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GO-SPACES AND GENERALIZATIONS OF METRIZABILITY

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PREFACE

In this monograph we discuss some generalizations of metrizability in connection with generalized ordered spaces (GO-spaces). The class of generalized ordered spaces was extensively studied by D.J. LUTZER [32] and M.J. FABER [19]. As a result of their work, the metrizability of GO-spaces is fairly well understood, and also other properties like paracompactness, perfect normality etc. are characterized in terms of the order structure of the space.

The generalizations of metrizability we discuss can be roughly divided into two groups: Those of the first group (like for instance stratifiable spaces or σ-spaces) imply metrizability in a GO-space, and will be treated in chapter V. Those of the second group, that contains p- and M-spaces, do not imply metrizability in a GO-space (not surprisingly, since they are also generalizations of compact spaces).

This monograph is organized as follows:
In the first chapter we list some results from the literature, introduce some techniques that will be used later on and discuss shortly the generalizations of metrizability that will be treated in the next chapters.
The second chapter is mainly about p- and M-spaces. We construct two quotient spaces of an arbitrary GO-space X that are metrizable if and only if X is a p-space or an M-space respectively, and derive some consequences from this fact. In chapter III we determine when the lexicographic product of two linearly ordered topological spaces is a p-space or an M-space respectively, making use of the results of the second chapter. In chapter IV we tackle the harder problem of characterizing generalized ordered Σ-spaces. Here we obtain only partial results. It turns out that only in a special case Σ-nets are "compatible" with the convexity structure of a GO-space, which makes Σ-spaces more difficult to handle.
In the fifth chapter we discuss the relations between the generalizations of metrizability of the first group mentioned above in a wider class of spaces. viz. the class of images of GO-spaces under various kinds of mappings. An interesting by-product is that we obtain some information about the question which spaces are images of a GO-space or a LOTS by for instance a closed or perfect mapping.

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CONTENTS

CHAPTER 0

BASIC CONCEPTS

This chapter has a preliminary character; it contains some facts about linearly ordered sets and some conventions about notation are made.

0.1. LINEARLY ORDERED SETS

A *linearly ordered set* is a pair $(X,\leq)$, where X is a set and $\leq$ a reflexive, anti-symmetric, transitive relation on X, such that

$$\forall x,y \in X : (x,y) \in \leq \quad \text{or} \quad (y,x) \in \leq$$

$\leq$ is called the *ordering* (or: the *order*) of $(X,\leq)$.

In the following a linearly ordered set $(X,\leq)$ will mostly be denoted by X, and instead of "$(x,y) \in \leq$" we shall always write $x \leq y$. With $x < y$ we mean that $x \leq y$ and $x \neq y$. This convention will also be applied to other cases: if for instance $(Y,\leq_2)$ is a linearly ordered set, and a and b belong to Y then $a <_2 b$ means that $a \leq_2 b$ and $a \neq b$.

If $(X,\leq)$ is a linearly ordered set and A is a subset of X then by $\leq$ an ordering $\leq_A$ is induced on A.

A subset C of a linearly ordered set X is called *convex* (*in* X) if for each $p,q \in C$ with $p \leq q$ the set

$$\{x \in X \mid p \leq x \leq q\}$$

is contained in C.

When it is made clear by the context, in which space the set C is convex, we will just say that the set C is convex.

Whenever A is a subset of X then a convex (in X) subset C of A is called a *convexity-component* of A (in X) if C is not a proper subset of any convex (in X) subset of A. Clearly each convex (in X) subset C of A is contained in a uniquely determined convexity-component of A.

For $p \in X$, we define

$$[p,\rightarrow[\ := \{x \in X \mid p \leq x\}; \quad]\leftarrow,p] := \{x \in X \mid x \leq p\}$$

and

$$]p,\rightarrow[\ := \{x \in X \mid p < x\}; \quad]\leftarrow,p[\ := \{x \in X \mid x < p\}.$$

Sets of the first type are called *closed half-lines* and sets of the second type are called *open half-lines*.

For $p,q \in X$, the sets

$$[p,q] := \{x \in X \mid p \leq x \leq q\}$$
$$[p,q[\ := \{x \in X \mid p \leq x < q\}$$
$$]p,q] := \{x \in X \mid p < x \leq q\}$$
$$]p,q[\ := \{x \in X \mid p < x < q\}.$$

are called *intervals*.

In particular, $]p,q[$ is called the *open interval* and $[p,q]$ the *closed interval between* p *and* q $(p \leq q)$.

Clearly all intervals are convex sets (in X). The converse need not be true.

REMARK. Let $(Y,\leq)$ be an ordered set with subset X. If necessary in order to avoid confusion, we shall use subscripts in denoting intervals, for instance, if $p,q \in Y$, then

$$[p,q]_Y := \{x \in Y \mid p \leq x \leq q\}$$

and

$$[p,q]_X := \{x \in X \mid p \leq x \leq q\} = [p,q]_Y \cap X.$$

If p and q are points of X such that $p < q$ and $]p,q[\ = \emptyset$ then p and q are said to be *neighbours* (*in* X); p is the *left neighbour of* q and q is called the *right neighbour of* p. A point $p \in X$ is said to be a *neighbour(point)* of X if there exists a point q in X such that p and q are neighbours in X.

If A is a subset of a linearly ordered set X then a point $p \in A$ is called *left endpoint* if $p \leq x$ for each $x \in A$; and $p \in A$ is called *right endpoint* of A if $x \leq p$ for each $x \in A$. A point p of A is said to be *endpoint* of A if it is left or right endpoint of A.

If $(X,\leq_1)$ and $(Y,\leq_2)$ are linearly ordered sets then we define the *lexicographic order* $\leq$ on (any subset of) the Cartesian product $X \times Y$ by

$$(x,y) \leq (x',y') \iff x <_1 x' \text{ or } (x = x' \text{ and } y \leq_2 y').$$

Obviously $(X\times Y,\leq)$ is again a linearly ordered set.

Whenever $(X,\leq_1)$ and $(Y,\leq_2)$ are linearly ordered sets, then a mapping $f\colon X \to Y$ is called *order preserving* if

$$x \leq_1 x' \Rightarrow f(x) \leq_2 f(x') \qquad (x,x' \in X).$$

0.2. SOME REMARKS ABOUT NOTATION

Let $X = (X,\leq)$ be a linearly ordered set. A subset A of X is said to be *cofinal* (resp. *coinitial*) in X if

$$X = \{x \in X \mid \exists a \in A:\ x \leq a\}$$

or

$$X = \{x \in X \mid \exists a \in X:\ a \leq x\} \qquad \text{respectively.}$$

Whenever $X = (X,\leq)$ is a linearly ordered set then $^*X$ denotes the same set with inverse order, and if μ is an ordinal number then $^*\mu$ denotes the inverse ordertype. The ordered set of all ordinals smaller than μ is denoted by $W(\mu)$.

An ordinal number μ is said to be *cofinal* in X if X contains a well-ordered subset A with ordinal number μ such that A is cofinal in X. Analogously we define: $^*\mu$ is *coinitial* in X.

The *cofinality* of X (denoted by cf(X)) is the least ordinal μ such that μ is cofinal in X, and the *coinitiality* of X (denoted by ci(X)) is the inverse ordertype of the smallest ordinal number μ such that $^*\mu$ is coinitial in X.

When we write for instance

- ξ is a (pseudo-) gap in X -

we mean that ξ is a pseudogap or a gap in X. Also, instead of a sentence like

- if p is the left endpoint of A then p is left-isolated and if p is the right endpoint of A then p is right isolated -

we frequently write:

- if p is left (right) endpoint of A then p is left-(right-) isolated -

If X is a set then $|X|$ denotes the cardinality of X. $\aleph_0$ is the cardinality of a countable set.

By $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the reals, the rationals, the integers and the positive integers respectively. ω_0 is the first infinite, ω_1 the first uncountable ordinal.

For all undefined terms and unproved statements in this treatise we refer to well-known textbooks as DUGUNDJI [15], ENGELKING [17], KELLEY [30] and NAGATA [38].

By convention all spaces are supposed to be T_1; all maps are continuous unless otherwise stated.

CHAPTER I

GENERALIZED ORDERED SPACES

1.1. LOTS's AND GO-SPACES

A *linearly ordered topological space* (abbreviated: LOTS) is a triple $(X,\leq,\lambda(\leq))$, where X is a set, $\leq$ is a linear order on X and $\lambda(\leq)$ is the topology on X generated by the open half-lines $]x,\rightarrow[$ and $]\leftarrow,x[$, where $x \in X$. The topology $\lambda(\leq)$ is called the *order topology*.

A topological space (X,τ) is said to be *orderable* if there exists an ordering $\leq$ on X such that $\lambda(\leq) = \tau$

If (X,τ) is a topological space and A is a subset of X then τ_A denotes the relative topology on A. If $(X,\leq,\lambda(\leq))$ is a LOTS and $A \subset X$ then in general $\lambda(\leq_A)$ and $\lambda(\leq)_A$ do not coincide, though it is clear that $\lambda(\leq_A)$ is contained in $\lambda(\leq)_A$.

A *generalized ordered space* (abbreviated: GO-space) is a triple $(X,\leq,\tau)$ where X is a set, $\leq$ a linear order on it, and τ a topology on X such that

(i) $\lambda(\leq) \subset \tau$

(ii) τ has a base consisting of convex subsets.

In the sequel we shall frequently, if no confusion is possible, denote a GO-space $(X,\leq,\tau)$ simply by X.

Clearly every LOTS is a GO-space and every subspace of a GO-space (and hence every subspace of a LOTS) is a GO-space. Conversely, every GO-space is a subspace of a LOTS, for instance if $(X,\leq,\tau)$ is a GO-space then

$$\begin{aligned} X^* := & \{(x,n) \in X \times \mathbb{Z} \mid n > 0 \text{ if }]\leftarrow,x] \in \tau\setminus\lambda(\leq)\} \cup \\ & \cup \{(x,n) \in X \times \mathbb{Z} \mid n < 0 \text{ if } [x,\rightarrow[\in \tau\setminus\lambda(\leq)\} \cup \\ & \cup \{(x,n) \in X \times \mathbb{Z} \mid n = 0\}. \end{aligned}$$

When we order X^* lexicographically by $\preceq$ then $(X^*,\preceq,\lambda(\preceq))$ is a LOTS and X is homeomorphic to the closed subspace $\{(x,n) \in X \times \mathbb{Z} \mid n = 0\}$ of X^*.

If no confusion is likely we will identify X with this subspace. Consequently, the class of all GO-spaces and the class of all subspaces of LOTS's coincide. Moreover, it is easy to see that if $(X,\leq,\tau)$ is a GO-space then

$$\lambda(\leq) = \tau \quad \text{if and only if} \quad X^* = X \times \{0\}.$$

There are two special cases in which $\lambda(\leq)$ equals τ, namely when X is connected and when X is compact. Note that when X is locally compact $\lambda(\leq)$ and τ need not be the same (consider the subset $\{0\} \cup]1,2]$ of the real line with relative order and relative topology).

Finally, we mention here the well-known fact that every LOTS and hence every GO-space is hereditarily collectionwise normal [45] (even monotonically normal; see chapter 5).

1.2. SOME PROPERTIES OF GO-SPACES

Let $X = (X,\leq,\tau)$ be a GO-space.

A *gap* in X is an ordered pair (A,B) of subsets of X such that

(i) $X = A \cup B$.

(ii) $a < b$ for all $a \in A$, $b \in B$.

(iii) A has no right endpoint and B has no left endpoint.

(iv) $A,B \in \tau$.

If both A and B are non-empty, (A,B) is called an *interior gap* of X. If either A or B is empty, we will speak of an *endgap*. In particular, (A,B) is a *left* (*right*) *endgap* if A (resp.B) is empty.

A *jump* in X is an ordered pair (A,B) of subsets of X such that

(i) $X = A \cup B$.

(ii) $a < b$ for all $a \in A$, $b \in B$.

(iii) A has a right and B has a left endpoint.

(iv) $A,B \in \tau$.

Clearly if (A,B) is a jump in X then the right endpoint of A and the left endpoint of B are neighbours.

An ordered pair (A,B) of subsets of X is called a *pseudogap* or *pseudojump* if

(i) $X = A \cup B$.

(ii) $a < b$ for all $a \in A$, $b \in B$.

(iii) (A has no right endpoint and B has a left endpoint) or (A has a right endpoint and B has no left endpoint).

(iv) $A,B \in \tau$.

(v) $A \neq \emptyset \neq B$.

A pseudogap (A,B) is a *left (right) pseudogap* or *right (left) pseudojump* if A (resp.B) has no right (left) endpoint. Note that in the definitions of "jump" and "gap" the topology actually does not play a role (in other words, condition (iv) is superflous since it is implied by condition (i), (ii) and (iii)) while on the other hand in the definition of pseudogap it is essential. Furthermore note that in our terminology a pseudogap always is an interior pseudogap.

The following theorems are well-known:

THEOREM 1.2.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

X *is connected* $\Longleftrightarrow$ X *has no interior gaps, nor pseudogaps and no jumps.*

THEOREM 1.2.2. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

X *is compact* $\Longleftrightarrow$ X *has no (interior and no end-) gaps and no pseudogaps.*

If $(X,\leq,\lambda(\leq))$ is a LOTS and $\xi = (A,B)$ is a gap in X then we can regard ξ as a "virtual" element of X satisfying $a < \xi < b$ for all $a \in A$, $b \in B$.

If we add all these gaps (inclusive endgaps) to X and give the resulting set the order topology with respect to its natural order, we obtain the *Dedekind compactification* of X, customarily denoted by X^+ (see [21]).

Now if $X = (X,\leq,\tau)$ is a GO-space then a compactification of X can be made in the following way, which originates from LUTZER [32].

First embed X in the LOTS X^* as described in 1.1. Then $X^+ = (X,\leq,\tau)^+$ is defined to be the closure of X in $(X^*)^+$. Note that X^+ is a LOTS, since it is closed and hence compact subset of $(X^*)^+$. We can look at X^+ in the case that X is a GO-space, as obtained by placing a point in each gap and each pseudogap, and ordening the resulting set with the natural order. As in the case that X is a LOTS we will call X^+ the Dedekind compactification of X, also when X is a GO-space.

If C is a convex subset of X and $\xi = (A,B)$ is a (pseudo-)gap in X then we say that C *covers* ξ if

$$C \cap A \neq \emptyset \neq C \cap B.$$

Furthermore, we define the following subsets of X:

$$E(X) := \{x \in X \mid [x,\rightarrow[\ \in \tau \text{ or }]\leftarrow,x] \in \tau\}$$

and

$$H(X) := \{x \in X \mid [x,\rightarrow[\ \in \tau\backslash\lambda(\leq) \text{ or }]\leftarrow,x] \in \tau\backslash\lambda(\leq)\}.$$

Also we say that a point $x \in X$ is *left-(right-) isolated* if $[x,\rightarrow[$ (resp. $]\leftarrow,x]$) belongs to τ.

A convex subset C of X is said to be *left-(right-) open* if it is a neighbourhood of all points from C that are not right (left) endpoint of C.

If A is a subset of X then A decomposes into a disjoint collection of convexity-components. It is easy to see that the convexity-components of A are closed in A (with respect to the topology $\lambda(\leq)_A$ and hence certainly with respect to the topology τ_A); hence they are closed in X if A is closed in X. Obviously if A is open in X then the convexity-components of A are also open in X.

In the sequel we shall frequently make use of the following facts about decompositions of GO-spaces. Some of these are known but they are proved here for the sake of completeness.

PROPOSITION 1.2.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space and let* $\mathcal{D}$ *be an equivalence relation on* X *such that the equivalence classes of* $\mathcal{D}$ *are convex closed sets. Then the quotient space* $X/\mathcal{D}$ *is itself a GO-space with respect to the obvious order* $\preceq$ *existing on the elements of* $X/\mathcal{D}$*, and the quotient topology* δ*. Moreover the quotient map* $\mathbb{P} : X \rightarrow X/\mathcal{D}$ *is closed and order preserving.*

PROOF.

(i) Suppose $y \in X/\mathcal{D}$ and that x is some element of $\mathbb{P}^{-1}(y)$. Then
$$\mathbb{P}^{-1}[\,]\leftarrow,y[\,] =]\leftarrow,x[\,\backslash\mathbb{P}^{-1}(y)$$
so $\mathbb{P}^{-1}[\,]\leftarrow,y[\,]$ is open in X. Consequently $]\leftarrow,y[$ belongs to δ. Since the same applies to $]y,\rightarrow[$ and open half-lines form a subbase for the order topology on $X/\mathcal{D}$, this proves that $\lambda(\preceq)$ is contained in δ.

(ii) Let U be any open (in δ) set containing the point y. Then $\mathbb{P}^{-1}[U]$

is open in X, as is the unique convexity-component O of $\mathbb{P}^{-1}[U]$ containing $\mathbb{P}^{-1}(y)$. Since $\mathbb{P}^{-1}[\mathbb{P}[O]] = O$, $\mathbb{P}[O]$ is a convex open neighbourhood of y, that is contained in U. (i) and (ii) together imply that $(X/\mathcal{D},\preceq,\delta)$ is a GO-space.

(iii) That the map $\mathbb{P}$ is order preserving is obvious. We claim that the decomposition corresponding to $\mathcal{D}$ is upper-semi-continuous; hence $\mathbb{P}$ is closed.

Suppose D is some element of $X/\mathcal{D}$ and let A be an open neighbourhood of D in X. Then a convex open neighbourhood B of D is constructed as follows: fix $x \in D$.

- If D is left-open then $B := [x,\rightarrow[\cup D$.
- If D is not left-open (and hence has a left endpoint ℓ that is not left-isolated) then there exists $x' < \ell$ such that $[x',\ell[\subset A$. Take $B :=]x',\rightarrow[\setminus\mathbb{P}^{-1}[\mathbb{P}(x')]$.

In both cases $B = \mathbb{P}^{-1}[\mathbb{P}[B]]$, and analogously a convex open neighbourhood C of D is constructed such that $C = \mathbb{P}^{-1}[\mathbb{P}[C]]$ and $C \cap [x,\rightarrow[\subset A \cap [x,\rightarrow[$. Consequently $B \cap C$ is an open neighbourhood of A such that

$$D \subset B \cap C \quad \text{and} \quad B \cap C = \mathbb{P}^{-1}[\mathbb{P}[B \cap C]].$$

This proves the claim. □

If X is a LOTS then the quotient space $X/\mathcal{D}$ is not necessarily a LOTS. However, this is the case under a special condition.

PROPOSITION 1.2.4. *Let $(X,\le,\lambda(\le))$ be a LOTS and let $\mathcal{D}$ be as in proposition 1.2.3. Suppose each equivalence class of $\mathcal{D}$ has a left and a right endpoint. Then the quotient topology δ and the order topology $\lambda(\preceq)$ coincide; hence $X/\mathcal{D}$ is a LOTS.*

PROOF. Suppose y is an element of $X/\mathcal{D}$ such that $[y,\rightarrow[$ belongs to δ, and such that y is not left endpoint of $X/\mathcal{D}$.

Then $\mathbb{P}^{-1}[[y,\rightarrow[]$ has a left endpoint ℓ, viz. the left endpoint of $\mathbb{P}^{-1}(y)$. Since $\mathbb{P}^{-1}[[y,\rightarrow[]$ is open and X is a LOTS, ℓ has a left neighbour ℓ^-. Then $\mathbb{P}(\ell^-)$ is left neighbour of y in $X/\mathcal{D}$ so $[y,\rightarrow[$ belongs to $\lambda(\preceq)$. Consequently X is a LOTS because it has no pseudogaps. □

COROLLARY. *If X is a LOTS then $X/\mathcal{D}$ is a LOTS when each $D \in X/\mathcal{D}$ is finite, or more generally, when each $D \in X/\mathcal{D}$ is compact.*

<u>DEFINITION</u>. Let $X = (X,\leq,\tau)$ be a GO-space and let $f\colon X \to Y$ be a continuous surjection. Then for each $y \in Y$ the fiber $f^{-1}(y)$ decomposes into convexity-components, which are called *convexity-components under* f. Let

$$\mathcal{F} := \{C \mid C \text{ is a convexity-component of any } f^{-1}(y), y \in Y\}$$

be the collection of all convexity-components under f. Clearly $\mathcal{F}$ is a decomposition of X into convex sets. Furthermore, each element of $\mathcal{F}$ is closed, since each fiber is closed, and the convexity-components of a closed set are closed. Let $\mathcal{D}$ be the equivalence relation corresponding to the decomposition $\mathcal{F}$.

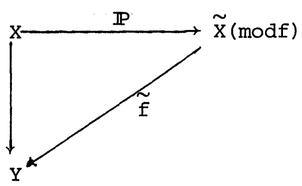
It follows from proposition 1.2.3 that the decomposition space $X/\mathcal{D}$ is a GO-space with respect to the obvious order and quotient topology. This GO-space is denoted by $\tilde{X}$ (mod f) or, when no danger of confusion arises, simply by $\tilde{X}$.

Furthermore, if $\mathbb{P}\colon X \to \tilde{X}$ is the quotient map, we define the (univalent) mapping

$$\tilde{f}\colon \tilde{X} \longrightarrow Y$$

by

$$\tilde{f} := f \circ \mathbb{P}^{-1}$$



Of course $\tilde{f}$ is continuous. Moreover, if f is either open, closed or (quasi-) perfect, the same is true for $\tilde{f}$.

By construction every convexity-component under the the map $\tilde{f}$ consists of a single point.

1.3. METRIZABILITY OF GO-SPACES

LUTZER [32] and FABER [19] have done much work on the metrizability of GO-spaces. Since we shall need some of their results furtheron, we shall mention them here and give a short proof for each of them.

The central point in almost each metrization theorem for GO-spaces is Bings theorem that a collectionwise normal Moore space is metrizable (see [7]). To make good use of this theorem, we need some definitions and a lemma.

If $\mathcal{U}$ is a cover of a topological space X, and $x \in X$, then the *star of* x *relative to* $\mathcal{U}$ is defined by

$$St(x,\mathcal{U}) := \cup \{U \in \mathcal{U} \mid x \in U\}.$$

A topological space X is said to be *developable* if it admits a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of open covers of X such that for each $x \in X$ the family $\{St(x,\mathcal{U}(n)) \mid n = 1,2,\ldots\}$ constitutes a local base at x. The sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ is called a *development* for X.

A regular developable space is called a *Moore space*.

A topological space X is said to have a G_δ*-diagonal* if the diagonal $\Delta := \{(x,x) \mid x \in X\}$ is a G_δ-set in $X \times X$. It was proved by CEDER [12] that X has a G_δ-diagonal if and only if X admits a sequence $(\mathcal{V}(n))_{n=1}^{\infty}$ of open coverings of X such that

$$\bigcap_{n=1}^{\infty} St(x,\mathcal{V}(n)) = \{x\} \quad \text{for each } x \in X.$$

Such a sequence $(\mathcal{V}(n))_{n=1}^{\infty}$ is called a *sequence of* G_δ*-coverings*. Obviously, every developable space has a G_δ-diagonal. For a LOTS the converse is true, since if a LOTS $X = (X,\leq,\lambda(\leq))$ has a G_δ-diagonal then a sequence $(\mathcal{V}(n))_{n=1}^{\infty}$ of G_δ-coverings can be chosen such that each $\mathcal{V}(n)$ consists of convex open sets, and such that $\mathcal{V}(n+1)$ refines $\mathcal{V}(n)$ $(n = 1,2,\ldots)$. Hence $(St(x,\mathcal{V}(n)))_{n=1}^{\infty}$ is a decreasing sequence of convex open sets, which then is a local base at x. This proves the following:

<u>THEOREM 1.3.1.</u> (LUTZER [31]) *Let* $X = (X,\leq,\lambda(\leq))$ *be a LOTS. Then*

$$X \text{ has a } G_\delta\text{-diagonal} \iff X \text{ is metrizable.}$$

<u>THEOREM 1.3.2.</u> *Let* $X = (X,\leq,\tau)$ *be a GO-space and let* $f\colon X \to Y$ *be a continuous surjection such that each convexity-component under* f *consists of one point. Then* X *has a* G_δ*-diagonal if* Y *has one.*

<u>PROOF.</u> Suppose Y has a G_δ-diagonal. Then there exists a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of open covers of Y such that

$$\bigcap_{n=1}^{\infty} St(y,\mathcal{U}(n)) = \{y\} \quad \text{for each } y \in Y.$$

For every $n \in \mathbb{N}$, $x \in X$ choose a convex open neighbourhood $I(x,n)$ of x such that $f[I(x,n)]$ is contained in some element of $\mathcal{U}(n)$. Define

$$\mathcal{J}(n) := \{I(x,n) \mid x \in X\}.$$

Then each $\mathcal{J}(n)$ is an open covering of X. Moreover

$$f[\bigcap_{n=1}^{\infty} St(x,\mathcal{J}(n))] \subset \bigcap_{n=1}^{\infty} f[St(x,\mathcal{J}(n))] \subset \bigcap_{n=1}^{\infty} St(f(x),\mathcal{U}(n)) =$$

$$= \{f(x)\} \quad (x \in X).$$

Hence $\bigcap_{n=1}^{\infty} St(x,\mathcal{J}(n))$ is a convex set which is mapped by f onto the point $f(x) \in Y$, so it is contained in the convexity-component of $f^{-1}(f(x))$ that contains x. Since each convexity-component under f consists of a single point, this implies that

$$\bigcap_{n=1}^{\infty} St(x,\mathcal{J}(n) = \{x\}.$$

It follows that X has a G_δ-diagonal. □

COROLLARY. *Let* $X = (X,\leq,\tau)$ *be a GO-space and let* $f: X \to Y$ *be a continuous surjection. Let* $\tilde{X}$ *and* $\tilde{f}$ *be defined as in section* 1.2. *Then* $\tilde{X}$ *has a* G_δ-*diagonal if* Y *has one.*

GO-spaces that have a G_δ-diagonal need not be metrizable, as is illustrated by the Sorgenfrey line (i.e. the set of the real numbers with the usual order, and all intervals of the form [a,b[for a base; see also [44]). However, they are first countable since it is easily shown that in a GO-space there is a countable local base at each point x which is such that {x} is a G_δ-set. Moreover a GO-space X having a G_δ-diagonal is metrizable if the set H(X) fulfils an extra condition described below.

A subset D of a topological space R is called *relatively discrete* if the relative topology on D is the discrete topology, and D is said to be *discrete* (*in* R) if it is both closed in R and relatively discrete. Obviously D is discrete (in R) if and only if $\{\{x\} \mid x \in D\}$ is a discrete collection of subsets of R. A subset D is σ-*discrete* (*in* R) if it is the union of countably many discrete (in R) subsets of R.

Obviously, a subset A of a GO-space $X = (X,\leq,\tau)$ is σ-discrete in X if and only if $A = \bigcap_{n=1}^{\infty} A(n)$, where for each $x \in X$, $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x,n)$ of x such that $O(x,n) \cap (A(n)\setminus\{x\}) = \emptyset$.

<u>THEOREM 1.3.3</u>. *Let* R *be a topological space and let* D *be a* σ-*discrete subset of* R. *If* $(\mathcal{U}(n))_{n=1}^{\infty}$ *is a sequence of open covers of* R *and if for each* $n \in \mathbb{N}$, $p \in D$, $V(p,n)$ *is an open neighbourhood of* p *then there exists a sequence* $(\mathcal{U}'(n))_{n=1}^{\infty}$ *of open covers of* R *such that*

a) $\mathcal{U}'(n)$ *refines* $\mathcal{U}(n)$ $(n = 1,2,\ldots)$.

b) *For each* $p \in D$ *there is a natural number* $n(p)$ *such that* $St(p,\mathcal{U}'(n)) \subset V(p,n)$ *for all* $n \geq n(p)$.

<u>PROOF</u>. Suppose $D = \bigcup_{n=1}^{\infty} D(n)$ where each $D(n)$ is discrete in R. Without loss of generality we may suppose that $D(n+1) \supset D(n)$ for each n. Then for each $p \in R$ and $n \in \mathbb{N}$ there exists an open neighbourhood $O(p,n)$ of p such that

(i) $O(p,n) \cap (D(n)\setminus\{x\}) = \emptyset$.

(ii) $O(p,n)$ is contained in some element of $\mathcal{U}(n)$.

(iii) $O(p,n) \subset V(p,n)$ whenever $p \in D$.

Take

$$\mathcal{U}'(n) = \{O(p,n) \mid p \in R\} \quad (n = 1,2,\ldots).$$

Then $(\mathcal{U}'(n))_{n=1}^{\infty}$ is a sequence of open covers of R with the following properties:

- $\mathcal{U}'(n)$ refines $\mathcal{U}(n)$ by (ii).
- Whenever p is some element of D let $n(p)$ be the first natural number such that $p \in D(n)$.

Now if $n \geq n(p)$ and p' is a point of X distinct from p, then from $O(p',n) \cap (D(n)\setminus\{p'\}) = \emptyset$ it follows that p is not an element of $O(p',n)$. Apparently we have

$$St(p,\mathcal{U}'(n)) = O(p,n) \subset V(p,n).$$

Consequently the sequence $(\mathcal{U}'(n))_{n=1}^{\infty}$ has properties a) and b). $\square$

<u>COROLLARY</u>. (FABER [19] th.3.2.) *Let* $X = (X,\leq,\tau)$ *be a GO-space. If there exists a sequence* $(\mathcal{U}(n))_{n=1}^{\infty}$ *of open covers of* X *such that*

(i) $\bigcap_{n=1}^{\infty} St(x,\mathcal{U}(n)) = \{x\}$ *for every* $x \in X$.

(ii) $X\setminus\{x \in X \mid (St(x,\mathcal{U}(n)))_{n=1}^{\infty}$ *is a local base at* $x\}$ *is* σ*-discrete in* X.

then X *is metrizable.*

<u>PROOF</u>. From (i) it follows that X is first countable. If $D := X\setminus\{x \in X \mid St(x,\mathcal{U}(n))_{n=1}^{\infty}$ is a local base at $x\}$ and $(V(p,n))_{n=1}^{\infty}$ is a decreasing local base at each point $p \in D$, then the sequence $(\mathcal{U}'(n))_{n=1}^{\infty}$ in theorem 1.3.3 is a development for X. Hence X is metrizable. □

Theorem 1.3.3 roughly says that if we have a sequence of open covers of a topological space R and a σ-discrete subset D of R then we can change the elements of the sequence a bit, such that the stars do not get bigger, and in each point of the set D the stars are eventually contained in pre-given sets. This we can use to give a short proof of (a reformulation of) Fabers metrization theorems.

<u>THEOREM 1.3.4</u>. (FABER [19] th.3.1 and 3.2). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*:

(i) X *is metrizable.*

(ii) X *has a* σ*-discrete dense subset and* E(X) *is* σ*-discrete.*

(iii) X *has a* G_δ*-diagonal and* H(X) *is* σ*-discrete.*

<u>PROOF</u>. (i)→(ii): Since X is metrizable, X has a σ-discrete open base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}(n)$ where each $\mathcal{B}(n)$ is a discrete family in X. For each $n \in \mathbb{N}$ construct a set D(n) by taking from each $B \in \mathcal{B}(n)$ the possible endpoints and one arbitrary point. It is easy to see that each D(n) is discrete. Therefore $D := \bigcup_{n=1}^{\infty} D(n)$ is a σ-discrete dense subset of X. Since D contains E(X) by construction, E(X) must be σ-discrete too.

(ii)→(iii): H(X) is σ-discrete since it is a subset of E(X). Suppose $D = \bigcup_{n=1}^{\infty} D(n)$ is dense in X, where for each $x \in X$, $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x,n)$ of x such that $O(x,n) \cap (D(n)\setminus\{x\}) = \emptyset$. Moreover we may suppose that $O(x,n+1) \subset O(x,n)$, that $O(x,n)$ is contained in $[x,\rightarrow[$ ($]\leftarrow,x]$ resp.) whenever x is left-(right-) isolated, and that $D(n+1) \supset D(n)$. Put

$$\mathcal{U}(n) := \{O(x,n) \mid x \in X\} \quad (n = 1,2,\ldots).$$

(I). Now take $x_0 \in X\setminus E(X)$ and $x' \neq x_0$, say $x' < x_0$. Then there exists $y \in]x',x_0[\cap D$ and $y' \in]y,x_0[\cap D$. Let n be such that y and y' belong to D(n).

Since no $O(x,n)$ contains both y and y', no $O(x,n)$ contains both x' and x_0. Hence $x' \notin St(x_0,\mathcal{U}(n))$. Because x' was arbitrary, this implies that $\bigcap_{n=1}^{\infty} St(x_0,\mathcal{U}(n)) = \{x_0\}$.

(II). Next, for each $x \in E(X)$ we clearly have that $\{x\} = \bigcap_{n=1}^{\infty} O(x,n)$. Hence, applying theorem 1.3.3 and (I) we obtain a sequence $(\mathcal{U}'(n))_{n=1}^{\infty}$ of open covers of X such that $\bigcap_{n=1}^{\infty} St(x,\mathcal{U}'(n)) = \{x\}$ for each $x \in X$; so X has a G_δ-diagonal.

(iii)→(i): Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of G_δ-coverings of X such that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$ and such that $\mathcal{U}(n)$ consists of convex sets. For each $x \in X$ let $(V(x,n))_{n=1}^{\infty}$ be a decreasing local base at x (see the remark after theorem 1.3.2). It is easy to see that since $(St(x,\mathcal{U}(n)))_{n=1}^{\infty}$ is a decreasing sequence of convex sets with intersection $\{x\}$, it must be a local base at each $x \in X\setminus H(X)$.

Using this fact, and applying theorem 1.3.3 we get that there exists a sequence $(\mathcal{U}'(n))_{n=1}^{\infty}$ of open covers of X such that $(St(x,\mathcal{U}'(n)))_{n=1}^{\infty}$ is a local base at each $x \in X$. Consequently X is metrizable. □

Analogous to the definition of Q-gap in a LOTS by GILLMAN and HENRIKSEN [21] we give the following definition for GO-spaces. This definition obviously coincides with that of [21] for any LOTS.

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space.

A (pseudo-)gap $\xi = (A,B)$ is said to be a Q_ℓ-*(pseudo-)gap* if there is a discrete subset L of A that is cofinal in A, and ξ is said to be a Q_r-*(pseudo-)gap* if there is a discrete subset R of B coinitial in B. The (pseudo-)gap ξ is said to be a Q-*(pseudo-)gap* if it is both a Q_ℓ- and a Q_r-(pseudo-)gap. Observe that if cf(A) (resp. ci(B)) is countable then $\xi = (A,B)$ certainly is a Q_ℓ-(pseudo-)gap (a Q_r(pseudo-)gap respectively). In particular any pseudo gap from the left (from the right) is a Q_r-pseudo-gap (Q_ℓ-pseudo gap).

In [21] GILLMAN and HENRIKSEN characterized paracompactness in LOTS's by using the notion of a Q-gap. Their result was generalized to GO-spaces by Faber:

<u>THEOREM 1.3.5</u>. ([19] th.2.4.6). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is paracompact} \Longleftrightarrow \textit{Each (pseudo-)gap in } X \textit{ is a } Q\textit{-(pseudo-)gap.}$$

<u>PROPOSITION 1.3.6</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space,* $\xi = (A,B)$ *a (pseudo-)gap in* X *that is not a* Q_ℓ*-(pseudo-)gap, and let* $\mathcal{U}$ *be an open cover of* X *consisting of convex sets. Then there is a point* $x \in A$ *such that* $\mathrm{St}(x,\mathcal{U}) \cap A$ *is cofinal in* A. [*Of course a similar proposition holds when* ξ *is not a* Q_r*-(pseudo-)gap*].

<u>PROOF</u>. Suppose that such a point does not exist. Then by (transfinite) induction we can construct an increasing sequence $(x(\eta))_{\eta<\beta}$ (where β is some limit ordinal) that is cofinal in A, such that

$$\bigcup \{\mathrm{St}(x(\alpha),\mathcal{U}) \mid \alpha < \eta\} \subset \,]\leftarrow,x(\eta)[\text{ for all } \eta < \beta.$$

Then each point of X is contained in some element $U \in \mathcal{U}$ which contains at most one $x(\eta)$ by the construction of the sequence. Hence the set $\{x(\eta) \mid \eta < \beta\}$ is discrete, which yields a contradiction. □

Using this fact and the characterization of paracompactness of theorem 1.3.5, we can give a short proof of the fact that a GO-space with a G_δ-diagonal is paracompact (see LUTZER [32]) and hence hereditarily paracompact since having a G_δ-diagonal is a hereditary property.

<u>THEOREM 1.3.7</u>. ([32]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ has a } G_\delta\textit{-diagonal} \Rightarrow X \textit{ is hereditarily paracompact.}$$

<u>PROOF</u>. Suppose $(\mathcal{U}(n))_{n=1}^{\infty}$ is a sequence of G_δ-coverings for X such that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$ and such that $\mathcal{U}(n)$ consists of convex sets $(n = 1,2,\ldots)$. Let $\xi = (A,B)$ be a (pseudo-)gap in X. Suppose ξ is not a Q-(pseudo-)gap, say not a Q_ℓ-(pseudo-)gap. By 1.3.6 there exists for each n an $x(n) \in A$ such that

$$\mathrm{St}(x(n),\mathcal{U}(n)) \cap A \quad \text{is cofinal in } A.$$

Since $(x(n))_{n=1}^{\infty}$ cannot be cofinal in A there is an $\ell \in A$ such that $x(n) < \ell$

for each n. Hence

$$\{\ell\} = \bigcap_{n=1}^{\infty} \mathrm{St}(\ell, \mathcal{U}(n)) \cap A \text{ is cofinal in } A; \text{ which is impossible.} \square$$

The next theorem will be used in the following

THEOREM 1.3.8. ([19] th.2.4.5). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*:

(i) X *is perfectly normal.*

(ii) *Every collection of mutually disjoint, convex open subsets of* X *constitues a* σ*-discrete family.*

(iii) *Each relatively discrete subset of* X *is* σ*-discrete* (*in* X).

Observe that by using property (iii) in the theorem above one can easily show that a perfectly normal GO-space is paracompact. (the result is due to LUTZER [32]). Moreover, a GO-space that has a σ-discrete dense subset obviously satisfies property (ii) and is hence perfectly normal. (see also [19] pp. 45 and 51).

1.4. SOME GENERALIZATIONS OF METRIZABILITY

In this section we shall list some generalizations of metrizability, that one frequently encounters in the literature, and investigate what they mean for a GO-space and a LOTS.

In the previous section we already encountered the notions of a developable space and that of a Moore space, which originate from the Alexandrov-Urysohn metrization theorem. Also spaces with a G_δ-diagonal are a direct generalization of the concept of a developable space. Another generalization of a metrizable space is obtained by deleting the triangle inequality in the definition of a metric. In this way we get the notion of a semi-metrizable space.

A topological space X is *semi-metrizable* if it admits a function

$$d: X \times X \longrightarrow \mathbb{R}_+$$

such that

(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x) \qquad (x,y \in X)$

(iii) $x \in \bar{A} \iff d(x,A)\ (:=\inf\{d(x,y) \mid y \in A\}) = 0 \qquad (x \in X; A \subset X)$.

Note that in a semi-metrizable space the set

$$S(a,\varepsilon) := \{x \in X \mid d(a,x) < \varepsilon\} \qquad (a \in X; \varepsilon > 0)$$

need not be open, but the following propositions remain true:

(a) If U is a open neighbourhood of x, then $x \in S(x,\varepsilon) \subset U$ for some $\varepsilon > 0$.

(b) If $\varepsilon > 0$, and $x \in X$, then $x \in S(x,\varepsilon)^{\circ}$.

The following is well-known and due to HEATH [23].

<u>THEOREM. 1.4.1.</u> *A topological space* X *is semi-metrizable if and only if, for every* $x \in X$ *a sequence* $(U(x,n))_{n=1}^{\infty}$ *of open neighbourhoods of* x *exists, such that*

(i) *if* $x \in U(x(n),n)$ *for* $n = 1,2,\ldots$ *then the sequence* $(x(n))_{n=1}^{\infty}$ *converges to* x.

(ii) $(U(x,n))_{n=1}^{\infty}$ *is a local base at* x.

By generalizing the concept of a σ-locally finite base which appears in the Nagata-Smirnov metrization theorem, we can obtain several generalizations of metrizable spaces. One of them is the notion of a stratifiable space.

Let X be a topological space. A collection $\mathcal{P}$ of ordered pairs (P_1,P_2) of subsets of X is called a *pairbase* if:

(i) P_1 is open and $P_1 \subset P_2$ for each $(P_1,P_2) \in \mathcal{P}$ and

(ii) for each $x \in X$ and each neighbourhood U of x there exists some $(P_1,P_2) \in \mathcal{P}$, such that $x \in P_1 \subset P_2 \subset U$.

Moreover $\mathcal{P}$ is called *cushioned* if for every subcollection $\mathcal{P}' \subset \mathcal{P}$

$$\overline{\bigcup \{P_1 \mid (P_1,P_2) \in \mathcal{P}'\}} \subset \bigcup \{P_2 \mid (P_1,P_2) \in \mathcal{P}'\}.$$

$\mathcal{P}$ is called σ-*cushioned* if it is the union of countably many cushioned subcollections.

A topological space is called *stratifiable* if it admits a σ-cushioned pairbase. Stratifiable spaces were introduced by CEDER [12] who called them M_3-spaces. Afterwards they were renamed by BORGES [8].

More convenient to work, is frequently the following characterization of stratifiable spaces (see [8]).

PROPOSITION 1.4.2. *A topological space* X *is stratifiable if and only if for every open set* $U \subset X$ *there exist closed sets* $U(n)$ $(n = 1,2,\dots)$ *such that*

(i) $\bigcup_{n=1}^{\infty} U(n) = U$

(ii) $U(n) \subset V(n)$ *whenever* $U \subset V$ $(n = 1,2,\dots)$

(iii) $\bigcup_{n=1}^{\infty} U(n)^0 = U$.

Every stratifiable space is hereditarily paracompact and perfectly normal [12].

A *Nagata space* X is a topological space with the property that for each $x \in X$ there exist sequences $(U(x,n))_{n=1}^{\infty}$ and $(S(x,n))_{n=1}^{\infty}$ of open neighbourhoods of x such that

(i) for each $x \in X$ $(U(x,n))_{n=1}^{\infty}$ is a local base at x.

(ii) $S(x,n) \cap S(y,n) \neq \emptyset \Rightarrow x \in U(y,n)$ $(x,y \in X; n \in \mathbb{N})$.

(Observe that it follows form (ii) that $S(x,n) \subset U(x,n)$, hence also $(S(x,n))_{n=1}^{\infty}$ is a local base at x).

The concept of a Nagata space originates from a metrization theorem due to NAGATA ([38] theorem VI-2.). CEDER [12] showed the following:

THEOREM 1.4.3. *Let* X *be a topological space. Then*

$$X \textit{ is a Nagata space} \Longleftrightarrow X \textit{ is first countable and stratifiable.}$$

A topological space X is called *semi-stratifiable* (CREEDE [14]) if for every open set $U \subset X$ there exist closed sets $U(n)$ $(n = 1,2,\dots)$ with

(i) $\bigcup_{n=1}^{\infty} U(n) = U$

(ii) $U(n) \subset V(n)$ whenever $U \subset V$.

Note that these are precisely the first two conditions in proposition 1.4.2. so semi-stratifiable spaces are a direct generalization of stratifiable spaces. Also, semi-stratifiable spaces can be characterized by conditions similar to those in 1.4.1 namely as follows:

THEOREM 1.4.4. (CREEDE [14]). *A topological space* X *is semi-stratifiable if and only if for each* $x \in X$ *a sequence* $(U(x,n))_{n=1}^{\infty}$ *of open neighbourhoods of* x *exists, such that*

(i) *if* $x \in U(x(n),n)$ *for* $n = 1,2,\ldots$ *then the sequence* $(x(n))_{n=1}^{\infty}$ *converges to* x

(ii) $\bigcap_{n=1}^{\infty} U(x,n) = \{x\}$.

Hence the following theorem now becomes obvious:

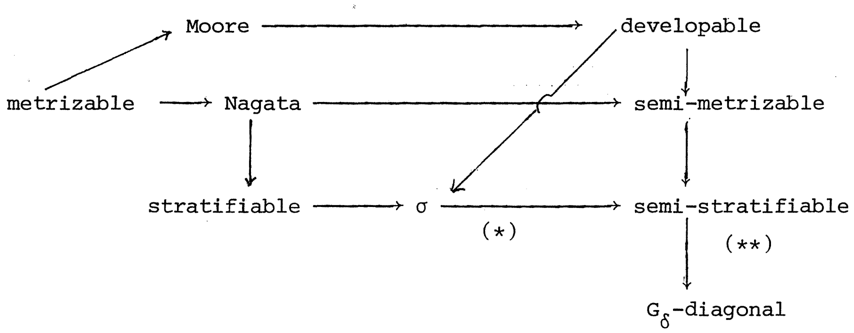
THEOREM 1.4.5. (CREEDE [14]). *A topological space is semi-metrizable if and only if it is first countable and semi-stratifiable.*

One can also weaken the concept of a σ-locally finite base in the following way:
A collection $\mathcal{B}$ of subsets of a topological space X is called a *net* for X if every open $U \subset X$ is the union of a subcollection of $\mathcal{B}$.

A topological space is called a *σ-space* if it admits a σ-locally finite net (see OKUYAMA [41]).

Every developable space, and each stratifiable space is a σ-space (see [11] resp. [24]). However there is an example due to BERNEY [6] of a semi-metrizable space that does not admit a σ-locally finite net. Finally, a regular σ-space is easily seen to be semi-stratifiable, since it has a closed σ-locally finite net.

Most of the results described here, can be found in the above mentioned literature. We sum up some of them in the following diagram (where "→" mean: implies).



(For implication (*) we need regularity, and for implication (**) the T_2-axiom).

Concentrating on GO-spaces and LOTS's, we see that in the class of all linearly ordered topological spaces, all these notions coincide, since the weakest property, that is: having a G_δ-diagonal, implies metrizability by theorem 1.3.1.

For the class of all GO-spaces this is not true, since a GO-space with a G_δ-diagonal need not be metrizable. All other properties however, do imply metrizability because of the next theorem:

THEOREM 1.4.6. (LUTZER [32]). *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is semi-stratifiable } \Rightarrow X \textit{ is metrizable.}$$

PROOF. X surely has a G_δ-diagonal so we only have to prove that $H(X)$ is σ-discrete in X (theorem 1.3.4).

For each $x \in X$, $n \in \mathbb{N}$ let $U(x,n)$ be a convex open set containing x such that the properties of 1.4.4 are fulfilled. Define

$$L := \{x \in H(X) \mid [x,\rightarrow[\ \in \tau\backslash\lambda(\leq)\}$$

and

$$R := \{x \in H(X) \mid]\leftarrow,x] \in \tau\backslash\lambda(\leq)\}.$$

Put

$$D(n) := \{x \in L \mid \forall x' < x : U(x',n) \subset]\leftarrow,x[\}.$$

Clearly $L = \bigcup_{n=1}^{\infty} D(n)$ (if some $x \in L$ should not belong to $\bigcup_{n=1}^{\infty} D(n)$ then for each n there exists an $x(n) < x$ such that $x \in U(x(n),n)$. Hence the sequence $(x(n))_{n=1}^{\infty}$ converges to x which is impossible); also from the defining property and the fact that $D(n) \subset L$ it follows that each $D(n)$ is relatively discrete. Since X is obviously perfectly normal, and so each relatively discrete subset is σ-discrete by theorem 1.3.8, each $D(n)$ and hence L is σ-discrete. Since the same applies to R, the theorem follows. □

1.5. MORE GENERALIZATIONS OF METRIZABILITY

All the generalizations described in section 1.4 have the property that they make a GO-space metrizable or very close to that. There is however another class of generalizations to be mentioned below, for which this does not hold true, in a way because the properties involved are generalizations not only of metrizability, but also of some covering property like (local) compactness or countably-compactness. We shall describe some of them here; in the next chapter we shall see what they mean for a GO-space.

If X and Y are topological spaces such that $X \subset Y$ then a *pluming for* X *in* Y is a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of coverings of X by sets open in Y such that

$$\bigcap_{n=1}^{\infty} St(x,\mathcal{U}(n)) \subset X \qquad \text{for every } x \in X.$$

A pluming $(\mathcal{U}(n))_{n=1}^{\infty}$ for X in Y is called a *strict pluming* if for every $x \in X$, $n \in \mathbb{N}$ there exists a natural number $n(x)$ such that

$$Cl_Y(St(x,\mathcal{U}(n(x)))) \subset St(x,\mathcal{U}(n)).$$

A completely regular space X is a (*strict*) p-*space* if it has a (strict) pluming in its Čech-Stone compactification, or equivalently in any of its Hausdorff compactifications.

p-Spaces were introduced by A.V. ARHANGELS'KIǏ, in [3]. Afterwards, internal characterizations of p-spaces and strict p-spaces were given by BURKE and STOLTENBERG in [10] and [11] respectively. Each locally compact T_2-space and each metrizable space is a p-space [3]. In [3] also, we find the following result:

THEOREM 1.5.1. *A topological space is a paracompact* p-*space if and only if it can be mapped onto a metrizable space by a perfect map.*

Hence p-spaces (and M-spaces too, as the following will show) are important in the light of Alexandrov's question in [1]: which spaces can be mapped onto nice spaces by nice maps?

Independently of p-spaces, M. MORITA introduced M-spaces in [35], in connection with product theory.

A topological space X is said to be a wΔ-*space* [9] (M-*space*) if it admits a (normal) sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of open covers of X with the following property:

if $x(n) \in St(x,\mathcal{U}(n))$ for $n = 1,2,\ldots$ then the sequence $(x(n))_{n=1}^{\infty}$ has a cluster point.

Here a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of covers of X is called *normal* if $\mathcal{U}(n+1)$ is a star-refinement of $\mathcal{U}(n)$ for each n.

A nice characterization of M-spaces is given by

THEOREM 1.5.2. ([35], th.6.1). *A topological space* X *is an* M-*space if and only if it can be mapped onto a metrizable space by a quasi-perfect map.*

COROLLARY. *The inverse image of an* M-*space under a quasi-perfect map is an* M-*space.*

It follows immediately from theorem 1.5.2 that each countably compact space and each metrizable space is an M-space. Moreover, theorem 1.5.1 and 1.5.2 together imply that a paracompact space is an M-space if and only if it is a p-space.

An important feature of (paracompact) p-spaces is that the product of countably many (paracompact) p-spaces is again a (paracompact) p-space. In the case of paracompactness this follows easily from theorem 1.5.1 and the fact that the product of perfect maps is perfect.

THEOREM 1.5.3. ([40] or [8]). *A Hausdorff space is metrizable if and only if it is a paracompact* p-*space with a* G_δ-*diagonal.*

CHABER ([13]) proved that a countably compact T_2-space with a G_δ-diagonal is metrizable. Combining this with theorem 1.5.2, 1.5.1 and 1.5.3, we find

THEOREM 1.5.4. (CHABER [13]). *A Hausdorff space is metrizable if and only if it is an* M-*space with a* G_δ-*diagonal.*

A property weaker than that of being a wΔ-space is quasi-completeness [22]. A topological space X is *quasi-complete* if it admits a sequence $(\mathcal{H}(n))_{n=1}^{\infty}$ of open covers of X such that a sequence $(x(n))_{n=1}^{\infty}$ clusters whenever it has the following property:

there is a $p \in X$ such that for every n there is a member of $\mathcal{H}(n)$ containing $\{p\} \cup \{x(k) \mid k \geq n\}$.

Finally, the concept of a Σ-space [37] belongs to this type of generalization and to the kind described in section 1.4, since each M-space and each regular σ-space is a Σ-space. Σ-spaces will be defined and discussed in chapter 4.

CHAPTER II

p- AND M-SPACES

2.1. p-SPACES

Since a completely regular space is a p-space iff it has a pluming in any of its Hausdorff compactifications, it seems natural to take for a GO-space the Dedekind compactification; a GO-space $X = (X,\leq,\tau)$ is a p-space if and only if it has a pluming in its Dedekind compactification X^+. This leads to the following proposition:

PROPOSITION 2.1.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then* X *is a* p*-space if and only if there exists a sequence* $(\mathcal{V}(n))_{n=1}^{\infty}$ *of convex open covers of* X *with the property that for each* $x \in X$ *and each (pseudo-)gap* $\xi = (A,B)$ *in* X *there is an* $n(=n(x,\xi))$ *such that* $St(x,\mathcal{V}(n))$ *does not cover the (pseudo-) gap* ξ.

PROOF. If X is a p-space, let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a pluming for X in X^+ and put

$$\mathcal{V}(n) := \{V \cap X \mid V \text{ is convexity-component of } U \text{ in } X^+ \text{ for some } U \in \mathcal{U}(n)\}$$

Conversely, if $(\mathcal{V}(n))_{n=1}^{\infty}$ is a sequence of open covers of X as described above, put

$$\mathcal{U}(n) := \{V_* \mid V \in \mathcal{V}(n)\}$$

where V_* is defined as $V \cup \{\xi \in X^+\setminus X \mid V \text{ covers } \xi\}$. Then $(\mathcal{U}(n))_{n=1}^{\infty}$ is a pluming for X in X^+. □

In the class of all GO-spaces the property of being a p-space can be characterized by means of the metrizability of a related space. This is made clear by theorem 2.1.3. First we need some definitions.

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space. Then $\mathcal{G}_X$ is the equivalence relation on X defined by

$$x\,\mathcal{G}_X\,y \iff \text{the closed interval between } x \text{ and } y \text{ is compact} \quad (x,y\in X).$$

The elements of the decomposition of X corresponding to this equivalence relation are precisely the convexity-components of X in X^+. They can also be thought of as maximal convex sets in X that do not cover any (pseudo-) gaps of X. The decomposition space $X/\mathcal{G}_X$ is denoted by gX and $g_X : X \longrightarrow gX$ is the quotient map.

When no confusion is possible we will drop the subscript X, and speak of the equivalence relation $\mathcal{G}$ and the map $g : X \longrightarrow gX$.

PROPOSITION 2.1.2. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the decomposition* gX *of* X *consists of convex, closed sets and* g *is a closed map. If* δ *is the identification topology on* gX *and* $\preceq$ *the natural order on* gX *inherited from* X, *then* $gX = (gX,\preceq,\delta)$ *is a GO-space.*

PROOF. That the decomposition gX consists of closed convex sets is trivial; the other assertions then follow from proposition 1.2.3. □

REMARK. In the sequel we will frequently use the same symbol for the ordering on X and on gX. Confusion does not seem likely.

Note that gX need not be a LOTS or, for that matter, an orderable space, even if X is a LOTS; for instance if $X := \mathbb{R} \times \mathbb{N}$ with lexicographic order and corresponding topology, then gX is homeomorphic to the Sorgenfrey line, which is well-known not to be orderable.

THEOREM 2.1.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a p-space} \iff gX \textit{ is metrizable.}$$

PROOF. $\Rightarrow$: Suppose that X is a p-space and let $(\mathcal{V}(n))_{n=1}^{\infty}$ be a sequence of open covers of X as described in proposition 2.1.1. Without loss of generality we may suppose that $\mathcal{V}(n+1)$ refines $\mathcal{V}(n)$ $(n = 1,2,\ldots)$. If $y \in gX$,

$n \in \mathbb{N}$, then

$$A(g^{-1}(y),n) := g^{-1}(y) \cup V(n,\ell(y)) \cup V(n,r(y))$$

where $V(n,\ell(y)) = \phi$ if $g^{-1}(y)$ is left-open in X; and if $g^{-1}(y)$ is not left-open, and $\ell(y)$ is the evidently existing left endpoint of $g^{-1}(y)$, then $V(n,\ell(y))$ is some element of $\mathcal{V}(n)$ containing $\ell(y)$; while $V(n,r(y))$ is defined analogously for the right side of $g^{-1}(y)$. Put

$$O'(y,n) := Y \backslash g[X \backslash A(g^{-1}(y),n)]$$

and

$$O(y,n) := \bigcap_{k=1}^{n} O'(y,k).$$

Observe that $O(y,n)$ is an open neighbourhood of y. Hence

$$\mathcal{O}(n) := \{O(y,n) \mid y \in gX\}$$

is an open cover of gX. By construction each $\mathcal{O}(n+1)$ refines $\mathcal{O}(n)$.

First we prove that $\bigcap_{n=1}^{\infty} St(y,\mathcal{O}(n)) = \{y\}$ for each $y \in gX$. To that purpose choose distinct point y_1 and y_2 in gX, for instance $y_1 < y_2$. Then there exists some (pseudo-)gap $\xi = (A,B)$ in X such that $g^{-1}(y_1) \subset A$ and $g^{-1}(y_2) \subset B$. Choose $x_1 \in g^{-1}(y_1)$ and $x_2 \in g^{-1}(y_2)$. Then there is a natural number n_0 such that

$$St(x_1,\mathcal{V}(n_0)) \subset A \quad \text{and} \quad St(x_2,\mathcal{V}(n_0)) \subset B \qquad (*).$$

It follows that y_1 does not belong to $St(y_2,\mathcal{O}(n_0))$. [For, suppose y_1 and y_2 belong to some $O(y,n_0) \in \mathcal{O}(n_0)$. Then $g^{-1}(y_1) \cup g^{-1}(y_2)$ is contained in $A(g^{-1}(y),n_0)$. Since $g^{-1}(y)$ does not cover the gap ξ, it is contained in either A or B, say in A. But then, by the construction of $A(g^{-1}(y),n_0)$, some element of $\mathcal{V}(n_0)$ intersecting $g^{-1}(y)$ must contain $g^{-1}(y_2)$ and hence x_2. Consequently $St(x_2,\mathcal{V}(n_0))$ covers the (pseudo-)gap ξ, in contradiction with (*).]

From this it follows that $(St(y,\mathcal{O}(n)))_{n=1}^{\infty}$ is a local base at all those points $y \in gX$ that do not belong to $H(gX)$. To prove that $(\mathcal{O}(n))_{n=1}^{\infty}$ is a development for gX, we only have to show that if $[y,\rightarrow[\; \in \delta\backslash\lambda(\leq)$ or

$]\leftarrow,y] \in \delta\backslash\lambda(\leq)$, then there exists an $n \in \mathbb{N}$ such that $St(y,\mathcal{O}(n)) \subset [y,\rightarrow[$ or $St(y,\mathcal{O}(n) \subset]\leftarrow,y]$ respectively (Recall that δ is the quotient topology and $\lambda(\leq)$ the order topology on gX).

Suppose $[y,\rightarrow[$ belongs to $\delta\backslash\lambda(\leq)$, the other case is treated analogously. Then $(g^{-1}[]\leftarrow,y[], g^{-1}[[y,\rightarrow[])$ is a (pseudo-)gap in X, because $g^{-1}[]\leftarrow,y[]$ is non-empty and has no right endpoint. Now fix a point $x \in g^{-1}(y)$ and let n be such that

$$St(x,\mathcal{V}(n)) \subset g^{-1}[[y,\rightarrow[].$$

Then clearly no $A(g^{-1}(y'),n)$ can contain x and meet $g^{-1}(]\leftarrow,y[)$ simultaneously. Hence, if $y \in O(y',n)$ then $O(y',n)$ must be contained in $[y,\rightarrow[$. [For, if $p < y$ and $p \in O(y',n)$ then $g^{-1}(p) \cup g^{-1}(y) \subset A(g^{-1}(y'),n)$, which is impossible].

Consequently $St(y,\mathcal{O}(n))$ is contained in $[y,\rightarrow[$. This implies that gX is developable, and hence metrizable.

$\Leftarrow$: Suppose gX is metrizable; let $(\mathcal{O}(n))_{n=1}^{\infty}$ be a sequence of open covers of gX such that for every $y \in gX$ the sequence $(St(y,\mathcal{O}(n)))_{n=1}^{\infty}$ is a local base at y. Without loss of generality we may suppose that each $\mathcal{O}(n)$ consists of convex sets. Put

$$\mathcal{V}(n) := \{g^{-1}[O] \mid O \in \mathcal{O}(n)\} \qquad (n = 1,2,\ldots).$$

Then each $\mathcal{V}(n)$ is an open cover of X, consisting of convex sets; we claim that the sequence $(\mathcal{V}(n))_{n=1}^{\infty}$ satisfies the properties of proposition 2.1.1. To prove this, choose $x_0 \in X$ and let $\xi = (A,B)$ be a (pseudo-)gap in X. Then $g^{-1}(g(x_0))$ is contained in either A or B; suppose the latter (the other case is treated analogously). Since $g^{-1}[g[B]] = B$, which is open in X, g[B] is an open neighbourhood of $g(x_0)$. Hence there is an $n_0 \in \mathbb{N}$ such that

$$St(g(x_0);\mathcal{O}(n_0)) \subset g[B].$$

This implies that $St(x_0,\mathcal{V}(n_0))$ is contained in B [For, suppose that some element $g^{-1}[O]$ of $\mathcal{V}(n_0)$ contains x_0 and intersects A. Then O is an element of $\mathcal{O}(n_0)$ containing $g(x_0)$ that is not contained in g[B]. Contradiction].

This proves the claim; the theorem follows. □

COROLLARY 1. *If* $X = (X,\leq,\tau)$ *is a GO-space then*

$$X \text{ is a p-space} \iff X^* \text{ is a p-space.}$$

PROOF. $g(X) \simeq g(X^*)$. □

COROLLARY 2. *If* $X = (X,\leq,\tau)$ *is a GO-space such that there is a (pseudo-) gap between each two distinct points of* X*, then*

$$X \text{ is a p-space} \iff X \text{ is metrizable.}$$

PROOF. The existence of (pseudo-)gaps between each pair of points from X implies that g is one-to-one. Hence gX is homeomorphic to X. □

Example of spaces that satisfy the properties of corollary 2 are for instance the Sorgenfrey line and the Michael line (i.e. the set of the real numbers with the usual order $\leq$ and the topology generated by $\lambda(\leq) \cup \{\{x\} \mid x$ is irrational$\}$; see [34]), which consequently are not p-spaces.

On the other hand, for a locally compact space X the quotient space gX has an especially nice form:

PROPOSITION 2.1.4. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is locally compact} \iff gX \text{ is a discrete space.}$$

PROOF. Obvious. □

PROPOSITION 2.1.5. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is a p-space} \Rightarrow H(X) \text{ is } \sigma\text{-discrete (in } X).$$

PROOF. Suppose X is a p-space. Whenever $x \in H(X)$ then $g(x)$ belongs to $E(gX)$. Since gX is metrizable $E(gX)$ is σ-discrete in gX. Moreover the mapping $g|H(X)$ is "at most two-to-one". Hence if $E(gX) = \bigcup_{n=1}^{\infty} A(n)$, where each $A(n)$ is discrete in gX, then

$$B(n) := g^{-1}[A(n)] \cap H(X)$$

is discrete in X and $H(X) = \bigcup_{n=1}^{\infty} B(n)$. □

In [16] it was proved that every separable LOTS is a paracompact p-space. As another application of theorem 2.1.3 we prove a generalization of this result:

THEOREM 2.1.6. *Let* $X = (X,\leq,\tau)$ *be a GO-space such that* $H(X)$ *is* σ*-discrete. Then*

$$X \text{ has a } \sigma\text{-discrete dense subset} \Rightarrow X \text{ is a paracompact p-space.}$$

PROOF. Let $D = \bigcup_{n=1}^{\infty} D(n)$ be a dense subset of X, where each D(n) is σ-discrete in X. That X is paracompact is a direct consequence of the remarks following theorem 1.3.8, so we only have to show that X is a p-space.

Let F be $g[H(X) \cup D]$ together with possible endpoints of gX. Then F is a dense subset of gX which is σ-discrete (in gX) since g is a closed map and consequently the image of a discrete set (in X) under g is discrete (in gX). We claim that E(gX) is contained in F:

Suppose y is some element of E(gX) say $[y,\rightarrow[$ is open in gX. If y is left endpoint of gX then we are done, so suppose this is not the case. Then $g^{-1}(y)$ has either a left endpoint ℓ, which consequently must belong to H(X) since ℓ cannot have a left neighbour by the construction of the quotient space gX; or $g^{-1}(y)$ has no left endpoint and consequently it has, as a convex set a non-empty interior, hence $g^{-1}(y) \cap D \neq \emptyset$.

In both cases we find that y belongs to F, so the claim is proven. It follows that E(gX) is σ-discrete; hence gX is metrizable by theorem 1.3.4. □

COROLLARY. *Let* $X = (X,\leq,\lambda(\leq))$ *be a LOTS with a* σ*-discrete dense subset. Then* X *is a paracompact* p*-space.*

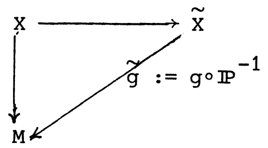
Note that in general a GO-space with a σ-discrete, dense subset (or even a separable GO-space) need not be metrizable as the Sorgenfrey line shows.

Paracompact p-spaces have especially nice properties. Recall that a completely regular space is a paracompact p-space if and only if it can be mapped onto a metrizable space by a perfect map. For GO-spaces there is a generalization, which runs as follows:

THEOREM 2.1.7. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then* X *is a paracompact* p*-space if and only if there is a metrizable GO-space* $Y = (Y,\preceq,\delta)$ *and a perfect, order preserving map* $f\colon X \rightarrow Y$ *onto* Y. *Furthermore, if* $\tau = \lambda(\leq)$

(i.e. if X *is a LOTS) then we can take* Y *such that* $\lambda(\preceq) = \delta$ *i.e. we can take* Y *to be a LOTS too.*

PROOF. We only have to prove the necessity, so suppose X is a paracompact p-space. Let $g: X \longrightarrow M$ be a perfect map from X onto M, where M is a metrizable space. Let $\tilde{X} = \tilde{X}$ (mod g) be the GO-space defined in section 1.2. Because g is perfect, the mapping $\tilde{g}: \tilde{X} \longrightarrow M$ is perfect, which implies that $\tilde{X}$ is a paracompact p-space.



Since M is metrizable it has a G_δ-diagonal. Hence by theorem 1.3.2 and its corollary, $\tilde{X}$ also has a G_δ-diagonal.

Consequently $\tilde{X}$ is metrizable by theorem 1.5.3.

The map $\mathbb{P}: X \longrightarrow \tilde{X}$ is order preserving and closed by proposition 1.2.3. Obviously it has compact fibers so $\mathbb{P}$ is perfect, which proves the first part of the theorem. Now, because f is perfect, every convexity-component under f is compact and hence has a left and a right endpoint. It follows from proposition 1.2.4 that $\tilde{X}$ is a LOTS if X is a LOTS; so the theorem is proved. □

Finally we state and prove here a theorem about the relation between p-spaces and strict p-spaces in the class of all GO-spaces. This result can also be derived with the help of some properties mentioned in [48]. We give here a direct proof:

THEOREM 2.1.8. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a strict p-space} \iff X \textit{ is a paracompact p-space.}$$

PROOF. ⇐ : Well-known (see for instance [3]).
⇒ : Suppose that X is not paracompact. Then there exists a (pseudo-)gap $\xi = (A,B)$ that is not a Q-(pseudo-)gap, say it is not a Q_ℓ-(pseudo-)gap. Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a strict pluming for X in X^+ and define

$$\mathcal{U}'(n) := \{O \subset X^+ \mid O \text{ is a convex, open subset of } X^+ \text{ that is contained in some element of } \mathcal{U}(n)\}.$$

Clearly each $\mathcal{U}'(n)$ is a cover of X consisting of convex, open sets in X^+. By proposition 1.3.6 there exists for each $n \in \mathbb{N}$ a point $x(n) \in A$ such that

$$\mathrm{St}(x(n),\mathcal{U}'(n)) \cap A \quad \text{is cofinal in } A.$$

Since ξ is not a Ω_{ℓ}-(pseudo-)gap, the sequence $(x(n))_{n=1}^{\infty}$ certainly is not cofinal in A. Hence, if we choose $x \in A$ such that $x(n) < x$ for all n, then

$$\xi \in \overline{\bigcap_{n=1}^{\infty} \mathrm{St}(x,\mathcal{U}'(n))} \subset \bigcap_{n=1}^{\infty} \overline{\mathrm{St}(x,\mathcal{U}(n))}$$

and $\bigcap_{n=1}^{\infty} \overline{\mathrm{St}(x,\mathcal{U}(n))}$ is equal to $\bigcap_{n=1}^{\infty} \mathrm{St}(x,\mathcal{U}(n))$ because $(\mathcal{U}(n))_{n=1}^{\infty}$ is a strict pluming. Hence $\xi \in \bigcap_{n=1}^{\infty} \mathrm{St}(x,\mathcal{U}(n))$, which is impossible. □

2.2. M-SPACES

Though p- and M-spaces have several common characteristics and coincide for paracompact spaces, in general p-spaces and M-spaces differ considerably. In [10] D.K. BURKE gives an example of a countably compact space (hence an M-space) which is not a p-space and an example of a locally compact space which is not an M-space nor a wΔ-space. Before looking closely at generalized ordered M-spaces, we give an a characterization of countably compact GO-spaces:

<u>DEFINITION</u>. Let $X = (X,\leq,\tau)$ be a GO-space and suppose $\xi = (A,B)$ is a (pseudo-)gap, possibly an endgap. Then ξ is said to be *countable from the left* if some strictly increasing, countably infinite sequence is cofinal in A, and ξ is said to be *countable from the right* if there is a strictly decreasing countably infinite sequence coinitial in B. The (pseudo-)gap ξ is said to be *countable* if it is countable from the left *or* from the right.

In fact, this definition of "countable (pseudo-)gap" is motivated by the following proposition.

<u>PROPOSITION 2.2.1</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent.*

(i) X *is countably compact.*

(ii) X *has no countable (pseudo-)gaps.*

PROOF. (i) $\to$ (ii) is trivial, so we only have to prove that (ii) implies (i). Suppose X has no countable (pseudo-)gaps and let $(x(n))_{n=1}^{\infty}$ be a sequence in X. In the compact space X^{+} this sequence surely has a cluster point x. If x does not belong to X and hence is some (pseudo-)gap (A,B) in X then obviously there is an increasing subsequence of $(x(n))_{n=1}^{\infty}$ cofinal in A or a decreasing subsequence of $(x(n))_{n=1}^{\infty}$ coinitial in B, so X should have a countable (pseudo-)gap. Consequently x belongs to X and the theorem follows. □

The proof of the next theorem was suggested by E.K. van Douwen. For a LOTS this result was also proved by N.V. VELICHKO ([48]) in a different way. In the following we will also give another proof which depends on the characterizations of p- and M-spaces.

THEOREM 2.2.2. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a p-space } \Rightarrow X \textit{ is an M-space.}$$

PROOF. Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of open covers of X satisfying the conditions of proposition 2.1.1. We will define a quasi-perfect mapping r from X onto a closed paracompact subspace Y of X as follows:
If $\xi = (A,B)$ is a (pseudo-)gap in X that is not a Q_{ℓ}-(pseudo-)gap, then for each $n \in \mathbb{N}$ there exists an x(n) in A such that $St(x(n),\mathcal{U}(n)) \cap A$ is cofinal in A by proposition 1.3.6.

Clearly the sequence $(x(n))_{n=1}^{\infty}$ cannot be cofinal in A so there is a point $\ell \in A$ such that $x(n) < \ell$ for every $n \in \mathbb{N}$. Hence $St(\ell,\mathcal{U}(n)) \cap A$ is cofinal in A for each n, and since $St(\ell,\mathcal{U}(n))$, like each element of $\mathcal{U}(n)$, is convex, the set $[\ell,\to[\cap A$ does not cover any (pseudo-)gaps of X.

Now for each (pseudo-)gap $q = (A(q),B(q))$ that is not a Q_{ℓ}-(pseudo-)gap choose a point $'q \in A(q)$ such that $['q,\to[\cap A(q)$ does not cover any (pseudo-)gaps of X.

For a (pseudo-)gap $p = (A(p),B(p))$ that is not a Q_r-(pseudo-)gap we define p' in an analogous way, with the additional condition that if the supremum (in X^{+}) of all $\ell \in B(p)$ such that $]\leftarrow,\ell] \cap B(p)$ does not cover any (pseudo-)gaps is a non-Q_{ℓ}-(pseudo-)gap q in X, then $p' := {}'q$. Define

$$Y := X \setminus \Big(\bigcup \{]'q,q[\mid q \text{ is a non-}Q_{\ell}\text{-(pseudo-)gap} \} \cup \bigcup \{]q,q'[\mid q \text{ is a non-}Q_{r}\text{-(pseudo-)gap.} \} \Big)$$

and a map $r : X \longrightarrow Y$ by:

$r(x) := {}'q$ if x belongs to $]'q,q[$ for some (pseudo-)gap q that is not a Q_ℓ-(pseudo-)gap.

$r(x) := q'$ if x belong to $]q,q'[$ for some (pseudo-)gap q that is not a Q_r-(pseudo-)gap.

$r(x) := x$ elsewhere.

It is easy to see that the map r is a well-defined, continuous order preserving surjection. Moreover the mapping r is a retraction and hence an identification; it then follows from proposition 1.2.3 that r is closed.

Consequently r is a quasi-perfect mapping since the fibers of r are either one-point sets or sets of the form $['q,q[$ (resp. $]q,q']$) where q is a non-Q_ℓ-(pseudo-)gap. (a non-Q_r-(pseudo-)gap respectively) which sets are countably compact by construction and proposition 2.2.1.

Y is a p-space since it is a closed subspace of X; finally Y is paracompact. For, suppose (A,B) is a (pseudo-)gap in Y which for instance, is not a Q_ℓ-(pseudo-)gap. Then

$$q := (r^{-1}[A], r^{-1}[B])$$

is a (pseudo-)gap in X. If F is a discrete set cofinal in $r^{-1}[A]$ then $r[F]$ is a discrete set cofinal in A. Hence q is not a Q_ℓ-(pseudo-)gap, so by construction A has a right endpoint, namely $r('q)$. Contradiction.

It follows that r is a quasi-perfect map from X onto the paracompact p-(=paracompact M-)space Y. Consequently X is an M-space by theorem 1.5.2 and its corollary. $\square$

The converse of theorem 2.2.2 is not true. The next example shows that there exists a (generalized) ordered M-space which is not a p-space.

<u>EXAMPLE</u>. Consider the following subset X of $W(\omega_1+1) \times ({}^*W(\omega_1) + W(\omega_1))$:

$$X := \{(x,y) \in W(\omega_1+1) \times ({}^*W(\omega_1) + W(\omega_1)) \mid y = 0 \in W(\omega_1) \text{ if } x \text{ is limit ordinal}\}$$

and supply X with the lexicographic order $\leq$.

Then $(X,\leq,\lambda(\leq))$ is a countably compact LOTS. (see for instance proposition 2.2.1.) but not a p-space: the space gX is homeomorphic to $W(\omega_1+1)$ and hence not metrizable.

In [50] we proved the fact that a GO-space is an M-space iff it is a wΔ-space, and LUTZER and BENNETT [5] afterwards proved the equivalence of the notions "quasi-complete" and "wΔ-space" for a GO-space. Here we prove both facts simultaneously.

THEOREM 2.2.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*:

(i) X *is an* M-*space*

(ii) X *is a* wΔ-*space*

(iii) X *is quasi-complete*.

PROOF. The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are valid in any space so we only have to prove that (iii) implies (i).

Let $\mathcal{D}$ be the collection of all maximal convex subsets of X that do not cover any countable (pseudo-)gap of X. Then $\mathcal{D}$ is a partition of X into closed convex subsets of X. For every $D \in \mathcal{D}$ we define a mapping $f_D : D \longrightarrow X$ as follows:

Denote the closure of D in X^+ by $[a,b]_{X^+}$ (which is possible since it is a compact, convex subset of X^+.) We have the following possibilities:

I $]a,b[_{X^+} \cap (X^+\setminus X) \neq \emptyset$; choose an interior (pseudo-)gap ξ $(=\xi_D)$ in D.

(i) $a \in X$; then define $f_D(x) := a$ for each $x \in [a,\xi[_X$.

(ii) a is a (pseudo-)gap in X that is not countable from the right; choose a point x_0 from $[a,\xi[_X$ and define $f_D(x) := x_0$ for each $x \in [a,\xi[_X$.

(iii) a is a (pseudo-)gap in X that is countable from the right and $]a,x_0[_{X^+}$ is contained in X for some $x_0 \in [a,\xi[_X$; put $f_D(x) := x$ for all x in $]a,x_0[_X$ and $f_D(x) := x_0$ if $x \in [x_0,\xi[_X$.

(iv) a is a (pseudo-)gap in X that is countable from the right and in $]a,\xi[_{X^+}$ the interior (pseudo-)gaps of $]a,\xi[_X$ are coinitial. Let $(\xi(n))_{n=1}^{\infty}$ be a strictly decreasing sequence of such (pseudo-)gaps i.e. of points in $X^+\setminus X$, such that $\xi(1) = \xi$ and $\lim_{n\to\infty} \xi(n) = a$.
Choose $x(n)$ from $]\xi(n+1),\xi(n)[_X$ and put $f_D(x) := x(n)$ for x in $]\xi(n+1),\xi(n)[_X$ $(n = 1,2,\ldots)$.

In a similar way we define f_D on $]\xi,b]_X$.

II $]a,b[_{X^+} \cap (X^+\setminus X) = \emptyset$.; fix an element $x_D \in D$. We consider three cases:

(i) $a \in X$; then define $f_D(x) := x$ for each x from $[a,x_D]_X$.

(ii) a is a (pseudo-)gap in X that is not countable from the right; then put $f_D(x) := x_D$ for every x from $[a,x_D]_X$.

(iii) a is a (pseudo-)gap in X that is countable from the right; then define $f_D(x) := x$ for each $x \in [a,x_D]_X$.

Analogously we define $f_D(x)$ for $x \in [x_D,b]_X$.

Clearly $f_D : D \longrightarrow X$ is a continuous map. Finally for every $x \in X$ put

$$f(x) := f_D(x) \quad \text{if} \quad x \in D, \quad \text{and} \quad Y := f[X].$$

Then $f: X \longrightarrow Y$ $(\subset X)$ is continuous; suppose not and let U be a convex open set in Y such that $f^{-1}[U]$ (which is convex in X since f is order preserving) is not open in X, say $f^{-1}[U]$ has a right endpoint x such that $]\leftarrow,x]_X$ is not open in X. Then x belongs to some $D \in \mathcal{D}$ and from the continuity of f_D it follows that x must be right endpoint of D. Note that hence $f_D(x) = x$, so x belongs to Y.

Since U is open in Y there exists a convex open set U' in X such that $U = U' \cap Y$. Obviously x belongs to U', so U' must contain a point x' to the right of x. Then there must be a countable (pseudo-)gap ξ between x and x', and because $U' \cap Y = U$, there is at most one such (pseudo-)gap, which hence is right neighbour of x in X^+; consequently $]\leftarrow,x]_X$ is open in X. Contradiction.

Since $f \mid Y = Y$, f is a retration; hence Y is a closed subset of X and f is an identification map. Then f is a closed mapping by proposition 1.2.3. By construction (and proposition 2.2.1) each fiber of f is countably compact. Hence f is a quasi-perfect mapping from X onto Y.

Since X is quasi-complete and Y is a closed subset of X, Y is also quasi-complete. We claim that Y is paracompact; since obviously a paracompact quasi-complete is an M-space, it then follows that X is an M-space by theorem 1.5.2 and its corollary.

To prove the claim, suppose that Y is not paracompact; hence Y has a (pseudo-)gap $\theta := (A,B)$ that is not a Q-(pseudo-)gap, say not a Q_ℓ-(pseudo-)gap. Then $\theta' := (f^{-1}[A],f^{-1}[B])$ is a (pseudo-)gap in X. If there exist a discrete set cofinal in $f^{-1}[A]$ then the same is true for A; consequently θ' is not a Q_ℓ-(pseudo-)gap. Now let $(\mathcal{H}(n))_{n=1}^{\infty}$ be a sequence of convex open covers of X such that a sequence $(x(n))_{n=1}^{\infty}$ clusters whenever it has the following property:

there is a $p \in X$ such that for every n there is a member of $\mathcal{H}(n)$ containing $\{p\} \cup \{x(k) \mid k \geq n\}$.

By proposition 1.3.6 there exists for each $n \in \mathbb{N}$ a point $a(n)$ in $f^{-1}[A]$ such that $St(a(n),\mathcal{H}(n)) \cap f^{-1}[A]$ is cofinal in $f^{-1}[A]$. Since the sequence $(a(n))_{n=1}^{\infty}$ cannot be cofinal in $f^{-1}[A]$ there exists a point $\ell \in f^{-1}[A]$ such that $a(n) < \ell$ for each $n \in \mathbb{N}$. Then the set

$$A' := [\ell,\rightarrow[\cap f^{-1}[A]$$

is countably compact; indeed, let $(x(n))_{n=1}^{\infty}$ be some sequence in A'. Because $(x(n))_{n=1}^{\infty}$ cannot be cofinal in $f^{-1}[A]$ either, there exists a point $\ell' \in f^{-1}[A]$ such that $\ell \leq x(n) \leq \ell'$ for each n.

Since $St(\ell,\mathcal{H}(n)) \cap f^{-1}[A]$ is cofinal in $f^{-1}[A]$ and each element of $\mathcal{H}(n)$ is convex, we can choose for each n a set $H(n) \in \mathcal{H}(n)$ such that $[\ell,\ell']_X \subset H(n)$, so

$$\{\ell\} \cup \{x(k) \mid k \geq n\} \subset H(n)$$

for each n; hence $(x(n))_{n=1}^{\infty}$ clusters.

Consequently $[\ell,\theta'[_X$ does not cover any countable (pseudo-)gaps, so it is contained in some $D \in \mathcal{D}$. Let $[a,b]_X$ denote the closure of D in X^+.

If $\theta' = b$ then in case I (resp. case II) f_D is defined on $]\xi_D,b]_X$ ($[x_D,b]_X$ respectively) as in subcase (ii), which is impossible, since this would imply that A has a right endpoint.

Hence θ' is an interior (pseudo-)gap of D, so case I applies in the definition of f_D.

a) If $\xi_D = \theta'$ then clearly A must have a right endpoint in each of the four subcases (i) $\rightarrow$ (iv). This is impossible

b) If $\theta' < \xi_D$, consider the possibilities in the definition of f_D, looking at $[a,\xi_D[_X$.
   - the subcases (i) and (ii) are excluded since $A \cap B = \emptyset$
   - subcase (iii) is excluded. First observe that $x_0 < \theta'$, since $]a,x_0]_X$ covers no (pseudo-)gaps. But $x_0 < \theta'$ implies that $A \cap B \neq \emptyset$ which is not true.
   - subcase (iv) is excluded since A has no right endpoint and $A \cap B = \emptyset$.

c) $\xi_D < \theta'$.

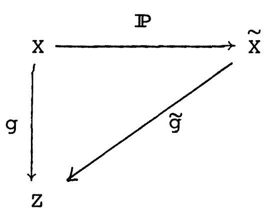
- subcase (i) and (ii) are excluded since $A \cap B = \emptyset$
- subcase (iii) is excluded. Observe that $\theta' < x_0$ and this implies that $A \cap B \neq \emptyset$, which is not true
- subcase (iv) is excluded for suppose not, and let $(x(n))_{n=1}^{\infty}$ be the sequence of points in $]\xi_D, b]_X$ used to define f_D. Then there is a first $x(n) > \theta'$ and hence A has a right endpoint $x(n-1)$ or $A \cap B \neq \emptyset$, which is impossible.

Hence all cases yield a contradiction; consequently the claim and the theorem are proved. □

The following is the analogue of theorem 2.1.7 for M-spaces.

<u>THEOREM 2.2.4</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then* X *is an* M-*space if and only if there is a metrizable GO-space* $Y = (Y,\preceq,\delta)$ *and a quasi-perfect, order preserving map* f *from* X *onto* Y.

<u>PROOF</u>. Again, we only have to prove the necessity of the condition. Suppose X is an M-space and let $g\colon X \longrightarrow Z$ be a quasi-perfect map from X onto the metrizable space Z. Let $\tilde{X} = \tilde{X}$ (mod g) and $\tilde{g}\colon \tilde{X} \longrightarrow Z$ be as defined in section 1.2. Then $\tilde{g}$ is quasi-perfect so $\tilde{X}$ is an M-space. Since Z is metrizable it has a G_δ-diagonal and hence $\tilde{X}$ has a G_δ-diagonal by theorem 1.3.2. Consequently $\tilde{X}$ is a metrizable GO-space by theorem 1.5.4. Obviously the quotient mapping $\mathbb{P} : X \longrightarrow \tilde{X}$ is quasi-perfect. Hence we may take $Y := \tilde{X}$ and $f := \mathbb{P}$. □



In contrast with the situation in 2.1.7, we cannot always take Y to be a LOTS if X is one. This is shown by the following example:

<u>EXAMPLE</u>. Let X be the ordered sum $W(\omega_1) +]0,1[$. If $f\colon X \longrightarrow Y$ is a quasi-perfect, order preserving map from the LOTS $(X,\leq,\lambda(\leq))$ onto some metrizable GO-space $(Y,\preceq,\delta)$ then $f[W(\omega_1)]$ is countably compact and hence compact since Y is metrizable. Consequently $f[W(\omega_1)]$ has a right endpoint y_0. Clearly $f^{-1}[]\leftarrow,y_0]] = W(\omega_1)$ so $]\leftarrow,y_0] \in \delta$. However, y_0 cannot have a right neighbour since this would contradict the fact that each fiber of f is countably compact. Hence $\lambda(\preceq)$ is not equal to δ, so Y is not a LOTS.

DEFINITION. Let $X = (X,\le,\tau)$ be a GO-space. Then $\mathcal{C}_X$ is the equivalence relation on X defined by

$$x\,\mathcal{C}_X y \iff \text{the closed interval between } x \text{ and } y \text{ is countably compact.} \quad (x,y\in X)$$

The elements of the decomposition of X corresponding to this equivalence relation are maximal convex sets that do not cover any countable (pseudo-) gaps of X. The decomposition space $X/\mathcal{C}_X$ is denoted by cX and $c_X\colon X \longrightarrow cX$ is the quotient map.

When no confusion is possible we shall drop the subscript X and speak of the equivalence relation $\mathcal{C}$ and the map $c\colon X \longrightarrow cX$. Also, we shall use the same symbol for the order on cX and the order on X.

PROPOSITION 2.2.5. *Let* $X = (X,\le,\tau)$ *be a GO-space. If* δ *is the identification topology on* cX *and* $\le$ *the natural order on* cX *inherited from* X *then* $cX = (cX,\le,\delta)$ *is a GO-space and* $c\colon X \longrightarrow cX$ *is a closed, order preserving map.*

PROOF. Follows directly from proposition 1.2.3. □

In [36], K. Morita introduced the notion of a countably-compactification of a space. He called a space S a *countably-compactification* of a space X if

(i) S is countably compact and contains X as a dense subspace.
(ii) Every countably compact, closed set in X is closed in S.

A. KATO ([29]) proved that not every normal M-space has a countably-compactification. On the other hand, we observe that every GO-space $X = (X,\le,\tau)$ has a generalized ordered countably-compactification X^c, which can be defined as follows:

$$X^c := X \cup \{\xi \in X^+\backslash X \mid \xi \text{ is a countable (pseudo-)gap}\} \quad (\subset X^+)$$

We give X^c the relative order with respect to the order on X^+, and the topology on X^c is generated by the subspace topology on X^c together with the collection

$\{[\xi,\rightarrow[\ \mid \xi \in X^c\backslash X$, and ξ is a (pseudo-)gap that is not countable from the left$\} \cup \{]\leftarrow,\xi] \mid \xi \in X^c\backslash X$ and ξ is a (pseudo-)gap that is not countable from the right}.

Clearly X^c is a GO-space, which contains X as a dense subspace, X^c is countably compact (see proposition 2.2.1) and it is not difficult to see by the same proposition, that every countably compact closed set F in X is closed in X^c.

Another, slightly different countably-compactification of X is obtained in the following way:

First compactify X by placing two points in each non-end gap, one point in each pseudogap or endgap, and giving the resulting set the topology corresponding to the obvious order on this set. The compactification thus obtained will be denoted by X^{++}. Define

$$X^{cc} := X \cup \{\xi \in X^{++}\setminus X \mid \xi \text{ is the limit of a (countable) sequence in } X\}.$$

then X^{cc} is a generalized ordered countably-compactification of X. Of course, X^c and X^{cc} are different as soon as X has an interior gap that is countable from both sides.

The next theorem shows that there is a close resemblance between generalized ordered p- and M-spaces.

<u>THEOREM 2.2.6</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then the following properties are equivalent*:

(i) X *is an* M-*space*

(ii) X *is a* wΔ-*space*

(iii) X *is quasi-complete*

(iv) X *has a pluming in* X^c

(v) X *has a pluming in* X^{cc}

(vi) *There exists a sequence* $(\mathcal{U}(n))_{n=1}^{\infty}$ *of convex open covers of* X *such that for each* $x \in X$ *and each countable (pseudo-)gap* $\xi = (A_\xi,B_\xi)$ *there exists an* $n = n(x,\xi) \in \mathbb{N}$ *such that* $St(x,\mathcal{U}(n))$ *does not cover the (pseudo-) gap* ξ.

(vii) cX *is metrizable*.

<u>PROOF</u>. We already know that (i), (ii), and (iii) are equivalent.

(ii) → (iv) Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of open coverings of X such that a sequence $(x(n))_{n=1}^{\infty}$ clusters whenever it has the following property:

- there is a $p \in X$ such that $x(n)$ belongs to $St(p,\mathcal{U}(n))$ for each $n \in \mathbb{N}$.
Without loss of generality we may assume that each $\mathcal{U}(n)$ consists of convex sets. Whenever U is a convex open subset of X, define

$$U_* := U \cup \{\xi \in X^c \setminus X \mid \exists u_1, u_2 \in U \quad \text{with } u_1 < \xi < u_2\}.$$

Then U_* is a convex open subset of X^c. Now put

$$\mathcal{V}(n) := \{U_* \mid U \in \mathcal{U}(n)\} \qquad (n = 1,2,\ldots).$$

Then $(\mathcal{V}(n))_{n=1}^{\infty}$ is a pluming for X in X^c. For suppose there exists a point $x \in X$ and a (pseudo-)gap $\xi = (A,B)$ in $X^c \setminus X$ (without loss of generality $x < \xi$) such that ξ belongs to $St(x,\mathcal{V}(n))$ for all n. Then for each $n \in \mathbb{N}$ there is a $U(n) \in \mathcal{U}(n)$ and $x(n) \in U(n)$ such that $x \in U(n)$ and $\xi < x(n)$. The (pseudo-)gap ξ is countable from the left or from the right; suppose the latter (the other case is treated almost analogously). Then it is possible to choose $x'(n)$ in B such that $x'(n+1) < x'(n) < x(n)$, for all n, and such that $(x'(n))_{n=1}^{\infty}$ is coinitial in B. Clearly $x'(n)$ belongs to $St(x,\mathcal{U}(n))$ for all n but $(x'(n))_{n=1}^{\infty}$ has no cluster point in X, contradicting the property of the sequence $(\mathcal{U}(n))_{n=1}^{\infty}$.
(ii) $\rightarrow$ (v) Completely analogous.
(iv) $\rightarrow$ (vi) and (v) $\rightarrow$ (vi) Whenever $(\mathcal{V}(n))_{n=1}^{\infty}$ is a pluming for X in X^c (resp. X^{cc}), put

$$\mathcal{V}'(n) := \{O \mid O \text{ is a convexity-component (in } X^c\text{, resp. } X^{cc}\text{) of some } V \in \mathcal{V}(n)\}$$

and

$$\mathcal{U}(n) := \{O \cap X \mid O \in \mathcal{V}'(n)\}.$$

Then $(\mathcal{U}(n))_{n=1}^{\infty}$ is a sequence of convex open covers of X, with the property named in (vi).
(vi) $\rightarrow$ (vii) Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of open covers as meant in (vi).
Without loss of generality we may assume that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$. For each $y \in cX$ and each $n \in \mathbb{N}$ put

$$A(c^{-1}(y),n) := c^{-1}(y) \cup U(\ell(y),n)' \cup U(r(y),n)'$$

where: $U(\ell(y),n)' := \phi$ if $c^{-1}(y)$ is left-open in X, and
$U(\ell(y),n)' := U(\ell(y),n) \cap]\leftarrow,\ell(y)]$ if $c^{-1}(y)$ is not left-open, $\ell(y)$ is the left endpoint of $c^{-1}(y)$ and $U(\ell(y),n)$ is some element of $\mathcal{U}(n)$ that contains $\ell(y)$
and $U(r(y),n)'$ is defined analogously for the right side of $c^{-1}(y)$.

Clearly $A(c^{-1}(y),n)$ is convex and open in X. Put

$$O'(y,n) := cX \backslash c[X\backslash A(c^{-1}(y),n)]$$

and

$$O(y,n) := \bigcap_{k=1}^{n} O'(y,k).$$

Since c is a closed mapping each $O(y,n)$ is an open neighbourhood of y. Define a convex open cover $\mathcal{O}(n)$ of cX for each $n \in \mathbb{N}$, by

$$\mathcal{O}(n) := \{O(y,n) \mid y \in cX\} \quad (n = 1,2,\ldots).$$

First we prove that $\bigcap_{n=1}^{\infty} St(y,\mathcal{O}(n) = \{y\}$ for each $y \in cX$. To that purpose choose $y_1,y_2 \in cX$ with $y_1 < y_2$. Then there is a countable (pseudo-)gap $\xi = (A,B)$ in X such that $c^{-1}(y_1) \subset A$ and $c^{-1}(y_2) \subset B$. Fix x_1 in $c^{-1}(y_1)$ and x_2 in $c^{-1}(y_2)$ and $n_0 \in \mathbb{N}$ such that $St(x_1,\mathcal{U}(n_0)) \subset A$ and $St(x_2,\mathcal{U}(n_0)) \subset B$.

This clearly implies that no $A(c^{-1}(y),n_0)$ can contain both $c^{-1}(y_1)$ and $c^{-1}(y_2)$; hence $y_1 \notin St(y_2,\mathcal{O}(n_0))$. Consequently, cX has a G_δ-diagonal.

Since for the metrizability of cX it is sufficient to prove that cX is a Moore space, and by the foregoing $(St(y,\mathcal{O}(n)))_{n=1}^{\infty}$ is a local base at all those points $y \in cX$ such that $y \notin H(cX)$, we only have to prove that if $]\leftarrow,y] \in \delta\backslash\lambda(\leq)$ or $[y,\rightarrow[\in \delta\backslash\lambda(\leq)$ then there exists an $n \in \mathbb{N}$ such that $St(y,O(n))$ is contained in $]\leftarrow,y]$ (in $[y,\rightarrow[$ respectively), where δ is the quotient topology and $\lambda(\leq)$ the order topology on cX.

Suppose that $[y,\rightarrow[$ belongs to $\delta\backslash\lambda(\leq)$ and $St(y,O(n)) \cap]\leftarrow,y[$ is non-empty for each $n \in \mathbb{N}$ (the other case is treated analogously). Then, for each $n \in \mathbb{N}$ there exists a $y(n) \in cX$ such that

$$y \in O(y(n),n) \quad \text{and} \quad O(y(n),n) \cap]\leftarrow,y[\neq \emptyset,$$

say $y'(n) \in O(y(n),n) \cap]\leftarrow,y[$.

Since $\bigcap_{n=1}^{\infty} St(y,\mathcal{O}(n)) = \{y\}$, the sequence $(y'(n))_{n=1}^{\infty}$ is cofinal in $]\leftarrow,y[$; so, if we choose $x(n)$ from $c^{-1}(y'(n))$ for each n, then $(x(n))_{n=1}^{\infty}$ is cofinal in $c^{-1}[]\leftarrow,y[]$ (Note that $]\leftarrow,y[$ has no right endpoint, because y has no left neighbour in cX). Hence in X the (pseudo-)gap $\xi := (c^{-1}[]\leftarrow,y[], c^{-1}[[y,\rightarrow[])$ is countable from the left. Choose $x \in c^{-1}(y)$ and a natural number n_0 such that $St(x,\mathcal{U}(n_0))$ is contained in $c^{-1}[[y,\rightarrow[]$.

Now, since $y \in O(y(n_0)),n_0)$, $c^{-1}(y)$ is contained in $A(c^{-1}(y(n_0)),n_0)$. Of course, y is not equal to $y(n_0)$. (For, if $y(n_0) = y$ then - since $c^{-1}(y)$ is left-open - it follows from the definition of $A(c^{-1}(y(n_0)),n_0)$ that $O(y(n_0),n_0)$ is contained in $[y,\rightarrow[$ which is impossible since $O(y(n_0),n_0) \cap]\leftarrow,y[$ is non-empty). But, if $y(n_0) < y$, then

$$x \in U(r(y(n_0)),n_0) \in \mathcal{U}(n_0)$$

and consequently, $U(r(y(n_0)),n_0)$ must be contained in $c^{-1}[[y,\rightarrow[]$; and if $y(n_0) > y$ then

$$x \in U(\ell(y(n_0)),n_0) \subset c^{-1}[[y,\rightarrow[]$$

and so $A(c^{-1}(y(n_0)),n_0)$ is contained in $c^{-1}[[y,\rightarrow[]$. Both cases lead to a contradiction; hence we are done.

(vii) $\rightarrow$ (ii) Since cX is metrizable, there is a development $(\mathcal{O}(n))_{n=1}^{\infty}$ for cX such that each $\mathcal{O}(n)$ consists of convex sets and $\mathcal{O}(n+1)$ refines $\mathcal{O}(n)$. For each $y \in cX$ choose, if possible, a decreasing sequence $(a(y,n))_{n=1}^{\infty}$ in $c^{-1}(y)$, that is coinitial in $c^{-1}(y)$, and an increasing sequence $(b(y,n))_{n=1}^{\infty}$ in $c^{-1}(y)$, that is cofinal in $c^{-1}(y)$, such that $a(y,1) = b(y,1)$ if both $(a(y,n))_{n=1}^{\infty}$ and $(b(y,n))_{n=1}^{\infty}$ exist.

Now, for every n and $O \in \mathcal{O}(n)$ we define a collection $\mathcal{U}(n,O)$ as follows: If O has a left endpoint ℓ such that $c^{-1}(\ell)$ has coinitiality $^*\omega_0$ but no right endpoint r with the property that $c^{-1}(r)$ has countably infinite cofinality,

$$\mathcal{U}(n,O) := \{]a(\ell,\nu+2),a(\ell,\nu)[\mid \nu = 1,2,\ldots\}\cup\{]a(\ell,2),\rightarrow[\cap c^{-1}[O]\}.$$

Analogously, if O has a right endpoint r such that $c^{-1}(r)$ has cofinality ω_0 but no left endpoint ℓ with countably infinite coinitiality of $c^{-1}(\ell)$, put

$$\mathcal{U}(n,O) := \{]b(r,\nu),b(r,\nu+2)[\mid \nu = 1,2,\ldots\} \cup \{c^{-1}[O] \cap]\leftarrow,b(r,2)[\}.$$

If O has a right endpoint r such that $c^{-1}(r)$ has cofinality ω_0 and a left-endpoint ℓ with the property that $c^{-1}(\ell)$ has coinitiality ${}^*\omega_0$

$$\mathcal{U}(n,O) := \{]a(\ell,\nu+2),a(\ell,\nu)[\mid \nu = 1,2,\ldots\} \cup \{]b(r,\nu),b(r,\nu+2)[\mid \nu = 1,2,\ldots\} \cup \{]a(\ell,2),b(r,2)[\}.$$

In all other cases, we take

$$\mathcal{U}(n,O) := \{c^{-1}[O]\}.$$

Clearly,

$$c^{-1}[O] = \bigcup \{K \mid K \in \mathcal{U}(n,O)\} \quad (O \in \mathcal{O}(n),\ n = 1,2,\ldots).$$

We define

$$\mathcal{U}(n) := \bigcup \{\mathcal{U}(n,O) \mid O \in \mathcal{O}(n)\} \quad (n = 1,2,\ldots).$$

Then every $\mathcal{U}(n)$ is an open cover of X consisting of convex sets. Suppose $x(n) \in St(x_0,\mathcal{U}(n))$ $(n = 1,2,\ldots)$ for some $x_0 \in X$. Then, for each $n \in \mathbb{N}$ there is a set $U(n) \in \mathcal{U}(n)$ such that

$$\{x_0,x(n)\} \subset U(n) \quad \text{and} \quad U(n) \in \mathcal{U}(n,O(n)) \quad \text{for some } O(n) \in \mathcal{O}(n).$$

Consequently $c(x(n))$ and $c(x_0)$ are contained in $O(n)$, so

$$c(x(n)) \in St((c(x_0),\mathcal{O}(n)) \quad (n = 1,2,\ldots).$$

Since $(\mathcal{O}(n))_{n=1}^{\infty}$ is a development for cX, the sequence $(c(x(n)))_{n=1}^{\infty}$ converges to $c(x_0)$. If infinitely many $x(n)$ do not belong to $c^{-1}(c(x_0))$ then the sequence $(x(n))_{n=1}^{\infty}$ clusters, since $c\colon X \to cX$ is a closed mapping; hence suppose that all $x(n)$ belong to $c^{-1}(c(x_0))$ and that $(x(n))_{n=1}^{\infty}$ has no cluster points. Then $(x(n))_{n=1}^{\infty}$ is cofinal or coinitial in $c^{-1}(c(x_0))$ since $c^{-1}(c(x_0))$ has no countable interior gap. Suppose the first, and, for the sake of con-convenience, put $c(x_0) = y_0$. Clearly, $c^{-1}(y_0)$ is a right-open set, hence

$]\!\leftarrow, y]$ is open in cX.

Choose $n_0 \in \mathbb{N}$ such that $St(y_0, \mathcal{O}(n_0))$ is contained in $\leftarrow, y_0]$. Furthermore, there is a $\nu_1 \geq n_0$ such that $b(y_0, \nu_1) \geq x_0$. Let n be greater than ν_1.

Now, since $x_0 \in U(n) \in \mathcal{U}(n, O(n))$ and hence $y_0 \in O(n)$, while moreover $O(n)$ is a subset of $]\!\leftarrow, y_0]$, each $O(n)$ must have y_0 as a right endpoint. Consequently, by the definition of $\mathcal{U}(n, O(n))$, each set $U(n)$ $(n \geq \nu_1)$ is a subset of $]\!\leftarrow, b(y_0, \nu_1+1)]$. Hence $x(n)$ belongs to $]\!\leftarrow, b(y, \nu_1+1)]$ in contradiction with the assumption that $(x(n))_{n=1}^{\infty}$ is cofinal in $c^{-1}(y_0)$.

Consequently, the sequence $(x(n))_{n=1}^{\infty}$ has a cluster point in cX. It follows that X is a $w\Delta$-space, which completes the proof. □

In contrast with corollary 1 of theorem 2.1.3, the corresponding proposition for M-spaces does not hold true: it is possible that X is an M-space, while X^* is not. For instance, if

$$X := W(\omega_2+1) \setminus \{\alpha \leq \omega_2 \mid cf(\alpha) = \omega_1\}$$

then X is countably compact, so X is an M-space, but X^* is not an M-space since cX is not first countable in the point $c(\omega_2)$.

<u>PROPOSITION 2.2.7</u>. *Let* $X = (X, \leq, \tau)$ *be a GO-space. If* X *is paracompact, then* $\mathcal{C} = \mathcal{G}$ *and* $gX = cX$.

<u>PROOF</u>. Obvious, since for any two points x and $x' \in X$, the closed interval between x and x' is paracompact and hence is compact if and only if it is countably compact. □

From 2.2.6 again follows that a generalized ordered p-space X is an M-space. Firstly, because a p-space X has a pluming in X^{++} and hence in X^{cc}, and secondly, because the mapping

$$f\colon gX \longrightarrow cX, \text{ defined by } f := c \circ g^{-1}$$

is a closed surjection. Obviously, this implies that cX is metrizable if gX is metrizable, for instance because semi-stratifiability is preserved by closed maps. (see [14]).

2.3. HEREDITARY PROPERTIES

In general, the property of being a p-space or M-space is not a hereditary one, though it is hereditary for closed subsets.

A.V. ARHANGELSKIĬ [4] already observed that the assumption that each subspace of a space X is a p-space is a very strong one; he proved that a space that is hereditarily Lindelöf and hereditarily a p-space, is metrizable. H.R. BENNETT and D.J. LUTZER [5] proved that a GO-space X is metrizable if each subspace is an M-space (a p-space respectively). We shall give here a new and shorter proof of this theorem, making use of the techniques, that are developed in the two previous sections. We start with two lemma's also due to Bennet and Lutzer.

LEMMA 2.3.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space that is hereditarily an M-space. Then* X *is hereditarily paracompact.*

PROOF. It is sufficient to prove that X is paracompact. Let $\xi = (A,B)$ be a (pseudo-)gap in X. We shall prove that ξ is a Q_ℓ-(pseudo-)gap.

Suppose A has no right endpoint, and is non-empty. Let $c\colon X \longrightarrow cX$ be the map defined in section 2.2. If $c[A]$ has no right endpoint, then $(c[A],c[B])$ is a (pseudo-)gap in cX. Since cX is surely metrizable, and hence paracompact, there must be a discrete subset D of $c[A]$ that is cofinal in $c[A]$. Obviously, this implies that there is a discrete set cofinal in A. Hence, consider the case that $c[A]$ has a right endpoint p. Then construct a strictly increasing sequence $(x(\alpha))_{\alpha<\beta}$ in $c^{-1}(p)$ that is cofinal in $c^{-1}(p)$ with the property that $x(\beta') = \lim_{\alpha<\beta'} x(\alpha)$ for each limit ordinal $\beta' < \beta$. Suppose that $\beta \geq \omega_1$, and look at the set

$$F := \{x(\alpha) \mid \alpha < \omega_1\}.$$

Obviously, F is homeomorphic to $W(\omega_1)$ which is well-known not to be a hereditary M-space (For instance, if $L := \{\alpha < \omega_1 \mid \alpha \text{ is isolated or } \alpha \text{ is a limit of limit ordinals}\}$ then $cL \simeq W(\omega_1)$), which is a contradiction. Hence β is a countable ordinal and we are done. Since we can prove analogously that ξ is a Q_r-(pseudo-)gap, X must be paracompact. □

COROLLARY. X *is hereditarily an* M-*space* $\Longleftrightarrow$ X *is hereditarily a* p-*space.*

LEMMA 2.3.2. *Let* $X = (X,\leq,\tau)$ *be a GO-space that is hereditarily an* M-*space. Then* X *is first countable.*

PROOF. Fix some $x \in X$, and suppose that x is not left-isolated. We shall prove that there is a (countable) sequence in $]\leftarrow,x[$ that converges to x.

If x is left endpoint of $c^{-1}(c(x))$ then $c(x)$ is not left-isolated in X. Since cX is metrizable, there is a sequence $(y(n))_{n=1}^{\infty}$ in $]\leftarrow,c(x)[$, that converges to $c(x)$. Choose $x(n)$ from $c^{-1}(y(n))$ $(n = 1,2,\ldots)$ then $(x(n))_{n=1}^{\infty}$ converges to x.

If x is not left endpoint of $c^{-1}(c(x))$ then construct an increasing transfinite sequence $(z(\alpha))_{\alpha<\beta}$ in $c^{-1}(c(x))$ converging to x, with the property that $z(\beta') = \lim_{\alpha<\beta'} z(\alpha)$ for each limit ordinal $\beta' < \beta$. Following the same reasoning as in 2.3.1, we find that β is countable. Analogously, we prove that there is a countable sequence in $]x,\rightarrow[$ converging to x, provided x is not right-isolated. Obviously, these two facts together imply that there is a countable local base at x. □

The following lemma seems to be the key-part of the proof:

LEMMA 2.3.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space that is hereditarily an* M-*space. Suppose* $\mathcal{D}$ *is an equivalence relation on* X *such that the equivalence classes of* $\mathcal{D}$ *are convex, closed sets, and such that*

(i) *each equivalence class is metrizable*

(ii) *the quotient space* $X/\mathcal{D}$ *is metrizable.*

Then X *is metrizable.*

PROOF. Denote the quotient space $X/\mathcal{D}$ by dX, and let $d: X \longrightarrow dX$ be the quotient map. Whenever $x \in X$, we shall denote $d^{-1}(d(x))$ by $\tilde{x}$. Define the following subsets of X:

$$K := \{x \in X \mid \tilde{x} = \{x\}\}.$$
$$L := \{x \in X \mid x \text{ is left endpoint of } \tilde{x} \text{ and } |\tilde{x}| > 1\}.$$
$$R := \{x \in X \mid x \text{ is right endpoint of } \tilde{x} \text{ and } |\tilde{x}| > 1\}.$$

Moreover,

$$A := L \cup \{x \in R \mid \tilde{x} \text{ has no left endpoint}\}.$$
$$B := R \cup \{x \in L \mid \tilde{x} \text{ has no right endpoint}\}.$$

The set $V := \{y \in dX \mid y \text{ is not isolated}\}$ is closed in dX, and hence a G_δ-set. Let $O(n)$ $(n = 1,2,\ldots)$ be open sets in dX such that $V = \bigcap_{n=1}^{\infty} O(n)$ and $O(n+1) \subset O(n)$, and put $U(n) := d^{-1}[O(n)]$. Now observe that if Z is a subset of X such that $d \mid Z$ is one-to-one then Z has a G_δ-diagonal, and hence is metrizable by theorem 1.5.4.

This implies that $K \cup A$ and $K \cup B$ are metrizable. Clearly A (resp. B) is contained in $E(K \cup A)$ $(E(K \cup B)$ respectively), so by 1.3.4, A,(B) is σ-discrete in $K \cup A$ $(K \cup B)$.

Consequently, A can be written as $\bigcup_{n=1}^{\infty} A(n)$ where $A(n+1) \supset A(n)$, and for each $x \in K \cup A$ and $n \in \mathbb{N}$ there exists an open (in X) convex neighbourhood $O(x,n)$ of x such that

$$O(x,n) \cap (A(n)\setminus\{x\}) = \emptyset,$$

and B can be written as $\bigcup_{n=1}^{\infty} B(n)$, where $B(n+1) \supset B(n)$, and for each $x \in K \cup B$ and $n \in \mathbb{N}$ there exists an open (in X) convex neighbourhood $U(x,n)$ of x, with

$$U(x,n) \cap (B(n)\setminus\{x\}) = \emptyset.$$

We may suppose that if x' belongs to $O(x,n)$ $(U(x,n)$ resp.) and $d(x') \neq d(x)$ then $\tilde{x}'$ is contained in $O(x,n)$ $(U(x,n)$ respectively), for there are at most two points $y \neq d(x)$ in dX such that $d^{-1}(y)$ meets $O(x,n)$ but is not contained in it. Subtracting $d^{-1}(y)$ from $O(x,n)$ for those y, we obtain a set with all the required properties. The same applies to $U(x,n)$.

We will now prove that X has a G_δ-diagonal, from which it follows immediately that X is metrizable.

Let $(\mathcal{V}(n))_{n=1}^{\infty}$ be a sequence of G_δ-coverings for dX and for each $n \in \mathbb{N}$, $y \in dX$ let $W(d^{-1}(y),n)$ be an open neighbourhood of $d^{-1}(y)$ that is mapped by d into some element of $\mathcal{V}(n)$, with the additional property that $W(d^{-1}(y),n+1) \subset W(d^{-1}(y),n)$.

Furthermore, for each $y \in dX$ let $(\mathcal{W}_y(n))_{n=1}^{\infty}$ be a sequence of G_δ-coverings for $d^{-1}(y)$. For each $x \in d^{-1}(y)$, $n \in \mathbb{N}$ choose an open (in $d^{-1}(y)$) neighbourhood $W_y(x,n)$ of x, contained in some element of $\mathcal{W}_y(n)$, such that $W_y(x,n+1) \subset W_y(x,n)$ and such that $W_y(x,n)$ contains no endpoints of $d^{-1}(y)$ except possibly x itself. In particular, this implies that $W_y(x,n)$ is open in X if x is an interior point of $\tilde{x}$.

Now, for $x \in X$, $n \in \mathbb{N}$ define $W(x,n)$ as follows:

- if $x \in \mathrm{Int}(\tilde{x})$ then $W(x,n) := W_{d(x)}(x,n)$
- if $x \notin \mathrm{Int}(x)$ then we have the following possibilities:
  - (i) $x \in K$
    $W(x,n) := U(x,n) \cap O(x,n) \cap U(n) \cap W(d^{-1}(d(x)),n)$.
  - (ii) $x \in L$
    $W(x,n) := [(O(x,n) \cap \,]\leftarrow,x]) \cup W_{d(x)}(x,n)] \cap U(n) \cap W(d^{-1}(d(x)),n)$.
  - (iii) $x \in R$
    $W(x,n) := [(U(x,n) \cap [x,\rightarrow[) \cup W_{d(x)}(x,n)] \cap U(n) \cap W(d^{-1}(d(x)),n)$.

Observe that in all cases: $W(x,n) \cap \tilde{x} = W_{d(x)}(x,n)$. (*)

Put

$$\mathcal{W}(n) := \{W(x,n) \mid x \in X\} \qquad (n = 1,2,\ldots)$$

then each $\mathcal{W}(n)$ is an open cover of X. We shall prove that $\bigcap_{n=1}^{\infty} \mathrm{St}(x,\mathcal{W}(n)) =$ $= \{x\}$ for each $x \in X$.

To this end fix x_1 and $x_2 \in X$, such that $x_1 \neq x_2$. Then there exists a natural number n, depending only on x_1 and x_2 such that each $W(x,n)$ misses either x_1 or x_2.

Let x be an arbitrary element of X. We have the following possible cases:

(I) $d(x_1) \neq d(x_2)$.
Take n such that $d(x_2) \notin \mathrm{St}(d(x_1),\mathcal{V}(n)))$.

Since $W(x,n)$ is contained in $W(d^{-1}(d(x)),n)$, and hence is mapped into some element of $\mathcal{V}(n)$, which cannot contain both $d(x_1)$ and $d(x_2)$, either x_1 or x_2 does not belong to $W(x,n)$.

(II) $d(x_1) = d(x_2)$ (Clearly x_1 and x_2 do not belong to K).

a) $d(x_1)$ is an isolated point of dX.

Take n such that $d(x_1)$ does not belong to $O(n)$ and $x_2 \notin \mathrm{St}(x_1,\mathcal{W}_{d(x_1)}(n))$.

If $d(x) = d(x_1)$ then (*) and the condition $x_2 \notin \mathrm{St}(x_1,\mathcal{W}_{d(x_1)}(n))$ imply that $W(x,n)$ does not contain both x_1 and x_2.

If $d(x) \neq d(x_1)$ then $W(x,n) \cap \tilde{x}_1 = \emptyset$ if $W(x,n)$ is contained in $\tilde{x}$, and if $W(x,n)$ is not contained in $\tilde{x}$, then $W(x,n) \subset U(n)$, because $d(x)$ is not isolated, so $W(x,n) \cap \tilde{x}_1$ is empty too.

b) $d(x_1)$ is not isolated point of dX.
We have three possible subcases.

1) $\tilde{x}_1$ has a left endpoint ℓ and no right endpoint.
Consequently $\ell \in A \cap B$.

Take n such that $\ell \in A(n) \cap B(n)$ and $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$. If $d(x) = d(x_1)$ then again (*) implies that $W(x,n)$ misses either x_1 or x_2; if $d(x) \neq d(x_1)$ then $W(x,n) \cap \tilde{x}_1 = \emptyset$ if $W(x,n)$ is contained in $\tilde{x}$. If $W(x,n)\setminus\tilde{x}$ is non-empty then $x \in K \cup A$ or $x \in K \cup B$, and hence $U(x,n)$ (or $O(x,n)$ respectively) is defined and contains $W(x,n)\setminus\tilde{x}$. Consequently, $W(x,n)$ misses ℓ, and hence $\tilde{x}_1$.

2) $\tilde{x}_1$ has a right endpoint r and no left endpoint.
The argument for this case is completely analogous to that for the preceding case.

3) $\tilde{x}_1$ has a left endpoint ℓ and a right endpoint r.
Fix n such that $\ell \in A(n)$, $r \in B(n)$ and $x_2 \notin St(x_1, \mathcal{W}_{d(x_1)}(n))$. Assume that $d(x) \neq d(x_1)$ and $W(x,n)$ is not contained in $\tilde{x}$. (else argue as under 1).). Then $W(x,n)\setminus\tilde{x}$ is contained in either $O(x,n)$ or $U(x,n)$; so it misses either ℓ or r, and hence $W(x,n) \cap \tilde{x}_1 = \emptyset$.

It follows that in all possible cases x_2 does not belong to $St(x_1, \mathcal{W}(n))$ for some n; so $\bigcap_{n=1}^{\infty} St(x, \mathcal{W}(n)) = \{x\}$ for each $x \in X$, which proves the lemma. □

<u>PROPOSITION 2.3.4</u>. *Let* $X = (X, \leq, \tau)$ *be a GO-space such that* $X = A \cup B$ *where* A *and* B *are dense, metrizable subspaces of* X. *Then* X *is metrizable.*

<u>PROOF</u>. We claim that a σ-discrete (in A) subset of A is σ-discrete in X. Indeed, let F be a discrete (in A) subset of A. Then F is relatively discrete, and since X is heriditarily collectionwise normal, there exists for each $x \in F$ a convex open (in X) neighbourhood $U(x)$ of X such that $U(x) \cap U(x') = \emptyset$ if $x \neq x'$. Since B is dense in X, $\{U(x) \cap B \mid x \in F\}$ is a disjoint collection of non-empty convex open subsets of B, which consequently can be written as $\bigcup_{n=1}^{\infty} \mathcal{O}(n)$, where each $\mathcal{O}(n)$ is a discrete collection in B, since B is surely perfectly normal (see theorem 1.3.8). Now put

$$F(n) := \{x \in F \mid U(x) \cap B \in \mathcal{O}(n)\} \quad (n = 1,2,\ldots).$$

Then each $F(n)$ is discrete in X, and $F = \bigcup_{n=1}^{\infty} F(n)$.

Hence each discrete subset of A is σ-discrete in X, and the same holds for a σ-discrete (in A) subset of A. Of course an analogous statement is true for a σ-discrete (in B) subset of B.

Now if D is a σ-discrete (in A) dense subset of A, then D is also a σ-discrete (in X) dense subset of X. Since $E(X) \subset E(A) \cup E(B)$, and both $E(A)$ and $E(B)$ are σ-discrete in X, $E(X)$ is σ-discrete in X. Hence X is metrizable by theorem 1.3.4. □

LEMMA 2.3.5. *Let* $X = (X,\leq,\tau)$ *be a compact GO-space that is hereditarily an M-space. Then* X *is metrizable.*

PROOF. Define an equivalence relation $\sim$ on X by

$x \sim x' \iff$ The closed interval between x and x' is metrizable.

It can be deduced easily from the fact that X is first countable, that each equivalence class is a closed convex, metrizable subspace of X. Since $X/\sim$ is compact it has no gaps or pseudogaps. Moreover $X/\sim$ has no jumps for because of the definition of the relation $\sim$, no neighbours can occur in $X/\sim$. Hence $X/\sim$ is connected by theorem 1.2.1. Suppose X is not metrizable. Then $X/\sim$ consists of more than one point; so it is easy to construct two disjoint dense subsets F and G of $X/\sim$ such that $F \cup G = X/\sim$. [By transfinite induction define $A(\alpha)$ and $B(\alpha)$ by choosing two points from each convexity component of $X/\sim \setminus \overline{\bigcup_{\eta<\alpha} (A(\eta) \cup B(\eta))}$ and assigning one to $A(\alpha)$ and one to $B(\alpha)$. If $\bigcup_{\alpha<\beta} (A(\alpha) \cup B(\alpha))$ is dense then the same holds for $\bigcup_{\alpha<\beta} A(\alpha)$ and $\bigcup_{\alpha<\beta} B(\alpha)$].

Since F and G are p-spaces too (note that the quotient map $\mathbb{P}$ is perfect, and use [27] or [20]), and $gF \simeq F$; $gG \simeq G$, since g_F (resp. g_G) is bijective and closed and hence a homeomorphism, F and G are metrizable by 2.2.6. It now follows from 2.3.4 that $X/\sim$ is metrizable.

Hence X is metrizable by lemma 2.3.2. □

THEOREM 2.3.6. (BENNETT and LUTZER [5]). *Let* $X = (X,\leq,\tau)$ *be a GO-space that is hereditarily an M-space. Then* X *is metrizable.*

PROOF. By lemma 2.3.3 it is sufficient to prove that each C of the decomposition cX of X is metrizable.

Let C be some element of cX (= gX). Whenever x and y ($x \leq y$) are two elements of C, the interval $[x,y]$ is compact and hence metrizable by lemma 2.3.5. Let $(\ell(n))_{n=1}^{\infty}$ be a non-increasing sequence coinitial in C and $(r(n))_{n=1}^{\infty}$ a non-decreasing sequence cofinal in C with $\ell(n) \leq r(n)$ for each

n. (Observe that such sequences exist: if, for instance, the cofinality of C is uncountable then take $x_0 \in C$, and it follows that $[x_0,\rightarrow[\cap C$ is countably compact and hence - because of paracompactness - compact, which is impossible). Then $C = \bigcup_{n=1}^{\infty}[\ell(n),r(n)]$ and $[\ell(n),r(n)]$ is metrizable for each n. It follows that C is metrizable which proves the theorem. □

CHAPTER III

LEXICOGRAPHIC PRODUCTS

Whenever $X = (X,\leq_1,\lambda(\leq_1)$ and $Y = (Y,\leq_2,\lambda(\leq_2))$ are LOTS's then by the lexicographic product of X and Y we mean the LOTS $(X,Y,\leq,\lambda(\leq))$, where X.Y is the cartesian product $X \times Y$ of X and Y, and $\leq$ is the lexicographic order on X.Y, viz.: $(x,y) \leq (x',y')$ if and only if $x <_1 x'$ or ($x=x'$ and $y \leq_2 y'$)

In this chapter we investigate when the lexicographic product of two ordered spaces is a p-space or an M-space. It turns out that in this connection the behaviour of p-spaces is similar to that of metrizable and perfectly normal spaces, for which we refer to ([19] Chapter 4.)

3.1. LEXICOGRAPHIC PRODUCTS AND p-SPACES

<u>THEOREM 1.3.1</u>. *Let $(X,\leq_1,\lambda(\leq_1))$ and $(Y,\leq_2,\lambda(\leq_2))$ be LOTS's such that Y has neither a left nor a right endpoint. Then*

$$X.Y \text{ is a p-space} \iff Y \text{ is a p-space.}$$

<u>PROOF</u>. Immediately clear, since X.Y is a disjoint union of open subsets of X.Y, each homeomorphic to Y. □

<u>THEOREM 1.3.2</u>. *Let $(X,\leq_1,\lambda(\leq_1)$ and $(Y,\leq_2,\lambda(\leq_2))$ be LOTS's and suppose Y has both a left and a right endpoint and no (interior) gaps. Then*

$$X.Y \text{ is a p-space} \iff X \text{ is a p-space.}$$

<u>PROOF</u>. Clearly, Y is a compact LOTS. Moreover, since Y has two endpoints the projection $\mathbb{P} : X.Y \longrightarrow X$, defined by

$$\mathbb{P}(x,y) := x,$$

is continuous and closed.

We now look at the following diagram:

$$\begin{array}{ccc} X.Y & \longrightarrow & X \\ g_{X.Y}\big\downarrow & & \big\downarrow g_X \\ g(X.Y) & \overset{f}{\dashrightarrow} & gX \end{array}$$

where $g_{X.Y}$ and g_X are the maps described in section 2.1. We define $f\colon g(X.Y) \longrightarrow gX$ by

$$f := g_X \circ \mathbb{P} \circ (g_{X.Y})^{-1}.$$

Then f is a (univalent) surjective continuous map. Moreover, f is bijective because Y is compact (hence, if (A,B) is a gap in X.Y, then ($\mathbb{P}[A]$, $\mathbb{P}[B]$) is a gap in X), and f is closed since $\mathbb{P}$ and g_X are closed.

Consequently, g(X.Y) is homeomorphic to gX. The theorem follows. □

<u>THEOREM 3.1.3</u>. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's, and suppose that* Y *has two endpoints and at least one interior gap. Then*

$$X.Y \textit{ is a p-space} \iff X \textit{ is } \sigma\textit{-discrete and } Y \textit{ is a p-space.}$$

<u>PROOF</u>. Denote the left endpoint of Y by ℓ, and the right endpoint by r.

$\Rightarrow$: Since Y is homeomorphic to the closed subspace $\{x\} \times Y$ of X.Y, where x is any element of X, Y must be a p-space.

Because Y is not compact, it has at least one gap $\xi = (A,B)$. In the following we adopt the convention that whenever θ is a gap in Y and x_0 is an element in X, then by (x_0,θ) we mean the following gap in X.Y:

$$(\{(x,y) \in X.Y \mid x <_1 x_0 \vee (x=x_0 \wedge y <_2 \theta)\},$$

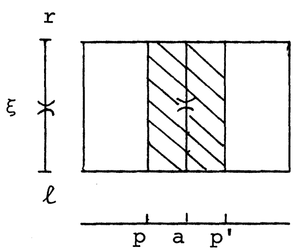
$$\{(x,y) \in X.Y \mid x_0 <_1 x \vee (x=x_0 \wedge \theta <_2 y)\}).$$

Since X.Y is a p-space, there exists a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of convex coverings of X.Y, with the property that for each $(x,y) \in X.Y$, and each gap θ in X.Y, there exists a natural number n such that $St((x,y),\mathcal{U}(n))$ does not cover the gap θ. Moreover, we may assume without loss of generality, that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$ $(n = 1,2,\ldots)$. Define for each $n \in \mathbb{N}$

$$A(n) := \{x \in X \mid (x,r) \notin St((x,\ell),\mathcal{U}(n))\}.$$

Then every $A(n)$ is discrete in X. For if $a \in X$ and a is not an endpoint of X then there exist (p,q) and (p',q') in X.Y and U and U' in $\mathcal{U}(n)$ such that $(p,q) < (a,\ell)$ and $(a,r) < (p',q')$ while

$$](p,q),(a,\ell)] \subset U \quad \text{and} \quad [(a,r),(p',q')[\subset U'.$$



Clearly $p < a < p'$.
Hence $]p,p'[$ is an open neighbourhood of a in X such that for every $x \neq a$ from $]p,p'[$ we have:

$$(x,r) \in St((x,\ell),\mathcal{U}(n)).$$

Consequently $]p,p'[\cap (A(n)\setminus\{x\}) = \emptyset$. The case that a is an endpoint of X is only formally different.

Next, $X = \bigcup_{n=1}^{\infty} A(n)$. For suppose $X \neq \bigcup_{n=1}^{\infty} A(n)$, say $x_0 \in X\setminus\bigcup_{n=1}^{\infty} A(n)$. Then (x_0,r) belongs to $St((x_0,\ell),\mathcal{U}(n))$ for each n; and hence $St((x_0,\ell),\mathcal{U}(n))$ covers the gap (x_0,ξ) for each n, which is a contradiction.

Consequently X is σ-discrete.

$\Leftarrow$: Since Y is a p-space, there exists a sequence $(\mathcal{B}(n))_{n=1}^{\infty}$ of open covers of Y with the properties of proposition 2.1.1. Because Y is a LOTS there exist, for each $y \in Y$ $(\ell \neq y \neq r)$ and each $n \in \mathbb{N}$, $a(n,y)$ and $b(n,y) \in Y$ such that

$$y \in]a(n,y),b(n,y)[\subset B \quad \text{for some } B \in B(n)$$

and for ℓ and r there exist $a(n,\ell)$ and $b(n,r)$ respectively in Y such that $\ell \in [\ell,b(n,\ell)[\subset B$ and $r \in]a(n,r),r] \subset B'$ for some $B,B' \in \mathcal{B}(n)$.

X is σ-discrete; say $X = \bigcup_{n=1}^{\infty} A(n)$ where $A(n+1) \supset A(n)$ and for each $x \in X$, $n \in \mathbb{N}$ there exist $c(n,x)$ and $d(n,x)$ in X with the property that

$$x \in \,]c(n,x),d(n,x)[\quad \text{and} \quad]c(n,x),d(n,x)[\,\cap\, (A(n)\setminus\{x\}) = \emptyset$$

if x is not an endpoint of X, and

$$x \in \,]\leftarrow,d(n,x)[\quad \text{and} \quad [x,d(n,x)[\,\cap\, (A(n)\setminus\{x\}) = \emptyset \quad \text{or}$$

$$x \in \,]c(n,x),\rightarrow[\quad \text{and} \quad]c(n,x),x] \,\cap\, (A(n)\setminus\{x\}) = \emptyset \quad \text{respectively,}$$

if x is left (resp. right) endpoint of X.

Observe that $\bigcap_{n=1}^{\infty}\,]c(n,x),d(n,x)[\, = \{x\}$ (if x is not an endpoint of X; in the other cases a similar assertion holds).

Now define for each $n \in \mathbb{N}$ and each $(x,y) \in X.Y$ an open interval $U((x,y),n)$ in X.Y, which contains (x,y), in the following way:

if $\ell \neq y \neq r$: $U((x,y),n) := \,](x,a(n,y)),(x,b(n,y))[$

if $y = r$: $U((x,y),n) := \,](x,a(n,r)),(x,r)] \cup [(x,r),(d(n,x),\ell)[$ if x is not the right endpoint of X and $U((x,y),n) :=$ $= \,](x,a(n,r)),(x,r)]$ if x is the right endpoint of X.

if $y = \ell$: $U((x,y),n) :=](c(n,x),r),(x,\ell)] \cup [(x,\ell)(x,b(n,\ell))[$ if x is not the left endpoint of X, and $U((x,y),n) :=$ $= [(x,\ell),(x,b(n,\ell))[$ if x is the left endpoint of X.

Put $\mathcal{U}(n) := \{U((x,y),n) \mid (x,y) \in X.Y\}$ for each $n \in \mathbb{N}$. Then each $\mathcal{U}(n)$ is an open cover of X.Y. We will prove that $(\mathcal{U}(n))_{n=1}^{\infty}$ satisfies the properties of proposition 2.1.1.

To that purpose choose a point $(x_0,y_0) \in X.Y$ and an $n_0 \in \mathbb{N}$ such that $x_0 \in A(n_0)$. Then for $n \geq n_0$ the following is true:

$$St((x_0,y_0),\mathcal{U}(n)) \subset \{x_0\} \times St(y_0,\mathcal{B}(n)) \cup [(x_0,r),(d(n,x_0),\ell)[\,\cup$$

$$\cup \,](c(n,x_0),r),(x_0,\ell)]$$

(if x_0 is not an endpoint of X; otherwise trivial alterations have to be made).

This clearly implies that $\bigcap_{n=1}^{\infty} St((x_0,y_0),\mathcal{U}(n)) \subset \{x_0\} \times Y$. Consequently we only have to show that for each gap in X.Y of the form $\theta = (x_0,\xi)$, where ξ is a gap in Y, there is an n such that $St((x_0,y_0),\mathcal{U}(n))$ does not cover θ.

Now let θ be such a gap in X.Y. Then there is an $n_1 \geq n_0$ such that $St(y_0,\mathcal{B}(n_1))$ does not cover ξ. It follows that $\theta = (x_0,\xi)$ is not covered by $\{x_0\} \times St(y_0,\mathcal{B}(n_1))$ so θ is not covered by $St((x_0,y_0),\mathcal{U}(n_1))$. Hence the theorem is proved. □

DEFINITION. Let $X = (X,\leq,\tau)$ be a GO-space. A subset $A \subset X$ is said to be *σ-ℓ-discrete (in X)* if $A = \bigcup_{n=1}^{\infty} A(n)$ where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x,n)$ of x such that $O(x,n) \cap \cap (A(n)\backslash\{x\}) \cap]\leftarrow,x] = \emptyset$.

A subset $A \subset X$ is called *σ-r-discrete (in X)* if $A = \bigcup_{n=1}^{\infty} A(n)$ where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x,n)$ of x such that $O(x,n) \cap (A(n)\backslash\{x\}) \cap [x,\rightarrow[= \emptyset$. (cf. FABER [ ]).

X is a *left-(right)* p-*space* if there exists a sequence $(\mathcal{O}(n))_{n=1}^{\infty}$ of coverings of X consisting of left-(right-) open convex sets in X, with the property that for each $x \in X$, and each (pseudo-)gap in X there is an $n \in \mathbb{N}$ such that $St(x,\mathcal{O}(n))$ does not cover ξ. (Note that if X is σ-ℓ-discrete (σ-r-discrete) then X is a left-(right-) p-space.)

THEOREM 3.1.4. *Let $X = (X,\leq,\tau)$ be a GO-space. Then*

$$X \textit{ is a p-space} \iff X \textit{ is both a left-p-space and a right-p-space.}$$

PROOF. $\Rightarrow$: Evident.
$\Leftarrow$: Let for each $n \in \mathbb{N}$, $\mathcal{O}(n)$ (resp.$\mathcal{V}(n)$) be a covering of X with left-(right-) open convex sets, with the property that for each $x \in X$, and each (pseudo-)gap ξ in X there is an $n \in \mathbb{N}$ such that $St(x,\mathcal{O}(n))$ ($St(x,\mathcal{V}(n))$ respectively) does not cover ξ. Without loss of generality we may suppose that $\mathcal{O}(n+1)$ defines $\mathcal{O}(n)$ and that $\mathcal{V}(n+1)$ defines $\mathcal{V}(n)$. Put

$$\mathcal{O}'(n) := \{int_X(O) \mid O \in \mathcal{O}(n)\} \text{ and } \mathcal{V}'(n) := \{int_X(V) \mid V \in \mathcal{V}(n)\}$$
$$(n = 1,2,\ldots).$$

Whenever $x \in K(n) := X\backslash(\bigcup\mathcal{O}'(n) \cup \bigcup\mathcal{V}'(n))$ put

$$U(x,n) := O(x,n) \cup V(x,n),$$

where $O(x,n)$ and $V(x,n)$ are fixed sets such that

$$x \in O(x,n) \in \mathcal{O}(n) \quad \text{and} \quad x \in V(x,n) \in \mathcal{V}(n).$$

Clearly $U(x,n)$ is open in X. Let

$$\mathcal{U}(n) := \mathcal{O}'(n) \cup \mathcal{V}'(n) \cup \{U(x,n) \mid x \in K(n)\} \qquad (n = 1,2,\ldots).$$

Then each $\mathcal{U}(n)$ is an open cover of X. Now suppose $x_0 \in X$ and $\xi = (A,B)$ is a (pseudo-)gap in X. Without loss of generality we may suppose that $x_0 \in A$. First choose n_0 such that $St(x_0,\mathcal{O}(n_0)) \cup St(x_0,\mathcal{V}(n_0))$ does not cover ξ. For each $n \in \mathbb{N}$ these obviously is at most one $x(n) \in K(n) \cap]x_0,\rightarrow[$ such that $x_0 \in U(x(n),n)$.. If such an $x(n)$ does not exist for $n = n_0$ then $St(x_0,\mathcal{U}(n_0))$ does not cover ξ and we are done. So suppose there exists an $x(n_0) \in K(n_0) \cap]x_0,\rightarrow[$ such that $x_0 \in U(x(n_0),n_0)$. Clearly $x(n_0)$ is an element of A. Fix $n_1 \geq n_0$ such that $St(x(n_0),\mathcal{O}(n_1)) \cup St(x(n_0),\mathcal{V}(n_1))$ is contained in A. If $x(n_1)$ does not exist we have again that $St(x_0,\mathcal{U}(n_1))$ does not cover ξ, and we are done. If $x(n_1)$ does exist, then $x_0 < x(n_1) \leq x(n_0)$ so $U(x(n_1),n_1)$ is contained in A because $St(x(n_0),\mathcal{V}(n_1))$ does not cover ξ. Hence $St(x_0,\mathcal{U}(n_1)$ does not cover ξ. □

THEOREM 3.1.5. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's, and suppose* Y *has a left (right) endpoint, no right (left) endpoint, and no interior gaps. Then*:

$$X.Y \text{ is a p-space} \iff \begin{array}{l} X \text{ is a left-(right-) p-space, and} \\ D := \{x \in X \mid x \text{ has no right (left) neighbour}\} \text{ is } \sigma\text{-}\ell\text{-discrete } (\sigma\text{-r-discrete}). \end{array}$$

PROOF. We only consider the case that Y has a left endpoint ℓ. By r we denote the right endgap of Y.

Note that Y is a p-space, since Y, having no interior gaps, is locally compact.

$\Rightarrow$: Suppose $(\mathcal{U}(n))_{n=1}^{\infty}$ is a sequence of convex open covers of X.Y with the property that for each $(x,y) \in X.Y$ and each gap ξ in X.Y there is an $n \in \mathbb{N}$ such that $St((x,y),\mathcal{U}(n))$ does not cover ξ.

For every $d \in D := \{x \in X \mid x \text{ has no right neighbour}\}$, there exists a gap in X.Y defined by the decomposition

$$(\{(x,y) \in X.Y \mid x \leq d\},\ \{(x,y) \in X.Y \mid d < x\})$$

which we denote by (d,r).

Then if we put $D(n) := \{d \in D \mid (d,r)$ is not covered by $St((d,\ell),\mathcal{U}(n))\}$, it is easy to see, that $D = \bigcup_{n=1}^{\infty} D(n)$, and that for each $x \in X$ and each $n \in \mathbb{N}$ a convex open neighbourhood $F(x,n)$ exists such that

$$F(x,n) \cap (D(n)\setminus\{x\}) \cap]\leftarrow,x] = \emptyset.$$

Hence D is σ-ℓ-discrete.

To prove that X is a left p-space, choose, for each $x \in X$ that is not left endpoint of X, and each $n \in \mathbb{N}$, points $a(n,x) \in X$, $b(n,x) \in Y$ such that $](a(n,x),b(n,x)),(x,\ell)]$ is contained in some $U \in \mathcal{U}(n)$, and put

$$O(x,n) :=]a(n,x),x] \quad \text{if } x \text{ is not left endpoint of } X,$$

$$O(x,n) := \{x\} \quad \text{if } x \text{ is left endpoint of } X,$$

and

$$O(n) := \{O(x,n) \mid x \in X\}.$$

Observe that each $O(x,n)$ is left-open in X, and that $\mathcal{O}(n)$ covers X. Take $x_0 \in X$ and let $\theta = (A,B)$ be a gap in X. Then $\theta' := (\mathbb{P}^{-1}[A],\mathbb{P}^{-1}[B])$ is a gap in X.Y, where $\mathbb{P} : X.Y \longrightarrow X$ is the projection on the first coordinate. Hence there is an $n \in \mathbb{N}$ such that $St((x_0,\ell),\mathcal{U}(n))$ does not cover θ'. Then $St(x_0,\mathcal{O}(n))$ does not cover θ. For, suppose $x_0 \in O(x,n)$, and $O(x,n)$ meets A and B. This clearly implies that (x_0,ℓ) belongs to $](a(n,x),b(n,x)),(x,\ell)]$, and that $](a(n,x),b(n,x)),(x,\ell)]$ meets both $\mathbb{P}^{-1}[A]$ and $\mathbb{P}^{-1}[B]$. Hence $St(x_0,\ell),\mathcal{U}(n))$ covers θ', which is a contradiction. Consequently, X is a left-p-space.

$\Leftarrow$: Suppose $D = \bigcup_{n=1}^{\infty} D(n)$, where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $F(x,n)$ of x such that

$$F(x,n) \cap (D(n)\setminus\{x\}) \cap]\leftarrow,x] = \emptyset$$

and let $(\mathcal{O}(n))_{n=1}^{\infty}$ be a sequence of left-open convex covers of X, with the property that for each $x \in X$ and each gap ξ in X there is an $n \in \mathbb{N}$ such that $St(x,\mathcal{O}(n))$ does not cover ξ.

For each x in X, that is not left endpoint of X, and each $n \in \mathbb{N}$ let $s(n,x)$ be an element of X such that

$$s(n,x) \in F(x,n) \cap O(x,n) \cap]\leftarrow,x[\text{ for some } O(x,n) \in \mathcal{O}(n) \text{ that contains } x$$

unless $F(x,n) \cap O(x,n) \cap]\leftarrow,x[$ is empty for every $O(x,n) \in \mathcal{O}(n)$ with $x \in O(x,n)$, in which case x has a left neighbour, which we then label $s(n,x)$. Define

$$U((x,\ell),n) := \{x\} \times Y \qquad \text{if } x \text{ is left endpoint of } X;$$

$$U((x,\ell),n) :=]s(n,x),\ell),(x,\ell)] \cup (\{x\} \times Y) \quad \text{for all other } x \in X.$$

and put

$$\mathcal{U}(n) := \{U(x,\ell),n) \mid x \in X\} \qquad (n = 1,2,\ldots).$$

Clearly each $\mathcal{U}(n)$ is an open covering of X.Y with convex sets. Take $(x_0,y_0) \in X.Y$ and suppose $\theta = (A,B)$ is a gap in X.Y. Because Y has no interior gaps, the sets $\mathbb{P}[A]$ and $\mathbb{P}[B]$ do not intersect. Moreover $\mathbb{P}[A] \cup \mathbb{P}[B] = X$, and $\mathbb{P}[B]$ has no left endpoint. There are two possibilities:

(i) $\mathbb{P}[A]$ has no right endpoint.
Then $(\mathbb{P}[A], \mathbb{P}[B])$ is a gap ξ in X. Choose $n \in \mathbb{N}$ such that ξ is not covered by $St(x_0,\mathcal{O}(n))$. Then θ is not covered by $St((x_0,y_0),\mathcal{U}(n))$. For, suppose (x_0,y_0) belongs to some $U((x,\ell),n)$ that covers θ; since Y has no interior gaps, we have $U((x,\ell),n) =](s(n,x),\ell),(x,\ell)] \cup \{x\} \times Y$. Evidently, $s(n,x)$ is not the left neighbour of x; so $[s(n,x),x]$ is contained in some $O \in \mathcal{O}(n)$ which covers ξ, in contradiction with our choice of n.

(ii) $\mathbb{P}[A]$ has a right endpoint d.
Then $\theta = (d,r)$, and d has no right neighbour, so d belongs to D. Let $n \in \mathbb{N}$ be such that $d \in D(n)$. Then the gap $\theta = (d,r)$ is not covered by any $U \in \mathcal{U}(n)$. For, if $x \leq_1 d$ then clearly $U((x,\ell),n)$ does not cover θ; and if $d <_1 x$ then (d is not the left neighbour of x and so) $d <_1 s(n,x)$, since $s(n,x)$ either belongs to $F(x,n)$ or is left neighbour of x, and consequently $\theta = (d,r)$ is not covered by $](s(n,x),\ell),(x,\ell)] \cup (\{x\} \times Y)$. It follows that $St(x_0,y_0),\mathcal{U}(n))$ does not cover θ.

Hence $(\mathcal{U}(n))_{n=1}^{\infty}$ is a sequence of convex open covers of X.Y with the properties of proposition 2.1.1; the theorem follows. □

THEOREM 3.1.6. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's, and suppose* Y *has at least one interior gap, a left (right) endpoint and no right (left) endpoint. Then*

X.Y *is a* p*-space* ⟺ X *is* σ-ℓ*-discrete* (σ-r*-discrete*), Y *is a* p*-space and if* X *contains neighbourpoints, and the interior gaps are cofinal (coinitial) in* Y[1] *then* Y *has cofinality* ω_0. *(coinitiality* $^*\omega_0$*)*.

PROOF. We only consider the case that Y has a left endpoint ℓ and no right endpoint. Whenever x belongs to X and ξ is an interior gap of Y, or when x is a point of X without right neighbour and r is the right endgap of Y, we will denote the corresponding gap in X.Y by (x,r). By $\mathbb{P}$: X.Y ⟶ X we mean the mapping that maps the point (x,y) from X.Y onto x. Note that $\mathbb{P}$ need not be continuous.

⇒: Since gY is homeomorphic to $g_{X.Y}[\{x\} \times Y]$ for any $x \in X$, and g(X.Y) is metrizable, gY is metrizable, so Y is a p-space by theorem 2.1.3. Let ξ be some fixed interior gap of Y, and let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of open covers of X.Y with the properties described in proposition 2.1.1. Then $X = \bigcup_{n=1}^{\infty} D(n)$, where

$$D(n) := \{x \in X \mid St((x,\ell),\mathcal{U}(n)) \text{ does not cover } (x,\xi)\}.$$

It follows that X is σ-ℓ-discrete. To prove the remaining assertion, let x^+ be the right neighbour of x in X and suppose the interior gaps are cofinal in Y. Then in g(X.Y) the point $g(x^+,\ell)$ is not left-isolated and g(X.Y) is metrizable; hence there is an increasing sequence $(g(x(n),y(n)))_{n=1}^{\infty}$ in g(X.Y) converging to $g(x^+,\ell)$ where (x(n),y(n)) are points from X.Y (n = 1,2,...). Clearly $\{(x(n),y(n)) \mid n \geq n_0\}$ belongs to, and is cofinal in

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1) "The interior gaps are cofinal in Y" means: $\forall y \in Y$ the half-line $]y,\rightarrow[$ has an interior gap. Analogous for "coinitial".

$\{x\} \times Y$ for some $n_0 \in \mathbb{N}$. Hence $\{y(n) \mid n \geq n_0\}$ is cofinal in Y.

$\Leftarrow$: Suppose $X = \bigcup_{n=1}^{\infty} D(n)$ such that $D(n) \subset D(n+1)$, and for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $F(x,n)$ of x such that $F(x,n) \cap (D(n)\setminus\{x\}) \cap\ ]\leftarrow,x] = \emptyset$. Furthermore, let $(\mathcal{O}(n))_{n=1}^{\infty}$ be a sequence of open covers of Y with the properties of proposition 2.1.1, such that $\mathcal{O}(n+1)$ refines $\mathcal{O}(n)$ for each n. We shall only consider the case that X has neighbourpoints and that the interior gaps are cofinal in Y, in which case there exists by assumption some increasing sequence $(y(n))_{n=1}^{\infty}$ cofinal in Y.

For each $y \in Y$, $n \in \mathbb{N}$, choose $O(y,n)$ such that $y \in O(y,n) \in \mathcal{O}(n)$ and whenever $(x,y) \in X.Y$ define $U((x,y),n)$ as follows:

$y \neq \ell$: $U((x,y),n) := \{x\} \times (O(y,n)\setminus\{\ell\}$

$y = \ell$: $U((x,y),n) :=$ -$\{x\} \times O(\ell,n)$ if x is left endpoint of X.

-$(\{x\} \times O(\ell,n)) \cup\ ](s(n,x),\ell),(x,\ell)]$ if $F(x,n) \cap$ $\cap\ ]\leftarrow,x[\ \neq \emptyset$, where $s(n,x)$ is some element of $F(x,n) \cap\ ]\leftarrow,x[$.

-$(\{x\} \times O(\ell,n)) \cup\ ](x^-,y(n)),(x,\ell)[$ if $F(x,n) \cap$ $\cap\ ]\leftarrow,x[\ = \emptyset$ and x^- is the left neighbour of x.

Put

$$\mathcal{U}(n) := \{U((x,y),n) \mid (x,y) \in X.Y\} \qquad (n = 1,2,\ldots)$$

then the sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ has the properties of proposition 2.1.1. To prove this, choose $(x_0,y_0) \in X.Y$ and let $\xi = (A,B)$ be a gap in X.Y.

Choose $n_0 \in \mathbb{N}$ such that $x_0 \in D(n_0)$ and $y_0 <_2 y(n_0)$. Then $x = x_0$ whenever (x_0,y_0) belongs to $U((x,y),n_0)$. Hence $St((x_0,y_0),\mathcal{U}(n_0))$ is contained in $U((x_0,\ell),n_0) \cup (\{x_0\} \times Y)$.

Consider the following three possible cases:

(i) $\mathbb{P}[A] \cap \mathbb{P}[B] = \emptyset$, and $\mathbb{P}[A]$ has no right endpoint. Then $(\mathbb{P}[A],\mathbb{P}[B])$ is a gap θ in X. If $x_0 \in \mathbb{P}[A]$ then clearly $St((x_0,y_0),\mathcal{U}(n_0))$ does not cover ξ. If $x_0 \in \mathbb{P}[B]$ choose $x_1 \in\ ]\leftarrow,x_0[\ \cap \mathbb{P}[B]$ and $n_1 \geq n_0$ such that $x_1 \in D(n_1)$. Then $St((x_0,y_0),\mathcal{U}(n_1))$ is contained in B, so it does not cover ξ.

(ii) $\mathbb{P}[A] \cap \mathbb{P}[B] = \emptyset$ and $\mathbb{P}[A]$ has a right endpoint x_1. If $x_0 \leq_1 x_1$ then clearly $St((x_0,y_0),\mathcal{U}(n_0))$ does not cover ξ; hence suppose $x_1 <_1 x_0$. Then x_1 is not the left neighbour of x_0, so ξ is not covered by

$St((x_0,y_0),\mathcal{U}(n_1))$ if n_1 is a natural number greater than n_0 such that $x_1 \in D(n_1)$.

(iii) $\mathbb{P}[A] \cap \mathbb{P}[B] \neq \emptyset$, say $x_1 \in \mathbb{P}[A] \cap \mathbb{P}[B]$. Then $\xi = (x_1,\theta)$ for some gap θ in Y; again the case is clear if $x_0 <_1 x_1$. If $x_1 \leq_1 x_0$ choose $n_1 \geq n_0$ such that θ is not covered by $St(y_0,\mathcal{O}(n_1))$ if $x_1 = x_0$, and such that $y(n_1) > \theta$ and $x_1 \in D(n_1)$ if $x_1 <_1 x_0$. In either case, ξ is not covered by $St((x_0,y_0),\mathcal{U}(n_1))$.

Consequently, in all three cases we can find an $n_1 \in \mathbb{N}$ such that ξ is not covered by $St((x_0,y_0),\mathcal{U}(n_1))$. Hence X.Y is a p-space by proposition 2.1.1. □

COROLLARY. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's and suppose* X *has no neighbourpoints,* Y *has a left (right) endpoint, and no right (left) endpoint. Then*

$$X.Y \text{ is a p-space} \iff X \text{ is } \sigma\text{-}\ell\text{-discrete } (\sigma\text{-r-discrete}) \text{ and } Y \text{ is a p-space.}$$

3.2. LEXICOGRAPHIC PRODUCTS AND M-SPACES

Since there is so close a resemblance between p-spaces and M-spaces, it is not surprising that the question when the lexicographic product of two LOTS's is an M-space can be answered in a way very similar to that for the analogous question for p-spaces. In most cases therefore, we will not give complete proofs, but leave it to the reader to make some obvious modifications in the corresponding theorem for p-spaces.

THEOREM 3.2.1. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's such that* Y *has neither a left nor a right endpoint. Then*

$$X.Y \text{ is an M-space} \iff Y \text{ is an M-space.}$$

PROOF. Obvious, since X.Y is the disjoint union of open subsets of X.Y, each homeomorphic to Y. □

THEOREM 3.2.2. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's and suppose* Y *has both a left and a right endpoint, and no countable gaps. Then*

$$X.Y \text{ is an M-space} \iff X \text{ is an M-space.}$$

<u>PROOF</u>. Arguing in a way analogous to that in the proof of theorem 3.1.2 we find that the map

$$f := c_X \circ \mathbb{P} \circ (c_{X.Y})^{-1}$$

is a homeomorphism between cX and c(X.Y), and the theorem follows immediately from 2.2.6. □

<u>THEOREM 3.2.3</u>. *Let* $X = (X,\leq_1,\lambda(\leq_1))$ *and* $Y = (Y,\leq_2,\lambda(\leq_2))$ *be LOTS's and suppose* Y *has two endpoints and at least one countable gap. Then*

$$X.Y \textit{ is an M-space} \iff X \textit{ is } \sigma\textit{-discrete and } Y \textit{ is an M-space.}$$

<u>PROOF</u>. Denote the left endpoint of Y by ℓ, and the right endpoint of Y by r.
⇒: Since Y is homeomorphic to the closed subspace $\{x\} \times Y$ of X.Y where x is any element of X, Y must be an M-space.

Furthermore, let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of open covers of X.Y such that a sequence $(x(n))_{n=1}^{\infty}$ in X.Y clusters whenever there exists a point $x \in X.Y$ such that $x(n) \in St(x,\mathcal{U}(n))$ for each n. Moreover we suppose that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$ and that $\mathcal{U}(n)$ consists of convex sets for each $n \in \mathbb{N}$. Put

$$X(n) := \{x \in X \mid (x,r) \notin St((x,\ell),\mathcal{U}(n))\}, \qquad (n = 1,2,\ldots).$$

Clearly $X = \bigcup_{n=1}^{\infty} X(n)$, and it is easy to check that each X(n) is discrete (in X).

⇐: This can be proved in a way analogous to the proof of 3.1.3. However, we give here a different proof, which makes use of the results of M.J. FABER [19] about the metrizability of a lexicographic product. To be able to do this, we first define what we mean with the lexicographic product of a LOTS and a GO-space: If $(X,\leq_1,\lambda(\leq_1))$ is a LOTS, and $(Y,\leq_2,\tau)$ is a GO-space then Y is homeomorphic with the subspace $Y \times \{0\}$ of Y^* (see 1.1).

We define $X.Y := \{(x,(y,0)) \mid x \in X, y \in Y\} \subset X.(Y^*)$ with the relative order and relative topology of the lexicographic product $X.(Y^*)$. Obviously, if Y is a LOTS, then this definition coincides with the ordinary definition of X.Y.

Now consider the following diagram.

$$\begin{array}{ccc} X.Y & \xrightarrow{\quad c_{X.Y} \quad} & c(X.Y) \\ \Big\downarrow f & \nearrow_{\; g:=c_{X.Y}\circ f^{-1}} & \\ X.cY & & \end{array}$$

where $f: X.Y \to X.cY$ is defined $f(x,y) := (x,(c(y),0))$.

One easily verifies that f is an identification map. We define $g: X.cY \twoheadrightarrow c(X.Y)$ by

$$g := c_{X.Y}\circ f^{-1}.$$

Clearly g is well-defined, and hence continuous. Since $c_{X.Y}$ is closed by proposition 1.2.3, g is closed too.

Now suppose X is σ-discrete, and Y is an M-space. Then cY is metrizable, and it follows easily that $(cY)^*$ is metrizable (see also [19]). By ([19] th. 4.4.2) $X.(cY)^*$ is metrizable, hence the subspace X.cY is metrizable. Since g is closed c(X.Y) must be metrizable. Consequently X.Y is an M-space. □

DEFINITION. A GO-space $X = (X,\leq,\tau)$ is a *left-(right-) M-space* if it admits a sequence $(\mathcal{U}(n))_{n=1}^{\infty}$ of convex left-(right-) open covers of X with the following property (*):

(*): If $\xi = (A,B)$ is a countable (pseudo-)gap in X and $x \in X$ then there exists an $n \in \mathbb{N}$ such that $St(x,\mathcal{U}(n))$ does not cover ξ.

REMARK. X is a left- (right-) M-space iff it admits a sequence $(\mathcal{O}(n))_{n=1}^{\infty}$ of convex left- (right-) open covers of X with property (**):

(**): If $x_0 \in X$, and $x(n) \in St(x_0,\mathcal{O}(n))$ for $n = 1,2,\ldots$, then the sequence $(x(n))_{n=1}^{\infty}$ clusters.

PROPOSITION 3.2.4. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

X is an M-space $\Longleftrightarrow$ *X is both a left-M-space and a right-M-space.*

PROOF. Compare the proof of 3.1.4. □

THEOREM 3.2.5(a). *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's, such that Y has a left (right) endpoint, a countable right (left) endgap, and no countable interior gaps. Then*

X.Y *is an* M-*space* $\Longleftrightarrow$ X *is a left-(right-)* M-*space and* D := $\{x \in X \mid x$ *has no right (left) neighbour*$\}$ *is* σ-ℓ-*discrete* (σ-r-*discrete*).

PROOF. This is completely analogous to the proof of theorem 3.1.4 (Replace everywhere "gap" by "countable gap"). □

THEOREM 3.2.5(b). *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be a LOTS's, and suppose* Y *has a left (right) endpoint, an uncountable right (left) endgap and no countable interior gaps. Then*

X.Y *is an* M-*space* $\Longleftrightarrow$ X *is a left-(right-)*M *space and* F := $\{x \in X \mid ci(]x,\to[) = {}^*\omega_0\}$ *is* σ-ℓ *discrete*. (*Resp.* F' := $\{x \in X \mid cf(]\leftarrow,x[) = \omega_0\}$ *is* σ-*r*-*discrete*).

PROOF. Again analogous to the proof of 3.1.5 with the exception that D is replaced by F (resp. F'). □

THEOREM 3.2.6. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's such that* Y *has a left (right) endpoint, a right (left) endgap and at least one countable interior gap. Then*

X.Y *is an* M-*space* $\Longleftrightarrow$ X *is* σ-ℓ-*discrete* (σ-r-*discrete*), Y *is an* M-*space, and if the countable interior gaps are cofinal (coinitial) in* Y *and* X *has neighbourpoints, then* Y *has cofinality* ω_0 *(coinitiality* ${}^*\omega_0$).

PROOF. Again, for the proof we refer to the corresponding theorem for p-spaces, namely theorem 3.16.

COROLLARY. *Let* $(X,\leq_1,\lambda(\leq_1))$ *and* $(Y,\leq_2,\lambda(\leq_2))$ *be LOTS's and suppose* X *has no neighbourpoints,* Y *has a left (right) endpoint and a countable right (left) endgap. Then*

X.Y *is an* M-*space* $\Longleftrightarrow$ X *is* σ-ℓ-*discrete* (σ-r-*discrete*) *and* Y *is an* M-*space*.

CHAPTER IV

Σ-SPACES

Σ-spaces were introduced by K. NAGAMI in [37]. They are a generalization both of M-spaces and of σ-spaces (in the case of T_3-spaces) and they play an important role in product theory. Some nice properties of Σ-spaces are illustrated by the fact that if a space X is the countable union of closed Σ-spaces, then X is also a Σ-space, and that Σ-spaces are preserved by quasi-perfect mappings both ways.

In this chapter we will describe the relations of generalized ordered Σ-spaces to M-spaces, and make a start with a characterization of Σ-spaces in the class of all GO-spaces.

4.1. Σ-SPACES VERSUS M-SPACES

We start with some definitions. Let X be a topological space and let $\mathcal{H}$ be a covering of X. Whenever $x \in X$, we put $C(x,\mathcal{H}) := \bigcap \{H \mid x \in H \in \mathcal{H}\}$. A *Σ-net* or *Σ-network* for X is a σ-locally finite closed cover $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ of X (where each $\mathcal{F}(n)$ is locally finite) such that for each $x \in X$

(i) the set $C(x,\mathcal{F})$ is countably compact (we will mostly write $C(x)$ instead of $C(x,\mathcal{F})$.)

(ii) $\mathcal{F}$ *contains an outer network for* $C(x)$ in X, i.e. if U is an open set containing $C(x)$ then there is an $F \in \mathcal{F}$ such that $C(x) \subset F \subset U$.

A space that admits a Σ-network is called a *Σ-space*.

If X has a Σ-network $\mathcal{F}$ such that each $C(x)$ is compact then $\mathcal{F}$ is called a *strong Σ-network*, and X a *strong Σ-space*.

Without loss of generality we may always suppose that

a) each $\mathcal{F}(n)$ covers X

b) $\mathcal{F}(n+1) \supset \mathcal{F}(n)$ $(n = 1,2,\ldots)$

c) each $\mathcal{F}(n)$ is closed under finite intersections.

In the sequel we will always make these assumptions. Note that $C(x,\mathcal{F}(n))$ is an element of $\mathcal{F}(n)$ in that case, and that condition (ii) can be rewritten as

(ii)' If U is an open set containing $C(x)$ then $C(x) \subset C(x,\mathcal{F}(n)) \subset U$ for some $n \in \mathbb{N}$.

From these definitions it is evident that each countably compact space and each regular σ-space (and a fortiori each metrizable space) is a Σ-space. Moreover, since the inverse image of a Σ-space under a quasi-perfect mapping is again a Σ-space, it is clear that each M-space is a Σ-space. Even for a GO-space the converse is not true, as the next example shows.

EXAMPLE. Let $X := \{(x,y) \in W(\omega_1+1) \times (\mathbb{Q} \cap]-1,1[) \mid y = 0$ if x is a limit ordinal$\}$ with the lexicographic order and corresponding topology. If we put

$$X(r) := \{(x,r) \in X \mid x \text{ is a non-limit ordinal}\} \cup \{(x,0) \in X \mid x \text{ is a limit ordinal}\} \quad (r \in \mathbb{Q} \cap]-1,1[)$$

then each $X(r)$ is a closed subset of X homeomorphic to $W(\omega_1+1)$ and hence a Σ-space, which implies that X, as the countable union of closed Σ-spaces, is a Σ-space itself. Since $cX \simeq X$, and X is not metrizable, X is not an M-space.

From the example above, we see that X can very well be a Σ-space without cX being metrizable, even if X and cX are homeomorphic. In fact, the following theorem shows that the quotient space cX is of little use when we want to know whether X is a Σ-space or not.

THEOREM 4.1.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a } \Sigma\textit{-space} \iff cX \textit{ is a } \Sigma\textit{-space}.$$

PROOF. Let C be some element of $X/\mathcal{C}$, where $\mathcal{C}$ is the equivalence relation used in defining cX, and let $x_0 \in C$. If C has countably infinite cofinality, let $(a(n))_{n=1}^{\infty}$ be a strictly increasing sequence cofinal in C such that $a(1) = x_0$ and put

$$C'(n) := [x_0,a(n)];$$

otherwise put

$$C'(n) := [x_0,\rightarrow[\cap C$$

for each n.

If C has countably infinite coinitiality, let $(b(n))_{n=1}^{\infty}$ be a strictly decreasing sequence coinitial in C such that $b(1) = x_0$, and put

$$C''(n) := [b(n),x_0];$$

else

$$C''(n) :=]\leftarrow,x_0] \cap C$$

for each n.

Next define

$$C(n) := C'(n) \cup C''(n)$$

and

$$X(n) := \bigcup \{C(n) \mid C \in X/\mathcal{C}\}.$$

Clearly each X(n) is a closed subset of X, X is the union of all X(n), and the mapping c maps eachs X(n) onto cX. Moreover, the restriction of c to X(n) is closed, because c is a closed mapping, and X(n) is a closed set; and by construction of X(n), the map $c \mid X(n)$ has countably compact fibers, so it is quasi-perfect.

Consequently, if cX is a Σ-space, then each X(n) is a Σ-space, which implies that X is a Σ-space.

Conversely, if X is a Σ-space then X(1) is a Σ-space so cX is a Σ-space. □

One of the difficulties in the handling of Σ-spaces is that a GO-space can be a Σ-space without having a *convex Σ-net* i.e. a Σ-network consisting of convex sets. This fact is illustrated by the following theorem.

<u>THEOREM 4.1.2</u>. *Let* X = (X,≤,τ) *be a GO-space. Then* X *has a Σ-network consisting of convex sets if and only if* X *is an* M*-space.*

PROOF. (i) Let X be an M-space. Then by theorem 2.2.4 there is a metrizable GO-space $Y = (Y,\preceq,\delta)$ and an order preserving, quasi-perfect map f from X onto Y. By [19], Y has a σ-discrete base $\mathcal{B}$ consisting of convex sets. Now put

$$\mathcal{F} := \{f^{-1}[\bar{B}] \mid B \in \mathcal{B}\}.$$

Then $\mathcal{F}$ is a σ-discrete closed cover of X, and each element of $\mathcal{F}$ is convex since f is order preserving. Moreover, it is not difficult to show that $\mathcal{F}$ is a Σ-network for X because f is quasi-perfect.

(ii) Suppose X has a Σ-net $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ consisting of convex sets. We will show that X is an M-space by proving that cX is metrizable.

- cX has a G_δ-diagonal.

For each n the collection $\{c[F] \mid F \in \mathcal{F}(n)\}$ consists of closed sets and is closure preserving. Hence, for each $y \in cX$ and each $n \in \mathbb{N}$, the set

$$U(y,n) := cX \setminus \bigcup \{c[F] \mid y \notin c[F] \text{ and } F \in \mathcal{F}(n)\}$$

is an open neighbourhood of y.

Put

$$\mathcal{U}(n) := \{U(y,n) \mid y \in cX\} \qquad (n = 1,2,\ldots)$$

and let y_1, y_2 be two distinct points of cX, for instance $y_1 < y_2$. Fix points $x_i \in c^{-1}(y_i)$ $(i = 1,2)$.

Since $C(x_i)$ is countably compact, and convex, $C(x_i)$ is contained in $c^{-1}(y_i)$. Let U_1 and U_2 be disjoint open neighbourhoods of $c^{-1}(y_1)$ and $c^{-1}(y_2)$ respectively, such that $c^{-1}[c[U_i]] = U_i$ $(i = 1,2)$.

Because $\mathcal{F}$ contains an outer network for $C(x_i)$ there is a natural number n such that $C(x_i,\mathcal{F}(n))$ is contained in U_i $(i = 1,2)$. Now let $U(y,n)$ be some element of $\mathcal{U}(n)$, where y is an element of cX. By the fact that U_1 and U_2 are disjoint, saturated sets containing $C(x_1,\mathcal{F}(n))$ and $C(x_2,\mathcal{F}(n))$ respectively, it follows that not both $C(x_1,\mathcal{F}(n))$ and $C(x_2,\mathcal{F}(n))$ can intersect $c^{-1}(y)$. Hence either $y \notin c[C(x_1,\mathcal{F}(n))]$ or $y \notin c[C(x_2,\mathcal{F}(n))]$, which implies that $y_1 \notin U(y,n)$ or $y_2 \notin U(y,n)$. Since this applies to any $U(y,n)$ we have that $y_2 \notin St(y_1,\mathcal{U}(n))$.

Consequently, cX has a G_δ-diagonal.

- The set $E(cX)$ is σ-discrete (in cX).

Put

$$L := \{y \in cX \mid [y,\rightarrow[\text{ is open in } cX\},$$

and choose, for each $y \in L$ a point $x(y)$ from $c^{-1}(y)$. If we put

$$L(n) := \{y \in L \mid C(x(y),\mathcal{F}(n)) \subset c^{-1}[[y,\rightarrow[]\}$$

then $L = \bigcup_{n=1}^{\infty} L(n)$; for, whenever $y \in L$, $c^{-1}[[y,\rightarrow[]$ is an open neighbourhood of $C(x(y))$ and $\mathcal{F}$ contains an outer network for $C(x(y))$.

Furthermore, each $L(n)$ is discrete in cX; take $y_0 \in cX$. If $c^{-1}(y_0)$ has a left (right) endpoint let $U(\ell)$ ($U(r)$ respectively) be a convex neighbourhood of that endpoint intersecting at most finitely many $F \in \mathcal{F}(n)$; otherwise let $U(\ell)$ (resp. $U(r)$) be empty. Put

$$U := c^{-1}(y_0) \cup U(\ell) \cup U(r)$$

and

$$O(y_0,n) := cX\backslash c[X\backslash U].$$

Then $O(y_0,n)$ is convex neighbourhood of y_0. Since $C(x(y),\mathcal{F}(n))$ and $C(x(y'),\mathcal{F}(n))$ are different if y and y' are different elements of $L(n)$, U contains at most finitely many $x(y)$ $(y \in L(n))$, so $O(y_0,n)$ contains only finitely many $y \in L(n)$. Hence L is σ-discrete. It follows immediately that $E(cX)$ is σ-discrete (in cX). The metrizability of cX now follows from theorem 1.3.4. □

As an application of this theorem, the next theorem shows that although even a paracompact LOTS with a Σ-network need not be an M-space (see the example above theorem 4.1.1) the situation changes if we look at perfectly normal spaces.

<u>THEOREM 4.1.3.</u> *Let* $X = (X,\leq,\tau)$ *be a GO-space that is perfectly normal. Then*

$$X \textit{ is a } \Sigma\textit{-space} \Rightarrow X \textit{ is an M-space}.$$

PROOF. Suppose X is a Σ-space and let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for X (where each $\mathcal{F}(n)$ is locally finite). For each $F \in \mathcal{F}$ let $(F^{(m)})_{m=1}^{\infty}$ be a decreasing sequence of open sets such that $F = \bigcap_{m=1}^{\infty} F^{(m)}$.

Now let n be some natural number. Then $\mathcal{F}(n)$ is a locally finite cover of X, and since each GO-space is collectionwise normal and countably paracompact (see [21]) there exists a locally finite open cover $\mathcal{U}(n) = \{U(F) \mid F \in \mathcal{F}(n)\}$ of X such that $F \subset U(F)$ for each $F \in \mathcal{F}(n)$ by [28].

Let F be some element of $\mathcal{F}(n)$. The open set $F^{(n)} \cap U(F)$ decomposes into a collection $\{O(F,\alpha) \mid \alpha \in A(F)\}$ of open convexity-components. Whenever $O(F,\alpha) \cap F$ is non-empty, put

$$F_\alpha := \bigcap \{G \mid G \text{ is a convex subset of X that contains } O(F,\alpha) \cap F\}.$$

(i.e. F_α is the "convex closure" of $O(F,\alpha) \cap F$) and

$$\mathcal{H}(F,n) = \{F_\alpha \mid O(F,\alpha) \cap F \neq \emptyset,\ \alpha \in A(F)\}.$$

Clearly, each F_α is a non-empty, convex subset of X. Observe that $F_\alpha \subset O(F,\alpha) \subset U(F)$ and that $\mathcal{H}(F,n)$ covers F. Moreover, (each $O(F,\alpha) \cap F$ and hence also) each F_α is closed as is easily seen, and $\mathcal{H}(F,n)$ is a discrete (in X) collection, (in fact the family $\{O(F,\alpha) \mid O(F,\alpha) \cap F \neq \emptyset,\ \alpha \in A(F)\}$ is discrete in X from which it follows immediately that $\mathcal{H}(F,n)$ is discrete). Put

$$\mathcal{F}'(n) := \bigcup \{\mathcal{H}(F,n) \mid F \in \mathcal{F}(n)\}.$$

Then $\mathcal{F}'(n)$ is locally finite, as can be deduced easily from the fact that each $\mathcal{H}(F,n)$ is discrete, and the fact that $\bigcup \{F_\alpha \mid F_\alpha \in \mathcal{H}(F,n)\} \subset U(F)$ for each fixed $F \in \mathcal{F}(n)$ and that $\mathcal{U}(n)$ is locally finite; furthermore each $\mathcal{F}'(n)$ consists of convex closed sets and covers X. We will prove that

$$\mathcal{F}' := \bigcup \{\mathcal{F}'(n) \mid n = 1,2,\ldots\}$$

is a (convex) Σ-network for X, which implies that X is an M-space.
- Take $x \in X$ and put

$$C'(x) := \bigcap \{F' \mid x \in F' \in \mathcal{F}'\}.$$

then $C'(x)$ is contained in $C(x) := \bigcap \{F \mid x \in F \in \mathcal{F}\}$; for, suppose $a \notin C(x)$. Then there exists an $F \in \mathcal{F}$ - say $F \in \mathcal{F}(n)$ - such that $x \in F$ and $a \notin F$. Choose $m \geq n$ such that $a \notin F^{(m)}$. Then a is not an element of any H from $\mathcal{H}(F,m)$. Since there is a $H \in \mathcal{H}(F,m) \subset \mathcal{F}'(m)$ that contains x, the point a does not belong to $C'(x)$. Consequently $C'(x)$ is countably compact, and since X is paracompact, $C'(x)$ is compact. Of course $C'(x)$ is convex.

- Let ℓ and r be the left and right endpoint of $C'(x)$ and take a convex open neighbourhood U of $C'(x)$. Then either there exists an $a < \ell$ such that $\emptyset \neq]a,\ell[\subset U \cap]\leftarrow,\ell[$, in which case we take n_1 so large that a does not belong to $C(x,\mathcal{F}'(n_1))$; or $[\ell,\rightarrow[$ belongs to τ. Then $]\leftarrow,\ell[\cap C(x)$ is compact, and hence is either empty or has a right endpoint z. If $]\leftarrow,\ell[\cap C(x)$ is empty, take n_1 such that $C(x,\mathcal{F}(n_1)) \subset [\ell,\rightarrow[$. Then $C(x,\mathcal{F}'(n_1))$ is also contained in $[\ell,\rightarrow[$; for, if we put $F := C(x,\mathcal{F}(n_1)) \in \mathcal{F}(n_1)$, then

$$x \in O(F,\alpha) \cap F \subset [\ell,\rightarrow[\quad \text{for some } \alpha \in A(F),$$

hence $x \in F_\alpha \subset [\ell,\rightarrow[$, which implies that $C(x,\mathcal{F}'(n_1)) \subset [\ell,\rightarrow[$. If $]\leftarrow,\ell[\cap C(x)$ has a right endpoint z, take n_1 such that $z \notin C(x,\mathcal{F}'(n_1))$.

Choosing n_2 analogously for the right side of $C'(x)$ and putting $n := \max(n_1,n_2)$, we have found a natural number n such that $C'(x) \subset$ $\subset C'(x,\mathcal{F}'(n_1)) \subset U$. Hence $\mathcal{F}'$ is a Σ-network for X. □

As a corollary to this last theorem we have that the Sorgenfrey line is not a Σ-space since it is perfectly normal and not an M-space (see chapter 2). Another way to prove this fact is by means of the following theorem of Nagami:

THEOREM 4.1.4. (NAGAMI). *Let X be a paracompact Σ-space. Then X is a σ-space if X has a G_δ-diagonal.*

COROLLARY. *Let $X = (X,\leq,\tau)$ be a GO-space with a G_δ-diagonal. Then X is a Σ-space if and only if X is metrizable.*

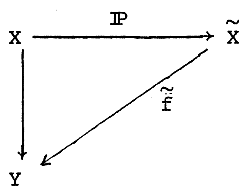
PROOF. Trivial, since a GO-space with a G_δ-diagonal is paracompact and GO-spaces that are σ-spaces are metrizable. □

DEFINITION. A *pre-σ-space* is a space which can be mapped onto a σ-space by a quasi-perfect map. In general there exist the implications M-space ⇒ pre-σ-space ⇒ Σ-space, none of which can be reversed (see [37]). For a GO-space we have the following:

PROPOSITION 4.1.5. *Let* X = (X,≤,τ) *be a GO-space. Then*

$$X \text{ is a pre-}\sigma\text{-space} \iff X \text{ is an } M\text{-space}.$$

PROOF. (for the ⇒ side only).
Suppose f: X ⟶ Y is a quasi-perfect surjection and Y is a σ-space. Since Y is surely a T_3-space, it has a G_δ-diagonal. Then the space $\tilde{X} = \tilde{X}$ (mod f) constructed from X and f as described in section 1.2 has a G_δ-diagonal too (th. 1.3.2). The mapping $\mathbb{P}$ is quasi-perfect and X is a Σ-space, so also $\tilde{X}$ has a Σ-network. Hence X is metrizable by theorem 4.1.4 and its corollary. It follows that X is an M-space. □



4.2. DECOMPOSITIONS OF Σ-SPACES

Since, by theorem 4.1.1. the decomposition space cX is not very helpful for the characterization of Σ-spaces, we shall now construct other decompositions of a GO-space X that are more useful in this respect. To this end, we want to make the elements of the decomposition bigger, such that the metrizability of the quotient space is implied by the fact that X is a Σ-space.

DEFINITION. A GO-space X = (X,≤,τ) is called a *countably-Σ-space* if it has a countable Σ-network i.e. a Σ-network consisting of countably many elements. Essentially, the countably-Σ-spaces are the $\Sigma(\aleph_0)$-spaces defined by NAGAMI [37].

Evidently, if $F = \{F_1, F_2, \ldots\}$ is a countable Σ-network for a space X, then we can write $F = \bigcup_{n=1}^{\infty} F(n)$ where each $F(n)$ is finite, and the $F(n)$'s still fulfil the extra conditions a),b) and c) from the beginning of this chapter.

DEFINITION. Suppose X = (X,≤,τ) is a GO-space. Then M_X is the equivalence relation on X, defined by

$$x \, \mathcal{M}_X \, y \iff \text{the closed interval between } x \text{ and } y \text{ admits a countable } \Sigma\text{-network} \quad (x,y \in X).$$

The decomposition space $X/\mathcal{M}_X$ is denoted by mX and

$$m_X\colon X \longrightarrow mX$$

is the quotient map.

When no confusion is likely, we will drop the subscript X on $\mathcal{M}_X$ and m_X.

The elements of the decomposition $X/\mathcal{M}$ are convex of course but not necessarily closed. Hence mX is not necessarily a T_1-space. The situation changes however, when X is a Σ-space.

<u>PROPOSITION 4.2.1</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. If* X *is a* Σ*-space then* $X/\mathcal{M}$ *is a decomposition of* X *into closed sets.*

<u>PROOF</u>. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for X, let M be some element of $X/\mathcal{M}$, $x \in M$, and suppose some point $a \in X$ belongs to $\bar{M}/M$. Without loss of generality we may suppose that $M \subset]a,\rightarrow[$. If there should exist a sequence $(a(n))_{n=1}^{\infty}$ in $]a,x]$ converging to a, then, by the definition of $\mathcal{M}$, each set $[a(n),x]$ has a countable Σ-network, since each a(n) belongs to M. Because the union of countably many closed sets with a countable Σ-network is again a countably-Σ-space, this implies that $[a,x]$ is a countably-Σ-space, so a belongs to M. Hence our assumption implies that such a sequence does not exist.

For each $n \in \mathbb{N}$ there exists a convex open neighbourhood U(n) of a that meets only finitely many elements of $\mathcal{F}(n)$, and such that $U(n+1) \subset U(n)$. By the foregoing the set $U = \bigcap_{n=1}^{\infty} U(n) \cap]a,x]$ is non-empty, say $p \in U$. Then $[p,x]$ has a countable Σ-network. Moreover $[a,p]$ has a countable Σ-network too, since $[a,p]$ meets only countable many elements of $\mathcal{F}$, and $\{[a,p] \cap F \mid F \in \mathcal{F}\}$ is a Σ-network for $[a,p]$. Consequently $[a,x]$ has a countable Σ-network. It follows that $a \in M$. Contradiction. □

<u>COROLLARY</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. If* X *is a* Σ*-space then* mX *is a GO-space. Moreover,* $m\colon X \longrightarrow mX$ *is an order preserving, closed map.*

<u>PROOF</u>. See proposition 1.2.3. □

<u>THEOREM 4.2.2</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \text{ is a } \Sigma\text{-space} \iff mX \text{ is metrizable and each } M \in X/\mathcal{M} \text{ is a } \Sigma\text{-space.}$$

<u>PROOF</u>. $\Rightarrow$: That each $M \in X/\mathcal{M}$ is a Σ-space is immediately clear from 4.2.1, so we still have to prove that mX is metrizable.

Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for X.

(i) mX has a G_δ-diagonal.

For each $y \in mX$, $n \in \mathbb{N}$ we define.

$$A(m^{-1}(y),n) := m^{-1}(y) \cup L(y,n) \cup R(y,n)$$

where - if $m^{-1}(y)$ has no left endpoint then $L(y,n) = \emptyset$

- if $m^{-1}(y)$ has a left endpoint ℓ, then $L(y,n)$ is the intersection of $]\leftarrow,\ell]$ and some convex open neighbourhood of ℓ that meets only finitely many elements of $\mathcal{F}(n)$,

and $R(y,n)$ is defined analogously for the right side of $m^{-1}(y)$.

Furthermore

$$U(y,n) := mX\backslash m[X\backslash A(m^{-1}(y),n)] \qquad (y\in mX; n\in\mathbb{N}).$$

Since $A(m^{-1}(y),n)$ is a convex, open neighbourhood of $m^{-1}(y)$, and m is a closed map, each $U(y,n)$ is a convex open neighbourhood of y, so

$$\mathcal{U}(n) := \{U(y,n) \mid y \in mX\}$$

is an convex open cover of mX for each $n \in \mathbb{N}$.

Claim: for each $y \in mX$ we have $\bigcap_{n=1}^{\infty} St(y,\mathcal{U}(n)) = \{y\}$.

Suppose not:

Then there exist distinct points y_1 and y_2 in mX (say $y_1 < y_2$) such that $y_2 \in St(y_2,\mathcal{U}(n))$ for each n, i.e. such that for each $n \in \mathbb{N}$ there is a point $z(n) \in mX$ with

$$\{y_1,y_2\} \subset U(z(n),n)$$

which implies

$$m^{-1}(y_1) \cup m^{-1}(y_2) \subset A(m^{-1}(z(n)),n).$$

Fix $x_1 \in m^{-1}(y_1)$ and $x_2 \in m^{-1}(y_2)$.

a) Suppose $z(n) < y_1$ for infinitely many $n \in \mathbb{N}$. Then for these n the following holds true:

$$R(z(n),n) \supset [x_1,x_2];$$

so $[x_1,x_2]$ meets finitely many elements of $\mathcal{F}(n)$. Since $\mathcal{F}(n) \subset \mathcal{F}(n+1)$, this implies that $[x_1,x_2]$ meets finitely many elements of $\mathcal{F}(n)$ for each $n \in \mathbb{N}$. Hence $[x_1,x_2]$ has a countable Σ-network, which contradicts our assumption that $y_1 \neq y_2$.

a') Analogously it is proven that $z(n) > y_2$ can happen for at most finitely many n.

From a) and a') it follows that all but finitely many $z(n)$ are in the interval $[y_1,y_2]$.

b) Suppose there is a $z \in [y_1,y_2]$ such that $z(n) = z$ for infinitely many $n \in \mathbb{N}$. Then, in the same way as in a) we can prove that $y_1 = z = y_2$ which again yields a contradiction. Hence such a point z does not exist.

c) Now fix $n_0 \in \mathbb{N}$ and choose $n_1 > n_0$ such that $z(n_1) \in]y_1,y_2[$. Since $A(m^{-1}(z(n_1),n_1)$ contains both $m^{-1}(y_1)$ and $m^{-1}(y_2)$, the set $m^{-1}(z(n_1))$ must have a left and a right endpoint.
Next choose $n_2 > n_1$ such that $z(n_2) \neq z(n_1)$ and $z(n_2) \in]y_1,y_2[$. Because $m^{-1}(z(n_1))$ is contained in either $L(z(n_2),n_2)$ or $R(z(n_2),n_2)$, it meets at most finitely many elements of $\mathcal{F}(n_2)$ and hence also of $\mathcal{F}(n_1)$. This implies that $A(m^{-1}(z(n_1)),n_1)$ and hence $[x_1,x_2]$ intersects only finitely many $F \in \mathcal{F}(n_1)$ and hence of $\mathcal{F}(n_0)$. Since n_0 was arbitrarily choosen, this again implies that $y_1 = y_2$, a contradiction.

Consequently the claim is proven, so mX has a G_δ-diagonal.

(ii) mX is metrizable.

Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a sequence of G_δ-coverings for mX such that $\mathcal{U}(n+1) \subset \mathcal{U}(n)$. Put

$$L := \{y \in mX \mid [y,\rightarrow[\in \delta\backslash\lambda(\leq) \text{ and } \forall n \in \mathbb{N}\ St(y,\mathcal{U}(n)) \cap]\leftarrow,y[\neq \emptyset\}$$

and

$$R := \{y \in mX \mid \,]\leftarrow,y] \in \delta\backslash\lambda(\leq) \text{ and } \forall n \in \mathbb{N}\ St(y,\mathcal{U}(n)) \cap \,]y,\rightarrow[\, \neq \emptyset\}$$

where δ is the topology and $\leq$ the order on mX.

Then in order to prove that mX is metrizable it suffices to show that L and R are σ-discrete (in mX), since the metrizability then follows from theorem 1.3.3 and its corollary.

We prove that L is σ-discrete in mX; for R the proof is analogous.

Choose, for each $y \in L$ a point $x(y)$ from $m^{-1}(y)$. Define

$$D(n) := \{y \in L \mid \exists y' < y\colon m^{-1}[\,]y',y[\,] \cap C(x(y),\mathcal{F}(n)) = \emptyset \text{ and } m^{-1}[\,]y',y[\,] \text{ meets uncountably many } F \in \mathcal{F}(n)\}.$$

a) $L = \bigcup_{n=1}^{\infty} D(n)$.

Take $y \in L$. Then $[y,\rightarrow[$ is open but y has no left neighbour and is not left endpoint of mX. Since $\bigcap_{n=1}^{\infty} St(y,\mathcal{U}(n)) = \{y\}$ but no $St(y,\mathcal{U}(n))$ is contained in $[y,\rightarrow[$ there must be a sequence $(y(n))_{n=1}^{\infty}$ that is cofinal in $]\leftarrow,y[$. Hence the sequence $(x(y(n)))_{n=1}^{\infty}$ is cofinal in $m^{-1}(]\leftarrow,y[)$.

Now $C(x(y))$ is countably compact, and hence $C(x(y)) \cap m^{-1}[\,]\leftarrow,y[\,]$ - which is also a countably compact set since $m^{-1}[\,]\leftarrow,y[\,]$ is closed - cannot be cofinal in $m^{-1}(]\leftarrow,y[)$ because of the previous remark (which implies that otherwise there would be a cofinal sequence in $C(x(y)) \cap m^{-1}[\,]\leftarrow,y[\,]$ which is impossible) and since $\mathcal{F}$ is a Σ-network for X, we have that for some $n = n_1 \in \mathbb{N}$:

$$C(x(y),\mathcal{F}(n_1)) \cap m^{-1}[\,]\leftarrow,y[\,] \text{ is not cofinal in } m^{-1}[\,]\leftarrow,y[\,].$$

Consequently there exists a $y' < y$ such that $m^{-1}[\,]y',y[\,] \cap C(x(y),\mathcal{F}(n_1)) = \emptyset$

Moreover, the fact that $y' < y$, implies that $m^{-1}[\,]y',y[\,]$ meets uncountably many $F \in \mathcal{F}$, so for some $n = n_2$, the set $m^{-1}[\,]y',y[\,]$ meets uncountably many $F \in \mathcal{F}(n_2)$. Put $n := \max(n_1,n_2)$, then $y \in D(n)$.

b) Each D(n) is discrete (in mX).

Fix $n \in \mathbb{N}$, $y \in mX$.

If $m^{-1}(y)$ has a left (right) endpoint, let $U(\ell)$ ($U(r)$ respectively) be a convex open enighthbourhood of that endpoint, that meets at most finitely many elements of $\mathcal{F}(n)$; let $U(\ell)$ ($U(r)$ respectively) be empty in the other case. Define

$$U := m^{-1}(y) \cup U(\ell) \cup U(r)$$

and

$$O(y,n) := mX\backslash m[X\backslash U].$$

Clearly $O(y,n)$ is an open neighbourhood of y. If y_1 and y_2 belong to $O(y,n) \cap D(n)$, such that $y_1 < y_2 < y$ then there exists a $y_2' < y_2$ such that

$$m^{-1}(]y_2',y_2[) \cap C(x(y_2),\mathcal{F}(n)) = \emptyset$$

and

$$m^{-1}[]y_2',y_2[] \text{ meets uncountably many } F \in \mathcal{F}(n).$$

Because of the construction of $O(y,n)$, the second condition implies that y_2' must be smaller than y_1. Then the first condition gives that $x(y_1)$ is not an element of $C(x(y_2),\mathcal{F}(n))$. Hence

$$C(x(y_1),\mathcal{F}(n)) \neq C(x(y_2),\mathcal{F}(n)).$$

Since the $C(x(y_i),\mathcal{F}(n))$'s are elements of $\mathcal{F}(n)$ intersecting U, and $U(\ell)$ and $U(r)$ intersect at most finitely many $F \in \mathcal{F}(n)$ this implies that $O(y,n)$ contains at most finitely many elements of $D(n)$. Consequently, $D(n)$ must be discrete in mX.

$\Leftarrow$: Suppose mX is metrizable; let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}(n)$ be a net for mX consisting of convex, closed sets, such that each $\mathcal{B}(n)$ is a discrete collection. (That we may suppose that the elements of $\mathcal{B}$ are convex follows from ([19],th.3.3)

$$Y(0) := \{y \in mX \mid \text{both } [y,\rightarrow[\text{ and }]\leftarrow,y] \text{ are not open}\}.$$

Then, whenever $y \in Y(0)$, the set $m^{-1}(y)$ has a left endpoint $a(y)$ and a right endpoint $b(y)$, and a countable Σ-network $\mathcal{F}^{(y)} := \{F_n^{(y)} \mid n = 1,2,\ldots\}$.
For the sake of convenience take $F_1^{(y)} = \emptyset$ for each $y \in Y(0)$. Define

$$F(n,B) := \bigcup \{F_n^{(y)} \cup \{a(y),b(y)\} \mid y \in Y(0) \cap B\} \cup \bigcup\{m^{-1}(y) \mid y \in B\backslash Y(0)\}. \quad (n\in\mathbb{N}, B\in\mathcal{B})$$

and

$$\mathcal{F}(n,m) := \{F(n,B) \mid B \in \mathcal{B}(m)\} \qquad (n,m\in\mathbb{N}).$$

Each $F(n,B)$ is contained in $m^{-1}[B]$, and it is easy to see that each $F(n,B)$ is closed in X, since distinct "parts" of $F(n,B)$ can accumulate only at endpoints $a(y)$, $b(y)$, or at endpoints of $m^{-1}(y)$, $y \in B\setminus Y(0)$. Hence each $\mathcal{F}(n,m)$ is a locally finite (even discrete) closed collection.

Next, since mX is metrizable, the set $mX\setminus Y(0)$ ($= E(mX)$) is σ-discrete in mX, say $mX\setminus Y(0) = \bigcup_{n=1}^{\infty} Y(n)$ where each $Y(n)$ is discrete (and closed) in mX.

For each $y \in mX\setminus Y(0)$ and for each $n \in \mathbb{N}$ let $\mathcal{F}_n^{(y)}$ be a locally finite closed collection of subsets of $m^{-1}(y)$ such that

$$\mathcal{F}^{(y)} := \bigcup_{n=1}^{\infty} \mathcal{F}_n^{(y)} \text{ is a } \Sigma\text{-network for } m^{-1}(y),$$

and put

$$\mathcal{G}(n,m) := \bigcup \{\mathcal{F}_n^{(y)} \mid y \in Y(m)\} \qquad (n,m\in\mathbb{N}).$$

Clearly, each $\mathcal{G}(n,m)$ is a locally finite closed collection in X.

We claim that

$$\mathcal{F} := \bigcup \{\mathcal{F}(n,m) \mid n,m \in \mathbb{N}\} \cup \bigcup \{\mathcal{G}(n,m) \mid n,m \in \mathbb{N}\}$$

is a Σ-network for X.
Take $x_0 \in X$ and put $y_0 := m(x_0)$.
(i) If $y_0 \in Y(0)$, then

$$C(x_0) := \bigcap \{F \mid x_0 \in F \in \mathcal{F}\} =$$
$$= \{a(y_0),b(y_0)\} \cup \bigcap \{F_n^{(y_0)} \mid x_0 \in F_n^{(y_0)} \in \mathcal{F}^{(y_0)}\}$$

(so $C(x_0)$ is contained in $m^{-1}(y_0)$).

Hence $C(x_0)$ is countably compact; and moreover, when U is an open neighbourhood of $C(x_0)$, then

$$U' := mX\setminus m[X\setminus(m^{-1}(y_0) \cup U)]$$

is an open neighbourhood of y_0 (since $a(y_0)$ and $b(y_0) \in C(x_0)$, $m^{-1}(y_0) \cup U$ is an open neighbourhood of $m^{-1}(y_0)$).

Consequently $y_0 \in B \subset U'$ for some $B \in \mathcal{B}$; and hence

$$m^{-1}[B] \subset m^{-1}[U'] \subset m^{-1}(y_0) \cup U,$$

so that $m^{-1}(y) \subset U$ for $y \in B\backslash\{y_0\}$. Furthermore, choose n such that

$$x_0 \in F_n^{(y_0)} \subset U \cap m^{-1}(y_0).$$

Then

$$C(x_0) \subset F(n,B) \subset U.$$

(ii) If y_0 is not an element of $Y(0)$, then y_0 belongs to some $Y(n)$ $(n \geq 1)$. Clearly $C(x_0) = \bigcap \{F \mid x_0 \in F \in \mathcal{F}^{(y_0)}\}$, which is countably compact. Whenever U is an open (in X) neighbourhood of $C(x_0)$ then $U \cap m^{-1}(y_0)$ is a neighbourhood of $C(x_0)$ in $m^{-1}(y_0)$. Hence for some $F \in \mathcal{F}^{(y_0)}$, say $F \in \mathcal{F}_m^{(y_0)}$ we have

$$C(x_0) \subset F \subset U \cap m^{-1}(y_0).$$

Because F belongs to $\mathcal{G}(n,m)$, our claim is proven. □

By using theorem 4.2.2 we can determine whether a given GO-space X is a Σ-space or not, provided we know when a GO-space has a countable Σ-network. The result is not very satisfactory however, since it is for from clear when a GO-space is a countably-Σ-space. The situation improves when we look at paracompact spaces, since we have the following theorem, due to Nagami:

<u>THEOREM 4.2.3</u>. *Let X be a paracompact Σ-space. Then*

X has a countable Σ-network ⟺ X is a Lindelof space.

Also, in a certain sense we may restrict ourselves to paracompact spaces, since we can embed each GO-space X as a dense subspace in a paracompact GO-space pX, where pX is a Σ-space if and only if X is a Σ-space. This will be proven in theorem 4.2.8. The space pX is constructed as follows:

Let $X = (X,\le,\tau)$ be a GO-space.
Let X^{++} be the ordered compactification of X described in section 2.2. Then pX is defined as the following subset of X^{++}:

$$pX := \{q \in X^{++}\backslash X \mid q \text{ is not a limit point of any discrete (in X) subset of X}\} \cup X.$$

Note that each point of $X^{++}\backslash X$, and hence each point of $pX\backslash X$ is either left- or right-isolated (both in X^{++} and in pX).

PROPOSITION 4.2.4. *Let* $X = (X,\le,\tau)$ *be a GO-space. Then* pX *is a paracompact subspace of* X^{++}.

PROOF. Let $\xi = (A,B)$ be a (pseudo-)gap in pX, then $\xi' := (X \cap A, X \cap B)$ is a (pseudo-)gap in X. Suppose A is non-empty and has no right endpoint; then the same is true for $X \cap A$. Let θ be the (obviously existing) right endpoint of $Cl_{X^{++}}(X \cap A)$. Since θ is not an element of pX, it must be limit point of some discrete (in X) subset D of X. Since θ is right-isolated in X^{++}, we may without loss of generality suppose that D is contained in $X \cap A$. Because $pX\backslash X$ contains no limit points of D, D is discrete in pX. Hence in A a discrete (in pX) subset is cofinal. It follows that pX is paracompact. □

Analogous to the notion of a countably-compactification, we call pX a *paracompactification* of X because of the following proposition:

PROPOSITION 4.2.5. *Let* $X = (X,\le,\tau)$ *be a GO-space. If* A *is a paracompact, closed subset of* X *then* A *is closed in* X.

PROOF. Suppose some point $q \in pX\backslash X$ is limit point of A. Without loss of generality q is right-isolated. Then $(]\leftarrow,q[\cap A,]q,\rightarrow[\cap A)$ is a (pseudo-) gap in A. Consequently there is a discrete (in A and hence in X) subset D of $]\leftarrow,q[\cap A$, cofinal in $]\leftarrow,q[\cap A$, because A is paracompact. This implies that q is limit point of the discrete subset D of X, a contradiction. □

To prove that the space pX has indeed the properties indicated above, we need the following two simple lemma's.

LEMMA 4.2.6. *Let* $X = (X,\leq,\tau)$ *be a GO-space and let* q *be some point of* $pX\setminus X$ *that is right-isolated. Put* $A :=]\leftarrow,q[\, \cap X$. *If* $A(1), A(2), \ldots$ *are closed (in* X*) cofinal subsets of* A *then* $\bigcap_{n=1}^{\infty} A(n)$ *is cofinal in* A. *(Of course an analogous statement is true for the case* q *is left-isolated).*

PROOF. (i) We first construct a (transfinite) increasing sequence $(x(\alpha))_{\alpha<\mu}$ in A that is cofinal in A, by transfinite induction. We start off with an arbitrary element $x(1)$ of A. If, for some ordinal η the sequence $(x(\alpha))_{\alpha<\eta}$ is already constructed, but $(x(\alpha))_{\alpha<\eta}$ is not cofinal in A, then let $x'(\eta)$ be some element of A such that $x(\alpha) < x'(\eta)$ for all $\alpha < \eta$. Then there exist $x(\eta,k) \in A(k)$ $(k = 1,2,\ldots)$ such that $x'(\eta) < x(\eta,k) < x(\eta,k+1)$. Since a countable sequence cannot be cofinal in A, we can choose $x(\eta) \in A$, such that $x(\eta,k) < (x,\eta)$ for all k.

(ii) Now fix some $x_0 \in A$. Since $F := \{x(\alpha) \mid x_0 < x(\alpha) \text{ and } \alpha < \mu\}$ is cofinal in A, it cannot be discrete in X, so there must exist a point $y_0 \in A \cap]x_0,\rightarrow[$ that is limit point of F. Then by construction of the sequence $(x(\alpha))_{\alpha<\mu}$, the point y_0 belongs to each $\overline{A(n)} = A(n)$. Consequently $y_0 \in \bigcap_{n=1}^{\infty} \overline{A(n)} = \bigcap_{n=1}^{\infty} A(n)$ so $\bigcap_{n=1}^{\infty} A(n)$ is cofinal in A. □

LEMMA 4.2.7. *Let* $X = (X,\leq,\tau)$ *be a GO-space, let* q *be a point of* $pX\setminus X$ *that is right-isolated, and put* $A :=]\leftarrow,q] \cap X$. *If* $\mathcal{F}$ *is a locally finite collection of subsets of* A *such that no* $F \in \mathcal{F}$ *is cofinal in* A, *then* $\bigcup\{F \mid F \in \mathcal{F}\}$ *is not cofinal in* A. *(Again, an analogous statement is true if* q *is left-isolated.)*

PROOF. Suppose $\bigcup\{F \mid F \in \mathcal{F}\}$ is cofinal in A. We construct an increasing sequence $(y(\alpha))_{\alpha<\mu}$ cofinal in A, and a subcollection $\{F(\alpha) \mid \alpha < \mu\}$ of $\mathcal{F}$, such that $y(\beta)$ belongs to $F(\beta)$ and $\bigcup\{F(\alpha) \mid \alpha < \beta\}$ is contained in $]\leftarrow,y(\beta)[$ for each $\beta < \mu$ as follows:

Suppose $y(\alpha)$ and $F(\alpha)$ have been choosen for all $\alpha < \eta < \mu$ such that the conditions above are fulfilled and that $(y(\alpha))_{\alpha<\eta}$ is not yet cofinal in A. Clearly $\bigcup\{F(\alpha) \mid \alpha < \eta\}$ is not cofinal in A. Consequently there must exist an $F \in \mathcal{F}$ and a point $y \in F$ such that

$$\bigcup\{F(\alpha) \mid \alpha < \eta\} \subset]\leftarrow,y[.$$

Put $y(\eta) := y$ and $F(\eta) := F$.

The set $\{y(\alpha) \mid \alpha < \mu\}$ constructed in this way can not be discrete, so it must have a limit point x_0. But then $\mathcal{F}$ is not locally finite at x_0. Contradiction. □

<u>THEOREM 4.2.8</u>. *Let* $X = (X,\leq,\tau)$ *be a GO-space. Then*

$$X \textit{ is a } \Sigma\textit{-space} \iff pX \textit{ is a } \Sigma\textit{-space.}$$

<u>PROOF</u>. $\Rightarrow$: Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for X, where each $\mathcal{F}(n)$ is a locally finite closed cover of X. We claim that

$$\overline{\mathcal{F}} := \{\overline{F} \mid F \in \mathcal{F}\}$$

is a Σ-network for pX. (All closures in this proof must be taken in pX).

Put

$$\overline{\mathcal{F}(n)} := \{\overline{F} \mid F \in \mathcal{F}(n)\}.$$

(i) Each $\overline{\mathcal{F}(n)}$ covers pX.

Take $q \in pX\backslash X$. Without loss of generality q is right-isolated. Denote $]\leftarrow,q[\cap X$ by A. Then by lemma 4.2.7 we have that $F \cap A$ is cofinal in A for some $F \in \mathcal{F}(n)$, so q belongs to $\overline{F} \in \overline{\mathcal{F}(n)}$

(ii) $\overline{\mathcal{F}(n)}$ is locally finite in pX for each n.

Clearly $\overline{\mathcal{F}(n)}$ is locally finite in each point of X. Let q be an element of $pX\backslash X$, which we again may suppose to be right-isolated, and $A :=]\leftarrow,q[\cap X$.

Now it can happen for at most finitely many elements of $\mathcal{F}(n)$ that $F \cap A$ is cofinal in A since, if $\{F_k(n) \mid k = 1,2,\ldots\}$ is a collection of different elements of $\mathcal{F}(n)$ such that $F_k(n) \cap A$ is cofinal in A, then by lemma 4.2.6 we have that $\bigcap_{k=1}^{\infty} F_k(n)$ is cofinal in A, and consequently $\mathcal{F}(n)$ is not locally finite in any point of this set.

By lemma 4.2.7 we have

$$\bigcup \{F \cap A \mid F \in \mathcal{F}(n) \text{ and } F \cap A \text{ is not cofinal in } A\}$$

is not cofinal in A. It follows that q has a neighbourhood U meeting only finitely many elements of $\overline{\mathcal{F}(n)}$. (namely those $\overline{F}$ such that $F \cap A$ is cofinal in A)

(iii) $\overline{C}(x) := \bigcap \{\overline{F} \mid x \in F \in \mathcal{F}\}$ is countably compact (and hence compact, since pX is paracompact and $\overline{C}(x)$ is closed in pX) for each $x \in pX$.

- First suppose $x \in X$. Then $\overline{C}(x) = \bigcap \{\overline{F} \mid x \in \overline{F} \in \overline{\mathcal{F}}\} = \bigcap \{\overline{F} \mid x \in F \in \mathcal{F}\}$ $= \overline{\bigcap \{F \mid x \in F \in \mathcal{F}\}} = \overline{C(x)}$.

[The fact that $\bigcap \{\overline{F} \mid x \in F \in \mathcal{F}\}$ and $\overline{\bigcap \{F \mid x \in F \in \mathcal{F}\}}$ coincide is proved as follows: if q belongs to $\bigcap \{\overline{F} \mid x \in F \in \mathcal{F}\}$, say q is right-isolated and put $A :=]\leftarrow,q[\cap X$, then since $\mathcal{F}$ is σ-locally finite, the collection $\{F \mid x \in F \in \mathcal{F}\}$ is countable; and for each member of this set we have that $F \cap A$ is cofinal in A, since $q \in \overline{F}$. Now, by 4.2.6 the set $\bigcap \{F \cap A \mid x \in F \in \mathcal{F}\}$ is cofinal in A and so q belongs to $\overline{\bigcap \{F \mid x \in F \in \mathcal{F}\}}$. Hence $\overline{C}(x)$ is equal to $\overline{C(x)}$]. Since $C(x)$ is countably compact, the same is true for $\overline{C(x)}$ *); so in this case the assertion follows.

- Next take $x \in pX \setminus X$; again we suppose that x is right-isolated and we denote $]\leftarrow,x] \cap X$ by A.

From part (ii) of this proof, we know that $\{F \mid x \in \overline{F} \in \overline{\mathcal{F}}\}$ is countable, and since each F from this set is cofinal in A, 4.2.6 then yields that $\bigcap \{F \cap A \mid x \in \overline{F} \in \overline{\mathcal{F}}\}$ is cofinal in A. Moreover, we conclude from lemma 4.2.7 that

$$K(n) := \bigcup \{F \cap A \mid F \in \mathcal{F}(n) \quad \text{and} \quad x \notin \overline{F}\}$$

is not cofinal in A for any $n \in \mathbb{N}$. Hence

$$K := \bigcup \{F \cap A \mid F \in \mathcal{F} \quad \text{and} \quad x \notin \overline{F}\} = \bigcup_{n=1}^{\infty} K(n)$$

is not cofinal in A since no countable set is cofinal in A. Now choose x' from $\bigcap \{F \mid x \in \overline{F} \in \overline{\mathcal{F}}\} \cap A$ such that K is contained in $]\leftarrow,x'[$. Then x belongs to $\overline{F}$ for $F \in \mathcal{F}$ if and only if x' belongs to $\overline{F}$. Hence $\overline{C}(x) = \overline{C}(x')$. Since x' is an element of X, $\overline{C}(x)$ is (countably) compact by the first part of the proof.

(iv) $\overline{\mathcal{F}}$ contains an outer network for $\overline{C}(x)$ $(x \in pX)$. Let U be an open neighbourhood of $\overline{C}(x)$. It follows from (iii) that without loss of generality we may take $x \in X$. Choose an open U' in pX such that $\overline{C}(x) \subset U' \subset \overline{U'} \subset U$. Since $\overline{C}(x) \cap X = \overline{C(x)} \cap X = C(x)$, we have $C(x) \subset U' \cap X$. Hence for some $F \in \mathcal{F}$

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*) As is easily verified, in each normal (and hence countably collectionwise normal) space the closure of a countably compact set is again countably compact.

$$C(x) \subset F \subset U' \cap X$$

and consequently

$$\overline{C}(x) = \overline{C(x)} \subset \overline{F} \subset \overline{U'} \subset U.$$

$\Leftarrow$: Suppose $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ is a strong Σ-network for pX such that each $\mathcal{F}(n)$ is a locally finite closed cover of pX. Put

$$\mathcal{F}(n) \cap X := \{F \cap X \mid F \in \mathcal{F}(n)\} \quad \text{and} \quad \mathcal{F} \cap X := \bigcup_{n=1}^{\infty} (\mathcal{F}(n) \cap X).$$

Then $\mathcal{F} \cap X$ is a Σ-network for X.

(i) Each $\mathcal{F}(n)$ is a locally finite closed cover of X.

(ii) For every $x \in X$:

$$C'(x) := \bigcap \{F \cap X \mid x \in F \cap X \in \mathcal{F} \cap X\} \quad \text{is countably compact.}$$

For, clearly $C'(x) = C(x) \cap X$ where $C(x) := \bigcap \{F \mid x \in F \in \mathcal{F}\}$. Every sequence $(x(n))_{n=1}^{\infty}$ in $C'(x)$ has a limit point in $C(x)$ since $C(x)$ is compact. This limit must belong to X, because no point of $pX \setminus X$ is limit point of a countable sequence in X. Consequently each sequence in $C'(x)$ has a limit point in $C'(x)$.

(iii) $\mathcal{F} \cap X$ contains an outer network for $C'(x)$ in X.

Let U be an open set in X containing $C'(x)$ and take an open set U' in pX such that $U' \cap X = U$.

Let $p \in \overline{C'(x)} \setminus C'(x)$. Suppose that p is right-isolated and put $A =]\leftarrow,p[\cap X$. The set $A \setminus U$ is not cofinal in A since this would imply by lemma 4.2.6 that $\emptyset = (A \setminus U) \cap (C'(x) \cap A)$ is cofinal in A. Hence for some $x_0 \in A$ we have that $]x_0,p]_{pX}$ is an open neighbourhood $U(p)$ of p in pX such that $U(p) \cap X \subset U$. When $p \in \overline{C'(x)} \setminus C'(x)$ is left-isolated we determine $U(p)$ in an analogous way.

Put

$$U'' := U' \cup \bigcup \{U(p) \mid p \in \overline{C'(x)} \setminus C'(x)\}$$

then U" is open in pX and $C(x) \setminus U''$ is compact. Whenever p belongs to $C(x) \setminus U''$ then p is not a point of $\overline{C'(x)}$. By lemma 4.2.6 there is a natural number $n(p)$ such that p is not an element of $C(x, \mathcal{F}(n(p)) \cap X)$, so there exist an open neighbourhood $O(p)$ of p in pX that does not meet $C(x,F(n(p)) \cap X\}$.

Since $C(x)\backslash U''$ is compact, there exist $p_1,p_2,\dots,p_k$ in $C(x)\backslash U''$ such that $C(x)\backslash U''$ is contained in $O(p_1) \cap\dots\cap O(p_k)$. Define

$$U''' := U'' \cup O(p_1) \cup\dots\cup O(p_k).$$

Because U''' is an open set in pX containing $C(x)$ we have for some $n_0 \in \mathbb{N}$ that $C(x) \subset C(x,F(n_0)) \subset U'''$. Take $n_1 := \max\{n_0,n(p_1),\dots,n(p_k)\}$. Since

$$C'(x) = C(x) \cap X \subset C(x,F(n_1)) \cap X \subset U''' \cap X$$

we also have

$$C'(x) \subset C(x,F(n_1)) \cap X \subset (U'' \cap X) \cup [(O(p_1)\cup\dots O(p_k)) \cap X]$$

and because $C(x,F(n_1)) \cap X$ is contained in $C(x,F(n(p_i)) \cap X$ for $1 \le i \le k$, $C(x,F(n_1)) \cap O(p_i) \cap X$ is void. This implies that

$$C'(x) \subset C(x,F(n_1)) \cap X \subset U''' \cap X = U. \quad \square$$

<u>COROLLARY</u>. *Let $X = (X,\le,\tau)$ be a GO-space. Then*

$$X \text{ is an M-space} \iff pX \text{ is an M-space.}$$

<u>PROOF</u>. Follows immediately from theorem 4.1.2 and (the proof of) 4.2.8. □

By theorem 4.2.8 we can confine ourselves to paracompact spaces when investigating which GO-spaces are Σ-spaces. A combination of 4.2.2 and 4.2.3 then suggests the following approach:

<u>DEFINITION</u>. Suppose $X = (X,\le,\tau)$ is a GO-space. Then L_X is the equivalence relation on X defined by:

$x L_X y \iff$ the closed interval between x and y is a Lindelöf space $(x,y \in X)$.

The decomposition space X/L_X is denoted by ℓX and

$$\ell_X \colon X \longrightarrow \ell X$$

is the quotient map.

When no confusion is likely we will drop the subscript X on L and ℓ.

PROPOSITION 4.2.9. *Let* $X = (X,\le,\tau)$ *be a GO-space. Then* ℓX *is a GO-space with regard to the obvious order, and the quotient topology, and* $\ell\colon X \to \ell X$ *is an order preserving closed map.*

PROOF. Obvious. □

PROPOSITION 4.2.10. *Let* $X = (X,\le,\tau)$ *be a paracompact GO-space. If each equivalence class of* L *has a* Σ*-network then* L *and* M *coincide.*

PROOF. Whenever x and y belong to X (x<y), we have: [x,y] has a countable Σ-network $\Rightarrow$ [x,y] is a Lindelöf-space, because of the paracompactness of X, and [x,y] is Lindelöf $\Rightarrow$ [x,y] has a countable Σ-network, because [x,y] has a Σ-network as a closed subset of some $L \in X/L$. This Σ-network is necessarily countable. □

THEOREM 4.2.11. *Let* $X = (X,\le,\tau)$ *be a paracompact GO-space. Then*

$$X \textit{ is a } \Sigma\textit{-space} \iff \ell X \textit{ is metrizable and each } L \in X/L \textit{ has a } \Sigma\textit{-network.}$$

PROOF. Follows directly from theorem 4.2.2 and proposition 4.2.10. □

The question that remains is: If X is a paracompact GO-space and L is some equivalence class of L_X, how can we determine whether L admits a Σ-network or not? We have not solved this problem, but it can be reduced to another one, in the following way:

If L has a Σ-network then for each $x,y \in L$ ($x\le y$) the closed interval [x,y] obviously is a Σ-space. The converse also is true: if each closed interval [x,y] contained in L has a Σ-network then L is a Σ-space.

Indeed, fix $x \in L$. If L has a right endpoint r then [x,r] has a Σ-network. If L has no right endpoint then, because of the paracompactness of X there must be a discrete subset D of L, cofinal in L. Without loss of generality we may suppose that D can be written as $\{x(\alpha) \mid \alpha < \omega_\mu\}$ for some ordinal ω_μ, such that $x(\alpha) < x(\beta)$ whenever $\alpha < \beta < \omega_\mu$; also we take $x(0) = x$.

Then for each $\alpha < \omega_\mu$, $[x(\alpha),x(\alpha+1)]$ has a countable Σ-network

$\mathcal{F}_\alpha$:= F(n,α) | n = 1,2,...}. Put

$$\mathcal{F}(n) := \{F(n,\alpha) \mid \alpha < \omega_\mu\} \quad \text{and} \quad \mathcal{F} := \cup_{n=1}^{\infty} \mathcal{F}(n).$$

Then $\mathcal{F}$ is a Σ-network for $L \cap [x,\rightarrow[$ as can be trivially verified. The same applies to $]\leftarrow,x] \cap L$; hence L is a Σ-space.

The only thing we still do not know is: when has the Lindelöf space [x,y] a (countable) Σ-network ($x,y \in L \in X/\mathcal{L}$) or: how can the Σ-property be characterized in Lindelöf GO-spaces?

4.3. HEREDITARY Σ-SPACES

Being a Σ-space is a property that is hereditary for closed subsets and for paracompact G_δ-subsets (see [37]) but not hereditary in general, not even for GO-spaces (For instance: the Alexandrov "double arrow space" [2] has the Sorgenfrey line as a subspace). In this section we consider generalized ordered, *hereditary Σ-spaces* i.e. spaces each of whose subspaces is a Σ-space.

PROPOSITION 4.3.1. *Let X = (X,≤,τ) be a GO-space that is a hereditary Σ-space. Then X is (hereditarily) paracompact.*

PROOF. Suppose X is not paracompact, and let (A,B) be a non-Q-(pseudo-)gap in X, say a non-Q_ℓ-(pseudo-)gap. Let $\mathcal{F} = \cup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for A, (where each $\mathcal{F}(n)$ is a locally finite closed cover of A). Then by lemma 4.2.7 there exists a point $x_0 \in A$ such that

$$x_0 \in F \Rightarrow F \text{ is cofinal in A} \qquad (F \in \mathcal{F}).$$

Now $C(x_0)$ is the intersection of countably many closed, cofinal subsets of A and hence is cofinal in A by lemma 4.2.6. Since $C(x_0)$ is countably compact, it is easy to define by transfinite induction a subset of $C(x_0)$ (and hence of X) that is homeomorphic to $W(\omega_1)$. Consequently X is not a hereditary Σ-space (see example IV in section 4.4). □

PROPOSITION 4.3.2. *Let X = (X,≤,τ) be a GO-space that is both a Σ-space and hereditarily paracompact. Then X is first countable.*

PROOF. Let $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$ be a Σ-network for X, where each $\mathcal{F}(n)$ is locally finite, and suppose some $x_0 \in X$ does not admit a countable local base, say x_0 is not left-isolated, and there is no countable sequence in $]\leftarrow,x_0[$ that converges to x_0. Let U(n) be a convex open neighbourhood of x_0 that intersects only finitely many $F \in \mathcal{F}(n)$ $(n \in \mathbb{N})$, then $\bigcap_{n=1}^{\infty} U(n)$ intersects only countably many elements of $\mathcal{F}$. Moreover, there exists a point $y \in \bigcap_{n=1}^{\infty} U(n) \cap]\leftarrow,x_0[$ and we have:

$$A := [y,x_0] \text{ has a countable } \Sigma\text{-network.}$$

Since A is paracompact, it is Lindelöf by 4.2.3. Now because of hereditary paracompactness there exists a discrete (in $A\setminus\{x_0\}$) cofinal subset of $A\setminus\{x_0\}$. Then D is not countable, and for every $z \in]y,x_0[$ we have

$$|[y,z] \cap D| = \aleph_0.$$

Hence $D' := D \cup \{x_0\}$ is homeomorphic to the space Y of example II in section 4.4, which is not a Σ-space. □

COROLLARY. *Let* $X = (X,\leq,\tau)$ *be a GO-space that is a hereditary* Σ*-space. Then* X *is first countable.*

LEMMA 4.3.3. *Let* $X = (X,\leq,\tau)$ *be a GO-space that is hereditarily a* Σ*-space. Suppose* $\mathcal{D}$ *is an equivalence relation on* X *such that the equivalence classes of* $\mathcal{D}$ *are closed, convex sets and such that*:

(i) *each equivalence class is metrizable*
(ii) *the quotient space* $X/\mathcal{D}$ *is metrizable.*

Then X *is metrizable.*

PROOF. Replace everywhere in the proof of lemma 2.3.3 (hereditarily) M by (hereditarily) Σ, and use 4.1.4. □

CONJECTURE 4.3.4. Let $X = (X,\leq,\tau)$ be a GO-space that is hereditarily a Σ-space. Then X is metrizable.

We were not able to prove this conjecture. However, we will show that it is equivalent to the following one:

CONJECTURE 4.3.5. Let $X = (X,\leq,\tau)$ be a Lindelöf GO-space that is a hereditarily Σ-space. Then X is hereditarily Lindelöf.

Indeed, to prove that conjecture 4.3.4 is true, it is sufficient to prove that each $L \in \ell X$ is metrizable if X is a hereditary Σ-space, and for that it is sufficient to prove that each convex subset L' of L with two endpoints (which is Lindelöf) is metrizable. Suppose conjecture 4.3.5 to be true; then L' is hereditarily Lindelöf and hece perfectly normal; so L' is an M-space by theorem 4.1.3. Since the same is true for each subspace of L', it follows that L' is hereditarily an M-space. Consequently L' is metrizable by theorem 2.3.6.

4.4. EXAMPLES

I. Let X be the set of the real numbers with the usual order $\leq$ and let τ be the topology generated by the order topology together with all sets $\{x\}$ (x irrational): i.e. $(X,\leq,\tau)$ is the Michael line [34]. Then X is not a Σ-space, for instance because each convex subset of X consisting of more than one point is not a Lindelöf space; hence ℓX is homeomorphic to X. Since X is not metrizable, it follows from 4.2.11 that X does not admit a Σ-network.

II. Let X' be the set $W(\omega_1)$ of all countable ordinals and $X := W(\omega_1+1)$ and supply X with the usual order and order topology. We define Y as the following subspace of X:

$$Y := \{x \in X \mid x \text{ is not a countable limit ordinal}\}.$$

Then Y is a non-compact Lindelöf-space. (in fact X is the Dedekind compactification of Y). Y is not a Σ-space, for suppose Y has a (necessarily countable) Σ-network $\mathcal{F}$. Put

$$\mathcal{F}' := \{F \in \mathcal{F} \mid F \cap X' \text{ is cofinal in } X'\}.$$

Then there exists an $\alpha \in Y$ such that, for each $F \in \mathcal{F}$: $\alpha \in F$ implies that F is an element of $\mathcal{F}'$.

By lemma 4.2.6 $K := \bigcap \{c\ell_{X'}(F) \mid F \in \mathcal{F}'\} \cap X'$ is a closed cofinal subset of X'. Fix a limit-ordinal $\beta \in K$. Because $C(\alpha)$ is compact, β does not belong to $c\ell_{X'}(C(\alpha))$, so $[\beta',\beta]_X \cap C(\alpha) = \emptyset$ for some F_0 containing α. However, F_0

must be an element of $\mathcal{F}'$ since α belongs to F_0, so β belongs to $\mathcal{C}\ell_X(F_0)$. Contradiction. Consequently, Y is not a Σ-space.

Note that though the map c: $Y \longrightarrow cY$ is not a homeomorphism, still Y and cY are homeomorphic.

III. Let X be as in example II, $X' := \{x \in X \mid x \text{ is a limit ordinal}\}$ and let Y be the space of all irrational numbers in the unit interval. Fix some $p_0 \in Y$. Define

$$Z := \{(x,y) \in X \times Y \mid x \in X' \Rightarrow y = p_0\}$$

with lexicographic order and order topology. Then c: $Z \longrightarrow cZ$ is a homeomorphism and Z is a non-metrizable Σ-space. For, if $\mathcal{B} = \{B(1),B(2),\ldots\}$ is a closed net for Y put

$$F(n) := [(X\backslash X') \times B(n)] \cup [X' \times \{p_0\}].$$

Then $\mathcal{F} := \{F(1),F(2),\ldots\}$ is a Σ-network for Z.

IV. A *stationary* subset of $W(\omega_1)$ is a subset that meets each closed, cofinal subset of $W(\omega_1)$. (See [18] for an extensive treatment). It is known (cf. [43]) that there exists a subset X of $W(\omega_1)$ that *bi-stationary* i.e. both X and $W(\omega_1)\backslash X$ are stationary in $W(\omega_1)$. Now let X be such a bi-stationary subset of $W(\omega_1)$. We claim that X is not a Σ-space.

Suppose X admits a Σ-network $\mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}(n)$, where each $\mathcal{F}(n)$ is locally finite in X. Whenever A is a cofinal subset of X then

$$A' := \{\alpha \in W(\omega_1) \mid \alpha \text{ is a limit point of } A\}$$

is closed and cofinal in $W(\omega_1)$. Hence both X and $W(\omega_1)\backslash X$ meet A'. It follows that a cofinal subset of X cannot be discrete in X, hence the right endgap of X is not a Q_ℓ-gap. Consequently there exist by 4.2.7 a point $\alpha_0 \in X$ such that

$$\alpha_0 \in F \Rightarrow F \text{ is cofinal in } X \qquad (F \in \mathcal{F})$$

and by lemma 4.2.6 also $C(x_0)$ is cofinal in X. But this implies by the argument above, that $C(x_0)'$ meets $W(\omega_1)\backslash X$, in contradiction with the countably-compactness of $C(x_0)$.

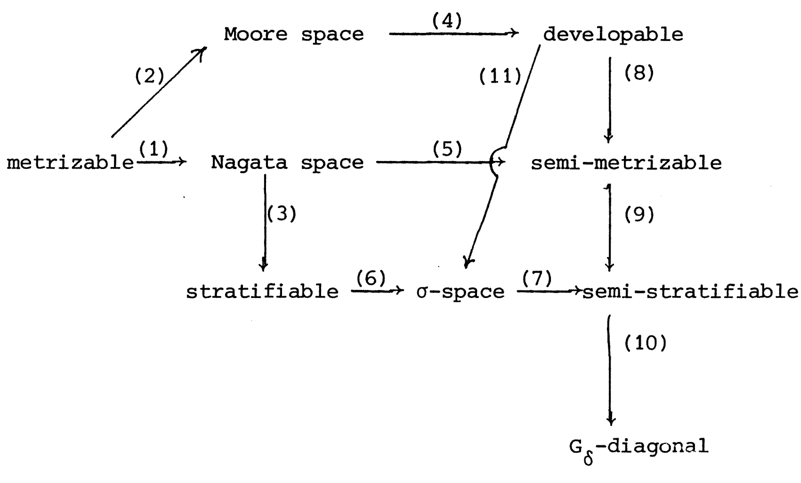
CHAPTER V

IMAGES OF GENERALIZED ORDERED SPACES

5.1. SOME LEMMAS AND EXAMPLES

In chapter 1 we encountered several generalizations of metrizability most of which imply metrizability in the class of GO-spaces. In this chapter we will investigate what the relations between these properties are for a more extended class of spaces, namely the class of images of a GO-space (or of a LOTS respectively) under various kinds of maps.

We will consider the following kinds of maps: open maps, closed maps finite-to-one open maps, perfect maps and open-and-closed maps. The properties under consideration are listed below in the diagram of section 1.5.



REMARK. In this chapter all σ-spaces are assumed to have a σ-locally finite *closed* net. Consequently, implication (7) is true without any separation axioms. In general implication (10) does not hold true, but it is valid for Hausdorff spaces.

LEMMA 5.1.1. *Every first countable T_0-space is the open image of a metrizable LOTS.*

PROOF. This is an immediate consequence of the two following results:

(i) (PONOMAROV [42]). If X is a first countable T_0-space, and Ω is the weight of X, then X is an open image of a subspace of the Baire space $N(\Omega)$.

(ii) (HERRLICH [26]). If Y is a metrizable space with Ind Y = 0 then Y is homeomorphic to a LOTS. □

LEMMA 5.1.2. *Every GO-space is the closed image of some LOTS.*

PROOF. Let $X = (X,\leq,\tau)$ be a GO-space, and let $X^* = (X^*,\preceq,\lambda(\preceq))$ be the LOTS defined in 1.1. Then the closed subspace $X \times \{0\}$ of X^* is homeomorphic to X, and it is easy to see that the mapping

$$\mathbb{P} : \quad X^* \longrightarrow X \times \{0\}$$

$$(x,n) \longmapsto (x,0)$$

is continuous and order preserving.
Moreover $\mathbb{P}$ is a retraction and hence an identification. Consequently, $\mathbb{P}$ is a closed mapping by proposition 1.2.3. □

It might be interesting to know the answer to the following

QUESTION: Is every GO-space the open image of a LOTS. ?

We shall now give some examples which we shall use later on in this chapter:

EXAMPLE A: The *Butterfly space* of McAuley [33]. This is a non-metrizable, first countable stratifiable (see, for instance [12]) space, and hence a Nagata space. Of course, by 5.1.1., it is the image of a LOTS under an open map.

In section 5.4., we shall see that it is not the closed image of a LOTS (or GO-space).

EXAMPLE B: The *Sorgenfrey line* S. [44]. By 5.1.1 and 5.1.2 respectively, S is the image of a LOTS under an open mapping and under a closed mapping, respectively. Such a closed mapping may be countable-to-one, as 5.1.2 shows

However, from what follows in section 5.5, we shall see that it can not be perfect.

An open mapping from a LOTS X onto the Sorgenfrey line cannot be countable-to-one, since the LOTS would be separable in that case and the product of two separable ordered spaces is Lindelöf [16], whereas $S \times S$ is not; neither can such a mapping be compact or open-and-closed (see 5.3 and 5.6 respectively).

EXAMPLE C: The *Tangent Circle space* of NIEMYTZKI [39]. This is an example of a non-metrizable Moore space. Again it is the open image of a LOTS, but it is not the closed image of any LOTS or GO-space, since it is not normal.

EXAMPLE D: Berney's example of a regular, Lindelöf semi-metric space which has no countable network [6]. Note that it is the open image of a LOTS by 5.1.1.

EXAMPLE E: Let Y be the unit interval, and put $A := \{\frac{1}{n} \mid n \in \mathbb{N}\}$. Give every point from $Y \setminus \{0\}$ its normal Euclidean neighbourhoods, and let $\{[0,x[\setminus A \mid x \in Y \setminus \{0\}\}$ be a neighbourhood base for the point 0. The topology τ generated by this neighbourhood system is easily seen to be compatible (as described in 1.4) whith the semi-metric d defined by:

$$d(x,0) = d(0,x) := 2 \text{ for each } x \in A$$

$$d(x,y) = |x-y| \text{ in all other cases.}$$

Then (Y,τ) is a non-regular, developable Hausdorff space (that Y is developable follows easily from 5.3.1). Clearly Y is not orderable, since it is not regular.

Furthermore Y is the finite-to-one, (in fact: at most two-to-one) open image of a LOTS:

Let X be the closed interval [0,2] minus the points of A, with the relativized metric topology. Note that X is a LOTS with respect to its usual order. Define

$$f: \quad X \longrightarrow Y$$

by

$$f(x) := x \quad \text{if } x \in [0,1] \cap X.$$

$$f(x) := x-1 \quad \text{if } x \in [1,2] \cap X.$$

Then f is a continuous, open and finite-to-one mapping.

5.2. OPEN IMAGES OF GO-SPACES

By lemma 5.1.1 we know that there are quite a few spaces that are the image of a LOTS under an open map. Hence we cannot expect much more implications among the properties we are interested in, in this class of spaces than there are in the class of all T_1-spaces.

LEMMA 5.2.1. *Suppose* $X = (X,\leq,\tau)$ *is a GO-space, and let* $f\colon X \longrightarrow Y$ *be a surjective, open mapping. Then there is a countable local base at each of those points* $y \in Y$ *for which* $\{y\}$ *is a* G_δ*-set.*

PROOF. Fix $y \in Y$ such that $\{y\}$ is a G_δ-set; if $\{y\}$ is an open set then there is nothing left to prove, so assume that y is not an isolated point, and let $(U(n))_{n=1}^{\infty}$ be a (decreasing) sequence of open sets in Y, such that $\bigcap_{n=1}^{\infty} U(n) = \{y\}$. Let x be a fixed point of $f^{-1}(y)$.

Now, for each $n \in \mathbb{N}$, choose a convex open set $I(n)$ in X, such that the following properties are fulfilled:

(i) $x \in I(n) \subset f^{-1}[U(n)]$.

(ii) $I(n+1) \subset I(n)$.

(iii) $I(n) \subset [x,\rightarrow[$ (resp. $]\leftarrow,x]$) if x is left-(right-) isolated

and put

$$A(n) := f[I(n)].$$

We will prove that $(A(n))_{n=1}^{\infty}$ is a local base at the point y. For that purpose, suppose that O is an open neighbourhood of y in Y. We define two neighbourhoods L and R of x as follows:

If x is left-isolated then $L := [x,\rightarrow[$; if x is not left-isolated then there exists some $a < x$ such that $]a,x[\subset f^{-1}[O]$. Moreover, we may suppose that a does not belong to $f^{-1}(y)$, (otherwise there would be a non-empty, open interval, with right endpoint x that is mapped onto y, contrary to the fact that y is not isolated.) Put

$$L :=]a,\rightarrow[.$$

Analogously, if x is not right-isolated there exist a point b in $]x,\rightarrow[\setminus f^{-1}(y)$ such that $]x,b[\ \subset f^{-1}[0]$. We define

$$R :=]\leftarrow,b[\quad \text{if} \quad x \text{ is not right-isolated,}$$

and

$$R :=]\leftarrow,x] \quad \text{if} \quad x \text{ is right-isolated.}$$

Observe that either a or b must exist, since x is not isolated. Because clearly $\bigcap_{n=1}^{\infty} I(n) \subset f^{-1}(n)$, there is an $n_0 \in \mathbb{N}$ such that a and b (if they are defined) do not belong to $I(n)$. Hence for $n \geq n_0$, we have

$$I(n) \subset L \cap R \subset f^{-1}[0]$$

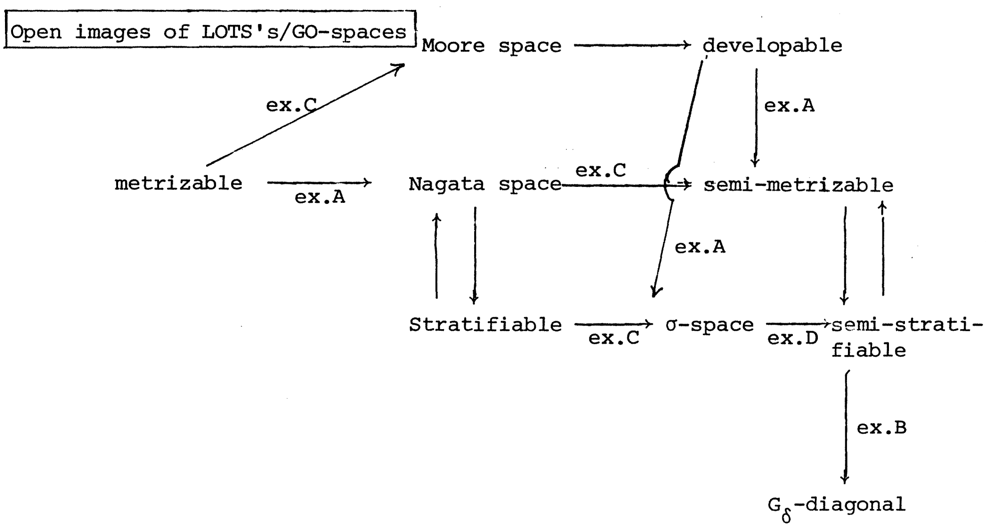
which implies:

$$A(n) = f[I(n)] \subset 0. \quad \square$$

THEOREM 5.2.2. *If* Y *is the open image of a GO-space and* Y *is semi-stratifiable (stratifiable), then* Y *is semi-metrizable (a Nagata space).*

PROOF. This follows immediately from lemma 5.2.1 and theorem 1.4.5 (theorem 1.4.3) since in a (semi-)stratifiable space every point is a G_δ. $\square$

Taking a look at our diagram, we see that only implications (3) and (9) can be reversed as theorem 5.2.2, and the examples corresponding to the non-reversible arrows show: (Here it makes no difference whether we consider open images of LOTS's or of GO-spaces)



5.3. OPEN, FINITE-TO-ONE IMAGES OF GO-SPACES

PROPOSITION 5.3.1. *Suppose* X *is a developable space, and* f: X ⟶ Y *is a finite-to-one, open mapping from* X *onto* Y. *Then* Y *is developable.*

PROOF. Let $(\mathcal{U}(n))_{n=1}^{\infty}$ be a development for X such that $\mathcal{U}(n+1)$ refines $\mathcal{U}(n)$ for each n. For every $x \in X$, and every $n \in \mathbb{N}$ choose a set $U(x,n) \subset X$ such that

$$x \in U(x,n) \in \mathcal{U}(n).$$

Furthermore, for each $y \in Y$ choose a point x(y) from $f^{-1}(y)$, and put

$$A(y,n) := f[U(x(y),n)],$$

and

$$\mathcal{A}(n) := \{A(y,n) \mid y \in Y\} \quad (n = 1,2,\ldots).$$

Then $St(z,\mathcal{A}(n)$ is contained in $\cup \{f[St(p,U(n))] \mid p \in f^{-1}(z)\}$ for each $z \in Y$, for if $z \in A(y,n) = f[U(x(y),n)]$, then p belongs to $U(x(y),n))$ for some $p \in f^{-1}(z)$, and hence $A(y,n)$ is contained in $f[St(p,\mathcal{U}(n)]$.

From this it can be deduced easily (using the finite-to-one- of f) that $(\mathcal{A}(n))_{n=1}^{\infty}$ is a development for Y. □

The following theorem was proved by Y. TANAKA in [47]. We give a proof of our own:

THEOREM 5.3.2. *Suppose* f: X ⟶ Y *is a finite-to-one, open mapping from a Hausdorff space* X *onto a semi-stratifiable space* Y. *Then* X *is semi-stratifiable.*

PROOF. For every natural number k let Q(k) be defined by

$$Q(k) := \{y \in Y \mid |f^{-1}(y)| = k\}$$

and

$$P(k) := f^{-1}[Q(k)].$$

Clearly $X = \bigcup_{n=1}^{\infty} P(k)$ and $Y = \bigcup_{n=1}^{\infty} Q(k)$.

Let $\le$ be a well-ordening of Y. By induction we will construct for every $y \in Y$ an open neighbourhood $M(y)$ of y, and open neighbourhoods $N(x)$ for each $x \in f^{-1}(y)$ such that the following properties are fulfilled:

(i) if x and $x' \in f^{-1}(y)$ then $f[N(x)] = f[N(x')] = M(f(x))$.

(ii) if $x, x' \in f^{-1}(y)$ and $x \neq x'$ then $N(x) \cap N(x') = \emptyset$.

(iii) if $x \in P(k)$ and z is the first element in $Q(k)$ (with respect to the well-order $\le$ on Y) such that $f(x) \in M(z)$ then there is (exactly one) $a \in f^{-1}(z)$ such that $N(x) = N(a)$ (and consequently $f[N(x)] = M(z)$)

Suppose $y \in Y$, then $y \in Q(k)$ for some k and $f^{-1}(y) = \{x(1), x(2), \dots x(k)\}$, let $M(y')$, and $N(x)$ be defined for every y' with $y' < y$, and every $x \in f^{-1}(y')$, such that (i) and (ii) are fulfilled.

If $y \notin \bigcup \{M(y') \mid y' \in Q(k) \text{ and } y' < y\}$ then take disjoint open neighbourhoods $V(x(1)), V(x(2)), \dots V(x(k))$ of $x(1), x(2), \dots, x(k)$, and put

$$M(y) := \bigcap_{i=1}^{k} f[V(x(i))]; \quad N(x(i)) := f^{-1}[M(y)] \cap V(x(i)) \quad (i = 1,2,\dots,k).$$

Clearly

$$N(x(i)) \cap N(x(j)) = \emptyset \quad \text{if } i \neq j,$$

and

$$f[N(x(i))] = M(y) \quad \text{for } i = 1,2,\dots,k.$$

If $y \in \bigcup \{M(y') \mid y' \in Q(k) \text{ and } y' < y\}$, let q be the first element in $Q(k)$ such that y belongs to $M(q)$ and put $f^{-1}(q) = \{p(1), \dots, p(k)\}$. Since $y \in M(q) = f[N(p(i))]$ for $i = 1,2,\dots,k$, every $N(p(i))$ contains an element from $f^{-1}(y)$. Because there are k disjoint $N(p(i))$, and $f^{-1}(y)$ has k elements, every $x(i)$ from $f^{-1}(y)$ is contained in exactly one $N(p(j(i)))$. We define

$$N(x(i)) := N(p(j(i))) \quad (i = 1,2,\dots,)$$

and

$$M(y) := M(q).$$

Clearly the $N(x)$ and $M(y)$ now fulfil the properties (i), (ii) and (iii) above.

Since Y is semi-stratifiable there exist for each $y \in Y$, a sequence $(U(y,i))_{i=1}^{\infty}$ of open neighbourhoods of y such that

(i) if $y \in U(y(i),i)$ for $i = 1,2,...$ then the sequence $(y(i))_{i=1}^{\infty}$ converges to y.

(ii) $\bigcap_{i=1}^{\infty} U(y,i) = \{y\}$ (cf. theorem 1.4.4).

Define

$$O(x,i) := f^{-1}[U(f(x),i)] \cap N(x), \qquad (x \in X; i = 1,2,...).$$

Then

$$\bigcap_{i=1}^{\infty} O(x,i) = N(x) \cap \bigcap_{i=1}^{\infty} f^{-1}[U(f(x),i)] =$$

$$= N(x) \cap f^{-1}[\bigcap_{n=1}^{\infty} U(f(x),i)] = N(x) \cap f^{-1}(f(x)) = \{x\}.$$

We claim that if $x \in O(x(i),i)$ for $i = 1,2,...,$ then the sequence $(x(i))_{i=1}^{\infty}$ converges to x, which together with the previous remark implies that X is semi-stratifiable.

To prove the claim, suppose that $x \in O(x(i),i)$ for all $i \in \mathbb{N}$ $(x,x(i) \in X)$, and let A be an open neighbourhood of x in X. Then $f(x)$ belongs to $U(f(x(i)),i)$ so the sequence $(f(x(i)))_{i=1}^{\infty}$ converges to $f(x)$. Since $A \cap N(x)$ is an open set containing x, and f is an open mapping, $f[A \cap N(x)]$ is an open neighbourhood of $f(x)$. Hence there is an $i_0 \in \mathbb{N}$ such that

$$i \geq i_0 \Rightarrow f(x(i)) \in f[A \cap N(x)].$$

Let i be an arbitrarily chosen natural number greater than i_0. Because $f(x(i)) \in f[A \cap N(x)]$ there is an $x'(i) \in A \cap N(x)$ such that $f(x(i)) =$ $= f(x'(i))$, so we have

$$x'(i) \in A \cap N(x) \subset N(x)$$

$$x \in O(x(i),i) \subset N(x(i)).$$

Suppose $x \in P(k)$, $x(i)$ (and $x'(i)$) belong to $P(\ell)$, then

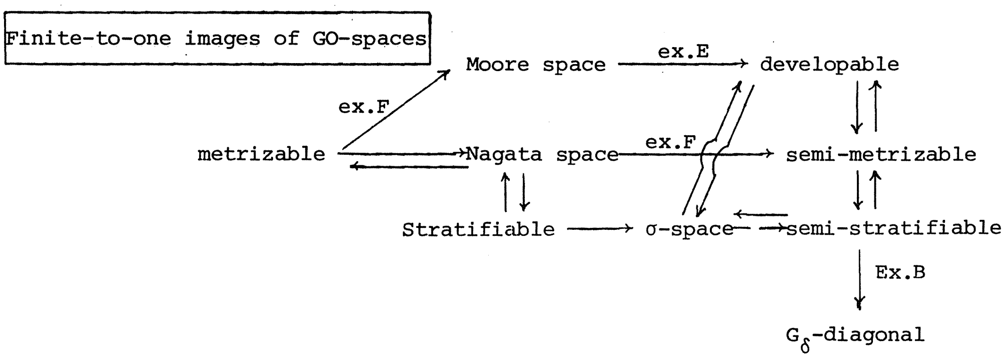
$$\begin{rcases} x \in N(x(i)) \quad \text{implies:} \quad k \geq \ell \\ x'(i) \in N(x) \quad \text{implies} \quad \ell \geq k \end{rcases} \text{ hence } k = \ell.$$

Let s be the first element in $Q(k)$ with $f(x) \in M(s)$ and let t be the first element in $Q(k)$ such that $f(x(i)) \in M(t)$. From $f(x(i)) = f(x'(i)) \in$ $\in f[N(x)] = M(s)$, it follows that $t \leq s$, while $f(x) \in f[N(x(i))] = M(t)$ implies that $s \leq t$; hence $s = t$.

Let a be the (unique) element in $f^{-1}(s)$ such that $N(x) = N(a)$, and let b the element in $f^{-1}(s)$ with $N(x(i)) = N(b)$. Since $x \in N(x) = N(a)$ and $x \in N(x(i)) = N(b)$, a and b must be equal, because otherwise $N(a) \cap N(b) = \emptyset$. This implies $x'(i) \in N(x) = N(x(i))$, hence $x'(i) = x(i)$. Apparently $x(i)$ belongs to A for every $i \geq i_0$, so $(x(i))_{i=1}^{\infty}$ converges to X. □

<u>REMARK</u>. In [47] it is also proved that a Hausdorff space that can be mapped onto a σ-space by a finite-to-one, open map is a σ-space itself. For another proof see [49]. However the result is not needed here.

Now, since a semi-stratifiable GO-space is metrizable, 5.3.1 and 5.3.2 together obviously imply that if X is the finite-to-one open image of a GO-space, then X is developable if X is semi-stratifiable. Moreover, if such a space Y is a Nagata space, it is developable, and hence metrizable, since a Nagata space certainly is collectionwise normal. Our diagram now looks as follows:



The example F, mentioned in the diagram is due to A.H. STONE [46] - who used it to show that a regular space, that is the image of a metrizable space by an open, compact map need not be metrizable - and it shows that the finite-to-one, open image of a metrizable LOTS can be a Moore space without being metrizable. We reproduce Stone's example in full detail.

EXAMPLE F: Let $\mathbb{N}$ be the set of the natural numbers, and let G be the set of all mappings of $\mathbb{N}$ into $\mathbb{N}$.
We take $X := G \cup (G \times \mathbb{N}) \cup \mathbb{N} \cup (\mathbb{N} \times G \times \mathbb{N})$.
The topology in X is generated by the following metric ρ:
$(g \in G, n, m, m' \in \mathbb{N})$

$$\rho[g,(g,n)] = \frac{1}{n} \quad ; \quad \rho[(g,n),(g,m)] = \frac{1}{n} + \frac{1}{m} \text{ if } n \neq m.$$

$$\rho[n,(n,g,m)] = \frac{1}{m} \; ; \quad \rho[(n,g,m),(n,g',m')] = \frac{1}{m} + \frac{1}{m'}, \text{ if } (g,m) \neq (g',m').$$

For every other pair (p,q) of points from X, $\rho[p,q] = 2$ if $p \neq q$, and $\rho[p,q] = 0$ if $p = q$.

ρ is easily verified to be a metric. Note that points of the form (g,m) and (n,g,m) are isolated, each having distance at least $\frac{1}{m}$ to any other point. If x belongs to X, and ε is a positive real number, we will denote by $S(x,\varepsilon)$ the collection of all points in X having distance less than ε to x.

X is strongly zero-dimensional (Ind X = 0). For, let A and B be disjoint closed subsets of X. Put

$$U := [A \cup \bigcup \{S(x,1) \mid x \in A \cap (G \cup \mathbb{N})\}]\setminus B,$$

then clearly U is an open set such that $A \subset U \subset X\setminus B$. Moreover U is closed, for suppose x is not isolated, and $x \notin U$. Hence $x \notin A$, so $S(x,\varepsilon) \cap A = \emptyset$ for some $\varepsilon (0<\varepsilon<1)$. Then also $S(x,\varepsilon) \cap U = \emptyset$, since $S(x,\varepsilon) \cap S(a,1) \neq \emptyset$ for some $a \in A \cap (G \cup \mathbb{N})$ would imply $\rho[x,a] < 2$, contradicting the fact that $\rho[x,a] = 2$ because both x and a are not isolated. Consequently, Ind X = 0 and X is metrizable, so X is homeomorphic to a LOTS by [26].

For every $g \in G$, and $n \in \mathbb{N}$, identify (g,n) and (n,g,g(n)). Let Y be the quotient space, and let f be the corresponding quotient map. Then f is finite-to-one (in fact at most two-to-one). and f is an open mapping because points of the form (g,n) and (n,g,m) are isolated in X.

One easily verifies that Y is regular, and by proposition 5.3.1, Y is developable, so Y is a Moore space. But Y is not metrizable, since it is not normal:

Suppose Y is normal; since the sets $f[\mathbb{N}]$ and $f[G]$ are disjoint and closed in Y, there exist disjoint open sets $U \supset f[\mathbb{N}]$ and $V \supset f[G]$. Now $\mathbb{N} \subset f^{-1}[U]$ so for every $n \in \mathbb{N}$ there is a natural number, which we shall call $h(n)$, such that

$$S\left(n, \frac{1}{h(n)-1}\right) \subset f^{-1}[U].$$

This implies that, for each $g \in G$, $f(n,g,h(n)) \in U$, because $\rho[n,(n,g,h(n))] = \frac{1}{h(n)} < \frac{1}{h(n)-1}$.
Since h can be considered as a function $h \in G$, we have in particular $f(n,h,h(n)) = f(h,n) \in U$ for every n, and so $(h,n) \in f^{-1}[U]$ for all $n \in \mathbb{N}$. But $h \in G \subset f^{-1}[V]$ which is open in X. Hence if n is large enough, $(h,n) \in S(h, \frac{1}{n-1}) \subset f^{-1}[V]$, which contradicts the fact that $U \cap V = \emptyset$.

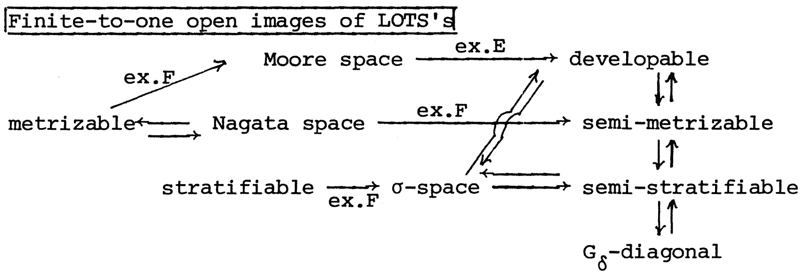
When we concentrate on those spaces that are the finite-to-one, open image of a LOTS, then we can go one step further:

THEOREM 5.3.3. *Suppose* $X = (X,\leq,\lambda(\leq))$ *is a LOTS and* $f: X \longrightarrow Y$ *is a finite-to-one, open continuous surjection. Then*

$$Y \textit{ has a } G_\delta\textit{-diagonal} \Rightarrow Y \textit{ is developable.}$$

PROOF. Consider the space $\tilde{X} = \tilde{X}$ (mod f) and the mapping $\tilde{f}: \tilde{X} \longrightarrow Y$. Since f is finite-to-one, we can apply proposition 1.2.4, to show that $\tilde{X}$ is a LOTS. Because X has a G_δ-diagonal by theorem 1.3.2, it follows that $\tilde{X}$ is metrizable. Since obviously $\tilde{f}$ also is a finite-to-one-open mapping, we conclude that Y is developable by 5.3.1. □

Hence for finite-to-one open images of LOTS's, we have the following diagram.



From what we have proved in this section, the following proposition now becomes obvious:

PROPOSITION 5.3.4. *Suppose* $X = (X,\leq,\tau)$ *is a GO-space (LOTS) and* $f: X \longrightarrow Y$ *is a finite-to-one, open surjection. Then* X *is metrizable if* Y *is semi-stratifiable (has a* G_δ*-diagonal).*

5.4. CLOSED IMAGES OF GO-SPACES

Observe that there is no need to differentiate between closed images of LOTS's and closed images of GO-spaces, since by 5.1.1 each space that is the closed image of a GO-space, is also the closed image of a LOTS. The converse is, of course, trivial.

In the foregoing sections, most of the "vertical" arrows in our diagram could be reversed. For closed images the situation is different, there exist easy counterexamples which show that the implications (3) and (9) cannot be reversed.

EXAMPLE G: Let $\mathbb{R}/\mathbb{Z}$ be the quotient space obtained from the reals by identifying the integers to a single point, and let $f: \mathbb{R} \longrightarrow \mathbb{R}/\mathbb{Z}$ be the quotient map. Then f is a closed mapping; hence $\mathbb{R}/\mathbb{Z}$ is stratifiable, since it is the closed image of a metrizable space (see [8]). However, since $\mathbb{R}/\mathbb{Z}$ is not first countable, it is neither semi-metrizable nor a Nagata space.

In contrast with this, implications (4), (5), (6) and (7) can be reversed, which is a direct consequence of the following theorems, the proofs of which can be found in [25].

DEFINITION. A space X is *monotonically normal* if it admits a function G that assigns to each ordered pair (H,K) of disjoint closed subsets of X an open set G(H,K) such that

(i) $H \subset G(H,K) \subset \overline{G(H,K)} \subset X\backslash K$.

(ii) if (H',K') is a pair of disjoint closed sets such that $H \subset H'$ and $K' \subset K$ then $G(H,K) \subset G(H',K')$.

Monotone normality is a hereditary property, and it implies collectionwise normality.

THEOREM 5.4.1. *Every LOTS (and hence every GO-space) is monotonically normal.*

THEOREM 5.4.2. *The closed image of a monotonically normal space is monotonically normal.*

THEOREM 5.4.3. *A semi-stratifiable space is stratifiable iff it is monotonically normal.*

Also, implication (2) can be reversed, since a collectionwise normal Moore space is metrizable, and a closed image of a GO-space certainly is collectionwise normal by the previous theorems.

THEOREM 5.4.4. *Let* $X = (X,\leq,\tau)$ *be a GO-space, let* Y *be a semi-metrizable space, and suppose* $f\colon X \longrightarrow Y$ *is a closed, surjective mapping such that each convexity-component under* f *consists of one point. Then* X *is metrizable.*

PROOF. We will show that X is semi-stratifiable, from which it follows that X is metrizable (see th. 1.4.6).

Certainly, Y has a G_δ-diagonal, so the same is true for X by theorem 1.3.2. Hence X is a first countable GO-space. Because Y is semi-metrizable there exists for each $y \in Y$, a sequence $(U(y,n))_{n=1}^{\infty}$ of open neighbourhoods of y such that

(i) if $y \in U(y(n),n)$ for $n = 1,2,\ldots,$ then the sequence $(y(n))_{n=1}^{\infty}$ converges to y

(ii) $(U(y,n))_{n=1}^{\infty}$ is a local base at y

(iii) $U(y,n+1) \subset U(y,n)$.

For each $x \in X$, and each $n \in \mathbb{N}$, choose a convex open subset $O(x,n)$ of X such that

- $x \in O(x,n)$
- $f[O(x,n)] \subset U(f(x),n)$
- $O(x,n+1) \subset O(x,n)$.

Because every convexity-component under f consists of a single point, it is clear that $\bigcap_{n=1}^{\infty} O(x,n) = \{x\}$.
For the sake of convenience, we denote $f^{-1}(f(x))$ by $\tilde{x}$. $(x \in X)$.

We claim that if $x \in X$, and $(x(n))_{n=1}^{\infty}$ is a sequence in X such that $x \in O(x(n),n)$ for all n, then the sequence $(x(n))_{n=1}^{\infty}$ converges to x, from which it follows that X is semi-stratifiable. To prove the claim, suppose that $x \in O(x(n),n)$ for $n = 1,2,\ldots$. Without loss of generality we may suppose that all $x(n) < x$. We first consider two special cases:

(I) $x(n) \notin \tilde{x}$ for $n = 1,2,\ldots$.

Let A be a neighbourhood of x, and suppose $(x(n))_{n=1}^{\infty}$ does not converge to x. Then there is a subsequence $(x(n(k)))_{k=1}^{\infty}$ of $(x(n))_{n=1}^{\infty}$ such that $x(n(k)) \notin A$. From $x \in O(x(n),n)$, it follows that

$$f(x) \in f[O(x(n),n)] \subset U(f(x(n)),n) \qquad (n = 1,2,\ldots).$$

Hence the sequence $(f(x(n)))_{n=1}^{\infty}$ and all its subsequences converge to $f(x)$ in Y. Note that each $f(x(n(k)))$ is distinct from $f(x)$. Because f is closed, the sequence $(x(n(k)))_{k=1}^{\infty}$ must have a cluster point p, which belongs to $\tilde{x}$. Clearly $p \neq x$, so we have $p < x$. Since every convexity-component under f consists of a single point, there exists a point b in the interval $]p,x[$ that is not mapped onto $f(x)$.

Then $]\leftarrow,b[$ is an open neighbourhood of p, which contains infinitely many $x(n(k))$. Because $O(x(n(k)),n(k))$ is convex, and $x(n(k)) < b < x$ for those $x(n(k))$, it follows that $b \in O(x(n(k)),n(k))$, hence $f(b) \in U(f(x(n(k))),n(k))$ for infinitely many k, which yields a contradiction, since $(f(x(n(k))))_{k=1}^{\infty}$ converges to $f(x)$, which is not equal to $f(b)$.

(II) $x(n) \in \tilde{x}$ for $n = 1,2,\ldots$.

a) We first prove that the half-line $[x,\rightarrow[$ is not an open set. For $n = 1,2,\ldots$ choose a point $x'(n) \in X$ such that $x'(n) \in]x(n),x[\setminus\tilde{x}$; this is possible, since all convexity-components under f consist of one point. Then $x'(n) \in O(x(n),x)$ by convexity, so

$$f(x'(n)) \in f[O(x(n),n)] \subset U(f(x(n)),n) = U(f(x),n).$$

Because $(U(f(x),n))_{n=1}^{\infty}$ is a decreasing local base at $f(x)$, this implies that $(f(x'(n)))_{n=1}^{\infty}$ converges to $f(x)$. Since f is closed, there is a subsequence $(x'(n(k)))_{k=1}^{\infty}$ of $(x'(n))_{n=1}^{\infty}$ converging to a point $p \in \tilde{x}$. It follows that $p = x$, since otherwise there would be a point $b \in]p,x[\setminus\tilde{x}$ belonging to $O(x(n(k)),n(k))$ for almost all k, and hence

$$f(b) \in U(f(x(n(k))),n(k)) = U(f(x),n(k)) \qquad \text{for all } k$$

which contradicts the fact that $(U(f(x),n))_{n=1}^{\infty}$ is a local base at $f(x)$.

Apparently $(x'(n(k)))_{k=1}^{\infty}$ converges to x and $x'(n(k)) < x$ for all k, so $[x,\rightarrow[$ cannot be open.

b) Let U be a convex open neighbourhood of x in X.
From a) we know that U contains a point $b \in]\leftarrow,x[\setminus\tilde{x}$. Now suppose that infinitely many $x(n(k))$ $(k = 1,2,\ldots)$ do not belong to U. Clearly we have $b \in O(x(n(k)),n(k))$, and hence

$$f(b) \in f[O(x(n(k)),n(k))] \subset U(f(x(n(k))),n(k)) = U(f(x),n(k)).$$

Consequently

$$f(b) \in \bigcap_{k=1}^{\infty} U(f(x),n(k)) = \{f(x)\}, \text{ which is a contradiction.}$$

(III) The general case now follows by combining (I) and (II). □

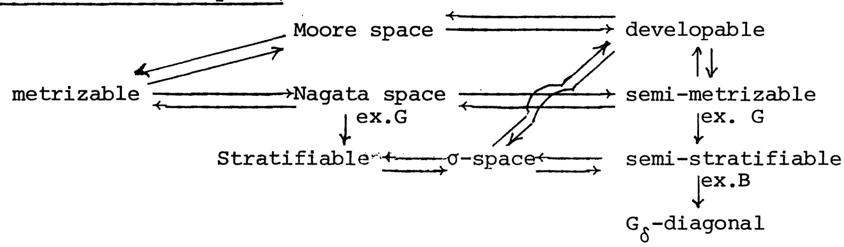
<u>THEOREM 5.4.5</u>. *Let $X = (X,\leq,\tau)$ be a GO-space, and let $f\colon X \longrightarrow Y$ be a closed, surjective mapping. Then*

$$Y \textit{ is semi-metrizable} \Rightarrow Y \textit{ is metrizable.}$$

<u>PROOF</u>. The space $\tilde{X} = \tilde{X}$ (mod f) is metrizable by 5.4.4. Hence Y is a first countable space, which is the closed image of the metrizable space $\tilde{X}$ under the map $\tilde{f}$. Consequently, Y is metrizable by Stone's theorem [46]. □

The diagram now looks as follows:

Closed images of LOTS's/GO-spaces



One might ask if the analogue of theorem 5.4.4 for semi-stratifiable spaces is also true, i.e., if f is a closed mapping from a GO-space X onto a semi-stratifiable space Y, and all convexity-components under f are one-point

sets, need X be metrizable?
However, the answer is no, as the next example shows.

EXAMPLE. Let X be the set of the real numbers, $\leq$ the usual order on X, and τ the topology obtained from the "Sorgenfrey topology" on X by adding each rational point as an open set. Then $(X,\leq,\tau)$ is a GO-space. Clearly $Irr := \{x \in X \mid x \text{ is irrational}\}$ is a closed set in X so if we identify all elements of Irr to one point, the quotient map

$$f: X \longrightarrow X/Irr \text{ is closed.}$$

Moreover, every convexity-component under f consists of a single point, and X/Irr is semi-stratifiable (but not first countable) since it is a countable T_1-space. But X is not metrizable, because it is separable and does not have a countable base.

5.5. PERFECT IMAGES OF GO-SPACES

THEOREM 5.5.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space, let* Y *be a semi-stratifiable space and suppose* f *is a perfect map from* X *onto* Y *such that each convexity-component under* f *consists of one point. Then* X *is metrizable.*

PROOF. The proof is very similar to the proof of theorem 5.4.4. The only part there, where we used the full force of the fact that $(U(y,n))_{n=1}^{\infty}$ is a local base at y was in case II a); everywhere else we only needed that $\bigcap_{n=1}^{\infty} U(y,n) = \{y\}$. Now let $(U(y,n))_{n=1}^{\infty}$ for each y be a sequence of neighbourhoods of y such that

(i) if $y \in U(y(n),n)$ for each n, then the sequence $(y(n))_{n=1}^{\infty}$ converges to x.

(ii) $\bigcap_{n=1}^{\infty} U(y,n) = \{y\}$

(iii) $U(y,n+1) \subset U(y,n)$.

Let $O(x,n)$, x and $x(n)$ be as in the proof of 5.4.4, case II. We will show that $[x,\rightarrow[$ is not an open set; the rest of the proof is identical with the proof of 5.4.4.

Suppose that $[x,\rightarrow[$ is open. Then $\emptyset \neq\ ]\leftarrow,x[\ \cap \tilde{x}$ is closed and hence compact, since f is perfect. Let x' be the right endpoint of $]\leftarrow,x[\ \cap \tilde{x}$, then there exists some $b \in\ ]x',x[$. Obviously $b \notin \tilde{x}$, but $b \in O(x(n),n)$, hence $f(b) \in U(f(x),n)$ for all n. Contradiction. □

COROLLARY. *Let* X *be a GO-space,* Y *a semi-stratifiable space, and suppose* $f: X \longrightarrow Y$ *is a perfect map from* X *onto* Y. *Then* Y *is metrizable.*

For perfect images of LOTS's the following is true:

THEOREM 5.5.2. *Let* $X = (X,\leq,\lambda(\leq))$ *be a LOTS, and let* $f: X \longrightarrow Y$ *be a perfect map from* X *onto* Y. *Then*

$$Y \textit{ has a } G_\delta\textit{-diagonal} \Rightarrow Y \textit{ is metrizable.}$$

PROOF. If $\tilde{X} = X$ (mod f), then $\tilde{X}$ is a LOTS by proposition 1.2.4. Moreover $\tilde{X}$ has a G_δ-diagonal by theorem 1.3.2, so $\tilde{X}$ is metrizable. Clearly $\tilde{f}: \tilde{X} \longrightarrow Y$ is perfect. Hence Y is metrizable. □

Consequently, the relations between the properties we are considering are in the class of all perfect images of LOTS's (GO-spaces) the same as in the class of all LOTS's (GO-spaces). This for instance implies that the Sorgenfrey line which is well-known not to be orderable, is not even the perfect image of a LOTS.

Note that the analogue of proposition 5.3.4 for perfect (or for finite-to-one, closed) mappings is not true:

EXAMPLE. Let X be the "*double arrow space*" of Alexandroff, i.e. the lexicographic product of the unit interval I and the discrete two point space $\{0,1\}$. Then X is a compact, non-metrizable LOTS, and the mapping $f: X \longrightarrow I$ defined by $f(x,i) := x$ $(x\in I; i\in\{0,1\})$ is a two-to-one, closed continuous mapping of X onto a metrizable space.

5.6. OPEN-AND-CLOSED IMAGES OF GO-SPACES

THEOREM 5.6.1. *Let* $X = (X,\leq,\tau)$ *be a GO-space, and let* $f: X \longrightarrow Y$ *be an open-and-closed surjective mapping. Then*

$$X \textit{ is semi-stratifiable} \Rightarrow X \textit{ is metrizable.}$$

PROOF. Follows immediately from theorem 5.2.2 and theorem 5.4.5. □

EXAMPLE H: Let $M = (M,\leq,\tau)$ be the Michael line (as described in section 2.1). *Then* $M^* = M \times \{0\} \cup \{(x,n) \mid x \text{ is irrational, } n \in \mathbb{Z}\}$ supplied with the lexicographic order and corresponding topology, is a LOTS that can be mapped onto M by an open-and-closed mapping, since the projection $\mathbb{P} : M^* \longrightarrow M$, defined by

$$\mathbb{P}(x,n) := x$$

is open-and closed.

From theorem 5.6.1 and example H, it follows that all properties, except the existence of a G_δ-diagonal are equivalent to metrizability, when we look at open-and-closed images of LOTS's and GO-spaces.

The space M in example H consists "almost entirely" of isolated points. We will show that in open-and-closed images of LOTS's, that do not have "too many" isolated points, having a G_δ-diagonal is equivalent to metrizability.

THEOREM 5.6.2. *Let* $X = (X,\leq,\lambda(\leq))$ *be a LOTS,* Y *a space having a* G_δ*-diagonal, and let* $f\colon X \longrightarrow Y$ *be an open and closed surjective mapping . Then*

$$I := \{y \in Y \mid y \text{ is isolated in } Y\} \text{ is } \sigma\text{-discrete} \iff Y \text{ is metrizable.}$$

PROOF.
$\Leftarrow$: Obvious.
$\Rightarrow$: We will show that $\tilde{X} = \tilde{X}\,(\mathrm{mod}\ f)$ is metrizable. Since $\tilde{f}\colon \tilde{X} \longrightarrow Y$ is also open-and-closed this is easily seen to imply the metrizability of Y.

If z is an isolated point of $\tilde{X}$ then $\tilde{f}(z)$ belongs to I. Conversely, if $\tilde{f}(z) \in I$, then z is a convexity-component of the open set $f^{-1}(\tilde{f}(z))$ in X, and hence z is an isolated point of $\tilde{X}$.

Now suppose $z \in H(\tilde{X})$, say $[z,\rightarrow[\ \in \tau\backslash\lambda(\leq)$ (where τ and $\leq$ are the topology and the order on $\tilde{X}$). Then $\mathbb{P}^{-1}[[z,\rightarrow[]$ is an open set in X, which cannot have a left endpoint, since this, and the fact that X is a LOTS would imply that z either is left endpoint of $\tilde{X}$ or has a left neighbour, contradictory to the fact that $z \in H(X)$. Consequently $\mathbb{P}^{-1}(z)$ has a non-empty interior; hence $f[\mathbb{P}^{-1}(z)] = \{f(z)\}$ is open.
It follows that

$$H(\tilde{X}) \subset \tilde{f}^{-1}[I].$$

Since I is an F_σ-set, the same is true for $\tilde{f}^{-1}[I]$, so $\tilde{f}^{-1}[I]$ is σ-discrete, and the same is true for $\tilde{H}(X)$. Moreover, $\tilde{X}$ has a G_δ-diagonal by 1.3.2. Hence $\tilde{X}$ is metrizable by theorem 1.3.4. □

COROLLARY. *The Sorgenfrey line S is not the image of a LOTS under a open-and-closed map.*

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