BRANCHING PROCESSES WITH CONTINUOUS STATE SPACE

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ACKNOWLEDGEMENTS

CONTENTS

CHAPTER 1

INTRODUCTION

In this monograph we shall be concerned with some asymptotic properties
of so-called branching processes. Talking about a branching process
$\{Z_n; n = 0,1,2,\ldots\}$ it is usual to think of Z_n as the number of individuals
in the nth generation of some population. Doing so, this number Z_n equals
the number of individuals produced by the Z_{n-1} individuals of the $(n-1)^{st}$
generation, that is, we can consider Z_n as a sum of Z_{n-1} random variables,
which are usually assumed to be independent and identically distributed.
Such processes are known as Galton-Watson processes. Many results about
these have been derived and we shall mention some of them below. For in-
stance, there is the so-called extinction or explosion theorem, which says
that $P(\lim_{n\to\infty} Z_n = 0 \text{ or } \infty) = 1$. It turns out that in a more detailed analysis
of asymptotic properties of Galton-Watson processes an important role is
played by the expectation of the so-called offspring distribution. This
offspring distribution is defined as the distribution of the random number
of individuals produced by one individual. We denote its expectation by m,
implying that under the usual assumption that $P(Z_0 = 1) = 1$, $m = EZ_1$. We can
distinguish four cases for Galton-Watson processes:
(1) subcritical processes, that is processes for which $m < 1$. In this case
    $P(\lim_{n\to\infty} Z_n = 0) = 1$, and if $EZ_1 \log Z_1 < \infty$ then $\lim_{n\to\infty} m^{-n}P(Z_n > 0) > 0$;
(2) critical processes, that is processes for which $m = 1$. In this case
    $P(\lim_{n\to\infty} Z_n = 0) = 1$, and if the variance $\sigma^2(Z_1)$ of Z_1 is finite, then
    $\lim_{n\to\infty} nP(Z_n > 0) = 2/\sigma^2(Z_1)$;
(3) supercritical processes, that is processes for which $1 < m < \infty$. In this
    case $P(\lim_{n\to\infty} Z_n = 0) < 1$. If $P(\lim_{n\to\infty} Z_n = 0) > 0$, then such a process,
    conditioned on $\{\lim_{n\to\infty} Z_n = 0\}$ can be considered as a subcritical pro-
    cess $\{\tilde{Z}_n; n = 0,1,2,\ldots\}$, with probability generating function of its
    offspring distribution given by $\tilde{f}(s) = \dfrac{f(rs)}{r}$, where f is the probability
    generating function of the offspring distribution of the original pro-
    cess and $r = P(\lim_{n\to\infty} Z_n = 0)$. Furthermore, there exists a sequence of

constants $\{a_n; \; n = 0,1,2,\ldots\}$ such that

$$P(0 < \lim_{n \to \infty} a_n Z_n < \infty \mid \lim_{n \to \infty} Z_n = \infty) = 1;$$

(4) explosive processes, that is processes for which $m = \infty$. In this case $P(\lim_{n \to \infty} Z_n = 0) < 1$. If $P(\lim_{n \to \infty} Z_n = 0) > 0$, then for such a process, conditioned on $\{\lim_{n \to \infty} Z_n = 0\}$, the same result as under 3 holds, whereas there exists no sequence of constants $\{a_n; \; n = 0,1,2,\ldots\}$ such that

$$P(0 < \lim_{n \to \infty} a_n Z_n < \infty \mid \lim_{n \to \infty} Z_n = \infty) = 1.$$

We can however construct a function L, such that

$$P(0 < \lim_{n \to \infty} e^{-n} L(1/Z_n) < \infty \mid \lim_{n \to \infty} Z_n = \infty) = 1.$$

In a paper of JIŘINA [1958] it was noticed, that the size of the population can be measured by other means than by the number of individuals, for instance by means of its weight or volume. Therefore, it is reasonable to consider branching processes with the non-negative real numbers as their state space. Such processes $\{Z_n; \; n = 0,1,2,\ldots\}$ are studied in this monograph. We pay particular attention to the correspondences and the differences between these processes and Galton-Watson processes. After some preliminaries in Chapter 2, we show that a necessary and sufficient condition for the existence of such processes is that the offspring distribution, defined as the distribution of the random quantity produced by a quantity of size 1, is infinitely divisible. After that we shall see in Chapter 3, that also for these processes $P(\lim_{n \to \infty} Z_n = 0 \text{ or } \infty) = 1$, and then the behaviour of the process on the events $\{\lim_{n \to \infty} Z_n = 0\}$ and $\{\lim_{n \to \infty} Z_n = \infty\}$ is further investigated. As a rule we can say that this behaviour, both on $\{\lim_{n \to \infty} Z_n = 0\}$ if $P(Z_1 = 0) > 0$ and on $\{\lim_{n \to \infty} Z_n = \infty\}$ is essentially the same as that for Galton-Watson processes. This correspondence is elaborated in Chapters 4, 5, 6 and 7, where we study successively the four cases mentioned above under 1, 2, 3 and 4 for Galton-Watson processes. The only remaining case is then the behaviour of the process on $\{\lim_{n \to \infty} Z_n = 0\}$ if $P(Z_1 = 0) = 0$. Notice that for Galton-Watson processes $P(Z_1 = 0) = 0$ implies that every individual produces at least one individual, whence it follows that the process cannot become extinct, that is $P(\lim_{n \to \infty} Z_n = 0) = 0$. However, this reasoning is not valid in the case we consider: if we take for instance $P(Z_1 = \frac{1}{2}) = 1$, then

it follows that $P(Z_n = (\tfrac{1}{2})^n) = 1$ for all $n = 1,2,3,\ldots$, and so $P(\lim_{n\to\infty} Z_n = 0) = 1$, while obviously $P(Z_1 = 0) = 0$. This example also shows that we can have that $P(\lim_{n\to\infty} Z_n = 0) \neq \lim_{n\to\infty} P(Z_n = 0)$, in contrast with Galton-Watson processes, for which these two expressions are always equal to each other. Now it turns out that we can use similar techniques for the study of the process on $\{\lim_{n\to\infty} Z_n = 0\}$ if $P(Z_1 = 0) = 0$ as we use for the study of the process on $\{\lim_{n\to\infty} Z_n = \infty\}$. We mentioned above that in the latter case the value of the parameter m is important and we get different results according as $m < \infty$ or $m = \infty$. If we consider the process on $\{\lim_{n\to\infty} Z_n = 0\}$ if $P(Z_1 = 0) = 0$, it is not anymore the parameter m which plays an essential role, but an other parameter comes in, to wit the almost sure infimum of Z_1, defined by $\inf\{x;\ P(Z_1 \leq x) > 0\}$. This parameter is denoted by a. A similar distinction as between the cases $m < \infty$ and $m = \infty$ will be shown to exist between the cases $a > 0$ and $a = 0$. As the most important results we have that if $a > 0$, then there exists a sequence of constants $\{a_n;\ n = 0,1,2,\ldots\}$ such that $P(0 < \lim_{n\to\infty} a_n Z_n < \infty \mid \lim_{n\to\infty} Z_n = 0) = 1$, whereas if $a = 0$ we can construct a function L such that

$$P(0 < \lim_{n\to\infty} e^{-n} L(1/Z_n) < \infty \mid \lim_{n\to\infty} Z_n = 0) = 1.$$

The cases $a > 0$ resp. $a = 0$ will be treated in Chapters 8 resp. 9. After this rather superficial introduction we shall now pass on to a more detailed approach of the problems sketched above.

CHAPTER 2

PRELIMINARIES

2.1. SOME PROBABILITY THEORY

In this section we formulate some concepts from probability theory we need in the sequel. Throughout this study all random variables are supposed to be defined on one probability triple $(\Omega, \mathcal{F}, P)$.

Let $\{X_t\}$ be a collection of random variables, indexed by a parameter t in some subset I of $\mathbb{R}$. Such a collection is called a stochastic process. We write $\{X_t;\ t \in I\}$ or $\{X(t);\ t \in I\}$. As a first example we consider a random walk.

DEFINITION 2.1.1. A *random walk* is a stochastic process $\{X_n;\ n = 0,1,2,\ldots\}$ such that $X_0 = 0$ and $X_n = \Sigma_{j=1}^n Y_j$, $n = 1,2,3,\ldots$, where $Y_1, Y_2, Y_3, \ldots$ are independent and identically distributed random variables.

From this definition we see that a random walk has the following two properties:
1. The process has stationary increments, that is, for any fixed, non-negative integer m, the increment $X(n+m) - X(n)$ has, for all $n = 0,1,2,\ldots$, the same distribution.
2. The process has independent increments, that is, the σ-field spanned by $\{X(n+m) - X(n),\ m = 0,1,2,\ldots\}$ is, for all $n = 1,2,3,\ldots$, independent of the σ-field spanned by $\{X(m),\ m = 0,1,\ldots,n\}$.

It is possible to generalize random walks to processes with index-set $[0,\infty)$ having properties analogous to 1 and 2 above, that is 1 and 2 do not only hold for all non-negative integers m and n, but for all s and $t \in [0,\infty)$. If the values of $X(1)$ are non-negative then these processes are called subordinators. More precisely,

DEFINITION 2.1.2. A stochastic process $\{X(t);\ t \in [0,\infty)\}$ is called a

subordinator, if it has stationary and independent increments, and if, for almost all sample paths $X(t,\omega)$:

1. $X(0,\omega) = 0$;
2. $X(\cdot,\omega)$ is right-continuous on $[0,\infty)$;
3. $X(\cdot,\omega)$ has finite left limits on $(0,\infty)$;
4. $X(1,\omega) \geq 0$.

It follows from this definition that for each $n = 1,2,3,\ldots$, $X(1) = \sum_{j=1}^{n} Y_{n,j}$, where $Y_{n,j} = X(\frac{j}{n}) - X(\frac{j-1}{n})$, and that $Y_{n,1}, Y_{n,2}, \ldots, Y_{n,n}$ are independent and identically distributed, implying that $X(1)$ has an infinitely divisible distribution. On the other hand, it is well-known that, given a non-negative random variable X having an infinitely divisible distribution, there is a unique subordinator $\{X(t); t \in [0,\infty)\}$ such that $X(1) \overset{d}{=} X$. (The notation $X \overset{d}{=} Y$ means that X and Y have the same distribution.) See e.g. BREIMAN [1968].

An important class of random variables appearing in the theory of stochastic processes is the class of so-called stopping times. Roughly speaking, a stopping time T only depends on the stochastic process up to time T. Let $\{X(t); t \in I \subset [0,\infty)\}$ be a stochastic process. We denote by $F(X(s), s \in [0,t] \cap I)$ the σ-field spanned by $\{X(s); s \in [0,t] \cap I\}$.

DEFINITION 2.1.3. Let $I \subset [0,\infty)$, $\{X(t); t \in I\}$ a stochastic process and T a random variable with values in I. T is called a *stopping time* for the process $\{X(t); t \in I\}$, if for every $t \geq 0$, $\{T \leq t\} \in F(X(s), s \in [0,t] \cap I)$.

It follows from Definition 2.1.2 that if we define $X_t(s)$ by $X_t(s) = X(t+s) - X(t)$, $s,t \in [0,\infty)$, then for any $t \in [0,\infty)$ the process $\{X_t(s); s \in [0,\infty)\}$ has the same distribution as the process $\{X(s); s \in [0,\infty)\}$ and is independent of the σ-field spanned by $\{X(s); s \in [0,t]\}$. This property is called the weak Markov property. It says that at any time $t > 0$, the process starts afresh. A similar property satisfied for any stopping time T is called the strong Markov property. Before stating the precise definition we notice the following. If $\{X(t); t \in I \subset [0,\infty)\}$ is a stochastic process and T a random variable having at most countably many values, which are moreover elements of I, then $X(T)$ is again a random variable. This is in general not true for an arbitrary random variable T. However, if for example T is a stopping time for a process $\{X(t), t \in [0,\infty)\}$ having almost all sample paths right-continuous, then $X(T)$ is also a random-variable.

See BREIMAN [1968]. If T is a stopping time for the process
$\{X(t); \ t \in I \subset [0,\infty)\}$, then we denote by $\mathcal{F}(X(s), \ s \in [0,T] \cap I)$ the σ-
field of events $B \in \mathcal{F}$ such that $B \cap \{T \leq t\} \in \mathcal{F}(X(s), \ s \in [0,t] \cap I)$.

DEFINITION 2.1.4. Let $I = [0,\infty)$ or $I = \{0,1,2,\ldots\}$, $\{X(t), \ t \in I\}$ a stocha-
stic process, having almost all sample paths right-continuous in case $I = [0,\infty)$. Then the property that for any stopping time T for the process
$\{X(t); \ t \in I\}$, the process $\{X_1(t), \ t \in I\}$, defined by $X_1(t) = X(T+t) - X(T)$,
has the same distribution as $\{X(t), \ t \in I\}$ and is independent of
$\mathcal{F}(X(s); \ s \in [0,T] \cap I)$, is called the *strong Markov property*.

The following result is well-known, see e.g. BREIMAN [1968].

LEMMA 2.1.5. *For random walks and for subordinators, the strong Markov*
property holds.

This section is closed with a lemma which will be applied in Chapter 3.

LEMMA 2.1.6. *Let* $\{X(t); \ t \in [0,\infty)\}$ *be a stochastic process such that, for*
almost all $\omega \in \Omega$, $X(t,\omega)$ *is a right continuous function of t for* $t \in [0,\infty)$,
and let $(Z_0, Z_1, \ldots, Z_n)$ *be a random vector such that* Z_n *is non-negative.*
Suppose that there exists a random variable M such that $|X(t)| \leq M$ *for all*
$t \in [0,\infty)$ *and that* $EM < \infty$, *and that furthermore* $(Z_0, Z_1, \ldots, Z_n)$ *and*
$\{X(t); \ t \in [0,\infty)\}$ *are independent. Then* $E(X(Z_n)|Z_0, Z_1, \ldots, Z_n) = [EX(t)]_{t=Z_n}$
a.s..

PROOF. Because almost all sample paths of the process $\{X(t); \ t \in [0,\infty)\}$
are right-continuous, we can write

$$X(Z_n) = \lim_{\ell \to \infty} \sum_{k=0}^{\infty} I_{[k/\ell, \ (k+1)/\ell)}(Z_n) X(\tfrac{k+1}{\ell}) \quad \text{a.s.,}$$

where I stands for the indicator function. Now $|X(t)| \leq M$ and $EM < \infty$, and
hence the dominated convergence theorem yields that

$$E(X(Z_n)|Z_0, Z_1, \ldots, Z_n) = \lim_{\ell \to \infty} E(\sum_{k=0}^{\infty} I_{[k/\ell, \ (k+1)/\ell)}(Z_n) X(\tfrac{k+1}{\ell}) | Z_0, Z_1, \ldots, Z_n) \quad \text{a.s..}$$

Furthermore,

$$|\sum_{k=0}^{N} I_{[k/\ell, \ (k+1)/\ell)}(Z_n) X(\tfrac{k+1}{\ell})| \leq M$$

for every positive integer N, and therefore, again by dominated convergence, we obtain

$$E(X(Z_n)|Z_0,Z_1,\ldots,Z_n) = \lim_{\ell\to\infty} \sum_{k=0}^{\infty} E(I_{[k/\ell,\,(k+1)/\ell)}(Z_n)X(\tfrac{k+1}{\ell})|Z_0,Z_1,\ldots,Z_n)$$

$$= \lim_{\ell\to\infty} \sum_{k=0}^{\infty} I_{[k/\ell,\,(k+1)/\ell)}(Z_n)E(X(\tfrac{k+1}{\ell})|Z_0,Z_1,\ldots,Z_n)$$

$$= \lim_{\ell\to\infty} \sum_{k=0}^{\infty} I_{[k/\ell,\,(k+1)/\ell)}(Z_n)EX(\tfrac{k+1}{\ell}) \quad a.s.,$$

where the last equality follows from the independence of $\{X(t);\ t \in [0,\infty)\}$ and $(Z_0,Z_1,\ldots,Z_n)$. Finally, the right-continuity of almost all sample paths of $\{X(t);\ t \in [0,\infty)\}$ allows another application of the dominated convergence theorem yielding that $E(X(Z_n)|Z_0,Z_1,\ldots,Z_n) = [EX(t)]_{t=Z_n}$ a.s.. □

2.2. SOME RESULTS ABOUT CUMULANT GENERATING FUNCTIONS

In the following chapters we shall often make use of cumulant generating functions. We therefore formulate some properties of these. Usually the cumulant generating function h of a possibly defective random variable X is defined for $s \in [0,\infty)$ by $h(s) = -\log E\,e^{-sX}$, with the convention that $e^{-s.\infty} = 0$ and $e^{-s.-\infty} = \infty$ for all $s \in [0,\infty)$. As long as we consider only non-negative, real values of s, and use the convention $-\log\infty = -\infty$, this function h is well-defined for every random variable X. But we shall also be concerned with $h(z)$ for complex values of z and then we encounter the well-known problem that the logarithm of a complex number is not univalent. Now we can try to define $h(z)$ as a continuous function, which satisfies $e^{-h(z)} = \phi(z)$, where $\phi(z)$, called the Laplace transform of X, is defined by $\phi(z) = E\,e^{-zX}$, with the conventions that $e^{-z.\infty} = 0$ if Re $z \geq 0$, $e^{-z.\infty} = \infty$ if Re $z < 0$ and $e^{-z.-\infty} = e^{z.\infty}$. This can be done for all complex z in a simply connected subset of the complex plane, if $\phi(z)$ is continuous and has no zeros in that subset. Now we are only dealing with non-negative random variables X, having an infinitely divisible, possibly defective distribution, and it follows from the theorems 5.3.1 and 8.4.1 in LUKACS [1970] that the Laplace transform $\phi(z)$ of such a random variable X has no zeros on $\{z; \text{Re } z \geq 0\}$ unless $P(X=\infty) = 1$. Furthermore, this $\phi(z)$ is continuous on $\{z; \text{Re } z \geq 0\}$ and hence we can define on $\{z; \text{Re } z \geq 0\}$ a univalent function $h(z)$, which is continuous

and satisfies $\phi(z) = e^{-h(z)}$ and which is real for $z \in [0,\infty)$, with the convention that $h(z) = \infty$ if $P(X=\infty) = 1$. This function $h(z)$ is written as $-\log E e^{-zX}$. This leads to the following definition.

DEFINITION 2.2.1. Let X be a non-negative random variable with an infinitely divisible, possibly defective distribution. Then the function

$$(2.2.1) \qquad h(z) = -\log E e^{-zX},$$

where z is a complex number with $\mathrm{Re}\, z \geq 0$, is called the *cumulant generating function* of X.

In this section h stands for the cumulant generating function of a non-negative random variable X with an infinitely divisible, possibly defective distribution, not concentrated in one point, and a for the first point of increase of the distribution function F(x) of X, that is

$$(2.2.2) \qquad a = \inf\{x; F(x) > 0\}.$$

Many properties of $h(z)$ can be deduced from the corresponding properties of the Laplace transform $\phi(z)$. First of all we notice that $\phi(z)$ is analytic for all z with $\mathrm{Re}\, z > 0$, and that $0 < |\phi(z)| < 1$ on $\{z; \mathrm{Re}\, z > 0\}$. The derivatives of $\phi(z)$ are given by

$$(2.2.3) \qquad \phi^{(n)}(z) = (-1)^n E X^n e^{-zX}, \qquad \mathrm{Re}\, z > 0, \ n = 1,2,3,\ldots,$$

with the convention that $\infty^n e^{-z \cdot \infty} = 0$ if $\mathrm{Re}\, z > 0$, $n = 1,2,3,\ldots$. This yields

LEMMA 2.2.2. *The function* $h(z)$ *is well-defined and continuous for all* $z \in \{z; \mathrm{Re}\, z \geq 0\}$ *and analytic for all* $z \in \{z; \mathrm{Re}\, z > 0\}$.

PROOF. From the remarks made before Definition 2.2.1 we know that $h(z)$ is well-defined and continuous for all z with $\mathrm{Re}\, z \geq 0$; $h(z)$ is analytic on $\{z; \mathrm{Re}\, z > 0\}$ since $0 < |\phi(z)| < 1$ and $\phi(z)$ is analytic on $\{z; \mathrm{Re}\, z > 0\}$. $\quad\square$

We shall now formulate some results on $h(s)$ for $s \in [0,\infty)$. The following property turns out to be very useful.

LEMMA 2.2.3.

(a) *If* $P(0 < X < \infty) > 0$, *then* $h(s)$ *is strictly increasing in* s, $0 \leq s < \infty$;

(b) $h(s)$ *is concave on* $[0,\infty)$.

PROOF. Part (a) is an easy consequence of the definition of h. Part (b) is obtained by an application of Schwarz' inequality, which states

$$E \exp(-\tfrac{1}{2}(s_1+s_2)X) \leq (E \exp(-s_1 X))^{\frac{1}{2}} (E \exp(-s_2 X))^{\frac{1}{2}}, \quad s_1, s_2 \in [0,\infty),$$

and therefore

$$h(\tfrac{1}{2}(s_1+s_2)) \geq \tfrac{1}{2}h(s_1) + \tfrac{1}{2}h(s_2)$$

for arbitrary $s_1, s_2 \in [0,\infty)$.   □

As is well-known the behaviour of $\phi(s)$ for small s provides us with some information on the distribution of X near ∞, and reversed. The translation of this fact in terms of h is stated in the following two lemmas.

LEMMA 2.2.4.

(a) $\lim_{s \downarrow 0} h(s) = h(0) = -\log P(X<\infty)$;

(b) *If* $P(X<\infty) = 1$, *then* $\lim_{s \downarrow 0} \dfrac{h(s)}{s} = h'(0) = EX \leq \infty$.

PROOF.

(a) By lemma 2.2.2 it follows that $h(0) = \lim_{s \downarrow 0} h(s)$; furthermore, the dominated convergence theorem implies that

$$\lim_{s \downarrow 0} h(s) = \lim_{s \downarrow 0} - \log E\, e^{-sX} = -\log E \lim_{s \downarrow 0} e^{-sX} = -\log P(X<\infty).$$

(b) By the concavity of h (Lemma 2.2.3(b)), $h'(0) = \lim_{s \downarrow 0} h'(s)$. Since $h(s) = -\log E\, e^{-sX}$, it follows from (2.2.3) that

$$h'(s) = -\frac{(E \exp(-sX))'}{E \exp(-sX)} = \frac{E\{X \exp(-sX)\}}{E \exp(-sX)},$$

and so, by part (a) of this lemma and again dominated convergence, $h'(0) = EX$.   □

LEMMA 2.2.5.

(a) $\lim_{s \to \infty} h(s) = -\log P(X=0)$;

(b) $\lim_{s\to\infty} \{h(s) - as\} = -\log P(X=a)$;

(c) $\lim_{s\to\infty} \dfrac{h(s)}{s} = a$.

PROOF. The proof of part (a) is analogous to the proof of Lemma 2.2.4(a). Part (b) follows by dominated convergence, because $h(s) - as = -\log E\, e^{-s(X-a)}$. Part (c) is proved by taking the limits as $s \to \infty$ and $\delta \downarrow 0$ in the inequalities

$$a \le -\frac{1}{s}\log E\, e^{-sX} \le -\frac{1}{s}\log E\, e^{-sX} I_{\{X\le a+\delta\}} \le a + \delta - \frac{1}{s}\log P(X\le a+\delta),$$

where I stands for the indicator function.   □

We have thus seen that $\dfrac{h(s)}{s} \to EX$ as $s \downarrow 0$ and $\dfrac{h(s)}{s} \to a$ as $s \to \infty$. Next we notice that $\dfrac{h(s)}{s}$ decreases from EX to a as s passes through $(0,\infty)$.

LEMMA 2.2.6. *The function* $\dfrac{h(s)}{s}$ *is strictly decreasing in* $s \in (0,\infty)$.

PROOF. This is immediate from the concavity of h and the fact that X is not concentrated in one point.   □

We terminate this section with a lemma which describes the connection between the cumulant generating functions of the random variables of a sub-ordinator. A similar result holds for random walks.

LEMMA 2.2.7. *Let* $\{X(t); t \in [0,\infty)\}$ *be a subordinator, and* $h(z,t)$ *the cumu-lant generating function of* $X(t)$. *Then*

$$h(z,t) = t \cdot h(z,1)$$

for all $t \ge 0$ *and all complex z with* $z \in \{z; \mathrm{Re}\, z \ge 0\}$.

PROOF. Because $X(t+s) = X(s) + \{X(t+s)-X(s)\}$, and, by Definition 2.1.2, $X(s)$ and $X(t+s) - X(s)$ are independent and $X(t+s) - X(s) \overset{d}{=} X(t)$,

(2.2.4) $h(z,t+s) = h(z,t) + h(z,s)$

for all s, $t \ge 0$ and all complex z with $z \in \{z; \mathrm{Re}\, z \ge 0\}$. Therefore, for all rationals $r = \dfrac{m}{n} \ge 0$, where $m \ge 0$ and $n > 0$ are integers,

$$h(z,r) = h(z,\tfrac{m}{n}) = m \cdot h(z,\tfrac{1}{n}) = \tfrac{m}{n}\, h(z,1) = r \cdot h(z,1).$$

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Now by (2.2.4), h(z,t) is non-decreasing in t ∈ [0,∞) for all z ∈ [0,∞),
and so h(z,t) = t·h(z,1) for all t ∈ [0,∞) and all z ∈ [0,∞). Since by Lemma
2.2.2 both h(z,t) and t·h(z,1) are analytic on {z;Re z > 0} and continuous on
{z;Re z ≥ 0}, also h(z,t) = t·h(z,1) for all t ∈ [0,∞) and all complex
z ∈ {z;Re z ≥ 0}. □

CHAPTER 3

GENERAL RESULTS

3.1. INTRODUCTION

Since long the theory of branching processes has been studied. For an interesting historical sketch we refer to JAGERS [1975]. The idea of these processes can be described in the following way. Consider individuals each of which produces a random number of new ones, called its direct descendants, such that the next two properties hold:
1. All the individuals act independently of each other.
2. The random numbers of produced individuals all have the same distribution.

One starts with a number Z_0 of individuals, which form the zeroth generation. Further, the number of individuals in the (n+1)st generation, Z_{n+1}, is the number of direct descendants of the Z_n individuals in the nth generation for n = 0,1,2,... . Such processes $\{Z_n;\ n = 0,1,2,...\}$ are called Galton-Watson processes after F. Galton and H. Watson, who studied these processes in the nineteenth century. For these processes all the Z_n are integer-valued. Now we want to generalize this to branching processes $\{Z_n;\ n = 0,1,2,...\}$ with Z_n non-negative, real-valued. To this end we notice that we can describe a Galton-Watson process more formally as follows. Let $Y_1,Y_2,Y_3,...$ be a sequence of independent and identically distributed random variables, Z_0 some given positive integer. Then we can define Z_1 to be the sum of the first Z_0 random variables of the sequence $\{Y_n;\ n = 1,2,3,...\}$. After that we can, conditionally given Z_1, define Z_2 to be the sum of the following Z_1 random variables of the sequence $\{Y_n;\ n = 1,2,3,...\}$, and so on. This leads to the next definition which has indeed the advantage of being easy generalizable to processes with Z_n real-valued.

DEFINITION 3.1.1. Let $\{X_n;\ n = 0,1,2,...\}$ be a random walk such that $P(X_1{=}k) = p_k$, $k = 0,1,2,...$, $\Sigma_{k=0}^{\infty} p_k = 1$, and let $Z_0 = C$ for some positive

integer C, $S_{-1} = 0$ and $S_n = \Sigma_{k=0}^n Z_k$, $n = 0,1,2,\ldots$, where $Z_{n+1} = X_{S_n} - X_{S_{n-1}}$,
$n = 0,1,2,\ldots$. Then the process $\{Z_n;\ n = 0,1,2,\ldots\}$ is called a *Galton-Watson process*; the distribution of X_1 is called the *offspring distribution* of the process $\{Z_n;\ n = 0,1,2,\ldots\}$.

From this definition we can indeed prove that Z_{n+1}, conditionally given Z_n, is equal to a sum of Z_n independent and indentically distributed random variables, and that the properties 1 and 2 mentioned at the beginning of this section hold. For let F_k be the σ-field spanned by $\{X_m,\ m = 0,1,\ldots,k\}$ for $k = 0,1,2,\ldots$. First of all we notice that since $S_{-1} = 0$ and $Z_0 = C$, both $\{S_{-1} \le k\}$ and $\{S_0 \le k\} \in F_k$ for all $k = 0,1,2,\ldots$, implying that S_{-1} and S_0 are stopping times for the process $\{X_n;\ n = 0,1,2,\ldots\}$. Now suppose that S_k is a stopping time for the process $\{X_n;\ n = 0,1,2,\ldots\}$ for $k = -1,0,1,\ldots,n-1$. This means, by the remark made before Definition 2.1.4 that X_{S_k} is well-defined for $k = -1,0,\ldots,n-1$. Furthermore, since $\{S_n \le k\} \subset \{S_{n-1} \le k\}$ and

$$S_n = \sum_{k=0}^{n} Z_k = C + \sum_{k=1}^{n} \{X_{S_{k-1}} - X_{S_{k-2}}\} = C + X_{S_{n-1}},$$

$\{S_n \le k\} = \{X_{S_{n-1}} \le k-C\} \cap \{S_{n-1} \le k\} \in F_k$ for all $k = 0,1,2,\ldots$, and therefore also S_n is a stopping time for the process $\{X_n;\ n = 0,1,2,\ldots\}$. Hence we obtain that X_{S_n} is well-defined and therefore the same is true for Z_{n+1}. Moreover, conditionally given Z_n,

$$Z_{n+1} = \sum_{j=1}^{Z_n} (X_{S_{n-1}+j} - X_{S_{n-1}+j-1}).$$

Since S_{n-1} is a stopping time, the same is true for $S_{n-1} + j$ for $j = 1,2,3,\ldots$, and because $Z_n \in F(X_j,\ j = 0,1,\ldots,S_{n-1})$, it follows from Lemma 2.1.5 that conditionally given Z_n, the random variables $X_{S_{n-1}+j} - X_{S_{n-1}+j-1}$, $j = 1,2,\ldots,Z_n$, are independent and identically distributed, also independent of the σ-field $F(X_j,\ j = 0,1,\ldots,S_{n-1})$, that is, the properties 1 and 2 above are satisfied.

Now the relation between random walks and subordinators explained in Section 2.1 makes it clear how to define a branching process $\{Z_n;\ n = 0,1,2,\ldots\}$ with Z_n real-valued.

DEFINITION 3.1.2. Let $\{X(t);\ t \in [0,\infty)\}$ be a subordinator and let $Z_0 = C$ for some positive real number C, $S_{-1} = 0$ and $S_n = \Sigma_{k=0}^n Z_k$, $n = 0,1,2,\ldots$,

where

(3.1.1) $Z_{n+1} = X(S_n) - X(S_{n-1})$, $n = 0,1,2,\ldots$.

Then the process $\{Z_n; n = 0,1,2,\ldots\}$ is called a *branching process with state space* $[0,\infty)$; the distribution of $X(1)$ is called the *offspring distribution* of the process $\{Z_n; n = 0,1,2,\ldots\}$.

Such processes were introduced by JIŘINA [1958]. Our definition is a slight modification of a definition used by ATHREYA [1974]. See also ATHREYA [1975].

From now on we suppose that $\{Z_n; n = 0,1,2,\ldots\}$ is a branching process with state space $[0,\infty)$, that $Z_0 = 1$, that Z_1 is a proper, non-degenerate and of course non-negative random variable having an infinitely divisible distribution and that h is the cumulant generating function of Z_1, unless stated otherwise.

Using a similar argument as the one following Definition 3.1.1 we can prove that S_n is a stopping time for the process $\{X(t); t \in [0,\infty)\}$ for all $n = -1,0,1,\ldots$, that Z_n is well-defined for $n = 0,1,2,\ldots$ and that, since $Z_0 = 1$,

(3.1.2) $S_n = 1 + X(S_{n-1})$, $n = 0,1,2,\ldots$.

Furthermore, we can deduce from Definition 3.1.2 the following lemma, which is often referred to as the *basic branching property*.

LEMMA 3.1.3. *For all complex* $z \in \{z; \operatorname{Re} z \geq 0\}$ *and all* $n = 0,1,2,\ldots$

(3.1.3) $E(e^{-zZ_{n+1}} \mid Z_0, Z_1, \ldots, Z_n) = e^{-h(z)Z_n}$ a.s..

PROOF. We know already that S_n is a stopping time for the process $\{X(t); t \in [0,\infty)\}$ for all $n = 0,1,2,\ldots$. Furthermore, it follows from (3.1.1) that $\{Z_n \leq x\} \in F(X(s); s \in [0,S_{n-1}])$, where $x \in (-\infty,\infty)$ and $n = 0,1,2,\ldots$, and so the Lemmas 2.1.5, 2.1.6 and 2.2.7 imply that a.s.

$$E(e^{-zZ_{n+1}} \mid Z_0, Z_1, \ldots, Z_n) = E(e^{-z\{X(S_{n-1}+Z_n) - X(S_{n-1})\}} \mid Z_0, Z_1, \ldots, Z_n)$$

$$= [E\{\exp(-z\{X(S_{n-1}+1) - X(S_{n-1})\})\}]^{Z_n} = [E\{\exp(-zZ_1)\}]^{Z_n} = e^{-h(z)Z_n},$$

where $z \in \{z; \text{Re } z \geq 0\}$ and $n = 0,1,2,\ldots$. $\quad\square$

3.2. NOTATION

In the sequel the following notation will be used:

$$h_n(z) = -\log E\, e^{-zZ_n}, \quad \text{with} \quad z \in \{z; \text{Re } z \geq 0\}, \quad n = 0,1,2,\ldots;$$

$c(s)$ resp. $c_n(s)$ are the inverses of $h(s)$ resp. $h_n(s)$,
$s \in [0,\infty)$, $n = 0,1,2,\ldots;$

$$c_{-n}(s) = h_n(s) \text{ and } h_{-n}(s) = c_n(s), \quad s \in [0,\infty), \quad n = 1,2,3,\ldots;$$

$$r = P(\lim_{n\to\infty} Z_n = 0);$$

$$m = EZ_1;$$

$$q_n = P(Z_n = 0);$$

$$q = \lim_{n\to\infty} q_n;$$

a = first point of increase of the distribution function of Z_1,
that is $a = \inf\{x;\ P(Z_1 \leq x) > 0\}$.

REMARK 3.2.1. Taking expectations in (3.1.3) yields that $h_n(z)$ is the nth iterate of $h(z)$. By Lemma 2.2.3(a) we know that $h(s)$ and therefore also $h_n(s)$ is strictly increasing in $s \in [0,\infty)$. Since by assumption $P(Z_1 < \infty) = 1$, it follows from Lemma 2.2.4(a) that $\lim_{s\downarrow 0} h(s) = 0$ and therefore also $\lim_{s\downarrow 0} h_n(s) = 0$ for all $n = 1,2,3,\ldots$. Furthermore, if $P(Z_1 = 0) = 0$, then by Lemma 2.2.5(a) $\lim_{s\to\infty} h(s) = \infty$ and so $\lim_{s\to\infty} h_n(s) = \infty$ for all $n = 1,2, 3,\ldots$. Thus in this case $c_n(s)$ is well-defined for all $s \in [0,\infty)$ and all $n = 1,2,3,\ldots$. However, if $P(Z_1 = 0) > 0$, then again by Lemma 2.2.5(a), $\lim_{s\to\infty} h(s) < \infty$, and so $\lim_{s\to\infty} h_n(s) = -\log P(Z_n = 0) < \infty$ for all $n = 1,2, 3,\ldots$. This means that in this case $c_n(s)$ is only well-defined for $s \in [0,-\log P(Z_n = 0))$.

REMARK 3.2.2. Since, by Definition 3.1.2, $\{Z_n = 0\} \subset \{Z_{n+1} = 0\}$, $q_n \leq q_{n+1}$ for all $n = 1,2,3,\ldots$, and so $q = \lim_{n\to\infty} q_n$ exists and equals $P(Z_n = 0$ from

some n on).

3.3. MAIN RESULTS

It turns out that the expectation of the offspring distribution and the first point of increase of its distribution function play an important role in the theory of branching processes. This is further explained in the following chapters. In this section we mention some results that hold in general, without any assumption about a or m. First of all we shall see that there is a simple relation between h and r.

THEOREM 3.3.1. $\lim_{n\to\infty} h_n(s) = -\log r$ *for all* $s \in (0,\infty)$. *Furthermore, either* (a) *or* (b) *or* (c) *holds:*

(a) $h(s) < s$ *for all* $s \in (0,\infty)$, *and* $\lim_{n\to\infty} h_n(s) = 0$;

(b) *there exists an* $s_0 \in (0,\infty)$ *such that* $h(s_0) = s_0$, *and* $\lim_{n\to\infty} h_n(s) = s_0$;

(c) $h(s) > s$ *for all* $s \in (0,\infty)$, *and* $\lim_{n\to\infty} h_n(s) = \infty$.

PROOF. Since $h(s)$ is concave by Lemma 2.2.3(b) and $\lim_{s\downarrow 0} h(s) = 0$ by Lemma 2.2.4(a), there is in case (b) exactly one solution $\in (0,\infty)$ of the equation $h(s) = s$. Moreover, it is obvious that we are always in one of the three cases mentioned, and that $\lim_{n\to\infty} h_n(s)$ equals, for all $s \in (0,\infty)$, the given value in each of the three cases. Now

$$(3.3.1) \qquad h_n(s) = -\log E\, e^{-sZ_n} \le -\log E\, e^{-sZ_n} I_{\{\lim_{n\to\infty} Z_n = 0\}} \quad , s \in (0,\infty),$$

where I stands for the indicator function. Because the right-hand side of (3.3.1) tends to $-\log r$ as $n \to \infty$,

$$(3.3.2) \qquad \lim_{n\to\infty} h_n(s) \le -\log r \quad \text{for all } s \in (0,\infty).$$

We shall now consider each of the three cases separately.

(a) Suppose $h(s) < s$ for all $s \in (0,\infty)$. Then by Lemma 2.2.4(b), $m \le 1$. It follows from (3.1.3) and Lemma 2.2.4(b) that $E(Z_{n+1} \mid Z_0, \ldots, Z_n) = E(Z_{n+1} \mid Z_n) = mZ_n \le Z_n$, that is $\{Z_n;\ n = 0,1,2,\ldots\}$ is a non-negative supermartingale. Hence Z_n converges almost surely to a finite limit Z_∞ as $n \to \infty$. Since $\lim_{n\to\infty} h_n(s) = 0$ for all $s \in (0,\infty)$, the continuity theorem for Laplace transforms (FELLER [1971]) yields

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(3.3.3) $P(Z_\infty = 0) = 1$,

or

$$-\log r = -\log P(\lim_{n \to \infty} Z_n = 0) = 0.$$

(b) Now suppose that there exists an $s_0 \in (0,\infty)$ such that $h(s_0) = s_0$. Then the sequence $\{e^{-s_0 Z_n};\ n = 0,1,2,\dots\}$ is a bounded martingale by Lemma 3.1.3. So $e^{-s_0 Z_n}$ converges a.s. to some random variable $X_\infty(s_0) \in [0,1]$ as $n \to \infty$ and

$$-\log E\, X_\infty(s_0)^{s/s_0} = \lim_{n \to \infty} h_n(s) = \begin{cases} s_0 & \text{if } s \in (0,\infty) \\ 0 & \text{if } s = 0 \end{cases}.$$

This means that

(3.3.4) $P(X_\infty(s_0) = 0) = 1 - P(X_\infty(s_0) = 1) = 1 - e^{-s_0}$,

or

$$-\log r = -\log P(X_\infty(s_0) = 1) = s_0.$$

(c) Finally, if $h(s) > s$ for all $s \in (0,\infty)$, then $-\log r = \infty$ by (3.3.2). $\quad\square$

In the following theorem the relation between q and $h(s)$ is described.

THEOREM 3.3.2.

(a) $q = 0$ *if and only if* $P(Z_1 = 0) = 0$;

(b) *If* $P(Z_1 = 0) > 0$, *then* $-\log q$ *is the maximal solution of the equation*
$h(s) = s$.

PROOF.

(a) Since $0 \le q_n \uparrow q$ as $n \to \infty$ and $P(Z_1 = 0) = q_1$, $q = 0$ implies that $P(Z_1 = 0) = 0$. If on the other hand $P(Z_1 = 0) = 0$, we have by Lemma 2.2.5(a) that $\lim_{s \to \infty} h(s) = -\log P(Z_1 = 0) = \infty$. It follows that $-\log P(Z_n = 0) = \lim_{s \to \infty} h_n(s) = \infty$, $n = 1,2,3,\dots$, and therefore $q = \lim_{n \to \infty} q_n = \lim_{n \to \infty} P(Z_n = 0) = 0$.

(b) $P(Z_1 = 0) > 0$ implies that $q_n \ge q_1 = P(Z_1 = 0) > 0$ for all $n = 1,2,3,\dots$,

and hence, again by Lemma 2.2.5(a), $h_n(\infty) := \lim_{s\to\infty} h_n(s) < \infty$, so that we may write, using the continuity of h (Lemma 2.2.2),

$$h(-\log q) + \log q = \lim_{n\to\infty}\{h(h_n(\infty)) - h_n(\infty)\} = \lim_{n\to\infty}\{h_{n+1}(\infty) - h_n(\infty)\}$$

$$= \lim_{n\to\infty}\{-\log q_{n+1} + \log q_n\} = 0.$$

Finally, if t is another solution of the equation h(s) = s, then $h_n(\infty) \geq h_n(t) = t$ for all n = 1,2,3,..., or $-\log q_n = h_n(\infty) \geq t$ for all n = 1,2,3,... whence $-\log q \geq t$. □

In theorem 3.3.1 we saw that $\lim_{n\to\infty} h_n(s) = -\log r$ for all $s \in (0,\infty)$. By the continuity theorem for Laplace transforms, this means that $Z_n \overset{d}{\to} Z$, where Z is a random variable with distribution P(Z = 0) = 1 - P(Z = ∞) = r. Actually, we can strengthen this to almost sure convergence. This result is the so-called "extinction or explosion theorem".

THEOREM 3.3.3.

$$P(\lim_{n\to\infty} Z_n = 0) = 1 - P(\lim_{n\to\infty} Z_n = \infty).$$

PROOF. From (3.3.3) and (3.3.4) we know already that the theorem holds true if r > 0. Now, if r = 0, Theorem 3.3.1 tells us that h(s) > s for all $s \in (0,\infty)$. By Lemma 2.2.5(c) this means that a ≥ 1, whence $P(Z_1 \geq 1) = 1$, and so, by (3.1.1), $Z_{n+1} \geq Z_n$ for all n = 0,1,2,... a.s.. Therefore $\lim_{n\to\infty} Z_n$ exists a.s.. Call this limit Z_∞. Then of course Z_∞ has the same distribution as the random variable Z mentioned above, that is $P(Z_\infty = \infty) = 1-r = 1$. □

Now that we know that $P(\lim_{n\to\infty} Z_n = 0 \text{ or } \infty) = 1$, the following step is to find a sequence of norming constants α_n such that $\alpha_n Z_n$ converges in some sense to a limit Z with P(0 < Z < ∞) > 0. Since $\alpha_n Z_n = 0$ for n large enough on the event A := {$Z_n = 0$ from some n on} and P(A) = q by Remark 3.2.2, it is clear that no such sequence {α_n; n = 1,2,3,...} can exist if q = 1, which happens, as we shall prove in the sequel, if and only if m ≤ 1 and $P(Z_1 = 0) > 0$. Suppose therefore that m > 1 or $P(Z_1 = 0) = 0$. Then, because $EZ_n = h_n'(0) = \{h'(0)\}^n = m^n$, a first guess for α_n might be $\alpha_n = m^{-n}$. There are however two objections against this α_n. First of all it is not clear what to do if m = ∞, and secondly, although {$m^{-n}Z_n$; n = 0,1,2,...} is a non-negative martingale if m < ∞, implying that $m^{-n}Z_n$ converges almost surely to some

random variable Z as n → ∞, it turns out that P(Z = 0 or ∞) = 1 in many cases. The corollary to the following lemma gives us a better choice for the α_n, although it may happen that also with this choice P($\lim_{n \to \infty} \alpha_n Z_n$ = 0 or ∞) = 1. However, as we shall see later on, in these cases there is no sequence {α_n; n = 1,2,3,...} of positive constants at all such that $\alpha_n Z_n$ converges a.s. and P($\lim_{n \to \infty} \alpha_n Z_n$ = 0 or ∞) < 1. Remark 3.2.1 and Theorem 3.3.1 explain the assumptions in the lemma.

LEMMA 3.3.4. *If* P(Z_1 = 0) = 0 *and* s ∈ [0,∞) *or* P(Z_1 = 0) > 0 *and* s ∈ [0,-log r], *then the sequence* {$e^{-c_n(s)Z_n}$; n = 0,1,2,...} *is a bounded martingale.*

PROOF. Although only stated there for Galton-Watson processes, this result was in fact first proved in HEYDE [1970]. It follows easily on substituting z = $c_n(s)$ in (3.1.3). □

COROLLARY 3.3.5. *If* P(Z_1 = 0) = 0 *and* s ∈ [0,∞) *or* P(Z_1 = 0) > 0 *and* s ∈ [0,-log r], *then* Y(s) := $\lim_{n \to \infty} c_n(s)Z_n$ *exists a.s.;* φ(z,s) := $-\log E\, e^{-zY(s)}$ *satisfies*

(3.3.5)   φ(z,s) = $\lim_{n \to \infty} h_n(zc_n(s))$,   z ∈ {z; Re z > 0}.

PROOF. These are all consequences of well-known results for bounded martingales. See e.g. LOEVE [1963].   □

From now on Y(s) stands for the a.s. limit of $c_n(s)Z_n$, φ(z,s) for its cumulant generating function, and ℓ(s) for the first point of increase of the distribution function of Y(s). When we talk about Y(s) it is taken for granted that P(Z_1 = 0) = 0 and s ∈ [0,∞), or that P(Z_1 = 0) > 0 and s ∈ [0,-log r]. In the next chapters the study of Y(s) will be continued. Now we pass on to the so-called total progeny of the branching process.

In Section 3.1 we defined $S_n = \Sigma_{k=0}^{n} Z_k$. Since each $Z_n \geq 0$, S := $\lim_{n \to \infty} S_n$ exists and equals $\Sigma_{k=0}^{\infty} Z_k$. This random variable S is called the *total progeny* of the process. In a Galton-Watson process, Z_n → 0 is equivalent to Z_n = 0 from some n on, and so S < ∞ on {Z_n → 0}. Furthermore, S = ∞ on {Z_n → ∞}. Since P(Z_n → 0 or ∞) = 1, this means that P(S < ∞) = P(Z_n → 0). If the state space is [0,∞) on the contrary, there is a possibility that Z_n → 0, but yet S = ∞. However, the next theorem shows that this happens only with probability 0. (The function {f(s)}[inv] or f[inv](s) is defined for functions f which are monotone for s ∈ I ⊂ ℝ, as follows: {f(s)}[inv] = g(s)

or $f^{inv}(s) = g(s)$, $s \in I$, if and only if $g(f(s)) = s$ for all $s \in I$).

THEOREM 3.3.6. *Let* $k(z)$ *be the cumulant generating function of* S. *Then:*
(a) k *satisfies the equation*

(3.3.6) $k(z) = z + h(k(z))$, $z \in \{z; \mathrm{Re}\, z \geq 0\}$;

(b) $k(s) = \{s - \log r - h(s - \log r)\}^{inv} - \log r$, $s \in [0,\infty)$ if $r > 0$;

   $k(s) = \infty$, $s \in [0,\infty)$ if $r = 0$;

(c) $P(S < \infty) = r$.

PROOF.

(a) Let $k_n(z)$ be the cumulant generating function of S_n. Then we have, using Lemma 3.1.3,

$$k_{n+1}(z) = -\log E\, e^{-zS_{n+1}} = -\log E\, \{E(e^{-zS_{n+1}} | Z_1)\}$$

$$= -\log\{e^{-z} E(Ee^{-zS_n})^{Z_1}\} = z + h(k_n(z)).$$

Now since $S_n \xrightarrow{a.s.} S$, $k_n(z) \to k(z)$ as $n \to \infty$ by the continuity theorem for Laplace transforms and so part (a) follows from the continuity of h.

(b) We distinguish again the same three cases as in Theorem 3.3.1.

1. Suppose $h(s) < s$ for all $s \in (0,\infty)$. Then $-\log r = 0$ by Theorem 3.3.1, and by the concavity of $h(s)$ (Lemma 2.2.3(b)), $s - h(s)$ is increasing for $s \in [0,\infty)$. Therefore $\{s - h(s)\}^{inv}$ is well-defined and from (3.3.6) we see that

(3.3.7) $k(s) = \{s - h(s)\}^{inv}$, $s \in [0,\infty)$.

2. Now suppose that there exists an $s_0 \in (0,\infty)$ such that $h(s_0) = s_0$. Then we know from Theorem 3.3.1 that $0 < -\log r = s_0 < \infty$. Let $\tilde{h}(s) = h(s - \log r) + \log r$, $s \in [0,\infty)$. Then $\exp(-\tilde{h}(s))$ is completely monotone since $\exp(-h(s))$ is so, and $\tilde{h}(0) = 0$. It follows therefore from Theorem XIII. 4.1 in FELLER [1971] that $\tilde{h}(s) = -\log E\, e^{-s\tilde{Z}_1}$, $s \in [0,\infty)$, where $\tilde{Z}_1$ is a proper, non-negative random variable; $\tilde{Z}_1$ is not concentrated in one point since Z_1 is not. Furthermore, because Z_1 has an infinitely divisible distribution, $\exp(-h(s)/n)$ is completely monotone for all

$n = 1,2,3,\ldots$, and hence the same is true for $\exp(-\tilde{h}(s)/n) = r^{-1/n}\exp(-h(s - \log r)/n)$, and therefore $\tilde{Z}_1$ has an infinitely divisible distribution. Let $\{\tilde{Z}_n; n = 0,1,2,\ldots\}$ be a branching process with $\tilde{Z}_0 = 1$, having the distribution of $\tilde{Z}_1$ as its offspring distribution. Since $\tilde{h}(s) < s$ for all $s \in (0,\infty)$, (3.3.7) yields

$$\tilde{k}(s) = \{s - \tilde{h}(s)\}^{inv}, \qquad s \in [0,\infty),$$

where $\tilde{k}(s) = -\log E\, e^{-s\tilde{S}}$ and $\tilde{S} = \Sigma_{k=0}^{\infty}\tilde{Z}_k$. Now let $g(s) = k(s) + \log r$, then by (3.3.6),

$$g(s) = s + h(k(s)) + \log r = s + h(g(s) - \log r) + \log r$$

$$= s + \tilde{h}(g(s)),$$

and hence

$$g(s) = \{s - \tilde{h}(s)\}^{inv} = \tilde{k}(s), \qquad s \in [0,\infty),$$

or

$$k(s) = g(s) - \log r = \{s - \tilde{h}(s)\}^{inv} - \log r$$

$$= \{s - \log r - h(s - \log r)\}^{inv} - \log r, \qquad s \in [0,\infty).$$

3. Finally, if $h(s) > s$ for all $s \in (0,\infty)$, then $r = 0$ by Theorem 3.3.1 and as in the proof of Theorem 3.3.3, $Z_{n+1} \geq Z_n$ for all $n = 0,1,2,\ldots$ a.s., so $P(S = \infty) = 1$. Since the cumulant generating function of this S equals ∞ for all $s \in [0,\infty)$ part (b) of the theorem is proved.

(c) By Lemma 2.2.4(a) $\lim_{s\downarrow 0} k(s) = -\log P(S < \infty)$, whence, if $r > 0$,

$$P(S < \infty) = \lim_{s\downarrow 0} e^{-k(s)} = \lim_{s\downarrow 0} e^{-\{s - \log r - h(s - \log r)\}^{inv} + \log r} = r.$$

If $r = 0$, then we know from part (b) that $P(S < \infty) = 0 = r$. $\qquad \square$

By (3.1.2) $S_n = 1 + X(S_{n-1})$, $n = 0,1,2,\ldots$. Furthermore $S_{-1} = 0$. This means that

(3.3.8) $S = 1 + X(S-)$ on $\{S < \infty\}$,

where we use the notation $X(t_0-)$ for $\lim_{t \uparrow t_0} X(t)$. It turns out however that the event $\{X(S) > X(S-)\}$ has probability 0 on $\{S < \infty\}$ and so, using the convention $X(\infty) = \lim_{t \to \infty} X(t)$, it follows that $S = 1 + X(S)$ a.s.. This is made precise in the next theorem.

THEOREM 3.3.7. $S = \Sigma_{k=0}^{\infty} Z_k$ *satisfies a.s. the equation*

(3.3.9) $S = 1 + X(S)$.

PROOF. Since S_n is a stopping time for the process $\{X(t); t \in [0, \infty)\}$ for all $n = -1, 0, 1, \ldots$, we know from Lemma 2.1.5 that

(3.3.10) $X(S_n + \varepsilon) - X(S_n) \overset{d}{=} X(\varepsilon)$ for all $n = -1, 0, 1, \ldots$ and all $\varepsilon > 0$.

This means that we have for every $\varepsilon > 0$ and $\delta > 0$

(3.3.11) $P(X(\varepsilon) > \delta) = a_n(\varepsilon, \delta) + b_n(\varepsilon, \delta)$,

where

$$a_n(\varepsilon, \delta) = P(X(S_n + \varepsilon) - X(S_n) > \delta \text{ and } S < \infty) \quad \text{and}$$

$$b_n(\varepsilon, \delta) = P(X(S_n + \varepsilon) - X(S_n) > \delta \text{ and } S = \infty).$$

Also, because S_n is non-decreasing in n,

(3.3.12) $X(S_n + \varepsilon) \overset{a.s.}{\longrightarrow} X(S + \varepsilon -)$ for all $\varepsilon > 0$ as $n \to \infty$ on $\{S < \infty\}$,

and

(3.3.13) $X(S_n) \overset{a.s.}{\longrightarrow} X(S-)$ as $n \to \infty$ on $\{S < \infty\}$.

Therefore, writing $a(\varepsilon, \delta) = P(X(S + \varepsilon -) - X(S-) > \delta \text{ and } S < \infty)$, we see by (3.3.12) and (3.3.13) that $\lim_{n \to \infty} a_n(\varepsilon, \delta) = a(\varepsilon, \delta)$ for every δ where $a(\varepsilon, \delta)$ is continuous and hence by (3.3.11), $\lim_{n \to \infty} b_n(\varepsilon, \delta) := b(\varepsilon, \delta)$ exists, is non-negative, and

(3.3.14) $P(X(\varepsilon) > \delta) = a(\varepsilon,\delta) + b(\varepsilon,\delta)$

for every $\delta > 0$ where $a(\varepsilon,\delta)$ is continuous. Finally, $\lim_{\varepsilon \downarrow 0} P(X(\varepsilon) > \delta) = 0$ by Definition 2.1.2 and, writing $a(\delta) = P(X(S) - X(S-) > \delta$ and $S < \infty)$, it follows that $\lim_{\varepsilon \downarrow 0} a(\varepsilon,\delta) = a(\delta)$ for every $\delta > 0$ where $a(\delta)$ is continuous, because $X(S+\varepsilon-) \xrightarrow{a.s.} X(S)$ as $\varepsilon \downarrow 0$ on $\{S < \infty\}$. Since every distribution function has at most countably many discontinuities, we see from (3.3.14), that if we choose a sequence $\{\varepsilon_n;\ n = 1,2,3,\ldots\}$ such that $\varepsilon_n \downarrow 0$ as $n \to \infty$, then $\lim_{n \to \infty} b(\varepsilon_n,\delta) := b(\delta)$ exists and is non-negative for all but at most countably many $\delta > 0$. Substituting this into (3.3.14) we obtain $0 = a(\delta) + b(\delta)$ for all but at most countably many $\delta > 0$, and hence that $X(S) = X(S-)$ a.s. on $\{S < \infty\}$. This together with (3.3.8) yields the required result on $\{S < \infty\}$. Since $\frac{X(t)}{t} \xrightarrow{a.s.} m$ as $t \to \infty$ by the strong law of large numbers, $X(\infty) = \infty$ a.s., and hence also $S = 1 + X(S)$ a.s. on $\{S = \infty\}$.   □

It follows from Theorem 3.3.7 that $S = \inf\{t \geq 0;\ X(t) = X(t-) \leq t - 1\}$ a.s.. For if $0 \leq t < S$ and $X(t) = X(t-)$, then $S_{n-1} \leq t < S_n$ for some $n = 0,1,2,\ldots$. This means that $X(t-) = X(t) \geq X(S_{n-1}) = S_n - 1 > t - 1$. The analog of this result for a Galton-Watson process is $S = \inf\{n \geq 0;\ X_n = n-1\}$, where $\{X_n;\ n = 0,1,2,\ldots\}$ is a random walk with $X_1 \overset{d}{=} Z_1$. In a paper of DWASS [1969], the distribution of both $S = \Sigma_{k=0}^{\infty} Z_k$ and $W = \{\inf n;\ X_n = n-1\}$ were derived and observed to be the same. The method used here makes it clear why this has to be so.

In Section 3.2 we defined a to be the first point of increase of the distribution function of Z_1. If $a \geq 1$ then $P(Z_1 \geq 1) = 1$ and hence, by (3.1.1), $Z_{n+1} \geq Z_n$ and $P(S = \infty) = 1$. If $a < 1$, then $P(Z_n \geq a^n) = 1$, since

$$\lim_{s \to \infty} \frac{h_n(s)}{s} = \lim_{s \to \infty} \prod_{k=1}^{n} \frac{h_k(s)}{h_{k-1}(s)} = a^n \qquad \text{if } P(Z_1 = 0) = 0$$

and

$$\lim_{s \to \infty} \frac{h_n(s)}{s} = 0 = a^n \qquad \text{if } P(Z_1 = 0) > 0,$$

and therefore $S \geq \lim_{n \to \infty} \Sigma_{k=0}^{n} a^k = \frac{1}{1-a}$. The next theorem tells us that $P(S < \frac{1}{1-a} + \varepsilon) > 0$ for all $\varepsilon > 0$.

THEOREM 3.3.8. *If* $a < 1$ *then the first point of increase of the distribution function of* S *equals* $\frac{1}{1-a}$. *Furthermore,* $P(S = \frac{1}{1-a}) = P(Z_1 = a)^{1/(1-a)}$.

PROOF; Because a < 1, either $h(s) < s$ for all $s \in (0,\infty)$ or there exists an $s_0 \in (0,\infty)$ such that $h(s_0) = s_0$, and therefore, by Theorem 3.3.1, $r > 0$. This means, because of Theorem 3.3.6(b), that

$$k(s) = \{s - \log r - h(s - \log r)\}^{\text{inv}} - \log r, \quad s \in [0,\infty),$$

or

(3.3.15) $\quad k(s - \log r - h(s - \log r)) + \log r = s, \quad s \in [0,\infty).$

Differentiating this we get

$$k'(s - \log r - h(s - \log r)) \cdot (1 - h'(s - \log r)) = 1,$$

and thus, using the concavity of $k(s)$ and $h(s)$ (Lemma 2.2.3(b)), and Lemma 2.2.5(c),

$$\lim_{s \to \infty} \frac{k(s)}{s} = \lim_{s \to \infty} k'(s) = \lim_{s \to \infty} k'(s - \log r - h(s - \log r))$$

$$= \lim_{s \to \infty} \frac{1}{1 - h'(s - \log r)} = \frac{1}{1-a}.$$

This together with Lemma 2.2.5(c) proves the first part. The second part follows on observing that

$$-\log P\left(S = \frac{1}{1-a}\right) = \lim_{s \to \infty} \left\{k(s) - \frac{s}{1-a}\right\}$$

by Lemma 2.2.5(b) and this together with (3.3.15) and again Lemma 2.2.5(b) implies

$$-\log P\left(S = \frac{1}{1-a}\right) = \lim_{s \to \infty}\left\{k(s - \log r - h(s - \log r)) - \frac{s - \log r - h(s - \log r)}{1-a}\right\}$$

$$= \lim_{s \to \infty}\left\{s - \log r - \frac{s - \log r - h(s - \log r)}{1-a}\right\}$$

$$= \lim_{t \to \infty} \frac{h(t) - at}{1-a} = \frac{-\log P(Z_1 = a)}{1-a}. \qquad \square$$

REMARK 3.3.9. We can understand this last result also as follows. $S = \dfrac{1}{1-a}$ if and only if $Z_n = a^n$ for all $n = 0,1,2,\dots$. So if $P(Z_1 = a) = 0$, then

$$P\left(S = \frac{1}{1-a}\right) = 0 = P(Z_1 = a)^{1/(1-a)}. \text{ If } P(Z_1 = a) > 0, \text{ then}$$

$$-\log E(e^{-sZ_{n+1}} \mid Z_n = a^n) = a^n h(s)$$

by (3.1.3), and because a^{n+1} is the first point of increase of the distribution function of Z_{n+1} conditioned on $Z_n = a^n$, Lemma 2.2.5(b) yields

$$-\log P(Z_{n+1} = a^{n+1} \mid Z_n = a^n) = \lim_{s \to \infty} \{a^n h(s) - a^{n+1} s\} =$$

$$= -a^n \log P(Z_1 = a).$$

This means that

$$P\left(S = \frac{1}{1-a}\right) = P(Z_n = a^n \text{ for all } n = 0,1,2,\ldots)$$

$$= P(Z_0 = 1) \prod_{n=0}^{\infty} P(Z_{n+1} = a^{n+1} \mid Z_n = a^n)$$

$$= \prod_{n=0}^{\infty} \{P(Z_1 = a)\}^{a^n} =$$

$$= \{P(Z_1 = a)\}^{\sum_{n=0}^{\infty} a^n} = P(Z_1 = a)^{\frac{1}{1-a}}.$$

We close this section with a discussion about the rate of convergence of the random variables Z_n in terms of ratios. More precisely, we want to determine a function f such that $Z_{n+1}/f(Z_n)$ converges in some sense to a positive and finite limit. This is not interesting in case $P(Z_1 = 0) > 0$ and $\lim_{n \to \infty} Z_n = 0$, because then $Z_n = 0$ from some n on, as we shall see in the following chapters. It turns out that we can choose f linear both if we consider the process on $\{Z_n \to \infty\}$ for $m < \infty$ and if we consider it on $\{Z_n \to 0\}$ for $a > 0$. This will be proved in Chapters 6 and 8. So we are left with the cases $a = 0$ and $P(Z_1 = 0) = 0$ on $\{Z_n \to 0\}$, and $m = \infty$ on $\{Z_n \to \infty\}$. Since the branching process is defined in terms of subordinators, we might expect some help from the paper of FRISTEDT and PRUITT [1971], in which the growth of subordinators was studied. They proved that under certain conditions

$$(3.3.16) \qquad \lim\inf \frac{W(t)}{f_\gamma(t)} = d \qquad \text{a.s.},$$

where the lim inf may be taken both for $t \downarrow 0$ and for $t \to \infty$, and where $\{W(t); t \in [0,\infty)\}$ is a subordinator and

$$(3.3.17) \qquad f_\gamma(t) = \frac{\log|\log t|}{b\left(\frac{\gamma \log |\log t|}{t}\right)}$$

with $b(s) = p^{inv}(s)$, $s \in [0,\infty)$ and $p(s) = -\log E\, e^{-sW(1)}$, $s \in [0,\infty)$, and d some constant $\in (0,\infty)$. Since $P(\lim_{n\to\infty} Z_n = 0$ or $\infty) = 1$, we can try to prove a result like (3.3.16) for $W(Z_n)/f_\gamma(Z_n)$. Then choosing $\{W(t);\ t \in [0,\infty)\}$ such that it has the same distribution as the subordinator by which the branching process is defined, but independent thereof, it follows that $W(Z_n) \overset{d}{=} Z_{n+1}$ for all $n = 0,1,2,\ldots$, and this might possibly lead to a result concerning $Z_{n+1}/f_\gamma(Z_n)$. In Chapters 7 and 9 however we shall see an example of a process $\{Z_n;\ n = 0,1,2,\ldots\}$ for which $\lim_{n\to\infty} Z_{n+1}/f_\gamma(Z_n) = \infty$ a.s., and for which the corresponding subordinator satisfies (3.3.16). This means that this method is not generally successful, and it is not clear how to choose a good norming function for an arbitrary branching process $\{Z_n;\ n = 0,1,2,\ldots\}$.

We shall now turn back to the quotient $W(Z_n)/f_\gamma(Z_n)$ mentioned above, for which the following results can be proved.

THEOREM 3.3.10. *Let* $\{W(t);\ t \in [0,\infty)\}$ *be a subordinator,* $p(s) = -\log E\, e^{-sW(1)}$ $s \in [0,\infty)$, $b(s) = p^{inv}(s)$, $s \in [0,\infty)$ *and*

$$f_\gamma(t) = \frac{\log|\log t|}{b\left(\frac{\gamma \log |\log t|}{t}\right)}\, , \qquad t \in (0,\infty).$$

Suppose that $P(Z_1 = 0) = 0$. *If* $\gamma > 1$ *then*

$$\liminf_{n \to \infty} \frac{W(Z_n)}{f_\gamma(Z_n)} \geq \gamma - 1 \qquad \text{a.s..}$$

PROOF. The proof is based on Lemma 4 in the paper of FRISTEDT and PRUITT [1971]. There they construct, for every $\beta < \gamma-1$, a sequence $\{t_k;\ k = 1,2,3,\ldots\}$ such that $t_k \downarrow 0$ as $k \to \infty$ and

$$(3.3.18) \qquad \liminf_{k \to \infty} \frac{W(t_{k+1})}{f_\gamma(t_k)} \geq \beta \qquad \text{a.s.,}$$

and a sequence $\{t_k';\ k = 1,2,3,\ldots\}$ such that $t_k' \to \infty$ as $k \to \infty$ and

$$(3.3.19) \qquad \liminf_{k \to \infty} \frac{W(t_k')}{f_\gamma(t_{k+1}')} \geq \beta \qquad \text{a.s. .}$$

We shall compare Z_n with these t_k resp. t_k'. Suppose

$$\omega \in A := \{\lim_{n \to \infty} Z_n = 0\} \cap \{Z_n > 0 \text{ for all } n = 0,1,2,\dots\}$$

$$\cap \{\liminf_{k \to \infty} \frac{W(t_{k+1})}{f_\gamma(t_k)} \geq \beta\}.$$

Because $P(Z_1 = 0) = 0$, $P(A) = r$. Define $t_0 = \infty$, then for every $n = 1,2,3,\dots$ there is an integer $k = k(n,\omega)$ such that $t_{k+1} < Z_n \leq t_k$. Hence we obtain

$$(3.3.20) \qquad \frac{W(Z_n)}{f_\gamma(Z_n)} \geq \frac{W(t_{k(n)+1})}{f_\gamma(t_{k(n)})} \qquad ,$$

because both $W(t)$ and $f_\gamma(t)$ are non-decreasing in t. (See FRISTEDT and PRUITT [1971]; we use the convention $f_\gamma(\infty) = \lim_{t \to \infty} f_\gamma(t)$.) Since $\lim_{n \to \infty} Z_n(\omega) = 0$,

$$(3.3.21) \qquad \lim_{n \to \infty} k(n,\omega) = \infty.$$

Therefore, by (3.3.18), (3.3.20) and (3.3.21)

$$\liminf_{n \to \infty} \frac{W(Z_n)}{f_\gamma(Z_n)} \geq \beta \qquad \text{a.s. on } \{Z_n \to 0\},$$

and since $\beta < \gamma-1$ was arbitrary, the result holds on $\{Z_n \to 0\}$. The proof for the case $\{Z_n \to \infty\}$ is similar, using the sequence $\{t_k'; k = 1,2,3,\dots\}$. $\square$

The same method yields a result for the lim sup.

THEOREM 3.3.11. *Let* $\{W(t); t \in [0,\infty)\}$ *and* $b(s)$ *be as in Theorem* 3.3.10, g *a positive function on* $(0,\infty)$, *such that* $\lim_{t \downarrow 0} g(t) = \lim_{t \to \infty} g(t) = 0$, α *a positive constant and*

$$f(t) = \frac{g(t)}{b(g(t)/t|\log t|^{1+\alpha})} , \qquad t \in (0,\infty).$$

Suppose that $P(Z_1 = 0) = 0$ *and that* $f(t)$ *is non-decreasing both for small* t *and for large* t. *Then*

$$\limsup_{n \to \infty} \frac{W(Z_n)}{f(Z_n)} = 0 \qquad \text{a.s.} \ .$$

REMARK 3.3.12. Writing

$$\chi(t) = \frac{g(t)}{t |\log t|^{1+\alpha}} , \qquad t \in (0,\infty),$$

we get

$$f(t) = \frac{\chi(t)}{b(\chi(t))} t |\log t|^{1+\alpha} .$$

Since $\dfrac{s}{b(s)} \uparrow EW(1)$ as $s \downarrow 0$ and

$$\frac{s}{b(s)} \downarrow \inf\{x ; P(W(1) \le x) > 0\} \qquad \text{as } s \to \infty$$

by the Lemmas 2.2.4(b), 2.2.5(c) and 2.2.6, and $t |\log t|^{1+\alpha}$ is increasing in t both for sufficiently small and for sufficiently large t, the conditions on f are fulfilled if $\chi(t) \downarrow 0$ for $t \to \infty$ and $\chi(t) \uparrow \infty$ for $t \downarrow 0$. As there exist functions g such that the corresponding χ has these properties and such that also $\lim_{t \downarrow 0} g(t) = \lim_{t \to \infty} g(t) = 0$, there are functions f which satisfy the conditions of the theorem.

PROOF OF THEOREM 3.3.11. Let $t_k = e^{-k}$, $k = 1,2,3,\ldots$, $\{v_k; k = 1,2,3,\ldots\}$ a sequence such that $\lim_{k \to \infty} v_k = \infty$, $v_k g(t_{k+1}) < 1$ for all $k = 1,2,3,\ldots$ and $\lim_{k \to \infty} v_k g(t_{k+1}) = 0$, and let $s_k = 1/(f(t_{k+1}) v_k)$, d any constant $\in (0,\infty)$ and p(s) as in Theorem 3.3.10. Then we can prove using Lemma 1 in FRISTEDT and PRUITT [1971],

$$(3.3.22) \qquad T := \sum_{k=1}^{\infty} P(W(t_k) \ge df(t_{k+1})) \le$$

$$T_1 := \sum_{k=1}^{\infty} \frac{t_k p(s_k)}{1 - e^{-s_k df(t_{k+1})}} .$$

Because $s_k f(t_{k+1}) = \dfrac{1}{v_k} \to 0$ as $k \to \infty$, we see that

$$(3.3.23) \qquad T_1 < \infty \text{ if and only if } T_2 := \sum_{k=1}^{\infty} \frac{t_k p(s_k)}{s_k df(t_{k+1})} < \infty ,$$

Now

$$T_2 = \frac{1}{d} \sum_{k=1}^{\infty} v_k t_k p\left(\frac{b(g(t_{k+1})/(t_{k+1} |\log t_{k+1}|^{1+\alpha}))}{v_k g(t_{k+1})} \right) \le$$

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$$\leq \frac{1}{d} \sum_{k=1}^{\infty} \frac{v_k t_k g(t_{k+1})}{v_k g(t_{k+1}) t_{k+1} |\log t_{k+1}|^{1+\alpha}} \quad ,$$

since $1/(v_k g(t_{k+1})) > 1$ and $p(t \cdot s) \leq t \cdot p(s)$ for $t > 1$ by the concavity of $p(s)$. So, substituting $t_k = e^{-k}$ we obtain

$$T_2 \leq \frac{1}{d} \sum_{k=1}^{\infty} \frac{e}{(k+1)^{1+\alpha}} < \infty, \qquad \text{because } \alpha > 0.$$

Thus, by (3.3.22) and (3.3.23), $T < \infty$. An application of the Borel-Cantelli lemma now yields

$$P(\lim \sup\{W(t_k) \geq df(t_{k+1})\}) = 0$$

and therefore, since $d \in (0,\infty)$ is arbitrary,

$$\lim_{k \to \infty} \sup \frac{W(t_k)}{f(t_{k+1})} = 0 \qquad \text{a.s. .}$$

In the same way we can prove, with $t'_k = e^k$, $k = 1,2,3,\ldots,$

$$\lim_{k \to \infty} \sup \frac{W(t'_{k+1})}{f(t'_k)} = 0 \qquad \text{a.s. .}$$

Then , using the same method as in the proof of Theorem 3.3.10, we get the required result.   □

CHAPTER 4

THE CASE $m < 1$

4.1. INTRODUCTION

As we know from the Lemmas 2.2.4 and 2.2.5, the values of m resp. a are determined by the behaviour of h(s) for small resp. large s. It turns out that in many theorems these m and a play an important role. We shall see that if $m \leq 1$, then $P(\lim_{n \to \infty} Z_n = 0) = 1$ and if $m > 1$, then $P(\lim_{n \to \infty} Z_n = 0) < 1$. Furthermore, as can easily be proved in a similar way as in JAGERS [1975], if we suppose that the variance of Z_1, $\sigma^2(Z_1)$, is finite, then $\lim_{n \to \infty} \sigma^2(Z_n) = 0$ if $m < 1$, whereas $\lim_{n \to \infty} \sigma^2(Z_n) = \infty$ if $m = 1$, indicating a different behaviour of the branching process in the cases $m < 1$ and $m = 1$. It also turns out that in many results it is essential that $\lim_{s \downarrow 0} \frac{h(s)}{s} < \infty$. This explains why we distinguish four cases for m, namely $m < 1$, $m = 1$, $1 < m < \infty$ and $m = \infty$. Similarly, since $\lim_{s \to \infty} \frac{h(s)}{s} = a$, the positivity of a is important. We therefore study the cases $a = 0$ and $a > 0$ separately.

Many proofs in the Galton-Watson process theory are based on the convexity of the probability generating function $f(s) := E(s^{Z_1} \mid Z_0 = 1) = \Sigma_{k=0}^{\infty} P(Z_1 = k \mid Z_0 = 1) s^k$ for $0 \leq s \leq 1$. If Z_1 is not integer-valued, then the function $E(s^{Z_1} \mid Z_0 = 1)$ is in general not convex. However we have at our disposal the cumulant generating function h(s) which is concave for $s \in [0, \infty)$, as we know from Lemma 2.2.3(b). We can therefore apply to this h(s) the techniques used for f(s) in the Galton-Watson process theory. For this reason we shall often not give a detailed proof, but only refer to the corresponding proof for the Galton-Watson process.

In this chapter we investigate the so-called *subcritical processes,* that is processes with $m < 1$. It turns out that if $P(Z_1 = 0) = 0$, the behaviour of the processes depends on the value of a. Results concerning that case are therefore mentioned in Chapters 8 and 9, and we mostly confine ourselves in this chapter to the case $P(Z_1 = 0) > 0$.

4.2. SOME LIMIT THEOREMS

First of all we look at the values of r and q.

THEOREM 4.2.1.

(a) r = 1;

(b) *if* $P(Z_1 = 0) > 0$, *then* q = 1.

PROOF. Since h'(0) = m by Lemma 2.2.4(b), we see that h'(0) < 1. Therefore, by the concavity of h(s) on [0,∞) (Lemma 2.2.3(b)), h(s) < s for all s ∈ (0,∞), and so part (a) follows from Theorem 3.3.1 and part (b) is an easy consequence of Theorem 3.3.2(b). □

Part (b) of Theorem 4.2.1 states that if $P(Z_1 = 0) > 0$, then $\lim_{n \to \infty} P(Z_n > 0) = 0$. It is natural to ask how fast this convergence is. An answer to this question is given in the following theorem, where we see that, just as in Theorem 4.2.3 below, the finiteness of $EZ_1 \log Z_1$ is important. In brief, this is caused by the fact that $EZ_1 \log Z_1 < \infty$ can be proved to be equivalent to $\int_0^\epsilon \frac{f(s)}{s} ds < \infty$ for any $\epsilon > 0$, where $f(s) = m - \frac{h(s)}{s}$. See the proof of Theorem 3 in SENETA and VERE-JONES [1968]. On the other hand, since by Lemma 2.2.5(a), $\lim_{s \to \infty} h(s) = -\log P(Z_1 = 0) < \infty$, we may write

$$h_n(s) = h(s) m^{n-1} \prod_{k=1}^{n-1} \left(1 - \frac{f(h_k(s))}{m}\right) \qquad \text{for every } s \in (0,\infty]$$

and n = 2,3,4,..., where we use the convention $h_n(\infty) = \lim_{s \to \infty} h_n(s)$. Therefore, $\lim_{n \to \infty} m^{-n} h_n(s)$ is positive together with

$$\prod_{k=1}^{\infty} \left(1 - \frac{f(h_k(s))}{m}\right),$$

that is if and only if

$$\sum_{k=1}^{\infty} f(h_k(s)) < \infty.$$

Furthermore, it follows from the concavity of h that $\{h'(h(s))\}^{k-1} h(s) \leq h_k(s) \leq m^{k-1} h(s)$, and therefore, for every s ∈ (0,∞] there exist $\delta_1 \in (0,1)$ and $\delta_2 \in (0,1)$ such that $\delta_1^k \leq h_k(s) \leq \delta_2^k$ for sufficiently large k. Now because integral comparison yields $\sum_{k=1}^{\infty} f(\delta^k) < \infty$ for any $\delta \in (0,1)$ if and only if $\int_0^\epsilon \frac{f(s)}{s} ds < \infty$ for any $\epsilon > 0$, we thus can associate $EZ_1 \log Z_1$ with

$\lim_{n\to\infty} m^{-n} h_n(s)$, and therefore, since $h_n(s)$ is the cumulant generating function of Z_n, with the limit behaviour of Z_n. The analysis in this chapter is thus in fact based on the Taylor expansion $h(s) = ms - f(s)s$.

THEOREM 4.2.2. *Suppose that* $P(Z_1 = 0) > 0$.

(a) *If* $EZ_1 \log Z_1 < \infty$, *then* $\lim_{n\to\infty} m^{-n} P(Z_n > 0) > 0$.

(b) *If* $EZ_1 \log Z_1 = \infty$, *then* $\lim_{n\to\infty} m^{-n} P(Z_n > 0) = 0$.

(c) *If* g *is a continuous, increasing function on* $[0,1)$ *such that* $g(0) = 0$
   and

$$(4.2.1) \qquad \int_0^1 \frac{g(y)f(y)}{y}\, dy < \infty,$$

   where

$$f(y) = m - \frac{h(y)}{y}, \qquad y \in (0,\infty),$$

   then

$$\lim_{n\to\infty} \{m^{-n} P(Z_n > 0)\}^{g(\delta^{n-1})} = 1, \qquad \text{for every } \delta \in (0,1).$$

PROOF. The proof of the parts (a) and (b) is analogous to the proof of Theorem (2.6.1) in JAGERS [1975]. It follows easily from the remark made above, since, writing $h_n(\infty) = \lim_{s\to\infty} h_n(s)$, $P(Z_n > 0) = 1 - P(Z_n = 0) = 1 - e^{-h_n(\infty)} \sim h_n(\infty)$ as $n \to \infty$ by Theorem 4.2.1(b). This means that $\lim_{n\to\infty} m^{-n} P(Z_n > 0) > 0$ if and only if $\lim_{n\to\infty} m^{-n} h_n(\infty) > 0$, which, as we saw above, in its turn is equivalent to $EZ_1 \log Z_1 < \infty$. For the proof of part (c) we notice that since $g(s)$ resp. $f(s)$ are non-negative and increasing in s by assumption resp. Lemma 2.2.6,

$$0 \le \sum_{k=1}^{n-1} g(\delta^k) f(\delta^k) \le \int_0^n g(\delta^x) f(\delta^x)\, dx = \int_1^{\delta^n} \frac{g(y)f(y)}{y \log \delta}\, dy$$

$$\le -\frac{1}{\log \delta} \int_0^1 \frac{g(y)f(y)}{y}\, dy \qquad \text{for every } \delta \in (0,1),$$

and thus, by (4.2.1) and the non-negativity of $g(s)\cdot f(s)$ for $s \in (0,1)$, $\sum_{k=1}^n g(\delta^k) f(\delta^k)$ has a finite limit as $n \to \infty$, for every $\delta \in (0,1)$. An application of the Kronecker Lemma (LOEVE [1963]) now yields

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$$\lim_{n \to \infty} g(\delta^n) \sum_{k=1}^{n} f(\delta^k) = 0$$

for every $\delta \in (0,1)$. Therefore, because $-\log(1-x) < 2x$ for small x and, by the continuity of g,

(4.2.2) $$\lim_{s \downarrow 0} g(s) = g(0) = 0,$$

(4.2.3) $$\lim_{n \to \infty} g(\delta^{n-1}) \sum_{k=1}^{n-1} \left\{ -\log\left(1 - \frac{f(\delta^k)}{m} \right) \right\} = 0$$

for every $\delta \in (0,1)$. Furthermore, a repeated application of the inequality $h(s) \le ms$ yields $h_k(s) \le m^k s$ for all $s \in (0,\infty)$ and all $k = 1,2,3,\ldots$, and so $\lim_{s \to \infty} h_k(s) = \lim_{s \to \infty} h_{k-1}(h(s)) \le \lim_{s \to \infty} m^{k-1} h(s) \le \delta^k$ for every $\delta \in (m,1)$ if k is large enough, since $\lim_{s \to \infty} h(s) = -\log P(Z_1 = 0) < \infty$. This together with (4.2.2) and (4.2.3) yields

(4.2.4) $$\lim_{n \to \infty} \lim_{s \to \infty} g(\delta^{n-1}) \sum_{k=1}^{n-1} \left\{ -\log\left(1 - \frac{f(h_k(s))}{m} \right) \right\} = 0$$

for every $\delta \in (m,1)$, and hence a fortiori for every $\delta \in (0,1)$. Iterating the equation

$$\frac{h(s)}{s} = m\left(1 - \frac{f(s)}{m} \right)$$

we obtain

$$h_n(s) = h(s) m^{n-1} \prod_{k=1}^{n-1} \left\{ 1 - \frac{f(h_k(s))}{m} \right\}$$

and therefore, since $\lim_{s \to \infty} |\log h(s)| < \infty$,

(4.2.5) $$\lim_{s \to \infty} \log h_n(s) =$$
$$= \lim_{s \to \infty} \log h(s) + (n-1)\log m + \lim_{s \to \infty} \sum_{k=1}^{n-1} \log\left(1 - \frac{f(h_k(s))}{m} \right).$$

Combining (4.2.2), (4.2.4) and (4.2.5) and remembering that $\lim_{s \to \infty} h(s) < \infty$, we can conclude that

$$\lim_{n \to \infty} \lim_{s \to \infty} g(\delta^{n-1}) \cdot (-\log h_n(s) + n \log m) = 0$$

for every $\delta \in (0,1)$. Finally, because $\lim_{n \to \infty} P(Z_n > 0) = 0$ by Theorem 4.2.1 (b), we have

$$\lim_{s \to \infty} h_n(s) = -\log\{1 - P(Z_n > 0)\} \sim P(Z_n > 0) \qquad \text{as } n \to \infty,$$

and therefore

$$\lim_{n \to \infty} \{P(Z_n > 0) m^{-n}\}^{g(\delta^{n-1})} = 1$$

for every $\delta \in (0,1)$. $\qquad \square$

Another interesting way to look at the behaviour of Z_n is to examine the distribution of Z_n conditioned on $Z_n > 0$, of course again in the case $P(Z_1 = 0) > 0$, since otherwise $P(Z_n > 0) = 1$ for all n. For Galton-Watson processes, there is the so-called "Yaglom-theorem". It turns out that this theorem is also true if the state space of the branching process is $[0,\infty)$. We can formulate it in the following way.

THEOREM 4.2.3. *Suppose* $P(Z_1 = 0) > 0$. *Then:*
(a) *The distribution of* Z_n *conditioned on* $\{Z_n > 0\}$ *converges weakly to some proper distribution.*
Let Z be a random variable having this limit distribution.
(b) $P(Z = 0) = 0$;
(c) $EZ < \infty$ *if and only if* $EZ_1 \log Z_1 < \infty$;
(d) *the cumulant generating function g of Z satisfies*

(4.2.6) $\qquad 1 - \exp(-g(h(s))) = m(1 - \exp(-g(s))), \qquad s \in [0,\infty)$;

(e) *if* $\tilde{g}$ *is the cumulant generating function of a random variable* $\tilde{Z}$ *for which* $P(\tilde{Z} \in (0,\infty)) = 1$, *such that* $1 - \exp(-\tilde{g}(h(s))) = m(1 - \exp(-\tilde{g}(s)))$, $s \in [0,\infty)$, *then* $\tilde{g}(s) = g(s)$ *for all* $s \in [0,\infty)$.

PROOF. Writing $h_n(\infty) = \lim_{s \to \infty} h_n(s)$, $n = 1,2,3,\ldots,$

$$g_n(s) = -\log E(e^{-sZ_n} \mid Z_n > 0), \qquad s \in [0,\infty)$$

and

$$\chi_n(s) = -\log\left(1 - \frac{h_n(s)}{h_n(\infty)}\right), \qquad s \in [0,\infty),$$

we see that

$$e^{-g_n(s)} = \frac{e^{-h_n(s)} - e^{-h_n(\infty)}}{1 - e^{-h_n(\infty)}} \sim e^{-\chi_n(s)} \qquad \text{as } n \to \infty, \ s \in [0,\infty),$$

since $\lim_{n\to\infty} h_n(s) = 0$ for all $s \in [0,\infty)$ by the Theorems 3.3.1 and 4.2.1, and, also by Theorem 4.2.1,

$$\lim_{n\to\infty} h_n(\infty) = \lim_{n\to\infty} -\log q_n = -\log q = 0.$$

In a similar way as in the proof of Theorem (2.6.2) of JAGERS [1975], we can prove that $h_n(s)/h_n(\infty)$, and therefore also $\chi_n(s)$, is increasing in n, and hence $\chi_n(s)$ converges to some limit function $g(s)$ as $n \to \infty$. This implies that also $\lim_{n\to\infty} g_n(s)$ exists and equals $g(s)$. The parts (a), (c) and (d) now follow as in JAGERS [1975]. In view of Lemma 2.2.5(a), part (b) is a consequence of the fact that $g(s) \geq \chi_n(s)$ for all $n = 1,2,3,\ldots$, and that $\lim_{s\to\infty} \chi_n(s) = \infty$ for all $n = 1,2,3,\ldots$. Finally, analogously to the proof of Theorem I.7.3 in ATHREYA and NEY [1972], we can prove that $1 - \exp(-g(s)) = c_1(1 - \exp(-\tilde{g}(s))) + c_2$ for some constants c_1 and c_2. By the Lemmas 2.2.4 (a) and 2.2.5(a), $g(0) = \tilde{g}(0) = 0$ and $\lim_{s\to\infty} g(s) = \lim_{s\to\infty} \tilde{g}(s) = \infty$, implying that $c_1 = 1$ and $c_2 = 0$. □

Functional equations like (4.2.6) appear often in branching process theory. In SENETA [1974] it is explained that there is an intimate relation between equations such as (4.2.6) and regularly varying functions. The next theorem is an example of that fact. As it provides a good insight in the reason of this relation, we give a proof of part (a), although it is the same as that of part (1) of Theorem 2 in SENETA [1974].

THEOREM 4.2.4. *Suppose* $P(Z_1 = 0) > 0$. *Let* Z *be as in Theorem* 4.2.3. *Then:*

(a) $m^{-n} P(Z_n > 0) \sim L_1(m^n)$ *as* $n \to \infty$;

(b) $\int_0^x P(Z > y) dy \sim L_2(1/x)$ *as* $x \to \infty$,

where L_1 *is a non-decreasing and* L_2 *a non-increasing function, each slowly varying at* 0, *such that, as* $s \downarrow 0$, $L_1(s) \downarrow \frac{1}{EZ}$ *and* $L_2(s) \uparrow EZ$.

PROOF. Writing $\psi(s) = 1 - e^{-g(s)}$, $s \in [0,\infty)$, (4.2.6) becomes

(4.2.7) $\psi(h(s)) = m\psi(s)$, $s \in [0,\infty)$.

Since $e^{-g(s)}$ is the Laplace transform of the proper random variable Z and $P(Z = 0) = 0$, $\psi(s)$ is continuous and increasing in s on $[0,\infty)$ from $1 - P(Z < \infty) = 0$ to $1 - P(Z = 0) = 1$. So we can take inverses in (4.2.7) to obtain

$$c(\psi^{inv}(s)) = \psi^{inv}(s/m), \qquad s \in [0,m).$$

Because $\lim_{s \downarrow 0} \frac{h(s)}{s} = m$ by Lemma 2.2.4(b), $\lim_{s \downarrow 0} \frac{c(s)}{s} = \frac{1}{m}$, and therefore

(4.2.8) $$\frac{\psi^{inv}(s/m)}{\psi^{inv}(s)} = \frac{c(\psi^{inv}(s))}{\psi^{inv}(s)} \to \frac{1}{m} \qquad \text{as } s \downarrow 0.$$

Further, since $\psi(s)$ is concave on $[0,\infty)$, $\psi^{inv}(s)$ is convex on $[0,1)$, and thus $\psi^{inv}(s)/s$ increases as s increases. So for $1 \le \lambda \le 1/m$,

$$1 \le \frac{\psi^{inv}(\lambda s)}{\lambda s} \cdot \frac{s}{\psi^{inv}(s)} \le \frac{\psi^{inv}(s/m)}{s/m} \cdot \frac{s}{\psi^{inv}(s)}, \qquad s \in (0,m).$$

Hence, using (4.2.8),

(4.2.9) $$\lim_{s \downarrow 0} \frac{\psi^{inv}(\lambda s)}{\psi^{inv}(s)} = \lambda$$

for all $\lambda \in [1,1/m]$, and we can iterate this to obtain (4.2.9) for all $\lambda > 0$. So we have proved that $\psi^{inv}(s) = s.L_1(s)$, where $L_1(s)$ varies slowly at 0. By the convexity of $\psi^{inv}(s)$, $L_1(s)$ is non-decreasing and because $\lim_{s \downarrow 0} \psi(s) = 0$,

$$\lim_{s \downarrow 0} \frac{\psi^{inv}(s)}{s} = \lim_{s \downarrow 0} \frac{s}{\psi(s)} = \lim_{s \downarrow 0} \frac{s}{g(s)} = \frac{1}{EZ} ,$$

by Lemma 2.2.4(b). Furthermore, L_1 is continuous on $(0,1)$, since ψ is continuous and strictly increasing. Iterating (4.2.7) gives

$$\psi(h_n(s)) = m^n \psi(s), \qquad s \in [0,\infty), \; n = 1,2,3,\ldots$$

or

$$h_n(s) = \psi^{inv}(m^n \psi(s)) = m^n \psi(s) L_1(m^n \psi(s)), \qquad s \in [0,\infty),$$

$$n = 1,2,3,\ldots .$$

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Letting $s \to \infty$ we obtain, in view of the continuity of L_1,

$$-\log P(Z_n = 0) = m^n L_1(m^n), \qquad n = 1,2,3,\ldots,$$

and as $\lim_{n\to\infty} P(Z_n = 0) = 1$ by Theorem 4.2.1(b), $-\log P(Z_n = 0) \sim P(Z_n > 0)$ as $n \to \infty$ and so part (a) of the theorem is proved. For the proof of part (b) we refer to part (2) of Theorem 2 in SENETA [1974]. $\quad\square$

CHAPTER 5

THE CASE m=1

5.1. INTRODUCTION

In this chapter we look at the so-called *critical processes,* that is processes with m = 1. It turns out that in this case we can exactly calculate some asymptotics of the process; see Theorem 5.2.2 below.

In Chapter 4 we saw that we could associate $\lim_{n\to\infty} m^{-n} h_n(s)$ with $EZ_1 \log Z_1$. However, if m = 1 and $P(Z_1 = 0) > 0$, it is not true that there exists for every $s \in (0,\infty]$ a $\delta \in (0,1)$ such that $h_k(s) \le \delta^k$ for sufficiently large k. In view of the discussion before Theorem 4.2.2 it is at least plausible that the association stated above does not exist in the case m = 1. We therefore do not find conditions on $EZ_1 \log Z_1$ in this chapter, and we have to use another term in the Taylor expansion for h(s) in the analysis. For this reason $\sigma^2(Z_1)$ appears in the conditions.

Again as in Chapter 4 the behaviour of the process if $P(Z_1 = 0) = 0$, depends on the value of a and results concerning this case can therefore be found in Chapters 8 and 9.

5.2. SOME LIMIT THEOREMS

To begin with we again calculate r and q. Their values are the same as in the subcritical case as we see in the following theorem, the proof of which is just like that of Theorem 4.2.1.

THEOREM 5.2.1.
(a) r = 1;
(b) *if* $P(Z_1 = 0) > 0$, *then* q = 1.

So again we have $\lim_{n\to\infty} P(Z_n > 0) = 0$ if $P(Z_1 = 0) > 0$, and the next theorem says how fast this convergence is.

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<u>THEOREM</u> 5.2.2. *Suppose* $P(Z_1 = 0) > 0$ *and* $\sigma^2 := \sigma^2(Z_1) < \infty$. *Then:*

(a) $\lim_{n \to \infty} nP(Z_n > 0) = 2/\sigma^2$;

(b) $\lim_{n \to \infty} E(Z_n/n \mid Z_n > 0) = \sigma^2/2$;

(c) $\lim_{n \to \infty} P(Z_n/n \leq u \mid Z_n > 0) = 1 - e^{-2u/\sigma^2}$, $\quad u \geq 0$.

<u>PROOF</u>. The proof is analogous to the proof of Theorem (2.4.2) in JAGERS [1975]. It is based on the relation

$$\lim_{n \to \infty} \frac{1}{n} \left(\frac{1}{h_n(s)} - \frac{1}{s} \right) = \frac{\sigma^2}{2} \; ,$$

holding uniformly in $0 < s \leq A$ for every $A > 0$. $\quad \square$

CHAPTER 6

THE CASE $1 < m < \infty$

6.1. INTRODUCTION

In this chapter we shall consider processes with $1 < m < \infty$. Such processes are called *supercritical*. The most important difference with the cases $m < 1$ and $m = 1$ is, that for supercritical processes $P(\lim_{n \to \infty} Z_n = 0) < 1$. This implies that $P(\lim_{n \to \infty} Z_n = \infty) > 0$. We therefore have to consider the behaviour of the process also on $\{\lim_{n \to \infty} Z_n = \infty\}$. Since $-\log r > 0$, the interval $(0, -\log r)$ is non-empty and Corollary 3.3.5 provides us with sequences of norming constants $\{c_n(s); n = 0, 1, 2, \ldots\}$ for the process $\{Z_n; n = 0, 1, 2, \ldots\}$, both if $P(Z_1 = 0) = 0$ and if $P(Z_1 = 0) > 0$. They are in fact only useful on $\{\lim_{n \to \infty} Z_n = \infty\}$, because $\lim_{n \to \infty} c_n(s) Z_n = 0$ on $\{\lim_{n \to \infty} Z_n = 0\}$, since $\lim_{n \to \infty} c_n(s) = 0$ for all $s \in (0, -\log r)$. On the other hand, it turns out that $P(0 < \lim_{n \to \infty} c_n(s) Z_n < \infty \mid \lim_{n \to \infty} Z_n = \infty) = 1$.

As we shall see, the process conditioned on $\{\lim_{n \to \infty} Z_n = 0\}$ can be considered as a subcritical branching process having the function $h(s - \log r) + \log r$ as the cumulant generating function of the offspring distribution. So if $P(Z_1 = 0) > 0$ we can apply the results of Chapter 4. If $P(Z_1 = 0) = 0$, the behaviour of the process again depends on the value of a, and will therefore be treated in Chapters 8 and 9.

6.2. THE BEHAVIOUR OF THE PROCESS ON $\{Z_n \to 0\}$.

THEOREM 6.2.1.

(a) $r < 1$;

(b) $r = 0$ *if and only if* $a \geq 1$;

(c) *if* $P(Z_1 = 0) > 0$ *then* $q = r$.

PROOF. The parts (a) and (b) follow from Theorem 3.3.1, since $\lim_{s \downarrow 0} \frac{h(s)}{s} = m$, $\lim_{s \to \infty} \frac{h(s)}{s} = a$ and $h(s)$ is concave on $[0, \infty)$ by the Lemmas 2.2.4(b),

2.2.5(c) and 2.2.3(b). Part (c) is obtained on observing that $\lim_{s \to \infty} h(s) < \infty$ by Lemma 2.2.5(a), and so, using Theorem 3.3.1 and the Lemmas 2.2.2 and 2.2.5(a)

$$-\log r = \lim_{n \to \infty} h_n(\lim_{s \to \infty} h(s)) = \lim_{n \to \infty} \lim_{s \to \infty} h_{n+1}(s)$$

$$= \lim_{n \to \infty} - \log q_{n+1} = -\log q. \qquad \Box$$

Let us now look at the process conditioned on $A := \{\lim_{n \to \infty} Z_n = 0\}$. Since $P(A) = r$, we suppose that $r > 0$, implying that $-\log r < \infty$. First of all we notice that, if we define the probability measure $\tilde{P}$ by $\tilde{P}(B) = P(B|A)$, and write $\tilde{E}$ for the expectation with respect to $\tilde{P}$, then for any Borel set B and for all random vectors X and Y

$$P(A|Y) \cdot \tilde{P}(X \in B|Y) = P(A \wedge \{X \in B\}|Y) \quad \text{a.s..}$$

This implies that a.s.

$$(6.2.1) \qquad P(A \mid Z_0, \ldots, Z_n) \cdot \tilde{E}(e^{-s Z_{n+1}} \mid Z_0, \ldots, Z_n)$$

$$= E(e^{-s Z_{n+1}} I_A \mid Z_0, \ldots, Z_n)$$

$$= E\{E(e^{-s Z_{n+1}} I_A \mid Z_0, \ldots, Z_{n+1}) \mid Z_0, \ldots, Z_n\}$$

$$= E\{e^{-s Z_{n+1}} P(A \mid Z_0, \ldots, Z_{n+1}) \mid Z_0, \ldots, Z_n\},$$

where I stands for the indicator function. In view of the basic branching property (3.1.3)

$$P(A \mid Z_0, \ldots, Z_n) = P(A \mid Z_n) = r^{Z_n} \quad \text{a.s..}$$

Hence we obtain from (6.2.1) that a.s.

$$\tilde{E}(e^{-s Z_{n+1}} \mid Z_0, \ldots, Z_n) = r^{-Z_n} \cdot E(e^{-(s-\log r) Z_{n+1}} \mid Z_0, \ldots, Z_n),$$

implying that

$$(6.2.2) \qquad -\log \tilde{E}(e^{-s Z_{n+1}} \mid Z_0, \ldots, Z_n) = \{h(s - \log r) + \log r\} Z_n \quad \text{a.s.}$$

by (3.1.3). Now define $\tilde{h}(s)$ by $\tilde{h}(s) = h(s - \log r) + \log r$, $s \in [0, \infty)$. From part 2 of the proof of Theorem 3.3.6(b) we know that $\tilde{h}(s)$ is the cumulant generating function of a proper, non-negative random variable $\tilde{Z}_1$, not concentrated in one point, having an infinitely divisible distribution. So if we define a subordinator $\{\tilde{X}(t); t \in [0, \infty)\}$, such that $\tilde{X}(1) \overset{d}{=} \tilde{Z}_1$, and if we construct a branching process $\{\tilde{Z}_n; n = 0,1,2,\ldots\}$ with the help of this subordinator as in Definition 3.1.2, with $\tilde{Z}_0 = 1$, then it follows from (6.2.2) that the processes $\{Z_n; n = 0,1,2,\ldots\}$ conditioned on A and $\{\tilde{Z}_n; n = 0,1,2,\ldots\}$ have the same distribution. Furthermore, since $\tilde{EZ}_1 = \tilde{h}'(0) = h'(-\log r) < 1$, and

$$\inf\{x; P(\tilde{Z}_1 \leq x) > 0\} = \lim_{s \to \infty} \frac{\tilde{h}(s)}{s}$$

$$= \lim_{s \to \infty} \frac{h(s - \log r) + \log r}{s} = \lim_{s \to \infty} \frac{h(s)}{s} = a,$$

we see that we can apply the results for subcritical processes to the process $\{Z_n; n = 0,1,2,\ldots\}$ conditioned on A, with m replaced by $h'(-\log r)$. Because

$$P(\tilde{Z}_1 = 0) = \lim_{s \to \infty} e^{-\tilde{h}(s)} = \frac{1}{r} \lim_{s \to \infty} e^{-h(s)} = \frac{1}{r} P(Z_1 = 0),$$

these results can be found in Chapter 4 in case $P(Z_1 = 0) > 0$ and in Chapters 8 and 9 in case $P(Z_1 = 0) = 0$.

6.3. THE BEHAVIOUR OF THE PROCESS ON $\{Z_n \to \infty\}$.

As already mentioned $\{c_n(s); n = 1,2,3,\ldots\}$ can serve as a sequence of norming constants for the random variables Z_n, $n = 1,2,3,\ldots$ if $s \in (0, -\log r)$. We shall now examine some properties of the random variable $Y(s)$, defined in Corollary 3.3.5 by $Y(s) = \lim_{n \to \infty} c_n(s) Z_n$. Throughout this section we suppose that $s \in (0, -\log r)$, unless stated otherwise. First of all we derive a functional equation which we shall often make use of. (Remember that $\phi(z,s)$ is the cumulant generating function of $Y(s)$.)

THEOREM 6.3.1. *For all z with* $\mathrm{Re}\, z \geq 0$

$$(6.3.1) \qquad \phi(mz,s) = h(\phi(z,s)).$$

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PROOF. Since $0 < c(s) < s$ for $s \in (0,-\log r)$ and $c(0) = 0$, $c_n(s) \downarrow 0$ as $n \to \infty$, and we see that

$$\lim_{n\to\infty} \frac{c_n(s)}{c_{n-1}(s)} = \lim_{n\to\infty} \frac{c_n(s)}{h(c_n(s))} = \lim_{s\downarrow 0} \frac{s}{h(s)} = \frac{1}{m}.$$

Hence, using (3.3.5) and Lemma 2.2.3(a), we get for all $u \in (0,\infty)$ and $\varepsilon \in (0,u)$,

$$h(\phi(u-\varepsilon,s)) = \lim_{n\to\infty} h(h_{n-1}((u-\varepsilon)c_{n-1}(s)))$$

$$\leq \lim_{n\to\infty} h(h_{n-1}(mu \frac{c_n(s)}{c_{n-1}(s)} c_{n-1}(s)))$$

$$= \lim_{n\to\infty} h_n(muc_n(s)) = \phi(mu,s)$$

$$= \lim_{n\to\infty} h(h_{n-1}(mu \frac{c_n(s)}{c_{n-1}(s)} c_{n-1}(s)))$$

$$\leq \lim_{n\to\infty} h(h_{n-1}((u+\varepsilon)c_{n-1}(s)))$$

$$= h(\phi(u+\varepsilon,s)),$$

since $h(t)$ is continuous for $t \in [0,\infty)$ by Lemma 2.2.2. Again using this same lemma, we see in the first place that $\phi(u,s)$ is continuous for $u \in [0,\infty)$ and therefore (6.3.1) is true for all $z \in [0,\infty)$. Furthermore, in view of the fact that $\mathrm{Re}\,\phi(z,s) > 0$ and both $\phi(z,s)$ and $h(z)$ are analytic on $\{z; \mathrm{Re}\,z > 0\}$, it follows that (6.3.1) holds for all $z \in \{z; \mathrm{Re}\,z > 0\}$, and so by the continuity of cumulant generating functions on $\{z; \mathrm{Re}\,z \geq 0\}$ also for all z with $\mathrm{Re}\,z \geq 0$. $\square$

Using (6.3.1) we shall now prove the already announced result, that $c_n(s)$ is a good norming on $\{\lim_{n\to\infty} Z_n = \infty\}$ in the sense that $P(0 < Y(s) < \infty \mid \lim_{n\to\infty} Z_n = \infty) = 1$.

THEOREM 6.3.2.

(a) $P(Y(s) = 0) = r$;

(b) $P(Y(s) < \infty) = 1$.

PROOF. From Lemma 2.2.5(a) we know that $-\log P(Y(s) = 0) = \lim_{u\to\infty} \phi(u,s)$. Since $h(t)$ is continuous for $t \in [0,\infty)$, (6.3.1) now yields

(6.3.2) $-\log P(Y(s) = 0) = \lim_{u \to \infty} \phi(u,s)$

$= \lim_{u \to \infty} h(\phi(u/m,s)) = h(\lim_{u \to \infty} \phi(u/m,s))$

$= h(-\log P(Y(s) = 0))$,

with the convention that $h(\infty) = \lim_{s \to \infty} h(s)$. Similarly we get

(6.3.3) $-\log P(Y(s) < \infty) = h(-\log P(Y(s) < \infty))$.

Because by (3.3.5) $Ee^{-Y(s)} = e^{-s}$, we see that $P(e^{-Y(s)} = 1) < 1$ and $P(e^{-Y(s)} > 0) > 0$, since $r < e^{-s} < 1$, and hence

(6.3.4) $-\log P(Y(s) = 0) > 0$ and $-\log P(Y(s) < \infty) < \infty$.

Furthermore, since $\lim_{n \to \infty} c_n(s) = 0$,

(6.3.5) $-\log P(Y(s) = 0) = -\log P(\lim_{n \to \infty} c_n(s) Z_n = 0)$

$\leq -\log P(\lim_{n \to \infty} Z_n = 0) = -\log r$,

and of course,

(6.3.6) $-\log P(Y(s) = 0) \geq -\log P(Y(s) < \infty)$.

Combining (6.3.2), (6.3.4) and (6.3.5), we obtain that $-\log P(Y(s) = 0) = -\log r$, that is $P(Y(s) = 0) = r$. Now using (6.3.3) and (6.3.6) it follows that $P(Y(s) < \infty) = r$ or 1. But if $P(Y(s) < \infty) = r$, then we should get $Ee^{-Y(s)} = r$, since $P(Y(s) = 0) = r$. However, $Ee^{-Y(s)} = e^{-s} > r$, and so $P(Y(s) < \infty) = 1$. $\square$

We shall now further investigate the distribution of $Y(s)$. It turns out that we can prove, making a repeated use of (6.3.1) that any sufficiently large power of the absolute value of the characteristic function of $Y(s)$ is integrable if $r = 0$. Then it follows from a result on Fourier inversion in FELLER [1971], and again some manipulation with (6.3.1) that $Y(s)$ has an absolutely continuous distribution if $r = 0$. The proof given here is analogous to the proof of Theorem 4 on page 34 of ATHREYA and NEY [1972],

where the theorem is stated for Galton-Watson processes.

THEOREM 6.3.3. *If r = 0, then the random variable Y(s) has an absolutely continuous distribution.*

PROOF. The result will be established with the help of the following lemmas.

LEMMA 6.3.4. $P(Y(s) = c) < 1$ *for all constants* $c \in (-\infty, \infty)$.

PROOF. Since $c_n(s) > 0$ and $Z_n \geq 0$ for all $n = 1,2,3,\ldots$, $P(Y(s) = c) = 0$ for all $c \in (-\infty, 0)$. Furthermore, we know from part (a) of Theorem 6.3.2 that $P(Y(s) = 0) = r$, which equals 0 by assumption. Now suppose $P(Y(s) = c) = 1$ for some $c \in (0, \infty)$. It follows then, in view of (6.3.1), that

$$\dot{m}cu = \phi(mu, s) = h(\phi(u, s)) = h(cu)$$

for all $u \in [0, \infty)$, and therefore $h(s) = ms$ for all $s \in [0, \infty)$. This means that $P(Z_1 = m) = 1$, which case is however excluded. □

Define $\psi(t, s)$ to be the characteristic function of $Y(s)$, that is $\psi(t, s) = Ee^{itY(s)}$, $t \in (-\infty, \infty)$. Then we have

LEMMA 6.3.5. $|\psi(t, s)| < 1$ *for all real* $t \neq 0$.

PROOF. Since the distribution of $Y(s)$ is non-degenerate, there exists a $\delta > 0$ such that $|\psi(t, s)| < 1$ for all $0 < |t| < \delta$. (See e.g. Lemma XV.1.4 of FELLER [1971].) This means that $\text{Re } \phi(-it, s) > 0$, and so using (6.3.1)

$$|\psi(mt, s)| = |e^{-\phi(-imt, s)}| = |e^{-h(\phi(-it, s))}|$$

$$= |Ee^{-\phi(-it, s)Z_1}| \leq E|e^{-\phi(-it, s)}|^{Z_1} = e^{-h(\text{Re}\phi(-it, s))} < 1$$

for all $0 < |t| < \delta$. This implies that $|\psi(t, s)| < 1$ for all $0 < |t| < m\delta$, and hence by iteration for all $0 < |t| < m^n\delta$. Since $1 < m < \infty$, we can conclude that $|\psi(t, s)| < 1$ for all real $t \neq 0$. □

The following lemma is the key step leading to the integrability of $|\psi(t, s)|^k$ for sufficiently large integers k. Define $\beta = \inf_{1 \leq |t| \leq m} \text{Re}\phi(it, s)$. By the continuity of $\phi(z)$ for $z \in \{\text{Re } z > 0\}$ and Lemma 6.3.5 we know that $\beta > 0$. Introducing furthermore $d = h(\beta) - \beta$ and $\delta = \dfrac{d}{\log m}$ we get

LEMMA 6.3.6.

$$\sup_{-\infty < t < \infty} |\psi(t,s)| \cdot |t|^{\delta} < \infty.$$

PROOF. Since $r = 0$, we know from Theorem 3.3.1 that $h(s) > s$ for all $s \in (0,\infty)$. Therefore, as $\beta > 0$, both $d > 0$ and $\delta > 0$. Because

$$|e^{-h_n(z)}| = |Ee^{-zZ_n}| \le E|e^{-z}|^{Z_n} = e^{-h_n(\text{Re } z)} \le e^{-h_n(\beta)}$$

for all z with $\text{Re } z \ge \beta$, and, again by Theorem 3.3.1, $\lim_{n\to\infty} h_n(\beta) = \infty$,

$$|e^{-h_n(z)}| \to 0 \qquad \text{as } n \to \infty,$$

uniformly for all z with $\text{Re } z \ge \beta$. Hence there exists for all $\varepsilon > 0$ an integer $N_0(\varepsilon)$ such that for all $n \ge N_0(\varepsilon)$

$$(6.3.7) \qquad \sup_{1 \le |t| \le m} |e^{-h_n(\phi(it,s))}| \le \varepsilon.$$

Now because of the fact that $r = 0$, $h(s) - s$ is increasing in s, and so for all z with $\text{Re } z \ge \beta$,

$$\text{Re } h(z) = -\log|Ee^{-zZ_1}| \ge -\log E|e^{-z}|^{Z_1} = h(\text{Re } z) \ge d + \text{Re } z > \beta.$$

Iterating this we get for all z with $\text{Re } z \ge \beta$,

$$(6.3.8) \qquad \text{Re } h_n(z) \ge nd + \text{Re } z > \beta, \qquad \text{for all } n = 1,2,3,\ldots .$$

Combining (6.3.7) and (6.3.8) yields

$$|\exp\{-h_{n+N_0}(\phi(it,s))\}| = \exp\{-\text{Re } h_n(h_{N_0}(\phi(it,s)))\}$$

$$\le \exp\{-nd - \text{Re } h_{N_0}(\phi(it,s))\} \le e^{-nd} \cdot \varepsilon,$$

for all $n = 1,2,3,\ldots$ and all t with $1 \le |t| \le m$, and so

$$(6.3.9) \qquad \sup_{1 \le |t| \le m} |\exp\{-h_{n+N_0}(\phi(it,s))\}| \le \varepsilon e^{-nd}$$

for all $n = 1,2,3,\ldots$. Hence, using (6.3.1), (6.3.9) and the definition of δ, we obtain

$$\sup_{m^{N_0+n} \leq |t| \leq m^{N_0+n+1}} |t|^{\delta} \cdot |\psi(t,s)| =$$

$$= \sup_{1 \leq |t| \leq m} |m^{N_0+n} t|^{\delta} \cdot |\psi(m^{N_0+n} t,s)|$$

$$= \sup_{1 \leq |t| \leq m} |m^{N_0+n} t|^{\delta} \cdot |\exp\{-\phi(-im^{N_0+n} t,s)\}|$$

$$= \sup_{1 \leq |t| \leq m} |m^{N_0+n} t|^{\delta} \cdot |\exp\{-h_{N_0+n}(\phi(-it,s))\}|$$

$$\leq (m^{N_0+n+1})^{\delta} \cdot \varepsilon \cdot e^{-nd}$$

$$= e^{d(n+N_0+1)} \cdot \varepsilon e^{-nd} = \varepsilon e^{(N_0+1)d}$$

for all $n = 1,2,3,\ldots$, implying that

(6.3.10) $\sup_{-\infty < t < \infty} |t|^{\delta} \cdot |\psi(t,s)| < \infty.$ □

Since δ is not necessarily greater than one, we cannot yet apply the Fourier inversion theorem to yield the absolute continuity of $Y(s)$. However, the fact that $k\delta > 1$ for sufficiently large k establishes the integrability of $|\psi(t,s)|^k$, that is the absolute continuity of $Y_1+Y_2+\ldots+Y_k$ for k sufficiently large where $Y_1,Y_2,\ldots,Y_k$ are independent and identically distributed random variables with the same distribution as $Y(s)$. This enables us to prove the absolute continuity as is made precise in Lemma 6.3.7. Before stating it we define for $n = 1,2,3,\ldots$ random variables $\bar{Y}^{(n)}, Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)},\ldots$ such that for every $n = 1,2,3,\ldots$, it holds that Z_n, $Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)},\ldots$ are independent and such that conditionally given Z_n, the random variables $\bar{Y}^{(n)}, Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)},\ldots$ are independent, with $Y_i^{(n)}$ distributed as $Y(s)$ for every $i = 1,2,3,\ldots$ and

(6.3.11) $E(e^{-t\bar{Y}^{(n)}} \mid Z_n) = E(e^{-tY(s)})^{Z_n - [Z_n]}$, $t \in [0,\infty)$,

where $[z]$ is the integer part of z. Finally we define $Y_0^{(n)}$ by $Y_0^{(n)} = m^{-n}\{Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_{[Z_n]}^{(n)} + \bar{Y}^{(n)}\}$, $n = 1,2,3,\ldots$.

Notice that since each Z_n has an infinitely divisible distribution, the same is true for $Y(s)$ being the limit of $c_n(s)Z_n$ as $n \to \infty$, and therefore, in view of (6.3.11), the distribution of $\bar{Y}^{(n)}$ given $Z_n = z$ is well-defined for every $z \geq 0$. A similar construction will be used in Chapters 7 and 9. For more details we refer to the proof of Theorem 9.2.13. By (6.3.11) and (6.3.1) we see that $Y_0^{(n)}$ and $Y(s)$ have the same Laplace transform, implying that

(6.3.12) $Y_0^{(n)} \overset{d}{=} Y(s)$ for every $n = 1,2,3,\ldots$.

LEMMA 6.3.7. *Suppose* $E \subset \mathbb{R}$ *has Lebesgue-measure zero. Then* $P(Y(s) \in E) = 0$.

PROOF. Because $\delta > 0$ there exists a positive integer k such that $k\delta > 1$. Since by (6.3.10) $\sup_{-\infty < t < \infty} |\psi(t,s)|^{\ell} \cdot |t|^{\ell\delta} < \infty$ for every positive ℓ, it follows from Theorem XV.3.3 of FELLER [1971], that $Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_{\ell}^{(n)}$ has an absolutely continuous distribution for every integer $\ell \geq k$. By (6.3.12),

$$P(Y(s) \in E) = P(Y_0^{(n)} \in E) = \int_{[0,\infty)} P(Y_0^{(n)} \in E \mid Z_n = z)dP_{Z_n}(z) =$$

$$= \int_{[0,k)} P(Y_0^{(n)} \in E \mid Z_n = z)dP_{Z_n}(z)$$

$$+ \int_{[k,\infty)} P(Y_0^{(n)} \in E \mid Z_n = z)dP_{Z_n}(z).$$

Putting in the definition of $Y_0^{(n)}$ it follows that

$$\int_{[k,\infty)} P(Y_0^{(n)} \in E \mid Z_n = z)dP_{Z_n}(z)$$

$$= \int_{[k,\infty)} P(m^{-n}\{Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_{[z]}^{(n)} + \bar{Y}^{(n)}\} \in E \mid Z_n = z)dP_{Z_n}(z).$$

Now since $\bar{Y}^{(n)}, Y_1^{(n)}, Y_2^{(n)}, Y_3^{(n)}, \ldots$ are, conditionally given Z_n, independent and $Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_\ell^{(n)}$ has an absolutely continuous distribution for every $\ell \geq k$, we see that $P(m^{-n}\{Y_1^{(n)} + Y_2^{(n)} + \ldots + Y_{[z]}^{(n)} + \bar{Y}^{(n)}\} \in E \mid Z_n = z) = 0$, because it is well-known that the sum of two independent random variables has an absolutely continuous distribution if one of these random variables has an absolutely continuous distribution. Hence we obtain

$$P(Y(s) \in E) = \int_{[0,k)} P(Y_0^{(n)} \in E \mid Z_n = z) \, dP_{Z_n}(z) \leq P(Z_n < k).$$

Since $r = 0$, $\lim_{n \to \infty} P(Z_n < k) = 0$ for all $k = 1,2,3,\ldots$, and so $P(Y(s) \in E) = 0.$ $\square$

This completes the proof of Theorem 6.3.3. $\square$

In Section 3.3 we defined $\ell(s)$ to be the first point of increase of the distribution function of $Y(s)$. In view of Theorem 6.3.2(a), obviously $\ell(s) = 0$ if $r > 0$. But $\ell(s) = 0$ also if $r = 0$. This follows since by Lemma 2.2.5(c) $\ell(s) = \lim_{n \to \infty} m^{-n} \phi(m^n, s)$. Now using (6.3.1) and (3.3.5) we see that $\phi(m^n, s) = h_n(\phi(1,s)) = h_n(s)$. This means that

$$\ell(s) = \lim_{n \to \infty} m^{-n} \phi(m^n, s) = \lim_{n \to \infty} m^{-n} h_n(s)$$

$$= \lim_{n \to \infty} s \prod_{k=1}^{n} \frac{h_k(s)}{m h_{k-1}(s)}.$$

Because $r = 0$, $\lim_{n \to \infty} h_n(s) = \infty$, and so

$$\lim_{n \to \infty} \frac{h_{n+1}(s)}{m h_n(s)} = \frac{a}{m} < 1,$$

implying that $\ell(s) = 0$.

In view of the remark made before Theorem 4.2.4 about the relation between functional equations and regularly varying functions, it is not at all astonishing that the following result follows from (6.3.1).

THEOREM 6.3.8. $\phi(u,s)$ *is regularly varying at* 0 *with exponent* 1 *as a function of* u.

PROOF. See the proof of Theorem 1 in SENETA [1974]. $\square$

Corollary 3.3.5 provides us with not only one but with a whole class of norming constants $c_n(s)$ for the random variables Z_n, since we can choose any $s \in (0, -\log r)$. One may ask if there is any relation between the limit random variables $Y(s)$ for different values of s. A positive answer to this question is given in the following theorem. Before stating it we notice that by the convexity of $c(s)$, $c_n(s)/c_n(t)$ is non-increasing in n whenever $0 < s \leq t < -\log r$, and hence converges to some limit $v(s,t)$. It follows immediately that $v(s,t) := \lim_{n \to \infty} c_n(s)/c_n(t)$ exists, also for $0 < t < s < -\log r$. Furthermore, $v(s,t) \in (0,\infty)$ for all $s,t \in (0,-\log r)$. For if $s \leq t$, then there exists a non-negative integer k such that $s \geq c_k(t)$. This means that

$$1 \geq v(s,t) = \lim_{n \to \infty} \frac{c_n(s)}{c_n(t)} \geq \lim_{n \to \infty} \frac{c_k(c_n(t))}{c_n(t)} = \lim_{u \downarrow 0} \frac{c_k(u)}{u} = m^{-k} > 0.$$

Similarly we can prove that $v(s,t) \in [1,\infty)$ if $s \geq t$.

THEOREM 6.3.9. *Suppose that both s and $t \in (0,-\log r)$. Then:*
(a) $Y(s) = v(s,t)Y(t)$ a.s.;
(b) $\phi(v(s,t),t) = s$.

PROOF. Part (a) is a consequence of the fact that $v(s,t) \in (0,\infty)$, whence

$$Y(s) = \lim_{n \to \infty} c_n(s)Z_n = \lim_{n \to \infty} \frac{c_n(s)}{c_n(t)} c_n(t)Z_n = v(s,t)Y(t) \quad \text{a.s.}$$

Part (b) follows on observing that $e^{-s} = Ee^{-Y(s)} = Ee^{-v(s,t)Y(t)} = e^{-\phi(v(s,t),t)}$, where the first equality is a consequence of (3.3.5). $\square$

Now that we know that all the random variables $Y(s)$ belong to the same class in the sense that every two have constant ratio, we shall have a closer look at this class of random variables which can occur as the limit of $c_n(s)Z_n$ as $n \to \infty$ for some $s \in (0,-\log r)$. The basic tool in this investigation is again the functional equation (6.3.1). It says that the cumulant generating function ϕ of a random variable Y belonging to the class we consider satisfies $\phi(ms) = h(\phi(s))$. So reasoning in a rather superficial way, we can, given a cumulant generating function ϕ, define $h(s)$ by $h(s) = \phi(m\phi^{inv}(s))$ for some $m \in (1,\infty)$. Then we can check if it is possible to define a branching process with the help of this $h(s)$. If so we can consider the limit random variable $\tilde{Y}(s)$ belonging to this process and, since its cumulant generating function $\tilde{\phi}(u,s)$ also satisfies $\tilde{\phi}(mu,s) = h(\tilde{\phi}(u,s))$ there

is some hope that $\tilde{\phi}$ and ϕ might be related. If we want to make this precise, there are of course many problems. First of all we know from Theorem 6.3.2 that $P(Y = 0) = r$ and $P(Y < \infty) = 1$, which implies that $\lim_{s \to \infty} \phi(s) = -\log P(Y = 0) = -\log r$ and that $\lim_{s \downarrow 0} \phi(s) = -\log P(Y < \infty) = 0$. So if $r > 0$, then $\phi^{inv}(s)$, and therefore also $h(s)$, is only well-defined for $s \in [0, -\log r)$. In the second place we have to make sure of the fact that $h(s)$ is a cumulant generating function of a random variable having an infinitely divisible distribution. This leads to the introduction of the following collection $F_{m,r}$, for any $m \in (1, \infty)$ and $r \in [0,1)$.

We say that a cumulant generating function ϕ of a non-negative, proper, non-degenerate random variable Y belongs to $F_{m,r}$ if and only if:

(1) $\lim_{u \to \infty} \phi(u) = -\log r$;

(2) $\phi^*(s) := \phi(m\phi^{inv}(s))$, $s \in [0, -\log r)$, can be continued analytically along the positive real line;

(3) $e^{-t\phi^*(s)}$ is completely monotone for every $t > 0$ as a function of s, where $\phi^*(s)$, $s \in [-\log r, \infty)$ is defined as the analytic continuation of $\phi^*(s)$, $s \in [0, -\log r)$.

(4) $\phi(u)$ is regularly varying at 0 with exponent 1.

We shall prove that a random variable Y can occur as the limit of $c_n(s)Z_n$ if and only if its cumulant generating function belongs to $F_{m,r}$. To this end we introduce furthermore the collections $H_{m,r}$ and $G_{m,r}$ for $m \in (1, \infty)$ and $r \in [0,1)$ as follows.

A cumulant generating function h of a proper, non-degenerate, non-negative random variable, having an infinitely divisible distribution, belongs to $H_{m,r}$ if and only if:

(1) $\lim_{s \downarrow 0} \dfrac{h(s)}{s} = m$;

(2) $\lim_{n \to \infty} h_n(s) = -\log r$.

A cumulant generating function ϕ belongs to $G_{m,r}$ if and only if there is a branching process $\{Z_n; n = 0,1,2,\ldots\}$ with state space $[0, \infty)$ such that $Z_0 = 1$ and $h(s) := -\log E \exp\{-sZ_1\} \in H_{m,r}$ and $\lim_{n \to \infty} c_n(s_0)Z_n$ has cumulant generating function ϕ, where $c_n(s)$ is the nth iterate of $h^{inv}(s)$ and $s_0 \in (0, -\log r)$.

THEOREM 6.3.10.

$$F_{m,r} = G_{m,r}.$$

PROOF. We shall first prove that $G_{m,r} \subset F_{m,r}$ and then that $F_{m,r} \subset G_{m,r}$.

(a) Suppose that $\phi \in G_{m,r}$. Then it follows from Theorem 6.3.2 and Lemma 6.3.4 that $\phi(u) = -\log E\,e^{-uY}$, where Y is a non-negative, proper, non-degenerate random variable and that $\lim_{u \to \infty} \phi(u) = -\log r$. Furthermore, by (6.3.1), $\phi^*(s) = \phi(m\phi^{inv}(s)) = h(s)$ for $s \in [0,-\log r)$. This means by Lemma 2.2.2 that $\phi^*(s)$ can be continued analytically along the positive real line and that $\phi^*(s) = h(s)$ for all $s \in [0,\infty)$. Since $h(s)$ is the cumulant generating function of a random variable having an infinitely divisible distribution, it is clear that $e^{-t\phi^*(s)}$ is completely monotone for every fixed $t > 0$ as a function of s. Finally, Theorem 6.3.8 yields that $\phi(u)$ is regularly varying at 0 with exponent 1. This proves that $G_{m,r} \subset F_{m,r}$.

(b) Now suppose that $\phi \in F_{m,r}$. Define $h(s)$ by $h(s) = \phi^*(s)$, $s \in [0,\infty)$. Then it follows from requirement 3 in the definition of $F_{m,r}$ that $h(s)$ is the cumulant generating function of some non-negative random variable X, having an infinitely divisible distribution. Since $h(s) = \phi(m\phi^{inv}(s))$ for $s \in [0,-\log r)$, $\lim_{s \downarrow 0} h(s) = \lim_{s \downarrow 0} \phi(m\phi^{inv}(s)) = 0$, and so by Lemma 2.2.4 (a), $P(X < \infty) = 1$. Now we shall prove that X is non-degenerate. Suppose that $P(X = c) = 1$ for some $c \in [0,\infty)$. Then $h(s) = c \cdot s$ for all $s \in [0,\infty)$, implying that $\phi(m\phi^{inv}(s)) = c \cdot s$ for all $s \in [0,-\log r)$. This means that

(6.3.13) $\phi(mt) = c\phi(t)$ for all $t \in [0,\infty)$.

Now by requirement 4 in the definition of $F_{m,r}$ we know that $\phi(u) = uL(u)$, where $L(u)$ is a slowly varying function at 0. It follows therefore from (6.3.13) that $m = c$, since $L(mt) \sim L(t)$ as $t \downarrow 0$. Hence we obtain that ϕ is linear, in contradiction with the fact that ϕ is the cumulant generating function of a proper, non-degenerate random variable. So X is also non-degenerate. Furthermore,

$$\lim_{s \downarrow 0} \frac{h(s)}{s} = \lim_{s \downarrow 0} \frac{\phi^*(s)}{s} = \lim_{s \downarrow 0} \frac{\phi(m\phi^{inv}(s))}{\phi(\phi^{inv}(s))}$$

$$= \lim_{s \downarrow 0} \frac{m\phi^{inv}(s)L(m\phi^{inv}(s))}{\phi^{inv}(s)L(\phi^{inv}(s))} = m.$$

Let $\{Z_n;\ n = 0,1,2,\ldots\}$ be a branching process having the distribution of X as its offspring distribution, and such that $Z_0 = 1$. By Theorem 3.3.1 we know that for all $s \in (0,\infty)$

$$(6.3.14) \qquad \lim_{n\to\infty} h_n(s) = -\log P(\lim_{n\to\infty} Z_n = 0),$$

and because $m > 1$, $\lim_{n\to\infty} h_n(s) > 0$. Choose some $s_0 \in (0, -\log P(\lim_{n\to\infty} Z_n = 0))$, then by Corollary 3.3.5, $\lim_{n\to\infty} c_n(s_0)Z_n$ exists a.s., with $c_n(s)$ the nth iterate of $h^{inv}(s)$. Defining $\tilde{\phi}(u,s_0)$ by

$$\tilde{\phi}(u,s_0) = -\log E\, e^{-u\, \lim_{n\to\infty} c_n(s_0)Z_n}, \qquad u \in [0,\infty),$$

it follows from (6.3.1) that $h(\tilde{\phi}(u,s_0)) = \tilde{\phi}(mu,s_0)$. Since by Theorem 6.3.2(b) $P(\lim_{n\to\infty} c_n(s_0)Z_n < \infty) = 1$, Lemma 2.2.4(a) implies that $\lim_{u\downarrow 0} \tilde{\phi}(u,s_0) = 0$. So because $r < 1$, there exists a $U > 0$ such that $\tilde{\phi}(u,s_0) < -\log r$ for all $u \in [0,U)$. Then we have $\tilde{\phi}(mu,s_0) = h(\tilde{\phi}(u,s_0)) = \phi^*(\tilde{\phi}(u,s_0)) = \phi(m\phi^{inv}(\tilde{\phi}(u,s_0)))$ for all $u \in [0,U)$, and so $\phi^{inv}(\tilde{\phi}(mu,s_0)) = m\phi^{inv}(\tilde{\phi}(u,s_0))$ for all $u \in [0,U/m)$. This means that $\phi^{inv}(\tilde{\phi}(u,s_0)) = b.u$, that is $\tilde{\phi}(u,s_0) = \phi(bu)$ for some $b \in (0,\infty)$ and for all $u \in [0,U/m)$, and hence for all u, since ϕ and $\tilde{\phi}$ are both cumulant generating functions. So

$$(6.3.15) \qquad \lim_{n\to\infty} c_n(s_0)Z_n \overset{d}{=} bY,$$

where Y is a random variable with cumulant generating function ϕ, and therefore $r = P(Y = 0) = P(\lim_{n\to\infty} c_n(s_0)Z_n = 0)$, implying that $P(\lim_{n\to\infty} Z_n = 0) = r$ by Theorem 6.3.2(a). This together with (6.3.14) yields that requirement 2 in the definition of $H_{m,r}$ is also fulfilled and we can conclude that $h \in H_{m,r}$. The proof is now finished, once we have established that $Y \overset{d}{=} \lim_{n\to\infty} c_n(s)Z_n$ for some $s \in (0, -\log r)$. To this end we define s by $s = \tilde{\phi}(1/b, s_0)$, with b as in (6.3.15). It follows then from Theorem 6.3.2 that $s \in (0, -\log r)$. Now writing $\tilde{\phi}^{inv}(s,s_0)$ for the inverse of $\tilde{\phi}(s,s_0)$ as a function of s, we have by Theorem 6.3.9 that a.s.

$$\lim_{n\to\infty} c_n(s)Z_n = \tilde{\phi}^{inv}(s,s_0) \lim_{n\to\infty} c_n(s_0)Z_n$$

$$= \tilde{\phi}^{inv}(\tilde{\phi}(1/b,s_0),s_0) \lim_{n\to\infty} c_n(s_0)Z_n = 1/b \lim_{n\to\infty} c_n(s_0)Z_n,$$

and so, by (6.3.15) $Y \overset{d}{=} \lim_{n\to\infty} c_n(s)Z_n$. This means that $\phi \in G_{m,r}$ and therefore $F_{m,r} \subset G_{m,r}$. $\square$

In Chapter 3 we introduced $S_n = \Sigma_{k=0}^{n} Z_k = \Sigma_{k=0}^{n} Z_{n-k}$. Since

$$\lim_{n \to \infty} \frac{c_n(s)}{c_{n-k}(s)} = \lim_{n \to \infty} \frac{c_n(s)}{h_k(c_n(s))} = \lim_{s \downarrow 0} \frac{s}{h_k(s)} = m^{-k}$$

for every integer k, one might ask if we can exchange limit and sum to obtain

$$\lim_{n \to \infty} c_n(s)S_n = \lim_{n \to \infty} \sum_{k=0}^{n} c_n(s)Z_{n-k} = \sum_{k=0}^{\infty} \lim_{n \to \infty} \frac{c_n(s)}{c_{n-k}(s)} c_{n-k}(s)Z_{n-k}$$

$$= \sum_{k=0}^{\infty} m^{-k}Y(s) = \frac{m}{m-1} Y(s) \quad \text{a.s.}.$$

The following theorem answers this question positively.

THEOREM 6.3.11.

$$\lim_{n \to \infty} c_n(s)S_n = \frac{m}{m-1} Y(s) \quad \text{a.s.}.$$

PROOF. Since

$$\lim_{n \to \infty} \frac{c_n(s)}{c_{n-k}(s)} = m^{-k}$$

and

$$c_n(s)S_n = c_n(s) \cdot \sum_{j=0}^{n} Z_j \geq c_n(s) \cdot \sum_{j=0}^{k} Z_{n-j}$$

for every integer $k \in [0,n]$,

$$\liminf_{n \to \infty} c_n(s)S_n \geq \lim_{n \to \infty} \sum_{j=0}^{k} \frac{c_n(s)}{c_{n-j}(s)} c_{n-j}(s)Z_{n-j} = Y(s) \cdot \sum_{j=0}^{k} m^{-j} \quad \text{a.s.}$$

for every non-negative integer k. So

$$(6.3.16) \qquad \liminf_{n \to \infty} c_n(s)S_n \geq \lim_{k \to \infty} Y(s) \cdot \sum_{j=0}^{k} m^{-j} = \frac{m}{m-1} Y(s) \quad \text{a.s.}.$$

Next we prove that $\limsup_{n \to \infty} c_n(s)S_n \leq \frac{m}{m-1} Y(s)$ a.s.. To this end we choose a $\delta > 0$ such that $\frac{1}{m} + \delta < 1$. Since

$$\lim_{k \to \infty} \frac{c_{k+1}(s)}{c_k(s)} = \frac{1}{m},$$

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there exists an integer $K_0 = K_0(\delta,s)$ such that $c_{k+1}(s)/c_k(s) \leq \frac{1}{m} + \delta$ for all $k \geq K_0$. Now choose $\omega \in A := \{\lim_{n\to\infty} c_n(s)Z_n = Y(s)\}$. It follows from Corollary 3.3.5 that $P(A) = 1$. Finally we choose an $\varepsilon > 0$. Then there exists an integer $N_0 = N_0(\varepsilon,s,\omega)$ such that $c_n(s)Z_n(\omega) \leq Y(s,\omega) + \varepsilon$ for all $n \geq N_0$. Let $L = \max(K_0,N_0)$. Then we have for all $n \geq L$,

$$c_n(s)S_n(\omega) = \sum_{k=0}^{L-1} c_n(s)Z_k(\omega) + \sum_{k=L}^{n} \frac{c_n(s)}{c_k(s)} c_k(s)Z_k(\omega)$$

$$\leq c_n(s) \sum_{k=0}^{L-1} Z_k(\omega) + \sum_{k=L}^{n} (\frac{1}{m} + \delta)^{n-k} (Y(s,\omega) + \varepsilon)$$

$$= c_n(s) \sum_{k=0}^{L-1} Z_k(\omega) + (Y(s,\omega) + \varepsilon) \frac{1-(\frac{1}{m} + \delta)^{n-L+1}}{1-(\frac{1}{m} + \delta)} \quad .$$

This means that

$$\limsup_{n \to \infty} c_n(s)S_n(\omega) \leq \frac{(Y(s,\omega) + \varepsilon)}{1 - \frac{1}{m} - \delta}$$

for all $\varepsilon > 0$, $0 < \delta < 1 - \frac{1}{m}$ and $\omega \in A$, and therefore $\limsup_{n\to\infty} c_n(s)S_n \leq Y(s)/(1-1/m) = \frac{m}{m-1} Y(s)$ a.s.. Combining this with (6.3.16) we obtain that

$$\lim_{n\to\infty} c_n(s)S_n = \frac{m}{m-1} Y(s) \qquad a.s.. \qquad \square$$

We close this section with a result, concerning the quotient Z_{n+1}/Z_n, conditioned on $\{Z_n \to \infty\}$. Since Z_{n+1} can be considered as a "sum" of Z_n independent and identically distributed random variables, all with expectation m, we might hope, in view of the law of large numbers, that Z_{n+1}/Z_n converges to m as $n \to \infty$. This is indeed proved in the next theorem.

THEOREM 6.3.12.

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = m \quad a.s.$$

on $\{Z_n \to \infty\}$.

PROOF. Choose some $s \in (0,-\log r)$. Then we know from Theorem 6.3.2 that $Y(s) \in (0,\infty)$ a.s. on $\{Z_n \to \infty\}$. Hence

$$\lim_{n \to \infty} \frac{Z_{n+1}}{Z_n} = \lim_{n \to \infty} \frac{c_{n+1}(s) Z_{n+1}}{c_n(s) Z_n} \cdot \frac{h(c_{n+1}(s))}{c_{n+1}(s)} =$$

$$= \frac{Y(s)}{Y(s)} \lim_{s \downarrow 0} \frac{h(s)}{s} = m \quad \text{a.s.}$$

on $\{Z_n \to \infty\}$. $\quad \square$

CHAPTER 7

THE CASE m=∞

7.1. INTRODUCTION

We shall now consider processes for which $m = \infty$. Such processes are called *explosive*. Again as in the previous chapters, the behaviour of the process on $\{Z_n \to 0\}$ is in fact determined by the value of a. We therefore make in this chapter only some brief remarks about that case and refer further to Section 6.2. The behaviour on $\{Z_n \to \infty\}$ is however completely different from what we have seen so far. It turns out that in many cases it is not useful anymore to normalize the random variables Z_n by a sequence of constants, as we did in Chapter 6, because for many explosive processes $P(0 < \lim_{n \to \infty} a_n Z_n < \infty) = 0$ for all sequences $\{a_n;\ n = 1,2,3,\ldots\}$ of positive and finite constants. We shall therefore study limits of $g_n(Z_n)$ for suitable, and hence non-linear, functions g_n. First of all we mention some weak convergence results, which will be extended to almost sure convergence results later on. The techniques used for this approach are well-known for Galton-Watson processes. Because similar arguments can and will be used in case $a = 0$, we shall not give all the details in this chapter, but confine ourselves to referring to Chapter 9. Furthermore, we pay some attention to the stochastic norming with the help of the function $f_\gamma(t)$, introduced in Chapter 3.

7.2. THE BEHAVIOUR OF THE PROCESS ON $\{Z_n \to 0\}$

In Section 6.2, where we studied the behaviour of supercritical processes $\{Z_n;\ n = 0,1,2,\ldots\}$ on $\{Z_n \to 0\}$, we did not use the finiteness of m, but only the fact that $m > 1$, implying that $h(s) > s$ for small s, and therefore that $r < 1$. For this reason we can again use the same techniques as in Section 6.2 with respect to the behaviour of the process on $\{Z_n \to 0\}$. This yields in particular, that if $r > 0$, then the process $\{Z_n;\ n = 0,1,2,\ldots\}$

conditioned on $\{Z_n \to 0\}$ is equivalent to a subcritical process
$\{\tilde{Z}_n; \; n = 0,1,2,\ldots\}$ with $\tilde{m} := E(\tilde{Z}_1 \mid \tilde{Z}_0 = 1) = h'(-\log r)$ and $\tilde{a} :=$
$\inf\{x; \; P(\tilde{Z}_1 \le x \mid \tilde{Z}_0 = 1) > 0\} = a$. Furthermore, we get analogous to Theorem 6.2.1

THEOREM 7.2.1.

(a) $r < 1$;

(b) $r = 0$ *if and only if* $a \ge 1$;

(c) *if* $P(Z_1 = 0) > 0$ *then* $q = r$.

7.3. THE BEHAVIOUR OF THE PROCESS ON $\{Z_n \to \infty\}$

In contrast with the behaviour of the process on $\{Z_n \to 0\}$, there is no
analogy with the supercritical case on the event $\{Z_n \to \infty\}$. We can for exam-
ple prove, in the same way as in SENETA [1969], that there is no sequence
of positive and finite constants $\{d_n; \; n = 1,2,3,\ldots\}$ such that $d_n Z_n$ converges
in distribution to a proper, non-degenerate limit as $n \to \infty$. One way to get
a hold on the process is now to look for some sequence of functions $\{g_n;$
$n = 0,1,2,\ldots\}$ such that $g_n(Z_n)$ converges in some sense to a proper, non-
degenerate limit as $n \to \infty$. As we saw above, these g_n cannot be linear func-
tions. A first step in this direction was made in DARLING [1970] for
Galton-Watson processes. He proved that under certain conditions, we can
take $g_n(x) = b_n \log(1+x)$, where $\{b_n; \; n = 1,2,3,\ldots\}$ is a sequence of posi-
tive constants. We can lift out a part of his proof to obtain the following
result to be used repeatedly in the sequel. See also SENETA [1973].

LEMMA 7.3.1. *Let* $\{f_n; \; n = 1,2,3,\ldots\}$ *be a sequence of cumulant generating*
functions of non-negative random variables X_n, $n = 1,2,3,\ldots$. *Suppose*
there exist a sequence $\{b_n; \; n = 1,2,3,\ldots\}$ *of positive constants and a*
distribution function w *such that* $\lim_{n \to \infty} b_n = 0$ *and*

(7.3.1) $\quad \lim_{n \to \infty} f_n\{-\log(1 - \exp(-t/b_n))\} = -\log w(t), \qquad 0 < t < \infty.$

Then $\lim_{n \to \infty} P(b_n \log(1 + X_n) \le t) = w(t)$, *for every* $t \in (0,\infty)$ *where* w *is*
continuous.

Using cumulant generating functions and their iterates as the basic
tool in deriving results concerning branching processes, one becomes more
and more aware of the saddening fact that examples are hard to give, be-
cause these iterates soon become very complicated. However, at this moment

we are able to present one.

EXAMPLE 7.3.2. Let $\{Z_n; \; n = 0,1,2,\ldots\}$ be a branching process having a strictly stable distribution, concentrated on $[0,\infty)$, as its offspring distribution. It follows from Section XIII.6 in FELLER [1971] that in this case $h(s) = ds^\alpha$, $s \in [0,\infty)$, where $d \in (0,\infty)$ and $\alpha \in (0,1)$ are constants; α is called the characteristic exponent of the stable distribution. In this example as well as in the following examples where we consider a branching process having a strictly stable distribution concentrated on $[0,\infty)$ as its offspring distribution, we suppose that $d = 1$, implying that

$$(7.3.2) \qquad h(s) = s^\alpha, \qquad s \in [0,\infty).$$

This means that $h_n(s) = s^{\alpha^n}$ for all $n = 1,2,3,\ldots$. Taking $b_n = \alpha^n$ and $f_n = h_n$ in (7.3.1) we obtain

$$f_n\{-\log(1 - \exp(-t/b_n))\} = \{-\log(1 - \exp(-t/\alpha^n))\}^{\alpha^n}$$

$$\sim \{\exp(-t/\alpha^n)\}^{\alpha^n} = e^{-t}, \qquad \text{as } n \to \infty, \text{ for all } t \in (0,\infty),$$

and hence it follows from Lemma 7.3.1 that

$$\alpha^n \log(1 + Z_n) \xrightarrow{d} Z,$$

where Z is a random variable with distribution function

$$w(t) = \begin{cases} 0 & , \quad t < 0 \\ \exp(-\exp(-t)), & \quad 0 \le t < \infty \end{cases}.$$

Since this distribution function is well-known in the extreme value theory, one might ask if it is possible to give any interpretation for Z in this context. For an answer to this question we refer to Example 7.3.11.

In general it is not so easy to decide whether or not the limit (7.3.1) exists, for $f_n = h_n$ and $\{b_n; \; n = 1,2,3,\ldots\}$ some sequence of constants. However, in SENETA [1973] a class of processes is described for which the corresponding cumulant generating functions satisfy (7.3.1). For the construction of this class he introduced the following function.

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(7.3.3) $f(t) = -\log\{1 - \exp(-h(-\log\{1 - e^{-t}\}))\}$, $t \in [0,\infty)$.

This function f was assumed to satisfy

(7.3.4) $f(t)$ is convex or concave on $[0,\infty)$

and

(7.3.5) $0 < \gamma := \lim_{t\to\infty} \frac{f(t)}{t} < 1$.

Furthermore introducing

(7.3.6) $\phi(t) = f(t - \log(1-r)) + \log(1-r)$, $t \in [0,\infty)$,

(7.3.7) $d(t) = \phi^{inv}(t)$, $t \in [0,\infty)$,

(7.3.8) $\rho(x) = 1/d(1/x)$, $x \in (0,\infty)$,

and the convention that the subscript n indicates the nth iterate, Seneta proved the following result, which is also true if the state space of the branching process is $[0,\infty)$.

THEOREM 7.3.3. *Suppose that* (7.3.4) *and* (7.3.5) *hold. Then for any fixed* $x \in (0,\infty)$,

(7.3.9) $-\log w(t,x) := \lim_{n\to\infty} h_n(-\log\{1 - \exp(-t/\rho_n(x))\})$

exists for all $t \in (0,\infty)$. *The function* w(t,x) *has the following properties:*

(7.3.10) $\lim_{t\downarrow 0} w(t,x) = r$ *and* $\lim_{t\to\infty} w(t,x) = 1$;

(7.3.11) $h(-\log w(t,x)) = -\log w(\gamma t,x)$, $t \in (0,\infty)$;

(7.3.12)   w(t,x) *is continuous and strictly increasing in* $t \in (0,\infty)$.

PROOF. The proof of (7.3.9), (7.3.10) and (7.3.12) is analogous to the proof of Theorem 1 in SENETA [1973]. Furthermore, the analog of relation (3.4) in SENETA [1974] yields that $\Delta(\gamma t,x) = \phi(\Delta(t,x))$, $t \in (0,\infty)$, where

$\Delta(t,x) = \log(1-r) - \log(1 - w(t,x))$. Hence we obtain from (7.3.6) and (7.3.3) that

$$-\log w(\gamma t,x) = -\log(1 - (1-r)e^{-\Delta(\gamma t,x)})$$

$$= -\log(1 - (1-r)e^{-\phi(\Delta(t,x))})$$

$$= -\log(1 - e^{-f(\Delta(t,x)-\log(1-r))})$$

$$= h(-\log(1 - (1-r)e^{-\Delta(t,x)}))$$

$$= h(-\log w(t,x)), \qquad t \in (0,\infty). \qquad \square$$

Until now the analysis was based on the cumulant generating function h_n of Z_n. For this reason we obtained only weak convergence results. We shall now try to extend this to almost sure convergence of $g_n(Z_n)$ for some sequence of functions $\{g_n;\ n = 1,2,3,\ldots\}$. This will be done using a sequence of random variables $\{U_n(x);\ n = 1,2,3,\ldots\}$, mentioned in PAKES [1976]. It turns out that these $U_n(x)$ constitute a martingale sequence, a fact which will be used to prove the almost sure convergence. These random variables $U_n(x)$ are defined by

$$(7.3.13) \qquad U_n(x) = \{1 - (1-r)\exp(-1/\rho_n(x))\}^{Z_n}, \qquad n = 1,2,3,\ldots;\ x \in (0,\infty),$$

with $\rho(x)$ as in (7.3.8) and $\rho_n(x)$ its nth iterate. Since

$$(7.3.14) \qquad \phi_n(1/\rho_n(x)) = 1/x$$

by (7.3.7) and (7.3.8), we see that $\phi(1/\rho_{n+1}(x)) = 1/\rho_n(x)$, and thus, using the basic branching property (3.1.3) and (7.3.6) and (7.3.3),

$$E(U_{n+1}(x) \mid U_1(x),\ldots,U_n(x))$$

$$= E\{(1 - (1-r)\exp\{-1/\rho_{n+1}(x)\})^{Z_{n+1}} \mid Z_n\}$$

$$= \{E(1 - (1-r)\exp\{-1/\rho_{n+1}(x)\})^{Z_1}\}^{Z_n}$$

$$= \{\exp[-h(-\log(1 - (1-r)\exp\{-1/\rho_{n+1}(x)\}))]\}^{Z_n} =$$

$$= \{1 - (1-r)\exp[-\phi(1/\rho_{n+1}(x))]\}^{Z_n}$$

$$= \{1 - (1-r)\exp(-1/\rho_n(x))\}^{Z_n} = U_n(x).$$

Furthermore obviously $0 \leq U_n(x) \leq 1$ for all $n = 1,2,3,\ldots$ and all $x \in (0,\infty)$. So $\{U_n(x); \ n = 1,2,3,\ldots\}$ is a bounded martingale, and therefore

(7.3.15) $\quad U_n(x) \xrightarrow{\text{a.s.}}$ some random variable $U(x)$, as $n \to \infty$,

and

$$\lim_{n\to\infty} EU_n(x)^\alpha = EU(x)^\alpha \qquad \text{for all } \alpha \in (0,\infty).$$

Particularly, since $EU_n(x)$ does not depend on n, by a well-known property of martingales, we obtain, using the definition of ϕ_n and (7.3.14)

(7.3.16) $\quad EU(x) = EU_n(x) = E\{1 - (1-r)\exp(-1/\rho_n(x))\}^{Z_n}$

$$= \exp\{-h_n(-\log(1 - (1-r)\exp\{-1/\rho_n(x)\}))\}$$

$$= 1 - (1-r)\exp\{-\phi_n(1/\rho_n(x))\}$$

$$= 1 - (1-r)\exp(-1/x).$$

With the help of this we can prove the following result, which will be used to show that there exists no sequence of positive constants $\{a_n; \ n = 1,2,3,\ldots\}$ such that $P(0 < \lim_{n\to\infty} a_n Z_n < \infty \mid \lim_{n\to\infty} Z_n = \infty) > 0$.

THEOREM 7.3.4. *Suppose* (7.3.4) *and* (7.3.5) *hold. Then, for any* $x \in (0,\infty)$, $\{\exp(-1/\rho_n(x))\} \cdot Z_n \xrightarrow{\text{a.s.}}$ *some random variable* $Z(x)$, *and*

(7.3.17) $\quad P(Z(x) = 0) = 1 - P(Z(x) = \infty) = 1 - (1-r)\exp(-1/x)$.

PROOF. From (7.3.13) and (7.3.15) we know that

$$Z_n \log(1 - (1-r)\exp\{-1/\rho_n(x)\}) \xrightarrow{\text{a.s.}} \log U(x) \qquad \text{as } n \to \infty.$$

Furthermore, it follows from the proof of Theorem 7.3.3 that $\rho_n(x) \downarrow 0$ as $n \to \infty$, implying that

$$\log(1 - (1-r)\exp\{-1/\rho_n(x)\}) \sim -(1-r)\exp(-1/\rho_n(x)) \qquad \text{as } n \to \infty$$

and so

$$(7.3.18) \qquad (1-r)\{\exp(-1/\rho_n(x))\}Z_n \xrightarrow{\text{a.s.}} -\log U(x) \qquad \text{as } n \to \infty.$$

Since obviously $-\log U(x) = 0$ on $\{Z_n \to 0\}$, we see that $Z_n \to \infty$ a.s. on $\{0 < -\log U(x) < \infty\}$. This means that

$$(7.3.19) \qquad (1 + Z_n)^{\rho_n(x)} = \left\{\frac{Z_n\{\exp(-1/\rho_n(x))\}(1-r)}{-\log U(x)}\right\}^{\rho_n(x)} \cdot$$

$$\cdot \left\{\frac{-\log U(x)}{1-r}\right\}^{\rho_n(x)} \left\{\frac{1+Z_n}{Z_n}\right\}^{\rho_n(x)} \cdot e \xrightarrow{\text{a.s.}} 1 \cdot 1 \cdot 1 \cdot e = e$$

as $n \to \infty$ on $\{0 < -\log U(x) < \infty\}$.

However, from Lemma 7.3.1 and Theorem 7.3.3 we know that $\rho_n(x)\log(1 + Z_n)$ converges weakly to some random variable $W(x)$, and that the distribution function of $W(x)$ is continuous on $(0,\infty)$. Combining this with (7.3.19) we can conclude that $P(0 < -\log U(x) < \infty) \leq P(\rho_n(x)\log(1 + Z_n) \to 1) = 0$. So we have, since $U_n(x) \in [0,1]$, and hence also $U(x) \in [0,1]$,

$$P(U(x) = 1) = P(-\log U(x) = 0) = 1 - P(-\log U(x) = \infty)$$

$$= 1 - P(U(x) = 0),$$

and therefore, in view of (7.3.16), $P(U(x) = 1) = 1 - (1-r)\exp(-1/x)$. This together with (7.3.18) proves the theorem. $\square$

This last result is more important than it might seem at first sight, because it says that not only for one sequence, but for a whole class of sequences of norming constants, to wit $\{\{\exp(-1/\rho_n(x))\}; n = 1,2,3,\ldots\};$ $x \in (0,\infty)\}$, it holds that $\lim_{n \to \infty} [\exp(-1/\rho_n(x))]Z_n = 0$ or ∞ almost surely. The fact that $\rho_n(x)$ increases from 0 to ∞ for every $n = 1,2,3,\ldots$ as x runs through $(0,\infty)$, as follows from the relations (7.3.3) up to and including

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(7.3.8), enables us to compare any sequence $\{a_n; n = 1,2,3,...\}$ with $\{\exp(-1/\rho_n(x)); n = 1,2,3,...\}$. This leads to the following theorem, which says that there is no sequence of positive constants $\{a_n; n = 1,2,3,...\}$ such that $P(0 < \lim_{n\to\infty} a_n Z_n < \infty \mid \lim_{n\to\infty} Z_n = \infty) > 0$. This result, together with the corresponding result in Chapter 9, was meant in the remark made before Lemma 3.3.4.

THEOREM 7.3.5. *Suppose* (7.3.4) *and* (7.3.5) *hold. Let* $\{a_n; n = 1,2,3,...\}$ *be a sequence of positive constants such that* $\lim_{n\to\infty} a_n = 0$ *and* $\lim_{n\to\infty} a_n Z_n$ *exists almost surely. Then* $P(\lim_{n\to\infty} a_n Z_n = 0$ *or* $\infty) = 1$.

PROOF. Obviously, there are only three possibilities for the sequence $\{a_n; n = 1,2,3,...\}$:

(1) for any $x \in (0,\infty)$ there exists a subsequence $\{a_{n_j(x)}; j = 1,2,3,...\}$ of $\{a_n; n = 1,2,3,...\}$ such that $a_{n_j(x)} < \exp(-1/\rho_{n_j(x)}(x))$ for all $j = 1,2,3,...$.

(2) for any $x \in (0,\infty)$ there exists a subsequence $\{a_{n_j(x)}; j = 1,2,3,...\}$ of $\{a_n; n = 1,2,3,...\}$ such that $a_{n_j(x)} > \exp(-1/\rho_{n_j(x)}(x))$ for all $j = 1,2,3,...$.

(3) there exist $0 < x_1 < x_2 < \infty$ such that $\exp(-1/\rho_n(x_1)) \leq a_n \leq \exp(-1/\rho_n(x_2))$ for all sufficiently large n.

We shall now investigate each of these cases separately. By assumption $\lim_{n\to\infty} a_n Z_n$ exists almost surely. Call this limit Z.

(1) Since $a_{n_j(x)} < \exp(-1/\rho_{n_j(x)}(x))$ for all $j = 1,2,3,...$ we obtain

$$0 \leq Z = \lim_{j\to\infty} a_{n_j(x)} Z_{n_j(x)} \leq \lim_{j\to\infty} [\exp(-1/\rho_{n_j(x)}(x))] Z_{n_j(x)} = Z(x)$$

for all $x \in (0,\infty)$ in view of Theorem 7.3.4. Hence $P(Z = 0) \geq P(Z(x) = 0) = 1 - (1-r)\exp(-1/x)$ by (7.3.17). Now letting $x \downarrow 0$ we see that $P(Z = 0) = 1$.

(2) Because $\lim_{n\to\infty} a_n = 0$, a similar argument as under (1) yields

$$1 - r = P(\lim_{n\to\infty} Z_n = \infty) \geq P(Z = \lim_{j\to\infty} a_{n_j(x)} Z_{n_j(x)} = \infty) \geq P(Z(x) = \infty)$$

$$= (1-r)\exp(-1/x)$$

and

$$r = P(\lim_{n\to\infty} Z_n = 0) \leq P(Z = 0) \leq P(Z(x) = 0) = 1 - (1-r)\exp(-1/x)$$

for all $x \in (0,\infty)$. Letting $x \to \infty$ we can conclude that $P(Z = \infty) = 1-r$ and $P(Z = 0) = r$.

(3) Since $1/\rho_n(x_2) \leq -\log a_n \leq 1/\rho_n(x_1)$, and the function $\phi(t)$, defined by (7.3.6) is increasing in t, because $h(t)$ is, (7.3.14) yields $0 < 1/x_2 \leq \phi_n(-\log a_n) \leq 1/x_1 < \infty$ for all sufficiently large n. This implies that the sequence $\{\phi_n(-\log a_n); n = 1,2,3,\ldots\}$ has a convergent subsequence $\{\phi_{n_j}(-\log a_{n_j}); j = 1,2,3,\ldots\}$. Call its limit A. Then obviously $A \in [1/x_2, 1/x_1]$. Furthermore, for any $\varepsilon \in (0,A)$

$$A - \varepsilon < \phi_{n_j}(-\log a_{n_j}) < A + \varepsilon,$$

for sufficiently large j. Now because $\rho_n(1/\phi_n(x)) = 1/x$ by (7.3.7) and (7.3.8), this implies

$$\exp(-1/\rho_{n_j}(\tfrac{1}{A+\varepsilon})) < a_{n_j} < \exp(-1/\rho_{n_j}(\tfrac{1}{A-\varepsilon}))$$

for sufficiently large j, and in a similar way as above, we see that

$$(1-r)\exp(-(A+\varepsilon)) = P(Z(\tfrac{1}{A+\varepsilon}) = \infty) \leq P(Z = \infty)$$

and

$$1 - (1-r)\exp(-(A-\varepsilon)) = P(Z(\tfrac{1}{A-\varepsilon}) = 0) \leq P(Z = 0).$$

Hence we obtain, letting $\varepsilon \downarrow 0$, $P(Z = \infty) = (1-r)e^{-A}$ and $P(Z = 0) = 1 - (1-r)e^{-A}$. $\square$

For an application of this last result we consider again the random variables $Y(s)$ introduced in Corollary 3.3.5. There we saw that $Y(s) = \lim_{n\to\infty} c_n(s)Z_n$ exists a.s. if $s \in (0,-\log r)$. Because $\lim_{n\to\infty} c_n(s) = 0$, it now follows that if (7.3.4) and (7.3.5) hold, then $P(Y(s) = 0) = 1 - P(Y(s) = \infty)$. Furthermore, the substitution $z = 1$ in (3.3.5) yields $Ee^{-Y(s)} = e^{-s}$ and hence $P(Y(s) = 0) = e^{-s}$ and $P(Y(s) = \infty) = 1 - e^{-s}$ for all $s \in (0,-\log r)$. One might ask if this last property holds true for all processes with $m = \infty$. A negative answer to this question is given in a paper of SCHUH and BARBOUR [1977]. There they divide the explosive Galton-Watson

processes into regular ones, for which $P(Y(s) = 0$ or $\infty) = 1$ for all $s \in$ $(0,-\log r)$ and irregular ones for which this is not true, and prove that there exist irregular processes. It turns out that their method can also be applied to explosive branching processes with state space $[0,\infty)$, but as a similar method will be used in the case $a = 0$, we shall at this moment only present a brief survey of the results analogous to the ones in the paper mentioned above and to some related results in other papers. First of all we can prove the following; see also GREY [1977].

THEOREM 7.3.6.

(a) *Let* L *be a non-increasing function on* $(0,\infty)$ *such that* $\lim_{x \downarrow 0} L(x) = \infty$ *and* $L(\infty) = \lim_{x \to \infty} L(x) = 0$, *and let* $\{a_n ; n = 0,1,2,\ldots\}$ *be a sequence of positive constants. Suppose that* $\lim_{n \to \infty} a_n L(c_n(s))$ *exists* $\in (0,\infty)$ *for all* $s \in (0,-\log r)$ *and is continuous on* $(0,-\log r)$. *Call this limit* $\psi(s)$. *Then there is a random variable* U *such that*

(7.3.20) $a_n L(1/Z_n) \xrightarrow{a.s.} U \qquad$ *as* $n \to \infty$,

where $U = 0$ *on* $\{Z_n \to 0\}$ *and* $U \in (0,\infty)$ *a.s. on* $\{Z_n \to \infty\}$;

(b) *Suppose furthermore that* L *is slowly varying at* 0 *and that* ψ *is strictly decreasing on* $(0,-\log r)$. *Then* $P(U \le t) = \exp(-\psi^{inv}(t))$, $t \in (t_0, t_1)$, *where* $t_0 := \lim_{t \uparrow -\log r} \psi(t)$ *and* $t_1 := \lim_{t \downarrow 0} \psi(t)$.

Later on we shall see that if L is continuous and strictly decreasing on $(0,A)$ for some $A \in (0,\infty)$, then $t_0 = 0$ and $t_1 = \infty$.

EXAMPLE 7.3.7. We can again apply this result to a branching process having a strictly stable distribution concentrated on $[0,\infty)$ with characteristic exponent $\alpha \in (0,1)$ as its offspring distribution. From (7.3.2) we know that in this case $h(s) = s^{\alpha}$, and hence $c_n(s) = s^{\alpha^{-n}}$. Furthermore, Theorem 3.3.1 implies that $-\log r = 1$, that is $r = e^{-1}$. Now choosing $a_n = \alpha^n$ and $L(s) =$ $\log(1 + 1/s)$, $s \in (0,\infty)$, we see that these a_n and L satisfy the conditions of Theorem 7.3.6(a) and (b) and that

$$\psi(s) = \lim_{n \to \infty} a_n L(c_n(s)) = \lim_{n \to \infty} \alpha^n \log(1 + s^{-\alpha^{-n}})$$
$$= \lim_{n \to \infty} \alpha^n \log s^{-\alpha^{-n}} = -\log s,$$

for $s \in (0,-\log r) = (0,1)$.

Hence it follows from (7.3.20) that $\alpha^n \log(1 + Z_n) \xrightarrow{\text{a.s.}} U$, where $P(U = 0) =$ $1/e$ and $P(U \leq t) = \exp(-\psi^{\text{inv}}(t)) = \exp(-\exp(-t))$ for $t \in (t_0, t_1) = (0, \infty)$, in agreement with the distribution function derived in Example 7.3.2. Moreover, it follows that $U \in (0, \infty)$ a.s. on $\{Z_n \to \infty\}$, a fact which will be used later on.

EXAMPLE 7.3.8. We can also apply this result to processes which satisfy (7.3.4) and (7.3.5), with $a_n = \rho_n(x)$ for any $x \in (0, \infty)$ and $L(s) = \log(1 + 1/s)$, $s \in (0, \infty)$, where we use the notation introduced before Theorem 7.3.3. For if we do so, then

$$a_n L(c_n(s)) = \rho_n(x) \log(1 + 1/c_n(s)) \sim -\rho_n(x) \log c_n(s)$$

$$= -\rho_n(x) \log\{-\log(1 - \exp\{-f_n^{\text{inv}}(-\log(1 - e^{-s}))\})\}$$

$$\text{for } s \in (0, -\log r),$$

with f as in (7.3.3). Now $-\log(1 - e^{-s}) \in (-\log(1-r), \infty)$ if $s \in (0, -\log r)$, and because $f(-\log(1-r)) = -\log(1-r)$, and f is convex or concave and $\lim_{t \to \infty} \frac{f(t)}{t} = \gamma < 1$, it follows that $f_n^{\text{inv}}(-\log(1 - e^{-s})) \to \infty$ as $n \to \infty$ for all $s \in (0, -\log r)$. Therefore

$$a_n L(c_n(s)) \sim -\rho_n(x) \log \exp\{-f_n^{\text{inv}}(-\log(1 - e^{-s}))\}$$

$$= \rho_n(x) f_n^{\text{inv}}(-\log(1 - e^{-s})) \qquad \text{as } n \to \infty.$$

Now in view of the relations mentioned before Theorem 7.3.3,

$$f_n^{\text{inv}}(t) = -\log(1-r) + 1/\rho_n(1/\{t + \log(1-r)\}),$$

and since $\rho_n(x) \downarrow 0$ as $n \to \infty$ this implies that

$$a_n L(c_n(s)) \sim \rho_n(x)/\rho_n(\{\log\{(1-r)/(1-\exp(-s))\}\}^{-1}) \qquad \text{as } n \to \infty$$

This last expression converges to some function $\psi(s,x)$, which is continuous and strictly decreasing as a function of s on $(0, -\log r)$ and satisfies $\lim_{s \uparrow -\log r} \psi(s,x) = 0$ and $\lim_{s \downarrow 0} \psi(s,x) = \infty$, as follows from Lemma 2.2 in

SENETA [1973]. An application of Theorem 7.3.6 therefore yields that
$\rho_n(x)\log(1+Z_n) \xrightarrow{\text{a.s.}}$ some random variable $U(x)$, where $U(x) = 0$ on $\{Z_n \to 0\}$
and $P(U(x) \le t) = \exp(-\psi^{\text{inv}}(t,x))$ for $t \in (0,\infty)$, thus extending the result,
obtained by combining Lemma 7.3.1 and Theorem 7.3.3, to almost sure conver-
gence.

The following step in the paper of SCHUH and BARBOUR [1977] is that
they construct, for every cumulant generating function $h(s)$, a function L
such that, with $a_n = e^{-n}$, $n = 0,1,2,\ldots$, the conditions of Theorem 7.3.6(a)
are satisfied. This yields

THEOREM 7.3.9. *There exists a function L such that* $e^{-n}L(1/Z_n) \xrightarrow{\text{a.s.}}$ *some*
random variable U *as* $n \to \infty$, *where* $U = 0$ *on* $\{Z_n \to 0\}$ *and* $U \in (0,\infty)$ *a.s. on*
$\{Z_n \to \infty\}$.

In connection with Theorem 7.3.6 we can prove a result, the analog of
which for Galton-Watson processes is established in COHN and PAKES [1978].

THEOREM 7.3.10. *Let* L, $\{a_n;\ n = 0,1,2,\ldots\}$, ψ *and* U *be as in Theorem 7.3.6,*
and suppose that the conditions of both part (a) *and part* (b) *of that theorem*
are satisfied and that furthermore $\alpha := \lim_{n\to\infty} a_n/a_{n-1}$ *exists* $\in (0,1)$ *and*
that L *is continuous and strictly decreasing on* $(0,A)$ *for some* $A \in (0,\infty)$.
Then

$$(7.3.21) \qquad U = \alpha^k \max\{U_1^{(k)}, U_2^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\} \quad \text{a.s.,}$$

for every non-negative integer k, *where* $[z]$ *is the integer part of* z;
$U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$ *are random variables all distributed as* U *and* $P(\bar{U}^{(k)} \le t \mid Z_k) =$
$= P(U \le t)^{Z_k - [Z_k]}$ *for all* $t \in (-\infty,\infty)$; *furthermore* $Z_k, U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$
are independent, and $\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$ *are, conditionally given* Z_k,
independent; finally the distribution function F *of* U *satisfies*

$$(7.3.22) \qquad -\log F(\alpha t) = h(-\log F(t)), \qquad t \in (-\infty,\infty).$$

It is interesting to compare this last result with relation (6.3.12)
which says that $Y(s)$ is distributed as $m^{-k}\{Y_1^{(k)} + Y_2^{(k)} + \ldots + Y_{[Z_k]}^{(k)} + \bar{Y}^{(k)}\}$,
with $\bar{Y}^{(k)}, Y_1^{(k)}, Y_2^{(k)}, Y_3^{(k)}, \ldots$ analogous to the random variables
$\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$ above. The sum in (6.3.12) is now replaced by a
maximum.

Using the notation of the Theorems 7.3.6 and 7.3.10 it follows that
under the conditions of Theorem 7.3.10, $-\log F(t) \in (0,-\log r)$ for
$t \in (t_0, t_1)$. By (7.3.22) this implies that also $-\log F(\alpha t) \in (0,-\log r)$ and
hence that $\alpha t \in (t_0, t_1)$. Since $\alpha \in (0,1)$, this is only possible if $t_0 = 0$.
On the other hand, if $\alpha t \in (t_0, t_1)$, then $-\log F(\alpha t) \in (0,-\log r)$ and thus
by (7.3.22) also $-\log F(t) \in (0,-\log r)$, that is $t \in (t_0, t_1)$. This implies
that $t_1 = \infty$. Since $-\log F(t) = \psi^{inv}(t)$ on (t_0, t_1) it follows therefore from
(7.3.22) that $\psi^{inv}(\alpha t) = h(\psi^{inv}(t))$, $t \in (0,\infty)$. We can consider this rela-
tion as the analog of relation (6.3.1), which says that $\phi(u,s)$, the cumu-
lant generating function of $Y(s)$, satisfies $\phi(mu,s) = h(\phi(u,s))$.

EXAMPLE 7.3.11. For a branching process with offspring distribution a
strictly stable distribution concentrated on $[0,\infty)$ with characteristic ex-
ponent $\alpha \in (0,1)$, Theorem 7.3.10 can be applied again with
$L(s) = \log(1 + 1/s)$, $s \in (0,\infty)$ and $a_n = \alpha^n$ as in Example 7.3.7. Then we ob-
tain that U is distributed as $\alpha^k \max\{U_1^{(k)}, U_2^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\}$, with the
random variables $\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, \ldots$ as in that theorem. Furthermore,
(7.3.22) becomes $-\log F(\alpha t) = (-\log F(t))^\alpha$, in agreement with the distri-
bution function derived in Example 7.3.7.

EXAMPLE 7.3.12. The result of Theorem 7.3.10 can also be applied to pro-
cesses which satisfy (7.3.4) and (7.3.5). From Example 7.3.8 we know al-
ready that with $a_n = \rho_n(x)$ for any $x \in (0,\infty)$ and $L(s) = \log(1 + 1/s)$,
$s \in (0,\infty)$, the conditions of Theorem 7.3.6 are fulfilled, implying that
$\rho_n(x)\log(1 + Z_n) \xrightarrow{a.s.} U(x)$ as $n \to \infty$. Furthermore, it follows from the re-
lations (7.3.5) up to and including (7.3.8) that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim_{n\to\infty} \frac{\rho_n(x)}{\rho_{n-1}(x)} = \lim_{t\downarrow 0} \frac{\rho(t)}{t} = \lim_{t\to\infty} \frac{t}{\phi^{inv}(t)}$$

$$= \lim_{t\to\infty} \frac{\phi(t)}{t} = \lim_{t\to\infty} \frac{f(t)}{t} = \gamma \in (0,1).$$

Since $L(s)$ is strictly decreasing on $(0,\infty)$, relation (7.3.22) holds and we
obtain that $-\log F(\gamma t, x) = h(-\log F(t,x))$, $t \in (-\infty,\infty)$, where $F(t,x)$ is the
distribution function of $U(x)$. Notice that for $t \in (0,\infty)$ this is just re-
lation (7.3.11), for as a consequence of Lemma 7.3.1 and Theorem 7.3.3 we
have that $\rho_n(x)\log(1 + Z_n)$ converges in distribution to some random vari-
able $Z(x)$, the distribution function $w(t,x)$ of which satisfies
$h(-\log w(t,x)) = -\log w(\gamma t, x)$, $t \in (0,\infty)$.

As already mentioned above, in Chapter 9 we shall give more details of the methods leading to these last results.

We now pass on to the study of the quotient $Z_{n+1}/f_\gamma(Z_n)$ introduced in Section 3.3, with

$$f_\gamma(t) = \frac{\log|\log t|}{c(\gamma t^{-1}\log|\log t|)} \; .$$

We might obtain some results by considering the events $\{Z_{n+1} \leq d\, f_\gamma(Z_n)\}$ for some constant d, and trying to apply a Borel-Cantelli-type lemma to them. The problem which arises then is not only the calculation of the probability of these events, but also the large measure of dependence between them. There is however a case for which this dependence does not exist, and this will be studied in the following example.

EXAMPLE 7.3.13. Let $\{Z_n;\ n = 0,1,2,\ldots\}$ be a branching process having a strictly stable distribution, concentrated on $[0,\infty)$ with characteristic exponent $\alpha \in (0,1)$ as its offspring distribution. In Lemma 1 of ATHREYA [1975] it is proved that in this case the random variables $Z_{n+1}/Z_n^{1/\alpha}$, $n = 0,1,2,\ldots$ are independent, and all distributed as Z_1. The argument leading to this statement is that the distribution of $Z_{n+1}/Z_n^{1/\alpha}$ conditionally given $Z_0,Z_1,\ldots,Z_n$ only depends on Z_n, and that it is the same as the distribution of

$$\frac{W(Z_n)}{Z_n^{1/\alpha}}$$

given Z_n, where $\{W(t);\ t \in [0,\infty)\}$ is a stochastic process independent of the subordinator defining the branching process $\{Z_n;\ n = 0,1,2,\ldots\}$, but with the same distribution as that subordinator. Because for such so-called stable processes $\{W(t);\ t \in [0,\infty)\}$ it holds true that

$$\frac{W(t)}{t^{1/\alpha}} \overset{d}{=} W(1) \qquad \text{for all } t \in (0,\infty),$$

it follows that both the conditional distribution of $Z_{n+1}/Z_n^{1/\alpha}$ given $Z_0,\ldots,Z_n$ and the unconditional distribution of $Z_{n+1}/Z_n^{1/\alpha}$ are the same as that of $W(1)$. This means that the random variables $Z_{n+1}/Z_n^{1/\alpha}$, $n = 0,1,2,\ldots$ are independent and all distributed as $W(1)$. Now introducing the events D_n by

$$D_n = \{Z_{n+1} \leq Z_n^{1/\alpha} \{\frac{B(\alpha)}{(1+\varepsilon)\log n}\}^{(1-\alpha)/\alpha}\}, \qquad n = 2,3,4,\ldots$$

where

$$B(\alpha) := (1-\alpha)^{\alpha/(1-\alpha)} \cdot (\cos \frac{\pi\alpha}{2})^{-1/(1-\alpha)} \in (0,\infty),$$

and ε any real number in $(-1,\infty)$, it follows that the events D_n, $n = 2,3,4,\ldots$ are independent. Furthermore, using the parts IV and VII of Theorem 2.1.7 in MIJNHEER [1975], we see that, with V a random variable having a standard normal distribution,

$$(7.3.23) \qquad P(D_n) = P\left(\frac{Z_{n+1}}{Z_n^{1/\alpha}} \leq \{\frac{B(\alpha)}{(1+\varepsilon)\log n}\}^{(1-\alpha)/\alpha}\right)$$

$$= P(W(1) \leq \{\frac{B(\alpha)}{(1+\varepsilon)\log n}\}^{(1-\alpha)/\alpha})$$

$$\sim (\frac{2}{\alpha})^{\frac{1}{2}} P(V \geq \{2(1+\varepsilon)\log n\}^{\frac{1}{2}})$$

$$\sim (\frac{1}{\alpha\pi})^{\frac{1}{2}}\{2(1+\varepsilon)\log n\}^{-\frac{1}{2}} e^{-(1+\varepsilon)\log n}$$

$$= \{2\pi\alpha(1+\varepsilon)\}^{-\frac{1}{2}}(\log n)^{-\frac{1}{2}} \cdot n^{-1-\varepsilon} \qquad \text{as } n \to \infty.$$

Hence it follows that

$$\sum_{n=2}^{\infty} P(D_n) \begin{cases} < \infty & \text{if } 0 < \varepsilon < \infty \\ = \infty & \text{if } -1 < \varepsilon \leq 0 \end{cases}.$$

Since the events D_n, $n = 2,3,4,\ldots$ are independent, we can apply the zero-one criterion of Borel to obtain

$$P(\limsup D_n) = \begin{cases} 0 & \text{if } 0 < \varepsilon < \infty \\ 1 & \text{if } -1 < \varepsilon \leq 0 \end{cases},$$

and therefore

$$(7.3.24) \qquad \liminf_{n \to \infty} \frac{Z_{n+1}(\log n)^{(1-\alpha)/\alpha}}{Z_n^{1/\alpha}} = B(\alpha)^{(1-\alpha)/\alpha} \qquad \text{a.s.}.$$

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Now we know from Example 7.3.7 that $\alpha^n \log(1+Z_n) \xrightarrow{\text{a.s.}} U$ as $n \to \infty$, and that $U \in (0,\infty)$ a.s. on $\{Z_n \to \infty\}$. This implies that also $\alpha^n \log Z_n \xrightarrow{\text{a.s.}} U$, and hence $\alpha^n \sim U/\log Z_n$ a.s. as $n \to \infty$ on $\{Z_n \to \infty\}$. Since $\lim_{n \to \infty} \alpha^n = 0$, this means that

$$(7.3.25) \qquad n \sim \frac{\log \log Z_n - \log U}{-\log \alpha} \sim \frac{\log \log Z_n}{-\log \alpha} \quad \text{a.s. as } n \to \infty \text{ on } \{Z_n \to \infty\}.$$

From this we can conclude that

$$(7.3.26) \qquad \log n \sim \log \log \log Z_n \quad \text{a.s.} \qquad \text{as } n \to \infty \text{ on } \{Z_n \to \infty\}.$$

Combining this with (7.3.24) we obtain

$$(7.3.27) \qquad \liminf_{n \to \infty} \frac{Z_{n+1} (\log \log \log Z_n)^{\frac{1}{\alpha} - 1}}{Z_n^{1/\alpha}} = B(\alpha)^{\frac{1}{\alpha} - 1} \quad \text{a.s.} \quad \text{on } \{Z_n \to \infty\}.$$

Turning back to $Z_{n+1}/f_\gamma(Z_n)$, we see that, since $c(s) = s^{1/\alpha}$,

$$(7.3.28) \qquad \frac{Z_{n+1}}{f_\gamma(Z_n)} = \frac{Z_{n+1} \, c\!\left(\dfrac{\gamma \log \log Z_n}{Z_n} \right)}{\log \log Z_n} =$$

$$= \frac{\gamma^{1/\alpha} Z_{n+1} (\log \log Z_n)^{(1-\alpha)/\alpha}}{Z_n^{1/\alpha}}$$

$$= \frac{\gamma^{\frac{1}{\alpha}} Z_{n+1} (\log \log \log Z_n)^{\frac{1}{\alpha} - 1}}{Z_n^{1/\alpha}} \cdot \left\{ \frac{\log \log Z_n}{\log \log \log Z_n} \right\}^{\frac{1}{\alpha} - 1} \to \infty$$

a.s. as $n \to \infty$ on $\{Z_n \to \infty\}$. This is one part of the result announced in Section 3.3, where we promised to give an example of a subordinator $\{W(t); \ t \in [0,\infty)\}$ with corresponding branching process $\{Z_n; \ n = 0,1,2,\ldots\}$ for which $\lim_{n \to \infty} Z_{n+1}/f_\gamma(Z_n) = \infty$ a.s. and for which (3.3.16) is satisfied, that is $\liminf W(t)/f_\gamma(t) = d \in (0,\infty)$ a.s., where the lim inf may be taken both for small and for large t, as can be found in FRISTEDT [1964]. The second part, to wit $\lim_{n \to \infty} Z_{n+1}/f_\gamma(Z_n) = \infty$ a.s. on $\{Z_n \to 0\}$, will be proved

in Chapter 9.

At first sight the difference between (7.3.28) and (3.3.16) might seem strange. However, a closer look at the method used above yields that we proved in fact $\lim_{n\to\infty} W_n(Z_n)/f_\gamma(Z_n) = \infty$ a.s. on $\{Z_n \to \infty\}$, where $\{\{W_n(t); t \in [0,\infty)\}; n = 1,2,3,\ldots\}$ is a sequence of independent subordinators all distributed as $\{W(t); t \in [0,\infty)\}$, whereas we observed in (3.3.16) only one subordinator. This does indeed give rise to an important difference, because the independence of the subordinators $\{W_n(t); t \in [0,\infty)\}$, $n = 1,2,3,\ldots$ enables us to prove, in a similar way as we proved (7.3.24),

$$(7.3.29) \qquad \liminf_{n \to \infty} \frac{W_n(t_n)}{t_n^{1/\alpha}} (\log n)^{(1-\alpha)/\alpha} = B(\alpha)^{(1-\alpha)/\alpha} \qquad a.s.$$

for any sequence $\{t_n; n = 1,2,3,\ldots\}$ of positive and finite constants, a result which is not true if we replace $W_n(t)$ by $W(t)$, even if $\lim_{n\to\infty} t_n = \infty$. For if we take $t_n = n$, then

$$\liminf_{n \to \infty} \frac{W(t_n)}{t_n^{1/\alpha}} (\log n)^{(1-\alpha)/\alpha}$$

$$= \liminf_{n \to \infty} \frac{W(t_n)}{f_\gamma(t_n)} \left(\frac{\log n}{\log \log n}\right)^{(1-\alpha)/\alpha} \cdot \gamma^{-1/\alpha} = \infty \qquad a.s.,$$

since

$$\liminf_{n \to \infty} \frac{W(t_n)}{f_\gamma(t_n)} \geq d > 0 \quad a.s..$$

It follows from (7.3.29) that if we consider a sequence $\{t_n; n = 1,2,3,\ldots\}$ for which there exists a function f such that $f(t_n) \sim \log n$ as $n \to \infty$, then

$$\liminf_{n \to \infty} \frac{W_n(t_n)}{t_n^{1/\alpha}} f(t_n)^{(1-\alpha)/\alpha} = B(\alpha)^{(1-\alpha)/\alpha} \qquad a.s..$$

In view of (7.3.26) this function f, in the example of a branching process with a strictly stable distribution with characteristic exponent $\alpha \in (0,1)$ as its offspring distribution, is given by $f(t) = \log \log \log t$.

We can use a similar method to obtain a result involving a lim sup. We shall see that this result is slightly different from (7.3.27) in the sense that the lim sup is a.s. 0 or ∞. This however is a well-known phenomenon in the theory of stable processes. Introducing the events E_n by

$$E_n = \{Z_{n+1} \geq d \ Z_n^{1/\alpha} f(n)^{1/\alpha}\}, \qquad n = 0,1,2,\ldots,$$

where d is any constant $\in (0,\infty)$ and f any non-negative function on $[0,\infty)$ such that $\lim_{n\to\infty} f(n) = \infty$ and

$$(7.3.30) \qquad f(a_n) \sim f(n) \qquad \text{as } n \to \infty$$

for any sequence of constants $\{a_n; n = 0,1,2,\ldots\}$ such that $a_n \sim n$ as $n \to \infty$, it follows that the events E_n, $n = 0,1,2,\ldots$ are independent, since the random variables $Z_{n+1}/Z_n^{1/\alpha}$, $n = 0,1,2,\ldots$ are independent.

Furthermore, part I of Theorem 2.1.7 in MIJNHEER [1975] yields

$$P(E_n) = P(W(1) \geq df(n)^{1/\alpha}) \sim \frac{A_1}{\pi\alpha} \frac{d^{-\alpha}}{f(n)} \qquad \text{as } n \to \infty,$$

where

$$A_1 := \Gamma(\alpha+1) \cdot (1 + \tan^2 (\frac{\pi\alpha}{2}))^{\frac{1}{2}} \cdot \sin \pi\alpha \in (0,\infty).$$

Hence we obtain

$$\sum_{n=0}^{\infty} P(E_n) \begin{cases} < \infty & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} < \infty \\ = \infty & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} = \infty \end{cases}.$$

The independence of the events E_n, $n = 0,1,2,\ldots$ allows again an application of the zero-one criterion of Borel, whence

$$P(\limsup E_n) = \begin{cases} 0 & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} < \infty \\ 1 & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} = \infty \end{cases}.$$

Since $d \in (0,\infty)$ was arbitrary, this implies that

$$\limsup_{n \to \infty} \frac{Z_{n+1}}{Z_n^{1/\alpha} f(n)^{1/\alpha}} = \begin{cases} 0 & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} < \infty \\ \infty & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} = \infty \end{cases} \quad \text{a.s.}.$$

In view of (7.3.25) and (7.3.30) we can conclude from this that

$$(7.3.31) \quad \limsup_{n \to \infty} \frac{Z_{n+1}}{Z_n^{1/\alpha} f\left(\frac{\log \log Z_n}{-\log \alpha}\right)^{1/\alpha}} = \begin{cases} 0 & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} < \infty \\ \infty & \text{if } \Sigma_{n=0}^{\infty} \frac{1}{f(n)} = \infty \end{cases}$$

$$\text{a.s. on } \{Z_n \to \infty\}.$$

CHAPTER 8

THE CASE a>0

8.1. INTRODUCTION

Now that we have studied branching processes for various values of m, the only thing that still rests is the behaviour of the process on $\{Z_n \to 0\}$ if $P(Z_1 = 0) = 0$. Of course, since we consider the process on $\{Z_n \to 0\}$, we assume that $P(Z_n \to 0) > 0$, implying that $-\log r < \infty$ and $a < 1$. Because the offspring distribution of a Galton-Watson process is concentrated on the non-negative integers, it follows that for such processes $P(Z_n \to 0) > 0$ implies that $P(Z_1 = 0) > 0$, and hence that the case we consider here does not occur in the theory of Galton-Watson processes.

In Chapters 6 and 7 we made a repeated use of the constants $c_n(s)$ for $s \in (0, -\log r)$. As we shall see, we can now use similar techniques with the help of the constants $c_n(s)$ for $s \in (-\log r, \infty)$. We know from Chapters 6 and 7 that there is a basic difference between supercritical and explosive branching processes on $\{Z_n \to \infty\}$, essentially caused by the fact that for supercritical processes $\lim_{n \to \infty} c_{n+1}(s)/c_n(s) = \frac{1}{m} > 0$, whereas $\lim_{n \to \infty} c_{n+1}(s)/c_n(s) = 0$ for explosive processes, with $s \in (0, -\log r)$. A similar difference will be shown to exist here. More precisely, we have to distinguish between the cases where, for $s \in (-\log r, \infty)$, $\lim_{n \to \infty} c_n(s)/c_{n+1}(s) > 0$ and where $\lim_{n \to \infty} c_n(s)/c_{n+1}(s) = 0$. In view of Lemma 2.2.5(c), these cases are $a > 0$ and $a = 0$, and we shall see that there is an intriguing parallel between the process $\{Z_n ; n = 0,1,2,\ldots\}$ on $\{Z_n \to 0\}$ if $0 < a < 1$ and the process $\{Z_n ; n = 0,1,2,\ldots\}$ on $\{Z_n \to \infty\}$ if $m < \infty$, and also between the process $\{Z_n ; n = 0,1,2,\ldots\}$ on $\{Z_n \to 0\}$ if $a = 0$ and the process $\{Z_n ; n = 0,1,2,\ldots\}$ on $\{Z_n \to \infty\}$ if $m = \infty$. Throughout this chapter we suppose that $0 < a < 1$, and hence that $-\log r < \infty$, and furthermore that $s \in (-\log r, \infty)$, unless stated otherwise.

8.2. MAIN RESULTS

As already mentioned in Section 8.1, we shall see that there is a close correspondence between this section and Section 6.3 where supercritical processes were studied on $\{Z_n \to \infty\}$, and it turns out that we can proceed along the same lines as we did there. First of all we consider the random variable $Y(s)$, defined in Corollary 3.3.5 by $Y(s) = \lim_{n \to \infty} c_n(s) Z_n$, and derive again a functional equation for its cumulant generating function. Notice that $P(Z_1 = 0) = 0$ since $a > 0$, and hence that $Y(s)$ is well-defined.

THEOREM 8.2.1. *For all z with* $\mathrm{Re}\, z \geq 0$

$$(8.2.1) \qquad \phi(az, s) = h(\phi(z, s)).$$

PROOF. Since $c(s) > s$ for all $s \in (-\log r, \infty)$, it follows that $\lim_{n \to \infty} c_n(s) = \infty$, and hence, using Lemma 2.2.5(c) we see that

$$\lim_{n \to \infty} \frac{c_n(s)}{c_{n-1}(s)} = \lim_{n \to \infty} \frac{c_n(s)}{h(c_n(s))} = \lim_{s \to \infty} \frac{s}{h(s)} = \frac{1}{a}.$$

The rest of the proof is now analogous to the proof of Theorem 6.3.1. $\quad\square$

Obviously $Y(s) = \infty$ on $\{Z_n \to \infty\}$, since $\lim_{n \to \infty} c_n(s) = \infty$. Using (8.2.1) we shall now prove that $Y(s) \in (0, \infty)$ a.s. on $\{Z_n \to 0\}$.

THEOREM 8.2.2.
(a) $P(Y(s) = 0) = 0$;
(b) $P(Y(s) < \infty) = r$.

PROOF. In a similar way as in the proof of Theorem 6.3.2 it follows that both $-\log P(Y(s) = 0)$ and $-\log P(Y(s) < \infty)$ are solutions of the equation $h(t) = t$, where $t \in [0, \infty]$, and that

$$(8.2.2) \qquad -\log P(Y(s) = 0) > 0 \quad \text{and} \quad -\log P(Y(s) < \infty) < \infty.$$

Furthermore, since $\lim_{n \to \infty} c_n(s) = \infty$,

$$(8.2.3) \qquad -\log P(Y(s) < \infty) = -\log P(\lim_{n \to \infty} c_n(s) Z_n < \infty)$$

$$\geq -\log P(\lim_{n \to \infty} Z_n = 0) = -\log r,$$

and

(8.2.4) $-\log P(Y(s) = 0) \geq -\log P(Y(s) < \infty)$.

Since the equation $h(t) = t$ has solutions $t = 0$, $t = -\log r$ and $t = \infty$, we obtain from (8.2.2) and (8.2.3) that $-\log P(Y(s) < \infty) = -\log r$, that is $P(Y(s) < \infty) = r$. Combining this with (8.2.4) it follows that $P(Y(s) = 0) = 0$ or r. But if $P(Y(s) = 0) = r$, then we would have $Ee^{-Y(s)} = r$, in contradiction with the fact that $Ee^{-Y(s)} = e^{-s} < r$, as follows from Corollary 3.3.5, and so $P(Y(s) = 0) = 0$.   □

In Section 6.3 relation (6.3.1) was further exploited to prove the absolute continuity of the distribution of $Y(s)$ for $s \in (0,\infty)$ if $r = 0$. This was established by observing that any sufficiently large power of the absolute value of the characteristic function $\psi(t,s)$ of $Y(s)$ is integrable. The reason that we could get a hold on $|\psi(t,s)|$ for large values of $|t|$ was in fact that we could write

$$|\psi(m^n t, s)| = |\exp\{-\phi(-itm^n, s)\}| = |\exp\{-h_n(\phi(-it,s))\}|,$$

and hence that we did indeed obtain information about large values of $|t|$, since $m > 1$. However, comparison of (6.3.1) with (8.2.1) shows that the role of m is now played by a, and since $a < 1$, we can not apply the same method as we did in Section 6.3. We can only prove the analog of Lemma 6.3.5.

LEMMA 8.2.3. *Suppose that a is an irrational number. Then $|\psi(t,s)| < r$ for all real $t \neq 0$.*

PROOF. Since by Theorem 8.2.2(b), $P(Y(s) < \infty) = r$, it follows that

$$|\psi(t,s)| = |Ee^{itY(s)} I_{\{Y(s) < \infty\}}| \leq P(Y(s) < \infty) = r$$

for all real t, where I stands for the indicator function. Now suppose $|\psi(t,s)| = r$ for some $t \neq 0$. Then (8.2.1) implies

$$\exp\{-h(\mathrm{Re}\phi(-it/a,s))\} = E|\exp\{-\phi(-it/a,s)\}|^{Z_1}$$

$$\geq |E\,\exp\{-\phi(-it/a,s)\}^{Z_1}| = \exp\{-\mathrm{Re}h(\phi(-it/a,s))\}$$

$$= \exp\{-\mathrm{Re}\phi(-it,s)\} = |\psi(t,s)| = r,$$

and so

$$h(\mathrm{Re}\phi(-it/a,s)) \leq -\log r,$$

whence also

(8.2.5) $\mathrm{Re}\ \phi(-it/a,s) \leq -\log r.$

On the other hand, because

$$|\psi(t,s)| = e^{-\mathrm{Re}\phi(-it,s)} \leq r \qquad \text{for all } t,$$

(8.2.6) $\mathrm{Re}\phi(-it/a,s) \geq -\log r,$

and a combination of (8.2.5) and (8.2.6) therefore yields, that $\mathrm{Re}\phi(-it/a,s) = -\log r$. Iterating this we get $\mathrm{Re}\phi(-it/a^n,s) = -\log r$ for all $n = 0,1,2,\ldots$. This means that the distribution of $Y(s)$ is concentrated on

$$\{\bigcap_{n=0}^{\infty} \bigcup_{k=-\infty}^{\infty} \{\frac{2k\pi}{t} a^n + d_n\}\} \cup \{\infty\},$$

where d_n are constants $\in (0,\infty)$. Now, since a is irrational, a standard argument yields $P(Y(s) = d(s)) = r$ for some constant $d(s)$, and by Theorem 8.2.2, $d(s) \in (0,\infty)$. This implies that $\phi(u,s) = -\log r + d(s)\cdot u$ for all $u \in [0,\infty)$. Introducing the function $\tilde{h}(x)$ by $\tilde{h}(x) = h(x-\log r) + \log r$, $x \in [0,\infty)$, it follows now from (8.2.1) that

$$\tilde{h}(d(s)u) = h(d(s)u - \log r) + \log r = h(\phi(u,s)) + \log r$$

$$= \phi(au,s) + \log r = ad(s)u \qquad \text{for all } u \in [0,\infty).$$

This however violates the assumption that the distribution of Z_1 is not concentrated in one point. So $|\psi(t,s)| < r$ for all real $t \neq 0$. $\qquad \square$

In Section 6.3 we saw that always $\ell(s) = 0$ for $s \in (0,-\log r)$, where $\ell(s)$ is the first point of increase of the distribution function of $Y(s)$. We shall prove below that this property does not hold in general if $s \in (\log r, \infty)$. Furthermore we shall see, that if $P(Z_1 = a) > 0$, then also $P(Y(s) = \ell(s)) > 0$.

THEOREM 8.2.4.

(a) $\ell(s) = 0$ *if and only if* $\int_{\varepsilon}^{\infty} \frac{h(s) - as}{s^2}\, ds = \infty$ *for any* $\varepsilon > 0$;

(b) $P(Y(s) = \ell(s)) = P(Z_1 = a)^{1/(1-a)}$.

PROOF.

(a) Iterating (8.2.1) it follows that for all $n = 1,2,3,\ldots$, $\phi(1,s) = h_n(\phi(a^{-n},s))$, and hence $c_n(\phi(1,s)) = \phi(a^{-n},s)$. This implies that $c_n(s) = \phi(a^{-n},s)$ for all $n = 1,2,3,\ldots$, since $\phi(1,s) = s$ by (3.3.5). Now using the fact that $a < 1$, Lemma 2.2.5(c) yields $\ell(s) = \lim_{n\to\infty} a^n \phi(a^{-n},s)$, and hence

$$(8.2.7) \qquad \ell(s) = \lim_{n\to\infty} a^n c_n(s).$$

Since

$$a^n c_n(s) = s \cdot \prod_{k=1}^{n} \frac{a c_k(s)}{c_{k-1}(s)} = s \prod_{k=1}^{n} \frac{a c_k(s)}{h(c_k(s))},$$

we see that

$$(8.2.8) \qquad \ell(s) = 0 \text{ if and only if } \prod_{k=1}^{\infty} \frac{h(c_k(s))}{a c_k(s)} = \infty, \text{ that is}$$

$$\text{if and only if } \sum_{k=1}^{\infty} \left\{ \frac{h(c_k(s))}{a c_k(s)} - 1 \right\} = \infty.$$

Now introducing $S(\delta)$ by

$$S(\delta) = \sum_{k=1}^{\infty} \left\{ \frac{h(\delta^k)}{a\delta^k} - 1 \right\},$$

where $\delta > 1$ is any constant, it follows that $S(\delta) = \infty$ if and only if

$$\sum_{k=1}^{\infty} \left\{ \frac{h(e^{\alpha k})}{e^{\alpha k}} - a \right\} = \infty,$$

where $\alpha := \log \delta > 0$. This implies that $S(\delta)$ is infinite together with

$$\int_{\varepsilon}^{\infty} \left\{ \frac{h(e^{\alpha x})}{e^{\alpha x}} - a \right\} dx,$$

or equivalently, together with

$$\int_{\varepsilon}^{\infty} \left\{ \frac{h(s)}{s} - a \right\} \frac{ds}{s} .$$

So we see that if $S(\delta) = \infty$ for some $\delta > 1$, then $S(\delta) = \infty$ for all $\delta > 1$, and also that this is equivalent with $\int_{\varepsilon}^{\infty} (h(s)-as)/s^2 \, ds = \infty$ for any $\varepsilon > 0$. In view of (8.2.8), part (a) is now proved once we have shown that

(8.2.9) $\qquad \delta_1^k \le c_k(s) \le \delta_2^k$ for some constants δ_1 and $\delta_2 \in (1,\infty)$, and all
$$\text{sufficiently large } k.$$

But this follows on observing that the function c, and therefore also c_k, is convex for every positive integer k, and so

$$\frac{s}{\{h'(-\log r)\}^k} = c_k'(-\log r) \cdot s \le c_k(s-\log r) + \log r$$

$$\le s \cdot \lim_{t \to \infty} c_k'(t) = \frac{s}{a^k} , \qquad s \in (0,\infty).$$

Since both $h'(-\log r)$ and $a \in (0,1)$, this implies (8.2.9).

(b) First of all we shall prove that the sets $\{Y(s) = \ell(s)\}$ and $\{Z_n = a^n$ for all $n = 0,1,2,\ldots\}$ differ only by a set of probability zero. For suppose that $Z_n = a^n$ for all $n = 0,1,2,\ldots$. Then it follows from (8.2.7) that $Y(s) = \lim_{n \to \infty} c_n(s) Z_n = \lim_{n \to \infty} a^n c_n(s) = \ell(s)$. On the other hand, the branching property (3.1.3) and Lemma 2.2.5(c) imply that $P(Z_{n+1} \ge a Z_n) = 1$ for every $n = 0,1,2,\ldots$, and therefore also $P(A) = 1$, where $A := \{Z_{n+1} \ge a Z_n$ for all $n = 0,1,2,\ldots\}$. Now suppose that $\omega \in A$ and $\omega \notin \{Z_n = a^n$ for all $n = 0,1,2,\ldots\}$. Then there is a positive integer n such that $Z_n(\omega) = a^n + \varepsilon(\omega)$, with $\varepsilon(\omega) > 0$. But this implies that $Z_{n+k}(\omega) \ge a^k Z_n(\omega) = a^k(a^n + \varepsilon(\omega))$ for every positive integer k, and hence

$$Y(s,\omega) = \lim_{k\to\infty} c_{n+k}(s) Z_{n+k}(\omega) \geq \lim_{k\to\infty} c_{n+k}(s) a^{n+k}(1 + \epsilon(\omega)/a^n)$$

$$= \ell(s)(1 + \epsilon(\omega)/a^n) > \ell(s).$$

This proves that $P(Y(s) = \ell(s)) = P(Z_n = a^n$ for all $n = 0,1,2,\ldots)$, and from Remark 3.3.9 we know that this is equal to $P(Z_1 = a)^{1/(1-a)}$. $\square$

In Section 4.2 we met with an integral analogous to

$$\int_{\epsilon}^{\infty} \frac{h(s) - as}{s^2} \, ds,$$

namely

$$\int_0^{\epsilon} \frac{ms - h(s)}{s^2} \, ds.$$

The finiteness of this last one could be proved to be equivalent to $EZ_1 \log Z_1 < \infty$. This is essentially done by writing both m and h(s) as integrals with respect to the distribution function of Z_1 and then applying Fubini's theorem. However, this technique cannot be used here, since the number a is, in contrast with m, not an expectation. The condition in Theorem 8.2.4(a) is therefore stated in terms of h(s) and not directly in terms of Z_1.

Appealing again to the paper of SENETA [1974] it is clear that we can prove from (8.2.1)

THEOREM 8.2.5. $\phi(u,s)$ *is regularly varying at ∞ with exponent 1 as a function of* u.

Our next aim is again the comparison of the random variables $Y(s)$ for different values of $s \in (-\log r, \infty)$. First of all we notice, that by the convexity of $c(s)$, the quotient $c_n(s)/c_n(t)$ is non-increasing in n for $-\log r < s \leq t < \infty$, and hence converges as $n \to \infty$. Call its limit $v(s,t)$. Then obviously

$$v(s,t) := \lim_{n\to\infty} \frac{c_n(s)}{c_n(t)} = \frac{1}{v(t,s)} \qquad \text{for } -\log r < t < s < \infty.$$

Furthermore, $v(s,t) \in (0,\infty)$ for all $s,t \in (-\log r, \infty)$. For if $s \leq t$, then there exists a non-negative integer k such that $s \geq h_k(t)$. This means that

$$1 \geq v(s,t) = \lim_{n \to \infty} \frac{c_n(s)}{c_n(t)} \geq \lim_{n \to \infty} \frac{c_n(h_k(t))}{c_n(t)} = \lim_{n \to \infty} \frac{h_k(c_n(t))}{c_n(t)}$$

$$= \lim_{u \to \infty} \frac{h_k(u)}{u} = a^k > 0.$$

Similarly we can prove that $v(s,t) \in [1,\infty)$ if $s \geq t$.

THEOREM 8.2.6. *Suppose that both* s *and* t $\in (-\log r, \infty)$. *Then*

(a) $Y(s) = v(s,t)Y(t)$ a.s.;

(b) $\phi(v(s,t),t) = s$.

PROOF. The proof is analogous to the proof of Theorem 6.3.9. □

In Section 6.3 we determined the class of possible limit distributions for $Y(s)$, $s \in (0, -\log r)$. The basic tool was the functional equation (6.3.1). It is therefore plausible that here we can use a similar method with (8.2.1) as the starting point. From Theorem 8.2.2 we know that $P(Y = 0) = 0$ and $P(Y < \infty) = r$, implying that $\lim_{s \downarrow 0} \phi(s) = -\log P(Y < \infty) = -\log r$ and $\lim_{s \to \infty} \phi(s) = -\log P(Y = 0) = \infty$, where ϕ is the cumulant generating function of any random variable Y, with distribution in the class of possible limit distributions. This leads to the introduction of the following collection $F_{a,r}$ for any $a \in (0,1)$ and $r \in (0,1]$.

We say that the cumulant generating function ϕ of a positive random variable Y belongs to $F_{a,r}$ if and only if:

(1) $\lim_{s \downarrow 0} \phi(s) = -\log r$;

(2) ϕ is non-linear;

(3) $\phi^*(s) := \phi(a\phi^{inv}(s))$, $s \in (-\log r, \infty)$, can be continued analytically along the positive real line;

(4) $e^{-t\phi^*(s)}$ is for every $t > 0$ completely monotone as a function of s, where $\phi^*(s)$, $s \in (0, -\log r]$ is defined as the analytic continuation of $\phi^*(s)$, $s \in (-\log r, \infty)$;

(5) $\lim_{s \downarrow 0} \phi^*(s) = 0$;

(6) $\phi(u)$ is regularly varying at ∞ with exponent 1.

This class $F_{a,r}$ turns out to be the class of cumulant generating functions of possible limits of $c_n(s)Z_n$. Before proving this we introduce the collections $H_{a,r}$ and $G_{a,r}$ for $a \in (0,1)$ and $r \in (0,1]$ as follows.

A cumulant generating function h of a proper, non-degenerate, non-negative random variable having an infinitely divisible distribution belongs to $H_{a,r}$ if and only if:

(1) $\lim_{s\to\infty} \frac{h(s)}{s} = a$;

(2) $\lim_{n\to\infty} h_n(s) = -\log r$.

A cumulant generating function ϕ belongs to $G_{a,r}$ if and only if there is a branching process $\{Z_n; \ n = 0,1,2,\ldots\}$ with state space $[0,\infty)$ such that $Z_0 = 1$ and $h(s) := -\log E \, e^{-sZ_1} \in H_{a,r}$ and $\lim_{n\to\infty} c_n(s_0)Z_n$ has cumulant generating function ϕ, where $c_n(s)$ is the nth iterate of $h^{inv}(s)$ and $s_0 \in (-\log r, \infty)$.

Notice that $c_n(s)$ is well-defined for all $s \in (-\log r, \infty)$, since $a > 0$.

THEOREM 8.2.7. $F_{a,r} = G_{a,r}$.

PROOF. The proof follows the same lines as that of Theorem 6.3.10. First we show that $G_{a,r} \subset F_{a,r}$, and then that $F_{a,r} \subset G_{a,r}$.

(a) Suppose that $\phi \in G_{a,r}$. Then we know from Theorem 8.2.2 that ϕ is the cumulant generating function of a positive random variable Y and that, in view of Lemma 2.2.4(a), $\lim_{s\downarrow 0} \phi(s) = -\log r$. This implies that if ϕ is linear, then $\phi(s) = -\log r + ds$ for some constant $d > 0$ and all $s \in [0,\infty)$. But then we should obtain $h(-\log r + ds) = -\log r + ads$, since ϕ satisfies (8.2.1), and hence that h is linear. This however violates the assumption that Z_1 is non-degenerate, and therefore ϕ is non-linear. Now, again using (8.2.1), we see that $\phi^*(s) = \phi(a\phi^{inv}(s)) = h(s)$ for $s \in (-\log r, \infty)$, and so $\phi^*(s)$ can be continued analytically along the positive real line and $\phi^*(s) = h(s)$ for all $s \in (0,\infty)$. This means that $e^{-t\phi^*(s)}$ is completely monotone for every $t > 0$ as a function of s, and that $\lim_{s\downarrow 0} \phi^*(s) = \lim_{s\downarrow 0} h(s) = 0$, since $P(Z_1 < \infty) = 1$. Finally, Theorem 8.2.5 yields that $\phi(u)$ is regularly varying at ∞ with exponent 1. This proves that $G_{a,r} \subset F_{a,r}$.

(b) Now suppose that $\phi \in F_{a,r}$. Define h(s) by $h(s) = \phi^*(s)$, $s \in (0,\infty)$. Then it follows from (4) and (5) in the definition of $F_{a,r}$ that h is the cumulant generating function of some proper, non-negative random variable X, having an infinitely divisible distribution. Furthermore, if $P(X = d) = 1$ for some constant $d \in [0,\infty)$, then $h(s) = ds$ for all $s \in [0,\infty)$, and hence $\phi(a\phi^{inv}(s)) = \phi^*(s) = ds$ for all $s \in (-\log r, \infty)$. This means that

(8.2.10) $d\phi(t) = \phi(at)$ for all $t \in [0,\infty)$.

But from (6) in the definition of $F_{a,r}$ we know that $\phi(u) = uL(u)$, with $L(u)$ slowly varying at ∞. Substituting this into (8.2.10) we obtain $dtL(t) = atL(at) \sim atL(t)$ as $t \to \infty$, and hence $a = d$. However this implies, again by (8.2.10), that ϕ is linear, in contradiction with requirement 2. So X is non-degenerate. Next we notice that

$$\lim_{s \to \infty} \frac{h(s)}{s} = \lim_{s \to \infty} \frac{\phi^*(s)}{s} = \lim_{s \to \infty} \frac{\phi(a\phi^{inv}(s))}{\phi(\phi^{inv}(s))}$$

$$= \lim_{s \to \infty} \frac{a\phi^{inv}(s)L(a\phi^{inv}(s))}{\phi^{inv}(s)L(\phi^{inv}(s))} = a.$$

Let $\{Z_n;\ n = 0,1,2,\ldots\}$ be a branching process having the distribution of X as its offspring distribution, and such that $Z_0 = 1$. Then by Theorem 3.3.1 and the fact that $a < 1$,

(8.2.11) $\lim_{n \to \infty} h_n(s) = -\log P(\lim_{n \to \infty} Z_n = 0) < \infty$.

Now choose some $s_0 \in (-\log P(\lim_{n \to \infty} Z_n = 0),\infty)$. Then by Corollary 3.3.5, $\lim_{n \to \infty} c_n(s_0)Z_n$ exists a.s., since $a > 0$, with $c_n(s)$ the nth iterate of $h^{inv}(s)$. Defining $\tilde{\phi}(u,s_0)$ by $\tilde{\phi}(u,s_0) = -\log E \exp\{-u \cdot \lim_{n \to \infty} c_n(s_0)Z_n\}$, $u \in [0,\infty)$, it follows from (8.2.1) that $h(\tilde{\phi}(u,s_0)) = \tilde{\phi}(au,s_0)$. Since by Theorem 8.2.2 $P(\lim_{n \to \infty} c_n(s_0)Z_n = 0) = 0$, Lemma 2.2.5(a) implies that $\lim_{u \to \infty} \tilde{\phi}(u,s_0) = \infty$. So because $r > 0$, there exists a $U \in (0,\infty)$ such that $\tilde{\phi}(u,s_0) > -\log r$ if $u \in (U,\infty)$. Hence we obtain that

$$\tilde{\phi}(au,s_0) = h(\tilde{\phi}(u,s_0)) = \phi^*(\tilde{\phi}(u,s_0)) = \phi(a\phi^{inv}(\tilde{\phi}(u,s_0)))$$

for all $u \in (U,\infty)$,

and so $\phi^{inv}(\tilde{\phi}(au,s_0)) = a\phi^{inv}(\tilde{\phi}(u,s_0))$ for all $u \in (U/a,\infty)$. Therefore $\phi^{inv}(\tilde{\phi}(u,s_0)) = bu$, that is $\phi(bu) = \tilde{\phi}(u,s_0)$ for some $b \in (0,\infty)$ and for all $u \in (U/a,\infty)$, and hence for all $u \in [0,\infty)$ since both ϕ and $\tilde{\phi}$ are cumulant generating functions. This means that

(8.2.12) $\lim_{n \to \infty} c_n(s_0)Z_n \overset{d}{=} bY,$

where Y is a random variable with cumulant generating function ϕ, and so $r = P(Y < \infty) = P(\lim_{n\to\infty} c_n(s_0)Z_n < \infty)$, implying that also $P(\lim_{n\to\infty} Z_n = 0) = r$ by Theorem 8.2.2(b). Combining this with (8.2.11) we see that requirement 2 in the definition of $H_{a,r}$ is fulfilled and therefore $h \in H_{a,r}$. Finally defining s by $s = \tilde{\phi}(1/b, s_0)$, it follows from Theorem 8.2.2 that $s \in (-\log r, \infty)$. Writing $\tilde{\phi}^{inv}(s, s_0)$ for the inverse of $\tilde{\phi}(s, s_0)$ as a function of s an application of Theorem 8.2.6 now yields

$$\lim_{n\to\infty} c_n(s)Z_n = \tilde{\phi}^{inv}(s, s_0) \cdot \lim_{n\to\infty} c_n(s_0)Z_n$$

$$= \tilde{\phi}^{inv}(\tilde{\phi}(1/b, s_0), s_0) \cdot \lim_{n\to\infty} c_n(s_0)Z_n$$

$$= 1/b \lim_{n\to\infty} c_n(s_0)Z_n \quad \text{a.s.},$$

and so by (8.2.12), $Y \overset{d}{=} \lim_{n\to\infty} c_n(s)Z_n$. This proves that $\phi \in G_{a,r}$, and hence the proof is complete. $\square$

Next we make some remarks about the total progeny of the process. This was defined in Section 3.3 by $S = \Sigma_{k=0}^{\infty} Z_k$. We saw there, that $S < \infty$ a.s. on $\{Z_n \to 0\}$, that is S_n converges as $n \to \infty$ to a finite limit for almost all $\omega \in \{Z_n \to 0\}$. It turns out that the $c_n(s)$ are useful norming constants for the difference $S - S_n$. A rather careless reasoning yields

$$c_{n+1}(s)(S - S_n) = \sum_{k=1}^{\infty} c_{n+1}(s)Z_{n+k}$$

$$= \sum_{k=1}^{\infty} \frac{c_{n+1}(s)}{c_{n+k}(s)} c_{n+k}(s)Z_{n+k} \xrightarrow{\text{a.s.}} \sum_{k=1}^{\infty} a^{k-1}Y(s) = \frac{Y(s)}{1-a} \quad ,$$

since

$$\lim_{n\to\infty} \frac{c_{n+1}(s)}{c_{n+k}(s)} = \lim_{n\to\infty} \frac{h_{k-1}(c_{n+k}(s))}{c_{n+k}(s)} = \lim_{s\to\infty} \frac{h_{k-1}(s)}{s} = a^{k-1}.$$

The following theorem shows that we may indeed interchange sum and limit as we did above.

THEOREM 8.2.8.

$$\lim_{n\to\infty} c_{n+1}(s) \cdot (S - S_n) = \frac{Y(s)}{1-a} \quad \text{a.s.} \qquad \text{on } \{Z_n \to 0\}.$$

PROOF. We shall prove, using dominated convergence, that the interchanging of sum and limit is allowed. To this end we have to show that there exists for every $k = 1,2,3,\ldots$ a random variable A_k such that

$$(8.2.13) \qquad c_{n+1}(s)Z_{n+k} \leq A_k \quad \text{a.s. on } \{Z_n \to 0\} \text{ for all } n = 1,2,3,\ldots,$$

and that

$$(8.2.14) \qquad \sum_{k=1}^{\infty} A_k < \infty \qquad \text{a.s. on } \{Z_n \to 0\}.$$

Now suppose that

$$\omega \in B := \{\lim_{n \to \infty} Z_n = 0\} \cap \{\lim_{n \to \infty} c_n(s)Z_n < \infty\}.$$

From Theorem 8.2.2 we know that $P(B) = r$. Furthermore, for all $n = 1,2,3,\ldots$,

$$c_{n+1}(s)Z_{n+k}(\omega) = \frac{h_{k-1}(c_{n+k}(s))}{c_{n+k}(s)} c_{n+k}(s)Z_{n+k}(\omega)$$

$$\leq \frac{h_{k-1}(c_{k+1}(s))}{c_{k+1}(s)} c_{n+k}(s)Z_{n+k}(\omega)$$

$$= \frac{c_2(s)}{c_{k-1}(c_2(s))} c_{n+k}(s)Z_{n+k}(\omega)$$

$$\leq \frac{c_2(s)}{c_{k-1}(c_2(s))} \sup_{n \geq k+1} \{c_n(s)Z_n(\omega)\}$$

$$\leq \delta^{k-1} \sup_{n \geq 1} \{c_n(s)Z_n(\omega)\}$$

for some constant $\delta \in (0,1)$ and all sufficiently large k, as follows from (8.2.9). Since $c_n(s)Z_n(\omega)$ converges to some finite limit as $n \to \infty$, obviously $\sup_{n \geq 1}\{c_n(s)Z_n(\omega)\} < \infty$. Hence we obtain that if we choose $A_k = \delta^{k-1} \sup_{n \geq 1}\{c_n(s)Z_n\}$, then these A_k satisfy (8.2.13) and (8.2.14), and so the theorem is proved. $\quad\square$

In Section 6.3 we studied the quotient Z_{n+1}/Z_n for supercritical processes on $\{Z_n \to \infty\}$. We proved there a kind of law of large numbers, in the

sense that $Z_{n+1}/Z_n \xrightarrow{\text{a.s.}} m$ as $n \to \infty$, while Z_{n+1} can be considered as a "sum" of Z_n independent and identically distributed random variables, all having expectation m. We shall now see that on $\{Z_n \to 0\}$ the number a takes over the role of m. This is not so surprising in view of Proposition 6.4 in FRISTEDT [1974]. There it is proved that $X(t)/t \xrightarrow{\text{a.s.}} a$ as $t \downarrow 0$, where $\{X(t); t \in [0,\infty)\}$ is a subordinator and a the first point of increase of the distribution function of $X(1)$. We can consider this a as the "rate of decrease" of the process.

THEOREM 8.2.9.

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = a \qquad \text{a.s. } on \; \{Z_n \to 0\}.$$

PROOF. Since, by Theorem 8.2.2, $Y(s) \in (0,\infty)$ a.s. on $\{Z_n \to 0\}$, it follows that

$$\lim_{n\to\infty} \frac{Z_{n+1}}{Z_n} = \lim_{n\to\infty} \frac{c_{n+1}(s)Z_{n+1}}{c_n(s)Z_n} \frac{h(c_{n+1}(s))}{c_{n+1}(s)}$$

$$= \frac{Y(s)}{Y(s)} \lim_{s\to\infty} \frac{h(s)}{s} = a \qquad \text{a.s. on } \{Z_n \to 0\}. \qquad \square$$

CHAPTER 9

THE CASE a = 0

9.1. INTRODUCTION

In this chapter we study the only remaining case, namely the behaviour of the branching process $\{Z_n;\ n = 0,1,2,\dots\}$ on $\{Z_n \to 0\}$ if $a = 0$ and $P(Z_1 = 0) = 0$. We mentioned already before, that we shall see that this behaviour closely parallels the behaviour of explosive processes on $\{Z_n \to \infty\}$. The fact that $\lim_{n\to\infty} c_{n+1}(s)/c_n(s) = \frac{1}{m} = 0$ for $s \in (0,-\log r)$ if $m = \infty$, is used as an essential argument leading to the results listed in Section 7.3. Since we have now that $\lim_{n\to\infty} c_n(s)/c_{n+1}(s) = a = 0$ for $s \in (-\log r, \infty)$, it is at least plausible that we can use similar techniques as in the papers mentioned in Chapter 7, to obtain results for the process $\{Z_n;\ n = 0,1,2,\dots\}$ if $a = 0$ and $P(Z_1 = 0) = 0$. This will be further elaborated in the following section.

9.2. MAIN RESULTS

As already indicated, the results in this section can and will be presented in the same order as their analogs in Section 7.3. First of all we notice, that we can prove, in a similar way as in SENETA [1969], that if $d_n Z_n$ converges in distribution to some random variable Z, where $\{d_n;\ n = 0,1,2,\dots\}$ is a sequence of positive and finite constants, then $P(Z < \infty) < r$ or $P(Z = 0) \geq r$. We shall come back to this norming by a sequence of constants later on. Now we pass on to a norming with the help of a sequence of functions $\{g_n;\ n = 0,1,2,\dots\}$, that is we look for functions g_n such that $g_n(Z_n)$ converges in some sense to a proper, non-degenerate limit. As a first step we prove a result, similar to Lemma 7.3.1.

LEMMA 9.2.1. *Let* $\{f_n;\ n = 1,2,3,\dots\}$ *be a sequence of cumulant generating functions of positive random variables* X_n, $n = 1,2,3,\dots$. *Suppose there*

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exist a sequence $\{b_n; n = 1,2,3,\ldots\}$ *of positive constants and a distribution function* w *such that* $\lim_{n\to\infty} b_n = 0$ *and*

(9.2.1) $\lim_{n\to\infty} f_n(de^{-t/b_n}) = -\log w(t)$, $t \in (-\infty,\infty)$,

for some constant $d \in (0,\infty)$. *Then* $\lim_{n\to\infty} P(b_n \log X_n < t) = w(t)$ *for every* $t \in (-\infty,\infty)$ *where* w *is continuous.*

PROOF. From (9.2.1) it follows that

$$v(t) := 1 - w(-\log t) = \lim_{n\to\infty} E(1 - \exp(-d \cdot t^{1/b_n} \cdot X_n))$$

for all $t \in (0,\infty)$.

Now define $a_n(\lambda)$ by

(9.2.2) $a_n(\lambda) = \lambda E \int_0^\infty e^{-\lambda t}(1 - \exp(-d \cdot t^{1/b_n} \cdot X_n))dt$,

for any $\lambda \in (0,\infty)$ and $n = 1,2,3,\ldots$. An application of Fubini's theorem and the dominated convergence theorem then yield that

(9.2.3) $a_n(\lambda) = \lambda \int_0^\infty e^{-\lambda t} E(1 - \exp(-d \cdot t^{1/b_n} \cdot X_n))dt \to \lambda \int_0^\infty e^{-\lambda t} v(t)dt$

$$= \int_0^\infty e^{-\lambda t} dv(t) \qquad \text{as } n \to \infty.$$

Next we shall prove that on the other hand

$$\lim_{n\to\infty} a_n(\lambda) = \lim_{n\to\infty} E \exp(-\lambda(dX_n)^{-b_n}) \qquad \text{for all } \lambda \in (0,\infty).$$

To this end we notice that, since by (9.2.2)

$$a_n(\lambda) = E\left\{ \int_0^\infty \lambda e^{-\lambda t} dt - \int_0^\infty \lambda \exp(-\lambda t - d \cdot t^{1/b_n} \cdot X_n)dt \right.$$

$$= 1 - \lambda E \int_0^\infty \exp(-\lambda t - d \cdot t^{1/b_n} \cdot X_n)dt,$$

the substitution $s = t(dX_n)^{b_n}$ yields (remember that $X_n > 0$ by assumption)

$$(9.2.4) \qquad a_n(\lambda) = 1 - \lambda E \int_0^\infty (dX_n)^{-b_n} \exp(-\lambda s (dX_n)^{-b_n} - s^{1/b_n}) ds.$$

For the calculation of this expression we introduce the following notation:

$$g(\beta,\gamma) = \int_0^\infty \exp(-\beta s - s^\gamma) ds, \qquad \beta \in (0,\infty), \ \gamma \in (0,\infty);$$

$$H_n(\lambda) = \lambda (dX_n)^{-b_n};$$

$$G_n(\lambda) = H_n(\lambda) g(H_n(\lambda), \ 1/b_n) - (1 - \exp\{-H_n(\lambda)\});$$

$$J_n(\lambda) = \int_0^1 H_n(\lambda) e^{-H_n(\lambda)s} (\exp(-s^{1/b_n}) - 1) ds;$$

$$K_n(\lambda) = \int_1^\infty H_n(\lambda) e^{-H_n(\lambda)s} \exp(-s^{1/b_n}) ds.$$

With this notation it follows that $G_n(\lambda) = J_n(\lambda) + K_n(\lambda)$, and by (9.2.4) that

$$(9.2.5) \qquad a_n(\lambda) = 1 - E(G_n(\lambda) + 1 - e^{-H_n(\lambda)}) = E(e^{-H_n(\lambda)} - G_n(\lambda)).$$

Furthermore, because $0 \leq xe^{-sx} \leq (es)^{-1}$ and $1 - e^{-x} \leq x$ for all $s \in (0,\infty)$ and $x \in (0,\infty)$,

$$0 \leq -J_n(\lambda) \leq \int_0^1 (es)^{-1} s^{1/b_n} ds = \frac{b_n}{e} \to 0 \qquad \text{as } n \to \infty,$$

and

$$0 \leq K_n(\lambda) \leq \int_1^\infty (es)^{-1} \exp(-s^{1/b_n}) ds$$

$$\leq \int_1^\infty \exp(-s^{1/b_n}) ds \to 0 \qquad \text{as } n \to \infty,$$

by dominated convergence. This means that $\lim_{n \to \infty} G_n(\lambda) = 0$. Again using dominated convergence we see that $\lim_{n \to \infty} E G_n(\lambda) = 0$, since

$$|G_n(\lambda)| = K_n(\lambda) - J_n(\lambda) \le \int_1^\infty \exp(-s^{1/b})ds + b/e < \infty$$

for all n = 1,2,3,..., where b:= $\sup_n b_n$. Substituting this into (9.2.5) yields the required result, that is

$$\lim_{n\to\infty} a_n(\lambda) = \lim_{n\to\infty} E \exp(-\lambda(dX_n)^{-b_n}) \qquad \text{for all } \lambda \in (0,\infty).$$

Combining this with (9.2.3), an application of the continuity theorem for Laplace-transforms implies that

$$\lim_{n\to\infty} P((dX_n)^{-b_n} \le t) = v(t)$$

for every t $\in$ (0,∞) where v is continuous, or equivalently, since $\lim_{n\to\infty} b_n = 0$,

$$\lim_{n\to\infty} P(-b_n \log X_n \le \log t) = v(t)$$

for every t $\in$ (0,∞) where v is continuous. Finally, remembering that v(t) = 1 - w(-log t), t $\in$ (0,∞), obvious calculations finish the proof. $\square$

EXAMPLE 9.2.2. We can apply this result to a branching process $\{Z_n; n = 0,1,2,...\}$ having a strictly stable distribution concentrated on [0,∞) with characteristic exponent $\alpha \in$ (0,1) as its offspring distribution. Choosing $b_n = \alpha^n$, $f_n = h_n$ and d = 1 in (9.2.1) we obtain, in view of (7.3.2),

$$f_n(\exp(-t/b_n)) = h_n(\exp(-t/\alpha^n)) = \{\exp(-t/\alpha^n)\}^{\alpha^n} = e^{-t}$$

$$\text{for } t \in (-\infty,\infty),$$

and Lemma 9.2.1 implies that

$$\alpha^n \log Z_n \xrightarrow{d} Z,$$

where Z is a random variable with distribution function w(t) = exp(-exp(-t)), t $\in$ (-∞,∞), thus extending Example 7.3.2.

Just as in Section 7.3 we shall now describe a class of processes for which the corresponding cumulant generating functions satisfy (9.2.1). To

this end we define the function $\bar{h}$ by

(9.2.6) $\bar{h}(s) = 1/h(1/s)$ if $s \in (0,\infty)$, $\bar{h}(0) = 0$ and $\bar{h}(\infty) = \infty$.

The function $\bar{h}$ is strictly increasing in s since h is, and

$$\lim_{s\downarrow 0} \frac{\bar{h}(s)}{s} = \lim_{t\to\infty} \frac{t}{h(t)} = \frac{1}{a} = \infty.$$

Moreover, $\bar{h}_n(s)$, the nth iterate of $\bar{h}(s)$, satisfies $\bar{h}_n(s) = 1/h_n(1/s)$. This gives rise to an imitation of the methods mentioned in Section 7.3 with $h(s)$ replaced by $\bar{h}(s)$. We therefore introduce the following functions.

(9.2.7) $\bar{c}(s) = \bar{h}^{inv}(s)$, $s \in [0,\infty]$;

(9.2.8) $f(t) = -\log\{1 - \exp(-\bar{h}(-\log\{1 - e^{-t}\}))\}$, $t \in [0,\infty)$.

Since $\bar{h}$ is strictly increasing, so is f. We assume that f satisfies

(9.2.9)   f is convex or concave on $[0,\infty)$

and

(9.2.10) $0 < \gamma := \lim_{t\to\infty} \frac{f(t)}{t} < 1$.

Furthermore, we define the number $\bar{r}$ and the functions ϕ, δ and ρ by

(9.2.11) $\bar{r} = \begin{cases} \exp(1/\log r) & \text{if } r < 1 \\ 0 & \text{if } r = 1 \end{cases}$;

(9.2.12) $\phi(t) = f(t -\log(1-\bar{r})) + \log(1-\bar{r})$, $t \in [\log(1-\bar{r}),\infty)$;

(9.2.13) $\delta(t) = \phi^{inv}(t)$, $t \in [\log(1-\bar{r}),\infty)$;

and

(9.2.14) $\rho(t) = 1/\delta(1/t)$, $t \in \begin{cases} (-\infty, \frac{1}{\log(1-\bar{r})}] \cup (0,\infty) & \text{if } \bar{r} > 0 \\ (0,\infty) & \text{if } \bar{r} = 0 \end{cases}$,

and use the convention that the subscript n indicates the nth iterate for n = 1,2,3,... . Notice that the relations (9.2.7), (9.2.8), (9.2.12), (9.2.13) and (9.2.14) also hold for the corresponding iterated functions. A closer look at the functions $\bar{h}$, f, ϕ, δ and ρ shows that if (9.2.9) holds, then ϕ and δ are convex or concave, and that

$$(9.2.15) \quad \bar{h}(0) = 0, \quad \bar{h}(-\log\bar{r}) = -\log\bar{r} \quad \text{and} \quad \lim_{s\to\infty} \bar{h}(s) = \infty;$$

$$f(0) = 0, \quad f(-\log(1-\bar{r})) = -\log(1-\bar{r}) \quad \text{and} \quad \lim_{t\to\infty} f(t) = \infty;$$

$$\phi(\log(1-\bar{r})) = \log(1-\bar{r}), \quad \phi(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} \phi(t) = \infty.$$

If f, and hence also ϕ, is concave, then

$$\lim_{n\to\infty} \delta_n(t) = \begin{cases} \infty & \text{if } t \in (0,\infty) \\ \log(1-\bar{r}) & \text{if } t \in [\log(1-\bar{r}),0) \end{cases},$$

whence

$$(9.2.16) \quad \lim_{n\to\infty} \rho_n(t) = \begin{cases} 0 & \text{if } t \in (0,\infty) \\ \dfrac{1}{\log(1-\bar{r})} & \text{if } t \in (-\infty, \dfrac{1}{\log(1-\bar{r})}] \end{cases} \quad \text{and} \quad \bar{r} > 0.$$

We shall now prove that for processes with m < ∞, and for which the corresponding function f satisfies (9.2.9) and (9.2.10), relation (9.2.1) holds, for $f_n = h_n$ and $b_n = \rho_n(x)$ for any $x \in (0,\infty)$. This implies that for such processes $\rho_n(x)\log Z_n$ converges in distribution to a random variable Z, which turns out to be proper and non-degenerate. There is a slight difference between this result and the result we get by combining Theorem 7.3.3 and Lemma 7.3.1, to wit $\rho_n(x)\log(1+Z_n)$ converges in distribution to a random variable Z, where $\rho_n(x)$ is as in Theorem 7.3.3. This last result is in fact only interesting on $\{Z_n \to \infty\}$, because $\rho_n(x) \downarrow 0$, and hence

$$\rho_n(x)\log(1+Z_n) \to 0 \qquad \text{as } n \to \infty \text{ on } \{Z_n \to 0\}.$$

However, the assertion

$$\rho_n(x)\log Z_n \xrightarrow{d} Z,$$

with $\rho_n(x)$ as in (9.2.16), is non-trivial both on $\{Z_n \to 0\}$ and on $\{Z_n \to \infty\}$.

The price we have to pay for this is that we have to make the extra assumption $m < \infty$, needed in proving (9.2.17) below for positive values of t. The proof of (9.2.17) for negative values of t is comparable with that of (7.3.9) for $t \in (0,\infty)$.

THEOREM 9.2.3. *Suppose that* $m < \infty$ *and that* (9.2.9) *and* (9.2.10) *hold. Then for any fixed* $x \in (0,\infty)$

$$(9.2.17) \qquad -\log w(t,x) := \lim_{n \to \infty} h_n\left(\frac{\exp(-t/\rho_n(x))}{1-\bar{r}}\right)$$

exists for all $t \in (-\infty,\infty)$. *The function* $w(t,x)$ *has the following properties:*

$$(9.2.18) \qquad \begin{aligned} &w(t,x) = 1 \ \textit{for all} \ t \in (0,\infty), \\ &w(0,x) = r \ \textit{and} \ \lim_{t \to -\infty} w(t,x) = 0; \end{aligned}$$

$$(9.2.19) \qquad h(-\log w(t,x)) = -\log w(\gamma t,x), \qquad t \in (-\infty,0);$$

$$(9.2.20) \qquad w(t,x) \ \textit{is continuous and strictly increasing in} \ t \in (-\infty,0).$$

PROOF. Analogously to a part of the proof of Theorem 1 in SENETA [1973] we can prove that, with ϕ as in (9.2.12), $\Delta(t,x) := \lim_{n \to \infty} \phi_n(t/\rho_n(x))$ exists for all $t \in [0,\infty)$. Furthermore, $\Delta(t,x)$ is continuous and strictly increasing in t, and satisfies

$$(9.2.21) \qquad \Delta(0,x) = 0 \quad \text{and} \quad \lim_{t \to \infty} \Delta(t,x) = \infty.$$

Now defining $a_n(t,x)$ by $a_n(t,x) = \{-\log(1-(1-\bar{r})\exp\{-t/\rho_n(x)\})\}^{-1}$, $t \in (0,\infty)$, $n = 1,2,3,\ldots$, the relations between h_n and $\bar{h}_n$ and between $\bar{h}_n$ and ϕ_n (see (9.2.6), (9.2.8) and (9.2.12)) imply that

$$(9.2.22) \qquad \lim_{n \to \infty} h_n(a_n(t,x))$$

$$= \lim_{n \to \infty} h_n(\{-\log[1 - (1-\bar{r})\exp(-t/\rho_n(x))]\}^{-1})$$

$$= \lim_{n \to \infty} \{\bar{h}_n(-\log[1 - (1-\bar{r})\exp(-t/\rho_n(x))])\}^{-1}$$

$$= \lim_{n \to \infty} \{-\log[1 - \exp\{-f_n(t/\rho_n(x) - \log(1-\bar{r}))\}]\}^{-1} =$$

$$= \lim_{n \to \infty} \{-\log[1 - (1-\bar{r})\exp\{-\phi_n(t/\rho_n(x))\}]\}^{-1}$$

$$= -\{\log[1 - (1-\bar{r})\exp(-\Delta(t,x))]\}^{-1}.$$

Furthermore defining $b_n(t,x)$ by

$$b_n(t,x) = \{\exp(t/\rho_n(x))\}/(1-\bar{r}), \qquad t \in (0,\infty), \ n = 1,2,3,\dots,$$

it follows from the inequality $-\log(1-x) \geq x$, holding for $x \in (0,1)$, that $a_n(t,x) \leq b_n(t,x)$ for every $t \in (0,\infty)$ and $n = 1,2,3,\dots$; also $a_n(t,x) \sim b_n(t,x)$ for every $t \in (0,\infty)$ as $n \to \infty$, since $\rho_n(x) \to 0$ as $n \to \infty$. This means that for every $\varepsilon > 0$ and every $t \in (0,\infty)$,

$$\frac{a_n(t+\varepsilon,x)}{b_n(t,x)} \sim \frac{b_n(t+\varepsilon,x)}{b_n(t,x)} = \exp(\varepsilon/\rho_n(x)) \to \infty \qquad \text{as } n \to \infty,$$

and so $a_n(t,x) \leq b_n(t,x) \leq a_n(t+\varepsilon,x)$ for sufficiently large n. Now using the continuity of $\Delta(t,x)$ as a function of t, together with (9.2.22), we get

$$\lim_{n \to \infty} h_n(b_n(t,x)) = \lim_{n \to \infty} h_n\left(\frac{\exp(t/\rho_n(x))}{1-\bar{r}}\right)$$

$$= -\{\log[1 - (1-\bar{r})\exp(-\Delta(t,x))]\}^{-1},$$

and so (9.2.17) is proved for all $t \in (-\infty,0)$ with

$$w(t,x) = \exp\{\log[1 - (1-\bar{r})\exp(-\Delta(-t,x))]\}^{-1}.$$

In particular, (9.2.21) implies that $\lim_{t \to -\infty} w(t,x) = 0$. The proof of (9.2.19) is analogous to that of (7.3.11); (9.2.20) follows in a similar way as the corresponding result in Theorem 1 in SENETA [1973]. Furthermore,

$$-\log w(0,x) = \lim_{n \to \infty} h_n(1/(1-\bar{r})) = -\log r$$

by Theorem 3.3.1. Now suppose that $t \in (0,\infty)$. First of all we notice that if $m \leq 1$, then $\lim_{n \to \infty} h_n(s) = 0$ for all $s \in [0,\infty)$ and because

$$\lim_{n \to \infty} \frac{\exp(-t/\rho_n(x))}{1-\bar{r}} = 0,$$

obviously

$$\lim_{n\to\infty} h_n\left(\frac{\exp(-t/\rho_n(x))}{1-\bar{r}}\right) = 0.$$

So we are left with the proof of (9.2.17) for $m \in (1,\infty)$ and $t \in (0,\infty)$. Since $r < 1$ in this case, and $r > 0$ because $a = 0$, (9.2.11) implies that $\bar{r} \in (0,1)$, whence

(9.2.23) $\log(1-\bar{r}) \in (-\infty,0)$.

In view of (9.2.10) and (9.2.15) this means that f, and therefore also ϕ, cannot be convex, and hence is concave by (9.2.9). By the relation between ϕ and h we know that

(9.2.24) $h_n(s) = 1/\bar{h}_n(1/s)$

$$= -\{\log[1 - (1-\bar{r})\exp\{-\phi_n(-\log[(1 - \exp(-1/s))/(1-\bar{r})])\}]\}^{-1}.$$

So because we want to know if

$$\lim_{n\to\infty} h_n\left(\frac{\exp(-t/\rho_n(x))}{1-\bar{r}}\right)$$

exists, we have to check whether $\lim_{n\to\infty} \phi_n(\alpha_n(t,x))$ exists, where $\alpha_n(t,x)$ is defined by

$$\alpha_n(t,x) = -\log[1 - \exp\{-(1-\bar{r})\exp(t/\rho_n(x))\}] + \log(1-\bar{r}),$$

$$t \in (0,\infty), \quad n = 1,2,3,\ldots .$$

Since $\lim_{n\to\infty} \rho_n(x) = 0$ by (9.2.16), it follows that $\lim_{n\to\infty} \alpha_n(t,x) = \log(1-\bar{r})$, whence $\alpha_n(t,x) \in (\log(1-\bar{r}),0)$ for all $n \geq$ some integer $N = N(t,x)$. The way to get a hold on $\phi_n(\alpha_n(t,x))$ for large n is now the following. Choose any $x_1 \in (-\infty, 1/\log(1-\bar{r}))$. Then (9.2.15) implies that

$$1/\rho_n(x_1) = \phi_n^{inv}(1/x_1) \in (\log(1-\bar{r}),0),$$

just as $\alpha_n(t,x)$. Now compare $\alpha_n(t,x)$ and $1/\rho_n(x_1)$ as follows. Define for every $n \geq N$ the integer $j_n = j_n(x,x_1,t)$ by

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(9.2.25) $1/\rho_{n+j_n+1}(x_1) < \alpha_n(t,x) \le 1/\rho_{n+j_n}(x_1)$,

where we use the conventions $1/\rho_{-n}(x) = \phi_n(1/x)$ and $\phi_{-n}(1/x) = 1/\rho_n(x)$ for positive integers n, and $\phi_0(1/x) = 1/\rho_0(x) = 1/x$. Such a j always exists, since $\lim_{n\to\infty} 1/\rho_n(x_1) = \log(1-\bar{r})$ and $\lim_{n\to-\infty} 1/\rho_n(x_1) = 0$ by (9.2.15) and (9.2.16). Below we shall prove that $\lim_{n\to\infty} j_n = \infty$, implying that

$$\log(1-\bar{r}) = \lim_{n\to\infty} 1/\rho_{j_n+1}(x_1) = \lim_{n\to\infty} \phi_n(1/\rho_{n+j_n+1}(x_1))$$

$$\le \lim_{n\to\infty} \phi_n(\alpha_n(t,x)) \le \lim_{n\to\infty} \phi_n(1/\rho_{n+j_n}(x_1))$$

$$= \lim_{n\to\infty} 1/\rho_{j_n}(x_1) = \log(1-\bar{r}),$$

that is

(9.2.26) $\lim_{n\to\infty} \phi_n(\alpha_n(t,x)) = \log(1-\bar{r})$.

First of all we notice that, since the function f(t), defined in (9.2.8), is concave, $\rho_{n+1}(t) < \rho_n(t)$ if $t \in (0,\infty)$ and $\rho_{n+1}(t) > \rho_n(t)$ if $t \in (-\infty, 1/\log(1-\bar{r}))$. Therefore, (9.2.25) implies that

$$1/\rho_{n+1+j_{n+1}+1}(x_1) < \alpha_{n+1}(t,x) < \alpha_n(t,x) \le 1/\rho_{n+j_n}(x_1),$$

whence

(9.2.27) $j_{n+1} \ge j_n - 2$, $n = N, N+1, N+2, \ldots$.

Defining p(t) by $p(t) = -\log(1 - (1-\bar{r})\exp(-1/t))$, $t \ne 0$, we see that

(9.2.28) $p(x_1) \in (-\log \bar{r}, \infty)$,

since $x_1 \in (-\infty, 1/\log(1-\bar{r}))$. Furthermore, it follows from the relations (9.2.7), (9.2.8), (9.2.12), (9.2.13) and (9.2.14) that

$$1/\rho_n(s) = \phi_n^{inv}(1/s) = \log(1-\bar{r}) - \log[1 - \exp\{-\bar{c}_n(p(s))\}],$$

$$1/s \in [\log(1-\bar{r}), \infty),$$

and so (9.2.25) yields

$$1/\rho_{n+j_n+1}(x_1) - \log(1-\bar{r}) = -\log[1 - \exp\{-\bar{c}_{n+j_n+1}(p(x_1))\}]$$

$$< \alpha_n(t,x) - \log(1-\bar{r}) = -\log[1 - \exp\{-(1-\bar{r})\exp(t/\rho_n(x))\}]$$

$$\leq 1/\rho_{n+j_n}(x_1) - \log(1-\bar{r}) = -\log[1 - \exp\{-\bar{c}_{n+j_n}(p(x_1))\}].$$

Therefore

$$(9.2.29) \qquad \bar{c}_{n+j_n}(p(x_1)) \leq (1-\bar{r})\{\exp(t/\rho_n(x))\} = (1-\bar{r})\left\{\frac{1-\bar{r}}{1-\exp\{-\bar{c}_n(p(x))\}}\right\}^t$$

$$< \bar{c}_{n+j_n+1}(p(x_1)).$$

Since $x \in (0,\infty)$

$$(9.2.30) \qquad p(x) \in (0,-\log \bar{r}).$$

and thus $\lim_{n\to\infty} \bar{c}_n(p(x)) = 0$ by (9.2.6) and (9.2.7). In view of (9.2.23) and (9.2.29) this means that $\lim_{n\to\infty} \bar{c}_{n+j_n}(p(x_1)) = \infty$, and then by (9.2.28) and again (9.2.6) and (9.2.7), it follows that

$$(9.2.31) \qquad \lim_{n\to\infty} (n+j_n) = \infty.$$

Now by (9.2.7) and (9.2.11) we have

$$(9.2.32) \qquad \lim_{n\to\infty} \frac{\bar{c}_{n+1}(s)}{\bar{c}_n(s)} = \lim_{n\to\infty} \frac{c_n(1/s)}{c_{n+1}(1/s)} = \begin{array}{ll} m & \text{if } 1/s \in (0,-\log r) \\ 0 & \text{if } 1/s \in (-\log r, \infty) \end{array}$$

$$= \begin{array}{ll} m & \text{if } s \in (-\log \bar{r}, \infty) \\ 0 & \text{if } s \in (0,-\log \bar{r}) \end{array}.$$

Combining this with (9.2.28) and (9.2.31) we see that

$$\lim_{n\to\infty} \frac{\bar{c}_{n+j_n}(p(x_1))}{\bar{c}_{n+j_n+1}(p(x_1))} = \frac{1}{m} \in (0,1).$$

Now because $\lim_{n\to\infty} \bar{c}_n(p(x)) = 0$, it follows that

$$(1+t)\log(1-\bar{r}) - t \cdot \log[1 - \exp\{-\bar{c}_n(p(x))\}]] \sim -t \cdot \log \bar{c}_n(p(x))$$

$$\text{as } n \to \infty.$$

So taking logarithms in (9.2.29) yields, in view of (9.2.28), (9.2.31) and (9.2.32),

$$\log \bar{c}_{n+j_n-1}(p(x_1)) \le -t \log \bar{c}_n(p(x)) < \log \bar{c}_{n+j_n+2}(p(x_1))$$

for sufficiently large n. Therefore

$$t \cdot \log \frac{\bar{c}_{n+1}(p(x))}{\bar{c}_n(p(x))} = -t \cdot \log \bar{c}_n(p(x)) - \{-t \cdot \log \bar{c}_{n+1}(p(x))\}$$

$$> \log \bar{c}_{n+j_n-1}(p(x_1)) - \log \bar{c}_{n+1+j_{n+1}+2}(p(x_1))$$

$$= \log \frac{\bar{c}_{n+j_n-1}(p(x_1))}{\bar{c}_{n+j_{n+1}+3}(p(x_1))} .$$

By (9.2.30) and (9.2.32), the left hand side of this expression has limit $-\infty$ as $n \to \infty$. The right hand side is at least $(j_{n+1} - j_n + 4)\log \frac{1}{m}$, in view of (9.2.27) and the fact that

$$\frac{\bar{c}_n(s)}{\bar{c}_k(s)} = \frac{c_k(1/s)}{c_n(1/s)} = \frac{c_{k-n}(c_n(1/s))}{c_n(1/s)} \ge (\frac{1}{m})^{k-n} \qquad \text{for } k \ge n$$

by the convexity of $c(s)$. So we can conclude that also

$$\lim_{n \to \infty}(j_{n+1} - j_n + 4)\log \frac{1}{m} = -\infty,$$

whence $\lim_{n \to \infty}(j_{n+1} - j_n) = \infty$, because $m \in (1,\infty)$. Therefore also $\lim_{n \to \infty} j_n = \infty$, implying that (9.2.26) holds. Substituting this into (9.2.24) we obtain

$$\lim_{n \to \infty} h_n\left(\frac{\exp(-t/\rho_n(x))}{1-\bar{r}}\right) = \lim_{n \to \infty} -\{\log[1-(1-\bar{r})\exp\{-\phi_n(\alpha_n(t,x))\}]\}^{-1} = 0,$$

and so we have proved (9.2.17) also for $t \in (0,\infty)$, with $w(t,x) = 1$. $\square$

Proceeding as in Section 7.3 we shall now try to derive results concerning almost sure convergence. To this end we define a sequence of random variables $\{U_n(x); n = 1,2,3,\ldots\}$ by

$$U_n(x) = \exp\left\{\frac{Z_n}{\log[1-(1-\bar{r})\exp(-1/\rho_n(x))]}\right\}, \quad n = 1,2,3,\ldots,$$
$$x \in (0,\infty).$$

with $\rho(x)$ as in (9.2.14) and $\rho_n(x)$ its nth iterate. Since $\rho_n(x) \in (0,\infty)$ for $x \in (0,\infty)$, we see that $U_n(x)$ is well-defined and satisfies $0 < U_n(x) \le 1$ for all $n = 1,2,3,\ldots$ and all $x \in (0,\infty)$. Moreover, the sequence $\{U_n(x); n = 1,2,3,\ldots\}$ turns out to be a martingale sequence, as can be proved in the following way with the help of the basic branching property (3.1.3) and the relations (9.2.6), (9.2.8), (9.2.12), (9.2.13) and (9.2.14).

$$E\{U_{n+1}(x) \mid U_1(x),U_2(x),\ldots,U_n(x)\}$$

$$= E\{\exp(Z_{n+1}/\log[1 - (1-\bar{r})\exp\{-1/\rho_{n+1}(x)\}]) \mid Z_n\}$$

$$= \{E \exp(Z_1/\log[1 - (1-\bar{r})\exp\{-1/\rho_{n+1}(x)\}])\}^{Z_n}$$

$$= \{\exp\{-h(-1/\log[1 - (1-\bar{r})\exp\{-1/\rho_{n+1}(x)\}])\}\}^{Z_n}$$

$$= \exp\{-Z_n/\bar{h}(-\log[1 - (1-\bar{r})\exp\{-1/\rho_{n+1}(x)\}])\}$$

$$= \exp\{Z_n/\log[1 - \exp\{-f(1/\rho_{n+1}(x)-\log(1-\bar{r}))\}]\}$$

$$= \exp\{Z_n/\log[1 - (1-\bar{r})\exp\{-\phi(1/\rho_{n+1}(x))\}]\}$$

$$= \exp\{Z_n/\log[1 - (1-\bar{r})\exp\{-1/\rho_n(x)\}]\} = U_n(x).$$

We have thus proved that $\{U_n(x); n = 1,2,3,\ldots\}$ is a bounded martingale, whence $U_n(x)$ converges almost surely to some random variable $U(x)$ as $n \to \infty$ and

$$EU(x) = EU_n(x) = E \exp\{Z_n/\log[1 - (1-\bar{r})\exp\{-1/\rho_n(x)\}]\}$$

$$= \exp\{-h_n(-1/\log[1 - (1-\bar{r})\exp\{-1/\rho_n(x)\}])\}$$

$$= \exp\{-1/\bar{h}_n(-\log[1 - (1-\bar{r})\exp\{-1/\rho_n(x)\}])\}$$

$$= \exp\{1/\log[1 - (1-\bar{r})\exp\{-\phi_n(1/\rho_n(x))\}]\}$$

$$= \exp\{1/\log[1 - (1-\bar{r})\exp\{-1/x\}]\}.$$

Remembering the discussion in Section 7.3 it is now clear that the following results hold.

THEOREM 9.2.4. *Suppose* (9.2.9) *and* (9.2.10) *hold. Then for any* $x \in (0,\infty)$, $\{\exp(1/\rho_n(x))\}Z_n \xrightarrow{\text{a.s.}}$ *some random variable* $Z(x)$, *and*
$$P(Z(x) = 0) = 1 - P(Z(x) = \infty) = \exp\{-1/\log[1 - (1-\bar{r})\exp\{-1/x\}]\}.$$

PROOF. Analogous to the proof of Theorem 7.3.4. □

THEOREM 9.2.5. *Suppose* (9.2.9) *and* (9.2.10) *hold. Let* $\{a_n; n = 1,2,3,\ldots\}$ *be a sequence of finite constants such that* $\lim_{n\to\infty} a_n = \infty$ *and* $\lim_{n\to\infty} a_n Z_n$ *exists almost surely. Then* $P(\lim_{n\to\infty} a_n Z_n = 0 \text{ or } \infty) = 1$.

PROOF. Analogous to the proof of Theorem 7.3.5. □

As a consequence of this last result we have that if (9.2.9) and (9.2.10) hold, then the random variable $Y(s)$, defined in Corollary 3.3.5 by $Y(s) = \lim_{n\to\infty} c_n(s)Z_n$, satisfies $P(Y(s) = 0 \text{ or } \infty) = 1$ for all $s \in (-\log r, \infty)$. For, since $P(Z_1 = 0) = 0$ by assumption, it follows from Corollary 3.3.5 that $Y(s)$ exists a.s.; furthermore $\lim_{n\to\infty} c_n(s) = \infty$ for every $s \in (-\log r, \infty)$. From (3.3.5) we know that $E \exp\{-Y(s)\} = e^{-s}$, and therefore $P(Y(s) = 0) = 1 - P(Y(s) = \infty) = e^{-s}$.

Our next aim is to derive almost sure convergence results such that the limit random variable has its values in $(0,\infty)$, at least on $\{Z_n \to 0\}$, since this is in fact the case we are interested in. This will be done using a technique developed in SCHUH and BARBOUR [1977] for explosive Galton-Watson processes. In short, this method is, to obtain sufficient conditions such that an almost sure convergence result holds, which is in agreement with the requirements just mentioned, and then to construct, for every branching process $\{Z_n; n = 0,1,2,\ldots\}$ with $P(Z_1 = 0) = 0$ and $a = 0$, a function, with the help of which we can prove that the sufficient conditions meant above are satisfied. For the processes we met so far in this chapter, it holds that $P(Y(s) = 0 \text{ or } \infty) = 1$ for all $s \in (-\log r, \infty)$. In the sequel we shall also consider the possibility that $P(0 < Y(s) < \infty) > 0$ for some $s \in (-\log r, \infty)$, and we introduce the following regularity concept.

DEFINITION 9.2.6. A point $s \in (-\log r, \infty)$ is called *regular* for the process $\{Z_n; n = 0,1,2,\ldots\}$ if $P(Y(s) = 0 \text{ or } \infty) = 1$ and *irregular* for the process otherwise.

First of all we shall have a closer look at the set of irregular points. It turns out that this set is open, as follows immediately from the next lemma. Before stating it we notice that we know already from the previous chapters that $c_n(s)/c_n(t)$ is non-increasing in n for $0 < s \leq t < \infty$, and hence converges to some limit as $n \to \infty$. This limit is again denoted by $v(s,t)$. Obviously,

$$v(s,t) := \lim_{n\to\infty} \frac{c_n(s)}{c_n(t)} = \frac{1}{v(t,s)} \qquad \text{for } 0 < t < s < \infty.$$

LEMMA 9.2.7. *If* s_i *is an irregular point for the process* $\{Z_n; n = 0,1,2,\ldots\}$ *then there exists an open interval* $I(s_i) = (s_1,s_2)$ *such that* $s_i \in I(s_i)$ *and all* $s \in I(s_i)$ *are irregular points for the process* $\{Z_n; n = 0,1,2,\ldots\}$; $v(s,s_i)$ *is a strictly increasing, continuous function of s on* (s_1,s_2); $v(s_1,s_i) = 0$ *and* $v(s_2,s_i) = \infty$; s_1 *and* s_2 *are both regular points for the process* $\{Z_n; n = 0,1,2,\ldots\}$.

PROOF. Since s_i is an irregular point for the process, $P(0 < Y(s_i) < \infty) > 0$. Hence the Lemmas 2.2.2 and 2.2.3(a) imply that $\phi(u,s_i)$, the cumulant generating function of $Y(s_i)$, is continuous and strictly increasing on $[0,\infty)$. Now define s_1 and s_2 by $s_1 = \phi(0,s_i)$ and $s_2 = \lim_{u\to\infty} \phi(u,s_i)$. Then it follows that $s_1 \geq -\log r$, because

$$\phi(0,s_i) = -\log P(Y(s_i) < \infty) \geq -\log P(\lim_{n\to\infty} Z_n = 0) = -\log r;$$

furthermore, the inverse of $\phi(u,s_i)$ as a function of u, written as $\phi^{inv}(u,s_i)$, is well-defined, continuous and strictly increasing on $[s_1,s_2)$ and

(9.2.33) $\phi^{inv}(s_1,s_i) = 0$ and $\lim_{s\uparrow s_2} \phi^{inv}(s,s_i) = \infty$.

Since $\phi(1,s_i) = s_i$ by (3.3.5), it is clear that $s_i \in (s_1,s_2)$. Next we shall prove that all $s \in I(s_i)$ are irregular points for the process. For suppose that $s \in (s_1,s_i)$ is regular. Then $P(Y(s) = 0) = 1 - P(Y(s) = \infty) = e^{-s}$, whence $\phi(u,s) = s$ for all $u \in [0,\infty)$. In view of (3.3.5) this means that $\lim_{n\to\infty} h_n(uc_n(s)) = s$ for all $u \in (0,\infty)$, and thus

$$h_n\left(\frac{c_n(t)}{c_n(s)} c_n(s)\right) = t < h_n(uc_n(s)) \qquad \text{for all } u \in (0,\infty),$$

all $t \in (-\log r, s)$ and all sufficiently large n. Hence we obtain that $c_n(t)/c_n(s) \le u$ for all $u \in (0,\infty)$, all $t \in (-\log r, s)$ and all sufficiently large n, that is $v(t,s) = 0$ for all $t \in (-\log r, s)$, and so a fortiori $v(t,s_i) = 0$ for all $t \in (-\log r, s)$. However, this implies that for all $\epsilon \in (0, s + \log r)$, and all $u \in (0, \phi^{inv}(s,s_i)]$, it holds that

$$s - \epsilon = h_n\left(\frac{c_n(s-\epsilon)}{c_n(s_i)} c_n(s_i)\right) \le \lim_{n\to\infty} h_n(uc_n(s_i))$$

$$= \phi(u,s_i) \le \phi(\phi^{inv}(s,s_i),s_i) = s,$$

whence $\phi(u,s_i) = s$ for all $u \in (0, \phi^{inv}(s,s_i)]$. But this contradicts the fact that $\phi(u,s_i)$ is strictly increasing on $[0,\infty)$. Therefore, all $s \in (s_1, s_i)$ are irregular points for the process. An analogous argument yields that $v(s,s_i) \ne 0$, that is $v(s,s_i) \in (0,1]$ for all $s \in (s_1, s_i)$. Similarly we can show that all $s \in (s_i, s_2)$ are irregular and that $v(s,s_i) \in [1,\infty)$ for all $s \in (s_i, s_2)$. Since

$$v(h(s_i),s_i) = \lim_{n\to\infty} \frac{h(c_n(s_i))}{c_n(s_i)} = \lim_{s\to\infty} \frac{h(s)}{s} = a = 0,$$

it follows that $s_1 \ge h(s_i) > -\log r$. Similarly we get $s_2 < \infty$. The following step we make is that we prove that $v(s,s_i) = \phi^{inv}(s,s_i)$ for all $s \in (s_1, s_2)$. To this end we notice that if $s \in (s_1, s_i]$, then $c_n(s)/c_n(s_i)$ decreases to $v(s,s_i)$ as $n \to \infty$. Hence

$$s = h_n\left(c_n(s_i) \frac{c_n(s)}{c_n(s_i)}\right) \ge h_n(c_n(s_i)v(s,s_i)).$$

Using the convexity of $c(s)$ we see from this that

$$1 \le \frac{s}{h_n(c_n(s_i)v(s,s_i))} \le \frac{c_n(s)}{c_n(h_n(c_n(s_i)v(s,s_i)))}$$

$$= \frac{c_n(s)}{c_n(s_i)v(s,s_i)} \to 1 \qquad \text{as } n \to \infty.$$

This proves that

$$s = \lim_{n\to\infty} h_n(v(s,s_i)c_n(s_i)) = \phi(v(s,s_i),s_i),$$

in view of (3.3.5), that is $v(s,s_i) = \phi^{inv}(s,s_i)$ for all $s \in (s_1,s_i]$. In a similar way we can prove the analogous result for all $s \in (s_i,s_2)$. Because $v(s,s_i)$ is non-decreasing as a function of s, it now follows that

$$0 \leq v(s_1,s_i) \leq v(s_1+\varepsilon,s_i) = \phi^{inv}(s_1+\varepsilon,s_i) \downarrow 0$$

as $\varepsilon \downarrow 0$ by (9.2.33) and hence $v(s_1,s_i) = 0$; similarly we get $v(s_2,s_i) = \infty$. We can use this to prove that s_1 and s_2 are regular points for the process. Namely, for all $u \in [1,\infty)$ it holds that

$$s_1 = h_n(c_n(s_1)) \leq \lim_{n\to\infty} h_n(uc_n(s_1))$$

$$= \lim_{n\to\infty} h_n\left(\frac{c_n(s_1)}{c_n(s_i)} uc_n(s_i)\right)$$

$$\leq \lim_{n\to\infty} h_n(\varepsilon uc_n(s_i)) = \phi(\varepsilon u,s_i) \to s_1 \qquad \text{as } \varepsilon \downarrow 0.$$

Because of (3.3.5) this means that $\phi(u,s_1) = \lim_{n\to\infty} h_n(uc_n(s_1)) = s_1$ for all $u \in [1,\infty)$, and thus for all $u \in [0,\infty)$, since ϕ is a cumulant generating function. Because the distribution corresponding to this cumulant generating function is given by $P(Y(s_1) = 0) = 1 - P(Y(s_1) = \infty) = e^{-s_1}$, and $s_1 \in (-\log r, \infty)$, it follows that s_1 is a regular point for the process. Similarly we can prove that s_2 is a regular point for the process. $\square$

Since $c_n(s)$ is increasing in s, $Y(s)$ is non-decreasing. This leads to the introduction of a very useful random variable, being the point where $Y(s)$ exceeds the level 1. More precisely, for every $\omega \in \Omega$ we define $T(\omega)$ by

(9.2.34) $\quad T(\omega) = \begin{cases} \inf\{s \in (-\log r,\infty) ; Y(s,\omega) > 1\} & \text{if } Y(s,\omega) > 1 \text{ for some } s \in (-\log r,\infty) \\ \infty & \text{else} \end{cases}$

T is a random variable, since

(9.2.35) $\quad \{T \leq t\} = \begin{cases} \emptyset & \text{if } t \in (-\infty, -\log r) \\ \lim_{\substack{s \downarrow t \\ s \text{ rational}}} \{Y(s) > 1\} & \text{if } t \in [-\log r, \infty) \end{cases}$

From Lemma 9.2.7 it follows that there exists a sequence $\{s_n; n = 1,2,3,\ldots\}$ of regular points such that $\lim_{n\to\infty} s_n = \infty$. Since $P(T = \infty) \leq P(Y(s_n) = 0)$ and $P(Y(s_n) = 0) = e^{-s_n}$ for all $n = 1,2,3,\ldots$, we see that $P(T = \infty) = 0$. Next we shall prove that $P(Y(s) = Y(t) \in (0,\infty)) = 0$ for all $-\log r < s < t < \infty$. For suppose that $P(Y(s) = Y(t) \in (0,\infty)) > 0$ for some $-\log r < s < t < \infty$. Then both s and t are irregular points, and hence Lemma 9.2.7 implies that $v(s,t) \neq 1$. On the other hand, for all $\omega \in \{\omega; Y(s,\omega) = Y(t,\omega) \in (0,\infty)\}$ it holds that

$$1 = \frac{Y(s,\omega)}{Y(t,\omega)} = \lim_{n\to\infty} \frac{c_n(s) Z_n(\omega)}{c_n(t) Z_n(\omega)} = v(s,t).$$

This proves that for almost all $\omega \in \Omega$ either (9.2.36) or (9.2.37) holds:

(9.2.36) $T(\omega) = -\log r;$

(9.2.37) $T(\omega) \in (-\log r, \infty),$ $Y(s,\omega) < 1$ on $(-\log r, T(\omega))$ and
                    $Y(s,\omega) > 1$ on $(T(\omega),\infty)$.

The random variable T will be used to prove the already announced almost sure convergence result.

THEOREM 9.2.8.

(a) *Let L be a non-decreasing function on* $[0,\infty)$, *such that* $\lim_{x\to\infty} L(x) = \infty$ *and* $\lim_{x\downarrow 0} L(x) = 0$, *and let* $\{a_n; n = 0,1,2,\ldots\}$ *be a sequence of positive constants. Suppose that* $\lim_{n\to\infty} a_n L(c_n(s))$ *exists* $\in (0,\infty)$ *for all* $s \in (-\log r, \infty)$ *and is continuous on* $(-\log r, \infty)$. *Call this limit* $\psi(s)$. *Then there is a random variable U such that*

(9.2.38) $a_n L(1/Z_n) \xrightarrow{\text{a.s.}} U$ *as* $n \to \infty$,

where $U = 0$ *on* $\{Z_n \to \infty\}$ *and* $U = \psi(T) \in (0,\infty)$ *a.s. on* $\{Z_n \to 0\}$, *with T as in* (9.2.34);

(b) *suppose furthermore that L is slowly varying at* ∞ *and that* ψ *is strictly increasing on* $(-\log r, \infty)$. *Then* $P(U \leq t) = 1 - \exp(-\psi^{\text{inv}}(t))$, $t \in (t_0, t_1)$, *where* $t_0 := \lim_{t\downarrow -\log r} \psi(t)$ *and* $t_1 := \lim_{t\to\infty} \psi(t)$.

REMARK 9.2.9. As a consequence of this theorem we have that the distribution function of U is given by

$$P(U \le t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0) \\ 1-r & \text{if } t \in [0, t_0) \\ 1 - \exp(-\psi^{inv}(t)) & \text{if } t \in [t_0, t_1) \\ 1 & \text{if } t \in [t_1, \infty) \end{cases}$$

Notice that this function is continuous on $(0, \infty)$. Later on we shall see that if L is continuous and strictly increasing on (A, ∞) for some $A \in [0, \infty)$, then $t_0 = 0$ and $t_1 = \infty$.

PROOF OF THEOREM 9.2.8. First of all we shall prove that $T = -\log r$ on $\{Z_n \to \infty\}$ and that $T \in (-\log r, \infty)$ a.s. on $\{Z_n \to 0\}$, where T is the random variable defined in (9.2.34). If $Z_n \to \infty$, then $Y(s) = \infty$ because $c_n(s) \to \infty$ for all $s \in (-\log r, \infty)$, and therefore $T = -\log r$ on $\{Z_n \to \infty\}$. The fact that $T \in (-\log r, \infty)$ a.s. on $\{Z_n \to 0\}$ follows now once we have proved that $P(T = -\log r) = 1-r$, because also $P(Z_n \to \infty) = 1-r$ while $P(Z_n \to 0 \text{ or } \infty) = 1$ and $P(T \in [-\log r, \infty)) = 1$. To this end we notice that it follows from Lemma 9.2.7 that there exists a sequence $\{s_n; n = 1,2,3,...\}$ of regular points such that $\lim_{n\to\infty} s_n = -\log r$. Now $P(T = -\log r) \le P(Y(s_n) = \infty) = 1 - e^{-s_n}$ for all $n = 1,2,3,...$, whence $P(T = -\log r) \le \lim_{n\to\infty}(1 - e^{-s_n}) = 1-r$. Since on the other hand $P(T = -\log r) \ge P(Z_n \to \infty) = 1-r$, this proves that $T \in (-\log r, \infty)$ a.s. on $\{Z_n \to 0\}$, implying that we can choose, for almost all $\omega \in \{Z_n \to 0\}$, numbers s and t such that $-\log r < s < T(\omega) < t < \infty$. By (9.2.37) this means that $c_n(s) < 1/Z_n(\omega) < c_n(t)$ for sufficiently large n. In view of the fact that we assumed that the function L, mentioned in the conditions of this theorem, is non-decreasing, and that a_n is positive for all $n = 0,1,2,...$, we obtain $a_n L(c_n(s)) \le a_n L(1/Z_n(\omega)) \le a_n L(c_n(t))$ for sufficiently large n. Finally the continuity of ψ on $(-\log r, \infty)$ yields that $\lim_{n\to\infty} a_n L(1/Z_n(\omega)) = \psi(T(\omega)) \in (0, \infty)$, that is $a_n L(1/Z_n) \xrightarrow{a.s.} U \in (0, \infty)$ on $\{Z_n \to 0\}$. Now suppose that $Z_n \to \infty$. Since $\lim_{n\to\infty} L(c_n(s)) = \lim_{x\to\infty} L(x) = \infty$ and $\lim_{n\to\infty} a_n L(c_n(s)) = \psi(s) < \infty$ for all $s \in (-\log r, \infty)$, and hence $\lim_{n\to\infty} a_n = 0$, the assumption $\lim_{x\downarrow 0} L(x) = 0$ implies that $\lim_{n\to\infty} a_n L(1/Z_n) = 0$ on $\{Z_n \to \infty\}$. This establishes part (a).

(b) We start with proving that the conditions of part (b) imply that all $s \in (-\log r, \infty)$ are regular points for the process $\{Z_n; n = 0,1,2,...\}$. This follows on observing that $v(s,t) = \lim_{n\to\infty} c_n(s)/c_n(t) = 0$ for all $-\log r < t < s < \infty$, because if $v(s,t) > 0$, then we would have by Corollary 1.2.1.2 in DE HAAN [1970] that $\lim_{n\to\infty} L(c_n(s))/L(c_n(t)) = 1$, in contradiction

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with the assumption that $\psi(s)$ is strictly increasing in s on $(-\log r, \infty)$.
Hence Lemma 9.2.7 implies that all $s \in (-\log r, \infty)$ are regular points. In
view of part (a) of this theorem and (9.2.35) we then obtain

$$P(U \leq t) = P(\psi(T) \leq t) = P(T \leq \psi^{inv}(t))$$

$$= \lim_{s \downarrow \psi^{inv}(t)} P(Y(s) = \infty) = 1 - \exp(-\psi^{inv}(t))$$

for all t such that $\psi^{inv}(t)$ is well-defined and $\psi^{inv}(t) \in (-\log r, \infty)$, that
is for all $t \in (t_0, t_1)$. $\square$

EXAMPLE 9.2.10. For an application of this last result we consider again a
branching process $\{Z_n; n = 0,1,2,\ldots\}$ having a strictly stable distribution
concentrated on $[0, \infty)$ with characteristic exponent $\alpha \in (0,1)$ as its off-
spring distribution. Since $h_n(s) = s^{\alpha^n}$, $c_n(s) = s^{\alpha^{-n}}$ and $-\log r = 1$, as we
know from the Examples 7.3.2 and 7.3.7, it follows that $L(s) = \log(1+s)$,
$s \in (0, \infty)$ and $a_n = \alpha^n$, $n = 0,1,2,\ldots$ satisfy the conditions of Theorem
9.2.8(a) and (b), with

$$\psi(s) = \lim_{n \to \infty} a_n L(c_n(s)) = \lim_{n \to \infty} \alpha^n \log(1 + s^{\alpha^{-n}}) = \log s,$$

$$s \in (-\log r, \infty) = (1, \infty).$$

Hence (9.2.38) implies that $\alpha^n \log(1 + 1/Z_n) \xrightarrow{a.s.} U$ as $n \to \infty$, where
$P(U = 0) = 1/e$ and

$$P(U \leq t) = 1 - \exp(-\psi^{inv}(t)) = 1 - \exp(-\exp t), \qquad t \in (0, \infty).$$

Because

$$\alpha^n \log(1 + 1/Z_n) \sim \alpha^n \log 1/Z_n = -\alpha^n \log Z_n$$

on $\{Z_n \to 0\}$ as $n \to \infty$ we see that $\alpha^n \log Z_n \xrightarrow{a.s.} -U$ on $\{Z_n \to 0\}$ as $n \to \infty$, where
$P(-U \leq t) = P(U \geq -t) = \exp(-\exp(-t))$, $t \in (-\infty, 0)$, in agreement with the dis-
tribution function derived in Example 9.2.2. Furthermore, $U \in (0, \infty)$ a.s.
on $\{Z_n \to 0\}$.

EXAMPLE 9.2.11. Just as in Example 7.3.8, we can also apply the result of Theorem 9.2.8 to processes which satisfy (9.2.9) and (9.2.10). Arguing as in Example 7.3.8 it follows that we can choose $a_n = \rho_n(x)$ for any $x \in (0,\infty)$ and $L(s) = \log(1+s)$, $s \in [0,\infty)$, where $\rho_n(x)$ is the nth iterate of the function $\rho(x)$ defined in (9.2.14). We then obtain that if (9.2.9) and (9.2.10) hold, then $\rho_n(x)\log(1 + 1/Z_n) \xrightarrow{a.s.}$ some random variable $U(x)$ as $n \to \infty$, where $U(x) = 0$ on $\{Z_n \to \infty\}$ and $P(U(x) \le t) = \exp(-\psi^{inv}(t,x))$, $t \in (0,\infty)$, with

$$\psi(t,x) = \lim_{n \to \infty} \rho_n(x)/\rho_n(\{\log[(1-\bar{r})/(1 - \exp\{-1/t\})]\}^{-1}).$$

This extends the weak convergence result we obtained by combining Lemma 9.2.1 and Theorem 9.2.3, to an almost sure convergence result.

We shall now construct a function L and a sequence $\{a_n ; n = 0,1,2,...\}$ which satisfy the conditions of Theorem 9.2.8(a). Let s_0 be any number $\in (-\log r, \infty)$. Since $(-\log r, \infty) = U_{n=-\infty}^{\infty} (h_{n+1}(s_0), h_n(s_0)]$, where we use the convention $h_{-n}(s) = c_n(s)$ for positive integers n, and since the sets $(h_{n+1}(s_0), h_n(s_0)]$ are disjoint for different values of n, it follows that there exist for all $x \in (-\log r, \infty)$ exactly one integer $n(x)$ and exactly one number $s(x) \in (h(s_0), s_0]$ such that $x = h_{n(x)}(s(x))$. Furthermore, we define the functions p and u by $p(s) = (s_0-s)/(s_0-h(s_0))$, $s \in [h(s_0), s_0]$ and $u(x) = n(x) + p(s(x))$, $x \in (-\log r, \infty)$. Finally we define L by

$$L(x) = \begin{cases} 0 & \text{if } x \in [0, -\log r] \\ \exp(-u(x)) & \text{if } x \in (-\log r, \infty) \end{cases},$$

and a_n by $a_n = e^{-n}$, $n = 0,1,2,...$. We shall prove that these L and a_n satisfy the conditions of Theorem 9.2.8(a). First of all we shall prove that $a_k L(c_k(x)) = L(x)$ for all $x \in (-\log r, \infty)$ and all $k = 1,2,3,...$. To this end we choose some $x \in (-\log r, \infty)$. Then by definition $s(x) = c_{n(x)}(x) \in (h(s_0), s_0]$. Since $n(h_{\ell}(s(x))) = \ell$ for every integer ℓ, it follows that $n(c_k(x)) = n(h_{-k+n(x)}(c_{n(x)}(x))) = n(h_{-k+n(x)}(s(x))) = -k + n(x)$; furthermore, $s(c_k(x)) = s(x)$. This implies that

$$a_k L(c_k(x)) = \exp(-k-u(c_k(x))) = \exp\{-k-n(c_k(x)) - p(s(c_k(x)))\}$$

$$= \exp\{-k+k-n(x) - p(s(x))\} = \exp(-u(x)) = L(x).$$

Next we shall prove that L is continuous on $(-\log r, \infty)$. Obviously L is continuous in x if $x \neq h_n(s_0)$ for some integer n. To prove the continuity of L in the points $h_n(s_0)$ it is sufficient to prove that $u(x)$ is right-continuous in s_0. Since $n(s_0) = 0$ and $s(s_0) = s_0$, it follows that $u(s_0) = 0$. Furthermore, $n(s_0+\varepsilon) = -1$ for all $\varepsilon \in (0,c(s_0)-s_0]$, and $s(s_0+\varepsilon) \downarrow h(s_0)$ as $\varepsilon \downarrow 0$. Therefore, $\lim_{\varepsilon \downarrow 0} u(s_0+\varepsilon) = -1 + p(h(s_0)) = -1 + 1 = 0 = u(s_0)$, proving the continuity of L on $(-\log r, \infty)$. Since $\lim_{x \to \infty} n(x) = -\infty$ and $\lim_{x \downarrow -\log r} n(x) = \infty$, we see that $\lim_{x \to \infty} L(x) = \infty$ and $\lim_{x \downarrow -\log r} L(x) = 0$. Since $L(x) = 0$ on $[0, -\log r]$, it follows that $\lim_{x \downarrow 0} L(x) = 0$, both if $r = 1$ and if $r < 1$. Finally we notice that $u(x)$ is non-increasing and hence $L(x)$ is non-decreasing on $(-\log r, \infty)$, and since $\lim_{x \downarrow -\log r} L(x) = 0$, also on $[0, \infty)$. So we have proved that L and $\{a_n; n = 0,1,2,\ldots\}$ satisfy the conditions of Theorem 9.2.8(a), with $\psi(s) = L(s)$ for $s \in (-\log r, \infty)$. We can formulate this as follows.

THEOREM 9.2.12. *There exists a function L such that* $e^{-n}L(1/Z_n)$ *converges almost surely to some random variable U as* $n \to \infty$, *where* $U = 0$ *on* $\{Z_n \to \infty\}$ *and* $U \in (0,\infty)$ *a.s. on* $\{Z_n \to 0\}$.

We shall now turn back again for a while to Theorem 9.2.8. It turns out that the random variable U, introduced in that theorem as $\lim_{n \to \infty} a_n L(1/Z_n)$ can be represented under certain conditions as a minimum of a random number of random variables.

THEOREM 9.2.13. *Let L,* $\{a_n; n = 0,1,2,\ldots\}$, ψ *and* U *be as in Theorem 9.2.8 and suppose that the conditions of both part (a) and part (b) of that theorem are satisfied, and that furthermore* $\alpha := \lim_{n \to \infty} a_n/a_{n-1}$ *exists* $\in (0,1)$, *and that L is strictly increasing and continuous on* (A,∞) *for some* $A \in [0,\infty)$. *Then*

$$(9.2.39) \qquad U = \alpha^k \min\{U_1^{(k)}, U_2^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\} \quad \text{a.s.,}$$

for every $k = 0,1,2,\ldots$, *where* $U_1^{(k)}, U_2^{(k)}, \ldots$ *are random variables all distributed as* U *and* $P(\bar{U}^{(k)} > t \mid Z_k) = P(U > t)^{Z_k - [Z_k]}$ *for all* $t \in (-\infty, \infty)$, *and where furthermore* $Z_k, U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$ *are independent and* $\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, U_3^{(k)}, \ldots$ *are conditionally given* Z_k *independent. The distribution function F of U satisfies*

$$(9.2.40) \qquad -\log(1 - F(\alpha t)) = h(-\log\{1 - F(t)\}), \qquad t \in (-\infty, \infty).$$

PROOF. From Remark 9.2.9 we know that F is continuous on $(0,\infty)$. Therefore, for every $t \in (0,\infty)$

(9.2.41) $\quad \{U \leq t\} = \lim_{n \to \infty} \{a_n L(1/Z_n) \leq t\} \quad$ a.s..

(The notation $A = B$ a.s. for two subsets A, B of Ω means that $P((A \cap B^C) \cup (A^C \cap B)) = 0$.) Since L is slowly varying at ∞, and $\lim_{x \to \infty} L(x) = \infty$ and $a_n \to 0$ as $n \to \infty$, as we know from the proof of Theorem 9.2.8(a), we see that $a_n L(bL^{inv}(t/a_n)) \sim a_n L(L^{inv}(t/a_n)) = t$ as $n \to \infty$ for any constant $b \in (0,\infty)$. Because t is a continuity point of F, this means that

$$\{U \leq t\} = \lim_{n \to \infty} \{a_n L(1/Z_n) \leq a_n L(bL^{inv}(t/a_n))\}$$

$$= \lim_{n \to \infty} \{1/Z_n \leq bL^{inv}(t/a_n)\}$$

$$= \lim_{n \to \infty} \{L^{inv}(t/a_n)Z_n \geq 1/b\} \quad \text{a.s.,}$$

and thus, since $b \in (0,\infty)$ is arbitrary, we obtain that

(9.2.42) $\quad L^{inv}(t/a_n)Z_n \xrightarrow{\text{a.s.}} \begin{cases} \infty & \text{on } \{U \leq t\} \\ 0 & \text{on } \{U > t\} \end{cases} \quad$ as $n \to \infty$.

Now remembering the definition of the branching process $\{Z_n; n = 0,1,2,\ldots\}$, it follows that we can consider Z_{n+k}, conditionally given Z_k, as a "sum" of Z_k independent random variables, each of which can be interpreted as the size of the nth generation of a branching process having the same offspring distribution as the process $\{Z_n; n = 0,1,2,\ldots\}$. More precisely, for any fixed non-negative integer k, we can write for every integer $n = 0,1,2,\ldots$

(9.2.43) $\quad Z_{n+k} = \sum_{j=1}^{[Z_k]} Z_{n,j}^{(k)} + \bar{Z}_n^{(k)},$

where $\{\{Z_{n,j}^{(k)}; n = 0,1,2,\ldots\}; j = 1,2,3,\ldots\}$ is a sequence of branching processes all distributed as $\{Z_n; n = 0,1,2,\ldots\}$, and $\{\bar{Z}_n^{(k)}; n = 0,1,2,\ldots\}$ is, conditionally given Z_k, a branching process with the same offspring distribution as $\{Z_n; n = 0,1,2,\ldots\}$, but with $P(\bar{Z}_0^{(k)} = Z_k - [Z_k] \mid Z_k) = 1$; furthermore Z_k, $\{Z_{n,1}^{(k)}; n = 0,1,2,\ldots\}$, $\{Z_{n,2}^{(k)}; n = 0,1,2,\ldots\}$, $\ldots$ are

independent and $\{\bar{z}_n^{(k)};\ n = 0,1,2,\ldots\}$, $\{z_{n,1}^{(k)};\ n = 0,1,2,\ldots\}$, $\{z_{n,2}^{(k)};\ n = 0,1,2,\ldots\}$, ... are, conditionally given Z_k, independent. An application of Theorem 9.2.8 now yields that for every $j = 1,2,3,\ldots$ it holds that $U_j^{(k)} := \lim_{n\to\infty} a_n L(1/z_{n,j}^{(k)})$ exists a.s. and is distributed as U. Furthermore, obviously $Z_k, U_1^{(k)}, U_2^{(k)}, \ldots$ are independent. Concerning $\{\bar{z}_n^{(k)};\ n = 0,1,2,\ldots\}$, we notice that if we consider a branching process $\{\bar{z}_n;\ n = 0,1,2,\ldots\}$ having the same offspring distribution as $\{Z_n;\ n = 0,1,2,\ldots\}$, but with $\bar{z}_0 = d$ for some constant $d \in (0,\infty)$, then

$$\bar{h}_n(s) := -\log E\, e^{-s\bar{z}_n} = -\log E(E(e^{-s\bar{z}_n} \mid \bar{z}_{n-1}))$$

$$= -\log E\, e^{-h(s)\bar{z}_{n-1}} = \bar{h}_{n-1}(h(s)) \qquad \text{for } n = 1,2,3,\ldots,$$
$$s \in [0,\infty).$$

Since $\bar{h}_0(s) := -\log E\, e^{-s\bar{z}_0} = ds$, this implies that $\bar{h}_n(s) = dh_n(s)$, and therefore $\bar{c}_n(s)$, its inverse, satisfies $\bar{c}_n(s) = c_n(s/d)$. Hence we obtain that

$$\bar{\psi}(s) := \lim_{n\to\infty} a_n L(\bar{c}_n(s)) = \lim_{n\to\infty} a_n L(c_n(s/d)) = \psi(s/d).$$

This means that $a_n L(1/\bar{z}_n)$ converges almost surely to some random variable $\bar{U}$, for which $P(\bar{U} = 0) = 1 - \exp(-\lim_{n\to\infty} \bar{h}_n(s)) = 1-r^d$, and

$$P(\bar{U} \le t) = 1 - \exp(-\bar{\psi}^{inv}(t)) = 1 - \exp(-d\psi^{inv}(t)), \qquad t \in (t_0, t_1),$$

that is $P(\bar{U} > t) = P(U > t)^d$ for all $t \in (-\infty,\infty)$. We can therefore conclude that $\bar{U}^{(k)} := \lim_{n\to\infty} a_n L(1/\bar{z}_n^{(k)})$ exists almost surely and that

$$P(\bar{U}^{(k)} > t \mid Z_k) = P(U > t)^{Z_k - \lfloor Z_k \rfloor} \qquad \text{for every } t \in (-\infty,\infty),$$

and it is clear that the random variables $\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, \ldots$ are, conditionally given Z_k, independent. Using a similar argument as the one leading to (9.2.42), it follows that $L^{inv}(t/a_n)z_{n,j}^{(k)}$ converges almost surely to zero or infinity as $n \to \infty$ for every $j = 1,2,3,\ldots$, and that conditionally given Z_k, also $L^{inv}(t/a_n)\bar{z}_n^{(k)}$ converges almost surely to zero or infinity as $n \to \infty$. The assumption that $\lim_{n\to\infty} a_n/a_{n-1} = \alpha$, together with (9.2.41) and (9.2.43) implies that

$$\{U \le t\} = \lim_{n\to\infty} \{a_{n+k} L(1/Z_{n+k}) \le t\}$$

$$= \lim_{n\to\infty} \{\alpha^k a_n L(1/Z_{n+k}) \le t\}$$

$$= \lim_{n\to\infty} \{1/Z_{n+k} \le L^{inv}(t/(\alpha^k a_n))\}$$

$$= \lim_{n\to\infty} \{L^{inv}(t/(\alpha^k a_n)) Z_{n+k} \ge 1\}$$

$$= \lim_{n\to\infty} \left\{ \sum_{j=1}^{[Z_k]} L^{inv}(t/(\alpha^k a_n)) Z_{n,j}^{(k)} + L^{inv}(t/(\alpha^k a_n)) \bar{Z}_n^{(k)} \ge 1 \right\}$$

a.s.

for every $t \in (0,\infty)$. This last expression is a sum consisting of a finite number of terms, each of which converges to zero or infinity almost surely as $n \to \infty$. In order that the limit of this sum is at least 1 it is therefore necessary and sufficient that at least one of the terms converges to infinity. So we get

$$\{U \le t\} = \lim_{n\to\infty}\{\max\{L^{inv}(t/(\alpha^k a_n)) Z_{n,1}^{(k)}, \ldots,$$

$$\ldots, L^{inv}(t/(\alpha^k a_n)) Z_{n,[Z_k]}^{(k)}, L^{inv}(t/(\alpha^k a_n)) \bar{Z}_n^{(k)}\} \ge 1\}$$

$$= \lim_{n\to\infty}\{\min\{1/Z_{n,1}^{(k)}, \ldots, 1/Z_{n,[Z_k]}^{(k)}, 1/\bar{Z}_n^{(k)}\} \le L^{inv}(t/(\alpha^k a_n))\}$$

$$= \lim_{n\to\infty}\{\min\{a_n L(1/Z_{n,1}^{(k)}), \ldots, a_n L(1/Z_{n,[Z_k]}^{(k)}),$$

$$a_n L(1/\bar{Z}_n^{(k)})\} \le t/\alpha^k\}$$

$$= \{\alpha^k \min\{U_1^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\} \le t\} \qquad \text{a.s.}$$

for every $t \in (0,\infty)$. Obviously this relation also holds for every $t \in (-\infty,0)$. We have thus proved that the sets $\{U \le t\}$ and $\{\alpha^k \min\{U_1^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\} \le t\}$ are a.s. equal both for $t \in (-\infty,0)$ and for $t \in (0,\infty)$, and therefore also for $t = 0$. This establishes the first part of the theorem. Taking $k = 1$ we obtain

$$1 - F(\alpha t) = P(U > \alpha t) = EP(\min\{U_1^{(1)}, \ldots, U_{[Z_1]}^{(1)}, \bar{U}^{(1)}\} > t \mid Z_1)$$

$$= E\{P(U > t)\}^{[Z_1] + Z_1 - [Z_1]} = EP(U > t)^{Z_1} = e^{-h(-\log\{1-F(t)\})},$$

thus proving (9.2.40).   ☐

Using the notation of Theorem 9.2.8 we see that, under the conditions of Theorem 9.2.13,

$$-\log(1 - F(t)) = \psi^{inv}(t) \in (-\log r, \infty) \qquad \text{for any } t \in (t_0, t_1).$$

By (9.2.40) this means that also $-\log(1 - F(\alpha t)) \in (-\log r, \infty)$, that is also $\alpha t \in (t_0, t_1)$. Since $\alpha \in (0,1)$ this is only possible if $t_0 = 0$. Conversely, if $\alpha t \in (t_0, t_1)$, then $-\log(1 - F(\alpha t)) \in (-\log r, \infty)$, and hence also $-\log(1 - F(t)) \in (-\log r, \infty)$, that is $t \in (t_0, t_1)$. This implies that $t_1 = \infty$. Now substituting $-\log(1 - F(t)) = \psi^{inv}(t)$ for $t \in (0,\infty)$ in (9.2.40), we obtain $\psi^{inv}(\alpha t) = h(\psi^{inv}(t))$ for all $t \in (0,\infty)$. This equation can be considered as the analog of (8.2.1), which says that $\phi(u,s)$, the cumulant generating function of $Y(s)$, satisfies $\phi(au,s) = h(\phi(u,s))$.

EXAMPLE 9.2.14. We can again apply this theorem to a branching process $\{Z_n; n = 0,1,2,\ldots\}$ having a strictly stable distribution concentrated on $[0,\infty)$ with characteristic exponent $\alpha \in (0,1)$ as its offspring distribution, with L and $\{a_n; n = 0,1,2,\ldots\}$ as in Example 9.2.10, since

$$\lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lim_{n \to \infty} \frac{\alpha^n}{\alpha^{n-1}} = \alpha \in (0,1)$$

and $L(s) = \log(1+s)$ is continuous and strictly increasing on $(0,\infty)$. Hence we obtain that for any fixed non-negative integer k, U is a.s. equal to

$$\alpha^k \min\{U_1^{(k)}, \ldots, U_{[Z_k]}^{(k)}, \bar{U}^{(k)}\},$$

with the random variables $\bar{U}^{(k)}, U_1^{(k)}, U_2^{(k)}, \ldots$ as in Theorem 9.2.13. Relation (9.2.40) becomes $-\log(1 - F(\alpha t)) = -\log(1 - F(t))^{\alpha}$, as fits in with the fact that $F(t) = 1 - \exp(-\exp t)$ for $t \in (0,\infty)$, as we know from Example 9.2.10.

EXAMPLE 9.2.15. For another application of Theorem 9.2.13 we consider again processes for which (9.2.9) and (9.2.10) are satisfied. In Example 9.2.11 we saw that with $a_n = \rho_n(x)$ for any $x \in (0,\infty)$ and $L(s) = \log(1+s)$, $s \in [0,\infty)$, the conditions of Theorem 9.2.8 are fulfilled and so

$$\rho_n(x) \log(1 + 1/Z_n) \xrightarrow{a.s.} U(x) \qquad \text{as } n \to \infty.$$

Furthermore, in a similar way as in Example 7.3.12 we get

$$\lim_{n \to \infty} \frac{\rho_n(x)}{\rho_{n-1}(x)} = \gamma \in (0,1).$$

Since $L(s)$ is strictly increasing in s on $(0,\infty)$, we can apply (9.2.40) to obtain that $-\log(1 - F(\gamma t,x)) = h(-\log\{1 - F(t,x)\})$, $t \in (-\infty,\infty)$, where $F(t,x)$ is the distribution function of $U(x)$. Now we know from Lemma 9.2.1 and Theorem 9.2.3 that if $m < \infty$ and (9.2.9) and (9.2.10) hold, then $\rho_n(x)\log Z_n$ converges in distribution as $n \to \infty$ to a random variable $Z(x)$, which has an atom of size $1-r$ at 0. Since $\rho_n(x)\log Z_n < 0$ on $\{Z_n \to 0\}$ and $\rho_n(x)\log Z_n > 0$ on $\{Z_n \to \infty\}$ for sufficiently large n, and $\rho_n(x)\log(1 + 1/Z_n) \sim -\rho_n(x)\log Z_n$ as $n \to \infty$ on $\{Z_n \to 0\}$ and $\rho_n(x)\log(1 + 1/Z_n) \xrightarrow{a.s.} 0$ as $n \to \infty$ on $\{Z_n \to \infty\}$, it follows that

$$I_{\{Z_n \to \infty\}} \cdot \rho_n(x)\log Z_n \xrightarrow{d} 0$$

and that

$$I_{\{Z_n \to 0\}} \cdot \rho_n(x)\log Z_n \xrightarrow{a.s.} -U(x) \qquad \text{as } n \to \infty,$$

where I stands for the indicator function. This means that

$$\rho_n(x)\log Z_n = I_{\{Z_n \to \infty\}} \cdot \rho_n(x)\log Z_n + I_{\{Z_n \to 0\}} \cdot \rho_n(x)\log Z_n \xrightarrow{d} -U(x)$$

$$\text{as } n \to \infty,$$

whence $Z(x) \overset{d}{=} -U(x)$. Therefore, $w(t,x)$, the distribution function of $Z(x)$, satisfies $w(t,x) = 1 - F(-t,x)$, and relation (9.2.40) becomes

$$-\log w(-\gamma t,x) = h(-\log w(-t,x)), \qquad t \in (-\infty,\infty).$$

For $t \in (0,\infty)$, this is just (9.2.19).

We close this section with a discussion on the norming of Z_{n+1} by a suitable function of Z_n, that is we try to find a sequence of functions $\{f_n; n = 0,1,2,\ldots\}$ such that $Z_{n+1}/f_n(Z_n)$ converges in some sense to a random variable W with $P(0 < W < \infty) > 0$. If we want to do so we encounter the same problems as in Section 7.3, where we studied this question for explosive processes, and just like we did there, we shall now only present

one example.

EXAMPLE 9.2.16. Let $\{Z_n; \ n = 0,1,2,\ldots\}$ be a branching process having a strictly stable distribution concentrated on $[0,\infty)$ with characteristic exponent $\alpha \in (0,1)$ as its offspring distribution. For this process we derived relation (7.3.24) which says that

$$\liminf_{n \to \infty} \frac{Z_{n+1}(\log n)^{(1-\alpha)/\alpha}}{Z_n^{1/\alpha}} = B(\alpha)^{(1-\alpha)/\alpha} \qquad \text{a.s..}$$

From Example 9.2.10 we know that $\alpha^n \log Z_n \xrightarrow{\text{a.s.}} -U \in (-\infty,0)$ as $n \to \infty$ on $\{Z_n \to 0\}$. In a similar way as in Example 7.3.13 it now follows that

$$\liminf_{n \to \infty} \frac{Z_{n+1}(\log \log | \log Z_n |)^{(1-\alpha)/\alpha}}{Z_n^{1/\alpha}} = B(\alpha)^{(1-\alpha)/\alpha} \qquad \text{a.s. on} \\ \{Z_n \to 0\}$$

and that $Z_{n+1}/f_\gamma(Z_n) \xrightarrow{\text{a.s.}} \infty$ as $n \to \infty$ on $\{Z_n \to 0\}$, where

$$f_\gamma(t) = \frac{\log|\log t|}{c(\gamma t^{-1}\log|\log t|)} \ .$$

Furthermore, we can prove, analogously to (7.3.31)

$$\limsup_{n \to \infty} \frac{Z_{n+1}}{f(\log|\log Z_n|/(-\log \alpha))^{1/\alpha} \cdot Z_n^{1/\alpha}} =$$

$$= \begin{cases} 0 & \text{if } \Sigma_{n=0}^{\infty} \dfrac{1}{f(n)} < \infty \\ \infty & \text{if } \Sigma_{n=0}^{\infty} \dfrac{1}{f(n)} = \infty \end{cases}$$

a.s. on $\{Z_n \to 0\}$, where f is any non-negative function on $[0,\infty)$ such that $\lim_{n \to \infty} f(n) = \infty$ and $f(n) \sim f(a_n)$ as $n \to \infty$ for any sequence of constants $\{a_n; \ n = 0,1,2,\ldots\}$ for which $a_n \sim n$ as $n \to \infty$.

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