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## PREFACE

Both the title and the subtitle of this collection of papers are somewhat inadequate. As there are here a number of papers on geometry, the title "Topological Structures II" is actually too narrow; it was chosen, nevertheless, to bring to mind that the 1978 symposium in Amsterdam was the second one in recent years<sup>\*)</sup> devoted (mainly) to topology. As for the subtitle, the designation "Proceedings" stretches and extends the usual meaning of that term, inasmuch not all of the papers in this volume proceed directly from the symposium.

Of the 34 papers presented here, 16 originate from lectures, and 2 from a problem session during the 1978 meeting. In addition, 8 papers have been contributed by participants to the symposium. Most of the remaining 8 papers have been solicited by the editors. In this manner, the valuable survey papers by M.G. BELL, D.W. CURTISS, M. HUŠEK, F.D. TALL and R.G. WOODS could be added to these "Proceedings".

The symposium on Topology and Geometry at the Free University in Amsterdam on October 31 and November 1 and 2, 1978, was sponsored by the Dutch Mathematical Society WISKUNDIG GENOOTSCHAP and financed by the Mathematics Department of the Free University, Amsterdam. In addition to the editors of these Proceedings, W.T. van EST and M.A. MAURICE took an essential part in the organization of the symposium. On the last day of the symposium there was a meeting of Dutch mathematicians working in the fields of topology and geometry.

Thanks are due, in the first place to all contributors to this volume and to the participants to the symposium; to them this volume owes its very existence. In the second place, we thank the Mathematics Department of the Free University in Amsterdam for its generous financial support, and the WISKUNDIG GENOOTSCHAP for sponsoring the symposium. Finally, we thank the Director of the Mathematical Centre for his consent to publish these volumes

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\*) The first symposium was held in 1973; its proceedings (entitled "*Topological Structures*") were published by the Mathematical Centre in Amsterdam as Mathematical Centre Tracts 52.

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as Mathematical Centre Tracts, and Mr. D. ZWARST and the Publication Service of the Mathematical Centre for the friendly and helpful way in which they did all those jobs necessary for the publication of these Proceedings.

P.C. Baayen

J. van Mill

## LOCAL DYNAMICAL SYSTEMS AND THEIR MORPHISMS

J.M. Aarts & J. de Vries

### 1. INTRODUCTION

1.1. The theory of local dynamical systems has its roots in the qualitative study of ordinary differential equations. It is an outgrowth of the "geometrical theory" initiated by H. POINCARÉ, A LYAPUNOV and G.D. BIRKHOFF. The latter actually laid the foundations of the theory of dynamical systems and the topological study of them (topological dynamics). The basic problems of this study were: 1) the study of solutions "globally", and 2) the study of solutions near singular points, both by means of topological methods. This can be done by either considering continuous actions of the additive group  $\mathbb{R}$  of the reals on metric spaces (see [15] or [5]) or, more abstractly, studying arbitrary topological groups, acting on arbitrary topological spaces (see [8] or [7]).

The relevance of group actions for the study of the behaviour of solutions of differential equations is based on the following fact. Consider an autonomous differential equation

$$(1.1) \quad \dot{x} = f(x)$$

where  $f$  is a continuous  $\mathbb{R}^n$ -valued function, defined on an open subset  $X$  of  $\mathbb{R}^n$ . Under suitable additional conditions, guaranteeing unicity and extendability of solutions, there exists for every  $x \in X$  a unique solution  $\pi_x$  of (1.1) such that  $\pi_x(0) = x$ . Here  $\pi_x$  is a function of  $\mathbb{R}$  into  $X$  for every  $x \in X$ , and it can easily be shown that the following conditions are fulfilled:

DS1. the mapping  $\pi: (x,t) \mapsto \pi_x t: X \times \mathbb{R} \rightarrow X$  is continuous;

DS2.  $\pi(x,0) = x$  for all  $x \in X$ ;

DS3.  $\pi(\pi(x,t),s) = \pi(x,t+s)$  for all  $x \in X$  and  $t,s \in \mathbb{R}$ .

In this way a continuous action  $\pi$  of the additive group  $\mathbb{R}$  on the space  $X$  is obtained such that the orbits of points of  $X$  under this action are just the

solution curves of equation (1.1). This motivates the definition of a (*global*) *dynamical system* as a triple  $\langle X, \mathbb{R}, \pi \rangle$ , where  $X$  is an arbitrary topological space<sup>\*</sup> and  $\pi$  is a mapping from  $X \times \mathbb{R}$  into  $X$ , satisfying the conditions DS1, DS2 and DS3. In this setting, most of the notions of classical dynamics can be defined and studied. For example, a point  $x \in X$  is called *periodic* whenever  $\pi_x(t) = x$  for some  $t \neq 0$ ; it is called a *rest point* whenever  $\pi_x(t) = x$  for all  $t \in \mathbb{R}$ . If  $X$  is a Hausdorff space and  $x \in X$  is a *moving periodic point* (i.e. periodic, but not a rest point), then it can easily be shown that the real number

$$(1.2) \quad p_x := \inf\{t \in \mathbb{R}: t > 0 \text{ and } \pi_x(t) = x\}$$

is non-zero; the number  $p_x$  is called the *primitive period* of  $x$ , and all periods of  $x$  (i.e. real numbers  $t$  with the property that  $\pi_x(t) = x$ ) are integral multiples of  $p_x$ . Another important notion is that of a limit set: if  $x \in X$ , then the (possibly empty) sets

$$(1.3) \quad A(x) := \bigcap_{t < 0} \overline{\pi_x(-\infty, t]}; \quad \Omega(x) := \bigcap_{t > 0} \overline{\pi_x[t, \infty)}$$

are called the *negative*<sup>†</sup> and *positive limit sets* of  $x$ , respectively. For the study of these and other notions, the reader is referred to [5] or [18], where the (topological) study of dynamical systems is exposed in a systematic way.

1.2. In order to cover also the study of solutions of equation (1.1) for the case that there is still unicity of solutions, but possibly no extension of solutions to all of  $\mathbb{R}$ , the concept of a *local dynamical system* (in the sequel to be abbreviated to *lds*) has to be used, a concept which has been introduced independently in [9] and [16], and which is closely related to the notion of an F-family introduced in [19] (see [3] for this relationship). An *lds* is a triple  $(X, D, \pi)$  where  $X$  is a topological space,  $D$  is a subset of  $X \times \mathbb{R}$  of the following special form

$$(1.4) \quad D = \bigcup_{x \in X} \{x\} \times J(x),$$

\* ) all spaces are supposed to be Hausdorff spaces.

† ) the  $A$  occurring in (1.3) is a capital alpha (symbolizing the begin, while  $\Omega$  symbolizes the end).



where  $J(x)$  is an open interval in  $\mathbb{R}$  containing 0 for every  $x \in X$ , and  $\pi: D \rightarrow X$  is a mapping satisfying the following conditions:

LDS1. Continuity axiom:  $D$  is open in  $X \times \mathbb{R}$  and  $\pi: D \rightarrow X$  is continuous;

LDS2. Identity axiom: if  $x \in X$ , then  $\pi(x,0) = x$ ;

LDS3. Group axiom: if  $x \in X$  and  $t, t+s \in J(x)$  and  $s \in J(\pi(x,t))$ , then  $\pi(\pi(x,t),s) = \pi(x,t+s)$ ;

LDS4. Maximality axiom: if  $(x,t) \in D$ , then  $J(\pi(x,t)) = J(x) - t$ .

(Note that in the case that  $J(x) = \mathbb{R}$  for all  $x \in X$  (that is,  $D = X \times \mathbb{R}$ ) we have a global dynamical system as defined in 1.1 above.) The space  $X$  and the mapping  $\pi$  are called the *phase space* and the *phase mapping* respectively. For every  $x \in X$ , the mapping  $\pi_x: t \mapsto \pi(x,t): J(x) \rightarrow X$  is called the *motion* of  $x$ , and the set  $\Gamma(x) := \pi_x[J(x)]$  is called the *orbit* of  $x$ . The definition of periodic point and rest point in an lds are formally the same as given in 1.1 above for global dynamical systems; in fact, for such points the difference between global and local systems disappears in the sense that if  $x$  is periodic, then  $J(x) = \mathbb{R}$  (an immediate consequence of the maximality axiom LDS4).

For the study of lds's, we refer to [11] and [17]. Most methods from the theory of global systems can be used in the context of local systems. The only difficulty is that  $\pi$  may be not defined on all of  $X \times \mathbb{R}$ , so that one always has to be careful in writing down " $\pi(x,t)$ " for a given  $x \in X$  and  $t \in \mathbb{R}$ ; in such situations the maximality axiom LDS4 usually is of great help. This has been illustrated above by the remark that  $J(x) = \mathbb{R}$  if  $x$  is periodic. Another illustration is as follows. If  $(X,D,\pi)$  is an lds, then for every  $x \in X$  the open interval  $J(x)$  will be denoted  $(\alpha(x), \omega(x))$  with  $-\infty \leq \alpha(x) < 0 < \omega(x) \leq \infty$ . Now the *negative* and *positive limit sets* of  $x \in X$  are defined by

$$A(x) := \overline{\{\pi_x(\alpha(x), t) : \alpha(x) < t \leq 0\}}$$

and

$$\Omega(x) := \overline{\{\pi_x[t, \omega(x)) : 0 \leq t < \omega(x)\}}.$$

It can be shown that in the definition of an lds the maximality axiom can be replaced by the condition

LSD4'. For every  $x \in X$ , if  $\alpha(x) > -\infty$  then  $A(x) = \emptyset$  and if  $\omega(x) < \infty$  then  $\Omega(x) = \emptyset$ .

(In particular, this guarantees that  $\mathbb{R}^- \subset J(x)$  if  $A(x) \neq \emptyset$  and  $\mathbb{R}^+ \subset J(x)$  if  $\Omega(x) \neq \emptyset$ , in which cases the first and the second equality in (1.3) can be used.) Another statement which can replace LDS4 such as to produce (together with LDS1 through LDS3) a set of axioms which is equivalent to LDS1 through LDS4 is:

LDS4". For every  $x \in X$ , if  $\alpha(x) > -\infty$  then  $\overline{\Gamma^-(x)}$  is not compact and if  $\omega(x) < \infty$  then  $\overline{\Gamma^+(x)}$  is not compact <sup>\*</sup>).

Proofs of the equivalence of LDS4, LDS4' and LDS4" under assumption of LDS1 through LDS3 are contained in [20] and in [11; Section IV.1]. This equivalence shows that our axiom system is equivalent to the axiom systems, given in [11], [17] and [20].

The name "maximality axiom" refers to the following fact: if  $(X, D, \pi)$  and  $(X, E, \rho)$  are lds's such that  $\pi|_{D \cap E} = \rho|_{D \cap E}$ , then it can be shown (using LDS4 in an essential way) that  $D = E$ , whence  $\pi = \rho$ . In particular, the domain of the phase mapping of an lds cannot be extended without violating axiom LDS4. In this context, it is interesting to observe that for any system  $(X, D, \pi)$  satisfying LDS1, LDS2 and LDS3 (a so-called *germ* of an lds; in fact, LDS1 may even slightly be weakened:  $D$  need not be open, but is required to be a neighbourhood of  $X \times \{0\}$  in  $X \times \mathbb{R}$ ) the domain  $D$  and the phase mapping  $\pi$  can be extended in a unique way so as to produce an lds. See [10] for details.

1.3. As was said above, the definition of an lds was motivated by autonomous ordinary differential equations (they can also be used as a model for solutions of non-autonomous equations; cf. [16] or [17]). In fact, if  $X$  is an open subset of  $\mathbb{R}^n$  and  $f: X \rightarrow \mathbb{R}^n$  is a continuous function such that the equation (1.1) has unique solutions (i.e. for every  $x \in X$  there is a unique solution  $\pi_x$  defined in a neighbourhood of 0 such that  $\pi_x(0) = x$ ), then let  $J(x)$  denote the maximal interval in  $\mathbb{R}$  to which this solution can be extended. If we denote, for every  $x \in X$ , this maximal solution by  $\pi_x: J(x) \rightarrow X$  and if we put  $D := \bigcup_{x \in X} \{x\} \times J(x)$ , and  $\pi(x, t) := \pi_x(t)$  for  $(x, t) \in D$ , then we obtain in this way an lds  $(X, D, \pi)$ . The orbits of this lds are just the solution curves of equation (1.1). We shall call this lds the *system, defined by the differential equation  $\dot{x} = f(x)$*  or: *defined by the vector-field  $f$  on  $X$* .

<sup>\*</sup>) Here  $\Gamma^\pm(x) := \pi_x[\mathbb{R}^\pm \cap J(x)]$ .

1.4. In differential equation theory, two differential equations on the same domain  $X$  in  $\mathbb{R}^n$ , say

$$\dot{x} = f(x), \quad \dot{x} = g(x),$$

( $f$  and  $g$  both continuous and giving unicity of solutions) are called *geometrically equivalent* if each solution of either system is a reparametrization of the other. For the corresponding lds's this means exactly that each orbit of either system is an orbit in the other system. It is well-known that this situation occurs iff

$$f(x) = k(x)g(x), \quad x \in X$$

for some continuous mapping  $k: X \rightarrow \mathbb{R}$  such that  $k(x) \neq 0$  for all  $x \in X$ . In particular, any lds defined by a differential equation is geometrically equivalent to a global system (Vinograd's theorem).

These results have their counterparts in arbitrary lds's<sup>\*)</sup>. Two lds's  $(X, D, \pi)$  and  $(Y, E, \rho)$  are called *geometrically equivalent* whenever there exists a homeomorphism  $h$  of  $X$  onto  $Y$  such that  $h[\Gamma_\pi(x)] = \Gamma_\rho(h(x))$  for all  $x \in X^\dagger$ . It can be shown that a homeomorphism  $h$  of  $X$  onto  $Y$  is a geometric equivalence iff there exists for every  $x \in X$  a homeomorphism  $\tau_x$  of  $J_\pi(x)$  onto  $J_\rho(h(x))$  such that

$$(1.5) \quad h \circ \pi(x, t) = \rho(h(x), \tau_x(t))$$

for all  $(x, t) \in D$ . See [11; Thm. VI. 1.14] for the case that  $X = Y$  and  $h = 1_X$  (also [4; Ch. 4] for global systems), [20] for the case that  $X$  and  $Y$  are Tychonoff spaces and [13] for the case that  $X$  and  $Y$  are  $T_1$ -spaces, cf. also 4.4 below. In [13] and [20] it is also shown that the mapping  $\tau: (x, t) \mapsto \tau_x(t): D \rightarrow \mathbb{R}$  is continuous on  $D \setminus (S_\pi \times \mathbb{R})$ . Here  $S_\pi$  denotes the set of rest points in the lds  $(X, D, \pi)$ . (Notice that in these papers, a geometrical equivalence is called an NS-isomorphism - after [15] - and a pair  $(h, \tau)$  satisfying the conditions mentioned above a GH-isomorphism - after [8].)

\*) For a generalization of Vinograd's theorem, cf. [6] and [22].

†) When dealing with more than one lds at a time, we shall use the symbol for the phase mapping as a subscript to distinguish the several sets which are associated with these systems.

Pairs  $(h, \tau)$  satisfying the above conditions are also called *phase space homeomorphisms with reparametrization* (cf. [12]); they include *equivariant homeomorphisms* (i.e. the case that  $\tau_x(t) = t$  for all  $(x, t) \in D$ ), which are related to isomorphisms of topological transformation groups [8] and also to the *conjugacy relation* of [4]: a reparametrization followed by an equivariant homeomorphism.

In the remainder of this paper, we shall discuss a notion of *morphism of lds's*, introduced by the authors in [1]. The definition is such that, first, important dynamical properties are preserved by morphisms and, second, in the resulting category of lds's the isomorphisms are just the phase space homeomorphisms with reparametrization. For general morphisms there are also counterparts for the theorems of HAJEK, URA and KIMURA mentioned above (cf. Theorem 3.4 below) as well as for the relationship with the conjugacy relation of BECK (cf. 3.3 below and the remarks after 3.3).

## 2. MORPHISMS OF LDS'S

All lds's are assumed to have Hausdorff phase spaces. The domain of the motion of a point  $x$  in an lds will consistently be denoted by  $J(x)$  or  $(\alpha(x), \omega(x))$ , where  $-\infty \leq \alpha(x) < 0 < \omega(x) \leq \infty$ . The set of periodic points will be denoted by  $P$ , and the set of rest points by  $S$ . All other notation will be as in Section 1. Recall, that the phase mapping will be used as a subscript in order to be able to distinguish between notions associated to different lds's.

2.1. A *morphism (of lds's)*  $\Phi$  from  $(X, D, \pi)$  to  $(Y, E, \rho)$  is a pair  $(\phi, \tau)$ , where  $\phi: X \rightarrow Y$  and  $\tau: D \rightarrow \mathbb{R}$  are functions satisfying the following conditions:

- M1: The mapping  $\phi: X \rightarrow Y$  is continuous;
- M2: For every  $x \in X$ , the mapping  $\tau_x: t \mapsto \tau(x, t): J_\pi(x) \rightarrow \mathbb{R}$  maps  $J_\pi(x)$  continuously into  $J_\rho(\phi(x))$  such that  $\tau_x(0) = 0$ ;
- M3: For all  $(x, t) \in D$ ,  $\phi \circ \pi(x, t) = \rho(\phi(x), \tau_x(t))$ ;
- M4: For every  $x \in X$ , the mapping  $\tau_x: J_\pi(x) \rightarrow J_\rho(\phi(x))$  is strictly increasing, and  $\tau: D \rightarrow \mathbb{R}$  is continuous.

Notation:  $\Phi: (X, D, \pi) \rightarrow (Y, E, \rho)$  or  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$ .

A morphism of the form  $(1_X, \tau): (X, D, \pi) \rightarrow (X, E, \rho)$  is called a *parameter-transformation*, and  $\tau$  is called a *reparametrization of  $\pi$  to  $\rho$* . If  $(\phi, \iota)$  is a morphism such that  $\iota(x, t) = t$  for all  $(x, t)$  in the domain of  $\iota$ , then  $\phi$  is

called an *equivariant mapping*, and  $(\phi, \tau)$  will be called an *equivariant morphism*.

If  $\Phi = (\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  is a morphism, then the symbol  $\Phi$  will also be used to denote the mapping  $(x, t) \mapsto (\phi(x), \tau(x, t)): D \rightarrow Y \times \mathbb{R}$ . By M2,  $\Phi$  maps  $D$  continuously into  $E$ , and M3 requires that the following diagram commutes:

$$(2.1) \quad \begin{array}{ccc} D & \xrightarrow{\quad \Phi \quad} & E \\ \pi \downarrow & & \downarrow \rho \\ X & \xrightarrow{\quad \phi \quad} & Y \end{array}$$

Using this point of view, namely, that a morphism of lds's is a mapping between the domains of the phase mappings, having a special form and special properties, the following definition of *composition of morphisms* is quite natural. Let for  $i = 1, 2$ ,  $\Phi_i = (\phi_i, \tau_i): (X_i, D_i, \pi_i) \rightarrow (X_{i+1}, D_{i+1}, \pi_{i+1})$  be morphisms of lds's. Then  $\Psi := \Phi_2 \circ \Phi_1$  also satisfies the conditions of a morphism of lds's, that is, there exist mappings  $\psi: X_1 \rightarrow X_3$  and  $\sigma: D_1 \rightarrow \mathbb{R}$  such that  $\Psi(x, t) = (\psi(x), \sigma(x, t))$  for all  $(x, t) \in D_1$ , and the conditions M1 through M4 are fulfilled. Actually,  $\psi$  and  $\sigma$  are given by

$$(2.2) \quad \psi := \phi_2 \circ \phi_1 \quad \text{and} \quad \sigma_x := (\tau_2)_{\phi_1(x)} \circ (\tau_1)_x \quad \text{for } x \in X.$$

In this way we have now defined a category whose objects are the lds's. The isomorphism in this category will be called *isomorphisms of lds's*. In [1], the following characterizations of isomorphisms have been obtained: a morphism  $\Phi = (\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  is an isomorphism iff  $\Phi$  is a homeomorphism of  $D$  onto  $E$ , iff  $\phi$  is a homeomorphism of  $X$  onto  $Y$  and for every  $x \in X$  the mapping  $\tau_x$  is a surjection of  $J_\pi(x)$  onto  $J_\rho(\phi(x))$ . See also Theorem 3.2 below.

2.2. REMARKS

1. The definition of a morphism given above is slightly redundant. It follows from 3.5 below that the conditions M1, M2 and M3 alone imply already that  $\tau$  is continuous on  $D \setminus (\phi^{-1}[S_\rho] \times \mathbb{R})$ , whereas a (quite natural) additional condition on  $\phi$  guarantees strict monotonicity of  $\tau_x$  for  $x \in X \setminus \phi^{-1}[S_\rho]$ .

2. At the time of preparing [1] and [2], the authors were not aware of the paper [14] of KIMURA, where also a category of all lds's is considered. The morphisms in [14], so-called *GH-morphisms*, are just pairs  $(\phi, \tau)$  satisfying the conditions M1, M2 and M3 of 2.1 above; several *types* of GH-morphisms are defined in [14] according to various continuity conditions for  $\tau$ . In fact, a pair  $(\phi, \tau)$  is a morphism (in the sense defined above) iff in the terminology of [14] it is a GH-morphism of type 4 with the additional property that  $\tau_x$  is strictly increasing for every  $x$ .
3. It follows from Theorem 3.4 below that it would be more natural to replace in M4 the condition that  $\tau_x$  is strictly increasing for every  $x \in X$  by the condition that  $\tau_x$  is injective (either strictly increasing or strictly decreasing) for every  $x \in X$ . Except for making the statements quite cumbersome, this would not essentially affect our results to be presented below. Therefore, we shall use the definition as given in 2.1 above. We want to emphasize here that the monotonicity condition in M4 is crucial for the preservation properties which we shall discuss now.

2.3. PROPOSITION. *Let  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  be a morphism of lds's. Then*

*$\phi[S_\pi] \subseteq S_\rho$  and  $\phi[P_\pi] \subseteq P_\rho$ . In addition, for every  $x \in X$  we have*

$$(i) \quad \phi[\Gamma_\pi^+(x)] \subseteq \Gamma_\rho^+(\phi(x)), \quad \phi[\Gamma_\pi^-(x)] \subseteq \Gamma_\rho^-(\phi(x)), \quad \text{so } \phi[\Gamma_\pi(x)] \subseteq \Gamma_\rho(\phi(x));$$

$$(ii) \quad \phi[A_\pi(x)] \subseteq A_\rho(\phi(x)) \quad \text{and} \quad \phi[\Omega_\pi(x)] \subseteq \Omega_\rho(\phi(x)).$$

*If  $\phi$  is a bijection, then we have even  $S_\pi = \phi^{-1}[S_\rho]$ ,  $P_\pi = \phi^{-1}[P_\rho]$  and for  $x \in P_\pi$  the primitive periods of  $x$  and  $\phi(x)$  are related by the equality  $p_{\phi(x)} = \tau_x(p_x)$ . Moreover, in this case we have equalities in (i) for every  $x \in X$ .*

PROOF. See [1; Section 4] for the general case and [2; 4.3] for the case that  $\phi$  is a bijection.  $\square$

2.4. REMARK. The difficult parts of Proposition 2.3 are the statements in (ii) and the case that  $\phi$  is a bijection. In the proof an essential use has been made of the following equality:

$$(2.3) \quad \tau_x(s+t) = \tau_x(s) + \tau_{\pi(x,s)}(t), \quad s, s+t \in J_\pi(x),$$

which is valid for all  $x \in X \setminus \phi^{-1}[S_\rho]$  and, by continuity of  $\tau$ , even for all  $x \in \overline{X \setminus \phi^{-1}[S_\rho]}$ .

2.5. COROLLARY. Let  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  be a morphism of lds's. If  $x \in X$  is positively Poisson stable (i.e.  $x \in \Omega_{\pi}(x)$ ) then so is  $\phi(x)$ . If  $x \in X$  is positively Lagrange stable (i.e.  $\overline{\Gamma_{\pi}^{+}(x)}$  is compact) then so is  $\phi(x)$ , and  $\phi[\overline{\Gamma_{\pi}^{+}(x)}] = \overline{\Gamma_{\rho}^{+}(\phi(x))}$ .  $\square$

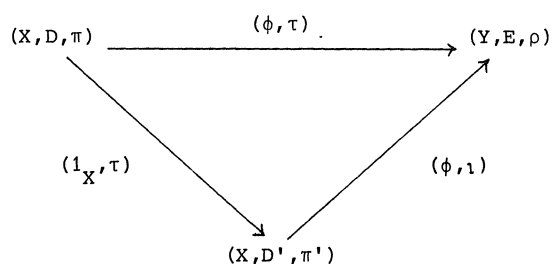
2.6. REMARK

1. Examples which illustrate the special role of condition M4 in the definition of morphism with respect to the preservation properties mentioned in 2.3 can be found in [1; 4.5].
2. Properties involving a certain distribution of the time parameter are in general not preserved by morphisms. In [1], an example is given where recurrence<sup>\*</sup> is not preserved by a reparametrization (which is an isomorphism!). Also an example is given, showing that morphisms need not preserve Liapunov stability of rest points. However, isomorphisms preserve Liapunov stability of rest points.

3. THE STRUCTURE OF MORPHISMS

The following proposition shows how the isomorphisms in our category are related to the conjugacy relation in [4]. For the quite straightforward proof, we refer to [1; 3.5].

3.1. PROPOSITION. Let  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  be a morphism such that  $\phi: X \rightarrow Y$  is a homeomorphism. Then there exists a commuting diagram of morphisms



where  $(\phi, \iota)$  is an equivariant morphism. Moreover,  $(1_X, \tau)$  is an isomorphism iff  $(\phi, \tau)$  is an isomorphism.  $\square$

---

\*) According to the definition in [17]; this is the same as (pointwise) almost periodicity in [8], almost recurrence in [18], and, for compact spaces, recurrence in [5] and [15].

Recall from 2.1 above that  $(\phi, \tau)$  and  $(1_X, \tau)$  are isomorphisms iff  $\phi$  is a homeomorphism of  $X$  onto  $Y$  and  $\tau_x: J_\pi(x) \rightarrow J_\rho(\phi(x))$  is a surjection for every  $x \in X$ . This latter condition turns out to be automatically fulfilled for every  $x \in X \setminus \phi^{-1}[S_\rho] = X \setminus S_\pi$  (under the assumption that  $\phi$  is a bijection); this is actually the basic observation upon which rests the proof of the second part of 2.3 above. However, it must be observed that for  $x \in \phi^{-1}[S_\rho]$  the values of  $\tau_x(t)$  are completely irrelevant; in particular, condition M3 in the definition of morphism is then trivially fulfilled. Therefore, we introduce the following *equivalence relation* between morphisms. Two morphisms  $(\phi, \tau)$  and  $(\phi', \tau')$  from  $(X, D, \pi)$  to  $(Y, E, \rho)$  will be called *equivalent* whenever  $\phi = \phi'$  and  $\tau_x = \tau'_x$  for every  $x \in X \setminus \phi^{-1}[S_\rho]$ . Notation:  $(\phi, \tau) \approx (\phi', \tau')$ .

In [2], examples are given which show that a morphism  $(\phi, \tau)$  which is equivalent to an isomorphism may not be an isomorphism itself. Morphisms, equivalent to isomorphisms can be characterized as follows:

**3.2. THEOREM.** *Let  $(\phi, \tau)$  be a morphism. The following conditions are equivalent:*

- (i)  $(\phi, \tau)$  is equivalent to an isomorphism;
- (ii)  $\phi$  is a homeomorphism of  $X$  onto  $Y$ .

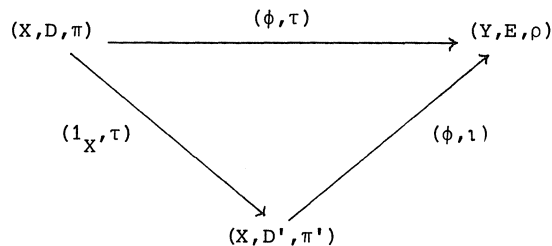
*In particular, if  $S_\pi$  is nowhere dense, then  $(\phi, \tau)$  is an isomorphism iff  $\phi$  is a homeomorphism of  $X$  onto  $Y$ .*

**PROOF.** Cf. [2; Theorem 4.5].  $\square$

Using the equivalence relation defined above we shall try to generalize the first part of 3.1 to general morphisms. First we state a lemma, whose proof can be found in [2; 5.4]:

**3.3. LEMMA.** *Let  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  be a morphism. The following conditions are equivalent:*

- (i)  $\tau$  satisfies the relation (2.3) for every  $x \in \overline{X \setminus S_\pi}$ ;
- (ii) there exists a commuting diagram of morphisms





where  $(\phi, \iota)$  is an equivariant morphism.  $\square$

Since the relation (2.3) is valid for all  $x \in X \setminus \overline{\phi^{-1}[S_\rho]}$  (see 2.4 above) the lemma seems of no help in the case that  $(\text{int } \phi^{-1}[S_\rho]) \setminus S_\pi \neq \emptyset$ . In that case, one can try to modify the mappings  $\tau_x$  for  $x \in (\text{int } \phi^{-1}[S_\rho]) \setminus S_\pi$  such as to make them to satisfy the relation (2.3), and still keeping  $\tau$  continuous on  $D$ , of course. Thus, we were able to prove that under certain additional conditions for the morphism  $(\phi, \tau): (X, D, \pi) \rightarrow (Y, E, \rho)$  there exist morphisms

$$(X, D, \pi) \xrightarrow{(1_X, \tau^*)} (X, D^*, \pi^*) \xrightarrow{(\phi, \iota)} (Y, E, \rho)$$

such that  $(\phi, \iota)$  is an equivariant morphism and the composition  $(\phi, \tau^*)$  of  $(1_X, \tau^*)$  and  $(\phi, \iota)$  is equivalent to  $(\phi, \tau)$ . That is, up to equivalence we obtain factorization of  $(\phi, \tau)$  as a reparametrization, followed by an equivariant morphism.

One of the (sufficient) conditions for such a factorization-up-to-equivalence of  $(\phi, \tau)$  is that  $X$  is metrizable and that  $\text{bnd } \phi^{-1}[S_\rho] \subseteq S_\pi$ . For a proof of this result and for other sufficient conditions for such factorizations, we refer to [2].

Our next, and final result essentially states that a morphism  $(\phi, \tau)$  is, up to equivalence, completely determined by its phase space mapping  $\phi$ . It is the generalization of a result for isomorphisms of URA [21] and KIMURA [13] to general morphisms. Roughly speaking, it says that an orbit preserving mapping  $\phi$  between the phase spaces of lds's is the space component of a morphism  $(\phi, \tau)$ . A similar result can be found in [14], where it is shown that every NS-morphism is a GH-morphism. Since our definition of a morphism requires more than the definition of a GH-morphism (the monotonicity condition M4), an orbit preserving mapping must be just a little bit more than an NS-morphism. This little bit more turns out to be local injectivity on orbits. Here is the definition:

If  $(X, D, \pi)$  and  $(Y, E, \rho)$  are lds's, then an *orbit-preserving mapping* from  $(X, D, \pi)$  to  $(Y, E, \rho)$  is a continuous mapping  $\phi: X \rightarrow Y$  such that

- OPM1. For every  $x \in X$ ,  $\phi[\Gamma_\pi(x)] \subseteq \Gamma_\rho(\phi(x))$ ;
- OPM2. For every  $x \in X \setminus S_\pi$  there is an arc  $\widehat{axb}$  in  $\Gamma_\pi(x)$  such that  $\phi$  is injective on  $\widehat{axb}$

Here  $\widehat{axb}$  means an *arc*, i.e. a topological embedding  $h$  of the unit interval  $[0, 1]$  into  $X$  such that  $h(0) = a$ ,  $h(1) = b$  and  $x = h(t)$  for some  $t$  with  $0 < t < 1$ ; as is often done, we identify such an embedding  $h$  with its

range  $h[0,1]$ .

It can be shown that for every morphism  $(\phi, \tau)$  of lds's,  $\phi$  is an orbit preserving mapping [1; Prop. 5.2]<sup>\*</sup>). Before we can state a converse of this statement, we have to make two notational conventions. The first is that for every invariant subset  $C$  in an lds  $(X, D, \pi)$  the restricted lds  $(X, D \cap (C \times \mathbb{R}), \pi|_{D \cap (C \times \mathbb{R})})$  will be denoted by  $(C, D_C, \pi_C)$ . It is useful to know that every component of an invariant subset is also invariant (cf. [11; IV 2.10 and IV. 2.13]). Our second convention is, that for an lds  $(Y, E, \rho)$  the *reverse system*  $(Y, E_*, \rho_*)$  is defined by  $E_* := \{(x, t) \in Y \times \mathbb{R} : (x, -t) \in E\}$  and  $\rho_*(x, t) := \rho(x, -t)$  for  $(x, t) \in E_*$ . It is easy to show that  $(Y, E_*, \rho_*)$  is an lds iff  $(Y, E, \rho)$  is. Now we can state our final result:

**3.4. THEOREM** *Let  $(X, D, \pi)$  and  $(Y, E, \rho)$  be lds's and let  $\phi: X \rightarrow Y$  be an orbit preserving mapping. Then there exists a unique continuous mapping  $\tau: D \setminus (\phi^{-1}[S_\rho] \times \mathbb{R}) \rightarrow \mathbb{R}$  such that for each component  $C$  of  $X \setminus \phi^{-1}[S_\rho]$  either*

$$(\phi|_C, \tau|_{D_C}): (C, D_C, \pi_C) \rightarrow (Y, E, \rho)$$

or

$$(\phi|_C, \tau|_{D_C}): (C, D_C, \pi_C) \rightarrow (Y, E_*, \rho_*)$$

*is a morphism of lds's.*

**PROOF.** Cf. [1; Theorem 5.7]. Part of the proof parallels the proof in [14], [13] or [20], but our proof is simpler because of our use of a result of J. and M. LEWIN (cf. [4; Theorem 1.25]). This result also guarantees the monotonicity condition for  $\tau$ .  $\square$

**3.5. Unicity** in the above theorem must be interpreted as follows:  $\tau$  is the unique mapping of  $D \setminus (\phi^{-1}[S_\rho] \times \mathbb{R})$  into  $\mathbb{R}$  such that  $(\phi|_{(X \setminus \phi^{-1}[S_\rho])}, \tau)$  satisfies the conditions M1, M2 and M3 of the definition of a morphism. In addition,  $\tau$  turns out to be continuous on  $D \setminus (\phi^{-1}[S_\rho] \times \mathbb{R})$  and condition OPM2 for  $\phi$  implies strict monotonicity of  $\tau_x$  for  $x \in X \setminus \phi^{-1}[S_\rho]$ .

In particular, this implies that for a pair  $(\phi, \tau)$  satisfying the conditions M1, M2 and M3 of the definition of a morphism,  $\tau$  must be continuous on  $D \setminus (\phi^{-1}[S_\rho] \times \mathbb{R})$ , whereas monotonicity of  $\tau_x$  for  $x \in X \setminus \phi^{-1}[S_\rho]$  is implied by

<sup>\*</sup>) Monotonicity of each  $\tau_x$  (i.e. condition M4) is essential in the proof.

requiring  $\phi$  to satisfy also condition OPM2.

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## HYPERSPACES OF FINITE SUBSETS

Murray G. Bell

### 1. INTRODUCTION

The topological analysis of a space can be furthered by a study of the global properties of the space's finite subsets. This study is pursued, quite profitably, by endowing the finite subsets of a space with a topology which is intimately connected to the topology of the space. In this paper, we survey and continue the investigations of two such topologies. Our main emphasis will be on problems of normality, paracompactness and metrizability.

Let  $X$  be a space with topology  $\tau$ .  $F(X)$  will denote the collection of all non-empty finite subsets of  $X$ .

If  $\mathcal{O}$  is a finite subcollection of  $\tau$ , then define  $\langle \mathcal{O} \rangle = \{F \in F(X) : F \subseteq \bigcup \mathcal{O} \text{ and } F \cap O \neq \emptyset \text{ for each } O \in \mathcal{O}\}$ . The collection  $\{\langle \mathcal{O} \rangle : \mathcal{O} \text{ is a finite subcollection of } \tau\}$  serves as a base for a topology on  $F(X)$ . When topologized in this fashion,  $F(X)$  is denoted by  $F\langle X \rangle$  and is referred to as the Vietoris hyperspace of finite subsets of  $X$ . This is exactly the topology that  $F(X)$  receives as a subspace of the hyperspace of non-empty compact subsets of  $X$  with the Vietoris topology  $[V]$ . Our basic reference for the Vietoris topology is  $[M_1]$ . For each positive integer  $n$ , we define  $F_n\langle X \rangle = \{F \in F\langle X \rangle : |F| \leq n\}$ .

If  $F \in F(X)$  and  $\mathcal{O} \in \tau$ , then define  $[F, \mathcal{O}] = \{G \in F(X) : F \subseteq G \subseteq \bigcup \mathcal{O}\}$ . The collection  $\{[F, \mathcal{O}] : F \in F(X) \text{ and } \mathcal{O} \in \tau\}$  serves as a base for a topology on  $F(X)$ . When topologized in this fashion,  $F(X)$  is denoted by  $F[X]$  and is referred to as the Pixley-Roy hyperspace of finite subsets of  $X$ . This kind of topology was introduced in  $[PR]$  in the special case of the real line and generalized in  $[vD]$ .

The author is not aware of any general study of the properties of  $F\langle X \rangle$ , other than those properties that  $F\langle X \rangle$ , for Hausdorff  $X$ , inherits upon being a dense  $F_\sigma$  subspace of the Vietoris hyperspace of all non-empty compact subspaces of  $X$ .

The matter of  $F[X]$  is quite different. It has been used as examples. [PR] used  $F[\text{real line}]$  as an easy example of a nonseparable Moore space of countable cellularity. [PT], assuming  $MA + \neg CH$ , used  $F[S]$ , where  $S$  is a subset of the real line of cardinality  $\omega_1$ , as an example of a nonseparable, metacompact, normal, Moore space of countable cellularity. [AP], assuming  $MA + \neg CH$ , showed that  $[F[S]]^\omega$  has all the preceding properties. [BE] used  $F[K]$  and  $F\langle K \rangle$ , where  $K$  is the Cantor space, to construct a first countable, sigma compact, nonseparable  $p$ -space of countable cellularity.  $F[X]$  has also been studied in its own right in [vD] and [L]. Normality, paracompactness and metrizability of subspaces of  $F[X]$ , for various spaces  $X$ , have been investigated in [BFL<sub>1</sub>], [BFL<sub>2</sub>], [PL], [PR] and [R]. Several of their results will appear in the following sections.

## 2. BASIC UNDERSTANDINGS

Our topological reference for undefined terms is [W].

All spaces considered are  $T_1$ .

## 3. FUNDAMENTAL PROPERTIES OF $F\langle X \rangle$ AND $F[X]$

It is easy to see that if  $S$  is a subspace of  $X$ , then the hyperspace topologies on  $F(S)$  coincide with the subspace topologies that  $F(S)$  receives from  $F\langle X \rangle$  and  $F[X]$ .  $X$  is always embedded in  $F\langle X \rangle$  (as the subspace  $F_1\langle X \rangle$ ) but only under very restricted conditions is  $X$  ever embeddable in  $F[X]$ .

PROPOSITION 3.1.  $F\langle X \rangle$  satisfies the following:

- (a)  $\mathcal{C}l_{F\langle X \rangle} \langle 0 \rangle = \langle \{ \mathcal{C}l_X 0 : 0 \in 0 \} \rangle$ .
- (b)  $F\langle S \rangle$  is closed (open) in  $F\langle X \rangle$  if and only if  $S$  is closed (open) in  $X$ .
- (c)  $F\langle X \rangle$  is  $T_2$ , regular, zero-dimensional and Tychonov if and only if  $X$  is  $T_2$ , regular, zero-dimensional and Tychonov respectively.
- (d) If  $X$  is  $T_2$ , then  $F_n\langle X \rangle$  is closed in  $F\langle X \rangle$  for each  $n$ .
- (e) The mapping  $h_n : X^n \rightarrow F_n\langle X \rangle$ , defined by  $h_n((x_1, x_2, \dots, x_n)) = \{x_1, x_2, \dots, \dots, x_n\}$ , is continuous and finite-to-one. If  $X$  is  $T_2$ , then  $h_n$  is also closed, i.e.,  $h_n$  is a perfect mapping. In general, only  $h_1$  and  $h_2$  are open mappings.

PROOFS. cf. [M<sub>1</sub>] or Problems 2.7.20 and 3.12.26 in [E].  $\square$

PROPOSITION 3.2.  $F[X]$  satisfies the following:

- (a) Each  $[F,0]$  is both open and closed in  $F[X]$ .
- (b) For each  $S \subseteq X$ ,  $F[S]$  is closed in  $F[X]$ . In addition,  $F[S]$  is open in  $F[X]$  if and only if  $S$  is open in  $X$ .
- (c)  $F[X]$  is zero-dimensional,  $T_2$ , hereditarily metacompact and is the union of countably many discrete subspaces - each subspace  $\{F \in F[X]: |F| = n\}$  is discrete.
- (d)  $F_n[X]$  is closed in  $F[X]$  for each  $n$ .
- (e)  $F[X]$  is a Moore space if and only if  $X$  is first countable.
- (f)  $F[X]$  is perfect (even hereditarily  $F_\sigma$ ) if and only if every point of  $X$  is a  $G_\delta$ .

PROOF. (a) through (e) were proved in [vD] and (f) was proved in [L]. □

We now look at various cardinal functions of  $F\langle X \rangle$  and  $F[X]$ .

A network for a space  $X$  is a collection  $N$  of subsets of  $X$  such that for each  $x \in X$  and each open neighbourhood  $O$  of  $x$ , there exists an  $N \in N$  such that  $x \in N \subseteq O$ . The net weight of a space  $X$ ,  $n\omega(X)$ , is the least cardinal of a network for  $X$ . The weight of a space  $X$ ,  $\omega(X)$ , is the least cardinal of a base for  $X$ . The density of a space  $X$ ,  $d(X)$ , is the least cardinal of a dense subspace of  $X$ . The Lindelöf number of a space  $X$ ,  $L(X)$ , is the least cardinal  $\kappa$  such that every open cover of  $X$  has a subcover of size  $\leq \kappa$ . The cellularity of a space  $X$ ,  $c(X)$ , is the least cardinal  $\kappa$  such that every disjoint collection of open sets has cardinality  $\leq \kappa$ .

PROPOSITION 3.3. The following hold for an infinite space  $X$ :

- (a)  $n\omega(F\langle X \rangle) = n\omega(X)$ ;  $n\omega(F[X]) = n\omega(X) \cdot |X|$ .
- (b)  $\omega(F\langle X \rangle) = \omega(X)$ ;  $\omega(F[X]) = \omega(X) \cdot |X|$ .
- (c)  $d(F\langle X \rangle) = d(X)$ ;  $d(F[X]) = |X|$ .
- (d) If  $X$  is  $T_2$ , then  $L(F\langle X \rangle) = \sup\{L(X^n): n < \omega\}$ ;  $L(F[X]) = |X|$ .

PROOF. The first three are easy to prove. The equations in (a) and (b) relating to  $F[X]$  appear in [L]. The equation in (c) relating to  $F[X]$  appears in [vD].

$L(F[X]) \leq |F[X]| = |X|$  and, since  $X$  is a closed discrete subspace of  $F[X]$  we also have that  $|X| \leq L(F[X])$ .

If  $X$  is  $T_2$ , then the mappings  $h_n: X^n \rightarrow F_n\langle X \rangle$  are perfect and so  $L(X^n) = L(F_n\langle X \rangle)$  for each  $n$ . If  $X$  is  $T_2$ , then each  $F_n\langle X \rangle$  is closed in  $F\langle X \rangle$ . Therefore,  $L(F_n\langle X \rangle) \leq L(F\langle X \rangle)$  for  $n$ . It follows that  $\sup\{L(X^n): n < \omega\} \leq L(F\langle X \rangle)$ .

Since  $F\langle X \rangle = \bigcup_{n < \omega} F_n \langle X \rangle$ , we have that  $L(F\langle X \rangle) \leq \sup\{L(X^n) : n < \omega\}$ .  $\square$

The determination of the cellular function of  $F\langle X \rangle$  and  $F[X]$  is more difficult. We only have that  $c(F\langle X \rangle) \leq \sup\{c(X^n) : n < \omega\}$  from [G] and that  $c(F[X]) \leq n\omega(X)$  from [L].

The spaces  $F_n \langle X \rangle$  can behave strangely, even for as simple a space as the closed unit interval  $I$ . In [BJ], it is shown that  $F_n \langle I \rangle$  is homeomorphic to  $I^n$  if and only if  $n = 1, 2$  and  $3$ .

#### 4. NORMALITY OF $F\langle X \rangle$ AND $F[X]$ .

Necessary and sufficient conditions on  $X$  in order that  $F\langle X \rangle$  or  $F[X]$  be normal are major unsolved problems. Is  $F\langle X \rangle$  normal if and only if  $X^n$  is normal for each  $n < \omega$ ? We mention two important instances where there are characterizations.

A space is a  $Q$ -set if every subset of it is a  $G_\delta$  set in it.

##### 4A. Subspaces of the Cantor Space $K = 2^\omega$ .

Since  $F\langle K \rangle$  is metrizable, for all subsets  $W$  of  $K$  we have that  $F\langle W \rangle$  is normal.

In [R] it is shown that for  $W \subseteq K$ ,  $F[W]$  is normal if and only if  $W^n$  is a  $Q$ -set for each  $n < \omega$ . Since  $K$  is not a  $Q$ -set, it follows that  $F[K]$  is not normal. From Proposition 3.2(b), we conclude that if a space  $X$  contains a copy of  $K$ , then  $F[X]$  is not normal.

##### 4B. Subspaces of a Souslin line $S$ .

We assume that  $S$  has no nontrivial separable subintervals and write  $S = \bigcup\{K_\alpha : \alpha < \omega_1\}$  where each  $K_\alpha$  is a Cantor space and  $K_\alpha \subseteq K_\beta$  for  $\alpha < \beta$ . In [R] it is shown that for  $X \subseteq S$ ,  $F[X]$  is normal if and only if

- (a)  $\{\alpha < \omega_1 : X \cap \bigcup_{\beta < \alpha} K_\beta \text{ is closed in } X\}$  contains a closed and unbounded subset of  $\omega_1$  and
- (b)  $(X \cap K_\alpha)^n$  is a  $Q$ -set for  $n < \omega$  and  $\alpha < \omega_1$ .

For subsets  $X$  of a Sorgenfrey-type space, in [PRZ] it is shown that  $F[X]$  is hereditarily normal if and only if the characters of all non-isolated points coincide. It is also shown there that for locally  $\check{C}$ ech-complete spaces  $X$ ,  $F[X]$  is normal if and only if  $X$  is scattered.



5. PARACOMPACTNESS AND METRIZABILITY OF  $F\langle X \rangle$  AND  $F[X]$

A space  $X$  is collectionwise Hausdorff if for every closed discrete subspace  $\{d_\alpha : \alpha \in A\} \subseteq X$ , there exists a disjoint collection  $\{O_\alpha : \alpha \in A\}$  of open subsets of  $X$  such that  $d_\alpha \in O_\alpha$  for each  $\alpha \in A$ . Moreover, if the collection  $\{O_\alpha : \alpha \in A\}$  can always be chosen to be a discrete collection, then  $X$  is strongly collectionwise Hausdorff. A space  $X$  is said to be collectionwise normal if for every discrete collection  $\{C_\alpha : \alpha \in A\}$  of closed subspaces of  $X$ , there exists a disjoint collection  $\{O_\alpha : \alpha \in A\}$  of open subsets of  $X$  such that  $C_\alpha \subseteq O_\alpha$  for each  $\alpha \in A$ . A space  $X$  is ultraparacompact if every open cover of  $X$  admits a disjoint open refinement. A space  $X$  is said to be ultrametrizable if the topology on  $X$  is generated by a metric  $d$  which satisfies the following: for every  $x, y$  and  $z$ ,  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ .

PROPOSITION 5.1. *If  $X$  is  $T_2$ , then  $F\langle X \rangle$  is paracompact if and only if  $X^n$  is paracompact for each  $n < \omega$ .*

PROOF. Assume  $F\langle X \rangle$  is paracompact. Since  $X$  is  $T_2$ ,  $F_n\langle X \rangle$  is closed in  $F\langle X \rangle$  and therefore paracompact. The mapping  $h_n : X^n \rightarrow F_n\langle X \rangle$  is perfect, hence  $X^n$  is paracompact.

Assume  $X^n$  is paracompact for each  $n < \omega$ . Let  $\{\langle O_F \rangle : F \in F\langle X \rangle\}$  be an open cover of  $F\langle X \rangle$  be basic open sets  $\langle O_F \rangle$  where  $F \in \langle O_F \rangle$ . By Theorem 20.7 in [W], it suffices to show that  $\{\langle O_F \rangle : F \in F\langle X \rangle\}$  has an open  $\sigma$ -locally finite refinement. Fix  $n < \omega$ . Since  $F_n\langle X \rangle$  is paracompact ( $h_n$  is closed), let  $b_n = \{\langle U_F \rangle \cap F_n\langle X \rangle : F \in F_n\langle X \rangle\}$  be an open locally finite refinement of  $\{\langle O_F \rangle \cap F_n\langle X \rangle : F \in F_n\langle X \rangle\}$  in  $F_n\langle X \rangle$  such that  $F \in \langle U_F \rangle$ ,  $\langle U_F \rangle \subseteq \langle O_F \rangle$  and  $|U_F| = |F|$ . Define  $b_n^* = \{\langle U_F \rangle : F \in F_n\langle X \rangle\}$ .

CLAIM.  $\{\langle U_F \rangle : F \in F_n\langle X \rangle\}$  is locally finite in  $F\langle X \rangle$ .

PROOF OF CLAIM. Let  $F \in F\langle X \rangle$ . Choose  $\{V(F, f) : f \in F\}$  such that

- (1) for each  $f \in F$ ,  $f \in V(F, f)$  is open in  $X$ ; and
- (2) if  $H \subseteq F$  and  $|H| \leq n$ , then  $\langle V(F, h) : h \in H \rangle \cap F_n\langle X \rangle$  intersects only finitely many  $\langle U_G \rangle \cap F_n\langle X \rangle$ 's for  $G \in F_n\langle X \rangle$ .

Then,  $F \in \langle V(F, f) : f \in F \rangle$  and this neighbourhood intersects only finitely many  $\langle U_G \rangle$ 's for  $G \in F_n\langle X \rangle$ . Observe that if  $\langle V(F, f) : f \in F \rangle \cap \langle U_G \rangle \neq \emptyset$  where  $|U_G| = |G| \leq n$ , then there exists an  $H \subseteq F$ ,  $|H| \leq n$  such that  $\langle V(F, h) : h \in H \rangle \cap \langle U_G \rangle \cap F_n\langle X \rangle \neq \emptyset$ . There can only be finitely many instances of this. End of proof of claim.

$\{b_n^* : n < \omega\}$  is an open  $\sigma$ -locally finite refinement of our original cover. Hence,  $F\langle X \rangle$  is paracompact.

PROPOSITION 5.2.  $F\langle X \rangle$  is metrizable if and only if  $X$  is metrizable.

PROOF. If  $F\langle X \rangle$  is metrizable, then  $F_1\langle X \rangle$  is metrizable. Since  $X$  is homeomorphic to  $F_1\langle X \rangle$ ,  $X$  is metrizable.

If  $X$  is metrizable, then the hyperspace of all non-empty compact subsets of  $X$  is metrizable  $[M_1]$  and hence  $F\langle X \rangle$  is metrizable.  $\square$

As we shall see, both paracompactness and metrizability of  $F[X]$  reduce to the question of when is  $F[X]$  collectionwise Hausdorff. By the following result in  $[BFL_2]$ , we see that if  $F[X]$  is paracompact or metrizable, then it is so in a strong way.

PROPOSITION 5.3.

- (a)  $F[X]$  is paracompact if and only if  $F[X]$  is ultraparacompact.
- (b)  $F[X]$  is metrizable if and only if  $F[X]$  is ultrametrizable.

It was proved in  $[BFL_2]$  that  $F[X]$  is paracompact if and only if  $F[X]$  is strongly collectionwise Hausdorff. We shall strengthen this result to just collectionwise Hausdorff. Independently of this author, T.C. Przymusiński has also proven this fact.

LEMMA 5.4. If  $F[X]$  is collectionwise Hausdorff, then there exists

$\{O(F) : F \in F[X]\}$  where each  $O(F) = \{O(F, f) : f \in F\}$  and

- (a) for each  $f \in F$ ,  $f \in O(F, f)$  and  $O(F, f)$  is open in  $X$ ;
- (b) if  $f_1, f_2 \in F$ , then  $f_1 \notin O(F, f_2)$  and  $f_2 \notin O(F, f_1)$ ;
- (c) if  $H \subseteq F$ ,  $h \in H$ ,  $f \in F$  and  $f \in O(H, h)$ , then  $O(F, f) \subseteq O(H, h)$ ;
- (d) if  $|F| = |G|$ ,  $F \subseteq UO(G)$ ,  $G \subseteq UO(F)$ ,  $F \not\subseteq UO(H)$  for any proper subset  $H$  of  $F$  and  $G \not\subseteq UO(H)$  for any proper subset  $H$  of  $G$ , then  $F = G$ .

PROOF. We construct the  $O(F)$ 's by induction on  $|F|$ . Assume that we have constructed the  $O(F)$ 's for all  $F$ 's with  $|F| < n$  so that (a) through (d) are satisfied.

$F_n[X] = U\{[H, UO(H)] : |H| < n\}$  is a closed discrete subspace of  $F[X]$ . Since  $F[X]$  is collectionwise Hausdorff, for each  $F$  with  $|F| = n$ , we can choose an open set  $V(F) \subseteq X$  such that  $F \subseteq V(F)$  and

$$\{[F, V(F)] : F \in F_n[X] - U\{[H, UO(H)] : |H| < n\}\}$$

is a disjoint collection. Let  $F$  have cardinality  $n$ . Since  $X$  is  $T_1$ , we can choose open sets  $W(F,f) \in X$  for each  $f \in F$  such that if  $\{f_1, f_2\} \subseteq F$ , then  $f_1 \notin W(F, f_2)$  and  $f_2 \notin W(F, f_1)$ . Finally, for each  $f \in F$ , define

$$O(F, f) = V(F) \cap W(F, f) \cap \{O(H, h) : h \in H \subseteq F \text{ and } f \in O(H, h)\}.$$

If  $\mathcal{O}(F) = \{O(F, f) : f \in F\}$ , then the collection  $\{\mathcal{O}(F) : |F| \leq n\}$  satisfies (a) through (d). The collection  $\{\mathcal{O}(F) : F \in F[X]\}$  is our required collection.  $\square$

In order to facilitate our investigation, we use the following definitions:

DEFINITION 5.5. A space  $X$  weakly separated if there exists a reflexive and antisymmetric relation  $\leq$  defined on  $X$  such that for every  $x \in X$ ,  $\{y \in X : y \leq x\}$  is open in  $X$ . If, in addition, the relation is a partial (total) order, then we say that  $X$  is partially (totally) separated. We call  $\leq$  a weak (partial, total) separation of  $X$ .

Weakly separated spaces were defined in [T]. Partially separated spaces were defined in [BFL<sub>2</sub>] (called acceptable partial orders). These are generalizations of right-separated spaces in which the relation is a well-ordering.

THEOREM 5.6. *The following are equivalent:*

- (a)  $F[X]$  is paracompact.
- (b)  $F[X]$  is collectionwise Hausdorff.
- (c)  $F\langle X \rangle$  is weakly separated.

PROOF.

(a) implies (b). This is obvious.

(b) implies (c). Let  $\{\mathcal{O}(F) : F \in F[X]\}$  be as in Lemma 5.4. Define a relation  $\leq$  on  $F\langle X \rangle$  by  $F \leq G$  if and only if  $F \in \langle \mathcal{O}(G) \rangle$ . This relation is reflexive since  $G \in \langle \mathcal{O}(G) \rangle$ . For every  $G \in F\langle X \rangle$ ,  $\{F : F \leq G\} = \langle \mathcal{O}(G) \rangle$  which is open in  $F\langle X \rangle$ . It remains to show that  $\leq$  is antisymmetric.

We first show that if  $|F| = |G|$ ,  $F \subseteq U\mathcal{O}(G)$  and  $G \subseteq U\mathcal{O}(F)$ , then  $F \cap G \neq \emptyset$ . We induct on the cardinality of  $F$  and  $G$ . If  $|F| = |G| = 1$ , then this is true by condition (d) of Lemma 5.4. Assume that for every pair  $\{H, K\}$  such that  $|H| = |K| < n$ , if  $H \subseteq U\mathcal{O}(K)$  and  $K \subseteq U\mathcal{O}(H)$ , then  $H \cap K \neq \emptyset$ . Let

$|F| = |G| = n$ ,  $F \subseteq U\mathcal{O}(G)$  and  $G \subseteq U\mathcal{O}(F)$ . If  $F = G$ , then clearly  $F \cap G \neq \phi$ . If  $F \neq G$ , then by condition (d) of Lemma 5.4., one of  $F$  or  $G$  must reduce. Without loss of generality, assume that there exists a proper subset  $H$  of  $F$  such that  $F \subseteq U\mathcal{O}(H)$ . Condition (c) of the same Lemma implies that  $U\mathcal{O}(F) \subseteq U\mathcal{O}(H)$ . There exists  $K \subseteq G$  such that  $|K| = |H|$  and  $H \subseteq U\mathcal{O}(K)$ . Since  $K \subseteq G \subseteq U\mathcal{O}(F) \subseteq U\mathcal{O}(H)$  and  $|H| = |K| < n$ , our inductive hypothesis implies that  $H \cap K \neq \phi$ , whence  $F \cap G \neq \phi$ .

We return to our proof of antisymmetry. Let  $F \leq G$  and  $G \leq F$ , i.e.,  $F \in \langle \mathcal{O}(G) \rangle$  and  $G \in \langle \mathcal{O}(F) \rangle$ . Condition (b) of Lemma 5.4 implies that for every  $f \in F - G$ , there exists a  $g \in G - F$  such that  $g \in \mathcal{O}(F, f)$ . Similarly, for every  $g \in G - F$ , there exists an  $f \in F - G$  such that  $f \in \mathcal{O}(G, g)$ . Thus, if at least one of  $F - G$  or  $G - F$  is non-empty, then there exists  $r \geq 1$ ,  $\{f_i: 1 \leq i \leq r\} \subseteq F - G$  and  $\{g_i: 1 \leq i \leq r\} \subseteq G - F$  such that  $g_1 \in \mathcal{O}(F, f_1)$ ,  $f_2 \in \mathcal{O}(G, g_1)$ ,  $\dots$ ,  $g_r \in \mathcal{O}(F, f_r)$  and  $f_1 \in \mathcal{O}(G, g_r)$ . This means that  $\{f_i: 1 \leq i \leq r\} \subseteq U\{\mathcal{O}(G, g_i): 1 \leq i \leq r\} \subseteq U\mathcal{O}(\{g_i: 1 \leq i \leq r\})$  and that  $\{g_i: 1 \leq i \leq r\} \subseteq U\{\mathcal{O}(F, f_i): 1 \leq i \leq r\} \subseteq U\mathcal{O}(\{f_i: 1 \leq i \leq r\})$ . By the preceding paragraph, we conclude that  $\{f_i: 1 \leq i \leq r\} \cap \{g_i: 1 \leq i \leq r\} \neq \phi$  which is a contradiction. Hence  $F - G = G - F = \phi$  and  $F = G$ .

(c) implies (a). Let  $\leq$  be a weak separation of  $F \langle X \rangle$ . Since  $F[X]$  is meta-compact, it suffices to show that  $F[X]$  is collectionwise normal. To this end, let  $\{H_\alpha: \alpha \in A\}$  be a discrete collection of closed subspaces of  $F[X]$ .

By induction on the cardinality of  $F$ , we can construct sets  $\mathcal{O}(F, f)$  for each  $F$  and  $f \in F$  such that

- (1)  $f \in \mathcal{O}(F, f)$  and  $\mathcal{O}(F, f)$  is open in  $X$ ;
- (2) if  $H \subseteq F$ ,  $h \in H$ ,  $f \in F$  and  $f \in \mathcal{O}(H, h)$ , then  $\mathcal{O}(F, f) \subseteq \mathcal{O}(H, h)$ ;
- (3) if  $H \subseteq F$ , then  $\langle \mathcal{O}(F, h): h \in H \rangle \subseteq \{G: G \leq H\}$ ;
- (4) if  $\alpha \in A$  and  $F \notin H_\alpha$ , then  $[F, U\{\mathcal{O}(F, f): f \in F\}] \cap H_\alpha = \phi$ .

For each  $F \in F[X]$ , define  $\mathcal{O}(F) = \{\mathcal{O}(F, f): f \in F\}$ . For each  $\alpha \in A$ , define  $U_\alpha = U\{[F, U\mathcal{O}(F)]: F \in H_\alpha\}$ . For each  $\alpha \in A$ ,  $U_\alpha$  is open in  $F[X]$  and  $H_\alpha \subseteq U_\alpha$ . We now show that  $U_\alpha \cap U_\beta = \phi$  for  $\alpha \neq \beta$ . This we do by induction on  $|F| + |G|$  where  $F \in H_\alpha$  and  $G \in H_\beta$ . Assume that for all  $F \in H_\alpha$  and for all  $G \in H_\beta$  such that  $|F| + |G| < n$ , we have that  $[F, U\mathcal{O}(F)] \cap [G, U\mathcal{O}(G)] = \phi$ . Let  $F \in H_\alpha$ ,  $G \in H_\beta$ , and  $|F| + |G| = n$ . Striving for a contradiction, assume that  $[F, U\mathcal{O}(F)] \cap [G, U\mathcal{O}(G)] \neq \phi$ , i.e.,  $F \subseteq U\mathcal{O}(G)$  and  $G \subseteq U\mathcal{O}(F)$ .

We first show that  $F \not\subseteq U\mathcal{O}(F \cap G)$ . If it were, then condition 2 would imply that  $F \in [F, U\mathcal{O}(F)] \subseteq [F \cap G, U\mathcal{O}(F \cap G)]$ . Since  $F \in H_\alpha$ , condition 4 implies that  $F \cap G \in H_\alpha$ . Hence,  $|F \cap G| < |G|$ , and our inductive hypothesis

implies that  $[F \cap G, UO(F \cap G)] \cap [G, UO(G)] = \phi$ , whence  $[F, UO(F)] \cap [G, UO(G)] = \phi$ . Contradiction. Similarly, we prove that  $G \notin UO(F \cap G)$ . Consequently,  $F - UO(F \cap G)$  and  $G - UO(F \cap G)$  are disjoint non-empty finite sets.

Condition 2 implies that  $F - UO(F \cap G) \subseteq U\{O(G, g) : g \in G - UO(F \cap G)\}$  and that  $G - UO(F \cap G) \subseteq U\{O(F, f) : f \in F - UO(F \cap G)\}$ . Thus there exists  $r \geq 1$ ,  $\{f_i : 1 \leq i \leq r\} \subseteq F - UO(F \cap G)$ , and  $\{g_i : 1 \leq i \leq r\} \subseteq G - UO(F \cap G)$  such that  $f_1 \in O(G, g_1), g_1 \in O(F, f_2), \dots, f_r \in O(G, g_r)$ , and  $g_r \in O(F, f_1)$ . Hence,  $\{f_i : 1 \leq i \leq r\} \in \langle O(G, g_i) : 1 \leq i \leq r \rangle$  and  $\{g_i : 1 \leq i \leq r\} \in \langle O(F, f_i) : 1 \leq i \leq r \rangle$ . Condition 3 implies that  $\{f_i : 1 \leq i \leq r\} \leq \{g_i : 1 \leq i \leq r\}$  and  $\{g_i : 1 \leq i \leq r\} \leq \{f_i : 1 \leq i \leq r\}$ . The antisymmetry of  $\leq$  implies that  $\{f_i : 1 \leq i \leq r\} = \{g_i : 1 \leq i \leq r\}$ . This is the contradiction that we were striving for.  $\square$

We mention a question posed in [PR]. Does there exist a space  $X$  such that  $F[X]$  is normal but not paracompact, i.e., not collectionwise Hausdorff? It is independent of the axioms of ZFC that an example exists of character  $\leq c$ .  $MA + \neg CH$  implies that  $F[S]$ , where  $S$  is a subset of the real line of cardinality  $\omega_1$ , is such a first countable example, whereas, according to a theorem of FLEISSNER [F],  $V = L$  implies that every normal space of character  $\leq c$  is collectionwise Hausdorff.

COROLLARY 5.7. *The following are equivalent:*

- (a)  $F[X]$  is metrizable.
- (b)  $X$  is first countable and  $F\langle X \rangle$  is weakly separated.

PROOF.

- (a) implies (b). First countability of  $F[X]$  implies first countability of  $X$ . Theorem 5.6 now applies because metric spaces are paracompact [ST].
- (b) implies (a).  $F[X]$  is paracompact by Theorem 5.6,  $F[X]$  is a Moore space by Proposition 3.2(e) and a paracompact Moore space is metrizable [BI].

$\square$

## 6. WEAK AND PARTIAL SEPARATION

It is clear from Section 5 that our main aim is to find "nice" necessary and sufficient conditions on  $X$  in order that  $F\langle X \rangle$  be weakly separated. Pursuant to this aim, we first investigate the general theory of separations on an arbitrary space  $X$ . To require that a space be weakly separated is a very strong restriction as we see by the following:

PROPOSITION 6.1. *If  $X$  is weakly separated then for every subspace  $S$  of  $X$ ,  $|S| = nw(S)$ .*

PROOF. Let  $\leq$  be a weak separation on  $X$ , let  $S$  be a subspace of  $X$ , and let  $N$  be a network for  $S$  of minimal cardinality. For each  $x \in S$ , choose  $N_x \in N$  such that  $x \in N_x \subseteq \{z \in X: z \leq x\}$ . The antisymmetry of  $\leq$  implies that for distinct  $x$  and  $y$  in  $S$ ,  $N_x \neq N_y$ . Hence,  $|S| \leq |N|$ , and the proposition is proved.  $\square$

Since  $X$  is a subspace of  $F\langle X \rangle$ , we see that if  $F\langle X \rangle$  is weakly separated, then for  $S \subseteq X$ ,  $|S| = nw(S)$ . Thus, for example, an uncountable space  $X$  which has a countable network cannot be weakly separated and  $F[X]$  cannot be collectionwise Hausdorff. On the other hand, spaces like the SORGENFREY Line [SO] or any subspace of an ordinal space clearly are totally separated. For any  $T_1$  space  $X$ ,  $F[X]$  is partially separated by defining  $F \leq G$  if  $G \subseteq F$ .

In the following proposition, the words "weakly separated" can be replaced (except where noted) by "partially separated" or "totally separated" and the same proofs carry over.

PROPOSITION 6.2.

- (a) *If  $X$  is weakly separated then every subspace of  $X$  is weakly separated.*
- (b) *If for every  $\alpha \in A$ ,  $X_\alpha$  is weakly separated then  $\prod_{\alpha \in A} X_\alpha$  is weakly separated. Not so for totally separated spaces.*
- (c) *If  $C$  is a closed subspace of  $X$  such that both  $C$  and  $X - C$  are weakly separated, then  $X$  is weakly separated.*
- (d) *If  $X = \bigcup_{n=1}^{\infty} C_n$  where each  $C_n$  is closed in  $X$  and are weakly separated, then  $X$  is weakly separated.*
- (e) *If each point of  $X$  has an open neighbourhood which is weakly separated, then  $X$  is weakly separated.*

PROOF. Proofs of (a) and (b) are straightforward. To prove (c) let  $\leq_1$  be a weak separation on a closed subspace  $C$  of  $X$  and let  $\leq_2$  be a weak separation on  $X - C$ . Define  $\leq$  on  $X$  by  $x \leq y$  if and only if at least one of the following obtains:

- (1)  $y \in C$  and  $x \notin C$ ;
- (2)  $y \in C$ ,  $x \in C$  and  $x \leq_1 y$ ;
- (3)  $y \notin C$ ,  $x \notin C$  and  $x \leq_2 y$ .

$\leq$  is reflexive and antisymmetric. If  $y \in C$ , then  $\{x \in X: x \leq y\} = \{x \in C: x \leq_1 y\} \cup (X - C)$  which is open in  $X$ . If  $y \notin C$ , then

$\{x \in X: x \leq y\} = \{x \in (X-C): x \leq_2 y\}$  which is open in  $X$ . Consequently,  $\leq$  is a weak separation on  $X$ .

To prove (d) let  $\leq_n$  be a weak separation on the closed subspace  $C_n$  of  $X$ . Define  $\leq$  on  $X$  by  $x \leq y$  if and only if at least one of the following obtains:

- (1) there exists an  $n$  such that  $y \in \bigcup_{i < n} C_i$  and  $x \notin \bigcup_{i < n} C_i$ ;
- (2) there exists an  $n$  such that  $y \in C_n - \bigcup_{i < n} C_i$ ,  $x \in C_n - \bigcup_{i < n} C_i$  and  $x \leq_n y$ .

$\leq$  is reflexive and antisymmetric. If  $y \in X$ , then there exists an  $n$  such that  $y \in C_n - \bigcup_{i < n} C_i$ . Thus,

$$\{x \in X: x \leq y\} = (X - \bigcup_{i \leq n} C_i) \cup \{x \in C_n - \bigcup_{i < n} C_i: x \leq_n y\}$$

which is open in  $X$  because the  $C_n$ 's are closed.

To prove (e) let  $\prec$  be any well-ordering of  $X$ . For each  $x \in X$ , choose an open neighbourhood  $O_x$  of  $x$  which has a weak separation  $\leq_x$ . Define  $\leq$  on  $X$  by  $x \leq y$  if and only if at least one of the following obtains:

- (1) there exists a  $v \in X$  such that  $x \in U\{O_\omega: \omega \prec v\}$  and  $y \notin U\{O_\omega: \omega \prec v\}$ ;
- (2) there exists a  $v \in X$  such that  $x \in O_v - U\{O_\omega: \omega \prec v\}$ ,  $y \in O_v - U\{O_\omega: \omega \prec v\}$  and  $x \leq_v y$ .

$\leq$  is reflexive and antisymmetric. If  $y \in X$ , then there exists a least (under  $\prec$ )  $v \in X$  such that  $y \in O_v - U\{O_\omega: \omega \prec v\}$ . Thus,  $\{x \in X: x \leq y\} = U\{O_\omega: \omega \prec v\} \cup \{x \in O_v: x \leq_v y\}$  which is open in  $X$ .  $\square$

**EXAMPLE 6.3.** Every discrete space is totally separated. Part (d) of Proposition 6.2 implies that every space which is the union of countably many closed discrete subspaces is totally separated. Thus from part (e) we see that every locally countable  $T_1$  space is totally separated. The MICHAEL line  $[M_2]$  is totally separated by part (c) because it is the union of a closed countable  $T_1$  subspace and an open discrete subspace.

**PROPOSITION 6.4.** *If  $X$  is  $T_2$ , then  $F\langle X \rangle$  is weakly separated if and only if for every positive integer  $n$ ,  $\{F \in F\langle X \rangle: |F| = n\}$  is weakly separated.*

**PROOF.** One direction follows directly from part (a) of Proposition 6.2. Since each  $F_n\langle X \rangle$  is closed in  $F\langle X \rangle$ , to show that  $F\langle X \rangle$  is weakly separated, it suffices to show that each  $F_n\langle X \rangle$  is weakly separated. This follows from the fact that each  $\{F \in F\langle X \rangle: |F| = n\}$  is weakly separated by  $n$  applications of part (c) of Proposition 6.2.  $\square$

PROPOSITION 6.5. *If  $X$  is partially separated, then  $F\langle X \rangle$  is weakly separated.*

PROOF. Let  $\leq$  be a partial separation on  $X$ . For every  $F \in F\langle X \rangle$  and  $f \in F$ , choose an open neighbourhood  $0(F, f)$  of  $f$  in  $X$  such that

- (1)  $\{f_1, f_2\} \subseteq F$  implies  $f_1 \notin 0(F, f_2)$  and  $f_2 \notin 0(F, f_1)$ ;
- (2)  $0(F, f) \subseteq \{h \in X: h \leq f\}$ .

Define  $F \leq^* G$  if and only if  $F \in \langle \{0(G, g): g \in G\} \rangle$ . The relation  $\leq^*$  is reflexive and for each  $G \in F\langle X \rangle$ ,  $\{F \in F\langle X \rangle: F \leq^* G\} = \langle \{0(G, g): g \in G\} \rangle$  which is open in  $F\langle X \rangle$ .

To prove antisymmetry, let  $F \leq^* G$  and  $G \leq^* F$ , i.e.,  $F \in \langle \{0(G, g): g \in G\} \rangle$  and  $G \in \langle \{0(F, f): f \in F\} \rangle$ . Condition 1 implies that for every  $f \in F - G$ , there exists a  $g \in G - F$  such that  $g \in 0(F, f)$ . Similarly, for every  $g \in G - F$ , there exists an  $f \in F - G$  such that  $f \in 0(G, g)$ . Thus, if at least one of  $F - G$  or  $G - F$  is non-empty, then there exists  $r \geq 1$ ,  $\{f_i: 1 \leq i \leq r\} \subseteq F - G$  and  $\{g_i: 1 \leq i \leq r\} \subseteq G - F$  such that  $g_1 \in 0(F, f_1)$ ,  $f_2 \in 0(G, g_1)$ , ...  $\dots, g_r \in 0(F, f_r)$  and  $f_1 \in 0(G, g_r)$ . Condition 2 implies that  $f_1 \leq g_r \leq f_r, \dots, \dots, f_2 \leq g_1 \leq f_1$ . Since  $\leq$  is acyclic, this is a contradiction. Hence,  $F - G = \phi$  and  $G - F = \phi$ , i.e.,  $F = G$ .  $\square$

COROLLARY 6.6. ([BFL<sub>2</sub>]). *If  $X$  is partially separated, then  $F[X]$  is paracompact. If, in addition,  $X$  is first countable, then  $F[X]$  is metrizable.*

REMARKS.

- (a) Corollary 6.6 has been generalized in [PR] as follows: Let  $(S, <)$  be a partially ordered set and suppose that  $X = \bigcup \{X_s: s \in S\}$ , where  $X_s \cap X_t = \phi$  for  $s \neq t$  and  $\bigcup \{X_t: t \leq s\}$  is open in  $X$  for every  $s \in S$ . If  $F[X_s]$  is paracompact for every  $s \in S$ , then  $F[X]$  is paracompact.
- (b) Partially separated spaces include all locally countable  $T_1$  spaces, all  $\sigma$ -closed discrete spaces, all scattered spaces, all subspaces of  $F[X]$  and many generalized-ordered spaces. All examples of spaces  $X$  for which  $F[X]$  is paracompact have turned out to be partially separated. In an upcoming paper on weakly separated spaces, we will show that there exists a first countable Tychonov space  $X$  for which  $F\langle X \rangle$  is weakly separated, but  $X$  is not partially separated. We also have an example of a  $T_1$  weakly separated space  $X$  for which  $F_2\langle X \rangle$  is not weakly separated. This example is not first countable, however.

PROBLEM 6.7. *If  $X$  is a first countable Tychonov space, is  $F[X]$  metrizable if and only if  $X$  is weakly separated?*



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## COMPLETENESS FOR NEARNESS SPACES

H.L. Bentley & H. Herrlich

### ABSTRACT AND INTRODUCTION.

For uniform spaces completeness is a well-defined and useful concept. We investigate, in the realm of nearness spaces, to which extent this concept can be extended in a sensible way to some wider range. For this purpose we first formulate 20 external resp. internal characterizations of completeness as well as some basic properties of completeness and completions for uniform spaces. Next we investigate these characterizations and properties in some wider range. Main results are:

- (1) Virtually the whole theory remains valid for regular nearness spaces.
- (2) Large parts of the theory remain valid for separated nearness spaces, the only notable exception being the fact that completions are no longer unique. For separated nearness spaces there exist two distinguished completions, a "largest" (= simple) one, which is a complete reflection but otherwise behaves rather badly, and a "smallest" (= strict) one, which behaves rather well and whose restriction to uniform spaces is the usual uniform completion, but which fails to be a complete reflection.
- (3) Beyond the range of separated nearness spaces, the various characterizations of completeness are no longer equivalent. The concept of completeness splinters into many different concepts and a natural theory of completeness ceases to exist.

In this paper we restrict attention to those nearness (in particular uniform) spaces, whose underlying topological spaces are Hausdorff.

## 1. UNIFORM SPACES

1.1. External Characterizations

We start with two external characterizations of complete spaces among uniform spaces:

- (A) A uniform space is complete iff it is *extension-closed*, i.e. closed in every uniform extension (equivalently: iff it has no proper dense extension).
- (B) A uniform space is complete iff it belongs to the epireflective hull of all complete metrizable uniform spaces in the category *Unif* (equivalently: iff it is isomorphic to a closed subspace of some product of completely metrizable uniform spaces).

1.2. Internal Tools

In order to provide internal characterizations of completeness among uniform spaces, we need to recall definitions and properties of several useful internal tools.

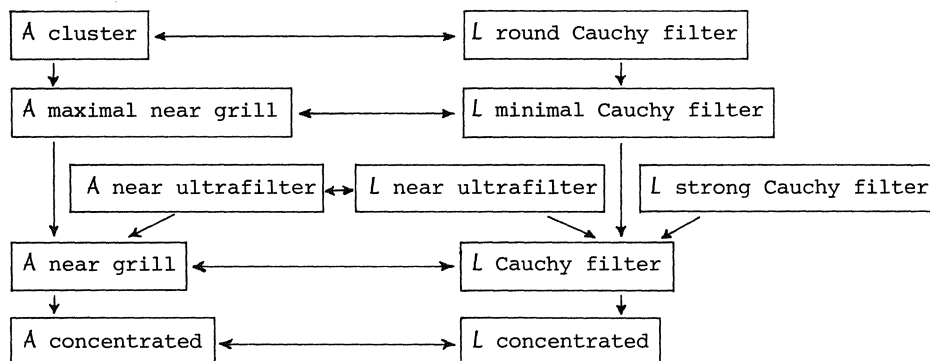
Let  $\underline{X} = (X, \mu)$  be a uniform (or more generally an arbitrary nearness) space - where  $\mu$  denotes the collection of all uniform covers of  $\underline{X}$ . A collection  $A$  of subsets of  $X$  is called

- *near*, provided every uniform cover contains some member, which meets every member of  $A$
- *micromeric*, provided for every uniform cover  $U$  there exist members  $U$  of  $U$  and  $A$  of  $A$  with  $A \subset U$
- *concentrated*, provided it is near and micromeric
- a *stack*, provided  $A \in A$  and  $A \subset B \subset X$  implies  $B \in A$
- a *grill*, provided  $X \in A$ ,  $\emptyset \notin A$  and  $(A \cup B \in A \iff A \in A \text{ or } B \in A)$ .

Micromeric filters are called *Cauchy filters*. Every Cauchy filter is near. An ultrafilter is near if and only if it is a Cauchy filter. A filter  $F$  on  $X$  is called a *strong Cauchy filter* (MORITA [6]), provided for every uniform cover  $U$  there exist members  $U$  of  $U$  and  $F$  of  $F$ , such that  $U$  is a uniform neighbourhood of  $F$ , i.e. such that there exists a uniform cover  $H$  with  $\text{star}(F, H) \subset U$ . Maximal (non-empty) near collections are called *clusters*. Minimal micromeric stacks (not containing the empty set) are called *round Cauchy filters*.

The operator  $sec$ , defined by  $sec A = \{B \subset X \mid B \cap A \neq \emptyset \text{ for all } A \in \mathcal{A}\}$ , is idempotent for stacks, reverses the order, interchanges the roles of filters and grills as well as of near collections and micromeric collections.

The following diagram exhibits the relations between the above concepts for arbitrary nearness spaces. Let  $\mathcal{A}$  be a stack and  $L = sec \mathcal{A}$  (hence  $\mathcal{A} = sec L$ ). Then:



For uniform spaces the names "strong Cauchy filter", "round Cauchy filter" and "maximal near grill" are superfluous, as the following results show:

- (C) In a uniform space, the clusters are precisely the maximal near grills, (hence) the round Cauchy filters are precisely the minimal Cauchy filters.
- (D) In a uniform space, the strong Cauchy filters are precisely the Cauchy filters.

Any of the above concepts can be used to describe the "holes" in a non-complete space. Particularly useful are clusters and minimal Cauchy filters, since for any "hole" in a uniform space the collection of all sets "near the hole" forms a cluster, and the collection of all sets "surrounding that hole" forms a minimal Cauchy filter. The following result sheds more light on the picture:

(E) For any stack  $\mathcal{A}$  in a uniform space  $(X, \mu)$ , the following conditions are equivalent:

- (1)  $\mathcal{A}$  is concentrated
- (2) there exists a (unique) cluster  $\mathcal{C}$  and a (unique) minimal Cauchy filter  $F$  with  $F \subset \mathcal{A} \subset \mathcal{C}$ .

If the above conditions hold, then

$$\mathcal{C} = \{C \subset X \mid \{C\} \cup \mathcal{A} \text{ is near}\}$$

and

$$F = \{F \subset X \mid \{F\} \cup \{X \setminus A \mid A \in \mathcal{A}\} \text{ is a uniform cover}\}.$$

Completeness means that certain collections  $\mathcal{A}$  converge resp. adhere. Here  $\mathcal{A}$  is said to *converge to*  $x$ , provided for any neighbourhood  $U$  of  $x$  there exists some member  $A$  of  $\mathcal{A}$  with  $A \subset U$ ; and  $\mathcal{A}$  is said to *adhere to*  $x$ , provided  $x$  is an adherence point of every member of  $\mathcal{A}$ . For uniform spaces we have the following useful result:

(F) For any concentrated collection  $\mathcal{A}$  and any point  $x$  in a uniform space, the following conditions are equivalent:

- (1)  $\mathcal{A}$  converges to  $x$
- (2)  $\mathcal{A}$  adheres to  $x$ .

### 1.3. Internal Characterizations

The tools, presented in the above section, suggest several possibilities to define completeness. The following result shows their equivalence for uniform spaces:

(G) For a uniform space the following conditions are equivalent:

- (1) every cluster adheres
- (2) every maximal near grill adheres
- (3) every near grill adheres
- (4) every concentrated collection adheres
- (5) every Cauchy filter adheres
- (6) every near ultrafilter adheres
- (7) every strong Cauchy filter adheres
- (8) every minimal Cauchy filter adheres
- (9) every round Cauchy filter adheres
- (10) every round Cauchy filter converges
- (11) every minimal Cauchy filter converges
- (12) every strong Cauchy filter converges
- (13) every near ultrafilter converges
- (14) every Cauchy filter converges
- (15) every concentrated collection converges
- (16) every near grill converges
- (17) every maximal near grill converges
- (18) every cluster converges.

#### 1.4. Completions

Here we formulate some fundamental results concerning completeness and completions of uniform spaces:

- (H) Completeness is productive, closed hereditary (and coproductive), and hence epireflective in the category *Unif* of uniform spaces.
- (I) Every uniform space  $\underline{X}$  has an essentially unique completion  $\gamma\underline{X}$ . The points of  $\gamma\underline{X}$  are in 1-1-correspondence to any of the following:
  - (1) all clusters of  $\underline{X}$
  - (2) all minimal Cauchy filters of  $\underline{X}$
  - (3) all concentrated collections in  $\underline{X}$ , modulo the equivalence  $A \sim L \iff A \cup L$  is near.
- (K) Completions preserve
  - (1) products
  - (2) embeddings
  - (3) total boundedness (= contiguity, = proximity)
  - (4) (large) uniform dimension
  - (5) uniform weight, i.e. the smallest cardinality of a base for the covering structure
  - (6) uniform separability degree, i.e. the smallest cardinal  $k$  such that there exists a base, all of whose members have cardinality  $\leq k$
  - (7) metrizability.

## 2. REGULAR NEARNESS SPACES

In a nearness space a uniform cover  $\hat{A}$  is called a *regular refinement* of a uniform cover  $L$ , provided for any member  $A$  of  $\hat{A}$  there exists a member  $B$  of  $L$ , which is a uniform neighbourhood of  $A$  (i.e. there exists a uniform cover  $\hat{C}$  with  $\text{star}(A, \hat{C}) \subset B$ ). A nearness space is called *regular* provided every uniform cover has a regular refinement. Since star-refinements are regular, any uniform space is a regular nearness space. Regular nearness spaces were first introduced by K. MORITA [6], who demonstrated that several results in §1 hold in this more general context. Moreover we have:

**THEOREM 2.1.** *With the exception of (B), all the results (A) - (K) of §1 remain valid for regular nearness spaces. Moreover, completions of regular nearness spaces preserve uniformity.*

Proofs can be found e.g. in [4, 5].

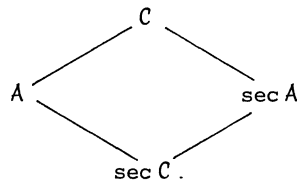
### 3. SEPARATED NEARNESS SPACES

A nearness space is called *separated*, provided for any concentrated collection  $A$ , the collection  $\{B \subset X \mid \{B\} \cup A \text{ near}\}$  is near (and hence the unique cluster containing  $A$ ). Separated nearness spaces were introduced in [4, 5]. Every regular nearness space is separated.

Essential parts of the theory of complete uniform spaces remain valid for separated nearness spaces. First, the technical Lemmas (C), (E) and (F) still hold in this context:

PROPOSITION 3.1. *Results (C), (E) and (F) hold for separated nearness spaces.*

PROOF. (C) follows immediately from the definition. (E)  $(2) \Rightarrow (1)$  is obvious. Vice versa, if  $A$  is a concentrated stack, then  $C = \{C \subset X \mid \{C\} \cup A \text{ near}\}$  is the unique cluster containing  $A$ . Since  $C$  is a near grill,  $\text{sec } C$  is a Cauchy filter and  $\text{sec } C \subset \text{sec}(\text{sec } C) = C$ . Therefore  $C$  is the unique cluster containing  $\text{sec } C$ . Hence  $\text{sec } C \subset \text{sec } A$  implies that  $C$  is the unique cluster containing  $\text{sec } A$ . Consequently  $\text{sec } C$  is the unique minimal Cauchy filter contained in  $\text{sec}(\text{sec } A) = A$ .



Finally,  $\text{sec } C = \{F \subset X \mid C \in C \Rightarrow F \cap C \neq \emptyset\} = \{F \subset X \mid (X-F) \notin C\} =$   
 $= \{F \subset X \mid \{X \setminus F\} \cup A \text{ not near}\} = \{F \subset X \mid \{F\} \cup \{X \setminus A \mid A \in A\} \text{ uniform cover}\}.$

(F) Let  $A$  be a concentrated collection. Without loss of generality we assume  $A$  to be a stack. If  $A$  converges to  $x$ , then the neighbourhood-filter  $L$  of  $x$  is contained in  $A$ . Hence the cluster  $C = \{C \subset X \mid \{C\} \cup L \text{ near}\}$  contains  $\{x\}$  and every member  $A$  of  $A$ , which implies that  $A$  adheres to  $x$ . Vice versa, let  $A$  adhere to  $x$ . Let  $L$  be the collection of all neighbourhoods of  $x$ , and  $C$  be the unique cluster containing  $L$ . Then  $\{x\}$  belongs to  $C$ . Hence, for every neighbourhood  $B$  of  $x$ , the complement  $X \setminus B$  cannot belong to  $C$ , i.e. there exists a uniform cover  $U$  none of its members meets simultaneously  $X \setminus B$  and

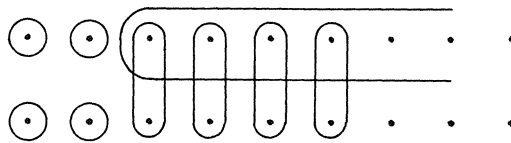


all members of  $L$ . Since  $A$  is micromeric, there exists a common member  $A$  of  $A$  and  $L$ . Since  $A$  meets every member of  $L$  it cannot meet  $X \setminus B$ , which implies that  $B \supset A$  is a member of  $A$ . Consequently  $A$  converges to  $x$ .

For the investigation of completeness in separated nearness spaces Morita's strong Cauchy filters are no longer a suitable tool, as the following example shows:

**EXAMPLE 3.2.** Consider the separated nearness space  $(X, \mu)$ , defined by  $X = \mathbb{N} \times \{0,1\}$  and

$$A \in \mu \iff \begin{cases} 1. \cup A = X \\ 2. \exists n \in \mathbb{N}, \forall m \geq n, \exists A \in A, \{(m,0), (m,1)\} \subset A \\ 3. \exists n \in \mathbb{N}, \exists A \in A, \{(m,1) \mid m \geq n\} \subset A \end{cases}$$



Then  $F = \{F \subset X \mid (\mathbb{N} \times \{1\}) \setminus F \text{ finite}\}$  is a non-convergent Cauchy filter, which is not strong. The strong Cauchy filters are precisely the fixed ultrafilters and hence convergent.

**THEOREM 3.3.** For separated nearness spaces, all the conditions of result (G), with the exception of (7) and (12), are equivalent.

Hence, for separated nearness spaces, each of the remaining 16 conditions of (G) describes internally the same phenomenon: completeness. Moreover, this concept coincides with the external one, given in (A):

**THEOREM 3.4.** Result (A) remains valid for separated nearness spaces.

Moreover, the following results have been shown in [1]:

**THEOREM 3.5.** Result (H) remains valid in the category *SepNear* of separated nearness space.

**THEOREM 3.6.** Every separated nearness space can be completed. Any two completions of a fixed separated nearness space are pointwise isomorphic. The points of the completions can be described via any of the conditions (1), (2) or (3) of (H). Among the completions of a fixed separated nearness

space are two distinguished ones:

- a "largest" (= simple) one, which provides the complete reflection in  $SepNear$
- a "smallest" (= strict) one.

THEOREM 3.7. *The strict completion preserves all the properties (1) - (7), exhibited in (K), as well as uniformity and regularity. The simple completion (= complete reflection) preserves none of them.*

For the latter, see the following example:

EXAMPLE 3.8. Consider the uniform space  $\underline{X} = (X, \mu)$ , defined by  $X = \mathbb{N}$  and

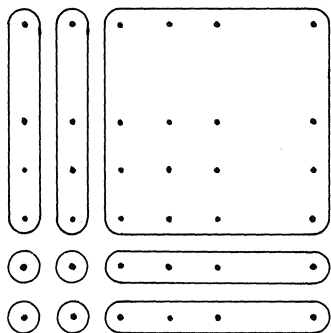
$$A \in \mu \iff \begin{cases} 1. UA = X \\ 2. \exists A \in \mathcal{A}, X \setminus A \text{ finite.} \end{cases}$$

$\underline{X}$  has (up to isomorphism), precisely one completion  $\gamma\underline{X} = (X', \mu')$ , which can be described as follows:  $X = \mathbb{N} \cup \{\omega\}$  and

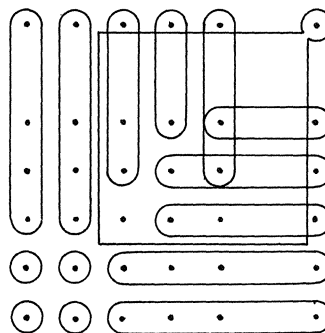
$$A \in \mu' \iff \begin{cases} 1. UA = X' \\ 2. \exists A \in \mathcal{A}, \omega \in A \text{ and } X' \setminus A \text{ finite.} \end{cases}$$

Then  $\gamma\underline{X} \times \gamma\underline{X}$  is the strict completion of  $\underline{X} \times \underline{X}$ . The simple completion  $(Y, \nu)$  of  $\underline{X} \times \underline{X}$  can be described as follows:  $Y = X' \times X'$  and

$$A \in \nu \iff \begin{cases} 1. UA = Y \\ 2. \forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \exists A \in \mathcal{A}, \\ \quad \{n\} \times (\{k \in \mathbb{N} \mid k \geq m\} \cup \{\omega\}) \subset A \\ 3. \forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \exists A \in \mathcal{A} \\ \quad (\{k \in \mathbb{N} \mid k \geq m\} \cup \{\omega\}) \times \{n\} \subset A \\ 4. \exists n \in \mathbb{N}, \exists A \in \mathcal{A}, \\ \quad (\{m \in \mathbb{N} \mid m \geq n\} \times \{m \in \mathbb{N} \mid m \geq n\}) \cup \{(\omega, \omega)\} \subset A. \end{cases}$$



Strict completion of  $\underline{X} \times \underline{X}$



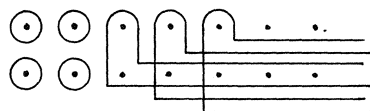
Simple completion of  $\underline{X} \times \underline{X}$

4. NEARNESS SPACES

For arbitrary nearness spaces (with underlying Hausdorff spaces) completeness splinters in many different concepts as the following examples demonstrate:

EXAMPLE 4.1. Consider the nearness spaces  $(X, \mu)$ , defined as follows:  
 $X = \mathbb{N} \times \{0,1\}$  and

$$A \in \mu \iff \begin{cases} 1. UA = X \\ 2. \exists n \in \mathbb{N}, \forall m \geq n, \exists A \in \mathcal{A}, \{(m,1)\} \cup \{(k,0) \mid k \geq m\} \subset A. \end{cases}$$



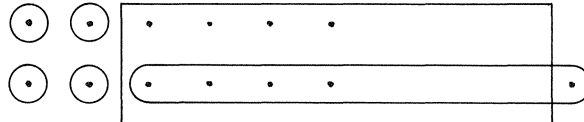
Then:

- (1)  $\mathcal{A} = \{A \subset X \mid A \cap (\mathbb{N} \times \{0\}) \text{ infinite}\}$  is a maximal near grill, but not a cluster.
- (2)  $\mathcal{L} = \{B \subset X \mid (\mathbb{N} \times \{0\}) \setminus B \text{ finite}\}$  is a minimal Cauchy filter, but not round (i.e. not a minimal Cauchy stack).
- (3)  $\mathcal{A} \cup \{\mathbb{N} \times \{1\}\}$  is concentrated, but not contained in any near grill.
- (4)  $\{C \subset X \mid \exists n \in \mathbb{N}, \{(n,1)\} \cup \{(m,0) \mid m \geq n\} \subset C\}$  is concentrated, but does not contain any Cauchy filter.

- (5) Every cluster adheres, but the maximal near grill  $A$  does not; every round Cauchy filter converges, but the minimal Cauchy filter  $L$  does not.

EXAMPLE 4.2. Consider the nearness space  $(X, \mu)$ , defined as follows:  
 $X = (\mathbb{N} \times \{0,1\}) \cup \{\omega\}$  and

$$A \in \mu \iff \begin{cases} 1. UA = X \\ 2. \exists n \in \mathbb{N}, \exists A \in \mathcal{A}, (\{m \in \mathbb{N} \mid m \geq n\} \times \{0,1\}) \subset A \\ 3. \exists n \in \mathbb{N}, \exists A \in \mathcal{A}, (\{m \geq n\} \cup \{\omega\}) \subset A \end{cases}$$

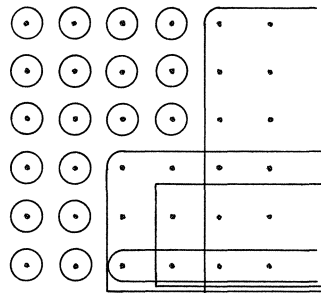


Then:

- (1) The collection  $\mathcal{A} = \{\mathbb{N} \times \{1\}\} \cup \{\{m,0 \mid m \geq n\} \mid n \in \mathbb{N}\}$  is concentrated and converges to  $\omega$ , but does not adhere.
- (2) The collection  $\mathcal{L} = \{\{m,i \mid m \geq n, i \in \{0,1\}\} \mid n \in \mathbb{N}\}$  is concentrated and adheres to  $\omega$ , but does not converge.

EXAMPLE 4.3. Consider the nearness space  $(X, \mu)$ , defined as follows:  
 $X = \mathbb{N} \times \mathbb{N}$  and

$$A \in \mu \iff \begin{cases} 1. UA = X \\ 2. \forall n \in \mathbb{N}, \exists m \in \mathbb{N}, \exists A \in \mathcal{A}, \\ \{k,i \mid k \geq m \text{ and } i \leq n\} \subset A \end{cases}$$



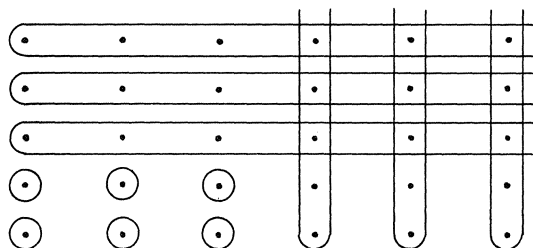
Then:

- (1) Every maximal near grill is a fixed ultrafilter, hence converges and adheres. So does every minimal Cauchy filter.
- (2) The Cauchy filter  $F = \{F \subset X \mid (\mathbb{N} \times \{0\}) \setminus F \text{ finite}\}$  neither converges nor adheres.

EXAMPLE 4.4. Consider the nearness space  $(X, \mu)$ , defined as follows:

$$X = \mathbb{N} \times \mathbb{N}$$

$$A \in \mu \iff \begin{cases} 1. UA = X \\ 2. \exists n \in \mathbb{N}, \forall m \geq n, \exists A \in \mathcal{A}, \{m\} \times \mathbb{N} \subset A \\ 3. \exists n \in \mathbb{N}, \forall m \geq n, \exists A \in \mathcal{A}, \mathbb{N} \times \{m\} \subset A. \end{cases}$$



Then:

- (1) Every Cauchy filter converges; every near grill adheres.
- (2) The collection  $\mathcal{A} = \{\mathbb{N} \times \{n\} \mid n \in \mathbb{N}\}$  is concentrated but neither converges nor adheres.

EXAMPLE 4.5. Consider the nearness space  $(X, \mu)$ , defined as follows:

$$X = [0,1] \text{ and}$$

$$A \in \mu \iff \begin{cases} 1. U\{\text{int } A \mid A \in \mathcal{A}\} = X \\ 2. \exists A \in \mathcal{A}, X \setminus A \text{ finite} \end{cases}$$

where  $\text{int}$  denotes the interior operator of the usual topology on  $[0,1]$ .

Then:

- (1) Every near ultrafilter converges and adheres.
- (2) The cluster, consisting of all infinite subsets of  $X$ , converges but does not adhere. The round Cauchy filter  $\{F \subset X \mid X \setminus F \text{ finite}\}$  adheres, but does not converge.

A systematic analysis of the various completeness concepts for arbitrary (Hausdorff) nearness spaces is still missing. Partial results are: Cluster completeness is productive [2] and allows the construction of a strict completion with completely satisfactory preservation properties [4, 5], but is neither closed hereditary nor reflective.

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## THE K-THEORY OF ALMOST SYMMETRIC FORMS

F.J.-B.J. Clauwens

### INTRODUCTION

To motivate this paper we first recall a few facts.

According to [W1, chapter 5] a normal map  $f$  between manifolds of dimension  $2k$  and fundamental group  $\pi$  gives rise to a (so-called quadratic) form  $\psi$  defined on some finitely generated free left  $B$  module  $V$ , where  $B$  denotes the integral group ring  $Z[\pi]$ . The appropriate equivalence class of  $\psi$  in  $L_{2k}(B)$  is the obstruction  $s(f)$  for changing  $f$  into a homotopy equivalence by surgery (for  $k > 2$ ).

According to [C] a closed manifold  $P$  of dimension  $2q$  and fundamental group  $\rho$  gives rise to a (so-called almost symmetric) form  $\sigma$  defined on some finitely generated free left  $A$  module  $K$ , where  $A$  is  $Z[\rho]$ . The main theorem there states that  $\sigma \otimes \psi$  represents the obstruction for doing surgery on  $\text{id}_P \times f$  if  $\psi$  does so for  $f$ .

In this paper we will study the algebra of almost symmetric forms; therefore we first recall the main things about quadratic forms from [W2].

Orientability considerations give rise to a homomorphism  $w: \pi \rightarrow \{\pm 1\}$ . The map  $-: B \rightarrow B$  defined by the formula  $\overline{\sum_n g} = \sum_n w(g)g^{-1}$  satisfies  $\overline{x+y} = \overline{x} + \overline{y}$ ,  $\overline{xy} = \overline{y} \overline{x}$  and  $\overline{\overline{x}} = x$ . For such an involuted ring  $B$  the dual  $V^d = \text{Hom}_B(V, B)$  of a left  $B$ -module  $V$  inherits the structure of a left  $B$ -module by  $(af)(v) = f(v)\overline{a}$ ; the canonical map  $\hat{\cdot}: V \rightarrow V^{dd}$  defined by  $\hat{x}(f) = \overline{f(x)}$  is an isomorphism provided  $V$  is finitely generated projective. A form  $\zeta$  on  $V$  can be viewed as a homomorphism  $V \rightarrow V^d$ ; then  $\zeta^* = \zeta^d \circ \hat{\cdot}: V \rightarrow V^{dd} \rightarrow V^d$  is one such too.

**DEFINITION.** Let  $\epsilon$  be a sign. An  $\epsilon$ -quadratic form over  $B$  consists of a finitely generated free left  $B$ -module  $V$  and a class of forms  $\psi$  on  $V$  defined up to the equivalence  $\psi \sim \psi + \zeta - \epsilon\zeta^*$ . It is called nonsingular if the symmetrisation  $\lambda = \psi + \epsilon\psi^*$  is an isomorphism  $V \rightarrow V^d$ . We call  $(W, \phi^d\psi\phi)$  isomorphic to

$(V, \psi)$  if  $\phi$  is a module isomorphism  $W \rightarrow V$ .

If  $F$  is f.g. free the quadratic form  $\psi$  on  $F \oplus F^d$  defined by  $\psi_F(x, f) = (f, 0)$  is nonsingular; any quadratic form of this isomorphism type is called standard. Now  $L_{2k}^{(-1)^k}(B)$  is defined as the quotient of the Grothendieck group of nonsingular  $(-1)^k$  quadratic forms over  $B$  by the subgroup generated by standard such forms.

DEFINITION. Let  $\eta$  be a sign,  $A$  an involuted ring. A nonsingular almost  $\eta$ -symmetric form over  $A$  consists of a finitely generated free left  $A$  module  $K$  and an isomorphism  $\sigma: K \rightarrow K^d$  such that  $\sigma^* = \eta\sigma(1+N)$ , where  $N$  is nilpotent (compare [C; §9]). Again  $\phi^d \sigma \phi$  is considered to be isomorphic to  $\sigma$  for any module isomorphism  $\phi$ .

#### ALMOST SYMMETRIC FORMS ARE QUADRATIC

Let  $A$  be an involuted ring,  $\eta = (-1)^q$ . We consider quadratic forms over the polynomial ring  $A[s]$  over  $A$  equipped with the involution  $\overline{\phantom{x}} = \eta \phantom{x} (1-s)$  such that  $\overline{\sum a_j s^j} = \sum \overline{a_j} (1-s)^j$ .

THEOREM 1. Any element in  $L_{2q}(A[s])$  can be represented by a quadratic form  $\Psi$  which is linear in  $s$ . Any such linear  $\Psi$  can be viewed as an almost  $(-1)^q$  symmetric form.

PROOF. Let the element be represented by a quadratic form  $\Psi = \sum \psi_i s^i$  of degree  $M$  in  $s$ . By the addition of a standard form and the use of an isomorphism we get (in matrix notation)

$$\begin{pmatrix} 1 & -s & \psi_M^* (1-s)^{M-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1+s & 1 & 0 \\ \psi_M s^{M-1} & 0 & 1 \end{pmatrix} = \begin{pmatrix} \psi - \psi_M s^M & 0 & -s \\ \psi_M s^{M-1} & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

a form of degree  $M-1$  if  $M \geq 2$ ; so we can make that  $M = 1$ .

We can get rid of the constant term by using the equivalence

$$\psi_0 + \psi_1 s \sim \psi_0 + \psi_1 s - \psi_0 (1-s) + \eta \psi_0^* s = (\psi_1 + \psi_0 + \eta \psi_0^*) s.$$

To prove the last clause we consider the linear  $\Psi = \psi_1 s$  and write  $\sigma$  for  $\eta \psi_1^*$ . Then the symmetrisation  $\Lambda = \Psi + \eta \Psi^*$  of  $\Psi$  becomes  $\sigma + (\eta \sigma^* - \sigma) s$  which is



invertible exactly if  $\sigma$  is invertible and  $\eta\sigma^{-1}\sigma^*-1$  is nilpotent. Q.E.D.

THEOREM 2. *There is a well defined biadditive pairing*

$$L_{2q}(A[s]) \times L_{2k}(B) \rightarrow L_{2q+2k}(A \otimes B)$$

which assigns to the quadratic form  $\Psi = \Sigma \Psi_i s^i$  over  $A[s]$  with symmetrisation  $\Lambda$  and the quadratic form  $\psi$  over  $B$  with symmetrisation  $\lambda$  the quadratic form

$$\lambda\Psi(\lambda^{-1}\psi) = \Sigma \Psi_i \otimes \lambda \cdot (\lambda^{-1}\psi)^i$$

over  $A \otimes B$ . In particular it extends the familiar product of a symmetric form with a quadratic form.

PROOF. Again write  $\eta = (-1)^q$ ,  $\epsilon = (-1)^k$ .

We start with the observation that for a general form  $\Gamma = \Sigma \Gamma_i s^i$  over  $A[s]$  we have

$$\begin{aligned} \lambda\Gamma^*(\lambda^{-1}\psi) &= \Sigma \Gamma_i^* \otimes \lambda(1-\lambda^{-1}\psi)^i = \Sigma \Gamma_i^* \otimes \lambda(\epsilon\lambda^{-1}\psi^*)^i \\ &= \epsilon \Sigma \Gamma_i^* \otimes \lambda^*((\lambda^*)^{-1}\psi^*)^i = \epsilon \Sigma \Gamma_i^* \otimes (\psi^*(\lambda^*)^{-1})^i \lambda^* \\ &= \epsilon\{\Sigma \Gamma_i \otimes \lambda(\lambda^{-1}\psi)^i\}^* = \epsilon\{\lambda\Gamma(\lambda^{-1}\psi)\}^* \end{aligned}$$

Hence the symmetrisation of the image is

$$\lambda\Psi(\lambda^{-1}\psi) + \epsilon\eta\{\lambda\Psi(\lambda^{-1}\psi)\}^* = \lambda\Psi(\lambda^{-1}\psi) + \eta\lambda\Psi^*(\lambda^{-1}\psi) = \lambda\Lambda(\lambda^{-1}\psi)$$

which is invertible since both  $\lambda$  and  $\Lambda$  are. Furthermore if we change  $\Psi$  into the equivalent  $\Psi + Z - \eta Z^*$  the image changes into

$$\begin{aligned} \lambda\Psi(\lambda^{-1}\psi) + \lambda Z(\lambda^{-1}\psi) - \eta\lambda Z^*(\lambda^{-1}\psi) &= \\ = \lambda\Psi(\lambda^{-1}\psi) + \lambda Z(\lambda^{-1}\psi) - \eta\epsilon\{\lambda Z(\lambda^{-1}\psi)\}^* & \end{aligned}$$

which is equivalent to  $\lambda\Psi(\lambda^{-1}\psi)$ .

If we change  $\Psi$  into the isomorphic  $\Phi^d\Psi\Phi$  the image changes into

$$\lambda \phi^d(\lambda^{-1}\psi)\Psi(\lambda^{-1}\psi)\Phi(\lambda^{-1}\psi) = \{\phi(\lambda^{-1}\psi)\}^d \lambda \Psi(\lambda^{-1}\psi)\Phi(\lambda^{-1}\psi)$$

which is isomorphic to  $\lambda \Psi(\lambda^{-1}\psi)$ . Finally if  $\Psi$  is standard then  $\lambda \Psi(\lambda^{-1}\psi)$  is also standard: in fact such a  $\Psi$  is induced from a quadratic form over  $A$  for which this statement is well-known. Since our pairing obviously respects direct sums we have proven that the class of the image in  $L_{2q+2k}(A \otimes B)$  is independent of the choice of the representing element for the class in  $L_{2q}(A[s])$ .

Now by Theorem 1 we may from now on assume that  $\Psi$  is of the type  $\sigma$ , where  $\sigma$  is nonsingular, almost  $\eta$  symmetric; so  $\lambda \Psi(\lambda^{-1}\psi)$  is just  $\sigma \otimes \psi$ .

Firstly if we change  $\psi$  by an isomorphism  $\phi$  into  $\phi^* \psi \phi$  then  $\sigma \otimes \psi$  changes by the isomorphism  $1 \otimes \phi$ .

Secondly the isomorphism  $K \otimes (F \otimes F^d) \cong (K \otimes F) \oplus (K \otimes F)^d$  which maps  $a \otimes (x, f)$  to  $(a \otimes x, \sigma(a) \otimes f)$  lets  $\sigma \otimes \psi_F$  correspond with  $\psi_{K \otimes F}$ . So standard forms are mapped to standard forms.

It remains to be shown that the equivalence  $\psi \sim \psi + \zeta - \varepsilon \zeta^*$  changes  $\sigma \otimes \psi$  into something in the same class; this will be a consequence of the following lemma.

**LEMMA.** For every integer  $p \geq 0$  there is an isomorphism  $\Phi_p$  and there are forms  $Z_p$  and  $H_p$  over  $A \otimes B$  such that

$$\Phi_p^d(\sigma \otimes \psi) \Phi_p = \sigma \otimes (\psi + \zeta - \varepsilon \zeta^*) + Z_p - \varepsilon \eta Z_p^* + H_p(N^{p+1} \otimes 1)$$

where  $N$  is  $\eta \sigma^{-1} \sigma^* - 1$  and thus nilpotent.

**PROOF.** We apply induction. For  $p = 0$  we take

$$\Phi_0 = 1, \quad Z_0 = -\sigma \otimes \zeta, \quad H_0 = -\varepsilon \sigma \otimes \zeta^*.$$

In general  $\Phi_p$  will be of the form  $1 + N \otimes \phi_1 + \dots + N^p \otimes \phi_p$  and  $H_p$  of the form  $\sigma \otimes \theta_{p0} + \sigma N \otimes \theta_{p1} + \dots$ . If we assume all this for  $p$  then  $\Phi_{p+1}^d(\sigma \otimes \psi) \Phi_{p+1}$  becomes

$$\begin{aligned} & \sigma \otimes (\psi + \zeta - \varepsilon \zeta^*) + Z_p - \varepsilon \eta Z_p^* + H_p(N^{p+1} \otimes 1) + \\ & + (N^d)^{p+1} \sigma \otimes \phi_{p+1}^d \psi + \sum_{j=1}^p (N^d)^{p+1} \sigma N^j \otimes \phi_{p+1}^d \psi \phi_j + \\ & + \sigma N^{p+1} \otimes \psi \phi_{p+1} + \sum_{j=1}^{p+1} (N^d)^j \sigma N^{p+1} \otimes \phi_j^d \psi \phi_{p+1}. \end{aligned}$$

Now we rewrite  $(N^d)^{p+1} \sigma \otimes \phi_{p+1}^d \psi + \sigma N^{p+1} \otimes \psi \phi_{p+1}$  as

$$\begin{aligned} & \{ (N^d)^{p+1} \sigma \otimes \phi_{p+1}^d \psi - \varepsilon \eta \sigma^* N^{p+1} \otimes \psi^* \phi_{p+1} \} + \\ & + \varepsilon (\eta \sigma^* - \sigma) N^{p+1} \otimes \psi^* \phi_{p+1} + \sigma N^{p+1} \otimes (\psi + \varepsilon \psi^*) \phi_{p+1} \end{aligned}$$

and we want the last term  $\sigma N^{p+1} \otimes \lambda \phi_{p+1}$  to cancel the first term  $\sigma N^{p+1} \otimes \theta_{p0}$  of  $H_p(N^{p+1} \otimes 1)$  hence we define  $\phi_{p+1} = -\lambda^{-1} \theta_{p0}$ . The first term we absorb in  $Z$  by defining  $Z_{p+1} = Z_p + (N^d)^{p+1} \sigma \otimes \phi_{p+1}^d \psi$ .

The remaining term  $\varepsilon \sigma N^{p+2} \otimes \psi^* \phi_{p+1}$  will be absorbed in  $H_{p+1}(N^{p+2} \otimes 1)$ , as are the remaining terms of  $H_p(N^{p+1} \otimes 1)$  and the  $\Sigma$ -terms. The last is possible because  $N^d \sigma$  can be rewritten as  $-\sigma N(1+N)^{-1}$ . So there exists  $\phi_{p+1}'$ ,  $Z_{p+1}$  and  $H_{p+1}$  of the right form. Q.E.D.

By viewing almost symmetric forms  $A$  as quadratic forms over  $A[s]$  and classifying the latter up to stable isomorphism we have defined an equivalence relation on them.

According to Theorem 2 this relation is sufficiently fine to admit the formulation of the product formula (for surgery obstructions). It is also sufficiently coarse to define a bordism invariant of algebraic symmetric Poincaré complexes in the sense of [R], hence one of geometric Poincaré complexes: As explained in [C] we can associate an almost  $(-1)^q$  symmetric form  $\sigma$  to a  $2q$ -dimensional algebraic symmetric Poincaré complex and then take its class in  $L_{2q}(A[s])$ . The result is well-defined on  $L^{2q}(A)$  since it can be seen as taking the tensor product with the element of  $L_0(Z[s])$  represented by  $\sigma = 1$ .

Both the inherent periodicity in  $q$  and the wealth of techniques available for  $L_{2q}$  make it probable that  $L_{2q}(A[s])$  is better suited for calculations than  $L^{2q}(A)$  is.

One could hope that an almost  $(-1)^q$  symmetric form is always equivalent to an honest  $(-1)^q$  symmetric one; the following example, due to A. Ranicki shows that this is not the case. However we will see that it is the case if the ring  $A$  contains a central element  $t$  such that  $t + \bar{t} = 1$  or if it is a Dedekind domain.

The two-dimensional torus  $T^2 = S^1 \times S^1$  gives rise to an element in  $L^2(A)$ , and hence to an element in  $L_2(A[s])$ , where  $A$  is the integral group ring of  $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$ . Suppose that this element could be represented by an anti-symmetric form  $\sigma$ ; then  $\sigma$  could be written as  $\phi - \phi^*$ ; the result  $\sigma \otimes \psi$  of its

action on a  $(-1)$ -quadratic form  $\psi$  would be equivalent to  $\phi \otimes (\psi - \psi^*)$ , hence would depend only on the symmetrisation  $\psi - \psi^*$  of  $\psi$ . In particular it would kill the Arf nontrivial element in  $L_2(\mathbb{Z})$ . On the other hand it follows from [SH] that multiplication with a circle induces a split injection on L-groups and hence the product with  $T^2$  gives a split injection  $L_2(\mathbb{Z}) \rightarrow L_4(A)$ .

#### SOME CALCULATIONS

**THEOREM 3.** *If there exists a central element  $t$  of  $Z$  such that  $t + \bar{t} = 1$  then the canonical map  $L_{2q}(A) \rightarrow L_{2q}(A[s])$  is an isomorphism.*

**PROOF.** The map  $A[s] \rightarrow A$  substituting  $t$  for  $s$  gives a left inverse so we must show that for any integer  $p \geq 0$  there is an isomorphism  $\phi_p$  and there are forms  $\zeta_p$  and  $\theta_p$  such that

$$\phi_p^d(\sigma s)\phi_p = \sigma t + \zeta_p - \eta\zeta_p^* + \sigma N^{p+1}\theta_p.$$

For  $p = 0$  we take  $\phi_0 = 1$ ,  $\zeta_0 = \sigma s(1-t)$ ,  $\theta_0 = (1-s)t$ . In general  $\phi_p$  will be of the form  $1 + \alpha_1 N + \dots + \alpha_p N^p$  and  $\theta_p = \theta_{p0} + \theta_{p1} N + \theta_{p2} N^2 + \dots$  where the  $\alpha_i$  and  $\theta_{ij}$  are polynomial in  $s$  and  $t$  with  $Z$  coefficients, hence central.

If we assume all this for  $p$  then  $\phi_{p+1}^d(\sigma s)\phi_{p+1}$  becomes

$$\begin{aligned} & \sigma t + \zeta_p - \eta\zeta_p^* + \sigma N^{p+1}\theta_p + \\ & + \bar{\alpha}_{p+1} (N^d)^{p+1} \sigma s + \sum_{j=1}^p \bar{\alpha}_{p+1} (N^d)^{p+1} \sigma s \alpha_j N^j + \\ & + \sigma s \alpha_{p+1} N^{p+1} + \sum_{j=1}^{p+1} \bar{\alpha}_j (N^d)^j \sigma s \alpha_{p+1} N^{p+1}. \end{aligned}$$

We rewrite  $\bar{\alpha}_{p+1} (N^d)^{p+1} \sigma s + \sigma s \alpha_{p+1} N^{p+1}$  as

$$\begin{aligned} & \{ \bar{\alpha}_{p+1} (N^d)^{p+1} \sigma s - \eta \sigma^* N^{p+1} \alpha_{p+1} (1-s) \} \\ & + (\eta \sigma^* - \sigma) N^{p+1} \alpha_{p+1} (1-s) + \sigma (s + (1-s)) \alpha_{p+1} N^{p+1} \end{aligned}$$

Then we let the last term cancel the first term of  $\sigma N^{p+1}\theta_p$  by defining

$$\begin{aligned} \alpha_{p+1} &= -\theta_{p0} \text{ and absorb the first term in } \zeta \text{ by defining} \\ \zeta_{p+1} &= \zeta_p + \bar{\alpha}_{p+1} (N^d)^{p+1} \sigma s. \end{aligned}$$

The middle term  $\sigma N^{p+2} \alpha_{p+1} (1-s)$  is absorbed in  $\sigma N^{p+2} \theta_{p+1}$ , as are the  $\Sigma$  terms and the remaining terms of  $\sigma N^{p+1} \theta_p$ . Q.E.D.

THEOREM 4.

$$L_0(Z[s]) \cong Z, \quad L_2(Z[s]) \cong (0).$$

PROOF. According to Theorem 1 we may restrict attention to  $\eta$ -quadratic forms  $\psi$  of the type  $\eta \sigma^* s$ , where  $\sigma$  is an almost  $\eta$ -symmetric form on some f.g. free  $Z$ -module  $K$ . Thus  $N = \eta \sigma^{-1} \sigma^* - 1$  satisfies  $N^e = 0$  for some  $e$ . Then  $N^{e-1} K$  is of finite index  $h$  in some direct summand  $L$  of  $K$ .

For  $x \in L^\perp = \{x \mid \sigma(x) = 0\}$  we have also  $\sigma(L)(x) = 0$  and vice versa, since  $\eta \sigma^* N^{e-1} = (\sigma + \sigma N) N^{e-1} = \sigma N^{e-1}$  implies

$$\sigma(N^{e-1} y)(x) = \eta \overline{\sigma(x)(N^{e-1} y)}, \quad \text{for } y \in K.$$

Furthermore  $L \subset L^\perp$  since  $\sigma N = -\eta N^d \sigma^*$  implies

$$\sigma(N^{e-1} y)(N^{e-1} x) = -\eta \overline{\sigma(N^e x)(N^{e-2} y)} = 0.$$

So  $\sigma$  induces a well-defined form  $\tilde{\sigma}$  on  $L^\perp/L$ , and  $\tilde{N} = \eta \tilde{\sigma}^{-1} \tilde{\sigma}^* - 1$  satisfies  $\tilde{N}(x+L) = Nx + L$  hence  $N^{e-1} K \subset L$  implies  $\tilde{N}^{e-1} = 0$ .

Now  $L \otimes Z[s]$  is a direct summand of  $K \otimes Z[s]$  which is isotropic for  $\psi$ . If  $x = \sum x_j s^j \in K \otimes Z[s]$  is in  $(L \otimes Z[s])^\perp$  for the symmetrisation  $\lambda = \sigma + \sigma N$  of  $\psi$  then we have for all  $\ell \in L$  that

$$0 = \lambda(\ell \otimes 1, \sum x_j s^j) = \sum \sigma(\ell, x_j) s^j + \sum \sigma(N\ell, x_j) (1-s) s^j = \sum \sigma(\ell, x_j) s^j$$

hence  $x_j \in L^\perp$ . We see that  $(L \otimes Z[s])^\perp / (L \otimes Z[s])$  is just  $(L^\perp/L) \otimes Z[s]$  and obviously the induced quadratic form  $\tilde{\psi}$  on it is just  $\eta \tilde{\sigma}^* s$ .

It is well known that  $\psi$  is stably equivalent to  $\tilde{\psi}$  and we have just seen that the latter is associated to an almost  $\eta$ -symmetric form  $\tilde{\sigma}$  which has a better  $e$ . We can go on inductively until  $e = 1$  which means that we get an  $\eta$ -symmetric form.

It is also well known [SE] that a  $(-1)$ -symmetric form is stably trivial and a  $(+1)$ -symmetric form stably isomorphic to some multiple  $m$  of the form (1) of rank one. Finally  $m$  can be detected by taking the signature of the

quadratic form over  $\mathbb{R}$  which we get by mapping  $s$  to  $\frac{1}{2}$ . Q.E.D.

Now some general remarks about torsion are necessary. If we start with a finite Poincaré complex  $P$  our module  $K$  gets a natural basis (see §6 of [C]).

The symmetrization  $\lambda$  of the associated quadratic form is  $\sigma(1+N_s)$  and according to Lemma 9 of [C] we have  $N^2 = 0$  and  $1 + N_s$  has a resolution by automorphisms  $1 + ((-1)^{1-\Sigma} \Sigma^* - 1)s$  of the  $E_i$  which are simple; in particular the isomorphisms involving  $N$  in the proofs of Theorems 2 and 3 are simple. So the torsion of  $\lambda$  lives in  $\tilde{K}_1(A) \subset \tilde{K}_1(A[s])$  and the appropriate  $L$  groups  $L_{2q}^X(A[s])$  have  $X = \text{Wh}(\rho)$  in the general case and  $(0)$  in the case of simple Poincaré complexes.

At the time this is written we do not have theorems as the above for the odd-dimensional case. Note however, that if we did, we could use the long exact sequence 9.4 of [R] for the  $L_n$  groups to calculate  $L_n(\mathbb{Z}[\rho][s])$  for  $\rho$  the cyclic group of prime order  $p > 2$ . If  $\omega$  denotes  $\exp(2\pi i/p)$  and  $F_p$  is the field of  $p$  elements, there are maps from  $\mathbb{Z}[\rho][s]$  to  $\mathbb{Z}[\omega][s]$  and  $\mathbb{Z}[s]$  and from these to  $F_p[s]$  satisfying all necessary conditions. Since  $K_2(F_p[s]) = 0$  according to Theorem 11 of [Q] and 9.13 of [M] the map

$$\tilde{K}_1(\mathbb{Z}[\rho][s]) \rightarrow \tilde{K}_1(\mathbb{Z}[\omega][s]) \oplus \tilde{K}_1(\mathbb{Z}[s])$$

is injective, so we may use the "simple"  $L$ -groups throughout and we get an exact sequence

$$\dots L_{n+1}(F_p[s]) \rightarrow L_n(\mathbb{Z}[\rho][s]) \rightarrow L_n(\mathbb{Z}[\omega][s]) \oplus L_n(\mathbb{Z}[s]) \rightarrow L_n(F_p[s]) \dots$$

But  $L_n(\mathbb{Z}[\omega][s]) \cong L_n(\mathbb{Z}[\omega])$  by Theorem 3, hence is known, and similarly  $L_n(F_p[s]) \cong L_n(F_p)$ .

The author has now calculated  $L_n(\mathbb{Z}[\rho][s])$  for  $\rho$  cyclic.

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## HYPERSPACES OF PEANO CONTINUA

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The hyperspace  $2^X$  of nonempty compact subsets of a metric continuum  $X$ , and the hyperspace  $C(X)$  of nonempty subcontinua of  $X$ , are topologized by the Hausdorff metric  $\rho(A,B) = \inf\{\epsilon > 0: A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}$ . In this survey paper we discuss some of the fundamental classical results for such hyperspaces, and some of the more recent work done on hyperspaces of Peano continua. We begin in §1 with a brief description of the general connectivity properties enjoyed by  $2^X$  and  $C(X)$ . The investigation of these properties was initiated by BORSUK and MAZURKIEWICZ [4], [23], [24] in the early 1930s, and continued by KELLEY [21] in 1942 and SEGAL [32] in 1959. A comprehensive treatment of these and many other topics in hyperspace theory is provided in the recently published monograph of NADLER [26]. In §2 we specialize to hyperspaces of Peano continua. The equivalence of local connectedness for  $X$ ,  $2^X$ , and  $C(X)$ , established by VIETORIS [37] and WAZEWSKI [38] in 1923, and the fundamental result of WOJDYSLAWSKI [41] in 1939 that such locally connected hyperspaces are absolute retracts, form the background for the topological characterization theorems of CURTIS, SCHORI and WEST [29], [12] in the early 1970s. We describe in broad outline those techniques from infinite-dimensional topology (the recognition of near-homeomorphisms, interior approximation by inverse sequences, and constructions involving  $Q$ -factors) which were crucial for obtaining these results, and which were in fact largely motivated by hyperspace problems. Relationships between topological properties and geometric positional properties of certain subspaces of  $2^X$  are discussed in §3, §4, and §5. Finally, in §6 we indicate how certain Peano compactifications may be used to establish topological characterization theorems for hyperspaces of non-compact spaces.

## §1. HYPERSPACES OF METRIC CONTINUA

The topology on  $2^X$  induced by the Hausdorff metric is also known as the *Vietoris finite topology*. The basic open sets are those of the form  $\langle V_1, \dots, V_k \rangle = \{F \in 2^X : F \subset V_1 \cup \dots \cup V_k \text{ and } F \cap V_i \neq \emptyset \text{ for each } i\}$ , where the  $V_i$  are open sets in  $X$ . It is easily shown that  $2^X$  and  $C(X)$  are continua.

THEOREM 1.1. [4].  $2^X$  and  $C(X)$  are arcwise-connected.

THEOREM 1.2. [21].  $2^X$  and  $C(X)$  are  $n$ -connected for all  $n$ , and locally  $n$ -connected for  $n > 0$ .

THEOREM 1.3. [21], [32].  $2^X$  and  $C(X)$  are acyclic in all dimensions.

The central concept underlying the proof of (1.1) is the existence of order arcs. An arc  $\alpha$  in  $2^X$  is an *order arc* if, for every  $A, B \in \alpha$ , either  $A \subset B$  or  $B \subset A$ . MAZURKIEWICZ [24] showed that every nondegenerate subcontinuum  $\alpha \subset 2^X$  with the above chain property is topologically an arc, hence an order arc. The endpoints of an order arc  $\alpha$  are the elements  $\cap\{A : A \in \alpha\}$  and  $\cup\{A : A \in \alpha\}$  of  $2^X$ , and if the endpoint  $\cap\alpha$  is in  $C(X)$ , then  $\alpha \subset C(X)$ . Borsuk and Mazurkiewicz essentially constructed an order arc between an arbitrary element  $A$  of  $2^X$  and the element  $X$ .

KELLEY [21] showed that there exists an order arc in  $2^X$  from  $A$  to  $B$  if and only if  $A \subset B$  and every component of  $B$  meets  $A$ . His proofs introduced into hyperspace theory the concept of Whitney maps. A map  $\omega : 2^X \rightarrow [0, \infty)$  is a *Whitney map* if  $\omega(\{x\}) = 0$  for every singleton  $\{x\}$  and  $\omega(A) < \omega(B)$  whenever  $A$  is a proper subset of  $B$ . Such maps exist for every metric continuum  $X$ , and have come to play an important unifying role in hyperspace theory (see [26]).

The higher dimensional connectivity properties for hyperspaces (global and local  $n$ -connectedness for  $n > 0$ ) are immediate consequences of the fact that, for a cell  $\sigma$  with  $\dim \sigma > 1$ , there exists a map  $r : \sigma \rightarrow C(\text{Bd}\sigma)$  such that  $r(p) = \{p\}$  for all  $p \in \text{Bd}\sigma$ . Thus any map  $f : \text{Bd}\sigma \rightarrow 2^X$  has an extension  $\tilde{f} : \sigma \rightarrow 2^X$  defined by  $\tilde{f}(y) = \cup\{f(p) : p \in r(y)\}$ , and if  $f$  maps into  $C(X)$ , so does  $\tilde{f}$ .

SEGAL [32] was the first to apply inverse limit techniques to hyperspaces. He showed that the hyperspace operation commutes with inverse limits: if  $X \approx \text{inv lim}(X_i, f_i)$ , where  $X$  and  $X_i$  are metric continua, and if  $f_i^* : 2^{X_{i+1}} \rightarrow 2^{X_i}$  and  $\tilde{f}_i : C(X_{i+1}) \rightarrow C(X_i)$  are the induced hyperspace maps, then

$2^X \approx \text{inv lim}(2^{X_i}, f_i^*)$  and  $C(X) \approx \text{inv lim}(C(X_i), \tilde{f}_i)$ . Since every metric continuum  $X$  has an inverse limit representation  $\text{inv lim}(X_i, f_i)$  with each  $X_i$  a finite connected polyhedron, and since each hyperspace  $2^{X_i}$  and  $C(X_i)$  is an AR (see §2), the hyperspace  $2^X$  and  $C(X)$  are represented as inverse limits of AR's, and can therefore be viewed as the intersections of nested sequences of AR's.

## 2. HYPERSPACES OF PEANO CONTINUA

THEOREM 2.1. [37], [38]. *The following conditions are equivalent:*

- (i)  $X$  is locally connected;
- (ii)  $2^X$  is locally connected;
- (iii)  $C(X)$  is locally connected.

The quickest proof for (2.1) is based on Kelley's criterion for the existence of order arcs. Thus, suppose  $X$  is locally connected. Then for nearby elements  $A$  and  $B$  of  $2^X$ , there exists an element  $C$  near each of  $A$  and  $B$  such that  $C \supset A \cup B$  and each component of  $C$  meets both  $A$  and  $B$ . Then the union of order arcs from  $A$  to  $C$  and from  $B$  to  $C$  provide a small-diameter path between  $A$  and  $B$ . For the converse, suppose  $2^X$  is locally connected, and consider nearby points  $a$  and  $b$  in  $X$ . There exists a small-diameter connected set  $M \subset 2^X$  containing  $\{a\}$  and  $\{b\}$ , and  $U\{M: M \in M\}$  is a small-diameter connected set in  $X$  containing  $a$  and  $b$ .

THEOREM 2.2. [41]. *The following are equivalent:*

- (i)  $X$  is locally connected;
- (ii)  $2^X$  is an AR;
- (iii)  $C(X)$  is an AR.

Perhaps the easiest proof for (2.2) is based on the Lefschetz-Dugundji characterization of ANR's by extension of partial realizations of polytopes [17]. The argument uses the local path-connectedness of  $2^X$  given by (2.1), the proof for local  $n$ -connectedness for  $n > 0$  (1.2), and the fact that an ANR which is  $n$ -connected for all  $n$  is an AR.

In [41] WOJDYSLAWSKI also asked whether  $2^X$  is homeomorphic to  $Q$  (the Hilbert cube) for every nondegenerate Peano continuum  $X$ . Earlier, MAZURKIEWICZ [23] had shown that for every nondegenerate metric continuum  $X$ ,  $2^X$  contains a copy of the Hilbert cube and is therefore infinite-dimensional. A special case of this question (is  $2^I \approx Q?$ ), and an analogous

question (is  $C(B^2) \approx Q$ ?), had been considered by Polish topologists in the 1920s. Note that a necessary condition for  $C(X) \approx Q$  is that  $X$  contain no free arcs, since  $C(I)$  is a 2-cell.

The question of dimension for the hyperspace  $C(X)$  of a Peano continuum  $X$  was answered by KELLEY [21]. If  $X$  is a finite graph, then  $C(X)$  is a finite polyhedron with  $\dim C(X) = \max\{\text{ord}_A[X]: A \in C(X)\}$ . And if  $X$  is not a finite graph,  $\dim C(X) = \infty$ . In the latter case,  $C(X)$  actually contains a copy of the Hilbert cube (see [8], [25]). DUDA [16] has investigated in considerable detail the polyhedral structure of the hyperspace  $C(X)$ , for  $X$  a finite graph.

Further evidence supporting the conjectures that  $2^X \approx Q$  and, for  $X$  containing no free arcs, that  $C(X) \approx Q$ , was given by GRAY [18], [19]. He showed that if  $X$  is nondegenerate, each point of  $2^X$  is unstable, and if  $X$  is a finite connected polyhedron with no free arcs, each point of  $C(X)$  is unstable. Of course, each point of  $Q$  is unstable (i.e. there exists arbitrarily small maps from  $Q$  into  $Q \setminus \{pt\}$ ).

**THEOREM 2.3.** [29], [30], [31]. *If  $X$  is a nondegenerate finite connected graph or dendron, then  $2^X \approx Q$ .*

The key concepts appearing in the proof are  $Q$ -factors, near-homeomorphisms, and 'interior approximation' via inverse limits. We briefly outline the proof that  $2^I \approx Q$  ( $I$  is the closed unit interval), indicating how these concepts are used.

A compact space  $X$  is a  $Q$ -factor if  $X \times Q \approx Q$ . Since  $X$  is a retract of  $X \times Q$ , it is necessary that  $X$  be an AR. EDWARDS (see [6]) later established that every compact metric AR is a  $Q$ -factor, but at the time the hyperspace results were obtained, geometric techniques of WEST [39] were used to show that certain finite-dimensional subspaces of  $2^I$  are  $Q$ -factors. In particular, for each positive integer  $n$  the subspace  $B_n = \{F \in 2^I: F \supset \{0,1\} \text{ and each component of } F \text{ has length at least } 1/n\}$  is a  $Q$ -factor.

The union  $\bigcup_1^\infty B_n$  is dense in the space  $2_{01}^I = \{F \in 2^I: F \supset \{0,1\}\}$ , and  $2_{01}^I \approx \text{inv lim}(B_n, f_n)$ , where each bonding map  $f_n: B_{n+1} \rightarrow B_n$  is a naturally defined 'fattening' map. A general interior approximation lemma that applies here and in subsequent hyperspace proofs, is stated in [12]:

**LEMMA 2.4.** *Let  $X$  be a compact metric space, and  $(X_i, f_i)$  an inverse sequence of maps and subcompacta of  $X$  such that*

- (i)  $X_i \rightarrow X$  (in  $2^X$ ),
  - (ii)  $d(f_i, \text{id}) < 2^{-i}$  for each  $i$ , and
  - (iii)  $\{f_i \circ \dots \circ f_j : j \geq i\}$  is an equi-uniformly continuous family for each  $i$ .
- Then  $X \approx \text{inv lim}(X_i, f_i)$ .

This lemma is used in conjunction with an approximation lemma of BROWN [5]:

**LEMMA 2.5.** *Let  $(X_i, f_i)$  be an inverse sequence such that each  $X_i$  is homeomorphic to a compact metric space  $Y$  and each  $f_i$  is a near-homeomorphism (i.e., a uniform limit of homeomorphisms). Then  $\text{inv lim}(X_i, f_i) \approx Y$ .*

The maps  $f_n : B_{n+1} \rightarrow B_n$  referred to above have the property that each map  $f_n \times \text{id} : B_{n+1} \times Q \rightarrow B_n \times Q$  is a near-homeomorphism. It follows from the approximation lemmas that  $2_{01}^I \times Q \approx \text{inv lim}(B_n \times Q, f_n \times \text{id}) \approx Q$ . Thus  $2_{01}^I$  is a  $Q$ -factor.

WEST [39] had previously shown that every countably infinite product of nondegenerate  $Q$ -factors is homeomorphic to  $Q$ . For each  $n$ , the subspace  $Y_n = \{F \in 2_{01}^I : F \supset \{\frac{1}{n}, \frac{1}{n+1}, \dots\}\}$  is homeomorphic to  $\prod_1^\infty 2_{01}^I$ , and therefore homeomorphic to  $Q$ . Appropriate near-homeomorphisms  $r_n : Y_{n+1} \rightarrow Y_n$  are constructed, and the approximation lemmas are used again to obtain  $2_{01}^I \approx \text{inv lim}(Y_n, r_n) \approx Q$ . Finally, the observation that  $2^I$  is the double cone over  $2_{01}^I$ , and the fact that  $\text{Cone } Q \approx Q$ , gives the result  $2^I \approx Q$ .

The verification of near-homeomorphisms in the above constructions involved the use of  $Q$ -factor decompositions [7], which are analogous to the simplicial subdivisions of complexes. CHAPMAN [6] later showed that a map of  $Q$  onto itself is a near-homeomorphism if (and only if) it is a CE-map (i.e., point-inverses have trivial shape).

**THEOREM 2.6.** [40]. *If  $X$  is a finite connected graph or dendron, then  $C(X) \times Q \approx Q$ ; if  $X$  is a dendron with a dense set of branch points, then  $C(X) \approx Q$ .*

One can view the hypothesis in the second part of (2.6) as follows: if the branch points of  $X$  are dense, each subcontinuum of  $X$  can expand (or contract) in infinitely many directions, thus each subcontinuum has non-finite order in  $X$ . Hence  $C(X)$  locally looks like an infinite product of intervals, and this is the key to the result  $C(X) \approx Q$ .

Finally, affirmative answers to the general conjectures concerning the hyperspaces of Peano continua were announced in [12]:

THEOREM 2.7. *Let  $X$  be a nondegenerate Peano continuum. Then  $2^X \approx Q$ ,  $C(X) \times Q \approx Q$ , and  $C(X) \approx Q$  if and only if  $X$  contains no free arcs.*

Details of the proof for the polyhedral case appear in [14], and for the general case in [15]. The proof is based on the earlier results (2.3) and (2.6) of SCHORI and WEST for the hyperspaces of finite connected graphs and dendra, and uses all the previously mentioned techniques. The main idea is to construct a sequence  $\{\Gamma_i\}$  of finite connected graphs in  $X$ , and maps  $f_i: 2^{\Gamma_{i+1}} \rightarrow 2^{\Gamma_i}$  which are near-homeomorphisms, such that the inverse sequence  $(2^{\Gamma_i}, f_i)$  is an interior approximation for  $2^X$  in the sense of (2.4), and therefore  $2^X \approx \text{inv lim}(2^{\Gamma_i}, f_i) \approx Q$ . In the case that  $X$  contains no free arcs the graphs  $\{\Gamma_i\}$  can be modified, by the addition of countably many 'stickers', to obtain a sequence  $\{\Gamma_i^*\}$  of local dendra with dense sets of branch points. West's techniques for dealing with the hyperspace of subcontinua of a dendron apply also to this situation, giving  $C(\Gamma_i^*) \approx Q$ . Near-homeomorphisms  $g_i: C(\Gamma_{i+1}^*) \rightarrow C(\Gamma_i^*)$  are then constructed such that  $C(X) \approx \text{inv lim}(C(\Gamma_i^*), g_i) \approx Q$ .

The graphs  $\Gamma_i$  may be obtained, when  $X$  is a polyhedron, as the 1-skeletons of a sequence  $\{K_i\}$  of subdivisions of  $X$ , with each  $K_{i+1}$  a refinement of  $K_i$  and mesh  $K_i \rightarrow 0$ . In the general case we must partition  $X$ , breaking it up into a finite number of small Peano subcontinua intersecting only along their boundaries (see [3]). Trees are constructed in each partition element, such that their union is a connected graph  $\Gamma$  which can be viewed as a 1-dimensional nerve of the partition. Thus, we construct a sequence  $\{P_i\}$  of partitions of  $X$ , with each  $P_{i+1}$  a refinement of  $P_i$  and mesh  $P_i \rightarrow 0$ , and a corresponding sequence  $\{\Gamma_i\}$  of nerves of the partitions. These nerves are the desired finite connected graphs in  $X$ .

In 1977, TORUNCZYK [34] obtained a suprisingly general characterization of Hilbert cube manifolds and the Hilbert cube. Specifically, a compact metric AR space  $X$  is homeomorphic to  $Q$  if for every  $\epsilon > 0$ , there exist maps  $f, g: X \rightarrow X$  with  $d(f, \text{id}) < \epsilon$ ,  $d(g, \text{id}) < \epsilon$ , and  $f(X) \cap g(X) = \emptyset$ . The proof uses Edward's  $Q$ -factor theorem together with many earlier techniques in  $Q$ -manifold theory. In turn, this simple characterization provides short, in some cases almost immediate, proofs of most previous theorems on identifying the Hilbert cube. The hyperspace theorems are no exception. In particular, for the hyperspace  $2^I$  the maps  $f, g: 2^I \rightarrow 2^I$  defined by  $f(A) = \bar{N}_\epsilon(A)$  and  $g(A) = (1-\epsilon)A \cup \{\text{sup } A\}$  have disjoint images, since no set  $f(A)$  has isolated points [36]. TORUNCZYK has given in [34] a short argument for  $2^X$  and  $C(X)$

using the existence of a convex metric on the Peano continuum  $X$  and a hyperspace lemma from [13].

§3. GROWTH HYPERSPACES

A nonempty closed subspace  $G$  of  $2^X$  satisfying the following condition is called a *growth hyperspace*: if  $A \in G$  and  $B \in 2^X$  such that  $B \supset A$  and each component of  $B$  meets  $A$ , then  $B \in G$ .

This condition was first studied by KELLEY [21], who showed that every growth hyperspace of a Peano continuum is an AR. The argument is the same as the outlines in §1 and §2 for  $2^X$  and  $C(X)$ , using the fact that if  $\alpha \subset 2^X$  is an order arc with the endpoint  $\bigcap\{A: A \in \alpha\}$  an element of  $G$ , then  $\alpha \subset G$ . Note that every growth hyperspace contains the element  $X$  of  $2^X$ .

Obviously both  $2^X$  and  $C(X)$  are growth hyperspaces. For  $A \in 2^X$  we have the following additional examples:

- (1)  $2_A^X = \{F \in 2^X: F \supset A\}$ ;
- (2)  $2^X(A) = \{F \in 2^X: F \cap A \neq \emptyset\}$ ;
- (3)  $C_A(X) = C(X) \cap 2_A^X$ ;
- (4)  $C(X;A) = C(X) \cap 2^X(A)$ ;
- (5)  $G_A(X) = \{F \in 2^X: F \supset A \text{ and each component of } F \text{ meets } A\}$ ;
- (6)  $C^n(X) = \{F \in 2^X: F \text{ has at most } n \text{ components}\}$ .

An *inclusion hyperspace* is a growth hyperspace satisfying the stronger condition: if  $A \in G$ , and  $B \in 2^X$  with  $B \supset A$ , then  $B \in G$ . Examples (1) and (2) above are inclusion hyperspaces.

**THEOREM 3.1.** [9]. *Let  $G$  be a nontrivial growth hyperspace of a Peano continuum  $X$ , such that either  $X$  contains no free arcs or  $G$  is an inclusion hyperspace. Then  $G \setminus \{X\}$  is a  $[0,1]$ -stable  $Q$ -manifold. Furthermore, the following are equivalent:*

- (i)  $G \approx Q$ ;
- (ii)  $G \setminus \{X\}$  is contractible;
- (iii)  $X$  is an unstable point in  $G$ .

If  $X$  is a nondegenerate and  $A \neq X$ , the inclusion hyperspaces  $2_A^X$  and  $2^X(A)$  are homeomorphic to  $Q$ . And if  $X$  contains no free arcs, the growth hyperspaces  $C_A(X)$ ,  $C(X;A)$ ,  $G_A(X)$ , and  $C^n(X)$  are homeomorphic to  $Q$ . For an example of a growth hyperspace not homeomorphic to  $Q$ , let  $A$  and  $B$  be proper closed subsets of  $X$  with  $A \cup B = X$ ; then  $2_A^X \cup 2_B^X$  is the union of two copies of  $Q$  intersecting in a point.

A more general result appears in [9]: for every  $Q$ -manifold  $M$  and nondegenerate Peano continuum  $X$ , there exists an inclusion hyperspace  $G$  of  $X$  such that  $G \setminus \{X\} \approx M \times [0,1)$ .

#### §4. THE HYPERSPACE OF COMPACT CONVEX SUBSETS

For  $X$  a compact convex set in a locally convex linear metric space, the hyperspace  $cc(X)$  is the space of nonempty compact convex subsets of  $X$ . NADLER, QUINN, and STAVRAKAS [27], [28] initiated the study of  $cc$ -hyperspaces, using techniques from functional analysis and convexity theory.

THEOREM 4.1. [27]. *If  $\dim X > 1$ , then  $cc(X) \approx Q$ .*

The proof consists in showing that  $\dim cc(X) = \infty$  and that  $cc(X)$  imbeds as a convex subset of Hilbert space  $\ell^2$ . The result follows from the classical theorem of KELLER [20] that every infinite-dimensional compact convex subset of  $\ell^2$  is a Hilbert cube. Many other interesting results, examples, and questions on  $cc$ -hyperspaces appear in [26].

A *convex growth hyperspace* is a nonempty closed subspace  $G$  of  $cc(X)$  satisfying the following condition: if  $A \in G$  and  $B \in cc(X)$  with  $B \supset A$ , then  $B \in G$ . Results for convex growth hyperspaces analogous to those for growth hyperspaces are given in [9].

#### §5. Z-SETS AND PSEUDO-BOUNDARIES IN HYPERSPACES

A closed subset  $A$  of a compact metric AR space  $Y$  is a *Z-set* in  $Y$  if for every  $\varepsilon > 0$  there exists a map  $f: Y \rightarrow Y \setminus A$  with  $d(f, id) < \varepsilon$ . An equivalent condition is that, for every open subset  $U$  of  $Y$ , the inclusion map  $U \setminus A \rightarrow U$  is a homotopy equivalence. Z-sets were introduced by ANDERSON [1], and have come to play a fundamental role in infinite-dimensional topology (see [6]). Obvious examples of Z-sets in the Hilbert cube  $I^\infty = \prod_1^\infty [-1,1]$  are the endslices  $W_i^+ = \pi_i^{-1}(1)$  and  $W_i^- = \pi_i^{-1}(-1)$ . Every closed subset  $A \subset I^\infty$  with the property that  $\pi_i(A)$  is a proper subset of  $[-1,1]$  for infinitely many indices  $i$ , is a Z-set. In particular, every compact subset of the *standard pseudo-interior*  $s = \prod_1^\infty (-1,1)$  is a Z-set in  $I^\infty$ . Anderson showed that the Hilbert cube is Z-set homogeneous: every homeomorphism between Z-sets in  $Q$  extends to a homeomorphism of  $Q$  onto itself.

We have the following examples of Z-sets in hyperspaces:



THEOREM 5.1. [10]. Let  $A$  be a nonempty closed subset of a Peano continuum  $X$ . Then  $2^X(A)$  (respectively,  $C(X;A)$ ) is a  $Z$ -set in  $2^X$  (respectively,  $C(X)$ ) if and only if  $A$  is locally non-separating in  $X$  (i.e., for every nonempty connected open subset  $U$  of  $X$ ,  $U \setminus A$  is nonempty and connected).

As another example, it can be shown that if  $X$  is nondegenerate, then  $C^n(X) = \{F \in 2^X : F \text{ has at most } n \text{ components}\}$  is a  $Z$ -set in  $2^X$ , for each  $n$ .

The subset  $\text{Bd}(I^\infty) = \bigcup_{i=1}^\infty (W_i^+ \cup W_i^-) = I^\infty \setminus s$  is the *standard pseudo-boundary* of  $I^\infty$ . In general, any subset  $B$  of a Hilbert cube  $Q$  such that  $(Q, B)$  is homeomorphic as a pair to  $(I^\infty, \text{Bd}(I^\infty))$  is called a *pseudo-boundary* of  $Q$ . The complement  $Q \setminus B$  is called a *pseudo-interior*; note that  $Q \setminus B \approx s \approx \ell^2$ . ANDERSON [2] gave the first non-trivial example  $\Sigma = \{(x_i) \in I^\infty : \sup |x_i| < 1\}$  of a pseudo-boundary, and a topological characterization of pseudo-boundaries as *cap-sets* (dense countable unions of  $Z$ -sets in  $Q$  with a certain compact absorption property). Using this, KROONENBERG [22] gave the following characterization:

LEMMA 5.2. Let  $\{K_i\}$  be an increasing sequence of subsets of  $Q$  such that

- (i) each  $K_i \approx Q$ ,
- (ii) each  $K_i$  is a  $Z$ -set in  $Q$ ,
- (iii) each  $K_i$  is a  $Z$ -set in  $K_{i+1}$ , and
- (iv) for each  $\epsilon > 0$  there exists a map  $f: Q \rightarrow K_i$  for some  $i$  such that  $d(f, \text{id}) < \epsilon$ .

Then  $\bigcup_{i=1}^\infty K_i$  is a pseudo-boundary in  $Q$ .

The first examples of pseudo-boundaries and pseudo-interiors in the hyperspaces  $2^X$  and  $C(X)$  were given by Kroonenberg, using the above characterization. Namely, the subspace of closed 0-dimensional sets in the interval  $I$ , the subspace of  $Z$ -sets in  $Q$ , and the subspace of connected  $Z$ -sets in  $Q$ , are pseudo-interiors in the hyperspaces  $2^I$ ,  $2^Q$ , and  $C(Q)$ , respectively. Using the fact that each countable union of  $Z$ -sets containing a pseudo-boundary is itself a pseudo-boundary. Kroonenberg showed further that the hyperspaces  $2^S = \{F \in 2^{I^\infty} : F \subset s\}$  and  $C(s) = \{F \in C(I^\infty) : F \subset s\}$  are pseudo-interiors in  $2^{I^\infty}$  and  $C(I^\infty)$ , respectively. Incidentally, these were the first examples given of hyperspaces homeomorphic to Hilbert spaces  $\ell^2$ . This technique was generalized in [10], resulting in a characterization of those non-compact spaces  $X$  which admit Peano compactifications  $\tilde{X}$  such that  $2^X$  is a pseudo-interior in  $2^{\tilde{X}}$  (see §6).

Mark Michael and the author have recently obtained a type of result which subsumes the above examples:

**THEOREM 5.3.** *If  $X$  is a Peano continuum and  $B \subset 2^X$  is a countable dense union of  $Z$ -set inclusion hyperspaces, then  $B$  is a pseudo-boundary in  $2^X$ . If  $X$  is a Peano continuum containing no free arcs, and  $B \subset 2^X$  (respectively,  $B \subset C(X)$ ) is a countable dense union of  $Z$ -set growth hyperspaces, then  $B$  is a pseudo-boundary in  $2^X$  (respectively,  $C(X)$ ).*

Thus for example, if  $\{F_i\}$  is a countable dense family of closed locally non-separating sets in  $X$ , it follows from (5.1) that  $\bigcup_{i=1}^{\infty} 2^X(F_i)$  is a pseudo-boundary in  $2^X$ , and if  $X$  contains no free arcs,  $\bigcup_{i=1}^{\infty} C(X;F_i)$  is a pseudo-boundary in  $C(X)$ . Another example is the subset  $\bigcup_{n=1}^{\infty} C^n(X)$ , which is a pseudo-boundary in  $2^X$  if  $X$  contains no free arcs.

There is a  $\sigma$ -finite-dimensional analogue of pseudo-boundaries. Consider  $I_f^{\infty} = \{(x_i) \in I^{\infty} : x_i = 0 \text{ for almost all } i\}$ , a countable dense union of finite-dimensional  $Z$ -sets in  $I^{\infty}$ . Any subset  $B$  of a Hilbert cube  $Q$  such that  $(Q, B) \approx (I^{\infty}, I_f^{\infty})$  is called an *fd-cap set* [2]. Anderson characterized these sets in terms of a finite-dimensional compact absorption property, and showed that the complement  $Q \setminus B$  is homeomorphic to  $s$ . (In the 'elliptic' Hilbert cube  $K = \{(y_i) \in \ell^2 : \sum_{i=1}^{\infty} i^2 y_i^2 \leq 1\}$ , the 'boundary' set  $B = \{(y_i) \in K : \sum_{i=1}^n i^2 y_i^2 = 1 \text{ for some } n\}$  is an fd-cap set, and it is easily seen that  $K \setminus B \approx s$ .)

**QUESTION 5.4.** Let  $\Gamma$  be a finite connected graph, and  $F$  a countable dense union of finite-dimensional  $Z$ -set growth hyperspaces in  $2^{\Gamma}$ . Is  $F$  an fd-cap set in  $2^{\Gamma}$ ?

Michael has obtained an affirmative answer for  $\Gamma = I$ . Thus for example,  $\bigcup_{n=1}^{\infty} C^n(I)$  is an fd-cap set in  $2^I$ . Relative to this example, it can in fact be shown that  $F = \{F \in 2^I : F \text{ is finite}\}$  is an fd-cap set in  $2^I$ .

## §6. HYPERSPACES OF NON-COMPACT SPACES

For an arbitrary metric space  $X$ , the Hausdorff metric on the hyperspace  $2^X$  of nonempty compact subsets induces the Vietoris finite topology (see §1). Wojdyslawski's characterization (2.2) of hyperspaces which are compact metric AR's applies also to topologically complete metric spaces, as noted by Tasmetov:

THEOREM 6.1. [33].  $2^X$  and  $C(X)$  are topologically complete metric AR's if and only if  $X$  is connected, locally connected, and topologically complete.

Hereafter,  $X$  denotes a connected, locally connected, and topologically complete metric space. There exist various topological characterization theorems for the hyperspaces of  $X$ , analogous to (2.7). The first two theorems stated below involve certain Peano compactifications  $\tilde{X}$  of  $X$ . A metric  $d$  on  $X$  has *Property S* if there exist finite covers of  $X$  by connected subsets with arbitrarily small diameters. Note that a space admitting a *Property S* metric is necessarily separable and locally connected.

LEMMA 6.2. [10].  $X$  has a Peano compactification  $\tilde{X}$  with a locally non-separating remainder  $\tilde{X} \setminus X$  if and only if  $X$  admits a metric with *Property S*.

$Q_0$  is the space  $Q \setminus pt$ . Since  $Q$  is homeomorphic to  $Cone\ Q$ ,  $Q_0 \approx Cone\ Q \setminus \{pt\} \approx Q \times [0,1)$ .

THEOREM 6.3. [10].  $2^X \approx Q_0$  if and only if  $X$  is locally compact but non-compact. Similarly,  $C(X) \approx Q_0$  if and only if  $X$  is locally compact, non-compact, and contains no free arcs.

Outline of Proof. A space  $X$  satisfying the above conditions admits a metric with *Property S*, and therefore has a Peano compactification  $\tilde{X}$  with a closed locally non-separating remainder. By (5.1), the hyperspace  $2^{\tilde{X}}(\tilde{X} \setminus X)$  is a  $Z$ -set copy of  $Q$  in  $2^{\tilde{X}}$ . Thus  $(2^{\tilde{X}}, 2^{\tilde{X}}(\tilde{X} \setminus X)) \approx (Q \times [0,1], Q \times \{1\})$  by  $Z$ -set homogeneity, and  $2^X = 2^{\tilde{X}} \setminus 2^{\tilde{X}}(\tilde{X} \setminus X) \approx Q \times [0,1)$ . The converse is immediate, since  $X$  has a closed embedding into  $2^X$ .

THEOREM 6.4. [10].  $X$  admits a Peano compactification  $\tilde{X}$  such that the pairs  $(2^{\tilde{X}}, 2^X)$  and  $(C(\tilde{X}), C(X))$  are homeomorphic to  $(I^\infty, s)$  if and only if  $X$  is nowhere locally compact and admits a metric with *Property S*.

Outline of Proof. A space  $X$  satisfying the above conditions has a Peano compactification  $\tilde{X}$  with the remainder  $\tilde{X} \setminus X$  a dense countable union  $\bigcup_{i=1}^\infty F_i$  of closed locally non-separating sets in  $\tilde{X}$ . Then  $\bigcup_{i=1}^\infty 2^{\tilde{X}}(F_i) \subset 2^{\tilde{X}}$  is a dense countable union of  $Z$ -set inclusion hyperspaces, and is therefore a pseudo-boundary in  $2^{\tilde{X}}$  (5.3). Thus  $2^X = 2^{\tilde{X}} \setminus \bigcup_{i=1}^\infty 2^{\tilde{X}}(F_i)$  is a pseudo-interior in  $2^{\tilde{X}}$ . The argument for  $C(X)$  is similar, using the fact that  $\tilde{X}$  contains no free arcs. Conversely, if either  $2^X$  or  $C(X)$  is a pseudo-interior in  $2^{\tilde{X}}$  or  $C(\tilde{X})$ , respectively, it can be shown that  $\tilde{X} \setminus X$  is locally non-separating in  $\tilde{X}$ , hence

$X$  admits a metric with Property S.

Since  $s \approx \ell^2$ , (6.4) gives sufficient (but not necessary) conditions for  $2^X$  and  $C(X)$  to be homeomorphic to  $\ell^2$ . The characterization of those spaces  $X$  with hyperspaces homeomorphic to Hilbert spaces takes the following simple form:

THEOREM 6.5. [11].  $2^X$  and  $C(X)$  are homeomorphic to  $\ell^2$  if and only if  $X$  is separable and nowhere locally compact.

The proof uses Toruńczyk's characterization of  $\ell^2$  as a separable, topologically complete metric AR with a general position property for countable families of compact subsets [35]. This characterization is applicable in a more general form to Hilbert spaces of arbitrary weight. Efforts to apply it to the hyperspace of a non-separable metric space  $X$  lead naturally to the following question:

QUESTION 6.6. Suppose that for some uncountable cardinal  $\alpha$ , every nonempty open subset of  $X$  has weight  $\alpha$ . Are  $2^X$  and  $C(X)$  homeomorphic to the Hilbert space  $\ell^2(\alpha)$  of weight  $\alpha$ ?

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## A MEASURE THAT KNOWS WHICH SETS ARE HOMEOMORPHIC

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Haar measure  $\mu$  on a compact group  $G$  has the pleasant property of being invariant under all autohomeomorphisms<sup>3)</sup> of  $G$  which are algebraically significant: left and right translations, and topological isomorphisms of  $G$  (in particular under inversion); the restriction to algebraically significant homeomorphisms is essential here, as an easy example with the circle group shows. Furthermore, up to a multiplicative constant,  $\mu$  is the only nonzero Borel measure on  $G$  which is invariant under all left translations.

We here announce the existence of the following "impossible" example, and sketch its construction. Details will appear in [vD].

EXAMPLE. There exists an infinite compact zero-dimensional homogeneous<sup>4)</sup> space  $bH$  in which every open set is an  $F_\sigma$ , which has a Borel measure  $\bar{\mu}$  satisfying

- (1) if  $X$  and  $Y$  are homeomorphic Borel sets in  $bH$ , then  $\bar{\mu}(X) = \bar{\mu}(Y)$ ,
- (2) if  $X$  and  $Y$  are open subsets which are both compact or both noncompact, then
  - (a) if  $\bar{\mu}(X) = \bar{\mu}(Y)$  then  $X$  and  $Y$  are homeomorphic, and
  - (b) if  $\bar{\mu}(X) \leq \bar{\mu}(Y)$  then  $X$  can be embedded in  $Y$  as an open subset;
- (3) up to a multiplicative constant  $\bar{\mu}$  is the only nonzero Borel measure on  $bH$  which is invariant under all autohomeomorphisms of  $bH$ .

REMARKS. (a): The consequence of (1) and (2) that

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- 1) I am indebted to Arthur Stone for indirectly suggesting this title.
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  - 3) An autohomeomorphism of a space  $X$  is a homeomorphism of  $X$  onto itself.
  - 4) A space  $X$  is called homogeneous if for all  $x, y \in X$  there is a autohomeomorphism<sup>3)</sup> of  $X$  sending  $x$  to  $y$ .

(\*)  $\bar{\mu}$  is a nonzero Borel measure invariant under all autohomeomorphisms

would have been trivial if the identity had been the only autohomeomorphism of  $bH$ . But (\*) is highly nontrivial since  $bH$  has many autohomeomorphisms, being homogeneous, and also because of (3). Of course (1) and (2) are even more nontrivial.

(b) The original motivation for finding  $bH$  is the question of Monk and Rubin, which occurs in an early version of [vDMR], of whether a Boolean algebra  $B$  must be homogeneous (in the Boolean algebraic sense, i.e.  $B$  is isomorphic to  $B \setminus b$  for all  $b \in B - \{0\}$ ) if its Stone space is homogeneous (in the topological sense). Our example shows that such a Boolean algebra can even be Hopfian (= not isomorphic to any proper quotient).

We construct  $bH$  as an easy to visualize compactification of a very special subgroup  $H$  of the circle group  $\mathbb{T}$ .  $H$  is among others a thick subgroup of  $\mathbb{T}$ , hence the Lebesgue (or Haar) measure  $\mu$  of  $\mathbb{T}$  induces a Borel measure  $\bar{\mu}$  on  $H$  according to the rule

$$\bar{\mu}(B) = \mu(B'), \text{ with } B' \text{ any Borel set in } \mathbb{T} \text{ with } B = H \cap B',$$

[H, p.74]. Then  $\bar{\mu}$  in turn induces  $\bar{\bar{\mu}}$  according to the rule

$$\bar{\bar{\mu}}(B) = \bar{\mu}(B \cap H).$$

[This explains the notation  $bH$  and  $\bar{\bar{\mu}}$ .]

In the remaining part of this announcement we indicate how one constructs a subgroup  $H$  of  $\mathbb{T}$  such that

(4) if  $X$  and  $Y$  are homeomorphic Borel subsets of  $H$ , then  $\bar{\mu}(X) = \bar{\mu}(Y)$ .

This condition will imply (1) above since we construct  $bH$  in such a way that if  $X$  and  $Y$  are homeomorphic subsets of  $bH$ , then there are countable sets  $C$  and  $D$  such that  $X \cap H - C$  and  $Y \cap H - D$  are homeomorphic, so that (4) and the definition of  $\bar{\bar{\mu}}$  imply

$$\bar{\bar{\mu}}(X) = \bar{\mu}(X \cap H) = \bar{\mu}(X \cap H - C) = \bar{\mu}(Y \cap H - D) = \bar{\mu}(Y \cap H) = \bar{\bar{\mu}}(Y),$$

if  $X$  and  $Y$  are Borel. Our construction of  $bH$  is a modification of the construction of the Alexandroff double arrow space, [AU, Ex. A<sub>7</sub>]. In order to

ensure that  $\bar{\mu}$  satisfies (2) and (3) we will build in additional properties in  $H$ , which tells us that the analogues of (2) and (3) hold for  $\bar{\mu}$ . These additional properties are easily built in, but it would take too much space to explain their function, hence we ignore them.

Before we proceed to the outline of the actual construction we explain what we are aiming for. Suppose (4) is false. Let  $X$  and  $Y$  be Borel sets of  $H$  with  $\bar{\mu}(X) > \bar{\mu}(Y)$  for which there is a homeomorphism  $f: X \rightarrow Y$ . There are  $G_\delta$ -subsets  $X'$  and  $Y'$  of  $\mathbb{T}$  with

$$X \subseteq X' \text{ and } \bar{\mu}(X) = \mu(X'), \text{ and } Y \subseteq Y' \text{ and } \bar{\mu}(Y) = \mu(Y').$$

Since  $\mathbb{T}$  is completely metrizable, being compact, there are by a classical result of LAVRENTIEFF, [L],  $G_\delta$ -subsets  $\tilde{X}$  and  $\tilde{Y}$  of  $\mathbb{T}$  with

$$X \subseteq \tilde{X} \subseteq X' \text{ and } Y \subseteq \tilde{Y} \subseteq Y'$$

such that  $f$  can be extended to a homeomorphism  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$ . Let  $X''$  be a Borel subset of  $\mathbb{T}$  such that  $X = H \cap X''$ . Since clearly  $\mu(\tilde{X} \cap X'') = \bar{\mu}(X) > \bar{\mu}(Y)$  we can find a compact  $D \subset \mathbb{T}$  with  $D \subset \tilde{X} \cap X''$  and  $\mu(D) > \mu(\tilde{Y})$ , hence  $\mu(D) > \mu(\tilde{f}^{-1}D)$ . Now  $H \cap D \subseteq H \cap X'' = X$ , hence  $\tilde{f}(x) = f(x) \in Y \subseteq H$  for all  $x \in H \cap D$ . Therefore our assumption that (4) is false will lead to a contradiction if we make sure that  $H$  is a *stiff* subgroup of  $\mathbb{T}$ , as defined next.

**DEFINITION.** A *compression* of  $\mathbb{T}$  is a homeomorphism  $f$  with  $\text{dom}(f)$  and  $\text{range}(f)$  compact subsets of  $\mathbb{T}$  satisfying  $\mu(\text{dom}(f)) > \mu(\text{range}(f))$ . A subset  $S$  of  $\mathbb{T}$  will be called *stiff* if for every compression  $f$  of  $\mathbb{T}$  there is an  $x \in S \cap \text{dom}(f)$  such that  $f(x) \notin S$ .

We observe that if  $S \subset \mathbb{T}$  is stiff, then  $S$  is *thick*, i.e.  $\mu(B) = 0$  for every Borel set  $B$  of  $\mathbb{T}$  that misses  $S$ . Indeed, if  $B \subset \mathbb{T}$  is Borel, and  $\mu(B) > 0$ , then there is a copy  $K$  of the Cantor Discontinuum with  $K \subseteq B$  and  $\mu(K) > 0$ . But  $K$  includes a copy  $C$  of itself with  $\mu(C) < \mu(K)$  (even with  $\mu(C) = 0$ ). A homeomorphism  $K \rightarrow C$  must be a compression, hence  $K \cap S \neq \emptyset$ .

We are now ready for the actual construction. We can list all compressions of  $\mathbb{T}$  as  $\langle f_\alpha: \alpha < \mathfrak{C} \rangle$ . With transfinite recursion on  $\alpha$  we will pick  $x_\alpha \in \text{dom}(f_\alpha)$ . For each  $\alpha < \mathfrak{C}$  we let  $H_\alpha$  be the subgroup of  $\mathbb{T}$  generated by  $\{x_\xi: \xi < \alpha\}$ . We want to make sure that for all  $\alpha \leq \mathfrak{C}$  we have

$$(*) \quad \text{if } \xi < \alpha \text{ then } f_\xi(x_\xi) \notin H_\alpha.$$

Then our stiff subgroup  $H$  will be  $H_{\mathbb{C}}$ . At stage  $\gamma$  of the construction put  $D = \text{dom}(f_\gamma)$ , and

$$S_k = \{x \in D: f_\gamma(x) \in x^k \cdot H_\gamma\}, \quad (k \in \mathbb{Z}),$$

$$S = \bigcup_{k \in \mathbb{Z}'} S_k, \quad \text{where } \mathbb{Z}' \text{ is the set of nonzero integers.}$$

We assume (\*) holds for  $\alpha = \gamma$ . Then an easy calculation shows that we can pick  $x_\gamma$  such that (\*) holds for  $\alpha = \gamma + 1$  if  $|D - S| = \mathbb{C}$ . We indicate the proof of this equality. For  $k \in \mathbb{Z}'$  define a continuous  $s_k: S_k \rightarrow H_\gamma$  by  $s_k(x) = f_\gamma(x) \cdot x^{-k}$ , and for  $C \subseteq H_\alpha$  define

$$C^\leftarrow = \{x \in D: s_k(x) \in C \text{ for some } k \in \mathbb{Z}'\}.$$

[Note that  $S = H_\gamma^\leftarrow$ ]. We claim that

$$(\dagger) D - C^\leftarrow \text{ is uncountably for all countable } C \subseteq H_\gamma.$$

In the proof one notes that  $s_k^\leftarrow\{x\} \cap s_\ell^\leftarrow\{y\}$  is countable for  $k, \ell \in \mathbb{Z}'$  and  $x, y \in H_\gamma$  with  $k \neq \ell$  or  $x \neq y$ , and that

$$f_\gamma^\rightarrow(s_k^\leftarrow\{y\}) = y \cdot \{z^k: z \in s_k^\leftarrow\{y\}\}, \quad \text{for } k \in \mathbb{Z}', y \in H_\gamma; \text{ and}$$

$$\mu(\{z^k: z \in B\}) \geq \mu(B), \quad \text{for } k \in \mathbb{Z}' \text{ and Borel } B \subseteq \mathbb{T}.$$

Let  $C \subseteq H_\gamma$  be countable. If in  $\Sigma$  we sum over  $k \in \mathbb{Z}'$  and  $y \in C$ , then

$$\mu(C^\leftarrow) = \Sigma \mu(s_k^\leftarrow\{y\}) \leq \Sigma \mu(f_\gamma^\rightarrow(s_k^\leftarrow\{y\})) = \mu(f_\gamma^\rightarrow C^\leftarrow) \leq \mu(f_\gamma^\rightarrow D),$$

since  $f_\gamma$  is an injection. [Note that each  $s_k^\leftarrow\{y\}$  and  $f_\gamma^\rightarrow(s_k^\leftarrow\{y\})$  is Borel, and in fact compact, since the former is closed in the compact set  $D$ .] Since  $f_\gamma$  is a compression, this proves  $(\dagger)$ .

Using  $(\dagger)$  one can construct a copy  $K$  of the Cantor Discontinuum with  $K \subseteq D$  such that  $s_k|_{K \cap S_k}$  is injective for  $k \in \mathbb{Z}'$ . Then  $|K \cap S| < \mathbb{C}$  since  $|H_\gamma| < \mathbb{C}$ , hence  $|D - S| = \mathbb{C}$  since  $|K| = \mathbb{C}$ .

The same argument shows that several other second countable locally compact groups, like the reals, have a stiff subgroup, but I do not know if every such group has a stiff subgroup.

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## ORDERABILITY OF GO-SPACES

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### SUMMARY

Two theorems are given answering the question when a GO-space is orderable. The first theorem implies that if  $Y$  is a GO-space of an ordered space  $X$  and  $Y$  is itself orderable then under some additional conditions  $\text{weight } Y \leq \text{weight } X$ . The second result shows that if a GO-space has no connected compactification then such a space has no topology induced by a dense order.

Let  $<$  be a given linear order on a set  $X$ . Then the order is said to be *dense* if for each  $x, y \in X$ ,  $x < y$ , there is a  $z \in X$  such that  $x < z < y$ . Open intervals are sets of the form

$$(a, b) = \{x \in X \mid a < x < b\},$$

$$(\leftarrow, a) = \{x \in X \mid x < a\}, \quad \text{and}$$

$$(b, \rightarrow) = \{x \in X \mid x > b\}.$$

Arrows are sets of the form

$$[a, b) = \{x \in X \mid a \leq x < b\},$$

with analogous definitions for  $(\leftarrow, a]$ ,  $[b, \rightarrow)$  and  $(a, b]$ .

A topological space is *orderable*, iff there exists a linear ordering  $<$  on  $X$ , such that the collection of all open intervals forms a base for the topology on  $X$  and then  $X$  is said to be an *ordered space* with respect to  $<$ . A topological space is a *generalized ordered space*  $X$  (abbreviation GO-space of  $X$ ) iff  $X = Y$  and the topology of  $Y$  is induced by a subbase consisting of all open sets and a subset of the family of arrows.

We define:

$$Y^- = \{x \mid \text{there is an } r \in X \text{ such that } (r, x] \text{ is open in } Y\}$$

$$Y^+ = \{x \mid \text{there is an } r \in X \text{ such that } [x, r) \text{ is open in } Y\}$$

Symbols  $w(x)$ ,  $L(X)$ ,  $hL(X)$ ,  $|X|$  mean respectively the weight of  $X$ , the Lindelöf number of  $X$ , the hereditarily Lindelöf number of  $X$ , and the cardinality of  $X$  (for the definitions see e.g., [3]).

**LEMMA 1.** *Let  $X$  be an ordered space. If  $\mathcal{P}$  is a family of open intervals, directed with respect to the inclusion and  $\{x\} = \cap \mathcal{P}$ , then  $\mathcal{P}$  is a local base at the point  $x$ .*

**PROOF.** Obvious.

**THEOREM 1.** *Let  $(X, J_1, <_1)$  and  $(X, J_2, <_2)$  be ordered spaces such that  $J_1 \subset J_2$ . Then  $w(X, J_2) \leq w(X, J_1) + hL(X, J_2)$ .*

**PROOF.** Let  $\mathcal{B}$  be the base for the topology of  $J_1$  consisting of open intervals in the sense of  $<_1$ , and assume that  $|\mathcal{B}| = w(X, J_1)$ . For each  $U \in \mathcal{B}$  we choose an open covering  $\mathcal{Q}_U$  of  $U$  consisting of open intervals in the sense of  $<_2$  such that

$$\bigcup \mathcal{Q}_U = U \quad \text{and} \quad |\mathcal{Q}_U| \leq hL(X, J_2).$$

Let  $\mathcal{Q}^* = \bigcup \{\mathcal{Q}_U \mid U \in \mathcal{B}\}$  and let  $\mathcal{Q}$  be the family of all finite intersections of elements of  $\mathcal{Q}^*$ . Then for each  $x \in X$  we can choose a subfamily  $\mathcal{P}(x)$  of the family  $\mathcal{Q}$  such that  $\cap \mathcal{P}(x) = \{x\}$  and  $\mathcal{P}(x)$  is directed with respect to the inclusion. By Lemma 1,  $\mathcal{Q}$  is a base for the space  $(X, J_2)$ . Since  $|\mathcal{Q}| \leq w(X, J_1) + hL(X, J_2)$ , we have that  $w(X, J_2) \leq w(X, J_1) + hL(X, J_2)$ .

**LEMMA 2.** *If  $Y$  is a GO-space of an ordered space  $X$ , where the topology of  $X$  is induced by a dense linear order, then  $hL(Y) = hL(X) + |Y^- \cap Y^+|$ .*

**PROOF.** Let  $\mathcal{U}$  be an arbitrary family of open sets of  $Y$ . We define

$$K = \mathcal{U} \setminus [\bigcup \{\text{int}_X U \mid U \in \mathcal{U}\} \cup (Y^- \cap Y^+)].$$

For each  $x \in K$  there exists a  $U_x \in \mathcal{U}$  and an arrow  $S_x = [x, r)$  or  $S_x = (r, x]$



such that  $S_x \subset U_x$ . From the definition of  $K$  it follows that if  $x \neq y$  and  $x, y \in K \cap Y^-$  or  $x, y \in K \cap Y^+$  then  $S_x \cap S_y = \emptyset$ . Since the linear order on  $X$  is dense,  $\text{int}_X[x, r] \neq \emptyset$  and  $\text{int}_Y(x, r] \neq \emptyset$ . Hence the families consisting of the sets of the form  $\text{int}_X S_x$ ,  $x \in K \cap Y^-$  and  $\text{int}_X S_x$ ,  $x \in K \cap Y^+$ , are families of mutually disjoint open subsets of  $X$  and therefore  $|K| = \text{hL}(X)$ . Since there is a subfamily  $U^*$  of the family  $U$  such that

$$U\{\text{int}_X U \mid U \in U^*\} = U\{\text{int}_X U \mid U \in U\}$$

and  $|U^*| \leq \text{hL}(X)$ , and since

$$|K \cup (Y^- \cap Y^+)| \leq \text{hL}(X) + |Y^- \cap Y^+|,$$

there exists a subfamily  $U^{**}$  of the family  $U$  such that  $UU^{**} = UU$  and  $|U^{**}| \leq \text{hL}(X) + |Y^- \cap Y^+|$  and therefore  $\text{hL}(Y) \leq \text{hL}(X) + |Y^- \cap Y^+|$ . Moreover  $\text{hL}(X) \leq \text{hL}(Y)$  and  $|Y^- \cap Y^+| \leq \text{hL}(Y)$ , and we obtain that  $\text{hL}(Y) = \text{hL}(X) + |Y^- \cap Y^+|$ .

**COROLLARY 1.** *Let  $Y$  be a GO-space of an ordered space  $X$ , with a dense linear order, and let  $|Y^- \cap Y^+| \leq w(X)$ . If moreover  $Y$  is orderable, then  $w(Y) \leq w(X)$ .*

**THEOREM 2.** *Let  $Y$  be a GO-space of an ordered space  $X$ , with a dense linear order. If  $|Y^- \cap Y^+| \leq w(X) < |Y^- \cup Y^+|$  then  $Y$  cannot be an orderable space.*

**PROOF.** It is clear that  $w(Y) \geq |Y^- \cup Y^+|$ . Assume that  $Y$  is an orderable space. Then by Corollary 1 we have that  $w(Y) \leq w(X)$  which contradicts  $w(X) < |Y^- \cup Y^+|$  (cf. [3]).

**COROLLARY 2.** *The Sorgenfrey line is not orderable. (See also [4], [5]).*

Let  $\mathbb{Z}$  be the set of integers in their natural ordering. For every ordinal  $\tau$  we define  $\mathbb{Z}^\tau$  to be the lexicographical product of  $\tau$  copies of  $\mathbb{Z}$ , i.e. the space of all sequences  $(z_\alpha \mid \alpha < \tau)$  in which  $(z_\alpha) < (z'_\alpha)$  iff for some  $\beta < \tau$  we have  $z_\beta < z'_\beta$  and for all  $\gamma \leq \beta$ :  $z_\gamma = z'_\gamma$ .

**DEFINITION.** A topological space  $Y$  is a  $\text{GO}^*$ -space iff  $Y$  is a GO-space and  $Y^- \cup Y^+ = Y$ .

Under the hypothesis  $2^\omega < 2^{\omega_1}$  Theorem 2 implies that each  $GO^*$ -space of  $\mathbb{Z}^{\omega_1}$  which is dense in itself cannot be orderable. Under the hypothesis  $2^\omega = 2^{\omega_1}$  Theorem 2 does not allow us to decide that a  $GO^*$ -space of  $\mathbb{Z}^{\omega_1}$  cannot be ordered. However, the following result will show that no  $GO^*$ -space of  $\mathbb{Z}^{\omega_1}$  can be ordered by a dense order.

**LEMMA 3.** *Let  $Z$  be a  $T_1$  regular space and let  $Y$  be a  $GO$ -space of a space  $X$  with a dense linear order. If  $Y \subset Z$  is a dense subspace of  $Z$  and*

$$(*) \quad cl_Z A \cap cl_Z (Y \setminus A) \neq \emptyset$$

for each arrow  $A \subset X$  that is clopen in  $Y$ , then for each interval  $(x', y') \subset X$  such that  $cl_Z (Y^- \cup Y^+) \supset (x', y')$  there is a point  $p \in Z$  and an interval  $[a, b] \subset (x', y')$ ,  $a < b$ , such that  $p \in cl_Z (x', a) \cap cl_Z (b, y')$ .

**PROOF.** Since  $cl_Z (Y^- \cup Y^+) \supset (x', y')$  we can find points  $x, y, t \in (x', y')$  with  $x < t < y$ , such that either  $(x, t]$  is clopen in  $Y$  and  $Y^-$  is dense in  $(x, t]$  or  $[t, y)$  is clopen in  $Y$  and  $Y^+$  is dense in  $[t, y)$ . Assume that  $[t, y)$  is clopen in  $Y$  and that  $Y^+$  is dense in  $[t, y)$ . (In the case that  $(x, t]$  is clopen in  $Y$  the proof is analogous.) Since  $Z$  is a regular space there is a point  $z \in [t, y) \cap (Y^- \cup Y^+)$  such that  $cl_Z [t, z) \subset U([t, y))$  where  $U([t, y))$  is the largest open set in  $Z$  such that  $U([t, y)) \cap Y = [t, y)$ . According to (\*) there is a point  $p \in cl_Z [t, z) \cap cl_Z (Y \setminus [t, z))$ . For each neighbourhood  $W$  of the point  $p$  we have

$$\emptyset \neq W \cap U([t, y)) \cap (Y \setminus [t, z)) = W \cap [t, y) \cap (Y \setminus [t, z)) = W \cap [z, y)$$

i.e.  $p \in cl_Z [z, y)$ . If  $[z, d)$  is open in  $Y$  for some  $d \in Y$ , (the case where  $(c, z]$  is open in  $Y$  for a suitable  $c$  is analogous) then since  $Z$  is a Hausdorff space, we can choose an interval  $[z, t')$ ,  $t' < t$ , such that  $[z, t') \cap W = \emptyset$  for some open neighbourhood  $W$  of the point  $p$ . Let  $a, b \in Y$  be such that  $[a, b] \subset (z, t')$ . Then  $p \in cl_Z (x, a)$  and  $p \in cl_Z (b, y)$ .

**DEFINITION.** A regular space  $X$  is *pseudobase-compact* (with respect to  $\mathcal{B}$ ) iff there exists a pseudobase  $\mathcal{B}$  (cf. [1]) of open sets for  $X$  such that for each centered subfamily  $\mathcal{C}$  of  $\mathcal{B}$  the intersection  $\bigcap \{cl_X U \mid U \in \mathcal{C}\} \neq \emptyset$ . Such a pseudobase  $\mathcal{B}$  is called a *compact pseudobase* (cf. [1]).

It is easily seen that the space  $\mathbb{Z}^{\mathbb{I}}$  has a compact pseudobase. The

following theorem is a generalization of Theorem 1 from [3].

**THEOREM 3.** *Let  $X$  be a pseudobase-compact ordered space with a dense linear order. Let  $Y$  be a GO-space of  $X$  with  $Y^+ \cap Y^- = \emptyset$  and  $Y^+ \cup Y^- \supset U \setminus F$  for some non-void open set  $U$  and some first category set  $F$  in  $X$ . Then there is no  $T_1$  regular space  $Z$  such that  $Y$  is a dense subspace of  $Z$  and*

$$(*) \quad cl_Z A \cap cl_Z (Y \setminus A) \neq \emptyset$$

for each arrow  $A$  that is clopen in  $Y$ .

**PROOF.** Suppose that there exists a  $T_1$  regular space  $Z$  satisfying the condition (\*) and  $Y$  is dense in  $Z$ . Let  $\mathcal{B}$  be a compact pseudobase for  $X$ . Recall that  $Y^+ \cup Y^- \supset U \setminus F$ . Let  $F = \cup \{F_n \mid n < \omega\}$  where  $F_n \subset F_m$  for  $n < m$  and  $F_n$  is nowhere dense in  $X$  for each  $n$ . According to Lemma 3 one can define decreasing sequences  $\{U_\xi \mid \xi < \lambda\}$  and  $\{V_\xi \mid \xi < \lambda\}$  such that:

$$(1) \quad U_0 \subset U \quad \text{and} \quad U_\xi \in \mathcal{B},$$

$$(2) \quad V_\xi \subset (x_\xi, y_\xi), \quad \xi < \lambda,$$

$$(3) \quad U_\xi \supset c_X V_\xi, \quad \xi < \lambda,$$

$$(4) \quad U_n \cap F_n = \emptyset, \quad n < \omega,$$

$$(5) \quad U_{\xi+1} \subset V_\xi, \quad \xi < \lambda,$$

$$(6) \quad \exists p_\xi: p_\xi \in cl_Z(x_\xi, x_{\xi+1}) \cap cl_Z(y_\xi, y_{\xi+1})$$

$$(7) \quad int_X \cap \{U_\xi \mid \xi < \lambda\} = \emptyset.$$

Let  $\{x_0\} = \cap \{U_\xi \mid \xi < \lambda\} = \cap \{cl_X V_\xi \mid \xi < \lambda\}$ . By Lemma 1 the open intervals  $V_\xi$ ,  $\xi < \lambda$ , form a base at the point  $x_0 \in X$ . Either there exists a  $z \in Y$  such that the arrow  $(z, x_0]$  is open in  $Y$  or there exists an  $y \in Y$  such that  $[x_0, y)$  is open in  $Y$ . Suppose that  $(z, x_0]$  is open in  $Y$  (the case where  $[x_0, y)$  is open is analogous). Since  $Z$  is regular there is a point  $t < x_0$  such that  $cl_Z(t, x_0] \subset U((z, x_0])$  where  $U((z, x_0])$  is the largest open set in  $Z$  such that  $U((z, x_0]) \cap Y = (z, x_0]$ . Let us choose an arbitrary

$y > x_0$ . There is a  $\xi < \lambda$  such that  $V_{\xi+1} \subset V_\xi \subset (t, y)$  and there is a point  $p_\xi \in Z$  such that  $p_\xi \in \text{cl}_Z(t, x_{\xi+1}) \cap \text{cl}_Z(y_{\xi+1}, y)$ . But  $p_\xi \in \text{cl}_Z(y_{\xi+1}, y)$  and  $p_\xi \in U((z, x_0])$  implies that  $U((z, x_0]) \cap (y_{\xi+1}, y) \neq \emptyset$  which contradicts  $U((z, x_0]) \cap Y = (z, x_0]$  and  $y_{\xi+1} > x$ .

COROLLARY 3. *Let  $X$  be a pseudobase-compact ordered space with a dense linear order. Let  $Y$  be a GO-space of  $X$  with  $Y^+ \cap Y^- = \emptyset$  and  $Y^+ \cup Y^- \supset U \setminus F$  for some non-void open set  $U$  and some first category set  $F$  in  $X$ . Then there exists no connected regular  $T_1$  space  $Z$  containing  $Y$  as a dense subspace.*

PROOF. If  $Z$  is a connected space then the condition (\*) holds for each arrow  $A$  that is clopen in  $Y$ .

COROLLARY 4. *There is no  $T_1$  regular connected space containing the Sorgenfrey line as a dense subspace, (cf. [2]).*

COROLLARY 5. *If  $Y$  is a  $GO^*$ -space of  $\mathbb{Z}^{\omega_1}$  then there exists no dense linear order inducing the topology of  $Y$ .*

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## FUNDAMENTAL GROUPS OF MANIFOLD SCHEMES

W.T. van Est

It is the purpose of this paper to describe briefly the notion of fundamental group for manifold schemes. Full details and remarks on the higher homotopy groups will appear elsewhere.

### 1. MANIFOLD SCHEMES

We recall briefly the notion of manifold scheme (cf. [2]) in the sheaf formulation.

Let  $P$  be a manifold<sup>\*\*)</sup> and  $\Gamma_P$  (or  $\Gamma$ ) the topological groupoid of germs of transitions, i.e. diffeomorphisms from open subsets of  $P$  to open subsets of  $P$  [2,6]; the set  $E_\Gamma$  of identities of  $\Gamma$  may be identified with  $P$  as a topological space.

A pair  $(P;T)$ , where  $T$  is an open subgroupoid of  $\Gamma$  with  $E_\Gamma \subset T$ , is called a *manifold scheme* with  $P$  as *page manifold* (or *chart manifold*) and  $T$  as *transition groupoid*; the components of  $P$  are called the *pages* (or *charts*).

Let  $P, P'$  be manifolds and let  $M$  denote the sheaf of germs of local maps  $P \rightarrow P'$ . A subsheaf  $\Lambda \subset M$  is said to be a *morphism* from the manifold scheme  $(P;T)$  to the manifold scheme  $(P';T')$  if

- (i)  $T \wedge T' \subset \Lambda$  <sup>\*\*\*)</sup>,
- (ii) the source map  $\alpha: \Lambda \rightarrow P$  is surjective,
- (iii) for any  $p \in P$ ,  $\alpha^{-1}(p)$  is a  $T'$ -orbit.

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\*) Part of this paper was also read in the "Journées de Géométrie Différentielle" May 16-20, (1978), at Schnepfenried, Haute Alsace.

\*\*) The differentiability class of manifolds and maps will be fixed throughout the first sections.

\*\*\*) To conform with the conventions on groupoids in [2] we consider *here* maps and germs of maps as right operators on spaces and germs of spaces.

$\Lambda$  is called an *immersion* if it is contained in the sheaf of germs of local immersions, it is called an *open immersion* if it belongs to the sheaf of germs of transitions  $P \rightarrow P'$ , it is called an *open embedding* if for any  $q \in P'$  the counter image  $\omega^{-1}(q)$  of the target map  $\omega$  is either empty or a  $T$ -orbit.

Composition of morphisms is the obvious one. The category of manifold schemes and morphisms will be denoted by  $MS$ .  $T$  is the identity morphism  $(P;T) \rightarrow (P;T)$ . Any manifold scheme  $(P;T)$  is  $MS$ -equivalent with a scheme  $(P';T')$  which has simply connected open subsets of  $\mathbb{R}^n$  (for suitable  $n$ ) as pages.

With a scheme  $A = (P;T)$  one associates a topological space  $Top(A)$  by factoring out  $P$  by the equivalence relation:  $p_0 \sim p_1$  if for some  $\tau \in T$ ,  $p_0 = \alpha(\tau)$ ,  $p_1 = \omega(\tau)$ , where  $\alpha$  and  $\omega$  denote the source and target map respectively.

A morphism  $\Lambda: A \rightarrow B$  leads naturally to a morphism  $Top(\Lambda): Top(A) \rightarrow Top(B)$  in the category of topological spaces.

A manifold scheme  $A$  is said to be *connected*, *compact* if  $Top(A)$  is connected, compact.

## 2. COHERENCE IN GROUPOIDS

For any set  $E$  we equip both  $E$  and  $E^2$  with a groupoid structure by defining  $e \cdot e = e$ ,  $e \in E$  in the first case and  $(e_0, e_1)(e_1, e_2) = (e_0, e_2)$  in the second case.

For any groupoid  $G$  with  $E$  as set of identities the map  $(\alpha, \omega): G \rightarrow E^2$  is a groupoid homomorphism, and  $(\alpha, \omega)|_E$  is just the diagonal homomorphism  $\Delta: E \rightarrow E^2$ .

$G$  is said to be *coherent* <sup>\*)</sup> if  $(\alpha, \omega)$  is surjective and to be *simply coherent* if  $(\alpha, \omega)$  is injective.

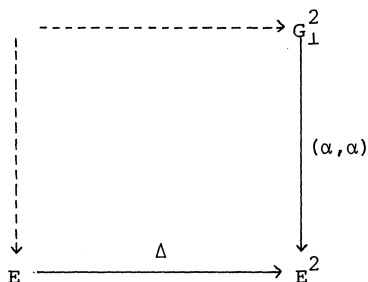
Any groupoid is a disjoint sum of coherent subgroupoids, the *components*.

Any component of a simply coherent groupoid is simply coherent.

For any  $e \in E$ , the group  $eGe = : G_e$  is called the *coherence group* at  $e$ . In a coherent groupoid the coherence groups are isomorphic.

Let  $G$  be a coherent groupoid and  $G_{ens}$  the underlying set. The map  $(\alpha, \omega): G_{ens}^2 \rightarrow E^2$  is obviously a homomorphism of groupoids. Any component of the fibred product  $G_{ens}^2 \times_{E^2} E$  that completes the diagram

\*) We prefer the adjective "coherent" instead of "connected" because we shall also be dealing with topological groupoids where confusion might arise.

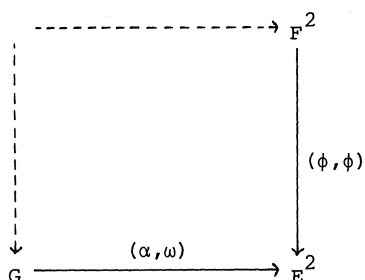


is called a *universal overlay*\*) of  $G$ .

With any  $e \in E \subset G$  there is associated a universal overlay  $\tilde{G}^e$  consisting of the pairs  $(f, g)$  with  $\alpha(f) = \alpha(g) = e$ , with multiplication  $(f, g)(g, h) = (f, h)$ .

The homomorphism  $\pi: \tilde{G}^e \rightarrow G$  with  $(f, g) \mapsto f^{-1}g$  is called the (*overlay*) *projection*.  $\tilde{G}^e$  is coherent and simply coherent.

For any map of sets  $\phi: F \rightarrow E$  the fibred product  $G \times_{E^2} F^2$  that completes the diagram



is called the *lift* of  $G$  under  $\phi$ .

Let  $G$  and  $H$  be coherent groupoids. A homomorphism  $\pi: H \rightarrow G$  is said to be a *weak fibre homomorphism* if (i)  $\pi$  is surjective and (ii)  $\pi^{-1}(e)$  is coherent for any  $e \in E_G$ .  $\pi$  is called a *fibre homomorphism* if (i)  $\pi$  is surjective and (ii)  $\pi$  induces a bijection  $E_H \rightarrow E_G$ .

### 3. TOPOLOGICAL GROUPOIDS; THE COMPONENT GROUPOID

Let  $G$  be a locally connected topological group,  $G_0$  the identity component of  $G$ , and  $\gamma: G \rightarrow G/G_0 =: \Pi(G)$  the quotient map.  $\Pi(G)$  is discrete,

\*) The term "overlay" is borrowed from R.H. FOX [5].

it is the group of the components of  $G$ ,  $\gamma$  is continuous, and furthermore the diagram  $G \xrightarrow{\gamma} \Pi(G)$  behaves functorially in the sense that there is a commutative diagram

$$\begin{array}{ccc}
 G & \xrightarrow{\gamma} & \Pi(G) \\
 \downarrow \phi & & \downarrow \Pi(\phi) \\
 H & \xrightarrow{\gamma} & \Pi(H)
 \end{array}
 \quad (*)$$

for any continuous homomorphism  $\phi$  in the category of l.c. topological groups.

Similarly one defines a functor  $\Pi$  from the category of l.c. topological groupoids to the subcategory of discrete groupoids and a continuous homomorphism  $\gamma: G \rightarrow \Pi(G)$  as follows:

Let for any  $g \in G$ ,  $[g]$  denote the connectedness component of  $g$ , and let  $\alpha[g] = [\alpha g]$ ,  $\omega[g] = [\omega g]$ . Then  $[G]$  together with the maps  $\alpha: [G] \rightarrow \{[e]; e \in E\}$  generates a free groupoid  $\Phi([G]; \alpha, \omega)$  with  $\{[e]; e \in E\}$  as set of identities. For any  $g \in G$  we continue to denote the canonical image of  $[g]$  in  $\Phi([G]; \alpha, \omega)$  again by  $[g]$ .

Define  $\Pi(G)$  to be the quotient of  $\Phi([G]; \alpha, \omega)$  by the relation  $[g_1][g_2] = [g_1g_2]$ , let  $\rho: \Phi([G]; \alpha, \omega) \rightarrow \Pi(G)$  be the quotient map, and put  $\gamma(g) = \rho([g])$ . Then  $G \xrightarrow{\gamma} \Pi(G)$  has the functorial property (\*).

A locally connected topological groupoid  $G$  is said to be *coherent*/*simply coherent* if  $\Pi(G)$  is coherent/*simply coherent*. The coherence group of  $\Pi(G)$  at  $\gamma(e)$  is said to be the *coherence group of  $G$  at the component  $[e]$* . A l.c. topological groupoid is the disjoint union of open coherent subgroupoids, the *coherence components*.

A map  $\sigma: F \rightarrow E$  of locally connected spaces is said to be a *spread*<sup>\*</sup> if (i)  $\sigma$  is surjective, (ii)  $\sigma$  is an open embedding on each of the components of  $F$ .

A locally connected space  $X$  is said to be *simply connected* if any overlay<sup>\*</sup>  $\tilde{X} \rightarrow X$  is an equivalence.

\*) "Overlay" and "spread" are introduced here in order to avoid the ambiguous word "covering".

Overlay means "covering space" (+ "covering map"). The terminology is partly borrowed from R.H. FOX [4,5].



For any l.c. topological groupoid  $G$  with  $E = E_G$  and any spread  $\sigma: F \rightarrow E$ , the lift  $G \times_{E^2} F^2$  of  $G$  by  $\sigma$  has a natural locally connected topology such that the natural projection  $p: G \times_{E^2} F^2 \rightarrow G$  is again a spread.

PROPOSITION 3.1. *For any spread  $\sigma: F \rightarrow E$  the morphism  $\Pi(p): \Pi(G \times_{E^2} F^2) \rightarrow \Pi(G)$  is a weak fibre morphism. If  $E$  is simply connected then  $\Pi(p)$  maps the coherence groups of  $\Pi(G \times_{E^2} F^2)$  isomorphically onto coherence groups of  $\Pi(G)$ .*

REMARK. The proposition results from more general propositions on the properties of  $\Pi$ .

For any coherent and locally connected groupoid  $G$  let  $\tilde{G}$  be the fibred product of a universal overlay  $\tilde{\Pi}(G) \xrightarrow{\pi} \Pi(G)$  and  $G \xrightarrow{\gamma} \Pi(G)$ . Then denoting the projection  $\tilde{G} \rightarrow G$  by  $p$ , we have

PROPOSITION 3.2. *There is an isomorphism  $\iota: \tilde{\Pi}(G) \rightarrow \Pi(\tilde{G})$  such that  $\Pi(p) \circ \iota = \pi$ , and  $\gamma: \tilde{G} \rightarrow \tilde{\Pi}(G) \xrightarrow{\iota} \Pi(\tilde{G})$ .*

$\tilde{G}$  is called the universal overlay of  $G$ , and is according to Proposition 3.2 simply coherent since  $\tilde{\Pi}(G)$  is simply coherent.

#### 4. FUNDAMENTAL GROUPS OF MANIFOLD SCHEMES

Let  $A = (P;T)$  be a manifold scheme.  $T$  is a locally connected topological groupoid.

PROPOSITION 4.1.  *$T$  is coherent iff  $A$  is connected. In general there is a 1-1 correspondence between the components of  $A$  and the coherence components of  $T$ .*

Proposition 3.1 leads to

PROPOSITION 4.2. *Let  $A$  be connected and  $P$  be simply connected. Then the coherence group of  $T$  is up to isomorphism an invariant of the equivalence class of  $A$  i.e. for any equivalent scheme  $(P',T')$  with  $P'$  simply connected,  $T$  and  $T'$  have isomorphic coherence groups.*

The coherence group of the transition groupoid of a connected manifold scheme with simply connected page manifold is therefore called the *fundamental group* of the manifold scheme; the fundamental group is up to isomorphism an equivalence invariant. As usual we denote the fundamental group by  $\pi_1$ .

Retaining the hypotheses and notations of Proposition 4.2, let  $\tilde{T}$  denote a universal overlay of  $T$  and  $\pi: \tilde{T} \rightarrow T$  be the overlay projection.  $\pi$  is such that it maps the components of  $E_{\tilde{T}} =: \tilde{P}$  homeomorphically onto the components of  $E_T$ . The canonical identification  $E_T \simeq P$  induces on  $E_{\tilde{T}}$  and hence on  $E_{\tilde{T}}$  a differential structure, which is such that  $\tilde{T}$  identifies in a natural fashion with an open subgroupoid of  $\Gamma_{\tilde{P}}$ .

$\tilde{A} = (\tilde{P}; \tilde{T})$  is called a *universal overlay of  $A = (P; T)$* .

PROPOSITION 4.3.  $\pi_1(\tilde{A}) = (1)$ . *The fundamental group of  $\tilde{A}$  acts in a natural fashion as an automorphism group of  $\tilde{A}$ .*

As explained in [2], for any manifold  $M$  and foliation  $F$  on  $M$  the quotient scheme  $M/F$  is well defined, and there is a natural MS-morphism  $p: M \rightarrow M/F$ . Supposing  $M$  and the leaves of  $F$  to be connected, one obtains

PROPOSITION 4.4.  $\pi_1(M/F)$  is a quotient of  $\pi_1(M)$ .

REMARK.

- 1) In order to specify the quotient map  $\pi_1(M) \rightarrow \pi_1(M/F)$ , one ought to choose a suitable open covering of  $M$  and relate the transition groupoid of this covering of  $M$  with the transition groupoid of  $M/F$ .
- 2) The quotient map  $\pi_1(M) \rightarrow \pi_1(M/F)$  is part of a short homotopy sequence.
- 3) A special case of manifold schemes are the  $Q$ -manifolds studied by R. BARRE [1], who also defined the notion of fundamental group of a  $Q$ -manifold and the notion of universal overlay (revêtement universel). It seems likely that the Barre construction and the manifold scheme construction can be proved to yield the same results in the case of a  $Q$ -manifold.

## 5. EXAMPLES

Let  $G$  be a set of transitions  $P \rightarrow P$  such that  $\text{id}_P \in G$ . By  $T_G$  we denote the groupoid generated by the germs of the  $g \in G$ .  $T_G$  is automatically open in  $\Gamma_P$ , and  $E_T \subset T_G$ . By abuse of notation the manifold scheme  $(P; T_G)$  will also be denoted by  $(P; G)$ .

A transition  $g$  is said to be *locally stable* if for some  $x \in P$ ,  $g_x = (\text{id}_P)_x$  (equality of germs).

A group  $G$  of diffeomorphisms is said to *act generically free* if  $\text{id}_P$  is the only locally stable element in  $G$ .

The examples below are given as illustrations of the general considerations in the preceding sections, they are manifold schemes with  $\mathbb{R}$  as page manifold. In these examples the fundamental group can be determined right away, or in case the set of transitions indicated generates a group of diffeomorphisms  $\mathbb{R} \rightarrow \mathbb{R}$  it may be determined by applying:

PROPOSITION 5.1. *Let  $G$  be a group of diffeomorphisms of a connected and simply connected manifold  $P$ , and let  $G_0$  denote the normal subgroup generated by the locally stable elements. Then  $\pi_1((P;G)) \simeq G/G_0$ . In particular if  $G$  acts generically free, then  $\pi_1((P;G)) \simeq G$ .*

In addition the universal overlay is indicated and furthermore a foliation  $F$  in a suitable manifold  $M$  such that  $M/F$  is the given scheme; the symbol  $\sqcup$  denotes the disjoint union of a collection of copies of  $\mathbb{R}$ , cf. the table of examples.

We proceed to make some comments on the examples in the table.

One should note that both  $\text{Top}(L_0)$  and  $\text{Top}(H_{-1})$  are homeomorphic to the real half line  $\{\xi; \xi \in \mathbb{R}, \xi \geq 0\}$ , however the difference in fundamental group indicates that  $L$  and  $H_{-1}$  are inequivalent schemes. Similarly  $\text{Top}(H_\alpha) \simeq \text{Top}(H_\alpha^*) \simeq \text{Top}(R_{\phi_+, \phi_-})$  whereas the associated schemes are inequivalent.

$\text{Top}(P_\alpha)$  is a "black hole" i.e. a topological space with the empty subspace and total space as the only open subspaces. It is known that  $P_{\alpha_1}$  and  $P_{\alpha_2}$  are equivalent iff  $\alpha_1$  and  $\alpha_2$  are related by a unimodular fractional linear transformation with integral coefficients. Therefore there are pairs of inequivalent  $P_\alpha$ 's although the underlying topological spaces  $\text{Top}(P_\alpha)$  are homeomorphic and their fundamental groups are isomorphic.

$P_{\mathbb{R}}$  is not a quotient scheme of a finite dimensional manifold by a foliation. (Since  $\text{Top}(P_{\mathbb{R}})$  is a single point, the foliation would consist of a single leaf; therefore the quotient scheme would then just be a 0-manifold consisting of a single point.)

The  $L_\beta$ 's admit the *lasso's* considered in [7] as a double overlay; we propose to call the  $L_\beta$ 's *non-orientable lasso's*; the  $H_{-1}$ , and  $H_\alpha$ 's might be called *elementary hyperbolic schemes*; the  $P_\alpha$ 's are the *Poincaré tori* of dimension 1 and rank 2 considered in [2]; the schemes  $R_{\phi_+, \phi_-}$  might be called *Reeb-schemes*.

EXAMPLES

A	$\pi_1$	$\tilde{A}$	M	F
$L_\beta (\beta \geq 0)$ : $(\mathbb{R}; \lambda)$ , $\lambda(\xi) = -\xi$ for $\xi > \beta$ .	$\mathbb{Z}$	$(\frac{\mathbb{Z}}{n \in \mathbb{Z}} \mathbb{R}_n; \tilde{\lambda})$ , $\xi_{n+1} = \tilde{\lambda}(\xi_n) = -\xi_n$ , $\xi_n > \beta$ , $\xi_n \in \mathbb{R}_n$ .	$(\mathbb{R}^2 - E_\beta) / (1, -1)$ $E_\beta = \{(x, y)   x^2 + y^2 \leq \beta^2\}$ $(1, -1)$ = group of scalar multiplications by 1 and -1.	foliation by the lines $y = \text{constant}$
$H_{-1} : (\mathbb{R}, \sigma)$ , $\sigma(\xi) = -\xi$ , $\xi \in \mathbb{R}$ .	$\mathbb{Z}_2$	$\mathbb{R}$	$\mathbb{R}^2 / S$ $S$ = group generated by $(x, y) \mapsto (x+1, -y)$	foliation by the lines $y = \text{constant}$
$H_\alpha (0 < \alpha < 1)$ : $(\mathbb{R}; \nu_\alpha)$ , $\nu_\alpha(\xi) = \alpha\xi$ , $\xi \in \mathbb{R}$ .	$\mathbb{Z}$	$\mathbb{R}$	$\mathbb{R}^2 - (0, 0)$	foliation by solutions of the diff. eqn. $(\log r)' = (\log \alpha) \log r$ $\varphi' = 2\pi$ $(r, \varphi)$ polar coordinates
$H_\alpha^* (0 < \alpha < 1)$ : $(\mathbb{R}; \nu_\alpha^*)$ , $\nu_\alpha^*(\xi) = \alpha\xi$ , $\xi \neq 0$ .	$\mathbb{Z} * \mathbb{Z} \cong F(a, b)$	$(\frac{\mathbb{Z}}{w \in F(a, b)} \mathbb{R}_w; \lambda_a, \lambda_b)$ $\xi_{wa} = \lambda_a(\xi_w) = \alpha\xi_w$ , $\xi_w > 0$ , $\xi_{wb} = \lambda_b(\xi_w) = \alpha\xi_w$ , $\xi_w < 0$ .	$\mathbb{R}^2 - \{(0, 0), (1, 0)\}$	same foliation as before
$P_\alpha (\alpha \text{ irrational})$ : $(\mathbb{R}; \tau_\alpha)$ , $\tau_\alpha(\xi) = \xi + 1$ , $\xi \in \mathbb{R}$ , $\tau_\alpha(\xi) = \xi + \alpha$ .	$\mathbb{Z} \oplus \mathbb{Z}$	$\mathbb{R}$	2 - torus	foliation by lines of slope $\alpha$ .
$R_{\varphi_+, \varphi_-}$ : $(\mathbb{R}; \varphi_+, \varphi_-)$ , $\xi < \varphi_+(\xi) < 0$ if $\xi < 0$ $\varphi_+(\xi) = \xi$ if $\xi \geq 0$ $\varphi_-(\xi) = \xi$ if $\xi \leq 0$ $0 < \varphi_-(\xi) < \xi$ if $\xi > 0$	(1)	$R_{\varphi_+, \varphi_-}$	$S^3$	Reeb-foliation, [10].
$P_{\mathbb{R}} : (\mathbb{R}; \tau_\beta)$ , $\beta \in \mathbb{R}$ $\tau_\beta(\xi) = \xi + \beta$	$\mathbb{R}$ as abstract group	$\mathbb{R}$		

6. LOCALLY HOMOGENEOUS SPACES

Let  $M$  be a connected real analytic manifold, and let  $X_M$  denote the sheaf of germs of local analytic vectorfields on  $M$ . Let  $\ell \subset X_M$  be a *transitive Lie algebra (of vectorfields) on  $M$* , i.e.  $\ell$  has the following properties for every  $m \in M$ : (i)  $\ell_m$  is a finite dimensional Lie subalgebra of  $X_m$ , (ii)  $\ell(m) = \{X(m); X \in \ell_m\}$  is the tangent space at  $m$ , (iii) there is a neighbourhood  $U$  of  $m$  such that the sections in  $\ell$  over  $U$  span  $\ell_y$  for every  $y \in U$ . The local diffeomorphisms obtained by integrating locally the local vectorfields that are sections in  $\ell$ , generate an open subgroupoid  $T_{\ell, M} \subset \Gamma_M$  with  $E_{\Gamma_M} \subset T_{\ell, M}$ .

The pair  $(M, T_{\ell, M})$  is called a *locally homogeneous space*. On the other hand  $(M; T_{\ell, M})$  is an object from MS. The difference in the two notions is of course due to the difference in the associated morphism concept.

Let  $U$  be a domain in  $M$ . By restricting  $\ell$  to  $U$  one gets an associated manifold scheme  $(U; T_{\ell, U})$ .

PROPOSITION 6.1. *If  $M$  is simply connected the "inclusion"  $T_{\ell, U} \cdot T_{\ell, M} : (U; T_{\ell, U}) \rightarrow (M; T_{\ell, M})$  is an equivalence of manifold schemes.*

Let  $m_0 \in M$  be chosen and let  $\mathfrak{f}_0$  be the isotropy subalgebra of  $\ell_{m_0} =: \ell_0$ . In the simply connected Lie group  $L$  generated by  $\ell_0$ ,  $\mathfrak{f}_0$  generates a Lie subgroup  $F$ . The collection of left cosets of  $F$  is a foliation  $F$  in  $L$ . If  $F$  is closed,  $F$  is a fibration of  $L$ , and hence  $L/F$  is just the coset manifold  $L/F$ . Therefore we write  $L/F$  instead of  $L/F$  in the general case also. In the case that  $F$  is closed in  $L$ ,  $L/F$  is up to equivariant equivalence the unique simply connected manifold which admits  $L$  as transitive Lie transformation group with  $F$  as isotropy subgroup for some point. Similarly one has in the general case

PROPOSITION 6.2. *Up to equivariant isomorphism  $L/F$  is the unique simply connected manifold scheme which admits  $L$  as transitive Lie automorphism group with  $F$  as isotropy subgroup for some point.*

By choosing a suitable neighbourhood  $U$  of  $m_0$  one finds by classical Lie theory that  $\pi_1((U; T_{\ell, U})) \simeq L$ . Let  $\tilde{A}$  be the universal overlay of  $A = (U; T_{\ell, U})$ . Then  $\tilde{A}$  admits  $L$  as a transitive Lie automorphism group with  $F$  as isotropy subgroup for some point. Combining this with Proposition 6.2 and 6.1 and with the universal overlay construction properties one obtains

THEOREM. Let  $M$  be a connected and simply connected analytic manifold, and  $\ell$  a transitive Lie algebra of vector fields on  $M$ ,  $\ell_0$  the isotropy subalgebra of some  $m_0 \in M$ . Then the following statements hold:

- (i)  $\pi_1(M; T_{\ell, M}) \simeq L$  where  $L$  is the simply connected Lie group generated by  $\ell_0$ .
- (ii)  $L$  operates transitively on the universal overlay  $\tilde{A}$  of the manifold scheme  $(M; T_{\ell, M})$  as a Lie automorphism group, with  $F$  as isotropy subgroup of some point. Therefore  $\tilde{A}$  is equivariantly equivalent with  $L/F$ .
- (iii) There is an open immersion  $\lambda: M \rightarrow L/F$  such that the Lie algebra  $\ell$  on  $M$  is the  $\lambda$ -counter image of the Lie algebra on  $L/F$  of the left translations by  $L$ .

COROLLARY. If in addition  $M$  is compact, then  $M \simeq L/F$ .

PROOF. Since  $\lambda: M \rightarrow L/F$  is an open immersion  $\lambda(M)$  is open and closed (because of the compactness) and therefore  $\lambda(M) = L/F$ . Again by the compactness of  $M$ ,  $\lambda: M \rightarrow L/F$  is an overlay of  $L/F$ , and since  $L/F$  is simply connected,  $M \simeq L/F$ .

REMARKS.

- (1) The corollary has been obtained by EHRESMANN [3]. Although the theorem is not stated by him, his procedure of "developing along a curve" is in fact a construction of  $\lambda$ , which is obtained here directly from the construction of the universal overlay. His considerations bear on the case that  $L/F$  is a Hausdorff manifold.
- (2) The construction of  $\lambda$  involves two steps. First a "natural" immersion  $\lambda_0: M \rightarrow \tilde{A}$  which is determined up to a decktransformation i.e. a translation by an element  $g \in L$ , and secondly an isomorphism  $\lambda_1: \tilde{A} \rightarrow L/F$  as homogeneous manifold schemes. The latter isomorphism is determined up to an automorphism of  $L/F$  (or  $\tilde{A}$ ) as homogeneous manifold scheme.
- (3) PALAIS [9] essentially obtained  $\lambda$  via the integration of a suitable involutive differential system.
- (4) The quotient scheme  $L/F$  is also a  $Q$ -manifold in the sense of R. BARRE [1].
- (5) The problem to give sufficient conditions for a locally homogeneous space to be enlargeable to a homogeneous space was also considered by G.D. MOSTOW in "The extensibility of local Lie groups of transformations and groups on surfaces", Ann. Math. 52, 606-636, (1950).

As an illustration to the theorem one might take  $\mathbb{R}^2$  as a homogeneous space under the action of some Lie transformation group  $L$ , e.g. the group of translations, and  $M$  to be the universal overlay of a non-simply connected domain  $D$  in  $\mathbb{R}^2$ . The transitive Lie algebra  $\mathfrak{l}$  is of course taken to be the lifting under the overlay projection  $p: M \rightarrow D$  of the Lie algebra of the infinitesimal transformations of  $L$ . The pair  $M, \mathfrak{l}$  satisfies then the hypotheses of the theorem. In this case  $\tilde{A}$  is just  $\mathbb{R}^2$ , and  $\lambda$  may be taken to be  $p$  followed by the inclusion map  $D \subset \mathbb{R}^2$ .

#### 6. ANALYTIC MANIFOLD SCHEMES OF DIMENSION 1

The simple connectedness of a Reeb-scheme (cf. §5) is a consequence of the fact that both  $\phi_+$  and  $\phi_-$  are locally stable. Obviously a Reeb-scheme cannot be realized within the analytic category. Thinking of Haefliger's theorem and the available examples one is led to conjecture:

PROPOSITION GH. *A compact connected 1-dimensional analytic manifold scheme has a non-trivial fundamental group.*

Exploiting the geometry of tree-manifolds (cf. [2]) (in particular the Helly property) and simple properties of analytic transitions one may reformulate the conjecture, and a somewhat more general one not involving compactness, as an enlargeability problem for local groups. This suggests to apply a criterion for enlargeability due to MALCEV [8]. There seems to be a chance that, by judiciously taking into account the geometry of tree-manifolds and of course the analyticity of the transitions, the criterion may be verified to hold.

Proposition GH entails.

THEOREM GH. *A compact connected 1-dimensional analytic manifold scheme has an infinite fundamental group.*

PROOF. Suppose the fundamental group of some such scheme  $A$  to be finite. Then the universal overlay  $\tilde{A}$  would lie finitely sheeted over  $A$ , and hence would still be compact (and of course connected analytic 1-dimensional). Therefore  $\tilde{A}$  would still have a non-trivial fundamental group which would contradict the simple connectedness of  $\tilde{A}$ .

As a consequence one would obtain by Proposition 4.4 Haefliger's theorem: *A compact connected analytic manifold which admits a codimension 1 analytic foliation has an infinite fundamental group.*

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Addendum: A proof of Proposition GH along the lines indicated can actually be obtained. The more general proposition states that a analytic manifold scheme of dimension 1 is simply connected iff it is a tree-manifold.



## UNDERLYING BOOLEAN ALGEBRAS OF TOPOLOGICAL SEMIFIELDS

J. Flachsmeyer

### INTRODUCTION

Soviet mathematicians have a good tradition in the investigation of ordered algebraic structures. Fundamental contributions in the theory of vector lattices have been made by L.V. Kantorovich and his school. Inspired partly by this work, M.YA. Antonowskii, V.G. Boltyanskii and T.A. Sarymsakov have developed in the last 20 years a new topological-algebraical object, which they called *topological semifield*. A theory of such topological semifields was subsequently built up mainly in the Soviet Union. Applications to probability theory and ergodic theory and to metrizations of normed spaces over topological semifields were made. But a satisfactory representation theory for topological semifields and for their underlying Boolean algebras is lacking. Some years ago the present author conjectured that the corresponding Boolean algebras are exactly the hyperstonian algebras. Now we show that our conjecture is equivalent to the Maharam Problem about submeasures. The key for this reduction is the approach by monotone Boolean pseudo-norms (Section 4) which has close analogies to the prenorms in the locally convex vector spaces.

Our paper is a continuation of our former work on Topologization of Boolean algebras [5]. At the same time it must be considered as a part of the announced work which was cited in [5] under [14]. Some missing details delayed its publication.

### 1. THE NOTION OF A TOPOLOGICAL SEMIFIELD

Roughly speaking a topological semifield is a topological lattice-ordered ring with sufficiently many invertible positive elements. The original axioms of M.YA. Antonowskii, V.G. Boltyanskii and T.A. Sarymsakov are

not based on the real numbers, but every semifield contains the reals as its so-called axis. It is more convenient to replace the original axioms, by the following equivalent system of axioms:

A *topological semifield* is a topological algebraical order-theoretical object  $S = (S, +, \cdot, \leq, \theta)$  satisfying the following conditions:

- (i)  $(S, +, \cdot)$  is a commutative real algebra with a unit element 1.
- (ii)  $(S, +, \cdot, \leq)$  is a Dedekind complete ordered algebra, (so in particular, it is a Dedekind complete vector lattice).
- (iii)  $(S, +, \cdot, \theta)$  is a topological algebra with a Hausdorff topology  $\theta$ .
- (iv) The positive cone  $\{x \in S: x \geq 0\}$  of  $S$  is the closure of the set  $K$  of all positive invertible elements. Moreover  $(K, \cdot)$  is a group and it satisfies:  $K + \text{cl}(K) \subset K$ .
- (v) In the set  $I(S)$  of all idempotents of  $S$  the following relation holds for nets:  $a_\alpha \downarrow 0 \Rightarrow a_\alpha \xrightarrow{\theta} 0$ .
- (vi) The positive cone of  $S$  (which equals  $\text{cl}(K)$ , by 4) is normal, i.e. the origin has a neighbourhood base consisting of solid sets  $U$ . (A set  $U$  is *solid* whenever for every  $y \in S$  one has  $y \in U$  iff there is an  $x \in U$  such that  $|y| \leq |x|$ .)
- (vii) For every neighbourhood  $W$  of the origin  $S$  there exists a neighbourhood  $V$  of the origin such that  $(I \cap V) S \subset W$ . This means that every neighbourhood  $W$  of zero absorbs  $S$  in some sense uniformly with respect to small idempotents.

REMARK. My student A. FRÖHLICH [8] has given a careful proof of the equivalence of the both systems of axioms. We drop this technical argumentation because of its length. He also discovered that an earlier version of my list of axioms contained a little gap.

Here we mention only a few examples of topological semifields.

(1) The products of reals in their canonical structure, i.e.  $\mathbb{R}^m$ ,  $m$  any cardinal, with the coordinate addition, multiplication, ordering and the Tichonov product topology. These semifields are called Tichonov semifields [2].

(2) The lattice-algebra  $\mathcal{C}(X)$  of all continuous real-valued functions on a hypertonian space  $X$  with respect to the topology of convergence in all hyperdiffuse measures on  $X$  (vid. Section 3 and [5]).

This is a subsemifield of the semifield  $\mathcal{C}^\infty(X)$  of all numerical continuous functions which take the value  $-\infty$ ,  $+\infty$  only on nowhere dense sets.

In this language, Example 1 is the special case of  $C^\infty(\beta M)$ , where  $\beta M$  is the Stone-Čech-compactification of a discrete space  $M$  of cardinality  $m$ .

2. THE NOTION OF ABS-TOPOLOGICAL BOOLEAN ALGEBRAS

The set  $I(S)$  of all idempotents of a topological semifield  $S$ , i.e.  $I(S) = \{x; x \in S, x^2 = x\}$ , forms a Boolean ring with respect to the following operations.

$$x \oplus y = x + y - 2xy$$

$$x \odot y = x \cdot y \quad (\text{vid. [5]})$$

This ring (or the corresponding canonical lattice) will be called the *underlying Boolean algebra* of the given semifield  $S$ . On  $I$  we shall consider the trace topology  $\mathcal{O}_I$  of  $\mathcal{O}$ .

We have:

- (i)  $(I, \oplus, \odot, \mathcal{O}_I)$  is an order complete Boolean ring with a compactible locally solid Hausdorff topology;
- (ii) The order convergence implies the convergence with respect to the topology  $\mathcal{O}$ .

A topological abstract Boolean algebra  $\underline{B}$  with properties (i) and (ii) we will call *ABS-Boolean algebra*. The three authors Antonowskii-Boltyanskii-Sarymsakov have shown that every ABS-Boolean algebra  $(\underline{B}, \mathcal{O})$  can be the underlying Boolean algebra of a (not uniquely determined) suitable topological semifield.

The question arises which Boolean algebras  $\underline{B}$  can be equipped with a topology  $\mathcal{O}$  such that  $(\underline{B}, \mathcal{O})$  is an ABS-Boolean algebra. The aim of our paper is to contribute to this problem.

The following statement can be immediately seen. Recall, that the  $\mathcal{O}$ -topology is the greatest topology  $\mathcal{P}$  for which  $\mathcal{O}$ -convergence of nets implies the convergence in  $\mathcal{P}$ ;  $\mathcal{O}$ -convergence of nets is defined as follows:  $a_\alpha \rightarrow a$  iff there are nets  $b_\beta \uparrow a$ ,  $c_\gamma \downarrow a$  such that  $b_\beta \leq a_\alpha \leq c_\gamma$  for all  $\alpha \geq \alpha(\beta, \gamma)$ .

**LEMMA 1.** *In every ABS-Boolean algebra  $(\underline{B}, \mathcal{O})$  the following inclusions hold: interval topology of  $\underline{B} \subset \mathcal{O} \subset \mathcal{O}$ -topology of  $\underline{B}$ .*

**PROOF.** Observe that  $\mathcal{O}$  is a Hausdorff topology. The multiplication in  $\underline{B}$  is continuous with respect to  $\mathcal{O}$ . Hence every order interval  $[a, b]$  is closed

with respect to  $\mathcal{O}$ , i.e.  $\mathcal{O}$  is interval topology. The other inclusion is clear from the definitions.  $\square$

COROLLARY 1. Let  $\underline{B}$  be a separable Boolean algebra.  $\underline{B}$  can be made into an ABS-Boolean algebra iff  $\underline{B}$  is atomic.

PROOF. Generalizing a result of FLOYD [6] we have shown in [4] that the order-topology of a separable Boolean algebra is a Hausdorff topology iff  $\underline{B}$  is atomic. On the other hand there is only one atomic complete separable Boolean algebra  $\underline{B} \simeq 2^{\aleph_0}$ , where  $\underline{2}$  is the Boolean algebra of two elements ( $\underline{2}$  is the ring  $(\mathbb{Z}/(2))$ ). But in  $2^{\aleph_0}$  Tichonov's product topology is an ABS-topology.  $\square$

COROLLARY 2. The Dedekind-MacNeille completion of a free generated Boolean algebra with infinite many free generators cannot be made into an ABS-algebra.

PROOF. We have shown in [4] that those Boolean algebras have no Hausdorff  $\mathcal{O}$ -topology.  $\square$

COROLLARY 3. (Theorem of Antonowskii-Boltyanskii-Sarymsakov). Let  $(\underline{B}, \mathcal{O})$  be an ABS-Boolean algebra. Then  $\mathcal{O}$  is the interval topology of  $\underline{B}$  iff  $\underline{B}$  is atomic.

In this case the interval topology of  $\underline{B}$ , the order topology of  $\underline{B}$  and the product topology of  $\underline{B}$  (namely  $\underline{B} \simeq 2^m$ ) coincide.

PROOF. KATĚTOV [9] has shown that the interval topology in a Boolean algebra  $\underline{B}$  is Hausdorff iff  $\underline{B}$  is atomic. (Another proof is given in our paper [4].) But for any atomic Boolean algebra all three mentioned topologies coincide.  $\square$

### 3. THE MAHARAM PROBLEM. HYPERTONIAN ALGEBRAS

Let us define a Maharam submeasure-algebra  $\underline{B}$  as follows:

- (i)  $\underline{B}$  is a complete Boolean algebra;
- (ii) There exists a strictly positive  $\mathcal{O}$ -continuous submeasure on  $\underline{B}$ , say

$\nu: \underline{B} \rightarrow \mathbb{R}$ , with the properties

- a)  $\nu(b) = 0 \iff b = 0$
- b)  $\nu(a) \leq \nu(b)$  for  $a \leq b$
- c)  $\nu(a \vee b) \leq \nu(a) + \nu(b)$
- d)  $b_\alpha \xrightarrow{\mathcal{O}} b \Rightarrow \nu(b_\alpha) \rightarrow \nu(b)$ .

REMARKS.

(1) A Maharam submeasure-algebra  $\underline{B}$  satisfies c.c.c. (countable chain condition), i.e. every set  $D$  of non-zero disjoint elements in  $\underline{B}$  is countable. The argumentation is given by the following well-known routine argument. There can be only a finite number of  $d_i \in D$  with  $v(d_i) \geq \epsilon > 0$ . Indeed, if  $v(d_i) \geq \epsilon$  for  $i = 1, \dots, n, \dots$ , then for  $a_n := \sup_{i \geq n} d_i$  one would have  $v(a_n) \geq \epsilon$  and otherwise  $a_n \downarrow 0$ , hence  $v(a_n) \rightarrow 0$ .

(2) For every Maharam submeasure algebra  $\underline{B}$  instead of  $d$ ) it is enough to require  $b_n \downarrow 0 \Rightarrow v(b_n) \rightarrow v(b)$  for ordinary sequences.

According to a well-known theorem of D. MAHARAM [10] a complete Boolean algebra  $\underline{B}$  is a Maharam submeasure algebra iff the order topology  $\mathcal{O}$  on  $\underline{B}$  is metrizable. Of course, every measure algebra  $(\underline{B}, \mu)$ , i.e.  $c$ ) is replaced by additivity, is a Maharam submeasure algebra. The famous unsolved *Maharam Problem* asked for the converse: Is every Maharam submeasure algebra  $\underline{B}$  a measure algebra?

Now we recall the notion of a hyperstonian space in the sense of DIXMIER [3]. Let be  $X$  a compact Hausdorff space and  $\mu$  positive Radon measure on  $X$ ,  $\mu \in M^+(X)$ . Then  $\mu$  is called *hyperdiffuse* (or normal) iff every nowhere dense Borel set has  $\mu$ -measure zero. A signed measure  $\mu \in M(X)$  is hyperdiffuse iff  $\mu^+$  and  $\mu^-$  are hyperdiffuse. Let be  $H(X) \subset M(X)$  the set of all hyperdiffuse measures on  $X$ . The space  $X$  is *hyperstonian* iff it has the following properties.

- (i)  $X$  is Stonian (= extremally disconnected compact Hausdorff space)
- (ii) The union  $U\{\text{supp } \mu : \mu \in H(X)\}$  is dense in  $X$ .

From a functional analytic point of view equivalent formulations of this notion are as follows:  $\mu \in M^+(X)$  is hyperdiffuse iff the corresponding linear functional  $\phi(\cdot) = \int \cdot d\mu : C(X) \rightarrow \mathbb{R}$  is order continuous, i.e.  $C(X) \ni f_\alpha \downarrow 0 \Rightarrow \phi(f_\alpha) \rightarrow 0$ , cf. [13]. The space  $X \in \text{Comp}$  is hyperstonian iff

- (i)  $C(X)$  is a Dedekind complete vector lattice and
- (ii)  $H(X)$  (as the set of functionals) is total over  $C(X)$ .

In the case of a hyperstonian space  $X$  the space  $C(X)$  is the Banach dual of  $H(X)$  is this natural pairing.

We shall call a Boolean algebra  $\underline{B}$  a *hyperstonian algebra* iff the Stone representation space  $X = \text{spec}(\underline{B})$  is hyperstonian. Without any look to the Stone representation space  $\text{spec}(\underline{B})$  one can say that  $\underline{B}$  is a hyperstonian Boolean algebra iff

- (i)  $\underline{B}$  is Dedekind complete and  
(ii) For every  $0 < b \in \underline{B}$  there is a nontrivial order continuous measure  $\mu: \underline{B} \rightarrow \mathbb{R}^+$  with support dominated by  $b$ :  $\text{supp } \mu \leq b$ .

It must be observed that in the book of D.F. FREMLIN [7] the notion of a Maharam algebra  $(\underline{B}, \nu)$  is defined as follows:

- (i)  $\underline{B}$  is a Dedekind complete Boolean algebra  
(ii) On  $\underline{B}$  there exists a numerical function  $\nu: \underline{B} \rightarrow [0, \infty]$  which is strictly positive, finite additive and monotonely continuous, i.e.  
 $a_{n+1} \leq a \Rightarrow \nu(a) = \sup \nu(a_n)$ , and which is semifinite, i.e. for every  $a \in \underline{B}$  with  $\nu(a) = \infty$  there is an  $b$  with  $0 < b < a$  and  $\nu(b) < \infty$ .

Now we can easily see: *The two notions of a hyperstonian Boolean algebra and a Maharam-algebra in the sense of Fremlin coincide.* Namely, take in a Maharam algebra  $\underline{B}$  a maximal disjoint set  $M \subset \underline{B}$  for which every  $b \in M$  has finite measure  $0 < \nu(b) < \infty$ .

(By Zorn's lemma such a set exists.) Then for every  $b' \leq b$ , for a suitable  $b \in M$ , the equation  $\nu_{b'}(a) := \nu(a \wedge b')$ ,  $a \in \underline{B}$ , defines an order continuous measure on  $\underline{B}$  with  $\text{supp } \nu_{b'} \leq b'$ . Because of the maximality of  $M$  every  $c \in \underline{B}$ ,  $c > 0$ , dominates a  $\text{supp } \nu_{b'}$  of a suitable  $b' \leq b \in M$ . Thus  $\underline{B}$  is hyperstonian. Conversely, let  $\underline{B}$  be hyperstonian. We take any maximal disjoint set  $M$  with the property that every  $b \in M$  is the support of a non-trivial 0-continuous measure  $\nu_b$ . Then let by definition  $\nu(a) = \sum \{\nu_b(a) : (b \in M)\}$ . This makes  $\underline{B}$  into a Maharam algebra.

Let  $\underline{B}$  be a hyperstonian algebra, or, what amounts to the same, a Maharam algebra. If  $\underline{B}_b$  denotes the ideal  $[0, b]$ , then one has

$$\underline{B} \simeq \prod_{b \in M} \underline{B}_b.$$

Here  $M$  is, as before, any maximal disjoint system. Observe that  $b = \text{supp } \nu_b$  for  $b \in M$ , so we have a decomposition of  $\underline{B}$  into supports of 0-continuous measures. Since  $(\underline{B}_b, \nu_b)$  is a measure algebra, we have the following

**PROPOSITION.** *The hyperstonian Boolean algebras are the same as the products of measure algebras.*

#### 4. BOOLEAN PSEUDO-NORMS AND THEIR RELATIONS TO ABS-ALGEBRAS

**DEFINITION.** If  $\underline{B}$  is a Boolean algebra, then a *Boolean pseudo-norm* on  $\underline{B}$  is a function  $p: \underline{B} \rightarrow \mathbb{R}$  such that

- (i)  $p(b) \geq 0$  and  $p(0) = 0$
- (ii)  $p(a \oplus b) \leq p(a) + p(b)$

We shall call a pseudo-norm *monotone* iff

- (iii)  $p(a) \leq p(b)$  for  $a \leq b$ .

REMARKS.

- (1) Not every Boolean pseudo-norm is automatically monotone, as an example on  $\underline{2}^2$  shows.
- (2) But for every bounded Boolean pseudo-norm  $q$  the function  $\mu(a) := \sup_{b \leq a} q(b)$  is a monotone pseudo-norm - the *monotone modification* of  $q$ .
- (3) Every *invariant pseudo-metric*  $\rho$  on  $\underline{B}$  defines a pseudo-norm  $p$  by  $p(a) := \rho(a, 0)$ . Conversely, every pseudo-norm  $p$  on  $\underline{B}$  gives by the definition  $\rho(a, b) := p(a \oplus b)$  an *invariant pseudo-metric* on  $\underline{B}$ .

PROPOSITION.  $p: B \rightarrow \mathbb{R}$  is a *monotone Boolean pseudo-norm* on  $\underline{B}$  iff  $p$  is a *subcontent* on  $\underline{B}$ , i.e.

- (i)  $p(a) \geq 0$  and  $p(0) = 0$
- (ii)  $p(a) \leq p(b)$  for  $a \leq b$
- (iii)  $p(a \vee b) \leq p(a) + p(b)$

PROOF.

- (1) Let  $p$  be a monotone pseudo-norm on  $\underline{B}$ . Then  $a \vee b = a \oplus (b \setminus a)$ . Thus  $p(a \vee b) \leq p(a) + p(b \setminus a) \leq p(a) + p(b)$ .
- (2) If  $p$  is a subcontent, then from  $a \oplus b \leq a \vee b$  it follows that  $p(a \oplus b) \leq p(a) + p(b)$ .  $\square$

THEOREM 1. Let  $\mathcal{O}$  be a topology on the Boolean algebra  $\underline{B}$ .

- (1)  $\mathcal{O}$  is a locally solid compactible topology iff  $\mathcal{O}$  is generated by a family  $P$  of monotone Boolean pseudo-norms  $p$ , i.e.

$$\mathcal{O} = \sup\{\mathcal{O}(p) : (p \in P)\}.$$

Here  $\mathcal{O}(p)$  is the corresponding pseudo-metric topology for  $p$ .

- (2) The locally solid compactible topology  $\mathcal{O}$  is Hausdorff iff the generating family  $P$  is separating.
- (3) The locally solid compactible topology  $\mathcal{O}$  is metrizable iff there is a strictly monotone Boolean pseudo-norm  $p$  with  $\mathcal{O} = \mathcal{O}(p)$ .

PROOF.

(1) Let be  $p$  any monotone Boolean pseudo-norm on  $\underline{B}$ . The corresponding pseudo-metric  $\rho$  defines the topology  $\mathcal{O}(p)$ . One gets a base for the neighbourhoods of the origin by

$$U_{p,\varepsilon} := \{a: a \in \underline{B}, p(a) < \varepsilon\}.$$

Each  $U_{p,\varepsilon}$  is solid. Furthermore, the addition and the multiplication are continuous with respect to  $\mathcal{O}(p)$ . Indeed, let be  $a_\alpha \xrightarrow[p]{\rightarrow} a$ ,  $b_\beta \xrightarrow[p]{\rightarrow} b$ , i.e.  $p(a_\alpha \oplus a) \rightarrow 0$ ,  $p(b_\beta \oplus b) \rightarrow 0$ .

Then  $p(a_\alpha \oplus a \oplus b_\beta \oplus b) \leq p(a_\alpha \oplus a) + p(b_\beta \oplus b)$  and  $p(a \otimes b \oplus a_\alpha \otimes b_\beta) = p(a_\alpha \otimes (b_\beta \oplus b) \oplus b \otimes (a_\alpha \oplus a)) \leq p(b_\beta \oplus b) + p(a_\alpha \oplus a)$ , i.e.  $a_\alpha \otimes b_\beta \xrightarrow[p]{\rightarrow} a \otimes b$ .

The properties local solidity and compactibility are invariant under taking the supremum topology. Thus  $\mathcal{O} = \sup\{\mathcal{O}(p): p \in P\}$  is a locally solid compatible topology.

Now let conversely  $\mathcal{O}$  be a locally solid compactible topology. The following argumentation is based on theorems of FREMLIN [7, 22c] and the author [5, Th. 5]. Due to the last theorem  $\mathcal{O}$  is the trace of a locally solid lattice-algebra topology  $P$  on  $\mathcal{C}(\text{spec}(\underline{B}))$  (containing  $\underline{B}$  as the Boolean ring of all idempotents). Now Fremlin has shown that  $P$  is generated by all its continuous so-called Riesz pseudo-norms  $q$ . But every Riesz-pseudo-norm  $q$  gives on  $\underline{B}$  a Boolean pseudo-norm  $p = q|_{\underline{B}}$ . The defining properties of a Riesz pseudo-norm  $q$  on a Riesz space  $E$  are the following: A function  $q: E \rightarrow \mathbb{R}$  is a Riesz pseudo-norm whenever

- (i)  $q(x) \geq 0$ ,
- (ii)  $q(x+y) \leq q(x) + q(y)$ ,
- (iii)  $q(x) \leq q(y)$  whenever  $|x| \leq |y|$ ,
- (iv)  $\lim_{\alpha \rightarrow 0} q(\alpha x) = 0$ .

(2) Of course,  $\mathcal{O}$  is Hausdorff iff  $P$  is separating, i.e. for every  $a > 0$  there exists an  $p \in P$  with  $p(a) > 0$ .

(3) If  $p$  is strictly positive, then it is obvious that  $\mathcal{O}(p)$  is metrizable. Conversely, let  $\mathcal{O}$  be a locally solid compatible metrizable topology. We have  $\mathcal{O} = \sup\{\mathcal{O}(p): p \in P\}$ , where  $P$  is the family of all its continuous monotone pseudo-norms on  $(\underline{B}, \mathcal{O})$ .

We take a countable basis  $U_1 \supset U_2 \supset \dots \supset U_n \dots$  of neighbourhoods of the origin. Then we can find  $p_n \in P$ ,  $p_n \neq 0$ , with



$$V_n := \{a: p_n(a) < 1\} \subset U_n \quad \text{for all } n.$$

Then  $\sup \mathcal{O}(p_n) = 0$ . Because of  $\bigcap_{n \geq 1} U_n = \{0\}$  the family of the  $p_n$  separates points. Now

$$p(\cdot) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{p_n(\cdot)}{p_n(1)}$$

is a monotone strictly positive Boolean pseudo-norm on  $\underline{B}$  with  $\mathcal{O}(p) = \sup \mathcal{O}(p_n) = 0$ .  $\square$

**THEOREM 2.** Let  $(\underline{B}, \mathcal{O})$  be a Dedekind complete Boolean algebra equipped with a Hausdorff topology  $\mathcal{O}$ . Then the following statements are equivalent:

- (1)  $(\underline{B}, \mathcal{O})$  is an ABS-Boolean algebra.
- (2) The topology  $\mathcal{O}$  is generated by a separating family  $P$  of  $\mathcal{O}$ -continuous monotone Boolean pseudo-norms:

$$\mathcal{O} = \sup\{\mathcal{O}(p): p \in P\}.$$

Recall that order continuity of  $p$  means:  $(*) b_\alpha \uparrow b \Rightarrow p(b_\alpha) \rightarrow p(b)$ . For a monotone Boolean pseudo-norm  $p$  the condition  $(*)$  is equivalent to:  $b_\alpha \downarrow 0$  (i.e.  $\inf b_\alpha = 0$ )  $\Rightarrow p(b_\alpha) \rightarrow 0$ .

**PROOF.**

(1) If  $\mathcal{O}$  is an ABS topology on  $\underline{B}$  then it is locally solid and compatible. Thus there exists by the last theorem a generating and separating family  $P$  of monotone Boolean pseudo-norms  $p$ . The following  $\mathcal{O}$  is smaller than the  $\mathcal{O}$ -topology; it follows that every  $p$  is  $\mathcal{O}$ -continuous.

(2) If every  $p \in P$  is an  $\mathcal{O}$ -continuous monotone pseudo-norm on  $\underline{B}$  then according to the preceding theorem  $\mathcal{O} = \sup \mathcal{O}(p): (p \in P)$  is a locally solid compatible topology. This is also smaller than the  $\mathcal{O}$ -topology and it is Hausdorff by the separating property of  $P$ .

The proof consists of an easy estimation, as follows. Suppose  $b_\alpha \uparrow b$ . Then there are monotone nets  $a_\beta \uparrow b$ ,  $c_\gamma \downarrow b$  with  $a_\beta \leq b_\alpha \leq c_\gamma$  for all  $\alpha \geq \alpha(\beta, \gamma)$ .

One has only to show:  $p(a_\beta) \uparrow p(b)$  and  $p(c_\gamma) \downarrow p(b)$ .

Part (i):  $c_\gamma \downarrow b$  means  $c_\gamma \oplus b \downarrow 0$  and therefore  $p(c_\gamma \oplus b) \downarrow 0$ . Now  $p(b) \leq p(c_\gamma) = p(b \oplus (c_\gamma \oplus b)) \leq p(b) + p(c_\gamma \oplus b)$ , thus  $p(c_\gamma) \rightarrow p(b)$ .

Part (ii):  $a_\beta \uparrow b$  means  $b \oplus a_\beta \downarrow 0$  and therefore  $p(b \oplus a_\beta) \downarrow 0$ . Now  
 $p(a_\beta) \leq p(b) \leq p(a_\beta) + p(b \oplus a_\beta)$ , thus  $p(a_\beta) \rightarrow p(b)$ .  $\square$

REMARKS.

(1) A monotone 0-continuous Boolean pseudo-norm  $p$  on any Boolean algebra is nothing but an 0-continuous submeasure on  $\underline{B}$ . (vid. the first proposition of this section).

(2) Every monotone 0-continuous Boolean pseudo-norm  $p$  on a complete Boolean algebra  $\underline{B}$  has a support, defined by

$$\text{supp } p := \text{complement of } \sup\{a: p(a) = 0\}.$$

The element  $\text{supp } p$  is characterized (under the assumption  $p \neq 0$ ) as the greatest element  $b \in B$  such that for every  $a$ ,  $0 < a \leq b$ ,  $p(a) > 0$ .

COROLLARY. Let  $\underline{B}$  be a Dedekind complete Boolean algebra. Then the following statements are equivalent:

- (1)  $\underline{B}$  is a Maharam submeasure algebra,
- (2)  $\underline{B}$  is an ABS-algebra with countable cellularity (i.e. c.c.c.),
- (3)  $\underline{B}$  is a metrizable ABS-algebra.

If the conditions are fulfilled. The topology in question must be the order-topology.

PROOF.

(1)  $\Rightarrow$  (2). If  $\underline{B}$  is a Maharam submeasure algebra by the strictly positive submeasure  $\nu$  then, as already remarked in section 3, c.c.c holds in  $\underline{B}$ . By the preceding theorem  $\hat{O}(\nu)$  is an ABS topology ( $\nu$  separates points!).

(2)  $\Rightarrow$  (3). For any ABS-algebra topology  $\hat{O}$  we have by the preceding theorem a generating family  $P$  of 0-continuous pseudo-norms  $p$ . If the corollary number of  $\underline{B}$  is  $m$ , we get a set  $P$  of cardinality  $m$ . Indeed, take a maximal set  $P$  for which the supports  $\text{supp } p$  are disjoint. Now a countable family  $P$  defines by

$$p(\cdot) := \sum \left\{ \frac{1}{2^n} \cdot \frac{p_n(\cdot)}{p_n(1)} : p_n \in P \right\}$$

a strictly positive monotone pseudo-norm on  $\underline{B}$  which is 0-continuous.

(3)  $\Rightarrow$  (1). If the ABS-topology  $\hat{O}$  is metrizable then we find by the last theorem a strictly positive monotone pseudo-norm  $p$  with  $\hat{O} = \hat{O}(p)$ . This  $p$  is

0-continuous.

For the final statement let  $p$  be any strictly positive monotone 0-continuous pseudo-norm on  $\underline{B}$ . Then  $\theta(p) = 0$ -topology. This was first shown by MAHARAM [10, Th. 1].  $\square$

REMARK. The equivalence of (1) and (2) in this corollary was stated as Theorem 1 in the paper [11] of T.A. SARYMSAKOV and A.N. ISLAMOV. In a forthcoming paper [12] of T.A. SARYMSAKOV and V.I. ČILIN this was proved similar to the invariant metrization theorem of Kakutani for topological groups. In [12], the coincidence of the 0-topology and ABS-topology on a Dedekind complete Boolean algebra with c.c.c. is ascribed to D.A. VLADIMIROV [14], but in fact it is one of the main statements of MAHARAM's paper [10].

THEOREM 3. Let  $(\underline{B}, \theta)$  be a Dedekind complete Boolean algebra equipped with a Hausdorff topology. Then the following statements are equivalent:

- (1)  $(\underline{B}, \theta)$  is an ABS-algebra.
- (2)  $(\underline{B}, \theta)$  is topological and Boolean isomorphic to a product

$$\prod \{ (\underline{B}_\alpha, \theta(p_\alpha)) : \alpha \in A \}$$

of Maharam submeasure algebras  $\underline{B}_\alpha$  with the submeasures  $p_\alpha \dots$ . The product structure is taken both for the Boolean structure and the topology.

PROOF.

(2)  $\Rightarrow$  (1). The product  $\prod \underline{B}_\alpha$  of the complete Boolean algebras  $\underline{B}_\alpha$  is again complete. For any net

$$(x^{(\beta)})_{\beta \in D} \text{ with } x^{(\beta)} = (x_\alpha^{(\beta)})_{\alpha \in A} \text{ and } x = (x_\alpha)_{\alpha \in A} \text{ we have:}$$

$$x^{(\beta)} \xrightarrow{0} x \text{ in } \prod \underline{B}_\alpha \iff x_\alpha^{(\beta)} \xrightarrow{0} x_\alpha \text{ in } \underline{B}_\alpha \text{ for all } \alpha \in A.$$

Now every Maharam submeasure algebra  $(\underline{B}_\alpha, p_\alpha)$  gives a metrizable ABS-algebra  $(\underline{B}_\alpha, \theta(p_\alpha))$ . Then  $\prod \{ \theta(p_\alpha) : \alpha \in A \}$  is a locally solid compatible Boolean topology because these properties are invariant under the product. Even more, every  $\hat{p}_\alpha(x) := p_\alpha(x_\alpha)$  is a monotone Boolean pseudo-norm on  $\prod \underline{B}_\alpha$  which is 0-continuous and

$$\prod \{ \theta(p_\alpha) \} = \sup \theta(\hat{p}_\alpha).$$

(1)  $\Rightarrow$  (2). For an ABS-algebra  $(B, 0)$  take a generating and separating family  $P$  of 0-continuous Boolean pseudo-norms. Then let  $Q \subset P$  be a maximal family for which the supports  $b_q$  of  $q \in Q$  are disjoint. It must be proved that  $Q$  is a generating family too. So let  $p \in P$ ,  $p \neq 0$ , and  $q(a_\beta) \rightarrow 0$  for all  $q \in Q$ . We shall deduce  $p(a_\beta) \rightarrow 0$ . Denote the support of  $p$  of  $b$ . Then only a countable number of the supports  $b_q$  intersect  $b$ , i.e. have  $b_q \wedge b > 0$ . Write these supports as  $b_1, b_2, \dots, b_n, \dots$ . For any  $\varepsilon > 0$  there is a  $k$  such that  $p(\sup_{n>k} b_n) < \varepsilon$ , because for  $c_m := \sup_{n>m} b_n$  we have that  $c_m \downarrow 0$  and  $p$  is 0-continuous. On the other hand  $q_0 := q_1 + q_2 + \dots + q_k$  is a strictly positive 0-continuous pseudo-norm on  $\sup_{n \leq k} b_n$ . The same is true for  $p$ , thus

$$a_\beta \xrightarrow{q_0} 0 \text{ and } p(a_\beta \wedge \sup_{n \leq k} b_n) \rightarrow 0$$

are equivalent.

Now  $a_\beta \wedge \sup_n b_n = (a_\beta \wedge \sup_{n \leq k} b_n) \oplus (a_\beta \wedge \sup_{n > k} b_n)$ . Hence

$$p(a_\beta) \leq p(a_\beta \wedge \sup_{n \leq k} b_n) + p(a_\beta \wedge \sup_{n > k} b_n),$$

$$p(a_\beta) \leq p(a_\beta \wedge \sup_{n \leq k} b_n) + \varepsilon. \quad \square$$

REMARK. Theorem 3 and the proposition at the end of Section 3 show that our conjecture on ABS-algebras which was mentioned in the introduction, namely, that they are just the hyperstonian algebras is equivalent to the affirmative answer to the Maharam problem.

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## ON EXTENDING HOMEOMORPHISMS ON THE CANTOR SET

A.Guttek

### 1. INTRODUCTION

B. Knaster stated the following problem: Let  $P$  and  $Q$  be closed and nowhere dense subsets of the Cantor set  $C$  and let  $h$  be a homeomorphism from  $P$  onto  $Q$ . Does there exist a homeomorphism  $h'$  from  $C$  onto itself which is an extension of  $h$ ? The answer is yes. The first proof, based on the notion of boolean algebra's, was presented by C. Ryll-Nardzewski in Wroclaw during the session of Polish Mathematical Society on 30th November 1951. A topological proof, given by M. Reichbach, allows some generalizations published by B. KNASTER and M. REICHBACH [2], and J. POLLARD [4]. J. Mioduszewski asked whether such a homeomorphism can be extended to a homeomorphism with a dense orbit. The aim of the paper is to answer this question.

### 2. DEFINITIONS AND PREPARATORY LEMMAS

The Cantor set is a perfect totally disconnected metric space. We assume that the Cantor set is given in the closed unit interval  $[0,1]$  by the usual ternary expansion, and that  $\mathcal{B}$  is the basis induced by this expansion, i.e. a basis of closed-open subsets such that if  $I$  and  $J$  are distinct and belong to  $\mathcal{B}$ , then  $I \cap J = \emptyset$  or one of them is included in the other and  $\text{diam} I = 3^n \cdot \text{diam} J$  for some integer  $n$ .

Let  $\text{dist}(A,B) = \inf\{|a-b| : a \in A \text{ and } b \in B\}$ , where  $A$  and  $B$  are non-void subsets of the Cantor set  $C$ . If  $A = \{p\}$ , then we put  $\text{dist}(p,B)$  instead of  $\text{dist}(\{p\},B)$ .

Let  $f$  be a one-to-one mapping from  $X$  onto itself; then  $f^0(x) = x$  and  $f^{k+1}(x) = f(f^k(x))$  for every integer  $k$ .

The set  $O_x = \{f^k(x) : k \text{ is an integer}\}$  is the orbit of a point  $x$ . If for some  $x$  the set  $O_x$  is dense in  $X$ , then we say that the mapping  $f$  has a dense orbit.

LEMMA 1. (BANACH [1], th. 1 p. 236, KURATOWSKI and MOSTOWSKI [3], th. 2 p. 183). *If A and B are sets and f and g are one-to-one, where  $f: A \rightarrow B$  and  $g: B \rightarrow A$ , then the sets A and B can be represented as unions of disjoint sets  $A = A_1 \cup A_2$  and  $B = B_1 \cup B_2$ , where  $f(A_1) = B_1$  and  $g(B_2) = A_2$ .*

LEMMA 2. (KNASTER and REICHBACH [3]). *Let P and Q be closed and nowhere dense subsets of the Cantor set C and let h be a homeomorphism from P onto Q. Then there exists a homeomorphism h' from C onto itself which is an extension of h.*

LEMMA 3. *Let A be a nowhere dense subset of the Cantor set C. Then there exists a nowhere dense subset D of C which is homeomorphic to C and such that A is a nowhere dense subset of D.*

LEMMA 4. (cf. KNASTER and REICHBACH [3]). *Let D be a nowhere dense non-void and closed subset of the Cantor set C, h be a homeomorphism from D onto itself and  $R = \{I_1, I_2, \dots\}$  be a family of closed-open pairwise disjoint subintervals of C such that  $UR = C - D$ . Then there exists a one-to-one mapping f from R onto itself such that any extension h' of h which maps homeomorphically every  $I_n$  onto  $f(I_n)$ , is a homeomorphism from C onto itself.*

PROOF. Since D is closed, nowhere dense and non-void, then R is infinite, and for each  $I_n$  of R there exists a point  $p_n$  of D such that  $\text{dist}(D, I_n) = \text{dist}(p_n, I_n)$ .

We define one-to-one mappings  $f_1$  and  $f_2$  from R into itself in such a way that both  $\text{dist}(h(p_n), f_1(I_n))$  and  $\text{dist}(h^{-1}(p_n), f_2(I_n))$  are not greater than  $\text{dist}(p_n, I_n)$ . By Lemma 1, there are disjoint subfamilies  $R_1$  and  $R_2$  of R such that  $R_1 \cup R_2 = R$  and the mapping f defined on R by  $f|_{R_1} = f_1|_{R_1}$  and  $f|_{R_2} = f_2^{-1}|_{R_2}$  is one-to-one and onto. Simple computation shows that f satisfies the conclusion of the lemma.

LEMMA 5. *Let  $D_0$  be a nowhere dense, non-void and closed subset of the Cantor set C,  $h_0$  be a homeomorphism from  $D_0$  onto itself and R be a family of closed-open pairwise disjoint subintervals of C such that  $UR = C - D_0$ . Then there exist a nowhere dense and closed subset D of C containing  $D_0$ , a homeomorphism h from D onto itself which is an extension of  $h_0$ , a family R' of closed-open and pairwise disjoint subintervals of C with  $UR' = C - D$ , and a one-to-one mapping f' from R' onto itself such that any extension h' of h which maps homeomorphically every  $I \in R'$  onto  $f'(I)$ , is a homeomorphism from C*



onto itself, and such that every orbit of  $f'$  is infinite. If  $R$  is a subfamily of the natural basis  $\mathcal{B}$  of  $C$ , then  $R'$  is contained in  $\mathcal{B}$  also.

PROOF. By Lemma 4, there exists a mapping  $f$  which describes some extensions of  $h_0$ . If every orbit of  $f$  is infinite, then we take  $D = D_0$ ,  $h = h_0$ ,  $R' = R$  and  $f' = f$ .

If there are finitely many finite orbits, then by changing  $f$  we can make them a part of an infinite orbit, and the resulting mapping  $f'$  satisfies the conclusion of the lemma.

If there are infinitely many finite orbits, then let  $J_n^i = f^n(J_0^i)$ , let  $B_i = \{J_0^i, \dots, J_{n_i}^i\}$  be the  $i$ -th finite orbit, and let  $a_n^i < b_n^i$  be endpoints of  $J_n^i$ , for  $i = 1, 2, \dots$ ; i.e.  $[a_n^i, b_n^i] \cap C = J_n^i$  and  $a_n^i, b_n^i \in C$ . Let  $\{J_n^i(k) : k \text{ is an integer}\}$  be a family of closed-open pairwise disjoint subintervals of  $J_n^i$  which belong to  $\mathcal{B}$ , the union of which is equal to  $J_n^i - \{a_n^i, b_n^i\}$ , and such that the sequence  $\{J_n^i(k) : k = 1, 2, \dots\}$  and  $\{J_n^i(k) : k = -1, -2, \dots\}$  converge to the points  $b_n^i$  and  $a_n^i$  respectively. Put  $D_i = \{a_0^i, \dots, a_{n_i}^i, b_0^i, \dots, b_{n_i}^i\}$  and let  $R_i$  be the family  $\{J_n^i(k) : n = 0, \dots, n_i\}$  and  $k$  is an integer. Define mappings  $f_i$  on  $D_i \cup R_i$  by setting  $f_i(a_n^i) = a_{n \oplus 1}^i$  and  $f_i(b_n^i) = b_{n \oplus 1}^i$ , where  $\oplus$  denotes addition modulo  $n_i + 1$ , and  $f_i(J_n^i(k))$  to be equal to  $J_{n+1}^i(k)$  for  $n = 0, \dots, n_i - 1$ , and to  $J_0^i(k+1)$  for  $n = n_i$ .

Put  $D = \{D_i : i = 0, 1, \dots\}$  and  $R' = \{R_i : i = 0, 1, 2, \dots\}$ , where  $R_0$  is the union of the infinite orbits of  $f$ . Let  $h$  be defined by  $h|_{D_0} = h_0$  and  $h|_{D_i} = f_i|_{D_i}$  for  $i = 1, 2, \dots$ . The mapping  $f'$  defined by  $f'|_{R_0} = f|_{R_0}$  and  $f'|_{D_i} = f_i|_{D_i}$  for  $i = 1, 2, \dots$  satisfies the conclusion of the lemma. Furthermore, if  $R \subset \mathcal{B}$ , then  $R' \subset \mathcal{B}$ .

### 3. THE MAIN RESULT

We prove the following:

THEOREM 1. Let  $P$  and  $Q$  be closed and nowhere dense subsets of the Cantor set  $C$ , and let  $h$  be a homeomorphism from  $P$  onto  $Q$ . Then there exists an extension  $h'$  of  $h$  such that  $h'$  is a homeomorphism from  $C$  onto itself and for some point  $c$  of  $C$  the set  $\{(h')^n(c) : n \text{ is an integer}\}$  is a dense subset of  $C$ .

Let us observe that because  $P \cup Q$  is a nowhere dense subset of  $C$ , then by Lemma 2 and Lemma 3 there exists a nowhere dense closed subset  $D$  of  $C$

which is homeomorphic to  $C$ , and which is such that  $P \cup Q$  is nowhere dense in  $D$ , and an extension  $h_D$  of  $h$  which is a homeomorphism from  $D$  onto itself. Thus, it suffices to prove the following:

**THEOREM 2.** *Let  $D$  be non-void, nowhere dense and closed subset of the Cantor set  $C$  and let  $h$  be a homeomorphism from  $D$  onto itself. Then there exists a homeomorphism  $h'$  from  $C$  onto itself which extends  $h$  and which has a dense orbit.*

**PROOF.** For every two subsets  $A$  and  $B$  of  $C$  put  $A < B$  if and only if  $a < b$  for every  $a \in A$  and  $b \in B$ . For every set  $I$  of the basis  $\mathcal{B}$  and for every positive integer  $n$  consider a partition of  $I$  into  $2^n$  disjoint subsets of diameters equal to  $3^{-n} \cdot \text{diam} I$ . Order these subsets as above and denote the  $i$ -th subset in this ordering by  $I(i, 2^n)$ . So,  $I(i, 2^n) < I(j, 2^n)$  if and only if  $i < j$ .

Let  $R$  be a subfamily of  $\mathcal{B}$  consisting of pairwise disjoint subsets such that the union  $UR$  is equal to  $C - D$ . Clearly,  $R$  is infinite

By Lemma 4 there exists a one-to-one mapping  $f$  from  $R$  onto itself such that any extension  $h'$  of  $h$  defined by  $h'|_D = h$  and  $h'|_I$  is for every  $I \in R$  a homeomorphism from  $I$  onto  $f(I)$ , is a homeomorphism from  $C$  onto itself. In virtue of Lemma 5, we can assume that every orbit of  $f$  is infinite. Let  $O_1, O_2, \dots$  be different orbits of  $f$ ,  $O_1 \cup O_2 \cup \dots = R$ . Choose for every orbit  $O_i$  a segment  $J_0^i$  and put  $J_k^i = f^k(J_0^i)$ . So,  $O_i = \{J_k^i : k \text{ is an integer}\}$ . Now, for every  $i$ , let  $\{m(i, j) : j \text{ is an integer}\}$  be a sequence of integers, such that  $m(i, j) + 2 < m(i, j+1)$  and the sequences  $\{J_{m(i, j)}^i : j = 1, 2, \dots\}$  and  $\{J_{m(i, j)}^i : j = -1, -2, \dots\}$  converge to some points  $a^i$  and  $b^i$  respectively.

If the mapping  $f$  has finitely many orbits, i.e.  $i = 1, \dots, N$ , then we define homeomorphisms  $h_i$  on  $UO_i$  by setting  $h_i$  to be linear and order-preserving mappings.

- (a) from  $J_m^i$  onto  $J_{m+1}^i$  if  $m+1 \neq m(i, j)$  for every integer  $j$ .
- (b) from  $J_m^i(2, 2)$ ,  $J_m^i(2, 4)$  and  $J_m^i(1, 4)$  onto  $J_{m(i, j-2)}^i(4, 4)$ ,  $J_{m(i, j-2)}^i(3, 4)$  and  $J_{m(i, j)}^i(1, 2)$  respectively if  $m+1 = m(i, j)$  for a certain  $j > 1$ ,
- (c) from  $J_m^i(2, 2)$  and  $J_m^i(1, 2)$  onto  $J_{m(i, -1)}^i(2, 2)$  and  $J_{m(i, 1)}^i(1, 2)$  respectively if  $m+1 = m(i, 1)$ ,
- (d<sub>1</sub>) from  $J_m^1(4, 4)$ ,  $J_m^1(3, 4)$  and  $J_m^1(1, 2)$  onto  $J_{m(1, j-2)}^1(2, 2)$ ,  $J_{m(i, j)}^1(1, 4)$  and  $J_{m(1, j)}^1(2, 4)$  if  $m+1 = m(1, j)$  for a certain  $j < 1$ ,

( $d_2$ ) from  $J_m^i(4,4)$ ,  $J_m^i(3,4)$  and  $J_m^i(1,2)$  onto  $J_m^i(i,j-2)(2,2)$ ,  $J_m^i(i,j)(2,4)$  and  $J_m^i(i,j)(1,4)$  if  $m+1 = m(i,j)$  for a certain  $j < 1$  and  $i = 2, \dots, N$ .

We say that a homeomorphism  $h$  is linear and order-preserving mapping from  $I$  onto  $J$ , where  $I$  and  $J$  belong to  $\mathcal{B}$ , if and only if for every positive integer  $j$  and  $k = 1, \dots, 2^j$  the image  $h(I(k, 2^j))$  is equal to  $J(k, 2^j)$ .

Note that an extension  $g$  of  $h$  defined by  $g|_D = h$  and  $g|_{U0_i} = h_i$ ,  $i = 1, \dots, N$ , is a homeomorphism from  $C$  onto itself. We will define an extension  $h'$  of  $h$  by changing the homeomorphism  $g$  on some subintervals of  $C$ .

CASE 1. The unique orbit of  $f$  is equal to  $R$ . We define  $h'$  by putting  $h'|_D = h$  and  $h'|_{UR} = h_1$ . Clearly,  $h'$  is continuous. Let  $d_0$  be the diameter of  $J_m^1(1,0)$ , and  $d_j$ ,  $j = 1, 2, \dots$ , be the supremum of the diameters of  $J_m^1(m(1,-j) \leq m < m(1,-j+1)$  and  $m(1,j-1) < m \leq m(1,j)$ , and of  $2^{-1} \cdot d_{j-1}$ . The sequence  $\{d_j: j = 1, 2, \dots\}$  converges to 0 when  $h$  converges to  $\infty$ .

Using induction we shall prove that for every positive integer  $n$  there exists a positive integer  $t_n$  such that

- (1) the family  $F_n = \{(h')^k(J_m^1(1,-n)(1,2)): k = 0, \dots, t_n\}$  consists of pairwise disjoint and closed-open subsets the diameters of which are less or equal to  $d_{n+1}$ ,
- (2)  $(h')^{t_n}(J_m^1(1,-n)(k, 2^j))$  is equal to  $J_m^1(1,n+1)(k, 2^j)$  and  $(h')^{m(1,n+1)-m(1,-n)}(J_m^1(1,n)(k+2^{j-1}, 2^j)) = J_m^1(1,-n-1)(k+2^{j-1}, 2^j)$  for  $k = 1, \dots, 2^{j-1}$  and  $j = 1, 2, \dots$ ,
- (3)  $\{(h')^{m(1,0)-m(1,-j)}(J_m^1(1,-j)(1,2)): j = 0, \dots, n\}$  is a decreasing sequence of closed-open sets,
- (4)  $J_m^1 - UF_n$  are equal to  $J_m^1(2^{n+1-j}, 2^{n+1-j})$  for  $m(1,-j-1) \leq m < m(1,-j)$ ,  $j = 0, \dots, n-1$  and for  $m(1,j) \leq m < m(1,j+1)$ ,  $j = 0, \dots, n$ .

It is easy to see that conditions (1) - (4) hold for  $n = 1$  and  $t_n = 2m(1,2) + m(1,1) - 2m(1,-1) - m(1,0)$ .

In fact, the family  $F_1$  consists of sets  $J_m^1(1,2)$  for  $m(1,-1) \leq m < m(1,0)$  and  $m = m(1,2)$ , of  $J_m^1(1,4)$  and  $J_m^1(2,4)$  for  $m(1,0) \leq m < m(1,2)$  and of  $J_m^1(3,4)$  for  $m(1,-1) \leq m < m(1,1)$ , and these sets are pairwise disjoint and their diameters are less than or equal to  $d_2$ .

Let  $k = 1, \dots, 2^{j-1}$  and  $j = 1, 2, \dots$ . We have the following:

$$(h')^{m(1,0)-1-m(1,-1)} (J_{m(1,-1)}^1(k, 2^j)) = J_{m(1,0)-1}^1(k, 2^j), \quad \text{by (a),}$$

$$h' (J_{m(1,0)-1}^1(k, 2^j)) = J_{m(1,0)}^1(k+2^{j-1}, 2^{j+1}), \quad \text{by (d}_1\text{),}$$

$$\begin{aligned} (h')^{m(1,2)-1-m(1,0)} (J_{m(1,0)}^1(k+2^{j-1}, 2^{j+1})) &= \\ &= J_{m(1,2)-1}^1(k+2^{j-1}, 2^{j+1}), \quad \text{by (a), (c),} \end{aligned}$$

$$h' (J_{m(1,2)-1}^1(k+2^{j-1}, 2^{j+1})) = J_{m(1,0)}^1(k+2^j, 2^{j+1}), \quad \text{by (b),}$$

$$\begin{aligned} (h')^{m(1,1)-1-m(1,0)} (J_{m(1,0)}^1(k+2^j, 2^{j+1})) &= \\ &= J_{m(1,1)-1}^1(k+2^j, 2^{j+1}), \quad \text{by (a),} \end{aligned}$$

$$h' (J_{m(1,1)-1}^1(k+2^j, 2^{j+1})) = J_{m(1,-1)}^1(k+2^j, 2^{j+1}), \quad \text{by (c),}$$

$$\begin{aligned} (h')^{m(1,0)-1-m(1,-1)} (J_{m(1,-1)}^1(k+2^j, 2^{j+1})) &= \\ &= J_{m(1,0)-1}^1(k+2^j, 2^{j+1}), \quad \text{by (a),} \end{aligned}$$

$$h' (J_{m(1,0)-1}^1(k+2^j, 2^{j+1})) = J_{m(1,0)}^1(k, 2^{j+1}), \quad \text{by (d}_1\text{),}$$

$$\begin{aligned} (h')^{m(1,2)-1-m(1,0)} (J_{m(1,0)}^1(k, 2^{j+1})) &= \\ &= J_{m(1,2)-1}^1(k, 2^{j+1}), \quad \text{by (a) and (c),} \end{aligned}$$

$$h' (J_{m(1,2)-1}^1(k, 2^{j+1})) = J_{m(1,2)}^1(k, 2^j), \quad \text{by (b).}$$

Because

$$\begin{aligned} m(1,0) - 1 - m(1,-1) + 1 + m(1,2) - 1 - m(1,0) + 1 + m(1,1) - 1 + \\ - m(1,0) + 1 + m(1,0) - 1 - m(1,-1) + 1 + m(1,2) - 1 - \\ - m(1,0) + 1 = \\ = 2 \cdot m(1,2) + m(1,1) - 2 \cdot m(1,-1) - m(1,0) = t_1, \end{aligned}$$

then

$$(h')^{t_1} (J_{m(1,-1)}^1 (k, 2^j)) = J_{m(1,2)}^1 (k, 2^j).$$

The other equality of (2) one can verify in the same way.

By two first steps of the computation above we have

$$(h')^{m(1,0)-1(1,1)} (J_{m(1,-1)}^1 (1,2)) = J_{m(1,0)}^1 (2,4) \subset J_{m(1,0)}^1 (1,2).$$

So condition (3) holds for  $n = 1$ .

To verify condition (4) we use the form of the family  $F_1$ , and we have that  $J_m^1 - UF_1$  is equal to  $J_m^1(4,4)$  for  $m(1,-1) \leq m < m(1,1)$ , and to  $J_m^1(2,2)$  for  $m(1,1) \leq m < m(1,2)$ .

Suppose that conditions (1) - (4) hold for some positive integer  $n$ . Then we show that conditions (1) - (4) hold when  $n+1$  is substituted for  $n$  and for  $t_{n+1} = 2t_n + 2m(1,n+2) - m(1,n+1) + m(1,-n) - 2m(1,-n-1)$ . We begin with condition (2). We have the following:

$$\begin{aligned} & (h')^{m(1,-n)-m(1,-n-1)} (J_{m(1,-n-1)}^1 (k, 2^j)) = \\ & = J_{m(1,-n)}^1 (k+2^{j-1}, 2^{j+1}) \quad \text{by (a) and (d}_1), \\ & (h')^{t_n} (J_{m(1,-n)}^1 (k+2^{j-1}, 2^{j+1})) = \\ & = J_{m(1,n+1)}^1 (k+2^{j-1}, 2^{j+1}) \quad \text{by assumption} \end{aligned}$$

that condition (2) holds for  $n$ ,

$$\begin{aligned} & (h')^{m(1,n+2)-m(1,n+1)} (J_{m(1,n+1)}^1 (k+2^{j-1}, 2^{j+1})) = \\ & = J_{m(1,n)}^1 (k+2^j, 2^{j+1}) \quad \text{by (a) and (b)}, \\ & (h')^{m(1,n+1)-m(1,-n)} (J_{m(1,n)}^1 (k+2^j, 2^{j+1})) = \\ & = J_{m(1,-n-1)}^1 (k+2^j, 2^{j+1}) \end{aligned}$$

by assumption that condition (2) holds for  $n$ ,

$$\begin{aligned} & (h')^{m(1,-n)-m(1,-n-1)} (J_{m(1,-n-1)}^1 (k+2^j, 2^{j+1})) = J_{m(1m-n)}^1 (k, 2^{j+1}) \\ & \quad \text{by (a) and (d}_1), \end{aligned}$$

$$(h')^{t_n} J_{m(1,-n)}^1(k, 2^{j+1}) = J_{m(1,n+2)}^1(k, 2^{j+1})$$

by assumption that condition (2) holds for  $n$ ,

$$(h')^{m(1,n+2)-m(1,n+1)} J_{m(1,n+1)}^1(k, 2^{j+1}) = J_{m(1,n+2)}^1(k, 2^j)$$

by (a) and (b). So, for  $t_{n+1}$  as above, we have the first equality of condition (2). The second one we can prove in the same way.

The family  $F_{n+1}$  consists of sets  $I(1,2)$ ,  $I(2,2)$ , for  $I \in F_n$ , of  $(J_m^1 - F_n)(1,2)$  if  $J_m^1$  intersects  $UF_n$ , and of  $J_m^1(1,2)$  and  $J_m^1(3,4)$  for  $m(1,-n-1) \leq m < m(1,-n)$ , of  $J_m^1(1,4)$  and  $J_m^1(2,4)$  for  $m(1,n) < m < m(1,n+1)$ , and of  $J_{m(1,n+1)}^1(1,2)$ . Thus condition (2) holds.

To verify condition (4) we use the form of the family  $F_{n+1}$ . To see that condition (3) holds we use the fact that

$$(h')^{m(1,-n)-m(1,-n-1)} J_{m(1,-n-1)}^1(1,4) \subset J_{m(1,-n)}^1(1,2).$$

From these conditions we conclude that the intersection

$$\cap \{ (h')^{m(1,0)-m(1,-n)} J_{m(1,-n)}^1(1,2) : n = 1, 2, \dots \}$$

is non-void and contains exactly one point, say  $p$ , and that the orbit  $O_p = \{(h')^n(p) : n \text{ is an integer}\}$  of this point is dense in  $C$ .

CASE 2. The mapping  $f$  has finitely many different orbits. Let  $O_1, \dots, O_N$  be the orbits. To obtain  $h'$  we use homeomorphisms  $h_i$ ,  $i = 1, \dots, N$ , defined by conditions (a) - (d<sub>2</sub>). We put  $h'|_D = h$ ,  $h'|_{(UO_i - J_{m(i,0)+1}^i(1,2))}$  to be equal to  $h_i$  restricted to the same set, and  $h'$  restricted to  $J_{m(i,0)+1}^i(1,2)$  to be linear and order-preserving mappings onto  $J_{m(i+1,0)+2}^{i+1}(1,2)$  for  $i = 1, \dots, N-1$ , and onto  $J_{m(1,0)+2}^1(1,2)$  for  $i = N$ .

Note that  $h'$  is a homeomorphism. By a similar induction as in Case 1, we can show that  $h'$  has a dense orbit.

CASE 3. The mapping  $f$  has infinitely many infinite orbits. Let  $O_i$ ,  $i = 1, 2, \dots$ , be these orbits. We reduce this case to Case 1.

For every positive integer  $n$  divide the Cantor set  $C$  into  $2^n$  pairwise disjoint subintervals with diameters equal to  $3^{-n}$ , and denote the  $k$ -th set of this partition by  $C(k, 2^n)$ . Order all pairs  $(k, 2^n)$ ,  $k = 1, \dots, 2^n$ , putting  $(k', 2^m) < (k, 2^n)$  if and only if  $2^m < 2^n$  or  $2^m = 2^n$  and  $k' < k$ . With every

pair  $(k, 2^n)$  join exactly one nonnegative integer  $i$ , and in result, if  $i > 0$ , an orbit  $O_i$ . We put  $i(1,2)$  to be the smallest positive integer  $i$  such that infinitely many subsets of  $C(1,2)$  belongs to  $O_i$ , if such an integer exists, and 0 otherwise. Suppose that we have done it for all pairs less than  $(k, 2^n)$ . We put  $i(k, 2^n)$  to be the smallest positive integer not chosen before, such that infinitely many subsets of  $C(k, 2^n)$  belong to  $O_i$ , if such an integer  $i$  exists, and 0 otherwise.

For every positive integer  $i$  we choose a sequence  $\{m(i,j) : j \text{ is an integer}\}$  of integers such that  $m(i,j) + 2 < m(i,j+1)$ , and whenever  $i = i(k, 2^n)$  for some positive integers  $k$  and  $n$ , then subsequences of the sets  $\{J_{m(i,j)}^i : j = 1, 2, \dots\}$  and  $\{J_{m(i,j)}^i : j = 0, -1, \dots\}$  converge to some points, say  $a^i$  and  $b^i$  respectively, and consist of subsets of  $C(k_1, 2^n)$  and  $C(k_2, 2^n)$  respectively, for some integers  $k_1, k_2$ , where  $k_1 = k$  or  $k_2 = k$ . We let  $e(i) = 1$  when  $k_1 = k$  and  $e(i) = -1$  otherwise. We can choose  $m(i,j)$  in such a way that every  $J_{m(i,j)}^i$  is contained in this  $C(t, 2^{|j|+i})$  which contains  $a^i$ , if  $j > 0$ , and  $b^i$ , if  $j \leq 0$ , where  $t$  is a positive integer not greater than  $2^{|j|+i}$ .

Using the mapping  $f$  and the family  $R$  we construct a family  $R'$  consisting of pairwise disjoint and closed-open intervals which belong to the natural basis  $\mathcal{B}$  such that  $c\mathcal{L}_C UR' = C$ , and a one-to-one mapping  $f'$  from  $R'$  onto itself such that  $R'$  is the unique orbit of  $f'$ .

Denote by  $a_m^i, b_m^i$  the endpoints of  $J_m^i$ , i.e.  $a_m^i < b_m^i, a_m^i, b_m^i \in C$  and  $[a_m^i, b_m^i] \cap C = J_m^i$ .

STEP 1. Consider  $i(1,2)$  and  $i(2,2)$ . At least one of these integers is positive. If both  $i(1,2)$  and  $i(2,2)$  are positive, then we define  $f'$  restricted to the partition  $\{J_m^i(2, 2^{k+1}), J_m^i(2^{k+1}-1, 2^{k+1}), a_m^i, b_m^i : k = 1, 2, \dots\}$  of  $J_m^i$  for  $i = i(1,2), i(2,2)$  and  $m(i, -3) \leq m < m(i, 3) - 1$  by setting

- (1a) the values of  $f'$  on  $J_m^i(2, 2^{k+1}), J_m^i(2^{k+1}-1, 2^{k+1}), a_m^i, b_m^i$  to be equal to  $J_{m+1}^i(2, 2^{k+1}), J_{m+1}^i(2^{k+1}-1, 2^{k+1}), a_{m+1}^i, b_{m+1}^i$  respectively, if  $m \neq m(i, e(i)) - 1$ ; and
- (1b) the values of  $f'$  on members of the partition of  $J_m^i(1, 2)$  and of  $J_m^i(2, 2)$  are equal to the corresponding members of the partition of  $J_m^i(2, 2)$  and of  $J_m^i(1, 2)$  respectively.

If  $i(1,2) = 0$  or  $i(2,2) = 0$ , then let  $i$  be equal to whichever is positive. If all segments  $J_m^i$  are contained in  $C(1,2)$  for  $i = i(1,2)$ , or in  $C(2,2)$  for  $i = i(2,2)$ , then we define  $f'$  as in (1a) for  $m(i, -3) \leq m < m(i, 3) - 1$ .

If this is not the case, then for  $m \neq m(i, -1)$ ,  $m(i, 1) - 1$  we define  $f'$  as before, and we put the values of  $f'$  on the members of the partition of  $J_m^i(i, 3) - 1$  to be equal to the corresponding members of the partition of  $J_m^i(i, -3)$ .

Let us observe that in this way we have defined two chains joining  $C(1, 2)$  and  $C(2, 2)$ , or, in a special case, one chain in  $C(1, 2)$  or in  $C(2, 2)$ .

Denote by  $A_1^1(s, 2^{k+1})$ ,  $s = 2, 2^{k+1} - 1$ , the corresponding segments, and by  $A_1^1(-\infty)$ ,  $A_1^1(+\infty)$  the corresponding points  $a_m^i$ ,  $b_m^i$  belonging to the first element of the chain which starts in  $C(1, 2)$ , and by  $B_1^1(s, 2^{k+1})$ ,  $B_1^1(-\infty)$ ,  $B_1^1(+\infty)$  the members of its last element. In a similar way we define  $A_1^2(s, 2^{k+1}), \dots, B_1^2(+\infty)$  for the other chain. The lower index denotes the number of the step (here  $n = 1$ ), and the upper index denotes the number of the segment of the partition of the Cantor set into  $2^n$  subsets, in which the chain has its first element.

Denote by  $D_1$  the set of points  $a_m^i$ ,  $b_m^i$ , and by  $R_1$  the family of segments on which  $f'$  has been defined.

STEP 2. This step allows to imagine the general one. Consider  $i(1, 4), \dots, i(4, 4)$ , and the segments  $C(s, 4)$ ,  $s = 1, \dots, 4$ , the members of which are points  $a_m^i$ ,  $b_m^i$  for  $i = i(1, 2), i(2, 2), i(1, 4), \dots, i(4, 4)$ . The idea of the construction is the following: we construct a chain from every such segment to the next one, and from the last segment to the first one in a similar way as before, but now we change  $f$  in  $C(1, 2)$  or in  $C(2, 2)$ , and we start to construct an orbit. We begin with  $C(1, 2)$ .

Suppose that  $i(1, 4)$  and  $i(2, 4)$  are positive. Then  $f'$  restricted to

$$\{J_m^i(1, 2) : m(i(1, 2), -5) \leq m < m(i(1, 2), -3) - 1\}$$

and to

$$\{J_m^i(1, 2) : m(i(1, 2), 3) \leq m < m(i(1, 2), 5) - 1\}$$

is equal to  $f$ , and the values of  $f'$  on  $J_m^i(1, 2)$  and on  $B_1^1(2, 2^2)$ ,  $B_1^2(2, 2^2)$ ,  $B_1^1(3, 2^2)$  are equal to  $A_1^1(2, 2^2)$ ,  $A_1^2(2, 2^2)$ ,  $A_1^1(3, 2^2)$  and  $A_1^2(3, 2^2)$  respectively, thus we pass twice by every chain constructed before.

Let us assume that both  $e(i(1, 4))$  and  $e(i(2, 4))$  are equal to 1 (in other cases  $f'$  is defined in a similar way). Thus, using the sets  $J_m^i$ ,  $i = i(1, 4), i(2, 4)$  and  $m(i, 1) \leq m < m(i, 2) - 1$ , we construct a chain from



$C(1,4)$  to  $C(2,4)$  as before, i.e. the values of  $f'$  on members of the partition of a given segment is a corresponding member of the partition of the next one, and on the partition of  $J_{m(i(1,4),2)-1}^{i(1,4)}$  corresponding member of the partition of  $J_{m(i(2,4),1)}^{i(2,4)}$ . From the sets  $\{J_m^i: m(i,2) \leq m < m(i,3)-1\}$  for  $i = i(1,4), i(2,4)$  we make the beginning, or the end of a chain which join  $C(4,4)$  with  $C(1,4)$  and  $C(2,4)$  with  $C(3,4)$ . The value of  $f'$  on  $B_1^2(3,2^2)$  is equal to  $J_{m(i(1,4),-3)}^{i(1,4)}$ , if it is contained in  $C(1,2)$ , and to  $A_1^1(2,2^3)$  otherwise, and then the value of  $f'$  on  $B_1^1(2,2^3)$  is equal to  $J_{m(i(1,4),-3)}^{i(1,4)}$ .

On the set

$$\{J_m^{i(1,4)}: m(i(1,4),-3) \leq m < m(i(1,4),1)-1\}$$

$f'$  is equal to  $f$ .

Because  $e(i(1,4)) = 1$ , we have that  $J_{m(i(1,4),1)}^{i(1,4)}$  is contained in  $C(1,2)$  and the value of  $f'$  on this segment is equal to  $J_{m(i(2,4),-3)}^{i(2,4)}$ . If  $J_{m(i(1,4),1)}^{i(1,4)}$  is contained in  $C(2,2)$  then we ought to use a member of the chain joining  $C(2,2)$  to  $C(1,2)$  having the biggest diameter and not used before, in our case  $A_1^2(2,2^3)$ .

We do a similar construction in  $C(2,2)$ . Then we come back to  $C(1,2)$  and we end the definitions of the chains. We map members of the partition of  $J_{m(i(2,4),3)-1}^{i(2,4)}$  on the segments  $A_1^1(s,2^{k+1})$  which are left.

Denote by  $D_2$  the set of points, and by  $R_2$  the set of segments on which  $f'$  has been defined. The set  $D \cup D_1 \cup D_2 \cup \dots$  is closed and nowhere dense, and  $g$  is defined on this set by  $g|D = h$  and  $g|D_j = f'|D_j$ ,  $j = 1, 2, \dots$  is a homeomorphism, and  $f'$  is defined on  $R' = R_1 \cup R_2 \cup \dots$  as  $f$  in Case 1, which finishes the proof.

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## A SPLITTING THEOREM FOR SURFACES

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We consider the situation that  $S$  is a closed surface with base point  $s$  and  $\pi$  its fundamental group. Suppose  $F$  is an infinite cyclic subgroup of  $\pi$  and that either

- a) there are subgroups  $G_0$  and  $G_1$  of  $\pi$  such that  $F = G_0 \cap G_1$  but  $F \neq G_0$ ,  $F \neq G_1$  and that  $\pi = G_0 *_F G_1$  (i.e.  $\pi$  is the free amalgamated product of  $G_0$  and  $G_1$  with amalgamation  $F$ ), or
- b) there is a subgroup  $G$  of  $\pi$  and  $t \in \pi$ , such that  $F \subset G$ , that  $\tau(f) = tft^{-1}$  defines a monomorphism  $F \rightarrow G$  and that  $\pi = GF^{\tau}$  (i.e.  $\pi$  equals the HNN extension of  $G$  by  $F$  by means of  $\tau$ ). ([3])

One can realize this situation geometrically as formulated below. We thank Robert Bieri for bringing the theorem to the attention of the first author. The theorem of which a proof is given here is known at least to Bieri.

### 1. THE THEOREM

THEOREM. *In the above situation there is a 2-sided simple closed curve  $C$  through  $s$  such that  $\pi_1(C, s) = F$ . Moreover given a small segment  $b$  transverse to  $C$  at  $s$  with endpoints  $b_0$  and  $b_1$  lying on either side of  $C$  and considering  $b$  as a 'fat' base point one may reach the following:*

- (i) *In case a, the complement of  $C$  has two components  $U_0$  and  $U_1$  and  $\pi_1(U_i, b_i) = G_i$  ( $i = 0, 1$ ).*
- (ii) *In case b, the complement of  $C$  has one component  $U$  and there is a path  $\alpha$  joining  $b_0$  to  $b_1$  in  $U$ ,  $\pi_1(U, b_0) = G$  and  $\alpha = t$  in  $\pi_1(S, b) = \pi$ .*

*The equalities of groups in the statement are induced by the subspace inclusions. The statements (i) and (ii) are intended to identify the factorizations of  $a$  and  $b$  as van Kampen diagrams for unions of subspaces  $U_0$  and  $U_1$  or  $U$  and a neighbourhood of  $C$ .*

We start the proofs by constructing an Eilenberg-MacLane complex  $X = K(\pi, 1)$  with a "codimension 1" subspace  $M = K(F, 1) \sim S^1$  as in [3, p. 60]. In case a (resp. b) take for  $i = 0, 1$  the mapping cylinder  $Y_i$  of the maps of Eilenberg-MacLane spaces corresponding to the monomorphisms  $F \subset G_i$  (resp.  $F \subset G$  and  $\tau$ ) and identify them along  $M = K(F, 1)$  (resp. along  $M = K(F, 1)$  and  $K(G, 1)$ ). Then  $M$  has a neighbourhood homeomorphic to  $M \times (-1, +1)$  such that  $M$  corresponds to  $M \times 0$ . Remark that thus transversality w.r.t.  $M$  is defined. Now one has the canonical homotopy class of (base point preserving) maps  $S \rightarrow X$ , both spaces having fundamental group  $\pi$ . Let  $f$  be a representative of this class.

CLAIM.  $f$  is homotopic to a map  $g: S \rightarrow X$ , transversal to  $M$ , so that  $g^{-1}(M) \cong S^1$  and  $g: g^{-1}(M) \rightarrow M$  is a homotopy equivalence.

In its proof we will not try to find a base point preserving homotopy. Therefore we will find a map  $g$  inducing an inner automorphism of  $\pi$ . By composing  $g$  with an appropriate diffeomorphism of  $S$  isotopic to the identity one can then find a map  $g$  in the canonical homotopy class. Moreover by arguments as in §3 one may deduce the theorem from the above claim.

We will make use of the following facts. Recall that a simple closed curve  $C$  in a surface  $V$  is *essential* if it does not bound a disc in  $V$ . We will say that  $C$  is *incompressible* if no proper power of  $C$  is zero in the fundamental group of  $V$ .

REMARK 1. ([1]). Let  $C$  be a simple closed curve in a surface. If  $C$  is null homotopic then  $C$  is the boundary of an embedded disc.

REMARK 2. (cf. [1, §2]). Let  $C_1$  and  $C_2$  be incompressible disjoint simple closed 2-sided curves in a surface. If some non-trivial powers of  $C_1$  and  $C_2$  are conjugate in the fundamental group, then  $C_1$  and  $C_2$  form the boundary of an embedded annulus.

REMARK 3. Let  $V$  be a surface and  $W \subset V$  be a closed connected codimension 0 submanifold with incompressible boundary components, then  $\pi_1 W \rightarrow \pi_1 V$  is injective.

REMARK. Let  $C$  be an essential simple closed curve in an orientable surface, then  $C$  represents an indivisible element in the fundamental group. The last remark can be deduced from intersection arguments in an appropriate covering of the surface.

## 2. THE PROOF OF THE CLAIM

Recall the map  $f: S \rightarrow X$ . Up to homotopy we may suppose that  $f$  is transverse to  $M$ . Consider  $T = f^{-1}(M)$ , then  $T$  is the disjoint union of finitely many simple closed two sided curves.

Suppose  $T$  has an inessential component. Using Remark 1, one can find an embedded disc  $D$  such that  $T \cap D = \partial D$ . It is now easy to find a homotopy of  $f$  with support in a neighbourhood  $U$  of  $D$  to a map  $g$  such that  $g(U) \cap M$  is empty ( $\pi_2(X) = 0$ ). This reduces the number of inessential components and this proves

LEMMA 1. *There is a homotopy of  $f$  to a map  $g$  such that  $g^{-1}(M)$  is the union of the essential components of  $f^{-1}(M)$ .*

For the further simplification we will use Stallings' notion of binding tie. Let  $C_1$  and  $C_2$  be different components of  $f^{-1}(M)$  and  $\gamma$  a path from  $C_1$  to  $C_2$  with interior disjoint from  $f^{-1}(M)$ . Then  $\gamma$  is a *binding tie* if  $f\gamma$  is homotopic in the complement of  $M$  and relative to its endpoints to a path in the collar of  $M$ . (Cf. [2]).

LEMMA 2. *If  $C_1$  and  $C_2$  are two essential components of  $f^{-1}(M)$  joined by a binding tie  $\gamma$ , there is a homotopy of  $f$  to a map  $g$  (transverse to  $M$ ) such that  $g^{-1}(M) = f^{-1}(M) - (C_1 \cup C_2)$ .*

PROOF. Let  $W^0$  be the component of  $S - f^{-1}(M)$  containing  $\gamma$  and let  $W = W^0 \cup C_1 \cup C_2$ . Since  $\gamma$  is a binding tie by Remarks 2 and 3,  $C_1$  and  $C_2$  bound an annulus in  $W$ , therefore  $W$  is an *annulus* itself. Now we can homotopy  $\gamma$  to an *embedded* binding tie. Embedded surgery methods enable us to find a homotopy of  $f$  with support in a neighbourhood of  $\gamma$  to a map  $g$  such that  $g^{-1}(M) = (f^{-1}(M) - (C_1 \cup C_2)) \cup C$  where  $C$  is the connected sum of  $C_1$  and  $C_2$  along  $\gamma$ . Since  $C$  bounds a disc it can be removed by Lemma 1. Of course Remark 2 implies that a priori any two essential components of  $f^{-1}(M)$  bound an annulus in  $M$ .

In the following section we will show that  $f^{-1}(M)$  cannot be empty and that whenever there are more than two essential components a binding tie can be found. The claim will then follow from Lemma 4.

## 3. FINDING A BINDING TIE

Recall the map  $f: S \rightarrow X$ . We may suppose that  $f^{-1}(M)$  consists of essential circles.

LEMMA 3.  $f^{-1}(M)$  is not empty and if  $f^{-1}(M)$  has more than 2 components and  $f: C \rightarrow M$  induces an isomorphism of fundamental groups for every component  $C$  of  $f^{-1}(M)$ , then there is a binding tie.

PROOF. Suppose that either  $f^{-1}(M)$  is empty or has more than 2 components but no binding tie. Then one may construct a non-trivial covering space  $p: \hat{X} \rightarrow X$  through which  $f$  factors. This would be absurd since then  $f_{\#}\pi_1 S$  would be contained in the proper subgroup  $p_{\#}\pi_1 \hat{X}$  of  $\pi_1 X$ .

The covering  $p$  can be taken as the union of a  $\pi_1 W$  covering of the component of  $X - M$  in which  $f(W)$  lies for every component  $W$  of  $S - f^{-1}(M)$  and of  $F'$  coverings of components of  $X - M$  where  $F' \subset F$  glued together above  $M$  in such a way that  $p$  is a covering and that  $f$  factors through  $p$ . If  $f^{-1}(M)$  is empty the degree of  $p$  is in case a at least  $\min_i ([G_i:F])$  and in case b it is infinite. If  $f^{-1}(M)$  is not empty the degree of  $p$  is at least the number of components of  $f^{-1}(M)$ .

LEMMA 4. Let  $C$  be an (essential) component of  $f^{-1}(M)$ , then  $f: C \rightarrow M$  induces an isomorphism of fundamental groups.

PROOF. If not, recall that the components of  $f^{-1}(M)$  are 2-sided and therefore orientation preserving. By Remark 4 the generator of  $F$  cannot be orientation preserving. Consider the orientation coverings  $p: \hat{S} \rightarrow S$ ,  $q: \hat{X} \rightarrow X$  and  $\hat{f}: \hat{S} \rightarrow \hat{X}$  of  $S$ ,  $X$  and  $f$ . Clearly  $\hat{M} = q^{-1}(M)$  is connected, and  $\hat{f}^{-1}(\hat{M}) = p^{-1}(f^{-1}(M))$  is a trivial degree 2 covering of  $f^{-1}(M)$ . Now the previous theory applies to  $\hat{f}: \hat{S} \rightarrow \hat{X}$  w.r.t.  $\hat{M}$ . By simplifying  $\hat{f}$  by removing pairs of components of  $\hat{f}^{-1}(\hat{M})$  one can find a homotopy of  $\hat{f}$  to a map  $g$  which does not meet  $\hat{M}$  at all. This contradicts Lemma 3.

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## A PRODUCT FORMULA FOR HIGHER ORDER COHOMOLOGY OPERATIONS

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### 1. INTRODUCTION AND NOTATION

It is the purpose of this note to exhibit a family of higher order cohomology operations that admit particularly "nice" Cartan-like formulae with a relatively low level of indeterminacy. That is, we shall define two  $q^{\text{th}}$  order cohomology operations,  $\Psi$  and  $\Omega$ , such that if  $\Omega(\chi)$  and  $\Omega(y)$  are defined then  $\Psi(\chi)$ ,  $\Psi(y)$  and  $\Psi(\chi \cup y)$  are defined and  $\Psi(\chi \cup y) = \Psi(\chi) \cdot y + \chi \cdot \Psi(y)$ , modulo the total indeterminacy. Let us begin by establishing some notation.

Throughout this paper,  $p$  shall denote some fixed prime number and  $F_p$  will be the category of homotopy CW complexes whose integral (co)homology is free of  $p$ -torsion. For the duration of this note, we shall not venture out of the confines of our category,  $F_p$ . By  $Q_p$ , we shall mean the subring of the rational numbers whose denominators are all relatively prime to  $p$ . Let  $m$  be the integer,  $(p-1)$ , and  $N$  some even natural number,  $2q$ .

In the notation of C.R.F. MAUNDER [4, 5], we shall give a chain complex of length  $2q$  such that  $\phi_*^{2q,q} = \Omega$  and  $\phi_*^{q,0} = \Psi$ . We shall, then, go on to prove the above mentioned Cartan-like formula for  $\Psi$ .

The author takes great pleasure in acknowledging the guidance and supervision of Prof. J.R. Hubbuck during the writing of [2] from which this note is excerpted.

### 2. THE $q^{\text{TH}}$ ORDER OPERATIONS

In this section we shall define the operation with which we will be working in terms of the chain complexes of C.R.F. MAUNDER [4]. We recall that "pyramids" of higher order operations may be associated with augmented chain complexes of the form:

$$(2.1) \quad C_N \xrightarrow{d_N} C_{N-1} \xrightarrow{d_{N-1}} \dots \rightarrow C_1 \xrightarrow{d_1} C_0 \xrightarrow{\epsilon} \text{HZ}_p^*(X),$$

where each  $C_n$  ( $0 \leq n \leq N$ ) is a locally finitely-generated free-graded left module over the mod  $p$  Steenrod algebra,  $A_p$ . The differentials,  $d_i$ , are all of degree zero and are such that  $d_i \circ d_{i+1} = 0$ , for all  $i$ ,  $1 \leq i \leq N-1$ . Associated with the complex (2.1) is an  $N^{\text{th}}$  order operation,  $\phi^{N,0}$ , which is defined on  $A_p$ -maps,  $\epsilon$ , such that  $\tilde{\phi}^{r,0}(\epsilon)$  is an equivalence class of maps  $C_r \rightarrow HZ_p^*(X)$  that includes the zero map, for all  $r$ ,  $0 < r < N$ . The equivalence relation used here is equality, modulo the indeterminacy of the operation. The image of  $\phi^{N,0}(\epsilon)$  is an equivalence class of maps  $C_N \rightarrow HZ_p^*(X)$ . The reader is encouraged to refer to [4] for a more detailed and thorough treatment of the construction of pyramids of higher order operations associated with such chain complexes.

To a given complex which defines a pyramid of operations, one may assign a dual complex, yielding higher order operations, Spanier-Whitehead dual to the original ones. In Theorem 4.3.1 of [4], MAUNDER defines such a complex, made up of locally finitely-generated free graded left  $A'_p$ -modules, where  $A'_p$  denotes the "anti"-Steenrod algebra (which agrees with  $A_p$  in  $F_p$ ). The dual chain complex to (2.1), for example, would take the form:

$$(2.2) \quad C_N^* \xleftarrow{d_N^*} C_{N-1}^* \xleftarrow{d_{N-1}^*} \dots \xleftarrow{d_1^*} C_0^* \xleftarrow{\epsilon^*} HZ_p^*(X).$$

Let us consider a particular example of a complex of the form (2.2). We denote by  $C^*(N,N)$  the complex defined in the following way. Each  $C_n^*$  is as in (2.2), for  $0 \leq n \leq N$ . For each such  $n$ ,  $C_n^*$  will be generated by the following set:  $\{c_n, c_{n,0}, c_{n,1}, \dots, c_{n,N-n}\}$  where the dimension of  $c_n$  is  $2Nm + N-n-1$  and that of  $c_{n,i}$  is  $N-n+2m(N-n-i)$ , for  $0 \leq i \leq N-n$ . We define the differentials of  $C^*(N,N)$  by:

$$(2.3.a) \quad d_n^*(c_n) = -\beta(c_{n+1}) + \sum_{i=n}^{N-1} p^{N-i+n}(c_{n+1, N-1-i})$$

and

$$(2.3.b) \quad d_n^*(c_{n,i}) = \beta(c_{n+1, i-1}) + \tau(c_{n+1, i}),$$

where  $\tau$  denotes the operation  $p^1\beta - \beta p^1$ .

**REMARK 2.4.** The chain complex,  $C^*(N,N)$ , is the dual of  $C(N,N)$  of [5], for a general prime,  $p$ .

Let us, now, define the two  $q^{\text{th}}$  order operations with which we will be working in terms of the chain complex, (2.2). We shall write  $\Omega$  for the operation  $\bar{\phi}_*^{N,q}$  and  $\Psi$  for  $\bar{\phi}_*^{q,0}$  where  $\bar{\phi}_*^{r,s}$  denotes an  $(r-s)^{\text{th}}$  order operation in the pyramid associated with the complex, (2.2). For the precise definition of  $\bar{\phi}_*^{r,s}$ , the reader is, once again, referred to [4].

Let  $\epsilon^*$  and  $\delta^*$  be two coaugmentations, that is to say maps from  $\text{HZ}_p^*(X)$  into  $C_0^*$ , for (2.2). These will be objects in the dual category corresponding to augmentations,  $\epsilon$  and  $\delta$ .

Heuristically speaking, a chain complex, in this sense, represents a "cohomologised" version of a Postnikov system. The augmentations, upon which higher order operations are defined, correspond to cohomology elements in  $\text{HZ}_p^*(X)$  which admit liftings in the appropriate Postnikov towers. With this in mind, we shall write  $\epsilon \cdot \delta$ , the product of two augmentations, for the augmentation corresponding to the cup product of the two elements,  $\chi$  and  $y$ , say, in  $\text{HZ}_p^*(X)$  which are represented, respectively, by  $\epsilon$  and  $\delta$ .

### 3. THE MAIN THEOREM

We are, now, in a position to state the main result of this paper:

**THEOREM 3.1.** *Let  $\epsilon^*$  and  $\delta^*$  be coaugmentations such that  $\Omega(\epsilon^*)$  and  $\Omega(\delta^*)$  are defined. Then,  $\Psi$  is defined on  $\epsilon^*$ ,  $\delta^*$  and  $\epsilon^* \cdot \delta^*$  and, modulo the total indeterminacy, one has:*

$$\Psi(\epsilon^* \cdot \delta^*) = \Psi(\epsilon^*) \cdot \delta^* + \epsilon^* \cdot \Psi(\delta^*).$$

In order to prove (3.1), we move to the context of [2]. It was shown, there, that  $\Omega$  and  $\Psi$  correspond to the  $q^{\text{th}}$  order operations,  $\bar{\phi}_N^{N,q}$  and  $\bar{\phi}_N^{q,0}$ , respectively. We indicate, briefly, how  $\bar{\phi}_q^{N,0}$  is defined, for any  $q' \geq N'$ .

For any  $X$  in  $F_p$ , one may identify the evenly graded  $\mathbb{Q}_p$ -cohomology of  $X$  with the associated graded group of the  $\mathbb{Q}_p$ -unitary  $K^0$ -theory of  $X$ . This follows directly from the fact that the differentials in the Atiyah-Hirzebruch spectral sequence are all torsion valued. Thus, using standard notation, we have, for any  $n \geq 0$ :

$$(3.2) \quad H^{2n}(X) \cong K_{2n}(X)/K_{2n+1}(X),$$

where zero- $\mathbb{Z}_2$  grading is understood for the  $K$ -theory and where all

coefficients are in  $\mathbb{Q}_p$  (we will continue to assume this, tacitly, in what follows, unless another coefficient group or K-theory grading is explicitly given). It is clear (see, for example, [3]) that one may define an isomorphism (indeed, several may be defined) of filtered  $\mathbb{Q}_p$ -modules:

$$(3.3) \quad J: H^{\text{ev}}(X) \xrightarrow{\cong} K(X)$$

such that  $J(H^{2n}(X)) \subseteq K_{2n}(X)$  and where the composition of  $J$  with the quotient map,  $I_{2n}: K_{2n}(X) \rightarrow K_{2n}(X)/K_{2n+1}(X)$ , is the identity map on  $H^{2n}(X)$ .

Let us denote by  $\text{ch}_t$ , the component of the Chern character in dimension  $2t$ . If  $u$  is an element of  $H^{2n}(X)$  and  $q'$  is any non-negative integer, then we may define a homomorphism of evenly graded  $\mathbb{Q}_p$ -cohomology groups by:

DEFINITION 3.4.

$$\theta_J^{q'} u = p^{q'} \text{ch}_{n+q', m} J(u) \in H^{2n+2q', m}(X).$$

The factor,  $p^{q'}$ , in (3.4) is the mod  $p$  component of the "Adams multiplier" ([1]) and it assures us that the image of  $\theta_J^{q'}$  lies in the image of the coefficient homomorphism:  $H^*(X) \rightarrow H\mathbb{Q}^*(X)$ . By definition of  $J$ , we have that  $\theta_J^0$  is the identity map. We define, now, another homomorphism,

$$\bar{\theta}_J^{q'}: H^{2n}(X) \rightarrow H^{2n+2q', m}(X),$$

by letting  $\bar{\theta}_J^{q'}$  be the formal inverse of  $\theta_J^{q'}$ . That is, for any  $q' \geq 0$ , we have:

$$(3.5) \quad \sum_{i=0}^{q'} \bar{\theta}_J^{q'-i} \theta_J^i = 0.$$

We, now, define an  $N'$ <sup>th</sup> order cohomology operation  $\bar{\theta}_q^{N', 0}$ , in terms of this homomorphism,  $\bar{\theta}_J^{q'}$ :

DEFINITION 3.6. Let  $J$  be as above and denote, by  $\rho'_*$ , the coefficient homomorphism,  $\rho'_*: H^* \rightarrow H\mathbb{Z}_p^*$ . Consider a  $\mathbb{Z}_p$ -cohomology class,  $\chi \in H\mathbb{Z}_p^{2n}(X)$  such that it is the  $\mathbb{Z}_p$  reduction of some  $\mathbb{Q}_p$ -class  $u \in H^{2n}(X)$ , that satisfies the following:

CONDITION \*:

$$\begin{aligned} \bar{\theta}_J^{q'} u &\equiv 0 \pmod{p^{N'-1}}, & \bar{\theta}_J^{q'-1} u &\equiv 0 \pmod{p^{N'-1}}, \\ \bar{\theta}_J^{q'-2} u &\equiv 0 \pmod{p^{N'-2}}, & \bar{\theta}_J^{q'-3} u &\equiv 0 \pmod{p^{N'-3}}, \\ \dots, \bar{\theta}_J^{q'-i} u &\equiv 0 \pmod{p^{N'-i}}, \dots, \bar{\theta}_J^{q'-(N'-1)} u &\equiv 0 \pmod{p}. \end{aligned}$$

We define an  $N'$ <sup>th</sup> order cohomology operation,

$$\bar{\theta}_{q'}^{N',0}: \text{HZZ}_p^{2n}(X) \rightarrow \text{HZZ}_p^{2n+2qm}(X)/Q,$$

such that  $\chi \rightarrow [\rho_*'(\bar{\theta}_J^{q'} u/p^{N'-1})]$ , where the outer square brackets indicate the equivalence class, modulo the indeterminacy  $Q$ ; this is generated by allowing  $J$  and  $u$  to range over all appropriate isomorphisms and  $Q_p$ -liftings of  $\chi$  that satisfy  $*$ .

REMARK 3.7. It can be shown, in fact, that the choice of  $J$  offers no real contribution to  $Q([2])$ . That is, all  $J$  may be thought of as equivalent with respect to determining the value of  $\bar{\theta}_{q'}^{N',0}$ .

Now, in order to prove (3.1), we shall have to introduce some new machinery which will allow us to determine the product behaviour of  $\Psi$ .

DEFINITION 3.8. Let  $M_i: H^{2n}(X) \times H^{2\ell}(X) \rightarrow H^{2n+2\ell+2im}(X)$  be the bilinear mapping defined by taking the component in dimension  $2n+2\ell+2im$  of  $(u,v) \rightarrow J^{-1}(J(u) \cdot J(v))$ , for  $i \geq 0$ .

It is clear that  $J^{-1}(J(u) \cdot J(y)) = \sum_{i \geq 0} M_i(u,v)$  and that  $M_0(u,v) = u \cup v$ . Now, using the fact that  $S_J^{q'}$  of [3] corresponds precisely to our homomorphism,  $\bar{\theta}_J^{q'}$  ([2]), we quote the following result of HUBBUCK (see 2.19 of [3]):

LEMMA 3.9. Let  $J$  be as above and let  $u$  and  $v$  be elements in  $H^{ev}(X)$ . Then, in  $KQ(X)$ , one has:

$$J\left\{ \sum_{q' \geq 0} \bar{\theta}_J^{q'}(u \cup v) \cdot p^{-q'} \right\} = J\left\{ \sum_{q' \geq 0} \bar{\theta}_J^{q'}(u) \cdot p^{-q'} \right\} \cdot J\left\{ \sum_{q' \geq 0} \bar{\theta}_J^{q'}(v) \cdot p^{-q'} \right\}.$$

Applying the definition (3.8) to (3.9) and restricting to dimension  $2q'm + \dim(u) + \dim(v)$ , one gets, in the above notation:

LEMMA 3.10.

$$\bar{\theta}_J^{q'}(u \cup v) = \sum_{r=0}^{q'} \sum_{i+j=q'-r} p^r M_r(\bar{\theta}_J^i(u), \bar{\theta}_J^j(v)).$$

We are, now, in position to proceed with

PROOF OF 3.1. Let us suppose that  $\chi$  and  $y$  are  $\mathbb{Z}_p$ -cohomology elements corresponding to  $\epsilon^*$  and  $\delta^*$ , respectively. The facts that  $\Omega(\epsilon^*)$  and  $\Omega(\delta^*)$  are defined imply that there exist isomorphisms,  $J$  and  $J'$ , as above, and  $\mathbb{Q}_p$ -liftings,  $u$  and  $v$ , of  $\chi$  and  $y$ , respectively, such that the following congruences are satisfied for the pairs  $(J, u)$  and  $(J', v)$ :

- (i)  $\bar{\theta}_J^{N-i}(u) \equiv \bar{\theta}_{J'}^{N-i}(v) \equiv 0(p^{q-1})$ ,  $0 \leq i \leq q+1$  and
- (ii)  $\bar{\theta}_J^{N-i}(u) \equiv \bar{\theta}_{J'}^{N-i}(v) \equiv 0(p^{N-i})$ ,  $q+2 \leq i \leq N-1$ .

By virtue of (3.7), we assume that  $J = J'$ . By (3.10), we may write, modulo  $p^q$ :

$$(3.11) \quad \bar{\theta}_J^N(u \cup v) = \sum_{i=0}^N \bar{\theta}_J^{N-i}(u) \cup \bar{\theta}_J^i(v) + \sum_{r=1}^{q-1} p^r \sum_{i=0}^{N-r} M_r(\bar{\theta}_J^{N-r-i}(u), \bar{\theta}_J^i(v)).$$

Since the pairs,  $(J, u)$  and  $(J, v)$  satisfy the congruences (i) and (ii), above, we may rewrite (3.11) as:

$$(3.12) \quad \bar{\theta}_J^N(u \cup v) = \bar{\theta}_J^N(u) \cup v + u \cup \bar{\theta}_J^N(v),$$

modulo  $p^q$ .

Moreover, by hypothesis, each individual summand on the right-hand side of (3.12) is divisible by  $p^{q-1}$ . The result now follows from (3.6) and the definition of  $\Psi$ .

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## MAPPINGS FROM PRODUCTS

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This contribution deals with factorizations of mappings from products and their applications, i.e. with the following situation: There is a continuous map  $f$  on a topological space  $X$  into another topological space  $Y$ ; if  $X$  is embedded into a product  $\prod\{X_i \mid i \in I\}$ , we look for small subsets  $J$  of  $I$  such that  $f$  depends on  $J$  (i.e., if  $x, y \in X$ ,  $\text{pr}_J x = \text{pr}_J y$ , then  $fx = fy$ ).

There are modifications of the above situation, for instance,  $f$  is less or more than continuous,  $X$  and  $Y$  may possess other structures, the factorized map may be requested to have a given property, etc.

In the first part of this contribution we shall look at topological conditions on the domain and the range of  $f$  which enables to factorize  $f$  (generalizations of results by Gleason, Comfort and Negrepointis, etc.). The last part shows some applications of the preceding results.

Now, we shall recall several concepts which will often be used in the sequel. For history and references see e.g. [HU<sub>1</sub>]. We shall suppose that all the spaces under consideration are completely regular although many results of this paper are valid for more general spaces (e.g. Hausdorff). If there is no other description of a product  $\prod X_i$  or letters  $\kappa, \alpha$ , we always mean that the index set of  $\prod X_i$  is  $I$  and that  $\kappa, \alpha$  are infinite cardinals. We shall say that a product  $\prod X_i$  is nontrivial if  $|I| \geq \omega$  and  $|X_i| \geq 2$  for all  $i \in I$ . By  $R(\prod A_i)$  for  $A_i \subset X_i$  we mean the set  $\{i \in I \mid A_i \neq X_i\}$ . A net  $\{x_a \mid a \in A\}$  converges to a set  $S$  in  $X$  if every neighbourhood  $U$  of  $S$  in  $X$  contains residually many  $x_a$ 's (i.e.,  $x_a \in U$  starting with some  $a_0$ ), it converges trivially to  $S$  if residually many  $x_a$ 's belong to  $S$ . A family  $\zeta$  of sets is said to be quasisdisjoint if  $\{S - \cap \zeta \mid S \in \zeta\}$  is disjoint; if  $\zeta$  is a family of finite sets and  $|\zeta|$  is regular uncountable, then there is a quasisdisjoint subfamily of  $\zeta$  with cardinality  $|\zeta|$ .

If  $Z$  is a subspace of  $X$ , then  $X$  is said to be pseudo- $(\kappa, \alpha)$ -compact with respect to  $Z$  if for every family  $\{U_\xi \mid \xi \in \kappa\}$  of non-void open sets in  $X$  there is a point in  $Z$  every neighbourhood of which intersects at least  $\alpha$  members

of  $\{U_\xi \mid \xi \in \kappa\}$ . We omit  $\alpha$  if  $\alpha = \omega$ , and the expression "with respect to  $Z$ " if  $Z = X$ . If, in the above definition, the sets  $U_\xi$  are replaced by points of  $X$ , we obtain the definition of  $(\kappa, \alpha)$ -compactness. A space  $X$  is said to have caliber  $(\kappa, \alpha)$  if for every family  $\{U_\xi \mid \xi \in \kappa\}$  of nonvoid open sets in  $X$  there is a subfamily  $\{U_\xi \mid \xi \in L\}$  with  $|L| = \alpha$  and  $\bigcap \{U_\xi \mid \xi \in L\} \neq \emptyset$ .

A subspace  $X$  of a product has the property  $V(\kappa)$  if every projection of  $X$  into a  $\alpha$ -fold subproduct,  $\alpha < \kappa$ , is a surjection; it is  $\kappa$ -invariant provided for every  $x, y \in X$  and  $J \subset I$  with  $|J| < \kappa$  there is a  $z \in X$  such that  $\text{pr}_J z = \text{pr}_J x$ ,  $\text{pr}_{I-J} z = \text{pr}_{I-J} y$ . For instance,  $\Sigma_\kappa$ -products have both the above properties ( $X$  is  $\Sigma_\kappa$ -product of  $X_i$  if, for some  $a \in \prod X_i$ ,  $|\{i \in I \mid \text{pr}_i x \neq \text{pr}_i a\}| < \kappa$  for every  $x \in X$ ).

#### 1. FACTORIZATIONS OF MAPPINGS

We shall start now with factorizations of mappings defined on the whole product.

**THEOREM 1.** [NU]. *The following conditions are equivalent for nontrivial  $\prod X_i$ ,  $|I| \geq \kappa$ :*

- (a)  $\prod X_i$  is pseudo- $\kappa$ -compact and pseudo- $\alpha$ -compact for some  $\alpha < \kappa$  in the case  $\text{cof } \kappa = \omega$ .
- (b) Every continuous real-valued function on  $\prod X_i$  depends on less than  $\kappa$  coordinates.

Thus pseudo- $\kappa$ -compactness is the largest property with respect to factorization of real-valued functions  $f$ . Of course, the range of  $f$  may be more general than reals. It was proved in [NU] that every space with  $\overline{G}_\delta$ -diagonal (i.e., the diagonal is an intersection of countably many of its closed neighbourhoods) may substitute reals in (b) (or spaces with  $\overline{G}_{<\kappa}$ -diagonal for regular  $\kappa$ ). We have proved, [HU<sub>1</sub>], that the class may still be enlarged to the class of spaces with weakly  $\kappa$ -inaccessible diagonal:

**DEFINITION.** A space  $X$  is said to have (weakly)  $\kappa$ -inaccessible diagonal, if for every family  $\{x_\xi \mid \xi \in \kappa\} \subset X \times X - \Delta_X$  and every regular  $\alpha$ ,  $\text{cof } \kappa \leq \alpha \leq \kappa$ , there is an open  $U \supset \Delta_X$  in  $X \times X$  and a cofinal  $S$  in  $\kappa$  with  $|S| = \alpha$  such that  $\{x_\xi \mid \xi \in S\} \cap U = \emptyset$  (or  $\{x_\xi \mid \xi \in S\} \cap \overline{U} = \emptyset$ , resp.).

Clearly, if  $\kappa$  is regular, then the above property is equivalent to the fact that only trivial well-ordered nets of length  $\kappa$  (weakly) converges to

the diagonal (or if  $M \subset X \times X - \Delta_X$ ,  $|M| = \kappa$ , then  $|M - U| = \kappa$  (or  $|M - \bar{U}| = \kappa$ , resp.) for an open  $U \supset \Delta_X$ ). See [HU<sub>2</sub>] for basic properties of spaces with  $\kappa$ -inaccessible diagonal. For instance, every subspace of a compact F-space (hence of extremally disconnected space) has weakly  $\omega$ -inaccessible diagonal. It is not known yet whether there are compact nonmetrizable spaces with  $\omega_1$ -inaccessible diagonal. It was proved in [BS], [S] that the spaces  $\beta(\kappa)$  have  $\omega_1$ -accessible diagonal.

**THEOREM 2.** [HU<sub>2</sub>]. *Each of the following conditions implies the next one:*

- (a) *Y has weakly  $\kappa$ -inaccessible diagonal.*
- (b) *Every continuous mapping on a pseudo- $\alpha$ -compact product ( $\alpha \leq \kappa$ ,  $\alpha$  regular) into Y depends on less than  $\kappa$  coordinates.*
- (c) *Y has  $\kappa$ -inaccessible diagonal.*

*If there is a base of closed neighbourhoods of the diagonal of Y, then all the three conditions are equivalent.*

An interesting corollary of Theorem 2 is the fact that every continuous mapping on a pseudocompact product into an extremally disconnected space depends on finitely many coordinates.

If we start with maps into  $2 = (0,1)$ , then we obtain almost the same results as above if we restrict our consideration to spaces projectively generated by 2, i.e. to zero-dimensional spaces. The only difference with Theorem 1 is the case  $\text{cof } \kappa = \omega$ :

**THEOREM 3.** [HU<sub>1</sub>]. *The condition (a) implies (b). If  $\prod X_i$  is nontrivial, zero-dimensional and  $|I| \geq \kappa$ , then (a) is equivalent to (b):*

- (a)  *$\prod X_i$  is pseudo- $\kappa$ -compact.*
- (b) *Every continuous function on  $\prod X_i$  into a discrete space depends on less than  $\kappa$  coordinates.*

We shall sketch the proofs: The proof that (b)  $\Rightarrow$  (a) in Theorems 1, 3 goes as follows. If  $\prod X_i$  is not pseudo- $\kappa$ -compact,  $\kappa \neq \omega$ ,  $\kappa$  regular, then a finite subproduct  $\prod \{X_i \mid i \in F\}$  is not pseudo- $\kappa$ -compact [NU], hence there are continuous  $g_\xi \in C(\prod \{X_i \mid i \in F\})$ , with discrete  $\{g_\xi^{-1}(R-(0)) \mid \xi \in \kappa\}$  and non-constant  $h_\xi \in C(X_\xi)$ ,  $\xi \in \kappa$ , for a  $\kappa \subset I - F$ . Then  $f x = \sum \{g_\xi(\text{pr}_F x) h_\xi(\text{pr}_\xi x) \mid \xi \in \kappa\}$  does not depend on less than  $\kappa$  coordinates. The same procedure can be used for the case  $\kappa = \omega$  if we know that a finite subproduct is not pseudocompact. If a zero-dimensional  $\prod X_i$  is not pseudocompact, then there is a continuous function on  $\prod X_i$  onto  $\omega$ , hence by (b),  $f$  factorizes via a

finite subproduct which cannot be pseudocompact (this process may be used also for the general case, instead of the above result from [NU], if we deal with mappings into metrizable spaces because pseudo- $\kappa$ -compactness can be characterized by mappings into metrizable hedgehogs; of course, the set  $F$  need not be finite then).

The proof of (b)  $\Rightarrow$  (c) in Theorem 2: If there is  $A = \{(a_\xi, b_\xi) \mid \xi \in \kappa\}$  in  $Y \times Y - \Delta_Y$  converging to  $\Delta_Y$ , we put  $X_{-1} = A \cup (\overline{A} \cap \Delta_Y)$  and define  $f: X_{-1} \times 2^\kappa \rightarrow Y$  to be equal at  $((y_1, y_2), \{h_\xi\})$  to  $y_1$  if  $y_1 = a_\xi$ ,  $h_\xi = 0$ , to  $y_2$  if  $y_1 = a_\xi$ ,  $h_\xi = 1$  or if  $y_1 = y_2$ . The  $f$  does not depend on less than  $\kappa$  coordinates. The implications (a)  $\Rightarrow$  (b) will follow from more general results in this paper.

## 2. X IS A PROPER SUBSET OF A PRODUCT

The case when  $X$  is a proper subset of a product is more complicated. Usually, the result depends on how  $X$  is embedded in the product:

**EXAMPLE 1.** For every nontrivial product  $\prod X_i$  and a space  $Y$  with  $|Y| \geq 2$  there is an  $X \subset \prod X_i$  and a continuous  $f: X \rightarrow Y$  such that  $f$  depends on no proper subset of  $I$ .

It suffices to suppose that  $X_i = Y = 2$  for all  $i$ . Every power  $2^I$  contains a discrete subspace  $X$  such that for each  $i \in I$  there are points  $x_i, y_i \in X$  with  $x_i \neq y_i$ ,  $\text{pr}_{I-(i)} x_i = \text{pr}_{I-(i)} y_i$  and  $(x_i, y_i) \cap (x_j, y_j) = \emptyset$  for  $i \neq j$ . Indeed, suppose that  $I$  is well-ordered without the last element and define the points  $x_i, y_i$  as follows:

$$\text{pr}_j x_i = \begin{cases} 0 & \text{if } j \neq i+1 \\ 1 & \text{if } j = i+1 \end{cases}$$

$$\text{pr}_j y_i = \begin{cases} 0 & \text{if } j \neq i, i+1 \\ 1 & \text{if } j = i, i+1. \end{cases}$$

The set  $X = \{x_i \mid i \in I\} \cup \{y_i \mid i \in I\}$  is discrete since the neighbourhood  $\text{pr}_i^{-1}(0) \cap \text{pr}_{i+1}^{-1}(1) \cap \text{pr}_{i+2}^{-1}(0)$  of  $x_i$  contains no other point of  $X$  and similarly the neighbourhood  $\text{pr}_i^{-1}(1) \cap \text{pr}_{i+1}^{-1}(1)$  of  $y_i$  contains no other point of  $X$ . Thus the map  $f$  that is 0 on all  $x_i$ 's and 1 on all  $y_i$ 's is continuous on  $X$  into  $Y$  and depends on no proper subset of  $I$ .

EXAMPLE 2. Take  $X$  to be  $2^{\omega_1}$  without two points  $p, q$ : then every continuous real-valued function on  $X$  depends on countably many coordinates (Theorem 5). Let  $Z$  be the one-point compactification of  $X$  and embed  $Z$  into  $2^{\omega_1}$ . There is a continuous real-valued function on  $X$  not depending on countably many coordinates in this last embedding. Indeed, take a function  $f$  on  $X$  which cannot be continuously extended onto  $Z$  - if  $f$  depends on countably many coordinates, it could be continuously extended onto the whole product because the image of  $X$  in a countable subproduct is compact ( $X$  is a pseudocompact) and the projection is quotient (no strictly finer topology of a compact metrizable one is pseudocompact).

The procedure from Example 2 can be generalized for every space which is not almost compact (has more than one compactification). Since every continuous mapping on an almost compact space is uniformly continuous, we get by [MI], [VI]:

THEOREM 4. *A space  $X$  is almost compact iff for every embedding of  $X$  into a product, every continuous mapping on  $X$  into a metrizable space depends on countably many coordinates.*

COROLLARY. *Every continuous mapping on an almost compact subspace of a product into a space with weight less than  $\kappa$  depends on less than  $\kappa$  coordinates.*

All the methods dealing with factorizations for general subspaces need some kind of openness of the subspace of the product. If we exclude trivial conditions (as e.g.  $X$  connected,  $Y$  disconnected or a projection  $X \rightarrow \text{pr}_j X$  is one-to-one) we shall see that  $X$  cannot contain certain discrete subspaces and  $Y$  cannot contain certain well-ordered nets. The next result is a direct generalization of the implication (a)  $\Rightarrow$  (b) in Theorem 1, [HU<sub>1</sub>] ([CN] for  $X$  dense in  $\prod X_i$ ).

THEOREM 5. *Suppose that a subspace  $X$  of a product  $\prod X_i$  is pseudo- $\kappa$ -compact with respect to  $X \cap \text{Int} \bar{X}$ , where  $\kappa$  is a regular cardinal. Then any continuous mapping on  $X$  into a space  $Y$ , the diagonal of which is intersection of less than  $\kappa$  of its closed neighbourhoods, depends on less than  $\kappa$  coordinates.*

PROOF. If an  $f \in C(X, Y)$  does not depend on less than  $\kappa$  coordinates, then we can construct transfinitely sets  $J_\xi \subset I$  and open canonical sets  $U_\xi, V_\xi$  in  $\prod X_i$  such that for all  $\xi \in \kappa$  and some open neighbourhood  $G$  of  $\Delta_Y$  in  $Y \times Y$

$$R(U_\xi) = R(V_\xi) \subset J_{\xi+1}, \quad J_\xi \subset J_\eta \text{ if } \xi \in \eta, \quad |J_\xi| < \kappa,$$

$$\text{pr}_{J_\xi} U_\xi = \text{pr}_{J_\xi} V_\xi, \quad f(X \cap U_\xi) \times f(X \cap V_\xi) \cap G = \emptyset.$$

Let  $p \in X \cap \text{Int } \bar{X}$  be an accumulation point of  $\{X \cap U_\xi \mid \xi \in \kappa\}$ ,  $W \subset \bar{X}$  an open canonical neighbourhood of  $p$  such that  $f(X \cap W) \times f(X \cap W) \subset G$ . Then there is a  $\xi \in \kappa$  such that  $W \cap U_\xi \neq \emptyset$ ,  $R(U_\xi) \cap R(W) \subset J_\xi$ . Consequently,  $W \cap V_\xi \neq \emptyset$ , hence  $X \cap W \cap V_\xi \neq \emptyset$  - a contradiction.

REMARKS.

- (1) The closure and interior in  $\text{Int } \bar{X}$  are always taken in the whole product  $\Pi X_i$ . The assumption on  $X$  means exactly that  $X \cap \text{Int } \bar{X}$  is pseudo- $\kappa$ -compact and  $\bar{X}$  is regularly closed (i.e.,  $X \subset \overline{\text{Int } \bar{X}}$ ).
- (2) The assumption on  $X$  coincides with pseudo- $\kappa$ -compactness of  $X$  if  $X \subset \text{Int } \bar{X}$ , i.e. if there is an open set  $G$  in the product such that  $X \subset G \subset \bar{X}$  (e.g., if  $X$  is open or dense in the product).
- (3) If  $\kappa$  is a singular cardinal, we obtain the same result if we suppose that  $X$  is pseudo- $\alpha$ -compact with respect to  $X \cap \text{Int } \bar{X}$  for a  $\alpha < \kappa$ .
- (4) The assumptions on  $X$  are satisfied if  $\text{d}X_i < \kappa$  and  $X$  is regularly closed.

We do not know whether Theorem 5 holds in the most general situation when  $Y$  has weakly  $\kappa$ -inaccessible diagonal. The next result shows that it almost holds. But first we present an important lemma (in the proof, we mean by continuous factorization the case when the factorized map is continuous):

LEMMA. In the category of regular spaces, for  $X \in \Pi X_i$ ,  $X \subset \text{Int } \bar{X}$ , we have

$$\text{pr}_J X = \lim_{\rightarrow} \{ \text{pr}_{I-F} X, \text{pr}_{I-F, I-F} \mid F, F' \text{ finite, } F, F' \subset I-J, F' \subset F \}.$$

PROOF. Take an  $f \in C(X, Y)$ ,  $Y$  regular, such that  $f$  depends continuously on every  $I-F$  for finite  $F \subset I-J$ . We must prove that  $f$  depends continuously on  $J$ . Take  $x, y \in X$  with  $\text{pr}_J x = \text{pr}_J y$  and assume that  $fx \neq fy$ . Thus  $f(X \cap U) \cap f(X \cap V) = \emptyset$  for some open canonical neighbourhoods  $U, V$  of  $x, y$  in  $\bar{X}$ . We may suppose that  $R(U) = R(V)$ ,  $\text{pr}_J U = \text{pr}_J V$ . Since  $f$  factorizes continuously via  $I-F$ , where  $F = R(U) - J$ , and  $\text{pr}_{I-F} x \in \overline{\text{pr}_{I-F}(X \cap V)}$ , we have got a contradiction. It remains to prove that the factorized map  $g$  on  $\text{pr}_J X$  is continuous. Assume that for some  $y \in X$ ,  $\{y_a \mid a \in A\} \subset X$  we have  $\text{pr}_J y_a \rightarrow \text{pr}_J y$ ,  $fy_a \not\rightarrow fy$ . Thus there is a canonical neighbourhood  $U$  of  $y$  in  $\bar{X}$  such that

$\text{pr}_J y_a \in \text{pr}_J U$  and  $fy_a \notin \overline{fU}$  for some  $a$ . If we put  $F = R(U) - J$ , then  $\text{pr}_{I-F} y_a \in \text{pr}_{I-F} U \subset \overline{\text{pr}_{I-F}(X \cap U)}$ , which contradicts the continuity of the factorized map on  $\text{pr}_{I-F} X$ .

REMARKS. For the above result it suffices to suppose less than  $X \subset \text{Int } \overline{X}$ , namely that for each  $x \in X$  there is a finite  $F \subset I - J$  and a neighbourhood  $U$  of  $x$  in the product such that  $\text{pr}_{I-F} V \subset \overline{\text{pr}_{I-F}(X \cap V)}$  for each cononical neighbourhood  $V$  of  $x$  in  $U$ . But the lemma is not true if we suppose only that  $X$  is regularly closed even in the case when  $X$  is compact:

EXAMPLE 3. Put  $X_0 = [0, 2]$ ,  $X_n = 2$  for  $n \in \omega - 0$ ,  $x_i \in \Pi X_n$ ,  $i = 1, 0$ , where  $\text{pr}_k x_i = 1$  for  $k = 0$ ,  $\text{pr}_k x_i = i$  for  $k > 0$ . Put  $X = A \cup B \cup (x_0, x_1)$ , where

$$A = \bigcup_n ([1 - 1/(n+1), 1 - 1/(n+2)] \times \{0 \mid k \leq n+1, k \neq 0\} \times \Pi\{x_k \mid k \geq n+2\}),$$

$$B = \bigcup_n ([1 + 1/(n+2), 1 + 1/(n+1)] \times \{1 \mid k \leq n+1, k \neq 0\} \times \Pi\{x_k \mid k \geq n+2\}).$$

Then  $X$  is compact, regularly closed in  $\Pi X_n$ , and the continuous mapping  $f = \text{pr}_1: X \rightarrow 2$  factorizes continuously via every  $\omega - F$ ,  $F$  finite,  $0 \notin F$ , but not via  $0$  (indeed, if  $\text{pr}_{\omega-F} x = \text{pr}_{\omega-F} y$ , then either  $x, y \in A$  or  $x, y \in B$  or  $x = y = x_i$  since  $\text{pr}_k x_0 \neq \text{pr}_k x_1$  for infinitely many  $k$ 's; clearly,  $fx_0 \neq fx_1$ ,  $\text{pr}_0 x_0 = \text{pr}_0 x_1$ ).

THEOREM 6. Let  $X$  be a pseudo- $\alpha$ -compact subspace of  $\Pi X_i$  with  $X \subset \text{Int } \overline{X}$ ,  $\alpha \leq \kappa$ ,  $\alpha$  regular, let  $Y$  have a weakly  $\kappa$ -inaccessible diagonal and let  $f \in C(X, Y)$ . If there is an  $A \subset I$  such that  $|A| \leq \kappa$  and  $F \cap A \neq \emptyset$  for every finite  $F \subset I$  with the property that  $f$  depends noncontinuously on  $I - F$ , then  $f$  depends on less than  $\kappa$  coordinates.

PROOF. There are two possibilities: either there is a disjoint family  $\{F_\xi \mid \xi \in \kappa\}$  of finite subsets of  $I$  such that  $f$  depends on no  $I - F_\xi$ , or there is a  $B \subset I - A$  with  $|B| < \kappa$  and such that  $f$  depends continuously on every  $I - F$ ,  $F$  finite,  $F \subset I - (A \cup B)$ . In this latter case,  $f$  depends continuously on  $A \cup B$  by the lemma, and we may suppose that  $I = \kappa$  in this case. If for each  $\xi \in \kappa$  there are points  $x_\xi, y_\xi$  in  $X$  such that  $\text{pr}_\xi x_\xi = \text{pr}_\xi y_\xi$ ,  $fx_\xi \neq fy_\xi$ , then there is a cofinal  $S$  in  $\kappa$  and an open  $G \supset \Delta_Y$  such that  $(fx_\xi, fy_\xi) \notin \overline{G}$  for  $\xi \in S$ . For  $\xi \in S$ , there are open canonical sets  $U_\xi, V_\xi$  in  $X$  such that  $x_\xi \in U_\xi, y_\xi \in V_\xi, \text{pr}_\xi U_\xi = \text{pr}_\xi V_\xi, R(U_\xi) = R(V_\xi), f(X \cap U_\xi) \times f(X \cap V_\xi) \cap G = \emptyset$ . If  $\alpha = |S| > \omega$ , there is a cofinal  $T$  in  $S$  such that  $\{R(U_\xi) \mid \xi \in T\}$  is

quasidisjoint with an intersection  $K$ . Let  $p$  be an accumulation point of  $\{U_\xi \mid T \ni \xi \supset K\}$  and  $W$  a canonical neighbourhood of  $p$  in  $\bar{X}$  such that  $f(X \cap W) \times f(X \cap W) \subset G$ . Then there is a  $\xi \in T$ ,  $\xi \supset K$  such that  $X \cap W \cap U_\xi \neq \emptyset$ ,  $R(W) \cap R(U_\xi) \subset K$ ; consequently,  $W \cap V_\xi \neq \emptyset$  and  $W \cap V_\xi \cap X \neq \emptyset$  - a contradiction. If  $\alpha = \omega$  and  $p$  is an accumulation point of  $\{U_\xi \mid \xi \in S\}$ ,  $W$  its canonical neighbourhood as above, then there is a  $\xi \in S$  such that  $X \cap W \cap U_\xi \neq \emptyset$ ,  $\xi \supset R(W)$ . Hence  $W \cap V_\xi \neq \emptyset$  - a contradiction.

It remains to establish the case where there is a cofinal  $S$  in  $\kappa$ ,  $|S| = \alpha$  and canonical open sets  $U_\xi, V_\xi$  in  $\bar{X}$  and an open  $G \supset \Delta_Y$  such that  $\text{pr}_{I-F_\xi} U_\xi = \text{pr}_{I-F_\xi} V_\xi$ ,  $R(U_\xi) = R(V_\xi)$ ,  $f(X \cap U_\xi) \times f(X \cap V_\xi) \cap G = \emptyset$ . If  $p$  is an accumulation point of  $\{U_\xi \mid \xi \in S\}$ ,  $W$  its canonical neighbourhood such that  $f(X \cap W) \times f(X \cap W) \subset G$ , then, for some  $\xi \in S$ ,  $W \cap U_\xi \neq \emptyset$ ,  $R(W) \cap F_\xi = \emptyset$ . Hence  $W \cap V_\xi \neq \emptyset$ , which is a contradiction.

**COROLLARY.** Let  $X$  be a pseudo- $\kappa$ -compact subspace of  $\Pi X_i$ ,  $\kappa$  regular,  $X \subset \text{Int } \bar{X}$ , and let  $Y$  have a weakly  $\kappa$ -inaccessible diagonal. Then in each of the following cases every  $f \in C(X, Y)$  depends on less than  $\kappa$  coordinates:

- (a)  $|I| = \kappa$ .
- (b)  $X \subset \text{Int } \text{Int } X$ .
- (c)  $X$  is  $\omega$ -invariant.

**PROOF.** In the case (a) we may put  $A = I$  in Theorem 6 and in the case (c)  $A = \emptyset$  (every factorization via a complement of finite set of  $f \in C(X, Y)$  is continuous). Suppose now that (b) holds and  $J \subset I$ . We shall prove that  $\text{pr}_J: X \rightarrow \text{pr}_J X$  is quotient in completely regular spaces. Let  $g: \text{pr}_J X \rightarrow R$  be such that  $g \circ \text{pr}_J$  is continuous and there are a net  $\{y_a\}$  converging to  $y$  in  $\text{pr}_J X$  and a neighbourhood  $V$  of  $fy$  in  $R$  missing all  $gy_a$ . Let  $x \in X$ ,  $\text{pr}_J x = y$ ,  $U$  a canonical neighbourhood of  $x$  with  $U \subset \text{Int } X$ ,  $(g \circ \text{pr}_J)(X \cap U) \subset V$ . There are  $y_a \in \text{pr}_J U$ ,  $x_a \in X$ , and a canonical neighbourhood  $W$  of  $x_a$  such that  $\text{pr}_J x_a = y_a$ ,  $(g \circ \text{pr}_J)(X \cap W) \cap (g \circ \text{pr}_J)(X \cap U) = \emptyset$ ,  $\text{pr}_J W \subset \text{pr}_J U$ . Since  $X \subset \text{Int } \bar{X}$  we find that  $\text{pr}_J(W \cap \text{Int } X)$  is nonempty open,  $\text{pr}_J(W \cap \text{Int } X) \subset \text{pr}_J U$  and, also,  $\text{pr}_J(U \cap \text{Int } X)$  dense in  $\text{pr}_J U$ . Thus  $\text{pr}_J(W \cap \text{Int } X) \cap \text{pr}_J(U \cap \text{Int } X) \neq \emptyset$  and there are points  $u \in X \cap W$ ,  $v \in X \cap U$  such that  $\text{pr}_J u = \text{pr}_J v$ ,  $g \text{pr}_J v \neq g \text{pr}_J u$  - a contradiction. Consequently, we may put again  $A = \emptyset$ .

In fact, we have proved that in (b) all the projections  $\text{pr}_J: X \rightarrow \text{pr}_J X$  are quotient in the category of regular spaces. The following easy examples show that the result is not true for nonregular spaces (thus  $\text{pr}_J: X \rightarrow \text{pr}_J X$



is not open in general) and that the assumption  $X \subset \overline{\text{Int Int } X}$  cannot be weakened to a similar form.

EXAMPLE 4.

- (i) Suppose  $X_0 = X_1 = [0,1]$ ,  $Y = [0,1]$  with the additional open set  $[0,1] - \{1/n \mid n=1,2,\dots\}$ ,  $X = \{(x,y) \mid \text{if } x = 1/n \text{ then } y = 1, \text{ if } x = 0 \text{ then } y = 0\}$ . Then  $f = \text{pr}_0: X \rightarrow Y$  factorizes via  $X_0$  noncontinuously.
- (ii) Suppose  $X_0 = [0,1]$ ,  $X_1 = 2$ ,  $X = \{(x,y) \mid \text{if } y = 0 \text{ then } x \text{ is rational, if } y = 1 \text{ then } x \text{ is irrational}\}$ ,  $Y = 2$ ,  $f(x,i) = i$  for  $(x,i) \in X$ . Then  $X \subset \overline{\text{Int } X}$  but the factorization via  $X_0$  is not continuous.
- (iii) Suppose  $X_0 = [0,\infty]$ ,  $X_1 = 2$ ,  $Y = 2$ ,  $X = \{(x,y) \mid \text{if } y = 0 \text{ then } x \in U\{[2n,2n+1] \mid n \in \omega\}, \text{ if } y = 1 \text{ then } x \in U\{[2n+1,2n+2] \mid n \in \omega\}\}$ ,  $f(x,i) = i$  for  $(x,i) \in X$ . Then  $X \subset \overline{\text{Int } X}$  but the factorization of  $f$  via  $X_0$  is not continuous.

REMARKS.  $X \subset \overline{\text{Int Int } X}$  iff  $X \subset \overline{\text{Int } X}$  and  $X \subset \overline{\text{Int } X}$  iff there is an open set  $G$  such that  $G \subset X \subset \overline{\text{Int } G}$ . This is true e.g. if  $X$  is open or contains an open dense set.

We know,  $[HU_3]$ , that if  $X$  has  $V(\kappa)$  then  $\text{pr}_J: X \rightarrow \text{pr}_J X$  are open provided  $|J| < \kappa$ . But almost the same example as (i) above ( $X_1 = [0,1]^{\omega_1}$ ,  $Y = X_1$  with the added open set  $Y - \{x_n\}$ , where  $\{x_n\}$  converges nontrivially to a point not in  $\{x_n\}$ ) shows that not all projections of  $X$  are quotient in Hausdorff spaces, and (ii) shows that not even in completely regular spaces

$$(X_0 = 2^{\omega_1}, X_1 = 2, X = \sum_{\omega} (0) \times (0) \cup \sum_{\omega} (1) \times (1)).$$

3. OTHER CONDITIONS ON  $X, Y$

Now, we shall look at other conditions on  $X, Y$ . Clearly, if we restrict the property for products, we may enlarge the class of ranges. We have used for products pseudo- $\kappa$ -compactness which is the most general property for our purpose if we consider real-valued functions. The corresponding class appeared to be the class of spaces with weakly  $\kappa$ -inaccessible diagonal. The pseudo- $\kappa$ -compactness means that there are no uniformly discrete subspaces of cardinality  $\kappa$ . We restrict this property by requiring that there is no closed discrete subspace of cardinality  $\kappa$ , which means  $\kappa$ -compactness. The next result can easily be deduced from results and remarks in  $[HU_2]$  and shows that the corresponding property for ranges in this new case is to

have  $\kappa$ -inaccessible diagonal. The proof is similar to that of Theorem 2.

THEOREM 7. *The following conditions on  $Y$  are equivalent:*

- (a) *Every  $\alpha$ -compact subspace of  $Y$  ( $\alpha \leq \kappa$ ,  $\alpha$  regular) has  $\kappa$ -inaccessible diagonal.*
- (b) *Every continuous mapping on a  $\alpha$ -compact product ( $\alpha \leq \kappa$ ,  $\alpha$  regular) into  $Y$  depends on less than  $\kappa$  coordinates.*

One can prove now similar assertions as before but for spaces which are  $\kappa$ -compact or have  $\kappa$ -inaccessible diagonal. We shall state only the following one:

THEOREM 8. *Let  $X$  be a  $\kappa$ -compact subspace of  $\prod X_i$ ,  $\kappa$  regular,  $X \subset \text{Int } \overline{X}$  and let  $Y$  have  $\kappa$ -inaccessible diagonal. Then every  $f \in C(X, Y)$  depends on less than  $\kappa$  coordinates provided one of the following conditions holds:*

- (a)  $X \subset \text{Int } \overline{\text{Int } X}$ .
- (b)  $X$  is  $\omega$ -invariant.

PROOF. Either there is a disjoint family  $\{L_\xi \mid \xi \in \kappa\}$  of finite subsets of  $I$  such that  $f$  depends on no  $I - L_\xi$  or there is a  $K \subset I$ ,  $|K| < \kappa$  such that for each finite  $L \subset I$  disjoint with  $K$ ,  $f$  depends on  $I - L$ . If the second case occurs and  $f$  depends continuously on those  $I - L$  (which are our cases (a), (b)), then by Lemma,  $f$  depends on  $K$  and we are ready. Suppose now that the first case occurs. For every  $\xi \in \kappa$  there are  $x_\xi, y_\xi \in X$  such that  $\text{pr}_{I - L_\xi} x_\xi = \text{pr}_{I - L_\xi} y_\xi$ ,  $fx_\xi \neq fy_\xi$ . There is an open neighbourhood  $G$  of  $\Delta_Y$  and a cofinal  $S$  in  $\kappa$  such that  $\{(fx_\xi, fy_\xi) \mid \xi \in S\} \cap G = \emptyset$ . There is an accumulation point  $p$  of  $\{x_\xi \mid \xi \in S\}$  and a canonical neighbourhood  $W$  of  $p$  such that  $f(X \cap W) \times f(X \cap W) \subset G$ . Choose a  $\xi \in S$  with  $R(W) \cap L_\xi = \emptyset$ ; consequently,  $y_\xi \in W$ , which is a contradiction.

REMARKS. Theorem 8 has corresponding formulation for singular  $\kappa$ . As for  $Y$ , one needs only suppose that every  $\kappa$ -compact subspace has  $\kappa$ -inaccessible diagonal. As for  $X$  one needs only suppose that certain projections are quotient.

In the second part of the proof no condition on how  $X$  is embedded in  $\prod X_i$  was needed. The next example shows that even in the case  $\kappa = \omega_1$ ,  $Y = 2$ , the general result without any condition on location of  $X$  in the product does not hold.

EXAMPLE 5. Put  $X = \omega_1 + \omega_1$  and embed it into  $2^{\omega_1}$  in such a way that  $|\bar{X} - X| = 1$ . Then  $X$  is countably compact and the map  $f: X \rightarrow 2$ , with  $f = 0$  on the first copy of  $\omega_1$  and 1 on the second one, does not depend on countably many coordinates.

We cannot construct a similar example for  $\kappa = \omega$ :

THEOREM 9. Let  $X$  be a  $(\alpha, \alpha)$ -compact subspace of a product  $\prod X_i$  for a regular  $\alpha \leq \kappa$ , where  $|I| = \kappa$ . Then every continuous mapping on  $X$  into a space with  $\kappa$ -inaccessible diagonal depends on less than  $\kappa$ -coordinates.

PROOF. If not, then for every  $\xi \in \kappa$  there are points  $x_\xi, y_\xi \in X$  such that  $\text{pr}_\xi x_\xi = \text{pr}_\xi y_\xi$  (we suppose that  $I = \kappa$ ),  $fx_\xi \neq fy_\xi$ . Then  $\{(fx_\xi, fy_\xi) \mid \xi \in S\} \cap G = \emptyset$  for some open  $G \supset \Delta_Y$ , cofinal  $S$  in  $\kappa$ ,  $|S| = \alpha$ . If  $p$  is a complete accumulation point of  $\{x_\xi \mid \xi \in S\}$ ,  $W$  its canonical neighbourhood such that  $f(X \cap W) \times f(X \cap W) \subset G$ , then we can find a  $\xi \in S$  such that  $x_\xi \in W$ ,  $R(W) \subset \xi$ , which is a contradiction because then  $y_\xi \in W$ .

#### 4. ANOTHER PROPERTY

Another property that is weaker than pseudo- $\kappa$ -compactness (and incomparable with  $\kappa$ -compactness) is to have caliber  $\kappa$ .

GLEASON proved (see [I]) that every  $f \in C(X, Y)$  depends on countably many coordinates provided  $X$  is an open subspace of a product of separable spaces and all the points of  $Y$  are  $G_\delta$ . MISCHENKO [MI] generalized this for open  $X$  with caliber  $\omega_1$  and HUSEK [HU<sub>3</sub>] for subspaces with  $V(\omega_1)$  in a product of separable spaces. The proofs can easily be modified for other cardinals. We shall show that the properties of  $X$  and  $Y$  may be requested to be more general without affecting the result. The most interesting case is, perhaps, again  $\kappa = \omega$ . We shall say that a space  $X$  has *pseudocaliber*  $\kappa$  if for every collection  $\mathcal{U}$  of  $\kappa$  open nonvoid sets in  $X$  there is a point  $p$ , a subcollection  $\{U_\xi \mid \xi \in \kappa\}$  of  $\mathcal{U}$  and  $u_\xi \in U_\xi$  such that  $\{u_\xi \mid \xi \in \kappa\}$  converges to  $p$ .

THEOREM 10. Let a product  $\prod X_i$  have pseudocaliber  $\kappa$ . If there is no nontrivial convergent well-ordered net of length  $\kappa$  in  $Y$ , then every  $f \in C(\prod X_i, Y)$  depends on less than  $\kappa$  coordinates.

PROOF. Suppose not. Then there are families  $\{J_\xi \mid \xi \in \kappa\}$  of subsets of  $I$ ,  $\{U_\xi \mid \xi \in \kappa\}$ ,  $\{V_\xi \mid \xi \in \kappa\}$  of canonical open sets in  $\prod X_i$  such that

$R(U_\xi) = R(V_\xi)$ ,  $J_\xi \cup R(U_\xi) \subset J_\eta$  for  $\xi \in \eta \in \kappa$ ,  $\text{pr}_{J_\xi} U_\xi = \text{pr}_{J_\xi} V_\xi$ ,  $f(U_\xi) \cap f(V_\xi) = \emptyset$ . By assumption, there are a point  $p$ ,  $S \subset \kappa$  with  $|S| = \kappa$  and  $u_\xi \in U_\xi$  for each  $\xi \in S$  such that  $\{u_\xi \mid \xi \in S\}$  converges to  $p$ . For  $\xi \in S$  let  $v_\xi \in V_\xi$  be a point with  $\text{pr}_{I-R(U_\xi)} v_\xi = \text{pr}_{I-R(V_\xi)} u_\xi$ . Then  $\{v_\xi \mid \xi \in S\}$  converges to  $p$  and, consequently, either  $\{fu_\xi \mid \xi \in S\}$  or  $\{fv_\xi \mid \xi \in S\}$  converges nontrivially to  $fp$  - a contradiction.

COROLLARY. Let  $X_i$  be sequentially compact spaces and suppose  $Y$  has no nontrivial convergent sequence. Then every  $f \in C(\prod X_i, Y)$  depends on finitely many coordinates.

THEOREM 11. If  $Y$  contains a nontrivial convergent sequence, then there is a product  $\prod X_i$  with pseudocaliber  $\omega$  and an  $f \in C(\prod X_i, Y)$  not depending on finitely many coordinates.

PROOF. Put  $X_{-1}$  to be a convergent sequence with its limit point, e.g.  $\omega + 1$ , and  $X_\xi = 2$  for  $\xi \in \omega + 1$ . Suppose that  $\{y_n \mid n \in \omega\}$  is a nontrivial sequence in  $Y$  converging to  $y_\omega$  and define  $f: \prod \{X_\xi \mid -1 \leq \xi \leq \omega\} \rightarrow 2$  as follows:

$$f\{x_\xi\} = y_{x_0} \quad \text{if} \quad x_{x_{-1}} = 0, \quad f\{x_\xi\} = y_\omega \quad \text{if} \quad x_{x_{-1}} = 1.$$

Then  $f \in C(\prod X_\xi, Y)$  and  $f$  does not depend on finitely many coordinates.

Clearly, Theorem 11 is true for other infinite cardinals without changing the proof and both the last theorems, with modified proofs, for some subspaces near to open ones. We shall not go into details in this case.

If the product has caliber  $\kappa$ , then the assumption on the range can be weakened. As examples we shall prove the next two results (the first result is, in fact, proved in [MI] by a different method). In the following,  $f \in C(\prod X_i, Y)$  depends at  $x$  on  $J$  if  $\text{pr}_J x = \text{pr}_J y$  implies  $fx = fy$ .

THEOREM 12. Let  $\prod X_i$  have caliber  $\kappa$  and let  $f \in C(\prod X_i, Y)$  depend on less than  $\kappa$  coordinates at each point of  $\prod X_i$ . Then  $f$  depends on less than  $\kappa$  coordinates.

PROOF. Suppose not, then we may construct  $\{J_\xi\}$ ,  $\{U_\xi\}$ ,  $\{V_\xi\}$  as in Theorem 10. If  $p \in \bigcap U_\xi$  and  $f$  depends at  $p$  on  $J$ ,  $|J| < \kappa$ , then there is an  $\eta \in \kappa$  such that  $J \cap \bigcup J_\xi = J \cap J_\eta$ . Consequently, there is a point  $z \in V_\eta$  with  $\text{pr}_J z = \text{pr}_J p$  - a contradiction.

THEOREM 13. *Suppose that  $M$  is metrizable,  $|M|$  nonmeasurable,  $\prod X_i$  countably compact. Then every  $f \in C(\prod X_i, \beta M)$  depends at each point of  $\prod X_i$  on countably many coordinates.*

PROOF. Choose an  $x \in \prod X_i$  and suppose that  $J = \{i \mid f y_i \neq f x \text{ for some } y_i \text{ with } \text{pr}_{I-(i)} y_i = \text{pr}_{I-(i)} x\}$  is uncountable. If  $f x \in M$ , then there is an infinite  $K \subset J$  and a neighbourhood  $U$  of  $f x$  such that  $f y_i \notin U$  for  $i \in K$  (since  $f x$  is  $G_\delta$  in  $\beta M$ ). If  $y$  is an accumulation point of  $\{y_i \mid i \in K\}$  then  $f y = f x$  because for every canonical neighbourhood  $G$  of  $y$  one can find  $i \in K - R(G)$  such that  $y_i \in G$ . Hence  $x \in G$ , but this is a contradiction. If  $f x \in \beta M - M$ , then there is again an infinite  $K \subset J$  and open  $U \ni f x$  such that  $f y_i \notin U$  because there is no nontrivial sequence in  $\beta M$  converging to  $f x$ .

COROLLARY. *If  $\prod X_i$  is countably compact and has caliber  $\omega_1$ , then every continuous  $f$  on  $\prod X_i$  into  $\beta M$ , where  $M$  is metrizable and  $|M|$  nonmeasurable, depends on countably many coordinates.*

#### 5. APPLICATIONS OF PRECEDING RESULTS

We shall look now at various applications of the preceding results (see [HU<sub>1,3</sub>]).

- (1) If  $X$  is a subspace with  $V(\omega_1)$  of a pseudo- $\omega_1$ -compact product  $\prod X_i$ , then  $X$  is  $C$ -embedded in the product (hence,  $\beta X = \prod \beta X_i$  provided  $\prod X_i$  is pseudo-compact or  $\forall X = \prod X_i$  provided all the  $X_i$  are realcompact).

If all the  $X_i$  are metrizable then  $X$  is  $C$ -embedded in  $\prod X_i$  whenever  $X$  is pseudocompact with respect to  $X \cap \text{Int } \bar{X}$ . The same result holds if all the  $X_i$  are metrizable separable and  $X$  is pseudocompact regularly closed.

The proofs of the above results go in the following way: every  $f \in C(X)$  depends on countably many coordinates, the factorized map is continuous on  $\text{pr}_J X$  and can be extended to  $\prod \{X_i \mid i \in J\}$ .

- (2) The assumption on pseudocompactness in the last result from (1) may be omitted. Indeed, if all the  $X_i$  are separable and  $X$  is regularly closed, then  $X$  depends as a set on a countable set  $J_1$  (the continuous map  $h: \prod X_i - (X - \text{Int } X) \rightarrow 2$ ,  $h x = 0$  if  $x \in \text{Int } X$ ,  $h x = 1$  if  $x \in \prod X_i - X$ , depends on such a set); if  $f \in C(X)$  then  $f$  depends on a countable set  $J_2$ , hence  $f$  depends continuously on  $J = J_1 \cup J_2$  and  $\text{pr}_J X$  is regularly closed in  $\prod \{X_i \mid i \in J\}$ . Thus every regularly closed subspace of a product of

metrizable separable spaces is C-embedded (a much stronger result without separability of  $X_1$  was proved in [SC] by a different method).

As a consequence of the proof we obtain the following result (recall from [SC] that a space is called k-normal if every regularly closed subset of X is  $C^*$ -embedded in X): A product of separable spaces is k-normal iff every countable subproduct is k-normal.

Clearly, the above mentioned results remain true if separability is replaced by a property ensuring pseudo- $\omega_1$ -compactness of open subspaces of the product. E.g., if  $\prod X_i$  has caliber  $\omega_1$ , A, B two disjoint regularly closed subsets, then  $\text{pr}_J A \cap \text{pr}_J B = \emptyset$  for a countable  $J \subset I$ .

(3) Every extremally disconnected continuous image of a product of sequentially compact spaces is finite (hence, every m-adic extremally disconnected compact space is finite). This result is a consequence of the Corollary to Theorem 11. In fact, one can prove more, namely that if Y is a continuous image of a product of sequentially compact spaces, then either Y is finite or contains a nontrivial convergent sequence.

(4) Let  $X_n$ ,  $n \in \omega$ , be finite compact F-spaces, A, B subsets of  $\prod X_n$  with  $|A| + |B| < 2^{2^\omega}$ . If A, B are not homeomorphic, then  $\prod X_n - A$ ,  $\prod X_n - B$  are not homeomorphic.

Indeed, if  $\prod X_n - A$  is homeomorphic to  $\prod X_n - B$ , then  $\prod X_n$  is the reflection of both spaces in the epireflective subcategory generated by  $\{X_n\}$ ; hence, A and B as remainders must also be homeomorphic. The fact about the reflection follows from Theorem 6 because under our conditions both spaces are pseudocompact, contain  $\Sigma_\omega$ -products and every continuous mapping on them into any  $X_n$  depends on finitely many coordinates, thus can be extended to the whole product.

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## SOME VERY SMALL CONTINUA

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### 0. INTRODUCTION

Given spaces  $X, Y$ , we shall say  $X \leq Y$  if  $Y$  embeds in a product of copies of  $X$ . This gives a preorder on the class of topological spaces. In what follows we shall pretend that " $\leq$ " is an order; it will be clear how to formalize what we say, but we feel that our informal approach is more perspicuous.

The properties of " $\leq$ " have been extensively studied (see [H], [P],[HP] for references). Our interest here is in a smaller class, the class of continua, and in particular the question of the existence of  $\leq$ -minimal continua and related questions.

In 1970 the first author asked whether or not the pseudoarc  $\mathbb{P}$  is minimal. We answer this question in the negative, but with a very nonmetrizable continuum. It is natural to ask whether  $\mathbb{P}$  is minimal among metric continua; we answer this question in the negative also.

CONVENTION 0.1. All given spaces are compact Hausdorff and have more than one point. In particular, continua are assumed to be nondegenerate.

DEFINITION 0.2.  $X \leq Y$  if  $C(X,Y)$  separates points of  $X$ .

Observe that for compact  $X$ , 0.2 is equivalent to the definition of  $\leq$  in the first paragraph above.

Clearly,  $\{0,1\} = 2 \leq X \leq [0,1]$  for all  $X$ ,  $X \leq 2$  iff  $X$  is zero-dimensional, and  $[0,1] \leq X$  iff  $X$  contains an arc.

### 1. THE MAIN RESULTS

Henceforth all given spaces are continua. Put  $\mathbb{H} = [0,\infty) \subseteq \mathbb{R}$ ,  $\mathbb{H}^* = \beta\mathbb{H} - \mathbb{H}$ .

THEOREM 1.1.  $\mathbb{H}^*$  is strictly smaller than any metric continuum.

This is the smallest of the continua promised in the title. The result is a corollary of

THEOREM 1.2. If  $f: \mathbb{H}^* \rightarrow K$  is nonconstant then  $\mathbb{H}^* \leq K$ .

A more general, and in one sense sharper, result on lower bounds is:

THEOREM 1.3. If  $\kappa$  is a cardinal, and  $A$  is a collection of at most  $\kappa$  continua, each of weight at most  $\kappa$ , then there is a continuum  $K$  of weight  $\kappa$  with  $K \leq H$  for each  $H \in A$ .

COROLLARY 1.3.1. Every set of continua is bounded below.

COROLLARY 1.3.2. If  $K_0$  and  $K_1$  are metric continua then there is  $K$  with  $K \leq K_0$  and  $K \leq K_1$ .

Since it is known [R] that there is a plane continuum incomparable with  $\mathbb{P}$ , 1.3.2 shows that  $\mathbb{P}$  is not minimal, indeed, that any minimally metric continuum is a minimum (i.e. a least metric continuum). We do not know whether such a beast exists.

## 2. PROOFS

PROOF OF THEOREM 1.1. By [AvEB] every metric continuum is the remainder in a compactification of  $\mathbb{H}$ , and hence an image of  $\mathbb{H}^*$ . Since every continuum has more than one point, the result follows from Theorem 1.2.  $\square$

PROOF OF THEOREM 1.2. For  $U$  open in  $\mathbb{H}$ , define

$$\hat{U} = \mathbb{H}^* - \text{cl}_{\beta\mathbb{H}}(\mathbb{H} - U).$$

We shall say that  $\langle U, V \rangle$  is an *alternating sequence of intervals* if

$$U = \bigcup_{n=1}^{\infty} \langle a_n, b_n \rangle, \quad V = \bigcup_{n=1}^{\infty} \langle c_n, d_n \rangle,$$

$a_n < b_n < c_n < d_n < a_{n+1}$  for each  $n$ , and  $\sup_{n < \omega} a_n = \infty$ .

The theorem follows at once from the following three observations:

- (a) If  $U, V$  are disjoint open sets of  $\mathbb{H}^*$ , then there is an alternating sequence  $\langle U, V \rangle$  of intervals with  $\hat{U} \subseteq U, \hat{V} \subseteq V$ .
- (b) If  $p, q$  are distinct points of  $\mathbb{H}^*$  then there is an alternating sequence  $\langle U, V \rangle$  of intervals with  $p \in \hat{U}, q \in \hat{V}$ .
- (c) All alternating sequences of intervals are the same; that is, for any two there is an autohomeomorphism of  $\mathbb{H}$  taking one to the other.  $\square$

The above may be neatly summarized by saying that  $\mathbb{H}^*$  is nearly homogeneous.

PROOF OF THEOREM 1.3. We leave the elementary verification of the following fact to the reader:

FACT. If  $H$  and  $K$  are continua,  $U$  and  $V$  are open subsets of  $H$  with  $\bar{U} \cap \bar{V} = \emptyset$ , and  $U'$  and  $V'$  are open proper subsets of  $K$  with  $U' \cup V' = K$ , then

$$H \times K - (U \times U' \cup V \times V')$$

is a continuum.

Fix  $\lambda \leq \kappa$  and  $A = \{K_\alpha : \alpha < \lambda\}$  as in the hypotheses of the theorem. Let  $j: \kappa \rightarrow \kappa^2 \times \lambda$  be a bijection such that if  $j(\alpha) = \langle \beta, \gamma, \delta \rangle$ , then  $\beta \leq \alpha$ ; we write  $j(\alpha) = \langle j_1(\alpha), j_2(\alpha), j_3(\alpha) \rangle$ . Fix, for each  $\alpha \leq \lambda$ , proper open subsets  $A_\alpha, B_\alpha$  of  $K_\alpha$  with  $A_\alpha \cup B_\alpha = K_\alpha$ .

We define inductively an inverse system  $\langle H_\alpha, f_{\alpha\beta}, \kappa \rangle$  of continua of weight at most  $\kappa$ . Given  $H_\alpha$ , let  $\{U_\alpha: \alpha < \kappa\}$  be an open basis for  $H_\alpha$  and let  $\{\langle V_\beta^\alpha, W_\beta^\alpha \rangle: \beta < \kappa\}$  enumerate the pairs of basic open sets of  $H_\alpha$  with disjoint closures.

- (a)  $H_0 = K_0$ ;
- (b) For  $\lambda$  a limit,  $H_\lambda = \lim_{\leftarrow} \langle H_\alpha, f_{\alpha\beta}, \lambda \rangle$ ;
- (c)  $H_{\alpha+1} = K_{j_3(\alpha)} \times H_\alpha - (A_{j_3(\alpha)} \times f_{\alpha j_1(\alpha)}^{-1}(V_{j_2(\alpha)}^{j_1(\alpha)}) \cup U_{j_3(\alpha)} \times f_{\alpha j_1(\alpha)}^{-1}(W_{j_2(\alpha)}^{j_1(\alpha)}))$ ,

and the  $f$ 's are defined as the restrictions of the appropriate projections.

We claim that  $K = \lim_{\leftarrow} \langle H_\alpha, f_{\alpha\beta}, \kappa \rangle$  satisfies the conclusion of the theorem. We need only show that there are point-separating maps from  $K$  into  $K_\alpha$  for  $\alpha < \lambda$ ; so fix  $\alpha < \lambda$  and distinct  $p, q \in X$ . For some  $\gamma < \kappa$ ,  $f_{\kappa\gamma}(p) \neq f_{\kappa\gamma}(q)$ , and so for some  $\xi < \kappa$ ,  $f_{\kappa\gamma}(p) \in V_\xi^\gamma$  and  $f_{\kappa\gamma}(q) \in W_\xi^\gamma$ . Let  $\eta = j^{-1}\langle \gamma, \xi, \alpha \rangle$ . Then

$$H_{\eta+1} = K_{\alpha} \times H_{\eta} - (A_{\alpha} \times f_{\eta\gamma}^{-1}(V_{\xi}^{\gamma}) \cup B_{\alpha} \times f_{\eta\gamma}^{-1}(W_{\xi}^{\gamma})),$$

and one sees directly that projection onto  $K_{\alpha}$  separates  $f_{\kappa, \eta+1}(p)$  from  $f_{\kappa, \eta+1}(q)$ .  $\square$

### 3. QUESTIONS AND REMARKS

As we have already noted we do not know whether there is a minimally metric continuum; one feels strongly however that there is not. One can ask also whether there is a minimal planar continuum.

We should note that one can easily construct, given a continuum  $K$ , a continuum  $H$  with  $H \not\leq K$ ; with Theorem 1.3 this shows that there are no minimal continua.

Let us also add the following information. If  $X$  is a hereditarily indecomposable (metric) continuum, then  $X \leq \mathbb{I}P$ ,  $[B_1]$ . There is also a hereditarily indecomposable continuum  $M_1$  in  $\mathbb{R}^3$  such that  $X \not\leq M_1$ , for every plane continuum  $X$ ,  $[C]$ ,  $[R]$ . Finally,  $\mathbb{I}H^*$  is indecomposable,  $[B_2]$ .

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## A FACTORIZATION THEOREM FOR GO-SPACES

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### ABSTRACT

A factorization theorem is given implying the possibility to factorize a continuous map  $f: X \rightarrow M$ , where  $X = (X, <, \tau)$  is a generalized ordered space and  $M$  is a metric space, such that  $f$  is equal to a composition  $X \xrightarrow{g} Z \rightarrow M$  of continuous maps, where  $Z = (Z, <, \bar{\tau})$  is a metric generalized ordered space and  $g: X \rightarrow Z$  is a monotonic map onto  $Z$ . In the case when  $X$  is a paracompact linearly ordered space, the space  $Z$  can be assumed to be linearly ordered.

### 1. PRELIMINARIES

Let  $X = (X, <)$  be a linearly ordered set. A subset  $C \subset X$  is called to be convex, whenever  $a, b \in C$  and  $a \leq b$  imply that  $\{x \in X: a \leq x \leq b\} \subset C$ . A convex set  $C \subset A$ ,  $A \subset X$ , is called a convexity-component of  $A$ , whenever  $C' \cap C \neq \emptyset$  implies  $C' \subset C$  for each convex set  $C' \subset A$ . Open intervals are sets of form

$$(a, b) = \{x \in X: a < x < b\}, \quad (\leftarrow, a) = \{x \in X: x < a\}, \quad (b, \rightarrow) \\ = \{x \in X: b < x\}.$$

A linearly ordered topological space  $X$  (abbreviated LOTS) is a triple  $X = (X, <, \lambda(<))$ , where  $(X, <)$  is a linearly ordered set on which a topology  $\lambda(<)$  is defined by the subbase of all sets  $(\leftarrow, a)$ ,  $(b, \rightarrow)$  with  $a, b \in X$ .

A generalized ordered space  $X$  (abbreviated GO-space) is a triple  $X = (X, <, \tau)$ , where  $\tau \supset \lambda(<)$  is a topology with a base consisting of convex subsets of  $X$ .

## 2. PARACOMPACTNESS OF GO-SPACES

The main result of this section is:

**THEOREM 1.** *If  $X$  is a generalized ordered space and if there exists a continuous map  $f: X \rightarrow M$  into a metric space  $M$  such that for each  $m \in M$ ,  $f^{-1}(m)$  is paracompact, then the space  $X$  is paracompact.*

The theorem is an easy consequence of two results: a theorem of ENGELKING and LUTZER [2]:

"A generalized ordered space  $X$  is not paracompact iff some closed subspace of  $X$  is homeomorphic to a stationary set in some regular uncountable cardinal  $\kappa$ ."

and the Pressing Down Lemma [4]:

"Suppose that  $S$  is a stationary subset in a regular uncountable cardinal number  $\kappa$  and let  $f: S \rightarrow \kappa$  be a function satisfying  $f(x) < x$  for each  $x \in S \setminus \{0\}$ . Then for some  $y \in \kappa$ ,  $f^{-1}(y)$  is a stationary subset of  $\kappa$ ."

A space  $X$  said to be  $m$ -space<sup>\*</sup> iff there exists a sequence  $\{P_n: n < \omega\}$  of open coverings of  $X$  such that for each  $x \in X$ ,  $\{x\} = \bigcap \{st(x, P_n): n < \omega\}$ .

The class of  $m$ -spaces contains all metric spaces.

The theorem will follow from a

**LEMMA.** *If  $f: S \rightarrow M$  is a continuous map, where  $S$  is a stationary set in some regular uncountable cardinal  $\kappa$  and  $M$  is a  $m$ -space, then there exists a  $\gamma < \kappa$  and an  $m \in M$  such that*

$$S \cap [\gamma, \kappa) \subset f^{-1}(m), \quad \text{i.e. } f|_{S \cap [\gamma, \kappa)}$$

*is constant.*

**Proof.** Let  $\{P_n: n < \omega\}$  be a sequence of open coverings of  $M$  such that for each  $x \in M$ ,  $\{x\} = \bigcap \{st(x, P_n): n < \omega\}$  and let  $S^d$  be the set of non-isolated points of  $S$ . The set  $S^d$  is also stationary in  $\kappa$ . For each  $n < \omega$  define a regressive function  $f_n: S^d \rightarrow S$  by putting  $f_n(x) \in S$  to be the first ordinal such that  $f_n(x) < x$  and  $[f_n(x), x] \cap S \subset f_n^{-1}[u]$  for some  $u \in P_n$ . Let us choose a  $y_n \in S$ ,  $n < \omega$ , such that  $f_n^{-1}(y_n)$  is stationary in  $\kappa$ . Choose  $\gamma < \kappa$  such that  $\gamma > y_n$  for each  $n < \omega$ . Then for each point  $x \in f_n^{-1}(y_n)$  ( $x > \gamma$ )

<sup>\*</sup>) usually a space with this property is said to have a  $G_\delta$ -diagonal



there exists a  $u \in P_n$  such that

$$[\gamma, x] \cap S \subset [y_n, x] \cap S \subset f^{-1}(u).$$

Since the set  $f^{-1}(y_n)$  is cofinal in  $\kappa$  we have

$$[\gamma, \kappa) \cap S \subset \text{st}(\gamma, f^{-1}(P_n)) = f^{-1}[\text{st}(f(\gamma), P_n)]$$

and hence

$$[\gamma, \kappa) \cap S \subset \bigcap_{n < \omega} f^{-1}[\text{st}(f(\gamma), P_n)] = f^{-1}[f(\gamma)].$$

COROLLARY (LUTZER [7]). *If  $f: S \rightarrow M$  is a continuous map, where  $S$  is a stationary set in some regular uncountable cardinal  $\kappa$  and  $M$  is a metric space ( $m$ -space), then  $|f[S]| < \kappa$ .*

PROOF. Let  $m_0 \in M$  and  $\gamma < \kappa$  be such that  $S \cap [\gamma, \kappa) \subset f^{-1}(m_0)$ . Then for each  $m \neq m_0$ ,  $f^{-1}(m) \cap S \subset [0, \gamma)$ . Hence  $|f[S]| \leq 1 + |\gamma| < \kappa$ .

PROOF OF THEOREM 1. Suppose that  $X$  is not paracompact. Then according to the Engelking-Lutzer theorem there is a stationary set  $S$  in some regular uncountable cardinal  $\kappa$  which is homeomorphic to a closed set in  $X$ . Assume that  $S \subset X$  and  $\text{cl}_X S = S$ . By the lemma there is an  $m \in M$  and there is a  $\gamma < \kappa$  such that  $S \cap [\gamma, \kappa) \subset f^{-1}(m)$ . But this implies that  $S \cap [\gamma, \kappa)$  is a paracompact space, because  $f^{-1}(m)$  is paracompact and  $S \cap [\gamma, \kappa)$  is closed, but this is impossible because  $S \cap [\gamma, \kappa)$  is not paracompact.

### 3. A FACTORIZATION THEOREM AND SOME OF ITS APPLICATIONS

THEOREM 2. *Let  $f: X \rightarrow M$  be a continuous map from a GO-space  $X = (X, <, \tau)$  into a metric space  $M$ . Then there exists a metric GO-space  $Z = (Z, <, T)$ ,  $\dim Z = 0$  whenever  $\dim X = 0$ , and continuous maps  $g: X \xrightarrow{\text{onto}} Z$ ,  $h: Z \rightarrow M$  such that  $f = h \circ g$  and  $g(x) \leq g(y)$  whenever  $x \leq y$ , for each  $x, y \in X$ . In addition, if the space  $X$  is a paracompact LOTS, the space  $Z$  is also a LOTS, i.e.  $T = \lambda(<)$ .*

We shall give some applications of Theorem 2 for metrizable and paracompactness of inverse images of metric spaces under continuous maps.

A continuous map  $f: X \rightarrow M$ , where  $X$  is a GO-space and  $M$  is a metric space, is said to be convexity-zero-dimensional (FABER [3]) if each convexity component of  $f^{-1}(m)$ ,  $m \in M$ , consists of a single point, and  $f$  is said to be convexity-paracompact if each convexity component of  $f^{-1}(m)$ ,  $m \in M$ , is paracompact.

COROLLARY 1. *If a GO-space  $X$  has a convexity-paracompact map into a metric space, then  $X$  is paracompact.*

COROLLARY 2. *If a GO-space  $X$  has a convexity-zero-dimensional map into a metric space, then  $X$  is paracompact and has a one-to-one continuous monotonic map onto a metric GO-space.*

COROLLARY 3. (FABER [3]). *If  $X$  is a LOTS and  $X$  has a convexity-zero-dimensional map into a metric space, then  $X$  is metrizable.*

PROOF OF THE COROLLARIES. Let  $f: X \rightarrow M$ ,  $X = (X, <, \tau)$  be a convexity-paracompact (convexity-zero-dimensional) map into a metric space  $M$ . According to Theorem 2 there exist a metrizable GO-space  $Z = (Z, <, \bar{\tau})$  and continuous maps  $g: X \xrightarrow{\text{onto}} Z$  and  $h: Z \rightarrow M$  such that  $f = h \circ g$  and  $g(x) \leq g(y)$  whenever  $x \leq y$ ,  $x, y \in X$ . Since  $g$  is a monotonic map,  $g^{-1}(z)$  is a convex set for each  $z \in Z$  and  $g^{-1}(z) \subset f^{-1}(h(z))$  because  $f = h \circ g$ . But each convex set in  $f^{-1}(h(z))$  is paracompact (a single point), hence  $g^{-1}(z)$  is paracompact for each  $z \in Z$ . In Section 2 it was proved that if  $g: X \rightarrow Z$  is a continuous map from a GO-space into a metric space such that  $g^{-1}(z)$  is paracompact for each  $z \in Z$ , then the space  $X$  is paracompact. Thus Corollary 1 is proved.

Now, if it is assumed that  $f$  is convexity-zero-dimensional then  $g^{-1}(z)$  consists of a single point. Then  $g: X \xrightarrow{\text{onto}} Z$  is a strictly monotonic continuous map from a paracompact GO-space  $X$  onto a metrizable GO-space  $Z$ . Thus Corollary 2 is proved.

If, in addition, it is assumed that  $f$  is convexity-zero-dimensional then  $g^{-1}(z)$  consists of a single point. Then  $g: X \xrightarrow{\text{onto}} Z$  is a strictly monotonic continuous map from a paracompact GO-space  $X$  onto a metrizable GO-space  $Z$ . Thus Corollary 2 is proved.

If, in addition, it is assumed that  $X$  is a LOTS, then it can be assumed that  $Z$  is also a LOTS, and then  $g: (X, <) \rightarrow (Z, <)$  being an isomorphism between the linearly ordered sets is a topological homeomorphism between the LOTS. Thus Corollary 3 is obvious.

QUESTION. Is in Theorem 2 the assumption on paracompactness of the LOTS  $X$  essential?

A family  $Q$  is a star-refinement of a family  $P$  if for each  $v \in Q$  there is a  $u \in P$  such that  $st(v, Q) \subset u$ , where  $st(v, Q) = \bigcup \{u \in Q : u \cap v \neq \emptyset\}$ .

LEMMA. Let  $\mathcal{P} = \{P_n : n < \omega\}$  be a family of open convex coverings of a GO-space  $X = (X, <, \tau)$  such that for each  $n < \omega$ ,  $P_{n+1}$  is a star-refinement of  $P_n$ . Put for each  $x, y \in X$ :  $[x] = \bigcap \{st(x, P_n) : n < \omega\}$  and  $[x] \leq [y]$  iff  $x \leq y$ .

The family  $X/P = \{[x] : x \in X\}$  is a closed partition of  $X$  into convex sets and  $X/P = (X/P, <, T)$  is a metrizable GO-space with a topology  $T = \{W \subset X/P : [x] \in W \text{ implies there is an } n < \omega \text{ such that } st([x], P_n^\#) \subset W\}$ , where  $P_n^\# = \{u^\# : u \in P_n\}$ ,  $u^\# = \{[x] \in u : x \in u\}$ . The map  $g : X \rightarrow X/P$ ,  $g(x) = [x]$ , is continuous and monotonic (i.e.  $x \leq y$  implies  $g(x) \leq g(y)$ ) and  $P_{n+1}$  is a refinement of  $g^{-1}[P_n^\#]$ ,  $n < \omega$ . Moreover, if each covering  $P_n$  consists of open intervals then  $T = \lambda(<)$  i.e.  $(X/P, <, T)$  is a LOTS.

PROOF. It was proved in [4] that: (1)  $X/P$  is a closed partition of  $X$ , (2)  $P_n^\#$  is a covering of  $X/P$ , (3)  $P_{n+1}^\#$  is a star-refinement of  $P_n^\#$ , (4)  $[x] \neq [y]$  implies  $[x] \notin st([y], P_n^\#)$  for some  $n < \omega$ , (5)  $P_{n+1}$  is a refinement of  $g^{-1}[P_n^\#]$ , where  $g(x) = [x]$ ,  $x \in X$ . The last condition implies that  $g$  is continuous.

According to the Alexandroff-Urysohn metrization theorem (see ENGELKING [1], p. 419) the conditions (1) - (4) imply that the space  $X/P$  with the topology  $T = \{W \subset X/P : [x] \in W \text{ implies there is a } n < \omega \text{ such that } st([x], P_n^\#) \subset W\}$  is metrizable.

Now, we shall show that  $(X/P, <, T)$  is a GO-space. First, let us notice that each set  $[x]$  is convex as an intersection of convex sets. Hence each set  $u^\#$ ,  $u \in P_n$ ,  $n < \omega$ , is convex, and hence each set  $st([x], P_n^\#)$  is convex. Now we shall show that each interval  $([x], [y])$  is open in the space  $(X/P, <, T)$ . Let  $[z] \in ([x], [y])$ . Then  $x < z < y$ . Since  $z \notin [x]$  and  $z \notin [y]$ , there exists a  $n < \omega$  such that  $z \notin st(x, P_n)$  and  $z \notin st(y, P_n)$ . This is equivalent to  $x, y \notin st(z, P_n)$ . From this and since  $st([z], P_n^\#)$  is convex and  $[x] < [y]$  we have  $st([z], P_n^\#) \subset ([x], [y])$ . Since  $st([z], P_n^\#)$  is convex and  $([x], [y]) \in T$  we have that  $int_T st([z], P_n^\#)$  is a convex set.

Assume now that each covering  $P_n \in \mathcal{P}$  consists of open intervals. We shall show that  $(X/P, <, T)$  is a LOTS i.e.  $\lambda(<) = T$ . Let  $W$  be a basic

neighbourhood of a point  $[x] \in X/P$ . It may be assumed that  $W = \text{st}([x], P_n^\#)$ . Let  $u \in P_n$  be such that  $\text{st}(x, P_{n+1}) \subset u$ . Then

$$[x] \in \text{int}_T \text{st}([x], P_n^\#) \subset \text{int}_T u^\# \subset u^\# \subset \text{st}([x], P_n^\#) \subset W,$$

i.e.  $[x] \in \text{int}_T u^\# \subset u^\# \subset W$ .

Assume that  $u = (\leftarrow, a)$  (for the cases  $u = (a, b)$  and  $u = (b, \rightarrow)$  the proof is similar). Then  $[x] \in (\leftarrow, [a])$ . We shall show that  $(\leftarrow, [a]) \subset u^\#$ . Suppose that there exists a point  $y < a$  such that  $[x] < [y] < [a]$  and  $(X \setminus u) \cap [y] \neq \emptyset$ . Since  $[y]$  is convex and  $u = (\leftarrow, a)$  we have  $a \in [y]$  but this is impossible because  $[a] \cap [y] = \emptyset$  since  $[y] < [a]$ .

PROOF OF THEOREM 2. Let  $f: X \rightarrow M$  be a continuous map between a GO-space  $X = (X, <, \tau)$  and a metric space  $M$ . Let  $\{Q_n: n < \omega\}$  be a family of open locally finite coverings of  $M$  such that each  $Q_n$  consists of sets of diameter  $< \frac{1}{n}$ . Then for each  $n < \omega$ ,  $f^{-1}[Q_n]$  is open locally finite covering of  $X$ . It is known that a space is normal iff each open locally finite covering has an open locally finite star-refinement. Since  $X$  is a normal space it is possible to choose a sequence  $P = \{P_n: n < \omega\}$  of open coverings of  $X$  consisting of convex sets (of open intervals whenever  $X$  is a paracompact LOTS) such that  $P_{n+1}$  is a star-refinement of  $P_n$  and  $P_n$  is a refinement of  $f^{-1}[Q_n]$  (and  $P_n$  is a disjoint family whenever  $\dim X = 0$ ).

Applying the lemma we obtain a metrizable GO-space  $Z = (X/P, <, T)$   $\dim Z = 0$  whenever  $\dim X = 0$ , see [5]) and a continuous map  $g: X \rightarrow Z$ ,  $g(x) = [x]$ , such that  $x \leq y$  implies  $g(x) \leq g(y)$ ,  $x, y \in X$ . Define  $h: Z \rightarrow M$ ,  $h([x]) = f(x)$ . The map  $h$  is continuous, because for each  $n < \omega$ ,  $P_{n+1}$  is a refinement of  $g^{-1}[P_n^\#]$  and of  $f^{-1}[Q_n]$ . It is clear that  $f = h \circ g$ . Thus, Theorem 2 is proved.  $\square$

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## SOME RESULTS ON CONTINUOUS SELECTIONS

E. Michael

The study of (continuous) selections deals with the following question: Given a set-valued map  $\phi: X \rightarrow 2^Y$  (where  $2^Y = \{S \subset Y: S \neq \emptyset\}$ ), under what conditions is there a *selection* for  $\phi$ , i.e. a continuous  $f: X \rightarrow Y$  such that  $f(x) \in \phi(x)$  for all  $x \in X$ . More generally, given a closed  $A \subset X$ , under what conditions can every selection for  $\phi|_A$  be extended to a selection for  $\phi$ , or at least to a selection for  $\phi|_U$  for some open  $U \supset A$ .

For technical reasons, we shall always assume that  $X$  is paracompact and that  $\phi$  is *lower semi-continuous*, or *l.s.c.* which means that  $\{x \in X: \phi(x) \cap V \neq \emptyset\}$  is open in  $X$  for every open  $V$  in  $Y$ .

The following are three relatively simple selection theorems; the first two were obtained in 1956 and the third, without proof, in 1974.

THEOREM 1. [1]. *Let  $X$  be paracompact,  $Y$  a Banach space, and  $\phi: X \rightarrow 2^Y$  l.s.c. such that  $\phi(x)$  is closed and convex for all  $x \in X$ . Then  $\phi$  has a selection.*

THEOREM 2. [1]. *Let  $X$  be paracompact with  $\dim X = 0$ ,  $Y$  a complete metric space, and  $\phi: X \rightarrow 2^Y$  l.s.c with  $\phi(x)$  closed in  $Y$  for all  $x \in X$ . Then  $\phi$  has a selection.*

THEOREM 3. [2]. *Let  $X$  be countable and regular,  $Y$  first-countable, and  $\phi: X \rightarrow 2^Y$  l.s.c. Then  $\phi$  has a selection.*

Observe that, in the preceding theorems, the hypotheses on  $X$  become progressively stronger while those on  $Y$  and the sets  $\phi(x)$  become weaker. By a suitable modification of their proofs, these theorems can now be combined into a single comprehensive - although somewhat complicated - result.

THEOREM 4. *Let  $X$  be paracompact,  $Y$  a Banach space,  $Z \subset X$  with  $\dim Z \leq 0$ ,  $C \subset X$  countable, and  $\phi: X \rightarrow 2^Y$  l.s.c such that  $\phi(x)$  is closed in  $Y$  for  $x \notin C$  and  $\overline{\phi(x)}$  is convex for  $x \notin Z$ . Then  $\phi$  has a selection.*

The conclusions of Theorems 1 - 4 can be strengthened a bit to assert that  $\phi$  has the *selection extension property*, or *SEP*, which means that, for all closed  $A \subset X$ , every selection for  $\phi|_A$  extends to a selection for  $\phi$ . This strengthened conclusion can be derived from the original one by a simple device: If  $g$  is a selection for  $\phi|_A$ , define  $\phi_g: X \rightarrow 2^Y$  by  $\phi_g(x) = \phi(x)$  if  $x \notin A$  and  $\phi_g(x) = \{g(x)\}$  if  $x \in A$ . It is not hard to check that this  $\phi_g$  is again l.s.c., hence  $\phi_g$  has a selection  $f$  by the original theorem, and this  $f$  is a selection for  $\phi$  which extends  $g$ .

Theorem 2 is merely the 0-dimensional version of a more complicated result, whose statement requires some more definitions: A space  $S$  is  $C^n$  (=  $n$ -connected) if every continuous image of an  $i$ -sphere ( $i \leq n$ ) in  $S$  is contractible in  $S$ . A collection  $S$  of subsets of a metric space  $Y$  is *uniformly equi-LC<sup>n</sup>* if to every  $\epsilon > 0$  corresponds a  $\delta > 0$  such that, for all  $S \in S$ , every continuous image of an  $i$ -sphere ( $i \leq n$ ) in  $S$  of diameter  $< \delta$  is contractible over a subset of  $S$  of diameter  $< \epsilon$ . Finally, we say that a map  $\phi: X \rightarrow 2^Y$  has the *selection neighbourhood extension property*, or *SNEP*, if for all closed  $A \subset X$ , every selection for  $\phi|_A$  extends to a selection for  $\phi|_U$  for some open  $U \supset A$ . We can now state the following generalization of Theorem 2.

**THEOREM 5.** [2, Theorem 1.2]. *Let  $X$  be paracompact with  $\dim X \leq n + 1$ ,  $Y$  a complete metric space, and  $\phi: X \rightarrow 2^Y$  l.s.c. with  $\phi(x)$  closed in  $Y$  for all  $x \in X$  and  $\{\phi(x): x \in X\}$  uniformly equi-LC<sup>n</sup>. Then  $\phi$  has the SNEP. If, moreover,  $\phi(x)$  is  $C^n$  for every  $x \in X$ , then  $\phi$  has the SEP.*

We can now generalize Theorem 4 by combining Theorem 5 with Theorems 1 and 3:

**THEOREM 6.** *Let  $X$  be paracompact,  $Y$  a Banach space,  $Z \subset X$  with  $\dim Z \leq n + 1$ ,  $C \subset X$  countable, and  $\phi: X \rightarrow 2^Y$  l.s.c. with  $\phi(x)$  closed in  $Y$  for  $x \notin C$ ,  $\overline{\phi(x)}$  convex for  $x \in Z$ , and  $\{\overline{\phi(x)}: x \in Z\}$  uniformly equi-LC<sup>n</sup>. Then  $\phi$  has the SNEP. If, moreover,  $\phi(x)$  is  $C^n$  for every  $x \in Z$ , then  $\phi$  has the SEP.*

We conclude this note with a result in which a local assumption yields a global conclusion. The map  $\phi$  is automatically l.s.c. in this result, since  $(\phi(x))^- = Y$  for all  $x \in X$ .

**THEOREM 7.** *Let  $X$  be paracompact with  $\dim X \leq n + 1$ ,  $Y$  an LC<sup>n</sup> complete metric space,  $C \subset X$  countable, and  $\phi: X \rightarrow 2^Y$  with  $\phi(x) = Y$  for  $x \notin C$  and*



$(\phi(x))^- = Y$  for  $x \in C$ . Then  $\phi$  has a selection.

Details regarding Theorems 4, 6 and 7 will be found in [4], [5] and [6].

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## A PROBLEM

E. Michael

PROBLEM. Suppose that  $f: X \rightarrow Y$  is a continuous map from a separable metric space  $X$  onto a compact metric space  $Y$ , that  $f^{-1}(y)$  is compact for all  $y \in Y$ , and that every countable compact subset of  $Y$  is the image of some compact subset of  $X$ . Must  $Y$  be the image of some compact subset of  $X$ ?

The following comments may be helpful

(1) I don't know the answer for even one uncountable compact metric space  $Y$  (even with  $X$  a subset of the plane).

(2) Without the hypothesis that each  $f^{-1}(y)$  is compact, the answer is "no": In [Duke Math. J. 26 (1959), 647-651; Example 4.1], I gave an example of a map  $f: X \rightarrow Y$ , where  $Y = I$ ,  $X$  is obtained from  $I \times I$  by removing one point from  $\{x\} \times I$  for each  $x \in I$ , and  $f$  is the projection onto the first coordinate, such that  $Y$  is not the image of any compact subset of  $X$ . (In [Duke Math. J. 36 (1969), 125-128], it is shown that  $X$  may even be chosen to be an  $F_\sigma$  in  $I \times I$ ). Since  $f$  is open, however, it follows from a result of H. Reiter [Math. Ann. 140 (1960), 417-421] (or E. Michael [Abstracts of Communications, International Congress of Mathematicians, Vancouver 1974]) that every countable subset of  $Y$  is the homeomorphic image of some subset of  $X$ .

(3) If it is only assumed that every convergent sequence in  $Y$  is the image of a compact  $C \subset X$ , then the answer is "no": Let  $f: X \rightarrow Y$  be as in (2) above. Let  $Z \subset X$  be obtained by removing a vertical open interval of length  $\frac{1}{4}$  about each point that was removed from  $I \times I$  to obtain  $X$ , and let  $g = f|_Z$ . Then  $g^{-1}(y)$  is compact for all  $y \in Y$ , and  $Y$  is not the image of any compact  $C \subset Z$ . But suppose  $x_n \rightarrow x_0$  is a convergent sequence in  $Y$ . Pick  $(x_0, s)$  and  $(x_0, t)$  in  $g^{-1}(x_0)$  such that  $|s-t| = \frac{1}{2}$ . Let  $K = \{x_n : n \geq 0\}$ , and let  $C = g^{-1}(K) \cap (K \times \{s, t\})$ . It is easy to check that  $C \subset Z$  is compact and  $g(C) = K$ .

(4) The problem is closely related to Question 1.9 in my paper [Illinois J. Math. 21 (1977), 716-733], but - unlike that question - is stated entirely in terms of simple, standard concepts. (A negative answer to our problem would imply a negative answer to Question 1.9; that follows from Theorem 6.5 (e), Remark 5.3, and Theorem 6.6 in the above paper.) It has recently been shown that the answer to Question 1.9 is affirmative if  $Y$  is a countable metric space, but that is no help in solving our problem.

A SIMPLE OBSERVATION CONCERNING THE EXISTENCE  
OF NON-LIMIT POINTS IN SMALL COMPACT F-SPACES

Jan van Mill

All spaces are completely regular and for all undefined terms we refer to [CN].

FRANKIEWICZ [F] has shown that under MA (a consequence of CH) in each compact extremally disconnected space  $X$  of weight  $2^\omega$  there is a point  $x \in X$  which is not a limit point of any countable discrete subset of  $X$ . For related results see [K<sub>1</sub>], [K<sub>2</sub>], [vM].

The aim of this note is to point out that under the stronger hypothesis CH a stronger result can quite easily be derived; apparently, this proof has been overlooked.

**THEOREM (CH):** *Let  $X$  be a compact F-space of weight  $2^\omega$ . Then there is an  $x \in X$  such that  $x \notin \bar{D}$  for each countable discrete  $D \subset X - \{x\}$ .*

**PROOF.** Striving for a contradiction, we assume that each point of  $X$  is a limit point of some countable discrete set. Let  $\{U_n : n \in \omega\}$  be a family of nonempty pairwise disjoint open  $F_\sigma$ 's of  $X$ . Define  $Y = \bigcap_{n < \omega} \bar{Y}_n$  and let  $\mathcal{B}$  be the collection of all nonempty open  $F_\sigma$ 's of  $Y$ . The family

$$E = \{\partial B : B \in \mathcal{B}\}$$

has clearly cardinality  $2^\omega$ . List  $E$  as  $\{E_\alpha : \alpha < \omega_1\}$  (by CH) and let  $\{F_\alpha : \alpha < \omega_1\}$  list the boundaries of the nonempty closed  $G_\delta$ 's of  $\bar{Y} - Y$ . Since each nonempty  $G_\delta$  in  $Y^* = \bar{Y} - Y$  has nonempty interior ([FG, 3.1]) by a straightforward induction ([R]) we can construct for each  $\alpha < \omega_1$  a nonempty open set  $V_\alpha$  of  $Y^*$  such that

- if  $\alpha < \kappa$  then  $\bar{V}_\kappa \subset V_\alpha$ ;
- $V_\alpha \cap (\bar{E}_\alpha \cup F_\alpha) = \emptyset$ .

(observe that  $\bar{E}_\alpha \cap Y^*$  is nowhere dense in  $Y^*$ , cf. [WO, 2.11]). Take  $x \in \bigcap_{\alpha < \omega_1} \bar{V}_\alpha$ . By assumption  $x \in \bar{D}$  for some countable discrete  $D \subset X - \{x\}$ . Since  $x$  is a P-point of  $Y^*$ , we may assume that  $D \cap Y^* = \emptyset$ . Moreover, since  $X$  is an F-space and  $D$  is countable, we may assume that  $D \cap (X - \bar{Y}) = \emptyset$ . List  $D$  as  $\{d_n : n < \omega\}$ . By assumption, each point of  $D$  is a limit point of some countable discrete set in  $X$ . Since  $D \subset Y$  and since  $Y$  contains a dense open  $F_\sigma$  of  $X$ , each point of  $D$  is a limit point of some countable discrete set in  $Y$ . Hence, there exists a family  $U = \{U_n^m : n, m < \omega\}$  of nonempty open  $F_\sigma$ 's of  $Y$  such that

$$\begin{aligned} & - d_n \in (U\{U_n^m : m < \omega\})^- - U\{U_n^m : m < \omega\}; \\ & - \text{if } k \neq n \text{ then } U\{U_k^m : m < \omega\} \cap U\{U_n^m : m < \omega\} = \emptyset. \end{aligned}$$

Put  $B = \bigcup_{n < \omega} \bigcup_{m < \omega} U_n^m$ . Then  $D \subset \partial B$ , and hence, by construction,  $x \notin \bar{D}$ ; a contradiction.  $\square$

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## ON SUPEREXTENSIONS AND HYPERSPACES

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### 0. INTRODUCTION

The *superextension*  $\lambda(X)$  (or  $\lambda X$ ) of a topological space  $X$  has been introduced by DE GROOT in [2]. Although its construction parallels the construction of Wallman compactifications, its properties are firmly distinct, and, in general,  $\lambda(X)$  is a much nicer space. For instance,  $\lambda(X)$  is a metric AR if (and only if)  $X$  is a metric continuum (cf. van MILL [4] or van de VEL [12]);  $\lambda(X)$  is a  $C^\infty$  and  $LC^\infty$  space if  $X$  satisfies certain weak assumptions, such as separability + path connectedness, or,  $\sigma$ -compactness + finite (homotopy) category (cf. van MILL & van de VEL [9]). Also  $\lambda(X)$  has the fixed point property if  $X$  is a connected normal  $T_1$ -space (cf. van de VEL [12]).

In all of these results, the *hyperspace*  $H(X)$  of a space  $X$  has been of invaluable help. The present paper is concerned with the relationship between the two kinds of topological extensions:  $\lambda$ ,  $H$ . We shall first prove that  $\lambda(X)$  is a subspace of  $H(H(X))$  for compact  $X$  (cf. Section 2). The proof of this nontrivial fact depends on the use of "compact" subbases, which were studied in van MILL & van de VEL [8]. With these techniques, we are able to derive more results at the time, e.g. that a certain "transversality" map in  $H(H(X))$  is continuous and that its fixed point set is exactly  $\lambda(X)$ . Also, we prove that a certain "convex closure operator" in  $H(H(X))$  is continuous. Finally, we use subbase convexity theory again to derive a retraction property of  $\lambda(X)$  in  $H(H(X))$ .

In view of the above facts, superextension theory can be looked upon as a kind of hyperspace theory. Both theories have also met with a same conjecture:  $H(X)$ , or  $\lambda(X)$ , is a Hilbert cube for suitable  $X$ . Concerning

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$H(X)$ , this conjecture has been settled in the affirmative by the work of CURTIS, SCHORI and WEST (cf. [1] and [11]). Concerning  $\lambda(X)$ , it has been proved by van MILL (cf. [4]) that  $\lambda[0,1]$  is a Hilbert cube, and (recently) that  $\lambda X$  is a Hilbert cube iff  $X$  is a nondegenerate metric continuum (cf. [7]). The proof of this result uses the above mentioned retraction property of  $\lambda(X)$  in  $H(H(X))$ .

### 1. COMPACT SUBBASES IN HYPERSPACES

The *hyperspace* of a  $T_1$  space  $X$  will be denoted by  $H(X)$ . If  $A_1, \dots, A_n$  are nonempty subsets of  $X$ , then we write

$$\langle A_1, \dots, A_n \rangle = \{D \in H(X) \mid D \subset \bigcup_{i=1}^n A_i \text{ and } D \cap A_i \neq \emptyset \text{ for each } i = 1, \dots, n\}.$$

With this notation, the family

$$H = H(X) = \{\langle C \rangle \mid C \in H(X)\} \cup \{\langle C, X \rangle \mid C \in H(X)\}$$

constitutes a closed subbase for  $H(X)$ .

If  $S$  is a closed subbase of  $X$ , then a nonempty subset  $C$  of  $X$  is called *S-convex* if  $C = \bigcap C$  for some  $C \subset S$ . We let  $H(X, S)$  denote the subspace of  $H(X)$ , consisting of all *S-convex* sets of  $X$ . We say that the closed subbase  $S$  is *compact* if; (i)  $H(X, S)$  is a normal  $T_1$  family, and; (ii) the space  $H(X, S)$  is compact.

Recall that a closed subbase  $S$  is *normal* if any two disjoint members of  $S$  can be separated by disjoint complements of members of  $S$ , and that  $S$  is  $T_1$  if for each  $S \in S$  and  $x \in X - S$  there is an  $S' \in S$  with  $x \in S' \subset X - S$ . See van MILL & van de VEL [8].

**THEOREM 1.1.** *Let  $X$  be compact  $T_1$ , and let  $S$  be a closed normal  $T_1$  subbase of  $X$  which is closed under formation of intersections. Then the following assertions are equivalent:*

- (a)  $S$  is a compact subbase;
- (b) the *S-convex* closure operator  $I_S: H(X) \rightarrow H(X, S)$  which sends  $C \in H(X)$  onto  $I_S(C) = \bigcap \{S \mid C \subset S \in S\}$ , is continuous;
- (c) the space  $H(X, S)$  admits a closed normal  $T_1$  subbase, consisting of all sets of type  $\langle C \rangle \cap H(X, S)$  or  $\langle C, X \rangle \cap H(X, S)$ , where  $C \in H(X, S)$ .



See [15], Theorem 2.6.

We now present a characterization of convexity in  $H(X)$ , relative to its canonical subbase  $H = H(X)$ . This result will be used to prove our basic result that  $H$  is actually a compact subbase for compact  $X$ .

Let  $A \subset H(X)$  be closed and nonempty, and let  $B \in H(X)$ . If  $B$  meets all members of  $A$ , then we call  $B$  a *transversal set of  $A$* . We let  $\perp(A)$  denote the collection of all transversal sets of  $A$ . With this notation, one can easily check the following formula on the convex closure operator  $I_H$ , related to the subbase  $H$  of  $H(X)$ :

$$I_H(A) = \bigcap \{ \langle B, X \rangle \mid B \in \perp(A) \} \cap \langle UA \rangle$$

**THEOREM 1.2.** *Let  $X$  be compact Hausdorff, and let  $A \subset H(X)$  be closed and nonempty. Then the following assertions are equivalent:*

- (i)  $A$  is  $H$ -convex;
- (ii) if  $B \in H(X)$  and if  $A \subset B \subset UA$  for some  $A \in A$ , then  $B \in A$ .

**PROOF.** Let  $A$  be  $H$ -convex, let  $B \in H(X)$ , and assume that  $A \subset B \subset UA$  for some  $A \in A$ . For each  $C \in \perp(A)$ , we have that  $C \cap A \neq \emptyset$ , and hence that  $C \cap B \neq \emptyset$ . Also,  $B \in \langle UA \rangle$ , whence  $B \in I_H(A) = A$  by the above formula.

Assume next that  $A$  satisfies condition (ii), and that there is a  $B \in I_H(A) - A$ . Then  $B \subset UA$ , and by (ii),  $\langle B \rangle \cap A = \emptyset$ .  $A$  being closed and  $\langle B \rangle$  being compact, there is an open set  $O \supset \langle B \rangle$  of  $H(X)$  of type

$$\bigcup_{k=1}^m \langle 0_1^k, \dots, 0_p^k \rangle, \quad 0_1^k \text{ open in } X,$$

which does not meet  $A$ . For each  $b \in B$  we put

$$0_b = \bigcap \{ 0_1^k \mid b \in 0_1^k, k = 1, \dots, m, l = 1, \dots, p \}.$$

In this way, we obtain but a finite number of different open sets of  $X$ , say  $0_1, \dots, 0_n$ . Writing  $I = \{1, \dots, n\}$ , we show that

$$\langle B \rangle \subset \bigcup \{ \langle 0_j \mid j \in J \mid \emptyset \neq J \subset I \} \subset O \quad (*)$$

In fact,

$$\langle 0_j \mid j \in J \rangle = \langle 0_{b_1}, \dots, 0_{b_r} \rangle$$

for some  $b_1, \dots, b_r \in B$ . Hence there is a  $k \in \{1, \dots, m\}$  such that

$$\{b_1, \dots, b_r\} \in \langle 0_1^k, \dots, 0_p^k \rangle.$$

Therefore, each  $0_j$  is contained in some  $0_1^k$ , and each  $0_1^k$  contains some  $0_j$ , whence

$$\langle 0_j \mid j \in J \rangle \subset \langle 0_1^k, \dots, 0_p^k \rangle.$$

The other half of (\*) is obvious, using  $B \subset \bigcup_{i=1}^n 0_i$ .

Let  $A \in \mathcal{A}$ . If  $A$  does not meet  $\bigcap_{i=1}^n X - 0_i$ , then  $A \subset \bigcup_{i=1}^n 0_i$ , and hence  $A \in \langle 0_j \mid j \in J \rangle$ , where  $J = \{i \mid A \cap 0_i \neq \emptyset\}$ , contradicting that  $A \cap \mathcal{O} = \emptyset$ . Hence  $\bigcap_{i \in I} X - 0_i$  is a transversal set of  $\mathcal{A}$  which does not meet  $B$ . This contradicts the fact that  $B$  is in  $I_H(\mathcal{A})$ .  $\square$

As a direct consequence of this theorem, it follows that  $\perp(A)$  is  $H$ -convex for each nonempty closed  $A \subset H(X)$ .

**THEOREM 1.3.** *Let  $X$  be compact Hausdorff. Then  $H = H(X)$  is a compact subbase of  $H(X)$ .*

**PROOF.** Let  $A \in \mathcal{H}(X)$  be nonconvex. Then by the previous theorem, there exists a  $B \in H(X)$  and an  $A_0 \in \mathcal{A}$  such that

$$A_0 \subset B \subset UA; \quad B \not\subset A.$$

Let  $\mathcal{O}, \mathcal{P}$  be disjoint open sets of  $H(X)$  such that  $B \in \mathcal{P}, A \subset \mathcal{O}$ . Then

$$B \in \langle 0_1, \dots, 0_n \rangle \subset \mathcal{P}$$

for some open sets  $0_1, \dots, 0_n$  of  $X$ . We assume that, among the latter,  $0_1, \dots, 0_p$  ( $p \leq n$ ) are all sets meeting  $A_0$ . Notice that  $p < n$ , and that  $A_0 \in \langle 0_1, \dots, 0_p \rangle$ . For each  $k$  with  $p < k \leq n$ , we choose  $b_k \in B \cap 0_k$ . As  $B \subset UA$ , there is an  $A_k \in \mathcal{A}$  with  $b_k \in A_k$ , and hence  $A_k \cap 0_k \neq \emptyset$ . Therefore,

$$V = \langle \mathcal{O} \rangle \cap \langle \langle 0_1, \dots, 0_p \rangle, \langle 0_{p+1}, X \rangle, \dots, \langle 0_n, X \rangle, H(X) \rangle$$

is a neighbourhood of  $A$  in  $\mathcal{H}(X)$ , no member of which is  $H$ -convex. In fact, if  $A' \in V$ , then there exist  $A'_0, A'_{p+1}, \dots, A'_n \in \mathcal{A}'$  such that

$$A'_0 \in \langle 0_1, \dots, 0_p \rangle; \quad A'_k \in \langle 0_k, X \rangle \quad \text{for } p < k \leq n.$$

Choose  $a'_k \in A'_k \cap 0_k$  for each  $p < k \leq n$ , and let  $B' = A'_0 \cup \{a'_{p+1}, \dots, a'_n\}$ .  
Then

$$A'_0 \subset B' \subset UA'; \quad B' \in \langle 0_1, \dots, 0_n \rangle \subset P; \quad A' \subset O,$$

whence  $B' \notin A'$ , and  $A'$  is not  $H$ -convex.

This shows that the space  $H(H(X), H)$  is compact, being a closed subspace of the compact space  $HH(X)$  (cf. MICHAEL [3]), and it remains to be verified that the family  $H(H(X), H)$  is normal and  $T_1$ :

Let  $A, B \subset H(X)$  be disjoint  $H$ -convex sets, say

$$A = \bigcap \{ \langle C, X \rangle \mid C \in \perp(A) \} \cap \langle A \rangle \quad (A = UA),$$

$$B = \bigcap \{ \langle D, X \rangle \mid D \in \perp(B) \} \cap \langle B \rangle \quad (B = UB).$$

Then  $A \cap B$  cannot meet all members of  $\perp(A) \cup \perp(B)$ , for otherwise  $A \cap B \in A \cap B$ . So e.g.  $A \cap B \cap C = \emptyset$ , where  $C \in \perp(A)$ .  $X$  being normal, there exist closed sets  $K, L$  in  $X$  with

$$A \cap C \subset K - L; \quad B \subset L - K; \quad K \cup L = X.$$

Hence,

$$A \subset \langle A \rangle \cap \langle C, X \rangle \subset \langle A \cap C, X \rangle \subset \langle K, X \rangle$$

$$B \subset \langle B \rangle \subset \langle L \rangle,$$

whereas  $A \cap \langle L \rangle = \emptyset$ ,  $B \cap \langle K, X \rangle = \emptyset$ , and  $\langle L \rangle \cup \langle K, X \rangle = H(X)$ . The  $T_1$ -property is obvious.  $\square$

Combining Theorems 1.1 and 1.3 yields:

COROLLARY 1.4. *Let  $X$  be compact Hausdorff. Then the convex closure operator*

$$I_H: HH(X) \rightarrow H(H(X), H)$$

*is continuous.*  $\square$

A *linked system* on a space  $X$  is a collection  $M \subset \mathcal{H}(X)$  such that any two members of  $M$  have a nonempty intersection. Equivalently,  $M \subset \perp(M)$ . A linked system  $M$  on  $X$  is *maximal* (or,  $M$  is an *mls*) if it is not properly contained in another linked system on  $X$ . The reader can verify that  $M$  is an mls iff  $M = \perp(M)$ .

COROLLARY 1.5. *Let  $X$  be compact Hausdorff. Then the transversality map  $\perp: \mathcal{H}(\mathcal{H}(X)) \rightarrow \mathcal{H}(\mathcal{H}(X))$  is continuous, and its fixed point set is exactly the collection  $\lambda(X)$  of all mls's on  $X$ .*

PROOF. As we noted before,  $\perp(A)$  is  $H$ -convex for each  $A \in \mathcal{H}\mathcal{H}(X)$ . Hence, the map  $\perp$  factors through the subspace  $\mathcal{H}(\mathcal{H}(X), H)$  of  $\mathcal{H}\mathcal{H}(X)$ . To prove continuity of  $\perp$ , it now suffices to use the closed subbase of  $\mathcal{H}(\mathcal{H}(X), H)$ , consisting of all sets of type  $\langle S \rangle$  or  $\langle S, H(X) \rangle$ , where  $S \subset \mathcal{H}(X)$  is  $H$ -convex (cf. Theorem 1.1(c)). For convenience, we write  $f = \perp$ , and we let

$$S = \cap \{ \langle B, X \rangle \mid B \in \perp(S) \} \cap \langle C \rangle \quad (S \neq \emptyset).$$

(i). *Computation of  $f^{-1}\langle S, H(X) \rangle$ .* Let  $A \in \mathcal{H}\mathcal{H}(X)$ . Then  $A \in f^{-1}\langle S, H(X) \rangle$  iff  $\perp(A) \cap S \neq \emptyset$ , iff for some  $B \in \perp(A)$ ,  $B \subset C$  and  $B$  meets all members of  $\perp(S)$ , iff  $C \in \perp(A)$ , iff  $A \subset \langle C, X \rangle$ . Hence:

$$f^{-1}\langle S, H(X) \rangle = \langle \langle C, X \rangle \rangle.$$

(ii) *Computation of  $f^{-1}\langle S \rangle$ .* Assume first that  $C \neq X$ . Then  $f^{-1}\langle S \rangle = \emptyset$ , since for each  $A \in f^{-1}\langle S \rangle$ ,  $X \in \perp(A) \subset S \subset \langle C \rangle$ , which is impossible. Assume now that  $C = X$ , and let  $\perp(A) \subset S$ . Then

$$\forall B \in \perp(S) \exists A \in \mathcal{A}: A \subset B \quad (*)$$

In fact, assume to the contrary that for some  $B \in \perp(S)$ ,  $A \cap (X - B) \neq \emptyset$  for all  $A \in \mathcal{A}$ . Fix  $a_A \in A - B$  for each  $A \in \mathcal{A}$ .  $X$  being regular, there exist disjoint open sets  $O_A, P_A$  of  $X$  with  $a_A \in O_A$  and  $B \subset P_A$ . By the compactness of  $\mathcal{A} \subset \mathcal{H}(X)$ , there exist  $A_1, \dots, A_n \in \mathcal{A}$  such that each  $A \in \mathcal{A}$  meets one of  $O_{A_1}, \dots, O_{A_n}$ . Let  $P = \bigcap_{i=1}^n P_{A_i}$ . Then each  $A \in \mathcal{A}$  meets the closed set  $X - P$ , whence  $X - P \in \perp(\mathcal{A})$ . However,  $B \cap (X - P) = \emptyset$ , contradicting that  $\perp(\mathcal{A}) \subset S \subset \langle B, X \rangle$ .

Conversely, if  $\mathcal{A} \in \mathcal{H}\mathcal{H}(X)$  satisfies (\*), then  $\perp(\mathcal{A}) \subset S$ . In fact, to each  $B \in \perp(S)$  we can assign an  $A \in \mathcal{A}$  with  $A \subset B$ . Hence, if  $D \in \perp(\mathcal{A})$ , then  $D$

meets each  $A \in \mathcal{A}$ , and hence it meets  $B$ , proving that

$$\perp(A) \subset \cap\{\langle B, X \rangle \mid B \in \perp(S)\} = S.$$

Using the formula (\*), it now follows that

$$f^{-1}\langle S \rangle = \cap\{\langle \langle B \rangle, H(X) \rangle \mid B \in \perp(S)\}.$$

In both cases (i) and (ii), we find that the inverse image is a closed set of  $HH(X)$ .  $\square$

## 2. SUPEREXTENSIONS

For a  $T_1$ -space  $X$ , the collection  $\lambda(X)$  of all maximal linked systems on  $X$  is given a topology, generated by the closed subbase

$$H(X)^+ = \{C^+ \mid C \in H(X)\},$$

where  $C^+ = \{M \in \lambda(X) \mid C \in M\}$ . With this topology,  $\lambda(X)$  is called *the super-extension of  $X$* . See VERBEEK [13] or van MILL [6] for details. Notice that  $\lambda(X)$  is compact.

The present section is mainly concerned with embedding and retraction properties of  $\lambda(X)$  in  $HH(X)$ .

**THEOREM 2.1.** *Let  $X$  be a compact Hausdorff space. Then  $\lambda(X)$  is a subspace of  $HH(X)$ .*

**PROOF.** As each  $M \in \lambda(X)$  is obviously a closed subfamily of  $H(X)$ , and satisfies  $M = \perp(M)$ , we find that  $M$  is  $H(X)$ -convex and hence that  $\lambda(X)$  is a subset of  $H(H(X), H)$ . We are again in a position to use the closed subbase of  $H(H(X), H)$  mentioned before, to prove that the inclusion mapping  $\lambda(X) \subset HH(X)$  is continuous. Let  $S$  be  $H$ -convex, say

$$S = \cap\{\langle B, X \rangle \mid B \in \perp(S)\} \cap \langle C \rangle.$$

(i)  $\langle S, H(X) \rangle \cap \lambda(X) = C^+$ :

In fact, as  $S \neq \emptyset$ , we have that  $C \cap B \neq \emptyset$  for each  $B \in \perp(S)$ . Therefore, an mls  $M$  is in  $\lambda(X) \cap \langle S, H(X) \rangle$  iff  $M \cap S \neq \emptyset$ , iff  $C \in M$ , iff  $M \in C^+$ .

(ii)  $\langle S \rangle \cap \lambda(X) = \emptyset$  if  $C \neq X$  and  $\langle S \rangle \cap \lambda(X) = \bigcap \{B^+ \mid B \in \perp(S)\}$  otherwise:  
 If  $C \neq X$ , then no mls  $M$  can satisfy  $M \subset S \subset \langle C \rangle$  since  $X \in M$ . Assuming  $C = X$  we have  $M \subset S$  iff for each  $B \in \perp(S)$  and for each  $M \in M$ ,  $B \cap M \neq \emptyset$ , iff  $\perp(S) \subset M$ , iff  $M \in \bigcap \{B^+ \mid B \in \perp(S)\}$ .  $\square$

Notice that the above computed traces on  $\lambda(X)$  are convex (or empty) relative to the canonical subbase of  $\lambda(X)$ .

A remarkable fact is that for metric compacta there is a direct proof of the above theorem without intervenience of compact subbases. Instead, we use the following metrizability result of VERBEEK [13]: if  $d$  is a metric on a compact space  $X$ , then the formula

$$\bar{d}(M, N) = \inf\{r \mid \forall M \in M: B_r(M) \in N\}$$

(where  $B_r(M) = \{x \mid d(x, M) \leq r\}$ ) defines a metric on  $\lambda(X)$ , compatible with its original topology. We notice that if  $X$  is compact metric, say with metric  $d$ , then  $HH(X)$  is metrized by the well-known Hausdorff metric, denoted by  $d_H$ .

We now prove the following result, adding some information to Theorem 2.1:

**THEOREM 2.2.** *Let  $(X, d)$  be a compact metric space. Then the inclusion mapping*

$$(\lambda(X), \bar{d}) \rightarrow (HH(X), (d_H)_H)$$

*is an isometry.*

**PROOF.** Let  $M, N \in \lambda(X)$  and let  $\bar{d}(M, N) = r$ . Hence, if  $N \in N$ , then  $B_r(N) \in M$  and consequently,  $d_H(N, M) \leq r$ . Similarly,  $d_H(M, N) \leq r$  for each  $M \in M$ , showing that  $(d_H)_H(M, N) \leq r$ .

Let  $s = (d_H)_H(M, N)$ . For each  $M \in M$  we can then find an  $N \in N$  such that  $d_H(M, N) \leq s$ , whence  $N \subset B_s(M)$  and  $B_s(M) \in N$ . Therefore,  $\bar{d}(M, N) \leq s$ .  $\square$

More information on the above (metric) embedding is presented in the next result.

Let  $L(X) \subset HH(X)$  denote the subspace of all closed linked systems on  $X$ . Then  $\lambda(X)$  is a subspace of  $L(X)$ . We now describe how to extend linked systems to maximal linked systems in a continuous way.

**THEOREM 2.3.** *Let  $X$  be a compact Hausdorff space. Then there is a continuous retraction*

$$h: L(X) \rightarrow \lambda(X)$$

*extending each linked system to a maximal linked system. If  $X$  is metrizable moreover, then  $h$  can be chosen such as to be a metric contraction.*

**PROOF.** Fix an  $x \in X$ . For each  $L \in L(X)$  we put

$$h'(L) = L \cup \{M \mid x \in M \in H(X) \text{ and } L \cup \{M\} \text{ is linked}\} \quad (*)$$

It has been proved in van MILL [5] that  $h'(L)$  is a linked system which is contained in a unique maximal linked system, which we denote by  $h(L)$ . This gives a mapping  $h: L(X) \rightarrow \lambda(X)$ , and we show that  $h$  has all the desired properties:

If  $T$  is a closed subbase of a space  $Y$ , then we let  $L(Y, T)$  denote the subspace of  $H(H(Y, T))$ , consisting of all closed linked systems  $L \subset H(Y, T)$ . With this notation, we have the following composition maps:

$$L(X) \xrightarrow{(\ )^+} L(\lambda(X), H(X)^+) \xrightarrow{\cap} H(\lambda(X), H(X)^+) \xrightarrow{p_x} \lambda(X) \quad (**)$$

The first map,  $(\ )^+$ , sends  $L \in L(X)$  ( $= (L(X, H(X)))$ ) onto

$$L^+ = \{L^+ \mid L \in L\},$$

where  $(\ )^+$  refers to the construction described at the beginning of this section. The second map is the *intersection operator*, sending  $M \in L(\lambda(X), H(X)^+)$  onto  $\cap M$ . It is easy to verify that  $\cap M \neq \emptyset$ . The third map is a restriction of the so-called *nearest point mapping* of  $\lambda(X)$ ,

$$p: \lambda(X) \times H(\lambda(X), H(X)^+) \rightarrow \lambda(X)$$

sending a pair  $(M, A)$  onto the unique point  $N \in \lambda(X)$  with the property that

$$I\{M, N\} \cap A = \{N\}.$$

(cf. van MILL & van de VEL [8]). In (\*\*),  $p_x$  denotes the map  $p(x, -)$  (regarding  $x \in X$  as a point of  $\lambda(X)$ ), and it has been proved in van de VEL [12]

that both constructions (\*) and (\*\*) coincide.

All mappings appearing in (\*\*) are continuous, see van MILL & van de VEL [8]. Hence  $h$  is continuous

Assume now that  $X$  is metrizable, say with a metric  $d$ . Using the induced metrics on the superextension  $\lambda(X)$  and on the various hyperspaces, we shall prove below that both  $\Omega$  and  $p_x$  are metric contractions. It remains to be verified that the first map,  $( )^+$ , is an isometry. But this is a straightforward consequence of the following elementary facts about  $\lambda(X)$ :

- (i)  $B_r(C)^+ = B_r(C^+)$  for each  $C \in H(X)$  and  $r \geq 0$ ;
- (ii)  $A \subset B$  iff  $A^+ \subset B^+$  for each  $A, B \in H(X)$ .

We now prove the contraction property of  $\Omega$  and  $p_x$  cited above. In order to simplify the argument, we give a proof which is valid for all spaces with a normal binary subbase, i.e. a closed normal subbase  $S$  such that for each linked system  $S' \subset S$  we have that  $\Omega S' \neq \emptyset$ .

As was shown in [8], there is also a *nearest point map*

$$p: X \times H(X, S) \rightarrow X$$

for such a subbase, satisfying a similar property as in the  $\lambda(X)$ -case, namely: for each  $x \in X$  and  $C \in H(X, S)$ ,  $I_S(x, p(x, C)) \cap C = \{p(x, C)\}$ , and  $p(x, C)$  is the unique point with this property.

In [10], a metric  $d$  on  $X$  (with a closed subbase  $S$ ) has been called  $S$ -convex provided that for each  $C \in H(X, S)$  and each  $r \geq 0$ ,  $B_r(C) \in H(X, S)$ . It is shown in [10] that the above mentioned metric  $\bar{d}$  on  $\lambda(X)$  is  $H(X)^+$ -convex, and that each metrizable space with a normal binary subbase  $S$  admits an  $S$ -convex metric.

**LEMMA.** *Let  $S$  be a normal binary subbase for  $X$  and let  $d$  be an  $S$ -convex metric on  $X$ . Then the intersection operator  $\Omega: L(X, S) \rightarrow H(X, S)$  is a metric contraction with respect to the metrics on  $L(X, S)$  and  $H(X, S)$  which are induced by  $d$ .*

**PROOF.** We first show that for each (nonempty) linked system  $A \in H(X, S)$  and for each  $r \geq 0$  the equality

$$B_r(\Omega A) = \Omega \{B_r(A) \mid A \in A\} \quad (*)$$

holds. The inclusion " $\subset$ " being obvious, take a point  $x$  in the right hand side of (\*). Then  $B_r(x)$  meets each  $A \in A$ , and since  $B_r(x)$  is  $S$ -convex, we



find that

$$B_r(x) \cap \Omega A \neq \emptyset$$

by the binarity of  $S$ . Hence  $x \in B_r(\Omega A)$ .

Now take  $L_1, L_2 \in L(X, S)$  such that  $(d_H)_H(L_1, L_2) \leq r$ . Then

$$\forall L_1 \in L_1 \exists L_2 \in L_2: d_H(L_1, L_2) \leq r$$

$$\forall L_2 \in L_2 \exists L_1 \in L_1: d_H(L_2, L_1) \leq r$$

and hence it easily follows that  $B_r(\Omega L_1) = \Omega \{B_r(L_1) \mid L_1 \in L_1\} \supset \Omega L_2$  by the formula (\*). Similarly  $B_r(\Omega L_2) \supset \Omega L_1$ , which proves that  $d_H(\Omega L_1, \Omega L_2) \leq r$ .  $\square$

The formula (\*) is also applied in the proof of the next result:

**LEMMA.** *Let  $S$  be a normal binary subbase for  $X$  and let  $d$  be an  $S$ -convex metric on  $X$ . Then for each  $x \in X$  the nearest point map*

$$p(x, -): H(X, S) \rightarrow X$$

*is a metric contraction.*

**PROOF.** Let  $A, B \in H(X, S)$  and assume that  $d_H(A, B) \leq r$ . Writing  $x_A = p(x, A)$  and  $x_B = p(x, B)$ , we show that  $d(x_A, x_B) \leq r$ . Indeed, since  $A \subset B_r(B)$ ,

$$\emptyset \neq B_r(x_A) \cap B \subset B_r(I_S(x, x_A)) \cap B;$$

whence by the construction of  $p$  (cf. the above remarks),  $x_B \in B_r(I_S(x, x_A))$ .

On the other hand  $B \subset B_r(A)$ , and consequently

$$x_B \in B_r(A) \cap B_r(I_S(x, x_A)) = B_r(A \cap I_S(x, x_A)) = B_r(x_A),$$

using formula (\*) and the construction of  $p$ .  $\square$

It has been proved in [10] that the nearest point map  $p$  is a metric contraction in the first variable too, and that  $p(x, A)$  is also metrically a nearest point of  $A$  with regard to  $x$ .

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## WALLMAN COMPACTIFICATIONS AND THE CONTINUUM HYPOTHESIS

J. van Mill, J. Vermeer

### 0. INTRODUCTION

All spaces are completely regular. In [10] UL'JANOV constructed a variety of compactifications which are not Wallman compactifications. In addition, combining these results with those of BANDT [1] he showed the following interesting theorem:

- (\*) CH is equivalent to the statement that every compactification of a separable space is a Wallman compactification.

Consequently, by applying constructions of ŠAPIRO [7] or STEINER & STEINER [9] it follows that under  $\neg$ CH there is a compactification  $\gamma\mathbb{N}$  of  $\mathbb{N}$  which is not a Wallman compactification. Since under CH every compactification of  $\mathbb{N}$  is a Wallman compactification (by (\*)) we even have that:

- (\*\*) CH is equivalent to the statement that every compactification of  $\mathbb{N}$  is a Wallman compactification.

Also, there is a theorem of HAGER [4] which states:

- (\*\*\*) Every compactification of a pseudo-compact space is a Wallman compactification.

At first glance, (\*\*) and (\*\*\*) do not give us any information concerning non pseudo-compact spaces.

In this note we will show that (\*\*) and (\*\*\*) imply the following two theorems.

THEOREM 1. CH is equivalent to the statement that there is a non pseudo-compact space all compactifications of which are Wallman compactifications.

THEOREM 2. [ $\neg$ CH]. For any space X the following assertions are equivalent

- (i) X is pseudo-compact

(ii) All compactifications of  $X$  are Wallman compactifications.

These two theorems imply that there is no honest (= not requiring additional set theoretic axioms) example of a non pseudo-compact space every compactification of which is a Wallman compactification.

## 1. THE THEOREMS

Recall that a Wallman compactification of a space  $X$  is a compactification  $\gamma X$  which has a closed base  $\mathcal{B}$  satisfying the following two conditions:

- (a)  $\mathcal{B}$  is closed under finite intersections and finite unions.
- (b) for all  $B \in \mathcal{B}$  we have that  $B = \text{cl}_{\gamma X}(B \cap X)$ .

(this can easily be derived from a theorem in STEINER [8]).

All our results follow from the following proposition, which is of independent interest.

**PROPOSITION 1.1.** *Let  $X$  be any space every compactification of which is a Wallman compactification. Let  $B$  be a closed subspace of  $X$ . If one of the following conditions is satisfied,*

- (i)  $X$  is normal
- (ii)  $B$  is a  $C$ -embedded copy of  $\mathbb{N}$  in  $X$ ,

*then every compactification of  $B$  is a Wallman-compactification.*

**PROOF.** Let  $\gamma B$  be any compactification of  $B$ .

Note that the closure operator in  $\beta X$  has the following two properties:

- (a) In both cases  $B$  is  $C^*$ -embedded in  $X$ , so  $\text{cl}_{\beta X} B = \beta B$ ,
- (b) If  $T$  is a closed subset of  $X$  such that  $T \cap B = \emptyset$ , then  $\text{cl}_{\beta X} B \cap \text{cl}_{\beta X} T = \emptyset$ .
  - When  $X$  is normal, this is clear;
  - When  $B$  is a  $C$ -embedded copy of  $\mathbb{N}$ , it follows from [3] (GILLMAN & JERISON, page 51 3L).

Let  $f: \beta B \rightarrow \gamma B$  be the unique map which extends the identity on  $B$ . Define

$$Z := \gamma B \cup (\beta X - \beta B)$$

Let  $\xi: \beta X \rightarrow Z$  be defined by

$$\begin{cases} \xi(x) = x & (x \in \beta X \setminus \beta B) \\ \xi(x) = f(x) & (x \in \beta B) \end{cases}$$

It is clear that  $Z$  supplied with the quotient-topology is a (Hausdorff)-compactification of  $X$ , say  $Z = \gamma_0 X$ , such that  $cl_Z B = \gamma B$ . By assumption,  $Z$  is a Wallman compactification of  $X$ . Let  $\mathcal{T}$  be a closed base for  $Z$  such that  $\mathcal{T}$  is closed under finite unions and finite intersections, while in addition  $cl_Z(T \cap X) = T$  for all  $T \in \mathcal{T}$ . Define

$$F = \{T \cap \gamma B \mid T \in \mathcal{T}\}.$$

It is clear that  $F$  is a closed base for  $\gamma B$  which is closed under finite unions and finite intersections. We claim that:

$$cl_{\gamma B}(F \cap B) = F$$

for all  $F \in \mathcal{F}$ , which suffices to prove the proposition.

Indeed, take  $F \in \mathcal{F}$ , say  $F = T \cap \gamma B$  and assume there is a point  $x$  such that  $x \in F - cl_{\gamma B}(F \cap B)$ . Since  $\mathcal{T}$  is a closed base for  $\gamma_0 X$  we may take  $T_0 \in \mathcal{T}$  such that  $x \in T_0$  and  $T_0 \cap cl_{\gamma B}(F \cap B) = \emptyset$ . Define  $T_1 = T \cap T_0$ .

Then

$$T_1 \cap B = T \cap T_0 \cap B = T_0 \cap F \cap B = \emptyset.$$

So, (b) implies that  $cl_{\beta X}(T_1 \cap X) \cap cl_{\beta X} B = cl_{\beta X}(T_1 \cap X) \cap \beta B = \emptyset$ . Therefore,  $cl_Z(T_1 \cap X) \cap \gamma B \subset \xi[cl_{\beta X}(T_1 \cap X)] \cap \xi[\beta B] = \emptyset$ . But this is a contradiction, since

$$x \in F \cap T_0 \cap \gamma B = T \cap T_0 \cap \gamma B = T_1 \cap \gamma B = cl_Z(T_1 \cap X) \cap \gamma B.$$

We conclude that  $\gamma B$  is a Wallman compactification.  $\square$

From this proposition the two theorems are immediately clear.

## 2. REMARKS

Recall that a compactification  $\gamma X$  of  $X$  is called a GA compactification provided that there is a closed subbase  $\mathcal{T}$  for  $\gamma X$  such that:

- ( $\alpha$ ) for each  $x \in \gamma X$  and  $T \in \mathcal{T}$  such that  $x \notin T$  there is a  $T_0 \in \mathcal{T}$  with  $x \in T_0$  and  $T_0 \cap T = \emptyset$ .

- ( $\beta$ ) for all disjoint  $T_0, T_1 \in \mathcal{T}$  there is a finite cover  $M$  of  $\gamma X$  by elements of  $\mathcal{T}$  such that each  $M \in M$  meets at most one of  $T_0$  and  $T_1$ .
- ( $\gamma$ ) for all  $T_0, T_1 \in \mathcal{T}$  with  $T_0 \cap T_1 \neq \emptyset$  we have that  $T_0 \cap T_1 \cap X \neq \emptyset$ .

(cf. van MILL [5]). In [5] and [6] it has been shown that if  $\gamma X$  is a compactification of  $X$  of weight at most  $2^{\omega}$  then  $\gamma X$  is a GA compactification. As remarked in the introduction,  $\neg CH$  implies that there is a compactification of  $\mathbb{N}$  which is not a Wallman compactification. Hence there is a consistent example of a GA compactification which is not a Wallman compactification. Whether there is a real example of such a compactification is unknown. In addition, it is unknown whether every compactification is a GA compactification.

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## WHEN COLORADO IS HOMEOMORPHIC TO UTAH

Charles F. Mills

In 1969, Maurice asked whether there is a compact order-homogeneous linearly ordered space (LOTS)  $X \neq I$  such that  $X^2 \simeq X^2 - (\alpha, 1]^2$  for some (equivalently, every)  $\alpha$  with  $0 < \alpha < 1$ . Here  $I$  is the closed unit interval, 0 and 1 are the endpoints of  $X$ , and in an order-homogeneous LOTS any two nondegenerate closed intervals are order-isomorphic.

In this note we show that the answer to Maurice's question is "no"; in fact, we prove a stronger theorem;

THEOREM. *If  $X$  is a compact connected LOTS with no separable intervals, then the only autohomeomorphisms of  $X^2$  have the form  $\phi \times \psi$ , possibly followed by a reflection  $\langle x, y \rangle \rightarrow \langle y, x \rangle$ , where  $\phi, \psi: X \simeq X$ .*

In other words, all autohomeomorphisms of  $X^2$  are "affine maps". To see that this answers Maurice's question we note that by conjugating the given homeomorphism between Colorado ( $\square$ ) and Utah ( $\sqcup$ ) with  $\text{id} \times \phi$  for some  $\phi \in \text{Aut } X$  we find a "non-affine" member of  $\text{Aut}(X^2)$ , that is, one violating the conclusion of the theorem. The astute reader will notice, however, that our proof of the theorem can be applied directly to Maurice's problem without making any use of homogeneity (except to prove that  $X$  is connected and contains no separable intervals).

PROOF OF THEOREM. First, a lemma;

LEMMA. *Let  $X$  satisfy the hypotheses of the theorem. Given  $\phi \in \text{Aut}(X^2)$ , there is a monotone  $f: X \rightarrow I$  and a  $\bar{\phi}: I^2 \simeq I^2$  such that the diagram*

$$\begin{array}{ccc}
 X^2 & \xrightarrow{\phi} & I^2 \\
 f^2 \downarrow & & \downarrow f^2 \\
 I^2 & \xrightarrow{\bar{\phi}} & I^2
 \end{array}$$

commutes. Moreover, given distinct  $x, y$  in  $X$ ,  $f$  can be chosen to separate  $x$  from  $y$ .

PROOF. We construct a sequence  $F_0 \subset F_1 \subset F_2 \subset \dots$  of finite subsets of  $X$ . Given  $F_n$ ,  $F_{n+1}$  defines a "cross-hatching" of  $X^2$  into  $(|F_n| + 1)^2$  rectangles. We take  $F_{n+1}$  fine enough so that if  $R$  is a rectangle of  $F_{n+1}$  then  $\phi(R)$  and  $\phi^{-1}(R)$  each meet at most four rectangles of  $F_n$ , and also so that between each two points of  $F_n$  is a point of  $F_{n+1}$ . Then there is a monotone map from  $\bigcup_{n \in \omega} F_n$  onto the rationals of the open unit interval, which extends to a monotone map from  $X$  onto  $I$  which satisfies our conclusion. Moreover, if  $x, y \in F_0$  then  $f$  separates  $x$  from  $y$ .  $\square$

To prove the theorem we now need only show that  $\bar{\phi}$  is an "affine" map", in the rather special sense we have been using; that is, that the collection of horizontal and vertical lines is invariant.

We note now that because  $X$  has no separable intervals there is a dense  $D \subset I$  such that if  $d \in D$  then  $f^{-1}(\{d\})$  is a nondegenerate interval. If  $\bar{\phi}$  is not "affine" then there is  $d \in D$ ,  $x \in I$  such that  $0 < x < 1$  and  $0 < d < 1$  and  $A = \bar{\phi}(\{d\} \times I)$  does not contain a horizontal or vertical line segment with  $\bar{\phi}(\langle d, x \rangle)$  in its interior (or similarly for  $I \times \{d\}$ , in which case we conjugate the following argument with a reflection across the diagonal).

Define  $B_0 = \{\langle y, z \rangle : y < \bar{\phi}_1 \langle d, x \rangle \text{ and } z < \bar{\phi}_2 \langle d, x \rangle\}$  where  $\bar{\phi}(\langle u, v \rangle) = \langle \phi_1 \langle u, v \rangle, \phi_2 \langle u, v \rangle \rangle$ , and define  $B_1, B_2$  and  $B_3$  similarly.

CASE 1. For all  $w < x$ ,  $\bar{\phi}(d \times D \cap (w, x)) \cap \bigcup_{i \in 4} B_i$  is infinite.

In this case we may pick, for some fixed  $i \in 4$ , a sequence  $d_n \nearrow x$ , with each  $d_n \in D$  and  $\bar{\phi}(\langle d, d_n \rangle) \in B_i$  for all  $n \in \omega$ . But then

$$\text{cl}_X f^{-1}\{\langle d, d_n \rangle : n \in \omega\} \cap f^{-1}(d) \times f^{-1}(x)$$

has more than one point while

$$\text{cl}_X \phi \circ f^{-1}\{\langle d, d_n \rangle : n \in \omega\} \cap \phi(f^{-1}(d) \times f^{-1}(x))$$

does not (for instance if  $i = 0$  it is  $\{\langle \min f^{-1} \bar{\phi}_1 \langle d, x \rangle, \min f^{-1} \bar{\phi}_2 \langle d, x \rangle \rangle\}$ ; otherwise one or both of the occurrences of "min" must be replaced by "max"), a contradiction.

CASE 2. Not case 1. That is, for some  $w < x$ ,  $\bar{\phi}\{d\} \times [w,x]$  is (say) a vertical line segment. By the above argument we may also assume that there is  $v < x$  such that  $\bar{\phi}(\{d\} \times [x,v])$  is horizontal line segment (where it vertical  $A$  would contain a vertical segment through  $\bar{\phi}\langle d,x \rangle$ , contrary to assumption). But then, more or less as above,

$$cl_{X^2} f^{-1}(\{d\} \times [x,v]) \cap cl_{X^2} f^{-1}(\{d\} \times [w,x])$$

has more than one point, unlike

$$cl_{X^2} \phi(f^{-1}(\{d\} \times [x,v])) \cap cl_{X^2} \phi(f^{-1}(\{d\} \times [w,x])). \quad \square$$

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