < (2^\lambda)^+ \le \exp_2\lambda < \kappa$$

for each $\alpha \in \kappa$ and 6.6 to conclude that

$$\rho(X) < (2^\lambda)^+,$$

and therefore

$$|X| \le 2^{\rho(X)} \le \exp_2\lambda. \qquad \dashv$$

The following result that we think is quite remarkable in itself will be used as an auxiliary result later; that explains its different character.

6.8. *If* $X \in \mathcal{T}$, κ *is an arbitrary cardinal, and* $w(Y) < \kappa$ *holds for each* $Y \in [X]^{\le\kappa}$ *then* $w(X) < \kappa$ *as well.*

PROOF. Let us assume first that κ is regular. The proof is then based on the following lemma.

LEMMA. *Let* $X \in \mathcal{T}$ *with* $\hat{z}(X) \le \kappa$, *where* κ *is regular. If* $\{Y_\alpha : \alpha \in \kappa\}$ *is an increasing chain of subspaces of* X *with* $Y = \cup\{Y_\alpha : \alpha \in \kappa\}$ *and* $\mathcal{B}$ *is a family of open sets in* X *such that for each* $\alpha \in \kappa$

$$\mathcal{B}\restriction Y_\alpha = \{B\cap Y_\alpha : B \in \mathcal{B}\}$$

is a base of Y_α*, then* $\mathcal{B}\restriction Y$ *is a base of* Y.

PROOF OF THE LEMMA. Assume, on the contrary, that $\mathcal{B}\restriction Y$ is a not a base of Y. Then there is a point $y \in Y$ and an open set G containing y such that if $B \in \mathcal{B}$ and $y \in B$ then $(B\cap Y)\setminus G \neq \emptyset$. We shall now define by transfinite induction a sequence of ordinals $\{\nu_\alpha : \alpha \in \kappa\} \subset \kappa$ and a sequence $\{y_\alpha : \alpha \in \kappa\}$ of points of Y with $y_\alpha \in Y_{\nu_\alpha}$ as follows. Let $\nu_0 \in \kappa$ be such that $y \in Y_{\nu_0}$, then by assumption there is a $B_0 \in \mathcal{B}$ with $y \in B_0 \cap Y_{\nu_0} \subset G\cap Y_{\nu_0}$. We then choose $y_0 \in (B_0\cap Y)\setminus G$. If $\alpha \in \kappa$ and we have chosen already $\{\nu_\beta : \beta \in \alpha\}$ and $\{y_\beta : \beta \in \alpha\}$, choose $\nu_\alpha \in \kappa$ in such a way that $\nu_\beta < \nu_\alpha$ if $\beta < \alpha$, this is possible because κ is regular. Then, by our assumption again, we have a $B_\alpha \in \mathcal{B}$ such that $y \in B_\alpha\cap Y_{\nu_\alpha} \subset G\cap Y_{\nu_\alpha}$, moreover we can choose $y_\alpha \in (B_\alpha\cap Y)\setminus G$. It is easy to see then that for $\beta < \alpha$ we have $y_\beta \notin B_\alpha$ because $y_\beta \notin G\cap Y_{\nu_\alpha}$, but that is impossible as then $\{y_\alpha : \alpha \in \kappa\}$ would be a left separated subset of X of type κ, while $\hat{z}(X) \leq \kappa$. $\dashv$

Now if X has the property $w(Y) < \kappa$ for each $Y \in [X]^\kappa$ then clearly X does not contain a left separated subset of type κ, i.e. $\hat{z}(X) \leq \kappa$. We shall now define for $\alpha \in \kappa$ sets $Y_\alpha \in [X]^{<\kappa}$ and families of open sets $\mathcal{B}_\alpha$ with $|\mathcal{B}_\alpha| < \kappa$ as follows. Put $Y_0 = \emptyset$ and $\mathcal{B}_0 = \emptyset$. If $\{Y_\beta : \beta \in \alpha\}$ and $\{\mathcal{B}_\beta : \beta \in \alpha\}$ have been defined then put first $Z_\alpha = \cup\{Y_\beta : \beta \in \alpha\}$. Since κ is regular we have $|Z_\alpha| < \kappa$, hence by assumption we can find a family of open sets $\mathcal{B}_\alpha \supset \cup\{\mathcal{B}_\beta : \beta \in \alpha\}$ such that $|\mathcal{B}_\alpha| < \kappa$ and $\mathcal{B}_\alpha\restriction Z_\alpha$ is a base of Z_α. Now, if $\mathcal{B}_\alpha$ is a base for X we are done hence we stop our construction. If not, then there is a point $p_\alpha \in X$ and an open set G_α containing p_α such that if $p_\alpha \in B \in \mathcal{B}_\alpha$ then $B\setminus G_\alpha \neq \emptyset$. Put $\mathcal{C}_\alpha = \{B \in \mathcal{B}_\alpha : p_\alpha \in B\}$ and for each $B \in \mathcal{C}_\alpha$ pick a point $p(B) \in B\setminus G_\alpha$. Then we let

$$Y_\alpha = Z_\alpha \cup\{p_\alpha\} \cup \{p(B) : B \in \mathcal{C}_\alpha\},$$

clearly $|Y_\alpha| < \kappa$.

Assume that this construction goes through for all $\alpha \in \kappa$ (if it does not we have established $w(X) < \kappa$). Then we can apply our lemma to X, the sequence $\{Y_\alpha : \alpha \in \kappa\}$ and the family $\mathcal{B} = \cup\{\mathcal{B}_\alpha : \alpha \in \kappa\}$ because trivially $\mathcal{B}_{\alpha+1}\upharpoonright Y_\alpha$ is a base of Y_α. Consequently $\mathcal{B}\upharpoonright Y$ is a base of Y, where clearly $|Y| \leq \kappa$ and therefore $w(Y) < \kappa$. But it is well-known that every base of a space Y contains a subfamily whose cardinality is the weight of Y and which is also a base of Y. Using this and the regularity of κ again we get that there is an $\alpha \in \kappa$ for which $\mathcal{B}_\alpha\upharpoonright Y$ is a base of Y. This however is impossible because $p_\alpha \in Y$ and by the choice of $\mathcal{B}_\alpha$, p_α and G_α for no $B \in \mathcal{B}_\alpha$ do we have $p_\alpha \in B \cap Y \subset G_\alpha \cap Y$, since if $p_\alpha \in B \in \mathcal{B}_\alpha$ then $p(B) \in Y_\alpha \cap (B\setminus G)$. This completes the proof in case κ is regular.

Let us now assume that κ is singular, in particular then κ is a limit cardinal. We first show that we actually have a $\lambda < \kappa$ such that $w(Y) < \lambda$ whenever $Y \in [X]^{\leq\kappa}$. Indeed if no such $\lambda < \kappa$ would exist then we could choose for each $\lambda < \kappa$ a subspace $Y_\lambda \in [X]^{\leq\kappa}$ with $w(Y_\lambda) \geq \lambda$. But then for $Y = \cup\{Y_\lambda : \lambda < \kappa\}$ we had $|Y| \leq \kappa$ and $w(Y) \geq \kappa$ (since $w(Y) \geq w(Y_\lambda) \geq \lambda$ for every $\lambda < \kappa$ and κ is limit), a contradiction. Thus choose such a $\lambda < \kappa$ and observe that then $w(Y) < \lambda^+$ holds for all $Y \in [X]^{\leq\lambda^+}$, hence in view of the first part of our proof and the regularity of λ^+ we get $w(X) < \lambda^+ < \kappa$. ⊣

<u>REMARK</u>. It is easy to see that the first half of the above proof (including the lemma), i.e. the case of regular κ, goes through if the weight is replaced by the π-weight. For the second part however this is no more true because the π-weight is not necessarily monotone. In fact 6.8 is false if κ is singular and w is replaced by π in it, as is shown by 7.13.

Having done most of the work in 6.8 now we get the following result on chains rather easily.

6.9. *If* $w(X_\alpha) < \lambda$ *for each* $\alpha \in \kappa$ *then* $nw(X) \leq \lambda$, *moreover if* $\lambda < \kappa$ *then even* $w(X) < \lambda$.

<u>PROOF</u>. If $\kappa \leq \lambda$ then as $nw(X_\alpha) \leq w(X_\alpha) < \lambda$ we get immediately from 6.2 that $nw(X) \leq \kappa.\lambda = \lambda$. If on the other hand $\lambda < \kappa$ then, as κ is regular, every $Y \in [X]^{\leq\lambda}$ is contained in some X_α, hence $w(Y) \leq w(X_\alpha) < \lambda$. Consequently 6.8 can be applied to conclude that $w(X) < \lambda$. ⊣

6.10. *If* $X \in T_2$ *and* $s(X_\alpha).\psi(X_\alpha) \leq \lambda$ *for each* $\alpha \in \kappa$ *then*

$$|X| \leq 2^\lambda.$$

PROOF. Since by 2.15a) we have $|X_\alpha| \leq 2^\lambda$ for all $\alpha \in \kappa$ it suffices to show that $\kappa \leq 2^\lambda$. To see this assume indirectly that $\kappa < 2^\lambda$, let us pick for each $\alpha \in \kappa$ a point $p_\alpha \in X_{\alpha+1} \setminus X_\alpha$ and consider the subspace $Y = \{p_\alpha : \alpha \in \kappa\}$. Note that for any $\alpha \in \kappa$ then $\psi(p_\alpha, \{p_\beta : \beta \leq \alpha\}) \leq$ $\leq \psi(p_\alpha, X_{\alpha+1}) \leq \lambda$, hence clearly $\psi_\ell(Y) \leq \lambda$. But we also have $s(Y) \leq s(X) \leq \lambda$ in view of 6.1 and $\lambda^+ \leq 2^\lambda < \kappa$, hence 2.19 implies

$$|Y| \leq 2^{s(Y) \cdot \psi_\ell(Y)} \leq 2^\lambda < \kappa,$$

a contradiction. ⊣

REMARK. I don't know whether 6.10 remains valid if we only assume $X \in T_1$ (compare this with 2.15a)).

6.11. *If* $X \in T_2$ *and for all* $\alpha \in \kappa$ *we have*

$$L(X_\alpha) \cdot \psi(X_\alpha) \cdot t(X_\alpha) \leq \lambda$$

then

$$|X| \leq 2^\lambda.$$

PROOF. From 2.27 we get $|X_\alpha| \leq 2^\lambda$ for each $\alpha \in \kappa$, hence our result immediately follows if $\kappa \leq 2^\lambda$. Thus in what follows we assume that $\kappa > 2^\lambda$ (which is clearly equivalent to $\kappa = (2^\lambda)^+$) and strive to obtain a contradiction. Let us start with a lemma that will be used repeatedly in the proof.

LEMMA. *If* $X \in T_2$, Y *is a subspace of* X *with* $L(Y) \leq \lambda$ *and* $p \in Y$, *then for every open set* U *in* X *containing* p *there is a family* $\mathcal{R}$ *of regular closed neighbourhoods of* p *in* X *such that* $|\mathcal{R}| \leq \lambda$ *and*

$$U \cap Y \supset \cap \mathcal{R} \cap Y.$$

PROOF OF THE LEMMA. Since $X \in T_2$ the intersection of all regular closed neighbourhoods of p is $\{p\}$, therefore their complements cover $X \setminus \{p\} \supset Y \setminus U$. But $L(Y \setminus U) \leq L(Y) \leq \lambda$, and as these complements are open we clearly have a family $\mathcal{R}$ of regular closed neighbourhoods of p in X with $|\mathcal{R}| \leq \lambda$ such that

$$Y \setminus U \subset \cup\{X \setminus R : R \in \mathcal{R}\},$$

i.e.

$$U \cap Y \supset \cap \mathcal{R} \cap Y. \quad \dashv$$

As a first consequence of this we prove that $\rho(X) > 2^\lambda$. **Assume, on** the contrary, that $\rho(X) \le 2^\lambda$ and consider any point $p \in X$ where e.g. $p \in X_{\alpha_0}$. Now for any $\alpha \in \kappa \setminus \alpha_0$ we have $p \in X_\alpha$ and $\psi(p, X_\alpha) \le \lambda$, hence we can choose a family of open neighbourhoods $\mathcal{U}_\alpha$ of p in X with $|\mathcal{U}_\alpha| \le \lambda$ and $\cap \mathcal{U}_\alpha \cap X_\alpha = \{p\}$. Since $L(X_\alpha) \le \lambda$ we obtain from our lemma for each $U \in \mathcal{U}_\alpha$ a family $\mathcal{R}_U$ of regular closed neighbourhoods of p in X such that $\cap \mathcal{R}_U \cap X_\alpha \subset U \cap X_\alpha$. Consequently if we put $\mathcal{R}_\alpha = \cup\{\mathcal{R}_U : U \in \mathcal{U}_\alpha\}$ then

$$\cap \mathcal{R}_\alpha \cap X_\alpha = \{p\}.$$

But if $\rho(X) = |RC(X)| \le 2^\lambda$ then from $\kappa < 2^\lambda$ we obtain the existence of an $\mathcal{R}_p \in [RC(X)]^{\le\lambda}$ and a set $a \in [\kappa]^\kappa$ such that $\mathcal{R}_\alpha = \mathcal{R}_p$ for every $\alpha \in a$. Then we have $\cap \mathcal{R}_p \cap X_\alpha = \{p\}$ for cofinally many $\alpha \in \kappa$, which is only possible if $\cap \mathcal{R}_p = \{p\}$. Since p was arbitrary we obtain then that

$$|X| \le [RC(X)]^{\le\lambda} \le (2^\lambda)^\lambda = 2^\lambda < \kappa,$$

a contradiction.

We shall call a set $Y \subset X$ bounded if there is an $\alpha \in \kappa$ with $Y \subset X_\alpha$ (clearly this is equivalent to $|Y| \le 2^\lambda$ if $\kappa > 2^\lambda$). Now if $F \subset X$ is closed and unbounded then taking $F_\alpha = F \cap X_\alpha$ we have $L(F_\alpha) \le$ $\le L(X_\alpha) \le \lambda$, hence the above result clearly applies to F as well, i.e. $\rho(F) > 2^\lambda$. Comparing this with 2.6d) we get the following important observation: for any set $A \in [X]^{\le\lambda}$ its closure $\bar{A}$ is bounded. Indeed, then $\rho(\bar{A}) \le 2^{|A|} \le 2^\lambda$.

The rest of the proof is divided into two parts according to whether $\psi(X) \le \lambda$ or $\psi(X) > \lambda$. In the first case the proof is quite similar to that of 2.27 with a few extra ingredients. Let us observe first of all that if $Y \subset X$ is bounded (i.e. $|Y| \le 2^\lambda$) and $p \in \bar{Y}$ then there is an $\alpha \in \kappa$ with $\{p\} \cup Y \subset X_\alpha$, hence $t(X_\alpha) \le \lambda$ implies the existence of a set $T \in [Y]^{\leq\lambda}$ with $p \in \bar{T}^{X_\alpha} \subset \bar{T}$. In other words

$$\bar{Y} = \cup\{\bar{T}: T \in [Y]^{\leq\lambda}\}.$$

Since we have already shown that $|\bar{T}| \leq 2^\lambda$ if $|T| \leq \lambda$, this implies then $|\bar{Y}| \leq 2^\lambda$. Thus we get that the closure of a bounded set is bounded, in particular $\bar{Y} \subset X_\alpha$ implies $L(\bar{Y}) \leq L(X_\alpha) \leq \lambda$. Now if $\psi(X) \leq \lambda$ then just as in the proof of 2.27 we can show that $\psi(\bar{Y},X) \leq 2^\lambda$, because if $\mathcal{V}_p$ is a ψ-base of p in X with $|\mathcal{V}_p| \leq \lambda$ for each $p \in \bar{Y}$ and $\mathcal{V} = \cup\{\mathcal{V}_p: p \in \bar{Y}\}$ then

$$\mathcal{U} = \{\cup\mathcal{V}': \mathcal{V}' \in [\mathcal{V}]^{\leq\lambda} \ \& \ \bar{Y} \subset \cup\mathcal{V}'\}$$

is a ψ-base of $\bar{Y}$ in X with $|\mathcal{U}| \leq (2^\lambda)^\lambda = 2^\lambda$. Now it is easy to see that in the proof of 2.26 we have not used the full assumption $t(X) < \kappa$ but only the existence of a cardinal $\tau < \kappa$ such that $a(p,S) \leq \tau$ if $|S| \leq \kappa$ and $p \in \bar{S}$ (with the notations used there) which as we have just shown above is satisfied here: $a(p,Y) \leq \lambda$ if $|Y| \leq 2^\lambda$ and $p \in \bar{Y}$. Consequently we see that every condition of (this modified) 2.26 is satisfied, consequently

$$d(X) \leq z(X) \leq 2^\lambda,$$

which is a contradiction as then X should be bounded.
Now consider the second case in which there is a point $p \in X$ with $\psi(p,X) > \lambda$, we can assume without loss of generality that $p \in X_0$. Next we define by transfinite induction ordinals $\alpha_\nu \in \kappa$, points $p_\nu \in X\setminus\{p\}$ and families $\mathcal{V}_\nu$ of regular closed neighbourhoods of p in X with $|\mathcal{V}_\nu| \leq \lambda$ as follows. Suppose $\mu \in \lambda^+$ and we have defined already α_ν, p_ν and $\mathcal{V}_\nu$ for $\nu \in \mu$. Then $\overline{\{p_\nu: \nu \in \mu\}}$ is bounded, hence we can choose $\alpha_\mu \in \kappa$ such that

$$\overline{\{p_\nu: \nu \in \mu\}} \subset X_{\alpha_\mu}.$$

Next we choose $\mathcal{V}_\mu$ as a family of regular closed neighbourhoods of p satisfying $|\mathcal{V}_\mu| \leq \lambda$ and

$$\cap\mathcal{V}_\mu \cap X_{\alpha_\mu} = \{p\}.$$

This is possible because by $\psi(p,X_{\alpha_\mu}) \le \lambda$ there is a family $\mathcal{U}$ of open neighbourhoods of p in X with $\cap\,\mathcal{U} \cap X_{\alpha_\mu} = \{p\}$ and $|\mathcal{U}| \le \lambda$, and then for each $U \in \mathcal{U}$ there is by our lemma a family $\mathcal{R}_U$ of regular closed neighbourhoods of p in X with

$$\cap\mathcal{R}_U \cap X_{\alpha_\mu} \subset U \cap X_{\alpha_\mu},$$

and then we can put

$$\mathcal{V}_\mu = \cup\{\mathcal{R}_U : U \in \mathcal{U}\}.$$

Since $\psi(p,X) > \lambda$, however, we must have

$$F_\mu = \cap\{\cup\{\mathcal{V}_\nu : \nu \le \mu\}\} \ne \{p\},$$

hence we can choose $p_\mu \in F_\mu \setminus \{p\}$. Having defined $S = \{p_\nu : \nu \in \lambda^+\}$ observe that S is "almost" a free sequence in the sense that for any $\mu \in \lambda^+$

$$\overline{\{p_\nu : \nu \in \mu\}} \cap \overline{\{p_\nu : \mu \le \nu < \lambda^+\}} \subset \{p\},$$

because

$$\overline{\{p_\nu : \nu \in \mu\}} \subset X_{\alpha_\mu},\ \overline{\{p_\nu : \mu \le \nu < \lambda^+\}} \subset F_\mu$$

and clearly

$$F_\mu \cap X_{\alpha_\mu} \subset \cap\mathcal{V}_\mu \cap X_{\alpha_\mu} = \{p\}.$$

But $|S| \le \lambda^+$ i.e. S is a bounded set, consequently $S \subset X_\alpha$ for some $\alpha \in \kappa$. Now as $\psi(p,X_\alpha) \le \lambda$, there is an open neighbourhood U of p for which

$$|S\setminus U| = \lambda^+,$$

or in other words we have an $a \in [\lambda^+]^{\lambda^+}$ with

$$S_0 = \{p_\nu : \nu \in a\} \subset X_\alpha \setminus U.$$

But then $p \notin \overline{S_0}$, hence clearly S_0 is a free sequence in X_α of length $|a| = \lambda^+$. But $L(X_\alpha) \le \lambda$ implies the existence of a complete accumulation point q of S_0 in X_α which clearly is impossible as $t(X_\alpha) \le \lambda$. This contradiction completes the proof. $\dashv$

<u>COROLLARY</u>. *If* $X \in T_3$, $L(X) \le \lambda$ *and* $\chi(X_\alpha) \le \lambda$ *for all* $\alpha \in \kappa$ *then* $|X| \le 2^\lambda$.

<u>PROOF</u>. Clearly it suffices to prove that $\chi(\bar{X}_\alpha) \le \lambda$ holds for each $\alpha \in \kappa$ because then 6.11 can be applied to the chain $\{\overline{X_\alpha} : \alpha \in \kappa\}$. But if $p \in \bar{X}_\alpha$ choose $\beta \in \kappa$ with $\{p\} \cup X_\alpha \subset X_\beta$, then using the regularity of X and 2.7a) we get

$$\chi(p, \bar{X}_\alpha) \le \chi(p, \bar{X}_\beta) = \chi(p, X_\beta) \le \lambda. \quad \dashv$$

It is not surprising that if one assumes that X in (*) is compact Hausdorff then a lot more can be said about its "cardinality behaviour". In the rest of this chapter we are going to study just this situation.

6.12. *If* $X \in C_2$ *and* $nw(X_\alpha) < \lambda$ *for each* $\alpha \in \kappa$ *then*

$$(nw(X) =) w(X) \le \lambda.$$

If, in addition, $\lambda < \kappa$ *then even* $w(X) < \lambda$.

<u>PROOF</u>. Clearly $X^2 = X \times X$ is the union of the chain $\{X_\alpha^2 : \alpha \in \kappa\}$, moreover $h(X_\alpha^2) \le nw(X_\alpha^2) < \lambda$ holds for every $\alpha \in \kappa$. Thus by 6.1 we have $h(X^2) \le \lambda$ and even $h(X^2) < \lambda$ if $\lambda < \kappa$. But from 2.10b) and 3.32 we obtain

$$w(X) = \psi_\Delta(X) = \psi(\Delta, X^2) \le h(X^2),$$

and our claims follow. $\dashv$

6.13. *If* $X \in C_2$ *and* $t(X_\alpha) < \lambda$ *for* $\alpha \in \kappa$ *then* $t(X) \le \lambda$, *and even* $t(X) < \lambda$ *if* $\lambda < \kappa$.

PROOF. Let us assume first that $\kappa \le \lambda$. Then by 3.12 it suffices to show that X does not contain a free sequence of length $\lambda^+ = \mu$. Assume on the contrary that $S = \{p_\alpha : \alpha \in \mu\}$ is such a free sequence. Since $\kappa < \mu = \lambda^+$ we can actually assume that S is bounded because otherwise we could just take an appropriate subsequence. For any $\nu \in \mu$ let us put

$$F_\nu = \overline{\{p_\alpha : \nu \le \alpha < \mu\}}$$

and

$$F = \cap\{F_\nu : \nu \in \mu\}$$

Then $F \neq \emptyset$ because X is compact hence we can choose a $p \in F$. Since S is free we clearly have

$$p \notin \overline{\{p_\alpha : \alpha \in \nu\}}$$

for every $\nu \in \mu$, while $p \in \bar{S}$ hence $a(p,S) \ge \mu$. This shows that if $\alpha \in \kappa$ is chosen in such a way that $\{p\} \cup S \subset X_\alpha$ then

$$\lambda^+ = \mu \le a(p,S) \le t(p,X_\alpha) < \lambda,$$

a contradiction.

Now assume $\lambda < \kappa$ and for each regular cardinal $\mu \le \lambda$ such that X contains a free sequence of length μ choose one, say S_μ. Observe that $\mu \le \lambda < \kappa$ implies that S_μ is then bounded. Let us select then a point p_μ to S_μ similarly as p was selected to S above. Now we have an $\alpha \in \kappa$ such that for every regular $\mu \le \lambda$ in question

$$S_\mu \cup \{p_\mu\} \subset X_\alpha.$$

According to our above observations then we have

$$\mu \le t(p_\mu, X_\alpha) \le t(X_\alpha)$$

for any such μ, consequently, as $t(X)$ is the sup of all these μ, we have $t(X) \le t(X_\alpha) < \lambda$. ⊣

We shall prove a similar result for the character but first we need an auxiliary result, which again is of independent interest.

6.14. a) *For any* $X \in T$ *and* $p \in X$ *if* $\chi(p,X) = \lambda$ *then there is a* $Y \in [X]^{\leq\lambda}$ *such that* $p \in Y$ *and* $\chi(p,Y) = \lambda$.

b) *If* $X \in C_2$, $t(X) < \lambda$ *and* $\chi(p,X) \geq \lambda$ *then there is a* $Y \in [X]^{\leq\lambda}$ *with* $p \in Y$ *and* $\chi(p,Y) \geq \lambda$.

PROOF.

a) Let $\{U_\alpha: \alpha \in \lambda\}$ be a neighbourhood base of p in X and put for each pair $\langle\alpha,\beta\rangle \in \lambda^2$

$$p_{\langle\alpha,\beta\rangle} = \begin{cases} \text{a member of } U_\alpha \setminus U_\beta \text{ if } U_\alpha \setminus U_\beta \neq \emptyset, \\ \\ p \text{ otherwise.} \end{cases}$$

Put $Y = \{p_{\langle\alpha,\beta\rangle}: \langle\alpha,\beta\rangle \in \lambda^2\}$, then $p = p_{\langle\alpha,\alpha\rangle} \in Y$, and trivially $|Y| \leq \lambda$. We claim that $\chi(p,Y) = \lambda$. Since $\chi(p,Y) \leq \lambda$ is obvious we only have to show that $\chi(p,Y) < \lambda$ is impossible. Assume, on the contrary that $\chi(p,Y) = \mu < \lambda$ and let $\{V_\nu: \nu \in \mu\}$ be neighbourhoods of p in X such that $\{Y \cap V_\nu: \nu \in \mu\}$ is a neighbourhood base of p in Y. For every $\nu \in \mu$ we can choose an $\alpha(\nu) \in \lambda$ such that

$$U_{\alpha(\nu)} \subset V_\nu.$$

Then $\{U_{\alpha(\nu)}: \nu \in \mu\}$ is not a neighbourhood base of p in X, consequently there is an $\alpha \in \lambda$ such that $U_{\alpha(\nu)} \setminus U_\alpha \neq \emptyset$ for all $\nu \in \mu$. But then

$$p_{\langle\alpha(\nu),\alpha\rangle} \in Y \cap (U_{\alpha(\nu)} \setminus U_\alpha) \neq \emptyset$$

for every $\nu \in \mu$, consequently $Y \cap (V_\nu \setminus U_\alpha) \neq \emptyset$, i.e. $Y \cap V_\nu \not\subset Y \cap U_\alpha$ for every $\nu \in \mu$, contradicting $\{Y \cap V_\nu: \nu \in \mu\}$ is a neighbourhood base of p in Y. $\dashv$

b) If there is an $S \in [X]^{<\lambda}$ such that $p \in S$ and $\chi(p,S) \geq \lambda$ then we are done, hence we assume in what follows that $S \in [X]^{<\lambda}$ and $p \in S$ imply $\chi(p,S) < \lambda$. We also restrict our attention to the case in which λ is regular, for the case of a singular λ will easily reduce to it. Also observe that $\chi(p,X) \geq \lambda > t(X)$ implies $\lambda > \omega$.

Now we define by transfinite induction points $p_\alpha \in X$ and families $\mathcal{B}_\alpha$ of open neighbourhoods of p in X with $|\mathcal{B}_\alpha| < \lambda$ as follows. Put $p_0 = p$ and $\mathcal{B}_0 = \emptyset$. If $\alpha \in \lambda\setminus\{0\}$ and p_β, $\mathcal{B}_\beta$ have been defined for all $\beta \in \alpha$ then put $S_\alpha = \{p_\beta : \beta \in \alpha\}$ and observe that

$$\chi(p,S_\alpha) = \chi(p,\bar{S}_\alpha) < \lambda.$$

Consequently we can choose a family of open neighbourhoods of p in X, say $\mathcal{B}_\alpha$, such that (0) $\cap\mathcal{B}_\alpha \cap \bar{S}_\alpha = \{p\}$, (i) $|\mathcal{B}_\alpha| < \lambda$, (ii) $\cup\{\mathcal{B}_\beta : \beta \in \alpha\} \subset \mathcal{B}_\alpha$. Using that $\omega < \lambda$ we may also assume that (iii) $\mathcal{B}_\alpha$ is closed under finite intersections, and (iv) for every $U \in \mathcal{B}_\alpha$ there is a $V \in \mathcal{B}_\alpha$ with $\bar{V} \subset U$ (in other words, (iii) and (iv) together say that $\mathcal{B}_\alpha$ is a regular filter base). Since $\psi(p,X) = \chi(p,X) \geq \lambda > |\mathcal{B}_\alpha|$ we have $\{p\} \neq \cap \mathcal{B}_\alpha$, hence we can choose a point $p_\alpha \in \cap \mathcal{B}_\alpha\setminus\{p\}$. Having completed the induction for all $\alpha \in \lambda$ put $Y = \{p_\alpha : \alpha \in \lambda\}$ and $\mathcal{B} = \cup\{\mathcal{B}_\alpha : \alpha \in \lambda\}$; clearly $\mathcal{B}$ is a regular filter base in X.
The regularity of λ and $t(X) < \lambda$ imply that

$$\bar{Y} = \cup\{\bar{S}_\alpha : \alpha \in \lambda\},$$

showing that since $\cap\mathcal{B}_\alpha \cap \bar{S}_\alpha = \{p\}$ for each $\alpha \in \lambda$, we have $\cap\mathcal{B}\cap\bar{Y} = \{p\}$. But then $\mathcal{B}\restriction\bar{Y} = \{B\cap\bar{Y} : B \in \mathcal{B}\}$ is a ψ-base of p in $\bar{Y}$ and at the same time a regular filter base in $\bar{Y}$, which in view of $\bar{Y} \in C_2$ then clearly implies that $\mathcal{B}\restriction\bar{Y}$ is actually a neighbourhood base of p in $\bar{Y}$. This shows $\chi(p,Y) = \chi(p,\bar{Y}) \leq \lambda$ but we claim that $\chi(p,\bar{Y}) = \lambda$. Assume on the contrary that $\chi(p,\bar{Y}) < \lambda$. Since $\mathcal{B}\restriction\bar{Y}$ is a neighbourhood base of p in $\bar{Y}$ then we can actually select a subfamily $\mathcal{C} \subset \mathcal{B}$ with $|\mathcal{C}| = \chi(p,\bar{Y}) < \lambda$ such that $\cap\mathcal{C}\cap\bar{Y} = \{p\}$. By the regularity of λ however then there is an $\alpha \in \lambda$ with $\mathcal{C} \subset \mathcal{B}_\alpha$, consequently we have $p_\alpha \in \cap \mathcal{B}_\alpha\setminus\{p\}$ and therefore $p_\alpha \in \cap\, \mathcal{C} \cap \bar{Y}\setminus\{p\}$ as well, a contradiction. This completes the proof for λ regular.

Now if λ is singular then we can apply the first part of our proof to obtain for every regular cardinal μ with $t(X) < \mu < \lambda$ a subspace $Y_\mu \in [X]^{\leq\mu}$ such that $\chi(p,Y_\mu) \geq \mu$. Thus if we put

$$Y = \cup\{Y_\mu : \mu = cf(\mu) \ \&\ t(X) < \mu < \lambda\}$$

then $|Y| \leq \lambda$ and $\chi(p,Y) \geq \lambda$ because

$$\chi(p,Y) \geq \chi(p,Y_\mu) \geq \mu$$

for every regular $\mu < \lambda$. $\dashv$

6.15. *If* $X \in C_2$ *and* $\chi(X_\alpha) < \lambda$ *for every* $\alpha \in \kappa$ *then* $\chi(X) \leq \lambda$, *moreover if* $\kappa > \lambda$ *then even* $\chi(X) < \lambda$.

PROOF. If $\kappa \leq \lambda$ then from 6.2 (iii) we get

$$\chi(X) = \psi(X) \leq \kappa.\lambda = \lambda.$$

Now, if $\kappa > \lambda$ then first of all $t(X_\alpha) \leq \chi(X_\alpha) < \lambda$ for all $\alpha \in \lambda$ implies by 6.13 that $t(X) < \lambda$. If $\chi(X) < \lambda$ failed, then for every successor cardinal μ with $t(X) < \mu \leq \lambda$ there would exist a point $p_\mu \in X$ with $\chi(p_\mu,X) \geq \mu$, hence using 6.14 a set $Y_\mu \in [X]^{\leq\mu}$ as well such that already $\chi(p_\mu,Y_\mu) \geq \mu$. By $\kappa > \lambda$ then there is an $\alpha \in \kappa$ such that $\{p_\mu\} \cup Y_\mu \subset X_\alpha$ for every μ in question. But then we have

$$\chi(X_\alpha) \geq \chi(p_\mu,Y_\mu) \geq \mu$$

for every such μ, that clearly implies $\chi(X_\alpha) \geq \lambda$, a contradiction. $\dashv$

6.16. *If* $X \in C_2$ *and* $t(X_\alpha).c(X_\alpha) \leq \lambda$ *for each* $\alpha \in \kappa$ *then* $w(X) \leq 2^\lambda$.

PROOF. Let us first consider the case in which $\kappa > \lambda^+$. Then from 6.13 and 6.1 we get $t(X) \leq \lambda$ and $c(X) \leq \lambda$, hence as $X \in C_2$, by 3.14a), $\pi\chi(X) \leq \lambda$ as well. Then we get from 2.37

$$w(X) \leq \rho(X) \leq \pi\chi(X)^{c(X)} \leq 2^\lambda.$$

Now assume that $\kappa \leq \lambda^+$. Then 6.13 yields us $t(X) \leq \lambda^+$, consequently

$$\pi\chi(\bar{X}_\alpha) \leq t(\bar{X}_\alpha) \leq t(X) \leq \lambda^+$$

for $\alpha \in \kappa$, since $\bar{X}_\alpha \in C_2$. From 2.6a) we get $c(\bar{X}_\alpha) = c(X_\alpha) \leq \lambda$, thus applying 2.37 to $\bar{X}_\alpha$ we obtain

$$nw(\bar{X}_\alpha) = w(\bar{X}_\alpha) \leq \rho(\bar{X}_\alpha) \leq \pi\chi(\bar{X}_\alpha)^{c(\bar{X}_\alpha)} \leq 2^\lambda.$$

But then applying 6.2 (i) to the chain $\{\bar{X}_\alpha : \alpha \in \kappa\}$ we have

$$w(X) = nw(X) \leq \kappa . 2^\lambda = 2^\lambda . \quad \dashv$$

CHAPTER 7

EXAMPLES

In this chapter we present examples that establish the sharpness of some of our earlier results. Since we have committed ourselves in this book not to use any tools going beyond the usual axioms of set theory, this chapter is necessarily very incomplete because, as it has turned out during the past decade, most of the interesting examples require just these kinds of metamathematical tools. Also the presentation of our examples is less self contained than that of the earlier chapters.

7.1. For any set S let us denote by $\mathcal{F}(S)$ the set of all *non-principal* ultrafilters on S. Fix an infinite cardinal κ, for any $n \in \omega$ put $\mathcal{P}_n = \mathcal{F}(\kappa \times \{n\})$, moreover $\mathcal{P} = \cup\{\mathcal{P}_n : n \in \omega\}$. It is well-known that $|\mathcal{P}| = |\mathcal{P}_n| = \lambda = \exp_2 \kappa$. Now, by a result of B. Pospíšil (cf [P 1939]) there is for each $n \in \omega$, an $u_n \in \mathcal{F}(\mathcal{P}_n)$ such that $\chi(u_n) = 2^\lambda$, i.e. the ultrafilter u_n has no base of size less than 2^λ. Finally let u be a member of $\mathcal{F}(\omega)$. We can then define an ultrafilter v on $\mathcal{P}$ as follows:

$$v = \{P \subset \mathcal{P} : \{n \in \omega : P \cap \mathcal{P}_n \in u_n\} \in u\}.$$

It is easy to see then that $\chi(v) = 2^\lambda$ holds too (for the details see [JK 1973]). Now put

$$X = (\kappa \times \omega) \cup \mathcal{P} \cup \{v\}$$

and define a topology on X as follows: every member of $\kappa \times \omega$ is isolated; if $p \in \mathcal{P}$ then all sets of the form $\{p\} \cup A$ where $A \in p$ form a neighbourhood base of p; all sets of the form

$$\{v\} \cup P \cup \cup\{f(p) : p \in P\}$$

constitute a neighbourhood base of v, where $P \in v$ and f is any choice function on P. It is easy to check that this gives a Hausdorff topology on X, $\kappa \times \omega$ is dense in X, $\mathcal{P}$ is a discrete subspace of X, moreover $\chi(v) = 2^{\lambda}$ clearly implies $\chi(v,X) = 2^{\lambda}$. Consequently we have, for any κ, an $X \in \mathcal{T}_2$ such that $d(X) = \kappa$, but

$$|X| = s(X) = \exp_2\kappa,$$

moreover

$$\chi(X) = w(X) = \exp_3\kappa. \qquad \dashv$$

7.2. Let κ be an arbitrary cardinal less than the first measurable cardinal μ (that is if it exists). We define the sets X_n for $n \in \omega$ by induction as follows: $X_0 = \kappa$, $X_{n+1} = F(X_n)$. Finally we put

$$X = \cup\{X_n : n \in \omega\}.$$

Our aim is to define a topology on X but to do that we have to establish certain facts about ultrafilters.

(i) If f is a choice function on $F(S)$ then there are finitely many members $u_1,\ldots,u_\ell$ of $F(S)$ such that

$$|S \setminus \bigcup_{j=1}^{\ell} f(u_j)| < \omega.$$

Indeed, if this was not the case then the family $\{S \setminus f(u) : u \in F(S)\}$ could be extended to a non-principal ultrafilter $v \in F(S)$, which is impossible because this would imply $(S \setminus f(v)) \in v$.

(ii) Let $u \in F(F(S))$ and put

$$u' = \cup\{\cap P : P \in u\}.$$

Then $u' \in F(S)$.

Let us put for $A \subset S$

$$\hat{A} = \{p \in F(S) : A \in p\}.$$

Clearly u' can also be defined as

$$u' = \{A \subset S: \hat{A} \in u\}.$$

That u' is an ultrafilter follows from the relationships

$$\hat{A} \cap \hat{B} = A \cap B$$

and

$$\widehat{S \setminus A} = F(S) \setminus \hat{A}.$$

That u' is non-principal is implied by the fact that if $s \in S$ then

$$\widehat{S \setminus \{s\}} = F(S).$$

Now, for any $u \in X_{n+1}$ we define $u^{(i)} \in X_{n+1-i}$ for $i \leq n$ by induction as follows:

$$u^{(0)} = u, \; u^{(i+1)} = (u^{(i)})'.$$

This is possible using (ii). We now define a topology on X as follows: All points of $X_0 = \kappa$ are isolated. If $u \in X_{n+1}$ then all sets of the form

$$V = \{u\} \cup \bigcup_{i=0}^{n} A^{(i)},$$

where $A^{(i)} \in u^{(i)}$ for $0 \leq i \leq n$, constitute a neighbourhood base of u. Clearly these form a filter, moreover if we put

$$B^{(i)} = A^{(i)} \setminus \{p \in X_{n-i}: A^{(i+1)} \notin p\},$$

then, by the definition of the operation u', we have $B^{(i)} \in u^{(i)}$ and

$$V' = \{u\} \cup \bigcup_{i=0}^{n} B^{(i)}$$

is a neighbourhood of u such that V is a neighbourhood of every $p \in V'$. This shows that we have indeed defined a topology τ on X. It is easy to see using (i) that from every τ-open cover of X_{n+1} we can choose finitely many members such that they cover all but finitely many members of X_n. From this it follows easily that X is Lindelöf, i.e. $L(X) = \omega$.

It is easy to see that (X,τ) is T_1, but in fact we show that $\psi(X) = \omega$. Since μ (if exists) is inaccessible, we get from $|X_0| = \kappa < \mu$ that $|X_n| < \mu$ as well. Consequently every member p of $X_{n+1} = F(X_n)$, as an ultrafilter, is not σ-complete. Thus if $u \in X_{n+1}$ we can choose for every $i < n+1$ a family

$$\{A_k^{(i)} : k \in \omega\} \subset u^{(i)}$$

such that

$$\cap\{A_k^{(i)} : k \in \omega\} = \emptyset.$$

Let us put

$$V_k = \{u\} \cup \bigcup_{i=0}^{n} A_k^{(i)} ,$$

then we have

$$\cap\{V_k : k \in \omega\} = \{u\},$$

showing that $\psi(u,X) = \omega$.

7.3. Put $I^* = I \times \{0,1\}$ (where $I = [0,1]$) and consider the lexicographic order $\prec$ on I^* (in other words I^* is obtained from I by "splitting" each point of I into two). Now I^* provided with the order topology determined by $\prec$ is a compact ordered space which, as is easy to see, satisfies $h(I^*) = z(I^*) = s(I^*) = \omega$. Now it is also easy to see that the set

$$\{\langle\langle x,0\rangle, \langle x,1\rangle\rangle : x \in I\}$$

is discrete in $I^* \times I^* = X$. Thus X is a compact Hausdorff space such that $d(X) = \chi(X) = \omega$ but

$$s(X) = 2^{\omega}.$$

From 2.11 we now conclude

$$K(X) = o(X) = 2^{2^{\omega}}. \qquad \dashv$$

7.4. Let λ be a singular cardinal with $\mu = cf(\lambda) > \kappa$ and suppose that $2^{\kappa} \geq \lambda$ (hence λ is not strong limit). Let X be a subset of $D(2)^{\kappa}$ with $|X| = \lambda$ and write X as a disjoint union

$$X = \cup\{X_{\alpha} : \alpha \in \mu\}$$

where $|X_{\alpha}| = \lambda_{\alpha} < \mu$ for every $\alpha \in \mu$. Let us consider the topology τ on X for which sets of the form

$$\{p\} \cup U \setminus \cup\{X_{\alpha} : \alpha \in a\}$$

constitute a neighbourhood base of $p \in X$, where U is open in the subspace topology of X (inherited from $D(2)^{\kappa}$) and $a \in [\mu]^{<\omega}$. Since τ is finer than this subspace topology on X we have $(X,\tau) \in \mathcal{H}$. Clearly every X_{α} is discrete in τ, hence

$$s(X) = \sup\{\lambda_{\alpha} : \alpha \in \mu\} = \lambda,$$

and consequently $h(X) = z(X) = \lambda$ as well.
Next we show that $\hat{\phi}(X) = \lambda$ for $\phi \in \{s,h,z\}$, hence X establishes *sup* $\neq$ *max* for ϕ on $\mathcal{H}$. It clearly suffices to show for this that every subset Y of X with $|Y| = \lambda$ is neither right nor left separated. Clearly if $Y \in [X]^{\lambda}$ then there is a $Y' \in [Y]^{\mu}$ with $|Y' \cap X_{\alpha}| \leq 1$ for each $\alpha \in \mu$. Put $S = \{\alpha \in \mu : |Y' \cap X_{\alpha}| = 1\}$ and for $\alpha \in S$ let $Y' \cap S = \{y_{\alpha}\}$. If Y' were e.g. right separated then we had open sets U_{α} and finite sets $a_{\alpha} \in [\mu]^{<\omega}$ such that

$$V_\alpha = \{y_\alpha\} \cup U_\alpha \setminus \cup \{x_\beta : \beta \in a_\alpha\}$$

is a right separating τ-neighbourhood of y_α. Applying Hajnal's theorem, 0.3, for the set mapping $\alpha \mapsto a_\alpha \cap S \in [S]^{<\omega}$ we may assume that S is also free with respect to this set mapping. Clearly then $y_\beta \in V_\alpha$ iff $y_\beta \in U_\alpha$ for $\alpha, \beta \in S$, hence $\{y_\alpha : \alpha \in S\}$ is also right separated in $D(2)^\kappa$, which is clearly impossible.
Now let us assume, in addition, that 2^δ is strictly increasing for cofinally many $\delta < \lambda$. In this case we have $\mathrm{cf}(2^{\underset{\smile}{\lambda}}) = \mathrm{cf}(\lambda) = \mu$. By 2.11

$$o(X) \geq 2^{\underset{\smile}{s(X)}} = 2^{\underset{\smile}{\lambda}},$$

while on the other hand $\hat{z}(X) = \lambda$ and $2^\kappa \geq \lambda$ imply

$$o(X) \leq |X|^{\underset{\smile}{\hat{z}(X)}} = \lambda^{\underset{\smile}{\lambda}} = 2^{\underset{\smile}{\lambda}}.$$

Thus we have $o(X) = 2^{\underset{\smile}{\lambda}}$, while $\mathrm{cf}(o(X)) = \mu$ implies

$$o(X)^\mu > o(X).$$

7.5. Let R be an arbitrary space and $\prec$ a well-ordering of R. We define two spaces R^ℓ and R^u on the same underlying set R as follows:
A basis for $R^\ell(R^u)$ consists of all sets of the form $G_x^\ell(G_x^u)$, where G is open in R, $x \in G$ and $G_x^\ell = \{y \in G : y \preceq x\}$ $(G_x^u = \{y \in G : x \preceq y\})$. Since $z \in G_x^\ell \cap G_y^\ell (z \in G_x^u \cap G_y^u)$ implies $(G \cap H)_z^\ell \subset G_x^\ell \cap G_y^\ell$ $((G \cap H)_z^u \subset G_x^u \cap G_y^u)$, both are indeed bases of some spaces whose topologies are obviously finer than that of R, hence in particular T_2 if R is so.

<u>PROPOSITION</u>
(i) $h(R^\ell) = |R|$ and $z(R^\ell) = z(R)$
(ii) $z(R^u) = |R|$ and $h(R^u) = h(R)$.

<u>PROOF</u>.
(i) $h(R^\ell) = |R|$ is trivial as $\prec$ right separates R^ℓ. To show $z(R^\ell) = z(R)$, let $S \subset R^\ell$ be left separated by a well-ordering $\lhd$, say. Just like in the proof of theorem 2.12, there is a subset $T \subset S$ with $|T| = |S|$ such that the two well-orderings $\prec$ and $\lhd$ coincide on T. But then T is obviously also left separated in the original space R, hence $|T| = |S| \leq z(R)$, which was to be shown.

The proof of (ii) is completely analogous.
Thus as we can have T_2-spaces R with $|R| = \exp h(R)$, we then have $z(R^u) = \exp(h(R^u))$, and as we can have ones with $|R| = \exp\exp z(X)$, then we have $h(R^{\ell}) = \exp\exp z(R^{\ell})$. ⊣

7.6. Let θ be a weakly inaccessible cardinal (i.e. regular and limit) and consider the product space

$$X = \times\{D(\kappa):\kappa < \theta\}.$$

By 5.10 we have $\theta \in cal(X)$, consequently

$$\hat{c}(X) \leq \theta.$$

But for each $\kappa < \theta$ the family $G^{(\kappa)} = \{pr_{\kappa}^{-1}(\{\alpha\}):\alpha \in \kappa\}$ is cellular in X with $|G^{(\kappa)}| = \kappa$, hence

$$c(X) = \hat{c}(X) = \theta,$$

showing that in this case *sup* $\neq$ *max* for c on X. ⊣

7.7. Let us denote by $\Sigma_{\kappa}(\lambda)$ the λ^{th} Σ_{κ}-power of D(2), i.e.

$$\Sigma_{\kappa}(\lambda) = \{f \in D(2)^{\lambda} : |\{\nu \in \lambda: f(\nu) = 1\}| \leq \kappa\}.$$

PROPOSITION. *A cardinal* α *is* not *a caliber of* $\Sigma_{\kappa}(\lambda)$ *if and only if*

(i) $cf(\alpha) = \omega$

or

(ii) $\kappa < \alpha \leq \lambda$

or

(iii) $\kappa < cf(\alpha) \leq \lambda$.

PROOF. Recalling that $\alpha \in cal(X)$ implies $cf(\alpha) \in cal(X)$ and $\omega \notin cal(\Sigma_{\kappa}(\lambda))$ as this space is Hausdorff, the if part follows if we show that (ii) implies $\alpha \notin cal(\Sigma_{\kappa}(\lambda))$. But for this consider the family

$$G = \{pr_\nu^{-1}(\{1\}):\nu \in \alpha\},$$

clearly $|G| = \alpha$ but for every $a \in [\alpha]^\alpha$

$$\Sigma_\kappa(\lambda) \cap \bigcap \{pr_\nu^{-1}(\{1\}): \nu \in a\} = \emptyset$$

since $\alpha > \kappa$. Thus $\alpha \notin cal(\Sigma_\kappa(\lambda))$.
Now assume that α does not satisfy either of the conditions (i)-(iii) and show that $\alpha \in cal(\Sigma_\kappa(\lambda))$. If $\lambda < cf(\alpha)$ then we have $d(\Sigma_\kappa(\lambda)) \le w(\Sigma_\kappa(\lambda)) = \lambda < cf(\alpha)$ and $\alpha \in cal(\Sigma_\kappa(\lambda))$ follows immediately.
If $\omega_1 \le cf(\alpha) \le \alpha \le \kappa$ then $\alpha \in \Sigma_\kappa(\lambda)$ easily follows from the fact that, by 5.11, $\alpha \in cal(D(2)^\alpha)$.
Finally it remains to check the case $\omega_1 \le cf(\alpha) \le \kappa \le \lambda < \alpha$. Now we can write

$$\alpha = \Sigma\{\alpha_\nu:\nu \in cf(\alpha)\},$$

where $\alpha_\nu < \alpha_\mu$ if $\nu \in \mu \in cf(\alpha)$ and each α_ν is a regular cardinal, $\alpha_\nu > \lambda$. Let $\{G_\beta:\beta \in \alpha\}$ be a family of elementary open sets in $\Sigma_\kappa(\lambda)$. Since there are only λ many elementary open sets in $\Sigma_\kappa(\lambda)$ (or $D(2)^\lambda$), for each $\nu \in cf(\alpha)$ there is a fixed elementary open set $G^{(\nu)}$ such that

$$|\{\beta \in \alpha_\nu:G_\beta = G^{(\nu)}\}| = \alpha_\nu.$$

Since by our earlier results $cf(\alpha) \in cal(\Sigma_\kappa(\lambda))$, we have a set $b\in[cf(\alpha)]^{cf(\alpha)}$ such that

$$\cap\{G^{(\nu)}:\nu \in b\} \neq \emptyset.$$

Now if we put $a = \cup\{\{\beta \in \alpha_\nu: G_\beta = G^{(\nu)}\}: \nu \in b\}$, then $|a| = \alpha$ and clearly $\cap\{G_\beta:\beta \in a\} \neq \emptyset$, hence $\alpha \in cal(\Sigma_\kappa(\lambda))$. $\dashv$

Let us now put for $\nu \in \omega_1$

$$R_\nu = \Sigma_{\omega_1}(\omega_\nu)$$

and

$$R = \times\{R_\nu : \nu \in \omega_1\}.$$

Then $\omega_{\omega_1} \in \mathrm{cal}(R_\nu)$ for each $\nu \in \omega_1$ by our above proposition. On the other hand if we consider the family

$$G = \{G_\mu^{(\nu)} = \{p \in R: p(\nu)(\mu) = 1\} : \nu \in \omega_1 \ \& \ \mu \in \omega_\nu\}$$

then for any $a \in [\cup\{\{\nu\} \times \omega_\nu : \nu \in \omega_1\}]^{\omega_2}$ there is a $b \in [a]^{\omega_2}$ and a fixed $\nu \in \omega_1$ such that

$$b \subset \{\nu\} \times \omega_\nu,$$

and clearly then

$$\cap\{G_\mu^{(\nu)} : \langle \nu\ \mu \rangle \in b\} = \emptyset.$$

This shows that ω_{ω_1} is not a caliber of R. Consequently we see that in 5.10 the condition on the regularity of λ cannot be removed. Moreover since $\omega_1 \in \mathrm{cal}(R_\nu)$ for each $\nu \in \omega_1$ we get from 5.10 that $\omega_1 \in \mathrm{cal}(R)$ showing that in general $\mathrm{cf}(\lambda) \in \mathrm{cal}(X)$ does not imply $\lambda \in \mathrm{cal}(X)$. ⊣

7.8. Let $F = \{0,1\}$ with the T_o topology in which 0 is isolated but 1 is not. Looking at the elementary open sets in F^κ it is obvious that $w(F^\kappa) \le \kappa$. On the other hand if $p_\xi \in F^\kappa$ is defined by

$$p_\xi(\mu) = \begin{cases} 0, & \text{if } \mu = \xi \\ 1, & \text{if } \mu \neq \xi, \end{cases}$$

then $\{p_\xi : \xi \in \kappa\}$ is clearly a discrete subspace of F^κ, hence we get

$$w(F^k) = h(F^k) = z(F^\kappa) = s(F^\kappa) = \kappa.$$

It is easy to see (using the same method as in the proof of 5.2b) that if $q \in F^\kappa$ is such that $q(\mu) = 0$ for each $\mu \in \kappa$ then $\pi\chi(q,F^\kappa) = \kappa$, hence $\chi(q,F^\kappa) = \kappa$ and therefore $\pi(F^\kappa) = \kappa$ holds as well. ⊣

7.9. Similarly as in 7.8 we can show that

$$s(D(2)^{\kappa}) = z(D(2)^{\kappa}) = h(D(2)^{\kappa}) = \chi(D(2)^{\kappa}) = w(D(2)^{\kappa}) = \kappa,$$

and from this and from 5.2b we also get

$$(D(2)^{\kappa}) = \pi\chi(D(2)^{\kappa}) = \kappa.$$

Finally, as $D(2)^{\kappa} \in T_3$ we obtain using 2.7b) that

$$d(D(2)^{\kappa}) \geq \log \kappa.$$

Now this together with 5.5a) then implies

$$d(D(2)^{\kappa}) = \log \kappa. \qquad \dashv$$

7.10. Let us put

$$X = \Sigma_c(c^+),$$

where $c = 2^{\omega}$. It is easy to see that $d(X) = c^+$, but we claim that X can be written as the union of a chain

$$X = \cup\{X_{\alpha} : \alpha \in c^+\}$$

such that $d(X_{\alpha}) = \omega$ for each $\alpha \in c^+$. Indeed, we can just write for $\alpha \in c^+$

$$X_{\alpha} = \{f \in X : \forall \nu \in (c^+ \setminus \alpha)(f(\nu) = 0)\}$$

then clearly X_{α} is homeomorphic to $D(2)^{\alpha}$, hence by 7.9.

$$d(X_{\alpha}) = d(D(2)^{\alpha}) = \omega + \log|\alpha| = \omega. \qquad \dashv$$

7.11. For any κ let p be a uniform ultrafilter on κ with $\chi(p) = 2^{\kappa}$ and let X be the space on $\kappa \cup \{p\}$ for which every $\alpha \in \kappa$ is isolated and sets of the form

$$\{p\} \cup A$$

with $A \in p$ are the neighbourhoods of p. Clearly $X \in T_5$, $w(X) = \chi(X) = 2^\kappa$, but if we put for $\alpha \in \kappa$

$$X_\alpha = \alpha \cup \{p\}$$

then X is the union of the increasing chain $\{X_\alpha : \alpha \in \kappa\}$ while X_α is discrete, hence $\chi(X_\alpha) = 1$ and $w(X_\alpha) = |\alpha| < \kappa$.

7.12. Let κ be an uncountable cardinal and

$$\{h_\alpha : \alpha \in \kappa\}$$

be an enumeration of $H(\kappa)$. By an easy induction one can define then a sequence of sets $S_\alpha \in [\kappa]^\omega$ for $\alpha \in \kappa$ such that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$ and $S_\alpha \cap D(h_\alpha) = \emptyset$.
Let us now define the points $p_\alpha \in D(2)^\kappa$ by

$$p_\alpha(\nu) = \begin{cases} h_\alpha(\nu), & \text{if } \nu \in D(h_\alpha); \\ 1, & \text{if } \nu \in S_\alpha \\ 0, & \text{if } \nu \in \kappa \setminus (S_\alpha \cup D(h_\alpha)) \end{cases}$$

Then $p_\alpha \supset h_\alpha$ for all $\alpha \in \kappa$ implies that $X = \{p_\alpha : \alpha \in \kappa\}$ is dense in $D(2)^\kappa$, consequently by 2.6a) and 5.10 we have $c(X) = c(D(2)^\kappa) = \omega$. It is also clear that $X \subset \Sigma_\omega(\kappa)$, which easily implies $t(X) \leq \omega$. Finally, we claim that $\psi(X) = \omega$ holds as well. Clearly it suffices to show for this that, for any $\alpha \in \kappa$, if $\beta \neq \alpha$ then there is a $\nu \in S_\alpha$ with

$$f_\beta(\nu) = 0 \neq 1 = f_\alpha(\nu).$$

But this is trivial since $S_\alpha \subset \kappa \setminus S_\beta$ and outside S_β the function f_β takes up the value 1 in at most finitely many places. We note that if $\kappa = (2^\omega)^+$, with some extra care we could construct X with the additional property that it be the union of an increasing chain $\{X_\alpha : \alpha \in (2^\omega)^+\}$ with $d(X_\alpha) = \omega$ for each $\alpha \in (2^\omega)^+$. ⊣

7.13. Let κ be a singular cardinal and τ the topology on κ^+ consisting of the sets

$$\kappa^+ \setminus (\alpha \cup F),$$

where $\alpha \le \kappa$ and $F \in [\kappa^+]^{<\omega}$. Clearly τ is a T_1 topology. Put $X = \langle \kappa, \tau \rangle$, we claim that $\pi(X) = \kappa^+$ but $\pi(Y) < \kappa$ whenever $Y \subset X$ and $|Y| \le \kappa$. Indeed, let the order type of Y (as a set of ordinals) be $\lambda+n$, where λ is limit and $n \in \omega$. Now if F denotes the set of the n last members of Y and $Y' = Y \setminus F$, then $t_p(Y') = \lambda$ with $|\lambda| \le \kappa$, hence there is a cofinal subset Z of Y' with $|Z| = cf(\lambda) < \kappa$. But clearly then the family $\mathcal{B} = \{Y \setminus (\alpha \cup F) : \alpha \in Z\} \cup P(F)$ is a π-base of Y with $|\mathcal{B}| = |Z| < \kappa$. That $\pi(X) = \kappa^+$, and even $d(X) = \kappa^+$, on the other hand is obvious. This example shows that for κ singular we can not replace the weight by π-weight in 6.8.

NOTES

Chapter 1. The reader should be warned that the notation of cardinal functions in the literature is not "standardized", the Russian authors especially use a system of notations different from ours: they denote e.g. by s(X) the *density* of X and use $\phi\phi(X)$ or $\bar{\phi}(X)$ where we use $\phi^*(X)$. There are differences with the notations of [EN 1977] as well, where τ is used instead of t to denote the *tightness* and $h\phi(X)$ is used instead of our $\phi^*(X)$.

Chapter 2. 2.7(b) is due to B. Efimov [EF 1968].
2.13 was proved independently by Šapirovskiĭ [ŠA 1972] and Hajnal and Juhász [HJ 1973].
For 2.15 see [HJ 1967]; proofs using the "closure" method were given in [PO 1974] for (b) and in [HO 1976] for (a).
The second half of 2.20 was proved in [HJ 1967], the first half in [ŠA 1972].
2.27 was proved in [ŠA 1974] in an entirely different way. Archangelskiĭ's theorem first appeared in [AR 1969].
2.28 was proved in [GW 1977].
2.29 appeared in [CH 1977].
2.30 is due to [ST 1972].
2.31 and 2.33 were proved by [BH 1976].
2.36 was proved in [BGW 1978].
2.37 is from [ŠA 1974].
2.38 was first proved by van Douwen [vD 1978], but the simple proof given here is from [FR 1979].

Chapter 3. The material in 3.1 to 3.10 is based on [ŠA 1975]
3.11 is due to [MI 1962].
3.12 is from [AR 1971].
3.13 was proved in [ŠA 1974].
The results of 3.14 were proven, as is mentioned in the main text, by Šapirovskiĭ, using a different method, see [ŠA 1976].
3.16 was published in [ČP 1938].
The method of proof of 3.18 given here is due to Gerlits and Nagy.
The results concerning K(X) are mainly from [BH 1976], except 3.33, which is from [GW 1977].

Chapter 4. 4.1 is a "classical" result of Erdős and Tarski, [ET 1943].
4.2 is proved in [HJ 1969a].
4.3 is from [HJ 1969b].
4.4 is proved in [KR 1977].
4.7-4.9 are taken from [J 1977].

Chapter 5. 5.5a) is due to Hewitt [HE 1946] and Pondiczery [PN 1944].
5.6 was proved in [KU 1959] for I finite. The lemma there is folklore.
5.8 is taken from [HJ 1972].
5.9 is proved by [MA 1972].
5.10 is due to Šanin, [ŠN 1948].
5.13 and 14 were announced in [AT 1978].
5.16 has precursors in [IS 1964], [EN 1966] and [MI 1966]. The strong version that we present here is due to Gerlits.

Chapter 6. 6.7 is due to Szentmiklossy.
6.8 is to appear in [HJ 1980a]; the special case of $X \in \mathcal{T}_3$ is proved in [TK 1978].

Chapter 7. Example 7.4 is from [RO 1975].
Example 7.7 is due to Gerlits.
7.13 was noticed by van Douwen.

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List of Symbols

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