TOPOLOGICAL SEMIGROUPS

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BY

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CONTENTS

INTRODUCTION AND SUMMARY

The present treatise has the goal of setting forth the basic aspects of the theory of topological semigroups.

A topological semigroup S is a semigroup endowed with a Hausdorff topology for which the mapping $(x,y) \to xy$ of $S \times S$ into S is continuous. There are many differences between topological groups and arbitrary topological semigroups. One striking difference is that we may introduce in any Hausdorff space S a continuous associative multiplication under which S is a topological semigroup. Hence it seems reasonable to study first those semigroups which are either algebraically or topologically easy to handle.

We will restrict our attention primarily to the theory of compact semigroups unless it requires no further effort to state a result for locally compact or more general topological semigroups.

In Chapter I we present a number of elementary concepts. The existence of maximal subgroups in a semigroup was noted first by Schwarz [12], Wallace [1] and Kimura [1]. It is of great interest to determine conditions under which a semigroup S will be a topological group. In particular it is important to find topological restrictions on a semigroup that are sufficient to insure that it will be a group. Some results of this kind stem from Koch and Wallace [6], Hudson and Mostert [3], Wallace [5]. Mostert [3] proved that if a semigroup S is locally compact and if H is a subgroup of S, then H is a topological group if and only if H is locally compact. The fundamental equivalence relations L,R and H, defined in section 1.1 were first introduced and studied by Green [1]. Wallace [12] examined them for topological semigroups and used them to prove that the kernel K of a compact semigroup S is a retract of S. In a compact semigroup these equivalences define upper semi-continuous decompositions. With some additional assumptions on S it is possible to give a completely topological definition of K, Wallace [6].

The structure theorem for completely simple semigroups was first proved by Suschkewitsch [1] in a special case. With the publication of his paper he really started the theory of semigroups. He showed that every finite semigroup contains a kernel and he determined the structure of finite simple semigroups. His results were extended by Rees [1] to completely simple semigroups. The only difficulty to prove this theorem for compact simple semigroups is that of selecting the various canonical mappings so that they are continuous.

We also introduce in section 1.3 the concept of the Rees factor semigroup. In general, congruences on a semigroup are not determined by any single congruence class as they are for groups. The congruence on a group, determined by its unit-component has a semigroup-theoretic version. If S is a locally compact semigroup such that each component is compact, then the component space of S can be made in an obvious way into a topological semigroup which is totally disconnected.

In section 1.4 the concept of a maximal ideal is introduced. With the aid of some results which involve maximal ideals one can prove for example the following theorems:

If S is compact with $S^2 = S$ and such that S has at most one idempotent, then S is a group.

If S is compact with unit u and if S is not a group, then S has a unique maximal proper ideal J and $J = S \setminus H(u)$.

Let S be a connected compact semigroup having at least one left unit and suppose that S is not right simple. Then every subgroup H(e), with e a left unit lies in the boundary of the maximal right ideal.

Section 1.5 is devoted to the study of open prime ideals in compact semigroups. It is proved that each open prime ideal P has the form $J_o(S \setminus \{e\})$, where e is a non minimal idempotent and $J_o(S \setminus \{e\})$ is the maximal ideal of S contained in $S \setminus \{e\}$.

The results of this section are due to Numakura [4] and for commutative semigroups to Schwarz [6].

In Chapter II we investigate the structure of some semigroups with zero or identity. The notion of nilpotent elements in a semigroup with zero was first introduced by Numakura [1]. He proved that if the set of nil-

potent elements of a locally compact semigroup S is not open, then O is a clusterpoint of the set of non-zero idempotents.
The characterization of minimal non-nil (left, right) ideals of a compact semigroup S with zero as the sets SeS (Se,eS) with e a non-zero primitive idempotent was given by Koch [1]. The complete determination of all possible completely O-simple semigroups was given by Rees. The Rees-theorem faithfully represents a completely O-simple semigroup as the semigroup of all matrices over a group with zero having at most one non-vanishing element and multiplication by means of a certain matrix.
In section 2.3 we give a topological extension of this theorem in the case of a compact O-simple semigroup. The essential difficulty of course, is that of finding a cross-section of the O-minimal left ideals. contained in a O-minimal right ideal.
In section 2.4 attention is given to connected semigroups, although we stick mainly to the realm of connected semigroups with an identity. The theorem of Faucett that if the minimal ideal K of a compact connected semigroup has a cutpoint, then every element of K is a left or right zero, has been generalized by Wallace [18] to relative ideals.
Mostert and Shields [8] have studied connected semigroups S with identity u in which the maximal subgroup containing u is open. They proved that this class includes the semigroups with identity on a manifold (theorem 2.4.9). This theorem is not true for general locally convex linear spaces.
Perhaps the most natural example of a compact connected semigroup is the closed unit interval I with the usual multiplication. Simple examples show that the space I admits many semigroup structures. These semigroups need not be abelian, may not have a zero element and may admit both idempotents and nilpotents.
In section 2.5 the semigroup structures with which the space I may be provided is analysed. The systematic study of I-semigroups was initiated by Faucett [2]. The general structure is given in theorem 2.5.4 and is due to Mostert and Shields [7]. It should be noted that nearly all theorems and proofs of section 2.5 and 2.6 generalize to arbitrary compact

connected linearly ordered topological spaces.

The object of section 2.6 is to characterize compact connected semigroups S with $S^2 = S$ on an interval. Partial results in this connection have been found by several authors. Cohen and Wade [4] have described compact connected semigroups with an identity and a zero, for which the underlying space is an interval. The class of compact connected interval semigroups with idempotent endpoints has been studied by Clifford [3], [4]. In addition the case when zero is an endpoint and $S = S^2$ has been described by Storey [1].

In this connection we also mention the work of Mostert and Shields [6], who gave a description of semigroups defined on the interval $[0,\infty)$ in which "zero" and "one" play their usual roles.

In Chapter III attention is given to compact commutative semigroups. Most of the results about compact monothetic semigroups are due to Koch [2] and Hewitt [1].

By a decomposition of a semigroup S we mean a partition of S into the union of disjoint subsemigroups. For this to be of any value the subsemigroups should be semigroups of some more restricted type than S. An example of such a decomposition is given by Schwarz [6] who proved that every compact commutative semigroup is a semilattice of subsemigroups containing exactly one idempotent.

We also study the embedding of a commutative cancellative semigroup in a group (Gelbaum, Kalisch, Olmsted [1]). The usual procedure for doing this, by means of ordered pairs is just like that of embedding an integral domain in a field. In fact it is easier, since there is only one binary operation to consider.

In section 3.3 characters on commutative semigroups are considered. The Pontryagin duality theorem asserts that a locally compact abelian group G can be identified in a natural way with its second dual. For discrete commutative semigroups S the Pontryagin duality holds if and only if S has an identity and is a union of groups. For compact abelian semigroups S a less complete result is obtained. Most of the results obtained in this section are due to Austin [1].

In the fourth chapter we are concerned with the theory of invariant and subinvariant measures on compact semigroups. In the theory of semigroups we are troubled for a lack of something like Haar measure. Without this we will be at a loss for representation theorems. A measure μ on a semigroup S will be called right invariant if for every Borel set $B \subset S$ and $a \in S$ for which Ba is also a Borel set, $\mu(Ba) = \mu(B)$ holds. μ is right subinvariant if $\mu(Ba) \leq \mu(B)$.

The investigation of subinvariant measures was suggested by Prof.dr. J. de Groot.

In section 4.1 it is proved that right invariant measures exist only if the minimal ideal K is a minimal left ideal. Right invariant means are also considered and it is proved that a mean is right invariant if and only if it is right subinvariant. If μ is the regular Borel measure determined by a right invariant mean, then μ has the property that $\mu(B) = \mu(B_a)$ where $B_a = \{x | x \in S, xa \in B\}$. The use of sets like B_a is typical. This set seems indicated as a replacement for the set Ba^{-1}, with which B_a should be identical were S a group. Furthermore the support of μ is the kernel K of S.

In section 4.2 we study subinvariant measures on simple semigroups. The principal result is contained in theorem 4.2.4 which states that if S is a compact simple mob such that $S = (Se_1 \cap E) \times H(e_1) \times (e_1 S \cap E)$, then S has a right subinvariant measure if and only if the compact space $e_1 S \cap E$ has a regular normed Borel measure μ such that $\mu(\{e\}) = \mu(\{e'\})$ for all points $e_1, e' \in e_1 S \cap E$. Some applications of this theorem to special kinds of semigroups are given.

Section 4.3 is devoted to the investigation of subinvariant measures on a certain class of semigroups, semigroups of type 0. This class contains the semigroups S with the property that Ua is open in S for all $a \in S$ and all open sets $U \subset S$.

A reasonably complete survey of the literature on the theory of topological semigroups is listed at the end of the treatise.

I wish to express my gratitude to the Mathematical Centre, Amsterdam, which gave me the opportunity to carry on the investigations which are dealt with in this treatise. I also wish to express my sincere thanks to Prof.dr. J. de Groot to whom I am deeply indebted. This tract would

never have been written but for his never failing and stimulating encouragement.
I am indebted to P.C. Baayen and M.A. Maurice for many stimulating discussions.

CONVENTIONS

In this section we explain some of the notation and terminology used throughout the text.

The empty set will be denoted by $\emptyset$. The symbols $\subset$ and $\supset$ mean ordinary inclusion between sets, they do not exclude the possibility of equality. If A and B are sets, then $A \setminus B$ will denote the set of points of A which do not belong to B. Mappings will be considered as left operators and written on the left of the argument. If f is a mapping of X into Y and $A \subset X$, $B \subset Y$, then

$$f(A) = \{ f(a) \mid a \in A\}, \quad f^{-1}(B) = \{x \in X \mid f(x) \in B\}.$$

A semigroup S is a non-void set together with an associative multiplication. We do not assume the existence of an identity or the validity of any cancellation law. Let A and B be subsets of a semigroup S. The symbol AB denotes the set $\{ab \mid a \in A, b \in B\}$. We write AA as A^2, AAA as A^3 etc.

If S_1 and S_2 are topological semigroups, then S_1 and S_2 are called isomorphic if there is a one-one correspondence between their elements which is a semigroup isomorphism and a space homeomorphism.

For further information on abstract semigroups see e.g. E.S. Ljapin [1] and A.H. Clifford and G.B. Preston [5].

If A is a subset of a topological space X, then $\bar{A}$ will denote the closure of A in X and A^o the interior of A in X. A covering $\mathfrak{A}$ of a space X is a refinement of a covering $\mathfrak{M}$ if each member of $\mathfrak{A}$ is a subset of a member of $\mathfrak{M}$.

A topological space will be called compact if every open covering of it has a finite subcovering.

A continuum is a compact connected Hausdorff space. A continuum is decomposable if it is the union of two proper subcontinua, otherwise it is indecomposable.

If X and Y are topological spaces, then $X \times Y$ will denote the product space.
We reserve the symbol E_n for Euclidean n-space.
For further topological concepts see J.L. Kelley [1].

If X is a locally compact space and C is the family of all compact subsets of X, then the family of Borel sets $\mathcal{B}$ in X is defined as the smallest σ-algebra of sets containing C. A Borel measure μ on X is an extended real valued non-negative and countably additive set function defined on $\mathcal{B}$, and such that $\mu(\emptyset) = 0$.
μ is called regular if for all $A \in \mathcal{B}$ we have both
$\mu(A) = \inf \{\mu(V). \mid V \text{ open and } A \subset V,\ V \in \mathcal{B}\}$ and
$\mu(A) = \sup \{\mu(F) \mid F \text{ is compact and } F \subset A\}$.
For further information on Borel measures and for some of the terminology and notation used in Chapter IV we refer to P.R. Halmos [1].

I SUBSEMIGROUPS

1.1. Subgroups and subsemigroups

Definition. A topological semigroup ("mob") is a space S together with a continuous function $f: S \times S \to S$ such that:

a) S is a Hausdorff space,

b) f is associative.

If we write $f(x,y) = xy$, then b) becomes the more familiar $(xy)z = x(yz)$ for all $x,y,z \in S$.

A mob may be thought of as a set of elements which is both an abstract semigroup and a Hausdorff space, the operation of the semigroup being continuous in the topology of the space.

Familiar examples are the topological groups and the closed unit interval with the usual multiplication and topology. Furthermore if X is any Hausdorff space, then a continuous associative multiplication may be introduced by

a) $xy = x$ all $x,y \in X$ or

b) $xy = y$ all $x,y \in X$.

Definitions. A subsemigroup of a mob S is a non-void set $A \subset S$ satisfying $A^2 \subset A$.

A non-void set $A \subset S$ is called a subgroup of S if $xA = Ax = A$ for all $x \in A$.

Of course this defines an abstract group in the customary sense. A, with the relative topology, then becomes a topological semigroup, although it need not be a topological group since the function g with $g(x) = x^{-1}$ $(x,x^{-1} \in A)$ need not be continuous.

1.1.1. Theorem. Let S be a mob with more than one element. Then S contains a submob S' such that $S' \neq S$.

Proof:

Suppose each submob S' of S is equal to S and let $a \in S$. Since Sa and aS are submobs of S we have $Sa = S = aS$.
Hence S is a group. If e is the identity of S, then $\{e\}$ is a submob of S, and $\{e\} \neq S$. This contradicts the assumption that each submob of S is equal to S.

1.1.2. Lemma. Let A be a submob of the mob S. Then $\bar{A}$ is a submob of S.

Proof:

Suppose for $x,y \in \bar{A}$, $xy \notin \bar{A}$. Then since $\bar{A}$ is closed, there exist neighbourhoods V of x and W of y such that $VW \cap \bar{A} = \emptyset$.
Since $x,y \in \bar{A}$, there is an $a_1 \in V \cap A$ and $a_2 \in W \cap A$.
This implies $a_1a_2 \in VW$ and $a_1a_2 \notin \bar{A}$ which is a contradiction.

1.1.3. Theorem. Each subgroup of a mob S is contained in a (unique!) maximal subgroup, and no two maximal subgroups of S intersect.

Proof:

Let A be a subgroup of S and e the identity of A.
Let A_o be the set of all $a \in S$ such that $ae = ea = a$ and such that there exists an element $a^{-1} \in S$ with $aa^{-1} = a^{-1}a = e$, $a^{-1}e = ea^{-1} = a^{-1}$.
Then it is immediately clear that A_o is a maximal subgroup of S containing A.
Suppose now that A_1 and A_2 are maximal subgroups of S and $a \in A_1 \cap A_2 \neq \emptyset$. Let e_1 and e_2 be the identities of A_1 and A_2 respectively, and let $aa_1^{-1} = e_1$, $aa_2^{-1} = e_2$.
Then $e_1aa_2^{-1} = e_1e_2 = aa_2^{-1} = e_2$ and $a_1^{-1}ae_2 = e_1e_2 = a_1^{-1}a = e_1$.
Hence $e_1 = e_2$.
Since A_1 is maximal, A_1 contains all a with $ae_1 = e_1a = a$ and $a^{-1}a = aa^{-1} = e_1$, $a^{-1}e_1 = a^{-1}$. Thus $A_1 = A_2$.

It may happen that a mob S contains no subgroups at all. Consider for example the open unit interval $I = (0,1)$ with the usual multiplication. I contains no subgroups. Or let N be the set of all positive integers with the discrete topology under addition. Then N contains no subgroups.

1.1.4. Lemma. Let S be a mob and let $A = \{a_\lambda\}_{\lambda \in \Lambda}$, $B = \{b_\lambda\}_{\lambda \in \Lambda}$ with $A \subset S$ and $\bar{B}$ a compact subset of S.

Then for every $a \in \bar{A}$ there exists a $b \in \bar{B}$ such that $ab \in \bar{C}$ with $C = \{a_\lambda b_\lambda\}_{\lambda \in \Lambda}$.

Proof:

Suppose that such a b does not exist. Then we have for every $b_\alpha \in \bar{B}$, $ab_\alpha \notin \bar{C}$. The continuity of multiplication implies the existence of neighbourhoods U_α of a and V_α of b_α such that $U_\alpha V_\alpha \cap \bar{C} = \emptyset$.
The set $\{V_\alpha\}_\alpha$ constitutes an open covering of the compact set $\bar{B}$.
There exists therefore a finite subcovering say $V_1, V_2, \ldots, V_n$.
Let $U = \bigcap_{i=1}^{n} U_i$. U is an open neighbourhood of a with $U\bar{B} \subset U \bigcup_{i=1}^{n} V_i$, and hence $U\bar{B} \cap \bar{C} = \emptyset$.
U however contains at least one element $a_{\lambda_o} \in A$, since $a \in \bar{A}$.
We have therefore $a_{\lambda_o} b_{\lambda_o} \in U\bar{B}$ and $a_{\lambda_o} b_{\lambda_o} \in \bar{C}$.
This contradiction proves the lemma.

1.1.5. Theorem. If S is a compact mob, then each maximal subgroup of S is closed.

Proof:

Let A be a maximal subgroup of S. Then $aA = Aa = A$ for all $a \in A$, hence $AA = A^2 = A$ and the continuity of multiplication implies that $\bar{A}\bar{A} = \bar{A}$. Thus $\bar{A}x \subset \bar{A}$ and $x\bar{A} \subset \bar{A}$ for all $x \in \bar{A}$.
On the other hand suppose $A \not\subset x\bar{A}$ for $x \in \bar{A}$. Then there is an $a_1 \in A$ with $a_1 \notin x\bar{A}$, and the continuity of multiplication together with the compactness of $\bar{A}$ imply the existence of a neighbourhood V of x such that $a_1 \notin V\bar{A}$. Since $x \in \bar{A}$ there is an $a_2 \in A \cap V$ and then $a_1 \in a_2 A$ leads to a contradiction.
Thus $A \subset x\bar{A}$ for all $x \in \bar{A}$, and hence $\bar{A} \subset x\bar{A}$.
Analogously we have $\bar{A} \subset \bar{A}x$.
Therefore $\bar{A}x = x\bar{A} = \bar{A}$ for all $x \in \bar{A}$, and $\bar{A}$ is a subgroup of S.

Since A is maximal we have $A = \overline{A}$.

If S is not compact, then the maximal subgroups of S may fail to be closed. Let S be the mob $[0,\infty)$ with the usual multiplication. $A = (0,\infty)$ is a maximal subgroup of S which is not closed.

1.1.6. Lemma. Let S be a locally compact mob and an abstract group. Let A be a countable subset of S and $x \in \overline{A}$. Then $x^{-1} \in \overline{A^{-1}}$.

Proof:

Let $B = \bigcup_{n=-\infty}^{\infty} (A \cup \{x\})^n$. Then B is a countable subgroup of S and the continuity of multiplication implies $\overline{B}^2 \subset \overline{B}$.

Let V be a compact neighbourhood of the identity and let $\overline{b} \in \overline{B}$. Since S is a group, $\overline{b}V$ is a neighbourhood of $\overline{b}$ and $\overline{b}V \cap B \neq \emptyset$.

This implies that $\overline{b} \in BV^{-1}$ and hence $\overline{B} \subset BV^{-1}$.

Thus $\overline{B} = \bigcup_{b \in B} [bV^{-1} \cap \overline{B}] = \bigcup_{b \in B} [b(V^{-1} \cap \overline{B})]$.

By 1.1.4 V^{-1} is closed since V is compact and hence $b(V^{-1} \cap \overline{B})$ is closed. Moreover $\overline{B}$ is a closed subset of S and hence locally compact. Baires category theorem implies that the interior relative to $\overline{B}$ of one of the sets $b(V^{-1} \cap \overline{B})$ is not empty.

Hence there exist an open set U with $\overline{B} \cap U \neq \emptyset$ and an element $b_o \in B$ such that $U \cap \overline{B} \subset b_o(V^{-1} \cap \overline{B})$.

Let $c \in B \cap U$. Then $xc^{-1}(U \cap \overline{B}) = xc^{-1}U \cap \overline{B}$ and $xc^{-1}U = U_o$ is open.

$U_o \cap A \subset U_o \cap \overline{B} \subset xc^{-1}(U \cap \overline{B}) \subset xc^{-1}b_oV^{-1}$.

Hence $(U_o \cap A)^{-1} \subset Vb_o^{-1}cx^{-1} = C$ with C compact.

Then by 1.1.4, there exists for every $a \in \overline{U_o \cap A}$ an element $b \in \overline{(U_o \cap A)^{-1}}$ with ab the identity.

Since $x \in \overline{U_o \cap A}$ it follows also that $x^{-1} \in \overline{(U_o \cap A)^{-1}} \subset \overline{A^{-1}}$.

1.1.7. Lemma. Let S be a locally compact mob and an abstract group. Let A be a compact subset of S. Then A^{-1} is compact.

Proof:

From 1.1.4 it follows that A^{-1} is closed.

Suppose that A^{-1} cannot be covered by a finite number of compact sets $x_i^{-1}V$, with V any compact neighbourhood of the identity, $x_i \in A$. Then

there is a sequence $\{x_n^{-1}\}_{n=1}^{\infty} \varepsilon A^{-1}$ such that $x_n^{-1} \not\in \bigcup_{i=1}^{n-1} x_i^{-1}V$.
Let $E_n = \{x_k \mid k \geq n\}$. Since A is compact, there exists a $y \varepsilon \bigcap_{n=1}^{\infty} \overline{E_n}$.
Since $y \varepsilon \overline{E_1}$, there is $x_m \varepsilon Vy$, whence $y^{-1} \varepsilon \overline{x_m^{-1}V}$.
Moreover $y \varepsilon \overline{E_{m+1}}$ implies by 1.1.6 $y^{-1} \varepsilon \overline{E_{m+1}^{-1}}$. Thus there is an $n > m$ such that $x_n^{-1} \varepsilon x_m^{-1}V$ which contradicts the choice of $\{x_n^{-1}\}_{n=1}^{\infty}$.

1.1.8. Theorem. Let S be a locally compact mob and an abstract group. Then S is a topological group.

Proof:

Let U be an open neighbourhood of the identity u of S and $\{V_\alpha\}_\alpha$ the collection of compact neighbourhoods of u.
Suppose that for every V_α, $V_\alpha^{-1} \not\subset U$. Then $V_\alpha^{-1} \cap S \setminus U \neq \emptyset$, and $\bigcap_\alpha V_\alpha^{-1} \cap S \setminus U \neq \emptyset$ since V_α^{-1} is compact.
But $\bigcap_\alpha V_\alpha^{-1} \cap S \setminus U \subset \bigcap_\alpha V_\alpha^{-1} = \{u\}$ implies that $u \varepsilon S \setminus U$, which is a contradiction.
Hence for every neighbourhood U of u there exists a neighbourhood V of u, such that $V^{-1} \subset U$. Therefore S is a topological group.

Let S be the additive group of real numbers. We define a topology in S by means of a base B consisting of all half open intervals $[a,b)$. S is a mob and an abstract group. S however is no topological group, for there is no neighbourhood U of 1, with $-U \varepsilon [-1,-\frac{1}{2})$.

Definition. An element e of a mob S is called an idempotent if $e^2 = e$.
We shall denote by E the set of idempotents in S.
If S contains an idempotent e, then $\{e\}$ is a subgroup of S, and is contained in a maximal subgroup.
By H(e) we shall denote the maximal subgroup of S containing the idempotent e.
An element 0 is termed the zero of S if $0x = x0 = 0$ for all $x \varepsilon S$. It is easily seen that the zero of S, if it exists is uniquely defined.
It is also immediately clear that it is an idempotent.
An element u is termed the identity of S if $ux = xu = x$ for all $x \varepsilon S$.

The identity of S, if it exists is uniquely defined and is an idempotent.

A mob S in which the product of any two elements is zero we term a <u>zero semigroup</u>.

1.1.9. <u>Lemma</u>. The set E of all idempotents of a mob is closed.

<u>Proof</u>:

If $E = \emptyset$ the lemma is trivial.

Suppose now $x \in \bar{E}$ and $x^2 \neq x$, then there exists a neighbourhood V of x such that $V^2 \cap V = \emptyset$.

Since $x \in \bar{E}$, there is an $e \in E \cap V$ and hence $e = e^2 \in V^2 \cap V$ which is a contradiction.

1.1.10. <u>Theorem</u>. Let S be a compact mob. Then S contains a subgroup and hence at least one idempotent.

<u>Proof</u>:

Let $a \in S$ and let K(a) denote <u>the set of cluster</u> points of the sequence $\{a^n\}_{n=1}^{\infty}$; $K(a) = \bigcap_{n=1}^{\infty} \overline{\{a^i \mid i \geq n\}}$.

Then since S is compact, K(a) is compact and the continuity of multiplication implies that K(a) is a commutative submob of S.

Suppose now $xK(a) \neq K(a)$, $x \in K(a)$. Then there exists $z \in K(a)$ such that $z \notin xK(a)$.

Therefore there are neighbourhoods V, O and U such that $VO \cap U = \emptyset$, $x \in V$, $K(a) \subset O$, $z \in U$.

Since $x, z \in K(a)$ there are $a^m \in V$ and $a^{n_i} \in U$ with $n_{i+1} > n_i > m$ $(i=1,2,\ldots)$.

Let b be a cluster point of the sequence $\{a^{n_i-m}\}_{i=1}^{\infty}$.

Then $b \in K(a) \subset O$ and hence there is a j such that $a^{n_j-m} \in O$.

Thus $a^m a^{n_j-m} = a^{n_j} \in VO$, a contradiction.

Hence $xK(a) = K(a)$. In the same way we prove $K(a)x = K(a)$, and it follows that K(a) is a subgroup of S.

<u>Corollary</u>. Let S be a mob and S' a compact submob. Then if S is an abstract group, S' is a subgroup.

Proof:

By 1.1.10 S' contains an idempotent which must be u (the identity of S).

Again by 1.1.10, applied to xS', $x \in S'$, there is an idempotent in xS'.
Thus $u \in xS'$ and $S' = uS' \subset xS'$.
Hence since $xS' \subset S'$, $xS' = S'$ for all $x \in S'$.
Analogously $S'x = S'$.

1.1.11. Lemma. Let G be a compact group and S a submob of G. If S is either open or closed, then S is a compact subgroup of G.

Proof:

If S is closed the preceding corollary implies that S is a subgroup of G.

Next let S be open. Then $\overline{S}$ is a closed submob of G and hence a subgroup of G. This implies that the identity u of G is contained in $\overline{S}$.
We now prove that $\overline{S}^{o} = S$. For let $x \in V \subset \overline{S}^{o}$, where V is a neighbourhood of x. Then there exists a neighbourhood O of u with $xO^{-1} \subset V$.
Since $u \in \overline{S}$ we have $O \cap S = W \neq \emptyset$ and $xW^{-1} \subset xO^{-1} \subset V$. Moreover xW^{-1} is open, hence $xW^{-1} \cap S \neq \emptyset$. Let $s \in xW^{-1} \cap S$, then $s = xw^{-1}$ and $x = sw$ with $w \in W = O \cap S$. Hence $x \in S$ and we have $\overline{S}^{o} \subset S$.
Since S is open we also have $S \subset \overline{S}^{o}$ and hence $S = \overline{S}^{o}$.
From this it follows that $S = \overline{S}^{o} = \overline{S}$, since any subgroup of a topological group having a non-void interior is an open and closed subgroup.

1.1.12. Theorem. Each locally compact submob S of a compact group G is a compact subgroup of G.

Proof:

Since $\overline{S}$ is a closed submob of G, $\overline{S}$ is a compact group. Furthermore S is a dense locally compact subset of $\overline{S}$, hence S is open in $\overline{S}$, so that S is a compact open subgroup of $\overline{S}$, i.e. $S = \overline{S}$.

Definition. If S is a mob and $a \in S$, then we shall denote by $\Gamma(a)$ the closure of the set $\{a^n\}_{n=1}^{\infty}$; i.e.

$$\Gamma(a) = \overline{\{a^n\}_{n=1}^{\infty}} .$$

From 1.1.10 it follows that if $\Gamma(a)$ is compact, it contains an idempotent. Moreover $\Gamma(a) = K(a) \cup \left(\{a^n\}_{n=1}^{\infty} \setminus K(a)\right)$ with $K(a)$ a group. Hence we see that $\Gamma(a)$ contains in that case exactly one idempotent.

1.1.13. Lemma. Let S be a mob and let A be a compact part of S, such that $Ax \subset A$, with $\Gamma(x)$ compact.
Then $\bigcap_{n=1}^{\infty} Ax^n = Ae$, with $e = e^2 \in \Gamma(x)$.

Proof:

Let $s \in \bigcap_{n=1}^{\infty} Ax^n$. Then $s = a_1x = a_2x^2 = \dots$, $a_i \in A$, $i=1,2,\dots$.
Hence it follows from 1.1.4 that there is an element $a \in \overline{\{a_i\}_{i=1}^{\infty}}$ such that $s = ae$; thus $\bigcap_{n=1}^{\infty} Ax^n \subset Ae$.
Now let $ae \notin Ax^k$. Then we can find a neighbourhood V of e such that $aV \cap Ax^k = \emptyset$. But since $e \in \Gamma(x)$, there is a $k_o \geq k$ such that $x^{k_o} \in V$ and hence $ax^{k_o} \notin Ax^k$. This is a contradiction since $Ax \subset A$ implies $Ax^{k_o} \subset Ax^k$.
Thus $Ae \subset Ax^k$ and $\bigcap_{n=1}^{\infty} Ax^n = Ae$.

1.1.14. Theorem. Let S be a mob and A a compact submob of S.
Then for every $a \in A$ there exists a unique maximal submob $A^* \subset A$ with the property $A^*a = A^*$; and $A^* = \bigcap_{n=1}^{\infty} Aa^n = Ae$ with $e = e^2 \in \Gamma(a)$.

Proof:

$(Ae)a = (\bigcap_{n=1}^{\infty} Aa^n)a \subset \bigcap_{n=1}^{\infty} Aa^{n+1} = Ae$.
Now let $x \in Ae$ and let $A_n = \{y \mid ya^n = x;\ y \in A\}$, $n=1,2,\dots$.
Then A_n is compact and $\bigcap_{n=2}^{k} A_na^{n-1} = A_ka^{k-1}$ for every k. Hence $\bigcap_{n=2}^{\infty} A_na^{n-1} \neq \emptyset$.
Let $y \in \bigcap_{n=2}^{\infty} A_na^{n-1} \subset \bigcap_{n=2}^{\infty} Aa^{n-1} = Ae$. Then $ya = x$ and thus $Ae \subset (Ae)a$.
It remains to show that Ae is the greatest submob $A^* \subset A$ such that $A^*a = A^*$.

Let A^* be any submob with this property, then $A^* = A^*a \subset Aa$ and hence $A^* = A^*a^n \subset Aa^n$, n=1,2,... .
Thus $A^* \subset \bigcap_{n=1}^{\infty} Aa^n = Ae$ and the theorem is proved.

Now let S be a mob and a ε S such that Γ(a) is compact.
Then since $\Gamma(a)a^n = \overline{\{a^i \mid i \geq n+1\}} = a^n\Gamma(a)$, we have
$K(a) = \bigcap_{n=1}^{\infty} \Gamma(a)a^n = \bigcap_{n=1}^{\infty} a^n\Gamma(a) = e\Gamma(a) = \Gamma(a)e$ with $e = e^2 \in \Gamma(a)$.
And thus K(a)a = K(a) = aK(a).

1.1.15. Theorem. Let S be a compact mob with two-sided cancellation (i.e. ax = bx implies a = b, a,b,x ε S and xa = xb implies a = b, a,b,x ε S).

Then S is a topological group.

Proof:

Let x ε S. Then xS ⊂ S and 1.1.13 implies that eS ⊂ xS ⊂ S, $e = e^2 \in \Gamma(x)$.
Since S has two-sided cancellation, the mapping ϕ: s → es, s ε S is a one-to-one continuous mapping of S onto eS.
On the other hand φ is the identity mapping on eS and hence eS = S = xS. Analogously we have S = Sx.

Let S be a mob and define $(a,b) \in \mathcal{L}$, a,b ε S to mean that {a} ∪ Sa = {b} ∪ Sb. Clearly $\mathcal{L}$ is an equivalence relation such that if $(a,b) \in \mathcal{L}$, then $(ac,bc) \in \mathcal{L}$ for all c ε S.
By L_a we shall mean the set of all elements of S which are $\mathcal{L}$ equivalent to a. Thus $L_a = \{b \mid \{a\} \cup Sa = \{b\} \cup Sb;\ b \in S\}$.
Dually we define $(a,b) \in \mathcal{R}$, a,b ε S to mean {a} ∪ aS = {b} ∪ bS and $R_a = \{b \mid \{a\} \cup aS = \{b\} \cup bS;\ b \in S\}$.
Finally we define $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $H_a = L_a \cap R_a$.

If e ε E, then $H(e) = H_e$.
For let x ε H(e), then x = ex = xe and $xx^{-1} = x^{-1}x = e$.
Hence {x} ∪ Sx ⊂ {e} ∪ Se; {x} ∪ xS ⊂ {e} ∪ eS and {e} ∪ Se ⊂ {x} ∪ Sx, {e} ∪ eS ⊂ {x} ∪ xS.

Thus $H(e) \subset H_e$.

Now let $x \in H_e$. Then since $x \in (\{e\} \cup Se) \cap (\{e\} \cup eS)$ we have $xe = ex = x$, and since $e \in (\{x\} \cup Sx) \cap (\{x\} \cup xS)$, x has a left and right inverse, hence $x \in H(e)$.

1.1.16. Lemma. If S is compact, then $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{H}$ are compact subsets of $S \times S$.

Proof:

Let $\mathcal{L} \neq S \times S$ and let $(x,y) \in S \times S \setminus \mathcal{L}$.

Then we may assume $x \notin Sy \cup \{y\}$ (or $y \notin Sx \cup \{x\}$).

Hence $\overline{V} \cap (Sy \cup \{y\}) = \emptyset$ for some open set V containing x, since S is regular and $Sy \cup \{y\}$ closed. Since S is compact there is an open set U containing y such that $\overline{V} \cap (SU \cup U) = \emptyset$.

Hence $(U \times V) \cap \mathcal{L} = \emptyset$ and we may infer that $\mathcal{L}$ is closed.

Similarly $\mathcal{R}$ is closed and hence $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ is closed.

1.1.17. Theorem. If S is compact then $H = \bigcup\{H(e) \mid e \in E\}$ is closed. If $x \in H$ let $\alpha(x)$ be the unit of the unique maximal subgroup containing x and let $\beta(x)$ be the inverse of x in this group. Then $\alpha: H \to E$ is a retraction and $\beta: H \to H$ is a homeomorphism.

Proof:

Let $\pi: S \times S \to S$ be the mapping defined by $\pi(x,y) = x$.

Then $H = \bigcup\{H(e) \mid e \in E\} = \pi(\mathcal{H} \cap S \times E)$.

Since π is continuous and $\mathcal{H}$ and E are closed, H is closed.

Furthermore let $B = \{(x,\beta(x)) \mid x \in H\}$ and $f : S \times S \to S$, $f(x,y) = xy$.

We now show that $B = \mathcal{H} \cap H \times H \cap f^{-1}(E)$.

For let $(x,\beta(x)) \in B$, then $x,\beta(x) \in H_e$ and $x\beta(x) = e$, hence $(x,\beta(x)) \in \mathcal{H} \cap H \times H \cap f^{-1}(E)$.

If on the other hand $(x,y) \in \mathcal{H} \cap H \times H \cap f^{-1}(E)$, then $xy = e \in E$ and $(x,y) \in \mathcal{H}$ hence $H_x = H_y$.

Furthermore $x,y \in H$ implies $H_x = H_{e_1}$ for some $e_1 \in E$.

Hence $H_x = H_y = H_{e_1} = H(e_1)$ and thus $xy = e_1$ and $y = \beta(x)$.

Since $\mathcal{H}$, $H \times H$ and $f^{-1}(E)$ are closed, B is compact.

Furthermore $\pi | B$ is one-to-one and continuous and hence topological. Thus $(\pi | B)^{-1} : x \to (x, \beta(x))$ is continuous, and we may infer that β is continuous.
α is continuous since $\alpha(x) = x\beta(x)$.

1.2. Ideals

Definitions. A non empty subset A of a mob S is called a left ideal if $SA \subset A$, a right ideal if $AS \subset A$ and an ideal if it is both a left and a right ideal.
A minimal left (right) ideal of S is a left (right) ideal containing no other left (right) ideal.
We shall denote by $\mathcal{L}(S)$ and $\mathcal{R}(S)$ respectively the collections of all minimal left and all minimal right ideals of S.
In general these may be empty collections.
The intersection of all ideals of S is called the kernel of S and denoted by K.
If K is non-empty it is clearly the smallest ideal of S.

1.2.1. Lemma. Let A be an ideal of a mob S. Then $\bar{A}$ is an ideal of S.
Proof:
Since $SA \subset A$ and $AS \subset A$, the continuity of multiplication implies $S\bar{A} \subset \bar{A}$ and $\bar{A}S \subset \bar{A}$.
Hence $\bar{A}$ is an ideal of S.

An analogous result holds for left and right ideals.
If $a \in S$ we let

$$J(a) = \{a\} \cup Sa \cup aS \cup SaS,$$
$$L(a) = \{a\} \cup Sa,$$
$$R(a) = \{a\} \cup aS.$$

Thus J(a) is the smallest ideal of S which contains a.
L(a) and R(a) are respectively the smallest left and right ideal of S which contain a.

If $A \subset S$ then we define $J_o(A)$ to be the null-set if A contains no ideal of S and $J_o(A)$ is the union of all ideals contained in A in the contrary case. $L_o(A)$ $(R_o(A))$ is the null-set if A contains no left (right) ideal of S and $L_o(A)$ $(R_o(A))$ is the union of all left (right) ideals contained in A in the contrary case.

It is clear that if $J_o(A) \neq \emptyset$, then $J_o(A)$ is the largest ideal of S contained in A.

Also if $L_o(A) \neq \emptyset$ and $R_o(A) \neq \emptyset$ then $L_o(A)$ is the largest left and $R_o(A)$ is the largest right ideal of S contained in A.

1.2.2. <u>Lemma</u>. If $A \subset S$ is closed, then $J_o(A)$, $L_o(A)$ and $R_o(A)$ are closed. If A is open and S compact, then $J_o(A)$, $L_o(A)$ and $R_o(A)$ are open.

<u>Proof</u>:

We only prove the lemma for $J_o(A)$.

Suppose $J_o(A) \neq \emptyset$, then since $J_o(A) \subset A$ we have $\overline{J_o(A)} \subset \bar{A}$.

Now $\overline{J_o(A)}$ is an ideal of S and hence $\overline{J_o(A)} \subset J_o(A)$ if $A = \bar{A}$.

Suppose now that S is compact and A is open.

Let $x \in J_o(A)$, then $\{x\} \cup xS \cup Sx \cup SxS \subset J_o(A) \subset A$ and there exists an open set V, $x \in V$, satisfying $V \cup VS \cup SV \cup SVS \subset A$.

Now this set is an ideal of S, hence is contained in $J_o(A)$.

Therefore $x \in V \subset J_o(A)$ completing the proof.

1.2.3. <u>Theorem</u>. Let S be a compact mob; then any proper ideal of S is contained in a maximal proper ideal of S, and each maximal proper ideal is open.

<u>Proof</u>:

If the ideal $I \neq S$, then 1.2.2 shows that $J_o(S \setminus \{x\})$ is an open proper ideal containing I for any $x \in S \setminus I$.

Let $\{T_\alpha\}_\alpha$ be a linearly ordered system of open proper ideals containing I.

If $S = \bigcup_\alpha T_\alpha = T$, then S is the union of a finite number of T_α's.

Since $\{T_\alpha\}_\alpha$ is linearly ordered, there is an α with $S = T_\alpha$, which is a contradiction.

Hence $T = \bigcup_{\alpha} T_{\alpha}$ is a proper ideal of S.
Using Zorn's lemma there is a maximal element in the collection of all open proper ideals containing I.
Each maximal proper ideal M is open, since $M = J_o(S \setminus \{x\})$, $x \notin M$.

An analogous result holds for left and right ideals.
Thus if S is compact, then any proper left (right) ideal of S is contained in a maximal proper left (right) ideal and each maximal proper left (right) ideal is open.

Corollary. If S is a compact connected mob and J a maximal proper ideal of S, then J is dense in S.
Proof:
Since J is open and $\bar{J}$ an ideal of S, the maximality of J and the connectedness of S imply $\bar{J} = S$.

Let S be the multiplicative semigroup of real numbers, with the usual topology. Then $\{0\}$ is the only proper ideal of S. Hence $\{0\}$ is a maximal proper ideal which is not open. Furthermore if $A = (-1,1)$ then $J_o(A) = \{0\}$.

1.2.4. Lemma. If S is a compact mob, then J(a) is compact for each $a \in S$. The same holds for L(a) and R(a).
Proof:
Since S is compact $\{a\}$, aS, Sa and SaS are compact subsets of S.

1.2.5. Theorem. Is S is a mob and S has a minimal left and minimal right ideal, then S has a minimal ideal K and

1) If A_1 and A_2 are both in $\mathcal{L}(S)$ or both in $\mathcal{R}(S)$ and $A_1 \cap A_2 \neq \emptyset$ then $A_1 = A_2$.
2) If $L \in \mathcal{L}(S)$ then $La = Sa = L$ for all $a \in L$.
   If $R \in \mathcal{R}(S)$ then $aR = aS = R$ for all $a \in R$.
3) $K = \bigcup \{L \mid L \in \mathcal{L}(S)\} = \bigcup \{R \mid R \in \mathcal{R}(S)\}$.

Proof:

1) If A_1 and A_2 are in $\mathcal{L}(S)$ and $A_1 \cap A_2 \neq \emptyset$, then $A_1 \cap A_2$ is a left ideal of S and thus $A_1 = A_1 \cap A_2 = A_2$.

2) If $a \in L$, La is a left ideal contained in L, hence $La \subset Sa \subset L$, which implies $La = Sa = L$.

The same argument holds for right ideals.

3) If $L_1 \in \mathcal{L}(S)$ and $a \in S$, then $L_1 a$ is a left ideal of S and $L_1 a \in \mathcal{L}(S)$.

For if L_o were a left ideal properly contained in $L_1 a$, then $L_1 \cap \{x \mid xa \in L_o\}$ would be a left ideal properly contained in L_1.

Thus $\bigcup\{L_1 a \mid a \in S\} = L_1 S$ is a union of minimal left ideals and is an ideal of S.

Now let I be any ideal of S, then $L_1 = IL_1 \subset I$, hence I contains L_1 and thus $L_1 S = \bigcup\{L_1 a \mid a \in S\}$, which must by definition be the kernel K of S.

Furthermore any $L_2 \in \mathcal{L}(S)$ must be contained in K.

So by 1) L_2 must be equal to $L_1 a$ for some $a \in S$.

Thus $K = \bigcup\{L \mid L \in \mathcal{L}(S)\}$.

In the same way we prove $K = \bigcup\{R \mid R \in \mathcal{R}(S)\}$.

Let S be the multiplicative semigroup of real numbers x, $0 < x < 1$, with the usual topology.

The kernel K of S is empty, since for any $a \in S$, the set $(0,a)$ is an ideal of S, and hence $K = \bigcap_{0<a<1} (0,a) = \emptyset$.

On the other hand let S be the cube in E_2, i.e.

$S = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$, and define a multiplication in S by $(x_1,y_1)\cdot(x_2,y_2) = (0,y_2)$. Then S is a compact mob and the minimal left ideals are precisely the points $(0,y)$, while the set $R = \{(0,y) \mid 0 \leq y \leq 1\}$ is the only minimal right ideal and $K = R$.

1.2.6. Theorem. If S satisfies the conditions of 1.2.5, then

1) If $L \in \mathcal{L}(S)$ and $R \in \mathcal{R}(S)$, then $L \cap R$ is a subgroup of S.

2) $\mathcal{L}(S) = \{Se \mid e \in K \cap E\}$, $\mathcal{R}(S) = \{eS \mid e \in K \cap E\}$.

3) $K = \bigcup\{H(e) \mid e \in K \cap E\}$ and for $e \in K \cap E$, $H(e) = eSe$.

Any pair $H(e_1)$, $H(e_2)$ of subgroups with $e_1, e_2 \in E \cap K$ are isomorphic.

Proof:

1) Choose $L \in \mathcal{L}(S)$ and $R \in \mathcal{R}(S)$.

Then $RL \subset L \cap R$, so $L \cap R \neq \emptyset$. Furthermore if $a \in L \cap R$, then $(L \cap R)a = L \cap R$ and $a(L \cap R) = L \cap R$.

For it is clear that $(L \cap R)a \subset L \cap R$, and if the inclusion were proper then $La = \bigcup\{(L \cap R)a \mid R \in \mathcal{R}(S)\} \neq \bigcup\{(L \cap R) \mid R \in \mathcal{R}(S)\} = L = La$ is a contradiction.

The equality $a(L \cap R) = (L \cap R)$ follows similarly and hence $L \cap R$ is a subgroup of S.

2) Let e be the unit element of $L \cap R$, then 1.2.5 implies $L = Le = Se$ and $R = eR = eS$.

3) $L \cap R = Se \cap eS \supset eSe = eL \supset e(L \cap R) = L \cap R$.

Hence $L \cap R = eSe$.

Now let $H(e)$ be the maximal subgroup containing $e \in E \cap K$.

Then $H(e) = eH(e)e \subset eSe = L \cap R$, so $H(e) = L \cap R = eSe$ and $K = \bigcup\{L \mid L \in \mathcal{L}(S)\} = \bigcup\{R \mid R \in \mathcal{R}(S)\} = \bigcup\{H(e) \mid e \in E \cap K\}$.

We shall now prove that any pair $H(e_1)$, $H(e_2)$ with $e_1, e_2 \in E \cap K$ are topologically isomorphic.

It is clear that if $H(e_1) \subset L$ and $H(e_2) \subset L$, then $e_2e_1 = e_2f = e_2$ for any $f \in E \cap L$.

Let ϕ: $H(e_1) \to L$ be defined by $\phi(x) = e_2x$ and suppose $e_2x \in H(f)$, $f \in E \cap L$.

Let $\bar{x}$ be the inverse of e_2x in $H(f)$. Thus $e_2x\bar{x} = \bar{x}e_2x = f$.

And so $e_2f = e_2^2x\bar{x} = f$, hence $f = e_2$.

It is clear then that ϕ is a map of $H(e_1)$ onto $H(e_2)$ and we easily verify that ϕ is a homomorphism.

If $e_2x = e_2y$, then $e_1e_2x = e_1e_2y$, so $e_1x = e_1y$ and $x = y$.

Hence ϕ is an isomorphism.

Since $\phi^{-1}(x) = e_1x$; $x \in H(e_2)$, ϕ and ϕ^{-1} are both continuous and $H(e_1)$ and $H(e_2)$ are isomorphic.

In the same way $H(e_1)$ and $H(e_2)$ both in R implies $H(e_1)$ and $H(e_2)$ isomorphic.

Suppose now $H(e_1) = L_1 \cap R_1$ and $H(e_2) = L_2 \cap R_2$, then $H(e_1)$ is isomorphic with $L_1 \cap R_2$ and $H(e_2)$ isomorphic with $L_1 \cap R_2$ and it follows that $H(e_1)$ and $H(e_2)$ are isomorphic.

1.2.7. Theorem. Let S be a compact mob. Then each left ideal of S contains at least one minimal left ideal of S and each minimal left ideal is closed. The same holds for right ideals.

Proof:

Let L be any left ideal of S and let T be the collection of all closed left ideals of S contained in L. T is partially ordered by inclusion and is non-void, since if $x \in L$, Sx is a closed left ideal contained in L.

Suppose $\{T_\alpha\}_\alpha$ is a linearly ordered subcollection of T.

Then $\bigcap_\alpha T_\alpha$ is non-empty since S is compact and so is an ideal in L.

Thus $\{T_\alpha\}_\alpha$ has a lower bound and Zorn's lemma assures the existence of a minimal L_o in T.

Now let L_1 be a left ideal contained in L_o and let $x \in L_1$.

Then Sx is a closed left ideal. Furthermore $Sx \subset L_1 \subset L_o$ and since L_o is minimal in T we have $Sx = L_o = L_1$. Thus L_o is a minimal left ideal.

The proof of the assertion for right ideals is completely analogous.

Corollary. Each compact mob S has a minimal ideal K, and if S is commutative, then K is a compact topological group.

Proof:

If S is commutative and J_1 and J_2 are minimal ideals then $J_1 \cap J_2$ is non empty since it contains J_1J_2.

Thus $J_1 = J_2$ and $J_1 \cap J_2 = J_1$ is a subgroup of S.

Since $K = J_1$, K is a subgroup of S. Furthermore K is compact and hence a topological group.

1.2.8. Lemma. Let S satisfy the conditions of theorem 1.2.5.

Then $K = (Se \cap E).eSe.(eS \cap E)$, $e \in E \cap K$.

Proof:

Since $e \in K$, we have $(Se \cap E).eSe.(eS \cap E) \subset K$.

Now let $k \in K$. Then $k \in H(f)$ with $f \in E \cap K$.

Suppose $H(g_1) = Se \cap fS$ and $H(g_2) = Sf \cap eS$, $g_1, g_2 \in E \cap K$.

Then $g_1 e = g_1$, $eg_2 = g_2$. Furthermore since $fS = g_1 S$ and $Sf = Sg_2$ we have $g_1 f = f$, $fg_2 = f$.

Hence $k = fkf = g_1 fkfg_2 = g_1 kg_2 = g_1 ekeg_2 \in (Se \cap E).eSe.(eS \cap E)$.

This implies that $K = (Se \cap E).eSe.(eS \cap E)$.

1.2.9. Theorem. Let S be a compact mob, and let K be the kernel of S and let $e \in K \cap E$.

Let $K^* = (Se \cap E) \times eSe \times (eS \cap E)$, with the multiplication $(x_1, y_1, z_1).(x_2, y_2, z_2) = (x_1, y_1 z_1 x_2 y_2, z_2)$.

Then K^* is a compact mob and K^* is isomorphic with K.

Proof:

According to lemma 1.2.8 $K = (Se \cap E).eSe.(eS \cap E)$.

Define ϕ: $K^* \to K$ by $\phi(x,y,z) = xyz$. Then ϕ is clearly a continuous mapping of K^* onto K.

Next let $x_1 y_1 z_1 = x_2 y_2 z_2$, with $x_1, x_2 \in Se \cap E$, $y_1, y_2 \in eSe$, $z_1, z_2 \in eS \cap E$.

Then since $x_1 S$ and $x_2 S$ are minimal right ideals with $x_1 S \cap x_2 S \neq \emptyset$, we have $x_1 S = x_2 S$ and thus $x_1 x_2 = x_2$.

Furthermore since $Se = Sx_1 = Sx_2$, $x_1 e = x_1$, $x_2 e = x_2$, $ex_1 = ex_2 = e$.

Hence $x_2 = x_1 x_2 = x_1 ex_2 = x_1 e = x_1$.

In the same way we prove $z_1 = z_2$.

Since $x_1 y_1 z_1 = x_1 y_2 z_1$ we have $ex_1 y_1 z_1 e = ex_1 y_2 z_1 e$, thus $ey_1 e = ey_2 e$ and so $y_1 = y_2$.

Hence ϕ is a one-to-one continuous map of K^* onto K.

Since $\phi(x_1, y_1, z_1).\phi(x_2, y_2, z_2) = x_1 y_1 z_1 x_2 y_2 z_2 = \phi(x_1, y_1 z_1 x_2 y_2, z_2) = \phi\{(x_1, y_1, z_1).(x_2, y_2, z_2)\}$ we have that K and K^* are topologically isomorphic mobs.

1.2.10. Theorem. If S is a compact mob, then the minimal ideal K is a retract of S.

Proof:

Define $f : S \to K$ by $f(x) = \alpha(xe).exe.\alpha(ex)$, $e \in E \cap K$, where $\alpha(xe)$ is

the unit of the unique maximal subgroup containing xe.
Then by theorem 1.1.17 and lemma 1.2.8 we may conclude that f is a continuous map of S into K.
Now let $k \in K$, then $k = g_1ekeg_2$, with $g_1 \in Se \cap E$ and $g_2 \in eS \cap E$ and $g_2e = e$, $eg_1 = e$.
$ke = g_1ekeg_2e = g_1eke \in g_1H(e) = H(g_1)$. Hence $\alpha(ke) = g_1$.
$ek = eg_1ekeg_2 = ekeg_2 \in H(e)g_2 = H(g_2)$. Hence $\alpha(ek) = g_2$.
$eke = eg_1ekeg_2e = eke$.
Thus if $k \in K$, then $f(k) = g_1ekeg_2 = k$ and f is a retraction of S onto K.

Corollary: If S is compact and S has the fixed point property then $K \subset E$.

Proof:

Let $e \in K$. Then $H(e) = eSe$ is a retract of S. Hence H(e) is a topological group with the fixed point property and thus $H(e) = e$.

1.2.11. Theorem. Let S be a compact mob and let $e \in E$. Then the following conditions are equivalent:

1) Se is a minimal left ideal.
2) SeS is the minimal ideal of S.
3) eSe is a maximal subgroup.

Proof:

1) → 2). If Se is a minimal left ideal, then $Se \subset K$ by theorem 1.2.5, hence $e \in K$. Since SeS is an ideal of S and $SeS \subset K$ we have $SeS = K$.
2) → 3). If $SeS = K$, then $e \in K$ and 1.2.6 implies that $H(e) = eSe$ is a maximal subgroup.
3) → 1). Let L be a left ideal contained in Se, and let $a \in L \cap eS$. Then since $a \in Se \cap eS = eSe$, there is an element $a^{-1} \in eSe$ such that $a^{-1}a = e$.
Hence $a^{-1}a = e \in a^{-1}L \subset L$. Thus $Se \subset L$ and $L = Se$.

Remark.

If the mob S contains a zero element 0, then theorem 1.2.5, 1.2.6, 1.2.9 and 1.2.11 become trivial, since then {0} is the minimal (left, right) ideal of S.

1.3. Simple semigroups

Definitions. A mob S is called (left, right) simple if S does not contain a proper (left, right) ideal.
The theory of simple mobs S with a zero element becomes trivial, because in this case S is simple if and only if $S = \{0\}$.
For this reason we introduce the notion of 0-simplicity.
A mob S with zero is called (left, right) 0-simple if $\{0\}$ is the only proper (left, right) ideal of S and $S^2 \neq \{0\}$.

If S is a mob with zero, such that $\{0\}$ is the only proper ideal, then either S is 0-simple or S is the zero semigroup of order two.
For evidently S is 0-simple or $S^2 = \{0\}$. In the latter case, if $S = \{0\}$, then $\{0\}$ is not a proper ideal of S, hence $S \neq \{0\}$.
But then if a is any element $\neq 0$ of S, $\{0,a\}$ is an ideal of S and so $S = \{0,a\}$.

1.3.1. Lemma. A necessary and sufficient condition for a mob S to be (0-)simple is that $SxS = S$ for all non-zero x of S.

Proof:
The condition is sufficient, since if I is a non-zero proper ideal of S and if $x \neq 0$, $x \in I$ we have $SxS \subset I$, which contradicts $SxS = S$.
Suppose now that S is (0-)simple and that the condition is not satisfied. Then there exists an element $x \neq 0$ such that $SxS = \{0\}$, since SxS is an ideal of S. Let X be the set of all such x. Then clearly $XS \subset X$ and $SX \subset X$. Thus X is an ideal of S which contains $x \neq 0$, hence $X = S$.
But then $S^3 = SXS = \{0\}$, so $S^2 = \{0\}$ a contradiction.

An analogous condition holds for a left or right (0-)simple mob, i.e. S is left (right) (0-)simple if and only if $Sx = S$ ($xS = S$) for all non-zero x of S.
Furthermore it follows that S is both left and right simple if and only if S is a group.

1.3.2. Theorem. If S is a right 0-simple mob, then $S \setminus \{0\}$ is a right simple submob of S.

Proof:

Suppose that $a,b \in S \setminus \{0\}$ and that $ab = 0$. Then the set of all x in S such that $ax = 0$ is a non-zero right ideal of S, hence coincides with S. But then $aS = \{0\}$, contrary to lemma 1.3.1.
Thus $S \setminus \{0\}$ is a submob of S. Since $aS = S = a(S \setminus \{0\}) \cup \{0\}$ for all $a \in S \setminus \{0\}$, it follows that $a(S \setminus \{0\}) = S \setminus \{0\}$, and we conclude that $S \setminus \{0\}$ is right simple.

Theorem 1.3.2 shows that there is no essential difference between right simple and right 0-simple, since every right 0-simple semigroup arises from a right simple semigroup by the adjunction of a zero element. (The topology, however, need not be the sum topology).
On the other hand we have that every simple mob with zero adjoined is a 0-simple mob. The converse however does not hold.

Definition. An idempotent e of a mob S is called primitive if $f^2 = f \in eSe$ implies $f = 0$ or $f = e$.
Definition. A mob S is called completely (0-)simple if S is (0-)simple and contains a non-zero primitive idempotent.

If S is a commutative (0-)simple mob then S is a group or a group with zero (i.e. $S = G \cup \{0\}$, where G is an abstract group and $0g = g0 = 0$ for all $g \in G$).
Furthermore we see that in the latter case S contains exactly one non-zero idempotent, hence S is completely (0-)simple.

Corollary. If $K \neq \emptyset$ is the kernel of a mob S, then K is a simple mob.
For since K is the minimal ideal of S and KaK an ideal of S contained in K for all $a \in K$, we have $KaK = K$, $a \in K$.
If moreover K is compact, then K is completely simple.
For let e and f be two idempotents in K, with $e \neq f$. Then either $Se \cap Sf = \emptyset$ or $eS \cap fS = \emptyset$ and hence either $fe \neq f$ or $ef \neq f$.

1.3.3. Lemma. If S is (0-)simple and e an idempotent of S, then eSe is (0-)simple.

Proof:

If eSe = {0}, then the lemma is trivial.

Suppose now eSe ≠ {0} and let exe be any non-zero element of eSe. Then since S is (0-)simple S = SexeS.

Hence eSe = eSexeSe = (eSe).exe.(eSe) and lemma 1.3.1 implies that eSe is (0-)simple.

1.3.4. Lemma. If S is (0-) simple and e is a primitive idempotent, then eSe is either a group or a group with zero.

Proof:

Since eSe is (0-)simple there exist non-zero elements $a_x, b_x \in eSe$ such that $a_x x b_x = e$ for any $x \neq 0$, $x \in eSe$.

Then $x b_x a_x$ and $b_x a_x x$ are non-zero idempotents in eSe.

Hence $x b_x a_x = e$ and $b_x a_x x = e$. This, however, implies that eSe \ {0} is a group.

1.3.5. Theorem. If S is completely (0-)simple, then all idempotents of S are primitive.

Proof:

Let e,f be two non-zero idempotents of S with e primitive.

Since S is simple there exist elements a,a' ε S such that aea' = f.

We may assume fa = ae = a, a'f = ea' = a'.

Furthermore (a'a)(a'a) = a'(aea')a = a'fa = a'a. Hence a'a is an idempotent contained in eSe, which implies a'a = e.

Now the correspondence x → x', where x ε eSe and x' ε fSf, which is defined by the equivalent relations x = a'x'a and x' = axa' is an algebraic isomorphism between eSe and fSf.

Hence fSf is a group or a group with zero, and thus f primitive.

Corollary. A completely (0-)simple mob S with identity u is either a group or a group with zero.

For by theorem 1.3.5 u is primitive and hence lemma 1.3.4 implies that $uSu = S$ is a group or a group with zero.

1.3.6. Theorem. A compact (0-)simple mob S is completely (0-)simple.

Proof:

Let $a \neq 0$ be any element of S. Then $a = bac$ for suitably chosen $b, c \in S$. Hence $a = b^n a c^n$, $n=1,2,\dots$.

Let $e = e^2 \in \Gamma(b)$. Then by lemma 1.1.4 there is an element $c' \in \{c^i \mid i=1,2,\dots\}$ such that $eac' = a$.

Hence $ea = a$ and $e \neq 0$.

Now let $f \neq 0$ be any idempotent in eSe. Then since eSe is a compact (0-)simple mob, we can again apply 1.1.4.

Hence there is an idempotent $g \in eSe$ and an element g' such that $gfg' = e$.

Since e is the identity of eSe we have $g = ge = ggfg' = gfg' = e$ and $fg' = efg' = gfg' = e$.

Henceforth $f = fe = ffg' = fg' = e$.

Thus e is the only idempotent $\neq 0$ contained in eSe and e must be primitive.

Let I be an ideal of the abstract semigroup S. Then the Rees quotient S / I is the abstract semigroup which consists of the set $S \setminus I$ together with an element 0.

The multiplication $\circ$ in S / I is defined in the following way

$$a \circ b = ab \quad \text{if } a,b,ab \in S \setminus I,$$
$$a \circ b = 0 \quad \text{if } ab \in I,$$
$$a \circ b = 0 \quad \text{if } a = 0 \text{ or } b = 0.$$

If S is a mob and I a closed compact ideal of S, then we can make S / I into a mob such that the natural map of S onto S / I is continuous. We take for S / I the space which we get from S by identifying I to a single point 0, with the quotient topology.

1.3.7. Theorem. Let S be a mob and let J and J^* be ideals of S with $J^* \subset J$ such that there is no ideal of S lying properly between them. Then J / J^* is either an abstract 0-simple semigroup or a zero semigroup.

Proof:

Since $J^* \cup J^2$ is an ideal of S and $J^* \subset J^* \cup J^2 \subset J$ we have $J^2 \subset J^*$ or $J^* \cup J^2 = J$.

If $J^2 \subset J^*$ then J / J^* is a zero semigroup.

Next let $J^* \cup J^2 = J$, then $J^* \cup J^3 = J$. Let I be an ideal of J / J^*, $I \neq \{0\}$. $I^* = (I \setminus \{0\}) \cup J^*$ is an ideal of J properly containing J^*.

Hence since $I^* \cup SI^* \cup I^*S \cup SI^*S$ is an ideal of S, we have $I^* \cup SI^* \cup I^*S \cup SI^*S = J$ and thus $JI^*J \cup JSI^*J \cup JI^*SJ \cup JSI^*SJ = JI^*J = J^3$.

This implies that $J^* \cup JI^*J = J^* \cup J^3 = J$.

On the other hand we have $J^* \cup JI^*J \subset I^*$, hence $I^* = J$ and it follows that J / J^* is a 0-simple semigroup.

Corollary. An ideal J of a mob S is a maximal proper ideal of S if and only if S / J is either a 0-simple semigroup or the zero semigroup of order two.

Proof:

It follows from theorem 1.3.7 that if J is maximal, then S / J is 0-simple or a zero semigroup.

Suppose now that $S \setminus J$ contains more than one element and that $(S / J)^2 = \{0\}$. Let $a \in S \setminus J$, then $J \cup \{a\}$ is a proper ideal of S containing J, which is a contradiction.

1.3.8. Theorem. Let J be a maximal proper ideal of the compact mob S. Then S / J is either the zero semigroup of order two or else completely 0-simple.

Proof:

Let S / J be 0-simple. Then by 1.3.1 we have $(S / J)a(S / J) = S / J$ for every $a \in S / J$, $a \neq 0$.

Thus $xay = a$ for suitably chosen $x, y \in S \setminus J$ and it follows that

$x^n a y^n = a$, $n=1,2,\dots$.
Hence $a = eay'$ with $e = e^2 \in \Gamma(x)$, $y' \in \Gamma(y)$.
Since $e \in S \setminus J$ we conclude that S / J contains a non-zero idempotent.
Now let $f^2 = f \in e.S/J.e$. Then since $e.S/J.e$ is 0-simple and $e.S/J.e$ isomorphic with eSe / eJe, it follows in the same way that there are elements a and b such that $e = afb$ with $a = af$.
Furthermore $aeb = afeb = afb = e$. Hence $e = a^n f b^n$, $n=1,2,\dots$.
Thus $e = gfb'$ with $g = g^2 \in \Gamma(a)$, $b' \in \Gamma(y)$.
Since $g = ge = ggfb' = e$ and $fb' = efb' = gfb' = e$, we have $f = fe = ffb' = e$.
Henceforth e is primitive and S / J completely 0-simple.

1.3.9. Lemma. Let S be a mob without zero having at least one minimal left ideal L. Then S is the sum of its minimal left ideals if and only if S is simple.

Proof:
Let S be simple. According to 1.2.5, the sum of all minimal left ideals of S is an ideal I of S and thus I = S.
Conversely if S is the sum of its minimal left ideals, then again by 1.2.5 S is its own minimal ideal and hence simple.

1.3.10. Theorem. Let H be a compact topological group and X and Y two compact Hausdorff spaces. Let $\phi: Y \times X \to H$ be a continuous function and denote by $[X,H,Y,\phi]$ the space $X \times H \times Y$ with the multiplication $(x_1,h_1,y_1)(x_2,h_2,y_2) = (x_1,h_1\phi(y_1,x_2)h_2,y_2)$. Then $[X,H,Y,\phi]$ is a compact simple mob.
On the other hand if S is a compact simple mob and $e \in S \cap E$, then S is isomorphic with $[Se \cap E, H(e), eS \cap E, \phi]$ where $\phi(e_1,e_2) = e_1e_2$, $e_1 \in eS \cap E$, $e_2 \in Se \cap E$.

Proof:
The second part of the theorem follows immediately from theorem 1.2.9.
Next let $[X,H,Y,\phi]$ be given. The multiplication defined in $[X,H,Y,\phi]$ is clearly continuous and associative.
Thus $[X,H,Y,\phi]$ is a compact mob.

Now let (x,h,y) and $(x',h',y') \in [X,H,Y,\phi]$. Choose elements $y_o \in Y$ and $x_o \in X$ and let h_o and h'_o be such that $h_o\phi(y_o,x)h\phi(y,x_o)h'_o = h'$. Then $(x',h_o,y_o)(x,h,y)(x_o,h'_o,y') = (x',h',y')$.
Hence $[X,H,Y,\phi](x,h,y)[X,H,Y,\phi] = [X,H,Y,\phi]$ for all (x,h,y) and henceforth $[X,H,Y,\phi]$ is simple.

1.3.11. Lemma. If S is a compact mob and A a (left, right) simple submob, then $\bar{A}$ is also a (left, right) simple mob.

Proof:

$\bar{A}$ is a submob of S, hence $\bar{A}x\bar{A} \subset \bar{A}$ for all $x \in \bar{A}$.
Now let A be simple and suppose there exists an $x \in \bar{A}$ such that $\bar{A}x\bar{A} \neq \bar{A}$.
Then there exist $y \in \bar{A}$, $y \notin \bar{A}x\bar{A}$ and neighbourhoods V of x and W of y such that $W \cap \bar{A}V\bar{A} = \emptyset$.
Since $y,x \in \bar{A}$ there are elements $a_1 \in A \cap V$ and $a_2 \in A \cap W$ with $a_2 \notin \bar{A}a_1\bar{A}$. This contradiction concludes the proof.
A similar argument applies to right and left simple mobs.

1.3.12. Theorem. Let S be a compact left simple mob. Then the right translation ρ_a: $x \to xa$ is a homeomorphism.

Proof:

According to theorem 1.2.6 $S = \bigcup\{H(e) \mid e \in E\}$, while from $Se = S$ for all $e \in E$ we infer that e is a right unit for S.
Now suppose $xa = ya$, $a \in H(e)$. Let a^{-1} be the inverse of a in H(e), then $xaa^{-1} = yaa^{-1}$, hence $xe = ye$ and thus $x = y$.
On the other hand since $(xa^{-1})a = x$ it follows that ρ_a is a mapping of S onto S.
If we recall that S is compact, it follows that ρ_a is a homeomorphism.

1.3.13. Theorem. Every left simple submob of a mob S is contained in a maximal left simple submob of S and each two maximal left simple submobs are disjoint.
If S is compact each maximal left simple submob is closed.

Proof:

Let S^* be a left simple submob of S and let T be the collection of all left simple mobs containing S^*. Let $\{T_\alpha\}_\alpha$ be a linearly ordered subcollection and $T^* = \bigcup_\alpha T_\alpha$.

Then T^* is left simple, for if $x \in T^*$, then $x \in T_\alpha$ for some α and hence $T_\alpha x = T_\alpha$. Thus since $T^* = \bigcup \{T_\beta \mid T_\alpha \subset T_\beta\}$ we have $T^* = \bigcup \{T_\beta x \mid T_\alpha \subset T_\beta\} = T^* x$.

Using Zorn's lemma there is a maximal element in the collection of all left simple mobs containing S^*.

Next let S_1 and S_2 be two maximal left simple submobs and suppose $x \in S_1 \cap S_2$. Let A be the mob generated by S_1 and S_2; i.e. A is the collection of all finite products $s_1 s_2 s_3 \ldots s_n$ with $s_i \in S_1$ or S_2, $i=1,2,\ldots,n$.

Let $y_1, y_1' \in S_1$ and $y_2 \in S_2$, then $S_1 y_1 = S_1 y_1' = S_1 x = S_1$ and $S_2 x = S_2$.

Hence $y_1' = s_o y_1$, $x = s_1 y_1$ and $y_2 = s_2 x$, $s_o, s_1 \in S_1$, $s_2 \in S_2$.

Thus $y_2 = s_2 s_1 y_1$ and we have $S_1 \subset Ay_1 \subset A$, $S_2 \subset Ay_1 \subset A$, and it follows that $A = Ay_1$ since Ay_1 is a submob of S containing S_1 and S_2.

In the same way we prove $A = Ay_2$ and thus that $A = Aa$ for every $a \in A$.

Since A is left simple and S_1 and S_2 are maximal, we have $S_1 = S_2 = A$.

Analogously it is possible to prove that every simple submob of a mob S is contained in a maximal simple submob. But here two maximal simple submobs may have a non empty intersection.

Let for instance $S = \{a_1, a_2, a_3, a_4, a_5\}$ with the following multiplication table

	a_1	a_2	a_3	a_4	a_5
a_1	a_5	a_1	a_5	a_3	a_5
a_2	a_5	a_2	a_5	a_4	a_5
a_3	a_1	a_1	a_3	a_3	a_5
a_4	a_2	a_2	a_4	a_4	a_5
a_5	a_5	a_5	a_5	a_5	a_5

Then S is a completely 0-simple mob with zero a_5. $S_1 = \{a_2, a_4\}$ and $S_2 = \{a_3, a_4\}$ are two maximal simple submobs with $S_1 \cap S_2 \neq \emptyset$.

1.3.14. Theorem. Let e be an idempotent of the compact mob S without zero, then the following conditions are equivalent:

1) e is primitive.
2) Se is a minimal left ideal.
3) SeS is the minimal ideal.
4) eSe is a maximal subgroup.
5) Each idempotent of SeS is primitive.

Proof:

1) $\rightarrow$ 2). If Se is not minimal, then there exists an idempotent f with $Sf \subset Se$ and Sf a minimal left ideal (1.2.6 and 1.2.7). Hence $fe = f$ and since $(ef)(ef) = eff = ef$, ef is an idempotent contained in eSe. Thus $ef = e$ and we have $e \in Sf$ which implies $Se \subset Sf$.

2) $\rightarrow$ 3) $\rightarrow$ 4) $\rightarrow$ 2). Theorem 1.2.11.

4) $\rightarrow$ 1). Since eSe is a group, eSe contains only one idempotent and it follows that e is primitive.

5) $\rightarrow$ 1). Trivial.

1) $\rightarrow$ 5). Let $f \in SeS$, then since $SeS = K$, we have $SfS = K$ and thus f primitive.

Remark.

In a compact mob without zero, idempotents are primitive if and only if they are contained in K.

1.3.15. Theorem. Let S be a compact simple mob and let S' be a locally compact submob of S.
Then S' is a locally compact simple mob.

Proof:

Since $S = \cup \{H(e) \mid e \in E\}$ we have $S' = \cup \{H(e) \cap S' \mid e \in E\}$.
Let $H^*(e) = H(e) \cap S' \neq \emptyset$. Then since $H(e)$ is compact, $H^*(e)$ is a locally compact submob of the compact group $H(e)$.
Theorem 1.1.12 implies that $H^*(e)$ is a compact group.

Now let L_α be a minimal left ideal of S and let $L_\alpha^* = L_\alpha \cap S' \neq \emptyset$. It is obvious that L_α^* is a left ideal of S'.
We now prove that L_α^* is a minimal left ideal of S'.
For let $L \subset L_\alpha^*$ be a left ideal of S'. Then since $L_\alpha = \cup \{H(e) \mid e \in E \cap L_\alpha\}$ we have $L_\alpha^* = \cup \{H^*(e) \mid e \in E \cap L_\alpha^*\}$ and consequently there is an idempotent $e' \in L_\alpha^*$ such that $L \cap H^*(e') \neq \emptyset$.
Since a group contains no proper left ideals we have $H^*(e') \subset L$ and hence $e' \in L$. Thus $S'e' \subset L$.
On the other hand we have $e' \in L_\alpha$ and it follows that e' is a right identity for L_α. Hence $L_\alpha^* = L_\alpha^* e' \subset S'e' \subset L$.
This proves that L_α^* is a minimal left ideal of S'.
Since $S' = \cup \{L_\alpha^* \mid L_\alpha \in \mathcal{L}(S)\}$ it follows by lemma 1.3.9 that S' is a simple submob.

Example. Let S be the additive group of real numbers mod 1 with the usual topology and let α be any irrational number, $0 < \alpha < 1$.
Then $S' = \{n\alpha\}_{n=1}^{\infty}$ is a submob of S. S' is not locally compact and not simple since $S' + \alpha \neq S'$.

1.4. Maximal ideals

We have seen in 1.3, that if S is a compact mob which contains properly a (left, right) ideal, then it contains a maximal proper (left, right) ideal J which is open.

1.4.1. Lemma. Let S be a compact mob and suppose E is contained in a maximal proper ideal J, then $S^2 \subset J$.

Proof:

It follows from 1.3.8 that S / J is either completely 0-simple or the zero semigroup of order 2.
Since $E \subset J$, S / J contains no idempotent other than 0 and hence $S / J \cong O_2$, i.e. $S = J \cup \{a\}$ with $a^2 \subset J$.
And thus $S^2 = J^2 \cup Ja \cup aJ \cup \{a^2\} \subset J$.

Corollary. Let S be compact with $S^2 = S$, then $SES = S$.
For if SES is a proper subset of S, we have since SES is an ideal that SES and hence E is contained in a maximal proper ideal. Lemma 1.4.1 then implies that $S = S^2 \subset J$; a contradiction.

1.4.2. Theorem. Let S be a compact mob with $S^2 = S$ and suppose that S has a unique idempotent.
Then S is a topological group.

Proof:
Let $e = e^2$, then $e \in K$ and K is a group. The preceding corollary implies that $S = SeS = K$, completing the proof.

Definition. A mob S has the (left, right) maximal property if there exists a maximal proper (left, right) ideal $(L^*, R^*) J^*$ containing every (left, right) ideal of S different from S.

1.4.3. Lemma. Let S be a mob and A a compact part of S. If $A \subset Ax$ with $\Gamma(x)$ compact, then $A = Ax = Ae$ with $e = e^2 \in \Gamma(x)$.

Proof:
$A \subset Ax \subset Ax^2 \subset \dots$.
Suppose now $Ax^k \not\subset Ae$ with $e = e^2 \in \Gamma(x)$. Then there is an $a \in A$ with $ax^k \notin Ae$, and there is a neighbourhood W of e such that $ax^k \notin AW$. But since e is a cluster point of $\{x^n\}_{n=1}^{\infty}$, there is a $k_o \geq k$ with $x^{k_o} \in W$. Hence $ax^k \notin Ax^{k_o}$, which is a contradiction.
We now have $A \subset Ax \subset Ae$, where $e^2 = e$; therefore $A = Ae$ and $A = Ax = Ae$.

It follows that for every $a \in K(x)$, we have $Aa = Ae = A$.
Now let $y \in \Gamma(x)$. Then since $K(x) = e\Gamma(x)$, we have $Ay = (Ae)y = A$.
Hence we have for all $y \in \Gamma(x)$, $A = Ay$.
Furthermore the mapping $\rho_y : a \rightarrow ay$, $a \in A$, is a homeomorphism.
ρ_y is clearly continuous and also one-to-one. For if $a_1y = a_2y$, then since $a_1e = a_1$ and $a_2e = a_2$, we have $a_1y = a_1(ey) = a_2(ey)$. Now let y^{-1} be the inverse in K(x) of (ey), then

$a_1(ey)y^{-1} = a_2(ey)y^{-1} \Longrightarrow a_1e = a_2e \Longrightarrow a_1 = a_2$.
Since A is compact it follows that ρ_y is a homeomorphism.

1.4.4. <u>Lemma</u>. Let S be a mob with a right unit element e and at least one proper left ideal. Then S has the left maximal property.

<u>Proof</u>:
Let L^* be the union of all proper left ideals. Then $L^* \neq \emptyset$ and L^* is a left ideal of S such that $e \notin L^*$.
For if $e \in L^*$, then $e \in L$ for some proper left ideal. But since e is a right unit, we have $S = Se \subset L$, a contradiction.
Therefore $L^* \neq S$, and it is obvious that L^* is the maximal left ideal of S.

We remark that 1.4.4 holds if right is replaced by left and vice versa. Also a similar argument shows that if S has a left or right unit and at least one proper ideal, then S has the maximal property.
From the proof of the lemma it also follows that in this case if S has a left unit, then R^* exists and $J^* \subset R^*$; if S has a right unit then $J^* \subset L^*$; and if S has a unit, then $J^* \subset L^* \cap R^*$.

1.4.5. <u>Theorem</u>. Let S be a compact mob. Then if L^* exists, there exists also J^* and we have $L^* = J^*$.
(The theorem also holds if L^* is replaced by R^*).

<u>Proof</u>:
Since for every $a \in S$, L^*a is a left ideal of S, we have $L^*a \subset L^*$ or $L^*a = S$.
Lemma 1.4.3 implies that if $L^*a = S$, then $Sa = S = Se$, and hence $L^*e = L^*$ with $e = e^2 \in \Gamma(a)$.
Now let K(a) be the set of cluster points of $\{a^n\}_{n=1}^{\infty}$. Then K(a) is a group and it follows from theorem 1.1.14, applied to $\Gamma(a)$, that $ae \in K(a)$.
Let a^* be the inverse of ae in K(a). Then we have $L^*a = S$ and thus $L^*aea^* = Sea^* \Longrightarrow L^*e = L^* = Sa^*$.
Since $e \in Sa^*$, we have $e \in L^*$ and hence $S = Se \subset L^*$, a contradiction.

Thus $L^* a \subset L^*$ for all $a \in S$.
But then it follows that $L^* S \subset L^*$. Hence L^* is an ideal of S which must be J^*, since every proper ideal of S is a proper left ideal of S and is contained in L^*.

1.4.6. Theorem. Let S be a compact mob and let P be the set of those elements $a \in S$ satisfying $aS = S$.
Then P is a closed submob of S and the left translation $\rho_a: x \to ax$, $a \in P$, $x \in S$, is a homeomorphism of S.
Furthermore $S \setminus P$ is an ideal of S and $P = \bigcup \{H(e) \mid e \in E \cap P\}$, while all $H(e)$, $e \in E \cap P$, are isomorphic.

Proof:
Let $a_1, a_2 \in P$ then $a_1 a_2 S = a_1 S = S$, and thus $a_1 a_2 \in P$.
To show that P is closed take $x \notin P$ and $y \notin xS$.
Then we can find an open set U, with $x \in U$ and such that $y \notin US$. Then $x \in U \subset S \setminus P$.
Now let $ax = ay$, $x \neq y$, $a \in P$, then $S = aS = eS$ with $e = e^2 \in \Gamma(a)$ and e is a left unit for S.
From $ex = x$ and $ey = y$ we infer the existence of an open set U including e such that $Ux \cap Uy = \emptyset$.
Since $e \in \Gamma(a)$, we know that some $a^n \in U$. But since $a^n x = a^n y$ we must have $x = y$.
Now let $ab \in P$, then $abS = S$ and lemma 1.4.3 implies that $bS = S$ and $b \in P$. But then since $abS = S = aS$, $a \in P$, and it follows that $S \setminus P$ is an ideal.
We now prove that $P = \bigcup\{H(e) \mid e \in E \cap P\}$.
Let $a \in P$, then $S = aS = eS$ with $e = e^2 \in \Gamma(a)$ and hence $e \in P$.
Now let $K(a)$ be the set of cluster points of $\{a^n\}_{n=1}^{\infty}$. Then $K(a) = e\Gamma(a) = \Gamma(a) \subset H(e)$, since e is a left unit for S.
Hence $a \in H(e)$ and since for each $h \in H(e)$, $a = hh^*$ for suitably chosen $h^* \in H(e)$ we have $H(e) \subset P$.
Therefore $P = \bigcup\{H(e) \mid e \in E \cap P\}$.
Now let $e, f \in E \cap P$ and let $\phi: H(e) \to H(f)$ be the mapping defined by $\phi(x) = xf$.

It is clear that $xf \in P$. Suppose now $xf \in H(g)$, $g \in E \cap P$ and let x^* be the inverse of xf in $H(g)$. Then $x^*xf = g$ and thus $gf = g$. But since g is a left unit we also have $gf = f$. Hence $f = g$.
Furthermore since for each $y \in H(f)$ we have $ye \in H(e)$ and $\phi(ye) = yef = yf = y$. We see that ϕ is onto. ϕ is one-to-one since if $x_1f = x_2f$, then $x_1fe = x_2fe$, which implies $x_1 = x_2$.
We can also easily verify that ϕ is a homomorphism.
Since $H(e)$ and $H(f)$ are both compact, it follows that ϕ is topological.

P is a right simple submob. For we know that $aS = S$, $a \in P$, and hence there exists b' such that $ab' = b$ for every $b \in P$.
Theorem 1.4.6 then implies that $b' \in P$ and thus $aP = P$.

1.4.7. Theorem. Let S be a compact mob and let $S \neq P \neq \emptyset$.
Then $S \setminus P$ is the maximal proper ideal J^* of S.

Proof:

$S \setminus P$ is an ideal of S. Let e be an idempotent contained in P.
Then e is a left unit of S and 1.4.4 implies that J^* exists and $S \setminus P \subset J^*$. Furthermore we see that since P is simple $P \cap J^*$ must be empty. Therefore $S \setminus P = J^*$.

Corollary. If S is compact with unit u and if S is not a group, then $J^* = S \setminus H(u)$.

Proof:

Since $S = uS$ we have $H(u) \subset P$. Now let $e \in E \cap P$, then e is a left identity of S and hence $eu = u = e$. Therefore $P = H(u)$ and $J^* = S \setminus P = S \setminus H(u)$.

1.4.8. Theorem. Let S be a compact mob and suppose that R^* exists.
Then R^* is open and if $S \setminus R^*$ has more than one element or if S is connected, then $S \setminus R^* = P$.

Proof:

Let $a \in S \setminus R^*$. Then since $aS \cup \{a\}$ is a right ideal of S not contained in R^*, we have $aS \cup \{a\} = S$.

Hence $aS = R^*$ or $aS = S$. If $S \setminus R^*$ has more than one element, aS cannot be equal to R^*, hence $aS = S$.
If S is connected, then $S = aS \cup \{a\}$ if and only if $a \in aS$ and hence if $aS = S$.
So we have in both cases $aS = S$ for $a \in S \setminus R^*$.
Moreover it is clear that if $x \in S$, with $xS = S$, then $x \in S \setminus R^*$ and hence $S \setminus R^* = P$.

Corollary. Let S be a compact connected mob with R^*. Then S contains at least one left unit element.

1.4.9. Theorem. The necessary and sufficient condition that a connected compact mob S contains R^* is S has at least one left unit element and is not right simple.

Proof:
The necessity of the condition follows from the definition of R^* and the above corollary.
That the condition is sufficient follows from lemma 1.4.4.

1.4.10. Theorem. Let S be a compact mob and suppose that $S \setminus L^*$ and $S \setminus R^*$ have more than one element.
Then 1) S has a unit u.
2) $L^* = R^* = J^*$.
3) $S \setminus L^* = H(u)$.

Proof:
According to theorem 1.4.9 S has a left unit e_1 and a right unit e_2.
Hence $e_1e_2 = e_1 = e_2$ is a unit element of S.
That $L^* = R^* = J^*$ follows from theorem 1.4.5 and since S contains a unit and S is no group, we have $H(u) = S \setminus J^* = S \setminus L^*$.

1.4.11. Theorem. Let S be a connected compact mob, having at least one left unit and suppose S is not right simple.
Then every subgroup H(e), with e a left unit lies in the boundary of the maximal right ideal R^*.

Proof:
Since R^* is open and $\overline{R^*}$ a right ideal of S, we have $\overline{R^*} = S$ and $S \setminus R^* = \bigcup\{H(e) \mid e \text{ a left unit}\} = \text{boundary } R^*$.

1.4.12. Theorem. Let S be a compact mob and suppose that J^* exists. Then if $S \setminus J^*$ has more than one element or if S is connected $S \setminus J^* = \{a \mid SaS = S\}$.

Proof:
Let $a \in S \setminus J^*$, then since $SaS \cup \{a\}$ is an ideal of S not contained in J^*, we have $SaS \cup \{a\} = S$.
Hence $SaS = J^*$ or $SaS = S$. If $S \setminus J^*$ has more than one element then $SaS \neq J^*$. If S is connected, then since SaS is closed and J^* is open, we again have $SaS \neq J^*$.
On the other hand it is clear that if $a \in S$, with $SaS = S$, we have $a \notin J^*$.

Corollary. A necessary and sufficient condition that a compact connected mob S contains J^* is S has at least one idempotent with $S = SeS$ and S not simple.

Proof:
If S contains J^*, then $S^2 = S$ and thus S / J^* completely 0-simple.
Hence $S \setminus J^*$ contains an idempotent e, and $S = SeS$.
If on the other hand S is not simple and $S = SeS$ for an idempotent $e \in E$, then if $Q = \{a \mid SaS = S\}$, we have $Q \neq \emptyset$ and $S \setminus Q \neq \emptyset$.
Furthermore it is clear that $S \setminus Q$ is an ideal of S and that $J^* = S \setminus Q$.

Let S be the closed interval of real numbers $[-1,1]$, with the usual topology. Define a multiplication on S in the following way

$$x \cdot y = xy \qquad \text{if } x \geq 0,\ y \geq 0,$$
$$x \cdot y = 0 \qquad \text{if } x \leq 0,\ y \geq 0 \text{ or } x \geq 0,\ y \leq 0,$$
$$x \cdot y = -xy \qquad \text{if } x \leq 0,\ y \leq 0,$$

where xy is the usual product of x and y.
With this multiplication S becomes a compact mob.
The sets $[-1,1)$ and $(-1,1]$ are both maximal ideals of S.
J^* however does not exist in S.

1.5. Prime ideals

Definitions. A (right, left) ideal P of a mob S is said to be prime if $A.B \subset P$ implies that $A \subset P$ or $B \subset P$, A and B being ideals of S. An ideal P^* is completely prime if $ab \in P^*$ implies that $a \in P^*$ or $b \in P^*$, $a,b \in S$.

An ideal which is completely prime is prime, but the converse is not generally true.

Let for instance $S = \{e_1, e_2, a, b, 0\}$ with multiplication table

	e_1	e_2	a	b	0
e_1	e_1	0	0	b	0
e_2	0	e_2	a	0	0
a	a	0	0	e_2	0
b	0	b	e_1	0	0
0	0	0	0	0	0

Then $\{0\}$ is a prime ideal which is not completely prime.

In the case of commutative mobs, however, this concepts coincide. For let P be a prime ideal in a commutative mob and let $ab \in P$. Then $(\{a\} \cup aS)(\{b\} \cup bS) = \{ab\} \cup abS \subset P$ and hence $\{a\} \cup aS \subset P$ or $\{b\} \cup bS \subset P$.

Thus $a \in P$ or $b \in P$.

1.5.1. Lemma. If P is a left ideal of S, then the following conditions are equivalent:

1) P is a prime left ideal.
2) If $aSbS \subset P$, then $a \in P$ or $b \in P$.
3) If $R(a)R(b) \subset P$, then $a \in P$ or $b \in P$.
4) If R_1, R_2 are right ideals of S such that $R_1R_2 \subset P$, then $R_1 \subset P$ or $R_2 \subset P$.

Proof:

1) $\rightarrow$ 2). Let $aSbS \subset P$.

Then $R(a)^2R(b)^2 \subset aSbS \subset P$.

Hence $J(a)^2 J(b)^2 = (R(a)^2 \cup SR(a)^2)(R(b)^2 \cup SR(b)^2) =$
$R(a)^2R(b)^2 \cup SR(a)^2R(b)^2 \subset P$.
Since $J(a)^2$ and $J(b)^2$ are ideals of S we have $J(a)^2 \subset P$ or $J(b)^2 \subset P$.
If $J(a)^2 \subset P$, then $J(a) \subset P$ and hence $a \in P$.
2) $\rightarrow$ 3). If $R(a)R(b) \subset P$, then $aSbS \subset P$, hence $a \in P$ or $b \in P$.
3) $\rightarrow$ 4). Let $R_1R_2 \subset P$ and suppose $a \in R_1 \setminus P$ and $b \in R_2 \setminus P$.
Since $R(a) \subset R_1$ and $R(b) \subset R_2$ we have $R(a)R(b) \subset P$, and thus $a \in P$ or $b \in P$ a contradiction.
Thus either $R_1 \subset P$ or $R_2 \subset P$.
4) $\rightarrow$ 1). Trivial.

A similar proof shows that lemma 1.5.1 holds, if we replace right by left and vice versa.
Condition 2 then becomes: If $SaSb \subset P$ then $a \in P$ or $b \in P$.
For two-sided ideals we have an analogous system of conditions.
Condition 2 then becomes: If $aSb \subset P$ then $a \in P$ or $b \in P$.

1.5.2. Theorem. Let S be a mob and suppose $E \neq \emptyset$ and let $e \in E$. Then each of $J_o(S \setminus \{e\})$, $R_o(S \setminus \{e\})$ and $L_o(S \setminus \{e\})$ is prime if it is not empty.

Proof:
Suppose that $a \notin J_o(S \setminus \{e\})$ and $b \notin J_o(S \setminus \{e\})$. Then since $J_o(S \setminus \{e\})$ is maximal $e \in J(a)$ and $e \in J(b)$. This implies that $e \in J(a)J(b)$ and hence $J(a)J(b) \not\subset J_o(S \setminus \{e\})$.
This shows that $J_o(S \setminus \{e\})$ is a prime ideal.
The statement for $R_o(S \setminus \{e\})$ and $L_o(S \setminus \{e\})$ can be proved in the same way.

If $E \neq \emptyset$, we can define a partial ordering in E as follows: for $e, f \in E$, $e \leq f$ if and only if $ef = fe = e$.
It is clear that the relation $\leq$ thus defined is reflexive and anti-symmetric.
Now let $e \leq f$ and $f \leq g$. Then $ef = fe = e$ and $fg = gf = f$.
Hence $eg = (ef)g = e(fg) = ef = e$ and $ge = gfe = fe = e$.
This implies that $e \leq g$ and the relation $\leq$ is transitive.

If S is a mob without zero, then the minimal elements of E are the primitive idempotents.

If S has a zero, then the non-zero primitive idempotents are the atoms of the partially ordered set E.

Furthermore, if S has a unit u, then u is the maximal element of E.

1.5.3. Lemma. Let P be an open prime right (left) ideal of a compact mob S. If A is a left (right) ideal of S which is not contained in P, then A contains an idempotent e with $Se \not\subset P$.

Proof:

Let P be an open prime right ideal and let $a \in A \setminus P$.

Then L(a) is a compact left ideal with $L(a) \subset A$, $L(a) \not\subset P$.

Now let $L_1 \supset L_2 \supset \ldots$ be a linearly ordered sequence of compact left ideals with $L_i \subset A$, $L_i \not\subset P$, $i=1,2,\ldots$.

If $L = \bigcap_{i=1}^{\infty} L_i$, then because P is open and the L_i compact, $L \not\subset P$. Now using Zorn's lemma there exists a minimal member L of the set of all compact left ideals L_α with $L_\alpha \subset A$, $L_\alpha \not\subset P$.

Next let $a \in L \setminus P$ and suppose $La \subset P$.

Then $(\{a\} \cup La)(\{a\} \cup La) \subset La \subset P$.

Hence by the dual of lemma 1.5.1 $\{a\} \cup La \subset P$; a contradiction. Thus $La \not\subset P$.

Since $La \subset L$ and L is minimal $La = L$.

Thus $L = La = Le$ with $e = e^2 \in \Gamma(a) \subset L$.

Since $Se = Se.e \in Le$ we have $Se = Le = L \not\subset P$.

Corollary. Let P be an open prime ideal of the compact mob S. If A is a right or left ideal of S not contained in P, then $A \setminus P$ contains a non-minimal idempotent.

Proof:

Let A be a left ideal.

Then it follows from lemma 1.5.3 that there exists $e \in A$ and $a \in A \setminus P$ with $a \in Se \not\subset P$.

Thus $ae = a$ and since P is an ideal, it would follow from $e \in P$ that $ae = a \in P$. Hence $e \in A \setminus P$.

Furthermore it is clear that $e \notin K$, since $K \subset P$.
If S is a mob without zero, then e is non-primitive and hence non-minimal.
If S has a zero, then since $K = \{0\}$, we have $e \neq 0$ and thus $e \geq 0$.

1.5.4. Theorem. If S is compact, then each open prime ideal $P \neq S$, has the form $J_o(S \setminus \{e\})$, e non-minimal.
If conversely e is a non-minimal idempotent, then $J_o(S \setminus \{e\})$ is an open prime ideal.

Proof:
Let P be an open prime ideal. Then we can find just as in lemma 1.5.3 a minimal ideal J_1, $J_1 \not\subset P$.
The above corollary shows that $J_1 \setminus P$ contains an idempotent e and hence $J_1 = J(e)$.
Now let $P^* = J_o(S \setminus \{e\})$, then P^* is an open prime ideal and $P \subset P^*$.
Again using lemma 1.5.3 if $P \neq P^*$, we can find an idempotent $f \in P^* \setminus P$ with $J_2 = J(f) \not\subset P$.
Since $e,f \notin P$, $J(e)J(f) = J_1J_2 \not\subset P$. Furthermore $J_1J_2 \subset J_1$ and since J_1 is minimal $J_1J_2 = J_1$.
Hence $J_1 = J_1J_2 \subset J_2 \subset P^*$; a contradiction.
Conversely if e is non-minimal, then $e \notin K$ and hence $J_o(S \setminus \{e\}) \neq \emptyset$ and consequently an open prime ideal.

1.6. Notes

Many of the theorems of chapter I are found in one or more of the following papers: Faucett, Koch and Numakura [3], Koch and Wallace [6], Numakura [1],[2], Schwarz [2],[10], Wallace [1],[2],[9] and Wright [1]. It is pointless to trace every source of every theorem and we will not attempt to do so. However, the following primary sources of results in chapter I may be of interest.
Let S be a mob and an abstract group. Under what conditions on S can we assert that S is a topological group? Some results of this kind stem from Montgomery [1], Ellis [1],[2] and Moriya [1]. The latter's

theorems were extended by Wallace [9]. Theorem 1.1.8 is due to Ellis [2]. Wallace [3],[4] also examined the structure of S related to its maximal subgroups. Theorems 1.1.14 and 1.1.15 first appear in Wallace [12]. Theorem 1.1.10 has been used by Wendel [1] to show Haar measure exists on a compact group.

Theorems 1.2.5 and 1.2.6 go back to Suschkewitsch [1] and Rees [1]. In this form, however, they are due essentially to Clifford [1]. For the case of a compact mob see Numakura [2]. Theorems 1.2.8 and 1.2.9 are topological extensions, Wallace [10], of a theorem of Rees-Suschkewitsch, Rees [1].

For the algebraic results of section 1.3 we refer to the monograph of Clifford and Preston [5]. Theorem 1.3.15 is a generalization to locally compact submobs of a theorem of Schwarz [10].

Maximal ideals have been studied by many authors. The results about the unique maximal ideals are due to Schwarz [2].

The statements of section 1.5 appear in Numakura [4].

II SEMIGROUPS WITH ZERO AND IDENTITY

2.1. Semigroups with zero

Definitions. Let S be a mob with 0 and a an element of S. If $a^n \to 0$ i.e. if for every neighbourhood U of 0 there exists an integer n_o such that $a^n \in U$ if $n \geq n_o$, then a is termed a nilpotent element.
We denote by N the set of all nilpotent elements of S.
An ideal (right, left) A of S with the property $A^n \to 0$ is called a nilpotent ideal.
A nil-ideal A is an ideal consisting entirely of nilpotent elements.
It is clear that every nilpotent ideal is a nil-ideal and that the join of a family of (right, left) nil-ideals is again a (right, left) nil-ideal of S.
Let S be the unit interval with the usual multiplication.
Then $I = [0,1)$ is an ideal consisting entirely of nilpotent elements.
I, however, is not a nilpotent ideal, since $I^n = I$ for all n.

2.1.1. Lemma. Every right (left) nil-ideal of S is contained in some nil-ideal of S.

Proof:
Let A be a right nil-ideal of S. Then SA is an ideal of S.
Suppose $x = sa \in SA$, and let U be any neighbourhood of 0.
Then there exists a neighbourhood V of 0 such that $sVa \subset U$.
As A is a right nil-ideal of S, $as \in A$, and $(as)^n \in V$ for $n \geq n_o$.
Hence if $m \geq n_o+1$ we have $(sa)^m = s(as)^{m-1}a \in sVa \subset U$.
Therefore SA is a nil-ideal of S, and hence $A \cup SA$ is a nil-ideal of S containing A.

Definition. The join R of all nil-ideals of a mob S with zero is called the radical of S.
According to lemma 2.1.1 R is a nil-ideal which contains every right and every left nil-ideal of S.

Hence R is the maximal right and the maximal left nil-ideal.
If S consists only of nilpotent elements, i.e. if S = R = N, then S is called a <u>nil-semigroup</u>.

2.1.2. <u>Theorem</u>. Let S be a mob with zero, with $\Gamma(a)$ compact for every $a \in S$. Then every (right, left) ideal of S is either a (right, left) nil-ideal or contains non-zero idempotents.

<u>Proof</u>:
Let a be a non-nilpotent element of the ideal I. Then $e = e^2 \in \Gamma(a)$ is not equal to zero. For if e = 0, then $K(a) = e\Gamma(a) = \{0\}$. Since K(a) is the set of cluster points of the sequence $\{a^n\}_{n=1}^{\infty}$, we would have $a^n \to 0$.
Furthermore $aK(a) = K(a)$ and thus $K(a) \subset I$, which implies $e \in I$.

<u>Corollary</u>. A compact mob is either a nil-semigroup or contains non-zero idempotents.

2.1.3. <u>Theorem</u>. Let e be a non-zero idempotent of the compact mob S with zero. Then the following conditions are equivalent:

1) $eSe \setminus N$ is a group.
2) e is primitive.
3) Se is a minimal non-nil left ideal.
4) SeS is a minimal non-nil ideal.
5) Each idempotent of SeS is primitive.

<u>Proof</u>:
1) → 2). If $eSe \setminus N$ is a group, then e is the only idempotent in $eSe \setminus \{0\}$, since no idempotent $\neq 0$ can be nilpotent.
2) → 3). Let L be a non-nil left ideal contained in Se.
Then there is an idempotent $f \in L$, $f \neq 0$. Since $f \in Se$ we have fe = f and (ef)(ef) = ef. Thus ef is a non-zero idempotent contained in eSe.
Since e is primitive ef = e. Thus $ef = e \in eL \subset L$, which implies L = Se.
3) → 4). Let I be a non-nil ideal, $I \subset SeS$.
Then there exist an idempotent $f \in I$, $f \neq 0$, and elements $a,b \in S$, such that aeb = f and bf = b.

Let $g = bae$, then $g^2 = baebae = bfae = bae = g$.
Furthermore $g \neq 0$, since otherwise $0 = gb = baeb = bf = b$.
Now $g \in Se$ and $g \in SfS$. Hence by 3) $Sg = Se \subset SfS$ and we conclude $SeS = SfS = I$.

4) $\rightarrow$ 5). Let f be a non-zero idempotent of SeS and let $g = g^2 \neq 0$, $g \in fSf$. Since $f,g \in SeS$, we have $SgS = SfS = SeS$ and $f \in SgS$. Hence $f = agb$, and we may assume $ag = a$, $gb = b$.
Since $gf = fg = g$, this implies $afb = agfb = agb = f$.
Hence $f = a^n gb^n$ and $f = g^* gb'$ with $g^{*2} = g^* \in \Gamma(a)$ and $b' \in \overline{\{b^n\}_{n=1}^{\infty}}$.
We note that $g^* g = g^*$, hence $g^* f = f = g^* gf = g^*$ and $f = g^* = g^* g = fg = g$.

5) $\rightarrow$ 1). Since every idempotent in SeS is primitive, e is primitive and hence $Se = L$ is a minimal non-nil left ideal. Now let $a \in eSe \setminus N$, then $a \in \{Se \cap eS\} \setminus N$.
Since L is minimal $a = ea \in La = L$. Hence there is $\bar{a} \in L$, such that $\bar{a}a = e$. Let $e\bar{a} = a'$, then $a' \in eSe$ and $a'a = e$. Furthermore $(aa')(aa') = aea' = aa'$, and aa' is an idempotent contained in $eSe \setminus N$, thus $aa' = e$.
So we can find for every $a \in eSe \setminus N$ an element $a' \in eSe$ such that $aa' = e = a'a$.
This implies that $eSe \setminus N$ is a group, since $a' \notin N$.
For if $a' \in N$, then $\bigcap_{n=1}^{\infty} S(a')^n = S.0 = \{0\}$ by lemma 1.1.13.
This is contradictory to $aa' = a^2(a')^2 = a^n(a')^n = e$.

Definition. A mob S with zero is said to be an <u>N-semigroup</u> if its nilpotent elements form an open set.

2.1.4. <u>Lemma</u>. Let S be a mob with zero, and let $a \in S$.
If a^n is nilpotent for some $n \geq 1$, then a itself is a nilpotent element.

<u>Proof</u>:
Let U be an arbitrary neighbourhood of 0. Then since $a^j 0 = 0$, there is a neighbourhood V of 0, such that $a^j V \subset U$ $(j=1,2,\ldots,n)$.
Since a^n is nilpotent, there exists an integer $k_o \geq 1$, such that $(a^n)^k \in V$ for $k \geq k_o$. Thus $a^j a^{nk} = a^{nk+j} \in U$, $j=1,2,\ldots,n$, $k \geq k_o$.

This implies that for $N > nk_o$, $a^N \in U$, hence a is nilpotent.

2.1.5. Theorem. If a mob S with 0 has a neighbourhood U of 0 which consists entirely of nilpotent elements, then S is an N-semigroup.

Proof:

Let $p \in N$, then there is an n such that $p^n \in U$. Therefore there is a neighbourhood V of p such that $V^n \subset U$. Hence every point of V^n is nilpotent.

Lemma 2.1.4 then implies that $V \subset N$.

2.1.6. Theorem. A locally compact mob S with 0 having a neighbourhood U of 0 which contains no non-zero idempotents is an N-semigroup.

Proof:

Since S is locally compact and Hausdorff, S is regular and we can find a neighbourhood W of 0, such that $\overline{W} \subset U$ and $\overline{W}$ compact. The continuity of multiplication and the compactness of $\overline{W}$ imply, that there is a neighbourhood V of 0 with $V\overline{W} \subset W$; $V \subset W$.

Hence $V^2 \subset V\overline{W} \subset W$ and $V^n \subset W$, $n=1,2,\ldots$.

Now the set $A = \bigcup_{i=1}^{\infty} V^i$ is a mob contained in W. Therefore $\overline{A}$ is a compact mob contained in U. Since $\overline{A}$ contains no non-zero idempotents, $\overline{A}$ is a nil-semigroup.

Hence V consists entirely of nilpotent elements, and by theorem 2.1.5 S is an N-semigroup.

Corollary. A locally compact mob with 0, which is not an N-semigroup contains a set of non-zero idempotents with clusterpoint 0.

2.1.7. Theorem. The radical of a compact N-semigroup is open.

Proof:

Since $R \subset N$, R is the largest ideal of S contained in N.

Hence $R = J_o(N)$ and $J_o(N)$ is open (1.2.2).

Let S be the half line $[0,\infty)$ under the usual multiplication of real numbers.
S is an N-semigroup, since $N = [0,1)$ is open. The radical of S, however, is not open, since $R = \{0\}$.

2.1.8. Theorem. Let S be a compact N-semigroup, which is not a nil-semigroup. Then any non-nil-ideal I of S contains a minimal non-nil-ideal M. Furthermore $R_M = M \cap R$ is the radical of M and R_M is a maximal proper ideal of M with M/R_M completely 0-simple.

Proof:

Let T be the collection of all closed non-nil-ideals of S contained in I. T is non-void since if $e = e^2 \neq 0 \in I$, SeS is a closed non-nil-ideal contained in I.
Now let $\{T_\alpha\}_\alpha$ be a linearly ordered subcollection of T.
Then $I_o = \bigcap_\alpha T_\alpha$ is non-empty, since S is compact.
Furthermore I_o is an ideal of S contained in I and $I_o \not\subset N$, since N is open and T_α compact, $T_\alpha \not\subset N$.
Thus $\{T_\alpha\}_\alpha$ has a lower bound and Zorn's lemma assures the existence of a minimal closed non-nil-ideal M in I.
Now let M^* be a non-nil-ideal contained in M. Then M^* contains a non-zero idempotent f and $SfS \subset M^* \subset M$.
Since SfS is a closed non-nil-ideal and M is minimal in T, we have $SfS = M^* = M$.
Thus M is a minimal non-nil-ideal and $M = SeS$ with e primitive.
Now we shall prove that $R_M = M \cap R$.
Since $M \cap R$ is a nil-ideal of M, we have $M \cap R \subset R_M$.
Furthermore $SR_MS \subset SMS \subset M$. If $SR_MS = M$, then $MSR_MSM = M^3 = M$, and therefore $M = MSR_MSM \subset MR_MM \subset R_M$. This contradicts the fact that M is a non-nil-ideal.
Hence SR_MS is an ideal of S properly contained in M.
This implies that SR_MS must be a nil-ideal, i.e. $SR_MS \subset R_M$.
Hence R_M is a nil-ideal of S, thus $R_M \subset M \cap R$.
Since there is no ideal of S lying properly between M and R_M, theorem 1.3.7 implies that M/R_M is either a 0-simple semigroup or a

zero semigroup. Since M contains a non-zero idempotent M / R_M is a 0-simple semigroup. Hence it follows from the corollary to 1.3.7 that R_M is a maximal proper ideal of M, and thus M / R_M completely 0-simple.

A similar proof shows that if L is a non-nil left ideal then L contains a minimal non-nil left ideal L_o.
Furthermore L_o contains no non-nil-ideals and the radical R_{L_o} of L_o is the maximal proper ideal of L_o.
Hence L_o / R_{L_o} is completely 0-simple.

<u>Corollary</u>. Let S be a compact mob with zero; then S contains a non-zero primitive idempotent if and only if there is a non-zero idempotent e with $eSe \setminus N$ closed.

<u>Proof</u>:

If e is primitive, then $eSe \setminus N$ is a maximal subgroup and hence closed. On the other hand if $eSe \setminus N$ is closed and $e \neq 0$, then $eSe \cap N$ is the set of nilpotent elements of eSe and hence eSe is a compact N-semigroup. We then conclude from theorem 2.1.8 that eSe contains a non-zero primitive idempotent. Hence so does S.

2.1.9. <u>Theorem</u>. Let e be a non-zero primitive idempotent of the compact mob S with zero. Then $Se \setminus N$ and $Se \cap N$ are submobs and $Se \setminus N$ is the disjoint union of the maximal groups $e_\alpha Se_\alpha \setminus N$ where e_α runs over the non-zero idempotents of Se.

<u>Proof</u>:

Suppose $a,b \in Se \setminus N$, then $a^n, b^n \in Se \setminus N$. Now let $ab \in N$.
Then since Se is a minimal non-nil left ideal, we know that
$Sa^n = Sb^n = Se$, $n=1,2,\ldots$. Hence $Sab = Sb^2 = Se$.
Thus $S(ab)^n = Se$, which implies $Se = \bigcap_{n=1}^{\infty} S(ab)^n = S0 = \{0\}$.
This is a contradiction, since $e \neq 0$.
Suppose now $a,b \in Se \cap N$ and $ab \notin N$. Then $(ab)^2 \notin N$ and hence $Sab = Se$.
Since $a \in Se$, we have $Sa \subset Se = Sab$.
Hence $Sa = Se = Saf$, with $f = f^2 \in \Gamma(b)$. Since $b \in N$, $f = 0$ and thus

$Se = Sa0 = \{0\}$, a contradiction.

Finally let $a \in Se \setminus N$. Then $Sa = Se$.

Let $f = f^2 \in \Gamma(a)$, then $Sf = Se = Sa$ and f is a right unit for Se.

Now let $K(a)$ be the set of cluster points of $\{a^n\}_{n=1}^{\infty}$. Then $K(a)$ is a group and $K(a) = f\Gamma(a) = \Gamma(a)f = \Gamma(a)$.

Hence $\Gamma(a)$ is a group and $Se \setminus N$ is the union of groups.

For any $e_\alpha = e_\alpha^2 \neq 0$, $e_\alpha \in Se$ we have $Se_\alpha = Se$, so that e_α is primitive and $e_\alpha Se_\alpha \setminus N$ a group.

Now the maximal group containing e_α is contained in $e_\alpha Se_\alpha$, and moreover since any group which meets N must be zero, we conclude that $e_\alpha Se_\alpha \setminus N$ is a maximal group.

2.2. <u>0-simple mobs</u>

As in 1.3 we call a mob S with zero 0-simple if $S^2 \neq \{0\}$ and $\{0\}$ is the only proper ideal of S. S is completely 0-simple if S is 0-simple and contains a non-zero primitive idempotent.

Hence if S is completely 0-simple S cannot be a nil-semigroup.

If on the other hand S is a non-nil-semigroup and if S is 0-simple, then every (right or left) nil-ideal of S is the zero ideal $\{0\}$, since every right or left nil-ideal of S is contained in some nil-ideal of S. Thus in this case $R = \{0\}$.

We shall call a (left, right) ideal I of a mob S with zero, 0-minimal if $I \neq \{0\}$, and $\{0\}$ is the only (left, right) ideal of S properly contained in I.

Hence every minimal non-nil left ideal of a non-nil 0-simple mob is a 0-minimal left ideal and conversely.

2.2.1. <u>Lemma</u>. Let L be a 0-minimal left ideal of a 0-simple mob S and let $a \in L \setminus 0$. Then $Sa = L$.

<u>Proof</u>:

Since Sa is a left ideal of S contained in L, it follows that $Sa = \{0\}$ or $Sa = L$.

If $Sa = \{0\}$, then $SaS = \{0\}$, in contradiction with $SaS = S$.

If S is compact, then every non-nil (left, right) ideal of S contains a non-zero idempotent. So in this case if L is a minimal non-nil left ideal of S, then there is an idempotent $e \in L$ with $Se = L$.

2.2.2. Lemma. Let L be a 0-minimal left ideal of a 0-simple mob S and let $s \in S$. Then Ls is either {0} or a 0-minimal left ideal of S.

Proof:

Assume $Ls \neq \{0\}$. Evidently Ls is a left ideal of S. Now let L_o be a left ideal of S contained in Ls, $L_o \subset Ls$.
Let A be the set of all $a \in L$ with $as \in L_o$.
Then $As = L_o$ and $A \subset L$. Furthermore $SAs \subset SL_o \subset L_o$ and $SA \subset SL \subset L$.
Hence $SA \subset A$ and A is a left ideal of S.
From the minimality of L it follows that either $A = \{0\}$ or $A = L$ and we have correspondingly $L_o = \{0\}$ or $L_o = Ls$.

2.2.3. Theorem. Let S be a 0-simple mob containing at least one 0-minimal left ideal. Then S is the union of all 0-minimal left ideals.

Proof:

Let A be the union of all 0-minimal left ideals of S. Clearly A is a left ideal of S and $A \neq \{0\}$. Now we show that A is also a right ideal.
Let $a \in A$ and $s \in S$. Then $a \in L$ for some 0-minimal left ideal L of S.
By lemma 2.2.2 $Ls = \{0\}$ or Ls is a 0-minimal left ideal.
Hence $Ls \subset A$ and $as \in A$. Thus A is a non-zero ideal of S, whence $A = S$.

Corollary. Let S be a compact 0-simple mob. Then S is the union of all 0-minimal left ideals of S.

Proof:

Since S is compact, S is completely 0-simple and hence contains a non-zero primitive idempotent e.
2.1.3 then implies that Se is a minimal non-nil left ideal.
Since minimal non-nil left ideals and 0-minimal ideals are the same in a compact 0-simple mob, Se is a 0-minimal left ideal and the corollary follows.

2.2.4. Lemma. Let L and R be 0-minimal left and right ideals of a 0-simple mob, such that $LR \neq \{0\}$. Then $RL = R \cap L$ is a group with zero and the identity e of $RL \setminus \{0\}$ is a primitive idempotent of S.

If $LR = \{0\}$, then $(R \cap L)^2 = \{0\}$ and in both cases we have $R \cap L \neq \{0\}$.

Proof:

Since LR is a non-zero ideal of S, we must have $LR = S$.
Furthermore $RL \neq \{0\}$, since $S = S^2 = LRLR$.
Now let $a \in RL \setminus \{0\}$, then $a \in L \setminus \{0\}$ and $a \in R \setminus \{0\}$, hence $Sa = L$ and $aR = \{0\}$ or $aR = R$ (lemma 2.2.1 and 2.2.2).
Since $S = LR = SaR$, it follows that $aR \neq \{0\}$. Consequently $aRL = RL$.
In the same way we can prove that $RLa = RL$.
From this we conclude that RL is a group with zero.
Now let e be the identity of RL. Then since $R = eS$ and $L = Se$ we have $R \cap L = eS \cap Se = eSe$ and $RL = eSSe = eSe$.
Since eSe is a group with zero, e is primitive.
If $LR = \{0\}$, then since $L \cap R \subset L$ and $L \cap R \subset R$, we have $(L \cap R)^2 \subset LR = \{0\}$ which implies $(L \cap R)^2 = \{0\}$. Moreover if $a \in L \setminus \{0\}$ and $b \in R \setminus \{0\}$, then $SaS = S$ and $SbS = S$, $Sa = L$ and $bS = R$. Hence $SbSSaS = S^2 \neq \{0\}$, and thus $bSSa \neq \{0\}$. Since $bSSa \subset L \cap R$. we have $L \cap R \neq \{0\}$.

2.2.5. Theorem. Let S be a 0-simple mob. Then S is completely 0-simple if and only if it contains at least one 0-minimal left and one 0-minimal right ideal. Moreover L is a 0-minimal left ideal of S if and only if $L = Se$ with e primitive.

Proof:

If S is completely 0-simple it contains a non-zero primitive idempotent e, and we have eSe a group with zero.
Now let L be a non-zero ideal contained in Se.
Then $SeSLS = SLS = S$ and hence $eSL \neq \{0\}$ and since $eSL \subset L \cap eS$ $L \cap eS \neq \{0\}$. Next let $a \in L \cap eS \setminus \{0\}$. Then $a \in eSe \setminus \{0\}$, and there is an a^{-1} such that $a^{-1}a = e$.

Hence $e = a^{-1}a \in L$ and $Se \subset L \subset Se$. Thus Se is a 0-minimal left ideal of S.

Dually we can prove that eS is 0-minimal.

Conversely assume that S contains at least one 0-minimal left ideal L and at least one 0-minimal right ideal R.

Since $SRS = S$, we have $SR \neq \{0\}$, and thus $s_1R \neq \{0\}$ for some $s_1 \in S$.

Since S is the union of 0-minimal left ideals, $s_1 \in L_1$ for some 0-minimal left ideal L_1 and evidently $L_1R \neq \{0\}$.

It then follows from lemma 2.2.4 that S contains a primitive idempotent e, with $eS = R$.

2.2.6. <u>Theorem</u>. Let S be a compact 0-simple mob and let e and f be non-zero idempotents of S. Then the maximal subgroups H(e) and H(f) containing e and f respectively are isomorphic compact groups.

<u>Proof</u>:

Since each idempotent $e \neq 0$ of S is primitive, Se and Sf are 0-minimal left ideals and eS and fS 0-minimal right ideals and it follows from lemma 2.2.4 that $eSe \setminus \{0\}$ and $fSf \setminus \{0\}$ are groups.

Since $H(e) \subset eSe \setminus \{0\}$, we have $H(e) = eSe \setminus \{0\}$ and $H(f) = fSf \setminus \{0\}$.

Furthermore $eS \cap Sf \neq \{0\}$. Now let $0 \neq a \in eS \cap Sf$. Then $ea = a = af$.

Since $eS = aS$ and $Sf = Sa$ (2.2.1), there exist a_1 and $a_2 \in S$ such that $e = aa_1$, $f = a_2a$.

Now let $b = fa_1e$, then $b \neq 0$ and $ab = afa_1e = aa_1e = ee = e$; $ba = fba = a_2aba = a_2ea = a_2a = f$.

From this it follows that $bS = fS$ and $Sb = Se$.

We now prove that the mappings $\phi: x \to bxa$ and $\psi: y \to ayb$ are mutually inverse one-to-one mappings of H(e) and H(f) upon each other.

For let $x \in H(e)$, then $bxa \in bS \cap Sa = fS \cap Sf = H(f) \cup \{0\}$.

Similarly $y \in H(f)$ implies $ayb \in aS \cap Sb = eS \cap Se = H(e) \cup \{0\}$.

And if $x \in H(e)$, then $a(bxa)b = exe = x$.

Moreover ϕ is an isomorphism, since $(bx_1a)(bx_2a) = bx_1ex_2a = bx_1x_2a$.

If we recall that both H(e) and H(f) are compact Hausdorff spaces it follows that ϕ is a topological isomorphism.

Corollary. Let S be a compact 0-simple mob. Then $S \setminus \{0\}$ is the disjoint union of sets $L_\alpha \cap R_\beta \setminus \{0\}$, where $L_\alpha \in \mathcal{L}'(S)$ and $R_\beta \in \mathcal{R}'(S)$, $\mathcal{L}'(S)$ and $\mathcal{R}'(S)$ being respectively the sets of all 0-minimal left and 0-minimal right ideals of S.
All sets $L_\alpha \cap R_\beta \setminus \{0\}$ are homeomorphic, while $L_\alpha \cap R_\beta \setminus \{0\}$ is either a maximal subgroup of S or $(L_\alpha \cap R_\beta)^2 = \{0\}$.

Proof:
Let L_1 and L_2 be two 0-minimal left ideals of S and suppose $0 \neq a \in L_1 \cap L_2$, then it follows from lemma 2.2.1 that $L_1 = Sa = L_2$.
Hence $L_1 \cap L_2 = \{0\}$ or $L_1 = L_2$. Analogously we have for 0-minimal right ideals $R_1 \cap R_2 = \{0\}$ or $R_1 = R_2$.
Thus $S \setminus \{0\}$ is the disjoint union of the sets $L_\alpha \cap R_\beta \setminus \{0\}$.
We know already that all sets $L_\alpha \cap R_\beta \setminus \{0\}$, with $L_\alpha \cap R_\beta \setminus \{0\}$ a group are homeomorphic and that in the other case $(L_\alpha \cap R_\beta)^2 = \{0\}$.
Now let $A = L_\alpha \cap R_\beta \setminus \{0\}$, with $L_\alpha = Se$ and let $a \in A$.
Then the mapping ϕ: $x \to ax$ is a homeomorphism of $H(e)$ onto A.
For if $ax_1 = ax_2$, then since $e = a^*a$ for suitable $a^* \in S$, we have $a^*ax_1 = a^*ax_2$, $ex_1 = ex_2$ and thus $x_1 = x_2$.
Furthermore ϕ is onto since for each $b \in A$ we have $b = as$ and hence $be = b = aese = ax$ with $x = ese \in H(e)$.
Since ϕ is continuous, ϕ is topological.

Corollary. Let S be a commutative compact 0-simple mob. Then S is a group with zero.

Proof:
By lemma 2.2.4 we have $S^2 = S = S \cap S$ is a group with zero, since S is both a 0-minimal left and right ideal.

2.2.7. Theorem. Let J be a maximal proper ideal of the compact mob S. Then the following conditions are equivalent:

1) $S \setminus J$ is the disjoint union of groups.
2) For each element of $S \setminus J$, there exists a unit element.
3) $a \in S \setminus J$ implies $a^2 \in S \setminus J$.
4) J is a completely prime ideal.

5) $S \setminus J$ contains an idempotent and the product of any two idempotents of $S \setminus J$ lies in $S \setminus J$.

Proof:

1) $\to$ 2). Obvious.

2) $\to$ 3). Let $a \in S \setminus J$ and $ax = xa = a$. Then $ae = ea = a$ for $e = e^2 \in \Gamma(x)$. Thus $e \in S \setminus J$ and S / J is completely 0-simple. Hence by lemma 2.2.4 $S \setminus J = \bigcup_\alpha H(e_\alpha) \cup \bigcup_\beta A_\beta$, with $A_\beta^2 \subset J$.
Furthermore $a \in H(e)$ which implies $a^2 \in H(e)$ and hence $a^2 \in S \setminus J$.

3) $\to$ 4). Let $a,b \in S \setminus J$ and suppose $ab \in J$. Then $I = \{x \mid x \in S,\ xb \in J\}$ is a left ideal with $J \subset I$.
Next let $x \in I$, $xs \notin I$, then $xsb \notin J$ and hence $(xsb)^2 \notin J$.
This implies that $bx \notin J$ and thus $bxbx \notin J$. From this it follows that $xb \notin J$ which is a contradiction.
Hence we have proved that I is an ideal of S containing J and we conclude that $I = S$.
Since $I = \{x \mid x \in S,\ xb \in J\}$ we have $b^2 \in J$, a contradiction.

4) $\to$ 5). This follows from the fact that $J = J_o(S \setminus \{e\})$ (1.5.4).

5) $\to$ 1). Since $e \in S \setminus J$, we have S / J completely 0-simple and $S \setminus J = \bigcup_\alpha H(e_\alpha) \cup \bigcup_\beta A_\beta$ with $A_\beta^2 \subset J$.
Now let $a \in A_\beta$, then $a \in Se$ and $a \in fS$ with $SefS \subset J$ or else it would follow from lemma 2.2.3 that $a \in Se \cap fS \setminus J = H(e_\alpha)$.
Since $ef \notin J$ we have, however, $SefS \not\subset J$. Thus $A_\beta = \emptyset$ and $S \setminus J$ is the union of groups.

From the theorem it immediately follows that $S \setminus J$ is a group if and only if $S \setminus J$ contains a unique idempotent.

2.2.8. Theorem. Let S be a compact 0-simple mob and S' a locally compact submob of S with $0 \notin S'$. Then S' is a simple submob.

Proof:

Since 0 is an isolated point of S (2.3.1), $\overline{S'}$ is a closed submob of S with $0 \notin \overline{S'}$.
Let $S = \bigcup \{L_\alpha \mid \alpha \in A\}$, L_α running through all 0-minimal left ideals.
Then $\overline{S'} = \bigcup \{L'_\alpha = L_\alpha \cap \overline{S'} \mid \alpha \in A\}$.

Clearly the closed set $L'_\alpha \neq \emptyset$ is a left ideal of $\overline{S'}$ and L'_α contains a non-zero primitive idempotent e_α.
Hence $\overline{S'}e_\alpha$ is a minimal left ideal of $\overline{S'}$ and since $L_\alpha e_\alpha = L_\alpha$ we have $L'_\alpha = L'_\alpha e_\alpha \subset \overline{S'}e_\alpha \subset L'_\alpha$. Hence $L'_\alpha = \overline{S'}e_\alpha$, and $\overline{S'}$ is the union of its minimal left ideals, which implies that $\overline{S'}$ is simple.
Since S' is a locally compact submob of the compact simple mob $\overline{S'}$, theorem 1.3.16 implies that S' is simple.

Let $S = \{e_1,e_2,a,b,0\}$ be the 0-simple mob given by the multiplication table

	e_1	a	e_2	b	0
e_1	e_1	0	0	b	0
a	a	0	0	e_2	0
e_2	0	a	e_2	0	0
b	0	e_1	b	0	0
0	0	0	0	0	0

Then $S' = \{e_1,a,0\}$ is a submob. S', however, is not simple since $\{0,a\}$ is a non-zero proper ideal of S'.

2.3. The structure of a compact (completely) 0-simple mob

We have seen in 2.2 that each compact 0-simple mob S is the union of all 0-minimal left (right) ideals.
Let $\{L^*_\alpha \mid \alpha \in A\}$ and $\{R^*_\beta \mid \beta \in B\}$ be the 0-minimal left and right ideals of S respectively.
Let $L_\alpha = L^*_\alpha \setminus \{0\}$, $R_\beta = R^*_\beta \setminus \{0\}$and $H_{\alpha\beta} = L_\alpha \cap R_\beta$.
Then it follows from 2.2.4 that $H_{\alpha\beta}$ is either a maximal subgroup of S or else $H^2_{\alpha\beta} = \{0\}$. If $H_{\alpha\beta}$ is a group, we shall denote by $e_{\alpha\beta}$ the identity of $H_{\alpha\beta}$.
Furthermore for every two different sets L_{α_1} and L_{α_2} (R_{β_1} and R_{β_2}) we have $L_{\alpha_1} \cap L_{\alpha_2} = \emptyset$ ($R_{\beta_1} \cap R_{\beta_2} = \emptyset$).

Now let $H = \bigcup \{H_{\alpha\beta} \mid H_{\alpha\beta}$ a group; $\alpha \in A,\ \beta \in B\}$ and
$H' = \bigcup \{H_{\alpha\beta} \mid H^2_{\alpha\beta} = \{0\};\ \alpha \in A,\ \beta \in B\}$.
Then S is the disjoint union of H, H' and {0}.

2.3.1. Lemma. Let S be a compact 0-simple mob. Then 0 is an isolated point of S.

Proof:

Let V be an open neighbourhood of 0 with $V \neq S$. Then since S is compact and $S0S = \{0\}$, there exists a neighbourhood W of 0 such that $SWS \subset V$. Since for all $a \neq 0$, $a \in S$ we have $SaS = S$, it follows that $W = \{0\}$, i.e. 0 is isolated.

2.3.2. Lemma. Let S be a compact 0-simple mob. Then, with the notation just introduced, the following assertions are true:

1) For each $\alpha \in A$ there exists a $\beta \in B$ such that $H_{\alpha\beta}$ is a group, and dually for each $\beta \in B$ there exists an $\alpha \in A$ such that $H_{\alpha\beta}$ is a group.
2) H and H' are both open and closed sets of S.

Proof:

1) For each $\alpha \in A$, there is a primitive idempotent $e_{\alpha\beta}$ such that $e_{\alpha\beta} \in L^*_\alpha$ (2.2.5). Hence $R^*_\beta = e_{\alpha\beta}S$ is a 0-minimal right ideal. Thus $e_{\alpha\beta} \in L_\alpha \cap R_\beta = H_{\alpha\beta}$ and since $e^2_{\alpha\beta} = e_{\alpha\beta} \in H^2_{\alpha\beta}$ we have $H^2_{\alpha\beta} \neq \{0\}$. The same argument applies to right ideals.

2) Suppose now for $a \in S$, $a^2 \neq 0$. Then there is an open set V with $a \in V$ such that $0 \notin V^2$. This implies that $V \cap H' = \emptyset$ and hence $a \notin \overline{H'}$. Moreover, we have for all $h \in H$, $h^2 \neq 0$, and hence $H \cap \overline{H'} = \emptyset$, which implies $H' = \overline{H'}$.

On the other hand $H \cup \{0\}$ is the set of all maximal subgroups of S, hence $H \cup \{0\}$ is closed.

If we recall that {0} is open, it follows that H must be closed.

2.3.3. Lemma. Let S be a compact 0-simple mob and let

$e_{\alpha\beta} = e^2_{\alpha\beta} \in L_\alpha \cap R_\beta$.

Then $L_\gamma e_{\alpha\beta} = L_\alpha$ if and only if $H_{\gamma\beta}$ is a group, and

$L_\gamma e_{\alpha\beta} = \{0\}$ if and only if $H^2_{\gamma\beta} = \{0\}$.

Proof:

Let $H_{\gamma\beta}$ be a group and suppose $e_{\gamma\beta} \in H_{\gamma\beta}$. Then since $R^*_\beta = e_{\gamma\beta}S = e_{\alpha\beta}S$ we have $e_{\gamma\beta}e_{\alpha\beta} = e_{\alpha\beta}$ and $e_{\alpha\beta}e_{\gamma\beta} = e_{\gamma\beta}$.

Furthermore $L^*_\gamma e_{\alpha\beta} = Se_{\gamma\beta}e_{\alpha\beta} = Se_{\alpha\beta} = L^*_\alpha$.

Hence $L_\alpha \subset L_\gamma e_{\alpha\beta}$.

Since for each $1_\gamma \in L_\gamma$, $1_\gamma e_{\alpha\beta}e_{\gamma\beta} = 1_\gamma e_{\gamma\beta} = 1_\gamma$ we have $0 \notin L_\gamma e_{\alpha\beta}$ and thus $L_\alpha = L_\gamma e_{\alpha\beta}$.

Now let $H^2_{\gamma\beta} = \{0\}$ and let $a \in H_{\gamma\beta}$. Then since $R^*_\beta = e_{\alpha\beta}S = aS$ we have $e_{\alpha\beta} = as_1$ for a suitably chosen $s_1 \in S$.

Hence $L_\gamma e_{\alpha\beta} \subset L^*_\gamma e_{\alpha\beta} = Sae_{\alpha\beta} = Sa^2 s_1 = \{0\}$, and thus $L_\gamma e_{\alpha\beta} = \{0\}$.

2.3.4. Lemma. Let S be a compact 0-simple mob and let C be the set of all $x \in S$ such that $xe_{\alpha\beta} \in L_\alpha$ for all $e_{\alpha\beta} \in E \cap L_\alpha$.

Then $C = \bigcup \{L_\gamma \mid \gamma \in A,\ H_{\gamma\beta} \subset H \text{ if } H_{\alpha\beta} \subset H \text{ for all } \beta \in B\}$.

Proof:

Let $x \in C$ and $x \in L_\gamma$, and let $e_{\alpha\beta} = e^2_{\alpha\beta} \in H_{\alpha\beta} \subset H$.

Furthermore $xe_{\alpha\beta} \in L_\alpha$ and it follows from lemma 2.3.3 that $L_\gamma e_{\alpha\beta} = L_\alpha$ and that $H_{\gamma\beta} \subset H$.

Hence $L_\gamma \subset C$ and $H_{\gamma\beta}$ is a group if $H_{\alpha\beta}$ is a group.

Suppose on the other hand that $\gamma \in A$ is such that for all β $H_{\gamma\beta}$ is a group if $H_{\alpha\beta}$ is a group.

Then for each β with $e_{\alpha\beta} \in E \cap L_\alpha$ we have $H_{\gamma\beta}$ a group and hence $L_\gamma e_{\alpha\beta} = L_\alpha$. Thus $L_\gamma \subset C$.

2.3.5. Lemma. Let S be a compact 0-simple mob and let D be the set of all $x \in S$ such that $xe_{\gamma\beta} = 0$ for all $e_{\gamma\beta} \in E$ with $H^2_{\alpha\beta} = \{0\}$.

Then $D = \bigcup \{L_\gamma \mid \gamma \in A,\ H_{\gamma\beta} \subset H' \text{ if } H_{\alpha\beta} \subset H' \text{ for all } \beta \in B\}$.

Proof:

Let $x \in D$, $x \in L_\gamma$ and let $H_{\alpha\beta} \subset H'$.

Since $H_{\alpha\beta} = L_\alpha \cap R_\beta$ and since R_β contains a non-zero idempotent there

is a γ^* such that $H_{\gamma^*\beta}$ is a group.
Then $xe_{\gamma^*\beta} = 0$ and it follows from lemma 2.3.3 that $L_\gamma e_{\gamma^*\beta} = \{0\}$ and hence that $L_\gamma \subset D$ and $H^2_{\gamma\beta} = \{0\}$.
On the other hand let L_γ be such that $H_{\gamma\beta} \subset H'$ if $H_{\alpha\beta} \subset H'$ and let $e_{\gamma^*\beta} \in E$ with $H^2_{\alpha\beta} = \{0\}$.
Then $H^2_{\gamma\beta} = \{0\}$ and lemma 2.3.3 implies that $L_\gamma e_{\gamma^*\beta} = \{0\}$ and hence $L_\gamma \subset D$.

2.3.6. Theorem. Let S be a compact 0-simple mob. Let $\alpha_o \in A$.

Then $\mathcal{K}_{\alpha_o} = \bigcup \{L_\alpha \mid \alpha \in A,\ H_{\alpha_o\beta}$ and $H_{\alpha\beta}$ both in H or both in H' for all $\beta \in B\}$

is an open and closed subset of S.

Proof:

Let E be the set of all idempotents in S. Then E is closed, and since S is compact, we have that $L_{\alpha_o} = L^*_{\alpha_o} \setminus \{0\} = Sa \setminus \{0\}$ with $a \in L_{\alpha_o}$ is closed, hence $E_{\alpha_o} = L_{\alpha_o} \cap E$ is closed.
For each $e_{\alpha_o\beta} \in E_{\alpha_o}$ let C^o_β be the set of all elements x of S such that $xe_{\alpha_o\beta} \in L^*_{\alpha_o}$ and let $C = \bigcap \{C_\beta \mid e_{\alpha_o\beta} \in E_{\alpha_o}\}$.
Since $L_{\alpha_o} = Se_{\alpha_o\beta}$ for all $e_{\alpha_o\beta} \in E_\alpha$, we have $C_\beta e_{\alpha_o\beta} = L_{\alpha_o}$ and $(S \setminus C_\beta)e_{\alpha_o\beta} = \{0\}$, and the continuity of multiplication implies that both C_β and $S \setminus C_\beta$ are closed. Thus C is closed.
Now let $x \in C$. Then $xe_{\alpha_o\beta} \neq 0$ for all $e_{\alpha_o\beta} \in E_{\alpha_o}$.
Since E_{α_o} is compact we can find a neighbourhood V of x such that $0 \notin VE_{\alpha_o}$ which implies $Ve_{\alpha_o\beta} \subset L_{\alpha_o}$ for all $e_{\alpha_o\beta} \in E_{\alpha_o}$ and thus $V \subset C$ and C is open.
Now let $H'_{\alpha_o} = \bigcup \{H_{\alpha_o\beta} \mid \beta \in B$ and $H^2_{\alpha_o\beta} = \{0\}\}$. Then $H'_{\alpha_o} = L_{\alpha_o} \cap H'$ and hence closed.
Furthermore let $E^*_{\alpha_o} = (E \cap H'_{\alpha_o} S) \setminus \{0\}$. Then $E^*_{\alpha_o}$ is compact and is the set of all idempotents $e_{\gamma\beta}$ such that $H^2_{\alpha_o\beta} = \{0\}$.
Let $D_{\gamma\beta}$ be the set of all elements x of S such that $xe_{\gamma\beta} = 0$, $e_{\gamma\beta} \in E^*_{\alpha_o}$ and let $D = \bigcap \{D_{\gamma\beta} \mid e_{\gamma\beta} \in E^*_{\alpha_o}\}$.
Then it follows that $(S \setminus D_{\gamma\beta})e_{\gamma\beta} = L_\gamma$ and hence both $D_{\gamma\beta}$ and $S \setminus D_{\gamma\beta}$ are closed.
Suppose now $x \in D$. Then $xE^*_{\alpha_o} = \{0\}$. Since $E^*_{\alpha_o}$ is compact and $\{0\}$ open,

we can find a neighbourhood V of x such that $VE^*_{\alpha_o} = \{0\}$, which implies $V \subset D$ and hence that D is open.

Now let $\mathcal{X}_{\alpha_o} = C \cap D$. Then $\mathcal{X}_{\alpha_o}$ is an open and closed subset of S.

From lemma 2.3.4 and 2.3.5 it follows that

$C = \bigcup \{L_\alpha \mid \alpha \in A,\ H_{\alpha\beta} \subset H \text{ if } H_{\alpha_o\beta} \subset H \text{ for all } \beta \in B\}$ and

$D = \bigcup \{L_\alpha \mid \alpha \in A,\ H_{\alpha\beta} \subset H' \text{ if } H_{\alpha_o\beta} \subset H' \text{ for all } \beta \in B\}$.

Hence $\mathcal{X}_{\alpha_o} = C \cap D = \bigcup \{L_\alpha \mid \alpha \in A,\ H_{\alpha\beta} \text{ and } H_{\alpha_o\beta} \text{ either both in } H \text{ or both in } H' \text{ for all } \beta \in B\}$.

Furthermore $\mathcal{X}_{\alpha_o} \neq \emptyset$, since clearly $L_{\alpha_o} \subset \mathcal{X}_{\alpha_o}$.

Since $\mathcal{X}_{\alpha_o} = \mathcal{X}_{\alpha_1}$ if $\mathcal{X}_{\alpha_o} \cap \mathcal{X}_{\alpha_1} \neq \emptyset$ and $S \setminus \{0\}$ is compact, we see that $S \setminus \{0\}$ is the union of a finite number of disjoint open and closed sets $\mathcal{X}_{\alpha_o}, \mathcal{X}_{\alpha_1}, \ldots, \mathcal{X}_{\alpha_n}$.

A similar argument shows that $S \setminus \{0\}$ is also the union of a finite number of disjoint open and closed sets, say $\mathcal{R}_{\beta_o}, \mathcal{R}_{\beta_1}, \ldots, \mathcal{R}_{\beta_m}$, where $\mathcal{R}_{\beta_j} = \bigcup \{R_\beta \mid \beta \in B,\ H_{\alpha\beta} \text{ and } H_{\alpha\beta_j} \text{ either both in } H \text{ or both in } H' \text{ for all } \alpha \in A\}$, $(j=0,1,\ldots,m)$.

From this it follows that $S \setminus \{0\}$ is the disjoint union of the sets $\mathcal{X}_{\alpha_i} \cap \mathcal{R}_{\beta_j}$, $i=0,1,\ldots,n$, $j=0,1,\ldots,m$, where each $\mathcal{X}_{\alpha_i} \cap \mathcal{R}_{\beta_j}$ is either contained in H or in H'.

2.3.7. <u>Theorem</u>. Let S be a compact 0-simple mob. Then $S \setminus \{0\}$ is homeomorphic to a topological product $Y_1 \times X \times Y_2$, where Y_1 and Y_2 are two compact Hausdorff spaces and X is homeomorphic to the underlying space of a maximal subgroup of S contained in $S \setminus \{0\}$.

<u>Proof</u>:

Let $H_{\alpha_o\beta_o}$ be a maximal subgroup of S and let $\mathcal{X}_{\alpha_o}, \ldots, \mathcal{X}_{\alpha_n}$ and $\mathcal{R}_{\beta_o}, \ldots, \mathcal{R}_{\beta_m}$ be decompositions of $S \setminus \{0\}$ as described in theorem 2.3.6.

For each set $\mathcal{X}_{\alpha_o} \cap \mathcal{R}_{\beta_j} \subset H'$ we choose an element $a_j \in L_{\alpha_o} \cap R_{\beta_j}$ and a set $\mathcal{X}_{\alpha_i} \cap \mathcal{R}_{\beta_j} \subset H$. (Such a set always exists by lemma 2.3.2).

Let $\phi_j(x) = xa_j$, $x \in L_{\alpha_i} \cap \mathcal{R}_{\beta_j}$.

We now prove that ϕ_j is a topological map of $L_{\alpha_i} \cap \mathcal{R}_{\beta_j}$ onto $L_{\alpha_o} \cap \mathcal{R}_{\beta_j}$, such that $\phi_j(L_{\alpha_i} \cap R_{\beta_k}) = L_{\alpha_o} \cap R_{\beta_k}$ for each $R_{\beta_k} \subset \mathcal{R}_{\beta_j}$.

For let $x \in L_{\alpha_i} \cap R_{\beta_k} = H_{\alpha_i\beta_k}$, then $xa_j \in R_{\beta_k}a_j \cap xL_{\alpha_o} \subset R^*_{\beta_k} \cap L^*_{\alpha_o}$.
Furthermore, if x^{-1} is the inverse of x in $H_{\alpha_i\beta_k}$ we have
$x^{-1}xa_j = e_{\alpha_i\beta_k}a_j \neq 0$, by the dual of lemma 2.3.3.
Hence $xa_j \neq 0$ and we have $\phi_j(L_{\alpha_i} \cap R_{\beta_k}) \subset L_{\alpha_o} \cap R_{\beta_k}$.
On the other hand if $y \in L_{\alpha_o} \cap R_{\beta_k}$, then $y \in Sa_j$ and thus $y = s_1a_j$,
which implies $y = e_{\alpha_i\beta_k}y = e_{\alpha_i\beta_k}s_1a_j = e_{\alpha_i\beta_k}s_1e_{\alpha_i\beta_j}a_j = xa_j$ with
$x = e_{\alpha_i\beta_k}s_1e_{\alpha_i\beta_j} \in H_{\alpha_i\beta_k}$.
Hence ϕ_j is a mapping of $L_{\alpha_i} \cap \mathcal{R}_{\beta_j}$ onto $L_{\alpha_o} \cap \mathcal{R}_{\beta_j}$.
Now let $x_1a_j = x_2a_j$, then since $e_{\alpha_i\beta_j} \in a_jS$ we have $e_{\alpha_i\beta_j} = a_js_2$ and
thus $x_1a_js_2 = x_2a_js_2 \Rightarrow x_1e_{\alpha_i\beta_j} = x_2e_{\alpha_i\beta_j} \Rightarrow x_1 = x_2$.
Thus ϕ is topological.
Now let $a^*_k = \phi_j(e_{\alpha_i\beta_k})$. Then $A_j = \bigcup \{a^*_k \mid R_{\beta_k} \subset \mathcal{R}_{\beta_j}\}$ is a closed set
of S such that $A_j \subset L_{\alpha_o}$ and $A_j \cap H_{\alpha_o\beta_k} = a^*_k$ for all $R_{\beta_k} \subset \mathcal{R}_{\beta_j}$.
Let $Y_1 = (E \cap L_{\alpha_o}) \cup \bigcup^{\rho} \{A_j \mid \mathcal{L}_{\alpha_o} \cap \mathcal{R}_{\beta_j} \subset H'\}$.
Y_1 is the union of a finite number of disjoint closed sets and hence
closed.

Moreover $Y_1 \subset L_{\alpha_o}$ and Y_1 has exactly one point in common with each
set $H_{\alpha_o\beta}$.
In the same way we construct a set
$Y_2 = (E \cap R_{\beta_o}) \cup \bigcup \{B_i \mid \mathcal{R}_{\beta_o} \cap \mathcal{L}_{\alpha_i} \subset H'\}$ such that Y_2 is closed
$Y_2 \subset R_{\beta_o}$ and Y_2 has exactly one point in common with each set $H_{\alpha\beta_o}$.
Now let $S^* = Y_1 \times H_{\alpha_o\beta_o} \times Y_2$ and let $\phi(y_1,h,y_2) = y_1hy_2$.
ϕ is a mapping of S^* onto $S \setminus \{0\}$.
For let $x \in S \setminus \{0\}$, then $x \in L_\alpha \cap R_\beta$ and let $Y_1 \cap R_\beta = y_1$ and
$Y_2 \cap L_\alpha = y_2$.
Then $R^*_\beta = y_1S$ and hence $x = y_1s_1$. Furthermore if $e \in L_\alpha$, then
$x = xe = y_1s_1e$.
Since $s_1e \in L^*_\alpha = Sy_2$ we have $s_1e = s_2y_2$ and thus $x = y_1s_1e = y_1s_2y_2 =$
$(y_1e_{\alpha_o\beta_o})s_2(e_{\alpha_o\beta_o}y_2) = y_1hy_2$, $h = e_{\alpha_o\beta_o}s_2e_{\alpha_o\beta_o} \in H_{\alpha_o\beta_o}$.
Furthermore ϕ is one-to-one.
For if $y_1hy_2 = y^*_1h^*y^*_2 = x \in L_\alpha \cap R_\beta$, then since y_1 and y^*_1 both in Y_1
and both in R_β, we have $y_1 = y^*_1$, and dually since y_2 and y^*_2 both in
Y_2 and both in L_α we have $y_2 = y^*_2$.
Moreover there exist elements s_1 and s_2 such that $e_{\alpha_o\beta_o} = s_1y_1$ and

$e_{\alpha_o\beta_o} = y_2 s_2$.
Hence $s_1 y_1 h y_2 s_2 = s_1 y_1 h^* y_2 s_2 \Rightarrow e_{\alpha_o\beta_o} h e_{\alpha_o\beta_o} = e_{\alpha_o\beta_o} h^* e_{\alpha_o\beta_o} \Rightarrow h = h^*$.
Furthermore $y_1 h y_2 \neq 0$ and ϕ is a one-to-one continuous mapping of S^* onto $S \setminus \{0\}$.
If we recall that $S \setminus \{0\}$ is compact, it follows that ϕ is topological.

2.3.8. Theorem. Let H be a compact topological group, and let $H^o = H \cup \{0\}$ be the group with zero arising from H by the adjunction of a zero element 0.
Let X_1 and X_2 be two compact Hausdorff spaces and ϕ a continuous mapping of $X_2 \times X_1$ into H^o, with $\phi(x_2, X_1) \cap H \neq \emptyset$ and $\phi(X_2, x_1) \cap H \neq \emptyset$ for all $x_1 \in X_1$ and $x_2 \in X_2$.
Denote by (X_1, H^o, X_2, ϕ) the space $X_1 \times H \times X_2 \cup \{0\}$ with a multiplication defined by

$$(y_1, h, y_2)(y_1^*, h^*, y_2^*) = \begin{cases} (y_1, h\phi(y_2, y_1^*)h^*, y_2^*) & \text{if } \phi(y_2, y_1^*) \neq 0 \\ 0 & \text{if } \phi(y_2, y_1^*) = 0 \end{cases}$$

and $s.0 = 0.s = 0$ for all $s \in (X_1, H^o, X_2, \phi)$.
Then (X_1, H^o, X_2, ϕ) is a compact 0-simple mob.

Proof:
The multiplication defined in (X_1, H^o, X_2, ϕ) is clearly continuous and associative.
Hence (X_1, H^o, X_2, ϕ) is a compact mob.
Now let (x_1, h, x_2) and $(x_1^*, h^*, x_2^*) \in X_1 \times H \times X_2$.
Choose an element $y_2 \in X_2$ such that $\phi(y_2, x_1) \in H$ and an element $y_1 \in X_1$ such that $\phi(x_2, y_1) \in H$. Finally let h_1 and h_2 be such that $h_1 \phi(y_2, x_1) h \phi(x_2, y_1) h_2 = h^*$.
Then $(x_1^*, h_1, y_2)(x_1, h_1, x_2)(y_1, h_2, x_2^*) = (x_1^*, h^*, x_2^*)$.
Hence we have proved that $(X_1, H^o, X_2, \phi) s (X_1, H^o, X_2, \phi) = (X_1, H^o, X_2, \phi)$ for all $s \neq 0$.
Thus (X_1, H^o, X_2, ϕ) is 0-simple.

2.3.9. Theorem. A compact mob is 0-simple if and only if it is isomorphic with a mob (X_1, H^o, X_2, ϕ).

Proof:

It follows from theorem 2.3.7 that if S is 0-simple S is homeomorphic with $Y_1 \times H_{\alpha_o\beta_o} \times Y_2 \cup \{0\}$, where $Y_1 \subset L_{\alpha_o}$ and $Y_2 \subset R_{\beta_o}$.
Now let η be the mapping of $Y_2 \times Y_1$ into $H_{\alpha_o\beta_o} \cup \{0\}$, defined by $\eta(y_2,y_1) = y_2y_1 \in R_{\beta_o}L_{\alpha_o} \subset H_{\alpha_o\beta_o} \cup \{0\}$.
Moreover η has the property that $\eta(y_2,Y_1) \cap H_{\alpha_o\beta_o} \neq \{0\}$ and $\eta(Y_2,y_1) \cap H^o_{\alpha_o\beta_o} \neq \{0\}$ for all $y_2 \in Y_2$ and $y_1 \in Y_1$.
Hence $(Y_1,H^o_{\alpha_o\beta_o},Y_2,\eta)$ is a compact 0-simple mob.
Since the mapping ϕ: $S \to (Y_1,H^o_{\alpha_o\beta_o},Y_2,\eta)$ with $\phi(y_1,h,y_2) = y_1hy_2$ and $\phi(0) = 0$ clearly is an algebraic isomorphism, we see that $S \cong (Y_1,H^o_{\alpha_o\beta_o},Y_2,\eta)$.
The "if" part of the theorem follows from theorem 2.3.8.

2.4. Connected mobs

2.4.1. Lemma. If S is a connected mob, then each minimal (left, right) ideal of S is connected.

Proof:

Let L be a minimal left ideal of S, then for any $a \in L$, $Sa = L$, and hence L is connected.

If K is the minimal ideal of S, then $K = SaS$ for each $a \in K$.

Hence $K = \bigcup \{Sas_\alpha \mid s_\alpha \in S\}$.

Since each Sas_α is connected and meets the connected set aaS, it follows that K is connected.

2.4.2. Lemma. If S is a connected mob, then each ideal of S is connected, provided S has a left or right unit.

Proof:

Let I be an ideal of S. Then $I = \bigcup_{x \in I} Sx$, if S contains a left unit.
Since each Sx meets aS with $a \in I$, we have that I is connected.

Example:

Let $S = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ with the usual topology.
For (x_1,y_1) and $(x_2,y_2) \in S$ define the product $(x_1,y_1).(x_2,y_2)$ to be

$(0, y_1 y_2)$.

Then S is a compact connected commutative mob.

Let $I = \{(x,y) \mid x = 0,1,\ 0 \leq y \leq 1\}$.

And $I^* = \{(x,y) \mid 0 \leq x < \frac{1}{4},\ \frac{3}{4} < x \leq 1,\ 0 \leq y \leq 1\}$.

Then I is a disconnected closed ideal, and I^* is a disconnected open ideal.

2.4.3. Theorem. If S is a connected mob and I an ideal of S, then one and only one component of I is an ideal of S.

Proof:

Let $I^* = SI \cup IS$. Then I^* is connected and the component of I which contains I^* is an ideal of S.

Furthermore it is readily seen, that this is the only component of S which is an ideal. We will call this ideal, the component ideal of I.

2.4.4. Lemma. Let S be a compact connected mob and U a proper open subset of S with $J_o(U) \neq \emptyset$.

Let C_o be the component ideal of $J_o(U)$, then $\overline{C}_o$ intersects $\overline{U} \setminus U$.

Proof:

If $\overline{C}_o \cap \overline{U} \setminus U = \emptyset$, then $\overline{C}_o \subset U$, and since $\overline{C}_o$ is an ideal, we have $\overline{C}_o \subset J_o(U)$ and $C_o = \overline{C}_o$.

Furthermore $J_o(U)$ is open and we can find an open set V, with $C_o \subset V \subset \overline{V} \subset J_o(U)$. Since C_o is a component of the compact set $\overline{V}$ of the connected set S, we have $C_o \cap \overline{V} \setminus V \neq \emptyset$ a contradiction.

Corollary. Let S be a compact connected mob and F a closed subset of $S \setminus K$ with the property that if $F \cap I \neq \emptyset$, then $F \subset I$ for any ideal I of S. Then if C is the component of $S \setminus F$ which contains K, then $F = \overline{C} \setminus C$.

Proof:

Since C is closed in $S \setminus F$ we have $\overline{C} \cap S \setminus F = C$, hence $F \supset \overline{C} \setminus C$.

Furthermore it follows from 2.4.4 that if C_o is the component ideal of $J_o(S \setminus F)$, then $K \subseteq C_o$ and $\overline{C}_o$ intersects $(\overline{S \setminus F}) \setminus (S \setminus F) \subset F$.

Hence $F \subset \overline{C}_o \subset \overline{C}$.

Since $F \cap C = \emptyset$ we have $F \subset \overline{C} \setminus C$.

2.4.5. <u>Theorem</u>. Let S be a compact connected mob, and $e \in E \setminus K$. If C is the component of $S \setminus H(e)$ which contains K, then $H(e) = \overline{C} \setminus C$.

<u>Proof</u>:

This follows immediately from the preceding corollary, if we take $F = H(e)$, and from the fact that if $H(e) \cap I \neq \emptyset$, then $H(e) \subset I$ for any ideal I of S.

It follows from theorem 2.4.5 that if S is a compact connected mob, then H(e), with $e \in E \setminus K$ can contain no inner points.

2.4.6. <u>Theorem</u>. Let S be a compact connected mob. If K is not the cartesian product of two non-degenerate connected sets, then either K is a group or the multiplication in K is of type (a) or (b).

(a) $xy = x$ all $x,y \in K$.

(b) $xy = y$ all $x,y \in K$.

<u>Proof</u>:

From theorem 1.2.9 it follows that K is homeomorphic to $(Se \cap E) \times H(e) \times (eS \cap E)$, and since K is connected, each of H(e), $(Se \cap E)$ and $(eS \cap E)$ must be connected.

Hence at least two of the factors must consist of single elements.

If $eS \cap E = Se \cap E = e$, then $K = eSe = H(e)$ a group.

If $eS \cap E = eSe = e$, then $K = Se$, and if $x,y \in K$ we have $xy = (xe)(ye) = x(eye) = xe = x$.

If $Se \cap E = eSe = e$, then the multiplication is of type (b).

<u>Corollary</u>. Let S be a compact connected mob. If K contains a cutpoint, then the multiplication in K is of type (a) or (b).

<u>Proof</u>:

If K contains a cutpoint, then K is not the cartesian product of two non-degenerate connected sets.

Hence from 2.4.6 it follows that K is a group or the multiplication is of type (a) or (b).
Since a compact connected group contains no cutpoints, the corollary follows.

2.4.7. <u>Theorem</u>. Let S be a compact connected mob, with $S^2 = S$.
Then each maximal proper ideal J is connected.

<u>Proof</u>:
Let $a \in S \setminus J$, then $J = J_o(S \setminus \{a\})$ and if C_o is the component ideal of J, then $a \in \overline{C_o}$.
Since $\overline{C}_o \cup J$ is an ideal of S, we have $\overline{C}_o \cup J = S$ and $S \setminus J \subset \overline{C}_o \setminus C_o$.
Furthermore we know that $SJ \cup JS$ is connected and $C_o \cap (SJ \cup JS) \neq \emptyset$ hence $JS \cup SJ \subset C_o$.
Since $S^2 = S$, we have $S^2 = (S \setminus J \cup J)S \subset \overline{C}_o$. Hence $\overline{C}_o = S$.
Since $C_o \subset J \subset \overline{C}_o$, we have J connected.

Definition. A <u>clan</u> is a compact connected mob with a unit element.

2.4.8. <u>Lemma</u>. Let B be the solid unit ball in Euclidian n-space and let f be a continuous mapping of B into itself, such that $|x - f(x)| < \frac{1}{2}$ for all $x \in B$. Then $0 \in f(B)$.

<u>Proof</u>:
Let $x = (x_1,\dots,x_n)$, $f(x) = (f_1(x),\dots,f_n(x))$.
We now consider the mapping $h(x) = (x_1,\dots,x_n) - (f_1(x),\dots,f_n(x))$.
This mapping transforms the ball $|x| \leq \frac{1}{2}$ into itself and hence by Brouwers fixed point theorem there is a point x^* for which $h(x^*) = x^*$, i.e. $(x_1^*,\dots,x_n^*) = (x_1^*,\dots,x_n^*) - (f_1(x^*),\dots,f_n(x^*))$. Hence $f(x^*) = 0$.

2.4.9. <u>Theorem</u>. Let S be a mob with unit element u having a Euclidean neighbourhood U of u.
Then H(u) is an open subset of S and is a Lie-group.

<u>Proof</u>:
We identify U with E_n and let $F_\varepsilon = \{x \mid |u - x| \leq \varepsilon\}$.
Since the multiplication on F_ε is uniformly continuous there is a δ

such that $|x - xy| < \frac{\varepsilon}{2}$, $|x - yx| < \frac{\varepsilon}{2}$, whenever $|u - y| < \delta$.
By lemma 2.4.8 $u \in F_\varepsilon y$ and $u \in yF_\varepsilon$, hence y has an inverse y^{-1} in F_ε and the mapping $y \rightarrow y^{-1}$ is continuous.
Therefore H(u) is a topological group and since it contains an open set, it must be open in S.
Furthermore H(u) is locally Euclidean and hence a Lie-group.

Corollary. If S is a clan having a Euclidean neighbourhood of the identity, then S is a Lie-group.

Proof:

By theorem 2.4.9 H(u) is open. Furthermore H(u) is closed since S is compact, and hence H(u) must be all of S.

Thus if S is a clan and S is an n-sphere, then S is a topological group and hence n=0,1 or 3.
In general a compact manifold which admits a continuous associative multiplication with identity, must be a group.

Corollary. Let S be a clan and F a closed subset of S, such that $S \setminus F$ is locally Euclidean. Then either S is a group or $H(u) \subset F$.

Proof:

Let $h \in H(u)$ and $h \notin F$. Then h has a Euclidean neighbourhood V. Since $h^{-1}V$ is a Euclidean neighbourhood of u, it follows from the preceding corollary that S is a group.
In case S is a subset of Euclidean space, then it follows that $H(u) \subset$ boundary of S or S a topological group.
If S contains interior points, then it cannot be a group and we have $H(u) \subset$ boundary of S.

Definition. A subset C of a space X is a C-set provided that $C \neq X$ and if M is a continuum with $C \cap M \neq \emptyset$, then $M \subset C$ or $C \subset M$.
It can easily be shown that if C is a C-set of a compact connected Hausdorff space, then the interior of C is empty and C is connected.
For let x be an interior point of C, then there is an open set V

with $x \in V \subset \overline{V} \subset C$.
Now let $y \in X \setminus C$. Then the component M of y in $X \setminus V$ has a non-empty intersection with the boundary of $X \setminus V \subset \overline{V}$.
Hence M is a continuum with $M \cap C \neq \emptyset$ and $C \not\subset M$, $M \not\subset C$.
Suppose now $C = C_1 \cup C_2$, with C_1 and C_2 both open and closed in C and $C_1 \cap C_2 = \emptyset$. Then if $x \in C_2$, there is an open set V in X such that $x \in V$, $V \cap C_1 = \emptyset$. Hence there is an open set U such that $x \in U \subset \overline{U} \subset V$.
If M is the component of $y \in C_1$ in $X \setminus U$, then M has a non-empty intersection with the boundary of $X \setminus U \subset \overline{U}$.
Since $y \in M$ and $x \notin M$, we have $M \subset C$ and hence there is a point x^* of C_2 in M. Since M is connected this is a contradiction and it follows that C is connected.

2.4.10. Theorem. Let G be a compact Lie-group which acts on a completely regular space X. Let $p \in X$ such that $g(p) \neq p$ unless g is the identity; $g \in G$.
Then there exists a closed neighbourhood N of p and a closed subset C of N, such that the orbit of every point of N has exactly one point in common with C.

Proof:
See Gleason: Proc. Amer. Math. Soc. 1, 1950, p.p. 35-43.

2.4.11. Lemma. Let G be a compact group and let U be an open neighbourhood of the identity.
Then U contains an invariant subgroup H of G, such that G / H is a Lie-group.

Proof:
See Montgomery-Zippin: Topological transformation groups, p. 99.

2.4.12. Theorem. Let S be a clan, S no group and G a compact invariant subgroup of $H(u) = H$, such that H / G is a Lie-group.
Then S contains a continuum M, such that M meets H and the complement of H, and such that $u \in M \cap H \subset G$.

Proof:

We can consider H as a transformation group acting on S.

Let $H' = H / G$ and S' the space of orbits of G. Then H' is a compact Lie-group acting on S'.

By theorem 2.4.10 there exists a closed neighbourhood N of $u' = uG$ and a closed set $C \subset N$ such that $nH' \cap C$ is a single point for each $n \in N$.

Now let S" be the space of orbits under H. Then we have the following canonical mappings $\alpha: S \to S'$, $\beta: S' \to S''$, $\gamma: S \to S''$, with $\gamma = \beta\alpha$.

Since α and γ are open maps, β is also open.

Let N^o be the interior of N, then βN^o is open and $\beta(u') \in \beta(N^o)$.

Let P be the component of $\beta(N)$ which contains $\beta(u')$.

Then P meets the boundary of $\beta(N)$ and hence P is non-degenerate.

Now let $\beta^* = \beta \mid C$. Then since $nH' \cap C$ is a single point for each $n \in N$, it follows that β^* is a homeomorphism between C and $\beta(N)$.

$\beta^{*-1}(P)$ is a continuum which meets H' only at $C \cap H'$ and hence $\beta^{*-1}(P)$ also meets the complement of H'.

Now let K be a component of $\alpha^{-1}\beta^{*-1}(P)$. Since α is an open mapping, we have $\alpha(K) = \beta^{*-1}(P)$. Hence K is a continuum which meets H and the complement of H and $K \cap H \subset \alpha^{-1}(c)$, where $c = C \cap H'$.

Now let $h \in K \cap H$, then $K \cap H \subset hG$. Suppose now $M = h^{-1}K$, then $u \in M \cap H$ and $M \cap H \subset G$.

If $k \in K$ and $k \notin H$, then $h^{-1}k \in M$, $h^{-1}k \notin H$, since $S \setminus H$ is an ideal of S, q.e.d.

2.4.13. Theorem. Let S be a clan which is no group.

Then the identity u of S belongs to no non-trivial C-set.

Proof:

Let $u \in C$, with C a C-set. We first prove $C \subset H(u)$.

If $x \in C$, then since xS is a continuum which meets C, we have $C \subset xS$ or $xS \subset C$.

If $u \in xS \cap Sx$, then x has an inverse and is thus included in H(u).

Now let $u \notin xS$, then $xS \subset C$; $xS \neq C$ and there is an open set V with $xS \subset V$; $C \setminus V \neq \emptyset$.

Since $xK \subset K$ we have $K \cap C \neq \emptyset$. If $u \in K$, then S is a group, hence

$u \notin K$ which implies $K \subset C$.

Now we can find an open set W, with $x \in W$, $WS \subset V$.

Since C contains no inner points, there exists a $y \in W \setminus C$ with $yS \subset V$. Clearly yS is a continuum which meets both C and $S \setminus C$ and $C \not\subset yS$, a contradiction.

Hence $u \in xS$ and $u \in Sx$ and thus $x \in H(u)$ which implies $C \subset H(u)$.

Now let U be a neighbourhood of u such that $C \not\subset U$.

By lemma 2.4.11 there is a subgroup $G \subset U$ such that H / G is a Lie-group and $C \not\subset G$.

Theorem 2.4.12 implies the existence of a continuum M such that $u \in M \cap H \subset G$ and such that M meets the complement of H. Hence $M \cap C \neq \emptyset$ and since $C \subset H$, M meets the complement of C. Thus $C \subset M$.

However, $M \cap H \subset G$ and $C \not\subset G$, which implies $C \not\subset M$, a contradiction.

<u>Example</u>:

Let $A = \{(x,y) \mid y = \sin \frac{1}{x}, \quad 0 < x \leq 1\}$,

$B = \{(2-x,y) \mid (x,y) \in A\}$,

$C = \{(0,y) \cup (2,y) \mid -1 \leq y \leq 1\}$,

and let $S = A \cup B \cup C$.

We will show that S does not admit the structure of a clan.

For suppose that S is a clan. Since S is not homogeneous, S cannot be a topological group and hence $S \neq H(u)$.

Then $S \setminus H(u) = J \neq \emptyset$ is the maximal proper ideal of S. Since J is open, dense and connected, we have $A \cup B \subset J$ and hence $u \in C$. But since C is the union of two C-sets u cannot be in C.

2.4.14. <u>Lemma</u>. Let S be a clan and C a non-trivial C-set of S. If g is an idempotent with $g \notin K$, then $g \notin C$.

<u>Proof</u>:

Suppose $g \in C$. Since gSg is a continuum we have $C \subset gSg$ or $gSg \subset C$.

g is the identity of the clan gSg and gSg is not a group since $g \notin K$.

Hence theorem 2.4.13 implies that $C \not\subset gSg$.

Now suppose $gSg \subset C$. Then $K \cap C \neq \emptyset$ and since $g \in C$, $C \setminus K \neq \emptyset$. Let U and V be neighbourhoods of K with $SK = K \subset U \subset \overline{U} \subset V$, while $g \notin V$.

Since S is compact, there is a neighbourhood W of K such that $SW \subset U$. $\overline{SW}$ is a continuum and hence $\overline{SW} \subset C$. Furthermore $W \subset \overline{SW}$ and this would imply that C contains inner points, a contradiction.

2.4.15. <u>Theorem</u>. Let S be a clan and C a non-trivial C-set of S, then $C \subset K$.

<u>Proof</u>:

From the proof of the preceding lemma it follows that if $K \cap C \neq \emptyset$, then $C \subset K$.

Suppose now $C \cap K = \emptyset$ and let $x \in C$ and U a neighbourhood of x with $C \setminus U \neq \emptyset$.

Let e be a minimal member of the partial ordered set E with $xe = x$.
e exists since $E_x = \{e \mid e^2 = e,\ xe = x\}$ is non-empty and compact.
Furthermore $e \notin K$, since $x \notin K$.
Hence $H(e) \neq eSe$ and we can find a neighbourhood V of e such that $xV \subset U$ and a continuum $M \subset eSe$ such that $e \in M \subset V$ and $M \cap \{eSe \setminus H(e)\} \neq \emptyset$. Since $x \in xM$ we have $xM \subset C$.
Let $m \in M \cap \{eSe \setminus H(e)\}$, then $C \subset xSm$. This implies that $x = xs_1m = xes_1em = xp$, with $p \in \{eSe \setminus H(e)\}$, since $eSe \setminus H(e)$ is an ideal of eSe. Hence $x = xf$ with $f = f^2 \in \Gamma(p) \subset eSe$ and thus $ef = fe = f$, i.e. $f \leq e$. But since e is minimal we have $f = e$.
Furthermore $pe = p = ep$ and thus $pf = p = pf$, which implies $p \in H(f) = H(e)$ a contradiction.

2.4.16. <u>Theorem</u>. If S is a clan and if K is a C-set, then K is a maximal subgroup of S.

<u>Proof</u>:

If $S = K$, then S is a group and the result follows.
If $S \neq K$, then K has no interior point since K is a C-set.
Let $\{a_\lambda \mid \lambda \in \Lambda\}$ be a directed set of points of $S \setminus K$ with $a_\lambda \to e$, where $e = e^2 \in K$.
Since $K \cap a_\lambda S \neq \emptyset$, $K \cap Sa_\lambda \neq \emptyset$ and $a_\lambda \in a_\lambda S \cap Sa_\lambda$ we have $K \subset a_\lambda S \cap Sa_\lambda$.
Hence $K \subset eS \cap Se = eSe$. But since $e \in K$ implies $H(e) = eSe$ we see that $K = H(e)$.

2.4.17. Theorem. If a clan is an indecomposable continuum, it is a group.

Proof:

If $S = K$, then S is a group.

Suppose now $S \neq K$. Then there exists an open set V with $K \subset V \subset \overline{V} \neq S$.

Let $J_o(V)$ be the maximal ideal of S contained in V, then $J_o(V)$ is open and connected, and $K \subset J_o(V) \subset \overline{J_o(V)} \neq S$.

Since $S = \overline{J_o(V)} \cup \overline{S \setminus \overline{J_o(V)}}$ and S is indecomposable we have $S \setminus \overline{J_o(V)}$ not connected.

Let $S \setminus \overline{J_o(V)} = A \cup B$, $A \cap B = \emptyset$, A, B open.

Then we have $\overline{J_o(V)} \cup A$ connected and $\overline{J_o(V)} \cup B$ connected and hence S not indecomposable, a contradiction.

2.5. I-semigroups

Definition. Let $J = [a,b]$ denote a closed interval on the real line. If J is a mob such that a acts as a zero-element and b as an identity, then J will be called an I-semigroup.

We will identify J usually with $[0,1]$, so that $0x = x0 = 0$ and $1x = x1 = x$ for all $x \in J$.

Example:

$J_1 = [0,1]$ under the usual multiplication.

$J_2 = [\frac{1}{2},1]$ with multiplication defined by $x \bullet y = \max(\frac{1}{2}, xy)$, where xy denotes the usual multiplication of real numbers.

$J_3 = [0,1]$ with multiplication defined by $x \bullet y = \min(x,y)$.

J_1 and J_2 have just the two idempotents zero and identity, but in J_3 every element is an idempotent.

Furthermore every non-idempotent element in J_2 is algebraically nilpotent, i.e. for every $x \in J_2$ there exists an n such that x^n is equal to zero.

2.5.1. Lemma. If J is an I-semigroup, $J = [0,1]$, then $xJ = Jx = [0,x]$ for all $x \in J$.

Proof:

Since xJ is connected and $0,x \in xJ$ we have $[0,x] \subset xJ$, and by the same argument $Jx \supset [0,x]$.

$J_o([0,x)) = J_o$ is open and connected and hence $x \in \bar{J}_o$ and $\bar{J}_o$ an ideal of J. Hence $Jx \subset J\bar{J}_o \subset \bar{J}_o \subset [0,x]$ and $xJ \subset [0,x]$.

Thus $xJ = Jx = [0,x]$.

Corollary. If J is an I-semigroup, then $x \leq y$ and $w \leq v$ implies $xw \leq yv$.

Proof:

Since $x \leq y$ there is a z such that $x = zy$. Hence $xw = z(yw) \leq yw$. In the same way we can prove $yw \leq yv$ and thus $xw \leq yv$.

2.5.2. Theorem. If J is an I-semigroup with just the two idempotents 0 and 1 and with no (algebraically) nilpotent elements, then J is isomorphic to J_1

Proof:

We first show that if $xy = xz \neq 0$, then $y = z$.

Assume $y < z$. Then by lemma 2.5.1 there is a w such that $y = zw$.

Hence $xy = xzw = xyw$ and thus $xy = (xy)w^n$ for every $n=1,2,\dots$.

Thus $xy = (xy)e$ with $e = e^2 \in \Gamma(w)$.

Since $1 \notin \Gamma(w)$, we have $e = 0$ and thus $xy = 0$, a contradiction.

We now prove that if $x \neq 0$, then x has a unique square root.

The function $f: J \to J$ defined by $f(x) = x^2$ is continuous and leaves 0 and 1 fixed. Hence f is a map of J onto J so that square roots exist for every element.

Assume $a^2 = b^2 \neq 0$ and let $a \leq b$. Then by lemma 2.5.1 $a^2 \leq ab \leq b^2$ and $ab = a^2$ which implies $a = b$.

This establishes that for $x \neq 0$, x has a unique square root and by induction that x has unique 2^n-th roots.

Let x_n be the 2^n-th root of $x \neq 0$ and for $r = p / 2^n$ define $x^r = x_n^p$.

Then it is easy to prove that $x^r x^s = x^{r+s}$, where r,s are positive dyadic rationals. Furthermore if $r < s$, then $x^r > x^s$. For by lemma 2.5.1 $x^r \geq x^s$ and if $x^r = x^s$, then $x^{r-s} = 1$, a contradiction.

This implies that $\lim x_n = 1$. For since $x_n < x_{n+1}$, $\lim x_n$ exists.

Assume $\lim x_n = y \neq 1$.

Then since $y^n \to 0$, there is an n_o such that $y^{n_o} < x$.

Hence $y < x_{n_o}$, a contradiction.

Now let $D = \{x^r \mid r$ a positive dyadic rational$\}$.

Then D is a commutative submob of J and $\overline{D} = J$.

Assume $\overline{D} \neq J$. Then there is an open interval $P \subset J \setminus \overline{D}$, $P = (a,b)$ and $b \in \overline{D}$.

Now since $x_n \to 1$, $x_n b \to b$ and $x_n b \leq b$ by lemma 2.5.1.

If $x_n b = b$, then $x_n = 1$, a contradiction. Hence $x_n b < b$ and $x_n b \in P$ for n sufficiently large.

Since $b \in \overline{D}$ and $x_n \in \overline{D}$, we have $x_n b \in \overline{D}$, a contradiction, and thus $\overline{D} = J$.

Now let $g: D \to J_1$ be defined by $g(x^r) = \frac{1}{2}^r$.

$g(D)$ is dense in J_1 and g is one-to-one continuous and order preserving.

Hence g can be extended to a topological isomorphism of J onto J_1.

2.5.3. Theorem. If J is an I-semigroup with just the two idempotents 0 and 1 and with at least one nilpotent element, then J is isomorphic to J_1.

Proof:

Let $d = \sup \{x \mid x^2 = 0\}$. Then $d \neq 0$, for let $y \neq 0$ be nilpotent, then $y^n = 0$, $y^{n-1} \neq 0$ for some $n > 1$.

Clearly $(y^{n-1})^2 = 0$. Hence $d \geq y^{n-1}$.

As was shown in theorem 2.5.2, d has a unique 2^n-th root, and if r and s are positive dyadic rationals, then $d^r < d^s$ if $r > s$ and $d^s \neq 0$, and $d^r d^s = d^{r+s}$.

Now let $D = \{d^r \mid r$ a positive dyadic rational$\}$. Then by the same type of argument used in the proof of theorem 2.5.2 we can prove that $\overline{D} = J$.

We define $g: D \to J_2$ by $g(d^r) = (\sqrt{\tfrac{1}{2}})^r$. Then g is one-to-one and continuous and is an isomorphism.

Moreover $g(D)$ is dense in J_2 and since g is order preserving it can be extended to an isomorphism of J onto J_2.

2.5.4. Theorem. Let J be an I-semigroup. Then E is closed and if $e,f \in E$, then $ef = \min(e,f)$.

The complement of E is the union of disjoint intervals. Let P be the closure of one of these. Then P is isomorphic to either J_1 or J_2. Furthermore if $x \in P$, $y \notin P$, then $xy = \min(x,y)$.

Proof:

Let $e,f \in E$, $e < f$. Then by lemma 2.5.1 $ee \leq ef$ and thus $e \leq ef$. Since $ef \leq e$ we have $e = ef$.

Now let $Q = [e,f]$. Then for any $(x,y) \in [e,f]$ we have $ee \leq xy \leq ff$. Hence Q is a submob of J.

Furthermore if $e \leq x$, then $e \geq ex \geq ee = e$ and hence $ex = e$. In other words e acts as a zero for $[e,1]$.

If $x \leq f$, then $x = fy$ and thus $fx = x$, which implies that f acts as an identity for $[0,f]$.

So we have in particular P an I-semigroup with only two idempotents and hence P is isomorphic either to J_1 or J_2.

If $x \in P$, $y \notin P$, $x \leq y$ then there is an $e \in E$ with $x \leq e \leq y$. Hence $xy = (xe)y = x(ey) = xe = x$.

From theorem 2.5.4 it follows that every I-semigroup is commutative.

2.5.5. Theorem. Let S be the closed interval $[a,b]$. If S is a mob such that a and b are idempotents and S contains no other idempotents, then S is abelian.

Proof:

Let $e \in E \cap K$. Then $e = a$ or b. Since S has the fixed point property $K \subset E$. Furthermore K is connected and thus $K = a$ or $K = b$.

If $K = a$, then a is a zero for S and g an identity since $gS = Sg = S$. Thus S is an I-semigroup and hence abelian.

2.5.6. Theorem. Let S be the closed interval $[a,b]$. If S is a clan such that both a and b are idempotents, then S is abelian if and only if S has a zero.

Proof:

Let S be commutative. Then K is a group and since S has the fixed point property, we see that K consists of only one element, a zero.

Now let S have a zero. If either a or b is the zero element, then the other is obviously a unit and the result follows from theorem 2.5.4.

Now let $a < 0 < b$. Then $S' = [a,0]$ is a submob of S. For suppose there exist $x,y \in S'$ with $xy \in (0,b]$. Then since a acts as a unit on S', we have $[x,xy] \in x[a,y]$.

Hence there is an $s^* \in [a,y]$ with $xs^* = 0$.

Since $[s^*,0] \in s^*S'$ we have $y = s^*q$ and $xy = xs^*q = 0$, a contradiction.

In the same way we can prove that $S'' = [0,b]$ is a submob of S and both S' and S'' are commutative since they are I-semigroups.

It also follows that the unit of S is either a or b.

Suppose b is the unit element. Then, in the same way as above, we can prove that $aS'' = S''a = [0,a]$.

Hence if $x'' \in S''$, then $ax'' = y''a = (y''a)a = a(x''a) = a(az'') = az'' = x''a$.

Furthermore if $x' \in S'$ and $x'' \in S''$, then $x'x'' = (x'a)x'' = x'(ax'') = (ax'')x' = (x''a)x' = x''x'$.

2.5.7. Theorem. Let S be the closed interval $[a,b]$. If S is a mob such that a and b are idempotents, then S is abelian if and only if S has a zero and $ab = ba$.

Proof:

If S is commutative, S has a zero by the same argument as in theorem 2.5.6 and obviously $ab = ba$.

Now let S have a zero and let $ab = ba$. Then again the result follows if either a or b is a zero.

If $a < 0 < b$, then $S' = [a,0]$ and $S'' = [0,b]$ are abelian submobs of S.

Suppose now $ab \in S'$, then $bS' = baS' = abS' = [ab,0] = S'b$ by lemma 2.5.1. Hence $bS = Sb = [ab,b]$ and $[ab,b]$ is an abelian submob by theorem 2.5.6.

To prove the theorem it suffices to show that if $x \in [a,ab]$ and $y \in [ab,b]$ then $xy = yx$.

Now $xy = (xa)(by) = (xab)y$ and $xab \in [ab,0]$.

Hence $(xab)y = y(xab) = y(xb) = (yb)(xb) = y(bxb) = ybbx = yx$.

2.6. Interval mobs with $S^2 = S$

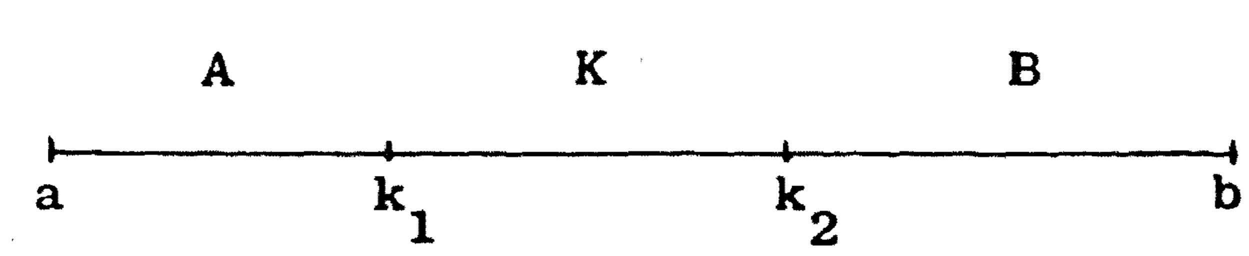
In what follows S will always be a mob on an interval $[a,b]$. The kernel of S is connected and hence either a point or an interval. We will assume that k_1 is the left hand endpoint of K and k_2 the right hand endpoint.

Furthermore K consists of either all left zeroes or all right zeroes of S. By passing to the product-dual of S if necessary, we can assume that the former is the case.

Throughout this paragraph, K will consist of all left zeroes of S.

Let A be the interval $[a,k_1]$ and B the interval $[k_2,b]$.

Then we have the following diagram for S.



Definition. Let T be a submob of S. A mapping f of T into S will be called <u>left invariant</u> if $f(x_1) = f(x_2)$, $x_1,x_2 \in T$, implies $f(tx_1) = f(tx_2)$ for all $t \in T$.

Notice that for instance all right translations $\rho_a: x \to xa$, $x \in T$, $a \in S$, are left invariant.

Furthermore all homomorphic mappings of T into S are left invariant and also all one-to-one mappings.

2.6.1. <u>Lemma</u>. Let S be an interval mob and let $T = [0,u]$ be an I-semigroup contained in S.

Let $f: T \to S$ be a continuous left invariant mapping.

Then for $t \in T$, either $f^{-1}(f(t)) = \{x \mid f(x) = f(t)\} = \{t\}$ or $f^{-1}(f(t))$ is an interval submob $[e,t^*]$ with $e^2 = e$.

Hence f is a continuous monotone mapping of T into S.

Proof:

Define an order relation in T by $x \prec y$ if $x \in [0,y]$ and suppose $f(x) = f(t)$, $x \neq t$.

Let $e = \inf \{x \mid f(x) = f(t),\ x \in T\}$ and $t^* = \sup \{x \mid f(x) = f(t),\ x \in T\}$.

Since f is continuous we have $f(e) = f(t^*) = f(t)$.

From 2.5.4 it follows that either $et^* = e$ or e, t^* both contained in a submob $P = [e_1, e_2]$ which is isomorphic to either J_1 or J_2.

If $et^* = e$, then $f(t^*) = f(e)$ implies $f(et^*) = f(e^2) = f(e)$. Since $e^2 \prec e$ it follows that $e^2 = e$.

In the other case there exists a $p \in P$ such that $e = pt^*$.

Thus $f(e) = f(pt^*) = f(t^*)$ and we have $f(p^n t^*) = f(t^*)$.

Since $p^n \to e_1$ the continuity of f implies that $f(e_1 t^*) = f(t^*) = f(e)$.

But $e_1 t^* = e_1$ and $e_1 \prec e$, so we have $e_1 = e$ and $e = e^2$.

Now for any $x \in [e, t^*]$, $x = qt^*$ for some $q \succ e$, and since $f(e) = f(t^*)$ we have $f(qe) = f(qt^*) = f(e)$.

This shows that $f^{-1}(f(t)) = [e, t^*]$ and hence f is monotone.

It follows from the above lemma that each continuous left invariant mapping of T determines a partition of T into disjoint closed intervals, such that the lower endpoint of each non-degenerate interval is an idempotent.

2.6.2. Lemma. Let S be an interval mob and T an I-mob contained in S. Let f be a continuous left invariant mapping of T into S. Let $g: T \times f(T) \to f(T)$ be given by $g(x,y) = f(x f^{-1}(y))$ and $h: f(T) \times f(T) \to f(T)$ by $h(x,y) = f(f^{-1}(x) f^{-1}(y))$. Then g and h are well defined and continuous.

Proof:

Let y' and $y'' \in f^{-1}(y)$ and $x \in T$, then $f(y') = f(y'')$ and hence $f(xy') = f(xy'')$.

If z' and $z'' \in f^{-1}(z)$ then $f(z'y') = f(z'y'') = f(y''z') = f(y''z'') = f(z''y'')$, since T is abelian and f left invariant. Hence h and g are well defined.

Now let U be a neighbourhood of $h(x,y)$. Since f is continuous $f^{-1}(U)$

is open and by continuity of multiplication in T, we have open intervals V^* and W^* containing $f^{-1}(x)$ and $f^{-1}(y)$ with $V^*W^* \subset f^{-1}(U)$.
Let $f^{-1}(x) = [e_1, x^*]$ and $f^{-1}(y) = [e_2, y^*]$ and let $x_1, x_2 \in V^*$, $y_1, y_2 \in W^*$ with $x_1 \prec e_1$, $x_2 \succ x^*$, $y_1 \prec e_2$, $y_2 \succ y^*$.
Then $f(x_1) \neq f(x) \neq f(x_2)$ and $f(y_1) \neq f(y) \neq f(y_2)$.
Since f is monotone we have $f(x_1) \neq f(x_2)$ and $f(y_1) \neq f(y_2)$.
Now let V be the open interval $(f(x_1), f(x_2))$ and W the open interval $(f(y_1), f(y_2))$.
Then $x \in V$, $y \in W$ and $h(V \times W) = f(f^{-1}(V) f^{-1}(W)) \subset f(V^*W^*) \subset ff^{-1}(U) = U$.
In the same way we prove that g is continuous.

Now let S be an interval mob $[a,b]$ with $S^2 = S$. Then either $S = K$ or S contains a maximal proper ideal M such that S / M is a completely 0-simple semigroup.
Since each maximal proper ideal is connected we have the following 4 cases.

2.6.3. <u>Lemma</u>. Let S be an interval mob $[a,b]$ with $S^2 = S$. Then one of the following cases holds:

1) $S = K$ and the multiplication is trivial.
2) S contains exactly one maximal ideal $M = (a,b)$.
   Then either i) $a^2 = a$, $b^2 = a$, $ab = ba = b$ (or dually $a^2 = b$, $b^2 = b$, $ab = ba = a$), or ii) $a^2 = a$, $b^2 = b$, $ab = a$, $ba = b$ (or dually $ab = b$, $ba = a$).
3) S contains exactly two maximal ideals $M_1 = [a,b)$, $M_2 = (a,b]$.
   Then $a^2 = a$, $b^2 = b$ and $b \neq ab \neq a$, $b \neq ba \neq a$.
4) S contains exactly one maximal ideal $M = (a,b]$.
   Then $a^2 = a$ and a is a left or right unit for S.
   Furthermore $ab \neq a \neq ba$, $b^2 \neq a$ (or dually $M = [a,b)$ ...)

<u>Proof</u>:
Since each maximal ideal of S is connected and dense in S, the maximal ideals in S can only be (a,b), $(a,b]$ and $[a,b)$ and we can only have the four cases mentioned.

If $M = (a,b)$, then $S / M = \{0,a,b\}$ is completely 0-simple and hence S / M a group with zero or S / M left (right) 0-simple.

Thus $a^2 = a$, $b^2 = a$, $ab = ba = b$ or $a^2 = a$, $b^2 = b$, $ab = a$, $ba = b$.

If $M = (a,b]$ is the only maximal ideal, then $S / M = \{0,a\}$ and $a^2 = a$.

Furthermore $b \in SaS = SaaS$.

If both $Sa \neq S$ and $aS \neq S$, then we have for instance $aS \subset Sa$ and $SaaS \subset SaSa \subset Sa$, a contradiction.

Hence we have either $Sa = S$ or $aS = S$ and thus a a left or right unit. Furthermore $a \not\in Ma$ and $a \not\in aM$, hence $ba \neq a \neq ab$.

Case 3 follows analogously.

Let S be $[-1,1]$ with the usual multiplication of real numbers. Then S belongs to case 2i).

If we define a multiplication by $x \bullet y = \frac{x}{|x|} |x| y$ then S belongs to case 2ii).

If we define a multiplication by $x \bullet y = \max \left(\frac{x}{|x|} \frac{y}{|y|}, 0 \right) \frac{x}{|x|} xy$ then S belongs to case 3).

If $S = [0,1]$ with the usual multiplication, then S belongs to case 4).

2.6.4. <u>Lemma</u>. Let $S = [a,b]$ be any interval mob with $a < k_1 \leq k_2 \leq b$ and let $a = a^2$. Then $A = [a,k_1]$ is an I-semigroup.

<u>Proof</u>:

Since $ak_1, k_1a \in K$ we have $[a,k_1] \subset Aa \cap aA$ and hence a is an identity for A.

Now let $x,y \in A$ and suppose $xy \not\in A$. Then $k_1 \in [x,xy] \subset x[a,y]$ and $k_1 = xt$, with $t \in [a,y]$. We also have $y \in [t,k_1] \subset t[a,k_1]$ and hence $y = tr$ with $r \in A$.

Thus $xy = xtr = k_1 r = k_o \in K$.

Since $k_1 \in [y,xy] \subset [a,x]y$ we have $k_1 = t'y$, $t' \in [a,x]$, and $x \in [t',k_1] \subset [a,k_1]t'$ which implies $x = r't'$, $r' \in A$.

Hence $xy = r't'y = r'k_1 = k_o$.

Thus $k_o = k_1 r = r'k_1$ and $k_o k_1 = k_o = k_1 k_o$. This implies that $k_o = k_1$, i.e. $xy \in A$.

Since k_1 is a zero element for A, A is an I-semigroup.

2.6.5. <u>Theorem</u>. Let $S = [a,b]$ be any interval $[a,b]$ with $a < 0 < b$. Let $A = [a,0]$ be any I-semigroup and f a homeomorphism of A onto $B = [0,b]$ with $f(0) = 0$.

Define a multiplication $\bullet$ on S as follows:

$x \bullet y = xy$, $u \bullet v = f^{-1}(u)f^{-1}(v)$, $x \bullet u = u \bullet x = f\big(xf^{-1}(u)\big)$,

$x,y \in A$, $u,v \in B$.

Then $(S,\circ)$ is a mob belonging to case 2i) and each such mob can be so constructed.

<u>Proof</u>:

Let $m\colon S \times S \to S$ be defined by $m(s_1,s_2) = s_1 \circ s_2$.

m is well defined since $m(0,s) = 0 \bullet s = 0s = f^{-1}(0)s = sf^{-1}(0) = m(s,0)$.

m is continuous since $m \mid A \times A$, $m \mid A \times B$, $m \mid B \times A$ and $m \mid B \times B$ are continuous. Furthermore m is commutative and the associativity of m follows from

$(x \circ y) \circ u = f\big(xyf^{-1}(u)\big) = f\Big(xf^{-1}\big(f\big(yf^{-1}(u)\big)\big)\Big) = x \circ (y \circ u)$.

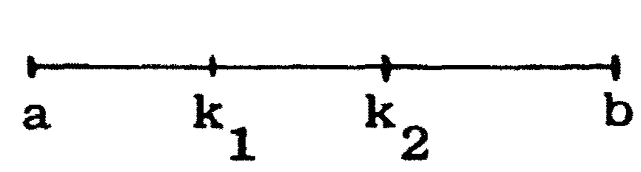
$(x \circ u) \circ v = f\big(xf^{-1}(u)\big) \bullet v = xf^{-1}(u)f^{-1}(v) = x \circ (u \circ v)$.

$(u \circ v) \circ w = \big(f^{-1}(u)f^{-1}(v)\big) \bullet w = f\big(f^{-1}(u)f^{-1}(v)f^{-1}(w)\big) = u \circ (v \circ w)$.

Thus $(S,\bullet)$ is an interval mob with $a^2 = a$ and $b^2 = f^{-1}(b)f^{-1}(b) = a$.

$b \bullet a = f\big(f^{-1}(b)a\big) = f(a) = b = a \circ b$.

Thus S belongs to case 2i).

Conversely if S is any interval mob $[a,b]$ belonging to case 2i), then we have S = 

with $a < k_1 \leq k_2 < b$.

Since $aS = Sa = S$, a is a unit element of S and lemma 2.6.4 implies that $A = [a,k_1]$ is an I-semigroup.

Let $k_1k_2 = k_1$. The mapping $f\colon x \to bx$, $x \in A$ is continuous and one-to-one, for if $bx_1 = bx_2$ then $b^2x_1 = b^2x_2$, i.e. $x_1 = x_2$. Furthermore $[k_2,b] \subset bA$. Hence $k_2 = bx$, $x \in A$, and thus $bk_1 = bxk_1 = k_2k_1 = k_2$. Since f is a monotone mapping we have $bA = B = [k_2,b]$, and since $k_1b = k_1$, $[k_1,b] \subset Ab$. This implies that $k_2 = xb$ with $x \in A$. Hence $k_2 = k_2b = xb^2 = x$ and $k_1 = k_2$.

Theorem 2.5.7 implies that S is abelian and hence f is a homeomorphism of A onto B with $xu = ux = f\big(xf^{-1}(u)\big)$ and $uv = f^{-1}(u)f^{-1}(v)$.

Definition: An $\underline{I_k\text{-mob}}$ is an interval mob $[a,k]$ with unit element a and $k \in K$.

It is clear that all I-semigroups are I_k-mobs with $K = \{0\}$.

Let $S = [-1,1]$ and define a multiplication on S by $x \circ y = xy$, $-x \circ y = -x$, $-x \circ -y = -x$, $x \circ -y = -xy$, where $x,y \in [0,1]$ and xy is the usual product of real numbers.

Then $(S,\circ)$ is an I_k-mob with non degenerate kernel $[-1,0]$.

2.6.6. Lemma. Let $S = [a,k_2]$ be any interval with $a < k_1 \leq k_2$. Let $A = [a,k_1]$ be an I-mob with unit a and f a continuous left invariant mapping of A onto $K = [k_1,k_2]$ with $f(k_1) = k_1$, $f(a) = k_2$.

Define a multiplication $\circ$ on S as follows:

$x \circ y = xy$, $k \circ S = k$, $x \circ k = f(xf^{-1}(k))$, $x,y \in A$, $k \in K$.

Then $(S,\circ)$ is an I_k-mob with kernel K and each such mob can be so constructed.

Proof:

Let m: $S \times S \to S$ be defined by $m(s_1,s_2) = s_1 \circ s_2$. Then m is well defined since $m(k_1,x) = k_1x = k_1$ and $m(k_1,k) = f(k_1(f^{-1}(k))) = f(k_1) = k_1$.

Furthermore $m(x,k_1) = f(xf^{-1}(k_1)) = f(k_1) = k_1 = xk_1$ and $m(k,k_1) = k$.

Lemma 2.6.2 implies that $m \mid A \times K$ is continuous, $m \mid A \times A$ and $m \mid K \times S$ are continuous and hence $(S,\circ)$ is an interval mob.

Moreover K is clearly the kernel of S and a the identity.

If on the other hand $[a,k_2]$ is an I_k-mob, with K all left zeroes, then $A = [a,k_1]$ is an I-semigroup and the mapping $\rho_{k_2} : x \to xk_2$ is clearly a left invariant mapping of A onto K with $x \circ k = \rho_{k_2}(x\rho_{k_2}^{-1}(k))$.

2.6.7. Theorem. Let $S = [a,b]$ with $a < k_1 \leq k_3 \leq k_4 \leq k_2 < b$ and let $[a,k_3] = A$ be an I_k-mob with kernel $[k_1,k_3]$ consisting of all left zeroes.

Let f be a homeomorphism of A onto $B = [k_4,b]$ with $f(k_3) = k_4$, $f(k_1) = k_2$, $f(a) = b$ and g a continuous mapping of $[k_3,k_4]$

into $[k_1, k_3]$ with $g(k_3) = g(k_4) = k_3$.

Define a multiplication $\circ$ on S by:

$x_1 \circ x_2 = x_1 x_2$, $y_1 \circ y_2 = f\big(f^{-1}(y_1) f^{-1}(y_2)\big)$, $x_1 \circ y_1 = x f^{-1}(y_1)$,

$y_1 \circ x_1 = f\big(f^{-1}(y_1) x_1\big)$, $k \circ s = k$, $s \circ k = s \circ g(k)$,

$x_1, x_2 \in A$, $y_1, y_2 \in B$, $k \in [k_3, k_4]$.

Then S belongs to case 2ii) and each such mob can be so constructed.

Proof:

We first show that the multiplication $m: S \times S \to S$, $m(s_1, s_2) = s_1 \circ s_2$ is well defined.

Since $k_3 \in K$ we have $k_3 \circ s = k_3$, and since $k_3 \in A$ we have $k_3 \circ s = k_3 s' = k_3$ with $s' \in A$.

On the other hand $s \circ k_3 = s \circ g(k_3) = s \circ k_3$ and m is well defined for k_3.

Analogously we have for k_4, since $k_4 \in K$, $k_4 \circ s = k_4$ and $k_4 \circ s = f\big(f^{-1}(k_4) s'\big) = f(k_3) = k_4$ with $s' \in A$ and $s \circ k_4 = s \circ g(k_4) = s \circ k_3 = s \circ f^{-1}(k_4)$.

Since $m \mid A \times A$, $m \mid B \times B$, $m \mid A \times B$, $m \mid B \times A$, $m \mid K \times S$ and $m \mid S \times K$ are continuous, m is continuous.

Furthermore elementary calculations show that m is associative.

Thus S is an interval mob with $a^2 = a$, $b^2 = f\big(f^{-1}(b) f^{-1}(b)\big) = f(a) = b$, $b \circ a = f(a \circ b) = f\big(a f^{-1}(b)\big) = b$, $a \circ b = a$ and S belongs to 2ii).

Conversely if $S = [a,b]$ belongs to 2ii) then both $[a,k_1]$ and $[k_2,b]$ are I-mobs. Furthermore $[a,k_1] \subset a[k_2,b]$ and hence there exists an $x \in [k_2,b]$ with $ax = k_1$.

But then $bax = bx = x = bk_1 \in K$, which implies $x = k_2$ and $ak_2 = k_1$.

In the same way we may prove $bk_1 = bak_2 = bk_2 = k_2$.

Now let $aS = [a,k_3]$ and $bS = [k_4,b]$.

Then $k_1 \le k_3$ and $k_4 \le k_2$ since $ak_1 = k_1$ and $bk_2 = k_2$.

Furthermore if $k_4 < k_3$, there is an $x < k_4$ with $k_4 < bx < k_3$ and we have $abx = bx = ax = x$, a contradiction.

Hence $k_1 \le k_3 \le k_4 \le k_2$ and $A = [a,k_3]$ is an I_k-mob.

The mapping $f_b: A \to B = [k_4,b]$ with $f(x) = bx$ is one-to-one and continuous, with $f(k_1) = k_2$.

f is onto since $bA = baS = bS = B$. Furthermore $f^{-1}(y) = ay$.

Let $g: [k_3,k_4] \to [k_1,k_3]$ be defined by $g(k) = ak$.

Then g is continuous and $g(k_3) = g(k_4) = k_3$.

We moreover have $y_1y_2 = b(y_1y_2) = (bay_1)y_2 = (bay_1a)y_2 = f\left(f^{-1}(y_1)f^{-1}(y_2)\right)$.

$xy = xay = xf^{-1}(y)$.

$yx = bayx = b(ayx) = f\left(f^{-1}(y)x\right)$.

$ks = k$, $sk = (sa)k = s(ak) = sg(k)$.

<u>Construction</u>:

Let $a < k_1 \leq k_2 < b$. We define a collection of mobs we call S(c) with $c \in (a,b)$ as follows:

1) $k_2 < c < b$.

Define a multiplication on $[a,k_1] = A$ and $[c,b] = B$, making them into I-semigroups with identity elements respectively a and b.

Let θ be an idempotent $a < \theta \leq k_1$ and f a left invariant mapping of A onto $[k_1,c]$ with $f(k_1) = k_1$, $f(\theta) = k_2$, $f(a) = c$.

Define a multiplication $\circ$ on S by

$$
\begin{aligned}
x_1 \circ x_2 &= x_1x_2 , & x_1,x_2 &\in A ,\\
x_1 \circ y_1 &= f\left(x_1f^{-1}(y)\right) , & y_1,y_2 &\in [k_1,c] ,\\
x_1 \circ z_1 &= x_1 \circ c , & z_1,z_2 &\in B ,\\
y_1 \circ x^* &= f\left(f^{-1}(y_1)x^*\right), & x^* &\in [a,\theta] ,\\
y_1 \circ x^o &= f\left(f^{-1}(y_1)\theta\right) , & x^o &\in [\theta,k_1] ,\\
y_1 \circ y_2 &= y_1 \circ f^{-1}(y_2) , & &\\
y_1 \circ z_1 &= y_1 \circ c , & &\\
z_1 \circ s &= c \circ s , & &\\
z_1 \circ z_2 &= z_1z_2 . & &
\end{aligned}
$$

To verify that "$\circ$" is associative and well defined on S is mainly routine and utilizes the associativity in A and B.

Thus S is a mob and it is straightforward to verify that $a^2 = a$, $b^2 = b$, $b \circ a = a \circ b = c$.

Hence S belongs to case 3.

2) $a < c < k_1$.

Then let $a < c < k_1 \leq k_3 \leq k_5 \leq k_6 \leq k_4 \leq k_2 < d < b$.

Let $C = [c,k_3]$ be an I_k-mob with kernel $[k_1,k_3]$ and f a homeomorphism of C onto $D = [k_4,d]$ with $f(c) = d$, $f(k_1) = k_2$, $f(k_3) = k_4$.

Let g_1 be a continuous mapping of $[k_5,k_6]$ into $[k_1,k_5]$ with $g_1(k_6) = k_3$, $g_1(k_5) = k_5$ and g_2 a continuous mapping of $[k_5,k_6]$ in $[k_6,k_2]$ with $g_2(k_6) = k_6$, $g_2(k_5) = k_4$, such that

i) $g_1(k) \in [k_3,k_5]$ if and only if $g_2(k) \in [k_6,k_4]$,

ii) $g_2(k) = fg_1(k)$ if $g_1(k) \in [k_1,k_3)$.

Now let $S^* = [c,d]$ be the mob of class 2ii) with kernel $[k_1,k_2]$, such that $cS^* = C$, $dS^* = D$ and

$dx = f(x)$ if $x \in C$,

$ck = k_3$ if $k \in [k_3,k_5] \cup [k_6,k_4]$,

$ck = cg_1(k)$ if $k \in [k_5,k_6]$.

Let $[a,c] = A$ and $[d,b] = B$ be two I-mobs with identity elements a and b respectively and h_1 a continuous left invariant mapping of A onto $[k_3,k_5]$ with $h_1(a) = k_5$, $h_1(c) = k_3$ and h_2 a left invariant mapping of B onto $[k_4,k_6]$ with $h_2(b) = k_6$, $h_2(d) = k_4$.

Define a multiplication $\circ$ on $S = [a,b]$ by

$x_1 \circ x_2 = x_1x_2$ if $x_1,x_2 \in A$, $x_1,x_2 \in B$ or $x_1,x_2 \in S^*$,

$s^* \circ x = s^*$ if $s^* \in S^*$, $x \in A \cup B$,

$$\left.\begin{aligned} x \circ y &= c \circ y \\ x \circ k &= h_1\left(xh_1^{-1}(k)\right) \\ x \circ k' &= x \circ g_1(k') \end{aligned}\right\} \quad x \in A,\ y \in S^* \cup B \setminus [k_3,k_6],\ k \in [k_3,k_5],\ k' \in [k_5,k_6],$$

$$\left.\begin{aligned} x \circ y &= d \circ y \\ x \circ k &= h_2\left(xh_2^{-1}(k)\right) \\ x \circ k' &= x \circ g_2(k') \end{aligned}\right\} \quad x \in B,\ y \in S^* \cup A \setminus [k_5,k_4],\ k \in [k_6,k_4],\ k' \in [k_5,k_6].$$

We again omit the proof that S is an interval mob with $a^2 = a$, $b^2 = b$, $a \circ b = c$, $b \circ a = d$ and that S belongs to case 3.

3) $k_1 \leq c \leq k_2$.

Then let $k_1 \leq c \leq k_3 \leq k_4 \leq d \leq k_2$ and let $A = [a,k_3]$ and $B = [k_4,b]$ be two I_k-mobs with unit a and b and kernel $[k_1,k_3]$ and $[k_4,k_2]$

respectively.

Let f_1 be a continuous mapping of $[k_3,k_4]$ into $[k_3,k_1]$ with $f_1(k_3) = k_3$, $f_1(k_4) = c$ and f_2 a continuous mapping of $[k_3,k_4]$ into $[k_4,k_2]$ with $f_2(k_4) = k_4$, $f_2(k_3) = d$.

Define a multiplication $\circ$ on S by

$$x_1 \circ x_2 = x_1x_2 \quad \text{if } x_1,x_2 \in A \text{ or } x_1,x_2 \in B,$$

$$\left.\begin{array}{l} x \circ y = x \circ c \\ y \circ x = y \circ d \end{array}\right\} \quad x \in A,\ y \in B,$$

$$\left.\begin{array}{l} k \circ s = k \\ x \circ k = xf_1(k) \\ y \circ k = yf_2(k) \end{array}\right\} \quad x \in A,\ y \in B,\ k \in [k_3,k_4].$$

Then $(S,\circ)$ is an interval mob belonging to case 3 with $a \circ b = c$, $b \circ a = d$.

2.6.8. <u>Theorem</u>. Let $S = [a,b]$ be a mob belonging to case 3, then $S \in S(c)$.

<u>Proof</u>:

Since $a^2 = a$, $b^2 = b$, we have $[a,k_1] = A$ and $[k_2,b]$ two I-semigroups.

Suppose now $c = ab \in (k_2,b)$.

Then $k_2 = ak_2$ and $bab = ab$, thus $ba \notin [k_1,k_2]$.

If $ba \in (a,k_1)$, then $k_1 \in [k_2,b]a$ which implies $k_1 = xa$, $x \in [k_2,b]$.

Hence $k_1k_2 = xak_2 = xk_2 = k_2$. Since k_1 is a left zero of S, we have $k_1 = k_2$ and by passing to the product dual of S we get the case $ab \in (a,k_1)$.

So we may assume $ba \in (k_2,b)$ and $bab = ab = ba = c$.

Furthermore $c^2 = abab = a^2b^2 = ab = c$ and hence $[c,b] = B$ an I-semigroup.

Since $Ab = [k_1,c]$, we have $k_2 = xb$, $x \in A$.

Now let $\theta = \max\{x \mid x \in A,\ xb = k_2\}$, then $\theta^2 \geq \theta$ and $\theta^2 b = \theta k_2 = \theta b k_2 = k_2^2 = k_2$, thus $\theta = \theta^2$.

Moreover if $x \in [a,\theta]$ then $xb \in [k_2,c]$, hence $xb = bxb = bx$ and $b\theta = \theta b = k_2$.

Thus for all $x' \in [\theta,k_1]$ we have $bx' = b\theta x' = k_2$ and the mapping $f\colon A \to [k_1,c]$ with $f(x) = xb$ is a left invariant mapping with

$f(k_1) = k_1$, $f(\theta) = k_2$, $f(a) = c$, which satisfies the conditions of construction 1).

Suppose now $c = ab \in (a,k_1)$, then $ab = aba$ and thus $ba \notin K$.

If $ba \in (a,k_1)$ we get the previous case by passing to the order dual of S. Hence we may assume $ba = d \in (k_2,b)$.

Since $c^2 = abab = ab^2 = c$, $d^2 = d$, $dc = d$, $cd = c$ and $[c,d] = S^*$ is a mob belonging to case 2ii).

Now let $cS = [c,k_3]$, $dS = [k_4,d]$ and suppose $aS = [a,k_5]$, $bS = [k_6,b]$.

Then $k_3 \leq k_5 \leq k_6 \leq k_4$, for if $k_5 = bk_5 = ak_5$, $k_5 = ck_5 = dk_5$ and thus $k_3 = k_4 = k_5 = k_6$.

If $k \in [k_3,k_5]$, $k_3 \in [a,c]k$ and $ck_3 = k_3 \in c[a,c]k = ck$, hence $c[k_3,k_5] = k_3$.

Analogously we have $d[k_6,k_4] = k_4$ and $c[k_6,k_4] = cd[k_6,k_4] = ck_4 = k_3$, $d[k_3,k_5] = k_4$.

Consider the function g_1: $[k_5,k_6] \to [k_1,k_5]$ and g_2: $[k_5,k_6] \to [k_6,k_2]$, defined by $g_1(k) = ak$, $g_2(k) = bk$.

If $ak \in [k_3,k_5]$, $bak = dak = dbk = k_4$ and $bk \in [k_6,k_4]$.

If $ak \in [k_1,k_3)$, $bak = dak \in [k_2,k_4)$ and $bk = dk = bak$.

By defining h_1: $[a,c] \to [k_3,k_5]$ and h_2: $[d,b] \to [k_6,k_4]$ through $h_1(x) = xk_5$, $h_2(y) = yk_6$, it is easy to complete the verification that $S \in S(c)$, $a < c < k_1$.

Finally let $k_1 \leq c \leq k_2$.

If $aS = [a,k_3]$, $bS = [k_4,b]$ and $ba = d$ we have $k_1 \leq c \leq k_3 \leq k_4 \leq d \leq k_2$.

For if $k = ak = bk$, $k = bak = abk = ba = ab$.

aS and bS are I_k-mobs with kernel $[k_1,k_3]$ and $[k_4,k_2]$ respectively.

Furthermore the mappings f_1: $[k_3,k_4] \to [k_1,k_3]$ and f_2: $[k_3,k_4] \to [k_4,k_2]$ with $f_1(k) = ak$, $f_2(k) = bk$, have the desired properties and it is straightforward to verify that $S \in S(c)$, $k_1 \leq c \leq k_2$.

Definition. Let T_1 and T_2 be submobs of an interval clan S. Two functions f and g on T_1 and T_2 respectively are called <u>comultiplicative</u> if and only if $f(T_1) = g(T_2)$ and $f(x_1) = g(y_1)$, $f(x_2) = g(y_2)$, imply $f(x_1x_2) = g(y_1y_2)$.

Suppose now $T_1 = [a_1,1]$ and $T_2 = [a_2,1]$, $a_1 = 0$ or $\frac{1}{2}$, $a_2 = 0$ or $\frac{1}{2}$,

isomorphic to either J_1 or J_2 and let f be a continuous left invariant mapping of T_1 into S with $f^{-1}[f(a_1)] = [a_1,r]$, $a_1 \leq r < 1$, $r \neq 0$.
Then we can construct to each $s \neq 0$, $a_2 \leq s < 1$, a continuous left invariant mapping g of T_2 into S such that g and f are comultiplicative and such that $g^{-1}[g(a_2)] = [a_2,s]$.
For let $g[a_2,s] = f[a_1,r] = f(r)$. Since $g((\sqrt{s})^2) = f((\sqrt{r})^2)$ we must have $g(\sqrt{s}) = f(\sqrt{r})$ and thus $g(s^{p/2^n}) = f(r^{p/2^n})$.
Since the set $\{s^{p/2^n} \mid p=0,1,\ldots;n=1,2,\ldots\}$ is dense in $[s,1]$ and g is order preserving with $g\{s^{p/2^n}\} = f\{r^{p/2^n}\}$ dense in $f(T_1)$, we can extend g to a continuous function of T_2 onto $f(T_1)$.
Moreover it is clear that each g is completely defined by the set $g^{-1}[g(a_2)]$.
If $r = 0$ then f is a one-to-one mapping of $[0,1]$ into S and we can find a comultiplicative continuous left invariant function g if and only if T_2 is isomorphic to J_1.
In this case g must be one-to-one and g is completely defined by the condition $f(x) = g(y)$, $x \in T_1$, $0 < x < 1$, $y \in T_2$, $0 < y < 1$.

Now let A be any I-semigroup, $A \subset S$, and let P be the set of all subsemigroups $[e_{\alpha_1},e_{\alpha_2}]$ of A with $[e_{\alpha_1},e_{\alpha_2}]$ isomorphic either to J_1 or J_2.
Let f be a continuous left invariant mapping of A into S and let $P_f = \{[e_{\alpha_1},e_{\alpha_2}] \mid [e_{\alpha_1},e_{\alpha_2}] \subset P,\ f(e_{\alpha_1}) \neq f(e_{\alpha_2})\}$.
Let E be the set of all idempotents of A.

2.6.9. <u>Lemma</u>. Let g be a continuous monotone mapping of A onto f(A) such that

1) $g(E) = f(E)$.
2) If $[e_{\alpha_1},e_{\alpha_2}] \in P_f$, then there is a $[e_{\beta_1},e_{\beta_2}] \in P$ with $g(e_{\beta_1}) = f(e_{\alpha_1})$, $g(e_{\beta_2}) = f(e_{\alpha_2})$, such that $g \mid [e_{\beta_1},e_{\beta_2}]$ and $f \mid [e_{\alpha_1},e_{\alpha_2}]$ are comultiplicative.

Then g is a left invariant mapping with f and g comultiplicative.
Conversely, every left invariant mapping g with f and g comultiplicative satisfies condition 1) and 2).

Proof:

Let g satisfy the conditions of the lemma. Define an order relation in A such that $x_1x_2 \le x_1$ for all $x_1, x_2 \in A$.

Now let $g^{-1}(g(x)) \ne x$ and let $y = \inf \{z \mid g(z) = g(x)\}$.

Suppose $y \notin E$. Then $y \in [e_1, e_2] \in P$ and $g(e_1) \ne g(y)$.

Hence there is a $[e_1^*, e_2^*] \in P_f$ with $f(e_1^*) = g(e_1)$, $f(e_2^*) = g(e_2)$, and $f \mid [e_1^*, e_2^*]$, $g \mid [e_1, e_2]$ comultiplicative.

If $g(y) = g(e_2) = f(e_2^*)$, then $g(y^n) = f(e_2^*) = g(y)$ hence $g(y) = g(e_1)$.

Now let $z \in [e_1^*, e_2^*]$ with $f(z) = g(y) = g(x)$, $z < e_2$.

Since $x > y$, we have $y = xt$ with $y \le t < e_2$.

Let $f(z^*) = g(t)$, $z^* < e_2^*$, then $g(y) = g(xt) = f(zz^*) = f(z)$.

But since $zz^* < z$ we have $g(y) = f(zz^*) = f(z) = f(e_1^*) = g(e_1)$.

Hence $g^{-1}(g(x)) = \{x\}$ or $g^{-1}(g(x)) = [e, x^*]$ and g is left invariant.

Now let $g(x) \in f(E)$ and let $g(x) = f(y)$. Then there exist e_1 and $e_2 \in E$ with $g(x) = g(e_1) = f(e_2) = f(y)$.

If $g(x_1) = f(y_1)$ and $x_1 \ge e_1$, then $y_1 \ge e_2$ and we have $g(xx_1) = g(e_1x_1) = g(e_1) = f(e_2) = f(e_2y_1) = f(yy_1)$.

If $x_1 \le e_1$, then $y_1 \le e_2$ and $g(xx_1) = g(e_1x_1) = g(x_1) = f(y_1) = f(y_1e_2) = f(yy_1)$.

If $g(x) \notin f(E)$ and $g(x) = f(y)$, then $x \in [e_1, e_2] \subset P$ and $y \in [e_1^*, e_2^*] \subset P_f$ with $g(e_1) = f(e_1^*)$, $g(e_2) = f(e_2^*)$.

Furthermore g and f are comultiplicative on $[e_1, e_2]$ and $[e_1^*, e_2^*]$.

Now let $g(x_1) = f(y_1)$ with $x_1 \ge e_2$, then $y_1 \ge e_2^*$ and $g(x_1x) = g(x) = f(y) = f(y_1y)$.

If $x_1 \le e_1$, then $y_1 \le e_1^*$ and $g(x_1x) = g(x_1) = f(y_1) = f(y_1y)$.

Let conversely g be a left invariant mapping with f and g comultiplicative. Then if $g(x) = f(e)$, we have $g(x^n) = f(e)$ and thus $g(e_1) = f(e)$ with $e_1 \in E$, $e_1 \le x$. Hence $g(E) = f(E)$.

If $[e_{\alpha_1}, e_{\alpha_2}] \in P_f$, then let $e_{\beta_1} = \max \{e \mid e \in E,\ g(e) = f(e_{\alpha_1})\}$ and $e_{\beta_2} = \min \{e \mid e \in E,\ g(e) = f(e_{\alpha_2})\}$.

Then $[e_{\beta_1}, e_{\beta_2}] \in P$ and g and f comultiplicative on $[e_{\alpha_1}, e_{\alpha_2}]$, $[e_{\beta_1}, e_{\beta_2}]$.

Construction:

Let $a < k_1 \leq k_2 \leq b$ and define a multiplication on $A = [a, k_1]$ making it into an I-mob with identity a. We define a collection of mobs we call $S(c,r)$ with $c, r \in (a,b]$ by extending the multiplication on A to S.

1) $k_2 \leq c \leq b$, $k_2 \leq r \leq c$.

Let θ and θ^* be idempotents with $a \leq \theta^* \leq \theta \leq k_1$ and f a left invariant continuous mapping of A onto $[k_1, b]$ with $f(k_1) = k_1$, $f(\theta) = k_2$, $f(\theta^*) = c$.

Let $t = \max\{x \mid f(x) = r, x \in A\}$ and $e = \max\{x \mid x = x^2, x \leq t\}$.

Suppose $f^*(x) = f(x)$, $x \in [\theta^*, \theta]$

$f^*(x) = c$, $x \in [a, \theta^*]$

and let g be a left invariant continuous mapping of $[a,\theta]$ onto $[k_2, c]$ with $f^*(x) = g(x)$, $x \geq e$, and f^* and g comultiplicative.

Define a multiplication $\circ$ on S by

$$
\begin{aligned}
x_1 \circ x_2 &= x_1 x_2 , & x_1, x_2 &\in A ,\\
x_1 \circ y_1 &= f(x_1 f^{-1}(y_1)) , & y_1, y_2 &\in [k_1, b] ,\\
y_1 \circ x^* &= f^{-1}(y_1) \circ g(x^*) , & x^* &\in [a, \theta] ,\\
y_1 \circ x' &= f(f^{-1}(y_1)\theta) , & x' &\in [\theta, k_1] ,\\
y_1 \circ y_2 &= y_1 \circ (t f^{-1}(y_2)) .
\end{aligned}
$$

1*) $k_2 \leq c \leq b$, $k_2 \leq r \leq c$.

Let θ and θ^* be as in 1) and f a left invariant continuous mapping of A onto $[k_1, c]$ with $f(k_1) = k_1$, $f(\theta) = k_2$.

$t = \max\{x \mid f(x) = r, x \in A\}$ and $e = \max\{x \mid x = x^2, x \leq t\}$.

Let g be a left invariant continuous mapping of $[a,\theta]$ onto $[k_2, b]$ with $g(\theta^*) = c$, $g(x) = f(x)$, $\theta \geq x \geq e$ and such that if

$g^*(x) = g(x)$, $x \in [\theta^*, \theta]$

$g^*(x) = c$, $x \in [a, \theta^*]$

then g^* and $f \mid [a,\theta]$ are comultiplicative.

Define a multiplication $\circ$ on S by

$$
\begin{aligned}
x_1 \circ x_2 &= x_1 x_2 , & x_1, x_2 &\in A ,\\
k \circ S &= k , & k &\in [k_1, k_2] ,\\
y_1 \circ x^* &= g(g^{-1}(y_1) x^*) , & y_1, y_2 &\in [k_2, b] , \quad x^* \in [a,\theta] ,\\
y_1 \circ x' &= k_2 , & x' &\in [\theta, k_2] ,
\end{aligned}
$$

$x_1 \circ k = f(x_1 f^{-1}(k))$,
$x_1 \circ y_1 = f(x_1) \circ g^{-1}(y_1)$,
$y_1 \circ y_2 = (g^{-1}(y_1)t) \circ y_2$.

2) $k_2 \leq c \leq b$, $a < r \leq k_1$, $k_1 = k_2$.
Let θ^* be an idempotent with $\theta^* \leq r$ and $e_1 = \min\{x \mid x = x^2, x \geq r\}$,
$e_2 = \max\{x \mid x = x^2, x \leq r\}$.
Let f be a left invariant continuous mapping of A onto $[k_1, b]$ with
$f(k_1) = k_1$, $f(\theta^*) = c$ and such that $f \mid [e_1, k_1]$ is one-to-one and
$[f^{-1}(f(e_1))]r = e_1$.
Let $f^*(x) = f(x)$, $x \in [\theta^*, k_1]$
$f^*(x) = c$, $x \in [a, \theta^*]$
and let g be a left invariant mapping of A onto $[k_1, c]$ with
$f^*(x) = g(x)$, $x \in [k_1, e_2]$ and f^* and g comultiplicative.
Define a multiplication $\circ$ on S by

$x_1 \circ x_2 = x_1 x_2$, $x_1, x_2 \in A$,
$x_1 \circ y_1 = f(x_1 f^{-1}(y_1))$, $y_1, y_2 \in [k_1, b]$,
$y_1 \circ x_1 = f^{-1}(y_1) \circ g(x_1)$,
$y_1 \circ y_2 = f^{-1}(y_1) f^{-1}(y_2) r$.

3) $a < c \leq k_1$; $k_2 \leq r \leq b$. $k_1 \leq k_3 \leq k_4 \leq k_2$.
Let $e_1 = \min\{x \mid x = x^2;\ x \geq c\}$. $e_2 = \max\{x \mid x = x^2;\ x \leq c\}$.
Let f be a continuous left invariant mapping of A onto $[k_2, b]$ with
$f(k_1) = k_2$, $f(c) = r$ and such that $f \mid [e_1, k_1]$ is one-to-one and
$[f^{-1}(f(e_1))]c = e_1$.
g is a continuous left invariant mapping of A onto $[k_1, k_3]$ with
$g(k_1) = k_1$.
Let h be a continuous mapping of $[k_1, k_3]$ onto $[k_4, k_2]$ with $h(k_1) = k_2$,
such that $hg(x) = h^* f(x)$ with h^* a continuous mapping of $[k_2, b]$ onto
$[k_2, k_4]$. Furthermore h has the following properties

i) $h^{-1}(h(x)) = \{x\}$ for $x \in [k_1, g(e_1))$
ii) if $g(e_1) \neq g(c)$ then $h(x) \neq h(y)$; $x \in [g(e_2), k_3]$, $y \in [g(e_1), g(e_2))$
iii) if $g(g^{-1}(x_i) \cdot c) \neq g(e_1)$; $x_i \in [g(e_1), g(e_2))$, $i=1,2$, then
$h(x_1) \neq h(x_2)$, $x_1 \neq x_2$.

Moreover let φ be a continuous mapping of $[k_3,k_4]$ in $[k_1,k_3]$ with $\varphi(k_3) = k_3$, $\phi(k_4) = g(c)$ and define

$$
\begin{array}{ll}
x_1 \circ x_2 = x_1x_2 & x_1,x_2 \in A \\
x_1 \circ k' = g(g^{-1}(k')x) & k' \in [k_1,k_3] \\
x_1 \circ k^{\circ} = x_1 \circ \varphi(k^{\circ}) & k^{\circ} \in [k_3,k_4] \\
x_1 \circ k^{*} = x_1 c \circ h^{-1}(k^{*}) & k^{*} \in [k_4,k_2] \\
x_1 \circ y_1 = x_1 cf^{-1}(y_1) & y_1,y_2 \in [k_2,b] \\
k \circ S = k & k \in [k_1,k_3] \\
y_1 \circ x_1 = f(x_1 f^{-1}(y_1)) & \\
y_1 \circ k = h(f^{-1}(y_1) \circ k) & \\
y_1 \circ y_2 = f(f^{-1}(y_1)f^{-1}(y_2)c). &
\end{array}
$$

4) $k_1 \leq c \leq r \leq k_2$.

Let $k_1 \leq c \leq k_3 \leq k_4 \leq r \leq k_2$ and f a left invariant continuous mapping of A onto $[k_2,b]$ with $f(k_1) = k_2$, g a continuous left invariant mapping of A onto $[k_1,k_3]$ with $g(k_1) = k_1$. Assume furthermore that there exist continuous mappings

$$
\left.\begin{array}{l}
h_1 : [k_3,k_4] \to [k_1,k_3] \\
h_2 : [k_1,k_3] \to [k_4,k_2] \\
h_2^{*} : [k_2,b] \to [k_4,k_2]
\end{array}\right\}
\begin{array}{l}
\text{with } h_1(k_4) = c,\ h_1(k_3) = k_3 \\
h_2(k_1) = k_2 \\
h_2 g(x) = h_2^{*} f(x).
\end{array}
$$

Define a multiplication $\circ$ on S by

$$
\begin{array}{ll}
x_1 \circ x_2 = x_2x_2 & x_1,x_2 \in A \\
x_1 \circ k' = g(g^{-1}(k')x_1) & k' \in [k_1,k_3] \\
x_1 \circ k^{\bullet} = x_1 \circ h_1(k^{\bullet}) & k^{\bullet} \in [k_3,k_4] \\
x_1 \circ k^{*} = x_1 \circ c & k^{*} \in [k_4,k_2]. \\
x_1 \circ y_1 = x_1 \circ c & y_1,y_2 \in [k_2,b] \\
k \circ S = k & k \in [k_1,k_2] \\
y_1 \circ x_1 = f(f^{-1}(y_1)x_1) & \\
y_1 \circ k = h_2(f^{-1}(y_1) \circ k) & \\
y_1 \circ y_2 = y_1 \circ c. &
\end{array}
$$

We omit the proof, that if $S \in S(c,r)$ then S belongs to case 4.

2.6.10. <u>Theorem</u>. Let $S = [a,b]$ be a mob belonging to case 4, then $S \in S(c,r)$.

Proof:

Let $(a,b]$ be the maximal ideal of S and let a be a left unit of S; $k_1 \neq k_2$.

Since $a = a^2$, we have $A = [a,k_1]$ an I-mob and $[k_2,b] \subset [a,k_1]\, b$ i.e. $k_2 = xb$, $x \in A$.

Let $\theta = \max \{x \mid xb = k_2,\ x \in A\}$ and suppose $r = b^2 \in [a,k_2]$.

Then $\theta b^2 = k_2 b = k_2$ and since $[a,k_1]\, b^2 = [k_1,b^2]$ it follows that $b^2 = k_2$. Hence $b^2 \in [k_2,b]$ and in the same way we prove $c = ba \in [k_2,b]$.

Furthermore $k_2 \in b\,[a,k_1]$, which implies $k_2 = bk_1$ and $\theta^2 b = \theta k_2 =$ $= \theta b k_1 = k_2$. Thus since θ is maximal, θ is an idempotent.

Let $\theta^* = \max \{x \mid xb = ba,\ x \in A\}$, then $\theta^* = \theta^{*2}$.

Moreover we have $b^2 \in [k_2,ba]\, b \subset [k_2,b^2]$, hence $b^2 \leq ba$.

Now let $t = \max \{x \mid xb = b^2;\ x \in A\}$ and

$t^* = \max \{x \mid bx = b^2;\ x \in A\}$.

Then $t \leq \theta$ and we have for each $x \in [a,\theta]$,

$$\left.\begin{array}{l} bxb = x'b^2 = x'tb = tx'b = tbx = b^2x \text{ and} \\ bxb = bxba = b^2x^* = bt^*x^* = bx^*t^* = xbat^* = xb^2 \end{array}\right\} \Rightarrow b^2x = xb^2 = bxb.$$

Suppose now $t \neq t^*$, and let $e \leq \theta$ be an idempotent with $t \geq e \geq t^*$, then $be = bt^*e = b^2e = eb^2 = etb = tb = bt^*$ and we have $e = t^*$.

Analogously we have if $e = e^2 \leq \theta$, $t^* \geq e \geq t$, $e = t$ and it follows that if $t \neq t^*$, t and t^* both in $[e_2,e_1]$, a subsemigroup of A isomorphic to J_1 or J_2.

Furthermore we have for all x, $\theta \geq x \geq e_1$, $bx = bt^*x = b^2x = xb^2 =$ $= xtb = xb$. Now let $f(x) = xb$; $g(x) = bx$ and $f^*(x) = xba$; $x \in [a,\theta]$.

Then f^* and g are comultiplicative left invariant mappings of $[a,\theta]$ onto $[k_2,ba]$.

Moreover we have $be_1 = e_1b = b^2e_1$ and $b^2e_2 = b^2$.

If $b^2 = b^2e_1 = be_1$, then $t = t^* = e_1$.

If $b^2 \neq be_1$, then there is an x, $e_2 \leq x \leq e_1$ with $b^2\,[e_2,x) = (be_1,b^2]$.

For each $y \in [e_2,x)$ we have $b^2y = byb = y^*b^2 = b^2y^*$ and thus $y = y^*$, i.e. $by = yb$.

Since for all $z \in [e_2,e_1)$ we have $z \in [e_2,x)^n$ we have $bz = zb$ and $t = t^*$. It is now easy to verify that $S \in S(c,r)$ with $k_2 \leq r \leq c \leq b$.

Now let a be a right unit of S and let $k_2 \leq c = ab \leq b$.

Let $\theta = \max \{x | xb = k_2, \ x \in A\}$.

If $r = b^2 \in [a_1, k_2]$, then $\theta b^2 = k_2 \in [a, k_1] b^2 = [b^2, k_1]$ and we have $b^2 = k_2$ or $b^2 \in A$ and $k_2 = k_1$.

If $b^2 \in [k_2, b]$, then we can prove in the same way as before that the mappings $f : x \to xb$; $g : x \to bx$; $g^* : x \to abx$, $x \in [a, \theta]$ satisfy the conditions of construction 1^* and hence $S \in S(c,r)$.

Now suppose $b^2 \in [a, k_1]$, $k_2 = k_1$ and let

$\theta^* = \max \{x | bx = ab, \ x \in A\}$; $b\theta^{*2} = ab\theta^* = a^2 b = ab$; hence $\theta^{*2} = \theta^*$.

Moreover we have $b^2 = bab = b^2 \theta^*$, which implies $b^2 \geq \theta^*$.

Let $e_1 = \min \{x | x = x^2, \ x \geq r\}$

$e_2 = \max \{x | x = x^2, \ x \leq r\}$.

If $e_1 \leq x_1$, $x_2 \leq k_1$ and $bx_1 = bx_2$, then $b^2 x_1 = b^2 x_2$ and hence $x_1 = x_2$.

Furthermore if $bx = be_1$, then $b^2 x = b^2 e_1 = e_1$, and for each $x \in A$ there exists an x^* and x^{**} such that $xb = bx^*$, $x^* b = bx^{**}$.

Hence $bx^* b = xb^2 = b^2 x^{**}$.

If $x > e_1$, then $x = x^{**}$ and hence $bx = xb$.

Moreover we have $b^2 e_1 = e_1$ and $b^2 e_2 = b^2$. If $b^2 \neq e_1$, then there is a y, $e_2 < y \leq e_1$ with $b^2 [e_2, y) = [b^2, e_1)$ and we have for each $x \in [e_2, y)$, $xb^2 = b^2 x^{**} = b^2 x$, i.e. $x = x^{**}$ and $bx = xb$. Since $[e_2, e_1) \subset \bigcup_{n=1}^{\infty} [e_2, y)^n$ we have $bx = xb$, $x \geq e_2$.

Now define f, g and f^* by $f(x) = bx$; $g(x) = xb$; $f^*(x) = abx$.

Then f^* and g are comultiplicative with $f^*(x) = g(x)$, $x \geq e_2$.

To verify that $S \in S(c,r)$ with $a < r \leq k_1 \leq c \leq b$ is now mainly routine.

Next we consider the case $c = ab \in (a, k_1]$.

We then have $r = b^2 \in b [a, k_1] = [k_2, b]$. Let e_1 and e_2 be defined as in construction 3).

If $e_1 \leq x_1$, $x_2 \leq k_1$ and $bx_1 = bx_2$, then $abx_1 = abx_2$ and hence $x_1 = x_2$.

Furthermore if $bx = be_1$, then $abx = abe_1 = e_1$.

Now let $aS = [a, k_3]$; $bS = [k_4, b]$.

Since $bk_1 = k_2$ and $ak_2 = k_1$ we have $k_1 \leq k_3 \leq k_2$, and $k_1 \leq k_4 \leq k_2$.

Now suppose $k_4 \leq k_3$. Since $bS = baS$ and $bk_1 = k_2$ there is a $k \in aS$ with

$bk = k_3$ and $bk^* \neq k_3$, $k^* < k$.
But $bk = k_3$ implies $abk = k_3 \leq k$ and $k = k_3$. Hence $k_3 \leq k_4$.
Now let f, g, h, h^* and ϕ be defined by

$$
\begin{array}{ll}
f(x) = bx & x \in A \\
g(x) = xk_3 & x \in A \\
h(k) = bk & k \in [k_1, k_3] \\
h^*(y) = yk_3 & y \in [k_2, b] \\
\phi(k') = ak' & k' \in [k_3, k_4] .
\end{array}
$$

These mappings satisfy the conditions of construction 3 and elementary calculations show that $S \in S(c,r)$.

Finally let $c = ab \in [k_1, k_2]$. We then have $r = b^2 = bab \in [k_1, k_2]$.
If $b^2 \leq ab$, then $b^2 = ab^2 = abb = ab$. Hence $ab \leq b^2$.
Furthermore if $aS = [a, k_3]$, $bS = [k_4, b]$, then $k_4 \geq k_3$ or else $k_4 = ak_4 = abx = ab = k_3$. Hence $k_1 \leq c \leq k_3 \leq k_4 \leq r \leq k_2$.
Let f, g, h_1, h_2 and h^* be defined by

$$
\begin{array}{ll}
f(x) = bx & x \in A \\
g(x) = xk_3 & x \in A \\
h_1(k) = ak & k \in [k_3, k_4] \\
h_2(k) = bk' & k' \in [k_1, k_3] \\
h_2^*(y) = yk_3 & y \in [k_2, b] .
\end{array}
$$

It can be easily verified that in this case $S \in S(c,r)$ with $k_1 \leq c \leq r \leq k_2$.

Notes

The concept of nil-ideals was introduced by Numakura [1]. An amplification of his results was given by Koch [1]. 0-simple mobs have been studied by many writers. Most of the results of section 2.2 are due to Clifford [2], Faucett, Koch, Numakura [3], Schwarz [10].

The results of section 2.3 seem to be new.

Wallace [11], Faucett [1], Koch and Wallace [8], Mostert and Shields [8] have all contributed to the theory of connected mobs. The position of C-sets in compact mobs was studied by Wallace [8] and Hunter [7].

I-semigroups were first studied by Faucett [2] who proved theorems 2.5.1, 2.5.2, 2.5.5 - 2.5.7. Mostert and Shields [7] extended their results and gave a complete characterization of an I-semigroup.

The contents of section 2.6 are an extension to mobs with $S = S^2$ of results by Cohen and Wade [4], Clifford [3], [4], Mostert and Shields [7], Philips [1].

III COMMUTATIVE SEMIGROUPS

3.1. Monothetic semigroups

Definition. A mob S is called monothetic if for some $a \in S$ the set $\{a^n\}_{n=1}^{\infty}$ is dense in S. The element a is called a generator of S. In the group case it is customary to use both positive and negative powers to define monotheticity; i.e. a group G is monothetic if for some $g \in G$ the set $\{g^n\}_{n=-\infty}^{\infty}$ is dense in G. However, it can easily be seen that the two notions agree in a compact group. For let g be an element of the compact group G such that $\{g^n\}_{n=-\infty}^{\infty}$ is dense in G. Since $\Gamma(g) = \overline{\{g^n\}_{n=1}^{\infty}}$ is a compact subsemigroup of G, theorem 1.1.11 implies that $\Gamma(g)$ is a compact subgroup of G. Hence $\{g^n\}_{n=-\infty}^{\infty} \subset \Gamma(g)$ i.e. $\Gamma(g) = G$. It is obvious that a monothetic mob is commutative.

3.1.1. Theorem. Let S be compact and monothetic with generator a. Then the cluster points of the sequence $\{a^n\}_{n=1}^{\infty}$ form a group K(a). K(a) is the minimal ideal of S and S contains exactly one idempotent, namely the unit of K(a).

Proof:
Since $K(a) = \bigcap_{n=1}^{\infty} \overline{\{a^i \mid i \geq n\}}$, 1.1.10 implies that K(a) is a compact group.
Every idempotent $e \in S$ must be a cluster point of $\{a^n\}_{n=1}^{\infty}$, hence $e \in K(a)$ and it follows that S contains exactly one idempotent. Now let K be the minimal ideal of S. Then $K = H(e)$ since e is the only idempotent in S and hence $K(a) \subset H(e) = K$. Now let $b \in H(e)$ and suppose b no cluster point of $\{a^n\}_{n=1}^{\infty}$. Then $b = a^n$ for some integer n and $a^n e = a^n$. For every neighbourhood W(b) there is a neighbourhood V(e) such that $b.V(e) \subset W(b)$. Hence $a^n V(e) \subset W(b)$.
Since V(e) contains arbitrarily high powers of a, W(b) contains arbitrarily high powers of a and $b \in K(a)$. Thus $K = K(a) = H(e)$.

3.1.2. Theorem. If $S = \Gamma(a)$ is compact, then K is a monothetic group.

Proof:

Since $\{a^n\}_{n=1}^{\infty}$ is dense in S, the set $\{a^n e\}_{n=1}^{\infty} = \{(ae)^n\}_{n=1}^{\infty}$ is dense in $Se = K$. Hence K is monothetic.

Corollary. If u is a unit for the compact monothetic mob S, then S is a group.

For in this case we have $K = H(u) = uSu = S$.

3.1.3. Theorem. A monothetic mob with unit u is either a finite group or is dense in itself.

Proof:

Let $S = \Gamma(a)$. If there are integers m and n with $a^m = a^n$, then S is finite and hence compact and the corollary implies that S is a group. In the other case if some element $s \in S$ is an isolated point, then $s = a^m$ for some integer m. Using the fact that $\{a^n\}_{n=1}^{\infty}$ clusters at u , we conclude that $\{a^{n+m}\}_{n=1}^{\infty}$ clusters at $ua^m = a^m$.

Corollary. Let S be compact and monothetic with generator a. If a is not an isolated point, then S is a topological group.

Proof:

Since $\{a^n\}_{n=1}^{\infty}$ clusters at a we have $\Gamma(a) = K(a) = S$.

3.1.4. Theorem. Let S be a compact monothetic mob with two distinct generators. Then S is a compact group.

Proof:

Let $S = \Gamma(a) = \Gamma(b)$, $a \neq b$. If either a or b is not an isolated point, then S is a group by the preceding corollary.
If both a and b are isolated, then $a = b^p$ and $b = a^q$ for some integers p and q. Hence $a = a^{pq}$ where $pq > 1$ and it follows from the preceding corollary that S is a group.

The structure of finite monothetic semigroups is quite simple. If S is such a mob, then $S = \{a, a^2, a^3, \ldots\}$ and there must be repetition among the powers of a.

Let p be the smallest positive integer such that $a^p = a^q$, $1 \leq q < p$.

Let r be the unique integer such that $q \leq r = n(p-q) \leq p-1$. Then the set $\{a^q, a^{q+1}, \dots, a^{p-1}\} = H$ is a cyclic group with unit element a^r.

Furthermore $S = H \cup \{a, a^2, \dots, a^{q-1}\}$.

3.1.5. Theorem. The only possible algebraic and topological structures for the compact monothetic mob $S = \Gamma(a)$ are the following:

1) All powers of a lie in $H(e) = K(a)$, in which case S is a compact monothetic group.
2) There is a positive integer q such that $a, a^2, \dots, a^q$ lie outside $H(e)$ and $a^{q+1}, a^{q+2}, \dots$ all lie in $H(e)$. In this case $S \setminus H(e) = \{a, a^2, \dots, a^q\}$ and all elements $a, a^2, \dots, a^q$ are isolated points in S.
3) All powers of a lie outside $H(e)$. In this case $S \setminus H(e) = \{a, a^2, \dots\}$ and all powers of a are isolated points.

Proof:

(1) and (3) are trivial.

(2) If a is not in $H(e) = K(a)$ and some power $a^p \in H(e)$, then we have $a^{p+r} = a^p \cdot a^r \in H(e)$, since $H(e)$ is an ideal. Hence there is a greatest power a^q such that $a^q \notin H(e)$.

3.1.6. Theorem. Let H be a compact monothetic group with unit e and let $b \in H$ be such that $\{b^n\}_{n=1}^{\infty}$ is dense in H.

Let q be a positive integer and let $a, a^2, \dots, a^q$ be q distinct objects not in H. Then there is one and only one way to make $S = H \cup \{a, a^2, \dots, a^q\}$ into a compact monothetic mob such that

1) H with its given topology and multiplication is an ideal of S.
2) $a^i . a^j = a^{i+j} \qquad i+j \leq q$.
3) $a . a^q \in H$.

Proof:

Define a multiplication in S by the rules

$$\left.\begin{array}{l} a^i x = b^i x \\ x a^i = x b^i \end{array}\right\} \text{ for } x \in H,\ i=1,2,\dots,q$$

$$a^i . a^j = a^{i+j} \quad i+j \leq q \qquad (*)$$

$$a^i . a^j = b^{i+j} \quad i+j > q$$

xy is as in H for $x,y \in H$.

Let S be topologized so that $a, a^2, \dots, a^q$ are isolated points and H has its original topology.

Since the continuity of multiplication is obvious, the fact that S is a mob is established simply by verifying that the associative law holds in all cases. Furthermore S evidently satisfies the conditions (1) → (3).

Now let $S = H \cup \{a, a^2, \dots, a^q\}$ be a mob which satisfies the conditions of the theorem.

Since $e \in H$ and H an ideal, we have $ea = (ea)e = ae \in H$. Let $b = ae$. Then for $x \in H$ and $i \leq q$ we have $a^i x = a^i(ex) = (a^i e)x = (ae)^i x = b^i x$ and analogously $xa^i = x(ea)^i = xb^i$. Next $a.a^q \in H$ implies that $a^{q+1} \in H$ and hence $a^{q+1}e = (ae)^{q+1} = b^{q+1}$. By finite induction we infer that $a^r = b^r$ for all $r > q$. Thus the multiplication in S is that given by (*) with $b = ae$. This shows that the algebraic structure of S is unique. Furthermore also the topological structure is unique. For since H is compact it must be closed in S and as $S \setminus H$ is finite and open, the points $a, a^2, \dots, a^q$ must all be isolated.

We now prove that with the multiplication defined by (*) $S = \Gamma(a)$. Since $ae = b$ and $a^i = b^i$ for $i > q$ it suffices to show that $\{b^{q+1}, b^{q+2}, \dots\}$ is dense in H.

If H is finite, then $\{b^{q+1}, b^{q+2}, \dots\} = H$.

If H is infinite, then since H has no isolated points, the removal of the finite set $\{b, b^2, \dots, b^q\}$ from $\{b^n\}_{n=1}^{\infty}$ does not affect its property of being everywhere dense in H.

3.1.7. Theorem. Let H be a compact monothetic group with unit e and let $\{b^n\}_{n=1}^{\infty}$ be dense in H.
Let $\{a^n\}_{n=1}^{\infty}$ be a countably infinite set of distinct elements not in H.

Then there is one and only one way to make $S = H \cup \{a^n\}_{n=1}^{\infty}$ into a compact monothetic mob, such that

1) H with its given multiplication and topology is an ideal of S

2) $a^i . a^j = a^{i+j} \qquad i,j = 1,2,3,\ldots$

<u>Proof</u>:

Define a multiplication in S by the rules

$$\left.\begin{array}{l} a^i x = b^i x \\ xa^i = xb^i \end{array}\right\} \text{ for } x \in H \text{ and } i=1,2,\ldots$$

$$a^i . a^j = a^{i+j} \qquad i,j = 1,2,3,\ldots \qquad (*)$$

xy is as in H for $x,y \in H$.

Checking the associative law is again a routine matter. Now let S be topologized as follows. Every point a^i is isolated. For $x \in H$ and an arbitrary neighbourhood U(x) in H define $U_n^*(x)$ as $U_n^*(x) = U(x) \cup \{a^i \mid i \geq n$ and $b^i \in U(x)\}$. The family of all sets $U_n^*(x)$ for all neighbourhoods U(x) in H and all positive integers n is a complete family of neighbourhoods of x in S.

It is easy to see that S with this topology is a Hausdorff space.

We now check the continuity of multiplication in S.

Given a product $a^i . a^j = a^{i+j}$, then multiplication is certainly continuous at a^i, a^j as a^i, a^j are isolated points.

Next consider a product xy where $x,y \in H$. Let U_n^* be a neighbourhood of xy. Then there are V(x) and W(y) in H such that $V(x).W(y) \subset U(xy)$. Hence $V_p^*(x)\ W_q^*(y) \subset U_n^*(xy)$ if $p+q \geq n$.

Finally we consider a product $a^i x = b^i x$, where $x \in H$. Let U_n^* be any neighbourhood of $b^i x$.

Since multiplication is continuous in H, there is a neighbourhood V(x) of x in H such that $b^i V(x) \subset U$. Furthermore if $a^r \in V_n^*(x)$, then $b^r \in V(x)$ and $b^i . b^r = b^{i+r} \in U$. Thus $a^{i+r} \in U_n^*$ and $a^i V_n^*(x) \subset U_n^*$.

Hence S is a topological semigroup.

Furthermore S is compact. For let C be any open covering of S. Then every $x \in H$ is contained in some $U_n^*(x) \in C$. Hence the neighbourhoods U(x) form an open covering of H.

Let $U(x_1),\ldots,U(x_r)$ be a finite subcovering of H and let

$U^*_{n_1}(x_1),\ldots,U^*_{n_r}(x_r)$ be the corresponding neighbourhoods in C.
Let $n = \max(n_1,\ldots,n_r)$, then $a^i \in \bigcup_{j=1}^{r} U^*_{n_j}(x_j)$ $i \geq n$.
This implies that the complement of the set $\bigcup_{j=1}^{r} U^*_{n_j}(x_j)$ is finite and C admits a finite subcovering.
Moreover we have $S = \Gamma(a)$ since every neighbourhood in S contains an element a^i.
Now let S be a compact monothetic mob $S = H \cup \{a,a^2,\ldots\}$, and let S satisfy the conditions (1) and (2).
Then it can be shown just as in 3.1.6 that the algebraic structure of S is unique and that the multiplication in S must be that given by (*) with $b = ae$.
We now show that the topological structure is unique. Let T^* be the topology for S described above, and let S have a topology T.
Since H is the minimal ideal of S, H must be the set of cluster points of $\{a^n\}_{n=1}^{\infty}$ and hence every point a^i, $i=1,2,\ldots$ must be isolated.
Now let U be an arbitrary neighbourhood of x in H. Then there is a neighbourhood U' in S such that $U' \cap H = U$ and there is a $V(x)$ in S such that $V(x)e \subset U'$.
Since $V(x)e \subset H$ we have $V(x)e \subset U$.
In particular if $a^i \in V(x)$, then $b^i = a^i e \subset U$ and $a^i \in U \cup \{a^i \mid b^i \in U\}$.
We therefore have $V(x) \subset U^*_1$. Moreover for every integer $n > 1$, we have $V(x) \setminus \{a,a^2,\ldots,a^{n-1}\} \subset U^*_n$. Hence every T^*-neighbourhood of x contains a T-neighbourhood.
Consequently every T^*-open set is also T-open, and the identity mapping of S onto itself is continuous in passing from the T topology to the T^* topology. However, since S is compact Hausdorff in both T and T^* topology, this mapping is a homeomorphism and $T = T^*$.

We can now summarize the preceding constructions.
Every compact monothetic mob S is one of the following types.
1) S is a compact monothetic group
2) S consists of an arbitrary compact monothetic group H, with generating element b, and a finite number of elements $a,a^2,\ldots,a^q$, for which $ae = b$ and $a^{q+1} \in H$. The algebraic and topological structure

are totally determined by q and the choice of b, as described in 3.1.6.

3) S consists of an arbitrary compact monothetic group H, with generating element b, and a countably infinite submob $\{a, a^2, \dots\}$ for which ae = b.

The algebraic and topological sutructe of S are totally determined by the choice of b as described in 3.1.7.

3.1.8. <u>Lemma</u>. Let S be a locally compact mob with a compact kernel $K \neq \emptyset$; then for any open V containing K, there is an open set J with $K \subset J \subset V$ and J an open submob of S.

<u>Proof</u>:

Let U be an open set having compact closure with $K \subset U \subset \overline{U} \subset V$. Since $K\overline{U} = K \subset U$, we may find an open set W with $K \subset W \subset U$ and $W\overline{U} \subset U$. Since $W \subset \overline{U}$, we have $W^2 \subset U$, $W^3 \subset U, \dots$ and hence $\bigcup_n W^n \subset U$.

Furthermore $\overline{\bigcup_n W^n}$ is a compact submob of S.

Now let $J = J_o(W)$ be the largest ideal of $\overline{\bigcup_n W^n}$ contained in W. Then J is a submob of S and J is open (1.2.2). Furthermore, since $K \subset W$ we have $K \subset J$.

3.1.9. <u>Theorem</u>. Let S be a locally compact monothetic mob, and suppose S has a kernel $K \neq \emptyset$; then S is compact.

<u>Proof</u>:

Since S is commutative, K is the unique minimal left and minimal right ideal, hence K is a group. Now let e be the unit of K, then K = Se and is a retract of S. Hence K is locally compact, and it follows that K is a topological group (1.1.8).

Next let a be a generator of S, then $\Gamma(a) = S$ and hence $\Gamma(ae) = Se$.

Thus K is monothetic with generator ae.

Then K must be either compact or a copy of the group of integers.

Since the group of integers is not generated by the positive powers of an element, K is compact.

Lemma 3.1.8 implies the existence of an open mob J with compact closure containing K.

Some power of a say a^r lies in J, hence $\Gamma(a^r) \subset \overline{J}$ and $\Gamma(a^r)$ is compact.

Since $S = \Gamma(a) = \{a^i\}_{i=1}^{r} \cup \{a^i\}_{i=1}^{r} \Gamma(a^r)$, S is compact.

3.2. Ideals in commutative mobs

We have seen in 1.2, that if S is a commutative compact mob, then K is a compact topological group. An analogous result holds for locally compact commutative mobs.

3.2.1. Theorem. Let S be a locally compact commutative mob which contains a minimal ideal K. Then K is a locally compact topological group.

Proof:

Since K is the unique minimal left and minimal right ideal $K = Ka = aK$ for every $a \in K$. Hence K is a group. Furthermore $K = Se$, with $e = e^2 \in K$, and hence K is a retract of S, thus closed and locally compact. This shows that K is a topological group.

Now let e be the identity of K. Then we have for every $e^* \in E$, $ee^* = e^*e = e$. Thus e is the minimal idempotent of S, and it follows that in a commutative compact mob there always exists a unique minimal idempotent.

Now let S be a mob and let $Z = \{x \mid xs = sx \text{ for all } s \in S\}$ be the centre of S. The continuity of multiplication implies that if $Z \neq \emptyset$ then Z is a closed submob of S.

Definition. A mob S is called normal if for every $x \in S$ we have $xS = Sx$.

3.2.2. Lemma. In a normal mob S the set of all idempotents E is contained in the centre Z of S.

Proof:

Let $e \in E$, then $eS = Se$ implies that $es_1 = s_2e$ and $s_1e = es_3$ for each $s_1 \in S$ and suitably chosen $s_2, s_3 \in S$.

But then $(es_1)e = s_2e = e(s_1e) = es_3$, and hence $es_1 = s_1e$.

3.2.3. Lemma. Let S be a compact mob with $E \subset Z$ and let $a,b \in S$. If $e_1 = e_1^2 \in \Gamma(a)$, $e_2 = e_2^2 \in \Gamma(b)$ then $e_1e_2 \in \Gamma(ab)$.

Proof:

It follows from 1.1.4 that $e = e_1e_2 \in \overline{\{ae_2, a^2e_2, \dots\}} = \Gamma(ae_2)$ and similarly that $e_1e_2 \in \Gamma(e_1b)$.

According to 1.1.14 we have $\Gamma(ae_2)e \subset H(e)$, $\Gamma(e_1b)e \subset H(e)$ and hence $ae_2e = ae \in H(e)$, $e_1be = eb \in H(e)$. Thus $aeeb = abe \in H(e)$.

Now let $f = f^2 \in \Gamma(ab)$. Since $fe \in \overline{\{(ab)^n e\}}_{n=1}^{\infty} \subset H(e)$ we have $fe = e$.

Moreover if x is the inverse of abf in the group H(f) then $abfx = abx = f$. This relation implies that $f = f^2 = fabx = afbx = a^2(bx)^2$, and by induction $f = a^n(bx)^n$ for every integer $n \geq 1$.

Thus $f = e_1b^*$, with $b^* \in \overline{\{(bx)^n\}}_{n=1}^{\infty}$. We have therefore $e_1f = f$ and similarly $e_2f = f$. These relations together with $fe = fe_1e_2 = e_1e_2$ imply $f = e_1e_2$.

This proves lemma 3.2.3.

Now let S be compact and let $P_\alpha = \{x \mid x \in S,\ e_\alpha \in \Gamma(x)\}$. Then $P_\alpha \cap P_\beta = \emptyset$ if $e_\alpha \neq e_\beta$ and S can be written as the class sum of the disjoint sets P_α. In general P_α need not be a submob of S. However if S satisfies the condition of lemma 3.2.3 (this is for instance the case if S is commutative) then each set P_α is a submob of S.

3.2.4. Theorem. Let S be a compact mob with $E \subset Z$. Then S is the union of disjoint submobs P_α, where each P_α contains exactly one idempotent.

Proof:

Let $a, b \in P_\alpha$, then $e_\alpha \in \Gamma(a)$, $e_\alpha \in \Gamma(b)$ and according to the previous lemma we have $e_\alpha = e_\alpha e_\alpha \in \Gamma(ab)$. Thus $ab \in P_\alpha$. Moreover it is clear that each P_α contains exactly one idempotent which proves our theorem.

3.2.5. Lemma. Let S be a compact mob and let $H(e_\alpha)$ be the maximal subgroup containing the idempotent e_α. Then $H(e_\alpha) \subset P_\alpha$ and $H(e_\alpha) = P_\alpha e_\alpha = e_\alpha P_\alpha$.

Proof:

Let $x \in H(e_\alpha)$. Then since $H(e_\alpha)$ is compact, we have $\Gamma(x) \subset H(e_\alpha)$, which implies $e_\alpha \in \Gamma(x)$. Thus $x \in P_\alpha$.

Furthermore we have for each $x \in P_\alpha$, $\Gamma(x)\, e_\alpha \subset H(e_\alpha)$, hence

$\bigcup_{x \in P_\alpha} \Gamma(x)e_\alpha \subset H(e_\alpha)$, and thus $P_\alpha e_\alpha \subset H(e_\alpha)$. Since $H(e_\alpha) \subset P_\alpha$ it follows that $P_\alpha e_\alpha = H(e_\alpha)$.

In the same way we can prove $H(e_\alpha) = e_\alpha P_\alpha$.

Corollary. Since each $H(e_\alpha)$ is closed we have $H(e_\alpha) = \overline{e_\alpha P_\alpha} = e_\alpha \overline{P}_\alpha$. Furthermore if e_α is a left or right identity of S, then $P_\alpha = H(e_\alpha)$ and P_α is a compact group.

3.2.6. Lemma. If S is a compact mob and $\overline{P}_\alpha \cap P_\beta \neq \emptyset$, $e_\alpha \neq e_\beta$, then $e_\beta \in \overline{P}_\alpha \setminus P_\alpha$ and $P_\alpha \cap \overline{P}_\beta = \emptyset$.

Proof:

Let $a \in \overline{P}_\alpha \cap P_\beta$ and let U and V be neighbourhoods of a and a^n respectively such that $U^n \subset V$, $n \geq 1$. Let $b \in U \cap P_\alpha$. Then $b^n \in \Gamma(b)$ and thus $\Gamma(b^n) \subset \Gamma(b)$. Hence $e_\alpha \in \Gamma(b^n)$, which implies $b^n \in P_\alpha$.

Since we also have $b^n \in V$ it follows that $a^n \in \overline{P}_\alpha$. Thus $\Gamma(a) \subset \overline{P}_\alpha$ and we have $e_\beta \in \Gamma(a) \subset \overline{P}_\alpha$. Since $e_\beta \notin P_\alpha$ it follows that $e_\beta \in \overline{P}_\alpha \setminus P_\alpha$.

The preceding corollary implies that $e_\alpha e_\beta \in H(e_\alpha)$, thus $e_\alpha e_\beta = e_\alpha e_\beta e_\alpha$ and $(e_\alpha e_\beta)(e_\alpha e_\beta) = e_\alpha e_\beta^2 = e_\alpha e_\beta$. Since $H(e_\alpha)$ contains only one idempotent we have $e_\alpha e_\beta = e_\alpha$ and analogously $e_\beta e_\alpha = e_\alpha$.

Suppose now that $P_\alpha \cap \overline{P}_\beta \neq \emptyset$, then it would follow in the same way that $e_\alpha e_\beta = e_\beta e_\alpha = e_\beta$, i.e. $e_\alpha = e_\beta$, a contradiction.

3.2.7. Theorem. If e_α is a maximal idempotent of the compact mob S, then P_α is closed.

Proof:

Let $x \in \overline{P}_\alpha \cap P_\beta$. Then $e_\beta \in \overline{P}_\alpha$ and it follows from lemma 3.2.6 that $e_\alpha e_\beta = e_\beta e_\alpha = e_\alpha$ i.e. $e_\alpha \leq e_\beta$.

Since e_α is maximal, $e_\alpha = e_\beta$ and the theorem is proved.

3.2.8. Theorem. Let S be a compact mob and let $\overline{P}_\alpha = S$. Then the kernel K of S is equal to $H(e_\alpha)$.

Proof:

Since $P_\beta \cap \overline{P}_\alpha \neq \emptyset$ for each e_β we have $e_\alpha e_\beta = e_\beta e_\alpha = e_\alpha$. Hence e_α is the minimal idempotent of S and it follows that $K = H(e_\alpha)$.

3.2.9. Theorem. Every open prime ideal P of a compact commutative mob S is a union of subsemigroups P_α.

$$P = \bigcup_\alpha P_\alpha$$

Proof:

Let $x \in P \cap P_\alpha$. Then since P_α is a mob we have $xe_\alpha \in P \cap P_\alpha$. On the other hand $xe_\alpha \in P_\alpha e_\alpha = H(e_\alpha)$, which implies the existence of an element x^* such that $x^* xe_\alpha = e_\alpha$. Hence $e_\alpha \in P$.
Next let $y \in P_\alpha \cap (S \setminus P)$. Since $S \setminus P$ is a closed submob, we have $\Gamma(y) \subset S \setminus P$ and thus $e_\alpha \in S \setminus P$, a contradiction.
Thus if $P \cap P_\alpha \neq \emptyset$, then $P_\alpha \subset P$ i.e. $P = \bigcup_\alpha P_\alpha$.

3.2.10. Theorem. If e_α is a non-minimal idempotent of the compact commutative mob S, then

$$J_o(S \setminus \{e_\alpha\}) = \bigcup \{P_\beta \mid e_\alpha e_\beta \neq e_\alpha \quad e_\beta \in E\}$$

is an open prime ideal of S.

Proof:

Theorem 1.5.5 implies that $J_o(S \setminus \{e_\alpha\})$ is an open prime ideal of S.
Furthermore we have for any idempotent $e_\beta \in J_o(S \setminus \{e_\alpha\})$,
$e_\alpha e_\beta \in J_o(S \setminus \{e_\alpha\})$ and thus $e_\alpha e_\beta \neq e_\alpha$.
Hence $J_o(S \setminus \{e_\alpha\} \subset \bigcup \{P_\beta \mid e_\alpha e_\beta \neq e_\alpha,\ e_\beta \in E\} = P$.

Now let $e_\alpha e_\beta \neq e_\alpha$, then for any idempotent $e_\gamma \in E$, we have $e_\alpha e_\gamma e_\beta \neq e_\alpha$, since $e_\alpha e_\gamma e_\beta = e_\alpha$ would imply $e_\alpha e_\beta = e_\alpha e_\gamma e_\beta = e_\alpha$. Thus if $x \in P$, $s \in S$ with $e_\beta \in \Gamma(x)$ and $e_\gamma \in \Gamma(s)$, then $e_\gamma e_\beta \in \Gamma(sx)$ with $e_\alpha e_\gamma e_\beta \neq e_\alpha$. Hence $sx \in P$ and P is an ideal not containing e_α.
This implies that $P \subset J_o(S \setminus \{e_\alpha\})$ and the theorem is proved.

Since by 1.5.4 every open prime ideal of S has the form $J_o(S \setminus \{e_\alpha\})$ we have also that every open prime ideal of the compact commutative mob S has the form $\bigcup \{P_\beta \mid e_\alpha e_\beta \neq e_\alpha,\ e_\beta \in E\}$. If $e_\alpha \leq e_\beta$ then

$$J_o(S \setminus \{e_\alpha\}) = \bigcup \{P_\gamma \mid e_\alpha e_\gamma \neq e_\alpha,\ e_\gamma \in E\} \subset \bigcup \{P_\gamma \mid e_\beta e_\gamma \neq e_\beta, e_\gamma \in E\}.$$

For if $e_\alpha e_\beta = e_\alpha$ and $e_\gamma e_\beta = e_\beta$, then $e_\gamma e_\alpha e_\beta = e_\gamma e_\alpha = e_\alpha e_\beta = e_\alpha$.
Hence $J_o(S \setminus \{e_\alpha\}) \subset J_o(S \setminus \{e_\beta\})$.
If on the other hand $J_o(S \setminus \{e_\alpha\}) \subset J_o(S \setminus \{e_\beta\})$, then

$e_\beta \in S\setminus J_o(S\setminus\{e_\alpha\})$ and hence $e_\alpha e_\beta = e_\alpha$ i.e. $e_\alpha \leq e_\beta$.

<u>Corollary</u>. If $J_o(S\setminus\{e_\alpha\})$ and $J_o(S\setminus\{e_\beta\})$ are two open prime ideals of the compact commutative mob S, then $J_o(S\setminus\{e_\alpha e_\beta\}) \subset J_o(S\setminus\{e_\alpha\}) \cap J_o(S\setminus\{e_\beta\})$, and there does not exist an open prime ideal P of S with $J_o(S\setminus\{e_\alpha e_\beta\}) \subset P \subset J_o(S\setminus\{e_\alpha\}) \cap J_o(S\setminus\{e_\beta\})$ with $P \neq J_o(S\setminus\{e_\alpha e_\beta\})$.

<u>Proof</u>:

Since $e_\alpha e_\beta \leq e_\alpha$ and $e_\alpha e_\beta \leq e_\beta$ we have $J_o(S\setminus\{e_\alpha e_\beta\}) \subset J_o(S\setminus\{e_\alpha\}) \cap J_o(S\setminus\{e_\beta\})$. Next let

$$J_o(S\setminus\{e_\alpha e_\beta\}) \subset P \subset J_o(S\setminus\{e_\alpha\}) \cap J_o(S\setminus\{e_\beta\}).$$

Then $P = J_o(S\setminus\{e_\gamma\})$ with $e_\alpha e_\beta \leq e_\gamma \leq e_\alpha$, $e_\alpha e_\beta \leq e_\gamma \leq e_\beta$.
Thus $e_\gamma e_\alpha = e_\gamma$ and $e_\gamma e_\beta = e_\gamma$ which implies $e_\gamma e_\alpha e_\beta = e_\gamma e_\beta = e_\gamma$. Hence $e_\gamma \leq e_\alpha e_\beta$. Since $e_\alpha e_\beta \leq e_\gamma$ we have $e_\gamma = e_\alpha e_\beta$ and $P = J_o(S\setminus\{e_\alpha e_\beta\})$.

Definition. A mob S is called <u>complete</u> if every element $a \in S$ has roots of every degree >0 in S, i.e. if for every $a \in S$ and $n > 0$ there exists $a_n \in S$ with $a = a_n^n$.

3.2.11. <u>Theorem</u>. In a compact commutative mob S the set of elements having roots of every degree > 0 forms a complete compact submob.

<u>Proof</u>:

Let $S_n = \{a^n | a \in S\}$ $n = 1,2,\ldots$

Then S_n is closed since S is compact and for a finite number of S_n's say $S_{n_1},\ldots,S_{n_k}$ we have

$$S_{n_1 n_2 \ldots n_k} \subset \bigcap_{i=1}^{k} S_{n_i}.$$

Hence $\bigcap_{n=1}^{\infty} S_n = S^* \neq \emptyset$.

Furthermore S^* is a closed submob of S since each S_n is a closed submob of S.

Now let $a \in S^*$, then $a = a_2^2 = a_3^3 = \ldots$ for suitable chosen $a_i \in S$.
Let $A_n = \{x | x \in S,\ x^n = a\}$. Then A_n is closed and $A_n \cap S_k \neq \emptyset$ since $a = a_{nk}^{nk}$ with $a_{nk}^k \in A_n \cap S_k$. Hence $A_n \cap S^* \neq \emptyset$. Thus $a \in S^*$ has roots of every degree in S^*. S^* also is a compact complete submob of S.

Moreover it is clear that S^* is the set of elements having roots of every degree.

3.2.12. Theorem. Let S be a complete compact mob and e an idempotent from S. Then H(e) is a complete compact group.

Proof:

Let $a \in H(e)$ and $a = a_n^n$. Then since $e \in \Gamma(a)$ and $a \in \Gamma(a_n)$ we have $e \in \Gamma(a) \subset \Gamma(a_n)$.

Hence $a_n e \in H(e)$ and $a_n e = ea_n e$. Thus $(a_n e)^n = a_n^n e = ae = a$.

This proves that H(e) is a complete group.

If U is an open subset of a mob S and x in S, then xU need not be open in S. If S is a compact connected commutative mob with this property, then it follows that S is a group. However the following theorem holds.

3.2.13. Theorem. Let S be a commutative mob with identity u. Then there is a stronger topology under which S is a mob such that

1) if U is open in S and $x \in S$, then xU open in S
2) the neighbourhoods at u are the same under these two topologies.

Proof:

Let T_1 denote the given topology of S and let $\{V_\alpha\}_{\alpha \in A}$ be a basis of open sets at u. Let $B = \{xV_\alpha \mid x \in S, \alpha \in A\}$, and define the topology T_2 on S by requiring that B be an open basis. We now verify that B is really a basis for a topology. Let xV_α, $yV_\beta \in B$ and let $z \in xV_\alpha \cap yV_\beta$.
We then have $z = xv_1 = yv_2$ where $v_1 \in V_\alpha$ and $v_2 \in V_\beta$.
The continuity of multiplication implies the existence of sets V_γ and V_δ such that $v_1V_\gamma \subset V_\alpha$ and $v_2V_\delta \subset V_\beta$. Choosing V_ε such that $V_\varepsilon \subset V_\gamma \cap V_\delta$ we have $zV_\varepsilon \subset zV_\gamma \cap zV_\delta \subset xv_1V_\gamma \cap yv_2V_\delta \subset xV_\alpha \cap yV_\beta$.
Hence given xV_α, $yV_\beta \in B$ with $z \in xV_\alpha \cap yV_\beta$, there exists a V_ε such that $zV_\varepsilon \subset xV_\alpha \cap yV_\beta$, which shows that B is an open basis for a topology.
We now show that multiplication is continuous in the T_2-topology.
Let $a,b \in S$ and $ab \in abV_\alpha$. If V_β is such that $V_\beta^2 \subset V_\alpha$, then $aV_\beta\, bV_\beta = abV_\beta^2 \subset abV_\alpha$.

Furthermore T_2 is stronger than T_1, because if $U \varepsilon T_1$ and $a \varepsilon U$, then there is a V_α such that $aV_\alpha \subset U$.
The T_2-topology of S obviously satisfies condition 1) and 2).

Definition. We shall call a mob S <u>embeddable</u> in a topological group G, if there is a submob S' of G such that S' is topologically isomorphic to S.

3.2.14. <u>Theorem</u>. Let S be a commutative mob with cancellation. If S has the property that U open implies aU open for each subset U of S and each $a \varepsilon S$, then S is embeddable in a topological group.

<u>Proof</u>:
Let $\mathfrak{R}$ be the relation in $S \times S$ defined by $(a,b)\mathfrak{R}(c,d)$ if and only if $ad = bc$. The fact that S is commutative and is a mob with cancellation implies that $\mathfrak{R}$ is an equivalence relation.
Let $G = S \times S/\mathfrak{R}$ be the family of equivalence classes with the quotient topology.
Each equivalence class A is a closed set of $S \times S$. For let $(a,b) \varepsilon A$ and let $(c,d) \varepsilon \bar{A}$. If $ad \neq bc$, then there are neighbourhoods $U(c)$ and $U(d)$ such that $aU(d) \cap bU(c) = \emptyset$. Hence for all $(x,y) \varepsilon U(c) \times U(d)$ we have $ay \neq bx$, i.e. $U(c) \times U(d) \cap A = \emptyset$, a contradiction.
Let P be the projection of $S \times S$ onto G. We now show that P is open.
Let $(a,b) \varepsilon S \times S$ and let $(a,b) \varepsilon U(a) \times U(b) = U$ with $U(a)$ and $U(b)$ open.
Let $U^* = P^{-1}(P(U))$ and $(x,y) \varepsilon U^*$. Then $(x,y)\mathfrak{R}(c,d)$, $(c,d) \varepsilon U$ and we have $(x,y)\mathfrak{R}(c,d)\mathfrak{R}(xc,yc) = (xc,xd)$.
Furthermore let $U(x) = \{x^* \mid x^*c \varepsilon xU(a)\}$ and

$U(y) = \{y^* \mid y^*c \varepsilon xU(b)\}$.

Then $U(x)$ and $U(y)$ are open and if $(x^*,y^*) \varepsilon U(x) \times U(y)$, then $x^*c = xp$, $y^*c = xq$, $(p,q) \varepsilon U(a) \times U(b)$.
Hence $(x^*,y^*)\mathfrak{R}(xp,xq)\mathfrak{R}(p,q)$ and $U(x) \times U(y) \subset U^*$. Since P is open and the relation $\mathfrak{R}$ is closed, G is a Hausdorff space.
Moreover G is a group if we define multiplication by $A \cdot B = C$, where C is the equivalence class of (a_1a_2, b_1b_2) with $(a_1,b_1) \varepsilon A$, $(a_2,b_2) \varepsilon B$.

We now show that G is a topological group.

Let U be a neighbourhood of $C = A \cdot B$. Then there are neighbourhoods $U(a_1a_2)$ and $U(b_1b_2)$ with $P(U(a_1a_2) \times U(b_1b_2)) \subset U$. Let $U(a_1), U(a_2), U(b_1)$ and $U(b_2)$ be such that $U(a_1)\ U(a_2) \subset U(a_1a_2)$ and $U(b_1)\ U(b_2) \subset U(b_1b_2)$. Hence $P(U(a_1) \times U(b_1))\ P(U(a_2) \times U(b_2)) \subset U$. Since P is open $P(U(a_1) \times U(b_1))$ and $P(U(a_2) \times U(b_2))$ are open neighbourhoods of A and B respectively, and it follows that multiplication is continuous.

Since $P(U(a) \times U(b)) = P(U(b) \times U(a))^{-1}$, the mapping $C \to C^{-1}$ is continuous and hence G is topological.

Now let $\alpha : S \to G$ be defined by $\alpha(a) = P(a^2, a)$. Then α is an isomorphism since S is commutative and satisfies the cancellation law.

Furthermore α is open since $\alpha(U(a)) = P(U(a)\ U(a) \times U(a))$ and α is continuous since if $V(a)V(a) \subset U(a^2)$ and $W(a) \subset V(a) \cap U(a)$, then $\alpha(W(a)) = P(W(a)W(a) \times W(a)) \subset P(U(a^2) \times U(a))$.

Hence α is topological and the theorem is proved.

3.3. <u>Characters of commutative mobs</u>

In this section S will always denote a commutative mob.

Definition. Let S be a mob and let χ be a complex valued continuous function on S such that

$$\chi(ab) = \chi(a)\chi(b) \quad \text{for all } a,b \in S.$$

If χ is also bounded and not identically zero, χ is called a <u>semicharacter</u> of S.

If the absolute value $|\chi(a)| = 1$ for all $a \in S$, χ is called a <u>character</u> of S.

If χ_α and χ_β are two semicharacters of S, the product $\chi_\alpha\chi_\beta$ is defined as the ordinary pointwise product

$$\chi_\alpha\chi_\beta(a) = \chi_\alpha(a)\chi_\beta(a) .$$

$\chi_\alpha\chi_\beta$ is either a semicharacter of S or is identically zero. Moreover if e is an idempotent $e \in S$, then $\chi(e^2) = \chi(e)\chi(e) = \chi(e)$ implies $\chi(e) = 0$ or $\chi(e) = 1$.

In particular, if S has an identity u, then $\chi(u) = 1$ for all semicharacters of S. Hence in this case the set of all semicharacters is a commutative semigroup $\hat{S}$. The set of all characters S^* of S clearly is an abelian group with identity element the unit character χ_1 and $\chi^{-1} = \bar{\chi}$.

3.3.1. Theorem. Let χ be a semicharacter of S and let

$$I(\chi) = \{a \mid |\chi(a)| < 1,\ a \in S\}$$

$$B(\chi) = \{a \mid |\chi(a)| = 1,\ a \in S\} .$$

Then $S = I(\chi) \cup B(\chi)$, while $I(\chi)$ is an open prime ideal of S if $I(\chi) \neq \emptyset$ and $B(\chi)$ a closed submob if $B(\chi) \neq \emptyset$.

Proof:

Suppose for $a \in S$, $|\chi(a)| = c > 1$. Then for every integer n we have $|\chi(a)|^n = |\chi(a^n)| = c^n > c > 1$. Since χ is bounded on S this relation leads to a contradiction. Hence $|\chi(a)| \leq 1$ for all $a \in S$ and $S = I(\chi) \cup B(\chi)$. Next suppose $I(\chi) \neq \emptyset$ and let $a \in I(\chi)$. Then

$$|\chi(as)| = |\chi(a)|\ |\chi(s)| \leq |\chi(a)| < 1.$$

Furthermore if $ab \in I(\chi)$, then $|\chi(ab)| < 1$ and hence $|\chi(a)| < 1$ or $|\chi(b)| < 1$ i.e. $I(\chi)$ is a prime ideal of S.
Since the function $|\chi|$ is continuous $I(\chi)$ is open. Moreover $B(\chi) = S \setminus I(\chi)$ and it follows that $B(\chi)$ is a closed submob of S.

Remark.

It follows from 3.2.9 that if S is compact, both $I(\chi)$ and $B(\chi)$ are unions of submobs P_α where

$$P_\alpha = \{x \mid x \in S,\ e_\alpha = e_\alpha^2 \in \Gamma(x)\} .$$

For every idempotent $e_\alpha \in I(\chi)$ we have $\chi(e_\alpha) = 0$ and for every idempotent $e_\beta \in B(\chi)$ we have $\chi(e_\beta) = 1$.
Thus $I(\chi) = \bigcup\{P_\alpha \mid \chi(e_\alpha) = 0\}$ and $B(\chi) = \bigcup\{P_\alpha \mid \chi(e_\alpha) = 1\}$.
Both sets $I(\chi)$ and $B(\chi)$ may be empty. $I(\chi)$ is empty if χ is a character of S.
Let S be the multiplicative semigroup of real numbers x, $0 \leq x \leq \frac{1}{2}$ with the usual topology. Then if χ is the semicharacter defined by $\chi(x) = x$

we have $B(\chi) = \emptyset$. In this case, we have in particular $B(\chi) = \emptyset$ for all semicharacters $\chi \neq \chi_1$.

3.3.2. Lemma. Let $\chi \in \hat{S}$ and define the null set $N(\chi)$ to be

$N(\chi) = \{a \mid \chi(a) = 0 \quad a \in S\}$.

If $N(\chi) \neq \emptyset$, then $N(\chi)$ is a closed prime ideal of S and

$\bigcup\{H(e_\alpha) \mid \chi(e_\alpha) = 0\} \subset N(\chi)$.

Proof:

If $a \in N(\chi)$, $s \in S$ we have $\chi(as) = \chi(a)\chi(s) = 0$, i.e. $as \in N(\chi)$.
Since $\chi(ab) = 0$ implies $\chi(a) = 0$ or $\chi(b) = 0$, $N(\chi)$ is a prime ideal and $N(\chi)$ is closed since χ is continuous.
Now let $\chi(e_\alpha) = 0$, then for every $h \in H(e_\alpha)$ we have $\chi(h) = \chi(he_\alpha) = \chi(h)\,\chi(e_\alpha) = 0$. Thus $H(e_\alpha) \subset N(\chi)$.

It is clear that if S is compact and $N(\chi)$ is given, both $I(\chi)$ and $B(\chi)$ are uniquely determined. Furthermore it follows from the next theorem that each semicharacter χ is uniquely determined by its values on $I(\chi)$ if $N(\chi) \neq I(\chi)$.

3.3.3. Theorem. Let I be an ideal of S and let χ be a semicharacter of I. Then there exists one and only one semicharacter ξ of S such that $\chi(x) = \xi(x)$ for all $x \in I$.

Proof:

Let $a \in I$ be an element with $\chi(a) \neq 0$.
If b is any element of S, we have $ba \in I$ and we define $\xi(b)$ by the relation

$$\xi(b) = \frac{\chi(ba)}{\chi(a)} .$$

The function ξ is clearly continuous and for every $b \in I$ we have

$$\xi(b) = \frac{\chi(ba)}{\chi(a)} = \frac{\chi(b)\chi(a)}{\chi(a)} = \chi(b).$$

Furthermore:

$$\xi(b)\xi(c) = \frac{\chi(ba)}{\chi(a)}\,\frac{\chi(ca)}{\chi(a)} = \frac{\chi(baca)}{\chi(a)\,\chi(a)} = \frac{\chi(bca)}{\chi(a)}\,\frac{\chi(a)}{\chi(a)} = \xi(bc).$$

Hence since ξ is bounded ξ is a semicharacter of S. Next let ξ_1 and ξ_2 be two semicharacters of S with $\xi_1(b) = \xi_2(b) = \chi(b)$ for all $b \in I$. Let $c \in S$, then $ac \in I$ and we have $\xi_1(ac) = \xi_2(ac)$, i.e. $\xi_1(a)\,\xi_1(c) = \xi_2(a)\,\xi_2(c)$.
Hence $\xi_1(c) = \xi_2(c)$ for every $c \in S$ and the theorem is proved.

Corollary. It follows from the proof of the theorem that if χ is any character of I, then there is only one character ξ of S such that ξ is an extension of χ.

Now let $N \neq S$ be a closed prime ideal of the mob S. Then there need not exist a semicharacter χ of S, such that $N(\chi) = N$.
Let, for instance S be the I-mob J_3. Then $\{0\}$ is a closed prime ideal and every element of J_3 is idempotent. Hence we have $\chi(a) = 0$ or $\chi(a) = 1$ for each $a \in J_3$. From the continuity of χ it now follows that J_3 has only one semicharacter, the unit character χ_1.

Let $N = N(\chi)$ be the null set of a semicharacter χ. Define S_N to be the set of all semicharacters $\xi \in \hat{S}$, such that $N(\xi) = N$.

Each S_N obviously is a semigroup. Furthermore if S is compact $S_\emptyset$ is the charactergroup S^* of S.
Indeed if $I(\chi) \neq \emptyset$, then $I(\chi)$ is an ideal of the compact mob S and hence contains an idempotent e, and we have $\chi(e) = 0$ which implies $N(\chi) \neq \emptyset$.

3.3.4. Theorem. Let S be a commutative mob. Then $\hat{S}$ is the union of disjoint semigroups S_{N_α}, where each S_{N_α} is a semigroup with cancellation. If S is compact, $S_\emptyset = S^*$.

Proof:
Let $\chi, \xi, \psi \in S_{N_\alpha}$ and suppose $\chi\xi = \chi\psi$. Then for every $a \in N_\alpha$ we have $\xi(a) = \psi(a) = 0$, and if $a \in S \setminus N_\alpha$, $\chi(a)\,\xi(a) = \chi(a)\,\psi(a)$ with $\chi(a) \neq 0$.
Hence $\xi(a) = \psi(a)$ for all $a \in S$.

Corollary. If S is connected and $N_\alpha \neq \emptyset$, then S_{N_α} cannot be finite, and S_{N_α} does not contain an idempotent.

Proof:

Let χ be an idempotent semicharacter $\chi \in S_{N_\alpha}$. Since χ can assume only two values 0 and 1, it would follow that $\chi(a) = 1$, $a \in S \setminus N_\alpha$. Hence N_α is a clopen set. This gives a contradiction with the connectedness of S. Next if S_{N_α} is finite, then S_{N_α} with the discrete topology is a compact mob and hence contains an idempotent.

Now let N_α and N_β be two null sets. If $N_\alpha \cup N_\beta \neq S$, then $N_\alpha \cup N_\beta$ is again a null set.

For if $\chi \in S_{N_\alpha}$ and $\psi \in S_{N_\beta}$, then $N_\alpha \cup N_\beta = \{x \mid \chi(x)\psi(x) = 0,\ x \in S\}$.

Hence $S_{N_\alpha} S_{N_\beta} \subset S_{N_\alpha \cup N_\beta}$.

It follows that $\hat{S}$ is a semigroup if and only if $S \neq N_\alpha \cup N_\beta$ for any two null sets N_α and N_β. This is for instance the case if S contains a unit element.

3.3.5. Theorem. Let $N \neq S$ be a clopen prime ideal of a mob S. Then N is a null set. Furthermore if S is compact, S_N is a group if and only if N is clopen and each $\chi \in S_N$ is of the form

$$\chi(x) = \begin{cases} 0 \text{ for } x \in N \\ \phi(x) \text{ for } x \in S \setminus N, \text{ where } \varphi \in (S \setminus N)^* . \end{cases}$$

Proof:

Let S_N be a group. Then S_N contains an idempotent χ and we have $N = N(\chi) = I(\chi)$. Therefore N is clopen. Conversely let $N \neq S$ be a clopen prime ideal. Then $S \setminus N$ is a closed submob. Let $\phi \in (S \setminus N)\hat{}$, $N(\phi) = \emptyset$. Then the function χ defined by

$$\chi(x) = \begin{cases} 0 & \text{for } x \in N \\ \phi(x) & \text{for } x \in S \setminus N \end{cases}$$

is a semicharacter of S.

It is clear that in this manner we obtain all semicharacters of S_N. If S is compact and $N(\phi) = \emptyset$, then $\phi \in S^*$. Hence in this case $S_N \cong (S \setminus N)^*$ and S_N is a group.

Corollary. Let S be finite. Then $\hat{S}$ is a union of disjoint groups.

Remark.

Now let S be a commutative mob such that S can be written as a union of groups. In such a mob every ideal of S is itself the union of max-

imal groups. Furthermore each $P_\alpha = \{x | x \in S,\ e_\alpha = e_\alpha^2 \in \Gamma(x)\}$ is identical with the maximal group $H(e_\alpha)$.
Hence if χ is any semicharacter then $I(\chi) = \bigcup \{P_\alpha |\ \chi(e_\alpha) = 0\} = \bigcup \{H(e_\alpha)\ |\chi(e_\alpha) = 0\} \subset N(\chi)$. Thus $I(\chi) = N(\chi)$ and $S_{N(\chi)}$ is a group.

Definition. Let S be a commutative mob and $\chi \in \hat{S}$. Let C be a compact subset of S, $\varepsilon > 0$ and define

$$U(C,\varepsilon,\chi) = \{\psi \in \hat{S}\ |\ |\psi(x) - \chi(x)| < \varepsilon \text{ for all } x \in C\}\ .$$

We now define a topology on $\hat{S}$ by requiring that the set $\{U(C,\varepsilon,\chi)\}$ be an open basis.
It is clear that if $\hat{S}$ is a semigroup, then $\hat{S}$ with this topology is a commutative mob.

3.3.6. Theorem. Let S be a discrete mob with identity, then $\hat{S}$ is a compact mob.

Proof:
Since all compact subsets of S are finite, the topology of $\hat{S}$ is its relative topology as a subspace of D^S with the product topology (D is the set of complex numbers z with $|z| \leq 1$). $\hat{S}$ clearly is a closed subset of D^S and hence compact.

3.3.7. Theorem. Let S be a compact mob and let $\hat{S}' = \bigcup \{S_N |\ S_N$ a group, $S_N \subset \hat{S}\}$.
Then $\hat{S}'$ is a discrete subspace of $\hat{S}$.

Proof:
Let $\chi \in \hat{S}'$ and suppose $\phi \neq \chi$, $\phi \in \hat{S}' \cap U(S,\frac{1}{2},\chi)$. Since $\phi \neq \chi$ we have $\phi(a) \neq \chi(a)$ for some $a \in S$. Furthermore we have $\phi(x) = 0$ or $|\phi(x)| = 1$ and $\chi(x) = 0$ or $|\chi(x)| = 1$ for all $x \in S$.
If either $\phi(a)$ or $\chi(a) = 0$, then $\phi \notin U(S,\frac{1}{2},\chi)$. Hence we have $|\phi(a)| = |\chi(a)| = 1$.
Suppose now $\phi(a) = e^{ix}$ and $\chi(a) = e^{iy}$, $y > x$. Then there is a positive integer n such that

$$|\phi(a^n) - \chi(a^n)| = |e^{inx} - e^{iny}| = |1 - e^{in(y-x)}| > \tfrac{1}{2}.$$

hus $\phi \notin U(S,\frac{1}{2},\chi)$ and we have $U(S,\frac{1}{2},\chi) \cap \hat{S}' = \{\chi\}$.

3.3.8. Lemma. Let S be a discrete mob with identity which is a union of groups. Then $\hat{S}^{\wedge}$ (the semigroup of semicharacters of $\hat{S}$) is a union of groups and is discrete.

Proof:

The remark to theorem 3.3.5 and theorem 3.3.6 imply that $\hat{S}$ is a compact mob which is the union of groups. Hence $\hat{S}^{\wedge} = \hat{S}^{\wedge\prime}$ is a union of groups and by theorem 3.3.7 $\hat{S}^{\wedge}$ is discrete.

Now let $a \in S$ and define $\tilde{a}$ by $\tilde{a}(\chi) = \chi(a)$, $\chi \in \hat{S}$.
It is obvious that each function $\tilde{a}$ is a semicharacter of $\hat{S}$.
Now let S be a discrete mob which is a union of groups. Then if $e_\alpha \neq e_\beta$ are two idempotents of S, we either have $e_\alpha e_\beta \neq e_\alpha$ or $e_\alpha e_\beta \neq e_\beta$. Hence there is a clopen ideal N such that $e_\alpha \in N$ and $e_\beta \notin N$ or vice versa. This implies the existence of a semicharacter $\chi \in \hat{S}$ such that $\chi(e_\alpha) \neq \chi(e_\beta)$.

3.3.9. Lemma. Let S be a discrete mob with identity such that S is a union of groups. Let $\mathfrak{A}$ be a clopen prime ideal of $\hat{S}$. Then there is one and only one idempotent $e \in S$ such that $\mathfrak{A} = N(\tilde{e})$.

Proof:

Since $\hat{S}$ is a union of groups S_{N_α} and $\hat{S}$ compact, each open ideal is of the form $\mathfrak{A} = \bigcup \{S_{N_\beta} \mid \varepsilon_\alpha \varepsilon_\beta \neq \varepsilon_\alpha\}$, where ε_β is the identity of S_{N_β} (3.2.9). If $\varepsilon_\alpha\varepsilon_\beta \neq \varepsilon_\alpha$, then $N_\beta \not\subset N_\alpha$ and hence $\mathfrak{A} = \bigcup \{S_{N_\beta} \mid N_\beta \not\subset N_\alpha\}$. Now let $\mathfrak{A}$ be closed, then there is an $x \notin N_\alpha$ such that $\chi(x) = 0$ for all $\chi \in \mathfrak{A}$. For let $x \notin N_\alpha$ and suppose there exists a $\chi \in \mathfrak{A}$ with $\chi(x) \neq 0$. Let C be any finite subset of S and let $\delta > 0$. Let $C \setminus N_\alpha = \{x_1, \ldots, x_n\}$ and $C \cap N_\alpha = \{x_{n+1}, \ldots, x_m\}$. Then $x_1 x_2 \ldots x_n \notin N_\alpha$ and there is a $\chi \in \mathfrak{A}$ such that $\chi(x_1 \ldots x_n) = \chi(x_1)\,\chi(x_2) \ldots \chi(x_n) \neq 0$.
Let $\phi = \varepsilon_\alpha \chi \overline{\chi}$. Then $\phi \in \mathfrak{A}$ and $\phi(x_i) = \underline{\varepsilon_\alpha}(x_i)$. $(i=1,2,\ldots,m)$.
Hence $\phi \in U(C,\delta,\varepsilon_\alpha) \cap \mathfrak{A}$ and thus $\varepsilon_\alpha \in \overline{\mathfrak{A}}$, which implies that $\mathfrak{A}$ is not closed.
Now let $x \notin N_\alpha$, $\chi(x) = 0$ for all $\chi \in \mathfrak{A}$ and let e_α be the idempotent such that $x \in H(e_\alpha)$. Then since N_α is a union of groups we have $e_\alpha \notin N_\alpha$ and $\chi(e_\alpha) = 0$ for all $\chi \in \mathfrak{A}$.

Hence $\tilde{e}_\alpha(\chi) = 0$ for all $\chi \in \mathfrak{A}$, i.e. $\mathfrak{A} \subset N(\tilde{e}_\alpha)$. On the other hand we have if $\chi \in N(\tilde{e}_\alpha)$, then $\chi(e_\alpha) = 0$, thus $N(\chi) \not\subset N_\alpha$ which implies $\chi \in \mathfrak{A}$. Thus $\mathfrak{A} = N(\tilde{e}_\alpha)$.
Now let $f \neq e_\alpha$ be an idempotent of S. Then there exists a semicharacter χ such that $\chi(e_\alpha) \neq \chi(f)$. Hence $N(\tilde{e}_\alpha) \neq N(\tilde{f})$ and the theorem is proved.

Remark.
It follows from the lemma that if $\mathfrak{A} = N(\tilde{e}_\alpha) = \bigcup \{S_{N_\beta} | N_\beta \not\subset N_\alpha\}$ is a clopen prime ideal of $\hat{S}$, then

$$N_\alpha = \bigcup \{H(e_\beta) \mid e_\beta e_\alpha \neq e_\alpha,\ e_\beta = e_\beta^2 \in S\} .$$

For let e_β be an idempotent $e_\beta \notin N_\alpha$, then $e_\beta e_\alpha \notin N_\alpha$ and $\chi(e_\beta e_\alpha) = \chi(e_\beta)\ \chi(e_\alpha) = 0$ for all $\chi \in \mathfrak{A}$. Hence since $e_\beta e_\alpha$ is an idempotent we have $e_\beta e_\alpha = e_\alpha$.
Thus $S \setminus N_\alpha = \bigcup \{H(e_\beta) \mid e_\beta e_\alpha = e_\alpha\}$ and it follows that $H(e_\alpha)$ is the minimal ideal of $S \setminus N_\alpha$.

3.3.10. Lemma. Let S be as in 3.3.9. Then S_{N_α} is topologically isomorphic with $(H(e_\alpha))^*$.

Proof:
Let $\phi \in S_{N_\alpha}$ and $\phi' = \phi \mid H(e_\alpha)$. Then 3.3.3 and 3.3.5 imply that the mapping $\phi \to \phi'$ is an isomorphism of S_{N_α} onto $(H(e_\alpha))^*$.
Furthermore $\phi(x) = \phi'(xe_\alpha)$, $x \in S \setminus N_\alpha$.
Now let C be a compact subset of $H(e_\alpha)$, then $U(C,\varepsilon,\phi)$ is mapped into $U(C,\varepsilon,\phi')$. On the other hand if C is a compact subset of S, then $(C \cap (S \setminus N_\alpha))e_\alpha = C'$ is a compact subset of $H(e_\alpha)$ and $U(C',\varepsilon,\phi')$ lies in the image of $U(C,\varepsilon,\phi)$.
Hence the mapping $\phi \to \phi'$ is a homeomorphism.

From lemma 3.3.9 and theorem 3.3.5 it now follows that $\hat{S}^\wedge = \bigcup \{\hat{S}_{N(\tilde{e})} \mid e = e^2 \in S\}$, where each $\hat{S}_{N(\tilde{e})}$ is a group and is the set of all semicharacters of $\hat{S}$ with null set $N(\tilde{e})$ and

$$\hat{S}_{N(\tilde{e})} \cong (\hat{S} \setminus N(\tilde{e}))^*$$

with $\hat{S} \setminus N(\tilde{e}) = \bigcup \{S_{N_\beta} \mid N_\beta \subset N_\alpha\}$.
Now let $\tilde{H}(e) = \{\tilde{x} \mid x \in H(e)\}$.

3.3.11. <u>Theorem</u>. Let S be a discrete commutative mob with identity which is the union of groups.
Then S is topologically isomorphic to $\hat{S}\hat{}$ under the natural mapping $x \to \tilde{x}$.

<u>Proof</u>:

Let $x \in H(e_\alpha)$ and $\chi \in \hat{S}$. Then $\chi(x) = 0$ if and only if $\chi(e_\alpha) = 0$ and it follows that $N(\tilde{x}) = N(\tilde{e}_\alpha)$. Thus $\tilde{H}(e_\alpha) \subset \hat{S}_{N(\tilde{e}_\alpha)}$. Now let $\phi \in \hat{S}_{N(\tilde{e}_\alpha)}$ and let $\phi' = \phi \mid S_{N_\alpha}$.
Then ϕ' is a character of S_{N_α} and lemma 3.3.10 implies that $S_{N_\alpha} \cong (H(e_\alpha))^*$ under the mapping $\chi \to \chi'$ with $\chi(x) = \chi'(e_\alpha x)$.
Thus the function $\chi' \to \phi'(\chi)$ is a character of $(H(e_\alpha))^*$. By the Pontrjagin duality theorem there exists an $x \in H(e_\alpha)$ such that $\phi'(\chi) = \tilde{x}(\chi') = \chi'(x) = \chi(x)$.
Hence $\phi' = \tilde{x} \mid S_{N_\alpha}$.
Since S_{N_α} is an ideal of $\hat{S} \setminus N(\tilde{e}_\alpha)$ it follows that $\phi = \tilde{x}$. Thus $\hat{S}_{N(\tilde{e}_\alpha)} = \tilde{H}(e_\alpha)$ and we have $\hat{S}\hat{} = \bigcup \{\tilde{H}(e_\alpha) \mid e_\alpha \in S\}$.

The converse of theorem 3.3.11 also holds.

3.3.12. <u>Theorem</u>. Let S be a discrete commutative mob, such that $\hat{S}$ is a mob and such that $S \cong \hat{S}\hat{}$ under the mapping $x \to \tilde{x}$. Then S is a mob with identity which is the union of groups.

<u>Proof</u>:

Since $\hat{S}\hat{}$ has an identity so does S.
Since the mapping $x \to \tilde{x}$ is one-to-one there existsto each pair $a,b \in S$ $a \neq b$, a $\chi \in \hat{S}$ with $\tilde{a}(\chi) \neq \tilde{b}(\chi)$ i.e. $\chi(a) \neq \chi(b)$.
Let $\chi(a) = r_a e^{t_a}$, $\chi(b) = r_b e^{t_b}$, then $t_a \neq t_b$ or $r_a \neq r_b$. If $t_a \neq t_b$ then the mapping
$$\chi^*(x) = \begin{cases} 0 \text{ if } \chi(x) = 0 \\ e^{t_x} \text{ if } \chi(x) \neq 0, \end{cases}$$
is a semicharacter of S such that $\chi^*(a) \neq \chi^*(b)$ and $|\chi^*(x)| = 0$ or $|\chi^*(x)| = 1$ for all $x \in S$.
If $r_a \neq r_b$ then let ϕ be any character of the multiplicative group of positive real numbers with $\phi(r_a) \neq \phi(r_b)$.

The mapping

$$\chi'(x) = \begin{cases} 0 & \text{if } \chi(x) = 0 \\ \phi(r_x) & \text{if } r_x \neq 0 \end{cases}$$ is a semicharacter of S such that $\chi'(a) \neq \chi'(b)$ and $|\chi'(x)| = 0$ or $|\chi'(x)| = 1$ for all $x \in S$.

Now let x' be the element such that $\tilde{x}' = \bar{\tilde{x}}$ (the complex conjugate of $\tilde{x}$) and let $e = xx'$.

Then if χ is such that $|\chi(x)| = 0$ or 1 for all $x \in S$, we have $\chi(e) = |\chi(x)|^2$ and hence $\chi(e) = 0$ or $\chi(e) = 1$. In both cases we have $\chi(ex) = \chi(e)\chi(x) = \chi(x)$.

Hence $ex = x$ and it follows that e is an idempotent with $x \in H(e)$. Thus S is a union of groups.

Now let S be a compact mob with identity which is the union of groups such that $\hat{S}$ separates points of S. Then $\hat{S}$ is a discrete mob with identity which is a union of groups and $\hat{S}\hat{}$ is a compact mob which also is the union of groups.

Now let $\alpha: x \to \tilde{x}$ be the natural mapping of S into $\hat{S}\hat{}$. Then α is a topological isomorphism of S into $\hat{S}\hat{}$.

α is clearly a homomorphism and α is one-to-one since for all $x \neq y$, $x,y \in S$ there is a $\chi \in \hat{S}$ such that $\chi(x) \neq \chi(y)$ i.e. $\tilde{x} \neq \tilde{y}$.

Next let C be a compact subset of $\hat{S}$, then C is finite, since $\hat{S}$ is discrete, $C = \{\chi_1, \chi_2, \ldots, \chi_n\}$ and let $\varepsilon > 0$. Let V be a neighbourhood of x in S such that $|\chi_i(x) - \chi_i(y)| < \varepsilon$ for all $y \in V$, $i = 1,2,\ldots,n$.

Then $\alpha(V) \subset U(C,\varepsilon,\tilde{x})$ and it follows that α is continuous. Since S is compact and $\hat{S}\hat{}$ a Hausdorff space, we have that α is topological.

3.3.13. <u>Lemma</u>. Let $e_\beta = e_\beta^2 \in S$ and let $E_\beta = \{e_\alpha \mid e_\alpha \leq e_\beta,\ J_o(S \setminus \{e_\alpha\})$ closed$\}$. Then $e_\beta \in \bar{E}_\beta$.

<u>Proof</u>:

Since the minimal idempotent of S belongs to E_β, E_β is nonvoid. Since S and $\alpha(S)$ are homeomorphic each neighbourhood U of e_β is of the form

$$U = \{x \mid |\chi_i(x) - \chi_i(e_\beta)| < \varepsilon \ , \quad i = 1,2,\ldots,n \ , \ \chi_i \in \hat{S}\} \ .$$

Let $\chi_i(e_\beta) = 1$ for $1 \leq i \leq k$ and $\chi_i(e_\beta) = 0$ for $k < i \leq n$. Let

$\chi_1\chi_2\ldots\chi_k = \chi$. Then $N(\chi)$ is a clopen prime ideal of S, hence $N(\chi) = J_o(S\setminus\{e_\alpha\})$ for some $e_\alpha \in S$. Furthermore we have $\chi(e_\beta) = 1$ and hence $N(\chi) \subset J_o(S\setminus\{e_\beta\})$ i.e. $e_\alpha \leq e_\beta$.
Since $\chi_i(e_\alpha) = 1$ $1 \leq i \leq k$ and $\chi_i(e_\alpha) = \chi_i(e_\alpha e_\beta) = 0$ $k < i \leq n$ we have $e_\alpha \in U$.

3.3.14. Lemma. Every idempotent of $\hat{S}^\wedge$ has the form $\tilde{e}_\alpha$, $e_\alpha = e_\alpha^2 \in S$.
Proof:
Let η_α be an idempotent such that $J_o(\hat{S}^\wedge\setminus\{\eta_\alpha\})$ is closed. Then it follows from 3.3.9 and the remark to 3.3.9 that
$J_o(\hat{S}^\wedge\setminus\{\eta_\alpha\}) = \bigcup\{\hat{S}_{\mathfrak{A}_\beta} \mid \eta_\beta\eta_\alpha \neq \eta_\alpha,\ \eta_\beta = \eta_\beta^2 \in \hat{S}^\wedge\}$, where η_β is the idempotent contained in $\hat{S}_{\mathfrak{A}_\beta}$.

$$\mathfrak{A}_\alpha = \bigcup\{S_{N_\beta} \mid \varepsilon_\beta\varepsilon_\alpha \neq \varepsilon_\alpha,\ \varepsilon_\beta = \varepsilon_\beta^2 \in \hat{S}\} = \bigcup\{S_{N_\beta} \mid N_\beta \not\subset N_\alpha\}.$$

Hence η_α is the characteristic function of

$$\hat{S}\setminus\mathfrak{A}_\alpha = \bigcup\{S_{N_\beta} \mid N_\beta \subset N_\alpha\}.$$

Since N_α is a clopen prime ideal of S we have $N_\alpha = J_o(S\setminus\{e_\alpha\})$. Hence $\eta_\alpha(\chi) = 1$ if and only if $\chi \in S_{N_\beta}$ with $N_\beta \subset N_\alpha$, i.e. if and only if $N(\chi) \subset J_o(S\setminus\{e_\alpha\})$. Thus $\eta_\alpha = \tilde{e}_\alpha^\beta$ and $\eta_\alpha \in \alpha(S)$.
From lemma 3.3.13, applied to the mob $\hat{S}^\wedge$ ($\hat{S}^\wedge$ is a compact mob with identity which is the union of groups and whose semicharacters separate points) it follows that each idempotent of $\hat{S}^\wedge$ is contained in the closure of $\alpha(S)$. Since $\alpha(S)$ is closed, the lemma follows.

3.3.15. Theorem. Let S be a compact mob with identity which is the union of groups, such that $\hat{S}$ separates points. Then S and $\hat{S}^\wedge$ are topologically isomorphic under the mapping $x \to \tilde{x}$.

Proof:
Since each idempotent of $\hat{S}^\wedge$ is of the form $\tilde{e}_\alpha$, we have
$\hat{S}^\wedge = \bigcup\{\hat{S}_{N(\tilde{e})} \mid e = e^2 \in S\}$.
Now let $\tilde{x} \in \tilde{H}(e) = \{\tilde{x} \mid x \in H(e)\}$. Then $\chi(x) = 0$ if and only if $\chi(e) = 0$, thus $N(\tilde{e}) = N(\tilde{x})$ and hence $\tilde{x} \in \hat{S}_{N(\tilde{e})}$. Next let $\phi \in \hat{S}_{N(\tilde{e})}$ and suppose that $J_o(\hat{S}^\wedge\setminus\{\tilde{e}\})$ is closed. Then ϕ is a character of

$$\hat{S} \setminus \mathfrak{a} = \bigcup \{ S_{N_\beta} \mid N_\beta \subset J_o(S \setminus \{e\}) = N \}.$$

Furthermore $\phi' = \phi | S_N$ is a character of S_N and by 3.3.5 we have $S_N \cong (S \setminus N)^*$.

Since H(e) is an ideal of $S \setminus N$ it follows from 3.3.3 that $(H(e))^* \cong (S \setminus N)^*$ under the mapping $\chi \to \chi'$ with $\chi(x) = \chi'(ex)$.

Thus the function $\chi' \to \phi'(\chi)$ is a character of $(H(e))^*$. Hence there exists an $x \in H(e)$ such that $\phi'(\chi) = \chi(x)$.

Hence $\phi' = \tilde{x} | S_N$ and by theorem 3.3.3 we have $\phi = \tilde{x}$ and $\hat{S}_{N(\tilde{e})} \subset \tilde{H}(e)$.

Finally let $\phi \in \hat{S}_{N(\tilde{e})}$, where $\tilde{e}$ is an arbitrary idempotent of $\hat{S}^{\wedge}$. Then by lemma 3.3.13 there is a net of idempotents $\tilde{e}_\alpha$, such that $J_o(\hat{S}^{\wedge} \setminus \{\tilde{e}_\alpha\})$ is closed and $\lim \tilde{e}_\alpha = \tilde{e}$. Moreover $\tilde{e}_\alpha \tilde{e} = \tilde{e}_\alpha$.

Then $\phi = \phi \tilde{e} = \lim \phi \tilde{e}_\alpha$ and since $\phi \tilde{e}_\alpha \in \hat{S}_{N(\tilde{e}_\alpha)} \subset \tilde{H}(e_\alpha) \subset \alpha(S)$ we have $\phi \in \alpha(S)$.

Since all groups $\hat{S}_{N(\tilde{e})}$ are disjoint and $\tilde{H}(e) \subset \hat{S}_{N(\tilde{e})}$ we have $\phi \in \tilde{H}(e)$.

If S is a compact mob with identity which is the union of groups, then the statement that $\hat{S}$ separates points is not necessarily true. If for instance $S \cong J_3$, then $\hat{S}$ contains only the unit character.

3.4. Notes

The study of monothetic mobs has been initiated by several authors. The results contained in section 1 are due to Numakura [2], theorem 3.1.1, Koch [2], theorem 3.1.2, 3.1.3, 3.1.4, 3.1.9 and Hewitt [1], theorem 3.1.5, 3.1.6, 3.1.7.

The structure theory for commutative compact mobs contained in section 2 is due largely to Schwarz [4], [6]. Theorems 3.2.12 and 3.2.13 were proved by Gelbaum, Kalisch and Olmsted [1].

In [2] Hewitt and Zuckerman proved theorem 3.3.11 for finite commutative mobs. The proof given here is based on Austen [1] who also proved theorems 3.3.12 - 3.3.15.

Semicharacters have also been studied by Schwarz [1], [5], [6]. He uses the term character and includes the zero character in his considerations.

IV. MEASURES ON COMPACT SEMIGROUPS

4.1. Invariant measures and means

Definitions: Let S be a compact mob. By a measure μ on S we shall mean a σ-additive, non-negative, real-valued regular set function defined on the Borel subsets of S, such that $\mu(S) = 1$.
The measure μ will be called right invariant if for every Borel set $B \subset S$ and $a \in S$ for which Ba is also a Borel set $\mu(Ba) = \mu(B)$ holds.
We will call the measure μ right subinvariant if for every Borel set $B \subset S$ and $a \in S$ for which Ba is also a Borel set, $\mu(Ba) \leq \mu(B)$ holds.

The property B a Borel set of S and a in S imply Ba a Borel set of S may fail in a semigroup. Let for instance $S \subset E_2$ be the set of all points of the closed square $0 \leq x \leq 1$, $0 \leq y \leq 1$,

$$S = \{(x,y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\},$$ with the relative topology.

Define a multiplication in S by

$$(x_1,y_1)\cdot(x_2,y_2) = (x_1,0).$$

The multiplication is continuous and associative, hence S is a compact mob.
It is known that in S there is a Borel subset B such that its projection $\pi(B)$ on the x-axis is not a Borel set (see C. Kuratowski. Topologie, p.368).
For any $(x,y) \in S$ we have $B(x,y) = \pi(B)$ and hence B is a Borel set, while $B(x,y)$ is not a Borel set.

For each element a of a compact group S left and right translations by a are homeomorphisms of S. Hence if B is a Borel set of S and $a \in S$, then Ba is a Borel set of S. A measure which is right invariant is right subinvariant, but the converse is not generally true. However, these concepts coincide in the case of compact groups.

For let B be a Borel set, then

$$\mu(B) \geq \mu(Ba) \geq \mu(Baa^{-1}) = \mu(B).$$

Moreover in this case such a right invariant measure is known to exist, namely the right Haar measure on the group.

4.1.1. Lemma. If a compact mob S has a right invariant measure μ , then S contains exactly one minimal left ideal, its kernel K, and $\mu(S \setminus K) = 0$.

Proof:

Let L be a minimal left ideal of S. Then L = Sx with $x \in L$ and hence $\mu(S) = \mu(Sx) = \mu(L)$. Thus $\mu(L) = 1$ and $\mu(S \setminus L) = 0$. Since this holds for any minimal left ideal and since no two minimal left ideals intersect, it follows that S contains only one minimal left ideal which must be the kernel of S.

Corollary. If a compact mob S has a right and a left invariant measure, then K is a group.

The converse of lemma 4.1.1 is not true. In fact, if S is a compact mob with zero, with $|S| \geq 2$, then S has no right nor left invariant measure. For in this case {0} is the only minimal left and right ideal. Hence if μ is a right invariant measure on S, we would have $\mu(\{0\}) = 1$. On the other hand we have for all $a \in S$, $a0 = 0$ and thus $1 = \mu(\{0\}) = \mu(\{a\}0) \leq \mu(\{a\})$. This contradicts the fact that $\mu(S \setminus \{0\}) = 0$.

Now let C denote the set of all $x \in S$ such that $\mu(U) \neq 0$ for each open set U about x. C is called the support of μ.
If $x \notin C$, then there is an open set U with $x \in U$, $\mu(U) = 0$. Hence $U \cap C = \emptyset$ and it follows that C is closed.

4.1.2. Lemma. If a compact mob S has a right invariant measure μ , then C is a closed right ideal of S with $C \subset K$, $\mu(C) = 1$.

Proof:

Since K is compact, $S \setminus K$ is open. Furthermore $\mu(S \setminus K) = 0$, according to lemma 4.1.1 and it follows that $C \subset K$.

Now let U be an open set such that $C \subset U$. Then $S \setminus U$ is compact and can be covered by a finite number of open sets V_i, $i = 1,\ldots,n$, with $\mu(V_i) = 0$. Hence $\mu(S \setminus U) \leq \mu(V_1) + \ldots + \mu(V_n) = 0$ and it follows that $\mu(U) = 1$. The regularity of μ implies that $\mu(C) = 1$.
We now prove that C is a right ideal. Since Ca is compact for all $a \in S$, we have $\mu(Ca) = \mu(C) = 1$. If $C \not\subset Ca$, then there is an $x \in C$, $x \notin Ca$ and hence a neighbourhood U of x with $U \cap Ca = \emptyset$. Since $\mu(U) > 0$ it would follow that $\mu(Ca) < 1$. Thus we have $C \subset Ca$ and by 1.4.3 $C = Ca$.

4.1.3. Theorem. If a compact mob S has a right invariant measure μ, then the support C of μ is the union of maximal subgroups H(e) with $e \in K$.

Proof:
Since S contains exactly one minimal left ideal, each minimal right ideal is a maximal subgroup and $K = \bigcup\{H(e) | e \in E \cap K\}$.
Since a group contains no proper right ideals we have either $C \cap H(e) = \emptyset$ or $H(e) \subset C$ and the theorem follows.

If S is a compact mob such that $(S \setminus K)S \not\supset K$ and such that $(S \setminus K)a$ is open for each $a \in S$, then a converse of lemma 4.1.1 is possible.

4.1.4. Theorem. Let S be a compact mob such that $(S \setminus K)a$ is open for each $a \in S$.
A necessary and sufficient condition that S has a right invariant measure is that K is a minimal left ideal of S and $K \not\subset (S \setminus K)S$.

Proof:
Let K be a minimal left ideal of S such that $K \not\subset (S \setminus K)S$. Then $K = \bigcup\{H(e) | e \in E \cap K\}$ and since $(S \setminus K)S$ is a right ideal of S we have for each $H(e) \subset K$ either $H(e) \subset (S \setminus K)S$ or $H(e) \cap (S \setminus K)S = \emptyset$.
Hence there is an $H(e) = H$ such that $H(e) \cap (S \setminus K)S = \emptyset$.
Now let ν be the normed Haar measure on H and let $\mu(B) = \nu(B \cap H)$ for each Borel set B of S. It is obvious that μ is a measure on S. We now prove that μ is right invariant.

Let B be a Borel set of S and $a \in S$. Then

$$Ba = (B \cap H)a \cup (B \cap S \setminus K)a \cup (B \cap K \setminus H)a.$$

Furthermore $(B \cap S \setminus K)a \subset (S \setminus K)a \subset (S \setminus K)S \subset S \setminus H$ and

$$(B \cap K \setminus H)a \subset (K \setminus H)a \subset K \setminus H.$$

Hence $Ba \cap H = (B \cap H)a \cap H = (B \cap H)a$ and we conclude that $\mu(Ba) = \nu(Ba \cap H) = \nu((B \cap H)a) = \nu((B \cap H)ea) = \nu(B \cap H) = \mu(B)$.
Now suppose on the other hand that S is a compact mob which has a right invariant measure μ. Then K is a minimal left ideal by lemma 4.1.1 and $\mu(S \setminus K) = 0$. If $K \subset (S \setminus K)S$, then the set $\{(S \setminus K)a\}_{a \in S}$ constitutes an open covering of the compact set K and we can find a finite subcovering $(S \setminus K)a_1, \ldots, (S \setminus K)a_n$. Since μ is right invariant we have

$$\mu(K) \leq \mu((S \setminus K)a_1) + \ldots + \mu((S \setminus K)a_n) = 0.$$

This contradiction completes the proof of the theorem.

It follows from 4.1.4 that a sufficient condition that a compact mob S has a right invariant measure is that K is a minimal left ideal and $K \not\subset (S \setminus K)S$. This condition however is not necessary.
Let for instance G be the additive group of real numbers mod 1 and let e be a symbol not representing any element of G. Extend the multiplication in G to one in $S = G \cup \{e\}$ by defining $ee = e$ and $eg = ge = g$ for every g in G. Now let S be topologized so that e is an isolated point and G has its original topology.
Then S is a compact mob with minimal ideal $K = G$ and $(S \setminus K)S = eS = S \supset K$. Let ν be the Haar measure defined on G and let μ be the measure on S defined by $\mu(B) = \nu(B \cap G)$ for each Borel set $B \subset S$.
Then μ is a right invariant measure.

Definitions. Let S be a compact mob and C(S) the set of all real valued continuous functions on S. For a fixed element $a \in S$ and $f \in C(S)$ let f_a be the function on S such that $f_a(x) = f(xa)$ for all $x \in S$.
Then f_a is called the <u>right translate</u> of f by a.
The <u>left translate</u> ${}_af$ is the function defined by ${}_af(x) = f(ax)$.

A <u>mean</u> M on C(S) is a real linear functional on C(S) having the property that

i) $M(f) \geq 0$ whenever $f \in C(S)$ and $f(x) \geq 0$ for all $x \in S$.

ii) $M(f) = 1$ if $f(x) = 1$ for all $x \in S$.

A <u>right (left) invariant mean</u> M on C(S) is a mean such that $M(f_a) =$ $= M(f)$ $(M({}_af) = M(f))$ for all $f \in C(S)$, $a \in S$.

4.1.5. <u>Theorem</u>. Let S be a compact mob. Then there is a right invariant mean M on C(S) if and only if the kernel K of S is a minimal left ideal.

<u>Proof</u>:

Suppose that L_1 and L_2 are two different minimal ideals of S. Then $L_1 \cap L_2 = \emptyset$ and there is an $f \in C(S)$ such that

$$f(x) = \begin{cases} 0 & \text{if } x \in L_1 \\ 1 & \text{if } x \in L_2. \end{cases}$$

If M is a right invariant mean on C(S), then we would have

$$M(f) = M(f_a) = \begin{cases} 0 & \text{if } a \in L_1 \\ 1 & \text{if } a \in L_2. \end{cases}$$

This contradiction proves the "only if" part of the theorem. Now let S be a compact mob, such that K is a minimal left ideal. Then $K = \bigcup \{H(e) \mid e \in E \cap K\}$, where each maximal subgroup H(e) is a minimal right ideal. Let I be the normed Haar integral on one of these groups, say $H = H(e_1)$, and let $M(f) = I(f')$, where $f' = f|H$. It is clear that M is a mean on C(S). We now prove that M is right invariant.

Let $x \in H$ and $a \in S$, then $xa = xe_1a \in H$, where $e_1a \in H$ and hence $f(xa) = f(xea)$ for all $x \in H$, $a \in S$; i.e. $f'_a = f'_{ea}$. Furthermore we have $I(f'_h) = I(f')$ for all $h \in H$ and we conclude that

$$M(f_a) = I(f'_a) = I(f'_{ea}) = I(f') = M(f).$$

4.1.6. <u>Theorem</u>. Let S be a compact mob and let M be a mean on C(S) such that $M(f_a) \leq M(f)$ for all $a \in S$ and $f \in C(S)$. Then M is right invariant.

Proof:

By the representation of linear functionals as integrals there is a regular Borel measure μ on S (as a space) such that $M(f) = \int_S f(x)d\mu$. Let L be a minimal left ideal of S and suppose $\mu(L) < 1$. Then by the regularity of μ we infer the existence of a compact subset $F \subset S$, $F \cap L = \emptyset$, with $\mu(F) > 0$. Now take $f \in C(S)$ such that $0 \leq f(x) \leq 1$; $f(x) = 1$ for $x \in L$ and $f(x) = 0$ for $x \in F$. Then we have for $a \in L$ $1 = M(f_a) \leq M(f) < 1$.

Hence we conclude that $\mu(L) = 1$ and since this holds for all minimal left ideals it follows that S contains exactly one minimal left ideal. Furthermore we have $M(f) = \int_L f(x)d\mu$.

Next let e be an idempotent of S contained in L. Then $L = Le = Se$ and e is a right identity of L. Moreover we have that for each $a \in S$, $ea \in L$. Since L is the union of maximal subgroups $H(e_\alpha)$, there exists an e_α such that $ea \in H(e_\alpha)$ and an element a^{-1} with $eaa^{-1} = e_\alpha$.

If we put $f_a = g$, then we have for all $x \in L$

$$g_{a^{-1}}(x) = g(xa^{-1}) = f_a(xa^{-1}) = f(xa^{-1}a) = f(xa^{-1}ea) = f(xe_\alpha) = f(x).$$

Hence $M(f) = \int_L f(x)d\mu = \int_L g_{a^{-1}}(x)d\mu \leq \int_L g(x)d\mu = \int_L f_a(x)d\mu = M(f_a)$.

Thus $M(f_a) = M(f)$ and the theorem is proved.

In the same way we can prove that M is right invariant if $M(f_a) \geq M(f)$.

From the proof of theorem 4.1.5 it follows that a right invariant mean on C(S) is not unique if the kernel K of S contains more than one minimal right ideal. The next theorem however states that a two-sided invariant mean on a compact mob is unique.

4.1.7. Theorem. Let S be a compact mob. Then the following conditions are equivalent.

1) K is a group.

2) S has a two-sided invariant mean.

3) S has a right and a left invariant mean.

Furthermore if M is a two-sided invariant mean, then $M(f) = \int_K f'd\mu$, where $\int_K f'd\mu$ is the Haar integral for the compact group K.

Proof:

1) → 2). From theorem 4.1.5 it follows that the Haar integral for K can be extended to a two-sided invariant mean on C(S).

2) → 3). Trivial.

3) → 1). Theorem 4.1.5.

Next let M be an invariant mean on C(S), then it follows that $M(f) = \int_S f d\mu$ where μ is a regular normed Borel measure and $\mu(K) = 1$. Hence $M(f) = \int_K f d\mu$, and since $\int_K f(xa) d\mu = \int_K f(x) d\mu$ it follows that $\int_K d\mu$ is the Haar integral for K.

Let B be a subset of S and a ε S. By B_a we will denote the set of all x ε S such that xa ε B.

$$B_a = \{x \mid x \in S,\ xa \in B\}$$

Since B_a is closed (open) if B is closed (open) and since $B_a \cap C_a = (B \cap C)_a$, $B_a \cup C_a = (B \cup C)_a$ it follows that B_a is a Borel set for each Borel set B of S and a ε S.

4.1.8. Theorem. Let S be a compact mob and M a right invariant mean on C(S). Then $M(f) = \int_S f(x) d\mu$, with $\mu(B) = \mu(B_a)$ for all Borel sets $B \subset S$ and $a \in S$.

Proof:

Since M can be represented as an integral we have $M(f) = \int f(x) d\mu$, where μ is a regular Borel measure on S.

Now let F be any closed set of S. Then given $\varepsilon > 0$, there is an open set V, $F \subset C$, such that $\mu(V) \leq \mu(F) + \varepsilon$. Let f ε C(S) be such that $0 \leq f(x) \leq 1$ for all x ε S and $f(x) = 1$, $x \in F$; $f(x) = 0$, $x \notin V$. Then we have

$$\mu(F_a) \leq \int f(xa) d\mu = \int f(x) d\mu \leq \mu(V) \leq \mu(F) + \varepsilon.$$

Since this holds for all ε we have $\mu(F_a) \leq \mu(F)$. Moreover we have $\mu(K) = 1$, $\mu(S \setminus K) = 0$. Hence $\mu(F) = \mu(F \cap K)$. Furthermore we have for all closed sets $F^* \subset K$ and a ε S an $a^{-1} \in K$ such that $F^* \subset F^*_{a^{-1}a}$. Hence $\mu(F^*) \leq \mu(F^*_{a^{-1}a}) \leq \mu(F^*_a) \leq \mu(F^*)$. Thus we have for all closed sets $F \subset S$

$\mu(F) = \mu(F \cap K) = \mu((F \cap K)_a) \leq \mu(F_a)$. This together with $\mu(F_a) \leq \mu(F)$ implies $\mu(F_a) = \mu(F)$. Since this holds for all closed sets it also holds for every Borel set and the theorem is proved.

4.2. Subinvariant measures on simple mobs

Let S be a compact simple mob. Then K = S and S clearly satisfies the condition of theorem 4.1.4. Hence it follows that a compact simple mob S has a right invariant measure if and only if S is left simple.
In this section we will establish necessary and sufficient conditions that a simple mob possess a right subinvariant measure.
It follows from theorem 1.3.10 that each compact simple mob S is isomorphic with the mob $(Se \cap E) \times H(e) \times (eS \cap E)$, with $e \in E \cap K$ and multiplication defined by

$$(x_1, h_1, y_1)(x_2, h_2, y_2) = (x_1, h_1 y_1 x_2 h_2, y_2).$$

4.2.1. Lemma. Let S be a compact simple mob. If S contains a finite number of minimal left ideals, then S has a right subinvariant measure.

Proof:
Let $S = (Se_1 \cap E) \times H(e_1) \times (e_1 S \cap E)$ with $|e_1 S \cap E| = n$ and let L be the minimal left ideal $(Se_1 \cap E) \times H(e_1) \times e_1$. Then S is isomorphic with the mob $S' = L \times (e_1 S \cap E)$ with multiplication defined by $(l_1, e)(l_2, e^*) = (l_1 e l_2, e^*)$.
We now identify S with S'.
Let μ_1 be a right invariant measure on L and μ_2 the measure on $(e_1 S \cap E)$ such that each point has measure $\frac{1}{n}$. Let $\mu = \mu_1 \times \mu_2$ be the product measure on S.
All that remains to be shown is that μ is right subinvariant. Let B be any Borel set of S. Then

$$B = (B_1 \times \{e_1\}) \cup (B_2 \times \{e_2\}) \cup \dots \cup (B_n \times \{e_n\}),$$

where $B_i \subset L$, $i = 1, \dots, n$ and $\{e_i\}_{i=1}^{n} = e_1 S \cap E$.
Now let $a = (l, e_j)$ be any element of S, then

$$Ba = (B_1 e_1 l \times \{e_j\}) \cup (B_2 e_2 l \times \{e_j\}) \cup \dots \cup (B_n e_n l \times \{e_j\}).$$

Thus $\mu(Ba) \leq \{\mu_1(B_1e_1 1)+\ldots+ \mu_1(B_ne_n 1)\}\frac{1}{n} = \{\mu_1(B_1)+\ldots+ \mu_1(B_n)\}\frac{1}{n}$ $= \mu(B)$.

4.2.2. Theorem. Let S be a compact simple mob $S = (Se_1 \cap E) \times H(e_1) \times (e_1S \cap E)$ such that $|e_1S \cap E| = n$. Then S has a right subinvariant measure and each such measure μ is a product measure $\mu = \mu_1 \times \mu_2 \times \mu_3$, where μ_1 is any regular normed Borel measure on $(Se_1 \cap E)$, μ_2 is Haar measure on $H(e_1)$ and μ_3 is the measure on $(e_1S \cap E)$ such that each point has measure $\frac{1}{n}$.

Proof:

From lemma 4.2.1 it follows that S has a right subinvariant measure μ and that we can identify S with the mob $L \times (e_1S \cap E)$, where $L = (Se_1 \cap E) \times H(e_1) \times \{e_1\}$.

Define ν on L by

$$\nu(B) = \mu(B \times (e_1S \cap E))$$

for each Borel set B of L and define μ_3 on $(e_1S \cap E)$ by

$$\mu_3(A) = \mu(L \times A).$$

Then it is clear that both μ_3 and ν are regular Borel measures. Furthermore $\mu_3(\{e_j\}) = \mu(L \times \{e_j\}) \geq \mu((L \times \{e_j\})(1,e_i)) = \mu(L \times \{e_i\}) = \mu_3(\{e_i\})$. Since this holds for all $j = 1,\ldots,n$ and $i = 1,\ldots,n$ it follows that $\mu_3(\{e_i\}) = \frac{1}{n}$, $i = 1,\ldots,n$.

Moreover for each $e_j \in (e_1S \cap E)$ there is an $1_j \in L$ such that $e_j 1_j = e_1$.

Hence

$$\mu((B \times \{e_j\})(1_j,e_i)) = \mu(B \times \{e_i\}) \leq \mu(B \times \{e_j\}),$$

and we conclude that $\mu(B \times \{e_1\}) = \mu(B \times \{e_j\})$ and so

$$\nu(B) = n.\,\mu(B \times \{e_1\}).$$

Now let $1 \in L$, then

$$\nu(B1) = n.\,\mu(B1 \times \{e_1\}) = n.\mu((B \times \{e_1\})(1,e_1)) \leq n.\mu(B \times \{e_1\}) = \nu(B).$$

Since there exist to each $1 \in L$ an 1^{-1} with $11^{-1} = e_1 \in (Se_1 \cap E)$ we have $\nu(B1) \geq \nu(B1.1^{-1}) = \nu(B)$.

Thus ν is a right invariant measure on L.

Finally μ clearly is the product measure $\nu \times \mu_3$, since for each $B \subset L$ and $A \subset eS \cap E$, $A = \{e_{j_i}\}_{i=1}^{k}$ we have $\mu(B \times A) = \mu(B \times \{e_{j_1}\}) + \ldots + \mu(B \times \{e_{j_k}\}) = \frac{k}{n} \nu(B) = \nu(B)\mu_3(A)$.
We now prove that the measure ν on L also is a product measure.
Since $L = (Se_1 \cap E) \times H(e_1)$ we have for each $l \in L$, $l = (e,h)$.
Define μ_1 on $Se_1 \cap E$ by

$$\mu_1(B) = \nu(B \times H(e_1))$$

and μ_2 on $H(e_1)$ by

$$\mu_2(A) = \nu((Se_1 \cap E) \times A),$$

where B and A are respectively Borel subsets of $(Se_1 \cap E)$ and $H(e_1)$.
It is obvious that μ_1 and μ_2 are regular Borel measures. Furthermore $\mu_2(Ah) = \nu((Se_1 \cap E) \times Ah) = \nu(((Se_1 \cap E) \times A)(e_1,h)) = \nu((Se_1 \cap E) \times A) = \mu_2(A)$. Hence since $\mu_2(H) = \nu(L) = 1$, μ_2 is actually the Haar measure on H.
Now let $B \subset Se_1 \cap E$ and define μ_B on H by $\mu_B(A) = \nu(B \times A)$. In a similar fashion it can be shown that μ_B is a regular Borel measure such that $\mu_B(Ah) = \mu_B(A)$ for all $h \in H$. Hence μ_B is a multiple of the Haar measure μ_2 and since $\mu_B(H) = \nu(B \times H)$ we have
$\nu(B \times A) = \mu_B(A) = \nu(B \times H)\ \mu_2(A) = \nu(B \times H)\ \nu((Se_1 \cap E) \times A)$.
We now define the product measure $\nu^* = \mu_1 \times \mu_2$ and we show that $\nu = \nu^*$.
Let B be a Borel set of $(Se_1 \cap E)$ and A a Borel set of $H(e_1)$. Then
$\nu^*(B \times A) = \mu_1(B) . \mu_2(A) = \nu(B \times H)\nu((Se_1 \cap E) \times A) = \nu(B \times A)$.

From theorem 4.2.2 it follows that right subinvariant measures are extremely non-unique. The measure μ is determined by the measure μ_1 on $(Se_1 \cap E)$. Since a regular normed Borel measure on a compact space is unique if and only if the space consists of a single point it follows that μ is unique if and only if each minimal left ideal is a group.

4.2.3. <u>Theorem</u>. Let S be a compact simple mob $S = (Se_1 \cap E) \times H(e_1) \times (e_1 S \cap E)$ such that S contains an infinite number of minimal left ideals. Then S has a right subinvariant measure if and only if the space $e_1 S \cap E$ has a normed regular Borel measure such that each point has measure zero.

Proof:

Let μ_1 be any normed regular Borel measure on $(e_1 S \cap E)$ such that each point has measure zero and let μ_2 be any regular normed Borel measure on $L = H(e_1) \times (Se_1 \cap E)$.

Then $\mu_2 \times \mu_1$ is a right subinvariant measure on S. For if B is a Borel set of S and $a \in S$, $a \in L \times \{e^*\}$, then $Ba \subset L \times \{e^*\}$ and hence if Ba is a Borel set we have $\mu(Ba) \leq \mu(L \times \{e^*\}) = \mu_2(L)\mu_1(\{e^*\}) = 0$.

From this we conclude that $\mu(Ba) \leq \mu(B)$.

Next suppose on the other hand that μ is a right subinvariant measure on S. Define μ_1 on $e_1 S \cap E$ by

$$\mu_1(B) = \mu(L \times B).$$

Then in a similar fashion as in the proof of theorem 4.2.2 it can be shown that $\mu_1(\{e\}) = \mu_1(\{e^*\})$ for all $e, e^* \in e_1 S \cap E$. Hence since $|e_1 S \cap E|$ is infinite and $\mu_1(e_1 S \cap E) = 1$ it follows that $\mu_1(\{e\}) = 0$ for all $e \in e_1 S \cap E$.

Example.

Let $S \subset E_2$ be the set $S = \{(x,y) | x = 0, \frac{1}{2}, \frac{1}{2^2}, \ldots; y = 0, \frac{1}{2}, \frac{1}{2^2}, \ldots\}$ with the relative topology.

Define a multiplication on S by

$$(x_1, y_1)(x_2, y_2) = (x_1, y_2).$$

The multiplication is continuous and associative, hence S is a compact mob. Since $S(x_1, y_1)S = S$ it follows that S is simple. Furthermore each set $\{(x,y) | x = \frac{1}{2^k};\ y = 0, \frac{1}{2} \ldots\}$ is a minimal left ideal and each set $\{(x,y) | x = 0, \frac{1}{2}, \ldots;\ y = \frac{1}{2^k}\}$ is a minimal right ideal. Furthermore each element of S is idempotent. Each minimal left ideal is a countable compact Hausdorff space and hence has no normed regular Borel measure such that each point has measure zero. Since this also holds for the minimal right ideals S has no right nor left subinvariant measure.

We can now summarize the preceding theorems.

4.2.4. Theorem. Let S be a compact simple mob with $S = (Se_1 \cap E) \times H(e_1) \times (e_1 S \cap E)$. Then S has a right subinvariant measure if and only if the compact space $e_1 S \cap E$ has a regular normed Borel measure μ such that $\mu(\{e\}) = \mu(\{e'\})$ for all points $e, e' \in e_1 S \cap E$.

Now let S be a compact 0-simple mob, then by theorem 2.3.9 S is isomorphic with a mob $S^* = Y_1 \times H(e) \times Y_2 \cup \{0\}$, where Y_1 is a compact set contained in a 0-minimal left ideal and Y_2 is a compact set contained in a 0-minimal right ideal. The multiplication in S^* is defined by

$$(y_1,h,y_2)(y_1^*,h^*,y_2^*) = \begin{cases} (y_1, hy_2y_1^*h^*, y_2^*) & \text{if } y_2y_1^* \neq 0 \\ 0 & \text{if } y_2y_1^* = 0. \end{cases}$$

and $s^*0 = 0s^* = 0$.

Now let μ be any right subinvariant measure on S. Then $\mu(\{0\}) = \mu(\{s\}0) \leq \mu(\{s\})$ for all $s \in S$ and hence $\mu(\{0\}) = 0$ if S is an infinite mob.

It is clear that each finite mob has a right subinvariant measure μ.
Let for instance μ be the measure defined by $\mu(\{s\}) = 1/n$ for all $s \in S$ if $|S| = n$.

4.2.5. Theorem. Let S be a compact 0-simple mob $S = Y_1 \times H(e) \times Y_2 \cup \{0\}$. Then if $|Y_2|$ is infinite S has a right subinvariant measure if and only if there exists a regular normed Borel measure μ_2 on Y_2 such that $\mu_2(\{y_2\}) = 0$ for all $y_2 \in Y_2$.

Proof:

Let μ be the Haar measure on the compact group H(e), μ_1 any normed regular Borel measure on Y_1 and let $\nu^* = \mu_1 \times \mu \times \mu_2$ be the product measure on $Y_1 \times H(e) \times Y_2$.

Furthermore let ν be the measure on S defined by $\nu(B) = \nu^*(B \setminus \{0\})$ for all Borel sets B of S.

ν is right subinvariant since

$\nu(B0) = \nu(\{0\}) = \nu^*(\emptyset) = 0 \leq \nu(B)$ and

$\nu(B(y_1,h,y_2)) \leq \nu(Y_1 \times H(e) \times \{y_2\} \cup \{0\}) = \nu^*(Y_1 \times H(e) \times \{y_2\}) =$
$= \mu_1(Y_1) \times \mu(H) \times \mu_2(\{y_2\}) = 0 \leq \nu(B)$.

If on the other hand ν is a right subinvariant measure on S, then ν_2 defined by $\nu_2(A) = \nu(Y_1 \times H(e) \times A)$ for all Borel sets $A \subset Y_2$ is a normed regular Borel measure on Y_2. $\nu_2(\{y_2\}) = \nu_2(\{y_2^*\})$ since there exists for each $y_2 \in Y_2$ a $y_1^* \in Y_1$ such that $y_2y_1^* \neq 0$.

Hence $\nu_2(\{y_2\}) = \nu(Y_1 \times H(e) \times \{y_2\}) \geq \nu((Y_1 \times H(e) \times \{y_2\})(y_1^*, e, y_2^*) =$
$= \nu(Y_1 \times H(e) \times \{y_2^*\}) = \nu_2(\{y_2^*\})$.

4.2.6. Theorem. Let S be an infinite compact 0-simple mob $S = Y_1 \times H(e) \times Y_2 \cup \{0\}$ such that $|Y_2| = n$. Then S has a right subinvariant measure and each such measure μ is such that $\mu(\{0\}) = 0$. Furthermore μ is a product measure on $Y_1 \times H(e) \times Y_2$, $\mu = \mu_1 \times \nu \times \mu_2$, where μ_1 is any normed regular Borel measure on Y_1, ν is the Haar measure on H(e) and $\mu_2(\{y_2\}) = \frac{1}{n}$ for all $y_2 \in Y_2$.

Proof:

Let μ be a right subinvariant measure. Define μ_1, ν and μ_2 respectively by

$$\mu_1(B) = \mu(B \times H(e) \times Y_2), \quad B \subset Y_1$$
$$\nu(A) = \mu(Y_1 \times A \times Y_2), \quad A \subset H(e)$$
$$\mu_2(C) = \mu(Y_1 \times H(e) \times C), \quad C \subset Y_2.$$

Then it follows in a similar fashion as in theorem 4.2.5 that $\mu_2(\{y_2\}) = \frac{1}{n}$ for all $y_2 \in Y_2$.
Since there exists for each $y_2 \in Y_2$ a $y_1^* \in Y_1$ and an $h_2 \in H(e)$ such that $y_2 y_1^* h_2 = e$ we have
$\mu(B \times H(e) \times \{y_2\}) \geq \mu(B \times H(e) \times \{y_2\})(y_1^*, h, y_2^*) = \mu(B \times H(e) \times \{y_2^*\})$ and analogously $\mu(Y_1 \times A \times \{y_2\}) \geq \mu(Y_1 \times A \times \{y_2^*\})$.
Hence $\mu(B \times H(e) \times \{y_2\}) = \frac{1}{n}\mu_1(B)$ and $\mu(Y_1 \times A \times \{y_2\}) = \frac{1}{n}\nu(A)$.
ν is the Haar measure on H(e) since
$\nu(Ah) = n\mu(Y_1 \times Ah \times \{y_2\}) = n\mu((Y_1 \times A \times \{y_2\})(y_1^*, h_2 h, y_2) \geq \nu(A)$.
It now follows in the same way as in theorem 4.2.2 that $\mu = \mu_1 \times \nu \times \mu_2$.
If we take for μ_1 the measure on Y_1 defined by $\mu_1(\{e\}) = 1$ and $\mu_1(Y_1 \setminus \{e\}) = 0$, then it follows just as in theorem 4.1.4 that $\mu_1 \times \nu \times \mu_2$ is a right subinvariant measure on S.

4.2.7. Lemma. Let S be a compact mob with a finite number of idempotents. Let $S \neq K$, $S = S^2$ and let J be a maximal ideal. Let S/J^* be the Rees semigroup S/J with the following topology. $S/J^* = S \setminus J \cup \{0\}$ where $\{0\}$ is an isolated point and $S \setminus J$ has the relative topology. Then S/J^* is a compact 0-simple mob.

Proof:

Since S/J has a finite number of idempotents and thus a finite number of 0-minimal left and right ideals we have

$S/J = a_1 S/J\, b_1 \cup a_1 S/J\, b_2 \cup \ldots \cup a_1 S/J\, b_n \cup a_2 S/J\, b_1 \cup \ldots \cup a_k S/J\, b_n \cup \{0\}$

where each $a_j S/J\, b_i$ either is a group with zero or a set

$$(a_j S/J\, b_i)^2 = \{0\} .$$

Let $(a_j\ S/J\ b_i) \setminus \{0\} = A_{ji}$. Then since $A_{ji} = a_j S b_i \setminus J$, A_{ji} is closed. Furthermore $A_{ji} \cap A_{kl} = \emptyset$, $(j,i) \neq (k,l)$ and hence $A_{ji} \cup J$ is open. Thus A_{ji} is open in S/J^*.

We now prove that multiplication is continuous in S/J^*. If $ab = c \neq 0$, then $a \in A_{ji}$, $b \in A_{kl}$ with $A_{ji}A_{kl} \cap J = \emptyset$. Next let V be an arbitrary neighbourhood of c in S and let $V(a)$ and $V(b)$ be neighbourhoods of a and b in S such that $V(a).V(b) \subset V$. Then $(V(a) \cap A_{ji})(V(b) \cap A_{kl}) \subset V \cap S \setminus J$. If $ab = 0$ $a \in A_{ji}$, $b \in A_{kl}$, then $A_{ji}A_{kl} = \{0\}$ and if $ab = 0$, with $a = 0$, then $0\ S/J^* = \{0\}$. Hence multiplication in S/J^* is continuous and S/J^* is a compact 0-simple mob.

4.2.8. Theorem. Let S be a compact mob with a finite number of idempotents. Then S has a right subinvariant measure.

Proof:

If $S = K$, then S is a compact simple mob and lemma 4.2.1 implies that S has a right subinvariant measure. If $S \neq S^2$, then let μ^* be any normed regular Borel measure on the set $S \setminus S^2$. Now define μ on S by $\mu(B) = \mu^*(B \cap (S \setminus S^2))$.

μ is a regular Borel measure since $S \setminus S^2$ is open. Furthermore μ is right subinvariant since $\mu(Ba) = \mu^*(Ba \cap S \setminus S^2) = \mu^*(\emptyset) = 0$. Finally let $S \neq K$, $S = S^2$. Then S contains a maximal proper ideal such that S/J is completely 0-simple and by lemma 4.2.7 S/J^* is a compact 0-simple mob.

By theorem 4.2.6 there exists a right subinvariant measure μ^* on S/J^* such that $\mu^*(\{0\}) = 0$.

Now let μ be the measure on S defined by $\mu(B) = \mu^*(B \cap S \setminus J)$ for all Borel sets B of S. μ is right subinvariant since

$\mu(Ba) = \mu^*(Ba \cap S \setminus J) = 0$ if $a \in J$ and
$\mu(Ba) = \mu^*(Ba \cap S \setminus J) = \mu^*((B \cap S \setminus J)a) \leq \mu^*(B \cap S \setminus J) = \mu(B)$ if $a \notin J$.

4.2.9. Theorem. Let S be a compact mob such that there is an $x \in S$ with $Sx = S$. Then S has a right subinvariant measure.

Proof:

The dual of theorem 1.4.7 implies that $Q = \{x | Sx = S\}$ is a closed submob of S. Furthermore Q is a left simple submob and $S \setminus Q$ is an ideal of S.

Now let μ^* be any right invariant measure on Q and define μ on S by $\mu(B) = \mu^*(B \cap Q)$ for all Borel sets B of S.

μ is right subinvariant since

$$\mu(Ba) = \mu^*(Ba \cap Q) = \mu^*((B \cap Q)a) = \begin{cases} 0 & \text{if } a \notin Q \\ \mu^*(B \cap Q) = \mu(B) & \text{if } a \in Q. \end{cases}$$

4.2.10. Theorem. Let S be a compact commutative mob. Then S has a two-sided subinvariant measure.

Proof:

If $S = K$, then S is a group and the Haar measure on S is invariant.
If $S \neq S^2$ then it can be shown in a similar fashion as in the proof of theorem 4.2.8 that S has a right subinvariant measure. Since S is commutative the measure clearly is left subinvariant.
If $S \neq K$, $S = S^2$, then S contains a maximal proper ideal such that S/J is completely 0-simple. Since S/J is commutative it follows that S/J is a group with zero and hence that $S \setminus J$ is a compact group.
The Haar measure on $S \setminus J$ can now be extended to a two-sided invariant measure on S.

4.2.11. Theorem. Let S be an interval mob $S = [a,b]$. Then S has a right or left subinvariant measure.

Proof:

If $S \neq S^2$, then any regular normed measure on $S \setminus S^2$ can be extended to a measure on S.

If $S = K$, then since K consists of either all left zeroes or all right zeroes of S, S is either right or left simple. Lemma 4.2.1 then implies that S has either a left or a right subinvariant measure. Finally if $S = S^2$, $S \neq K$, then according to lemma 2.6.3 S contains a maximal ideal J such that the Rees semigroup S/J has a finite number of idempotents. It now follows from lemma 4.2.7 and theorem 4.2.8 that S has a right subinvariant measure.

4.3. Subinvariant measures on a certain class of mobs

Definition. A compact mob S with a minimal left ideal L such that for each open set U of S and each element $a \in S \setminus L$. Ua is open in S will be called a mob of type 0.

It is clear that all finite mobs are of type 0, in fact all compact mobs S such that Ua is open for all open sets $U \subset S$ and $a \in S$ are of type 0. This class contains the compact groups and all simple mobs with a finite number of minimal left ideals.
Let S be the set $\{\frac{1}{2^n};\ n = 1,2,\ldots\} \cup \{0\}$ with the natural topology, and the usual multiplication of rational numbers, then S is of type 0. We will show in this section that if S is a mob of type 0, then S has a right subinvariant measure. If S is a left simple mob then according to 4.2.1 S has a right subinvariant measure. Hence we will now restrict our attention to mobs of type 0 with $S \neq L$.

4.3.1. Lemma. Let S be a mob of type 0 and let U be an open set of S such that Ua is open in S, $a \in S \setminus L$. If C is a compact set, $C \subset Ua$, then there is a compact set $D \subset U$ with $C = Da$.

Proof:
Let $C \subset Ua$ and $D' = \{x \mid x \in S,\ xa \in C\}$. Then D' is compact and $(D' \cap U)a = C$. For each point $x \in D' \cap U$ there is a neighbourhood V(x) of x with $\overline{V(x)} \subset U$. Since S is of type 0, each set V(x)a is open and the set $\{V(x)a \mid x \in D' \cap U\}$ constitutes a covering of C. Let $V(x_1)a, \ldots, V(x_n)a$ be a finite subcovering of C and let $D = \bigcup_{i=1}^{n} \overline{V(x_i)} \cap D'$. Then D is compact, $D \subset U$ and $Da = C$.

Now let S be a mob of type 0, $S \neq L$. Let $\{V_\alpha \mid \alpha \in A\}$ be the set of all coverings of S such that $V_\alpha = \{O_{\alpha\beta} \mid \beta \in B_\alpha\}$, where each $O_{\alpha\beta}$ is an open set of S such that for every $a \in S \setminus L$ there is an $O_{\alpha\beta'} \in V_\alpha$ such that $O_{\alpha\beta} a \subset O_{\alpha\beta'}$.

4.3.2. Lemma. Let S be a mob of type 0, $S \neq L$ and let J be an open left ideal of S, $J \neq S$.

For each compact set $C \subset S$ and $\alpha \in A$ let

$$\lambda_\alpha(C) = \frac{\text{smallest number of } O_{\alpha\beta}\text{'s that will cover } C \setminus J}{\text{smallest number of } O_{\alpha\beta}\text{'s that will cover } S \setminus J.}$$

Then λ_α is a non-negative, finite monotone and subadditive function defined on the set of all closed subsets of S.

Moreover $\lambda_\alpha(Ca) \leq \lambda_\alpha(C)$ for all $a \in S$, $C \subset S$.

Proof:

Since $L \neq S$, it follows from 1.2.3 that S contains an open left ideal J, with $J \neq S$. Furthermore $C \setminus J$ and $S \setminus J$ are compact sets, hence $0 \leq \lambda_\alpha(C) \leq 1$. $\lambda_\alpha(L) = 0$ and $\lambda_\alpha(S) = 1$. Moreover it is clear that λ_α is monotone and subadditive. If $a \in L$, then since L is a left ideal $Ca \subset L$ and hence $\lambda_\alpha(Ca) = 0 \leq \lambda_\alpha(C)$.

Next let $a \in S \setminus L$ and let $O_{\alpha\beta_1}, \ldots, O_{\alpha\beta_n}$ be a finite subcovering of $C \setminus J$. Let $O_{\alpha\beta_i'} \in V_\alpha$ be such that $O_{\alpha\beta_i} a \subset O_{\alpha\beta_i'}$, $i = 1,2,\ldots,n$. Then $O_{\alpha\beta_1'}, \ldots, O_{\alpha\beta_n'}$ is a covering of $(C \setminus J)a = Ca \setminus J$.

Hence $\lambda_\alpha(Ca) \leq \lambda_\alpha(C)$.

Now let C^* denote the set of all closed subsets of S. To each $C \in C^*$ we make correspond the closed interval $I_C = [0,1]$. Let $I = \prod_{C \in C^*} I_C$ be the product of all these intervals. I is a compact Hausdorff space whose points are real-valued functions f defined on C^*, such that $0 \leq f(C) \leq 1$ for all $C \in C^*$.

Furthermore for each covering V_α we have $\lambda_\alpha \in I$.

Now let $\Lambda(\alpha) = \{ \lambda_{\alpha^*} \mid V_{\alpha^*}$ a refinement of V_α , $\alpha^* \in A\}$.

4.3.3. Lemma. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be any finite subset of A. Then there is an $\alpha_0 \in A$ such that $\lambda_{\alpha_0} \in \Lambda(\alpha_1) \cap \ldots \cap \Lambda(\alpha_n)$.

Proof:

Let $p \in S$, then there is a set $O_{\alpha_i\beta_i} \in V_{\alpha_i}$ such that $p \in O_{\alpha_i\beta_i}$. Hence $p \in \bigcap_{i=1}^{n} O_{\alpha_i\beta_i} = O_p$, where O_p is open in S.

For each $a \in S$ there are open sets O_p^a and $O(a) \subset S$ such that $O_p^a\, O(a) \subset O_{pa}$. The set $\{O(a) \mid a \in S\}$ is an open covering of S.

Let $O(a_1),\ldots,O(a_n)$ be a finite subcovering and let $U_p = \bigcap_{i=1}^{n} O_p^{a_i} \cap O_p$.

Since each U_p is open in S, we have that the set of all U_p, $p \in S$ together with all $U_p a$ $a \in S \setminus L$ constitutes a covering V_{α_o} of S such that V_{α_o} is a refinement of V_{α_i}, $i = 1,2,\ldots,n$.

Thus $\Lambda_{\alpha_o} \in \Lambda(\alpha_1) \cap \ldots \cap \Lambda(\alpha_n)$.

4.3.4. Lemma. Let $\lambda \in \bigcap\{\overline{\Lambda(\alpha)} \mid \alpha \in A\}$. Then λ is a non-negative finite monotone, additive and subadditive set function on the class C^* of all compact sets, with the property that $\lambda(C) \geq \lambda(Ca)$, $C \in C^*$, $a \in S$. Moreover $\lambda(L) = 0$ and $\lambda(S) = 1$.

Proof:

Since the class of all sets $\Lambda(\alpha)$ has the finite intersection property according to 4.3.3, the compactness of I implies that there is a point $\lambda \in I$ with

$$\lambda \in \bigcap \{\overline{\Lambda(\alpha)} \mid \alpha \in A\}.$$

Furthermore it is clear that $0 \leq \lambda(C) \leq 1$.

Next let $C \in C^*$ and let π_C be the projection of I onto I_C, i.e. $\pi_C(f) = f(C)$. Then π_C is a continuous function and the set $\Phi_{C,D} = \{f \mid \pi_C(f) \leq \pi_D(f)\}$ is closed, $C,D \in C^*$. If $C \subset D$, then $\lambda_\alpha \in \Phi_{C,D}$ for all $\alpha \in A$ and hence $\Lambda(\alpha) \subset \Phi_{C,D}$. Since $\Phi_{C,D}$ is closed it follows that $\lambda \in \Phi_{C,D}$ and thus that $\lambda(C) \leq \lambda(D)$.

The proof of the subadditivity of λ is entirely similar to the above argument. We just take for $\Phi_{C,D}$ the set $\Phi_{C,D} = \{f \mid \pi_{C\cup D}(f) \leq \pi_C(f) + \pi_D(f)\}$.

We now show that λ is additive.

If C and D are two compact sets such that $C \cap D = \emptyset$, then there is an $\alpha \in A$ such that $V_\alpha = \{O_{\alpha\beta} \mid \beta \in B_\alpha\}$ is a covering of S with the property that if $C \cap O_{\alpha\beta} \neq \emptyset$ then $D \cap O_{\alpha\beta} = \emptyset$. For let $a,p \in S$. We choose

an open set $O \subset S$ such that $pa \in O$ and such that either $O \cap C = \emptyset$ or $O \cap D = \emptyset$. The continuity of multiplication implies the existence of two sets U_p^a and $U(a)$ open in S with $p \in U_p^a$, $a \in U(a)$ and $U_p^a\, U(a) \subset O$. Since the set $\{U(a) \mid a \in S,\ p \text{ fixed}\}$ is an open covering of S there is a finite subcovering $U(a_1), \dots, U(a_n)$.
Let $U(p)$ be an open set with $p \in U(p)$ such that either $U(p) \cap C = \emptyset$ or $U(p) \cap D = \emptyset$ and let $O_p = U(p) \cap U_p^{a_1} \cap \dots \cap U_p^{a_n}$.
The set O_p has the following properties:
1) O_p is open
2) $O_p \cap C = \emptyset$ or $O_p \cap D = \emptyset$
3) $O_p a \cap C = \emptyset$ or $O_p a \cap D = \emptyset$ for all $a \in S$.
From this it follows that there is an $\alpha \in A$ such that

$$\{O_p \mid p \in S\} \cup \{O_p a \mid p \in S,\ a \in S \setminus L\} = V_\alpha .$$

Thus if $C, D \in C^*$ and $C \cap D = \emptyset$, then there is an α such that $\lambda_\alpha(C \cup D) = \lambda_\alpha(C) + \lambda_\alpha(D)$. Moreover if V_{α^*} is a refinement of V_α we have $\lambda_{\alpha^*}(C \cup D) = \lambda_{\alpha^*}(C) + \lambda_{\alpha^*}(D)$. Now let $\Phi_{C,D} = \{f \mid \pi_{C \cup D}(f) = \pi_C(f) + \pi_D(f)\}$. $C \cap D = \emptyset$. $\Phi_{C,D}$ is closed and there is an $\alpha \in A$ such that $\Lambda(\alpha) \subset \Phi_{C,D}$ and hence $\alpha \in \overline{\Lambda(\alpha)} \subset \Phi_{C,D}$.

Thus $\lambda(C \cup D) = \lambda(C) + \lambda(D)$.
Finally we have that for all $C \in C^*$ and $a \in S$ $\lambda(Ca) \leq \lambda(C)$, since if $\Phi_C = \{f \mid \pi_{Ca}(f) \leq \pi_C(f)\}$ then $\Lambda(\alpha) \in \Phi_C = \overline{\Phi}_C$ and hence $\lambda \in \Phi_C$.
Since for all $\alpha \in A$ we have $\lambda_\alpha(L) = 0$ and $\lambda_\alpha(S) = 1$ it is clear that $\lambda(L) = 0$, $\lambda(S) = 1$.

Now let O^* denote the set of all open subsets of S. We define a function λ_* on O^* by

$$\lambda_*(O) = \sup\ \{\lambda(C) \mid C \subset O,\ C \in C^*\} .$$

4.3.5. <u>Lemma</u>. λ_* is monotone, countably subadditive and countably additive. Moreover if Oa is open for an open set O of S, then $\lambda_*(Oa) \leq \lambda_*(O)$.

<u>Proof</u>:
If $U, O \in O^*$ and $U \subset O$, $C \subset U$, $C \in C^*$, then $\lambda(C) \leq \lambda_*(O)$. Hence

$\sup\{\lambda(C)|C \subset U,\ C \in C^*\} = \lambda_*(U) \leq \lambda_*(O)$ and it follows that λ_* is monotone.

Next let $U,O \in O^*$ and let $C \in C^*$, $C \subset U \cup O$. Then there are closed sets D and E such that $D \subset U$, $E \subset O$ and $C = D \cup E$.

Hence according to the subadditivity of λ we have

$\lambda(C) \leq \lambda(D) + \lambda(E) \leq \lambda_*(U) + \lambda_*(O)$.

Thus $\sup\{\lambda(C)|C \subset U \cup O,\ C \in C^*\} = \lambda_*(U \cup O) \leq \lambda_*(U) + \lambda_*(O)$.

By induction it now follows that λ_* is finitely subadditive.

Now suppose that $C \subset \bigcup_{i=1}^{\infty} O_i$, with $O_i \in O^*$ $i = 1,2,\ldots$. Since C is compact there is a positive integer n such that $C \subset \bigcup_{i=1}^{n} O_i$ and hence

$$\lambda(C) \leq \lambda_*(\bigcup_{i=1}^{n} O_i) \leq \sum_{i=1}^{n} \lambda_*(O_i) \leq \sum_{i=1}^{\infty} \lambda_*(O_i).$$

This implies that

$$\sup\{\lambda(C)|C \subset \bigcup_{i=1}^{\infty} O_i,\ C \in C^*\} = \lambda_*(\bigcup_{i=1}^{\infty} O_i) \leq \sum_{i=1}^{\infty} \lambda_*(O_i).$$

Suppose now that $U,O \in O^*$ and that $U \cap O = \emptyset$. Then if $C,D \in C^*$ $C \subset U$, $D \subset O$ we have $C \cap D = \emptyset$ and according to the additivity of λ it follows that

$\lambda(C) + \lambda(D) = \lambda(C \cup D) \leq \lambda_*(U \cup O)$ and hence that $\lambda_*(U) + \lambda_*(O) =$
$= \sup\{\lambda(C)|C \subset U,\ C \in C^*\} + \sup\{\lambda(D)|D \subset O,\ D \in D^*\} \leq \lambda_*(U \cup O)$. Since λ_* is subadditive it follows that $\lambda_*(U) + \lambda_*(O) = \lambda_*(U \cup O)$ and by induction that λ_* is finitely additive.

If $\{O_i\}_{i=1}^{\infty}$ is a sequence of disjoint open sets $O_i \in O^*$, then $\lambda_*(\bigcup_{i=1}^{\infty} O_i) \geq \lambda_*(\bigcup_{i=1}^{n} O_i) = \sum_{i=1}^{n} \lambda_*(O_i)$. Since this holds for all n we have $\lambda_*(\bigcup_{i=1}^{\infty} O_i) \geq \sum_{i=1}^{\infty} \lambda_*(O_i)$ and the countable additivity follows from the countable subadditivity.

Finally we prove that $\lambda_*(Oa) \leq \lambda_*(O)$ if $Oa \in O^*$.

$\lambda_*(Oa) = \sup\{\lambda(C)|C \subset Oa,\ C \in C^*\}$. If $a \in S \setminus L$, then according to lemma 4.3.1 there is a compact set $D \in C^*$ such that $D \subset O$ and $Da = C$.

Hence $\lambda(C) = \lambda(Da) \leq \lambda(D) \leq \lambda_*(O)$.

Thus $\sup\{\lambda(C) \mid C \subset Oa,\ C \in C^*\} = \lambda_*(Oa) \leq \lambda_*(O)$.

If $a \in L$, then $Oa \subset L$ and hence

$$\lambda_*(Oa) = \sup \{\lambda(C) \,|\, C \subset Oa \subset L,\ C \in C^*\} = 0 \leq \lambda_*(O).$$

Note that $\lambda_*(S) = \lambda(S) = 1$ and that $\lambda_*(J) = 0$.

Now let μ^* be the function defined on all subsets of S such that if $E \subset S$

$$\mu^*(E) = \inf \{\lambda_*(O) \,|\, E \subset O,\ O \in O^*\}.$$

The function μ^* is an outer measure, since clearly

1) $\mu^*(\emptyset) = 0$.

2) If $E \subset F \subset S$ and $F \subset O$, $O \in O^*$, then
$\mu^*(E) \leq \lambda_*(O)$ and thus $\mu^*(E) \leq \mu^*(F)$.

3) If $\{E_i\}_{i=1}^{\infty}$ is any sequence of sets, then there is an $O_i \in O^*$ such that
$\lambda_*(O_i) \leq \mu^*(E_i) + \varepsilon/2^i$ for any $\varepsilon > 0$. $E_i \subset O_i$.
Hence $\mu^*(\bigcup_{i=1}^{\infty} E_i) \leq \lambda_*(\bigcup_{i=1}^{\infty} O_i) \leq \sum_{i=1}^{\infty} \lambda_*(O_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i) + \varepsilon$.

Since ε is arbitrary this implies the countable subadditivity of μ^*.

4.3.6. <u>Theorem</u>. Let S be a mob of type O, $L \neq S$. Then S has a right subinvariant measure.

<u>Proof</u>:

Let μ be the set function defined for all Borel sets $B \subset S$ by $\mu(B) = \mu^*(B)$. Then μ is a regular Borel measure.

We first prove that every closed set is μ^*-measurable.

Let $O \in O^*$ and $C \in C^*$. Suppose that $D \in C^*$, $D \subset O \cap S \setminus C$ and $E \in C^*$, $E \subset O \cap S \setminus D$.

Then $D \cap E = \emptyset$ and $D \cup E \subset O$. Hence

$$\mu^*(O) = \lambda_*(O) \geq \lambda(D \cup E) = \lambda(D) + \lambda(E).$$

Thus $\mu^*(O) \geq \lambda(D) + \sup \{\lambda(E) \,|\, E \subset O \cap S \setminus D,\ E \in C^*\} = \lambda(D) + \lambda_*(O \cap S \setminus D)$
$= \lambda(D) + \mu^*(O \cap S \setminus D) \geq \lambda(D) + \mu^*(O \cap C)$.

From this it follows that

$$\mu^*(O) \geq \sup \{\lambda(D) \,|\, D \subset O \cap S \setminus C,\ D \in C^*\} + \mu^*(O \cap C) =$$
$$= \lambda_*(O \cap S \setminus C) + \mu^*(O \cap C) = \mu^*(O \cap S \setminus C) + \mu^*(O \cap C).$$

If A is any subset of S and $A \subset O$ then

$$\mu^*(A) = \inf\{\lambda_*(O) \mid A \subset O,\ O \in O^*\} \geq \mu^*(O \cap S \setminus C) + \mu^*(O \cap C) \geq$$
$$\mu^*(A \cap S \setminus C) + \mu^*(A \cap C).$$

Hence C is measurable and therefore all Borel sets are measurable.

The fact that μ is regular follows from

$$\mu(B) = \mu^*(B) = \inf\{\lambda_*(O) \mid B \subset O,\ O \in O^*\} = \inf\{\mu(O) \mid B \subset O,\ O \in O^*\}.$$

Finally we prove that μ is right subinvariant.

If $a \in S \setminus L$ and Ba a Borel set for a Borel set B, then

$$\mu(Ba) = \inf\{\lambda_*(O) \mid Ba \subset O,\ O \in O^*\} \leq \inf\{\lambda_*(Oa) \mid B \subset O,\ O \in O^*\} \leq$$
$$\leq \inf\{\lambda_*(O) \mid B \subset O,\ O \in O^*\} = \mu(B).$$

If $a \in L$, then $Ba \subset L \subset J$ and hence

$$\mu(Ba) \leq \mu(J) = \lambda_*(J) = 0 \leq \mu(B).$$

μ is normed, since $\mu(S) = \lambda_*(S) = \lambda(S) = 1$.

4.3.7. <u>Theorem</u>. Let G be a locally compact group with zero and let S be a compact subsemigroup of G with non-empty interior. Then S has a right subinvariant measure.

<u>Proof</u>:

If $0 \notin S$, then S is a compact group according to the corollary to theorem 1.1.10. Hence in this case S has a right invariant measure.

Now let $0 \in S$ and let μ be the right Haar measure defined on $G \setminus \{0\}$.

Let V be any open set of G, $0 \in V$, such that $S \setminus V$ has a non-empty interior.

According to lemma 1.2.2 there is an open ideal J of S with $0 \in J \subset S \cap V$.

We now define a measure ν on S by $\nu(B) = \dfrac{\mu(B \setminus J)}{\mu(S \setminus J)}$ for all Borel sets B of S.

Since J, B and S are Borel sets of G and since $\mu(S \setminus J) > 0$ it is clear that ν is a normed regular Borel measure on S.

Furthermore if $a \in S \setminus J$, then $(B \setminus J)a = Ba \setminus Ja \supset Ba \setminus J$, hence

$$\nu(Ba) = \frac{\mu(Ba \setminus J)}{\mu(S \setminus J)} \leq \frac{\mu(Ba \setminus Ja)}{\mu(S \setminus J)} = \frac{\mu((B \setminus J)a)}{\mu(S \setminus J)} = \frac{\mu(B \setminus J)}{\mu(S \setminus J)} = \nu(B).$$

If $a \in J$, then $Ba \subset J$ and hence

$$\nu(Ba) = 0 \leq \nu(B).$$

Remark.

If S is a mob such that S has a right subinvariant measure μ and I any closed ideal of S, with $\mu(I) < 1$, then we can define a right subinvariant measure ν on the Rees semigroup S/I with the quotient topology by

$$\nu(B) = \frac{\mu(B \setminus \{0\})}{\mu(S \setminus I)} \qquad B \text{ a Borel set of } S/I.$$

4.3.8. Theorem. Let G be a compact transformation semigroup of continuous open homomorphisms of a compact mob S of type 0 into itself such that $\tau(J) \subset J$ for an open proper left ideal containing L and for all $\tau \in G$. Then there is a right subinvariant measure μ on S such that $\mu(\tau B) \leq \mu(B)$ for each $\tau \in G$ and each Borel set $B \subset S$ such that $\tau(B)$ is a Borel set.

Proof:

Let $V_\alpha = \{O_{\alpha\beta} \mid \beta \in B_\alpha\}$ be a covering of S with sets $O_{\alpha\beta}$ such that if $a \in S \setminus L$ then there is an $O_{\alpha\beta^a}$ such that $O_{\alpha\beta}\, a \subset O_{\alpha\beta^a}$. Next let $p \in S$. For each $\tau \in G$ there are $O_{\alpha\beta_1}$ and $O_{\alpha\beta_2}$ such that $\tau(p) \in O_{\alpha\beta_1}$ and $p \in O_{\alpha\beta_2}$.

Since the mapping $(p,\tau) \to \tau(p)$ is continuous simultaneously in $p \in S$ and $\tau \in G$ there are neighbourhoods O^p_τ and V^p_τ such that $p \in O^p_\tau \subset O_{\alpha\beta_2}$, $\tau \in V^p_\tau \subset G$ and $V^p_\tau(O^p_\tau) \subset O_{\alpha\beta_1}$.

The set $\{V^p_\tau \mid \tau \in G\}$ constitutes an open covering of G.

Let $V^p_{\tau_1}, \ldots, V^p_{\tau_n}$ be a finite subcovering and let $O_p = O^p_{\tau_1} \cap \ldots \cap O^p_{\tau_n} \subset O_{\alpha\beta_2}$.

O_p has the property that for each $\tau \in G$, there is an open set $O_{\alpha\beta} \in V_\alpha$ such that $\tau(O_p) \subset O_{\alpha\beta}$.

The covering $V_{\alpha'} = \{O_p, O_p a, \tau(O_p), \tau(O_p)a \mid p \in S,\ a \in S \setminus L,\ \tau \in G\}$ is a refinement of V_α.

Furthermore if $O \in V_{\alpha'}$, $a \in S \setminus L$ and $\tau \in G$, then there are $O' \in V_{\alpha'}$ and $O'' \in V_{\alpha'}$ such that $Oa = O'$ and $\tau O = O''$ or $\tau O \subset L$.

For $\tau(O_p a) = \tau(O_p)\,\tau a$. If $\tau a \in L$, then $\tau(O_p a) \subset L$, and if $\tau a \notin L$, then $\tau(O_p a) \in V_{\alpha'}$.

Finally we have $\tau_1(\tau_2(O_p)) = \tau_1\tau_2(O_p) \in V_{\alpha'}$.

Now let J be the open left ideal containing L such that $\tau(J) \subset J$ and let $\lambda_{\alpha'}$ be defined as in lemma 4.3.2. Then we have for each compact set $C \subset S$ and $\tau \in G$, $\lambda_{\alpha'}(C) \geq \lambda_{\alpha'}(\tau C)$.

Let I be as in lemma 4.3.3 and $\Phi = \{f \mid f \in I,\ f(C) \geq f(\tau C),\ C \in C^*,$
$\tau \in G\}$.

Then Φ is closed and for each covering V_α there is a refinement $V_{\alpha'}$ such that $\lambda_{\alpha'} \in \Phi$.

Hence $\Phi \cap \Lambda(\alpha) \neq \emptyset$ and it follows that $\bigcap_\alpha \overline{\Lambda(\alpha)} \cap \Phi \neq \emptyset$. We now choose $\lambda \in \bigcap_\alpha \overline{\Lambda(\alpha)} \cap \Phi$. For this choice of λ we have $\lambda(C) \geq \lambda(\tau C)$ for all $C \in C^*$, $\tau \in G$.

Finally let μ be the right subinvariant measure induced by λ. Then μ has the desired property.

For let O be an open set of S and let $\tau \in G$. For each $C \subset \tau(O)$ there is a $D \in C^*$ with $\tau(D) = C$. For each point $p \in D \cap O$ there is a neighbourhood V_p with $\overline{V}_p \subset O$. The set $\{\tau(V_p) \mid p \in D \cap O\}$ is an open covering of C. Let $\tau(V_{p_1}), \ldots, \tau(V_{p_n})$ be a finite subcovering and let

$D' = \bigcup_{i=1}^{n} \overline{V}_{p_i} \cap D$. D' is compact $D' \subset O$ and $\tau(D') = C$.

Hence $\lambda_*(\tau(O)) = \sup\{\lambda(C) \mid C \subset \tau O,\ C \in C^*\} =$

$\sup\{\lambda(\tau D') \mid D' \subset O,\ D' \in C^*\} \leq \sup\{\lambda(D) \mid D \subset O,\ D \in C\}$.

Finally since $\mu(\tau(B)) = \inf\{\lambda_*(O) \mid \tau B \subset O,\ O \in O^*\} \leq$

$\leq \inf\{\lambda_*(\tau O) \mid B \subset O,\ O \in O^*\} \leq \inf\{\lambda_*(O) \mid B \subset O,\ O \in O^*\} = \mu(B)$,

for each Borel set τB, the theorem is proved.

Remark.

If S is a compact mob such that Ua is open for each open set $U \subset S$ and each $a \in S$, then 4.3.8 holds for any compact transformation semigroup of continuous open homomorphisms. For in this case we can define $\lambda_{\alpha'}$ by

$$\lambda_{\alpha'}(C) = \frac{\text{smallest number of } O_{\alpha',\beta}\text{'s that will cover } C}{\text{smallest number of } O_{\alpha',\beta}\text{'s that will cover } S},$$ i.e.

we let J be the empty set.

4.4. Notes

Invariant measures on semigroups were first investigated by Schwarz [7], [8], [11] and Rosen [1].

Let $\mathfrak{M}(S)$ be the convolution semigroup of normalized regular non-negative Borel measures on a compact topological semigroup S. Schwarz [8] studied the structure of $\mathfrak{M}(S)$ in the case that S is a finite semigroup, and if right invariant measures exist on S, the role of such measures in $\mathfrak{M}(S)$.

In this connection we also mention the work of Wendel [1], Collins [1], [2], [3], [4] and Glicksberg [1] who investigated the structure of idempotent measures μ, $\mu \in \mathfrak{M}(S)$. Right invariant measures on compact mobs S in which the implication, U open in S and $a \in S \Longrightarrow Ua$ open in S, holds were studied by Schwarz [7]. He established necessary and sufficient conditions that such a mob possess a right invariant measure.

We prove in section 4.2 that on such a mob right subinvariant measures always exist.

Rosen [1] established necessary and sufficient conditions for a compact semigroup to possess an invariant mean. Theorem 4.1.6 however seems to be new.

In section 4.2 we are concerned with right subinvariant measures on (0-) simple compact semigroups. We establish necessary and sufficient conditions that such a mob possess a right subinvariant measure. The structure of such a measure is then determined.

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