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ON THE ASYMPTOTIC ANALYSIS OF LARGE-SCALE OCEAN CIRCULATION

W.P.M. DE RUIJTER

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CHAPTER 1

INTRODUCTION

Over the past thirty years a considerable amount of publications has been devoted to the theory of the large scale ocean circulation. One can get an impression of the development of the theory by consulting for instance the review papers of STOMMEL (1957) and VERONIS (1973) or the book of STOMMEL (1965). The book edited by ROBINSON (1963) provides further a collection of pioneering studies on the wind-driven ocean circulation. Among the more general textbooks on ocean modelling we mention KRAUSS (1973) and KAMENKOVICH (1977). In all of these studies numerous references to the relevant existing literature can be found.

One of the very first papers that gave an explanation of an important feature of the general ocean circulation by a very simple mathematical model is due to STOMMEL (1948). In his linear model the input of wind energy at the surface of the ocean is dissipated by bottom friction which is assumed to be proportional to the velocity of the flow. The water is supposed to be homogeneous. For a squared ocean basin Stommel solved the resulting second order elliptic boundary value problem and then showed that the westward intensification of the stream line pattern is a consequence of the variation of the Coriolis force with latitude.

Munk, in 1950, assumed lateral friction to be the main dissipating mechanism. He showed the dominant influence of the curl of the wind stress on the general circulation. His model is also linear but the density field is inhomogeneous. A description of the density structure however was not necessary because the equations of motion were integrated between an assumed level of no stresses and the surface. This model has been the starting point of a number of theoretical investigations of the wind-driven ocean circulation (examples of which can be found in ROBINSON'S (1963) book).

Inertial effects, not included by Munk and Stommel, were brought in by several authors in an attempt to explain features of the western boundary currents that didn't follow from the linear transport theories (e.g.

CARRIER & ROBINSON, 1962, SPILLANE & NIILER, 1975). An example of such a phenomenon is the separation of the Gulf Stream from the coast at an "unexpected" latitude.

For the description of the detailed three dimensional circulation the knowledge of the density distribution is needed. This is described by a nonlinear diffusion equation and has only been succesfully tackled under severe restrictive assumptions concerning the density distributions (e.g. PEDLOSKY, 1969, RATTRAY & WELANDER, 1975).

Starting from the general equations for the motion of viscous fluid flow we will formulate in chapter 1 the basic equations and boundary conditions that describe the steady state circulation in the ocean. After scaling these equations it will be shown by a systematic analysis under what assumptions various models result. If the nonlinear inertial terms are neglected and the system is integrated from the bottom to the surface of the ocean the linear transport model results. A transport stream function ψ can then be defined for which a boundary value problem will be formulated. The differential equation for the transport stream function takes the form

$$(1.1) \qquad L_E\psi \equiv E\, L\psi + \frac{\partial\psi}{\partial x} = h(x,\phi)$$

where L is a fourth order elliptic differential operator in the two space variables x and ϕ. In the main part of this study the assumptions underlying the vorticity equation (1.1) will be supposed to hold. The mathematical model described by (1.1) is in essence Munk's model except that in our study full account is taken of the geometry of the globe, i.e. (1.1) is a differential equation on a manifold. Our objective is to study in detail the ocean circulation governed by (1.1) in various subdomains of the globe, and thus for various shapes of the coastal boundaries. A motivation for the study is to find out how much of the general features of large scale ocean circulation is reproduced in the relatively simple model (1.1). We shall find, in various problems, surprisingly strong agreement between our theory and observational data. A second motivation for our study is that the analysis of (1.1) in various subdomains of the globe leads to nontrivial mathematical problems.

The parameter E in equation (1.1) is the so called (lateral) Ekman number. For the oceanographic application $E = O(10^{-6})$. This makes the problems we deal with of singular perturbation type. To construct approximations of a solution of equation (1.1) with boundary conditions the method of matched asymptotic expansions will be applied. An extensive treatment of this method has recently been given by ECKHAUS (1979). Various methods to prove the asymptotic validity of approximations that have been constructed by the formal methods have been described there as well. In chapter 3 of our study a short description of the method of construction along the lines of Eckhaus will be given.

An important factor that influences the structure of the approximations is the geometry of the boundary of the domain $\mathcal{D}$ under consideration. A crucial role is played by the characteristics of the unperturbed part $\frac{\partial}{\partial x}$ of the operator L_E. It appears that the ocean domain can be divided in strips $\Omega = \{(x,\phi) \in \mathcal{D} | \phi_1 + \alpha \leq \phi \leq \phi_2 - \beta\}$ where α and β are arbitrary small positive numbers and $\phi = \phi_i$ $(i = 1,2)$ are characteristics that are tangent to or partly coincide with the boundary. To the east and west the subdomains Ω are bounded by smooth parts of the continental boundary $(\partial\mathcal{D})$ such that each characteristic intersects the eastern and western boundary once. For such regions formal asymptotic approximations of the solution of (1.1) with boundary conditions can be constructed by straightforward application of the method of matched asymptotic expansions. This will be carried out in chapter 4 where we will also sketch how the results of BESJES (1973) can be applied to prove the asymptotic validity of the approximations on the strips Ω. The simple model for a strip Ω already shows that the geometry of the eastern boundary is reflected throughout the ocean basin. This is not the case with the western boundary. Its variability causes only local changes of the stream line pattern.

If the eastern boundary is such that a characteristic $\phi = c$ touches that boundary the method of constructing a local approximation in a neighbourhood of such a characteristic is based on the concept of a free boundary layer that propagates westward into the interior of the domain. When a part of the eastern boundary coincides with a characteristic a lateral boundary layer along such a section exists that leaves the coast and develops into the interior as a free boundary layer. The differential equations describing those boundary layers are of diffusion type. This fact plays an important role in the methods of constructing asymptotic approximations. The chapters 5, 6 and 7 are predominantly devoted to the development of such methods.

In chapter 5 an asymptotic analysis is made of the wind-driven circulation around the Antarctic continent (the so-called Antarctic Circumpolar Current, see figure 1.1). Between Antarctica and the southern tip of South America the flow can encircle the globe unobstructed. The interaction with currents in the meridionally closed basins more northward takes place through a viscous free boundary layer along the characteristic that separates the two essentially differing regions. It turns out that a local analysis of the region near the southern tip of South America is necessary to determine the approximations in the free boundary layer and in the Antarctic region. The value of the total Antarctic Circumpolar Transport that comes out of the model calculations appears to be of the same order of magnitude as values that have been calculated from observational data.

The models in chapter 6 provide an analysis of the flow in the part of the world ocean basin that contains the South African peninsula. First a (two dimensional) model in which the African continent is represented by a straight line will be developed. As a result a simple explanation can be given of the turning of the Agulhas Current south of Africa as related to the positions of the extrema and the zeros of the wind stress curl with respect to the southern tip of the African continent. An explanation of the relatively low transport of the Brasil Current (see fig. 1.1) is obtained as well.

If a more realistic curved geometry of the South African continent is used this turns out not to alter the first approximation of the free boundary layer to the west of the southern tip. It is reflected in the appearance of an extra boundary layer near that southern tip and a higher order of magnitude of the second approximation in the free boundary layer.

If the geometry of the continent is such that the southern coast coincides with a characteristic $\phi = c$ the first approximation in the free boundary layer is different from that in the line shaped case. This and related problems will be analysed in chapter 7. For the oceanographic application the model with a corner shaped eastern boundary leads to interesting results. We find the formation of intense eastern boundary currents that leave the coast and propagate into the interior. Such eastern boundary currents are for instance the Guinee Current and the Flinders Current which originates in the corner formed by the South Australian and West Tasmanian coasts.

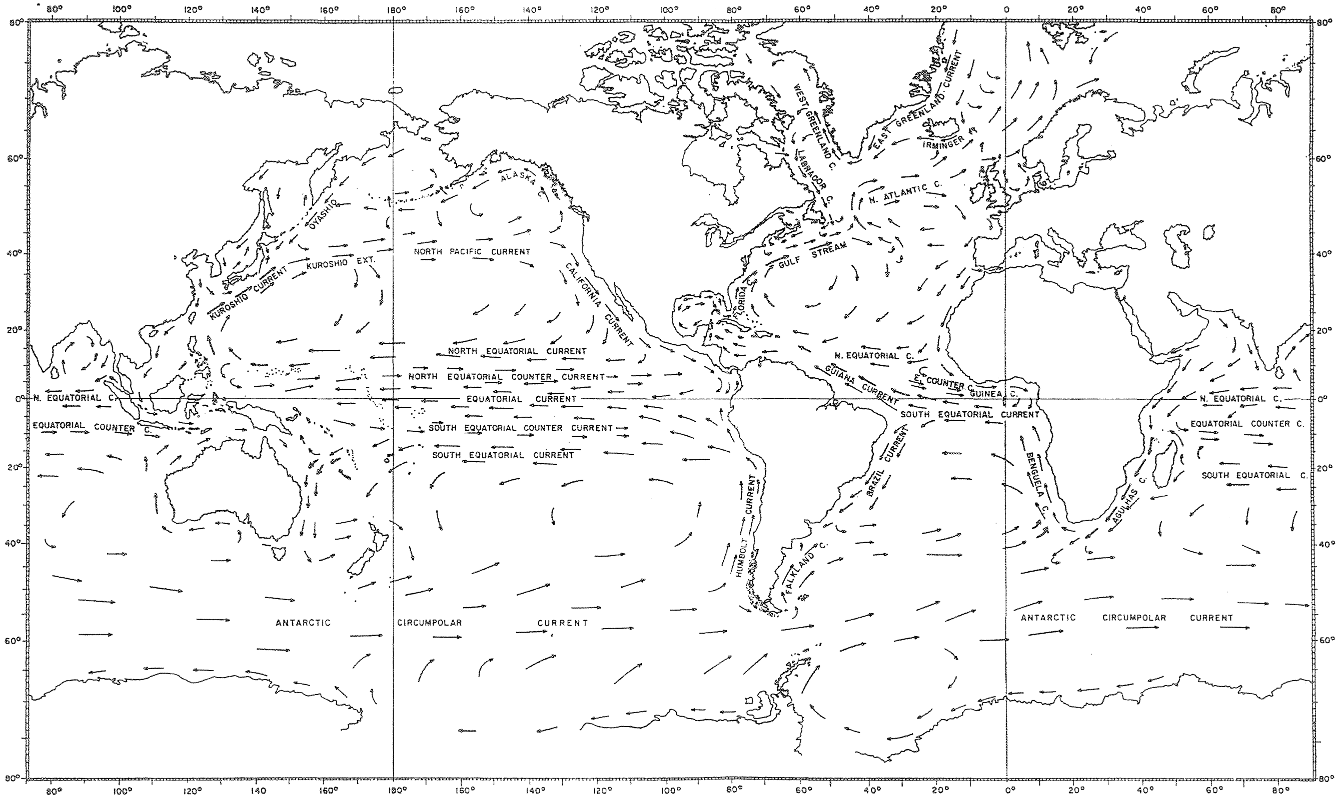

fig. 1.1. Major features of the surface circulation of the oceans.
(After Mc Lellan, 1965).

CHAPTER 2

MODELS OF THE LARGE-SCALE OCEAN CIRCULATION

After formulating the basic equations that describe the in general turbulent motion in the ocean these shall be averaged. It leads to a set of equations for the mean part of the flow. The influence of the turbulence on the mean flow is represented by the "Reynolds stress" tensor which takes a simple form based on the introduction of so called eddy viscosity coefficients. The equations and boundary conditions can then be scaled in order to obtain an impression of the relative importance of the different mechanisms that govern the motion. In a systematic way (among other things by a small parameter analysis) various models shall be derived from the full non-dimensional system. From this analysis it will become clear which are the basic assumptions underlying different models. Vertical integration (between the bottom and the surface of the ocean) of the equations of motion and neglection of the inertial terms leads to a linear transport model. For the so called transport stream function ψ a singularly perturbed fourth order elliptic boundary value problem results. The main part of this study will then be devoted to the asymptotic analysis of such a problem for different geometries of the ocean domain.

2.1. The basic equations

We start from the general equations describing the motion of viscous fluid flow. The conservation law of mass is expressed by the equation of continuity:

$$(2.1) \qquad \frac{\partial \rho}{\partial t} + \vec{v}.\nabla\rho + \rho\nabla.\vec{v} = 0.$$

Here, $\vec{v}$ is the fluid velocity, ρ its density and ∇ is the gradient operator.

In the case of constant density for every volume element ($\frac{d\rho}{dt} \equiv 0$) (2.1) reduces to the simple equation:

(2.2) $\nabla.\vec{v} = 0$

In most cases (2.2) (often referred to as the incompressibility condition) is a good approximation for seawater.

The conservation law of momentum leads to the so called momentum equation. For an incompressible, viscous fluid and referred to a coordinate system that is fixed to the earth (so it rotates with the angular velocity of the earth) it reads:

(2.3) $$\rho(\frac{\partial\vec{v}}{\partial t} + \vec{v}.\nabla\vec{v}) + 2\rho\vec{\Omega}\times\vec{v} = -\nabla p + \rho\vec{g} + \mu\Delta\vec{v}$$

A systematic derivation of this equation, which in fact is a version of the Navier-Stokes equation, can be found for instance in KRAUSS (1973). The first term on the lefthand side represents the mass acceleration, the second one the Coriolis force. This is an apparent force which appears in the equation because the coordinate system is rotating with the angular velocity $\vec{\Omega}$ of the earth. On the righthand side ∇p represents the pressure gradient and $\rho\vec{g} = (0,0,-\rho g)$ is the actual force of gravity (that is the gravitational acceleration with a small modification from the centrifugal force). Viscous effects enter the equation through the term $\mu\Delta\vec{v}$, where μ is the molecular viscosity coefficient and Δ the Laplace operator.

The density of seawater depends on the other thermodynamic variables. Therefore, a functional relationship between these quantities has to be added to the equations of motion. In general the pressure p, the temperature T and the salinity S (the mass of dissolved solids per unit mass of seawater) can be chosen to describe the state of seawater. This leads to the equation of state:

(2.4) $$\rho = \rho(p,T,S).$$

Only empirical approximations of this function are known (e.g. FOFONOFF, 1962; WILSON & BRADLEY, 1968). An approximation which is often used in oceanography is the linear equation of state:

(2.4') $$\rho = \rho_0\{1 - a(T-T_0) - b(S-S_0)\}.$$

a and b are constants and T_0 and S_0 are reference values. If T^* is defined as $T^* = T + \frac{b}{a}.S$ (the "apparent" temperature) (2.4') reads:

(2.4") $\rho = \rho_0\{1 - a(T^* - T_0^*)\}.$

Finally, we add the equation of diffusion for the salinity S:

(2.5) $$\frac{\partial S}{\partial t} + \vec{v},\nabla S = \nu_S \Delta S$$

and the equation of heat conduction:

(2.6) $$\frac{\partial T}{\partial t} + \vec{v},\nabla T = \nu_T \Delta T.$$

Here ν_S and ν_T are the diffusion coefficients.

Together the equations (2.1),(2.3),(2.4),(2.5) and (2.6) provide a closed system for the unknown variables $\vec{v}$, ρ, p, T and S.

2.2. Averaging of the equations of motion for fluctuating fields

In general the motion in the ocean is turbulent. The velocity field can be represented in the form

(2.7) $$\vec{v} = \vec{u} + \vec{v}'$$

where $\vec{u}$ is the mean part and $\vec{v}'$ is the fluctuating part of the velocity. The average is taken over the set of possible realisations of a motion under consideration (under macroscopically identical conditions). Substituting the representation (2.7) into the equations of motion and taking the mean of it leads, with the neglect of molecular viscosity, to the following set of equations:

(2.8) $$\nabla,\vec{u} = 0$$

(2.9) $$\frac{\partial \vec{u}}{\partial t} + \vec{u},\nabla\vec{u} + 2\vec{\Omega}\times\vec{u} = -\frac{1}{\rho}\nabla p + \vec{g} + \vec{F},$$

where in tensor notation,

$$F_i = -\frac{1}{\rho}\frac{\partial}{\partial x_j}\rho\,\overline{v_i' v_j'} \equiv \frac{1}{\rho}\frac{\partial}{\partial x_j}\tau_{ij}$$

(the overbar indicates averaging).

The symmetric tensor $\tau_{ij} = -\rho\overline{v_i' v_j'}$ is called the "Reynolds stress" tensor after O. REYNOLDS (1895), who first introduced the above procedure. It thus describes the influence of the turbulence on the mean flow.

The functional form of the Reynolds stress tensor is not known. To get around this difficulty in many theoretic oceanographical studies the components of τ_{ij} are assumed to be linear expressions in the first derivatives of the components of the mean velocity $\vec{u}$. The proportionality coefficients are the so called eddy viscosity coefficients. Under the assumption that the resulting tensor of eddy coefficients is axisymmetric about the vertical direction rather simple expressions for the Reynolds stress have been derived by KAMENKOVICH (1967). The expressions contain three coefficients of turbulent viscosity, A_H, A_V and A. A_H and A_V are the so called horizontal and vertical eddy viscosity coefficients.

In an analogous way the equations of heat conduction and salinity diffusion can be averaged. In this case two coefficients of turbulent diffusion appear, describing the different mixing in the vertical from that in the horizontal direction.

2.3. The equations of motion in spherical coordinates

Because in this study we are especially interested in the large scale characteristics of the ocean circulation we will formulate the equations of motion in spherical coordinates. Moreover we will only study stationary flow: $\vec{u} = \vec{u}(\lambda,\phi,r)$, where λ is the longitude $(0 \le \lambda < 2\pi)$, positive in eastward direction, ϕ the latitude $(-\frac{\pi}{2} \le \phi \le \frac{\pi}{2})$ and r the distance to the center of the earth $(0 \le r \le a)$.

With $\nabla = (\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}, \frac{1}{r}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial r})$ the momentum equation becomes:

$$(2.10) \qquad \vec{u}.\nabla u - \frac{uv}{r}\,\mathrm{tg}\phi + \frac{uw}{r} + 2\Omega w\cos\phi - 2\Omega v\sin\phi = -\frac{1}{\rho r\cos\phi}\frac{\partial p}{\partial\lambda} + F^{(\lambda)}$$

$$(2.11) \qquad \vec{u}.\nabla v + \frac{u^2}{r}\,\mathrm{tg}\phi + \frac{vw}{r} + 2\Omega u\sin\phi = -\frac{1}{\rho r}\frac{\partial p}{\partial\phi} + F^{(\phi)}$$

$$(2.12) \qquad \vec{u}.\nabla w - \frac{u^2+v^2}{r} - 2\Omega u\cos\phi = -\frac{1}{\rho}\frac{\partial p}{\partial r} - g + F^{(r)}.$$

For the equation of continuity we obtain:

$$(2.13) \qquad \frac{1}{r\cos\phi}\{\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi) + 2w\cos\phi\} + \frac{\partial w}{\partial r} = 0,$$

and the equations of salt diffusion and heat conduction read:

(2.14) $\vec{u},\nabla S = \sigma$

(2.15) $\vec{u},\nabla T = Q$

In these equations u,v,w are the velocity components along λ,ϕ and r, respectively. $F^{(\lambda)}$, $F^{(\phi)}$ and $F^{(r)}$ represent the Reynolds stress terms, σ and Q the effects of turbulent diffusion.

2.4. The boundary conditions

For steady state circulation the main driving forces act upon the surface of the sea. Very important is the wind stress, leading to the boundary condition:

(2.16) $A_V(\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}) = (\tau^\lambda, \tau^\phi)$ in $r = a$.

Here τ^λ, τ^ϕ are the eastward and northward components of the wind stress $\vec{\tau}$ respectively.

The ocean surface is assumed to be level and it is given by $r = a$. In consequence at the upper boundary we impose:

(2.17) $w = 0$

Further there is the differential heating at the surface of the ocean, also leading to differences between precipitation and evaporation. In the boundary conditions this is reflected in either a given flux of heat and salinity or given temperature and salinity distributions themselves.

The lateral boundaries and the bottom of the ocean basin can in most cases be considered to be insulated:

(2.18) $\frac{\partial T}{\partial n} = \frac{\partial S}{\partial n} = 0$ at the bottom and coasts.

Here n denotes the direction normal to the boundary.

For the velocity at rigid boundaries the so called condition of no slip holds:

(2.19) $u = v = w = 0.$

2.5. The basic equations in nondimensional form

To get an impression of the relative importance of the different mechanisms that govern the steady state ocean circulation it is useful to put the basic equations and the boundary conditions in nondimensional form. For that purpose characteristic values of the appearing dependent and independent variables have to be defined. For instance the characteristic length scale L of the motion has to be of the order of the width of the ocean basin if it is the large scale circulation that we intend to analyse.

The nondimensional variables (marked by primes) are introduced as follows:

$$r = a + Hr'; \quad (u,v) = V(u',v'); \quad w = Ww';$$

$$\rho = \rho_0(1 + \frac{L\Omega V}{Hg}\rho'); \quad p = \rho_0 L\Omega V p' - \rho_0 gHr'; \tag{2.20}$$

$$\vec{\tau} = \tau_m \vec{\tau}'; \quad T = T_0 T'; \quad S = S_0 S'.$$

(p' is called the reduced pressure).
If we substitute these in the equations and drop the primes there results:

$$R\{\frac{u}{(1+\varepsilon r)\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{v}{1+\varepsilon r}\frac{\partial u}{\partial\phi} + \frac{\delta}{\varepsilon} w \frac{\partial u}{\partial r} - \frac{uv}{1+\varepsilon r}\mathrm{tg}\phi + \frac{\delta}{1+\varepsilon r} uw\} + \delta 2w\cos\phi - 2v\sin\phi = - \frac{\mu}{(1+\varepsilon r)(1+\alpha\rho)\cos\phi}\frac{\partial p}{\partial\lambda} + F^{(\lambda)} \tag{2.21}$$

$$R\{\frac{u}{(1+\varepsilon r)\cos\phi}\frac{\partial v}{\partial\lambda} + \frac{v}{1+\varepsilon r}\frac{\partial v}{\partial\phi} + \frac{\delta}{\varepsilon} w \frac{\partial v}{\partial r} + \frac{u^2}{1+\varepsilon r}\mathrm{tg}\phi + \frac{\delta}{1+\varepsilon r} vw\} + 2u\sin\phi = - \frac{\mu}{(1+\varepsilon r)(1+\alpha\rho)}\frac{\partial p}{\partial\phi} + F^{(\phi)} \tag{2.22}$$

$$\varepsilon R\{\delta(\frac{u}{(1+\varepsilon r)\cos\phi}\frac{\partial w}{\partial\lambda} + \frac{v}{1+\varepsilon r}\frac{\partial w}{\partial\phi} + \frac{\delta}{\varepsilon} w \frac{\partial w}{\partial r}) - \frac{u^2+v^2}{1+\varepsilon r}\} - \varepsilon 2u\cos\phi = - \frac{1}{1+\alpha\rho}(\frac{\partial p}{\partial r} + \rho) + \varepsilon F^{(r)} \tag{2.23}$$

$$\frac{1}{(1+\varepsilon r)\cos\phi}\{\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi) + \delta 2w\cos\phi\} + \frac{\delta}{\varepsilon\mu}\frac{\partial w}{\partial r} = 0 \tag{2.24}$$

$$\frac{u}{(1+\varepsilon r)\cos\phi}\frac{\partial S}{\partial\lambda} + \frac{v}{1+\varepsilon r}\frac{\partial S}{\partial\phi} + \frac{\delta}{\varepsilon\mu} w \frac{\partial S}{\partial r} = \sigma \tag{2.25}$$

$$\frac{u}{(1+\varepsilon r)\cos\phi}\frac{\partial T}{\partial\lambda} + \frac{v}{1+\varepsilon r}\frac{\partial T}{\partial\phi} + \frac{\delta}{\varepsilon\mu} w \frac{\partial T}{\partial r} = Q. \tag{2.26}$$

The parameters that appear in these equations are: the Rossby number

$$R = \frac{V}{\Omega a}\,, \tag{2.27}$$

which is a measure for the ratio between the inertial and Coriolis forces;

$$\alpha = \frac{L\Omega V}{Hg}\,, \tag{2.28}$$

and the aspect ratios:

$$\mu = \frac{L}{a}; \quad \varepsilon = \frac{H}{L}; \quad \delta = \frac{W}{V} \tag{2.29}$$

(here, H is a characteristic length scale of the vertical motion).

To determine the numerical values of these parameters the characteristic scale factors must be known. In modelling the large scale ocean circulation the orders of magnitude of the relevant scales are (see for instance KRAUSS, 1973):

$$\begin{aligned} &H = 5\times10^{3}\text{m.} \quad \text{(the average depth of the ocean);}\\ &L = a = 6.4\times10^{6}\text{m.}; \quad V = 10^{-1}\text{m sec}^{-1};\\ &\tau_m = 7.8\times10^{-2}\,\text{kg m}^{-1}\text{sec}^{-2}; \quad \rho_0 = 10^{3}\text{kg m}^{-3}\\ &\Omega = 10^{-4}\text{sec}^{-1}; \quad g = 9.8\text{m sec}^{-2}; \quad W = 7.9\times10^{-5}\text{m sec}^{-1};\\ &A_V = 6.1\,\text{kg m}^{-1}\text{sec}^{-1}; \quad A_H = 10^{7}\text{kg m}^{-1}\text{sec}^{-1}. \end{aligned} \tag{2.30}$$

This leads to the values:

$$R = 2.1\times10^{-4}; \quad \delta = \varepsilon = 7.9\times10^{-4};$$

$$\alpha = 1.3\times10^{-3}; \quad \mu = 1 \quad \text{(in general: } \mu \le 1\text{)}.$$

The non-dimensional version of the boundary condition (2.16) reads:

$$\sqrt{E_V}\,\left(\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}\right) = (\tau^{\lambda}, \tau^{\phi}) \quad \text{in } r = 0, \tag{2.31}$$

where E_V is the vertical Ekman number, defined by:

(2.32) $$E_V = \frac{A_V}{\rho_0 \Omega H^2} \qquad (\text{so } V = \frac{\tau_m}{(\rho_0 \Omega A_V)^{\frac{1}{2}}}).$$

It is a measure for the ratio between the frictional and the Coriolis forces. Substitution of the above given scale factors gives:

$$E_V = 2.44 \times 10^{-6}.$$

2.6. Reduction of the basic equations to simpler form

The small value of ε expresses the relative shallowness of the ocean. We will exploit this fact by approximating in the system (2.21) through (2.26) terms of the form $\frac{1}{1+\varepsilon r} f(\lambda,\phi,r)$ by $f(\lambda,\phi,r)$.

If terms $\frac{1}{1+\alpha\rho} f(\lambda,\phi,r)$ are approximated by $f(\lambda,\phi,r)$ the so called Boussinesq approximation results. Employing both approximations and neglecting $O(\varepsilon)$ and $O(\delta)$ terms with respect to $O(1)$ terms of the same kind the following system of equations results:

(2.33) $$R\{\vec{u},\nabla u - uvtg\phi\} - 2v\sin\phi = -\frac{\mu}{\cos\phi}\frac{\partial p}{\partial\lambda} + E_V \frac{\partial^2 u}{\partial r^2} + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial u}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 u}{\partial\lambda^2} + \frac{\cos2\phi}{\cos^2\phi}u - \frac{2\sin\phi}{\cos^2\phi}\frac{\partial v}{\partial\lambda}\}$$

(2.34) $$R\{\vec{u},\nabla v + u^2 tg\phi\} + 2u\sin\phi = -\mu\frac{\partial p}{\partial\phi} + E_V \frac{\partial^2 v}{\partial r^2} + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial v}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 v}{\partial\lambda^2} + \frac{\cos2\phi}{\cos^2\phi}v + \frac{2\sin\phi}{\cos^2\phi}\frac{\partial u}{\partial\lambda}\}$$

(2.35) $$0 = \frac{\partial p}{\partial r} + \rho$$

(2.36) $$\frac{1}{\cos\phi}(\frac{\partial u}{\partial\lambda} + \frac{\partial}{\partial\phi}(v\cos\phi)) + \frac{1}{\mu}\frac{\partial w}{\partial r} = 0$$

(2.37) $$\vec{u},\nabla S = \gamma_V^S \frac{\partial^2 S}{\partial r^2} + \gamma_H^S\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial S}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 S}{\partial\lambda^2}\}$$

(2.38) $$\vec{u},\nabla T = \gamma_V^T \frac{\partial^2 T}{\partial r^2} + \gamma_H^T\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial T}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 T}{\partial\lambda^2}\}$$

where now $\nabla = (\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}, \frac{\partial}{\partial\phi}, \frac{1}{\mu}\frac{\partial}{\partial r})$.

In the frictional terms the horizontal Ekman number E appears. It is defined by

(2.39) $$E = \frac{A_H}{\rho_0 \Omega a^2} .$$

With the scale factors as given in (2.30) the numerical value of E is the same as that of E_V: $E = 2.44 \times 10^{-6}$. In the sequel the two Ekman numbers will be set equal: $E_V = E$.

The parameters that appear in the diffusion terms of equations (2.37) and (2.38) are given by:

$$\gamma_V^S = \frac{\nu_V^S a}{VH^2} ; \quad \gamma_H^S = \frac{\nu_H^S}{Va} ; \quad \gamma_V^T = \frac{\nu_V^T a}{VH^2} ; \quad \gamma_H^T = \frac{\nu_H^T}{Va} ,$$

where the ν_V and ν_H are the vertical and horizontal eddy diffusion coefficients, respectively.

If the diffusion coefficients and the equation of state (e.g. 2.4") are such that it can be combined with the equations of heat conduction and salt diffusion to give one equation of diffusion for the density we get:

(2.40) $$\vec{u},\nabla\rho = \gamma_V \frac{\partial^2\rho}{\partial r^2} + \gamma_H\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,\frac{\partial\rho}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2\rho}{\partial\lambda^2}\},$$

with

$$\gamma_V = \frac{\nu_V a}{VH^2} ; \quad \gamma_H = \frac{\nu_H}{Va} .$$

The vertical component of the momentum equation (2.35) represents the well known hydrostatic approximation. It is applicable in almost all oceanic cases and we will use it throughout this study.

2.7. The f-plane and β-plane approximations

In the preceeding section the basic equations have been formulated in spherical coordinates, which in most cases makes them difficult to analyse. Depending on the scale of the motion various approximations of the equations can be deduced.

A measure for the scale is the parameter μ, the ratio between the characteristic length of the motion (L) and the earth radius (a).

If the scale of the motion considered is small with respect to the earth radius, i.e. if $\mu \ll 1$, this can be exploited in the equations by performing the following transformation:

$$\mu x = \cos\phi_0(\lambda-\lambda_0) \tag{2.41}$$

$$\mu y = \phi-\phi_0$$

$$z = r.$$

Here ϕ_0 and λ_0 are reference latitude and longitude. Moreover we expand the metric factors in the equations in Taylor series around $\phi = \phi_0$. Performing these operations the equations (2.33) through (2.36) together with (2.40) take the following form:

$$R^*\{(\vec{u},\nabla u) - \mu uv(tg\phi_0 + \mu y \frac{1}{\cos^2\phi_0} + \ldots)\} - \tag{2.42}$$
$$- (2\sin\phi_0 + 2\mu y\cos\phi_0 + \ldots)v = - (1 + \mu y tg\phi_0 + \ldots) \frac{\partial p}{\partial x} + E_V \frac{\partial^2 u}{\partial z^2} +$$
$$+ E^*\{(1 + 2\mu y tg\phi_0 + \ldots) \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \mu(tg\phi_0 + \frac{\mu y}{\cos^2\phi_0} + \ldots) \frac{\partial u}{\partial y} +$$
$$+ \mu^2(\frac{\cos 2\phi_0}{\cos^2\phi_0} + \ldots)u - \mu(2tg\phi_0 + \ldots) \frac{\partial v}{\partial x}\}$$

$$R^*\{(\vec{u},\nabla v) + \mu u^2(tg\phi_0 + \ldots)\} + (2\sin\phi_0 + 2\mu y\cos\phi_0 + \ldots)u = \tag{2.43}$$
$$= - \frac{\partial p}{\partial y} + E_V \frac{\partial^2 v}{\partial z^2} + E^*\{(1 + 2\mu y tg\phi_0 + \ldots) \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} -$$
$$- \mu(tg\phi_0 + \ldots) \frac{\partial v}{\partial y} + \mu^2(\frac{\cos 2\phi_0}{\cos^2\phi_0} + \ldots)v + \mu(2tg\phi_0 + \ldots) \frac{\partial u}{\partial x}\}$$

$$0 = \frac{\partial p}{\partial z} + \rho \tag{2.44}$$

$$(1 + \mu y tg\phi_0 + \ldots) \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - \mu(tg\phi_0 + \ldots)v = 0 \tag{2.45}$$

$$\vec{u},\nabla\rho = \gamma_V^* \frac{\partial^2\rho}{\partial z^2} + \gamma_H^*\{(1 + 2\mu y tg\phi_0 + \ldots) \frac{\partial^2\rho}{\partial x^2} + \tag{2.46}$$
$$+ \frac{\partial^2\rho}{\partial y^2} - \mu(tg\phi_0 + \ldots) \frac{\partial\rho}{\partial y}\}$$

where now $\nabla = ((1 + \mu y tg\phi_0 + \ldots) \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$,

$$R^* = \frac{R}{\mu} = \frac{V}{\Omega L}\ ; \quad E^* = \frac{E}{\mu^2} = \frac{A_H}{\rho_0\Omega L^2}\ ;$$

$$\gamma_H^* = \mu\gamma_V = \frac{\nu_V L}{VH^2}\ ; \ \gamma_H^* = \frac{\gamma_H}{\mu} = \frac{\nu_H}{VL}\ .$$

Thus, R^*, E^*, γ_V^* and γ_H^* are the Rossby and Ekman numbers and the eddy diffusion coefficients based on the horizontal scale of the motion. In the sequel we will omit the stars.

If in these equations μ is set equal to zero the so called f-plane approximation emerges:

$$(2.47) \qquad R.\vec{u},\nabla u - fv = -\frac{\partial p}{\partial x} + E_V \frac{\partial^2 u}{\partial z^2} + E\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$(2.48) \qquad R.\vec{u},\nabla v + fu = -\frac{\partial p}{\partial y} + E_V \frac{\partial^2 v}{\partial z^2} + E\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$(2.49) \qquad 0 = \frac{\partial p}{\partial z} + \rho$$

$$(2.50) \qquad \nabla,\vec{u} = 0$$

$$(2.51) \qquad \vec{u},\nabla\rho = \gamma_V \frac{\partial^2 \rho}{\partial z^2} + \gamma_H\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right)$$

where $f = 2 \sin\phi_0$ is the Coriolis parameter and $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$. Formally this system is equal to that of flow in a plane layer of fluid, rotating with constant angular velocity. It is for instance used to study the local influence on the flow of the continental slopes that engirdle the ocean basin (e.g. PEDLOSKY, 1974 a,b; KILLWORTH, 1973).

The approximation of the Coriolis parameter by f is based on the hypothesis that

$$2\mu y \cos \phi_0 \ll 2 \sin \phi_0.$$

Consequently, this approximation cannot be used to study processes near the equator. In that case ($\phi_0 = 0$) the leading term will be $2\mu y$, which brings back in the momentum equation the variability of the Coriolis parameter. Moreover, the ratio between the nonlinear acceleration terms and the Coriolis force is not small any more. The same holds for the ratio between viscous and Coriolis terms. For instance if L = 64 km (and the other scale factors as in (2.30)) we find $\mu = 10^{-2}$, $\frac{R}{\mu}$ $(= \frac{V_a}{\Omega L^2}) = 2.1$ and $\frac{E}{\mu} = 2.4$. These values indicate that in that case the nonlinear and viscous terms cannot be neglected with respect to the Coriolis acceleration. A review of the interesting problem of modelling of equatorial currents has been given by MOORE & PHILANDER (1977).

In many studies of motion on "intermediate" scale ($\mu = O(10^{-1})$) the system (2.42) through (2.46) is reduced to describe the so called β-plane approximation. In this approximation the variability of the Coriolis parameter with latitude is retained while the earth curvature is neglected. This is accomplished by neglecting all $O(\mu)$-terms except for the second term in the Taylor expansion of the Coriolis parameter. The resulting equations are of the same form as (2.47)-(2.51). However, in this case the Coriolis parameter is defined by:

$$f = 2\sin\phi_0 + \beta y \tag{2.52}$$

where $\beta = 2\mu\cos\phi_0$.

A lot of studies of the wind-driven ocean circulation are based on the β-plane concept (e.g. MUNK, 1950; CARRIER and ROBINSON, 1962; PEDLOSKY, 1968; SPILLANE and NIILER, 1975); the motion is then considered to be of intermediate scale. However, if we consider for instance the horizontal extent of the path of the Gulf Stream, this leads to an $O(1)$ value of μ. This means that in this approximation $O(1)$ terms are neglected with respect to other $O(1)$ terms and conclusions drawn from such model calculations are difficult to extend to the real oceanic case.

To study the large scale ocean circulation the complete system (2.33)-(2.40) has to be retained, with $\mu = 1$. In fact, most parts of the wind-driven ocean circulation belong to this category. Another example of world wide oceanic flow is the thermohaline circulation, mainly caused by the differential heating at the surface.

2.8. The geostrophic approximation

In regions of the ocean with low accelerations and away from frictional influences an approximation of the momentum equation is obtained by setting $E_V = E = R = 0$. With $\mu = 1$ (and without employing the Boussinesq approximation) we then get:

$$2v\sin\phi = \frac{1}{(1+\alpha\rho)\cos\phi}\frac{\partial p}{\partial\lambda} \tag{2.53}$$

$$2u\sin\phi = -\frac{1}{1+\alpha\rho}\frac{\partial p}{\partial\phi} \tag{2.54}$$

$$0 = \frac{\partial p}{\partial r} + \rho. \tag{2.55}$$

Together with the continuity and diffusion equations the system describes the motion of geostrophic hydrostatic flow.

The rate of change of the horizontal velocity components with depth can easily be obtained in terms of the density gradient:

$$(2.56) \qquad \frac{\partial v}{\partial r} = - \frac{1}{(1+\alpha\rho).\sin 2\phi} \left(\frac{\alpha}{1+\alpha\rho} \frac{\partial \rho}{\partial r} \frac{\partial p}{\partial \lambda} + \frac{\partial \rho}{\partial \lambda}\right).$$

Now, the density field is called barotropic if the equidensity surfaces coincide with the isobaric surfaces. In non dimensional terms, when p denotes the reduced pressure, this means

$$\rho = \rho(\alpha p - r).$$

If the variable s is defined by $s \equiv \alpha p - r$, we obtain

$$(2.57) \qquad \frac{\partial \rho}{\partial r} = \frac{d\rho}{ds} . (\alpha \frac{\partial p}{\partial r} - 1) = - (1 + \alpha\rho) \frac{d\rho}{ds}$$

$$(2.58) \qquad \frac{\partial \rho}{\partial \lambda} = \alpha \frac{d\rho}{ds} \frac{\partial p}{\partial \lambda} .$$

Substitution of (2.57) and (2.58) in (2.56) yields:

$$(2.59) \qquad \frac{\partial v}{\partial r} = 0$$

and analogously $\frac{\partial u}{\partial r} = 0$. Consequently, if the density field is barotropic, the horizontal velocity components of the geostrophic flow are uniform with depth. A trivial but frequently used example of a barotropic density field is obtained if the model ocean is assumed to be homogeneous.

In the case the equidensity and the isobaric surfaces intersect the density field is called baroclinic. Because $\alpha \ll 1$ it follows from (2.56) that the rate of change with depth of the horizontal velocity components is mainly caused by the horizontal density differences.

For the geostrophic approximation to hold the inertial and frictional forces have to be negligible. In almost all cases in the interior of the ocean this can be assumed. Near the bottom and lateral boundaries and at the surface the viscous and acceleration effects will become more important and have to be retained in the equations.

From (2.53) and (2.54) it is immediately clear that in general it will be impossible to apply the boundary conditions to solutions of the geostrophic equations. For that boundary layers have to be analysed. Well known examples are formed by the so called Ekman layers at the surface and the bottom of the oceans.

In the open ocean there can also occur narrow regions with important frictional or inertial effects. An example has already been mentioned earlier in this chapter: near the equator the Coriolis parameter approaches zero and the acceleration and lateral friction terms cannot be neglected to describe the interior flow. In a three dimensional study GILL (1971) has shown the existence of an equatorial boundary layer with an important undercurrent. Observations of a subsurface current with high velocities, the so called equatorial undercurrent, are an empirical confirmation of this theoretical fact (e.g. BUBNOV and YEGORIKHIN, 1978).

Other examples of free boundary layers in the ocean will be analysed in the following chapters.

2.9. The thermohaline circulation

The part of the ocean circulation that is driven by the differences in heating between equatorial and polar regions is called the thermohaline circulation. The influence of particularly the horizontal density gradient on the structure of the flow has already been pointed out in the preceeding section. So far, solutions of the full nonlinear system (2.33) through (2.36) with (2.40) at which both the wind stress and for instance the density distribution at the surface can be imposed haven't been found. Several approximations of the equations have been derived, based on special assumptions concerning the density field. A few will be mentioned here.

If in (2.40) the diffusion is neglected ($\gamma_V=\gamma_H=0$) a purely advective model results. WELANDER (1971) used this assumption, together with the geostrophic hydrostatic approximation of the momentum equation. As a consequence he could only impose the surface density distribution and no wind stress.

JOHNSON (1971) assumes the vertical diffusion of heat to dominate the horizontal diffusion, $\gamma_H = 0$. The momentum equation is linearised $(R = 0)$ and also lateral friction is neglected $(E = 0)$. The vertical diffusion is confined to an $0(\gamma_V^{\frac{1}{3}})$ thermocline layer, with an Ekman sub-boundary layer at the surface. However, of the resulting nonlinear diffusion equation for the thermocline, only similarity solutions are known. If these are used, it is again impossible to impose both the wind and temperature boundary conditions at the surface.

A linearisation of (2.40) has been proposed by BARCILON and PEDLOSKY (1967 a,b) and applied to the oceanic case by PEDLOSKY (1969). They assume the density to vary linearly with depth with a perturbation around this linear state:

$$\rho(\lambda,\phi,r) = \frac{\Delta\rho_V}{\rho_0 \cdot \alpha} \cdot r + \frac{\Delta\rho_H}{\rho_0 \cdot \alpha} \cdot \rho^*(\lambda,\phi,r),$$

where $\Delta\rho_V$ and $\Delta\rho_H$ are characteristic values of the vertical and horizontal density differences in the ocean respectively. If p^* is defined by

$$p^* = p + \frac{\Delta\rho_V}{\rho_0 \alpha} \cdot \frac{1}{2} r^2,$$

the hydrostatic equation (2.35) takes its original form:

$$\frac{\partial p^*}{\partial r} = - \frac{\Delta\rho_H}{\rho_0 \alpha} \rho^*.$$

The diffusion equation becomes:

$$R\{\vec{u}, \nabla\rho^* + \frac{\Delta\rho_V}{\Delta\rho_H} \cdot w\} = \frac{\nu_V}{\Omega D^2} \frac{\partial^2 \rho^*}{\partial r^2} + \frac{\nu_H}{\Omega a^2} \{\frac{1}{\cos\phi} \frac{\partial}{\partial\phi}(\cos\phi \frac{\partial\rho^*}{\partial\phi}) + \frac{1}{\cos^2\phi} \frac{\partial^2\rho^*}{\partial\lambda^2}\}.$$

Under the assumption $\frac{\Delta\rho_V}{\Delta\rho_H} \gg 1$, which indicates a strongly stratified ocean, this leads to the linearised equation of diffusion:

$$R \frac{\Delta\rho_V}{\Delta\rho_H} \cdot w = \frac{\nu_V}{\Omega D^2} \frac{\partial^2\rho^*}{\partial r^2} + \frac{\nu_H}{\Omega a^2} \{\frac{1}{\cos\phi} \frac{\partial}{\partial\phi}(\cos \frac{\partial\rho^*}{\partial\phi}) + \frac{1}{\cos^2\phi} \frac{\partial^2\rho^*}{\partial\lambda^2}\} \qquad (2.60)$$

For the oceanic case the main assumption, $\frac{\Delta\rho_V}{\Delta\rho_H} \gg 1$, is not fulfilled. In fact $\Delta\rho_V = 0(\Delta\rho_H)$.

An important advantage of the linearisation is that now both the boundary condition for the density and the surface wind stress can be applied.

2.10. Vertical integration of the equations

In many studies the main interest is to determine the large scale characteristics of the horizontal transport in the ocean. For that purpose an important expedient can be formed by the vertical integration (from bottom to surface) of the equations of motion (2.33) through (2.36).

Using the surface and bottom boundary conditions the following set of equations results:

$$(2.61)\qquad R.\int_{-1}^{0}(\vec{u},\nabla u - uvtg\phi)dr - 2V\sin\phi = -\frac{1}{\cos\phi}\frac{\partial P}{\partial\lambda} + \sqrt{E_V}(\tau^{\lambda}-\tau_b^{\lambda}) + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial U}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 U}{\partial\lambda^2} + \frac{\cos2\phi}{\cos^2\phi}.U - \frac{2\sin\phi}{\cos^2\phi}\frac{\partial V}{\partial\lambda}\}$$

$$(2.62)\qquad R.\int_{-1}^{0}(\vec{u},\nabla v + u^2 tg\phi)dr + 2U\sin\phi = -\frac{\partial P}{\partial\phi} + \sqrt{E_V}(\tau^{\phi}-\tau_b^{\phi}) + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial V}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2 V}{\partial\lambda^2} + \frac{\cos2\phi}{\cos^2\phi}.V + \frac{2\sin\phi}{\cos^2\phi}\frac{\partial U}{\partial\lambda}\}$$

$$(2.63)\qquad \frac{\partial U}{\partial\lambda} + \frac{\partial}{\partial\phi}(V\cos\phi) = 0.$$

The diffusion equation becomes:

$$(2.64)\qquad \int_{-1}^{0}\vec{u},\nabla\rho dr = \gamma_V(\frac{\partial\rho}{\partial r}(0) - \frac{\partial\rho}{\partial r}(-1)) + \gamma_H\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial\rho}{\partial\phi}) + \frac{1}{\cos^2\phi}\frac{\partial^2\rho}{\partial\lambda^2}\}.$$

Here

$$(U,V,P) = \int_{-1}^{0}(u,v,p)dr.$$

At the coasts the boundary conditions read:

$$(2.65)\qquad U = V = 0.$$

In the momentum equation bottom stress terms enter, defined by:

$$\sqrt{E_V}(\frac{\partial u}{\partial r},\frac{\partial v}{\partial r})(r=-1) = (\tau_b^{\lambda},\tau_b^{\phi}).$$

These terms are not given by the boundary conditions. To calculate the bottom friction an analysis of the full three dimensional problem is necessary and the advantage of the vertical integration is lost.

In many two dimensional studies (e.g. MUNK, 1950; MUNK and CARRIER, 1950) the additional assumption is made that the bottom stress can be neglected with respect to the applied wind stress. This assumption seems to be confirmed by observations and analysis of near bottom currents (e.g. CALLAHAN, 1971 and 1972).

If the model ocean is barotropic such an assumption can not be justified in all cases. In particular at latitudes where the flow can encircle the globe without meeting meridional barriers the bottom and wind stresses appear to be of the same order of magnitude (see chapter 4).

A curious result of the vertical integration is a decoupling of the diffusion equation (2.64) from the other equations in the system. No information concerning the vertical density and velocity distribution is needed to analyse (2.61)-(2.63). This appears to be a consequence of the use of the Boussinesq approximation. For instance, if in (2.61) the integrated pressure term is replaced by the original term there results:

$$\int_{-1}^{0} \frac{1}{1+\alpha\rho} \frac{\partial p}{\partial \lambda} \, dr.$$

In this way the decoupling is undone: the vertical density distribution must be known and for that the knowledge of the three dimensional flow field is necessary. In the remainder of this study we will retain the simplification brought in by the Boussinesq approximation.

A more serious problem is constituted by the nonlinear terms in the momentum equation. In their study on the wind-driven ocean circulation CARRIER and ROBINSON (1962) get round this difficulty by replacing these terms by

$$(2.66) \qquad R.(\vec{U},\nabla U - UV tg\phi)$$

and

$$(2.67) \qquad R.(\vec{U},\nabla V + U^2 tg\phi).$$

This means that they assume the flow field to be uniform with depth. The approximation becomes less crude if the vertical integration is carried out accross a relatively thin top layer of the ocean, whereas the deeper part is assumed to be in rest (or, alternatively, if there is no momentum transport into the deeper layer).

The main qualitative result of Carrier and Robinsons study is the need for the western boundary current (e.g. the Gulf Stream) to separate from the coast. This is an observed phenomenon, which couldn't so far be predicted by the linear transport theories.

Other nonlinear analytical studies have been presented based on the same assumptions ((2.66)-(2.67)), e.g. FOFONOFF (1954), CHARNEY (1955),

MORGAN (1956), SPLILLANE & NIILER (1975). KUO (1975) has shown that if friction is introduced the free inertial jet at middle latitude no longer exists.

A serious difficulty in modelling the ocean circulation is the fact that the Rossby and Ekman numbers are of equal order of magnitude. Moreover, in regions with high accelerations viscous effects can become important. As a consequence in such boundary layers both the frictional and the nonlinear terms have to be retained. Analytical solutions of this combined problem have not yet been obtained.

2.11. The linear transport model

If we assume that the nonlinear terms in (2.61) and (2.62) can be neglected, the linear transport model results. Equation (2.63) enables us to define a transport stream function ψ by

$$(2.68) \qquad -\frac{\partial\psi}{\partial\phi} = U; \qquad \frac{\partial\psi}{\partial\lambda} = V\cos\phi.$$

Elimination of P from (2.61) and (2.62) then leads to the differential equation for the transport stream function:

$$E\{\frac{\partial^4\psi}{\partial\phi^4} - 2\mathrm{tg}\phi\,\frac{\partial^3\psi}{\partial\phi^3} - \mathrm{tg}^2\phi\,\frac{\partial^2\psi}{\partial\phi^2} - \mathrm{tg}\phi\,(2+\frac{1}{\cos^2\phi})\,\frac{\partial\psi}{\partial\phi} + \frac{2}{\cos^2\phi}\,\frac{\partial^4\psi}{\partial\lambda^2\partial\phi^2} +$$

$$+\frac{1}{\cos^4\phi}\,\frac{\partial^4\psi}{\partial\lambda^4} + \frac{4}{\cos^4\phi}\,\frac{\partial^2\psi}{\partial\lambda^2} + \frac{2\sin\phi}{\cos^3\phi}\,\frac{\partial^3\psi}{\partial\lambda^2\partial\phi}\} - 2\frac{\partial\psi}{\partial\lambda} = -\sqrt{E_V}\,.\,T(\lambda,\phi;E),$$

where

$$T(\lambda,\phi;E) = -\frac{1}{\cos\phi}\left[\frac{\partial}{\partial\phi}\{(\tau^\lambda-\tau_b^\lambda)\cos\phi\} - \frac{\partial}{\partial\lambda}(\tau^\phi-\tau_b^\phi)\right]$$

is the vertical component of the curl of the difference between wind stress and bottom stress. With

$$\Delta = \frac{1}{\cos\phi}\,\frac{\partial}{\partial\phi}\,(\cos\phi\,\frac{\partial}{\partial\phi}) + \frac{1}{\cos^2\phi}\,\frac{\partial^2}{\partial\lambda^2}$$

equation (2.68) can be written in the form

$$(2.69) \qquad E\{\Delta^2\psi + 2\Delta\psi\} - 2\,\frac{\partial\psi}{\partial\lambda} = -\sqrt{E_V}\,T.$$

The boundary conditions are

(2.70) $$\frac{\partial\psi}{\partial\phi} = \frac{\partial\psi}{\partial\lambda} = 0 \quad \text{along the coasts.}$$

An analogous derivation can be carried out in case that the β-plane approximation is used. With the transport stream function ψ defined by

$$U = -\frac{\partial\psi}{\partial y} ; \qquad V = \frac{\partial\psi}{\partial x}$$

the following vorticity equation results:

(2.71) $$E\Delta^2\psi - \beta\frac{\partial\psi}{\partial x} = -\sqrt{E_V}\, T,$$

where

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad T(x,y;E) = \frac{\partial}{\partial x}(\tau^y - \tau_b^y) - \frac{\partial}{\partial y}(\tau^x - \tau_b^x)$$

(and $\beta = 2\mu\cos\phi_0$). Now we have the boundary conditions:

$$\frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial y} = 0 \text{ at the coasts.}$$

In his well known paper on the wind-driven ocean circulation MUNK (1950) solved the transport equation (2.71) for a squared ocean basin by separation of variables.

Both equation (2.69) and (2.71) is of the structure

(2.72) $$L_E\psi \equiv E.L\psi + \frac{\partial\psi}{\partial t} = h(t,s;E)$$

where L is a fourth order elliptic differential operator in the two space variables t and s. The horizontal Ekman number E is a small parameter multiplying the term with the highest order derivatives. This makes the problems of singular perturbation type. Therefore, to construct approximations of the solutions of the equations that satisfy the boundary conditions the method of matched asymptotic expansions can be applied (see for instance ECKHAUS, 1979 and the next chapter).

The boundary conditions that the solutions of the vorticity equations have to satisfy are of Neumann type. As a consequence the transport stream function can be determined up to an arbitrary additive constant. The resulting stream line pattern doesn't depend on the choice of this constant.

Consider a part of an ocean basin as given in figure 2.1. The continents A and B are disconnected. The boundary conditions (2.70) at the boundary of

$B(=\partial B)$ can be replaced by:

$$\psi = k, \quad \frac{\partial \psi}{\partial n} = 0 \quad \text{on } \partial B,$$

where $\frac{\partial}{\partial n}$ is the derivative normal to the boundary and k an arbitrary constant.

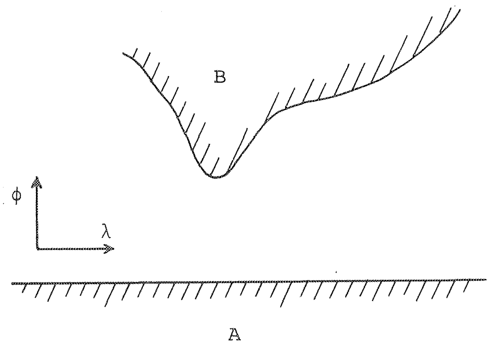

Figure 2.1. A part of a model ocean basin in which the continents A and B are disconnected.

Along ∂A the boundary conditions then read:

$$\psi = k + t_{AB}, \quad \frac{\partial \psi}{\partial n} = 0$$

where $t_{AB}(\equiv \int_{\partial A}^{\partial B} U = \psi_A - \psi_B)$ is the total transport through the strait between A and B. The value of t_{AB} has to emerge from the analysis. An example of such an analysis can be found in the model of the Antarctic Circumpolar Current (chapter 5).

CHAPTER 3

THE METHOD OF ANALYSIS

3.1. Introduction

The main problems dealt with in this study are of a type already described in §2.11: determine on a domain $\mathcal{D}$ (an asymptotic approximation of) the solution of the boundary value problem (3.1),(3.2):

$$L_E\psi \equiv E.L\psi + \frac{\partial\psi}{\partial x_1} = h(x_1,x_2) \tag{3.1}$$

$$\psi = f(x_1,x_2), \quad \frac{\partial\psi}{\partial n} = 0 \quad \text{for } (x_1,x_2) \in \partial\mathcal{D}. \tag{3.2}$$

$\partial\mathcal{D}$ denotes the boundary of $\mathcal{D}$ and $\frac{\partial}{\partial n}$ is the normal derivative to that boundary. The fourth order operator L is assumed to be elliptic and E is a small positive parameter. Thus the above formulated boundary value problem is of singular perturbation type.

The emphasis will be on the construction of approximations for different types of domains. Methods to prove the asymptotic validity of the (formal) approximations are applicable in most cases on subdomains of $\mathcal{D}$ only.

A powerful tool in the construction of approximations of (3.1) and (3.2) is the method of matched asymptotic expansions. We will only briefly outline the method here. In ECKHAUS (1979) an extensive treatment of this method has been given and for further reading we refer to that publication.

3.2. Definitions

The main concepts of asymptotic analysis used in this study will be stated here.

A real function $\delta(E)$ is called an order function if there exists an $E_0 > 0$ such that on $(0,E_0]$ δ is positive and continuous whereas $\lim_{E\downarrow 0} \delta(E)$ exists.

Landau's order symbols are introduced to be able to express the order of magnitude of a function in some domain in terms of the small parameter E:

$\psi(x_1,x_2;E) = O(\delta)$ on $\mathcal{D}$ if there exist constants E_1 and k such that $\|\psi\| \le k.\delta$ for $0 < E \le E_1$, $(x_1,x_2) \in \mathcal{D}$.

$$\psi = o(\delta) \quad \text{on } \mathcal{D} \text{ if } \lim_{E\downarrow 0} \frac{\|\psi\|}{\delta} = 0.$$

$$\psi = O_s(\delta) \quad \text{on } \mathcal{D} \text{ if } \psi = O(\delta) \text{ and } \psi \ne o(\delta)$$

(Obviously the order of magnitude depends on the choice of the norm $\|.\|$).

Suppose $\psi = O_s(1)$ on some subdomain $\mathcal{D}_s \subset \mathcal{D}$. A function ψ_{as} is called an asymptotic approximation of ψ on $\mathcal{D}_s$ if

(3.3) $$\psi - \psi_{as} = o(1) \quad \text{on } \mathcal{D}_s.$$

We will also need the concept of local asymptotic approximation. In most of the applications that we study a local analysis near a special point or curve must be carried out to investigate the behaviour of the considered function near such a lower dimensional subset S of the domain $\mathcal{D}$.

Let $(x_1,x_2) \mapsto (s_1,s_2)$ be a transformation of coordinates such that S is represented by a part of $\{s_1=0\}$ if S is a curve and by $\{s_1=s_2=0\}$ if it denotes a point. Coordinates $\vec{\xi}$ are called local (stretched) coordinates if $\vec{\xi} = (\xi_1,\xi_2)$ with $\xi_1 = \frac{s_1}{\delta_1(E)}$, $\xi_2 = \frac{s_2}{\delta_2(E)}$, where $\delta_1 = o(1)$ and $\delta_2 = O_s(1)$ if S is a curve and δ_1 and δ_2 are both $o(1)$ if S represents a point. Two local variables $\vec{\xi}_1$ and $\vec{\xi}_2$ are called equidimensional if in these variables the corresponding components both have $\delta_i = o(1)$ or $O_s(1)$.

If the transformation to equidimensional local variables is denoted by $T_{\vec{\xi}}$ the function ψ transforms to $T_{\vec{\xi}}\psi \equiv \psi^*(\vec{\xi};E)$. Suppose now $\psi^* = O_s(1)$ on some domain $\mathcal{D}^*$. A function ψ^*_{as} is called a local asymptotic approximation of ψ on $\mathcal{D}^*$ if

(3.4) $$\psi^* - \psi^*_{as} = o(1) \quad \text{on } \mathcal{D}^*.$$

We now return to the (linear) differential equation (3.1). Suppose $h = O_s(1)$. $\Phi(x_1,x_2;E)$ will be called a formal asymptotic approximation of a solution $\psi(x_1,x_2;E)$ of (3.1) on a subdomain $\mathcal{D}_s \subset \mathcal{D}$ if

(3.5) $\quad L_E\Phi - h = o(1) \quad \text{for } (x_1,x_2) \in \mathcal{D}_s.$

Let $\Phi^{(m)}$ be a regular asymptotic series (i.e.

$$\Phi^{(m)}(x_1,x_2;E) = \sum_{n=0}^{m} \delta_n(E)\phi_n(x_1,x_2)$$

with $\delta_{n+1} = o(\delta_n)$ and $\phi_n = \mathcal{O}(1)$ for $(x_1,x_2) \in \mathcal{D}_s$, $0 \le n \le m-1$). $\Phi^{(m)}$ is called a formal regular asymptotic expansion of ψ in $\mathcal{D}_s$ if for each $q = 1,\ldots,m$

(3.6) $\quad L_E\Phi^{(q)} - h = \mathcal{O}(\tilde{\delta}_q)$

where $\tilde{\delta}_0 = o(1)$, $\tilde{\delta}_q = o(\tilde{\delta}_{q-1})$ $(q = 1,\ldots,m)$.

3.3. The method of construction

Only a short description of the method of constructing formal asymptotic expansions of the solution of (3.1) and (3.2) will be given. (3.1) suggests a regular asymptotic expansion of the form

(3.7) $\quad \Phi^{(m)} = \sum_{n=0}^{m} E^n\phi_n$

$\Phi^{(m)}$ is a formal asymptotic expansion if:

(3.8) $\quad \dfrac{\partial\phi_0}{\partial x_1} = h$

(3.9) $\quad \dfrac{\partial\phi_n}{\partial x_1} = - L\phi_{n-1} \quad \text{for } 1 \le n \le m$

(3.8) and (3.9) are first order differential equations so in general it will be impossible to apply the boundary conditions for ψ ((3.2)) on $\Phi^{(m)}$. One of these boundary conditions can be imposed, but only along a part Γ_r of the boundary $\partial\mathcal{D}$. The choice of the boundary condition to be imposed on $\Phi^{(m)}$ is not an arbitrary one. In the type of problems that we will consider here it appears that if a wrong choice of boundary conditions has been made it is impossible to satisfy the remaining boundary conditions by boundary layers.

If singularities have been introduced in $\Phi^{(m)}$ these too must be removed by boundary layers. The analysis of such a layer at S (see §3.2) proceeds in terms of local coordinates $\vec{\xi}$. Suppose for instance that S is a point ($s_1=s_2=0$). A family of (equidimensional) local variables is then introduced by

$\vec{\xi} = (\xi_\nu, \xi_\mu)$, where

(3.10) $$\xi_\nu = \frac{s_1}{E^\nu} \; ; \quad \xi_\mu = \frac{s_2}{E^\mu} \quad (\nu, \mu > 0).$$

Under the transformation $(x_1, x_2) \mapsto (\xi_\nu, \xi_\mu)$ the operator L_E transforms to $L_E^{\vec{\xi}}$ which (after Taylor expansion of the non constant coefficients) is of the form

(3.11) $$L_E^{\vec{\xi}} = \sum_{n=0}^{\infty} \delta_n^*(E)(L_E)_n^{\vec{\xi}}.$$

The principal part $(L_E)_0^{\vec{\xi}}$ is called the degeneration of L_E in the $\vec{\xi}$-variable.

Let $\vec{\xi}_1 = (\frac{s_1}{E^{\nu_1}}, \frac{s_2}{E^{\mu_1}})$ and $\vec{\xi}_2 = (\frac{s_1}{E^{\nu_2}}, \frac{s_2}{E^{\mu_2}})$ both be elements of (3.10). The corresponding degenerations of L_E then read: $(L_E)_0^{\vec{\xi}_1}$ and $(L_E)_0^{\vec{\xi}_2}$. $(L_E)_0^{\vec{\xi}_2}$ is said to be contained in $(L_E)_0^{\vec{\xi}_1}$ if

(3.12) $$((L_E)_0^{\vec{\xi}_1})_0^{\vec{\xi}_2} = (L_E)_0^{\vec{\xi}_2}.$$

A degeneration $(L_E)_0^{\vec{\xi}}$ is significant within the family (3.10) if it is not contained in any other degeneration.

In a similar way the concept of significant approximations of a function $\psi(x_1, x_2; E)$ can be introduced. Suppose

$$T_{\vec{\xi}}\psi \equiv \psi^*(\vec{\xi}; E) = \sum_{n=0}^{m} \tilde{\delta}_n \phi_n^{\vec{\xi}} + O(\tilde{\delta}_{m+1})$$

with order functions $\tilde{\delta}_n$ such that $\tilde{\delta}_{n+1} = o(\tilde{\delta}_n)$. The local asymptotic expansion of ψ up to order m in the coordinates $\vec{\xi}$ is then denoted by

(3.13) $$E_{\vec{\xi}}^{(m)}(\psi) = \sum_{n=0}^{m} \tilde{\delta}_n \phi_n^{\vec{\xi}}.$$

$E_{\vec{\xi}_1}^{(m)}$ is said to be contained in $E_{\vec{\xi}_2}^{(\ell)}(\psi)$ if

(3.14) $$E_{\vec{\xi}_1}^{(m)}(E_{\vec{\xi}_2}^{(\ell)}(\psi)) = E_{\vec{\xi}_1}^{(m)}(\psi).$$

An expansion $E_{\vec{\xi}}^{(m)}(\psi)$ is a significant local approximation of ψ within the family (3.10) of equidimensional local coordinates and within the expansions of order m if it is not contained in any other local asymptotic expansion.

For the problems treated in this study we assume the so-called "property of inclusion" (ECKHAUS, 1973) to hold:

If a degeneration $(L_E)_0^{\vec{\xi}_2}$ *is contained in a significant degeneration* $(L_E)_0^{\vec{\xi}_1}$ *then there are* m *and* n *such that* $E^{(m)}_{\vec{\xi}_2}(\psi)$ *is contained in* $E^{(n)}_{\vec{\xi}_1}(\psi)$.

Obviously it is important to determine those μ and ν (in (3.10)) for which the related degenerations of L_E are significant. A significant approximation must then be represented in the (μ,ν)-plane by one of the significant degenerations.

Local asymptotic approximations can then be constructed by substitution of formal local asymptotic expansions in the differential equation

$$L_E^{\vec{\xi}}\psi^* = h^*(\vec{\xi};E) \tag{3.15}$$

which results if (3.1) is transformed to local coordinates. This also leads to a recursive system of differential equations for the subsequent terms of the expansion.

If S is a part of the boundary $\partial\mathcal{D}$ boundary conditions along S can be applied. However, in most cases this is not sufficient to determine the local approximation uniquely. If S is contained in the interior of the domain, additional relations to obtain unique local approximations must also be derived.

Such additional relations are the so-called matching relations which can be obtained in various ways. A profound discussion on the hypotheses underlying different matching principles can be found in ECKHAUS (1979) where the other concepts introduced in this section are extensively treated as well.

In many cases treated in the sequel we can apply the property of inclusion to reach the desired relations. Different significant approximations are represented in the (μ,ν)-plane by different pairs (μ,ν). Now it appears that for (μ_i,ν_i) on the straight line connecting two "significant pairs" the degenerations $(L_E)_0^{\vec{\xi}_i}$ are contained in both significant degenerations. By inclusion we can then obtain the matching relations.

3.4. Free boundary layers

The boundaries of the ocean are very irregular. Different types of irregularities lead to different mathematical problems. An additional complication comes from the fact that the domain $\mathcal{D}$ consists of a part of a spherical surface. In figure 3.1 a sketch of such a domain is given.

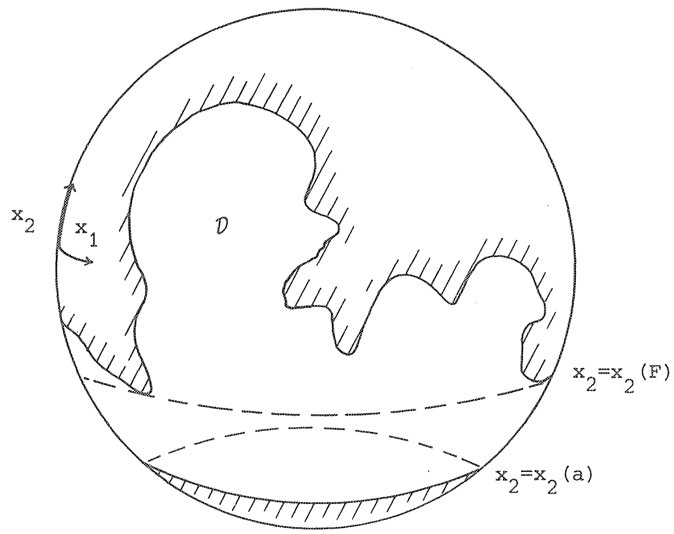

fig. 3.1. A part of the domain $\mathcal{D}$.

The characteristics of the "unperturbed" part $(\frac{\partial}{\partial x_1})$ of the operator L_E are the parallelcircles x_2 = constant. A special role is played by those characteristics that are tangent to or partly coincide with the boundary. The solution of the reduced equation (3.8) that satisfies along an eastern boundary (e.b.) the condition $\phi_0(x_{e.b.}) = \gamma(x_2)$ reads:

$$(3.16) \qquad \phi_0(x_1,x_2) = \int_{x_1(e.b.)}^{x_1} h(t,x_2)dt + \gamma(x_2).$$

The expression (3.16) strongly depends on the properties of such an eastern boundary. For a smooth boundary the division in eastern and western boundaries (w.b.) takes place in points where a characteristic is tangent to the coast. Let x be a point of the boundary $\partial\mathcal{D}$ and $n = (n_1,n_2)$ the normal to the boundary that points into the basin $\mathcal{D}$. Then x is an element of a western boundary if the zonal component n_1 of n is positive. x is an element of an eastern boundary if $n_1 < 0$. Lateral boundary layers along the eastern and western coasts are necessary to satisfy the remaining boundary conditions. The analysis of such boundary layers can be found in §4.3. The terms of the approximations in these regions are solutions of ordinary differential equations. At the points where the tangents to $\partial\mathcal{D}$ are characteristics the lateral boundary layers turn out to become singular. Obviously different local approximations hold near these points. The asymptotic analysis in the neighbourhood of such points of contact has already been outlined in §3.3.

We will now examine more closely the influence of the boundary configuration on the approximation in the interior of the domain $\mathcal{D}$. In figure 3.2 a part of a domain has been sketched (in Mercator projection) that contains an essential "irregularity" that can appear.

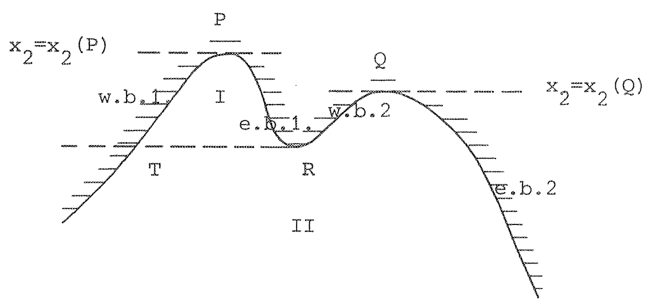

fig. 3.2.

In the points P, Q and R the characteristics are tangent to the boundary. The resulting division in eastern and western boundaries has been given in the figure. The characteristic that passes through R divides the basin in two subregions I and II. If we suppose that the boundary conditions are homogeneous (3.16) gives for the first approximation in region I:

$$\phi_0^I(x_1,x_2) = \int_{x_1(e.b.1)}^{x_1} h(t,x_2)dt. \tag{3.17}$$

The approximation in region II satisfying the boundary condition along e.b.2 reads:

$$(3.18) \qquad \phi_0^{II}(x_1,x_2) = \int_{x_1(e.b.2)}^{x_1} h(t,x_2)dt.$$

In general $\phi_0^I \neq \phi_0^{II}$ along RT. Accordingly in a neighbourhood of the line RT a regular expansion of the form (3.7) does not exist. Therefore such a neighbourhood must be investigated separately. Examples of such investigations will be presented in the sequel. It appears that a so called free boundary layer develops westward into the interior along RT. This layer brings about the transition between the regular approximations in I (3.17) and II (3.18). The partial differential equations for the terms of the approximation in this region are of the parabolic type.

A boundary configuration of the type given in figure 3.3 also leads to a free boundary layer in the interior of the domain. A part of the coast coincides with a characteristic. (In fig. 3.3 RQ is a so called eastern characteristic boundary). Consequently the regular approximation in the interior of the domain is discontinuous along RT. Also in this case a free boundary layer emerges.

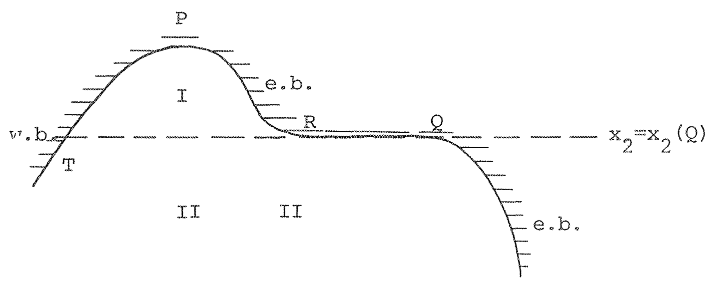

fig. 3.3.

The essential difference with the preceding case is that along RQ a lateral boundary layer exists that is of a different type than the boundary layers along the eastern and western boundaries in fig. 3.2. The equations for the terms of the local approximation in this new layer are parabolic partial differential equations. Therefore this type of lateral boundary layers is called parabolic (the terminology has been proposed by ECKHAUS, 1968). The origin of the time like variable is in $x_1(Q)$. The parabolic boundary layer thus originates in Q, develops westward and separates from the coast in R to establish the free boundary layer.

If a part of a western boundary of the ocean changes into a western characteristic boundary the regular approximation in the interior of the domain remains continuous. A lateral parabolic boundary layer is only necessary to bring the flow to rest along such a characteristic boundary.

Discontinuities in the regular approximation (3.16) can also come from discontinuous boundary conditions, while non-differentiabilities in the boundary conditions manifest themselves in the higher order terms of the regular expansion. Accordingly, also this type of non-uniformities of the regular approximation can lead to the formation of free boundary layers.

A very special type of problem arises in a part of the domain which has no meridional boundaries (those who can see the dark side of the globe in fig. 3.1 can imagine that this is the case for $x_2(a) < x_2 < x_2(F)$). If for such a region a regular expansion (3.7) exists then the boundary condition must be replaced by a periodicity condition. Solving for ϕ_0 shows that this can only be satisfied if

$$(3.19) \qquad \int_0^{2\pi} h(t,x_2)\,dt = 0 \qquad (\forall x_2(a) < x_2 < x_2(F))$$

(where h is the inhomogeneous term in the reduced equation (3.8)). In general this condition is not fulfilled. Therefore, for $x_2 < x_2(F)$ a different type of asymptotic approximation must be constructed. The transition with the regular expansion for $x_2 > x_2(F)$ is then again brought about through a free boundary layer (along $x_2 = x_2(F)$).

In chapter 5 we treat a case in which (3.19) holds and a problem with $h = h(x_2)$. There result two entirely different models of the so-called Antarctic Circumpolar Current: under the assumption of a homogeneous ocean the condition (3.19) appears to be satisfied, for a baroclinic ocean it is not.

3.4. Asymptotic validity

The problems we deal with in this study fall within a class for which in several cases proof of the asymptotic validity of the approximations can be given by applying results of BESJES (1973). In ECKHAUS, (1977,1979) the main results have been summarised and a short description of Besjes method can be found there too.

For second order linear elliptic singular perturbation problems methods to obtain asymptotic estimates have, among others, been derived by ECKHAUS and DE JAGER (1966). Their method is based on the maximum principle which they use to obtain estimates of the solutions of such second order singularly perturbed elliptic boundary value problems. However, such a principle is not available for higher order elliptic problems. Therefore Besjes method completely differs from that of Eckhaus and De Jager.

His main tools are the so called a priori or Schauder's estimates (see for instance AGMON, DOUGLIS & NIRENBERG, 1959). For our purpose the central theorem reads:

THEOREM 3.1. *Let $\mathcal{D}$ be a bounded domain (in the two dimensional Euclidean space) with a smooth boundary $\partial\mathcal{D}$ and let L_E be given by $L_E = E.L_{2m} + L_0$, where L_{2m} is a linear uniformly strongly elliptic partial differential operator of order 2m, independent of E, of which all the coefficients are elements of $C^{\ell-2m+\alpha}$ ($\ell \geq 2m$, $0 < \alpha < 1$) and $L_0 = \frac{\partial}{\partial x_1} + g(x_1,x_2)$ with $[g]_0^{\mathcal{D}} > C > 0$. Consider the Dirichlet problem*

$$E.L_{2m}\psi + L_0\psi = h \qquad \text{in } \mathcal{D}$$

$$\frac{\partial^j \psi}{\partial n^j} = 0, \qquad j = 0,1,\ldots,m-1 \qquad \text{on } \partial\mathcal{D}$$

where $\frac{\partial}{\partial n}$ is the derivative normal to $\partial\mathcal{D}$. Assume $\psi \in C^{2m+\alpha}(\mathcal{D})$. Then for any $j = 0,1,\ldots,\ell$:

$$[\psi]_j^{\mathcal{D}} \leq C_1 E^{\frac{\ell-j-2m+2}{2m-2}} |h|_{\ell-2m+\alpha}^{\mathcal{D}} + C_2 E^{\frac{j+1}{2m-2}} \left(\int_{\mathcal{D}} h^2 dx\right)^{\frac{1}{2}}$$

(where C_1, C_2 are constants).

In the theorem the following notations have been used:

$$[\psi]_j^{\mathcal{D}} \equiv \max_{|s|=j} \sup_{\vec{x}\in\mathcal{D}} |\psi^{(s)}(\vec{x})| \qquad (j = 0,1,\ldots)$$

where s is a multi index: $s = (s_1,s_2) \in \mathbb{N}^2$, $|s| = s_1+s_2$; $\psi^{(s)}(\vec{x})$ denotes derivatives of order s. $\psi \in C^{k+\alpha}(\mathcal{D})$ $(0 < \alpha < 1)$ means: $\psi \in C^k(\bar{\mathcal{D}})$ (that is:

ψ is continuously differentiable up to order k in $\bar{\mathcal{D}}$) and

$$[\psi]_{k+\alpha} \equiv \max_{|s|=k} \sup_{\vec{x},\vec{y}\in\mathcal{D}} \frac{|\psi^{(s)}(\vec{x})-\psi^{(s)}(\vec{y})|}{|\vec{x}-\vec{y}|^{\alpha}} < \infty$$

(where $|\vec{x}-\vec{y}|$ is the Euclidean distance between $\vec{x}$ and $\vec{y}$).

$$|h|_k^{\mathcal{D}} \equiv \sum_{j=1}^{k} [h]_j^{\mathcal{D}}$$

and

$$|h|_{k+\alpha}^{\mathcal{D}} \equiv |h|_k^{\mathcal{D}} + [h]_{k+\alpha}^{\mathcal{D}}$$

(these are so called Hölder norms).

Now in the operator L_E in (3.1) we have m = 2 and $L_0 = \frac{\partial}{\partial x_1}$, which is not of the desired form. If we define $\tilde{\psi} \equiv \psi.e^{cx_1}$ ($c \in \mathbb{R}, \neq 0$) the problem for $\tilde{\psi}$ has the right form to apply the theorem. After a very complex analysis Besjes has succeeded to prove the asymptotic validity of the approximations on subregions of $\mathcal{D}$ that are bounded to the north and south by characteristics of L_0. In §4.3 a sketch of such an analysis is given. It leads to a proof of the asymptotic validity of the formal approximations that have been constructed of the solution of an elementary ocean circulation model.

CHAPTER 4

SOME ELEMENTARY MODELS IN THE THEORY OF THE LARGE-SCALE OCEAN CIRCULATION

In this section we treat some elementary ocean circulation models that form part of more extensive models that shall be analysed in the later chapters. In §4.1 we investigate the (three-dimensional) flow of homogeneous water in a zonal channel that encircles the globe. The geometry of the model ocean basin in §4.2 is such that there do exist meridional boundaries. In both cases the driving wind stress at the surface is purely zonal. As a result of the essentially different geometries we find in the interior of the zonal channel, (that is below the surface (Ekman) boundary layer) $O(1)$ velocities and consequently an $O(1)$ bottom stress. In the meridionally closed basins the interior velocities and the bottom stress turn out to be $O(\sqrt{E})$. Both models form part of a homogeneous model of the Antarctic Circumpolar Current that shall be investigated in chapter 5. In that model the ocean basin can be divided in subregions where different approximations of the solution of the equations of motion with boundary conditions hold. For two of those subregions the results of §4.1 and §4.2 can then be applied.

In §4.3 we shall construct approximations of the solution of the linear transport model (§2.11) for a part of the ocean domain that is meridionally closed by smooth boundaries. Moreover a sketch of the proof of the asymptotic validity of the approximations is added. In each of the following chapters the results of this elementary model will be used in one way or another.

4.1. A three-dimensional model of the flow in a zonally unobstructed homogeneous ocean

Let the boundaries of the ocean basin coincide with the latitude circles $\phi = \phi_0$ and $\phi = \phi_1$ (see figure 4.1). At the surface we impose a zonal wind stress:

$$(4.1) \qquad \sqrt{E_V}\left(\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r}\right) = (\tau(\phi), 0) \quad \text{in } r = 0,$$

while the other boundary conditions are given in (2.17) and (2.19).

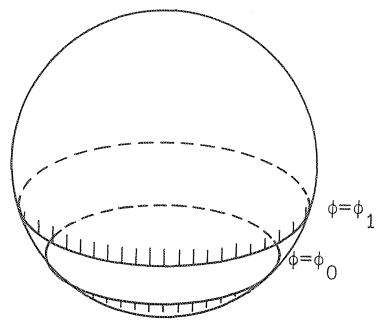

Fig. 4.1. The geometry of the zonally unobstructed ocean basin.

The linear equations of motion for this model problem can be obtained from (2.33) through (2.36) by setting R = 0, μ = 1 and ρ = 0.

From the hydrostatic approximation we can immediately conclude that in this homogeneous case the reduced pressure field is uniform with depth.

In view of the geometry and the given zonal wind stress we look for a solution of the problem which is independent of the longitude λ. The resulting system of governing equations reads (with $E_V = E$):

$$-2v \sin\phi = E\frac{\partial^2 u}{\partial r^2} + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial u}{\partial\phi}) + \frac{\cos 2\phi}{\cos^2\phi}\cdot u\} \tag{4.2}$$

$$2u \sin\phi = -\frac{dp}{d\phi} + E\frac{\partial^2 v}{\partial r^2} + E\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\frac{\partial v}{\partial\phi}) + \frac{\cos 2\phi}{\cos^2\phi} v\} \tag{4.3}$$

$$\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(v\cos\phi) + \frac{\partial w}{\partial r} = 0. \tag{4.4}$$

In order to construct an asymptotic approximation of the solution of this problem we introduce the following formal expansions:

$$(4.5)\quad \begin{cases} u(\phi,r;E) = u^{(1)}(\phi,r) + E^{1/2}u^{(2)}(\phi,r) + \ldots \\ v(\phi,r;E) = Ev^{(1)}(\phi,r) + E^{3/2}v^{(2)}(\phi,r) + \ldots \\ w(\phi,r;E) = E^{1/2}w^{(1)}(\phi,r) + \ldots \\ p(\phi;E) = p^{(1)}(\phi) + \ldots \end{cases}$$

The leading terms of these expansions must satisfy the equations:

$$(4.6)\quad -2v^{(1)}\sin\phi = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,\frac{\partial u^{(1)}}{\partial\phi}) + \frac{\cos2\phi}{\cos^2\phi}\,u^{(1)}$$

$$(4.7)\quad 2u^{(1)}\sin\phi = -\frac{dp^{(1)}}{d\phi}$$

$$(4.8)\quad \frac{\partial w^{(1)}}{\partial r} = 0.$$

Consequently $u^{(1)} = u^{(1)}(\phi)$; $v^{(1)} = v^{(1)}(\phi)$; $w^{(1)} = w^{(1)}(\phi)$. In order to determine these functions we analyse so-called Ekman boundary layers at the surface and the bottom.

(-) The surface Ekman layer

Let the local variable ζ be defined by

$$\zeta = \frac{z}{\sqrt{E}}.$$

With the expansions:

$$(4.9)\quad \begin{cases} \bar{u}(\phi,\zeta;E) = \bar{u}^{(1)}(\phi,\zeta) + E^{1/2}\bar{u}^{(2)}(\phi,\zeta) + \ldots \\ \bar{v}(\phi,\zeta;E) = \bar{v}^{(1)}(\phi,\zeta) + E^{1/2}\bar{v}^{(2)}(\phi,\zeta) + \ldots \\ \bar{w}(\phi,\zeta;E) = E^{1/2}\bar{w}^{(1)}(\phi,\zeta) + \ldots \\ \bar{p}(\phi;E) = p(\phi;E) \end{cases}$$

the equations for the leading terms read:

$$(4.10)\quad -2\bar{v}^{(1)}\sin\phi = \frac{\partial^2\bar{u}^{(1)}}{\partial\zeta^2}$$

$$(4.11)\quad 2\bar{u}^{(1)}\sin\phi = -\frac{dp^{(1)}}{d\phi} + \frac{\partial^2\bar{v}^{(1)}}{\partial\zeta^2}$$

(4.12) $$\frac{\partial}{\partial\phi}(\bar{v}^{(1)}\cos\phi) + \cos\phi\,\frac{\partial\bar{w}^{(1)}}{\partial\zeta} = 0.$$

The solutions have to match with the interior and satisfy the surface boundary conditions:

(4.13) $$\frac{\partial\bar{u}^{(1)}}{\partial\zeta} = \tau(\phi);\quad \frac{\partial\bar{v}^{(1)}}{\partial\zeta} = 0 \quad\text{and}\quad \bar{w}^{(1)} = 0 \text{ in } \zeta = 0.$$

For $\bar{v}^{(1)}$ the following equation can be obtained:

(4.14) $$\frac{\partial^4\bar{v}^{(1)}}{\partial\zeta^4} + 4\sin^2\phi\,\bar{v}^{(1)} = 0$$

which is easy to solve. As a result we find (with $a \equiv \sqrt{|\sin\phi|}$):

(4.15) $$\bar{v}^{(1)}(\phi,\zeta) = \frac{\tau}{2a}\,e^{a\zeta}.(\sin a\zeta - \cos a\zeta)$$

(4.16) $$\bar{u}^{(1)}(\phi,\zeta) = \frac{\tau}{2a}\,e^{a\zeta}.(\sin a\zeta + \cos a\zeta) + u^{(1)}(\phi)$$

(4.17) $$\bar{w}^{(1)}(\phi,\zeta) = \frac{1}{\cos\phi}\,\frac{\partial}{\partial\phi}\left[\cos\phi\,\frac{\tau}{2\sin\phi}(1 - e^{a\zeta}\cos a\zeta)\right].$$

From the matching of the vertical velocity components we get in the interior (that is below the Ekman layer):

(4.18) $$w^{(1)}(\phi) = \frac{1}{\cos\phi}\,\frac{d}{d\phi}\left(\cos\phi.\,\frac{\tau}{f}\right) \qquad (f = 2\sin\phi).$$

This is the so called "Ekman suction" that comes out of the surface boundary layer and penetrates, unchanged, the bottom Ekman layer.

(-) The bottom Ekman layer

With $\xi = \frac{z+1}{\sqrt{E}}$ as the stretched variable and the expansions

(4.19) $$\begin{cases} \hat{u}(\phi,\xi;E) = \hat{u}^{(1)}(\phi,\xi) + E^{1/2}\hat{u}^{(2)}(\phi,\xi) + \ldots \\ \hat{v}(\phi,\xi;E) = \hat{v}^{(1)}(\phi,\xi) + E^{1/2}\hat{v}^{(2)}(\phi,\xi) + \ldots \\ \hat{w}(\phi,\xi;E) = E^{1/2}\hat{w}^{(1)}(\phi,\xi) + \ldots \\ \hat{p}(\phi,\xi;E) = p(\phi;E) \end{cases}$$

the differential equations for the first terms are identical to (4.7),(4.8) and (4.9). Now, the bottom boundary conditions have to be imposed:

$$\hat{u}^{(1)} = \hat{v}^{(1)} = \hat{w}^{(1)} = 0 \quad \text{in } \xi = 0. \tag{4.20}$$

The solutions $\hat{u}^{(1)}$, $\hat{v}^{(1)}$ that match with the interior can then easily be computed. One finds:

$$\hat{u}^{(1)}(\phi,\xi) = u^{(1)}(\phi)(1 - e^{-a\xi}\cos a\xi) \tag{4.21}$$

$$\hat{v}^{(1)}(\phi,\xi) = u^{(1)}(\phi).e^{-a\xi}\sin a\xi \tag{4.22}$$

and, from the continuity equation:

$$\hat{w}^{(1)}(\phi,\xi) = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\{\frac{\cos\phi}{2a}.u^{(1)}(\phi).[e^{-a\xi}(\sin a\xi + \cos a\xi) - 1]\}. \tag{4.23}$$

Matching of the vertical velocities leads to the condition

$$\frac{d}{d\phi}(u^{(1)}(\phi).\frac{\cos\phi}{2\sqrt{|\sin\phi|}}) = \frac{d}{d\phi}(\tau\frac{\cos\phi}{2\sin\phi}) \tag{4.24}$$

and thus

$$u^{(1)}(\phi) = \frac{\tau(\phi)}{\sqrt{|\sin\phi|}} + k\frac{\sqrt{|\sin\phi|}}{\cos\phi}.$$

The constant k will be determined by the demand that the total north-south transport, integrated along a parallel of latitude is zero. Because the flow field is uniform in the λ-direction this reduces to:

$$\int_{-1}^{0} vdr = 0. \tag{4.25}$$

Substituting the above given expansions we get:

$$\frac{u^{(1)}}{2\sqrt{|\sin\phi|}} - \frac{\tau(\phi)}{2\sin\phi} + 0(\sqrt{E}) = 0,$$

so k = 0 and

$$u^{(1)}(\phi) = \frac{\tau(\phi)}{\sqrt{|\sin\phi|}}. \tag{4.26}$$

From (4.6) and (4.26) $v^{(1)}$ can now be calculated.

In an analogous fashion terms of higher order (of E) can be determined:

$$(4.27)\qquad u^{(2)}(\phi) = -\frac{1}{a}\left\{\frac{1}{\cos\phi}\frac{d}{d\phi}\left(\cos\phi\,\frac{d}{d\phi}\left(\frac{\tau}{a}\right)\right) + \frac{\cos 2\phi}{\cos^2\phi}\left(\frac{\tau}{a}\right)\right\}$$

$$(4.28)\qquad \bar{u}^{(2)}(\phi,\zeta) = u^{(2)}(\phi)$$

$$(4.29)\qquad \bar{v}^{(2)}(\phi,\zeta) = 0$$

$$(4.30)\qquad \hat{u}^{(2)}(\phi,\xi) = u^{(2)}(\phi)(1 - e^{-a\xi}\cos a\xi)$$

$$(4.31)\qquad \hat{v}^{(2)}(\phi,\xi) = u^{(2)}(\phi).e^{-a\xi}\sin a\xi .$$

At this stage the bottom stress can be determined:

$$(4.32)\qquad \tau_b^{\lambda}(\phi;E) = \frac{\partial}{\partial\xi}(\hat{u}^{(1)} + \sqrt{E}\,\hat{u}^{(2)} + \ldots)(\phi,0) =$$

$$= \tau(\phi) + \sqrt{E}\left\{-\frac{1}{\cos\phi}\frac{d}{d\phi}\left(\cos\phi\,\frac{d}{d\phi}\left(\frac{\tau}{a}\right)\right) - \frac{\cos 2\phi}{\cos^2\phi}.\frac{\tau(\phi)}{a}\right\} + \ldots$$

$$(4.33)\qquad \tau_b^{\phi}(\phi;E) = \tau(\phi) + \sqrt{E}\left\{-\frac{1}{\cos\phi}\frac{d}{d\phi}\left(\cos\phi\,\frac{d}{d\phi}\left(\frac{\tau}{a}\right)\right) - \frac{\cos 2\phi}{\cos^2\phi}.\frac{\tau(\phi)}{a}\right\} + \ldots .$$

It appears that in this homogeneous model there are 0(1) velocities in the interior and a resulting 0(1) stress at the bottom.

The approximations constructed so far do not satisfy the boundary conditions along $\phi = \phi_0$ and $\phi = \phi_1$. For that lateral boundary layers have to be introduced. The analysis near ϕ_0 is identical to that along ϕ_1. There is an $0(E^{\frac{1}{4}})$ boundary layer that brings the zonal velocity (u) to rest, with an $0(E^{\frac{1}{2}})$ sub-boundary layer where the boundary conditions for v and w can be applied. The most simple way to find this boundary layer structure is to derive one differential equation for the north-south velocity v from (4.2) and (4.3). The local coordinates $\eta_{\frac{1}{4}} = \frac{\phi-\phi_i}{E^{\frac{1}{4}}}$ and $\eta_{\frac{1}{2}} = \frac{\phi-\phi_i}{E^{\frac{1}{2}}}$ (i = 0,1) then correspond to significant degenerations of the differential equation for v (compare with chapter 3).

It is straightforward to determine the approximations in these boundary layers and we will not present it here. (The interested reader is refered to FANDRY, 1971). A result for the $0(E^{\frac{1}{4}})$-layer along ϕ_0 which will be used in one of the following chapters reads:

$$(4.34)\qquad u(\eta_{\frac{1}{4}};E) = \frac{\tau(\phi_0)}{\sqrt{|\sin\phi_0|}}\{1 - \exp[-(|\sin\phi_0|^{\frac{1}{4}}\eta_{\frac{1}{4}}]\} + 0(E^{\frac{1}{4}})$$

in the interior of the ocean. Again Ekman layers at the top and the bottom exist. These are of the same structure as those analysed earlier in this section.

One of the most striking features of the homogeneous model is the fact that the approximation in the region outside the lateral boundary layers has uniquely been determined independently of the flow near the coasts. Another important result is the O(1) velocity and velocity gradient in the bottom Ekman layer, which doesn't agree with observation (e.g. CALLAHAN, 1971,1972).

Unfortunately, a three dimensional baroclinic model of the flow in a zonally unobstructed basin is very difficult to analyse. The main reason is the nonlinearity of the diffusion equation, of which no suitable solution is known (see also the section on the thermohaline circulation). Therefore, to get an impression of the large scale horizontal transport in the baroclinic ocean we will use throughout this study the linear transport model, as formulated in the former section, with the bottom stress neglected ($\tau_b^\lambda = \tau_b^\phi = 0$).

4.2. A three dimensional model of the flow in a homogeneous ocean with meridional coasts

We will now investigate the flow in an ocean basin where the coasts coincide with parallel and latitude circles. The main interest is the order of magnitude of the flow in the interior of the ocean and the resulting stress near the bottom. Again a purely zonal wind stress distribution is given at the surface. The analysis of this model has been performed by PEDLOSKY (1968) within the β-plane approximation. No essential differences appear when the analysis is carried out on the sphere. Therefore the results will merely be stated.

(-) The surface Ekman layer.

The analysis of the Ekman layers proceeds in almost the same way as in the preceding section. The only difference is that the solutions have to match with an interior solution which is not purely zonal. This is caused by the fact that now boundary conditions along the meridional coasts must be satisfied whereas in the zonal model of §4.1 the interior velocity had to be periodic (in λ). Consequently the Ekman layers depend on λ too. Using the same notation as in §4.1 we now have:

$$\bar{u}^{(1)}(\lambda,\phi;\zeta) = \frac{\tau}{2a} e^{a\zeta}(\sin a\zeta + \cos a\zeta) \tag{4.35}$$

(4.36) $$\bar{v}^{(1)}(\lambda,\phi;\zeta) = \frac{\tau}{2a} e^{a\zeta}(\sin a\zeta - \cos a\zeta).$$

The resulting Ekman suction into the interior is:

(4.37) $$\bar{w}^{(1)}(\lambda,\phi;\zeta) \to \frac{1}{2\cos\phi}\frac{d}{d\phi}(\tau \text{ cotg } \phi) \quad \text{as } \zeta \to -\infty.$$

(-) The interior of the ocean.

Below the surface Ekman layer the interior velocity is now $O(\sqrt{E})$:

(4.38) $$\begin{cases} u(\lambda,\phi,r;E) = \sqrt{E}(u^{(1)}(\lambda,\phi) + \ldots) \\ v(\lambda,\phi,r;E) = \sqrt{E}(v^{(1)}(\lambda,\phi) + \ldots) \\ w(\lambda,\phi,r;E) = \sqrt{E}(w^{(1)}(\lambda,\phi,r) + \ldots) \\ p(\lambda,\phi,r;E) = \sqrt{E}(p^{(1)}(\lambda,\phi) + \ldots). \end{cases}$$

In these representations already the fact has been used that the $O(\sqrt{E})$ motion is geostrophic and hydrostatic. Consequently (see §2.8) the horizontal velocities are uniform with depth.

Elimination of the pressure and use of the continuity equation yields:

(4.39) $$\frac{\partial w^{(1)}}{\partial r} = - \text{cotg}(\phi).v^{(1)}.$$

With (4.37) this gives:

(4.40) $$w^{(1)}(\lambda,\phi,r) = - r.\text{cotg}(\phi).v^{(1)}(\lambda,\phi) + \frac{1}{2\cos\phi}\frac{d}{d\phi}(\tau \text{ cotg } \phi).$$

The unique determination of $v^{(1)}$ and $w^{(1)}$ proceeds by the analysis of the bottom Ekman layer. Because in the interior u and v are $O(\sqrt{E})$ the horizontal velocities in the bottom layer will be $O(\sqrt{E})$ as well. This leads to an $O(E)$ vertical velocity (from the continuity equation). Therefore $w^{(1)}$ (4.40) has to satisfy the boundary condition at the bottom:

$$w^{(1)}(\lambda,\phi,-1) = 0, \quad \text{giving}$$

(4.41) $$v^{(1)}(\lambda,\phi) = \frac{-\sin\phi}{2\cos^2\phi}\frac{d}{d\phi}(\tau \text{ cotg } \phi)$$

(4.42) $$w^{(1)}(\lambda,\phi,r) = \frac{1}{2\cos\phi}\frac{d}{d\phi}(\tau \text{ cotg } \phi).(r+1).$$

From (2.53) and (2.54) (with $\alpha = 0$) $u^{(1)}$ can be determined up to a function of ϕ. To determine that function $O(E^{1/3})$ lateral boundary layers along the meridional coasts have to be analysed. It appears that in the case of a purely zonal wind stress ($\tau^{\phi} \equiv 0$) $u^{(1)}$ already satisfies the boundary condition along the eastern coast. This bears a resemblance to the east-west transport in a two-dimensional transport model (see §4.3).

However, if the meridional component of the wind stress is nonzero at the eastern coast, the zonal velocity is brought to rest through an $O(E^{\frac{1}{2}})$ sub boundary layer, where also important upwelling occurs. This boundary layer disappears if only the horizontal transport characteristics are investigated: in the surface and bottom Ekman layers and in the interior of the $O(E^{\frac{1}{2}})$-layer the east-west velocity is nonzero whereas the net zonal transport is zero. For the ingenious construction of these lateral boundary layers we refer to PEDLOSKY (1968).

The flow in the above model differs essentially from that in a zonally unobstructed ocean. Firstly, the existence of meridional boundaries prevents the possibility of O(1) horizontal velocity components below the surface Ekman layer. The resulting bottom stress is much smaller than the wind stress at the surface: $|\frac{\vec{\tau}_b}{\vec{\tau}}| = O(\sqrt{E})$.

Secondly there is a clear east-west asymetry in the circulation in this model ocean. This is comparable to the results of the early transport models for ocean basins where the meridional coasts are along latitude circles (e.g. STOMMEL, 1948; MUNK, 1950). For general coastal boundaries a transport model will be analysed in the following section.

4.3. The transport model for an ocean with "general" coastal boundaries

Let the western coast of our model ocean be given by a function of latitude: $\lambda_w = g(\phi)$. The eastern coast will be described by $\lambda_e = f(\phi)$. A solution of the transport equation (2.69) then has to satisfy the boundary conditions:

$$\psi = \frac{\partial \psi}{\partial n} = 0 \quad \text{along} \quad \lambda = f(\phi) \quad \text{and} \quad \lambda = g(\phi). \tag{4.43}$$

The functions f and g describing the boundary are assumed to be four times differentiable. An example of an appropriate boundary has been drawn in fig. 4.2.

fig. 4.2. An example of sufficiently smooth ocean boundaries to apply the method of section 4.3.

It is possible now to apply the method of matched asymptotic expansions in a straightforward manner. In this section we will carry out the analysis in detail. Suppose that in the interior of the ocean, that is away from the boundaries, the solution of (2.69) and (4.43) can be represented by a regular expansion in powers of E:

$$\psi(\lambda,\phi;E) = \sqrt{E}\{\psi^{(0)}(\lambda,\phi) + E.\psi^{(1)}(\lambda,\phi) + \ldots\}.$$

Substitution of this expansion in (2.69) and collecting terms with equal power of E leads for $\psi^{(0)}$ to the equation:

$$(4.44) \qquad 2\,\frac{\partial\psi^{(0)}}{\partial\lambda} = T(\phi)$$

(analogously equations for the higher order terms can be found). The solution of (4.44) that satisfies the boundary condition $\psi^{(0)}(\lambda = f(\phi)) = 0$ reads:

$$(4.45) \qquad \psi^{(0)}(\lambda,\phi) = \frac{T(\phi)}{2}(\lambda - f(\phi)).$$

However, because the order of the differential equation (4.44) is too low we cannot apply the other boundary conditions. For this we have to analyse boundary layers along the coasts. The choice of the applied boundary condition on $\psi^{(0)}$ is not arbitrary. If one imposes another boundary condition it turns out to be impossible to complete the approximation by boundary layers.

(i) Boundary layer along $\lambda = f(\phi)$

Let the local variable ξ be defined by

$$\xi(\lambda,\phi;E) = \frac{f(\phi)-\lambda}{E^{1/3}} . \tag{4.46}$$

The choice of $\xi(\lambda,\phi;E)$ corresponds to the so-called "significant degeneration" of the operator L_E under transformation to (equidimensional) local variables (see chapter 3). Performing the transformation leads, among other things, to the requirement that f has to be at least a four times differentiable function. The transformed equation reads:

$$-2\frac{\partial\psi}{\partial\xi} = E^{5/6}\, T(\phi) + [f'(\phi)^2 + \frac{1}{\cos^2\phi}]^2 . \frac{\partial^4\psi}{\partial\xi^4} + E^{7/6}.G(\psi) \quad (' = \frac{d}{d\phi}) \tag{4.47}$$

$G(\psi)$ is an expression containing, among other things, the fourth order derivative of f. Let the local approximation be represented by

$$\bar{\psi}(\xi,\phi;E) = E^{5/6}\bar{\psi}^{(1)}(\xi,\phi) + E^{7/6}\bar{\psi}^{(2)}(\xi,\phi) + \ldots .$$

Then $\bar{\psi}^{(1)}$ has to satisfy the differential equation:

$$\frac{\partial^4\bar{\psi}^{(1)}}{\partial\xi^4} + \alpha^3(\phi)\frac{\partial\bar{\psi}^{(1)}}{\partial\xi} = -\, T(\phi) \tag{4.48}$$

where

$$\alpha^3(\phi) = 2.[f'(\phi)^2 + \frac{1}{\cos^2\phi}]^{-2}.$$

The boundary conditions are:

$$\bar{\psi}^{(1)}(\xi=0) = \frac{\partial\bar{\psi}^{(1)}}{\partial\xi}(\xi=0) = 0. \tag{4.49}$$

Moreover $\bar{\psi}^{(1)}$ has to satisfy a matching condition with the solution in the interior of the ocean. In this case it can not be formulated in a simple limit form. The matching has been carried out by imposing an asymptotic matching principle (for technical details see ECKHAUS, 1979). The solution reads:

$$\bar{\psi}^{(1)}(\xi,\phi) = \frac{T(\phi)}{2}\{\frac{1}{\alpha}(1-e^{-\alpha\xi})-\xi\}. \tag{4.50}$$

Calculation of higher order terms of the expansion proceeds analogously and will therefore not be presented here.

(ii) Boundary layer along $\lambda = g(\phi)$

The analysis of the western boundary layer can be carried out in a way completely analogous to that of the eastern one. For the local variable we find:

$$\zeta(\lambda,\phi;E) = \frac{\lambda-g(\phi)}{E^{1/3}}. \tag{4.51}$$

The expansion of the local approximation now takes the form:

$$\tilde{\psi}(\zeta,\phi;E) = \sqrt{E}\{\tilde{\psi}^{(0)}(\zeta,\phi) + E^{1/3}\tilde{\psi}^{(1)}(\zeta,\phi) + \ldots\}$$

Now $\tilde{\psi}^{(0)}$ has to satisfy the differential equation:

$$\frac{\partial^4\tilde{\psi}^{(0)}}{\partial\zeta^4} - \beta^3(\phi)\frac{\partial\tilde{\psi}^{(0)}}{\partial\zeta} = 0 \tag{4.52}$$

where

$$\beta^3(\phi) = 2[g'(\phi)^2 + \frac{1}{\cos^2\phi}]^{-2},$$

with boundary conditions

$$\tilde{\psi}^{(0)}(\zeta=0) = \frac{\partial\tilde{\psi}^{(0)}}{\partial\zeta}(\zeta=0) = 0.$$

The matching condition can in this case be formulated as:

$$\lim_{\zeta\to\infty}\tilde{\psi}^{(0)}(\zeta,\phi) = \lim_{\lambda\to g(\phi)}\psi^{(0)}(\lambda,\phi). \tag{4.53}$$

This leads to the solution:

$$\tilde{\psi}^{(0)}(\zeta,\phi) = \frac{T(\phi)}{2}(f(\phi)-g(\phi))\{\frac{2}{\sqrt{3}}e^{-\frac{\beta\zeta}{2}}\sin(\frac{\beta\sqrt{3}}{2}\zeta+\frac{\pi}{3})-1\}. \tag{4.54}$$

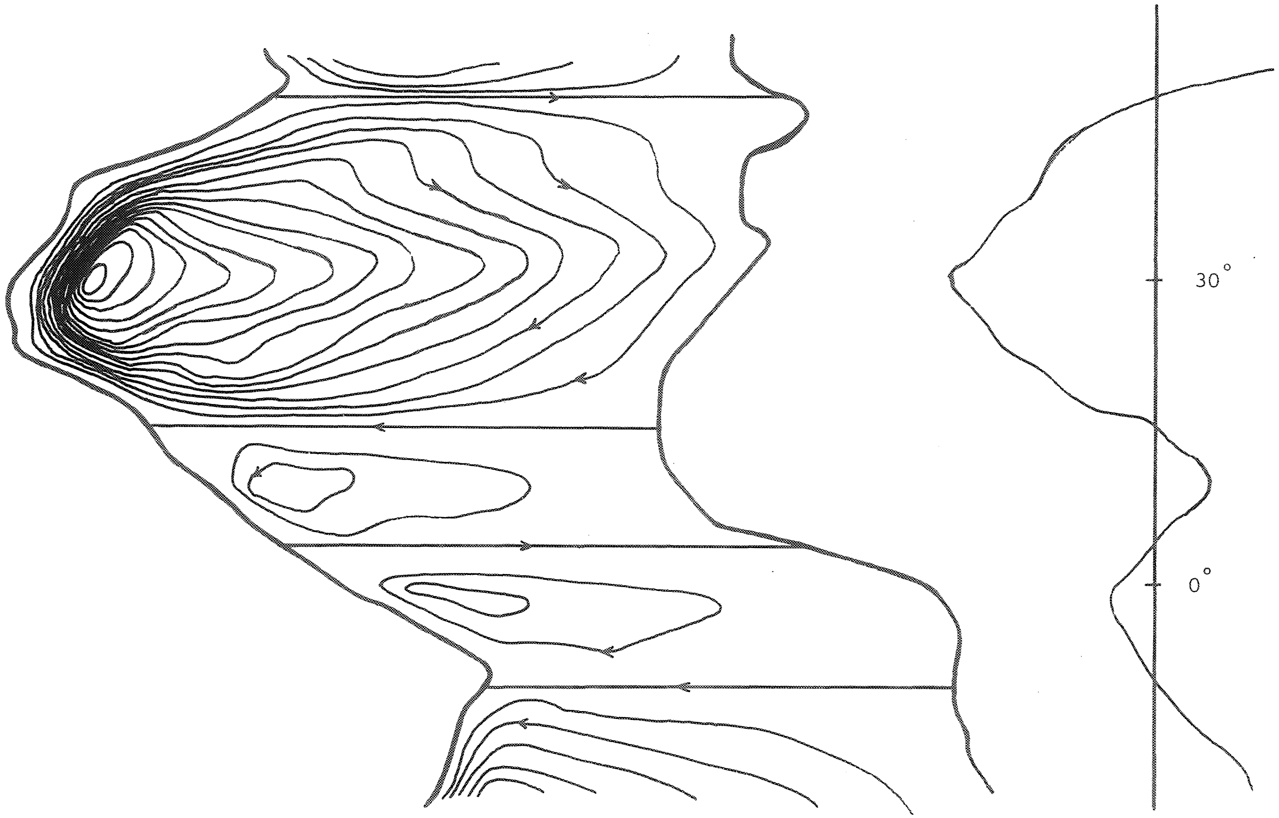

Fig. 4.3. The stream line pattern in a section of the Atlantic Ocean as calculated by the simple method of §4.3. It is given by (4.5), (4.50) and (4.54). The applied wind stress curl is given on the righthand side of the figure and is an approximation of the actual one.

It represents the well-known narrow intense western boundary current (e.g. Gulfstream, Agulhas Current).

In figure 4.3 the resulting stream lines have been drawn for a part of the Atlantic Ocean. The wind stress curl is an approximation of the actual one and has been calculated by EVANSON and VERONIS (1975) from the HELLERMAN (1967) wind stress data. It comes from a zonal average of the observed mean wind stress.

The very strong influence on the flow pattern of the wind stress curl can at once be seen on the picture. Another curious feature is the role played by the east coast of the basin. Its shape is more or less "reflected" in the interior of the ocean, even in the western boundary layer as can be seen in the expression for the approximation (4.54). The western coast doesn't play such a crucial role in the over all picture of the flow.

It appears that "irregularities" of the eastern boundary are felt in a larger part of the basin. Contrary to that irregularities of a west coast produce only local changes in the stream line pattern.

The very important role of the eastern coast is also reflected in the method of constructing approximations. Different types of boundary irregularities leading to the formation of free boundary layers have already been mentioned in chapter 3. In the chapters 5, 6 and 7 we will treat examples of such behaviour.

The early model of MUNK (1950) follows (on the sphere) from our results by taking f and g to be constant functions. To obtain the triangular model of MUNK and CARRIER (1950) we have to take $f(\phi) = c+a.\phi$ $(a < 0,\ c > 0)$ and $g(\phi) = b.\phi$ $(b > 0)$ with the range of ϕ such that no intersection takes place.

4.4. Asymptotic validity

The results of Besjes (§3.4) will now be applied to the model of the preceding section. We have seen that no analysis near northern and southern boundaries is necessary to determine uniquely the formal approximation in the parts of the basin limited to the north and to the south by parallel-circles. This feature also appears in the proof of the correctness of the formal approximations. The proof can be given for a zonal strip of the ocean bounded above and below by parallelcircles $\phi = \phi_0$, $\phi = \phi_1$ (which are characteristics of the unperturbed part of the operator $L_E \equiv E(\Delta^2+2\Delta) - 2\frac{\lambda}{\partial\lambda})$ and with smooth western and eastern boundaries where the boundary conditions are satisfied.

In figure 4.4 such a strip $S(=\{\phi_1 \le \phi \le \phi_0\} \cap \mathcal{D})$ has been put in the domain $\mathcal{D}$. The characteristic $\phi = \phi_1$ is tangent to the boundary in A.

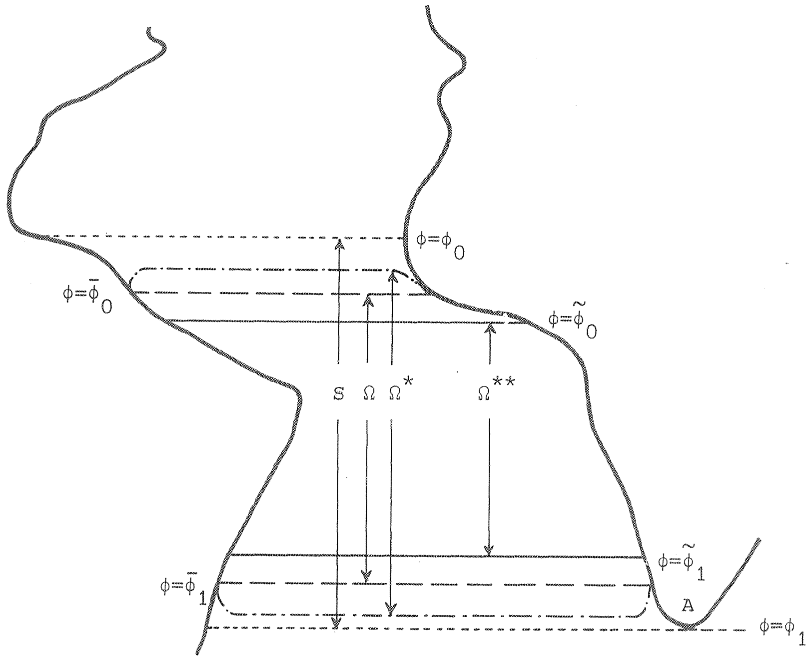

fig. 4.4.

Let Ω be a subdomain of S bounded to the north by $\bar{\phi}_0$, to the south by the parallelcircle $\bar{\phi}_1$ $(\phi_1 < \bar{\phi}_1 < \bar{\phi}_0 < \phi_0)$ and to the east and west by the boundary of $\mathcal{D}$. In an analogous way $\Omega^{**} \subset \Omega$ is defined:

$$\Omega^{**} = \{\tilde{\phi}_1 \le \phi \le \tilde{\phi}_0\} \cap \mathcal{D} \quad \text{where} \quad \bar{\phi}_1 < \tilde{\phi}_1 < \tilde{\phi}_0 < \bar{\phi}_0.$$

The purpose is to estimate the order of magnitude of the remainder $R \equiv \frac{1\psi}{\sqrt{E}} - \psi^{(0)} - \tilde{\psi}^{(0)}$ in Ω^{**}. Substitution of $\psi = \sqrt{E}(R + \psi^{(0)} + \tilde{\psi}^{(0)})$ in the transport equation (2.69) leads to a differential equation for R with (homogeneous) boundary conditions on $\partial\Omega \cap \partial\mathcal{D}$. To this problem theorem 3.1 (see §3.4) can not yet be applied. The conditions of the theorem are not fulfilled because the boundary of Ω is not smooth enough. To overcome this difficulty a domain Ω^* can be introduced such that $\Omega \subset \Omega^* \subset S$ and the boundaries of Ω^* are sufficiently smooth (see fig. 4.4).

If a C^∞-function $\zeta(\phi)$ is defined such that

$$\begin{aligned} 0 \le \zeta \le 1 \quad & (\forall\phi) \\ \zeta \equiv 1 \quad & \text{if } \tilde{\phi}_1 \le \phi \le \tilde{\phi}_0 \\ \zeta \equiv 0 \quad & \text{if } \phi < \bar{\phi}_1 \quad \text{or} \quad \phi > \bar{\phi}_0 \end{aligned}$$

a Dirichlet problem for ζR on Ω^* can be derived. To that problem theorem 3.1 and the results of Besjes theory can be applied. It leads to estimates of $[\zeta R]_j^{\Omega^*}$ $(j = 0,1,\ldots,\ell,\ \ell \ge 4)$. Now $[R]_j^{\Omega^{**}} = [\zeta R]_j^{\Omega^{**}} \le [\zeta R]_j^{\Omega^*}$ so estimates of R on Ω^{**} are obtained as well. There results:

$$[R]_j^{\Omega^{**}} \le C_1 E^{\frac{\ell-j-1}{2}} |T|_{\ell-4+\alpha}^{\Omega^*} + C_2 E^{\frac{\ell-j}{2}} \mu [R]_\ell^{\Omega^*} + C_3 \{E^{\frac{\ell-j}{2}} \mu^{-\frac{\ell-1+\alpha}{1-\alpha}} +$$
$$+ E^{-\frac{j}{2}}\}.\omega^{-1}.\sqrt{\int_{\Omega^*} |R|^2 dx} + C_4\{E^{\frac{\ell-j}{2}} \mu^{-\frac{\ell-1+\alpha}{1-\alpha}} + E^{-\frac{j}{2}}\}.\omega.[R]_1^{\Omega^*}$$

$(j = 0,1,\ldots,\ell;\ \ell \ge 4;\ \mu \in (0,\mu_0)$ arbitrary; $\omega \in (0,\omega_0)$ arbitrary). To investigate $\int_{\Omega^*} |R|^2 dx$ an elaborate analysis analogous to the one carried out by Besjes can be performed. With an optimal choice of the free parameters μ and ω as functions of E finally (for our purpose) the main result is obtained:

$$\frac{1\psi}{\sqrt{E}} - \psi^{(0)} - \tilde{\psi}^{(0)}]_0^{\Omega^{**}} \le c.E^{1/3}.$$

We remark that this is an estimate in the supremum norm. Moreover it holds for any domain Ω^{**} with $\phi_1 < \tilde{\phi}_1 < \tilde{\phi}_0 < \phi_0$. This provides an implicit proof of the statement that a free boundary layer propagates along a characteristic of L_0.

Let now the ocean basin be divided in separate subdomains that are bounded to the north and to the south by characteristics that are tangent to or (partly) coincide with the boundary. To each of the independent subdomains the above result applies. Therefore by using the theory of BESJES (1973) the asymptotic validity of the formal approximations that have been constructed in §4.3 has been shown for such strips. Only on arbitrary small neighbourhoods of the separating characteristics the approximations do not hold.

An exception arises if a strip encircles the globe that is if there is no eastern and western boundary. However, we have seen that in that case also the formal construction of the preceding section does not hold.

CHAPTER 5

THE ANTARCTIC CIRCUMPOLAR CURRENT

5.1. Introduction

The large scale circulation in the so-called Southern Ocean differs markedly from that in separate ocean basins in that it does not meet meridional barriers as it flows in eastward direction. Because the driving wind stress at the surface is mainly directed zonally the resulting circulation in the Antarctic Circumpolar Current (A.C.C.) also is chiefly zonal.

To the north of this zonal channel the flow is obstructed by the continents and consequently exhibits a gyre like pattern. A model for flow in basins with meridional barriers has been given in §4.3. Between the meridionally closed basins and the circumpolar ocean there is an intermediate region of flow that passes the South American peninsula near the southern tip and brings about the interaction between the two completely different regions.

An early study of the dynamics of the A.C.C. has been carried out by MUNK and PALMÉN (1951). In a baroclinic ocean model the A.C.C. is regarded as an eastward flow on a plane tangent to the earth at the South Pole, driven by a constant eastward wind stress which is balanced by lateral friction against the Antarctic Continent and an imaginary wall to the north (for which the 45°S latitude was taken). With a lateral eddy viscosity A_H of $10^7\,kg\,m^{-1}\,sec^{-1}$ they computed a transport of a hundred times that observed, which is of an order of $10^8\,m^3\,sec^{-1}$ (e.g. BRYDEN and PILLSBURY, 1977; NOWLIN et.al., 1977). From these calculations Munk and Palmén concluded that lateral friction could not provide enough dissipation to balance the wind stress. They suggest that the main balancing mechanism is the stress against the bottom, combined with zonal pressure gradients, caused by the presence of submarine ridges along parallel circles that pass through Drake Passage.

Several bottom frictional models have been developed since then. In a two-dimensional model GILL (1968) assumes as in STOMMEL (1948) the bottom stress to be proportional to the velocity of the flow. Moreover he uses a model ocean basin which incorporates the South American peninsula. The same geometry has been used by FANDRY (1971) in a three-dimensional flat bottom model of the A.C.C. on the β-plane. He assumes the water to be homogeneous which leads in the zonal part of the flow to bottom stresses which are equal in magnitude to the applied surface wind stress (see also §4.1). However, the observations of CALLAHAN (1971,1972) suggest that bottom stress is of minor importance.

The effects of submarine ridges on the transport characteristics have (among others) been studied by HILL and JOHNSON (1975) in a homogeneous model and by SMITH and FANDRY (1978) in a two layer analogue of the model of FANDRY (1971). They all indicate a reduction of transport caused by the presence of such ridges.

A three-dimensional thermocline model of the A.C.C. has been developed by DEVINE (1972), who assumes the A.C.C. to be frictionless, except for a small region near Drake Passage.

In all these models it is assumed that a free shear layer brings about the interaction between the zonally unobstructed flow and the flow in the meridionally closed basins more northward. However, the stream line pattern in the zonal channel can be calculated without the knowledge of the shear layer, and this is one of the reasons that a mathematical analysis of that free boundary layer has not been carried out in those studies.

For a description of numerical A.C.C. models we refer to Mc.WILLIAMS et.al. (1978).

In the present investigation we take up again the baroclinic ocean model. However, instead of postulating an imaginary wall at the north, as in MUNK and PALMÉN (1951), we introduce a more realistic geometry including the South American peninsula. Using the techniques of singular perturbation analysis (see chapter 3) we construct asymptotic approximations of the solution of the problem. The process of constructing the asymptotic approximations which will occupy us through most of this chapter is a nontrivial application of the method of singular perturbation analysis exhibiting some rather unusual aspects.

It turns out that in the present model an analysis of the free boundary layer is necessary to determine the solution in the zonal channel. The shear layer originates at the western side of the tip of South America and develops in westward direction along the parallel circle through this southern point. Moreover, a local analysis of the region near Cape Horn has to be carried out in order to complete the solution of the problem.

The resulting transport turns out to be of the same order of magnitude as the observed. This leads to the conclusion that lateral friction can indeed be one of the main dissipating mechanisms in the A.C.C. system.

In order to make clear what are the essential differences between the baroclinic model and the homogeneous bottom frictional A.C.C. model, the main results of FANDRY (1971) will also be discussed (section 5.3).

5.2. Formulation of the model

We consider stationary flow in an ocean of uniform depth, driven by a purely zonal wind stress acting at the surface. The equations of motion are those of the linear transport model that have been derived in chapter 2. For convenience these will be repeated here (with $\tau^\lambda \equiv \tau(\phi)$, $\tau^\phi \equiv 0$):

(5.1) $$-2V\sin\phi = -\frac{1}{\cos\phi}\frac{\partial P}{\partial\lambda} + \sqrt{E_V}(\tau(\phi) - \tau_b^\lambda) + E\{\Delta U + \frac{\cos 2\phi}{\cos^2\phi}U - \frac{2\sin\phi}{\cos^2\phi}\frac{\partial V}{\partial\lambda}\}$$

(5.2) $$2U\sin\phi = -\frac{\partial P}{\partial\phi} - \sqrt{E_V}\,.\,\tau_b^\phi + E\{\Delta V + \frac{\cos 2\phi}{\cos^2\phi}V + \frac{2\sin\phi}{\cos^2\phi}\frac{\partial U}{\partial\lambda}\}$$

(5.3) $$\frac{\partial U}{\partial\lambda} + \frac{\partial}{\partial\phi}(V\cos\phi) = 0.$$

If $U \equiv -\frac{\partial\psi}{\partial\phi}$, $V\cos\phi \equiv \frac{\partial\psi}{\partial\lambda}$ the differential equation for the transport stream function reads:

(5.4) $$E(\Delta^2\psi + 2\Delta\psi) - 2\frac{\partial\psi}{\partial\lambda} = -\sqrt{E_V}\,T(\lambda,\phi;E).$$

The most simple geometry containing the characteristic features of the Southern Ocean has been drawn in figure 5.1.

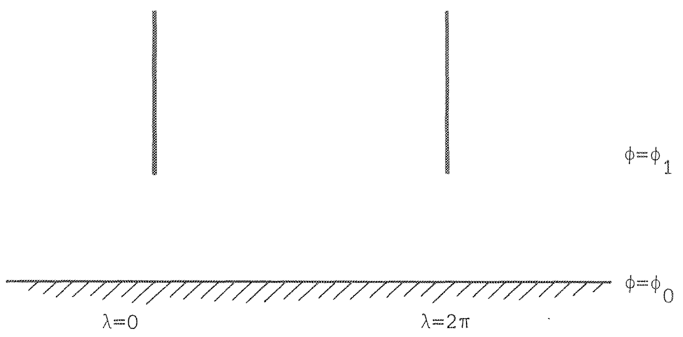

Fig. 5.2. The geometry of the model Southern Ocean.

The South American peninsula is represented by a flat plane perpendicular to the ocean floor, situated along the meridian $\lambda = 0$ (which coincides with $\lambda = 2\pi$) with the southern tip at $\phi = \phi_1$ ($\approx -57^\circ$). The coast line of Antarctica is approximated by the parallel $\phi = \phi_0$ ($\approx -65^\circ$). The same geometry has been used by GILL (1968) and FANDRY (1971) who both carried out their analysis within the β-plane approximation.

At the coasts the following boundary conditions will be imposed:

$$\text{(5.5)} \qquad \psi = 0;\ \frac{\partial\psi}{\partial\lambda} = 0 \quad \text{along } \lambda = 0,\ \phi \geq \phi_1 \quad \text{and } \lambda = 2\pi,\ \phi \geq \phi_1$$

$$\psi = B;\ \frac{\partial\psi}{\partial\phi} = 0 \quad \text{along } \phi = \phi_0.$$

Here the (unknown) constant B represents the total transport through the gap between South America and Antarctica (see §2.11) and has to be determined from the analysis.

5.3. The homogeneous model

A three dimensional analysis of a homogeneous A.C.C. is not very difficult to perform and has, within the β-plane approximation, been carried out by FANDRY (1971). His main results will be formulated for the transport model and somewhat extended in this section. A part of the analysis has already been given in §4.1.

Following Fandry the basin of the Southern Ocean can be divided in various subareas in which the solution of the transport equation behaves quite differently (fig. 5.2)

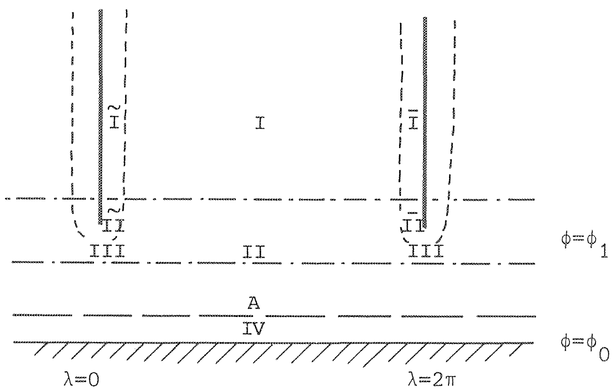

fig. 5.2. Division of the Southern Ocean in subareas for the homogeneous model (schematically).

(-) Region A. In this part of the ocean the flow doesn't meet meridional barriers. The results of §4.1 are applicable in this case. The transport can be described by a purely zonal solution of (5.1) through (5.3):

$$V^A \equiv 0 \tag{5.6}$$

$$\frac{1}{\cos\phi}\frac{d}{d\phi}(\cos\phi\frac{dU^A}{d\phi}) + \frac{\cos 2\phi}{\cos^2\phi}U^A = \frac{1}{\sqrt{E}}(\tau(\phi) - \tau_\lambda^b) \tag{5.7}$$

(where we have used: $E_V = E$).

With the expression (4.33) for the bottom stress the inhomogeneous term of (5.7) reduces to:

$$\frac{1}{\cos\phi}\frac{d}{d\phi}(\cos\phi\frac{d}{d\phi}(\frac{\tau(\phi)}{\sqrt{-\sin\phi}})) + \frac{\cos 2\phi}{\cos^2\phi}\cdot\frac{\tau(\phi)}{\sqrt{-\sin\phi}}.$$

The resulting east-west transport is:

$$U^A(\phi;E) = \frac{\tau(\phi)}{\sqrt{-\sin\phi}} + O(\sqrt{E}) \tag{5.8}$$

which can easily be obtained from the three-dimensional approximation by integrating it from bottom to surface. It is surprising that in the homogeneous model the transport in region A can be determined independently of the flow near the Antarctic coast and the currents more northward.

(-) Region I, the interior region in which the influence of lateral friction is zero up to order E. Here the results of §4.2 do apply. The order of magnitude of the transport is smaller than that in the Antarctic Region (A):

$$U^I(\lambda,\phi;E) = O(\sqrt{E}) \quad \text{and} \quad V^I(\lambda,\phi;E) = O(\sqrt{E}).$$

(-) Region IV. The $O(E^{\frac{1}{4}})$ viscous boundary layer where, from (4.34), we get for the leading term of the east-west transport:

$$U^{IV}(\eta;E) = \frac{\tau(\phi_0)}{\sqrt{-\sin\phi_0}}\{1 - \exp[-(-\sin\phi_0)^{\frac{1}{4}}\eta]\} + O(E^{\frac{1}{4}}) \tag{5.9}$$

(here $\eta = \frac{\phi-\phi_0}{E^{\frac{1}{4}}}$).

(-) Region II. The transition from the parallel transport in A to the much smaller λ-dependent transport in I is brought about in a viscous free boundary layer along $\phi = \phi_1$. For $\phi > \phi_1$ again the lateral boundary layers $\widetilde{II}$ and $\overline{II}$ serve to bring the flow to rest at the coasts.

Let again η be the stretched variable: $\eta = \frac{\phi-\phi_1}{E^{1/4}}$. The O(1) component of the east-west transport $U_{II}^{(0)}$ has to satisfy the differential equation

$$(5.10)\qquad \frac{\partial^4 U_{II}^{(0)}}{\partial \eta^4} - a.\frac{\partial^2 U_{II}^{(0)}}{\partial \eta^2} + \frac{\partial U_{II}^{(0)}}{\partial x} = 0 \qquad \left(\text{where } x = \frac{2\pi-\lambda}{2},\ a = (-\sin\phi_1)^{1/2}\right).$$

The second term represents the contribution of the bottom stress which now turns out to be

$$(5.11)\qquad \tau_\lambda^{II}(x,\eta;E) = a.U_{II}^{(0)}(x,\eta) + O(E^{1/4}).$$

Because U^I is $O(\sqrt{E})$ the matching condition to the north is simply:

$$U_{II}^{(0)} \to 0 \qquad \text{as } \eta \to \infty.$$

To the south there is the condition:

$$U_{II}^{(0)} \to a.\tau(\phi_1) \qquad \text{as } \eta \to -\infty.$$

We have slightly extended Fandry's analysis by calculating the solution of this problem for $U_{II}^{(0)}$ by means of Fourier transformation. The result is:

$$(5.12)\qquad U_{II}^{(0)}(x,\eta) = a.\tau(\phi_1)\left\{\frac{1}{2} - \frac{1}{\pi}\int_0^\infty \frac{\sin(\lambda\eta)}{\lambda}\exp[-(\lambda^4 + a\lambda^2).x]d\lambda\right\}.$$

REMARKS: (-) the fact that the bottom stress is proportional to the transport is a consequence of the assumption in this model that the ocean is homogeneous. This gives in the interior of the ocean, that is between the Ekman layers, horizontal components of the flow which are uniform with depth and have O(1) magnitude in this region (see also chapter 2).

(-) the east-west component of flow in the interior of region II is described by the same differential equation (5.10). As a consequence the solution of the transport equation (5.10) immediately solves this part of a three-dimensional analysis as well.

(-) in the three-dimensional analysis an additional $O(E^{1/2})$ upwelling layer is necessary to bring about the transition of the north-south and the vertical velocity components. In this study the main interest is in the large scale horizontal transport and the detailed vertical structure has been integrated out. However, now that a solution of (5.10) with matching conditions has been found there are no essential difficulties left to describe the three-dimensional structure of the free boundary layer.

(-) Region III, the region near the southern tip of South America. An analysis of this region will not be carried out for the homogeneous model for two reasons:

(1) The transport in the other regions of the Southern Ocean has already uniquely been determined. An analysis of region III is therefore not necessary to obtain global results.

(2) The solution in II is not valid across the line $\lambda = 0\ (2\pi)$, $\phi < \phi_1$. Making the model complete with respect to this fact can proceed almost analogously to a local analysis which is *necessary* in the nonhomogeneous model. Therefore the method of analysis will be developed there.

5.4. The baroclinic model

It has already been pointed out earlier that the bottom stress can be assumed negligible in the baroclinic ocean model.

The structure of the approximations in the different ocean regions will turn out to differ markedly from the results in the homogeneous case. Even the division of the Southern Ocean in subareas is different (fig. 5.3).

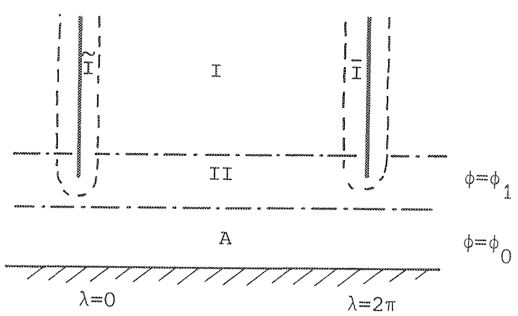

fig. 5.3. Division of the Southern Ocean in subareas for the baroclinic case. A: purely zonal flow;
I: interior region, no lateral friction up to O(E)
II: free viscous boundary layer.

The region A extends now to the Antarctic coast $\phi = \phi_0$. The flow is described by a purely zonal exact solution of the vorticity equation. No boundary layer is needed to bring the flow to rest along $\phi = \phi_0$ because the boundary conditions along the Antarctic coast can be imposed on the exact zonal solution.

It can be easily verified that this solution reads:

$$(5.13) \qquad \psi^A(\phi) = \frac{1}{\sqrt{E}} \Phi(\phi) + A.\varphi(\phi) + B,$$

where

$$\Phi(\phi) = \frac{1}{2} \int_{\phi_0}^{\phi} \{(\sin s - \sin\phi).\tau(s).f(s)\cos(s) +$$

$$+ (\sin(s) \ln(tg(\frac{1}{2}s + \frac{\pi}{4})) - \sin(\phi) \ln(tg(\frac{1}{2}\phi + \frac{\pi}{4}))).\tau(s)\cos^2(s)\}ds.$$

$$\varphi(\phi) = \sin\phi.(\frac{f(\phi_0)}{\cos\phi_0} - \ln(tg(\frac{1}{2}\phi + \frac{\pi}{4}))) +$$

$$+ \sin\phi_0.(-\frac{f(\phi_0)}{\cos\phi_0} + \ln(tg(\frac{1}{2}\phi_0 + \frac{\pi}{4}))),$$

(with $f(\phi) \equiv tg(\phi) + \cos(\phi) \ln(tg(\frac{1}{2}\phi + \frac{\pi}{4}))$).

MUNK and PALMÉN (1951) introduced a wall at about 45°S to determine the values of A and B. This led to a calculated transport which exceeds by several orders of magnitude the observed transport.

In what follows the constants A and B will be determined by matching the zonal solution with the free boundary layer approximation (region II). A remarkable difference with the homogeneous model is the fact that the analysis of the shear layer will be necessary to determine the zonal solution.

In region I the influence of lateral friction is zero up to order E. The approximation for this region has been calculated in §4.3. If we substitute $\lambda_w = 0$ and $\lambda_e = 2\pi$ the approximation in I is obtained:

$$(5.14) \qquad \psi_I(\lambda,\phi;E) = \sqrt{E}\{\psi_I^{(0)}(\lambda,\phi) + E.\psi_I^{(1)}(\lambda,\phi) + \ldots\}$$

with

$$(5.15) \qquad \psi_I^{(0)}(\lambda,\phi) = \frac{T(\phi)}{2}.(\lambda-2\pi).$$

In the same way the boundary layers $\bar{I}$ and $\tilde{I}$ can be obtained.

5.5. The free viscous boundary layer in the baroclinic ocean model

We now turn to the analysis of region II (fig. 5.2). In this shear layer the appropriate stretching of the north-south variable is defined by:

$$\eta = \frac{\phi-\phi_1}{E^{\frac{1}{4}}} . \qquad (5.16)$$

For convenience we introduce $x = \frac{2\pi-\lambda}{2}$; hence in this layer $0 \leq x < \pi$; $-\infty < \eta < \infty$. Introducing the expansion:

$$\psi_{II}(x,\eta;E) = \frac{1}{\sqrt{E}} \{\psi_{II}^{(0)}(x,\eta) + E^{\frac{1}{4}}\psi_{II}^{(1)}(x,\eta) + \ldots\}$$

we obtain

$$\frac{\partial^4\psi_{II}^{(0)}}{\partial\eta^4} + \frac{\partial\psi_{II}^{(0)}}{\partial x} = 0 \qquad (5.17)$$

$$\frac{\partial^4\psi_{II}^{(1)}}{\partial\eta^4} + \frac{\partial\psi_{II}^{(1)}}{\partial x} = 2\mathrm{tg}\phi_1 \frac{\partial^3\psi_{II}^{(0)}}{\partial\eta^3} . \qquad (5.18)$$

In an analogous way equations for the higher order terms of the expansion can be obtained.

We shall make use of similarity solutions for the homogeneous problem $\psi_{\eta\eta\eta\eta} + \psi_x = 0$ of the form $x^{n/4}.w_n(\eta/x^{\frac{1}{4}})$. These have been calculated and analysed by GILL and SMITH (1970). The functions $w_n(y)$ satisfy the differential equation

$$4\frac{d^4w_n}{dy^4} - y\frac{dw_n}{dy} + n.w_n = 0$$

(where $y = \frac{\eta}{x^{\frac{1}{4}}}$). For each n four independent solutions exist:

(i) a polynomial solution:

$$P_n(y) = \begin{cases} \displaystyle\sum_{m=0}^{[\frac{n}{4}]} \frac{(-1)^m.y^{n-4m}}{(n-4m)!m!} & \text{if } n \geq 0 \\ \displaystyle\sum_{m=0}^{\infty} \frac{(-1)^m y^m}{m!} . \Gamma(\frac{m-n}{4}) & \text{for } n < 0; \end{cases} \qquad (5.19)$$

(ii) an exponentially growing solution (as $y \to \pm\infty$);

(iii) the solution $Jo_n(y)$, an oscillating function that diminishes at an exponential rate as $y \to \infty$, exhibits exponential decay as $y \to -\infty$ if n is negative but grows algebraically as $y \to -\infty$ if n is positive;

(iv) $Ko_n(y)$, which is also an oscillating function with exponential decay as $y \to \infty$ and exponential growth as $y \to -\infty$.

These solutions have many useful properties such as:

$$(5.20)\quad (i)\quad \frac{d^m w_n}{dy^m} = w_{n-m} \qquad (m = 1,2,\dots)$$

$$(ii)\quad 4w_{n-4} = y.w_{n-1} - n.w_n$$

$$(iii)\quad P_n(-y) = (-1)^n P_n(y)$$

$$(iv)\quad Jo_n(-y) = (-1)^n.\{-Jo_n(y) + 2P_n(y)\}.$$

In our case the values of n with the proper linear combinations of the above solutions are determined by the matching conditions:

to the north ψ_{II} has to match ψ_I, to the south it has to match the parallel ψ_A which moreover is of a much larger order of magnitude than ψ_I.

To analyse the matching we transform ψ_A and ψ_I to x,η-coordinates and then expand for small E. Putting

$$(5.21)\qquad A = \frac{1}{\sqrt{E}}(a_0 + E^{\frac{1}{4}}a_1 + \dots); \qquad B = \frac{1}{\sqrt{E}}(b_0 + E^{\frac{1}{4}}b_1 + \dots)$$

we obtain:

$$(5.22)\qquad \psi^A := \frac{1}{\sqrt{E}}\{[\Phi(\phi_1) + a_0\varphi(\phi_1) + b_0] + E^{\frac{1}{4}}.[\eta.(\Phi'(\phi_1) + a_0\varphi'(\phi_1)) + a_1\varphi(\phi_1) + b_1] + E^{\frac{1}{2}}.[\frac{\eta^2}{2}(\Phi''(\phi_1) + a_0\varphi''(\phi_1)) + \eta.a_1\varphi'(\phi_1) + a_2\varphi(\phi_1) + b_2] + \dots\}$$

$$(5.23)\qquad \psi_I := \sqrt{E}\{-T(\phi_1).x - E^{\frac{1}{4}}.\eta.T'(\phi_1)x + \dots\}.$$

The problem for $\psi_{II}^{(0)}$ is thus described by equation (5.17) and the conditions that the solution tends to zero for large positive η and to $\Phi(\phi_1) + a_0\varphi(\phi_1) + b_0$ as $\eta \to -\infty$. This gives

$$(5.24)\qquad \psi_{II}^{(0)} = \alpha_0^{(0)}.Jo_0(y). \quad (\text{with } y = \frac{\eta}{x^{\frac{1}{4}}}).$$

With (5.20-iv) it follows that $\psi_{II}^{(0)} \to 2\alpha_0^{(0)}$ as $\eta \to -\infty$, so we get:

$$2\alpha_0^{(0)} = \Phi(\phi_1) + a_0\varphi(\phi_1) + b_0. \tag{5.25}$$

For $\psi_{II}^{(1)}$ equation (5.18) becomes

$$\frac{\partial^4\psi_{II}^{(1)}}{\partial\eta^4} + \frac{\partial\psi_{II}^{(1)}}{\partial x} = 2tg\phi_1.\alpha_0^{(0)}.x^{-3/4}.Jo_{-3}(\frac{\eta}{x^{\frac{1}{4}}}) \tag{5.26}$$

with the conditions that $\psi_{II}^{(1)} \to 0$ as $\eta \to \infty$ and

$$\psi_{II}^{(1)} \to a_1\varphi(\phi_1) + b_1 + \eta.(\Phi'(\phi_1) + a_0\varphi'(\phi_1)) \quad \text{as } \eta \to -\infty.$$

Now the solution reads:

$$\psi_{II}^{(1)} = \alpha_0^{(1)}Jo_0(y) + \alpha_1^{(1)}x^{\frac{1}{4}}Jo_1(y) + 2tg\phi_1.\alpha_0^{(0)}.x^{\frac{1}{4}}.Jo_{-3}(y), \tag{5.27}$$

and the matching leads to the relations

(5.28) a. $2\alpha_1^{(1)} = \Phi'(\phi_1) + a_0\varphi(\phi_1)$

b. $2\alpha_0^{(1)} = a_1\varphi(\phi_1) + b_1.$

In an analogous way we get:

$$\psi_{II}^{(2)} = \alpha_0^{(2)}Jo_0 + \alpha_1^{(2)}x^{\frac{1}{4}}Jo_1 + \alpha_2^{(2)}x^{\frac{1}{2}}Jo_2 + \psi_p^{(2)} \tag{5.29}$$

with $\psi_p^{(2)}$ a particular solution which doesn't come into the matching. The relations are now:

a. $2\alpha_2^{(2)} = \Phi'' + a_0\varphi''$

b. $2\alpha_1^{(2)} = a_1\varphi'$

c. $2\alpha_0^{(2)} = a_2\varphi + b_2.$

Proceeding in this way the equation for $\psi_{II}^{(4)}$ is:

$$\frac{\partial^4\psi_{II}^{(4)}}{\partial\eta^4} + \frac{\partial\psi_{II}^{(4)}}{\partial x} = 2tg\phi_1\frac{\partial^3\psi_{II}^{(3)}}{\partial\eta^3} + tg^2\phi_1\frac{\partial^2\psi_{II}^{(2)}}{\partial\eta^2} + tg\phi_1.(2+\frac{1}{\cos^2\phi_1})\frac{\partial\psi_{II}^{(1)}}{\partial\eta} - T(\phi_1) + \{\text{terms that are not relevant for matching}\} \tag{5.30}$$

with solution:

(5.31)
$$\psi_{II}^{(4)} = \sum_{n=0}^{4} \alpha_n^{(4)} x^{n/4} Jo_n(y) + x\{(2tg\phi_1.\alpha_3^{(3)} + tg^2\phi_1.\alpha_2^{(2)} + tg\phi_1.(2+\frac{1}{\cos^2\phi_1}).\alpha_1^{(1)}).Jo_0(y) - T(\phi_1)\} + \psi_p^{(4)}.$$

As $\eta \to \infty$ $\psi_{II}^{(4)}$ behaves like $-T(\phi_1).x$ so it matches nicely the northern solution ψ_I. As $\eta \to -\infty$

$$\psi_{II}^{(4)} \sim 2\alpha_4^{(4)}.(\frac{\eta^4}{4!} - x) + 2\alpha_3^{(4)}.\frac{\eta^3}{3!} + 2\alpha_2^{(4)}.\frac{\eta^2}{2!} + 2\alpha_1^{(4)}.\eta + 2\alpha_0^{(4)} + [4tg\phi_1.\alpha_3^{(3)} + 2tg^2\phi_1.\alpha_2^{(2)} + 2tg\phi_1(2+\frac{1}{\cos^2\phi_1}).\alpha_1^{(1)} - T(\phi_1)].x.$$

This gives the relations

(5.32) a. $2\alpha_4^{(4)} = \Phi^{(iv)} + a_0\varphi^{(iv)}$

b. $2\alpha_3^{(4)} = a_1\varphi'''$

c. $2\alpha_2^{(4)} = a_2\varphi''$

d. $2\alpha_1^{(4)} = a_3\varphi'$

e. $2\alpha_0^{(4)} = a_4\varphi + b_4.$

Because the solution has to become x-independent (for $\eta \to -\infty$) we have the relation:

f. $$-2\alpha_4^{(4)} + 4tg\phi_1.\alpha_3^{(3)} + 2tg^2\phi_1.\alpha_2^{(2)} + 2tg\phi_1.(2+\frac{1}{\cos^2\phi_1}).\alpha_1^{(1)} - T(\phi_1) = 0.$$

Here

(5.33)
$$2\alpha_3^{(3)} = \Phi''' + a_0\varphi'''.$$

It now looks as if the relations (5.28a), (5.29a), (5.32 a and f) and (5.33) determine uniquely the $\alpha_i^{(i)}$ (i = 1,2,3,4) and a_0. This is however not the case because if one works out the left hand side of (5.32f) one finds that this relation is automatically fulfilled. Also calculation of further terms in the expansion of ψ_{II} and matching does not lead to the unique determination of the remaining constants. We shall find that a local analysis of the region near the southern tip of South America is necessary in order to determine fully the solution.

If $\eta > 0$ again we need boundary layers near $x = 0$ and $x = \pi$ to bring the flow to rest.

Along $x = 0$ we find for the approximation that matches both ψ_{II} and $\bar{\psi}_I$:

$$\bar{\psi}_{II}(\xi,\eta;E) = \sqrt{E}\{E^{1/3}\frac{T(\phi_1)}{2}\{\frac{1}{\alpha(\phi_1)}(1 - e^{-\alpha(\phi_1).\xi}) - \xi\} + O(E^{2/3}\} \tag{5.34}$$

where $\xi = \frac{2x}{E^{1/3}}$ (compare with §4.3).

The western boundary layer ($\widetilde{II}$) again exhibits intense north-south flow. With $\zeta = \frac{2(\pi-x)}{E^{1/3}}$ the approximation that matches with ψ_{II} and $\tilde{\psi}_I$ reads:

$$\tilde{\psi}_{II}(\zeta,\eta;E) = \frac{1}{\sqrt{E}}.\alpha_0^{(0)}Jo_0(\frac{\eta}{\pi^{\frac{1}{4}}})\{1 - \frac{2}{\sqrt{3}}e^{-\frac{\beta(\phi_1)\zeta}{2}}\sin(\frac{\beta(\phi_1)\sqrt{3}}{2}\zeta + \frac{\pi}{3})\} + O(E^{-\frac{1}{4}}). \tag{5.35}$$

For $\eta < 0$ there is no coast where the flow has to come to rest, but there is the demand that the solution and all the derivatives appearing in the governing equations are continuous across $x = 0,\pi$. The approximations $\psi_{II}^{(i)}$ do not satisfy that condition. This is another reason to analyse the Cape Horn region.

5.6. The region near the southern tip of South America

It is convenient to place the point $\phi = \phi_1$; $x = 0$ (π) at the center of the coordinate system, which is achieved by the transformation

$$t = \begin{cases} -x & \text{when } x \le \frac{\pi}{2} \\ \pi-x & \text{when } x > \frac{\pi}{2}, \end{cases} \tag{5.36}$$

so $-\frac{\pi}{2} \le t < \frac{\pi}{2}$.

The southern tip of the peninsula is now at $\phi = \phi_1$, $t = 0$, and to analyse the region near this point we introduce the local variables:

$$(5.37)\qquad \tau = \frac{t}{E^{1/3}}\ ; \qquad \mu = \frac{\phi-\phi_1}{E^{1/3}}\ .$$

If the formal expansion of the approximation $\psi_{III}(\tau,\mu;E)$ has the form:

$$(5.38)\qquad \psi_{III}(\tau,\mu;E) = \frac{1}{\sqrt{E}}\{\psi_{III}^{(0)}(\tau,\mu) + E^{1/4}.\psi_{III}^{(1)}(\tau,\mu) + \ldots\}$$

then the equation for $\psi_{III}^{(0)}$ reads:

$$(5.39)\qquad \frac{\partial^4\psi_{III}^{(0)}}{\partial\mu^4} + \frac{1}{2\cos^2\phi_1}\frac{\partial^4\psi_{III}^{(0)}}{\partial\tau^2\partial\mu^2} + \frac{1}{16\cos^4\phi_1}\frac{\partial^4\psi_{III}^{(0)}}{\partial\tau^4} - \frac{\partial\psi_{III}^{(0)}}{\partial\tau} = 0,$$

with the conditions that the solution matches with ψ_{II}, $\overline{\psi}_{II}$ and $\tilde{\psi}_{II}$, and is zero at $\tau = 0$, $\mu \geq 0$.

Solving this equation with matching conditions is not the simplest problem one can think of. Fortunately it turns out that it is not necessary to calculate the solution of (5.39), if the main interest is not the local behaviour of the stream line pattern very near the tip of South America, but the influence of this small region on the global characteristics of the flow. In other words: the as yet unknown constants in the boundary layer approximation ψ_{II} and the Antarctic solution ψ_A can be determined without the explicit knowledge of ψ_{III}.

We proceed to the analysis of the process of matching, as described in ECKHAUS (1979) (see also chapter 3) and introduce "intermediate" variables defined by:

$$(5.40)\qquad \tau_i = \frac{t}{E^{4\alpha-1}}\ ; \qquad \eta_i = \frac{\phi-\phi_1}{E^{\alpha}}\ , \qquad \text{with } \frac{1}{4} < \alpha < \frac{1}{3}.$$

Suppose that the so-called intermediate solutions, $\psi_{int}(\tau_i,\eta_i;E)$, can be expanded as follows:

$$(5.41)\qquad \psi_{int}(\tau_i,\eta_i;E) = \frac{1}{\sqrt{E}}\{\psi_{int}^{(0)}(\tau_i,\eta_i) + E^{\frac{1}{4}}\psi_{int}^{(1)}(\tau_i,\eta_i) + \ldots\},$$

then the equation that $\psi_{int}^{(0)}$ must satisfy, reads:

$$(5.42)\qquad \frac{\partial^4\psi_{int}^{(0)}}{\partial\eta_i^4} - \frac{\partial\psi_{int}^{(0)}}{\partial\tau_i} = 0, \qquad \text{where } |\tau_i| > 0,\ \eta_i < 0.$$

On the basis of the so-called overlap hypothesis (property of inclusion, §3.2) we now demand that in the intermediate limit (that is: $\lim\limits_{\substack{(\tau_i,\eta_i)\text{ fixed}\\ E\to 0}}$)

the difference of ψ_{int} and ψ_{III} and that of ψ_{int} and ψ_{II} go to zero, that is:

(5.43) $$\lim_{\substack{(\tau_i,\eta_i)\text{fixed}\\ E\to 0}} [\psi^{(0)}_{int} + E^{\frac14}\psi^{(1)}_{int} + \ldots - \psi^{(0)}_{III} - E^{\frac14}\psi^{(1)}_{III} \ldots] = 0$$

and

(5.44) $$\lim_{\substack{(\tau_i,\eta_i)\text{fixed}\\ E\to 0}} [\psi^{(0)}_{int} + E^{\frac14}\psi^{(1)}_{int} + \ldots - \psi^{(0)}_{II} - E^{\frac14}\psi^{(1)}_{II} - \ldots] = 0.$$

We shall now concentrate on the analysis of matching condition (5.44). Transforming $\psi^{(0)}_{II}$ given by (5.24) to intermediate coordinates gives:

(5.45) $$\psi^{(0)}_{II} := \begin{cases} \alpha^{(0)}_0 Jo_0\left(\frac{\eta_i}{(-\tau_i)^{\frac14}}\right) & \text{for } \tau_i < 0 \\ \alpha^{(0)}_0 Jo_0\left(\frac{\eta_i E^{\alpha-\frac14}}{(\pi - E^{4\alpha-1}\tau_i)^{\frac14}}\right) & \text{for } \tau_i > 0. \end{cases}$$

Performing the limits we find

(5.46) $$\psi^{(0)}_{int}(\tau_i,\eta_i) = \begin{cases} \alpha^{(0)}_0 Jo_0\left(\frac{\eta_i}{(-\tau_i)^{\frac14}}\right. & \text{for } \tau_i < 0 \\ \alpha^{(0)}_0 Jo_0(0) & \text{for } \tau_i > 0. \end{cases}$$

One may verify by substitution that $\psi^{(0)}_{int}$ thus defined indeed satisfies the differential equation for the intermediate approximation (5.42). However, we must define a continuation of $\psi^{(0)}_{int}$ on the whole domain $-\infty < \tau_i < \infty$, $\eta_i < 0$, and require it to be a continuous function, four times differentiable with respect to η_i and one time with respect to τ_i. Straightforward analysis shows that there does not exist a sub-boundary-layer along the line $t = 0$, $\eta_i < 0$ that satisfies the continuity and differentiability conditions on $\bar{\tau} = 0$, $\eta_i < 0$ and matches with (5.46) as $\bar{\tau} \to \pm\infty$. (Here $\bar{\tau} \equiv \frac{t}{\delta(E)}$, with $\delta(E) = \mathcal{O}(E^{4\alpha-1})$). Therefore $\psi^{(0)}_{int}$ itself must satisfy the continuity and differentiability requirements along $\tau_i = 0$, $\eta_i < 0$.

Because of the properties of Jo_0 (see §5.5) we get

$$\lim_{\substack{\tau_i\uparrow 0\\ \eta_i<0}} \alpha^{(0)}_0 Jo_0\left(\frac{\eta_i}{(-\tau_i)^{\frac14}}\right) = 2\alpha^{(0)}_0 \quad \text{and } \alpha^{(0)}_0 Jo_0(0) = \alpha^{(0)}_0.$$

So, imposing the continuity demand we get:

$$(5.47) \qquad \alpha_0^{(0)} = 0.$$

The consequences are that

$$(5.48) \qquad \psi_{II}^{(0)} = 0$$

and

$$(5.49) \qquad b_0 = -\Phi(\phi_1) - a_0\varphi(\phi_1).$$

Let us therefore consider the problem for the next terms in the approximations. After transformation to intermediate coordinates $\psi_{II}^{(1)}$ reads:

$$\psi_{II}^{(1)} := \begin{cases} \alpha_1^{(1)}.E^{\alpha-\frac{1}{4}}.(-\tau_i)^{\frac{1}{4}}.Jo_1\left(\frac{\eta_i}{(-\tau_i)^{\frac{1}{4}}}\right) + \alpha_0^{(1)}.Jo_0\left(\frac{\eta_i}{(-\tau_i)^{\frac{1}{4}}}\right) & \text{if } \tau_i < 0 \\ \alpha_1^{(1)}.(\pi-E^{4\alpha-1}\tau_i)^{\frac{1}{4}}.Jo_1\left(\frac{\eta_i.E^{\alpha-\frac{1}{4}}}{(\pi-E^{4\alpha-1}\tau_i)^{\frac{1}{4}}}\right) + \alpha_0^{(1)}Jo_0\left(\frac{\eta_i.E^{\alpha-\frac{1}{4}}}{(\pi-E^{4\alpha-1}\tau_i)^{\frac{1}{4}}}\right) & \text{if } \tau_i > 0. \end{cases}$$

Taking the limit we find:

$$(5.50) \qquad \psi_{int}^{(1)}(\tau_i,\eta_i) = \begin{cases} \alpha_0^{(1)}Jo_0\left(\frac{\eta_i}{(-\tau_i)^{\frac{1}{4}}}\right) & \text{if } \tau_i < 0 \\ \alpha_1^{(1)}\pi^{\frac{1}{4}}Jo_1(0) + \alpha_0^{(1)} & \text{if } \tau_i > 0;\ (Jo_0(0) = 1), \end{cases}$$

which satisfies again (5.43).

With the continuity demand in $\tau_i = 0$ ($\eta_i < 0$) we now get the relation:

$$(5.51) \qquad \alpha_0^{(1)} = \alpha_1^{(1)}\pi^{\frac{1}{4}}Jo_1(0).$$

The differentiability requirements on $\psi_{int}^{(1)}$ can easily be shown to be satisfied. This follows from the nice properties of the $Jo_n(y)$ as $y \to -\infty$, $n < 0$.

In what follows we shall also need an analogous result for $\psi_{int}^{(2)}$. The calculations proceed as before and we find:

$$(5.52)\qquad \psi^{(2)}_{int}(\tau_i,\eta_i) = \begin{cases} \alpha_0^{(2)} Jo_0\left(\dfrac{\eta_i}{(-\tau_i)^{1/4}}\right) & \text{if } \tau_i < 0 \\ \alpha_2^{(2)}\pi^{1/2}Jo_2(0) + \alpha_1^{(2)}\pi^{1/4}Jo_1(0) + \alpha_0^{(2)} & \text{if } \tau_i > 0 \end{cases}$$

and the relation:

$$(5.53)\qquad \alpha_0^{(2)} = \alpha_2^{(2)}\pi^{1/2}Jo_2(0) + \alpha_1^{(2)}\pi^{1/4}Jo_1(0).$$

5.7. Closure of the solution

To complete the solution we will use the "continuity of pressure" condition. It can be derived from equation (5.1) and compensates for the loss of information in deriving the vorticity equation from (5.1) through (5.3).

Integration of equation (5.1) over a closed parallel in the region $\phi_0<\phi<\phi_1$ and substitution of the stream function leads to the condition:

$$(5.54)\qquad \sqrt{E}\int_0^{2\pi}\left\{\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\,\frac{\partial^2\psi}{\partial\phi^2}\right) + \frac{\cos 2\phi}{\cos^2\phi}\frac{\partial\psi}{\partial\phi}\right\}d\lambda = 2\pi.\tau.$$

Outside the free boundary layer this condition turns out to be identically fulfilled. It is possible to impose this condition if we integrate along a parallelcircle that traverses the intermediate region. The path of integration can be divided in parts such that in each part a proper local approximation is valid.

By the transformation (5.36) the path of integration becomes $-\frac{\pi}{2}< t < \frac{\pi}{2}$. We divide it in three parts:

$$(-\tfrac{\pi}{2},\ -k.E^{4\alpha-1});\ (-k.E^{4\alpha-1}, l.E^{4\alpha-1}) \quad\text{and}\quad (l.E^{4\alpha-1},\ \tfrac{\pi}{2})$$

where k and l are arbitrary positive constants and $\frac{1}{4}<\alpha<\frac{1}{3}$. The approximation ψ_{II} is valid in the first and last parts and ψ_{int} in $(-k.E^{4\alpha-1}, l.E^{4\alpha-1})$.

Substituting the expansions and performing the transformation $\eta_i = \frac{\phi-\phi_1}{E^\alpha}$ we get the condition:

$$(5.55)\qquad \int_{-\frac{\pi}{2}}^{-k.E^{4\alpha-1}} \frac{\partial^3\psi^{(1)}_{II}}{\partial\eta_i^3}\,dt + \int_{-k.E^{4\alpha-1}}^{l.E^{4\alpha-1}} \frac{\partial^3\psi^{(1)}_{int}}{\partial\eta_i^3}\,dt + \int_{l.E^{4\alpha-1}}^{\pi/2} \frac{\partial^3\psi^{(1)}_{II}}{\partial\eta_i^3}\,dt + O(E^\alpha) = 0.$$

Grouping terms that are of the same order of magnitude and equating each group to zero leads to a new set of relations. The first one is:

$$\alpha_0^{(1)} Jo_1(0) + \alpha_1^{(1)} \pi^{\frac{1}{4}} Jo_2(0) = 0. \tag{5.56}$$

With the relation $\alpha_0^{(1)} - \alpha_1^{(1)} \pi^{\frac{1}{4}} Jo_1(0) = 0$ which has been derived in §5.6 we get the result:

$$\alpha_0^{(1)} = 0 \quad \text{and} \quad \alpha_1^{(1)} = 0. \tag{5.57}$$

As a consequence we now have:

$$\psi_{II}^{(1)} \equiv 0 \tag{5.58}$$

and, with (5.28)a,b):

$$a_0 = -\frac{\Phi'(\phi_1)}{\varphi'(\phi_1)} \tag{5.59}$$

$$b_1 = -a_1 . \varphi(\phi_1). \tag{5.60}$$

With a_0 also b_0 (5.49) can be calculated:

$$b_0 = -\Phi(\phi_1) + \frac{\Phi'(\phi_1).\varphi(\phi_1)}{\varphi'(\phi_1)}, \tag{5.61}$$

so the leading term of the Antarctic solution ψ_A has now uniquely been determined.

This, however, is still not the case with the leading term of the expansion of the free boundary layer ψ_{II}. For that we have to equate the next term in the expansion of (5.55) to zero. As a result we get the relation:

$$\alpha_0^{(2)} Jo_1(0) + \alpha_1^{(2)} \pi^{\frac{1}{4}} Jo_2(0) + \alpha_2^{(2)} \pi^{\frac{1}{2}} Jo_3(0) = 0. \tag{5.62}$$

Now $\alpha_2^{(2)} = \frac{1}{2}(\Phi''(\phi_1) + a_0\varphi''(\phi_1))$ is, with (5.59), a known constant while $\alpha_0^{(2)}$ and $\alpha_1^{(2)}$ can at this stage be calculated with the relations (5.53) and (5.62). Using the fact that $Jo_2(0) = 0$ this gives:

$$\alpha_0^{(2)} = -\frac{\alpha_2^{(2)} . \pi^{\frac{1}{2}} Jo_3(0)}{Jo_1(0)} \tag{5.63}$$

$$(5.64)\qquad \alpha_1^{(2)} = -\frac{\alpha_2^{(2)}.\pi^{\frac{1}{4}} Jo_3(0)}{Jo_1^2(0)} .$$

With (5.29b) we then have:

$$(5.65)\qquad a_1 = \frac{2\alpha_1^{(2)}}{\varphi'(\phi_1)}, \quad \text{so} \quad b_1 = -\frac{2\alpha_1^{(2)}.\varphi(\phi_1)}{\varphi'(\phi_1)} .$$

The leading term of ψ_{II} is now determined, and calculation of further terms in the expansions can take place along the same lines.

Before proceeding to the discussion and analysis of our results some additional remarks on the structure of the solution for the free boundary layer must be made. To be specific consider for example $\psi_{II}^{(2)}$. From the condition that $\psi_{II}^{(2)} \to 0$ as $\eta \to \infty$ and that $\psi_{II}^{(2)}$ behaves algebraically with respect to η as $\eta \to -\infty$ we have concluded that $\psi_{II}^{(2)}$ was of the form:

$$\psi_{II}^{(2)} = \sum_{n=0}^{2} \alpha_n^{(2)} x^{\frac{n}{4}}.Jo_n\left(\frac{\eta}{x^{\frac{1}{4}}}\right).$$

However, one could also write

$$\psi_{II}^{(2)} = \sum_{n=-\infty}^{2} \alpha_n^{(2)} x^{\frac{n}{4}} Jo_n$$

because for negative n, $Jo_n \to 0$ as $y \to \pm\infty$. However, this nonuniqueness can fortunately be removed. In fact it is not difficult to show that the $\alpha_n^{(2)}$ have to be zero for $n < 0$. This is done by substitution of the last form in the continuity of pressure condition and working out the integrals.

5.8. Discussion of the results

Like most oceanographic models the model analysed in this chapter is of course a simplified description of the ocean circulation. One can therefore not expect detailed small scale correct results. What one can expect are correct qualitative and quantitative global results. It is with this in mind that we analyse further our results.

Let us first look at the expressions that have been derived in §5.7 for the first two terms in the expansion of the total Antarctic Circumpolar Transport (B):

$$b_0 = -\Phi(\phi_1) - a_0\varphi(\phi_1) \quad \text{and} \quad b_1 = -a_1\varphi(\phi_1).$$

Substituting these in the expansion of the Antarctic solution (5.22) gives the result that $\psi^A(\phi_1) = O(1)$ so the first two terms satisfy the "boundary condition" $\psi^A(\phi_1) = 0$. Moreover the first term of the expansion of $U^A(\equiv -\frac{\partial\psi^A}{\partial\phi})$ satisfies $U^A(\phi_1) = 0$ and the effect of the viscous free boundary layer appears (in first approximation) to be the same as that of a rigid wall, placed along the parallelcircle $\phi = \phi_1$ ($\approx -57^\circ$) and not further northward at $\phi = -45^\circ$ as MUNK and PALMÉN (1951) supposed. This explains already to some extent the erroneous numerical results of Munk & Palmén.

In calculating a numerical value of the Antarctic circumpolar transport we take the wind stress to be:

$$(5.66) \qquad \tau(\phi) = \sin(6\phi + \frac{\pi}{10}).$$

This represents the prevailing westerlies over the Southern Ocean and the westward wind stress very near the Antarctic coast (e.g. EVANSON & VERONIS, (1975). Then we get the following values:

$$b_0 = 4.6 \times 10^{-5}; \quad b_1 = 2.8 \times 10^{-3}$$

which are small because of the narrowness of the gap between Antarctica and South America. With $E = 2.44 \times 10^{-6}$ this gives:

$$B \approx 0.1.$$

In m.k.s. units the transport thus calculated is approximately $3.2 \times 10^8 m^3 sec^{-1}$ which is the same order of magnitude as values that have been calculated from observational data (e.g. BRYDEN and PILLSBURY, 1977, give a value of $2.6 \times 10^8 m^3 sec^{-1}$).

Based on formula (5.8) also a value for the transport through Drake Passage in a homogeneous ocean model can be calculated. If the same wind stress (5.66) is applied as in the baroclinic model the resulting approximate value of the transport is $0.9 \times 10^8 m^3 sec^{-1}$. Accordingly neither of the two models can be rejected or prefered on the basis of the calculated transport only. Both lead to the right order of magnitude.

In figure 5.4 the stream line pattern in the free boundary layer has been sketched for the baroclinic ocean. The calculations are based on the formulas (5.13) and (5.29) with the appearing constants as in §5.7. There is a clear asymmetry between the northern and southern part of the region, caused by the geometry and the resulting asymmetric matching conditions for the shear layer. Near x = 0 (figure 5.5) a crowding of northern stream lines takes place whereafter they wind themselves around Cape Horn ($\phi = \phi_1$; $x = 0(\pi)$) to form the intense northward current which agrees with the Falkland Current. After leaving the region near the east coast of South America ($\tilde{II}$) the current again flows eastward and completes the stream lines in the vicinity of the southern tip.

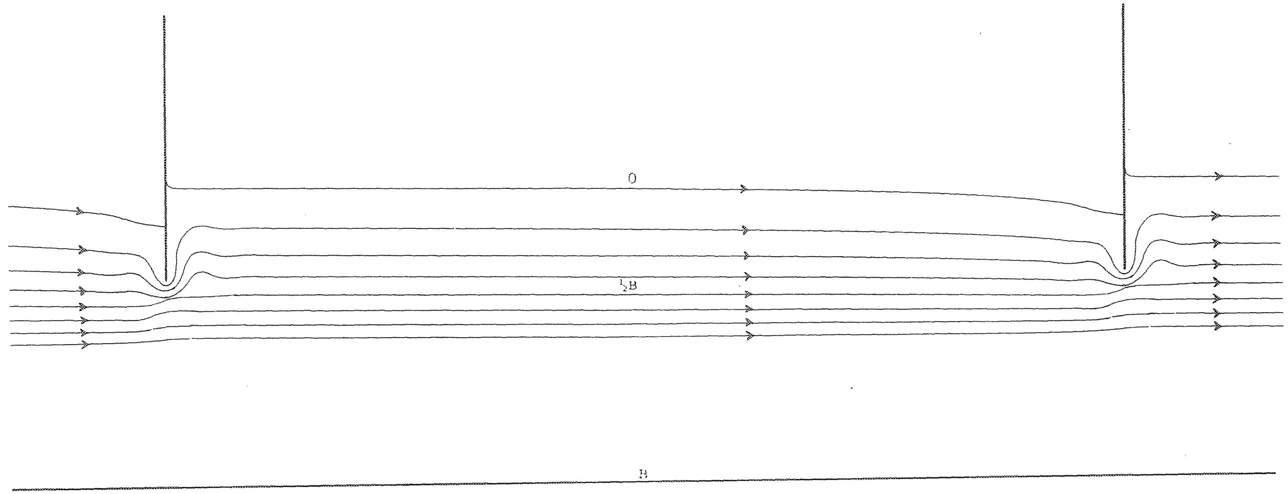

Fig. 5.4. The stream line pattern in the viscous free boundary layer.
Latitude has been exaggerated ten times.

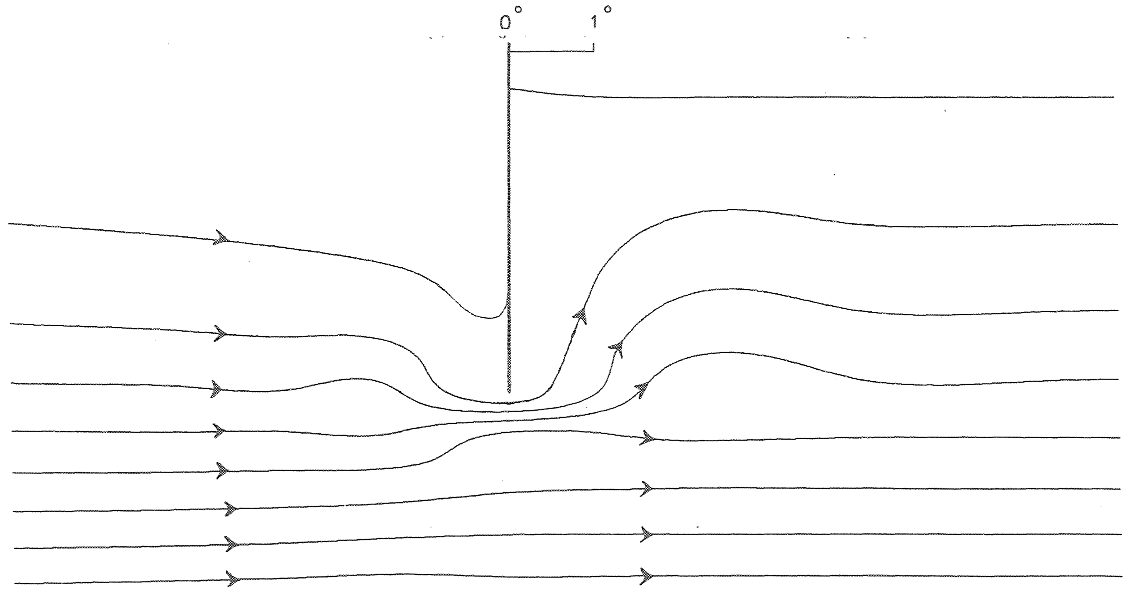

Fig. 5.5. Transport stream lines near the southern tip of South America. The north-south scale is not exagerrated.

CHAPTER 6

THE RETURN AGULHAS CURRENT

6.1. Formulation of the model

In this chapter we will use the free boundary layer concept to analyse one of the curious phenomena in the ocean circulation i.e. the turning of the Agulhas Current south of Africa (see fig. 1.1). For that we use the most simple geometry containing the essential features of that part of the world ocean basin.

The South African peninsula is represented by a straight line (in the transport model) and the ocean is bounded by straight coasts to the east and west. In figure 6.1 this geometry has been sketched.

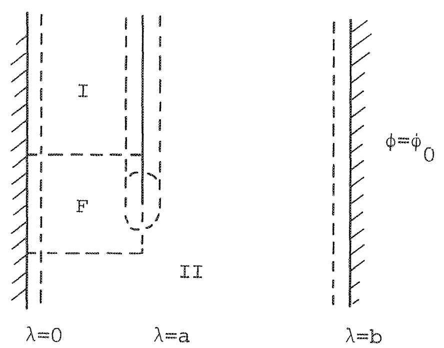

fig. 6.1. The model ocean in which the basin is divided by a continent. The basin is divided in subregions where different approximations of the solution of the associated boundary value problem hold.

We start from the assumptions leading to the linear transport model of §2.11. The equation for the transport stream function then reads ((2.69) with $E_V = E$):

$$E\{\Delta^2\psi + 2\Delta\psi\} - 2\frac{\partial\psi}{\partial\lambda} = -\sqrt{E}.T(\phi,\lambda). \tag{6.1}$$

Moreover the bottom stress is assumed to be negligible. The applied wind stress is purely zonal. Thus the inhomogeneous term represents the curl of the wind stress. It has the simple form:

$$T(\phi) = -\frac{1}{\cos\phi}\frac{d}{d\phi}(\tau(\phi)\cos\phi).$$

The boundary conditions are

$$\psi = \frac{\partial\psi}{\partial\lambda} = 0 \quad \text{along the coasts.} \tag{6.2}$$

6.2. Construction of approximations

In figure 6.1 the basin is divided in different subareas in which the solution of (6.1) and (6.2) will exhibit different behaviour.

Considering the regions I ($\{0 < \lambda < a;\ \phi > \phi_0\}$) and II ($\{0 < \lambda < b;\ \phi < \phi_0\} \cup \{a < \lambda < b;\ \phi \geq \phi_0\}$) the results of §4.3 can immediately be applied. This leads to:

$$\psi_I(\lambda,\phi;E) = \sqrt{E}.T(\phi).\ \frac{\lambda-a}{2} + O(E) \tag{6.3}$$

$$\psi_{II}(\lambda,\phi;E) = \sqrt{E}.T(\phi).\ \frac{\lambda-b}{2} + O(E), \tag{6.4}$$

with $O(E^{1/3})$ boundary layers along the coasts. The western boundary layer along $\lambda = a^+$, $\phi > \phi_0$ reads:

$$\tilde{\psi}(\zeta,\phi;E) = \sqrt{E}.T(\phi)\left(\frac{b-a}{2}\right)\left\{\frac{2}{\sqrt{3}}\,e^{-\frac{\beta\zeta}{2}}\sin\left(\frac{\beta\sqrt{3}\zeta}{2}+\frac{\pi}{3}\right)-1\right\} + O(E^{5/6}),$$

with

$$\zeta = \frac{\lambda-a}{E^{1/3}}, \qquad \beta^3 = \frac{2}{\cos^4\phi}.$$

Around $\lambda = a^+$, $\phi = \phi_0$ an $E^{1/3}\times E^{1/3}$ boundary layer emerges. The leading term of the local approximation must satisfy a differential equation which is of the same structure as equation (7.1) (with E = 1). If $\zeta = \frac{\lambda-a}{E^{1/3}}$ and $\tau = \frac{\phi-\phi_0}{E^{1/3}}$ we will denote this approximation by

$$\psi^*(\zeta,\tau;E) = \sqrt{E}\{\psi^*_{(0)}(\zeta,\tau) + \ldots\}.$$

(-) The free boundary layer (F)

The interaction between the regions I and II takes place through a viscous shear layer along the characteristic $\phi = \phi_0$ $(0 < \lambda < a)$. The appropriate local coordinate is:

$$\eta = \frac{\phi-\phi_0}{E^{1/4}} .$$

Let the local expansion be

(6.5) $$\psi_F(\lambda,\eta;E) = \sqrt{E}.\psi_F^{(0)}(\lambda,\eta) + E.\psi_F^{(1)}(\lambda,\eta) + \ldots .$$

The differential equation for $\psi_F^{(0)}$ reads

(6.6) $$\frac{\partial^4\psi_F^{(0)}}{\partial\eta^4} - 2\frac{\partial\psi_F^{(0)}}{\partial\lambda} = -T(\phi_0).$$

The solution has to satisfy the matching conditions:

(6.7) $$\psi_F^{(0)}(\lambda,\eta) \to T(\phi_0).\frac{\lambda-a}{2} \qquad \text{as } \eta \to \infty$$

(6.8) $$\psi_F^{(0)}(\lambda,\eta) \to T(\phi_0).\frac{\lambda-b}{2} \qquad \text{as } \eta \to -\infty.$$

Along $\lambda = a$ we impose the "initial" condition:

(6.9) $$\psi_F^{(0)}(a,\eta) = g(\eta)$$

where

$$g(\eta) = \begin{cases} 0 & \text{for } \eta \geq 0 \\ \psi^-(\eta) & \text{for } \eta < 0 \end{cases}$$

The function ψ^- is defined such that it equals $\psi^*_{(0)}(0,\eta.E^{-1/12})$ if $\eta = O(E^{1/12})$ (so $\phi - \phi_0 = O(E^{1/3})$) and $\psi_{II}^{(0)}(a,\phi_0)$ otherwise.

A fundamental solution $e(\lambda,\eta)$ of the homogeneous differential equation $\frac{\partial^4 e}{\partial\eta^4} - 2\frac{\partial e}{\partial\lambda} = 0$ can easily be constructed by means of Fourier transformation. It reads:

$$(6.10) \qquad e(\lambda,\eta) = \frac{1}{\pi}\int_0^\infty \cos(\eta t)e^{-t^4.(\frac{a-\lambda}{2})}dt \qquad (\lambda < a,\ -\infty < \eta < \infty),$$

(so $\lim_{\lambda\uparrow a} e(\lambda,\eta) = \delta(\eta)$ where $\delta(\eta)$ is the Dirac δ-distribution). This leads to the solution of the problem formulated by (6.7) through (6.9):

$$(6.11) \qquad \psi_F^{(0)}(\lambda,\eta) = \int_{-\infty}^{\infty} g(\eta')e(\lambda,\eta-\eta')d\eta' + T(\phi_0).\frac{\lambda-a}{2} =$$

$$= \frac{1}{\pi}\int_{-\infty}^{0}\psi^-(\eta')\int_0^\infty \cos((\eta-\eta')t).e^{-t^4(\frac{a-\lambda}{2})}dtd\eta' + T(\phi_0).\frac{\lambda-a}{2}\ .$$

If the integration interval $(-\infty,0)$ is divided in $(-\infty,-k.E^{1/12}) \cup (-k.E^{1/12},0)$, where k is an arbitrary positive constant, (6.11) can be written in the form:

$$(6.12) \qquad \psi_F^{(0)}(\lambda,\eta) = \frac{1}{\pi}\int_{-\infty}^{0}\psi_{II}^{(0)}(a,\phi_0)e(\lambda,\eta-\eta')d\eta' +$$

$$+ \frac{1}{\pi}\int_{-k.E^{1/12}}^{0}(\psi^*_{(0)}(0,\eta'E^{-1/12})-\psi_{II}^{(0)}(a,\phi_0)).e(\lambda,\eta-\eta')d\eta' +$$

$$+ T(\phi_0).\frac{\lambda-a}{2}.$$

The contribution to the free boundary layer of the $E^{1/3}\times E^{1/3}$-region turns out to be $O(E^{1/12})$:

$$(6.13) \qquad \frac{1}{\sqrt{E}}\psi_F(\lambda,\eta;E) = \frac{1}{\pi}\int_{-\infty}^{0}\psi_{II}^{(0)}(a,\phi_0)\int_0^\infty \cos((\eta-\eta')t)e^{-t^4(\frac{a-\lambda}{2})}dtd\eta' +$$

$$+ T(\phi_0).\frac{\lambda-a}{2} + O(E^{1/12}) \equiv \psi_F^{(0)*}(\lambda,\eta) + O(E^{1/12}).$$

An $O(E^{1/3})$ western boundary layer serves to impose the boundary conditions along $\lambda = 0$. The structure of this boundary layer is the same as that of the corresponding layers in the regions I and II. Using an analogous notation we find:

$$(6.14) \qquad \frac{1}{\sqrt{E}}\tilde{\psi}_F(\zeta,\eta;E) = \tilde{\psi}_F^{(0)}(\zeta,\eta) + E^{1/3}\tilde{\psi}_F^{(1)}(\zeta,\eta) + \ldots$$

with

$$(6.15) \qquad \tilde{\psi}_F^{(0)}(\zeta,\eta) = \psi_F^{(0)}(0,\eta)\cdot\{1 - \frac{2}{\sqrt{3}}e^{-\frac{\beta\zeta}{2}}\sin(\frac{\beta\sqrt{3}}{2}\zeta + \frac{\pi}{3})\}$$

(where now $\beta^3 = \frac{2}{\cos^4\phi_0}$, $\zeta = \frac{\lambda}{E^{1/3}}$).

6.3. Discussion of the results

We are now in a position to apply different wind stresses in our model ocean. It turns out that the positions of the maximum and the zeros of the wind stress curl with respect to the southern tip of the African continent are of particular importance. In the figures (6.2) through (6.5) subsequently these positions are shifted a little. The resulting changes in the flow pattern are dramatic.

If the position of the curl maximum is above $\phi = \phi_0$ (fig. 6.3), two maxima of the transport stream function appear, one on each side of the African Continent. To the south of Africa there is a saddle point, which represents a place in the ocean where there is no (vertically averaged) motion. A part of the flow through the western boundary layer along $\lambda = a$ is returned to the east, where the saddle point serves as the "retroflexion" point. The position of this point depends on the wind stress distribution and it is not "fixed" to South Africa (see fig. 6.3 and 6.4).

It is very remarkable that such a wandering retroflexion point has indeed been observed recently in satellite infra-red images of the region under consideration (e.g. HARRIS, et.al. (1978), HARRIS and VAN FOREEST (1978)).

The wind stress curl in fig. 6.3 qualitatively resembles most the actual one (e.g. EVANSON and VERONIS, 1975). It gives a picture of the Agulhas Current and the Return Agulhas Current. A part of the Agulhas transport passes the South African coast and turns north westward into the Atlantic. After crossing the ocean it is deflected southward where it constitutes the Brasil Current. It is obvious that the transport of the Brasil Current is, under these wind stress conditions, much smaller than that of the Agulhas Current. The above theory gives a simple explanation of that reduction of transport as coupled to the turning of (a part of) the Agulhas Current.

If the maximum of the wind stress curl is situated south of ϕ_0 the full transport of the western boundary layer along $\lambda = a$ ($\phi > \phi_0$) is deflected westward through the free boundary layer (see fig. 6.5). In that case the transport of the Brasil Current would exceed the Agulhas transport. In fig. 6.2 one of the zeros of the wind curl coincides with the latitude ϕ_0. The (first approximation of the) free boundary layer then disappears ($T(\phi_0) = 0$). The zero stream line that passes through the southern tip of the continent divides the basin in three separate parts that do not interact.

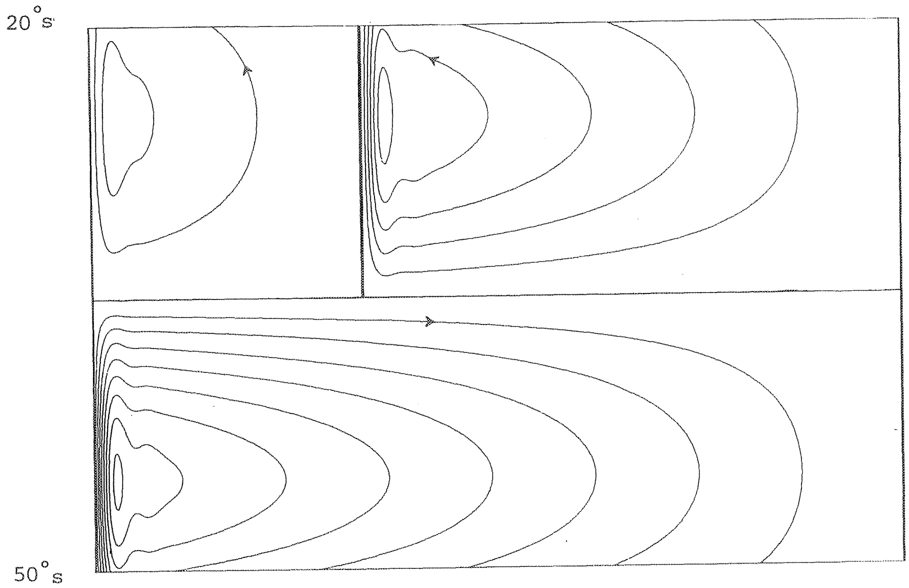

fig. 6.2. $T(\phi) = \sin(9(\phi - \frac{\pi}{36}))$.

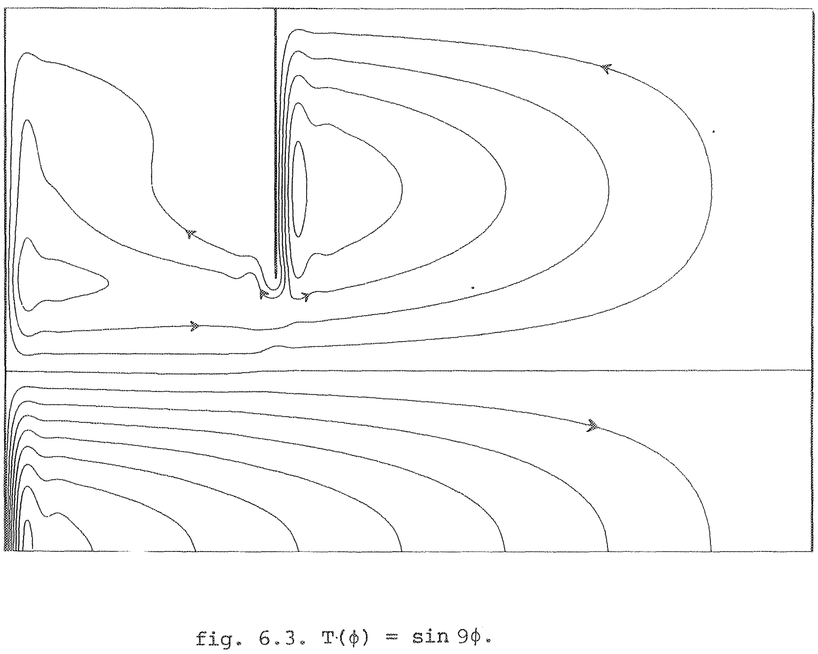

fig. 6.3. $T(\phi) = \sin 9\phi$.

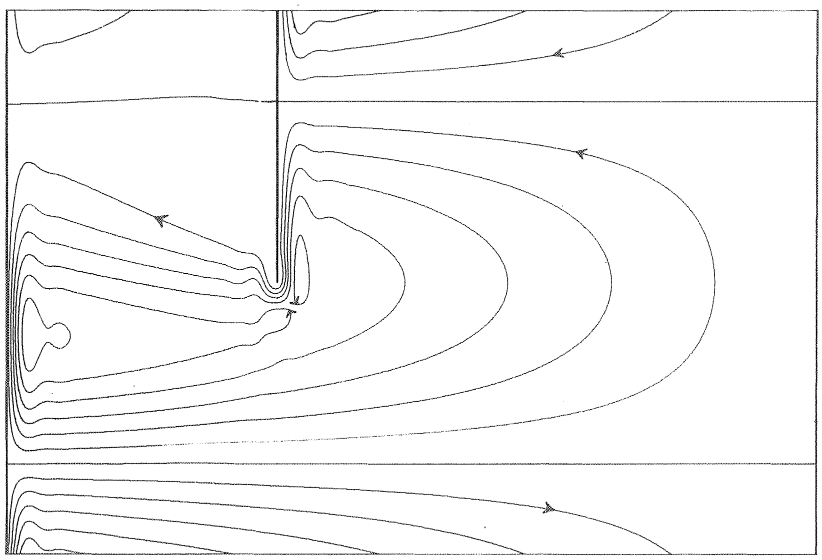

fig. 6.4. $T(\phi) = \sin(9(\phi + \frac{\pi}{36}))$.

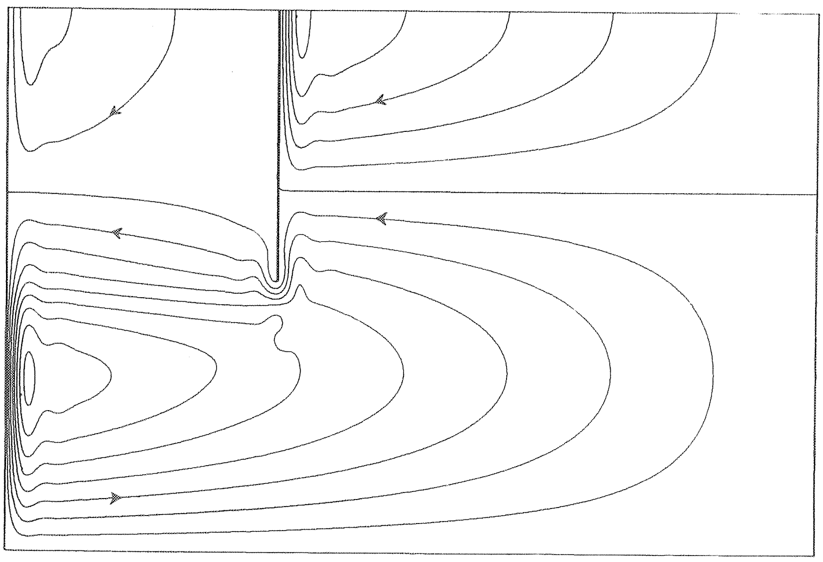

fig. 6.5. $T(\phi) = \sin(9(\phi + \frac{\pi}{18}))$.

6.4. The influence of a more realistic shape of the South-African continent

We shall analyse now the influence on the free boundary layer of a more realistic geometry of the African continent. As in the line shaped case a free boundary layer propagates westward along the characteristic through the southern tip ($\phi = \phi_0$) of the peninsula. Instead of an "initial" condition (6.9) along the line $\lambda = a$ for the first approximation such a condition must now be satisfied along the curved continent (for $\eta > 0$). A comparison will be made between this problem and one in which the same initial condition is applied at the line $\lambda = a$. The difference with the model of §6.1 and 6.2 then appears in the initial condition for $\eta \leq 0$. It now reads: $\psi(\lambda = a, \eta \leq 0) = \psi^{-}(\eta)$ where the function ψ^{-} is determined by the approximation more eastward in the basin ($\lambda > a$). The structure of this approximation differs from that when the continent is a line. Due to the curvature of the boundary an extra boundary layer appears near the southern tip of which the $E^{1/3} \times E^{1/3}$-layer (see §6.2) is a sub boundary layer. If the continent is a second order parabola this "intermediate" boundary layer appears to be $0(E^{1/7} \times E^{2/7})$. Outside this boundary layer the initial condition (ψ^{-}) is determined by the regular approximation for $\lambda > a$. We will show that the influence of the curvature (i.e. of the intermediate boundary layer) on the approximation in the free boundary layer does not appear in the leading term. It is reflected in the order of magnitude of the second approximation which is $0(E^{1/12})$ for a line shaped continent and $0(E^{1/28})$ if the continent is a second order parabola.

Let the boundary of the continent be given by

$$\phi = \phi_0 + g(\lambda - a) \tag{6.16}$$

where the function g has the properties:

$$g(0) = g'(0) = \ldots = g^{(2n-1)}(0) = 0 \qquad (n \geq 1)$$
$$g^{(2n)}(0) \neq 0$$

(which expresses that the tangency of the characteristic in $\lambda = a$, $\phi = \phi_0$ is of order 2n);

$$g'(x) \begin{cases} < 0 & \text{if } x < 0 \\ > 0 & \text{if } x > 0. \end{cases}$$

As a consequence a local approximation of the boundary is given by

(6.17) $\phi = \phi_0 + (\frac{\lambda-a}{k})^{2n}$, with $k > 0$ a constant.

In the variables of the free boundary layer we get:

(6.16') $E^{\frac{1}{4}}.\eta = g(\lambda-a) \quad (\lambda < a)$

and, for instance for $0 \le \eta < N$, (6.17) yields:

(6.17') $\lambda - a = -k.E^{1/8n}.\eta^{1/2n}$.

We define the "thickness" $\delta(E)$ of the continent relative to the width of the free boundary layer by the absolute value of $\lambda - a$ in (6.17') for $\eta = 1$, so $\delta(E) = k.E^{1/8n}$.

The above considerations lead us to the following problem for the first approximation of the free boundary layer:

(6.18) $$\frac{\partial^4 \psi_F^{(0)}}{\partial \eta^4} - 2\frac{\partial \psi_F^{(0)}}{\partial \lambda} = -T(\phi_0).$$

with the "initial condition":

(6.19) $$\begin{cases} \psi_F^{(0)}(a - E^{1/8n}.f(\eta),\eta) = 0 & \text{for } \eta > 0 \\ \psi_F^{(0)}(a,\eta) = \psi^-(\eta) & \text{for } \eta \le 0. \end{cases}$$

Moreover $\psi_F^{(0)}$ has to satisfy the matching conditions (6.7) and (6.8). We take f to be a bounded and monotonically decreasing function with $f(0) = 0$, $f'(\eta) \to \infty$ as $\eta \downarrow 0$.

To determine the function ψ^- a local analysis of the region near $\lambda = a$, $\eta = 0$ with $\lambda > a$ will be carried out in the sequel.

The relations (6.19) are rather unusual conditions. We will compare this problem with the one described by the differential equation (6.18) but with the initial condition given on the line $\lambda = a$:

(6.20) $$\psi^*(a,\eta) = \begin{cases} 0 & \text{for } \eta > 0 \\ \psi^-(\eta) & \text{for } \eta \le 0. \end{cases}$$

Such a problem has already been solved in §6.2 where the solution is given in (6.11). From that formula the value of ψ^* along $\lambda = a - E^{1/8n}.f(\eta)$ $(\eta > 0)$ can be calculated. The result is:

$$\psi^*(a - E^{1/8n}.f(\eta),\eta) = E^{1/8n}.R(\eta) \qquad (\eta > 0), \tag{6.21}$$

where $R(\eta)$ is a bounded function.

For the difference of ψ^* and $\psi_F^{(0)}$ we then get the initial value problem:

$$\left(\frac{\partial^4}{\partial\eta^4} - 2\frac{\partial}{\partial\lambda}\right)(\psi^* - \psi_F^{(0)}) = 0$$

with

$$(\psi^* - \psi_F^{(0)})(a - E^{1/8n}.f(\eta),\eta) = E^{1/8n}.R(\eta) \qquad (\eta > 0)$$

$$(\psi^* - \psi_F^{(0)})(0,\eta) = 0 \qquad (\eta \leq 0),$$

and the requirement that $\psi^* - \psi_F^{(0)} \to 0$ as $\eta \to \pm\infty$. The problem thus formulated makes it plausible to assume that for the region under consideration $(0 < \lambda < a,\ -\infty < \eta < \infty)$:

$$\psi_F^{(0)} = \psi^* + O(E^{1/8n}), \tag{6.22}$$

with

$$\psi^*(\lambda,\eta) = \frac{1}{\pi}\int_{-\infty}^{0} \psi^-(\eta') \int_0^\infty \cos((\eta-\eta')t)e^{-t^4\left(\frac{a-\lambda}{2}\right)} dt d\eta' + T(\phi_0).\frac{\lambda-a}{2}\ . \tag{6.23}$$

Let us now turn to the analysis of the boundary layers near $\lambda = a$, $\phi = \phi_0$. This analysis is necessary to gather information about the structure of the function ψ^-. For convenience we take for the local geometry a parabola to which the tangency of the characteristic is of order 2 (so $n = 1$ in (6.17)).

If $t \equiv \cos\phi_0.(\lambda-a)$ and the metric factors are expanded in Taylor series around $\phi = \phi_0$ the vorticity equation (6.1) becomes:

$$E\{\Delta^2\psi + (4(\phi-\phi_0)\mathrm{tg}\phi_0 + \ldots)\frac{\partial^4\psi}{\partial\phi^2\partial t^2} + \frac{\partial^4\psi}{\partial t^4} + \text{lower order terms}\} - $$
$$- 2\cos\phi_0\frac{\partial\psi}{\partial t} = \sqrt{E}.T(\phi) \tag{6.24}$$

where now $\Delta = \dfrac{\partial^2}{\partial\phi^2} + \dfrac{\partial^2}{\partial t^2}$.

A natural system of curvilinear orthogonal coordinates is achieved by introducing parabolic coordinates (u,v):

$$(6.25)\qquad \begin{cases} \phi - \phi_0 - \frac{1}{2}u_0^2 = \frac{1}{2}(v^2 - u^2) \\ t = uv \end{cases}$$

with the prescriptions

$t < 0 \leftrightarrow \quad u > 0$ and $v < 0$

$t > 0 \leftrightarrow \quad u > 0$ and $v > 0$.

If we choose $u_0 = \frac{k\cos\phi_0}{\sqrt{2}}$ the boundary of the continent ((6.17)) coincides with the parabola $u = u_0$. After transformation to parabolic coodinates (6.24) takes the form:

$$(6.26)\qquad E\{\frac{\partial^4\psi}{\partial u^4} + 2\frac{\partial^4\psi}{\partial u^2\partial v^2} + \frac{\partial^4\psi}{\partial v^4} + \ldots\} - 2(u^2+v^2)\cos\phi_0.\{v\frac{\partial\psi}{\partial u} + u\frac{\partial\psi}{\partial v}\} =$$

$$= \sqrt{E}\,.\,T(\phi_0 + \tfrac{1}{2}(u_0^2 - u^2 + v^2))(u^2+v^2)^2$$

with the boundary conditions:

$$(6.27)\qquad \psi = \frac{\partial\psi}{\partial u} = 0 \quad \text{for } u = u_0.$$

A local analysis of the region near $\phi = \phi_0$, $\lambda = a$ now comes down to an analysis near $u = u_0$, $v = 0$. To that end we transform (6.26) and (6.27) to local coordinates (ξ,τ) defined by:

$$(6.28)\qquad u - u_0 = E^{\nu}\xi; \qquad v = E^{\mu}\tau.$$

Performing the transformation yields:

$$(6.29)\qquad L\psi \equiv E^{1-4\nu}\frac{\partial^4\psi}{\partial\xi^4} + E^{1-2\mu-2\nu}\frac{\partial^4\psi}{\partial\xi^2\partial\tau^2} + E^{1-4\mu}\frac{\partial^4\psi}{\partial\tau^4} + \{\text{non-relevant terms}\} -$$

$$- 2\cos\phi_0\{(u_0+E^{\nu}.\xi)^2 + E^{2\mu}.\tau^2\}\{E^{\mu-\nu}.\tau\frac{\partial\psi}{\partial\xi} + (E^{-\mu}.u_0 + E^{\nu-\mu}.\xi)\frac{\partial\psi}{\partial\tau}\} =$$

$$= \sqrt{E}\{(u_0+E^{\nu}.\xi)^2 + E^{2\mu}.\tau^2\}^2.T(\phi_0 - E^{\nu}u_0\xi - E^{2\nu}\frac{\xi^2}{2} + E^{2\mu}\frac{\tau^2}{2}).$$

In figure 6.6 in the (ν,μ)-plane the positions of the significant degenerations of the operator L in (6.29) have been marked. These significant degenerations are:

(6.30) a. $\mu=\nu=\frac{1}{3}$: $-2\cos\phi_0 . u_0^3 \frac{\partial}{\partial\tau} + \frac{\partial^4}{\partial\xi^4} + 2\frac{\partial^4}{\partial\xi^2\partial\tau^2} + \frac{\partial^4}{\partial\tau^4}$

b. $\mu=\nu=0$: $\tau\frac{\partial}{\partial\xi} + (u_0+\xi)\frac{\partial}{\partial\tau}$

c. $\mu=\frac{1}{7},\ \nu=\frac{2}{7}$: $-2\cos\phi_0 . u_0^2 . (\tau\frac{\partial}{\partial\xi} + u_0\frac{\partial}{\partial\tau}) + \frac{\partial^4}{\partial\xi^4}$

d. $\mu=\frac{1}{3},\ \nu=0$: $-2\cos\phi_0 . (u_0+\xi)^3\frac{\partial}{\partial\tau} + \frac{\partial^4}{\partial\tau^4}$

e. $\mu=0,\ \nu=\frac{1}{3}$: $-2\cos\phi_0 . (u_0^2+v^2)\frac{\partial}{\partial\xi} + \frac{\partial^4}{\partial\xi^4}$

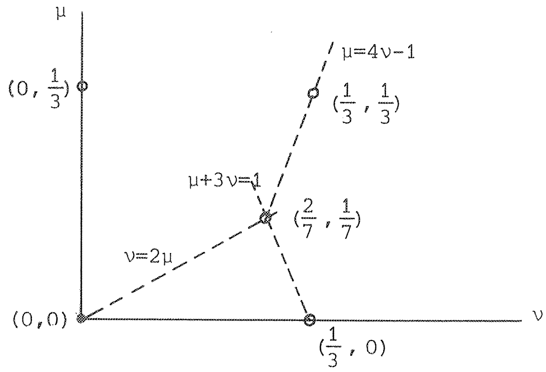

fig. 6.6.

In figure 6.7 the corresponding boundary layers have been drawn schematically. The degeneration (e) $\mu = 0$, $\nu = \frac{1}{3}$ represents the "standard" western boundary layer if $v > 0$ (see fig. 6.7).

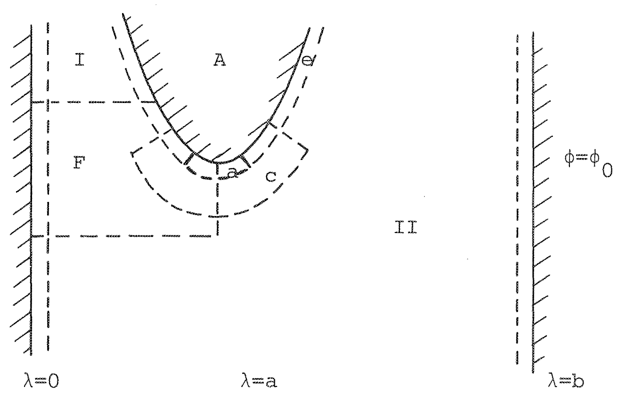

fig. 6.7. The division of the ocean basin in subareas. In regions a, c and e the corresponding degenerations of the vorticity equation hold (see (6.3)).

It brings the flow of region II to rest along the coast of continent A. If we transform the regular approximation ψ_{II} (6.4) to the local variables of this lateral boundary layer the leading term becomes

$$\sqrt{E}.T(\phi_0 + \frac{1}{2}v^2)(\frac{a-b}{2}). \tag{6.31}$$

For the western boundary layer (e) this suggests the expansion

$$\tilde{\psi}(\xi,v;E) = \sqrt{E}.(\tilde{\psi}^{(0)}(\xi,v) + E^{1/3}\tilde{\psi}^{(1)}(\xi,v) + \ldots). \tag{6.32}$$

The first approximation then must satisfy the differential equation

$$\frac{\partial^4\tilde{\psi}^{(0)}}{\partial\xi^4} - 2\cos\phi_0\; v(u_0^2+v^2)\;\frac{\partial\tilde{\psi}^{(0)}}{\partial\xi} = 0 \tag{6.33}$$

with the boundary conditions:

$$\tilde{\psi}^{(0)} = \frac{\partial\tilde{\psi}^{(0)}}{\partial\xi} = 0 \quad \text{in} \quad \xi = 0.$$

Matching with $\psi_{II}^{(0)}$ comes down to the condition

(6.34) $$\lim_{\xi\to\infty} \tilde{\psi}^{(0)}(\xi,v) = T(\phi_0+\tfrac{1}{2}v^2)(\tfrac{a-b}{2})$$

(compare with 6.31).

The solution reads:

(6.35) $$\tilde{\psi}^{(0)}(\xi,v) = T(\phi_0+\tfrac{1}{2}v^2)(\tfrac{a-b}{2})\{1-\tfrac{2}{\sqrt{3}}\,e^{-d(v)\cdot\xi}\sin(d(v)\sqrt{3}\xi+\tfrac{\pi}{3})\}$$

(where $d(v) = (2\cos\phi_0\cdot v\cdot(u_0^2+v^2))^{1/3}$).

The function given in (6.35) is a boundary layer function only if $v > 0$ ($\leftrightarrow \lambda > a$). Therefore the approximation is strictly confined to the region $v > 0$. Near $v = 0$ a different local approximation brings $\psi_{II}^{(0)}$ to rest. The appropriate local variables can be read off in figure 6.6: for $(\mu,\nu) = (\frac{1}{7},\frac{2}{7})$ the significant degeneration (6.30-c) results with a possible sub boundary layer at $(\mu,\nu) = (\frac{1}{3},\frac{1}{3})$.

We will not try to determine the corresponding local approximations but only give a qualitative description of the analysis. For $v > 0$ ($\tau > 0$) the $\mathcal{O}(E^{2/7}\times E^{1/7})$ "intermediate" boundary layer has to match with both ψ_{II}, $\tilde{\psi}$ and the $\mathcal{O}(E^{1/3}\times E^{1/3})$ sub boundary layer. Along the line $\nu = 2\mu$ (fig. 6.7), for $0 < \mu < \frac{1}{7}$, both the differential operator that determines $\psi_{II}^{(0)}$ and that of the intermediate boundary layer degenerate to

$$\tau\frac{\partial}{\partial\xi} + u_0\frac{\partial}{\partial\tau}\,.$$

Therefore, if the property of inclusion holds (ch. 3), matching of the two approximations takes place along this line.

The same reasoning leads to the conclusion that

(-) the matching condition between the western $((\frac{1}{3},0))$ and the intermediate boundary layers holds along the line $3\nu + \mu = 1$ for $0 < \mu < \frac{1}{7}$, where the common degeneration is $-2\cos\phi_0\cdot u_0^2\cdot\tau\frac{\partial}{\partial\xi} + \frac{\partial^4}{\partial\xi^4}$,

(-) the intermediate and its sub boundary layer must match along $\mu - 4\nu = -1$ for $\frac{1}{7} < \mu < \frac{1}{3}$, where the common degeneration is $-2\cos\phi_0\cdot u_0^3\cdot\frac{\partial}{\partial\tau} + \frac{\partial^4}{\partial\xi^4}$.

We now divide the integration interval in (6.23) in $(-\infty,-k\cdot E^{\frac{1}{28}})\cup(-k\cdot E^{\frac{1}{28}},0)$, with k an arbitrary positive constant. For $\eta \in (-k\cdot E^{\frac{1}{28}},0)$ (which corresponds to $u - u_0 = O(E^{\frac{2}{7}})$) ψ^- represents the intermediate boundary layer, so we can write

$$(6.36)\qquad \psi^* = \frac{1}{\pi}\int_{-\infty}^{0} \psi_{II}^{(0)}(a,\phi_0)\int_0^{\infty}\cos\{(\eta-\eta')t\}e^{-t^4(\frac{a-\lambda}{2})}\,dtd\eta' +$$

$$+\frac{1}{\pi}\int_{-k.E^{\frac{1}{28}}}^{0}(\psi^{int}(\eta')-\psi_{II}^{(0)}(a,\phi_0))\int_0^{\infty}\cos\{(\eta-\eta')t\}.e^{-t^4(\frac{a-\lambda}{2})}\,dtd\eta' +$$

$$+ T(\phi_0).\ \frac{\lambda-a}{2}\ .$$

(ψ^{int} denotes the first approximation of the intermediate layer). Consequently:

$$(6.37)\qquad \psi^* = \frac{1}{\pi}\int_{-\infty}^{0}\psi_{II}^{(0)}(a,\phi_0)\int_0^{\infty}\cos\{(\eta-\eta')t\}e^{-t^4(\frac{a-\lambda}{2})}\,dtd\eta + T(\phi_0).\ \frac{\lambda-a}{2} + + O(E^{\frac{1}{28}}).$$

Comparison with (6.13) shows that the leading term of the free boundary layer for the parabolic continent equals the one for a line shaped continent! The distinction appears in the order of magnitude of the remainder term.

Because $\psi_{II}^{(0)}(a,\phi_0)$ is a constant, after some manipulation (6.37) can be rewritten in the form:

$$(6.38)\qquad \psi^* = \frac{T(\phi_0)}{2}\Big[\frac{b-a}{\pi}\int_0^{\infty}\frac{\sin\eta t}{t}\,e^{-t^4(\frac{a-\lambda}{2})}dt + \lambda - \frac{a+b}{2}\Big] + O(E^{\frac{1}{28}}) \equiv \Psi + O(E^{\frac{1}{28}}).$$

Combining this result with (6.22) we find that

$$(6.39)\qquad \psi_F = \Psi + O(E^{\frac{1}{28}}).$$

From (6.22) it follows that if the order of tangency (2n) in $\lambda = a$, $\phi = \phi_0$ increases the remainder term becomes more and more important. As $n \to \infty$ the southern coast approaches a characteristic boundary. The analysis for such a configuration is given in the following chapter.

CHAPTER 7

THE CIRCULATION IN OCEAN BASINS WHEN A PART OF THE BOUNDARY IS CHARACTERISTIC

7.1. A basin with a corner shaped eastern boundary

Let the boundaries of a model ocean be given by:

$$\lambda = 0 \qquad \text{western coast}$$

$$\left.\begin{array}{lll} \lambda = \lambda_0 & \text{for} & \phi > \phi_0 \\ \lambda = \lambda_1 & \text{for} & \phi < \phi_0 \\ \phi = \phi_0 & \text{for} & \lambda_0 \le \lambda \le \lambda_1 \end{array}\right\} \quad \text{eastern coast}$$

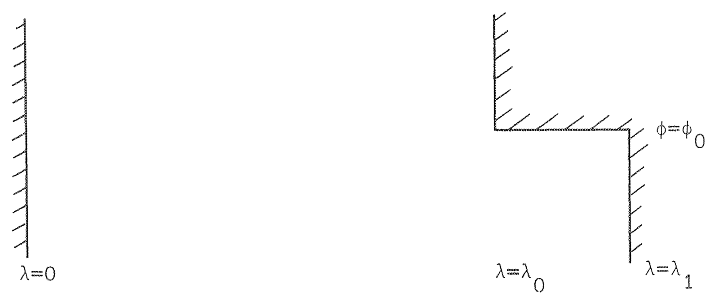

fig. 7.1. A model of an ocean basin including a corner shaped eastern boundary.

A domain with this geometry can for instance be regarded as a simple model of a section of the Atlantic Ocean including the Gulf of Guinee (see fig. 7.1). The west coast $\lambda = 0$ then describes the American coast. Though the boundaries are not realistic this geometry contains the essential mathematical difficulties involved in realistic geometries.

The eastern boundary cannot be given by a function of ϕ only. Part of it coincides with a characteristic of the unperturbed part of the operator L_E (2.72). Therefore the method (and results) of §4.3 can not be applied to this case. In this section the method of constructing approximations of the solution of equation (2.69) satisfying the boundary conditions

$$\psi = \frac{\partial\psi}{\partial n} = 0 \quad \text{along the coasts} \tag{7.1}$$

will be extended such that this type of geometry can be included.

For this purpose we divide the ocean basin in a northern and a southern part, separated by the parallel circle $\phi = \phi_0$. In the northern half, $\phi > \phi_0$, the method of §4.3 applies to give for the approximation in the interior:

$$\psi_N(\lambda,\phi;E) = \sqrt{E}\{\psi_N^{(0)}(\lambda,\phi) + O(E)\}, \quad \text{with}$$

$$\psi_N^{(0)}(\lambda,\phi) = T(\phi_0).\frac{\lambda-\lambda_0}{2} \qquad (\phi > \phi_0). \tag{7.2}$$

Along $\lambda = 0$ and $\lambda = \lambda_0$ the $O(E^{1/3})$ lateral boundary layers are given by (4.50) and (4.54) with $f(\phi) \equiv \lambda_0$ and $g(\phi) \equiv 0$.

In an analogous way an approximation for $\phi < \phi_0$ can be written down:

$$\psi_S(\lambda,\phi;E) = \sqrt{E}\{\psi_S^{(0)}(\lambda,\phi) + O(E)\}, \quad \text{where}$$

$$\psi_S^{(0)}(\lambda,\phi) = \frac{T(\phi)}{2}(\lambda-\lambda_1) \qquad (\phi < \phi_0). \tag{7.3}$$

From (7.2) and (7.3) we immediately conclude that $\psi_N^{(0)}$ together with $\psi_S^{(0)}$ cannot represent an approximation for the full interior of the basin. Along the line $\phi = \phi_0$ (for $0 < \lambda < \lambda_0$) a discontinuity appears.

We will remove this discontinuity by introducing a free boundary layer along $\phi = \phi_0$, $0 < \lambda < \lambda_0$ where also the viscous terms in the transport equation (2.69) play an important role. In fact this shear layer "originates" in the corner point (λ_1,ϕ_0), develops westward along the coast where it serves to bring the interior flow ψ_S to rest and leaves the coast in (λ_0,ϕ_0).

It is along these lines that we shall develop the construction: first the lateral ("parabolic") boundary layer along the coast will be analysed. This yields an "initial" condition (where λ serves as the time-like variable) along $\lambda = \lambda_0$, $\phi < \phi_0$ which the free boundary layer has to satisfy.

(i) The parabolic boundary layer

We define for the boundary layer the following stretching of the north-south variable (which corresponds to a significant degeneration of the operator L_E, see chapter 3):

(7.4) $$\eta = \frac{\phi-\phi_0}{E^{1/4}}$$

and, for convenience:

$$x = \frac{\lambda_1-\lambda}{2} .$$

As a consequence we have $\eta < 0$ and $0 < x < \frac{\lambda_1-\lambda_0}{2}$. For the solution in this region we assume the following expansion:

$$\psi_p(x,\eta;E) = \sqrt{E}\{\psi_p^{(0)}(x,\eta) + E^{1/4}\psi_p^{(1)}(x,\eta) + \ldots\}.$$

Inserting this expansion in the (transformed) transport equation leads to the following problem for the leading term:

(7.5) $$\frac{\partial^4\psi_p^{(0)}}{\partial\eta^4} + \frac{\partial\psi_p^{(0)}}{\partial x} = - T(\phi_0)$$

with boundary conditions:

(7.6) $$\psi_p^{(0)}(\eta=0) = \frac{\partial\psi_p^{(0)}}{\partial\eta}(\eta=0) = 0 \quad \text{and} \quad \psi_p^{(0)}(x=0) = 0.$$

Moreover $\psi_p^{(0)}$ has to satisfy a matching condition which in this case simply reads:

(7.7) $$\lim_{\eta\to-\infty} \psi_p^{(0)}(x,\eta) = - T(\phi_0)x$$

For the solution of this problem we again make use of the similarity solutions for the homogeneous differential equation $\psi_{\eta\eta\eta\eta} + \psi_x = 0$ of the form $x^{n/4}.w(\frac{\eta}{x^{1/4}})$ that have been analysed by GILL and SMITH (1970, see also §5.5). Using their notation we find:

(7.8) $$\psi_p^{(0)}(x,\eta) = - T(\phi_0).x.[Jo_4(\frac{-\eta}{x^{1/4}}) + Ko_4(\frac{-\eta}{x^{1/4}}) + 1].$$

$Jo_4(y)$ and $Ko_4(y)$ are both oscillating functions, infinitely differentiable and diminishing at an exponential rate as $y \to \infty$ (see fig. 7.2). The same properties hold for the derivatives.

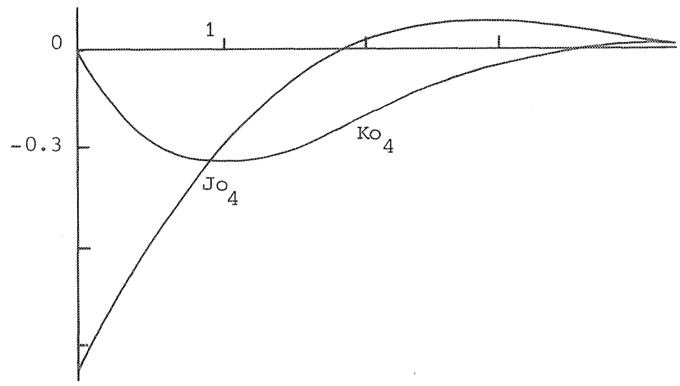

Fig. 7.2. The functions Jo_4 and Ko_4 for positive values of the argument.

In an analogous way higher order terms of the approximation can be calculated.

The solution (7.8) does not satisfy the boundary condition $\frac{\partial\psi}{\partial x}(x=0)=0$. For that again an $O(E^{1/3})$ boundary layer along x = 0 can be constructed in a fully analogous way to §4.3. As a result we find:

$$\bar{\psi}_p(\xi,\eta;E) = E^{5/6}\bar{\psi}_p^{(1)}(\xi,\eta) + O(E^{7/6}), \quad \text{where } \xi = \frac{2x}{E^{1/3}} \text{ and}$$

(7.9)

$$\bar{\psi}_p^{(1)}(\xi,\eta) = \frac{T(\phi_0)}{2}\{\frac{1}{\alpha(\phi_0)}(1-e^{-\alpha(\phi_0)\cdot\xi})-\xi\}.$$

It can easily be verified that $\bar{\psi}_p^{(1)}$ also matches with $\bar{\psi}^{(1)}$ (4.50). Now (7.9) doesn't satisfy the boundary conditions along $\eta = 0$. To solve this problem an analysis has to be added of the corner region near $\phi = \phi_0$, $\lambda = \lambda_1$ where the appropriate local variables are given by:

$$\xi = \frac{2x}{E^{1/3}} \quad \text{and} \quad \mu = \frac{\phi-\phi_0}{E^{1/3}} .$$

The leading term of the local expansion then must satisfy the following differential equation:

$$\frac{\partial^4\psi}{\partial\mu^4} + \frac{2}{\cos^2\phi_0}\frac{\partial^4\psi}{\partial\xi^2\partial\mu^2} + \frac{1}{\cos^4\phi_0}\frac{\partial^4\psi}{\partial\xi^4} + 2\frac{\partial\psi}{\partial\xi} = 0$$

with matching and boundary conditions.

We will not tackle this problem. It would only give some insight in the local behaviour of the solution. However, approximations in the other parts of the basin already have been constructed without the a priori knowledge of the corner flow.

(ii) The free boundary layer

For the shear layer we find the same stretched variable $\eta = \frac{\phi-\phi_0}{E^{1/4}}$ as for the parabolic layer. However, the variable η now ranges between $-\infty$ and ∞ while $\frac{\lambda_1-\lambda_0}{2} < x < \frac{\lambda_1}{2}$. If

$$\psi_F(x,\eta;E) = \sqrt{E}\{\psi_F^{(0)}(x,\eta) + O(E^{1/4})\},$$

again the differential equation (7.5) is found for the leading term of the expansion. Now there are matching conditions to the south and to the north which simply read:

$$(7.10)\qquad \begin{cases} \lim\limits_{\eta\to\infty} \psi_F^{(0)}(x,\eta) = T(\phi_0).(\frac{\lambda_1-\lambda_0}{2} - x) \\ \lim\limits_{\eta\to-\infty} \psi_F^{(0)}(x,\eta) = -T(\phi_0).x. \end{cases}$$

In $x = \frac{\lambda_1-\lambda_0}{2}$ we impose the boundary condition

$$(7.11)\qquad \psi_F^{(0)}(x = \frac{\lambda_1-\lambda_0}{2}) = \begin{cases} 0 & \text{for } \eta \geq 0 \\ \psi_p^{(0)}(\frac{\lambda_1-\lambda_0}{2}, \eta) & \text{for } \eta < 0 \end{cases}$$

where $\psi_p^{(0)}(x,\eta)$ has been calculated in the foregoing.

The problem for $\psi_F^{(0)}$ is identical to the one formulated and solved in §6.1. The solution reads:

$$(7.12)\qquad \psi_F^{(0)}(x,\eta) = \frac{1}{\pi}\int_{-\infty}^{\infty} g(\eta')\int_0^{\infty} \cos((\eta-\eta')t)e^{-t^4(x-\frac{\lambda_1-\lambda_0}{2})}dt d\eta' - T(\phi_0).x$$

where

$$g(\eta') \equiv \begin{cases} T(\phi_0).\frac{\lambda_1-\lambda_0}{2} & \text{for } \eta' \geq 0 \\ \psi_p^{(0)}(\frac{\lambda_1-\lambda_0}{2}, \eta') + T(\phi_0).\frac{\lambda_1-\lambda_0}{2} & \text{for } \eta' < 0. \end{cases}$$

Moreover it is easy to show the following properties for $\eta < 0$:

$$\lim_{x \downarrow \frac{\lambda_1-\lambda_0}{2}} \frac{\partial^4 \psi_F^{(0)}}{\partial \eta^4} = \lim_{x \uparrow \frac{\lambda_1-\lambda_0}{2}} \frac{\partial^4 \psi_p^{(0)}}{\partial \eta^4}$$

and

$$\lim_{x \downarrow \frac{\lambda_1-\lambda_0}{2}} \frac{\partial \psi_F^{(0)}}{\partial x} = \lim_{x \uparrow \frac{\lambda_1-\lambda_0}{2}} \frac{\partial \psi_p^{(0)}}{\partial x} .$$

With these additional properties the functions $\psi_p^{(0)}$ and $\psi_F^{(0)}$ together provide an approximation on the whole interval $0 < x < \frac{\lambda_1}{2}$ (for $\eta < 0$).

Again a lateral boundary layer along $x = \frac{\lambda_1}{2}$, $-\infty < \eta < \infty$ has to be analysed. It leads to the solution:

$$\text{(7.13)} \qquad \tilde{\psi}_F^{(0)}(\zeta,\eta) = \psi_F^{(0)}(\frac{\lambda_1}{2}, \eta)\{1 - \frac{2}{\sqrt{3}} e^{\frac{-\beta(\phi_0).\zeta}{2}} \sin(\frac{\beta(\phi_0)\sqrt{3}}{2} \zeta + \frac{\pi}{3})\}.$$

Along $x = \frac{\lambda_1-\lambda_0}{2}$, $\eta > 0$ it is easy to verify that $\frac{\partial \psi_F^{(0)}}{\partial x} \neq 0$. The boundary layer that brings the flow to rest along this part of the eastern coast is of the same structure as (7.9). If now $\xi_0 \equiv \frac{\lambda_0-\lambda}{E^{1/3}}$ we find:

$$\text{(7.14)} \qquad \bar{\psi}_F^{(1)}(\xi_0,\eta) = \frac{T(\phi_0)}{2} \{\frac{1}{\alpha(\phi_0)} (1 - e^{-\alpha(\phi_0).\xi_0}) - \xi_0\}.$$

Along $\eta = 0$, $\xi_0 > 0$ a singularity appears which has to be removed by a local analysis of an $O(E^{1/3} \times E^{1/3})$ region near the second corner point in the boundary: $\phi = \phi_0$, $\lambda = \lambda_0$. This analysis will not be carried out too because it has no consequences for the global stream line pattern.

In figure (7.3) a picture is drawn showing schematically the complete set of boundary layers needed to construct a uniform asymptotic approximation of the solution of the transport equation (2.69) satisfying the boundary conditions (7.1).

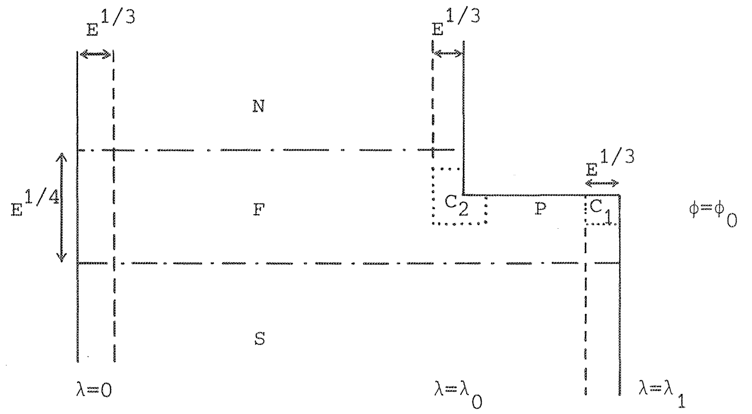

Fig. 7.3. The subdivision of the model ocean in northern (N) and southern (S) interior regions, with an $O(E^{1/4})$ free viscous boundary layer (F) in between. This layer "originates" in the $O(E^{1/3} \times E^{1/3})$ corner region C1, develops as a "parabolic" boundary layer (P) along the coast and leaves the coast through the corner region C2. Along $\lambda = 0$; $\lambda = \lambda_0$ & $\phi > \phi_0$; $\lambda = \lambda_1$ & $\phi < \phi_0$ $O(E^{1/3})$ lateral bcundary layers serve to bring the flow to rest at the coasts.

7.2. Discussion and conclusions

For three different functions describing the wind stress curl a sketch of the resulting stream line pattern is given in figure 7.4. The longitude and latitudes determining the coast line geometry have been chosen to be $\phi_0 = \frac{\pi}{36}$, $\lambda_0 = \frac{\pi}{4}$ and $\lambda_1 = \frac{13\pi}{36}$. Figures 7.4b and c are obtained from 7.4a by applying a phaseshift on the curl (northward and southward respectively).

It is clear from the figure that the "perceptibility" of the shear layer depends on the properties of the given wind stress curl. In particular the position of the zeros with respect to the axis $\phi = \phi_0$ of the boundary layer is important. When such a zero approaches the latitude $\phi = \phi_0$ the free boundary layer becomes less important and the westward drift current (or eastward, depending on the sign of the curl) will dominate the picture.

For reasons of comparison in figure (7.5) the transport stream lines have been drawn for a model ocean basin where the eastern and western coasts coincide with latitude circles.

A first conclusion from these calculations and pictures is the following: in this simple linearized model the shape of the eastern coast can be a determining factor in the formation of a strong "eastern" boundary current (which corresponds with the parabolic boundary layer). Moreover this coastal shape is "reflected" into the interior through the viscous shear layer and in this way it remodels the path of the drift current in the interior.

To compare with observations we analyse further the figures. Figure 7.4b fits best in the Gulf of Guinee type of situation because qualitatively the given $T(\phi)$ resembles most the actual wind stress curl (e.g. EVANSON and VERONIS, 1975). In reality the period of the curl is about two thirds of the one chosen here. For the sake of clearness we have exaggerated the period a little in the figure.

The strong eastern boundary current represents the Guinee current while the eastward current in the interior can be considered to be the Equatorial Counter Current. It can not be looked upon as a pure drift current, as is the case in fig. 7.5b, because its path is altered by the viscous effects of the free shear layer.

The intense northward current along the western coast of the basin can be identified with the Guiana Current. As a result of the linear analysis the full Guiana Current leaves the coast to form the Equatorial Counter Current. This doesn't agree with observation where it overshoots partly the latitude of zero wind stress curl and links up with the Florida Current (see for instance SVERDRUP et.al. 1942). The remaining part turns eastward. In this way the transport of the Equatorial Counter Current is reduced. Obviously too much simplifications have been made to obtain this phenomenon from the theory developed here.

In fig. 7.4c the driving wind stress curl is obtained from 7.4a by a phaseshift to the north. Moreover its direction has been reversed. The resulting flow pattern can be regarded then as a (simple) model of the southern part of the Indian Ocean. The strong southward current along the western coast can now be looked upon as the Agulhas Current. Tasmania is "fixed" to Australia (by the continental shelf) and in this way it forms the corner like shape at the east coast of our basin.

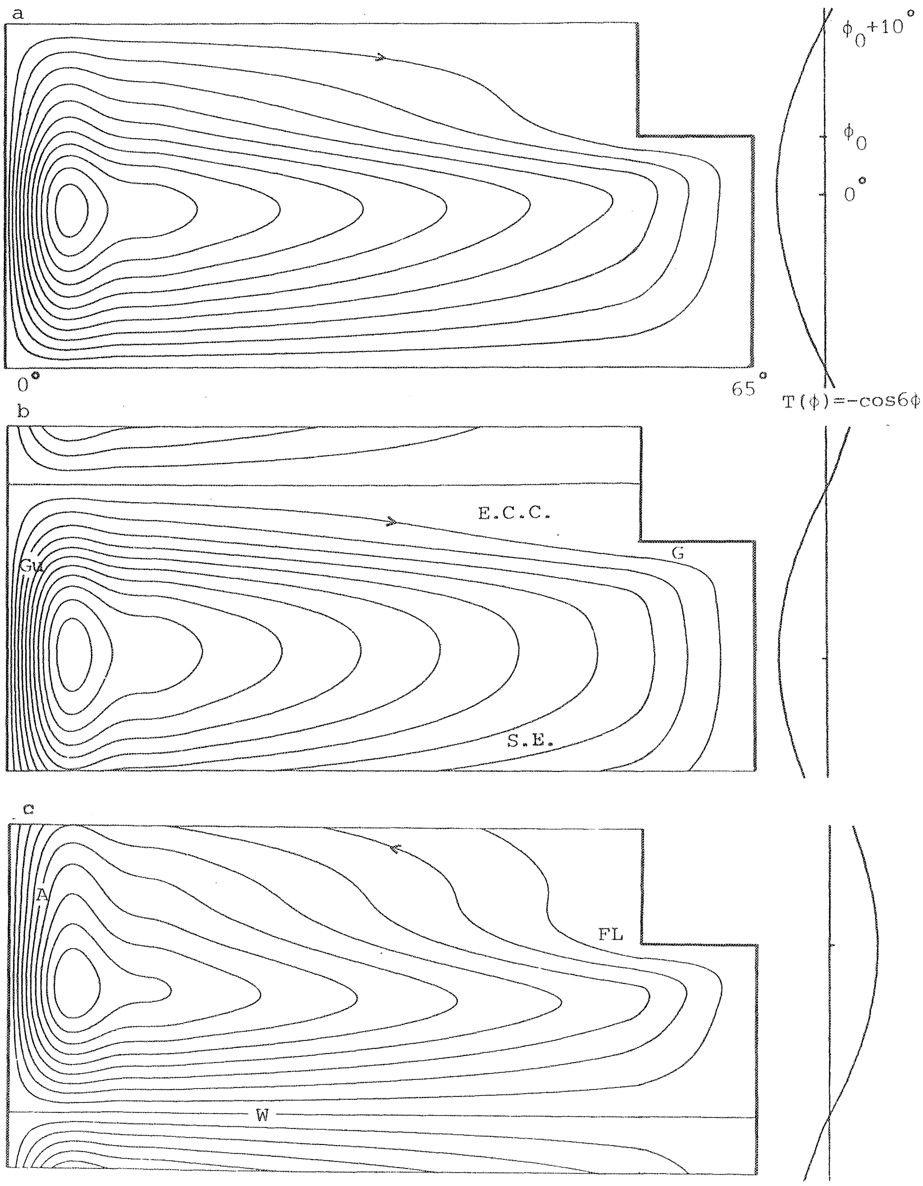

Fig. 7.4. The stream line pattern for three different wind stress curls T(φ). G: Guinee Current; Gu: Guiana Current; E.C.C.: Equatorial Counter Current; S.E.: South Equatorial Current; Fl.: Flinders Current; A.: Agulhas; W.: West Wind Drift.

a

b

c

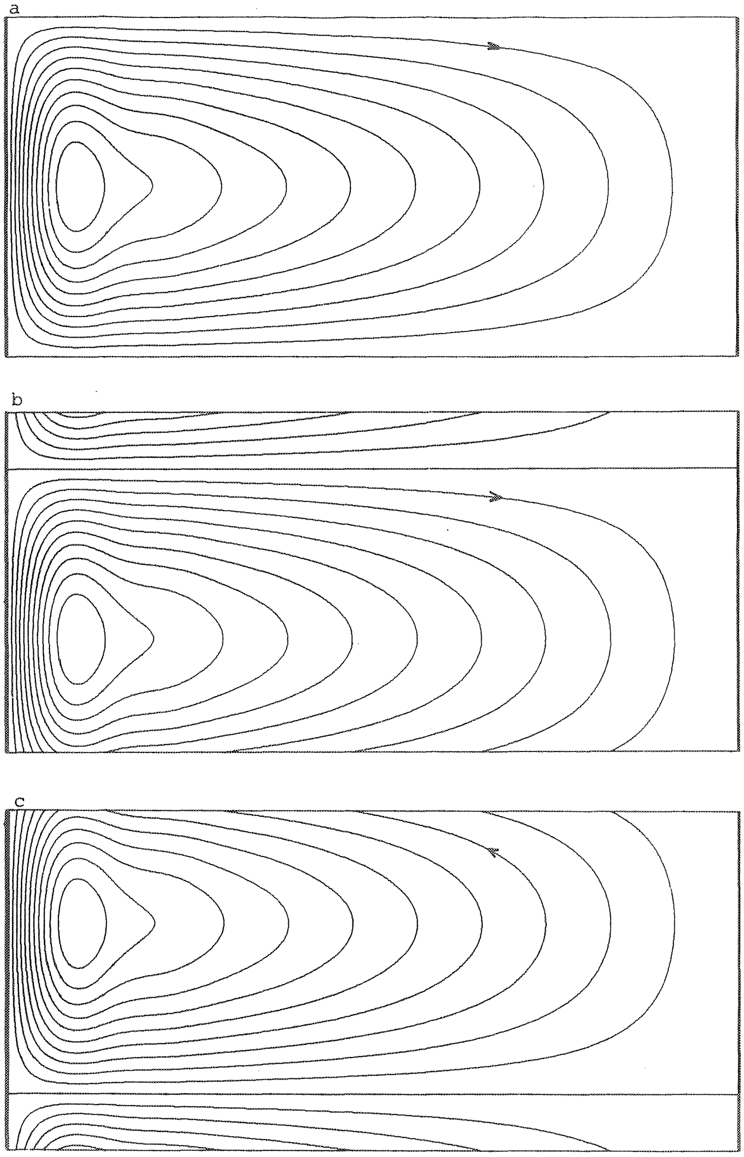

Fig. 7.5. The stream lines when both the eastern and western coast coincide with latitude circles. In a., b. and c. the same wind stress curls have been applied as in the corresponding parts of fig. 7.4.

Because in this region the maximum of the curl almost coincides with the latitude of the South Australian Coast there is a large "onshore" (inviscid) transport. This must all be turned westward in a narrow intense eastern boundary layer, the so-called Flinders Current (Bye, 1968).

Of course in the model the geometry has been simplified. Especially the western coast is not realistic: the African continent doens't extend as far southward as it does here. However, this doesn't alter the picture east of the African coast. This is a consequence of the diffusive character of the parabolic and free boundary layers: the solutions originate in the corner point at the eastern coast and develop westward. Accordingly in a model including a more realistic South African coastal shape the stream line pattern east of Africa will remain the same. In $\lambda = 0$, $\phi < \phi_0$, that is south of Africa, the free boundary layer solution acts as an "initial" condition for a free shear layer that develops westward into the Atlantic Ocean.

The techniques developed here can be used again to construct an approximation for that part of the ocean system.

7.3. The interaction of the flow in ocean basins divided by a rectangular continent

We will apply the method of the former section to construct approximations in an ocean basin with a geometry as in figure 7.5. This geometry resembles very much that of §6.1. However, now the continent A has a southern coast which coincides with a characteristic of the unperturbed part of the operator L_E (2.60). We will show in this section that this leads to different approximations not only near the southern coast but also in the free boundary layer to the west.

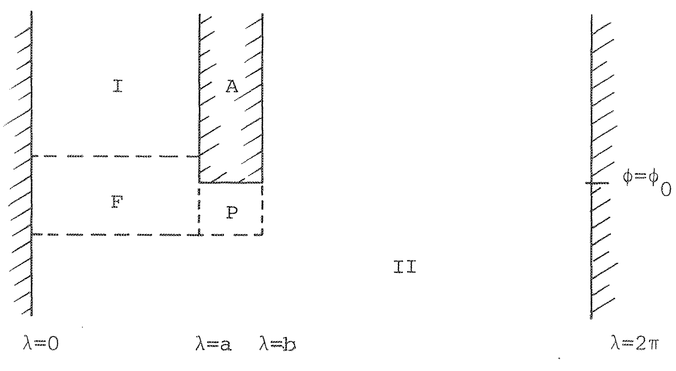

Fig. 7.6. The geometry of the model ocean with a division in subareas.

The division of the domain in subareas has been drawn in figure 7.5. In the regions I and II we have the approximations:

(7.15) $$\psi_I(\lambda,\phi;E) = \sqrt{E}.T(\phi).\frac{\lambda-a}{2} + O(E) \qquad (0 < \lambda < a \;\&\; \phi > \phi_0)$$

(7.16) $$\psi_{II}(\lambda,\phi;E) = \sqrt{E}.T(\phi).\frac{\lambda-2\pi}{2} + O(E) \qquad (\{0 < \lambda < 2\pi \;\&\; \phi < \phi_0\} \cup \{b < \lambda < 2\pi \;\&\; \phi > \phi_0\}).$$

To apply the boundary conditions $\psi = \frac{\partial\psi}{\partial\phi} = 0$ at $\phi = \phi_0$, $a < \lambda < b$ again a parabolic boundary layer along that coast has to be introduced. The discontinuity along $\phi = \phi_0$, $0 < \lambda < a$ will be removed by a free boundary layer that develops westward.

(-) The parabolic boundary layer

Putting

(7.17) $$\eta = \frac{\phi-\phi_0}{E^{1/4}} \qquad (\text{so } \eta < 0),$$

$$x = \frac{b-\lambda}{2} \qquad (0 < x < \frac{b-a}{2})$$

and

$$\psi_p(x,\eta;E) = \sqrt{E}.\psi_p^{(0)}(x,\eta) + \ldots$$

the equation for $\psi_p^{(0)}$ reads:

(7.18) $$\frac{\partial^4\psi_p^{(0)}}{\partial\eta^4} + \frac{\partial\psi_p^{(0)}}{\partial x} = -T(\phi_0)$$

with boundary conditions:

(7.19) $$\psi_p^{(0)} = \frac{\partial\psi_p^{(0)}}{\partial\eta} = 0 \quad \text{in } \eta = 0, \quad 0 < x < \frac{b-a}{2}$$

and a matching condition

(7.20) $$\psi_p^{(0)} \to -T(\phi_0)(x + \frac{2\pi-b}{2}) \quad \text{as } \eta \to -\infty$$

whereas there is also the condition

(7.21) $$\psi_p^{(0)} \to -T(\phi_0).\frac{2\pi-b}{2} \quad \text{as } x \downarrow 0 \quad (\eta < 0).$$

The solution of this problem can easily be constructed and reads:

(7.22) $$\psi_p^{(0)}(x,\eta) = -T(\phi_0).[x\{Jo_4(-\frac{\eta}{x^{1/4}}) + Ko_4(-\frac{\eta}{x^{1/4}}) + 1\} + + \frac{2\pi-b}{2}\{Jo_0(-\frac{\eta}{x^{1/4}}) + Ko_0(-\frac{\eta}{x^{1/4}}) + 1\}].$$

(For the Jo_i and Ko_i-functions see §5.5).

(-) The free boundary layer

The construction of the free shear layer can proceed along the same lines as in the former section. With

$$\eta = \frac{\phi-\phi_0}{E^{1/4}}, \quad \bar{x} = \frac{a-\lambda}{2} \qquad (\text{so } -\infty < \eta < \infty,\ \bar{x} > 0)$$

the problem for the first approximation becomes:

(7.23) $$\frac{\partial^4\psi_F^{(0)}}{\partial\eta^4} + \frac{\partial\psi_F^{(0)}}{\partial\bar{x}} = -T(\phi_0)$$

(7.24) $$\psi_F^{(0)}(\bar{x}=0) = \begin{cases} 0 & \text{for } \eta \geq 0 \\ \psi_p^{(0)}(\frac{b-a}{2}, \eta) & \text{for } \eta < 0 \end{cases}$$

(7.25) $$\psi_F^{(0)} \to -T(\phi_0).\bar{x} \qquad \text{as } \eta \to \infty$$

(7.26) $$\psi_F^{(0)} \to -T(\phi_0).(\bar{x} + \frac{2\pi-a}{2}) \qquad \text{as } \eta \to -\infty.$$

The solution of this problem reads:

(7.27) $$\psi_F^{(0)}(\bar{x},\eta) = \frac{1}{\pi}\int_{-\infty}^{0} \psi_p^{(0)}(\frac{b-a}{2}, t) \int_{0}^{\infty} \cos\{s(\eta-t)\}e^{-s^4\bar{x}}ds\,dt - T(\phi_0).\bar{x}.$$

Along the meridional coasts again $O(E^{1/3})$ lateral boundary layers can easily be analysed. The parabolic boundary layer $\psi_p^{(0)}$ has a singularity in $x = 0$, $\eta = 0$. Consequently an $E^{1/3} \times E^{1/3}$ region near that singular point appears where a different local approximation of the solution of this model problem holds.

(-) Conclusions

We have shown that when the southern boundary of the continent A coincides with a characteristic the first approximation in the free shear layer is not only dependent on the matching conditions but also on the initial condition at the rim of the parabolic boundary layer. In this way the information about the processes along the southern coast of A is reflected in the interior of the basin.

In the case of a parabolic shaped southern boundary the leading term does not contain such information. As has been shown in the preceding chapter the influence of the continent is contained then in the remainder term (see 6.38) the order of magnitude of which depends among other things on the order of tangency of the characteristic to the southern tip.

If in this section the thickness $(\frac{b-a}{2})$ of the continent is set dependent on E, for instance $\frac{b-a}{2} = E^{\beta}$, the local coordinates (7.17) for the parabolic boundary layer should be replaced by

$$\eta_{\beta} = \frac{\phi-\phi_0}{E^{\frac{1+\beta}{4}}} ; \qquad x_{\beta} = \frac{b-\lambda}{2.E^{\beta}} . \tag{7.28}$$

Accordingly the north-south extend of this layer is thinner than in the case $b-a=0(1)$. The initial condition (7.24) for the free boundary layer must then be altered in agreement. The result is again that the leading term of the free boundary layer is the same as that for the case of a line shaped continent (treated in chapter 6).

REFERENCES

AGMON, S., A. DOUGLIS & L. NIRENBERG, 1959, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. Comm. Pure Appl. Math., 12, 623-727.

BARCILON, V. & J. PEDLOSKY, 1967a, *Linear theory of rotating stratified fluid motions*. J. Fluid Mech., 29, 1-16.

– , 1967b, *A unified linear theory of homogeneous and stratified rotating fluids*. J. Fluid Mech., 29, 609-621.

BESJES, J.G., 1975, *Singular perturbation problems for linear elliptic differential operators of arbitrary order. II. Degeneration to first order operators*. J. Math. Anal. Appl., 49, 324-346.

BRYDEN, H.L. & R. DALE PILLSBURY, 1977, *Variability of deep flow in the Drake Passage from year-long current measurements*. J. Phys. Ocean., 7, 803-810.

BUBNOV, V.A. & V.D. YEGORIKHIN, 1978, *Investigation of the large scale current structure in the tropical Atlantic Ocean along* 23°30'W. Oceanology of the Acad. of Sci. of the U.S.S.R., English edition vol. 17, 631-635.

BYE, J.A.T., 1968, *The hydrology of the Southern Ocean*. 1. The Australian Section-North. Horace Lamb Centre Res. Pap., No.26; 64pp.

CALLAHAN, J.E., 1971, *Velocity structure and flux of the Antarctic Circumpolar Current south of Australia*. J. Geophys. Res. 76, 5859.

– , 1972, *The structure and circulation of deep water in the Antarctic*. Deep Sea Res., 19, 563-575.

CARRIER, G.F. & A.R. ROBINSON, 1962, *On the theory of the wind-driven ocean circulation*. J. Fluid Mech., 12, 49-80.

CHARNEY, J.G., 1955, *The Gulf Stream as an inertial boundary layer*. Proc. Nat. Acad. Sci. U.S., 41, 731-740.

DEVINE, M., 1972, *Some aspects of the dynamics of the Antarctic Circumpolar Current*. J. Geophys. Res., 77, 5987-5992.

ECKHAUS, W., 1968, *On the structure of certain asymptotic boundary layers*. Rep. Math. Inst. Delft Univ. of Techn.

– , 1973, *Matched asymptotic expansions and singular perturbations*. Amsterdam etc., North-Holland publ. comp.

– , 1977, *Formal approximations and singular perturbations*. SIAM Rev., 19, 593-633.

– , 1979, *Asymptotic analysis of singular perturbations*. North-Holland, Amsterdam.

ECKHAUS, W. & E.M. DE JAGER, 1966, *Asymptotic solutions of singular perturbation problems for linear differential equations of elliptic type*. Arch. Rat. Mech. Anal., 23, 26-86,

EVANSON, A.J. & G. VERONIS, 1975, *Continuous representation of wind stress and wind stress curl over the world ocean*. J. Nar. Res., 33, (supplement), 131.

FANDRY, C.B., 1971, *A three dimensional model of the Antarctic Circumpolar Current*. Paper no.35, Geophys. Fluid Dyn. Lab., Monash Univ., Clayton, Vic. 3168, Australia.

FOFONOFF, N.P., 1962, *Physical properties of seawater, in The Sea,* (ed. M.N. Hill), vol. 1, pp.3-30, New York, Interscience.

GILL, A.E., 1968, *A linear model of the Antarctic Circumpolar Current*. J. Fluid Mech., 32, 465.

– , 1971, *The Equatorial Current in a homogeneous ocean*. Deep Sea Res., 18, 421-431.

HARRIS, T.F.W. & D. VAN FOREEST, 1978, *The Agulhas Current in March 1969*. Deep Sea Res., 25, 549-561.

HARRIS, T.F.W., R. LEGECKIS & D. VAN FOREEST, 1978, *Satellite infrared images in the Agulhas Current system*. Deep Sea Res., 25, 543-548.

HELLERMAN, S., 1967, *An updated estimate of the wind stress over the world ocean*. Mon. Weather Rev., 95, 607-626.

HILL, R.B. & J.A. JOHNSON, 1975, *A three dimensional theory of coastal currents and upwelling over a continental shelf*. Tellus, 27, 249-258.

JOHNSON, J.A., 1971. *On the wind-driven circulation of a stratified ocean*. J. Mar. Res., 29, 197-213.

KAMENKOVICH, V.M., 1967, *On the coefficients of eddy diffusion and eddy viscosity in large-scale oceanic and atmospheric motions*. Izv. Akad. Nauk. USSR. Atmos. Oceanic. Phys. 3, 1326.

- , 1977, *Fundamentals of ocean dynamics*. Elsevier Oceanography Series, 16. Elsevier Scient. Publ. Comp. Amsterdam-Oxford-New York.

KILLWORTH, P.D., 1973, *On the circulation of a homogeneous ocean induced by the presence of continental slopes*. J. Phys. Ocean., 3, 3-15.

KRAUSS, W., 1973, *Methods and results of theoretical oceanography* I: *Dynamics of the homogeneous and the quasi-homogeneous ocean*. Gebrüder Borntraeger, Berlin-Stuttgart.

KUO, H.S., 1975, *A note on the effects of bottom friction and nonlinearity on the wind-driven ocean circulation*. J. Meteor., 7, 79-93.

Mc.LELLAN, H.J., 1965, *Elements of physical oceanography*. Pergamon Press. Oxford-London-Edinburgh-New York-Paris-Frankfurt.

Mc.WILLIAMS, J.C., W.R. HOLLAND & J.H.S. CHOW, 1978, *A description of numerical Antarctic Circumpolar Currents*. Dynamics of Atmospheres and Oceans, 2, 213-291.

MOORE, D.W. & S.G.H. PHILANDER, 1977, *Modelling of the tropical oceanic circulation. In: The Sea*, vol. 6, E.D. Goldberg, Ed., Wiley, chap. 8, pp.319-361.

MORGAN, G.W., 1956, *On the wind-driven ocean circulation*. Tellus, 8, 301-320.

MUNK, W.H., 1950, *On the wind-driven ocean circulation*. J. Meteor., 7, 79-93.

MUNK, W.H. & G.F. CARRIER, 1950, *The wind-driven circulation in ocean basins of various shapes*. Tellus, 2, 158-167.

MUNK, W.H. & E. PALMÉN, 1951, *Note on the dynamics of the Antarctic Circumpolar Current*, Tellus, 3, 53.

NOWLIN, Jr. W.D., Th. WHITWORTH & R. DALE PILLSBURY, 1977, *Structure and transport of the Antarctic Circumpolar Current at Drake Passage from short term measurements*. J. Phys. Ocean., 7, 788-802.

PEDLOSKY, J., 1968, *An overlooked aspect of the wind driven ocean circulation*. J. Fluid Mech., 32, 809-821.

– , 1969, *Linear theory of the circulation of a stratified ocean*. J. Fluid Mech., 35, 185-205.

– , 1974a, *On coastal jets and upwelling in bounded basins*. J. Phys. Ocean., 4, 3-18.

– , 1974b, *Longshore currents, upwelling and bottom topography*. J. Phys. Ocean., 4, 214-226.

RATTRAY, M., Jr. & P. WELANDER, 1975, *A quasi-linear model of the combined wind-driven and thermohaline circulations in a rectangular β-plane ocean*. J. Phys. Ocean., 5, 585-602.

REYNOLDS, O., 1895, *On the dynamical theory of incompressible viscous fluids and the determination of the criterion*. Phil. Trans. R. Soc. (A), 186, 123-164.

ROBINSON, A.R., 1963, *Wind-driven ocean circulation*. Blaisdell Publ. Comp., New York-Toronto-London. A division of Random House.

SMITH, N.R., & C.B. FANDRY, 1978, *Combined effects of wind stress and topology in a two layer model of the Southern Ocean*. Deep Sea Res., 25, 371-391.

SPILLANE, M. & P.P. NIILER, 1975, *On the theory of the strong, midlatitude wind-driven ocean circulation*. I. Geophys. Fluid Dyn., 7, 43-66.

STOMMEL, H., 1948, *The westward intensification of wind-driven ocean currents*. Trans. Amer. Geophys. Union, 29, 202-206.

– , 1957, *A survey of ocean current theory*. Deep Sea Res., 4, 149-184.

– , 1965, *The Gulf Stream (2nd edition)*. Univ. of Calif. Press, Berkeley and Los Angeles.

VERONIS, G., 1973, *Large scale ocean circulation*. Adv. Appl. Mech., 13, 1-92.

WELANDER, P., 1971, *Some exact solutions to the equations describing an ideal fluid*. J. Mar. Res., 29, 60-68.

WILSON, W. & D. BRADLEY, 1968, *Specific volume of sea water as a function of temperature, pressure and salinity*. Deep Sea Res., 15, 355-363.

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