MATHEMATICAL CENTRE TRACTS 103

CLASSIFYING INFINITELY DIVISIBLE DISTRIBUTIONS BY FUNCTIONAL EQUATIONS

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CHAPTER 1

INTRODUCTION AND PRELIMINARIES

1.1. *Introduction and summary*

The theory of infinitely divisible probability distributions plays an important role in theoretical problems, such as in the study of limit theorems, more so than in practical situations, though applications do occur, especially in statistical modelling (cf. Katti (1977), Thorin (1977) and Ahmad & Abouammoh (1977)). The first stage of its development ended around 1950; the basic properties, such as canonical representations, derived especially by P. Lévy and I.A. Khintchine, and the important applications in the theory of limit distributions of sums of independent random variables, have been formulated, for instance, in the books by Lévy (1937) and by Gnedenko & Kolmogorov (1949). In the next two decades research on this field has been carried out along many lines; especially, much attention has been paid to factorization problems and stable distributions, as is apparent from the survey paper by Fisz (1962) and from the books by Linnik (1960) and Lukacs (1970). For more recent information we refer to Petrov (1972).

During the last ten years more research has been done on the often difficult problem how to decide whether a given probability distribution is infinitely divisible or not. On the one hand new methods of constructing infinitely divisible distributions have been introduced; for instance, the methods of compounding and mixing are very useful, as has been shown by Steutel (1970), Kelker (1972) and others. On the other hand many necessary and (or) sufficient conditions for infinite divisibility have been obtained in terms of the probabilities themselves, rather than in terms of the corresponding characteristic functions, the most obvious tool in this field; this is evident from the survey paper by Steutel (1973).

In this monograph this tendency is continued in the following sense: most of the classes of infinitely divisible probability distributions that we introduce, are characterized by means of functional equations for the probabilities themselves; furthermore, we study properties of distribution functions and densities in these classes, like asymptotic behaviour, absolute continuity, complete monotonicity, etc.

Our starting point is the "gap" between the class C_0 of compound geometric distributions on $\mathbb{N}_0$ and the class C_1 of compound Poisson distributions on $\mathbb{N}_0$, i.e. (cf. Feller (1968), ch. XII) the class of all infinitely divisible distributions on $\mathbb{N}_o$ with factors in $\mathbb{N}_o$. It is known that $\mathcal{C}_o \subsetneq \mathcal{C}_1$ (cf. Lukacs (1970), ch. 5). Furthermore, the classes $\mathcal{C}_o$ and $\mathcal{C}_1$ can be characterized by means of recurrence relations as follows (cf. Steutel (1970) and Katti (1967)): a probability distribution $\{p_n\}_o^\infty$ on $\mathbb{N}_o$ with $p_o > 0$ is in $\mathcal{C}_o$ iff there exist nonnegative quantities $r_n(0)$ $(n \in \mathbb{N}_o)$ such that

$$(1.1.1) \qquad p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_o) \;;$$

similarly, $\{p_n\}$ is in $\mathcal{C}_1$ iff there exist nonnegative quantities $r_n(1)$ $(n \in \mathbb{N}_o)$ such that

$$(1.1.2) \qquad (n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(1) \qquad (n \in \mathbb{N}_o) \;.$$

Now, in order to fill the gap between $\mathcal{C}_o$ and $\mathcal{C}_1$ we interpolate between (1.1.1) and (1.1.2) by means of a set of recurrence relations of the following form:

$$(1.1.3) \qquad c_n(\alpha)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(\alpha) \qquad (n \in \mathbb{N}_o) \;,$$

where $c_n(0) = 1$, $c_n(1) = n+1$ $(n \in \mathbb{N}_o)$ and $c_n(\alpha)$ is nondecreasing in both n and $\alpha \in [0,1]$. Introducing for $0 < \alpha < 1$ the class $\mathcal{C}_\alpha$ as the set of distributions $\{p_n\}_o^\infty$ with $p_o > 0$ and satisfying (1.1.3) with nonnegative $r_n(\alpha)$'s, we wish to choose $c_n(\alpha)$ in such a way that the $\mathcal{C}_\alpha$'s yield a *classification* of $\mathcal{C}_1$, i.e. such that $\mathcal{C}_\alpha$ depends monotonically on $\alpha \in [0,1]$. The most obvious choices for $c_n(\alpha)$ do not have this property, but in chapter 2 we show that the choice

$$(1.1.4) \qquad c_n(\alpha) = (1-\alpha^{n+1})/(1-\alpha) \qquad (n \in \mathbb{N}_o;\ 0 \le \alpha \le 1)$$

produces classes $\mathcal{C}_\alpha$ that give a classification of $\mathcal{C}_1$. Rather surprisingly, perhaps, we did not find any other. It would seem that these $\mathcal{C}_\alpha$'s are "closer" to $\mathcal{C}_o$ than to $\mathcal{C}_1$, but as $\bigcup_{\alpha<1} \mathcal{C}_\alpha$ is dense in $\mathcal{C}_1$ in the sense of weak convergence, the situation is not too bad. Also in this chapter we briefly consider some other classifications.

Furthermore, the classes $\mathcal{C}_\alpha$ give rise to a number of other interesting observations. The equations defining the probability generating functions of distributions in $\mathcal{C}_\alpha$ suggest several other classes of decomposable distributions; these are studied in chapter 3. One of these gives rise to discrete analogues of the well known concepts of self-decomposability and stability

(cf. Lukacs (1970), ch. 5 and Feller (1971), ch. XVII), concepts which were restricted to absolutely continuous distributions.
In chapter 4 we investigate the recurrence relations (1.1.3) with $c_n(\alpha)$ given by (1.1.4) for sequences $\{p_n\}$ that are not necessarily probability distributions. Several properties can be proved that are analogous to properties of the sequences studied by Kaluza (1928) and De Bruijn & Erdős (1951). Also, we show a fruitful relation with renewal theory; it turns out that the case $\alpha = 0$ is strongly related to the renewal sequences (cf. Kingman (1972)), while for $0 < \alpha < 1$ the bounded solutions of (1.1.3) with nonnegative $r_n(\alpha)$'s can be considered as delayed renewal sequences. Although, especially from these relations, several properties can be obtained, it turns out that the case $0 < \alpha < 1$ is often difficult to handle; in many respects this case seems to inherit the difficulties of both the cases $\alpha = 0$ and $\alpha = 1$.
In chapter 5 the classification of $\mathcal{C}_1$, obtained by means of the classes $\mathcal{C}_\alpha$, is extended to all infinitely divisible distributions on $[0,\infty)$, by replacing the system of recurrence relations (1.1.3) for p_n by the analogous functional equation for the distribution function. As, contrary to the discrete case, we also have to consider distributions on $[0,\infty)$ without a jump at zero, the proofs are more delicate and the analogy with the discrete case is not perfect. At this point it is interesting to note that the resulting classes determine a limiting class $\mathcal{F}_\infty$ that can be considered as the analogue of $\mathcal{C}_0$ for distributions on $[0,\infty)$, just as the class of all infinitely divisible distributions on $[0,\infty)$ is the analogue of $\mathcal{C}_1$. A good deal of chapter 5 is devoted to investigating the structure and properties of this class $\mathcal{F}_\infty$. It turns out that the absolutely continuous elements of $\mathcal{F}_\infty$ contain the standard p-functions of Kingman (1972) as a subclass. Finally, in the last section of chapter 5 we briefly discuss the classification of the infinitely divisible distributions on $\mathbb{R}$, and on $[0,\infty)^2$, by means of functional equations.

The remainder of the present chapter contains definitions and preliminary results. After some notations and conventions in section 2, in section 3 we introduce the concepts of absolute and complete monotonicity, which we shall use frequently. The concept of infinite divisibility and its basic properties are introduced in section 4, where also some attention is paid to compound distributions. In sections 5 and 6 we study the infinitely divisible distributions on $\mathbb{N}_0$ and on $[0,\infty)$, respectively, in more detail. Finally, in section 7 we give a survey of canonical representations for infinitely divisible distributions and the relations between them.

1.2. *Notations and conventions*

First we give a list of general symbols and notations, which we shall use throughout this monograph.

$\mathbb{N}$	: the set $\{1,2,3,\ldots\}$ of natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	: the set of integers.
$\mathbb{R}$	: the set of real numbers.
$\mathbb{C}$	: the set of complex numbers.
A^2	: the cartesian product of the set A with itself; for instance: $\mathbb{R}^2$, $\mathbb{N}_0^2$, $[0,\infty)^2$.
(a,b)	: the open interval $\{x \in \mathbb{R} \mid a < x < b\}$.
$[a,b]$	: the closed interval $\{x \in \mathbb{R} \mid a \le x \le b\}$; similarly $(a,b]$, $[a,b)$.
$\#(A)$	: the cardinality of the set A.
1_A	: the indicator function of the set A.
$\emptyset$	: the empty set; $\sum_{n\in\emptyset} a_n := 0$, $\prod_{n\in\emptyset} a_n := 1$.
$\delta_{i,j}$	: the Kronecker symbol, i.e. $\delta_{i,i} = 1$ and $\delta_{i,j} = 0$ if $i \neq j$.
$f(x+)$	$:= \lim_{h\downarrow 0} f(x+h)$, $f(x-) := \lim_{h\downarrow 0} f(x-h)$, $f(\infty) := \lim_{x\to\infty} f(x)$, $f(-\infty) := \lim_{x\to-\infty} f(x)$.
$\square$	: indicates the end of the proof.

We shall frequently make use of generating functions, Laplace transforms and Fourier transforms; we shall use the following notation for these. If $a_n \in \mathbb{C}$ for $n \in \mathbb{N}_0$, then the *generating function* (gf) of the sequence $\{a_n\}_0^\infty$ is denoted by the corresponding capital, so

$$A(z) = \sum_{n=0}^{\infty} a_n z^n ,$$

for those $z \in \mathbb{C}$ for which the power series converges. A *probability generating function* (pgf) is the gf P of a probability distribution $\{p_n\}_0^\infty$ on $\mathbb{N}_0$. Such distributions will be called *lattice distributions;* their pgf's are always defined for $|z| \le 1$.
If $U \not\equiv 0$ is a function on $\mathbb{R}$ that is nonnegative, nondecreasing and right-continuous, and if $\ell(U)$, the *left extremity* of U, defined by

$$\ell(U) := \inf\{x \in \mathbb{R} \mid U(x) > 0\} ,$$

is finite, then the *Laplace-Stieltjes transform* (LST) $\hat{U}$ of U is defined by

$$\hat{U}(\tau) := \int_{(-\infty,\infty)} e^{-\tau x} dU(x) = \int_{[\ell(U),\infty)} e^{-\tau x} dU(x) ,$$

for those $\tau \in \mathbb{R}$ for which the integral is finite. If U is a distribution function, then $\hat{U}$ is called the *probability Laplace-Stieltjes transform* (PLST) of U. The corresponding small letter u will be used for the (probability) density function of U in case of absolute continuity; the ordinary (*probability*) *Laplace transform* ((P)LT) of u is then also denoted by $\hat{U}$, so

$$\hat{U}(\tau) = \int_{(\ell(U),\infty)} e^{-\tau x} u(x)\, dx .$$

Finally, if U is a right-continuous, nondecreasing and bounded function on $\mathbb{R}$ with $U(-\infty) = 0$, then the *Fourier-Stieltjes transform* (FST) $\tilde{U}$ of U is defined by

$$\tilde{U}(t) := \int_{(-\infty,\infty)} e^{itx} dU(x) ,$$

which exists for all $t \in \mathbb{R}$. Analogous to the LT, we define the ordinary *Fourier transform* (FT). If F is a distribution function on $\mathbb{R}$, then the FST $\tilde{F}$ is called the *characteristic function* (chf) of F. As $\tilde{F}$ is continuous and $\tilde{F}(0) = 1$, there exists a neighbourhood of the origin where $\tilde{F}$ is different from zero. So, the principal branch of the logarithm of $\tilde{F}$, denoted by $\log \tilde{F}(t)$, can be defined uniquely in that neighbourhood.

Besides the abbreviations gf, pgf, LST, PLST, LT, PLT, FST, FT and chf, just introduced, we shall use the following:

rv	random variable
df	distribution function
pdf	probability density function
n-div	n-divisible, n-divisibility
inf div	infinitely divisible, infinite divisibility
abs mon	absolutely monotone, absolute monotonicity
comp mon	completely monotone, complete monotonicity
$\stackrel{d}{=}$	equal in distribution
iff	if and only if.

If U and V are nonnegative, nondecreasing and right-continuous functions on $\mathbb{R}$, then the convolution $U * V$ of U and V is defined by

$$(U * V)(x) := \int_{(-\infty,\infty)} U(x-y)\,dV(y) = \int_{(-\infty,\infty)} V(x-y)\,dU(y) \qquad (x \in \mathbb{R}),$$

which is again a nonnegative, nondecreasing and right-continuous function. For $n \in \mathbb{N}$ the n-fold convolution of U with itself is denoted by U^{*n}. If F is an inf div df, then for $\gamma > 0$ $F^{*\gamma}$ denotes the df with chf $\tilde{F}^{\gamma}$.
If $\{a_n\}_0^\infty$ is a sequence with $a_0 \neq 0$ and with gf A, then for $\gamma > 0$ the sequence $\{a_n^{*\gamma}\}_0^\infty$ is defined by its gf as follows:

$$\sum_{n=0}^{\infty} a_n^{*\gamma} z^n = A(z)^\gamma .$$

Unless stated otherwise, throughout this monograph we only consider probability distributions on $\mathbb{R}$ that are not concentrated at zero. For instance, as in case of a lattice distribution $\{p_n\}_0^\infty$ we often take $p_0 > 0$ (cf. section 5), it is then tacitly assumed that $0 < p_0 < 1$.
If $\{p_n\}_0^\infty$ is a lattice distribution in some class $\mathcal{C}$ with pgf P, then we shall also say that $P \in \mathcal{C}$. Similar conventions hold for $\hat{F}$ and $\tilde{F}$.
Finally, if $\mathcal{C}$ is a class of probability distributions, then a family $(\mathcal{C}_t \mid t \in T)$ of subclasses of $\mathcal{C}$ is said to define a *classification* of $\mathcal{C}$, if T can be totally ordered in such a way that the classes $\mathcal{C}_t$ are nondecreasing in the ordering of T. It follows that for $t_1 < t_2 < \ldots < t_n$ the classes $\mathcal{C}_{t_1}, \mathcal{C}_{t_2} \backslash \mathcal{C}_{t_1}, \ldots, \mathcal{C}_{t_n} \backslash \mathcal{C}_{t_{n-1}}, \mathcal{C} \backslash \mathcal{C}_{t_n}$ form a partition of $\mathcal{C}$.

1.3. *Absolute and complete monotonicity*

In the sequel we shall characterize several classes of probability distributions by making use of the concepts of absolute and complete monotonicity (cf. Widder (1946), ch. IV and Feller (1971), ch. VII and XIII). Since we only need absolute monotonicity on intervals of the form $[0,\rho)$ and complete monotonicity on $(0,\infty)$, we usually do not mention these intervals. We start with considering absolutely monotone functions.

DEFINITION 1.3.1. A function R is said to be *absolutely monotone on* $[0,\rho)$ if it is continuous there and possesses derivatives of all orders on $(0,\rho)$ with

$$\left(\frac{d}{dz}\right)^n R(z) \geq 0 \qquad (n \in \mathbb{N}_0;\ 0 < z < \rho) .$$

R is said to be *absolutely monotone* (abs mon) if there exists $\rho > 0$ such that R is abs mon on $[0,\rho)$.

When proving the abs mon of a function, we shall often utilize the following characterization (cf. Widder (1946), ch. IV).

THEOREM 1.3.2. A function R is abs mon on $[0,\rho)$ iff there exist $r_n \geq 0$ $(n \in \mathbb{N}_0)$ such that

$$(1.3.1) \qquad R(z) = \sum_{n=0}^{\infty} r_n z^n \qquad (0 \leq z < \rho) ;$$

in this case the quantities r_n are given by $r_n = R^{(n)}(0+)/n!$ $(n \in \mathbb{N}_0)$.

Thus, an abs mon function R on $[0,\rho)$ can be extended analytically to the disk $|z| < \rho$.
There exists a number of simple properties of abs mon functions that we shall use in the sequel without further comment; the following lemma contains some of them.

LEMMA 1.3.3.
(i) R is abs mon iff $R(0) \geq 0$ and $R'(z)$ is abs mon.
(ii) If R is abs mon, then so are $R(\alpha z)$ and $R(z) - R(\alpha z)$ for all $\alpha \in (0,1)$.
(iii) If R and S are abs mon, then so are $R(z) + S(z)$ and $R(z)S(z)$.
(iv) If R_n is abs mon on $[0,\rho)$ $(n \in \mathbb{N})$ and if $R(z) = \lim_{n\to\infty} R_n(z)$ exists for $z \in [0,\rho)$, then R is abs mon on $[0,\rho)$.
(v) If R is abs mon on $[0,\rho)$ and if S is abs mon on $[0,\sigma)$ with $S(z) < \rho$ $(0 \leq z < \sigma)$, then $R(S(z))$ is abs mon on $[0,\sigma)$. For instance:
  (a) If S is abs mon, then $\exp[S(z)]$ is abs mon;
  (b) If S is abs mon with $S(z) < 1$ in some interval $[0,\sigma)$, then $\{1 - S(z)\}^{-1}$ is abs mon.

The following lemma, and simple extensions of it, will be used particularly in chapter 3.

LEMMA 1.3.4. Let P be a pgf with $P(0) > 0$, and let Q be a pgf. If the function R, defined by

$$R(z) := Q(z)/P(z) ,$$

is abs mon, then R coincides, at least in $|z| \leq 1$, with a pgf.

PROOF. As R is abs mon, by theorem 1.3.2 there exist $\rho > 0$ and $r_n \geq 0$ $(n \in \mathbb{N}_o)$ such that

$$(1.3.2) \qquad R(z) = \sum_{n=0}^{\infty} r_n z^n \qquad (|z| < \rho) .$$

Since $P(0) > 0$, we may assume that $P(z) \neq 0$ for $|z| < \rho$, and hence, if P and Q are the pgf's of $\{p_n\}_o^\infty$ and $\{q_n\}_o^\infty$, respectively, then

$$q_n = \sum_{k=0}^{n} r_k p_{n-k} \qquad (n \in \mathbb{N}_o) .$$

Summing over n we get

$$1 = \sum_{n=0}^{\infty} \sum_{k=0}^{n} r_k p_{n-k} = \sum_{k=0}^{\infty} r_k \sum_{n=k}^{\infty} p_{n-k} = \sum_{k=0}^{\infty} r_k ,$$

i.e. $\{r_n\}_o^\infty$ is a probability distribution. Now, let A denote the set of poles of R in $|z| \leq 1$, then, as P has finitely many zeros in $|z| \leq 1$, we have $\#(A) < \infty$, while by analytic continuation we see that the equality in (1.3.2) holds in $\{|z| \leq 1\}\backslash A$. However, since $\Sigma r_n z^n$ is bounded in $|z| \leq 1$, we necessarily have $A = \emptyset$, and the lemma is proved. □

Finally, we state the continuity theorem for pgf's, which we shall need several times. It can be found in Feller (1968), ch. XI.

THEOREM 1.3.5. Suppose that for every $n \in \mathbb{N}$ the sequence $\{p_k(n)\}_{k=0}^\infty$ is a probability distribution with pgf P_n.

(i) If $p_k := \lim_{n\to\infty} p_k(n)$ exists for all $k \in \mathbb{N}_o$, then $P(z) := \lim_{n\to\infty} P_n(z)$ exists for all $z \in [0,1]$, while

$$(1.3.3) \qquad P(z) = \sum_{k=0}^{\infty} p_k z^k \qquad (0 \leq z < 1) .$$

If in addition $\{p_k\}_o^\infty$ is a probability distribution, then P is the pgf of $\{p_k\}$ (in fact, as is easily shown, $P(z) := \lim_{n\to\infty} P_n(z)$ exists for $|z| \leq 1$ and (1.3.3) holds for $|z| \leq 1$).

(ii) If $P(z) := \lim_{n\to\infty} P_n(z)$ exists for all $z \in (0,1)$, then $p_k := \lim_{n\to\infty} p_k(n)$ exists for all $k \in \mathbb{N}_o$, while (1.3.3) holds. If in addition P is left-continuous in $z = 1$, then $\{p_k\}_o^\infty$ is a probability distribution with pgf P.

Next, we consider completely monotone functions.

DEFINITION 1.3.6. A function φ on $(0,\infty)$ is said to be *completely monotone* (comp mon) if φ possesses derivatives of all orders on $(0,\infty)$ with

$$(-1)^n \left(\frac{d}{d\tau}\right)^n \varphi(\tau) \geq 0 \qquad (n \in \mathbb{N}_o;\ \tau > 0) \ .$$

The comp mon functions can be represented as LST's; this result is known as Bernstein's theorem (see e.g. Feller (1971), ch. XIII).

THEOREM 1.3.7. A function φ on $(0,\infty)$ is comp mon iff there exists a nonnegative, right-continuous and nondecreasing function U with $\ell(U) \geq 0$ such that $\varphi = \hat{U}$, i.e. such that

$$\varphi(\tau) = \int_{[0,\infty)} e^{-\tau x} dU(x) \qquad (\tau > 0) \ .$$

In the following lemma we summarize the principal properties of comp mon functions (cf. Feller (1971), ch. XIII); they will be used without further comment.

LEMMA 1.3.8.

(i) φ is comp mon iff $-\varphi'(\tau)$ is comp mon and $\varphi(\infty) \geq 0$.

(ii) If φ is comp mon, then so are $\varphi(\lambda\tau)$, $\varphi(\tau+\lambda)$ and $\varphi(\tau) - \varphi(\tau+\lambda)$ for all $\lambda > 0$.

(iii) If φ and ψ are comp mon, then so are $\varphi + \psi$ and $\varphi\psi$.

(iv) If φ_n is comp mon ($n \in \mathbb{N}$) and if $\varphi(\tau) := \lim_{n\to\infty} \varphi_n(\tau)$ exists for $\tau > 0$, then φ is comp mon.

(v) If R is abs mon on $[0,\rho)$ and if φ is comp mon with $\varphi(\tau) < \rho$ $(\tau > 0)$, then $R(\varphi(\tau))$ is comp mon. For instance, if φ is comp mon, then $\exp[\varphi(\tau)]$ is comp mon, and if in addition $\varphi(\tau) < 1$ $(\tau > 0)$, then $\{1 - \varphi(\tau)\}^{-1}$ is comp mon.

(vi) If φ and ψ' are comp mon and if $\psi(0+) \geq 0$, then $\varphi(\psi(\tau))$ is comp mon. For instance, if ψ' is comp mon and $\psi(0+) \geq 0$, then $\exp[-\psi(\tau)]$ and $\{1 + \psi(\tau)\}^{-1}$ are comp mon.

We also mention two relations between the LST $\hat{U}$ (comp mon if $\ell(U) \geq 0$) and the function U, which we shall use repeatedly.

LEMMA 1.3.9. Let $U \not\equiv 0$ be a nonnegative, right-continuous and nondecreasing function on $\mathbb{R}$ with $\ell(U) > -\infty$ and such that $\hat{U}(\tau)$ exists for $\tau > \tau_o$. Then

$$(1.3.4) \qquad U(\ell(U)) = \lim_{\tau\to\infty} \hat{U}(\tau)e^{\ell(U)\tau} ,$$

and, if $\ell(U) \geq 0$,

$$(1.3.5) \qquad U(0) = \lim_{\tau\to\infty} \hat{U}(\tau) .$$

If $\tau_o \leq 0$, then also

$$(1.3.6) \qquad U(\infty) = \lim_{\tau\downarrow 0} \hat{U}(\tau) .$$

PROOF. In view of the definition of $\hat{U}$ we can write

$$\hat{U}(\tau)e^{\ell(U)\tau} = U(\ell(U)) + \int_{(\ell(U),\infty)} e^{-\tau(x-\ell(U))}dU(x) ,$$

from which (1.3.4) follows by the dominated convergence theorem. Similarly we obtain (1.3.5). Finally, applying the monotone convergence theorem, we see that

$$\lim_{\tau\downarrow 0} \hat{U}(\tau) = \int_{[\ell(U),\infty)} dU(x) = \lim_{x\to\infty} U(x) = U(\infty) . \qquad \square$$

Finally, we give a definition of comp mon for sequences and a representation of such sequences, which is due to Hausdorff (cf. Feller (1971), ch. VII).

DEFINITION 1.3.10. A sequence $\{a_n\}_o^\infty$ of real numbers is called comp mon if

$$(-1)^n\Delta^n a_k \geq 0 \qquad (n,k \in \mathbb{N}_o) ,$$

where $\Delta a_k := a_{k+1} - a_k$, $\Delta^o a_k := a_k$ and $\Delta^n := \Delta(\Delta^{n-1})$.

THEOREM 1.3.11. A sequence $\{a_n\}_o^\infty$ is comp mon iff there exists a finite measure ν on $[0,1]$ such that

$$(1.3.7) \qquad a_n = \int_{[0,1]} x^n \nu(dx) \qquad (n \in \mathbb{N}_o) .$$

1.4. *Definition and basic properties of infinitely divisible distributions on* $\mathbb{R}$

The concept of infinite divisibility can be introduced as follows.

DEFINITION 1.4.1. For $n \in \mathbb{N}$ a rv X is said to be *n-divisible* (n-div) if there exist independent and identically distributed rv's $X_{n,1},\ldots,X_{n,n}$ such that

$$X \overset{d}{=} X_{n,1} +\ldots+ X_{n,n} .$$

A rv X is said to be *infinitely divisible* (inf div) if X is n-div for all $n \in \mathbb{N}$.

In fact, inf div is a property of the *distribution* of X; therefore we call the df, pdf, chf, etc., corresponding to an inf div rv X, inf div too. Thus, a chf $\tilde{F}$ is inf div iff for every $n \in \mathbb{N}$ there exists a df F_n such that

$$\tilde{F}(t) = \{\tilde{F}_n(t)\}^n \qquad (t \in \mathbb{R}) .$$

Next we list a number of basic properties of inf div distributions that we need in the following chapters; they can be found in Lukacs (1970). The first three of them have obvious analogues for pgf's and PLST's.

THEOREM 1.4.2. If $\tilde{F}$ and $\tilde{G}$ are inf div chf's, then $\tilde{F}\tilde{G}$ is an inf div chf.

THEOREM 1.4.3 (Closure theorem). A chf which is the limit of a sequence of inf div chf's, is inf div.

THEOREM 1.4.4. A nonvanishing chf $\tilde{F}$ is inf div iff $\tilde{F}^\gamma$ is a chf for all $\gamma > 0$ (or for all $\gamma = 1/n$, $n \in \mathbb{N}$, or for all $\gamma = 2^{-n}$, $n \in \mathbb{N}$).

THEOREM 1.4.5. An inf div chf has no real zeros.

THEOREM 1.4.6. If the rv X is nondegenerate and bounded, then X is not inf div.

THEOREM 1.4.7 (Lévy canonical representation). A function φ on $\mathbb{R}$ is an inf div chf iff φ has the form

$$\varphi(t) = \exp[ita - \tfrac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \{e^{itx} - 1 - \frac{itx}{1+x^2}\} dM(x)] \qquad (t \in \mathbb{R}) , \tag{1.4.1}$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and M is a right-continuous function on $\mathbb{R}\setminus\{0\}$ with the following properties: M is nondecreasing on $(-\infty,0)$ and on $(0,\infty)$, $M(-\infty) =$ $= M(\infty) = 0$, and

$$\int_{(-1,1)\setminus\{0\}} x^2 dM(x) < \infty . \tag{1.4.2}$$

REMARK 1.4.8. If a chf $\tilde{F}$ has a representation of the form (1.4.1), where M violates the monotonicity condition of the theorem, then F is not inf div.

The canonical representation (1.4.1) can be somewhat modified to obtain other well known representations. For instance, in the Lévy-Khintchine representation an inf div chf $\tilde{F}$ has the form

$$\tilde{F}(t) = \exp[ita + \int_{(-\infty,\infty)} \{e^{itx} - 1 - \frac{itx}{1+x^2}\} \frac{1+x^2}{x^2} d\Theta(x)] \qquad (t \in \mathbb{R}) , \tag{1.4.3}$$

where $a \in \mathbb{R}$ and Θ is a right-continuous, nondecreasing and bounded function on $\mathbb{R}$ with $\Theta(-\infty) = 0$ (for $x = 0$ the integrand is defined by continuity: $-\tfrac{1}{2}t^2$). The canonical representations (1.4.1) and (1.4.3) are generalizations of the following representation, due to Kolmogorov, which is valid only for chf's of inf div distributions with finite second moment:

$$\tilde{F}(t) = \exp[ita + \int_{(-\infty,\infty)} \{e^{itx} - 1 - itx\} \frac{1}{x^2} dK(x)] \qquad (t \in \mathbb{R}) , \tag{1.4.4}$$

where $a \in \mathbb{R}$ and K is a right-continuous, nondecreasing and bounded function on $\mathbb{R}$ with $K(-\infty) = 0$. We prefer the Lévy canonical representation, as it has the clearest relations with the canonical representations known for inf div distributions on $[0,\infty)$ and on $\mathbb{N}_0$, which are special cases; this will be clarified in section 7.

Simple examples of inf div distributions are provided by the degenerate, Poisson, negative-binomial (and hence geometric), gamma (and hence exponential), normal and Cauchy distributions; their inf div is easily verified from their chf's. Considerably harder to prove is the inf div of the log-normal and the Student distributions; this has recently been done by Thorin (1977) and Grosswald (1976), respectively.
There are several methods to construct new inf div distributions from given ones; the best known are convolution, compounding and mixing. As an example of the method of mixing we state the following theorem of Feller (1971), ch. XVII (see also Steutel (1970)), and we note that in section 6 mixtures of exponential distributions are considered.

THEOREM 1.4.9. If G and H are inf div df's on $[0,\infty)$ and $\mathbb{R}$, respectively, then

$$\hat{G}(-\log \tilde{H}(t)) = \int_{[0,\infty)} \tilde{H}(t)^x dG(x) \qquad (t \in \mathbb{R}) \tag{1.4.5}$$

is an inf div chf.

COROLLARY 1.4.10. If G is an inf div df on $[0,\infty)$, then the following mixture of normal chf's is inf div:

$$\int_{[0,\infty)} \exp[-t^2 x] dG(x) \qquad (t \in \mathbb{R}) . \tag{1.4.6}$$

Finally, we pay some attention to compound distributions. Here we use the terminology of Feller; such distributions are also called generalized distributions by some authors (cf. Gurland (1957) and Johnson & Kotz (1969)).

DEFINITION 1.4.11. A probability distribution is called a *compound* distribution if its chf $\tilde{F}$ can be written in the form

$$\tilde{F}(t) = P(\tilde{G}(t)) \qquad (t \in \mathbb{R}) , \tag{1.4.7}$$

where P is a pgf and G is a df.

A rv X with chf $\tilde{F}$ given by (1.4.7) can be represented as

$$X \overset{d}{=} Y_1 + Y_2 + \ldots + Y_N ,$$

where $N, Y_1, Y_2, \ldots$ are independent, N has a lattice distribution with pgf P and $Y_1, Y_2, \ldots$ are identically distributed with df G.

EXAMPLE 1.4.12.

(i) A *compound Poisson* chf $\tilde{F}$ is a chf of the form

$$(1.4.8) \qquad \tilde{F}(t) = \exp[\mu(\tilde{G}(t) - 1)] \qquad (t \in \mathbb{R}) ,$$

where $\mu > 0$ and G is a df.

(ii) A *compound geometric* chf $\tilde{F}$ is a chf of the form

$$(1.4.9) \qquad \tilde{F}(t) = \frac{1 - p}{1 - p\tilde{G}(t)} \qquad (t \in \mathbb{R}) ,$$

where $0 < p < 1$ and G is a df.

REMARK 1.4.13. For a chf $\tilde{F} \not\equiv 1$ of the form (1.4.8) or (1.4.9) it is possible to choose the df G in such a way that G is continuous at zero. *We shall always do so*; the representations (1.4.8) and (1.4.9) are then unique, and we will refer to them as compound-Poisson-(μ,G) and compound-geometric-(p,G) distributions, respectively.

The compound Poisson and the compound geometric (more general: compound negative-binomial) distributions are known to be inf div (cf. Lukacs (1970), ch. 5). In fact, this is a consequence of the following property of compound distributions.

LEMMA 1.4.14. If P is an inf div pgf with $P(0) > 0$, then for all df's G the compound chf $\tilde{F}(t) = P(\tilde{G}(t))$ is compound Poisson and hence inf div.

PROOF. As we shall see in theorem 1.5.1, if P is an inf div pgf with $P(0) > 0$, then P is compound Poisson, so

$$P(z) = \exp[\mu(Q(z) - 1)] \qquad (|z| \le 1) ,$$

with $\mu > 0$ and Q is a pgf with $Q(0) = 0$. It follows that

$$\tilde{F}(t) = P(\tilde{G}(t)) = \exp[\mu(Q(\tilde{G}(t)) - 1)] ,$$

i.e. F is compound-Poisson-(μ,H), with $\tilde{H}(t) := Q(\tilde{G}(t))$. □

In sections 5 and 6 compound distributions on $\mathbb{N}_0$ and on $[0,\infty)$ will be considered in more detail. We conclude this section with De Finetti's observation, that every inf div distribution can be obtained as the weak limit of compound Poisson distributions (cf. Lukacs (1970), ch. 5).

THEOREM 1.4.15. A chf $\tilde{F}$ is inf div iff $\tilde{F}$ has the form

$$(1.4.10) \qquad \tilde{F}(t) = \lim_{n\to\infty} \exp[\mu_n(\tilde{G}_n(t) - 1)] \qquad (t \in \mathbb{R}) \ ,$$

where $\mu_n > 0$ and G_n is a df $(n \in \mathbb{N})$. In this case we may take $\mu_n = n$ and $G_n = F^{*1/n}$ $(n \in \mathbb{N})$.

1.5. *Infinitely divisible lattice distributions*

Let $\{p_n\}_o^\infty$ be a lattice distribution, i.e. a probability distribution on $\mathbb{N}_o$. When investigating the inf div of $\{p_n\}$, we shall always require that $0 < p_o < 1$; the condition "$p_o > 0$" ensures that, in case of inf div of $\{p_n\}$, the distribution $\{p_n^{*1/k}\}_{n=0}^\infty$ (with pgf $P(z)^{1/k}$) is again a distribution on $\mathbb{N}_o$. It is not an essential restriction: for all $\gamma \in \mathbb{R}$, $P(e^{it})$ is an inf div chf iff $e^{it\gamma}P(e^{it})$ is an inf div chf. Further we note that log $P(z)$ and $P(z)^\gamma$ $(\gamma \in \mathbb{R})$ are always uniquely defined in a neighbourhood of zero if $p_o > 0$.

For an inf div pgf P we have the following representation theorem (cf. Feller (1968), ch. XII).

THEOREM 1.5.1. A pgf P, with $0 < P(0) < 1$, is inf div iff P is compound Poisson, i.e. iff P has the form

$$(1.5.1) \qquad P(z) = \exp[\mu(Q(z) - 1)] \qquad (|z| \le 1) \ ,$$

where $\mu > 0$ and Q is a pgf with $Q(0) = 0$. The representation (μ,Q) is unique.

COROLLARY 1.5.2. An inf div pgf P with $P(0) > 0$ has no zeros in the closed unit disk.

Feller (1968) reformulates theorem 1.5.1 to obtain a criterion for inf div. We shall now do so in a slightly different way, using the concept of absolute monotonicity (cf. definition 1.3.1). Additionally we obtain a slightly different representation for inf div pgf's, which is sometimes more convenient.

THEOREM 1.5.3. A pgf P, with $0 \le P(0) < 1$, is inf div iff the function R_1, defined by

(1.5.2) $\quad R_1(z) := P'(z)/P(z)$,

is abs mon, or, equivalently, iff there exist nonnegative quantities $r_n(1)$ $(n \in \mathbb{N}_0)$ satisfying

$$(1.5.3) \qquad \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty ,$$

such that P has the form

$$(1.5.4) \qquad P(z) = \exp\left[\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1}(z^{n+1} - 1)\right] \qquad (|z| \le 1) .$$

PROOF. If P is inf div with $0 < P(0) < 1$, then P has the form (1.5.1) and hence $R_1(z) = \mu Q'(z)$ is abs mon.

Next, let R_1 be abs mon. Then there exist $\rho > 0$ and $r_n(1) \ge 0$ $(n \in \mathbb{N}_0)$ such that

$$(1.5.5) \qquad P'(z)/P(z) = \sum_{n=0}^{\infty} r_n(1) z^n \qquad (|z| < \rho) .$$

Integrating this equation from 0 to z $(|z| < \rho)$, we obtain

$$(1.5.6) \qquad \log\{P(z)/P(0)\} = \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} z^{n+1} \qquad (|z| < \rho) .$$

From (1.5.5) we get the following relations:

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(1) \qquad (n \in \mathbb{N}_0) ,$$

from which by the nonnegativity of the $r_n(1)$'s it can be shown (cf. lemma 1.5.6) that (1.5.3) holds. Hence the power series in (1.5.6) is convergent for $|z| \le 1$, and by analytic continuation it follows that the equality in (1.5.6) holds for $|z| \le 1$. Taking $z = 1$, one sees that

$$(1.5.7) \qquad -\log P(0) = \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} ;$$

hence P takes the form (1.5.4).

Finally, if P has the form (1.5.4) with nonnegative $r_n(1)$'s satisfying (1.5.3), then defining

(1.5.8) $$\mu := \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} \quad \text{and} \quad Q(z) := \frac{1}{\mu} \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} z^{n+1} \qquad (|z| \le 1) ,$$

we see that P takes the form (1.5.1) and hence is inf div. □

REMARK 1.5.4. To some quantities we add an index 1 or 0 in order to fit them in the more general notation of the next chapter. For instance: R_1, $r_n(1)$, and, presently, R_0 and $r_n(0)$.

The sequence $\{r_n(1)\}$ from the preceding theorem is uniquely determined by P, its gf R_1 satisfies (1.5.2). Therefore $\{r_n(1)\}$ is called the *canonical sequence* of the inf div pgf P; its relation with the Lévy canonical representation (a,σ^2,M) for $P(e^{it})$ will be shown in section 7.
From theorem 1.5.3 one easily verifies the following theorem, due to Katti (1967), which gives a characterization of the inf div lattice distributions in terms of the p_n's themselves.

COROLLARY 1.5.5. A lattice distribution $\{p_n\}_0^\infty$ with $0 < p_0 < 1$ is inf div iff there exist nonnegative quantities $r_n(1)$ $(n \in \mathbb{N}_0)$ such that

(1.5.9) $$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(1) \qquad (n \in \mathbb{N}_0) .$$

It is useful to consider the recurrence relations (1.5.9) in some more detail.

LEMMA 1.5.6.

(i) If $\{p_n\}_0^\infty$ is a lattice distribution with $p_0 > 0$, then there exists a unique sequence $\{r_n(1)\}_0^\infty$ satisfying (1.5.9); its gf R_1 has a positive radius of convergence, while for $|z|$ sufficiently small

(1.5.10) $$R_1(z) = P'(z)/P(z) .$$

If, in addition, all $r_n(1)$'s are nonnegative, then necessarily

(1.5.11) $$\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty .$$

(ii) If $\{r_n(1)\}_0^\infty$ is a sequence of nonnegative numbers satisfying (1.5.11), then there exists a unique lattice distribution $\{p_n\}_0^\infty$, with $p_0 > 0$, satisfying (1.5.9).

PROOF.

(i) Evidently, the first $n+1$ equations in (1.5.9) determine $r_0(1), r_1(1), \ldots$ $\ldots, r_n(1)$. As $p_o > 0$, the function R_1, defined by (1.5.10), is analytic in a neighbourhood of zero, and therefore has a power-series expansion with coefficients $r_0, r_1, \ldots$, say. But from (1.5.10) it follows that the r_n's satisfy (1.5.9), so $r_n = r_n(1)$ $(n \in \mathbb{N}_o)$, and R_1 is the gf of the sequence $\{r_n(1)\}$.

If, in addition, all $r_n(1)$'s are nonnegative, then we can write

$$1 - p_o = \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} p_k r_{n-k}(1) =$$

$$= \sum_{k=0}^{\infty} p_k \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1+k} \geq p_o \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} ,$$

and hence

$$\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} \leq \frac{1-p_o}{p_o} < \infty .$$

(ii) Clearly, there exists at most one probability distribution $\{p_n\}_o^\infty$ with $p_o > 0$, satisfying (1.5.9) for given $r_n(1)$. Now, if $r_n(1) \geq 0$ $(n \in \mathbb{N}_o)$ and if (1.5.11) holds, then it is seen that the function P defined by (1.5.4) is abs mon with $P(1) = 1$, i.e. P is a pgf. It follows that if R_1 is the gf of $\{r_n(1)\}$, then P satisfies (1.5.10), i.e. the coefficients p_n of P satisfy (1.5.9). □

The following result about zeros of an inf div $\{p_n\}_o^\infty$ can be derived from corollary 1.5.5 (cf. Steutel (1970)).

THEOREM 1.5.7. If $\{p_n\}$ is an inf div lattice distribution with $0 < p_o < 1$, then for all $n \in \mathbb{N}_o$ and all $k \in \mathbb{N}_o$ the following implication holds:

$$[p_n > 0 \text{ and } p_k > 0] \Rightarrow p_{n+k} > 0 .$$

Consequently, if $p_1 > 0$ then $p_n > 0$ for all $n \in \mathbb{N}_o$.

Next we turn to the compound geometric lattice distributions, i.e. (cf. example 1.4.12(ii)) distributions with pgf P of the form

$$(1.5.12)\qquad P(z) = \frac{1-p}{1-pQ(z)} \qquad (|z| \leq 1) ,$$

where $0 < p < 1$ and Q is a pgf with $Q(0) = 0$. These distributions are inf div (cf. lemma 1.4.14) and have properties similar to those of the compound Poisson lattice distributions.

THEOREM 1.5.8. A pgf P with $0 < P(0) < 1$ is compound geometric iff the function R_o, defined by

$$(1.5.13) \quad R_o(z) := \frac{1}{z}\{1 - P(0)/P(z)\} ,$$

is abs mon.

PROOF. The necessity of the condition immediately follows from (1.5.12). So, let R_o be abs mon, i.e. there exist $\rho > 0$ and $r_n(0) \geq 0$ ($n \in \mathbb{N}_o$) such that R_o has a power-series representation for $|z| < \rho$ with coefficients $r_n(0)$. Then from (1.5.13) it follows that

$$(1.5.14) \quad P(z)/P(0) = \{1 - \sum_{n=0}^{\infty} r_n(0) z^{n+1}\}^{-1} \qquad (|z| < \rho) ,$$

and that

$$p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_o) .$$

From these relations it can be shown (cf. lemma 1.5.10) that $\sum_{n=0}^{\infty} r_n(0) < 1$, and hence the right-hand side of (1.5.14) is an analytic function on $|z| \leq 1$. It follows that the equality in (1.5.14) holds for $|z| \leq 1$. Taking $z = 1$ we see that

$$(1.5.15) \quad P(0) = 1 - \sum_{n=0}^{\infty} r_n(0) ,$$

and hence P takes the form (1.5.12) if we define

$$(1.5.16) \quad p := \sum_{n=0}^{\infty} r_n(0) \quad \text{and} \quad Q(z) := \frac{1}{p} \sum_{n=0}^{\infty} r_n(0) z^{n+1} \qquad (|z| \leq 1) . \qquad \square$$

From theorem 1.5.8 one obtains the following analogue of corollary 1.5.5 (cf. Steutel (1970)).

COROLLARY 1.5.9. A lattice distribution $\{p_n\}_o^\infty$ with $0 < p_o < 1$ is compound geometric iff there exist nonnegative quantities $r_n(0)$ ($n \in \mathbb{N}_o$) such that

$$(1.5.17) \quad p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_o) .$$

The following lemma is the analogue of lemma 1.5.6 for the recurrence relations (1.5.17).

LEMMA 1.5.10.

(i) If $\{p_n\}_0^\infty$ is a lattice distribution with $p_0 > 0$, then there exists a unique sequence $\{r_n(0)\}_0^\infty$ satisfying (1.5.17); its gf R_0 has a positive radius of convergence, while for $|z|$ sufficiently small

(1.5.18) $$R_0(z) = \frac{1}{z}\{1 - P(0)/P(z)\} .$$

If, in addition, all $r_n(0)$'s are nonnegative, then necessarily

(1.5.19) $$\sum_{n=0}^{\infty} r_n(0) < 1 .$$

(ii) If $\{r_n(0)\}_0^\infty$ is a sequence of nonnegative numbers satisfying (1.5.19), then there exists a unique lattice distribution $\{p_n\}_0^\infty$, with $p_0 > 0$, satisfying (1.5.17).

PROOF.

(i) The proof of the first part is similar to that of lemma 1.5.6. If $r_n(0) \geq 0$ for all $n \in \mathbb{N}_0$, then we can write

$$1 - p_0 = \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} p_k r_{n-k}(0) = \sum_{n=0}^{\infty} r_n(0) ,$$

from which (1.5.19) follows.

(ii) If $r_n(0) \geq 0$ for all n and if (1.5.19) holds, then it is seen that the function P, defined by (1.5.14) with $P(0)$ given by (1.5.15), is abs mon with $P(1) = 1$. It follows that P is the pgf of a lattice distribution $\{p_n\}_0^\infty$ that satisfies (1.5.17). The uniqueness of $\{p_n\}$ is evident from (1.5.17). □

To conclude this section we mention two more classes of inf div lattice distributions: the classes of comp mon and log-convex lattice distributions. Comp mon has been introduced in definition 1.3.10; from theorem 1.3.11 one easily deduces the following lemma.

LEMMA 1.5.11. A lattice distribution $\{p_n\}_0^\infty$ is comp mon iff $\{p_n\}$ is a mixture of geometric distributions, i.e. iff there exists a df G on $[0,1)$ such that

$$(1.5.20) \quad p_n = \int_{[0,1)} (1-p)p^n dG(p) \qquad (n \in \mathbb{N}_0) \ .$$

Log-convexity can be introduced as follows.

DEFINITION 1.5.12. A lattice distribution $\{p_n\}_0^\infty$ is said to be *log-convex* if

$$(1.5.21) \quad p_n^2 \le p_{n-1}p_{n+1} \qquad (n \in \mathbb{N}) \ .$$

Let us define the following four classes of lattice distributions $\{p_n\}_0^\infty$ with $p_0 > 0$:

- $\{p_n\} \in \mathcal{A}$ if $\{p_n\}$ is comp mon,
- $\{p_n\} \in \mathcal{B}$ if $\{p_n\}$ is log-convex,
- $\{p_n\} \in \mathcal{C}_0$ if $\{p_n\}$ is compound geometric,
- $\{p_n\} \in \mathcal{C}_1$ if $\{p_n\}$ is compound Poisson, i.e. if $\{p_n\}$ is inf div.

Then the family $(\mathcal{A},\mathcal{B},\mathcal{C}_0,\mathcal{C}_1)$ defines a classification (cf. the end of section 2) of $\mathcal{C}_1$, as will be apparent from the following relations (cf. Kaluza (1928), Goldie (1967), Steutel (1970) and Warde & Katti (1971)).

THEOREM 1.5.13. $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}_0 \subset \mathcal{C}_1$, where all inclusions are strict.

REMARK 1.5.14. "$\mathcal{C}_0 \subset \mathcal{C}_1$" also easily follows from theorems 1.5.3 and 1.5.8 by use of the following relation between R_0 and R_1:

$$(1.5.22) \quad R_1(z) = \{1 - zR_0(z)\}^{-1} \frac{d}{dz}[zR_0(z)] \ .$$

1.6. *Infinitely divisible distributions on* $[0,\infty)$

The inf div distributions on $[0,\infty)$ can be characterized in the following way (cf. Feller (1971), ch. XIII).

THEOREM 1.6.1. A positive function φ on $[0,\infty)$ is the PLST $\hat{F}$ of an inf div df F on $[0,\infty)$ iff $\varphi(0) = 1$ and the function φ_0, defined by

$$(1.6.1) \quad \varphi_0(\tau) := -\frac{d}{d\tau}\log\varphi(\tau) \qquad (\tau > 0) \ ,$$

is comp mon, or, equivalently, iff there exists a right-continuous, nondecreasing function K_0 on $\mathbb{R}$, satisfying

$$(1.6.2) \qquad \int_{(1,\infty)} \frac{1}{x}\, dK_o(x) < \infty ,$$

such that φ has the form

$$(1.6.3) \qquad \varphi(\tau) = \exp\Big[\int_{[0,\infty)} (e^{-\tau x} - 1)\frac{1}{x}\, dK_o(x) \Big] \qquad (\tau \geq 0) .$$

We can (and will) choose the function K_o such that K_o vanishes on $(-\infty,0)$. As we then also have $\hat{K}_o = \varphi_o$, with φ_o given by (1.6.1), the function K_o is uniquely determined by $\varphi = \hat{F}$; K_o is called the *canonical function* of the inf div df F. Its relation with the Lévy canonical representation (a,σ^2,M) will be shown in the next section.

Before giving some properties of K_o, we state a characterization of the inf div df's on $[0,\infty)$ in terms of the df's themselves, which has been used by Steutel (1970), and can be obtained by inverting the expression for $\varphi' = \hat{F}'$ in (1.6.3).

THEOREM 1.6.2. A df F on $[0,\infty)$ is inf div iff there exists a right-continuous, nondecreasing function K_o such that

$$(1.6.4) \qquad \int_{[0,x]} y\, dF(y) = \int_{[0,x]} F(x-y)\, dK_o(y) \qquad (x \geq 0) .$$

COROLLARY 1.6.3. A pdf f on $(0,\infty)$ is inf div iff there exists a right-continuous, nondecreasing function K_o such that

$$(1.6.5) \qquad xf(x) = \int_{[0,x]} f(x-y)\, dK_o(y) \qquad (\text{almost all } x > 0) .$$

Now we can prove the following properties of the canonical function K_o.

LEMMA 1.6.4. Let F be an inf div df with $\ell(F) \geq 0$ and canonical function K_o. Then

(i) $K_o(0) = \ell(F)$;

(ii) $\int_{(0,\infty)} \frac{1}{x}\, dK_o(x) < \infty$ iff $F(\ell(F)) > 0$, in which case the following relation holds:

$$(1.6.6) \qquad \int_{(0,\infty)} \frac{1}{x}\, dK_o(x) = -\log F(\ell(F)) ;$$

(iii) K_o is bounded iff $\mu_1 := \int_{(0,\infty)} x\, dF(x) < \infty$, in which case

$$(1.6.7) \qquad \int_{[0,\infty)} dK_o(x) = \mu_1 .$$

PROOF. Define the df G by $G(x) := F(x + \ell(F))$ $(x \in \mathbb{R})$, then G is again inf div, with $\ell(G) = 0$ and canonical function L_o, say. According to theorem 1.6.2, we have for all $x > 0$

$$G(x)L_o(0) \le \int_{[0,x]} G(x-y)\,dL_o(y) = \int_{[0,x]} y\,dG(y) \le xG(x) ,$$

and hence, as $\ell(G) = 0$, $L_o(0) = 0$. Using the representation (1.6.3) for $\hat{G}$, we can write

$$\hat{F}(\tau) = e^{-\ell(F)\tau}\hat{G}(\tau) = \exp[-\ell(F)\tau + \int_{(0,\infty)} (e^{-\tau x} - 1)\frac{1}{x}\, dL_o(x)] \qquad (\tau \ge 0) .$$

But as $\hat{F}$ can also be represented by (1.6.3), the uniqueness of the canonical function implies $K_o(0) = \ell(F)$. In view of (1.6.3) we can now write

$$\log\{\hat{F}(\tau)e^{\ell(F)\tau}\} = \int_{(0,\infty)} (e^{-\tau x} - 1)\frac{1}{x}\, dK_o(x) \qquad (\tau \ge 0) ,$$

from which, letting $\tau \to \infty$ and using (1.3.4) and the dominated convergence theorem, we obtain part (ii) of the lemma. Finally, using (1.3.6) and the fact that $\hat{K}_o = \varphi_o$ with φ_o given by (1.6.1), we obtain part (iii) as follows:

$$\int_{[0,\infty)} dK_o(x) = \lim_{\tau \downarrow 0} \hat{K}_o(\tau) = \lim_{\tau \downarrow 0} -\hat{F}'(\tau)/\hat{F}(\tau) = \mu_1 \ (\le \infty) . \qquad \square$$

Part (iii) of the preceding lemma can be generalized to obtain necessary and sufficient conditions for the existence of higher moments of inf div distributions on $[0,\infty)$. This has already been done by Wolfe (1971b) for general inf div distributions, but in our case the proof is very simple and we obtain a relation with the class $\mathcal{C}_1$ of inf div lattice distributions.

THEOREM 1.6.5. Let F be an inf div df on $[0,\infty)$ with canonical function K_o. Then for all $n \in \mathbb{N}_o$

$$(1.6.8) \qquad \mu_{n+1} := \int_{[0,\infty)} x^{n+1}dF(x) < \infty \Leftrightarrow \nu_n := \int_{[0,\infty)} x^n dK_o(x) < \infty ,$$

in which case

$$(1.6.9)\qquad \mu_{n+1} = \sum_{k=0}^{n} \binom{n}{k} \mu_k \nu_{n-k} \ ;$$

if $\mu_n < \infty$ for all $n \in \mathbb{N}$, then $\{\mu_n/n!\}_0^\infty$ satisfies the recurrence relations (1.5.9) for C_1, with $r_n(1) = \nu_n/n!$ $(n \in \mathbb{N}_0)$.

PROOF. If U is a nonnegative, right-continuous, nondecreasing function with $\ell(U) \geq 0$, then obviously for all $n \in \mathbb{N}_0$

$$(1.6.10)\qquad \lim_{\tau \downarrow 0} (-1)^n \left(\frac{d}{d\tau}\right)^n \hat{U}(\tau) = \int_{[0,\infty)} x^n dU(x) \qquad (\leq \infty)\ .$$

Since the canonical function K_0 of an inf div df F on $[0,\infty)$ satisfies $-\hat{F}'(\tau) = \hat{F}(\tau)\hat{K}_0(\tau)$ (cf. theorem 1.6.1 or (1.6.4)), we can write

$$(1.6.11)\qquad (-1)^{n+1}\left(\frac{d}{d\tau}\right)^{n+1}\hat{F}(\tau) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} \left\{\left(\frac{d}{d\tau}\right)^k \hat{F}(\tau)\right\}\left\{\left(\frac{d}{d\tau}\right)^{n-k}\hat{K}_0(\tau)\right\}\ .$$

Now, using the fact that $\mu_k < \infty$ $(k = 0,1,\ldots,n)$ if $\mu_{n+1} < \infty$ and the same property of $\{\nu_n\}_0^\infty$, and letting $\tau \downarrow 0$ in (1.6.11) (cf. (1.6.10)), we see the assertions of the theorem to be true. □

If F is an inf div df on $[0,\infty)$ with $F(0) > 0$, then the representation (1.6.3) for $\hat{F}$ can be simplified as follows.

THEOREM 1.6.6. A df F on $[0,\infty)$ is inf div with $F(0) > 0$ iff F is compound Poisson, i.e. iff $\hat{F}$ has the form

$$(1.6.12)\qquad \hat{F}(\tau) = \exp[\mu(\hat{G}(\tau) - 1)] \qquad (\tau \geq 0)\ ,$$

where $\mu > 0$ and G is a df with $G(0) = 0$.

PROOF. Let F be an inf div df on $[0,\infty)$ with $F(0) > 0$. Then $\ell(F) = 0$, and if K_0 is the canonical function of F, it follows by lemma 1.6.4(ii) that

$$\mu := \int_{(0,\infty)} \frac{1}{x}\, dK_0(x) < \infty\ .$$

Now if we define

$$G(x) := \mu^{-1} \int_{(0,x]} \frac{1}{y}\, dK_0(y) \qquad (x \geq 0)\ ,$$

then G is a df with $G(0) = 0$, and it is easily seen that the representation (1.6.3) for $\hat{F}$ can be rewritten in the form (1.6.12).
Conversely, it is well known (and trivial) that a PLST of the form (1.6.12) is inf div with $F(0) > 0$. □

The compound geometric distributions on $[0,\infty)$, which are also compound Poisson (cf. lemma 1.4.14), can be characterized by a functional equation similar to (1.6.4) (cf. Steutel (1970)). We use a notation that will be clarified in chapter 5.

THEOREM 1.6.7. A df F on $[0,\infty)$ is compound geometric, i.e. has a PLST $\hat{F}$ of the form

$$(1.6.13)\qquad \hat{F}(\tau) = \frac{1-p}{1-p\hat{G}(\tau)} \qquad (\tau \geq 0)\ ,$$

where $0 < p < 1$ and G is a df with $G(0) = 0$, iff $F(0) > 0$ and there exists a right-continuous, nondecreasing function K_∞ such that

$$(1.6.14)\qquad F(x) - F(0) = \int_{[0,x]} F(x-y)\,dK_\infty(y) \qquad (x \geq 0)\ .$$

From (1.6.14) one easily proves the following result about moments of compound geometric df's on $[0,\infty)$ (cf. (the proof of) theorem 1.6.5).

THEOREM 1.6.8. Let F be a compound geometric df on $[0,\infty)$, and let K_∞ be the function in theorem 1.6.7. Then for all $n \in \mathbb{N}$

$$(1.6.15)\qquad \mu_n := \int_{[0,\infty)} x^n dF(x) < \infty \Leftrightarrow \nu_n := \int_{[0,\infty)} x^n dK_\infty(x) < \infty\ ,$$

in which case

$$(1.6.16)\qquad \mu_n = \sum_{k=0}^{n} \binom{n}{k} \mu_k \nu_{n-k}\ ;$$

if $\mu_n < \infty$ for all $n \in \mathbb{N}$, then $\{\mu_n/n!\}_0^\infty$ satisfies the recurrence relations (1.5.17) for C_o, with $r_n(0) = F(0)^{-1}\nu_{n+1}/(n+1)!$ $(n \in \mathbb{N}_o)$.

Finally, as in the discrete case (cf. section 5), we consider the following classes:

$\mathcal{D}$: the class of df's on $[0,\infty)$ with a comp mon density,

$\mathcal{E}$: the class of df's on $[0,\infty)$ with a log-convex density.

Here log-convexity is defined as follows.

DEFINITION 1.6.9. A positive pdf f on $(0,\infty)$ is said to be *log-convex* if log f is convex, i.e. if

$$(1.6.17)\quad f(\lambda x + (1-\lambda)y) \le f(x)^{\lambda} f(y)^{1-\lambda} \qquad (x > 0,\ y > 0,\ 0 < \lambda < 1)\ .$$

In view of Bernstein's theorem (theorem 1.3.7) one easily verifies that the following characterization of the class $\mathcal{D}$ holds.

LEMMA 1.6.10. A pdf f on $(0,\infty)$ is comp mon iff f is a mixture of exponential distributions, i.e. iff there exists a df G on $(0,\infty)$ such that

$$(1.6.18)\quad f(x) = \int_{(0,\infty)} \mu e^{-\mu x} dG(\mu) \qquad (x > 0)\ .$$

The df's in the classes $\mathcal{D}$ and $\mathcal{E}$ are inf div; this has been proved by Goldie (1967) and by Steutel (1970), respectively. In fact, denoting the class of all inf div df's on $[0,\infty)$ by $\mathcal{F}_0$, we have the following partial analogue of theorem 1.5.13.

THEOREM 1.6.11. $\mathcal{D} \subset \mathcal{E} \subset \mathcal{F}_0$.

In chapter 5 we shall introduce an analogue of the class $\mathcal{C}_0$ for distributions on $[0,\infty)$; this class, called $\mathcal{F}_\infty$, will fill the gap between $\mathcal{E}$ and $\mathcal{F}_0$, i.e. it will have the property that

$$\mathcal{E} \subset \mathcal{F}_\infty \subset \mathcal{F}_0\ .$$

It follows that the family $(\mathcal{D},\mathcal{E},\mathcal{F}_\infty,\mathcal{F}_0)$ defines a classification of $\mathcal{F}_0$, which can be considered as an analogue of the classification of $\mathcal{C}_1$, defined by $(\mathcal{A},\mathcal{B},\mathcal{C}_0,\mathcal{C}_1)$ (cf. section 5), for distributions on $[0,\infty)$.

1.7. *Properties of and relations between canonical representations*

In theorem 1.4.7 the chf $\widetilde{F}$ of an inf div df F on $\mathbb{R}$ has been characterized by the Lévy canonical representation (a,σ^2,M):

$$(1.7.1a)\quad \widetilde{F}(t) = \exp[ita - \tfrac{1}{2}\sigma^2 t^2 + \int_{\mathbb{R}\setminus\{0\}} \{e^{itx} - 1 - \frac{itx}{1+x^2}\} dM(x)] \qquad (t \in \mathbb{R})\ ,$$

where $a \in \mathbb{R}$, $\sigma^2 \geq 0$ and M is a right-continuous function on $\mathbb{R}\backslash\{0\}$ with the following properties: M is nondecreasing on $(-\infty,0)$ and on $(0,\infty)$, $M(-\infty) = M(\infty) = 0$ and

(1.7.1b) $$\int_{(-1,1)\backslash\{0\}} x^2 dM(x) < \infty .$$

The chf $\tilde{F}$ of an inf div df F on $[0,\infty)$ can be represented by means of its canonical function K_o (cf. theorem 1.6.1):

(1.7.2a) $$\tilde{F}(t) = \exp\Big[\int_{[0,\infty)} \{e^{itx} - 1\}\frac{1}{x}\, dK_o(x)\Big] \qquad (t \in \mathbb{R}) ,$$

where K_o is a right-continuous, nondecreasing function, vanishing on $(-\infty,0)$ and satisfying

(1.7.2b) $$\int_{(1,\infty)} \frac{1}{x}\, dK_o(x) < \infty .$$

Finally, the chf $\tilde{F}$ of an inf div df F on $\mathbb{N}_o$ with $F(0) > 0$ has the following form (cf. theorem 1.5.3):

(1.7.3a) $$\tilde{F}(t) = \exp\Big[\sum_{n=0}^{\infty} \{e^{it(n+1)} - 1\}\frac{1}{n+1}\, r_n(1)\Big] \qquad (t \in \mathbb{R}) ,$$

where $\{r_n(1)\}_o^\infty$ is a sequence of nonnegative numbers, satisfying

(1.7.3b) $$\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty .$$

In fact, the three classes of df's, considered in (1.7.1a), (1.7.2a) and (1.7.3a), respectively, define a classification of the class of all inf div df's on $\mathbb{R}$. Now we want to investigate under what conditions on the canonical quantities an inf div df belongs to one of the subclasses and what relations exist between the canonical representations. Not all of this is new, but it seems useful to collect the available information, together with a few additions, and, sometimes, simpler proofs.

First we modify (1.7.2a) in such a way that we get a representation for all inf div df's F with $\ell(F) > -\infty$.

THEOREM 1.7.1. A function φ on $\mathbb{R}$ is the chf $\tilde{F}$ of an inf div df F with $\ell(F) > -\infty$ iff φ has the form

$$(1.7.4a) \qquad \varphi(t) = \exp[it\gamma + \int_{(0,\infty)} \{e^{itx} - 1\}dN(x)] \qquad (t \in \mathbb{R}) \ ,$$

where $\gamma \in \mathbb{R}$ and N is a right-continuous, nondecreasing function on $(0,\infty)$ with $N(\infty) = 0$ and satisfying

$$(1.7.4b) \qquad \int_{(0,1]} x dN(x) < \infty \ .$$

The representation (γ,N) is unique, and necessarily $\ell(F) = \gamma$.

PROOF. Let F be an inf div df with $\ell(F) > -\infty$. The df F_1, defined by $F_1(x) := F(x + \ell(F))$ $(x \in \mathbb{R})$, is inf div with $\ell(F_1) = 0$, and hence $\tilde{F}_1$ has the form (1.7.2a) with $K_o(0) = \ell(F_1) = 0$ (cf. lemma 1.6.4(i)). It follows that

$$\tilde{F}(t) = \exp[it\ell(F) + \int_{(0,\infty)} \{e^{itx} - 1\}\frac{1}{x} dK_o(x)] \qquad (t \in \mathbb{R}) \ .$$

Because of (1.7.2b) we can define a function N on $(0,\infty)$ by

$$N(x) := -\int_{(x,\infty)} \frac{1}{y} dK_o(y) \qquad (x > 0) \ .$$

Then N is right-continuous and nondecreasing, and satisfies (1.7.4b):

$$\int_{(0,1]} x dN(x) = \int_{(0,1]} dK_o(x) = K_o(1) < \infty \ ,$$

while $\tilde{F}$ takes the form (1.7.4a) with $\gamma = \ell(F)$. The representation (γ,N) is unique as K_o is unique.
Conversely, if a function φ has the form (1.7.4a), then $\varphi_1(t) := e^{-it\gamma}\varphi(t)$ has the form (1.7.2a) with K_o defined by (cf. (1.7.4b))

$$K_o(x) := \int_{(0,x]} y dN(y) \qquad (x > 0) \ .$$

By theorem 1.6.1 and lemma 1.6.4(i) it follows that φ_1 is the chf of an inf div df F_1 with $\ell(F_1) = 0$, and hence $\varphi(t) = e^{it\gamma}\varphi_1(t)$ is the chf of an inf div df F with $\ell(F) = \gamma$. □

COROLLARY 1.7.2. If F is an inf div df with $\ell(F) \geq 0$, then the following relation holds between its canonical function K_o and its representation

(γ,N) from (1.7.4a):

$$(1.7.5)\qquad K_o(x) = \gamma + \int_{(0,x]} y\,dN(y) \qquad (x \geq 0)\ .$$

Using theorem 1.7.1 we can give necessary and sufficient conditions for an inf div df F to have a support that is bounded from below, i.e. $\ell(F) > -\infty$. This has also been done by Baxter & Shapiro (1960) (see also Feller (1971), Ch. XVII), but our method of making use of the representation (1.7.2a) for inf div df's on $[0,\infty)$, instead of using only (1.7.1a), simplifies matters. Also, the expression for $\ell(F)$ to be given in (1.7.7) follows much more directly than in Tucker (1961).

THEOREM 1.7.3. Let F be an inf div df with Lévy canonical representation (a,σ^2,M). Then $\ell(F) > -\infty$ iff $\sigma^2 = 0$, $M \equiv 0$ on $(-\infty,0)$ and

$$(1.7.6)\qquad \int_{(0,1]} x\,dM(x) < \infty\ ,$$

in which case necessarily

$$(1.7.7)\qquad \ell(F) = a - \int_{(0,\infty)} \frac{x}{1+x^2}\,dM(x)\ .$$

PROOF. Let $\ell(F) > -\infty$. Then by theorem 1.7.1 $\tilde{F}$ has the representation (γ,N) from (1.7.4a) with, because of (1.7.4b),

$$\int_{(0,\infty)} \frac{x}{1+x^2}\,dN(x) < \infty\ .$$

It follows that $\tilde{F}$ can be written in the form

$$\tilde{F}(t) = \exp[it\{\gamma + \int_{(0,\infty)} \frac{x}{1+x^2}\,dN(x)\} + \int_{(0,\infty)} \{e^{itx} - 1 - \frac{itx}{1+x^2}\}dN(x)],$$

from which by the uniqueness of the representation (a,σ^2,M) it is seen that $\sigma^2 = 0$, $M \equiv 0$ on $(-\infty,0)$ and

$$(1.7.8)\qquad M \equiv N \text{ on } (0,\infty),\ a = \gamma + \int_{(0,\infty)} \frac{x}{1+x^2}\,dN(x)\ .$$

Conversely, if (a,σ^2,M) satisfies the conditions of the theorem, then the integral in (1.7.7) is finite. Hence the representation (1.7.1a) for $\tilde{F}$ can be written in the form (1.7.4a) with γ given by the right-hand side of (1.7.7) and $N \equiv M$ on $(0,\infty)$. By theorem 1.7.1 it now follows that $\ell(F) > -\infty$, and as $\ell(F) = \gamma$, we have (1.7.7). □

COROLLARY 1.7.4. If F is an inf div df with $\ell(F) \geq 0$, then the following relations hold between its canonical function K_o and the Lévy representation (a,σ^2,M):

$$(1.7.9)\qquad K_o(x) = \{a - \int_{(0,\infty)} \frac{x}{1+x^2}\,dM(x)\} + \int_{(0,x]} y\,dM(y) \qquad (x \geq 0)\ ,$$

and, conversely,

$$(1.7.10)\qquad a = \int_{[0,\infty)} \frac{1}{1+x^2}\,dK_o(x)\ ,\quad M(x) = -\int_{(x,\infty)} \frac{1}{y}\,dK_o(y) \qquad (x > 0)\ .$$

PROOF. Use the relations (1.7.5) and (1.7.8). □

Using the same technique as in the proofs of lemma 1.6.4(ii) and theorem 1.6.6, we obtain a generalization of these results to all inf div df's F with $\ell(F) > -\infty$.

THEOREM 1.7.5. Let F be an inf div df with Lévy representation (a,σ^2,M) and with $\ell(F) > -\infty$. Then $F(\ell(F)) > 0$ iff M is bounded, in which case

$$(1.7.11)\qquad -\log F(\ell(F)) = M(0+)\ ,$$

and F is a shifted compound Poisson df on $[0,\infty)$, i.e. $\tilde{F}$ has the form

$$(1.7.12)\qquad \tilde{F}(t) = e^{it\ell(F)}\exp[\mu(\tilde{G}(t) - 1)] \qquad (t \in \mathbb{R})\ ,$$

where $\mu > 0$ and G is a df with $G(0) = 0$.

COROLLARY 1.7.6. If F is an inf div df with $\ell(F) > -\infty$, then F is continuous at $\ell(F)$ iff F is continuous everywhere.

PROOF. If F is continuous at $\ell(F)$, then by the preceding theorem M is unbounded. Now, by a result of Blum & Rosenblatt (1959) (our theorem 1.7.9) it follows that F is continuous everywhere. □

We now consider the class of inf div lattice distributions $\{p_n\}_0^\infty$ with $p_0 > 0$ as a subclass of the class of inf div distributions on $[0,\infty)$. Comparing the canonical representations (1.7.2a) and (1.7.3a), and the conditions (1.7.2b) and (1.7.3b), we are immediately led to the following result.

THEOREM 1.7.7. Let F be an inf div df on $[0,\infty)$ with canonical function K_0. Then F is the df of a lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$ iff K_0 is a step function with discontinuities restricted to $\mathbb{N}$; in this case the following relations exist between K_0 and the canonical sequence $\{r_n(1)\}_0^\infty$ of $\{p_n\}$:

$$(1.7.13)\qquad K_0(x) = \sum_{n=1}^{\infty} r_{n-1}(1)\,1_{[n,\infty)}(x) \qquad (x \in \mathbb{R}) ,$$

and, conversely,

$$(1.7.14)\qquad r_n(1) = K_0(n+1) - K_0(n) \qquad (n \in \mathbb{N}_0) .$$

COROLLARY 1.7.8. Let F be an inf div df with Lévy representation (a,σ^2,M). Then F is the df of a lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$ iff $\sigma^2 = 0$, M is a step function with discontinuities restricted to $\mathbb{N}$ and

$$(1.7.15)\qquad a = \int_{(0,\infty)} \frac{x}{1+x^2}\, dM(x) \ ;$$

in this case the following relations exist between M and the canonical sequence $\{r_n(1)\}_0^\infty$ of $\{p_n\}$:

$$(1.7.16)\qquad M(x) = -\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1}\, 1_{(0,n+1)}(x) \qquad (x > 0) ,$$

and, conversely,

$$(1.7.17)\qquad r_n(1) = (n+1)\{M(n+1) - M(n)\} \qquad (n \in \mathbb{N}_0) .$$

The first part of this corollary is a special case of part (i) of the following result, due to Blum & Rosenblatt (1959).

THEOREM 1.7.9. Let F be an inf div df with Lévy representation (a,σ^2,M). Then

(i) F is discrete iff $\sigma^2 = 0$, M is bounded and M is a step function,

(ii) F is continuous iff $\sigma^2 > 0$ or M is unbounded.

Also, for an inf div df to be absolutely continuous, necessary and sufficient conditions are known (cf. Tucker (1964)). We confine ourselves to the following simple sufficient conditions (cf. Tucker (1962) and Fisz & Varadarajan (1963)), and return to this in chapter 5.

THEOREM 1.7.10. Let F be an inf div df with Lévy representation (a,σ^2,M). If $\sigma^2 > 0$ or if there exists $\varepsilon > 0$ such that M is unbounded and absolutely continuous on $(-\varepsilon,0)$ or on $(0,\varepsilon)$, then F is absolutely continuous.

Finally, we consider two more classes of inf div df's for which the representation (1.7.1a) has a simpler form. Let F be an inf div df without a normal component, i.e. with $\sigma^2 = 0$ in its Lévy representation (a,σ^2,M). The function M satisfies (cf. (1.7.1b))

$$(1.7.18) \qquad \int_{(-1,1)\setminus\{0\}} x^2 dM(x) < \infty ,$$

and now we want to consider the special cases where

$$(1.7.19) \qquad \int_{(-1,1)\setminus\{0\}} |x| dM(x) < \infty ,$$

or

$$(1.7.20) \qquad \int_{(-1,1)\setminus\{0\}} dM(x) < \infty \text{ (i.e. M is bounded)} .$$

If $M \equiv 0$ on $(-\infty,0)$, then, according to theorems 1.7.3 and 1.7.5, (1.7.19) and (1.7.20) give rise to the inf div df's F with $\ell(F) > -\infty$ and to the shifted compound Poisson distributions on $[0,\infty)$, respectively. In the general case we have analogous results, the first of which we shall need in the last section of chapter 5.

THEOREM 1.7.11. Let F be an inf div df with Lévy representation (a,σ^2,M). Then there exist inf div df's F_1 and F_2 on $[0,\infty)$ such that

$$(1.7.21) \qquad \tilde{F}(t) = \tilde{F}_1(t)\tilde{F}_2(-t) \qquad (t \in \mathbb{R})$$

iff $\sigma^2 = 0$ and (1.7.19) holds.

PROOF. Let F satisfy (1.7.21) with F_1 and F_2 inf div df's on $[0,\infty)$. Then by theorem 1.7.1 for $j = 1,2$ $\tilde{F}_j$ has the representation (γ_j,N_j) given by

(1.7.4a), and hence

$$\tilde{F}(t) = \exp[it(\gamma_1 - \gamma_2) + \int_{(0,\infty)} (e^{itx} - 1)dN_1(x) + \int_{(0,\infty)} (e^{-itx} - 1)dN_2(x)]. \tag{1.7.22}$$

Now by (1.7.1a) it follows that the Lévy representation (a,σ^2,M) of F satisfies

$$M(x) = \begin{cases} N_1(x) & , \text{ if } x > 0 \\ -N_2(-x-), & \text{if } x < 0 \end{cases}, \quad \sigma^2 = 0, \quad a = \gamma_1 - \gamma_2 + \int_{\mathbb{R}\setminus\{0\}} \frac{x}{1+x^2}\, dM(x) , \tag{1.7.23}$$

so that $\sigma^2 = 0$ and M satisfies (1.7.19), because of (1.7.4b).
The converse can be shown in a similar way; we choose the df's F_1 and F_2 in (1.7.21) by giving their representations (γ_1,N_1) and (γ_2,N_2), where we note that N_1 and N_2 are completely determined by M, but that the nonnegative quantities γ_1 and γ_2 only need to satisfy the equality in (1.7.23). □

COROLLARY 1.7.12. Let F be an inf div df with Lévy representation (a,σ^2,M). Then there exists an inf div df F_1 on $[0,\infty)$ such that

$$\tilde{F}(t) = \tilde{F}_1(t)\tilde{F}_1(-t) \qquad (t \in \mathbb{R}) \tag{1.7.24}$$

iff $a = \sigma^2 = 0$, $M(x) = -M(-x-)$ $(x > 0)$ and

$$\int_{(0,1]} x dM(x) < \infty ,$$

in which case $\tilde{F}$ can be given the following form:

$$\tilde{F}(t) = \exp[2 \int_{(0,\infty)} (\cos tx - 1)dM(x)] \qquad (t \in \mathbb{R}) . \tag{1.7.25}$$

THEOREM 1.7.13. Let F be an inf div df with Lévy representation (a,σ^2,M). Then F is a shifted compound Poisson df, i.e. $\tilde{F}$ has the form

$$\tilde{F}(t) = e^{it\gamma}\exp[\mu(\tilde{G}(t) - 1)] \qquad (t \in \mathbb{R}) , \tag{1.7.26}$$

where $\gamma \in \mathbb{R}$, $\mu > 0$ and G is a df continuous at zero, iff $\sigma^2 = 0$ and (1.7.20) holds. The representation (γ,μ,G) in (1.7.26) is unique.

PROOF. If $\sigma^2 = 0$ and (1.7.20) holds, then it is easily verified that (1.7.1a) takes the form (1.7.26) by putting

$$\gamma := a - \int_{\mathbb{R}\setminus\{0\}} \frac{x}{1+x^2}\, dM(x)\,,\quad \mu := \int_{\mathbb{R}\setminus\{0\}} dM(x)\,,\quad G(x) := \frac{1}{\mu} M(x) + 1_{[0,\infty)}(x)\,. \tag{1.7.27}$$

Conversely, if F satisfies (1.7.26), then, comparing this equation with (1.7.1a), we see that

$$a = \gamma + \mu \int_{(-\infty,\infty)} \frac{x}{1+x^2}\, dG(x)\,,\quad \sigma^2 = 0,\quad M(x) = \mu\{G(x) - 1_{[0,\infty)}(x)\}\,, \tag{1.7.28}$$

so that $\sigma^2 = 0$ and M is bounded.

Finally, suppose that $\tilde{F}$ can be represented by (1.7.26) in two ways: by (γ_1,μ_1,G_1) and (γ_2,μ_2,G_2), say. Then necessarily

$$it\gamma_1 + \mu_1(\tilde{G}_1(t) - 1) = it\gamma_2 + \mu_2(\tilde{G}_2(t) - 1) + 2k\pi i \qquad (t \in \mathbb{R};\ k \in \mathbb{Z})\,.$$

Taking t = 0 and using the continuity of $\tilde{G}_1$ and $\tilde{G}_2$, we get

$$it(\gamma_1 - \gamma_2) = \mu_2(\tilde{G}_2(t) - 1) - \mu_1(\tilde{G}_1(t) - 1) \qquad (t \in \mathbb{R})\,. \tag{1.7.29}$$

It follows that

$$|\gamma_1 - \gamma_2|\,|t| \le 2(\mu_1 + \mu_2) \qquad (t \in \mathbb{R})\,,$$

and hence $\gamma_1 = \gamma_2$. Now, as the jump of a df G at zero is given by (cf. Lukacs (1970), ch. 3)

$$G(0) - G(0-) = \lim_{T\to\infty} \frac{1}{2T} \int_{(-T,T)} \tilde{G}(t)\,dt\,, \tag{1.7.30}$$

and as G_1 and G_2 are supposed to be continuous at zero, it follows from (1.7.29) with $\gamma_1 = \gamma_2$ that $\mu_1 = \mu_2$, and hence $G_1 = G_2$. □

Combining theorems 1.7.9(ii) and 1.7.13, we obtain the following result.

COROLLARY 1.7.14. An inf div df F that is not continuous, has a chf $\tilde{F}$ of the form (1.7.26).

CHAPTER 2

CLASSIFICATION OF THE INFINITELY DIVISIBLE LATTICE DISTRIBUTIONS

In section 1.5 we introduced four classes of inf div lattice distributions: the classes $\mathcal{C}_1$, $\mathcal{C}_o$, $\mathcal{B}$ and $\mathcal{A}$ of compound Poisson (i.e. inf div), compound geometric, log-convex and comp mon distributions, respectively. Also, we noted that, as in this order these classes contain increasingly special distributions (cf. theorem 1.5.13), they define a classification (cf. the end of section 1.2) of the inf div lattice distributions. The distributions in $\mathcal{A}$ and $\mathcal{B}$, and to some extent in $\mathcal{C}_o$, are often easily recognized as such. Now, in order to get a better understanding of the class $\mathcal{C}_1 \backslash \mathcal{C}_o$, we want to classify the distributions in this class.
To obtain such a classification, in section 1 we introduce classes of lattice distributions characterized by means of recurrence relations. These relations depend on certain sequences $\{c_n^*(\alpha)\}$, and the main problem will be to choose $c_n^*(\alpha)$ in such a way that the resulting classes yield a classification of $\mathcal{C}_1$ (section 2). This turns out to be possible by choosing a special sequence $\{c_n^*(\alpha)\}$; in section 3 the classes $\mathcal{C}_\alpha$ $(0 \le \alpha \le 1)$, resulting from this special choice, are shown to be increasing with α. It turns out that an important role is played here by the compound geometric distributions. Properties of the $\mathcal{C}_\alpha$'s are proved in section 4, where also some examples are given. In section 5 two other classifications of $\mathcal{C}_1$ are discussed. Specifically, we consider the classes of compound negative-binomial distributions, which turn out to have properties similar to those of the $\mathcal{C}_\alpha$'s.

2.1. *Interpolating between $\mathcal{C}_0$ and $\mathcal{C}_1$*

The class $\mathcal{C}_1$ of inf div (i.e. compound Poisson) lattice distributions $\{p_n\}_o^\infty$, with $0 < p_o < 1$, can be characterized as follows (cf. corollary 1.5.5): $\{p_n\} \in \mathcal{C}_1$ iff there exist nonnegative quantities $r_n(1)$ $(n \in \mathbb{N}_o)$ such that

$$(2.1.1) \qquad (n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(1) \qquad (n \in \mathbb{N}_o) .$$

Similarly, in corollary 1.5.9 the class $\mathcal{C}_o$ of compound geometric lattice distributions $\{p_n\}_o^\infty$ is characterized by the nonnegativity of quantities $r_n(0)$ satisfying

$$(2.1.2) \qquad p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_o) .$$

We know that $C_o \subset C_1$ (cf. theorem 1.5.13), and the recurrence relations (2.1.1) and (2.1.2) now suggest the idea of generalizing them in order to obtain classes of lattice distributions that fill the gap between C_o and C_1. Proceeding in such a way, for $0 \le \alpha \le 1$ we consider sequences $\{c_n^*(\alpha)\}_o^\infty$ with the following properties:

$$(2.1.3)\qquad \begin{cases} c_n^*(0) = 1,\ c_n^*(1) = n+1 & (n \in \mathbb{N}_o) \\ c_n^*(\alpha) \text{ is nondecreasing in both } n \text{ and } \alpha\ . \end{cases}$$

For any choice of $\{c_n^*(\alpha)\}$ satisfying (2.1.3) we introduce classes C_α^* $(0 \le \alpha \le 1)$ as follows.

DEFINITION 2.1.1. For $0 \le \alpha \le 1$ a lattice distribution $\{p_n\}_o^\infty$, with $p_o > 0$, is said to be in the class C_α^* if there exist nonnegative quantities $r_n^*(\alpha)$ $(n \in \mathbb{N}_o)$ such that

$$(2.1.4)\qquad c_n^*(\alpha)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}^*(\alpha) \qquad (n \in \mathbb{N}_o)\ .$$

REMARK. The symbol $*$ is added to the quantities $c_n^*(\alpha)$, $r_n^*(\alpha)$ and C_α^* to enable us to use the notation without $*$ for the special situation that will be considered from section 3 on. Further, as already noted in section 1.2, if $\{p_n\} \in C_\alpha^*$ with pgf P, then we shall also say that $P \in C_\alpha^*$.

Clearly, taking $\alpha = 0$ and $\alpha = 1$ in definition 2.1.1, we get $C_o^* = C_o$ and $C_1^* = C_1$. For these cases the recurrence relations (2.1.4) have been considered in detail in section 1.5. Now, before choosing a special sequence $\{c_n^*(\alpha)\}$, we discuss some general properties of (2.1.4) with an arbitrary $\alpha \in [0,1]$. We need the following notation (cf. (2.1.3)):

$$(2.1.5)\qquad c^*(\alpha) := \lim_{n\to\infty} c_n^*(\alpha) \qquad (\le \infty;\ 0 \le \alpha \le 1)\ ,$$

and for a given lattice distribution $\{p_n\}_o^\infty$

$$(2.1.6)\qquad A_\alpha^*(z) := \sum_{n=0}^{\infty} c_n^*(\alpha)p_{n+1}z^n \qquad (|z| < 1;\ 0 \le \alpha \le 1)\ .$$

LEMMA 2.1.2. Consider a fixed sequence $\{c_n^*(\alpha)\}_0^\infty$ satisfying (2.1.3), and a fixed $\alpha \in [0,1]$.

(i) Let $\{p_n\}_0^\infty$ be a lattice distribution with $p_0 > 0$. Then there exists a unique sequence $\{r_n^*(\alpha)\}_0^\infty$ satisfying (2.1.4), and its gf has a positive radius of convergence. If $r_n^*(\alpha) \geq 0$ for all $n \in \mathbb{N}_0$ (i.e. if $\{p_n\} \in C_\alpha^*$), then

$$(2.1.7) \qquad \sum_{n=0}^{\infty} r_n^*(\alpha)/c_n^*(\alpha) < \infty \;;$$

if furthermore $c^*(\alpha)$ is finite, then

$$(2.1.8) \qquad \sum_{n=0}^{\infty} r_n^*(\alpha) < c^*(\alpha) \;.$$

(ii) Let $\{r_n^*(\alpha)\}_0^\infty$ be a sequence of nonnegative numbers satisfying (2.1.7), or, if $c^*(\alpha)$ is finite, (2.1.8). Then there exists a unique lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$, satisfying (2.1.4).

PROOF.

(i) Let $\{p_n\}_0^\infty$ be a lattice distribution with $p_0 > 0$. Then, rewriting (2.1.4) in the form

$$p_0 r_n^*(\alpha) = c_n^*(\alpha) p_{n+1} - \sum_{k=1}^{n} p_k r_{n-k}^*(\alpha) \qquad (n \in \mathbb{N}_0) \;,$$

we see that the sequence $\{r_n^*(\alpha)\}_0^\infty$, recursively defined by this equation, is the unique solution of (2.1.4). Further, as $p_0 > 0$, the function R_α^*, defined by

$$R_\alpha^*(z) := A_\alpha^*(z)/P(z) \;,$$

is analytic in $|z| < \varepsilon$ for some $\varepsilon > 0$. It follows that the coefficients in the power-series expansion of R_α^* satisfy (2.1.4), and hence are equal to the $r_n^*(\alpha)$'s. So, for $|z| < \varepsilon$ the gf of $\{r_n^*(\alpha)\}$ coincides with R_α^* and therefore has a positive radius of convergence.

If all $r_n^*(\alpha)$'s are nonnegative, then we can write

$$1 - p_0 = \sum_{n=0}^{\infty} p_{n+1} = \sum_{n=0}^{\infty} c_n^*(\alpha)^{-1} \sum_{k=0}^{n} p_k r_{n-k}^*(\alpha) =$$

$$= \sum_{k=0}^{\infty} p_k \sum_{n=0}^{\infty} r_n^*(\alpha)/c_{n+k}^*(\alpha) \;.$$

From this it follows that

$$\sum_{n=0}^{\infty} r_n^*(\alpha)/c_n^*(\alpha) \le (1-p_o)/p_o < \infty ,$$

while, in case $c^*(\alpha)$ is finite, by the monotonicity of $c_n^*(\alpha)$ we conclude that

$$\sum_{n=0}^{\infty} r_n^*(\alpha) \le (1-p_o)c^*(\alpha) < c^*(\alpha) .$$

(ii) Let $\{r_n^*(\alpha)\}_o^\infty$ be a sequence of nonnegative numbers satisfying (2.1.7). As A_α^*, defined by (2.1.6), is not expressible in terms of P, it is not possible to use the method applied in the cases $\alpha = 0$ and $\alpha = 1$; we have to work with the recurrence relations themselves. First consider the quantities γ_k, for $k \in \mathbb{N}_o$ defined by

$$\gamma_k := \sum_{n=0}^{\infty} r_n^*(\alpha)/c_{n+k}^*(\alpha) .$$

Because of the monotonicity of $c_n^*(\alpha)$ the γ_k's form a nonincreasing sequence of nonnegative numbers. Using the dominated convergence theorem one easily sees that $\lim_{k\to\infty} \gamma_k = 0$, if $c^*(\alpha)$ is infinite, and that $\lim_{k\to\infty} \gamma_k < 1$, if $c^*(\alpha)$ is finite and (2.1.8) holds. It follows that in both cases it is possible to choose $\gamma < 1$ and $K \in \mathbb{N}$ such that $\gamma_k \le \gamma$ for all $k > K$. Next we note that for any choice of $p_o > 0$ (2.1.4) has exactly one solution $\{p_n\}_o^\infty$ with, necessarily, $p_n \ge 0$ for all n. Considering such a solution, for $N > K$ we can write

$$\sum_{n=1}^{N+1} p_n = \sum_{n=0}^{N} c_n^*(\alpha)^{-1} \sum_{k=0}^{n} p_k r_{n-k}^*(\alpha) = \sum_{k=0}^{N} p_k \sum_{n=k}^{N} r_{n-k}^*(\alpha)/c_n^*(\alpha) \le$$

$$\le \sum_{k=0}^{N} p_k\gamma_k \le \sum_{k=0}^{K} p_k\gamma_o + \sum_{k=K+1}^{N} p_k\gamma \le \gamma_o \sum_{k=0}^{K} p_k + \gamma \sum_{k=1}^{N+1} p_k .$$

Hence, as $\gamma < 1$,

$$\sum_{n=1}^{N+1} p_n \le \frac{\gamma_o}{1-\gamma} \sum_{k=0}^{K} p_k \qquad (N > K) ,$$

from which it follows that for all solutions $\{p_n\}_o^\infty$ of (2.1.4) with $p_o > 0$ we have

$$\sum_{n=0}^{\infty} p_n < \infty .$$

Finally, every solution $\{p_n\}$ can be written as $p_n = u_n p_o$ $(n \in \mathbb{N}_o)$, where $\{u_n\}_o^\infty$ denotes the solution of (2.1.4) with starting value 1. By choosing $p_o = \{\Sigma u_n\}^{-1}$, the corresponding solution $\{p_n\}_o^\infty$ becomes a probability distribution. □

REMARK. Part (i) of the preceding lemma enables us to speak of the sequence $\{r_n^*(\alpha)\}_o^\infty$, and its gf R_α^*, *corresponding to* a given lattice distribution $\{p_n\}_o^\infty$ with $p_o > 0$. In the sequel we shall do so without further comment.
Part (ii) enables us to construct examples of distributions in C_α^*, by starting from an *arbitrary* sequence $\{r_n^*(\alpha)\}_o^\infty$ of nonnegative numbers satisfying (2.1.7) or (2.1.8).

In order to obtain a classification of the inf div lattice distributions, we wish to choose the sequence $\{c_n^*(\alpha)\}$ in such a way that

(2.1.9) $$C_\alpha^* \subset C_\beta^* \text{ if } \alpha < \beta \qquad (0 \le \alpha < 1,\ 0 < \beta \le 1) ,$$

and we shall say that in this case $\{c_n^*(\alpha)\}$ satisfies (2.1.9). In the next section we shall investigate if sequences satisfying (2.1.9) do indeed exist. Here we give a few more properties that hold for *every* choice of $\{c_n^*(\alpha)\}$.

THEOREM 2.1.3. Let $\{p_n\} \in C_o$. Then the $r_n^*(\alpha)$'s corresponding to $\{p_n\}$ satisfy

(2.1.10) $$r_n^*(\alpha)/c_n^*(\alpha) \ge r_n(0) \qquad (0 \le \alpha \le 1;\ n \in \mathbb{N}_o) .$$

Furthermore, if $c_{n+1}^*(\alpha)/c_n^*(\alpha)$ is nondecreasing in α for all n, then so are both $r_n^*(\alpha)/c_n^*(\alpha)$ and $r_n^*(\alpha)$.

PROOF. In the first place we note that for the $r_n^*(\alpha)$'s corresponding to a general $\{p_n\}$ the following relation holds:

(2.1.11) $$\sum_{k=0}^{n} c_k^*(\alpha) p_{k+1} r_{n-k}^*(\beta) = \sum_{k=0}^{n} c_k^*(\beta) p_{k+1} r_{n-k}^*(\alpha) \qquad (0 \le \alpha,\ \beta \le 1;\ n \in \mathbb{N}_o) ;$$

in fact, this is a consequence of the following relation (cf. (2.1.6)):

$$A_\alpha^*(z) R_\beta^*(z) = A_\beta^*(z) R_\alpha^*(z) \quad (= P(z) R_\alpha^*(z) R_\beta^*(z)) .$$

Now, let $\{p_n\} \in \mathcal{C}_o$, so $r_n(0) \geq 0$ for all n. Then, applying (2.1.11) with $\beta = 0$, we can write

$$p_o r_n^*(\alpha) = c_n^*(\alpha) p_{n+1} - \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}^*(\alpha) =$$

$$= c_n^*(\alpha) p_{n+1} - \sum_{k=0}^{n-1} c_k^*(\alpha) p_{k+1} r_{n-1-k}(0) \ ,$$

so that

$$p_o\{r_n^*(\alpha) - c_n^*(\alpha) r_n(0)\} =$$

$$= c_n^*(\alpha) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) - \sum_{k=0}^{n-1} c_k^*(\alpha) p_{k+1} r_{n-1-k}(0) =$$

$$= \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) \{c_n^*(\alpha) - c_k^*(\alpha)\} \ .$$

Now by the monotonicity of $c_n^*(\alpha)$ it follows that (2.1.10) holds.
Next, let us consider sequences $\{c_n^*(\alpha)\}$ for which $c_{n+1}^*(\alpha)/c_n^*(\alpha)$ is nondecreasing in α for all n, or, equivalently,

$$(2.1.12) \quad c_n^*(\beta) c_k^*(\alpha) \geq c_n^*(\alpha) c_k^*(\beta) \qquad (0 \leq \alpha < \beta \leq 1;\ n \in \mathbb{N}_o;\ k \in \{0,1,..,n\}).$$

Applying first (2.1.4) and then twice (2.1.11) with $\beta = 0$ and $\alpha = 0$, respectively, we can write

$$p_o\{c_n^*(\alpha) r_n^*(\beta) - c_n^*(\beta) r_n^*(\alpha)\} =$$

$$= c_n^*(\beta) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}^*(\alpha) - c_n^*(\alpha) \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}^*(\beta) =$$

$$= \sum_{k=0}^{n-1} p_{k+1} r_{n-1-k}(0) \{c_n^*(\beta) c_k^*(\alpha) - c_n^*(\alpha) c_k^*(\beta)\} \ ,$$

which is nonnegative on account of (2.1.12) for $\alpha < \beta$. It follows that $r_n^*(\alpha)/c_n^*(\alpha)$ is nondecreasing in α for all n. Finally, from this we obtain the same property for $r_n^*(\alpha)$, using the monotonicity of $c_n^*(\alpha)$ in α. □

From the preceding theorem it follows that definition 2.1.1 can only generate classes $\mathcal{C}_\alpha^*$ with $\mathcal{C}_o$ as a subclass. We state this as a corollary.

COROLLARY 2.1.4. For every choice of $\{c_n^*(\alpha)\}$ and for all $\alpha \in [0,1]$: $\mathcal{C}_o \subset \mathcal{C}_\alpha^*$.

To conclude this section we prove a simple property of C_α^*, which is already known for $\alpha = 0$ and $\alpha = 1$.

THEOREM 2.1.5. If $0 \le \alpha \le 1$ and $\{p_n\} \in C_\alpha^*$, then $\{p_n^{(\gamma)}\}_0^\infty \in C_\alpha^*$ for all $\gamma \in (0,1)$, where

$$p_n^{(\gamma)} := \gamma^n p_n / P(\gamma) \qquad (n \in \mathbb{N}_0) .$$

PROOF. It is easy to see that, if $\{p_n\}$ satisfies the recurrence relations (2.1.4), then $\{p_n^{(\gamma)}\}$ satisfies (2.1.4) with $r_n^*(\alpha)$ replaced by $\gamma^{n+1} r_n^*(\alpha)$. As these quantities are nonnegative if the $r_n^*(\alpha)$'s are, the theorem follows. □

2.2. *The choice of* $c_n^*(\alpha)$

Clearly, there are many sequences $\{c_n^*(\alpha)\}_0^\infty$ satisfying (2.1.3). Examples of such sequences are easily obtained by a simple interpolation between $c_n^*(0) = 1$ and $c_n^*(1) = n+1$. Rather more sophisticated, one may try to interpolate between $C_0^*(z) = (1-z)^{-1}$ and $C_1^*(z) = (1-z)^{-2}$, where C_α^* denotes the gf of $\{c_n^*(\alpha)\}$. Of both methods we give three examples.

EXAMPLE 2.2.1. The following sequences $\{c_n^*(\alpha)\}$ are easily seen to satisfy (2.1.3):

(i) $\quad c_n^*(\alpha) = 1 + \alpha n$,

(ii) $\quad c_n^*(\alpha) = (1+n)^\alpha$,

(iii) $\quad c_n^*(\alpha) = (1+\alpha n)^\alpha$,

(iv) $\quad C_\alpha^*(z) = (1-z)^{-1-\alpha}$, so $c_n^*(\alpha) = \binom{\alpha+n}{n}$,

(v) $\quad C_\alpha^*(z) = (1-z)^{-1}(1-\alpha z)^{-1}$, so $c_n^*(\alpha) = 1 + \alpha + \ldots + \alpha^n = (1-\alpha^{n+1})/(1-\alpha)$,

(vi) $\quad C_\alpha^*(z) = (1-z)^{-1}(1-\alpha z)^{-\alpha}$, so $c_n^*(\alpha) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k}\alpha^k$.

Next we want to investigate which (if any) of the sequences $\{c_n^*(\alpha)\}$, listed above, satisfy the desired monotonicity property (2.1.9). Not able to give useful sufficient conditions for (2.1.9), we look for simple necessary conditions, in order to check them for the $c_n^*(\alpha)$'s in example 2.2.1. To this end we introduce a special distribution, which is in a way a critical one.

DEFINITION 2.2.2. For a given sequence $\{c_n^*(\alpha)\}$ satisfying (2.1.3), the *boundary distribution* with parameters $\alpha \in [0,1]$ and $\gamma \in (0,c^*(\alpha))$ is the lattice distribution $\{\tilde{p}_n\}_0^\infty$, defined by

$$(2.2.1) \qquad \tilde{p}_n = \tilde{p}_o \gamma^n \prod_{k=0}^{n-1} c_k^*(\alpha)^{-1} \qquad (n \in \mathbb{N}) .$$

This distribution $\{\tilde{p}_n\}$ can be obtained by choosing in lemma 2.1.2(ii)

$$(2.2.2) \qquad r_o^*(\alpha) = \gamma, \; r_n^*(\alpha) = 0 \qquad (n \in \mathbb{N}) ,$$

which (see also example 3 on p. 57) explains the name "boundary distribution". It follows that

$$(2.2.3) \qquad \{\tilde{p}_n\}_o^\infty \in C_\alpha^* .$$

Computing the $r_n^*(\alpha)$'s corresponding to the boundary distribution $\{\tilde{p}_n\}_o^\infty$ with parameters α_o and γ, we obtain

$$r_n^*(\alpha) = c_n^*(\alpha)\gamma^{n+1} \prod_{\ell=0}^{n} c_\ell^*(\alpha_o)^{-1} - \sum_{k=1}^{n} r_{n-k}^*(\alpha)\gamma^k \prod_{\ell=0}^{k-1} c_\ell^*(\alpha_o)^{-1} ,$$

from which it successively follows that

$$r_o^*(\alpha) = \gamma ,$$
$$r_1^*(\alpha) = \gamma^2\{c_1^*(\alpha)/c_1^*(\alpha_o) - 1\} ,$$
$$r_2^*(\alpha) = \gamma^3 c_1^*(\alpha_o)^{-1}\{c_2^*(\alpha)/c_2^*(\alpha_o) - c_1^*(\alpha) + c_1^*(\alpha_o) - 1\} .$$

Now, if $\{c_n^*(\alpha)\}$ satisfies (2.1.9), then, as $\{\tilde{p}_n\} \in C_{\alpha_o}^*$, the $r_n^*(\alpha)$'s have to be nonnegative for all $\alpha \in [\alpha_o,1]$. This condition is satisfied for n = 0 and n = 1, but not generally for n = 2. Working out the conditions

$$r_2^*(1) \ge 0 \text{ and } \frac{d}{d\alpha} r_2^*(\alpha)\Big|_{\alpha=\alpha_o} \ge 0 ,$$

respectively, we get the following lemma (where α_o has been replaced by α).

LEMMA 2.2.3. For $\{c_n^*(\alpha)\}$ to satisfy (2.1.9) the following conditions are necessary:

$$(2.2.4) \qquad c_1^*(\alpha)c_2^*(\alpha) \ge 3(c_2^*(\alpha) - 1) \qquad (0 < \alpha < 1)$$

and

(2.2.5) $\frac{d}{d\alpha}\, c_2^*(\alpha) \geq c_2^*(\alpha)\frac{d}{d\alpha}\, c_1^*(\alpha) \qquad (0 < \alpha < 1)$,

where in the latter case c_1^* and c_2^* are supposed to be differentiable.

For the examples of $\{c_n^*(\alpha)\}$ given in example 2.2.1 it is not difficult to check whether or not they satisfy (2.2.4) and (2.2.5). Therefore we state the mostly negative results without proofs.

LEMMA 2.2.4.

(i) $c_n^*(\alpha) = 1 + \alpha n$ violates (2.2.4) iff $\frac{1}{2} < \alpha < 1$, and (2.2.5) iff $\frac{1}{2} < \alpha < 1$.

(ii) $c_n^*(\alpha) = (1 + n)^{\alpha}$ violates (2.2.4) if $0.7427 < \alpha < 1$, and (2.2.5) iff ${}^2\log {}^2\log 3$ $(\simeq 0.6645) < \alpha < 1$.

(iii) $c_n^*(\alpha) = (1 + \alpha n)^{\alpha}$ violates (2.2.5) if $0.7573 < \alpha < 1$.

(iv) $c_n^*(\alpha) = \binom{\alpha+n}{n}$ violates (2.2.4) iff $\sqrt{3} - 1$ $(\simeq 0.7321) < \alpha < 1$, and (2.2.5) iff $\frac{1}{2}(\sqrt{5} - 1)$ $(\simeq 0.6180) < \alpha < 1$.

(v) $c_n^*(\alpha) = (1 - \alpha^{n+1})/(1-\alpha)$ satisfies both (2.2.4) and (2.2.5).

(vi) $c_n^*(\alpha) = \sum_{k=0}^{n} \binom{\alpha+k-1}{k}\alpha^k$ violates (2.2.4) if $0.9344 < \alpha < 1$, and (2.2.5) if $0.8907 < \alpha < 1$.

It is a pity that $(1+n)^{\alpha}$ and $\binom{\alpha+n}{n}$ $(\sim n^{\alpha}/\Gamma(\alpha+1),\ n \to \infty)$ do not have the required properties, as they seem to be more interesting, especially with regard to asymptotic behaviour, than the only choice left: $c_n^*(\alpha) = (1-\alpha^{n+1})/(1-\alpha)$, for which we have $c_n^*(\alpha) \sim (1-\alpha)^{-1}$ $(n \to \infty)$. It would be interesting to know whether other choices of $\{c_n^*(\alpha)\}$ exist, satisfying (2.1.3) and (2.1.9), with a more attractive limit behaviour, e.g. $c_n^*(\alpha) \sim n^{\alpha}$ $(n \to \infty)$. It might be possible to obtain more interesting $c_n^*(\alpha)$'s by weakening somewhat the condition (2.1.3). Dropping, for example, the monotonicity of $c_o^*(\alpha)$, we may consider $c_n^*(\alpha)$'s like $(n+\alpha)^{\alpha}$ and the one obtained by means of "fractional differentiation" as follows:

$$C_\alpha^*(z) = z^{\alpha-1}\left(\frac{d}{dz}\right)^{\alpha}[z(1-z)^{-1}] = z^{\alpha-1}\left(\frac{d}{dz}\right)^{\alpha}\left[\sum_{n=1}^{\infty} z^n\right] =$$

$$= z^{\alpha-1}\sum_{n=1}^{\infty}\frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}\, z^{n-\alpha} ,$$

so

$$(2.2.6) \qquad c_n^*(\alpha) = \frac{(n+1)!}{\Gamma(n+2-\alpha)} = \frac{1}{\Gamma(1-\alpha)}\binom{n+1-\alpha}{n+1}^{-1} .$$

It turns out, however, that also these two choices fail to satisfy (2.1.9); if $c_n^*(\alpha) = (n+\alpha)^{\alpha}$, then computing[1] $r_3^*(1)$ corresponding to the boundary distribution $\{\tilde{p}_n\}_0^{\infty}$ with parameters α and γ, we obtain, for instance for $\alpha = 0.6$ and $\gamma = 1$: $r_3^*(1) \simeq -0.0089$; if $c_n^*(\alpha)$ is given by (2.2.6), then $r_2^*(1)$, corresponding to $\{\tilde{p}_n\}$, turns out to satisfy: $r_2^*(1) < 0$ iff $3-\sqrt{5}$ ($\simeq 0.7639$) $< \alpha < 1$.

Although (2.2.6) does not satisfy (2.1.9), it suggests a class of other possible choices of $c_n^*(\alpha)$. If we rewrite (2.2.6) as

$$(2.2.7) \qquad c_n^*(\alpha) = \frac{n+1}{\Gamma(1-\alpha)} \int_{(0,1)} (1-u)^{-\alpha} u^n du \qquad (n \in \mathbb{N}_0;\ 0 \le \alpha < 1) ,$$

then by (2.1.4) it follows that the gf R_α^* of $\{r_n^*(\alpha)\}_0^{\infty}$, corresponding to a given lattice distribution $\{p_n\}_0^{\infty}$, is given by

$$(2.2.8) \qquad R_\alpha^*(z) = \frac{1}{\Gamma(1-\alpha)} \int_{(0,1)} (1-u)^{-\alpha} \frac{P'(uz)}{P(z)} du \qquad (0 \le \alpha < 1) .$$

Now (2.2.7) and (2.2.8) can be generalized in the following way:

$$(2.2.9) \qquad c_n^*(\alpha) = c_0^*(\alpha)(n+1) \int_{(0,1)} g_\alpha(u) u^n du \qquad (n \in \mathbb{N}_0;\ 0 \le \alpha < 1)$$

and

$$(2.2.10) \qquad R_\alpha^*(z) = c_0^*(\alpha) \int_{(0,1)} g_\alpha(u) \frac{P'(uz)}{P(z)} du \qquad (0 \le \alpha < 1) ,$$

where for all $\alpha \in [0,1)$ g_α is a pdf on $(0,1)$ with the property that

$$g_0(u) = 1_{(0,1)}(u) \text{ and } \lim_{\alpha \uparrow 1} g_\alpha(u) = \delta_1(u)$$

(δ_1 is Dirac's delta function at 1). It is not clear though, under what conditions R_β^* would be abs mon if R_α^* is abs mon and $\alpha < \beta$. Rather curiously, as

$$\frac{1-\alpha^{n+1}}{1-\alpha} = (n+1) \int_{(\alpha,1)} \frac{1}{1-\alpha} u^n du \qquad (n \in \mathbb{N}_0;\ 0 \le \alpha < 1) ,$$

---

[1] My thanks are due to L.G.F.C. van Bree, who did the programming needed here and in some other cases.

the only remaining example of $c_n^*(\alpha)$ from lemma 2.2.4, i.e. $c_n^*(\alpha) = (1-\alpha^{n+1})/(1-\alpha)$, is of the form (2.2.9) with g_α the uniform pdf on $(\alpha,1)$. We shall not pursue this in this monograph.

We now restrict our attention to the only remaining candidate, $c_n^*(\alpha) = (1-\alpha^{n+1})/(1-\alpha)$. This choice, which we shall consider in detail in the next section, is supported by the following lemma.

LEMMA 2.2.5. If $c_1^*(\alpha)$ and $c_2^*(\alpha)$ are linear and quadratic functions, respectively, then for (2.2.4) and (2.2.5) to hold it is necessary and sufficient that

$$c_1^*(\alpha) = 1+\alpha,\ c_2^*(\alpha) = 1+\alpha+\alpha^2 \qquad (0 \le \alpha \le 1) .$$

PROOF. Because of (2.1.3) c_1^* and c_2^* have necessarily the form

$$c_1^*(\alpha) = 1+\alpha,\ c_2^*(\alpha) = 1+(2-b)\alpha+b\alpha^2 ,$$

where $-2 \le b \le 1$, $b \ne 0$. The sufficiency of the condition has already been shown in lemma 2.2.4. Suppose therefore that (2.2.4) holds. Then it is easy to see that

$$b \ge (2\alpha-1)/\{\alpha(2-\alpha)\} \qquad (0 \le \alpha \le 1) ,$$

from which it follows that $b \ge 1$. Hence $b = 1$, and the lemma is proved. □

2.3. *The classes C_α; basic properties*

From now on we shall consider the $c_n^*(\alpha)$ from lemma 2.2.4(v), and we shall show that it satisfies indeed (2.1.9). For this special case we use the notation without *, so we have

$$c_n(\alpha) = \frac{1-\alpha^{n+1}}{1-\alpha} = \sum_{k=0}^{n} \alpha^k \qquad (n \in \mathbb{N}_0;\ 0 \le \alpha \le 1) ,$$

while the corresponding classes C_α are defined as follows.

DEFINITION 2.3.1. A lattice distribution $\{p_n\}_0^\infty$, with $p_0 > 0$, is said to be in the class C_α if there exist $r_n(\alpha) \ge 0$ $(n \in \mathbb{N}_0)$ such that

$$(2.3.1) \qquad \frac{1-\alpha^{n+1}}{1-\alpha}\, p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(\alpha) \qquad (n \in \mathbb{N}_0) .$$

Applying lemma 2.1.2 to our special situation, we easily obtain the following properties of (2.3.1). We only note that $c(\alpha)$ and A_α, defined by (2.1.5) and (2.1.6), respectively, are now given by

$$c(\alpha) = \frac{1}{1-\alpha}\ ,\quad A_\alpha(z) = \frac{P(z) - P(\alpha z)}{(1-\alpha)z} \qquad (0 \le \alpha < 1;\ |z| \le 1)\ .$$

LEMMA 2.3.2. Consider the recurrence relations (2.3.1) for a fixed $\alpha \in [0,1)$.

(i) For every lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$ there exists a unique sequence $\{r_n(\alpha)\}_0^\infty$ satisfying (2.3.1). The gf R_α of $\{r_n(\alpha)\}$ has a positive radius of convergence, while in a neighbourhood of zero

$$(2.3.2) \qquad (1-\alpha)zR_\alpha(z) = 1 - P(\alpha z)/P(z)\ .$$

Furthermore, if $r_n(\alpha) \ge 0$ for all n (i.e. if $\{p_n\} \in \mathcal{C}_\alpha$), then

$$(2.3.3) \qquad \sum_{n=0}^{\infty} r_n(\alpha) < \frac{1}{1-\alpha}\ .$$

(ii) For every sequence $\{r_n(\alpha)\}_0^\infty$ with $r_n(\alpha) \ge 0$ for all n and satisfying (2.3.3) there exists a unique lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$, satisfying (2.3.1).

As noted in section 1, we shall speak of the sequence $\{r_n(\alpha)\}$ *corresponding to* $\{p_n\}$, when considering the solution of (2.3.1) for $\{r_n(\alpha)\}$. We shall refer to its gf R_α as the R_α*-function* of $\{p_n\}$ or P. It is now obvious that we have the following useful characterization of $\mathcal{C}_\alpha$ in terms of pgf's (cf. definition 1.3.1).

LEMMA 2.3.3. For $0 \le \alpha < 1$ a pgf P, with $P(0) > 0$, is in $\mathcal{C}_\alpha$ iff the R_α-function of P, given by (2.3.2), is abs mon.

Taking $\alpha = 0$ in the preceding lemma, we get theorem 1.5.8, while the first part of theorem 1.5.3 is a limiting case in the following sense:

$$\lim_{\alpha\uparrow 1} R_\alpha(z) = \lim_{\alpha\uparrow 1} \frac{1}{P(z)}\,\frac{P(z)-P(\alpha z)}{z - \alpha z} = \frac{P'(z)}{P(z)} = R_1(z)\ .$$

As an illustration of the usefulness of lemma 2.3.3 we prove corollary 2.1.4 in our situation using gf's. Rewriting (2.3.2) for $\alpha = 0$ as

$$(2.3.4) \qquad P(z) = P(0)/\{1 - zR_0(z)\}\ ,$$

and substituting this in (2.3.2), we obtain for $0 < \alpha < 1$ the following relation between the R_0- and R_α-function of a pgf P:

$$(2.3.5) \qquad (1-\alpha)R_\alpha(z) = \{R_0(z) - \alpha R_0(\alpha z)\}/\{1-\alpha z R_0(\alpha z)\} \ .$$

It follows that R_α is abs mon if R_0 is abs mon. Hence we have proved

$$\mathcal{C}_0 \subset \mathcal{C}_\alpha \qquad (0 \le \alpha \le 1) \ .$$

We now turn to the general monotonicity property (2.1.9). As a first step we prove that for every $\alpha \in [0,1)$ the distributions in $\mathcal{C}_\alpha$ are inf div.

THEOREM 2.3.4. For all $\alpha \in [0,1)$ the following inclusion holds:

$$\mathcal{C}_\alpha \subset \mathcal{C}_1 \ .$$

PROOF. Let $0 \le \alpha < 1$ and $P \in \mathcal{C}_\alpha$. Then according to (2.3.2) and (2.3.3) for $|z| \le 1$ we can write

$$(2.3.6) \qquad P(z) = \{1 - (1-\alpha)zR_\alpha(z)\}^{-1}P(\alpha z) \ .$$

Iterating this equation we obtain for every $n \in \mathbb{N}$

$$P(z) = P(\alpha^n z)\prod_{k=0}^{n-1}\{1-(1-\alpha)\alpha^k z R_\alpha(\alpha^k z)\}^{-1} \qquad (|z| \le 1) \ .$$

Taking $z = 1$ we see that

$$P(\alpha^n) = \prod_{k=0}^{n-1}\{1-(1-\alpha)\alpha^k R_\alpha(\alpha^k)\} \ ,$$

and so

$$P(z) = \frac{P(\alpha^n z)}{P(\alpha^n)}\prod_{k=0}^{n-1}\frac{1-(1-\alpha)\alpha^k R_\alpha(\alpha^k)}{1-(1-\alpha)\alpha^k z R_\alpha(\alpha^k z)} \qquad (|z| \le 1;\ n \in \mathbb{N}) \ .$$

From this, by letting $n \to \infty$, the following expression for P in R_α is obtained:

$$(2.3.7) \qquad P(z) = \prod_{k=0}^{\infty}\frac{1-(1-\alpha)\alpha^k R_\alpha(\alpha^k)}{1-(1-\alpha)\alpha^k z R_\alpha(\alpha^k z)} \qquad (|z| \le 1) \ ,$$

where it is easily verified that the infinite product is absolutely convergent. Now define $\pi_k := (1-\alpha)\alpha^k R_\alpha(\alpha^k)$ and $Q_k(z) := zR_\alpha(\alpha^k z)/R_\alpha(\alpha^k)$. Then P takes the form

$$P(z) = \prod_{k=0}^{\infty} \frac{1-\pi_k}{1-\pi_k Q_k(z)} \qquad (|z| \le 1) ,$$

where $\pi_k \in [0,1)$ because of (2.3.3) and Q_k is a pgf because of the abs mon of R_α (lemma 2.3.3). It follows that P is the limit of a sequence of products of pgf's in $C_o \subset C_1$, and hence by theorems 1.4.2 and 1.4.3 we conclude that $P \in C_1$. □

Equation (2.3.7) entails a representation theorem for C_α (cf. the representations (1.5.1) and (1.5.12) for C_1 and C_o, respectively).

THEOREM 2.3.5. For $0 < \alpha < 1$ a pgf P, with $P(0) > 0$, is in C_α iff P has the form

$$P(z) = \prod_{k=0}^{\infty} \frac{1-pQ(\alpha^k)}{1-pQ(\alpha^k z)} , \tag{2.3.8}$$

where $p \in (0,1)$ and Q is a pgf with $Q(0) = 0$. The representation (p,Q) is unique.

PROOF. If $P \in C_\alpha$, then P can be represented by (2.3.7), which takes the form (2.3.8) if we put $p = (1-\alpha)R_\alpha(1)$ and $Q(z) = zR_\alpha(z)/R_\alpha(1)$. Conversely, a pgf P of the form (2.3.8) has an R_α-function given by

$$R_\alpha(z) = \frac{p}{1-\alpha}\,\frac{1}{z}\,Q(z) , \tag{2.3.9}$$

which, as $Q(0) = 0$, is an abs mon function. Hence by lemma 2.3.3 $P \in C_\alpha$. The uniqueness of the representation (p,Q) follows from (2.3.9) and the uniqueness of R_α. □

In order to prove the general monotonicity property (relation (2.1.9)) by using lemma 2.3.3, we need a convenient relation between the R_α- and R_β-function of a pgf P. Such a relation is rather hard to find, but the following alternative proof of theorem 2.3.4 suggests relation (2.3.13), which is the key to the proof of the main theorem. On account of (2.3.2) we can write for $0 \le \alpha < 1$ (cf. (1.5.2))

$$(1-\alpha)\frac{d}{dz}[zR_\alpha(z)] = -\frac{d}{dz}[P(\alpha z)/P(z)] = \frac{P(\alpha z)}{P(z)}\{R_1(z) - \alpha R_1(\alpha z)\} ,$$

and so

$$(2.3.10)\qquad R_1(z) - \alpha R_1(\alpha z) = (1-\alpha)\frac{P(z)}{P(\alpha z)}\frac{d}{dz}[zR_\alpha(z)] \;.$$

Now, let $0 \le \alpha < 1$ and $P \in C_\alpha$. Then R_α is abs mon, which by (2.3.6) implies the abs mon of $P(z)/P(\alpha z)$. It follows that the left-hand side of (2.3.10), and hence R_1, is abs mon too, and so $P \in C_1$.

REMARK. Equating the coefficients of z^n in (2.3.10) we obtain the following analogue of (2.1.10): if $\{p_n\} \in C_\alpha$, then the $r_n(\alpha)$'s corresponding to $\{p_n\}$ satisfy (cf. corollary 2.3.8 and its proof)

$$(2.3.11)\qquad r_n(\alpha)/c_n(\alpha) \le r_n(1)/(n+1) \qquad (n \in \mathbb{N}_o) \;.$$

In the proof of the main theorem we shall need the following lemma.

LEMMA 2.3.6. If $P \in C_1$, then $P(z)/P(\gamma z)$ is abs mon for $0 \le \gamma \le 1$. In fact, $P(\gamma)P(z)/P(\gamma z)$ is a pgf in C_1.

PROOF. If $P \in C_1$, then P has the form (1.5.1), from which it follows that

$$P(\gamma)P(z)/P(\gamma z) = \exp[\mu\{Q(z) - Q(\gamma z) + Q(\gamma) - 1\}] \;.$$

For $0 \le \gamma \le 1$ this is again of the form (1.5.1), so $P(\gamma)P(z)/P(\gamma z) \in C_1$. □

THEOREM 2.3.7. C_α is nondecreasing in $\alpha \in [0,1]$, i.e. for all $\alpha,\beta \in [0,1]$

$$C_\alpha \subset C_\beta \text{ if } \alpha < \beta \;.$$

PROOF. The theorem has been already proved in the case $0 \le \alpha \le 1$, $\beta = 1$; so, in view of lemma 2.3.3, it remains to show that, if $0 \le \alpha < \beta < 1$ and if R_α is abs mon, then R_β is abs mon. To do so, from (2.3.2) we subtract the same equality with z replaced by βz, and we obtain

$$z(1-\alpha)\{R_\alpha(z) - \beta R_\alpha(\beta z)\} = \frac{P(\alpha\beta z)}{P(\beta z)} - \frac{P(\alpha z)}{P(z)} \;,$$

or, dividing by $P(\alpha z)$,

$$(2.3.12)\qquad \frac{z(1-\alpha)}{P(\alpha z)}\{R_\alpha(z) - \beta R_\alpha(\beta z)\} = \frac{P(\alpha\beta z)}{P(\alpha z)P(\beta z)} - \frac{1}{P(z)} \;.$$

The right-hand side of (2.3.12), and hence the left-hand side of (2.3.12), is symmetric in α and β, from which it follows that for $0 \le \alpha < 1$ and

$0 \leq \beta < 1$ we have

$$(2.3.13) \quad R_\beta(z) - \alpha R_\beta(\alpha z) = \frac{1-\alpha}{1-\beta}\frac{P(\beta z)}{P(\alpha z)}\{R_\alpha(z) - \beta R_\alpha(\beta z)\} .$$

Now suppose that $\alpha < \beta$ and $P \in \mathcal{C}_\alpha$. Then by theorem 2.3.4 $P \in \mathcal{C}_1$, and from lemma 2.3.6 we conclude that $P(z)/P(\{\alpha/\beta\}z)$, and so $P(\beta z)/P(\alpha z)$, is abs mon. From the abs mon of R_α we obtain the abs mon of $R_\alpha(z) - \beta R_\alpha(\beta z)$, so that by (2.3.13) it follows that $R_\beta(z) - \alpha R_\beta(\alpha z)$, with coefficients $(1-\alpha^{n+1})r_n(\beta)$, is abs mon. Hence R_β is abs mon, and the theorem is proved. □

Looking somewhat more precisely at the identity (2.3.13), we obtain a generalization of the inequalities (2.1.10) and (2.3.11).

COROLLARY 2.3.8. If $0 \leq \alpha_o < 1$ and if $P \in \mathcal{C}_{\alpha_o}$, then $r_n(\alpha)/c_n(\alpha)$, and hence $r_n(\alpha)$, is nondecreasing in $\alpha \in [\alpha_o, 1]$ for all $n \in \mathbb{N}_o$, i.e.

$$(2.3.14) \quad r_n(\alpha)/c_n(\alpha) \leq r_n(\beta)/c_n(\beta) \qquad (n \in \mathbb{N}_o;\ \alpha_o \leq \alpha \leq \beta \leq 1) .$$

PROOF. Let $P \in \mathcal{C}_{\alpha_o}$ and, in view of (2.3.11), $\alpha_o \leq \alpha \leq \beta < 1$. Then by theorem 2.3.7 also $P \in \mathcal{C}_\alpha$, and hence (cf. the proof of theorem 2.3.7) $P(\beta z)/P(\alpha z)$ can be written as

$$P(\beta z)/P(\alpha z) = \sum_{n=0}^{\infty} s_n(\alpha,\beta) z^n ,$$

where $s_n(\alpha,\beta) \geq 0$ for all n and $s_o(\alpha,\beta) = 1$. Now, equating the coefficients of z^n in (2.3.13), we see that

$$c_n(\alpha) r_n(\beta) = \sum_{k=0}^{n} c_k(\beta) r_k(\alpha) s_{n-k}(\alpha,\beta) \qquad (n \in \mathbb{N}_o) ,$$

from which (2.3.14) follows. Finally, as $c_n(\alpha) \leq c_n(\beta)$ $(n \in \mathbb{N}_o)$, we conclude from (2.3.14) that $r_n(\alpha) \leq r_n(\beta)$ $(n \in \mathbb{N}_o)$. □

We restate theorem 2.3.7 as a property that generalizes a result of Goldie (1967), who proved, in fact, our result in case $\alpha = 0$ and $\beta = 1$.

COROLLARY 2.3.9. If for a given distribution $\{p_n\}_o^\infty$, with $p_o > 0$, the quantities $r_n(\alpha)$, for a fixed $\alpha \in [0,1)$ and $n \in \mathbb{N}_o$ defined by

$$(2.3.15)\quad (1+\alpha+\alpha^2+\ldots+\alpha^n)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(\alpha) \qquad (n \in \mathbb{N}_0)\ ,$$

are all nonnegative, then also the quantities $r_n(\beta)$, defined by (2.3.15) with α replaced by β, are nonnegative for all $n \in \mathbb{N}_0$ and all β with $\alpha < \beta \le 1$.

2.4. *Further properties and examples*

In view of theorem 2.3.7 the classes C_α define a classification (cf. the end of section 1.2) of C_1. In this section we shall compare the properties of the separate C_α's with those known for C_1. To this end the characterization of C_α, given by lemma 2.3.3, is frequently used, and, for notational convenience, we denote by $R_\alpha^{(\gamma)}$ the R_α-function of a pgf, P_γ, depending on a parameter γ.
We start with a generalization of the closure property of C_1 (theorem 1.4.3).

THEOREM 2.4.1. For $0 \le \alpha \le 1$ the class C_α is closed under weak convergence, i.e. a pgf P, for which there exist pgf's $P_n \in C_\alpha$ $(n \in \mathbb{N})$ such that $P(z) = \lim_{n\to\infty} P_n(z)$ $(0 \le z \le 1)$, is in C_α.

PROOF. Let $0 \le \alpha < 1$, and let P be a pgf for which $P(z) = \lim_{n\to\infty} P_n(z)$ with $P_n \in C_\alpha$ $(n \in \mathbb{N})$. Then $R_\alpha^{(n)}$ is abs mon for all n, and as on account of (2.3.2) the R_α-function of P satisfies $R_\alpha(z) = \lim_{n\to\infty} R_\alpha^{(n)}(z)$, it follows by lemma 1.3.3(iv) that R_α is abs mon, too. Hence $P \in C_\alpha$. □

In the following theorem we state some properties of C_α that are well known, or trivial, if all C-classes are replaced by C_1.

THEOREM 2.4.2. For $0 \le \alpha \le 1$ the class C_α has the following properties.
(i) If $P \in C_\alpha$ and $0 \le \gamma \le 1$, then $P(\gamma z)/P(\gamma) \in C_\alpha$.
(ii) If $P \in C_\alpha$ and $0 \le \gamma \le 1$, then $P(z)^\gamma \in C_\alpha$.
(iii) If $P \in C_\alpha$ and $0 \le \gamma \le 1$, then $P_\gamma(z) := P(\gamma)P(z)/P(\gamma z) \in C_\alpha$.
(iv) If $P \in C_\alpha$, $n \in \mathbb{N}$ and $\alpha^{1/n} \le \gamma \le 1$, then

$$(2.4.1)\quad P_{n,\gamma}(z) := \prod_{k=0}^{n-1} P(\gamma^k z)/P(\gamma^k) \in C_\gamma\ .$$

PROOF. In all four cases we apply lemma 2.3.3.
(i) In fact, this result has already been proved in theorem 2.1.5, but it also follows from the following obvious relation:

$$(2.4.2)\qquad R_\alpha^{(\gamma)}(z) = \gamma R_\alpha(\gamma z)\ .$$

(ii) For the R_α-function $R_\alpha^{(\gamma)}$ of P^γ we have

$$\frac{d}{dz}[(1-\alpha)zR_\alpha^{(\gamma)}(z)] = -\frac{d}{dz}[(P(\alpha z)/P(z))^\gamma] =$$

$$= -\gamma(P(\alpha z)/P(z))^{\gamma-1}\frac{d}{dz}[P(\alpha z)/P(z)]\ ,$$

and so

$$(2.4.3)\qquad \frac{d}{dz}[zR_\alpha^{(\gamma)}(z)] = \gamma(P(z)/P(\alpha z))^{1-\gamma}\frac{d}{dz}[zR_\alpha(z)]\ .$$

Now, if $P \in \mathcal{C}_\alpha$, then $P \in \mathcal{C}_1$, and by lemma 2.3.6 we know that $P(\alpha)P(z)/P(\alpha z)$, and hence $\{P(\alpha)P(z)/P(\alpha z)\}^{1-\gamma}$ for $0 \le \gamma \le 1$, is a pgf in $\mathcal{C}_1$. It follows that $\{P(z)/P(\alpha z)\}^{1-\gamma}$ is abs mon for $0 \le \gamma \le 1$. As R_α is abs mon too, we obtain the abs mon of $R_\alpha^{(\gamma)}$ from (2.4.3).
(iii) First, note that, if $P \in \mathcal{C}_\alpha$ then for $0 \le \gamma \le 1$ P_γ is indeed a pgf (cf. lemma 2.3.6). Calculating the R_α-function of P_γ we obtain

$$(1-\alpha)zR_\alpha^{(\gamma)}(z) = 1 - \frac{P(\alpha z)}{P(\alpha\gamma z)}\,\frac{P(\gamma z)}{P(z)} = \frac{P(\gamma z)}{P(\alpha\gamma z)}\Big\{\frac{P(\alpha\gamma z)}{P(\gamma z)} - \frac{P(\alpha z)}{P(z)}\Big\}\ ,$$

and so

$$(2.4.4)\qquad R_\alpha^{(\gamma)}(z) = \frac{P(\gamma z)}{P(\alpha\gamma z)}\{R_\alpha(z) - \gamma R_\alpha(\gamma z)\}\ .$$

As according to lemma 2.3.6 $P(\gamma z)/P(\alpha\gamma z)$ is abs mon, we obtain the abs mon of $R_\alpha^{(\gamma)}$ from that of R_α, using (2.4.4).
(iv) We calculate the R_γ-function of $P_{n,\gamma}$, and we get

$$(2.4.5)\qquad R_\gamma^{(n,\gamma)}(z) = c_{n-1}(\gamma)R_{\gamma^n}(z)\ .$$

By hypothesis, this is abs mon for $\gamma^n = \alpha$, and hence by theorem 2.3.7 for $\gamma^n \ge \alpha$. □

Taking $\alpha = 0$ in part (iv) of the preceding theorem, we see that for every $n \in \mathbb{N}$ the product of the first n factors in the canonical representation (2.3.8) of a pgf in $\mathcal{C}_\alpha$ also is a pgf in $\mathcal{C}_\alpha$. We state this as a corollary.

COROLLARY 2.4.3. If $0 \le \alpha \le 1$, $n \in \mathbb{N}$, $0 \le p < 1$ and if Q is a pgf with $Q(0) = 0$, then

$$\prod_{k=0}^{n-1} \frac{1 - pQ(\alpha^k)}{1 - pQ(\alpha^k z)} \in C_\alpha \;. \tag{2.4.6}$$

Next we prove a property of the C_α's, showing that, though many inf div distributions are in $C_1 \setminus \bigcup_{0\le\alpha<1} C_\alpha$ (cf. example 2 on p. 56), the situation is not too bad: we prove that $\bigcup_{0\le\alpha<1} C_\alpha$ is dense in C_1 in the sense of weak convergence.

THEOREM 2.4.4. If $P \in C_1$, then there exist an increasing sequence $\{\alpha_n\}_1^\infty$, with $\lim_{n\to\infty} \alpha_n = 1$, and a sequence of pgf's $P_n \in C_{\alpha_n}$ $(n \in \mathbb{N})$, such that

$$P(z) = \lim_{n\to\infty} P_n(z) \qquad (|z| \le 1) \;. \tag{2.4.7}$$

PROOF. Let $P \in C_1$. Then by theorem 1.5.1 P has the form

$$P(z) = \exp[\mu(Q(z) - 1)] \;, \tag{2.4.8}$$

where $\mu > 0$ and Q is a pgf with $Q(0) = 0$. Take $\alpha_n = 1 - n^{-2}$, and for $n > \mu$ define the pgf's P_n by

$$P_n(z) := \prod_{k=0}^{n-1} \frac{1 - \{\mu/n\}Q(\alpha_n^k)}{1 - \{\mu/n\}Q(\alpha_n^k z)} \;.$$

From corollary 2.4.3 we know that $P_n \in C_{\alpha_n}$. We rewrite P_n as

$$P_n(z) = \{1 + \tfrac{\mu}{n}(Q(z) - 1)\}^n \prod_{k=0}^{n-1} \{1 + g_{k,n}(z)\} \;, \tag{2.4.9}$$

with

$$g_{k,n}(z) := \frac{n - \mu Q(\alpha_n^k)}{n - \mu Q(\alpha_n^k z)} \, \frac{1}{1 + \{\mu/n\}(Q(z) - 1)} - 1 =$$

$$= \mu \, \frac{1 - Q(\alpha_n^k) + Q(\alpha_n^k z) - Q(z) + \{\mu/n\}Q(\alpha_n^k z)(Q(z) - 1)}{[n - \mu Q(\alpha_n^k z)][1 + \{\mu/n\}(Q(z) - 1)]} \;.$$

It follows that for $|z| \le 1$, n sufficiently large and $k < n$ we have

$$|g_{k,n}(z)| \leq \frac{2\mu}{n}\{|1-Q(\alpha_n^k)| + |Q(\alpha_n^k z) - Q(z)| + \frac{2\mu}{n}\} .$$

Since for $|z| \leq 1$ and $k < n$

$$|Q(\alpha_n^k z) - Q(z)| = |\sum_{m=0}^{\infty} q_m(\alpha_n^{km} - 1)z^m| \leq \sum_{m=0}^{\infty} q_m(1-\alpha_n^{km}) =$$

$$= 1 - Q(\alpha_n^k) \leq 1 - Q(\alpha_n^n) ,$$

we obtain an upperbound for $|g_{k,n}(z)|$ that is independent of k and z:

$$|g_{k,n}(z)| \leq \frac{4\mu}{n}\{1 - Q(\alpha_n^n) + \frac{\mu}{n}\} .$$

Finally, from $\lim_{z\uparrow 1} Q(z) = 1$ and $\lim_{n\to\infty} \alpha_n^n = \lim_{n\to\infty} (1-n^{-2})^n = 1$ it follows that $\lim_{n\to\infty} Q(\alpha_n^n) = 1$, and so, for n sufficiently large,

(2.4.10) $\quad \forall_{k\in\{0,1,\ldots,n\}} \forall_{|z|\leq 1} \; |g_{k,n}(z)| \leq \varepsilon_n$, with $\varepsilon_n = o(\frac{1}{n})$ $(n \to \infty)$.

Now, observing that for n sufficiently large

$$|\log\{1 + g_{k,n}(z)\}| \leq -\log\{1 - |g_{k,n}(z)|\} \leq$$

$$\leq |g_{k,n}(z)|/\{1 - |g_{k,n}(z)|\} \leq 2|g_{k,n}(z)| ,$$

we can estimate as follows:

$$|\log \prod_{k=0}^{n-1} \{1 + g_{k,n}(z)\}| \leq \sum_{k=0}^{n-1} |\log\{1+g_{k,n}(z)\}| \leq 2 \sum_{k=0}^{n-1} |g_{k,n}(z)| \leq$$

$$\leq 2n\varepsilon_n,$$

which by (2.4.10) becomes arbitrarily small for $n \to \infty$. Hence

$$\lim_{n\to\infty} \prod_{k=0}^{n-1} \{1 + g_{k,n}(z)\} = 1 ,$$

and (2.4.7) immediately follows from (2.4.8) and (2.4.9). □

Before turning to some examples, we mention a characterization of $\mathcal{C}_\alpha$ in terms of $\mathcal{C}_0$.

THEOREM 2.4.5. Let P be a pgf, with $P(0) > 0$, let $0 < \alpha < 1$ and let P_α be defined by

(2.4.11) $P_\alpha(z) := P(\alpha)P(z)/P(\alpha z)$.

Then $P \in C_\alpha$ iff the function P_α is a pgf in C_0.

PROOF. First we note that if $P \in C_\alpha$ then by lemma 2.3.6 P_α is a pgf. Next, consider the case that P_α is a pgf. Then the R_0-function of P_α is given by

$$R_0^{(\alpha)}(z) = \frac{1}{z}\{1 - P_\alpha(0)/P_\alpha(z)\} = (1-\alpha)R_\alpha(z) \ .$$

Hence $R_0^{(\alpha)}$ is abs mon iff R_α is abs mon, from which the theorem follows. □

Using theorem 2.3.7 we can now improve theorem 2.4.2(iii) in the following way.

COROLLARY 2.4.6. If $P \in C_\alpha$ and if $0 \le \gamma \le 1$, then P_γ, defined by (2.4.11), is a pgf, while $P_\gamma \in C_\alpha$ for $0 \le \gamma < \alpha$, and $P_\gamma \in C_0$ for $\alpha \le \gamma \le 1$.

Theorem 2.4.5 is not only convenient to prove properties of the C_α's, e.g. the properties in theorem 2.4.2(i), (ii) and (iv), but it also provides examples of distributions in C_α. This will be clear from the following obvious reformulation.

COROLLARY 2.4.7. Let Q be a pgf with $Q(0) > 0$, and let $0 < \alpha < 1$. Then $Q \in C_0$ iff $P(z) := \prod_{k=0}^{\infty} Q(\alpha^k z)/Q(\alpha^k)$ is a pgf in C_α.

Combining corollaries 2.4.3 and 2.4.7, we get for all $\alpha \in (0,1)$

$$(2.4.12) \qquad Q \in C_0 \Rightarrow \prod_{k=0}^{n-1} Q(\alpha^k z)/Q(\alpha^k) \in C_\alpha \qquad (n \in \mathbb{N} \cup \{\infty\}) \ .$$

This will be used presently to construct examples of distributions in C_α. Note that on account of theorem 1.5.13 for Q we may take the pgf of any comp mon or log-convex distribution.

Now, we list a number of examples of (inf div) lattice distributions in the various classes C_α. If the proofs of the statements made are straightforward, we only give brief indications.

1. Consider the *Poisson distribution* with parameter $\mu > 0$

$$p_n = \frac{\mu^n}{n!} e^{-\mu} \qquad (n \in \mathbb{N}_0), \text{ or } P(z) = \exp[\mu(z-1)] .$$

Calculating the $r_n(\alpha)$'s corresponding to $\{p_n\}$ we see that

$$\exp[\mu(z-1)] \notin \bigcup_{0\le\alpha<1} \mathcal{C}_\alpha \qquad (\mu > 0) . \tag{2.4.13}$$

2. The *negative-binomial distribution* with parameters $p \in [0,1)$ and $u > 0$ is defined by

$$p_n = \binom{n+u-1}{n}(1-p)^u p^n \qquad (n \in \mathbb{N}_0), \text{ or } P(z) = \left(\frac{1-p}{1-pz}\right)^u . \tag{2.4.14}$$

For $u = 1$ $\{p_n\}$ is comp mon, hence by theorem 2.4.2(ii) we have

$$\left(\frac{1-p}{1-pz}\right)^u \in \mathcal{C}_0 \qquad (0 < u \le 1;\ 0 \le p < 1) . \tag{2.4.15}$$

However, if u exceeds 1, the classes $\mathcal{C}_\alpha$ $(0 < \alpha < 1)$ are skipped over:

$$\left(\frac{1-p}{1-pz}\right)^u \notin \bigcup_{0\le\alpha<1} \mathcal{C}_\alpha \qquad (u > 1;\ 0 \le p < 1) . \tag{2.4.16}$$

To prove this, we put $u = 1 + \varepsilon$ with $\varepsilon > 0$ and obtain for the R_α-function of P

$$(1-\alpha)zR_\alpha(z) = 1 - \left(\frac{1-pz}{1-p\alpha z}\right)^{1+\varepsilon} = 1 - \left\{1 - \frac{p(1-\alpha)z}{1-p\alpha z}\right\}^{1+\varepsilon} .$$

Expanding this in a power series in z and putting $x = (1-\alpha)/\alpha$, we see that for $n \in \mathbb{N}$

$$(1-\alpha)(p\alpha)^{-n} r_{n-1}(\alpha) = (-1)^{n-1} \sum_{k=1}^{n} \binom{1+\varepsilon}{k}\binom{-k}{n-k} x^k =$$

$$= \sum_{k=1}^{n} (-1)^{k-1} \binom{1+\varepsilon}{k}\binom{n-1}{k-1} x^k .$$

For the case $0 < \varepsilon \le 1$ it follows that

$$(1-\alpha)(p\alpha)^{-n} r_{n-1}(\alpha) = (1+\varepsilon)x - (1+\varepsilon)\varepsilon \sum_{k=2}^{n} \frac{1}{k(k-1)} \binom{k-2-\varepsilon}{k-2}\binom{n-1}{k-1} x^k \le$$

$$\le (1+\varepsilon)x - \tfrac{1}{2}(n-1)(1+\varepsilon)\varepsilon x^2 ,$$

which tends to $-\infty$ for $n \to \infty$. Hence (2.4.16) is proved for $1 < u \le 2$.

Next, suppose $u > 2$ and that $\{(1-p)/(1-pz)\}^u \in \mathcal{C}_\alpha$ for some $\alpha \in [0,1)$. Then according to theorem 2.4.2(ii) it would follow that for all $v \in (1,2]$

$$\left(\frac{1-p}{1-pz}\right)^v = \left\{\left(\frac{1-p}{1-pz}\right)^u\right\}^{v/u} \in \mathcal{C}_\alpha ,$$

but this contradicts the result above.

3. Consider the *boundary distribution* $\{\tilde{p}_n\}$ with parameters $\alpha_o \in [0,1]$ and $\gamma \in (0,(1-\alpha_o)^{-1})$ (cf. definition 2.2.2); it satisfies

$$(2.4.17)\quad \tilde{p}_n = \tilde{p}_o\gamma^n \prod_{k=0}^{n-1} c_k(\alpha_o)^{-1} \qquad (n \in \mathbb{N}) .$$

For $\alpha_o = 0$ we get the geometric distribution with parameter γ, and for $\alpha_o = 1$ the Poisson distribution with parameter γ, which has been considered in example 1. Now, take $\alpha_o \in (0,1)$. By (2.2.2) and (2.2.3) we know that $\{\tilde{p}_n\} \in \mathcal{C}_{\alpha_o}$ with $R_{\alpha_o} \equiv \gamma$. Hence by (2.3.7) we have for the pgf $\tilde{P}$ of $\{\tilde{p}_n\}$

$$(2.4.18)\quad \tilde{P}(z) = \prod_{k=0}^{\infty} \frac{1-(1-\alpha_o)\gamma\alpha_o^k}{1-(1-\alpha_o)\gamma\alpha_o^k z} \qquad (|z| \le 1) .$$

In order to check whether $\tilde{P} \in \mathcal{C}_\alpha$ for some $\alpha < \alpha_o$, we calculate

$$(1-\alpha)zR_\alpha(z) = 1 - \prod_{k=0}^{\infty} \frac{1-(1-\alpha_o)\gamma\alpha_o^k z}{1-(1-\alpha_o)\gamma\alpha_o^k \alpha z} ,$$

from which it follows that for $m \in \mathbb{N}$

$$(1-\alpha_o^m)zR_{\alpha_o^m}(z) = 1 - \prod_{k=0}^{m-1} \{1-(1-\alpha_o)\gamma\alpha_o^k z\} .$$

As this is a polynomial in z of degree m, we must have $r_n(\alpha_o^m) = 0$ for $n = m, m+1, \ldots$, or, equivalently,

$$\forall_{n\in\mathbb{N}}\, \forall_{m\in\{1,\ldots,n\}}\; r_n(\alpha_o^m) = 0 .$$

From (2.4.17) and the recurrence relations (2.3.1), however, we see that $r_n(\alpha)$ is a polynomial in α of degree n with leading coefficient $\gamma^{n+1} \prod_{k=0}^{n} c_k(\alpha_o)^{-1}$, so that

$$(2.4.19)\qquad r_n(\alpha) = \gamma^{n+1} \prod_{k=1}^{n} \frac{\alpha - \alpha_o^k}{c_k(\alpha_o)} \qquad (n \in \mathbb{N}_o;\ 0 \le \alpha < 1)\ .$$

From this we obtain (take $n = 1$) for every choice of the parameters α_o and γ

$$(2.4.20)\qquad \{\tilde{p}_n\} \in C_\alpha \Leftrightarrow \alpha \ge \alpha_o\ .$$

4. The *logarithmic distribution* with parameter $\theta \in (0,1)$, given by

$$(2.4.21)\qquad p_n = \frac{1}{b}\,\frac{1}{n+1}\,\theta^{n+1} \qquad (n \in \mathbb{N}_o),\ \text{or}\ P(z) = -\frac{1}{bz}\log(1-\theta z)\ ,$$

where $b := -\log(1-\theta)$, is comp mon, and hence in C_o. The following related distribution, which will be called the *semi-logarithmic distribution* with parameters $a > 0$ and $\theta \in (0,1)$, is more interesting:

$$(2.4.22)\qquad p_o = \frac{a}{1+a}\ ,\ p_n = \frac{1}{(1+a)b}\,\frac{1}{n}\,\theta^n \qquad (n \in \mathbb{N})\ ,$$

where again $b := -\log(1-\theta)$. Its pgf P is given by

$$(2.4.23)\qquad P(z) = \frac{a}{1+a}\{1 - \frac{1}{ab}\log(1-\theta z)\}\ .$$

This distribution is also considered by Katti (1967), who proves that $\{p_n\}$ is inf div iff $ab \ge 1$, and by Steutel (1970), who notes that $\{p_n\}$ is log-convex (and so in C_o) if $ab \ge 2$. For the case $1 \le ab \le 2$ we now prove the following result:

$$(2.4.24)\qquad \{p_n\} \in C_\alpha \Leftrightarrow ab \ge \frac{2}{1+\alpha} \qquad (0 \le \alpha \le 1)\ .$$

To this end we consider the recurrence relations (2.3.1) rather than the relation (2.3.2) for pgf's, and obtain

$$c_n(\alpha)\frac{\theta^{n+1}}{n+1} = ab\ r_n(\alpha) + \sum_{k=1}^{n} \frac{\theta^k}{k}\, r_{n-k}(\alpha) \qquad (n \in \mathbb{N}_o)\ ,$$

or, defining $\bar{r}_n(\alpha) := \theta^{-n-1} r_n(\alpha)$,

$$(2.4.25)\qquad 1 = \frac{n+1}{c_n(\alpha)}\{ab\ \bar{r}_n(\alpha) + \sum_{k=1}^{n} \frac{1}{k}\,\bar{r}_{n-k}(\alpha)\} \qquad (n \in \mathbb{N}_o)\ .$$

For $n = 1$ we get $ab\ \bar{r}_1(\alpha) = \frac{1}{2}(1+\alpha) - (ab)^{-1}$; hence the condition in the right-hand side of (2.4.24) is necessary. To prove its sufficiency, from (2.4.25) we subtract the same relation with n replaced by $n-1$, and obtain

$$ab\,\bar{r}_n(\alpha) = \sum_{k=2}^{n} \left[\frac{1}{k-1}\,\frac{n}{n+1}\,\frac{c_n(\alpha)}{c_{n-1}(\alpha)} - \frac{1}{k}\right]\bar{r}_{n-k}(\alpha) +$$

$$+ \left[\frac{n}{n+1}\,\frac{c_n(\alpha)}{c_{n-1}(\alpha)}\,ab - 1\right]\bar{r}_{n-1}(\alpha) \;.$$

By the monotonicity of $c_n(\alpha)$ in n the coefficients of the $\bar{r}_{n-k}(\alpha)$'s are all positive; therefore it is sufficient to prove that under the condition in (2.4.24) the following holds:

$$ab \geq \frac{n+1}{n}\,\frac{1-\alpha^n}{1-\alpha^{n+1}} =: d_n(\alpha) \qquad (n \in \mathbb{N};\; 0 < \alpha < 1) \;.$$

This immediately follows from the fact that $d_1(\alpha) = 2/(1+\alpha)$ and that $d_n(\alpha) \leq d_1(\alpha)$ for all $n \in \mathbb{N}$ and all $\alpha \in (0,1)$. The last assertion is easily verified by noting that for the function f, defined on (0,1) by $f(\alpha) := 2(1-\alpha^{n+1})/(n+1) - (1+\alpha)(1-\alpha^n)/n$, one has $f(0) > 0$, $f(1) = 0$, $f' \leq 0$ on (0,1).

5. A product of two geometric pgf's can be written as

$$\text{(2.4.26)} \quad P(z) = \frac{1-p}{1-pz}\,\frac{1-p\gamma}{1-p\gamma z}\,, \text{ or } p_n = (1-p)(1-p\gamma)c_n(\gamma)p^n \qquad (n \in \mathbb{N}_0)\;,$$

where $0 \leq p < 1$ and $0 \leq \gamma \leq 1$. From the R_α-function of P we find

$$r_o(\alpha) = (1+\gamma)p,\; r_n(\alpha) = \{c_{n+1}(\gamma)\alpha - \gamma c_{n-1}(\gamma)\}p^{n+1}\alpha^{n-1} \qquad (n \in \mathbb{N})\,.$$

Now, the zero $\alpha_n := \gamma c_{n-1}(\gamma)/c_{n+1}(\gamma)$ of $r_n(\alpha)$ is nondecreasing in n and $\lim_{n\to\infty} \alpha_n = \gamma$, from which it follows that

$$\text{(2.4.27)} \quad \frac{1-p}{1-pz}\,\frac{1-p\gamma}{1-p\gamma z} \in \mathcal{C}_\alpha \Leftrightarrow \alpha \geq \gamma \qquad (0 \leq p < 1;\; 0 \leq \gamma \leq 1) \;.$$

6. For a product of three geometric pgf's we find in a similar way

$$\text{(2.4.28)} \quad \frac{1-p}{1-pz}\,\frac{1-p\gamma}{1-p\gamma z}\,\frac{1-p\gamma\delta}{1-p\gamma\delta z} \in \mathcal{C}_{\max(\gamma,\delta)} \qquad (0 \leq p < 1;\; 0 \leq \gamma \leq 1;\; 0 \leq \delta \leq 1)\,.$$

The following examples are special cases of (2.4.12).

7. $$\prod_{k=0}^{n-1} \frac{1-p\alpha^k}{1-p\alpha^k z} \in \mathcal{C}_\alpha \qquad (0 \leq p < 1;\; n \in \mathbb{N} \cup \{\infty\}) \;.$$

Note that for $n = \infty$ we get the boundary distribution from example 3.

8. $$\prod_{k=0}^{n-1} \frac{\gamma - \log(1-\theta\alpha^k z)}{\gamma - \log(1-\theta\alpha^k)} \in C_\alpha \qquad (0 < \theta < 1;\ \gamma \geq 2;\ n \in \mathbb{N} \cup \{\infty\}) .$$

9. Lamperti (1958) notes that tan z/(z tan 1) is a pgf in C_o. Hence

$$\prod_{k=0}^{n-1} \frac{\tan(\alpha^k z)}{z \tan \alpha^k} \in C_\alpha .$$

10. It is easy to see that the quotient of two geometric pgf's with parameters p and q, respectively, is again a pgf iff $p \geq q$. In that case it belongs necessarily to C_o. Hence we have

$$\prod_{k=0}^{n-1} \frac{1-p\alpha^k}{1-p\alpha^k z} \Big/ \frac{1-p\gamma\alpha^k}{1-p\gamma\alpha^k z} \in C_\alpha \qquad (0 \leq p < 1;\ 0 \leq \gamma \leq 1;\ n \in \mathbb{N} \cup \{\infty\}).$$

The characterization of C_α by C_o, given in theorem 2.4.5, only holds for $\alpha < 1$. In the case $\alpha = 1$, however, there also exists such a characterization; in fact, we have the following analogue of theorem 2.4.5 and corollary 2.4.7.

THEOREM 2.4.8. Let P be a pgf with $P(0) > 0$. Then
(i) $P \in C_1$ iff $\{1 - \log P(z)\}^{-1}$ is a pgf in C_o,
(ii) $P \in C_o$ iff $\exp[1 - P(z)^{-1}]$ is a pgf in C_1.

PROOF. The theorem immediately follows from the following two observations. If P is compound-Poisson-(μ,Q), then

$$\{1 - \log P(z)\}^{-1} = \{1 - \mu(Q(z) - 1)\}^{-1} = \{1 + \mu - \mu Q(z)\}^{-1} ,$$

which is compound-geometric-(p,Q) with $p = \mu/(1+\mu)$.
Conversely, if P is compound-geometric-(p,Q), then

$$\exp[1 - P(z)^{-1}] = \exp[1 - (1 - pQ(z))/(1-p)] ,$$

which is compound-Poisson-(μ,Q) with $\mu = p/(1-p)$. □

Using simple properties of C_1, from the preceding theorem we obtain some operations, under which C_o is closed. Closure under operations on the p_n's themselves will be discussed in section 4.1.

THEOREM 2.4.9.

(i) If $P \in \mathcal{C}_o$ and $\mu > 0$, then $\mu/\{\mu - \log P(z)\} \in \mathcal{C}_o$.

(ii) If $P \in \mathcal{C}_o$ and $\gamma > 0$, then $P(z)/\{\gamma + (1-\gamma)P(z)\} \in \mathcal{C}_o$.

(iii) If $P \in \mathcal{C}_o$ and $0 \le p \le 1$, then $p + (1-p)P(z) \in \mathcal{C}_o$.

(iv) If $P_1 \in \mathcal{C}_o$ and $P_2 \in \mathcal{C}_o$, then $P_1(z)P_2(z)/\{P_1(z) + P_2(z) - P_1(z)P_2(z)\} \in \mathcal{C}_o$.

PROOF.

(i) If $P \in \mathcal{C}_o$, then by theorem 1.4.4 we have $P^{1/\mu} \in \mathcal{C}_1$ for all $\mu > 0$. The result follows by application of theorem 2.4.8(i).

(ii) Using theorem 2.4.8, we see that if $P \in \mathcal{C}_o$, then $Q(z) := \exp[1 - P(z)^{-1}]$, and hence for all $\gamma > 0$ Q^γ, is a pgf in $\mathcal{C}_1$. It follows that

$$\{1 - \log Q(z)^\gamma\}^{-1} = \{1 - \gamma(1 - P(z)^{-1})\}^{-1} = \frac{P(z)}{\gamma + (1-\gamma)P(z)}$$

is a pgf in $\mathcal{C}_o$.

(iii) Let $P \in \mathcal{C}_o$ and $0 \le p \le 1$. Then for the R_o-function $R_o^{(p)}$ of $p + (1-p)P$ we have

$$zR_o^{(p)}(z) = 1 - \frac{p + (1-p)P(0)}{p + (1-p)P(z)} = (1-p)\frac{P(z)}{p + (1-p)P(z)}\, zR_o(z) \; ,$$

which is abs mon because of (ii). Hence $p + (1-p)P \in \mathcal{C}_o$.

(iv) If $P_j \in \mathcal{C}_o$ $(j = 1,2)$, then also $Q_j(z) := \exp[1 - P_j(z)^{-1}] \in \mathcal{C}_1$ $(j = 1,2)$. It follows that $Q_1Q_2 \in \mathcal{C}_1$, and hence the following function is a pgf in $\mathcal{C}_o$:

$$\{1 - \log Q_1(z)Q_2(z)\}^{-1} = \{1 - [1 - P_1(z)^{-1}] - [1 - P_2(z)^{-1}]\}^{-1} =$$

$$= P_1(z)P_2(z)/\{P_1(z) + P_2(z) - P_1(z)P_2(z)\} \; . \qquad \Box$$

As in the case $\alpha < 1$, it is possible, by use of theorem 2.4.8, to construct examples of inf div distributions from specially chosen distributions in $\mathcal{C}_o$, and conversely. In the following theorem we mention a general class of inf div distributions, obtained in such a way.

THEOREM 2.4.10. For every df G on $[0,1)$ the function Q, defined by

$$(2.4.29) \quad Q(z) := \exp\Big[1 - \Big\{\int_{[0,1)} \frac{1-p}{1-pz}\, dG(p)\Big\}^{-1}\Big] \; ,$$

is a pgf in $\mathcal{C}_1$.

PROOF. According to lemma 1.5.11 the pgf of a comp mon lattice distribution $\{p_n\}_0^\infty$ has the form

$$P(z) = \sum_{n=0}^{\infty} \int_{[0,1)} (1-p)p^n dG(p) z^n = \int_{[0,1)} \frac{1-p}{1-pz} dG(p) ,$$

where G is a df on $[0,1)$. As by theorem 1.5.13 $P \in \mathcal{C}_0$, the theorem follows from theorem 2.4.8(ii). □

Next we list a few examples, which are all obtained by using theorem 2.4.8.

1. Considering the semi-logarithmic distribution (cf. (2.4.22)), we get

$$\exp\left[\frac{-b - \log(1-\theta z)}{ab - \log(1-\theta z)}\right] \in \mathcal{C}_1 \quad \text{iff } ab \geq 2 ,$$

where $a > 0$, $0 < \theta < 1$ and $b := -\log(1-\theta)$.

2. As noted in example 9, p. 60, $\tan z/(z \tan 1)$ is a pgf in $\mathcal{C}_0$. Hence

$$\exp[1 - z \tan 1/\tan z] \in \mathcal{C}_1 .$$

3. Taking the semi-logarithmic distribution with $ab = 1$, we get

$$\left\{1 - \log \frac{1 - \log(1-\theta z)}{1 - \log(1-\theta)}\right\}^{-1} \in \mathcal{C}_0 .$$

4. As $\frac{1+pz}{1+p}\frac{1-p}{1-pz}$ is a pgf in $\mathcal{C}_1$ (and not in $\mathcal{C}_\alpha$ for $\alpha < 1$), we have

$$\left\{1 - \log \frac{1-p}{1+p}\frac{1+pz}{1-pz}\right\}^{-1} \in \mathcal{C}_0 .$$

Finally, we mention a property of $\mathcal{C}_0$ that ensures the product of a geometric pgf and a pgf from a large subclass of $\mathcal{C}_0$ to be again in $\mathcal{C}_0$. Note, that in general the product of two $\mathcal{C}_0$-pgf's does not belong to $\mathcal{C}_0$ (cf. (2.4.27)). This property is suggested by replacing the exponential distribution by its discrete analogue, the geometric distribution, in a similar property of the analogue of $\mathcal{C}_0$ for distributions on $[0,\infty)$, the class $\mathcal{F}_\infty$ (cf. theorem 5.4.19).

THEOREM 2.4.11. If p, p_1 and $p_2 \in [0,1)$ and if Q is a pgf, then

$$(2.4.30) \quad P(z) := \frac{1-p_1}{1-p_1 z}\frac{1-p}{1-p\{(1-p_2)/(1-p_2 z)\}Q(z)} \in \mathcal{C}_0 \quad \text{if } p_1 \leq p_2 .$$

PROOF. It is no restriction to suppose that $Q(0) = 0$, as a pgf P of the form (2.4.30) can be written in the same form with different p and Q, such that $Q(0) = 0$. For the R_o-function of P we can write now

$$zR_o(z) = 1 - \frac{P(0)}{P(z)} = 1 - (1-p_1z)\{1-p\,\frac{1-p_2}{1-p_2z}\,Q(z)\} =$$

$$= p_1z + p(1-p_2)\frac{1-p_1z}{1-p_2z}\,Q(z) =$$

$$= p_1z + p(1-p_2)Q(z) + p(p_2-p_1)z\,\frac{1-p_2}{1-p_2z}\,Q(z) \quad ,$$

which is an abs mon function, if $p_1 \le p_2$. Hence $P \in C_o$. □

2.5. *Other classifications*

The classes C_α, introduced by means of recurrence relations, were seen to consist of infinite products of certain compound geometric pgf's. We will now start from classes of similar pgf's, the classes C_o^u of the *compound negative-binomial* pgf's with parameter $u \ge 0$ (cf. (2.4.14)). Also these classes define a classification of C_1, but there is no characterization by means of recurrence relations of type (2.1.4).

DEFINITION 2.5.1. For $u \ge 0$ a lattice distribution $\{p_n\}_o^\infty$, with $p_o > 0$, is said to be in the class C_o^u if its pgf P has the form

(2.5.1) $\quad P(z) = \{\frac{1-p}{1-pQ(z)}\}^u \quad ,$

where $0 \le p < 1$ and Q is a pgf with $Q(0) = 0$.

Clearly, C_o^o only consists of the degenerate distribution at zero, $C_o^1 = C_o$ and $C_o^u \subset C_1$ for all $u \ge 0$. The following obvious criterion is useful.

LEMMA 2.5.2. Let P be a pgf with $P(0) > 0$, and let $u > 0$. Then $P \in C_o^u$ iff $P^{1/u}$ is a pgf in C_o.

It is now easily shown that the classes C_o^u define a classification of the inf div lattice distributions.

THEOREM 2.5.3. C_o^u is nondecreasing in $u \in [0,\infty)$, i.e. for all u and $v \in [0,\infty)$

$$C_o^u \subset C_o^v \text{ if } u < v .$$

PROOF. Let $0 < u < v$ and $P \in C_o^u$. Then $P^{1/u}$ is a pgf in C_o, so that according to theorem 2.4.2(ii) also $P^{1/v} = (P^{1/u})^{u/v}$ is a pgf in C_o. Hence $P \in C_o^v$.□

In fact, interpolation between $(1-p)/(1-pQ(z)) \in C_o$ and $\exp[\mu(Q(z)-1)] \in C_1$ is most easily achieved by considering pgf's of the form

$$P_u(z) := \left\{\frac{1-\mu/u}{1-(\mu/u)Q(z)}\right\}^u \qquad (u > \mu) .$$

Using these pgf's we easily see that, like $\bigcup_{0\leq\alpha<1} C_\alpha$, $\bigcup_{u>0} C_o^u$ is dense in C_1. We state this in the following theorem.

THEOREM 2.5.4. If $P \in C_1$, then there exists a sequence of pgf's $P_n \in C_o^n$ ($n \in \mathbb{N}$) such that $P(z) = \lim_{n\to\infty} P_n(z)$ for $|z| \leq 1$.

Although little can be said about the relation between the C_o^u's and the C_α's (cf. (2.4.16)), the C_o^u's possess the same sort of properties the C_α's have. In this respect the functions $P^{1/u}$ for C_o^u, and $P(z)/P(\alpha z)$ for C_α, play analogous roles. The following properties of C_o^u are very similar to those of C_α (cf. theorem 2.4.2), and are easily verified by using lemma 2.5.2 and theorem 2.4.2 with $\alpha = 0$.

THEOREM 2.5.5. Let $u > 0$.

(i) If a pgf P is the limit of a sequence of pgf's $P_n \in C_o^u$, then also $P \in C_o^u$.

(ii) If $P \in C_o^u$ and $0 \leq \gamma \leq 1$, then $P(\gamma z)/P(\gamma) \in C_o^u$.

(iii) If $P \in C_o^u$ and $\gamma \geq 0$, then $P^\gamma \in C_o^{\gamma u}$.

(iv) If $P \in C_o^u$ and $0 \leq \gamma \leq 1$, then $P(\gamma)P(z)/P(\gamma z) \in C_o^u$.

Next, we briefly consider classes of gf's that turn out to be closely related to the classes C_o^u. They are suggested by the characterization of C_1 by means of the abs mon of $R_1(z) = P'(z)/P(z)$ (cf. theorem 1.5.3) and the following characterization of C_o.

LEMMA 2.5.6. A pgf P, with $P(0) > 0$, is in C_o iff $-\frac{d}{dz}P(z)^{-1} = P'(z)/P(z)^2$ is an abs mon function.

PROOF. According to theorem 1.5.8 P is in C_o iff $R_o(z) = z^{-1}\{1-P(0)/P(z)\}$ is abs mon. Clearly, this is equivalent to the abs mon of $\frac{d}{dz}[zR_o(z)]$, and the lemma is proved. □

DEFINITION 2.5.7. For $\gamma \in \mathbb{R}$ a gf P, with $P(0) > 0$ and $P(1) = 1$, is said to be in the class H_γ if the function S_γ, defined by

$$(2.5.2) \qquad S_\gamma(z) := P'(z)/P(z)^{1+\gamma} ,$$

is abs mon.

The function S_γ in (2.5.2) can be written in the following form:

$$(2.5.3) \qquad S_\gamma(z) = \begin{cases} \dfrac{d}{dz} \log P(z), & \text{if } \gamma = 0 \\ -\dfrac{1}{\gamma}\dfrac{d}{dz} P(z)^{-\gamma}, & \text{if } \gamma \neq 0 . \end{cases}$$

Although we do not require the coefficients of a gf $P \in H_\gamma$ to be nonnegative, the classes H_γ, with $\gamma \geq 0$, turn out to contain only pgf's.
Considering first the case $\gamma = 0$, we solve (2.5.3) for P and obtain

$$(2.5.4) \qquad P(z) = P(0)\exp[\int_{(0,z]} S_o(u)du] .$$

It follows that if $P \in H_o$ then by lemma 1.3.3(v) P is a pgf. Hence, as $S_o = R_1$, we conclude that

$$(2.5.5) \qquad H_o = C_1 .$$

Denoting the coefficients in the power-series expansion of S_γ by $s_n(\gamma)$ ($n \in \mathbb{N}_o$), we obtain from (2.5.2) for all $\gamma \in \mathbb{R}$ a characterization of H_γ by means of relations that, like (2.1.4), generalize the known recurrence relations (2.1.1) for C_1, but are much less attractive than (2.1.4).

LEMMA 2.5.8. For $\gamma \in \mathbb{R}$ a sequence $\{p_n\}_o^\infty$, with $p_o > 0$ and $\Sigma p_n = 1$, is in H_γ iff there exist $s_n(\gamma) \geq 0$ ($n \in \mathbb{N}_o$) such that

$$(2.5.6) \qquad (n+1)p_{n+1} = \sum_{k=0}^{n} p_k^{*(1+\gamma)} s_{n-k}(\gamma) \qquad (n \in \mathbb{N}_o) .$$

The case $\gamma < 0$ is not very interesting, as is apparent from the following properties, which we give without their, simple, proofs.

LEMMA 2.5.9. If $\gamma < 0$, then a gf P, with $P(0) > 0$ and $P(1) = 1$, is in H_γ iff the function $P^{|\gamma|}$ is abs mon.

LEMMA 2.5.10.

(i) For all $\gamma < 0$: $C_1 = H_o \subset H_\gamma$.

(ii) For $n \in \mathbb{N}$: $H_{-1/n} = \{\text{pgf } P \mid P(0) > 0 \text{ and } P \text{ is n-div}\}$.

(iii) For $n \in \mathbb{N}$: $H_{-n} = \{\text{gf } P \mid P(0) > 0 \text{ and } P^n \text{ is a pgf}\}$.

The case $\gamma > 0$ is more interesting, because $(H_\gamma \mid \gamma \geq 0)$ turns out to define a classification of C_1.

THEOREM 2.5.11. For $\gamma > 0$ the following relation holds:

$$H_\gamma = C_o^{1/\gamma} .$$

In fact, if $\gamma > 0$ then a gf P, with $P(0) > 0$ and $P(1) = 1$, is in H_γ iff P^γ is a pgf in C_o.

PROOF. Let $\gamma > 0$, and let P be a gf with $P(0) > 0$ and $P(1) = 1$.
First we show that if $P \in H_\gamma$ then P^γ is a pgf. So, let S_γ be abs mon. Solving (2.5.3) for P^γ, we obtain

$$P(z)^\gamma = P(0)^\gamma\{1 - \gamma P(0)^\gamma \int_{(0,z]} S_\gamma(u)du)\}^{-1} ,$$

from which by lemma 1.3.3(v) we see that P^γ is abs mon. Since the S_o-function of P can be written as

$$S_o(z) = P'(z)/P(z) = S_\gamma(z)P(z)^\gamma ,$$

it follows that S_o is abs mon, and hence by (2.5.5) that P is an inf div pgf. In view of theorem 1.4.4 we conclude that also P^γ is a pgf.
Next, consider the case that P^γ is a pgf. From (2.5.3) the following relation between the S_1-function $S_1^{(\gamma)}$ of P^γ and the S_γ-function S_γ of P is easily seen to hold:

$$S_1^{(\gamma)}(z) = \gamma S_\gamma(z) ,$$

from which by definition it follows that $P \in H_\gamma$ iff $P^\gamma \in H_1$. On the other hand by lemma 2.5.6 we have $P^\gamma \in C_o$ iff $P^\gamma \in H_1$, and hence the theorem is proved. □

Thus, the H_γ's yield the same classification of C_1 as the C_o^u's. For instance, it follows that a gf P in H_γ, with $\gamma > 0$, has the (canonical) representation (2.5.1) with $u = 1/\gamma$, and that the compound negative-binomial lattice distributions with parameter $u = 1/\gamma$ can be characterized by means of the relations (2.5.6).

CHAPTER 3

DECOMPOSITIONS OF LATTICE DISTRIBUTIONS

The classes $\mathcal{C}_\alpha$ from the previous chapter lead us to consider in general the pgf's P that have $P(\alpha z)/P(\alpha)$ as a factor. These pgf's, which will be called α-decomposable[1], are briefly studied in section 1 together with the somewhat larger class of the so-called α-factorizable pgf's P, which are of the form $Q(z)Q(\alpha z)/Q(\alpha)$ with Q a pgf.
In view of a notational analogy to the well-known self-decomposable distributions (on $\mathbb{R}$; cf. the end of section 1), also called distributions of class $\mathcal{L}$, it is then natural to consider (this is done in section 2) the classes of pgf's that are α-decomposable and α-factorizable, respectively, for *all* $\alpha \in (0,1)$. The former class turns out to coincide with $\mathcal{C}_1$, while the latter contains $\mathcal{C}_1$ as a proper subset, but nevertheless shares a number of basic properties with $\mathcal{C}_1$.
In section 3 we introduce a new class of inf div lattice distributions, which is a close analogue of the class of (absolutely continuous) self-decomposable distributions on $[0,\infty)$; in fact, we consider pgf's P that have $P(1-\alpha+\alpha z)$ as a factor for all $\alpha \in (0,1)$. As a subclass of this we obtain a lattice analogue of the (absolutely continuous) stable distributions.
Finally, in section 4 we briefly consider pgf's P that have $1-\alpha+\alpha P(z)$ as a factor. This gives rise to a new characterization of $\mathcal{C}_0$.

3.1. α-*decomposable and* α-*factorizable lattice distributions*

Clearly, the characterization of $\mathcal{C}_\alpha$, given by theorem 2.4.5, can be reformulated as follows: if P is a pgf with $P(0) > 0$ and if $0 < \alpha < 1$, then

$$(3.1.1) \qquad P \in \mathcal{C}_\alpha \Leftrightarrow \exists_{P_\alpha \in \mathcal{C}_0} \; P(z) = \frac{P(\alpha z)}{P(\alpha)} P_\alpha(z) \; .$$

Dropping the condition that the factor P_α be in $\mathcal{C}_0$, we can generally consider pgf's P that have $P(\alpha z)/P(\alpha)$ as a factor. To this end we give the following definition.

DEFINITION 3.1.1. For $0 < \alpha < 1$ a pgf P, with $P(0) > 0$, is said to be α-*decomposable* (α-dec) if there exists a pgf P_α such that

1) This concept is not related to the "α-decompositions" considered in chapter 10 of Lukacs (1970).

$$(3.1.2) \qquad P(z) = \frac{P(\alpha z)}{P(\alpha)} P_\alpha(z) \qquad (|z| \le 1) .$$

The following property of C_α, similar to (3.1.1) (cf. theorem 2.4.2(iv), with $\alpha = 0$ and $n = 2$):

$$(3.1.3) \qquad Q \in C_0 \Rightarrow P(z) := \frac{Q(\alpha z)}{Q(\alpha)} Q(z) \in C_\alpha ,$$

raises the following question: is the converse of (3.1.3) also true, i.e. can every $P \in C_\alpha$ be represented as $Q(z)Q(\alpha z)/Q(\alpha)$ with $Q \in C_0$? This problem, which will be solved in lemma 3.1.8, lead us to consider the α-factorizable pgf's, introduced as follows.

DEFINITION 3.1.2. For $0 < \alpha < 1$ a pgf P, with $P(0) > 0$, is said to be *α-factorizable* (α-fact) if there exists a pgf ${}^\alpha P$ such that

$$(3.1.4) \qquad P(z) = \frac{{}^\alpha P(\alpha z)}{{}^\alpha P(\alpha)} {}^\alpha P(z) \qquad (|z| \le 1) .$$

First, let us consider the α-dec pgf's. The factor P_α in (3.1.2) is uniquely determined by P, as in a neighbourhood of zero we have

$$(3.1.5) \qquad P_\alpha(z) = P(\alpha)P(z)/P(\alpha z) .$$

For a general (not necessarily α-dec) pgf P with $P(0) > 0$, the function in the right-hand side of (3.1.5) is always defined in a neighbourhood of zero, and throughout sections 1 and 2 we shall denote this function by P_α. Applying the mapping $P \to P_\alpha$ to P_β, we see that, with an obvious notation,

$$(3.1.6) \qquad (P_\beta)_\alpha = (P_\alpha)_\beta \qquad (0 < \alpha < 1,\ 0 < \beta < 1) .$$

In view of lemma 1.3.4 we have the following criterion for a pgf to be α-dec.

LEMMA 3.1.3. If $0 < \alpha < 1$ and if P is a pgf with $P(0) > 0$, then P is α-dec iff P_α is abs mon.

Looking for a general representation of α-dec pgf's, we solve (3.1.2) for P and we see that P has the form of an infinite product. We now introduce the following notation: if P is a pgf with $P(0) > 0$, then

$$(3.1.7) \qquad \Pi_\alpha P(z) := \prod_{k=0}^{\infty} P(\alpha^k z)/P(\alpha^k) \qquad (0 < \alpha < 1) .$$

It is not difficult to prove that this infinite product is absolutely and uniformly convergent in $|z| \le 1$, and that $\Pi_\alpha P$ is again a pgf, with

$$(3.1.8) \qquad \Pi_\beta(\Pi_\alpha P) = \Pi_\alpha(\Pi_\beta P) \qquad (0 < \alpha < 1;\ 0 < \beta < 1) .$$

Furthermore it is seen that $\Pi_\alpha P_\beta$, defined similarly as (3.1.7), is absolutely and uniformly convergent in some neighbourhood of zero, with

$$(3.1.9) \qquad \Pi_\alpha P_\beta = (\Pi_\alpha P)_\beta \text{ and } \Pi_\alpha P_\alpha = P .$$

Now the following representation lemma for α-dec pgf's is easily verified.

LEMMA 3.1.4. If $0 < \alpha < 1$ and if P is a pgf with $P(0) > 0$, then P is a α-dec iff P has the form $P = \Pi_\alpha Q$, where Q is a pgf. The representation is unique: $Q = P_\alpha$.

Turning to the α-fact case, we give the following characterization.

LEMMA 3.1.5. Let $0 < \alpha < 1$ and let P be a pgf with $P(0) > 0$. Then P is α-fact iff there exists a pgf Q with $Q(0) > 0$ such that

$$(3.1.10) \qquad P_\alpha = Q_{\alpha^2} ,$$

or, equivalently, iff $\Pi_{\alpha^2} P$ is α-dec; in this case the factor ${}^{\alpha}P$ of P is given by

$$(3.1.11) \qquad {}^{\alpha}P(z) = \Pi_{\alpha^2} P_\alpha(z) = \prod_{k=0}^{\infty} \frac{P(\alpha^{2k} z)/P(\alpha^{2k})}{P(\alpha^{2k+1} z)/P(\alpha^{2k+1})} .$$

PROOF. Let P be α-fact. Then, iterating (3.1.4) once, we get

$${}^{\alpha}P(z) = P(z)\,{}^{\alpha}P(\alpha)/{}^{\alpha}P(\alpha z) = {}^{\alpha}P(\alpha^2 z)P(z)/P(\alpha z) .$$

From this and the fact that $P(\alpha) = {}^{\alpha}P(\alpha^2)$ (take $z = \alpha$ in (3.1.4)), we obtain (3.1.10) with $Q = {}^{\alpha}P$. By (3.1.9) it now follows that

$$(\Pi_{\alpha^2} P)_\alpha = \Pi_{\alpha^2} P_\alpha = \Pi_{\alpha^2} Q_{\alpha^2} = Q ,$$

and hence by lemma 3.1.3 $\Pi_{\alpha^2} P$ is α-dec.

Conversely, let $\Pi_{\alpha^2}P$ be α-dec. Then $Q := \Pi_{\alpha^2}P_\alpha$ is a pgf, for which

$$(3.1.12)\quad Q_{\alpha^2} = (\Pi_{\alpha^2}P_\alpha)_{\alpha^2} = \Pi_{\alpha^2}(P_\alpha)_{\alpha^2} = \Pi_{\alpha^2}(P_{\alpha^2})_\alpha = (\Pi_{\alpha^2}P_{\alpha^2})_\alpha = P_\alpha .$$

As $P(\alpha) = Q(\alpha^2)$, it follows that for all $n \in \mathbb{N}$

$$P(z) = \{Q(z)/Q(\alpha^2 z)\}P(\alpha z) = P(\alpha^n z)\prod_{k=0}^{n-1} Q(\alpha^k z)/Q(\alpha^{k+2}z) =$$

$$= P(\alpha^n z)\frac{Q(z)Q(\alpha z)}{Q(\alpha^n z)Q(\alpha^{n+1}z)} .$$

Letting $n \to \infty$, we get

$$P(z) = P(0)Q(z)Q(\alpha z)/Q(0)^2 ,$$

from which it is seen that $P(0)/Q(0)^2 = Q(\alpha)$. It follows that P is α-fact with factor ${}^\alpha P = Q$. □

Thus we have shown that the factor ${}^\alpha P$ of an α-fact pgf P is uniquely determined by P. Furthermore it follows that the class of α-dec pgf's is a proper subset of the class of α-fact pgf's.

LEMMA 3.1.6. If a pgf P with $P(0) > 0$ is α-dec, then P is α-fact. In fact, P is α-dec iff P is α-fact with α^2-dec factor ${}^\alpha P$.

PROOF. Let P be α-dec. Then P_α, and hence $\Pi_{\alpha^2}P_\alpha$, is a pgf, so that $\Pi_{\alpha^2}P$ is α-dec. By lemma 3.1.5 it now follows that P is α-fact and its factor ${}^\alpha P = \Pi_{\alpha^2}P_\alpha$ is α^2-dec because of (3.1.12).
Conversely, if P is α-fact with α^2-dec factor ${}^\alpha P$, then by (3.1.10) $P_\alpha = ({}^\alpha P)_{\alpha^2}$ is a pgf, and hence P is α-dec. □

From lemma 3.1.4 and (3.1.4) it is evident that the classes of α-dec and α-fact pgf's are too large to have much structure. Therefore we want to restrict them, and a first way to do so is by imposing conditions on the factors P_α and ${}^\alpha P$, respectively.
The most obvious condition is the inf div of P_α and ${}^\alpha P$. It turns out, however, that for all $\alpha \in (0,1)$ we then get the same class, namely $\mathcal{C}_1$, both in the α-dec and α-fact case.

THEOREM 3.1.7. Let P be a pgf with $P(0) > 0$. Then

(i) $P \in \mathcal{C}_1$ iff for some, and then for all, $\alpha \in (0,1)$ P is α-dec with factor $P_\alpha \in \mathcal{C}_1$;

(ii) $P \in \mathcal{C}_1$ iff for some, and then for all, $\alpha \in (0,1)$ P is α-fact with factor ${}^\alpha P \in \mathcal{C}_1$.

PROOF. First, let $P \in \mathcal{C}_1$. In lemma 2.3.6 it is shown that then for all $\alpha \in (0,1)$ P_α is an inf div pgf. Hence for all $\alpha \in (0,1)$ P is α-dec with factor $P_\alpha \in \mathcal{C}_1$. Now, by lemma 3.1.6 it follows that P is also α-fact for all $\alpha \in (0,1)$, while in view of (3.1.11) the factor ${}^\alpha P$ is the limit of a sequence of inf div pgf's, so by the closure theorem ${}^\alpha P \in \mathcal{C}_1$.
Next, let P be α-dec with factor $P_\alpha \in \mathcal{C}_1$, for a fixed $\alpha \in (0,1)$. Then applying the closure theorem once more, from lemma 3.1.4 we see that $P \in \mathcal{C}_1$.
Finally, if P is α-fact with factor ${}^\alpha P \in \mathcal{C}_1$ for a fixed $\alpha \in (0,1)$, then the inf div of P immediately follows from (3.1.4). □

Next, let us consider the cases that P_α and ${}^\alpha P$ are compound geometric. In view of (3.1.1) we have

(3.1.13) $\quad P \in \mathcal{C}_\alpha \Leftrightarrow P$ α-dec with $P_\alpha \in \mathcal{C}_0$.

Using this, one easily verifies that the classes $\mathcal{C}_\alpha$ can also be obtained as follows:

(3.1.14) $\quad P \in \mathcal{C}_\alpha \Leftrightarrow P$ α-fact with ${}^\alpha P \in \mathcal{C}_{\alpha^2}$.

Now, as for $0 < \alpha < 1$ $\mathcal{C}_0$ is a proper subset of $\mathcal{C}_{\alpha^2}$ (cf. section 2.4), from (3.1.14) it follows that the question that led us to consider the α-fact pgf's, has to be answered in the negative. In fact, we can say the following about the set of α-fact pgf's with compound geometric factor ${}^\alpha P$.

LEMMA 3.1.8. If $0 < \alpha < 1$, then $\mathcal{C}_0 \subsetneqq \{\alpha\text{-fact } P \mid {}^\alpha P \in \mathcal{C}_0\} \subsetneqq \mathcal{C}_\alpha$.

PROOF. In view of (3.1.14) we only need to prove the first part of the lemma. So let $P \in \mathcal{C}_0$. Then by theorem 3.1.7(ii) P is α-fact, and, as from corollary 2.4.7 we see that

$$\Pi_{\alpha^2} P \in \mathcal{C}_{\alpha^2} \subset \mathcal{C}_\alpha ,$$

it follows from (3.1.13) that the factor ${}^\alpha P = \Pi_{\alpha^2} P_\alpha = (\Pi_{\alpha^2} P)_\alpha$ is in $\mathcal{C}_0$.

Now, take $Q(z) := (1-p)/(1-pz)$, and define the pgf P by

$$P(z) := Q(z)Q(\alpha z)/Q(\alpha) \quad ,$$

then P is α-fact with factor ${}^{\alpha}P = Q \in \mathcal{C}_0$. In view of (2.4.27), however, we see that $P \in \mathcal{C}_\alpha \setminus \bigcup_{\beta<\alpha} \mathcal{C}_\beta$, and hence $P \notin \mathcal{C}_0$. □

Of course, many other conditions might be imposed on P_α or ${}^{\alpha}P$ to obtain subclasses of the classes of α-dec and α-fact pgf's. We mention one result: the classes $\mathcal{A}_{\alpha,\beta}$, say, of α-dec pgf's with factor $P_\alpha \in \mathcal{C}_\beta$ define a classification of $\mathcal{C}_1$; $\mathcal{A}_{\alpha,\beta}$ is nondecreasing in both α and β, with

$$\mathcal{A}_{\alpha,\beta} = \mathcal{A}_{\beta,\alpha} \supset \mathcal{C}_\alpha \cup \mathcal{C}_\beta \; ,$$

as can be seen from the following characterization:

$$P \in \mathcal{A}_{\alpha,\beta} \Leftrightarrow 1 - \frac{P(\alpha z)P(\beta z)}{P(z)P(\alpha\beta z)} \text{ abs mon } .$$

A second way to restrict the classes of α-dec and α-fact pgf's is to require α-dec and α-fact for *all* $\alpha \in (0,1)$. This is suggested by a notational analogy to the so-called self-decomposable chf's: a chf $\tilde{F}$ is said to be *self-decomposable* (self-dec) if for every $\alpha \in (0,1)$ there exists a chf $\tilde{F}_\alpha$ such that

(3.1.15) $\quad \tilde{F}(t) = \tilde{F}(\alpha t)\tilde{F}_\alpha(t) \qquad (t \in \mathbb{R})$.

These chf's were introduced by Lévy and Khintchine (cf. Lévy (1937)). For a survey of their main properties we refer to Lukacs (1970), ch. 5. Here we only mention that the class of self-dec distributions contains the well-known *stable* distributions and is itself a proper subclass of the class of all inf div distributions.

3.2. *Totally decomposable and totally factorizable lattice distributions*

DEFINITION 3.2.1. Let P be a pgf with $P(0) > 0$. Then P is said to be *totally decomposable* (tot dec) if P is α-dec for all $\alpha \in (0,1)$. Similarly, P is called *totally factorizable* (tot fact) if P is α-fact for all $\alpha \in (0,1)$.

We start with studying the tot dec pgf's. From theorem 3.1.7(i) we know that every inf div pgf is tot dec. It will turn out that the converse is also true. Though simpler proofs exist (to be given later), we shall show this in a way, very similar to the proof of the inf div of a self-dec chf (cf.

Lukacs (1970), ch. 5), thus showing that the analogy, noted at the end of section 1, is not merely notational. Proceeding in this way, we first prove the following lemma.

LEMMA 3.2.2. A tot dec pgf has no zeros in the closed unit disk $|z| \leq 1$.

PROOF. We need the following inequality:

$$(3.2.1) \qquad |Q(\alpha z)| \leq |Q(z)| + 1 - Q(\alpha) \qquad (Q \text{ is pgf; } 0 < \alpha < 1;\ |z| \leq 1),$$

which can be proved as follows:

$$|Q(\alpha z)| \leq |Q(z)| + |Q(z) - Q(\alpha z)| \leq |Q(z)| + \sum_{n=0}^{\infty} q_n (1 - \alpha^n) |z|^n \leq$$
$$\leq |Q(z)| + 1 - Q(\alpha) .$$

Suppose the assertion of the lemma not to be true. Then we can find $\rho \in (0,1]$ and $z_o \in \mathbb{C}$ with $|z_o| = \rho$ such that

$$P(z_o) = 0 \text{ and } P(z) \neq 0 \qquad (|z| < \rho) .$$

From (3.1.2) it now follows that for all $\alpha \in (0,1)$ the factor P_α of P satisfies

$$P_\alpha(z_o) = 0 \text{ and } P_\alpha(z) \neq 0 \qquad (|z| < \rho) .$$

As $|\tfrac{1}{2}z_o| < \rho$, we have on the one hand

$$\lim_{\alpha \uparrow 1} P_\alpha(\tfrac{1}{2}z_o) = \lim_{\alpha \uparrow 1} P(\alpha)P(\tfrac{1}{2}z_o)/P(\tfrac{1}{2}\alpha z_o) = 1 ,$$

whereas on the other hand from the inequality (3.2.1) it follows that

$$|P_\alpha(\tfrac{1}{2}z_o)| \leq |P_\alpha(z_o)| + 1 - P_\alpha(\tfrac{1}{2}) \leq 1 - P_\alpha(0) = 1 - P(\alpha) ,$$

which tends to zero as $\alpha \uparrow 1$. Thus we have obtained a contradiction, and the lemma is proved. □

In the self-decomposable case one makes use of the theorem (see e.g. Gnedenko & Kolmogorov (1949)) that, if a chf φ can be written as

$$\varphi(t) = \lim_{n\to\infty} \prod_{k=1}^{n} \varphi_{n,k}(t) \qquad (t \in \mathbb{R}) ,$$

where the $\varphi_{n,k}$'s form an infinitesimal system of chf's, then φ is inf div.

Here a system $(\varphi_{n,k} \mid n \in \mathbb{N};\ k \in \{1,\ldots,n\})$ of chf's is called *infinitesimal* if

$$\lim_{n\to\infty} \sup_{1\le k\le n} |\varphi_{n,k}(t) - 1| = 0 \qquad (t \in \mathbb{R}) .$$

Translating this to pgf's, we obtain the following result.

THEOREM 3.2.3. A pgf P, with $P(0) > 0$, is inf div iff there exist pgf's $P_{n,k}$ $(n \in \mathbb{N};\ k \in \{1,\ldots,n\})$ satisfying

$$\lim_{n\to\infty} \inf_{1\le k\le n} P_{n,k}(0) = 1 , \tag{3.2.2}$$

such that

$$P(z) = \lim_{n\to\infty} \prod_{k=1}^{n} P_{n,k}(z) \qquad (|z| \le 1) . \tag{3.2.3}$$

We are now ready to prove the announced result.

THEOREM 3.2.4. A pgf P, with $P(0) > 0$, is inf div iff P is tot dec.

PROOF. As already noted, in view of theorem 3.1.7(i) we only need to show that a tot dec pgf is inf div. So let P be tot dec. Then by lemma 3.2.2 it is seen that for the factors P_α $(0 < \alpha < 1)$ relation (3.1.5) holds for all $|z| \le 1$. If we define the pgf's $P_{n,k}$ for $n \in \mathbb{N}$ and $k \in \{1,\ldots,n\}$ by

$$P_{n,k}(z) := P_{(k-1)/k}(\tfrac{k}{n} z)/P_{(k-1)/k}(\tfrac{k}{n}) \qquad (|z| \le 1) ,$$

then it follows that

$$P_{n,k}(z) = \frac{P(\frac{k}{n} z)/P(\frac{k}{n})}{P(\frac{k-1}{n} z)/P(\frac{k-1}{n})} \qquad (|z| \le 1) ,$$

and hence

$$P(z) = \prod_{k=1}^{n} P_{n,k}(z) \qquad (n \in \mathbb{N};\ |z| \le 1) .$$

Now from theorem 3.2.3 we obtain the inf div of P, as soon as we have proved (3.2.2), which in our case is equivalent to

$$\lim_{n\to\infty} \inf_{1\le k\le n} P(\tfrac{k-1}{n})/P(\tfrac{k}{n}) = 1 . \tag{3.2.4}$$

Because of the uniform continuity of P on $[0,1]$, for all $\varepsilon > 0$ one can find $N \in \mathbb{N}$ such that

$$\forall_{n\geq N}\ \forall_{k\in\{1,\ldots,n\}}\ \left|P(\tfrac{k-1}{n}) - P(\tfrac{k}{n})\right| < \varepsilon\ ,$$

or

$$\forall_{n\geq N}\ \forall_{k\in\{1,\ldots,n\}}\ \left|P(\tfrac{k-1}{n})/P(\tfrac{k}{n}) - 1\right| < \varepsilon/P(0)\ .$$

From this (3.2.4), and hence the theorem, immediately follows. □

The following two corollaries are obtained by using theorem 3.1.7(i) (or lemma 2.3.6) and lemma 3.1.3, respectively.

COROLLARY 3.2.5. The factors P_α $(0 < \alpha < 1)$ of a tot dec pgf P are all inf div.

COROLLARY 3.2.6. Let P be a pgf with $P(0) > 0$. Then $P \in \mathcal{C}_1$ iff for all $\alpha \in (0,1)$, or, equivalently, for all $\alpha \in (1-\varepsilon,1)$ (some $\varepsilon > 0$), the function $P(z)/P(\alpha z)$ is abs mon.

The characterization of $\mathcal{C}_1$, just given, can also be proved as follows. The necessity of the condition has already been shown in lemma 2.3.6. Suppose therefore that $\varepsilon > 0$ and that $P(z)/P(\alpha z)$ is abs mon for all $\alpha \in (1-\varepsilon,1)$. As $P(z)/P(\alpha z)$ is equal to 1 for $z = 0$, for all $\alpha \in (1-\varepsilon,1)$ the function

$$(3.2.5)\qquad T_\alpha(z) := \frac{1}{1-\alpha}\,\frac{1}{z}\{P(z)/P(\alpha z) - 1\}$$

is abs mon. But then so is the following function:

$$\lim_{\alpha\uparrow 1} T_\alpha(z) = \lim_{\alpha\uparrow 1} \frac{1}{P(\alpha z)}\,\frac{P(z) - P(\alpha z)}{z - \alpha z} = P'(z)/P(z)\ ,$$

from which by theorem 1.5.3 it follows that $P \in \mathcal{C}_1$.
In view of lemma 3.1.3 this alternative proof now yields a simpler proof of theorem 3.2.4. Furthermore we note that another proof of this theorem can be given along the lines of the proofs of theorems 3.3.3 and 3.4.5.
The R_α-function of a pgf P can be expressed in the function T_α from (3.2.5) as follows:

$$(3.2.6)\qquad R_\alpha(z) = T_\alpha(z)P(\alpha z)/P(z)\ .$$

Now, apart from (3.1.13), from this relation it can be clearly seen that $\mathcal{C}_\alpha$ is a subclass of the class of α-dec pgf's, the latter being characterized by the

abs mon of T_α.
Finally, denoting the coefficients in the power-series expansion of T_α by $t_n(\alpha)$ $(n \in \mathbb{N}_0)$, from corollary 3.2.6 we get the following characterization of $\mathcal{C}_1$ by means of recurrence relations (cf. those defining the classes $\mathcal{C}_\alpha$).

LEMMA 3.2.7. A lattice distribution $\{p_n\}_0^\infty$, with $p_0 > 0$, is in $\mathcal{C}_1$ iff for all $\alpha \in (0,1)$ (or $\alpha \in (1-\varepsilon,1)$, some $\varepsilon > 0$) there exist nonnegative quantities $t_n(\alpha)$ $(n \in \mathbb{N}_0)$ such that

$$\text{(3.2.7)} \qquad \frac{1-\alpha^{n+1}}{1-\alpha}\, p_{n+1} = \sum_{k=0}^{n} \alpha^k p_k t_{n-k}(\alpha) \qquad (n \in \mathbb{N}_0) .$$

We now turn to the tot fact pgf's. By theorem 3.1.7(ii) we know that all inf div pgf's are tot fact. Although an α-fact pgf is more general than an α-dec pgf (cf. lemma 3.1.6), and the tot dec pgf's coincide with the inf div pgf's, all tot fact pgf's might be inf div. In order to decide this, we study the tot fact pgf's in more detail, and start with the behaviour of the factor ${}^{\alpha}P$ for $\alpha \uparrow 1$ and $\alpha \downarrow 0$.

THEOREM 3.2.8. A tot fact pgf P is 2-div; in fact, the factor ${}^{\alpha}P$ of P satisfies

$$\text{(3.2.8)} \qquad \lim_{\alpha \uparrow 1} {}^{\alpha}P(\alpha^n z) = P(z)^{\frac{1}{2}} \qquad (|z| \le 1;\ n \in \mathbb{N}_0) ,$$

and furthermore

$$\text{(3.2.9)} \qquad \lim_{\alpha \downarrow 0} {}^{\alpha}P(z) = P(z) \text{ and } \lim_{\alpha \downarrow 0} {}^{\alpha}P(\alpha^n z) = P(0) \qquad (|z| \le 1;\ n \in \mathbb{N}) .$$

PROOF. Let P be tot fact. Taking $z = \alpha$ in (3.1.4), we get ${}^{\alpha}P(\alpha^2) = P(\alpha)$, and hence $\lim_{\alpha \uparrow 1} {}^{\alpha}P(\alpha^2) = 1$. As ${}^{\alpha}P(\alpha^2) \le {}^{\alpha}P(\alpha) \le 1$ for $0 < \alpha < 1$, it follows that $\lim_{\alpha \uparrow 1} {}^{\alpha}P(\alpha) = 1$, and hence

$$\lim_{\alpha \uparrow 1} {}^{\alpha}P(z)\, {}^{\alpha}P(\alpha z) = P(z) \qquad (|z| \le 1) .$$

Now, let $\{p_k\}_0^\infty$ and $\{p_k(\alpha)\}_0^\infty$ be the lattice distributions with pgf P and ${}^{\alpha}P$, respectively. Then by the continuity theorem for pgf's (theorem 1.3.5) we have

$$(3.2.10)\quad \lim_{\alpha\uparrow 1} \sum_{j=0}^{k} \alpha^j p_j(\alpha) p_{k-j}(\alpha) = p_k \qquad (k \in \mathbb{N}_o)\ .$$

Taking $k = 0$ in this relation, we see that $\lim_{\alpha\uparrow 1} p_o(\alpha) = p_o^{\frac{1}{2}}$. Suppose that $p_n(1) := \lim_{\alpha\uparrow 1} p_n(\alpha)$ exists for $n = 1,\ldots,k-1$; then from (3.2.10) it follows that also

$$\lim_{\alpha\uparrow 1} p_k(\alpha) = \tfrac{1}{2} p_o^{-\frac{1}{2}} \{p_k - \sum_{j=1}^{k-1} p_j(1) p_{k-j}(1)\}$$

exists, and hence (mathematical induction): $p_k(1) = \lim_{\alpha\uparrow 1} p_k(\alpha)$ exists for all $k \in \mathbb{N}_o$, while $\{p_k(1)\}_o^\infty$ satisfies

$$\sum_{j=0}^{k} p_j(1) p_{k-j}(1) = p_k \qquad (k \in \mathbb{N}_o)\ .$$

Applying the continuity theorem in the other direction, we conclude that

$$\lim_{\alpha\uparrow 1} {}^{\alpha}P(z) = Q(z) \qquad (|z| \le 1)\ ,$$

where $Q(z) := \Sigma p_n(1) z^n$ is a pgf satisfying $Q(z)^2 = P(z)$. Hence P is 2-div, and (3.2.8) is proved for $n = 0$. Replacing z by $\alpha^n z$ in (3.1.4), we get

$$(3.2.11)\quad {}^{\alpha}P(\alpha^{n+1} z) = {}^{\alpha}P(\alpha) P(\alpha^n z) / {}^{\alpha}P(\alpha^n z)\ ,$$

from which by use of mathematical induction (3.2.8) follows for all $n \in \mathbb{N}_o$. Finally, let $\alpha \downarrow 0$. Taking $z = \alpha$ and $z = \alpha^2$ in (3.1.4) successively, one sees that

$$\lim_{\alpha\downarrow 0} {}^{\alpha}P(\alpha^2) = P(0) \text{ and } \lim_{\alpha\downarrow 0} {}^{\alpha}P(\alpha^3) / {}^{\alpha}P(\alpha) = 1\ .$$

As for $0 < \alpha < 1$ we have ${}^{\alpha}P(\alpha^3)/{}^{\alpha}P(\alpha) \le {}^{\alpha}P(\alpha^2)/{}^{\alpha}P(\alpha) \le 1$, it follows that

$$\lim_{\alpha\downarrow 0} {}^{\alpha}P(\alpha^2) / {}^{\alpha}P(\alpha) = 1\ ,$$

and hence

$$\lim_{\alpha\downarrow 0} {}^{\alpha}P(\alpha) = P(0)\ .$$

Observing that for $0 < z \le 1$ and $0 < \alpha \le z$ we have ${}^{\alpha}P(\alpha^2) \le {}^{\alpha}P(\alpha z) \le {}^{\alpha}P(\alpha)$, we conclude that for all $z \in (0,1]$, and hence by the continuity theorem for all $|z| \le 1$,

$$\lim_{\alpha \downarrow 0} {}^{\alpha}P(\alpha z) = P(0) .$$

The second part of (3.2.9) is now easily obtained from (3.2.11) by mathematical induction, while the first part follows from (3.1.4). □

In view of the preceding theorem one might conjecture that the tot fact pgf's coincide with the 2-div ones. Consider, however, the pgf $P(z) = (1+z)^2/4$; obviously, P is 2-div, but P is not α-fact for any $\alpha \in (0,1)$. In fact, if P would be α-fact, then ${}^{\alpha}P$ is necessarily of the form ${}^{\alpha}P(z) = c_{\alpha}(1 + \gamma_{\alpha}z)$ with $c_{\alpha} > 0$, $\gamma_{\alpha} > 0$. But if we look at the zeros of P, then we see that γ_{α} has to satisfy: $\gamma_{\alpha} = \alpha\gamma_{\alpha} = 1$, which is impossible for $\alpha \in (0,1)$.
A similar observation with respect to the zeros of P yields a property that the tot fact pgf's share with the inf div ones (cf. theorem 1.4.6).

THEOREM 3.2.9. If P is a tot fact pgf, then the corresponding lattice distribution $\{p_n\}$ has an infinite support.

PROOF. Let P be tot fact and suppose that the support of $\{p_n\}_0^{\infty}$ is finite. Then P, and its factors ${}^{\alpha}P$ $(0 < \alpha < 1)$, are polynomials. Let z_o be a zero of the factor ${}^{\frac{1}{2}}P$, then, as $p_o > 0$, $z_o \neq 0$, and from (3.1.4), which now holds for all $z \in \mathbb{C}$, it follows that $P(z_o) = P(2z_o) = 0$. Taking $\alpha = 1/k$ $(k = 2,3,\ldots)$ in (3.1.4), we then see that ${}^{1/k}P(z_o) = 0$ or ${}^{1/k}P(z_o/k) = 0$, and hence

$$P(kz_o) = 0 \text{ or } P(z_o/k) = 0 \qquad (k \in \mathbb{N}) .$$

As $z_o \neq 0$, we have thus obtained an infinite sequence of different zeros of P, which contradicts the fact that P is a polynomial. It follows that the support of $\{p_n\}$ has to be infinite. □

REMARK 3.2.10. If P is α-fact for only a finite number of α's, then the support of $\{p_n\}$ may be finite. To show this, let $\alpha_1,\ldots,\alpha_n \in (0,1)$, and let P_n be the pgf that is equal to a polynomial with A_n as its set of zeros, where A_n is recursively defined by

$$A_k := A_{k-1} \cup \{\alpha_k^{-1}x \mid x \in A_{k-1}\} \qquad (k \in \{1,\ldots,n\}),\ A_o := \{-1\} .$$

Then P_n is of degree 2^n, and it is easily seen that P_n is α_k-fact for all $k \in \{1,\ldots,n\}$.

Apart from being 2-div and having infinite support, the tot fact pgf's share a third property with the inf div pgf's, viz. having no zeros in the *open* unit disk $|z| < 1$ (cf. corollary 1.5.2). It is not clear if a tot fact pgf may have zeros for $|z| = 1$.

THEOREM 3.2.11. If P is a tot fact pgf, then $P(z) \neq 0$ for all $|z| < 1$.

PROOF. Let P be tot fact, and suppose the assertion not to be true. Then we can find $\rho \in (0,1)$ and $z_o \in ¢$ with $|z_o| = \rho$, such that

$$P(z_o) = 0 \text{ and } P(z) \neq 0 \qquad (|z| < \rho) .$$

From (3.1.4) it follows that for all $\alpha \in (0,1)$ the factor ${}^{\alpha}P$ of P satisfies

$${}^{\alpha}P(z_o) = 0 \text{ and } {}^{\alpha}P(z) \neq 0 \qquad (|z| < \rho) .$$

Now, take $\alpha \in (\rho,1)$, then $P(z_o/\alpha)$ is well defined, and according to (3.1.4)

$$P(z_o/\alpha) = {}^{\alpha}P(z_o/\alpha)\,{}^{\alpha}P(z_o)/{}^{\alpha}P(\alpha) = 0 ;$$

hence

$$P(\gamma z_o) = 0 \qquad (1 \leq \gamma \leq 1/\rho) .$$

However, as P is analytic on $|z| \leq 1$, this would imply that $P \equiv 0$, so we have obtained a contradiction, and hence $P(z) \neq 0$ for all $|z| < 1$. □

If P is tot fact, then the coefficients $p_k(\alpha)$, say, in the power-series expansion of the function ${}^{\alpha}P$, given by (3.1.11), have to be nonnegative for all $\alpha \in (0,1)$. Now, from (3.1.4) we easily see that

$$(3.2.12) \qquad [\forall_{\alpha\in(0,1)}\ p_2(\alpha) \geq 0] \Leftrightarrow 4p_op_2 \geq p_1^2 .$$

We compare this condition with the condition $r_1(1) \geq 0$ (cf. corollary 1.5.5), necessary for P to be inf div, for which we have

$$r_1(1) \geq 0 \Leftrightarrow 2p_op_2 \geq p_1^2 .$$

As the latter condition is more restrictive and as the same phenomenon seems to occur at further coefficients, we are led to look for examples of tot fact pgf's that are not inf div.
To this end the following characterization of an α-fact pgf (and hence of a tot fact pgf) is useful.

LEMMA 3.2.12. Let P be a pgf with $P(0) > 0$, and define the sequence $\{a_n\}_1^\infty$ by

$$(3.2.13)\qquad \log\{P(z)/P(0)\} = \sum_{n=1}^{\infty} a_n z^n .$$

Then for $0 < \alpha < 1$ P is α-fact iff the function $Q^{(\alpha)}$, defined by

$$(3.2.14)\qquad Q^{(\alpha)}(z) := \exp\Big[\sum_{n=1}^{\infty} \frac{a_n}{1+\alpha^n} z^n\Big] ,$$

is abs mon, in which case necessarily $Q^{(\alpha)}(z) = {}^{\alpha}P(z)/{}^{\alpha}P(0)$.

PROOF. Let P be a pgf with $P(0) > 0$, and let $0 < \alpha < 1$. Then, using (3.2.13), we can write

$$P(z)/P(0) = \exp\Big[\sum_{n=1}^{\infty} a_n z^n\Big] = \exp\Big[\sum_{n=1}^{\infty} \frac{a_n}{1+\alpha^n} z^n + \sum_{n=1}^{\infty} \frac{a_n}{1+\alpha^n}(\alpha z)^n\Big] ,$$

from which it follows that P can be written as (cf. (3.2.14))

$$(3.2.15)\qquad P(z)/P(0) = Q^{(\alpha)}(z)Q^{(\alpha)}(\alpha z) .$$

Now, if P is α-fact with factor ${}^{\alpha}P$, then $P(0)\,{}^{\alpha}P(\alpha) = \{{}^{\alpha}P(0)\}^2$, which, together with (3.2.15), implies that $Q^{(\alpha)}(z) = {}^{\alpha}P(z)/{}^{\alpha}P(0)$; hence $Q^{(\alpha)}$ is abs mon.

Conversely, let $Q^{(\alpha)}$ be abs mon. Then similarly to the proof of lemma 1.3.4 it follows from (3.2.15) that $Q^{(\alpha)}(z)$ is convergent for $|z| \le 1$, with $Q^{(\alpha)}(1)Q^{(\alpha)}(\alpha) = 1/P(0)$. Now, defining ${}^{\alpha}P(z) := Q^{(\alpha)}(z)/Q^{(\alpha)}(1)$, we see from (3.2.15) that P is α-fact with factor ${}^{\alpha}P$. □

For a pgf P to be inf div it is necessary and sufficient that the quantities a_n, defined by (3.2.13), are all nonnegative (cf. theorem 1.5.3). Hence, in view of lemma 3.2.12, in order to obtain a tot fact pgf that is not inf div, we need an example of a power series Q with at least one negative coefficient such that $\exp[Q(z)]$ is abs mon. The simplest example of this type is provided by a polynomial of degree 4, as was shown by Lévy (1937), ch. VII. We state his result as a lemma.

LEMMA 3.2.13. For given $a > 0$, $c > 0$ and $d > 0$ there exists a (unique) real number $m(a,c,d) > 0$, such that $F(z) := \exp[az - bz^2 + cz^3 + dz^4]$ is abs mon iff $b \le m(a,c,d)$. Furthermore, in this case the pgf $P(z) := F(z)/F(1)$ is indecomposable iff $b = m(a,c,d)$.

Now we are ready to prove the existence of tot fact pgf's that are not inf div.

THEOREM 3.2.14. The class C_1 is a proper subset of the class of tot fact pgf's.

PROOF. Choose arbitrary positive numbers a_1, a_3 and a_4, and define $m(\alpha)$ by

$$m(\alpha) := m(a_1/(1+\alpha), a_3/(1+\alpha^3), a_4/(1+\alpha^4)) \qquad (0 \le \alpha \le 1)$$

(cf. the preceding lemma). It is easy to show that the function $m(a,c,d)$ is nondecreasing in a, c and d, and hence $m(\alpha)$ is nonincreasing in $\alpha \in [0,1]$. Now, take $\varepsilon > 0$ such that $\varepsilon \le m(1)$, then $\varepsilon \le m(0)$ and by lemma 3.2.13 it follows that

$$(3.2.16) \quad P(z)/P(0) = \exp[a_1 z - \varepsilon z^2 + a_3 z^3 + a_4 z^4]$$

defines a pgf. As its R_1-function is not abs mon, we have (cf. theorem 1.5.3) $P \notin C_1$. But by the monotonicity of $m(\alpha)$, we have

$$\varepsilon/(1+\alpha^2) < \varepsilon \le m(1) \le m(\alpha) \qquad (0 < \alpha < 1) \; ,$$

and hence, applying lemma 3.2.13 once more, we see that

$$\exp[\frac{a_1}{1+\alpha} z - \frac{\varepsilon}{1+\alpha^2} z^2 + \frac{a_3}{1+\alpha^3} z^3 + \frac{a_4}{1+\alpha^4} z^4]$$

is an abs mon function for all $\alpha \in (0,1)$. From lemma 3.2.12 it now follows that P is tot fact, and thus the theorem is proved. □

REMARK 3.2.15. In view of the preceding proof one can replace the condition $\varepsilon \le m(1)$ by the weaker condition

$$\varepsilon \le \inf_{0 \le \alpha < 1} (1+\alpha^2) m(\alpha) \; .$$

It follows that if the function $m(\alpha)$ is such that

$$2m(1) \le (1+\alpha^2) m(\alpha) \qquad (0 \le \alpha < 1) \; ,$$

then one could choose $\varepsilon = 2m(1)$, in which case the pgf P from (3.2.16) is such that its factor $P(z)^{\frac{1}{2}}$ is *indecomposable* (cf. the last part of lemma 3.2.13).

In order to construct real examples of tot fact pgf's that are not inf div, it is necessary to have an expression, or at least a positive lower bound, for the function $m(a,c,d)$. Lévy does not give any information of this type, but Lukacs (1970), ch. 8 notes that if $Q(z) := 1 + az - bz^2 + cz^3 + dz^4$ is such that $b \le \frac{1}{2}a^2$ and Q^2 and Q^3 are abs mon, then $\exp[Q(z)]$ is abs mon. Elaborating this, one easily verifies the following lemma.

LEMMA 3.2.16. For given $a > 0$, $c > 0$ and $d > 0$ the function $\exp[az - bz^2 + cz^3 + dz^4]$ is abs mon, if $b \le \min\{a^2/3, c/a, ad/(2c), c^2/(3d)\}$.

It follows that, for instance, the function P defined by

$$P(z) = \exp[6z - z^2 + 6z^3 + 6z^4 - 17] ,$$

is a tot fact pgf that is not inf div.

To conclude this section, we return to chf's. Analogous to self-dec chf's (cf. the end of section 1) we defined tot dec pgf's. Now, having introduced tot fact pgf's, we could reverse matters and consider totally factorizable chf's, i.e. chf's $\tilde{F}$ that satisfy

$$(3.2.17) \quad \tilde{F}(t) = \tilde{F}_\alpha(\alpha t)\tilde{F}_\alpha(t) \qquad (t \in \mathbb{R};\ 0 < \alpha < 1) ,$$

where $\tilde{F}_\alpha$ is a chf. Or, in terms of rv's: X is said to be tot fact if

$$(3.2.18) \quad X \overset{d}{=} X_\alpha + \alpha X'_\alpha \qquad (0 < \alpha < 1) ,$$

where X_α and X'_α are independent rv's with the same distribution. As an $\mathbb{N}_0$-valued rv $X \not\equiv 0$ cannot satisfy (3.2.18), we see that an inf div chf is not necessarily tot fact (cf. theorem 3.1.7(ii)). Furthermore, proceeding in a way similar to lemmas 3.1.5 and 3.1.6, we can show that the class of self-dec chf's is a proper subset of the class of tot fact chf's, and the question arises whether the latter class is a subset of the class of inf div chf's, i.e. whether all tot fact chf's are inf div. We will not go further into this now. We only note that e.g. the rectangular distribution is α-fact in this sense for infinitely many α's; in fact, the chf of the rectangular distribution on $(-1,+1)$ can be written as (cf. Lukacs (1970), ch. 6)

$$\tilde{F}(t) = \frac{\sin t}{t} = \prod_{k=1}^{\infty} \cos(t/2^k) ,$$

from which it is seen that $\tilde{F}$ is 2^{-n}-fact for all $n \in \mathbb{N}$.

3.3. *Discrete self-decomposability and stability*

As noted at the end of section 1, a chf $\tilde{F}$ is said to be self-decomposable (self-dec) if $\tilde{F}$ satisfies

$$(3.3.1)\qquad \tilde{F}(t) = \tilde{F}(\alpha t)\tilde{F}_\alpha(t) \qquad (t \in \mathbb{R};\ 0 < \alpha < 1)\ ,$$

with $\tilde{F}_\alpha$ a chf. For the corresponding rv's this means that

$$(3.3.2)\qquad X \overset{d}{=} \alpha X + X_\alpha \qquad (0 < \alpha < 1)\ ,$$

where X and X_α are independent. Vervaat (1978) considers equations of the form (3.3.2), where also the factor α is a rv. Clearly, except $X \equiv 0$, no $\mathbb{N}_0$-valued rv X can satisfy (3.3.2); in fact, all nondegenerate self-dec distributions are known to be absolutely continuous (cf. Fisz & Varadarajan (1963)). Now in this section we propose analogues of the concepts of self-decomposability and stability (cf. (3.3.20)) for lattice distributions. A slightly condensed version of the results of the present section can be found in Steutel & Van Harn (1978). It turns out that the discrete self-dec distributions and the discrete stable distributions share the basic properties with their continuous counterparts. The discrete self-dec distributions, for instance, are unimodal (cf. Wolfe (1971a) and Yamazato (1978)), and the discrete stable distributions are very similar to their continuous analogues on $(0,\infty)$.

Looking for analogues of (3.3.2) that operate within the set of $\mathbb{N}_0$-valued rv's, we consider equations of the form

$$(3.3.3)\qquad X \overset{d}{=} \alpha \circ X + X_\alpha \qquad (0 < \alpha < 1)\ ,$$

where the operation $\circ$ is such that $\alpha \circ X$ is an $\mathbb{N}_0$-valued rv. In terms of pgf's:

$$(3.3.4)\qquad P(z) = (T_\alpha P)(z)P_\alpha(z) \qquad (|z| \le 1;\ 0 < \alpha < 1)\ ,$$

where the operator T_α is such that $T_\alpha P$ is a pgf. Now we want to choose $\alpha \circ X$ or $T_\alpha P$ in such a way that they have properties as in ordinary scalar multiplication. We mention three examples, which satisfy

$$T_0 P = 1,\ T_1 P = P,\ T_\alpha(T_\beta P) = T_{\alpha\beta} P\ .$$

EXAMPLE 3.3.1.

(i) $\quad (T_\alpha P)(z) = P(\alpha z)/P(\alpha)$.

(ii) $\quad (T_\alpha P)(z) = 1-\alpha+\alpha P(z)$, or $\alpha \circ X = X_1 + \ldots + X_N$,

where $P(N = 1) = 1 - P(N = 0) = \alpha$, $X_k \overset{d}{=} X$ $(k \in \mathbb{N})$, all rv's being independent.

(iii) $\quad (T_\alpha P)(z) = P(1-\alpha+\alpha z)$, or $\alpha \circ X = N_1 + \ldots + N_X$,

where $N_k \overset{d}{=} N$ $(k \in \mathbb{N})$ with N as in example (ii), all rv's being independent.

It is easy to see that only examples (i) and (iii) satisfy

$$T_\alpha(PQ) = (T_\alpha P)(T_\alpha Q) \quad ,$$

and that only examples (ii) and (iii) satisfy

$$T_\alpha(\gamma P + (1-\gamma)Q) = \gamma T_\alpha P + (1-\gamma)T_\alpha Q \qquad (0 \le \gamma \le 1)$$

and

$$(T_\alpha P)'(1) = \alpha P'(1) \quad .$$

Thus, example (iii) seems to be most similar to ordinary scalar multiplication. Indeed, it will turn out that using this example in (3.3.4), we obtain a class of lattice distributions that can be considered as the lattice analogue of the class of self-dec distributions. When using example (i) in (3.3.4), we get the class C_1 (cf. theorem 3.2.4), and, as we shall see in the next section, the class of pgf's P that satisfy (3.3.4) with $T_\alpha P$ given by example (ii), coincides with C_o.

DEFINITION 3.3.2. A pgf P, with $P(0) > 0$, is said to be *discrete self-decomposable* if P satisfies

(3.3.5) $\quad P(z) = P(1-\alpha+\alpha z)P_\alpha(z) \qquad (|z| \le 1;\ 0 < \alpha < 1)$,

with P_α a pgf.

REMARK 3.3.3. A relation that suggests the analogy of discrete self-dec pgf's to self-dec chf's in another respect, is the relation that $H(z) :=$ $:= P(1-z)$ satisfies:

(3.3.6) $\quad H(z) = H(\alpha z)H_\alpha(z) \qquad (|1-z| \le 1;\ 0 < \alpha < 1)$,

with $H_\alpha(z) := P_\alpha(1-z)$.

In the present section the notation P_α is no longer used for the function in (3.1.5); it will now denote the following function (cf. (3.3.5)):

$$P_\alpha(z) := P(z)/P(1-\alpha+\alpha z) \ , \tag{3.3.7}$$

which is always defined in a neighbourhood of zero.
Before establishing the canonical representation of the discrete self-dec pgf's, we state an auxiliary lemma.

LEMMA 3.3.4. If P is a pgf, then

$$\lim_{x\uparrow 1} (1-x)P'(x) = 0 \ .$$

PROOF. As P' is nondecreasing on $[0,1)$, we can write for all $x \in [0,1)$

$$0 \le (1-x)P'(x) \le \int_{(x,1)} P'(y)dy = 1 - P(x) \ ,$$

which tends to zero as $x \uparrow 1$. Hence the lemma is proved. □

THEOREM 3.3.5. A pgf P, with $0 < P(0) < 1$, is discrete self-dec iff P has the form

$$P(z) = \exp[-\mu \int_{(z,1)} \frac{1-Q(u)}{1-u}\, du] \ , \tag{3.3.8}$$

where $\mu > 0$ and Q is a pgf with $Q(0) = 0$; the representation (μ, Q) is unique. Equivalently, P is discrete self-dec iff P is inf div and has a canonical sequence $\{r_n(1)\}$ (cf. theorem 1.5.3) that is nonincreasing.

PROOF. Let P be discrete self-dec, i.e. let the function P_α, defined by (3.3.7), be a pgf. For $0 \le z < 1$ we can write

$$P(1-\alpha+\alpha z) - P(z) = (1-\alpha)(1-z)P'(z) + o(1-\alpha) \qquad (\alpha \uparrow 1) \ ,$$

and hence

$$P_\alpha(z) = \{1 + (1-\alpha)(1-z)P'(z)/P(z) + o(1-\alpha)\}^{-1} \qquad (\alpha \uparrow 1) \ .$$

Let $\gamma > 0$, and take α_n such that $\gamma/(1-\alpha_n) = n \in \mathbb{N}$, i.e. $\alpha_n = 1 - \gamma/n$. Then for all $n \in \mathbb{N}$ $\{P_{\alpha_n}(z)\}^{\gamma/(1-\alpha_n)}$ is a pgf, for which

$$S_\gamma(z) := \lim_{n\to\infty} \{P_{\alpha_n}(z)\}^{\gamma/(1-\alpha_n)} = \lim_{n\to\infty} \{1 + \frac{\gamma}{n}(1-z)\frac{P'(z)}{P(z)} + o(\frac{1}{n})\}^{-n} = \tag{3.3.9}$$
$$= \exp[-\gamma(1-z)P'(z)/P(z)] \ .$$

Since $S_\gamma(z) \to 1$ as $z \uparrow 1$ (cf. lemma 3.3.4), it is seen by theorem 1.3.5 that S_γ is a pgf for all $\gamma > 0$. As $\{S_1\}^\gamma = S_\gamma$, it follows that $S := S_1$ is an inf div pgf, and hence (theorem 1.5.1) there exist $\mu > 0$ and a pgf Q with $Q(0) = 0$ such that S is compound-Poisson-(μ,Q). From (3.3.9) it is now seen that the R_1-function of P is given by

$$(3.3.10) \qquad R_1(z) = P'(z)/P(z) = \frac{-\log S(z)}{1-z} = \mu\,\frac{1-Q(z)}{1-z}\ ,$$

which yields (3.3.8). Furthermore, if Q is the pgf of $\{q_n\}_0^\infty$, from (3.3.10) it follows that the $r_n(1)$'s corresponding to P are given by

$$(3.3.11) \qquad r_n(1) = \mu \sum_{k=n+1}^{\infty} q_k \qquad (n \in \mathbb{N}_0)\ ;$$

hence $r_n(1) \geq 0$ for all $n \in \mathbb{N}_0$ (so $P \in \mathcal{C}_1$) and $r_n(1)$ is nonincreasing. Conversely, let $P \in \mathcal{C}_1$ with a nonincreasing canonical sequence $\{r_n(1)\}$, and let $\alpha \in (0,1)$. In view of the second part of theorem 1.5.3, for the function P_α we can write

$$(3.3.12) \qquad P_\alpha(z) = \exp[-\int_{(z,1-\alpha+\alpha z)} R_1(u)\,du]\ ,$$

so that

$$\frac{d}{dz}\log P_\alpha(z) = R_1(z) - \alpha R_1(1-\alpha+\alpha z)\ .$$

This function has a power-series expansion with the following coefficients:

$$r_n(1) - \alpha^{n+1}\sum_{k=n}^{\infty}\binom{k}{n}(1-\alpha)^{k-n}r_k(1) \geq r_n(1)\{1-\alpha^{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}(1-\alpha)^k\} =$$

$$= r_n(1)\{1-\alpha^{n+1}\sum_{k=0}^{\infty}(-1)^k\binom{-n-1}{k}(1-\alpha)^k\} = 0\ ,$$

where we have used the fact that $r_n(1)$ is nonincreasing. It follows that $\log\{P_\alpha(z)/P_\alpha(0)\}$, and hence P_α, is abs mon. Finally, $P_\alpha(z) \to 1$ as $z \uparrow 1$, i.e. P_α is a pgf, so P is discrete self-dec. □

COROLLARY 3.3.6. The factors P_α $(0 < \alpha < 1)$ of a discrete self-dec pgf P are all inf div.

COROLLARY 3.3.7. If $\{p_n\}_0^\infty$ is discrete self-dec with $0 < p_0 < 1$, then $p_n > 0$ for all $n \in \mathbb{N}_0$.

PROOF. If p_1 would vanish, then $r_o(1) = p_1/p_o = 0$, and hence, as $r_n(1)$ is nonincreasing, $r_n(1) = 0$ for all $n \in \mathbb{N}_o$, which is only possible if $p_o = 1$. It follows that $p_1 > 0$, but then $p_n > 0$ for all $n \in \mathbb{N}_o$ by theorem 1.5.7. □

The unimodality of discrete self-dec distributions is a corollary to the following theorem (cf. corollary 1.5.5).

THEOREM 3.3.8. Let $\{p_n\}_o^\infty$ and $\{r_n\}_o^\infty$ be sequences of real numbers with $p_o > 0$, $p_n \geq 0$ $(n \in \mathbb{N})$, r_n nonincreasing and such that

$$(3.3.13) \qquad (n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \qquad (n \in \mathbb{N}_o)\ .$$

Then $\{p_n\}_o^\infty$ is *unimodal*, i.e. $p_n - p_{n-1}$ changes sign at most once (put $p_{-1} = 0$); $\{p_n\}_o^\infty$ is nonincreasing iff $r_o \leq 1$.

PROOF. Our proof is suggested by the proof of Wolfe (1971a) for self-decomposable densities on $(0,\infty)$. First we introduce the sequences $\{d_n\}_o^\infty$ and $\{\lambda_n\}_o^\infty$ by

$$d_n := p_n - p_{n-1},\ \lambda_n := r_n - r_{n+1} \qquad (n \in \mathbb{N}_o)\ .$$

Because of the monotonicity of $\{r_n\}$ we have $\lambda_n \geq 0$ $(n \in \mathbb{N}_o)$, while

$$(3.3.14) \qquad r_n = r_o - \sum_{k=0}^{n-1} \lambda_k \qquad (n \in \mathbb{N})\ .$$

From (3.3.13) we obtain by subtraction

$$(3.3.15) \qquad (n+1)d_{n+1} = (r_o - 1)p_n - \sum_{k=0}^{n-1} \lambda_k p_{n-k-1} \qquad (n \in \mathbb{N}_o)\ ,$$

and hence $d_n \leq 0$ for $n \in \mathbb{N}$ iff $r_o \leq 1$. Now let $r_o > 1$, and suppose that there exist $n_1 \in \mathbb{N}$ and $n_2 := n_1 + m$ (some $m \in \mathbb{N}$) such that

$$(3.3.16) \qquad d_1 > 0, d_2 \geq 0, \ldots, d_{n_1} \geq 0, d_{n_1+1} < 0, \ldots, d_{n_2} \leq 0, d_{n_2+1} > 0\ .$$

Then we have, putting $p_{n-j} = 0$ if $j > n$,

$$(3.3.17) \qquad \begin{array}{ll} p_{n_1-j} \leq p_{n_1} & (j = 1,2,\ldots,m) \\ p_{n_1-j} \leq p_{n_2-j} & (j = m+1, m+2, \ldots)\ . \end{array}$$

From (3.3.15) and (3.3.16) it is seen that on the one hand

$$(3.3.18)\qquad (n_1+1)d_{n_1+1} = (r_o-1)p_{n_1} - \sum_{k=0}^{n_1-1} \lambda_k p_{n_1-k-1} < 0\ ,$$

and on the other hand

$$(3.3.19)\qquad (n_2+1)d_{n_2+1} = (r_o-1)p_{n_2} - \sum_{k=0}^{n_2-1} \lambda_k p_{n_2-k-1} > 0\ .$$

From (3.3.19) it follows that

$$(r_o-1)p_{n_2} > \sum_{k=0}^{m-1} \lambda_k p_{n_2} + \sum_{k=m}^{n_2-1} \lambda_k p_{n_2-k-1}\ ,$$

and hence, because of (3.3.14),

$$(r_m-1)p_{n_2} > \sum_{k=m}^{n_2-1} \lambda_k p_{n_2-k-1}\ .$$

Now, using this and (3.3.17), we can estimate in the following way:

$$\sum_{k=0}^{n_1-1} \lambda_k p_{n_1-k-1} \le \sum_{k=0}^{m-1} \lambda_k p_{n_1} + \sum_{k=m}^{n_2-1} \lambda_k p_{n_2-k-1} <$$

$$< (r_o-r_m)p_{n_1} + (r_m-1)p_{n_2} < (r_o-r_m)p_{n_1} + (r_m-1)p_{n_1} = (r_o-1)p_{n_1}\ ,$$

which contradicts (3.3.18). It follows that (3.3.16) is impossible. □

COROLLARY 3.3.9. A discrete self-dec distribution $\{p_n\}_o^\infty$ is unimodal; it is nonincreasing iff $r_o(1) = p_1/p_o \le 1$. Equivalently, an inf div lattice distribution with $p_o > 0$ is unimodal if its canonical sequence $\{r_n(1)\}$ is nonincreasing; it is nonincreasing iff in addition $r_o(1) \le 1$.

REMARK 3.3.10. The unimodality of discrete self-dec distributions can be used to give a slightly simpler proof of the unimodality of self-dec distributions on $[0,\infty)$. The latter distributions have a Lévy function M (cf. theorem 1.4.7) that is absolutely continuous with $xM'(x)$ nonincreasing on $(0,\infty)$, i.e. (cf. corollary 1.7.4) its K_o-functions are concave on $[0,\infty)$. Now, such a K_o-function can be approximated by step functions for which the step heights form a nonincreasing sequence, and hence (cf. theorem 1.7.7) by making the lattice finer it can be seen that a self-dec distribution on $[0,\infty)$

is the limit of discrete self-dec distributions. This procedure amounts to a more drastic discretization than that used by Wolfe (1971a).

REMARK 3.3.11. In theorem 3.3.8 the r_n's are not supposed to be all nonnegative, i.e. we seem to find a sufficient condition for unimodality of more general sequences than inf div lattice distributions. For nonnegative p_n's, however, r_n nonincreasing implies $r_n \geq 0$ $(n \in \mathbb{N}_o)$, which can be shown as folllows.

First, let $\{p_n\}$ be nondecreasing. Suppose that there exists $n_o \in \mathbb{N}$ such that $r_{n_o} < 0$, then for all $n > n_o$ we have

$$(n+1)p_{n+1} \leq \sum_{k=0}^{n_o-1} r_k p_{n-k} \leq r_o n_o p_n ,$$

which contradicts the fact that $\{p_n\}$ is nondecreasing.

If $\{p_n\}$ is not nondecreasing, then, as $\{p_n\}$ is unimodal, $\{p_n\}$ is bounded: $p_n \leq M$, say, for all $n \in \mathbb{N}_o$. Suppose that not all r_n's are nonnegative, then there exist $c > 0$ and $n_o \in \mathbb{N}$ such that $r_n \leq -c$ for all $n \geq n_o$. It follows that

$$(n+1)p_{n+1} \leq r_o \sum_{k=0}^{n_o-1} p_{n-k} - c \sum_{k=0}^{n-n_o} p_k .$$

Now, if $\Sigma p_n = \infty$, then we can choose $N \in \mathbb{N}$ such that for n sufficiently large

$$(n+1)p_{n+1} \leq r_o n_o M - cN < 0 ;$$

similarly, if $\Sigma p_n =: \ell < \infty$, then we can choose $\varepsilon > 0$ and $\delta > 0$, such that for n sufficiently large

$$(n+1)p_{n+1} \leq r_o n_o \varepsilon - c(\ell - \delta) < 0 .$$

Thus in both cases we have obtained a contradiction to the fact that $p_n \geq 0$ for all $n \in \mathbb{N}_o$.

The class of self-dec distributions contains a very important subclass, viz. the class of stable distributions, which can be introduced as follows (cf. Feller (1971), ch. VI): a rv X is said to be (strictly) *stable* with exponent γ (necessarily $\in (0,2]$) if

$$(3.3.20)\quad (s_1+s_2)^{1/\gamma}X \stackrel{d}{=} s_1^{1/\gamma}X_1 + s_2^{1/\gamma}X_2 \qquad (s_1 > 0;\ s_2 > 0)\ ,$$

where X_1 and X_2 are independent rv's with the same distribution as X. We rewrite (3.3.20) as

$$(3.3.21)\quad X \stackrel{d}{=} \alpha X_1 + (1-\alpha^\gamma)^{1/\gamma}X_2 \qquad (0 < \alpha < 1)\ ,$$

or in terms of chf's

$$(3.3.22)\quad \tilde{F}(t) = \tilde{F}(\alpha t)\tilde{F}((1-\alpha^\gamma)^{1/\gamma}t) \qquad (t \in \mathbb{R};\ 0 < \alpha < 1)\ .$$

Now, analogous to the definition of discrete self-dec distributions, we introduce discrete stable distributions by replacing αX_1 in (3.3.21) by $\alpha \circ X_1$ as given by example 3.3.1(iii), and similarly for the other term in (3.3.21). In terms of pgf's we then obtain the following definition.

DEFINITION 3.3.12. A pgf P with $0 < P(0) < 1$ is said to be (strictly) *discrete stable* with exponent $\gamma > 0$ if it satisfies

$$(3.3.23)\quad P(z) = P(1-\alpha+\alpha z)P(1 - (1-\alpha^\gamma)^{1/\gamma}(1-z)) \qquad (|z| \le 1;\ 0 < \alpha < 1).$$

Comparing the defining relations (3.3.5) and (3.3.23), we see that, as in the continuous case, the following theorem holds.

THEOREM 3.3.13. A discrete stable distribution is discrete self-dec, and hence unimodal.

Next we establish the canonical representation of a discrete stable distribution.

THEOREM 3.3.14. A pgf P with $0 < P(0) < 1$ is discrete stable with exponent γ iff P has the form

$$(3.3.24)\quad P(z) = \exp[-\mu(1-z)^\gamma] =: Q_\gamma^\mu(z) \qquad (|z| \le 1)\ ,$$

where $\mu > 0$. The exponent γ necessarily satisfies $0 < \gamma \le 1$.

PROOF. Let P be discrete stable with exponent γ. As for $0 \le z < 1$ we can write

$$P(1-\alpha+\alpha z) - P(z) = (1-\alpha)(1-z)P'(z) + o(1-\alpha) \qquad (\alpha \uparrow 1)\ ,$$

it follows from (3.3.23), with $u_\alpha := (1-\alpha^\gamma)^{1/\gamma}$, that

$$\lim_{\alpha\uparrow 1} \frac{1-P(1-u_\alpha(1-z))}{1-\alpha} = \lim_{\alpha\uparrow 1} \frac{1}{P(1-\alpha+\alpha z)} \frac{P(1-\alpha+\alpha z) - P(z)}{1-\alpha} =$$

$$= (1-z)P'(z)/P(z) \;,$$

or, equivalently,

$$(3.3.25)\quad \lim_{\alpha\uparrow 1} \frac{1-P(1-u_\alpha(1-z))}{u_\alpha^\gamma} = \frac{1}{\gamma}(1-z)P'(z)/P(z) \qquad (0 \le z < 1) \;.$$

As for $0 \le z < 1$ we have $\alpha \uparrow 1$ iff $v := u_\alpha(1-z) \downarrow 0$, we can rewrite (3.3.25) in the form

$$(3.3.26)\quad \lim_{v\downarrow 0} \frac{1-P(1-v)}{v^\gamma} = \frac{1}{\gamma}(1-z)^{1-\gamma}P'(z)/P(z) \qquad (0 \le z < 1) \;.$$

As the left-hand side, and hence the right-hand side, of (3.3.26) is independent of $z \in [0,1)$, we get, by taking $z = 0$ in (3.3.26),

$$(3.3.27)\quad P'(z)/P(z) = \frac{p_1}{p_o}(1-z)^{\gamma-1} \qquad (0 \le z < 1) \;.$$

Integrating this equation and observing that the resulting function is analytic in $|z| \le 1$, we see that P has the form (3.3.24), with μ given by

$$(3.3.28)\quad \mu = \gamma^{-1}p_1/p_o = r_o(1)/\gamma \;.$$

Conversely, if $P = Q_\gamma^\mu$, then it is easily verified that P satisfies (3.3.23), i.e. P is discrete stable.
Finally, as $P'(1) > 0$ (possibly infinite) unless $P(0) = 1$, from (3.3.27) it is seen that $0 < \gamma \le 1$. □

COROLLARY 3.3.15. A discrete stable distribution $\{p_n\}_o^\infty$ has a finite first moment iff its exponent γ is equal to 1, in which case $\{p_n\}$ is Poissonian.

REMARK 3.3.16. From (3.3.27) we see that the canonical sequence $\{r_n(1)\}$ of Q_γ^μ is given by

$$(3.3.29)\quad r_n(1) = \mu\gamma(-1)^n\binom{\gamma-1}{n} = \mu\gamma\binom{n-\gamma}{n} \qquad (n \in \mathbb{N}_o) \;.$$

As $r_n(1) \ge 0$ for all n (Q_γ^μ is inf div), we see also from (3.3.29) that necessarily $0 < \gamma \le 1$.

The discrete stable pgf's, i.e. the pgf's Q_γ^μ $(0 < \gamma \le 1;\ \mu > 0)$ from (3.3.24), are quite similar to the PLST's of the stable distributions on $[0,\infty)$ (cf. Feller (1971), ch. XIII): a df F on $[0,\infty)$ is stable with exponent γ (necessarily $\in (0,1]$) iff $\hat{F}$ has the form

$$(3.3.30)\quad \hat{F}(\tau) = \exp[-\mu\tau^\gamma] =: \hat{G}_\gamma^\mu(\tau) \qquad (\tau \ge 0)\ ,$$

where $\mu > 0$. Rather curiously, in the discrete case the Poisson distribution replaces the degenerate one (cf. corollary 3.3.15).
A df F on $[0,\infty)$ is said to be in the *domain of attraction* of G_γ^μ if there exist α_n $(n \in \mathbb{N})$ such that

$$(3.3.31)\quad \lim_{n\to\infty} \{\hat{F}(\alpha_n\tau)\}^n = \hat{G}_\gamma^\mu(\tau) \qquad (\tau \ge 0)\ ,$$

or, if $X_1, X_2, \ldots$ are independent rv's with df F and if Y_γ has df G_γ^μ,

$$\lim_{n\to\infty} \alpha_n(X_1 + \ldots + X_n) \overset{d}{=} Y_\gamma\ .$$

As shown in e.g. Feller (1971), ch. XIII, only a stable PLST appears as a limit like in the left-hand side of (3.3.31). Furthermore the α_n's necessarily satisfy

$$(3.3.32)\quad \alpha_n \sim n^{-1/\gamma} \qquad (n \to \infty)\ .$$

Now, similarly, a lattice distribution $\{p_n\}_0^\infty$ with $p_0 > 0$ is said to be in the *domain of discrete attraction* of Q_γ^μ if there exist α_n $(n \in \mathbb{N})$ such that

$$(3.3.33)\quad \lim_{n\to\infty} \{P(1-\alpha_n+\alpha_n z)\}^n = Q_\gamma^\mu(z) \qquad (|z| \le 1)\ .$$

Taking α_n such that (3.3.32) holds, we see that Q_γ^μ belongs to its own domain of attraction. Furthermore we have the following property.

THEOREM 3.3.17. Every lattice distribution with finite first moment belongs to the domain of discrete attraction of Q_1^μ, i.e. of the Poisson distribution.

PROOF. Let P be a pgf with $P'(1) =: \mu < \infty$, and let $z \in [0,1)$. Then for all $n \in \mathbb{N}$ there exists $\xi_n \in (1-\frac{1}{n}(1-z), 1)$ such that

$$1 - P(1-\tfrac{1}{n}(1-z)) = \tfrac{1}{n}(1-z)P'(\xi_n)\ .$$

Now, as $\lim_{n\to\infty} P'(\xi_n) = \mu$, it follows that for $0 \le z < 1$, and hence by the continuity theorem for all $|z| \le 1$,

$$\lim_{n\to\infty} \{P(1-\frac{1}{n}(1-z))\}^n = \lim_{n\to\infty} \{1-\frac{\mu}{n}(1-z)+o(\frac{1}{n})\}^n = \exp[\mu(z-1)] \ . \qquad \square$$

A general theory of attraction could easily be developed. The domains of discrete attraction, however, are completely determined by their continuous counterparts. In fact, for every $\gamma \in (0,1]$ and $\mu > 0$ we have

$$Q_\gamma^\mu(1-\tau) = \hat{G}_\gamma^\mu(\tau) \qquad (\tau \ge 0) \ ,$$

and as for every $\tau \ge 0$

$$\{P_X(1-\alpha_n\tau)\}^n = \{E \exp[X \log(1-\alpha_n\tau)]\}^n \sim$$

$$\sim \{E \exp[-\alpha_n \tau X]\}^n = \{\hat{F}_X(\alpha_n\tau)\}^n \qquad (n \to \infty) \ ,$$

where X is an $\mathbb{N}_0$-valued rv with pgf P_X and df F_X, it follows that

$$\lim_{n\to\infty} \{P_X(1-\alpha_n+\alpha_n z)\}^n = Q_\gamma^\mu(z) \Leftrightarrow \lim_{n\to\infty} \{\hat{F}_X(\alpha_n\tau)\}^n = \hat{G}_\gamma^\mu(\tau) \ ,$$

i.e. an $\mathbb{N}_0$-valued rv X is in the domain of discrete attraction of Q_γ^μ iff it is in the domain of attraction of G_γ^μ.

3.4. *α-decomposable(1) lattice distributions*

In this section we briefly consider pgf's P that have $1-\alpha+\alpha P(z)$ as a factor (cf. example 3.3.1(ii)). Proceeding as in sections 1 and 2 for α-dec pgf's (which have $P(\alpha z)/P(\alpha)$ as a factor), we obtain properties that are similar to those of the α-dec pgf's. The main purpose is to give an analogue of theorem 3.2.4 for $\mathcal{C}_0$.

DEFINITION 3.4.1. For $0 < \alpha < 1$ a pgf P with $P(0) > 0$ is said to be *α-decomposable(1)* (α-dec(1)) if there exists a pgf P_α such that

$$(3.4.1) \qquad P(z) = \{1-\alpha+\alpha P(z)\}P_\alpha(z) \qquad (|z| \le 1) \ .$$

P is called *totally-decomposable(1)* if P is α-dec(1) for all $\alpha \in (0,1)$.

For an arbitrary pgf P with $P(0) > 0$ and for $\alpha \in (0,1)$ we denote by P_α the following function (cf. (3.4.1)):

$$(3.4.2)\qquad P_\alpha(z) := P(z)/\{1-\alpha+\alpha P(z)\}\ ,$$

which is always defined in some neighbourhood of zero. Obviously, we have the following criterion (cf. lemma 1.3.4) and representation lemma (solve (3.4.1) for P).

LEMMA 3.4.2. For $0 < \alpha < 1$ a pgf P with $P(0) > 0$ is α-dec(1) iff P_α is abs mon.

LEMMA 3.4.3. If $0 < \alpha < 1$ and if P is a pgf with $P(0) > 0$, then P is α-dec(1) iff P has the form

$$(3.4.3)\qquad P(z) = \frac{1-\alpha}{1-\alpha Q(z)}\,Q(z) \qquad (|z| \le 1)\ ,$$

where Q is a pgf. The representation is unique: $Q = P_\alpha$.

Thus, an α-dec(1) pgf P has a compound geometric factor. It turns out that the factor P_α of P is in $\mathcal{C}_o$ iff $P \in \mathcal{C}_o$. We state this in the following lemma, which is easily verified by using theorem 2.4.9(ii) and the following relation between the R_o-functions R_o and $R_o^{(\alpha)}$ of P and P_α, respectively:

$$R_o(z) = \{1 + \frac{\alpha}{1-\alpha}\,P(0)\}R_o^{(\alpha)}(z)\ .$$

LEMMA 3.4.4. If P is a pgf with $P(0) > 0$, then $P \in \mathcal{C}_o$ iff for some, and then for all, $\alpha \in (0,1)$ P is α-dec(1) with factor $P_\alpha \in \mathcal{C}_o$.

By expressing P_β in P_α one obtains the following implication:

$$(3.4.4)\qquad [P\ \alpha\text{-dec}(1),\ 0 < \beta < \alpha < 1] \Rightarrow P\ \beta\text{-dec}(1)\ .$$

It follows that the classes of α-dec(1) pgf's are decreasing in α. Now the limiting class ($\alpha \uparrow 1$) turns out to be $\mathcal{C}_o$, i.e. we have the following analogue of theorem 3.2.4 for $\mathcal{C}_o$.

THEOREM 3.4.5. A pgf P with $P(0) > 0$ is in $\mathcal{C}_o$ iff P is tot-dec(1).

PROOF. In view of lemma 3.4.4 we only need show that a tot-dec(1) pgf is in $\mathcal{C}_o$. So let P be tot-dec(1), i.e. let P_α be a pgf for all $\alpha \in (0,1)$. Rewrite P_α as follows

$$P_\alpha(z) = \{1 + (1-\alpha)[P(z)^{-1} - 1]\}^{-1}\ ,$$

let $\gamma > 0$ and take $\alpha_n = 1 - \gamma/n$ ($n \in \mathbb{N}$). Then it follows that $\{P_{\alpha_n}(z)\}^{\gamma/(1-\alpha_n)}$ is a pgf ($n \in \mathbb{N}$), which satisfies

$$S_\gamma(z) := \lim_{n\to\infty} \{P_{\alpha_n}(z)\}^{\gamma/(1-\alpha_n)} = \lim_{n\to\infty} \{1 + \tfrac{\gamma}{n}[P(z)^{-1} - 1]\}^{-n} =$$

$$= \exp[-\gamma(P(z)^{-1} - 1)] \; .$$

Since $S_\gamma(z) \to 1$ as $z \uparrow 1$, by theorem 1.3.5 it is seen that S_γ is a pgf for all $\gamma > 0$, and hence

$$S_1(z) = \exp[1 - P(z)^{-1}]$$

is an inf div pgf. From theorem 2.4.8(ii) it now follows that $P \in \mathcal{C}_o$. □

COROLLARY 3.4.6. The factors P_α ($0 < \alpha < 1$) of a tot-dec(1) pgf are all in $\mathcal{C}_o$.

Finally, we note that (cf. sections 1 and 2) we might consider α-fact(1) pgf's, i.e. pgf's P of the form $\{1-\alpha+\alpha Q(z)\}Q(z)$, but we shall not do so here.

CHAPTER 4

THE CLASSES C_α IN RELATION TO RENEWAL THEORY

In chapter 2 we introduced the classes $\mathcal{C}_\alpha$ $(0 \leq \alpha \leq 1)$. Most of their properties, given there, were most easily formulated in terms of pgf's. Now, in the present chapter, we want to derive properties of the probabilities themselves; in particular, we consider inequalities, asymptotic behaviour and closure properties.
The recurrence relations by means of which $\mathcal{C}_o$ can be characterized (cf. corollary 1.5.9), are very similar to those defining the so-called renewal sequences (class $\mathcal{R}_o$). This class $\mathcal{R}_o$ has many well-known properties, which are often easily obtained by a probabilistic interpretation (cf. Feller (1968), ch. XIII and Kingman (1972), ch. 1). Now, using the relation between $\mathcal{C}_o$ and $\mathcal{R}_o$, we obtain similar properties for $\mathcal{C}_o$ (section 1). In section 2 we consider the case $0 < \alpha < 1$. We introduce classes $\mathcal{R}_\alpha$ of generalized renewal sequences that are related to the $\mathcal{C}_\alpha$'s in the same way as $\mathcal{R}_o$ is related to $\mathcal{C}_o$, and we investigate to what extent the properties of $\mathcal{R}_o$ can be extended to the $\mathcal{R}_\alpha$'s. Specifically, we look for a probabilistic interpretation of the sequences in $\mathcal{R}_\alpha$. We do find interpretations, but these are rather complicated and yield only few results. Finally, in section 3 we briefly consider an extension $\mathcal{R}_1$ of $\mathcal{C}_1$.

4.1. *The class* $\mathcal{C}_o$ *and discrete-time renewal theory*

The class $\mathcal{C}_o$ of compound geometric lattice distributions $\{p_n\}_o^\infty$ can be characterized by means of the recurrence relations

$$(4.1.1a) \quad p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_o) ,$$

where the sequence $\{r_n(0)\}_o^\infty$ satisfies

$$(4.1.1b) \quad r_n(0) \geq 0 \ (n \in \mathbb{N}_o),\ r(0) := \sum_{n=0}^{\infty} r_n(0) < 1 .$$

Now, the recurrence relations (4.1.1a) are very similar to those defining the so-called renewal sequences (cf. Kingman (1972), ch. 1): a sequence $\{u_n\}_o^\infty$, with $u_o = 1$, is said to be a *renewal sequence* (or of class $\mathcal{R}_o$) if it satisfies

$$(4.1.2a)\qquad u_n = \sum_{k=1}^{n} f_k u_{n-k} \qquad (n \in \mathbb{N}) ,$$

where the sequence $\{f_n\}_1^\infty$ is such that

$$(4.1.2b)\qquad f_n \geq 0 \ (n \in \mathbb{N}),\ f := \sum_{n=1}^{\infty} f_n \leq 1 ;$$

in this case $\{u_n\}$ is called the renewal sequence *associated with* $\{f_n\}_1^\infty$. In fact, we have the following relation between $\mathcal{C}_o$ and $\mathcal{R}_o$.

LEMMA 4.1.1.

(i) If $\{p_n\} \in \mathcal{C}_o$ and if $u_n := p_n/p_o$ $(n \in \mathbb{N}_o)$, then $\{u_n\} \in \mathcal{R}_o$ and $\{u_n\}$ is associated with $\{f_n\}_1^\infty$ given by $f_n = r_{n-1}(0)$ $(n \in \mathbb{N})$.

(ii) If $\{u_n\} \in \mathcal{R}_o$, associated with $\{f_n\}$, then

a) If $f < 1$ and $p_n := (1-f)u_n$ $(n \in \mathbb{N}_o)$, then $\{p_n\} \in \mathcal{C}_o$ with $r_n(0) = f_{n+1}$ $(n \in \mathbb{N}_o)$;

b) If $f = 1$ and $p_n := \gamma^n u_n/U(\gamma)$, where $\gamma \in (0,1)$, then $\{p_n\} \in \mathcal{C}_o$ with $r_n(0) = \gamma^{n+1} f_{n+1}$ $(n \in \mathbb{N}_o)$.

PROOF.

(i) immediately follows by comparing (4.1.1a) with (4.1.2a), and (4.1.1b) with (4.1.2b).

(ii) Let $\{u_n\} \in \mathcal{R}_o$, associated with $\{f_n\}$. If $f < 1$, then, as we shall see in theorem 4.1.2(i), $\sum_{n=0}^{\infty} u_n = (1-f)^{-1}$ is finite. Hence (cf. (i)), if $p_n := (1-f)u_n$, then $\{p_n\} \in \mathcal{C}_o$. If $f = 1$, then $\sum_{n=0}^{\infty} u_n = \infty$, but $U(\gamma) = \sum_{n=0}^{\infty} u_n\gamma^n < \infty$ for all $\gamma \in (0,1)$ (cf. lemma 4.1.3). It follows that $\{p_n\}_o^\infty$, with $p_n := \gamma^n u_n/U(\gamma)$, is a probability distribution that satisfies (4.1.1a) with $r_n(0) = \gamma^{n+1} f_{n+1} \geq 0$. Hence $\{p_n\} \in \mathcal{C}_o$. □

The recurrence relations (4.1.2a) have been studied by several authors, e.g. by Kaluza (1928), De Bruijn & Erdős (1951), Lamperti (1958), Kendall (1967), Feller (1968) and Kingman (1972). The main properties of these relations will be given in the remainder of this section. Using lemma 4.1.1 we can transfer most of these properties to $\mathcal{C}_o$, but we shall do so only if this is of special interest.

In the next section we want to investigate to what extent the properties that can be obtained for $\mathcal{C}_o$, can be extended to $\mathcal{C}_\alpha$ $(0 < \alpha < 1)$. As, to this end, the sequences in C_α will be related to the so-called delayed renewal sequences, we consider recurrence relations that are slightly more general than those in (4.1.2a).

Our starting point is the theory of recurrent events of Feller (1968), ch. XIII, but we prefer a formulation similar to the one usual for renewal theory in continuous time (cf. Feller (1971), ch. XI). Let us introduce two sequences $\{b_n\}_o^\infty$ and $\{f_n\}_1^\infty$ satisfying

$$(4.1.3) \qquad b_n \geq 0 \;(n \in \mathbb{N}_o),\; 0 < b := \sum_{n=0}^{\infty} b_n < \infty,\; f_n \geq 0 \;(n \in \mathbb{N}),\; 0 < f := \sum_{n=1}^{\infty} f_n < \infty \;.$$

Then the (discrete) *renewal equation associated with* $\{b_n\}$ *and* $\{f_n\}$, is defined by

$$(4.1.4) \qquad v_n = b_n + \sum_{k=1}^{n} f_k v_{n-k} \qquad (n \in \mathbb{N}_o) \;.$$

It is called *pure* if $b_n = \delta_{o,n}$, and *delayed* otherwise. Clearly, the renewal equation has a unique solution $\{v_n\}_o^\infty$, and its gf V satisfies

$$(4.1.5) \qquad V(z) = B(z)/\{1 - F(z)\} \;,$$

or, if $\{u_n\}_o^\infty$ is the solution of the *pure* renewal equation associated with $\{f_n\}$,

$$(4.1.6) \qquad V(z) = B(z)U(z) \;.$$

The *period* d of a sequence $\{a_n\}_o^\infty$ of real numbers (not all a_n $(n \in \mathbb{N})$ zero) is defined as follows:

$$(4.1.7) \qquad d := \gcd\{n \in \mathbb{N} \mid a_n \neq 0\} \;.$$

Here the value of a_o is not relevant; the period of a sequence $\{a_n\}_1^\infty$ is also defined by (4.1.7). Of course, (4.1.7) is equivalent to the definition of Feller (1968), who defines the period of $\{a_n\}$ by

$$d := \max\{k \in \mathbb{N} \mid \forall_{n\in\mathbb{N}} [a_n \neq 0 \Rightarrow k|n]\} \;.$$

If $d = 1$, then the sequence $\{a_n\}$ is called *aperiodic*. Now we are ready to state the following basic result, which is known as the *renewal theorem* (cf. Feller (1968), ch. XIII).

THEOREM 4.1.2. Suppose (4.1.3) and that $\{f_n\}_1^\infty$ is aperiodic. Then the solution $\{v_n\}_o^\infty$ of the renewal equation associated with $\{b_n\}$ and $\{f_n\}$ has the following properties:

(i) $f < 1$ iff $v := \sum_{n=0}^{\infty} v_n < \infty$, in which case $v = b/(1-f)$;

(ii) If $f = 1$, then $v_\infty := \lim_{n\to\infty} v_n$ exists, and $v_\infty = b/\mu$ with $\mu := \sum_{n=1}^{\infty} nf_n$ $(\leq \infty)$;

(iii) If $f > 1$, then $\lim_{n\to\infty} x_o^n v_n = B(x_o)/\{xF'(x_o)\}$, where $x_o \in (0,1)$ is such that $F(x_o) = 1$.

We shall frequently use the following characterization of the *bounded* solutions of the renewal equation.

LEMMA 4.1.3. The solution $\{v_n\}_o^\infty$ of the renewal equation (4.1.4) is bounded iff $f \leq 1$, in which case $v_n \leq b$ for all $n \in \mathbb{N}_o$.

PROOF. Let $\{v_n\}$ be bounded. Then its gf V exists, and does not vanish, on $(0,1)$. Hence by (4.1.5) we have

$$F(x) = 1 - B(x)/V(x) \leq 1 \qquad (0 < x < 1) \quad ,$$

from which, letting $x \uparrow 1$, we see that $f = F(1) \leq 1$.
Conversely, let $f \leq 1$. Considering first the solution $\{u_n\}_o^\infty$ of the pure renewal equation associated with $\{f_n\}$, from (4.1.4), using mathematical induction, we see that $u_n \leq 1$ for all $n \in \mathbb{N}_o$. Now, equation the coefficients of z^n in (4.1.6), we obtain

$$v_n = \sum_{k=0}^{n} b_k u_{n-k} \leq \sum_{k=0}^{n} b_k \leq b \quad ,$$

and the lemma is proved. □

We are particularly interested in certain bounded solutions of (4.1.4), which frequently occur in probability theory, viz. the delayed renewal sequences, as they will provide a probabilistic interpretation of the sequences in C_α. If $\{b_n\}$ and $\{f_n\}$ are sequences satisfying (4.1.3) with $b \leq 1$ and $f \leq 1$, then the solution $\{v_n\}_o^\infty$ of the renewal equation associated with $\{b_n\}$ and $\{f_n\}$ is called the *delayed renewal sequence associated with* $\{b_n\}$ *and* $\{f_n\}$. The set of such sequences is denoted by R. We note that a renewal sequence (cf. the beginning of this section) can be considered as a delayed renewal sequence for which $b_n = \delta_{o,n}$. Hence we have

(4.1.8) $\quad R_o \subset R$,

and in order to distinguish the R_o-sequences from those in $R\backslash R_o$ we occasionally call them *pure* renewal sequences.

The (delayed) renewal sequences owe their name to their interpretation in renewal processes. For ease of reference we first give the definition of such processes for discrete time.

DEFINITION 4.1.4. Let $\{b_n\}_o^\infty$ and $\{f_n\}_1^\infty$ be two sequences satisfying (4.1.3) with $b \le 1$ and $f \le 1$. If $T_o, T_1, T_2, \ldots$ are mutually independent, $\mathbb{N}_o$-valued (possibly defective) rv's such that $\{b_n\}$ is the distribution of T_o and $\{f_n\}$ that of T_k for $k \in \mathbb{N}$, then the sequence $\{S_n\}_o^\infty$, defined by

$$S_n = \sum_{k=0}^{n} T_k \qquad (n \in \mathbb{N}_o) ,$$

is called the (discrete-time) *renewal process associated with* $\{b_n\}$ *and* $\{f_n\}$. The rv S_n is called the n-th renewal epoch, and T_k the k-th life time. If $f = 1$, then the mean recurrence time μ is defined by

$$\mu := ET_1 = \sum_{n=1}^{\infty} nf_n \qquad (\le \infty) .$$

Finally, $\{S_n\}$ is called *pure* if $b_n = \delta_{o,n}$ (*delayed* otherwise), and *persistent* if $b = f = 1$ (*transient* otherwise).

THEOREM 4.1.5. A sequence $\{v_n\}_o^\infty$ is in R iff there exists a (discrete-time) renewal process $\{S_n\}_o^\infty$ such that

(4.1.9) $\qquad v_n = P(\exists_{k \in \mathbb{N}_o} S_k = n) \qquad (n \in \mathbb{N}_o) .$

If $\{v_n\}$ is associated with $\{b_n\}$ and $\{f_n\}$, then $\{S_n\}$ is associated with the same sequences, and conversely. Finally, $\{v_n\}$ is pure ($\in R_o$) iff $\{S_n\}$ is pure.

In addition to the interpretation of $\{v_n\} \in R$, just given, there exists another one in terms of a Markov chain. A (discrete-time) Markov chain with stationary transition probabilities is a sequence $\{X_n\}_o^\infty$ of rv's, taking values in a countable state space S and satisfying the Markov property

$$P(X_n = j \mid X_o, X_1, \ldots, X_{n-1}) = P(X_n = j \mid X_{n-1}) \qquad (n \in \mathbb{N};\ j \in S),$$

such that the transition probabilities

$$p_{ij} := P(X_n = j \mid X_{n-1} = i) \qquad (i,j \in S)$$

do not depend on n. In fact, we have the following characterization of (delayed) renewal sequences (cf. Kingman (1972)).

THEOREM 4.1.6.

(i) A sequence $\{u_n\}_o^\infty$ is in $\mathcal{R}_o$ iff there exist a Markov chain $\{X_n\}_o^\infty$ and a state $j \in S$ such that

(4.1.10) $\quad u_n = P(X_n = j \mid X_o = j) \qquad (n \in \mathbb{N}_o)$.

The sequence $\{f_n\}$, $\{u_n\}$ is associated with, is then the recurrence time distribution of $j \in S$:

(4.1.11) $\quad f_n = P(X_1 \neq j,\dots,X_{n-1} \neq j,\ X_n = j \mid X_o = j) \qquad (n \in \mathbb{N})$.

(ii) A sequence $\{v_n\}_o^\infty$ is in $\mathcal{R}\backslash\mathcal{R}_o$ iff there exist a Markov chain $\{X_n\}_o^\infty$ and two distinct states $i,j \in S$ such that

(4.1.12) $\quad v_n = P(X_{n+1} = j \mid X_o = i) \qquad (n \in \mathbb{N}_o)$.

The sequences $\{b_n\}$ and $\{f_n\}$, $\{v_n\}$ is associated with, then satisfy (4.1.11) and

(4.1.13) $\quad b_n = P(X_1 \neq j,\dots,X_n \neq j, X_{n+1} = j \mid X_o = i) \qquad (n \in \mathbb{N}_o)$.

These interpretations can be used to obtain the following inequalities and closure properties for $\mathcal{R}_o$ (cf. Kingman (1972)). An analytic proof of (i) in a slightly less general situation has been given by De Bruijn & Erdős (1951), but for (ii) and (iii) such proofs seem nonexistent.

THEOREM 4.1.7.

(i) If $\{u_n\} \in \mathcal{R}_o$, then $u_n u_k \leq u_{n+k} \leq u_n u_k + 1 - u_n$ for all $n,k \in \mathbb{N}_o$.

(ii) If $\{u_n\} \in \mathcal{R}_o$ and $k \in \mathbb{N}$, then $\{u_{kn}\}_{n=0}^\infty \in \mathcal{R}_o$.

(iii) If $\{u_n\} \in \mathcal{R}_o$ and $\{v_n\} \in \mathcal{R}_o$, then also $\{u_n v_n\} \in \mathcal{R}_o$.

By lemma 4.1.1 we successively obtain from the preceding theorem the following results for $\mathcal{C}_o$. Only part (i) of the first corollary immediately follows from (4.1.1a) by mathematical induction (cf. lemma 4.1.3):

$$p_{n+1} = \sum_{k=0}^{n} r_k(0) p_{n-k} \le p_o \sum_{k=0}^{n} r_k(0) \le p_o \sum_{k=0}^{\infty} r_k(0) < p_o \; .$$

COROLLARY 4.1.8. The following inequalities hold for $\{p_n\} \in \mathcal{C}_o$:

(i) $p_n < p_o$ for all $n \in \mathbb{N}$;

(ii) $p_n p_k \le p_o p_{n+k} \le p_n p_k + p_o(p_o - p_n)$ for all $n,k \in \mathbb{N}_o$;

(iii) $p_o(p_1/p_o)^n \le p_n \le p_o - (p_o - p_1)(p_1/p_o)^{n-1}$ for all $n \in \mathbb{N}$.

COROLLARY 4.1.9. If $\{p_n\} \in \mathcal{C}_o$ and $k \in \mathbb{N}$, then $\{c_k p_{kn}\}_{n=0}^{\infty} \in \mathcal{C}_o$, where c_k is a suitable norming constant. In particular, if $k = 2$, then in terms of pgf's

$$(4.1.14) \quad \{P(z^{\frac{1}{2}}) + P(-z^{\frac{1}{2}})\}/(1 + P(-1)) \in \mathcal{C}_o \; .$$

COROLLARY 4.1.10. If $\{p_n\} \in \mathcal{C}_o$ and $\{q_n\} \in \mathcal{C}_o$, then also $\{cp_n q_n\} \in \mathcal{C}_o$ (c is a norming constant).

In view of a result of Lamperti (1958), theorem 4.1.7(iii), which also holds if $\mathcal{R}_o$ is replaced by $\mathcal{R}$, can be generalized as follows.

THEOREM 4.1.11. If $\{u_n\} \in \mathcal{R}_o$ and $\{v_n\} \in \mathcal{R}_o$, then $\{w_n\} \in \mathcal{R}_o$ for all $\alpha \ge 0$, $\beta \ge 0$, $\gamma \ge 0$ (not all zero), where w_n is defined by

$$(4.1.15) \quad w_n := (\alpha+\beta+\gamma)^{-n} \sum_{i+j+k=n} \frac{n!}{i!j!k!} \alpha^i \beta^j \gamma^k u_{i+j} v_{i+k} \qquad (n \in \mathbb{N}_o) \; .$$

PROOF. According to the result of Lamperti, $\{(\alpha+\beta+\gamma)^n w_n\}$ is a solution of the pure renewal equation. It easily follows that then also $\{w_n\}$ is a solution of the pure renewal equation, and as $u_n \le 1$ and $v_n \le 1$ (lemma 4.1.3), we see that $w_n \le 1$. Applying lemma 4.1.3 in the reverse direction, we conclude that $\{w_n\} \in \mathcal{R}_o$. □

REMARK 4.1.12. If $\alpha \ne 0$ or $\beta \ne 0$ and if $\Sigma u_n < \infty$, then $\{w_n\}$, multiplied with a suitable constant, is a lattice distribution in $\mathcal{C}_o$:

$$w_n = (\alpha+\beta+\gamma)^{-n} \sum_{\ell=0}^{n} \binom{n}{\ell} u_\ell \gamma^{n-\ell} \sum_{j=0}^{\ell} \binom{\ell}{j} \alpha^j \beta^{\ell-j} v_{n-\ell+j} \le$$

$$\le (\alpha+\beta+\gamma)^{-n} \sum_{\ell=0}^{n} \binom{n}{\ell} u_\ell (\alpha+\beta)^\ell \gamma^{n-\ell} \; ,$$

and hence

$$\sum_{n=0}^{\infty} w_n \le \sum_{\ell=0}^{\infty} u_\ell(\alpha+\beta)^\ell \sum_{n=\ell}^{\infty} \binom{n}{\ell}\gamma^{n-\ell}(\alpha+\beta+\gamma)^{-n} =$$

$$= \sum_{\ell=0}^{\infty} u_\ell(\alpha+\beta)^\ell \sum_{m=0}^{\infty} \binom{-\ell-1}{m}(-\gamma)^m(\alpha+\beta+\gamma)^{-\ell-m} =$$

$$= \sum_{\ell=0}^{\infty} u_\ell(\alpha+\beta)^\ell \left(1-\frac{\gamma}{\alpha+\beta+\gamma}\right)^{-\ell-1}(\alpha+\beta+\gamma)^{-\ell} = \frac{\alpha+\beta+\gamma}{\alpha+\beta}\sum_{\ell=0}^{\infty} u_\ell .$$

Of course, by lemma 4.1.1 we can also obtain properties of R_o from known properties of C_o. We give one example on mixtures of renewal sequences; the result seems to be new, but in fact it is a special case of theorem 4.1.7 (iii) (take $v_n = p\delta_{o,n} + (1-p)$ there; then $\{v_n\} \in R_o$, as it is associated with $\{f_n\}$, given by $f_n = (1-p)p^{n-1}$).

THEOREM 4.1.13. If $\{u_n\} \in R_o$ and $0 < p < 1$, then also $\{w_n\} \in R_o$, where

$$w_n := p\delta_{o,n} + (1-p)u_n \qquad (n \in \mathbb{N}_o) . \tag{4.1.16}$$

PROOF. Let $\{u_n\} \in R_o$ and $0 < p < 1$. Then for $\{w_n\}$, defined by (4.1.16), we have $w_o = 1$ and $0 \le w_n \le 1$ $(n \in \mathbb{N})$, and its gf W satisfies

$$W(z) = p + (1-p)U(z) \qquad (|z| < 1) .$$

Choose $\gamma \in (0,1)$, and define the lattice distributions $\{p_n\}$ and $\{q_n\}$ by

$$p_n = \gamma^n u_n/U(\gamma),\ q_n = \gamma^n w_n/W(\gamma) \qquad (n \in \mathbb{N}_o) .$$

Then, considering their pgf's, we have

$$Q(z) = W(\gamma z)/W(\gamma) = \frac{p + (1-p)U(\gamma z)}{p+(1-p)U(\gamma)} = p_1 + (1-p_1)P(z) ,$$

where $p_1 := p/\{p + (1-p)U(\gamma)\} \in (0,1)$. Since $\{p_n\} \in C_o$ (lemma 4.1.1(ii)), from theorem 2.4.9(iii) it now follows that also $\{q_n\} \in C_o$, and hence, by lemma 4.1.1(i), $\{\gamma^n w_n\} \in R_o$. But then $\{w_n\}$ satisfies the pure renewal equation, and as $w_n \le 1$, it follows by lemma 4.1.3 that $\{w_n\} \in R_o$. □

Next we consider the asymptotic behaviour of a renewal sequence. Of course, the strong result in this area has been already given in the renewal theorem (theorem 4.1.2), of which parts (i) and (ii) can be applied. If, however,

$u_\infty := \lim_{n\to\infty} u_n = 0$, one can ask how fast u_n tends to zero. The following theorem (cf. Kingman (1972)) gives an answer to this question.

THEOREM 4.1.14. Let $\{u_n\} \in \mathcal{R}_o$ be associated with $\{f_n\}_1^\infty$. Then the period d of $\{u_n\}_o^\infty$ is equal to that of $\{f_n\}_1^\infty$, and

(4.1.17) $\quad \#(\{n \in \mathbb{N} \mid u_{nd} = 0\}) < \infty$.

Furthermore $\theta := \lim_{n\to\infty} u_{nd}^{1/(nd)}$ exists in $(0,1]$, and

(4.1.18) $\quad u_n \leq \theta^n \qquad (n \in \mathbb{N}_o)$.

COROLLARY 4.1.15. If $\{u_n\} \in \mathcal{R}_o$, then the radius of convergence ρ of the gf U of $\{u_n\}_o^\infty$ is finite. Furthermore, if $v_n := \rho^n u_n$ $(n \in \mathbb{N}_o)$, then $\{v_n\}_o^\infty \in \mathcal{R}_o$.

PROOF. We have

$$\rho = \{\limsup_{n\to\infty} u_n^{1/n}\}^{-1} .$$

Since in view of theorem 4.1.14

$$\limsup_{n\to\infty} u_n^{1/n} = \limsup_{n\to\infty} u_{nd}^{1/(nd)} = \lim_{n\to\infty} u_{nd}^{1/(nd)} = \theta > 0 ,$$

it follows that

(4.1.19) $\quad \rho = \theta^{-1} < \infty$.

Now, if $v_n := \rho^n u_n$, from (4.1.18) it is seen that $v_n \leq 1$ $(n \in \mathbb{N}_o)$, and, as $\{v_n\}$ obviously satisfies the renewal equation, we conclude by lemma 4.1.3 that $\{v_n\} \in \mathcal{R}_o$. $\square$

Now, applying the renewal theorem to the sequence $\{v_n\}$, thus obtained, we can improve on the second part of theorem 4.1.14 in some sense. We state the results in the following theorem. They are interesting only if $\rho > 1$, which implies that $\sum_{n=0}^{\infty} u_n < \infty$, i.e. in fact we consider $\mathcal{C}_o$.

THEOREM 4.1.16. Let $\{u_n\} \in \mathcal{R}_o$, associated with $\{f_n\}_1^\infty$ and with radius of convergence ρ. Then $\rho < \infty$ and $F(\rho) \leq 1$, while

(i) $F(\rho) < 1$ iff $U(\rho) < \infty$, in which case $\lim_{n\to\infty} \rho^n u_n = 0$;

(ii) If $F(\rho) = 1$ and $\{f_n\}_1^\infty$ is aperiodic, then $\lim_{n\to\infty} \rho^n u_n = 1/\mu$, where μ is given by

$$\mu := \sum_{n=1}^{\infty} nf_n\rho^n = \rho F'(\rho) \qquad (\leq \infty) .$$

REMARK 4.1.17. Theorem 4.1.16 and the second part of theorem 4.1.14 have been proved previously by De Bruijn & Erdős (1951) in a slightly less general situation; they consider general solutions of the pure renewal equation, but suppose the f_n's to be all positive. For this case they show furthermore that the quantities ρ and θ can be expressed in terms of the f_n's as follows:

$$(4.1.20) \quad \rho = \theta^{-1} = \sup\{x \geq 0 \mid F(x) \leq 1\} .$$

However, this relation holds for all renewal sequences, as will be shown in a more general situation in the next section.

The class $\mathcal{R}_o$ contains two easily recognizable subclasses, viz. the class of comp mon sequences $\{u_n\}_o^\infty$ (cf. definition 1.3.10) with $u_o = 1$, and the class of bounded Kaluza sequences. Here a sequence $\{a_n\}_o^\infty$ is called a *Kaluza sequence* if $a_o = 1$, $a_n \geq 0$ $(n \in \mathbb{N})$ and $\{a_n\}_o^\infty$ is log-convex, i.e.

$$a_n^2 \leq a_{n-1}a_{n+1} \qquad (n \in \mathbb{N}) .$$

Note that in this case $a_n = 0$ for all $n \in \mathbb{N}$ or $a_n > 0$ for all $n \in \mathbb{N}$, and that $\{a_n\}$ is bounded iff it is nonincreasing. Kaluza (1928) proved the following theorem, the analogue of which for $\mathcal{C}_o$ has been already given in theorem 1.5.13.

THEOREM 4.1.18. A comp mon sequence $\{u_n\}_o^\infty$ with $u_o = 1$ is a bounded Kaluza sequence, and a bounded Kaluza sequence is in $\mathcal{R}_o$.

The bounded Kaluza sequences can be characterized as a subclass of $\mathcal{R}_o$ in two somewhat similar ways. The first one is given by Kendall (1967), the second one seems to be new.

THEOREM 4.1.19. Let $\{u_n\}_0^\infty$ be a sequence of positive numbers. Then $\{u_n\}$ is a bounded Kaluza sequence iff for all $t > 0$ (or, equivalently, for all $t = 1/m$, $m \in \mathbb{N}$) $\{u_n^t\} \in \mathcal{R}_0$, in which case $\{u_n^t\}$ is also a bounded Kaluza sequence for all $t > 0$.

Translating this result for $\{p_n\} \in \mathcal{C}_0$, we have to consider $\gamma^n p_n$ for some $\gamma \in (0,1)$ rather than p_n, since Σp_n^t is not necessarily finite for all $t > 0$.

THEOREM 4.1.20. Let $\{u_n\}_0^\infty$ be a sequence of positive numbers. For $\ell \in \mathbb{N}_0$ define the sequence $\{u_n(\ell)\}_0^\infty$ by

$$u_n(\ell) := u_{\ell+n}/u_\ell \qquad (n \in \mathbb{N}_0) \ .$$

Then $\{u_n\}$ is a bounded Kaluza sequence iff $\{u_n(\ell)\} \in \mathcal{R}_0$ for all $\ell \in \mathbb{N}_0$, in which case $\{u_n(\ell)\}$ is also a bounded Kaluza sequence for all $\ell \in \mathbb{N}_0$.

PROOF. Let $\{u_n\}$ be a bounded Kaluza sequence and take $\ell \in \mathbb{N}_0$ fixed. As $\{u_n\}_0^\infty$ is nonincreasing, $u_n(\ell)$ is bounded by 1. Furthermore, $\{u_n(\ell)\}$ is log-convex, as is seen from ($n \in \mathbb{N}$)

$$u_n(\ell)^2 = u_\ell^{-2} u_{\ell+n}^2 \le u_\ell^{-2} u_{\ell+n-1} u_{\ell+n+1} = u_{n-1}(\ell) u_{n+1}(\ell) \ .$$

Conversely, let $\{u_n(\ell)\} \in \mathcal{R}_0$ for all $\ell \in \mathbb{N}_0$. Taking $\ell = 0$, we see that $\{u_n\} \in \mathcal{R}_0$, and hence $\{u_n\}$ is bounded. The sequence $\{f_n(\ell)\}_1^\infty$, with which $\{u_n(\ell)\}$ is associated, consists of nonnegative numbers. Now, calculating $f_2(\ell)$, we get

$$f_2(\ell) = u_2(\ell) - u_1(\ell)^2 = u_\ell^{-2}\{u_\ell u_{\ell+2} - u_{\ell+1}^2\} \ ,$$

from which it follows that the nonnegativity of $f_2(\ell)$ for all $\ell \in \mathbb{N}_0$ implies the log-convexity of $\{u_n\}_0^\infty$. Hence $\{u_n\}$ is a bounded Kaluza sequence. □

REMARK 4.1.21. If we define the sequence $\{u_n(\ell)\}$ somewhat differently, viz.

$$u_0(\ell) = 1, \ u_n(\ell) = u_{\ell+n} \qquad (n \in \mathbb{N}) \ ,$$

then we get a weaker result: $\{u_n\}$ is a bounded Kaluza sequence iff $\{u_n(\ell)\}_0^\infty$ is a bounded Kaluza sequence for all $\ell \in \mathbb{N}_0$.

Finally, we consider the occurrence of a compound geometric lattice distribution in a stochastic process. By lemma 4.1.1(i), theorems 4.1.5 and 4.1.6

(i) yield, rather artificially, an interpretation of $\{p_n\} \in C_o$ in a pure, transient renewal process and in a Markov chain, respectively. In the first case, however, we can say a little more. Using the notation of definition 4.1.4, we denote the (pure) renewal sequence corresponding to $\{S_n\}$ (cf. theorem 4.1.5) by $\{u_n\}_o^\infty$, and define

$$N := \#(\{n \in \mathbb{N} \mid S_n < \infty\}) ,$$

i.e. N is the total number of renewal epochs ($S_o = 0$ not counted). Then N is finite with probability one:

$$P(N=\infty) = P(\forall_{n\in\mathbb{N}}\, S_n < \infty) = \lim_{n\to\infty} P(\forall_{k\le n}\, T_k < \infty) = \lim_{n\to\infty} f^n = 0 .$$

In fact, N has a geometric distribution with parameter f:

$$P(N=k) = P(S_k < \infty;\ T_{k+1} = \infty) = (1-f)f^k \qquad (k \in \mathbb{N}_o) .$$

Now we can define the *duration* D of the process $\{S_n\}$ as $D = S_N$. Its distribution is easily calculated:

$$P(D = n) = \sum_{k=0}^{\infty} P(S_N = n;\ N = k) = \sum_{k=0}^{\infty} P(S_k = n;\ T_{k+1} = \infty) =$$

$$= \sum_{k=0}^{\infty} P(S_k = n)(1-f) = (1-f)u_n \qquad (n \in \mathbb{N}_o) ,$$

from which by lemma 4.1.1 we obtain the following interpretation of $\{p_n\} \in C_o$.

THEOREM 4.1.22. Let $\{p_n\}_o^\infty$ be a lattice distribution. Then $\{p_n\} \in C_o$ iff there exists a pure, transient renewal process $\{S_n\}_o^\infty$ such that $\{p_n\}$ is the distribution of the duration D of the process.

4.2. *Generalized renewal sequences; classes* R_α $(0 < \alpha < 1)$

In this section we study the distributions in C_α for $0 < \alpha < 1$, by considering slightly more general classes R_α. As these are defined in such a way that there exists a relation between C_α and R_α (to be given in lemma 4.2.4), similar to that between C_o and R_o, we can (and will) confine ourselves to studying the R_α's. As in section 1, the properties of R_α can easily be translated into properties of C_α. We define the following classes R_α, employing the notation used for the C_α's rather than that used for R_o.

DEFINITION 4.2.1. For $0 < \alpha < 1$ a sequence $\{u_n\}_0^\infty$ of real numbers with $u_0 = 1$ is said to be in the class R_α if there exist nonnegative quantities $r_n(\alpha)$ $(n \in \mathbb{N}_0)$ satisfying

$$(4.2.1) \qquad r(\alpha) := \sum_{n=0}^{\infty} r_n(\alpha) \le \frac{1}{1-\alpha},$$

such that

$$(4.2.2) \qquad \frac{1-\alpha^{n+1}}{1-\alpha} u_{n+1} = \sum_{k=0}^{n} r_k(\alpha) u_{n-k} \qquad (n \in \mathbb{N}_0).$$

We could consider more general classes by dropping condition (4.2.1). Many results to be proved for R_α will also hold in that case. But in view of probabilistic interpretations and because for a sequence $\{u_n\}$ that satisfies (4.2.2) and has a gf with positive radius of convergence, we can choose $\gamma \in (0,1)$ such that $\{\gamma^n u_n\}$ is bounded and still satisfies (4.2.2), we will only consider *bounded* sequences $\{u_n\}$. This results in the classes R_α, as is seen from the following lemma.

LEMMA 4.2.2. Let $0 < \alpha < 1$ and let $\{u_n\}_0^\infty$ be a sequence with $u_0 = 1$ and satisfying (4.2.2) with nonnegative $r_n(\alpha)$'s $(n \in \mathbb{N}_0)$. Then $\{u_n\}$ is bounded iff (4.2.1) holds (i.e. $\{u_n\} \in R_\alpha$), in which case for all $k \in \mathbb{N}_0$

$$(4.2.3) \qquad u_k \le \prod_{\ell=1}^{k} (1-\alpha^\ell)^{-1} \qquad \left(\le \prod_{\ell=1}^{\infty} (1-\alpha^\ell)^{-1} < \infty\right).$$

PROOF. Let $\{u_n\}$ be bounded. Then its gf U exists on $[0,1)$. Taking gf's in (4.2.2) we get

$$(4.2.4) \qquad \frac{U(z) - U(\alpha z)}{(1-\alpha)z} = U(z)R_\alpha(z) \qquad (0 \le z < 1),$$

and hence

$$zR_\alpha(z) = \{1 - U(\alpha z)/U(z)\}/(1-\alpha) \le 1/(1-\alpha) \qquad (0 \le z < 1).$$

Letting $z \uparrow 1$, by the monotone convergence theorem we see that $r(\alpha) \le 1/(1-\alpha)$. Conversely, let $r(\alpha) \le 1/(1-\alpha)$. Suppose (4.2.3) to be true for $k = 1,\dots,n$. Then

$$u_{n+1} = \frac{1-\alpha}{1-\alpha^{n+1}} \sum_{k=0}^{n} r_k(\alpha) u_{n-k} \le \frac{1-\alpha}{1-\alpha^{n+1}} \sum_{k=0}^{n} r_k(\alpha) \prod_{\ell=1}^{n} (1-\alpha^\ell)^{-1} \le \prod_{\ell=1}^{n+1} (1-\alpha^\ell)^{-1},$$

which is less than

$$\prod_{\ell=1}^{\infty} (1-\alpha^{\ell})^{-1} = \exp\Big[\sum_{\ell=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}\alpha^{\ell j}\Big] \le \exp[\alpha(1-\alpha)^{-2}] < \infty . \qquad \square$$

REMARK 4.2.3. The inequality (4.2.3) cannot easily be improved, as is seen from the following example (cf. the boundary distribution from definition 2.2.2). Define $\{\tilde{u}_n\}_o^{\infty}$ as the sequence in $\mathcal{R}_\alpha$ corresponding to $r_o(\alpha) = 1/(1-\alpha)$, $r_n(\alpha) = 0$ $(n \in \mathbb{N})$.
Then from (4.2.2) it follows that

$$(4.2.5) \qquad (1-\alpha^{n+1})\tilde{u}_{n+1} = \tilde{u}_n \qquad (n \in \mathbb{N}_o) ,$$

and hence

$$(4.2.6) \qquad \tilde{u}_n = \prod_{k=1}^{n} (1-\alpha^k)^{-1} \qquad (n \in \mathbb{N}_o) .$$

One easily verifies that $\mathcal{C}_\alpha$ is related to $\mathcal{R}_\alpha$ in the following way (cf. lemma 4.1.1).

LEMMA 4.2.4. Let $0 < \alpha < 1$.
(i) If $\{p_n\} \in \mathcal{C}_\alpha$ and $u_n := p_n/p_o$ $(n \in \mathbb{N}_o)$, then $\{u_n\} \in \mathcal{R}_\alpha$.
(ii) If $\{u_n\} \in \mathcal{R}_\alpha$ and $p_n := \gamma^n u_n/U(\gamma)$, where $0 < \gamma < 1$, then $\{p_n\} \in \mathcal{C}_\alpha$.

It may be noted that if $r(\alpha) < 1/(1-\alpha)$, then in (ii) we can take $\gamma = 1$. This follows from the following property of $\mathcal{R}_\alpha$.

THEOREM 4.2.5. Let $0 < \alpha < 1$ and let $\{u_n\} \in \mathcal{R}_\alpha$. Then $u := \sum_{n=0}^{\infty} u_n$ is finite iff $r(\alpha) < 1/(1-\alpha)$, in which case

$$(4.2.7) \qquad u = U(\alpha)/\{1 - (1-\alpha)r(\alpha)\} .$$

PROOF. Let $0 < \alpha < 1$ and let $\{u_n\} \in \mathcal{R}_\alpha$. Then relation (4.2.4) holds for all $|z| < 1$, and can be written in the form

$$U(z) = \{1 - (1-\alpha)zR_\alpha(z)\}^{-1}U(\alpha z) \qquad (|z| < 1) .$$

Now, letting $z \uparrow 1$ and applying the monotone convergence theorem concludes the proof. $\square$

Studying the $\mathcal{R}_\alpha$'s further, we first observe that many properties of the $\mathcal{C}_\alpha$'s can be extended to the $\mathcal{R}_\alpha$'s. For instance, using lemmas 4.2.4 and 4.2.2, one easily shows that

$$(4.2.8) \qquad \mathcal{R}_\alpha \subset \mathcal{R}_\beta \qquad (0 \le \alpha < \beta < 1) \ .$$

Specifically, it follows that every $\mathcal{R}_\alpha$ contains $\mathcal{R}_0$ as a subclass, i.e. the sequences in $\mathcal{R}_\alpha$ can be considered as *generalized* renewal sequences. Now, in the remainder of this section, we mainly investigate to what extent the properties of $\mathcal{R}_0$, summarized in the preceding section, can be generalized to the larger classes $\mathcal{R}_\alpha$. To this end we first state a useful relation of $\mathcal{R}_\alpha$ with the (pure) renewal sequences, and next a more direct relation between $\mathcal{R}_\alpha$ and certain *delayed* renewal sequences.

THEOREM 4.2.6. Let $0 < \alpha < 1$, and let $\{u_n\}_0^\infty$ be a sequence with $u_0 = 1$ and gf U. Define the sequence $\{u_n(\alpha)\}_0^\infty$ by its gf $U_\alpha(z) := U(z)/U(\alpha z)$, or recursively by

$$(4.2.9) \qquad u_n = \sum_{k=0}^{n} \alpha^k u_k u_{n-k}(\alpha) \qquad (n \in \mathbb{N}_0) \ .$$

Then $\{u_n\} \in \mathcal{R}_\alpha$ iff $\{u_n(\alpha)\} \in \mathcal{R}_0$.

PROOF. Let $\{u_n\} \in \mathcal{R}_\alpha$ and take $\gamma \in (0,1)$. Then by lemma 4.2.4 the relation between $\mathcal{C}_\alpha$ and $\mathcal{C}_0$, given by theorem 2.4.5, implies that $\{\gamma^n u_n(\alpha)\} \in \mathcal{R}_0$. It follows that $\{u_n(\alpha)\}$ satisfies the pure renewal equation. Furthermore, since $\{u_n\}$ is bounded (lemma 4.2.2) and as by (4.2.9) we have $u_n(\alpha) \le u_n$ $(n \in \mathbb{N}_0)$, $\{u_n(\alpha)\}$ is bounded, and on account of lemma 4.1.3 we conclude that $\{u_n(\alpha)\} \in \mathcal{R}_0$.

Conversely, let $\{u_n(\alpha)\} \in \mathcal{R}_0$. Then, similarly, it can be shown that $\{\gamma^n u_n\} \in \mathcal{R}_\alpha$ for all $\gamma \in (0,1)$, and hence $\{u_n\}$ satisfies (4.2.2) with nonnegative $r_n(\alpha)$'s. Now, in view of lemma 4.2.2, for $\{u_n\}$ to be in $\mathcal{R}_\alpha$ it is sufficient that $\{u_n\}$ is bounded. To show this, we prove (4.2.3), or, equivalently, $u_n \le \tilde{u}_n$ (cf. remark 4.2.3). By (4.2.5) and mathematical induction, $\{\tilde{u}_n\}$ is seen to satisfy

$$(4.2.10) \qquad \tilde{u}_n = \sum_{k=0}^{n} \alpha^k \tilde{u}_k \qquad (n \in \mathbb{N}_0) \ .$$

Hence, assuming that $u_k \le \tilde{u}_k$ for $k = 0,1,\ldots,n-1$, by (4.2.9) and the fact that $u_n(\alpha) \le 1$ $(n \in \mathbb{N}_0)$ we can estimate u_n as follows:

$$u_n = (1-\alpha^n)^{-1} \sum_{k=0}^{n-1} \alpha^k u_k u_{n-k}(\alpha) \le (1-\alpha^n)^{-1} \sum_{k=0}^{n-1} \alpha^k \tilde{u}_k =$$

$$= (1-\alpha^n)^{-1} \tilde{u}_{n-1} = \tilde{u}_n \ .$$

It follows that $\{u_n\}$ is bounded, and the theorem is proved. □

THEOREM 4.2.7. Let $0 < \alpha < 1$.

(i) If $\{u_n\} \in \mathcal{R}_\alpha$ and if $v_n := u_n/U(\alpha)$ $(n \in \mathbb{N}_0)$, then $\{v_n\}_0^\infty \in \mathcal{R}$, associated with $\{b_n\}_0^\infty$ and $\{f_n\}_1^\infty$ given by

(4.2.11) $\quad b_n = \alpha^n v_n \ (n \in \mathbb{N}_0), \quad f_n = (1-\alpha) r_{n-1}(\alpha) \ (n \in \mathbb{N})$.

(ii) If $\{v_n\} \in \mathcal{R}$ is associated with $\{b_n\}$ and $\{f_n\}$ such that $b_0 > 0$ and $b_n = \alpha^n v_n$ $(n \in \mathbb{N}_0)$, and if $u_n := v_n/v_0$ $(n \in \mathbb{N}_0)$, then $\{u_n\} \in \mathcal{R}_\alpha$.

PROOF. Let $\{u_n\} \in \mathcal{R}_\alpha$ and $v_n := u_n/U(\alpha)$ $(n \in \mathbb{N}_0)$. Then by definition 4.2.1 $\{v_n\}_0^\infty$ is easily seen to satisfy the renewal equation (4.1.4) with $\{b_n\}$ and $\{f_n\}$ given by (4.2.11). It follows that $b = 1$ and $f = (1-\alpha)r(\alpha) \le 1$, and hence that $\{v_n\} \in \mathcal{R}$.
Finally, (ii) can be shown in a similar way. □

REMARK 4.2.8. For $\{v_n\} \in \mathcal{R}$, associated with $\{b_n\}$ and $\{f_n\}$, the condition $b_n = \alpha^n v_n$ $(n \in \mathbb{N}_0)$ can be reformulated as a relation between $\{b_n\}$ and $\{f_n\}$:

(4.2.12) $\quad (1-\alpha^n) b_n = \sum_{k=1}^{n} \alpha^k f_k b_{n-k} \qquad (n \in \mathbb{N})$

(i.e. in case $v_0 > 0$ and $V(\alpha) = 1$: $\{b_n\}_0^\infty \in \mathcal{C}_\alpha$ with corresponding $r_n(\alpha)$'s given by $r_n(\alpha) = (1-\alpha)^{-1}\alpha^{n+1} f_{n+1}$). This can be seen from (4.1.5) (replace z by αz).

REMARK 4.2.9. Using theorem 4.2.6 one easily proves the following implication:

(4.2.13) $\quad U \in \mathcal{R}_0 \Rightarrow W(z) := U(z)U(\alpha z) \in \mathcal{R}_\alpha$.

Now, in view of (4.1.6), the sequences $\{w_n\}$ in $\mathcal{R}_\alpha$, thus obtained, are of the form $w_n = U(\alpha)v_n$, where $\{v_n\} \in \mathcal{R}$ is such that the sequences $\{b_n\}$ and $\{f_n\}$, that $\{v_n\}$ is associated with, satisfy $B(z) = \{1-F(\alpha)\}/\{1-F(\alpha z)\}$, i.e.

$$(4.2.14)\qquad b_n = (1-F(\alpha))\,\alpha^n \sum_{k=0}^{n} f_n^{*k} \qquad (n \in \mathbb{N}_o)\ .$$

Relation (4.2.14) seems somewhat more tractable than (4.2.12), but, unfortunately, not all sequences in R_α can be obtained in this way, i.e. the converse of (4.2.13) is not true. Indeed, every $W \in R_\alpha$ can be written as $U(z)U(\alpha z)$, but generally U will belong to R_{α^2} and not necessarily to R_o (cf. (3.1.14) and lemma 3.1.8).

The relation between R_α and R in theorem 4.2.7 yields interpretations of $\{u_n\} \in R_\alpha$ in a renewal process and Markov chain (cf. theorems 4.1.5 and 4.1.6(ii)). J. Wijngaard (personal communication) gave a third interpretation, where u_n appears as the expected number of particles at time n in a complicated "Markov branching-process". Although theorem 4.2.7 will be applied when we consider the asymptotic behaviour of $\{u_n\} \in R_\alpha$, these interpretations do not yield easy proofs of inequalities or closure properties for R_α of the kind one has for R_o. For instance, if $\{u_n^{(i)}\} \in R_\alpha$ $(i = 1,2)$ and $v_n^{(i)} := u_n^{(i)}/U_i(\alpha)$ $(i = 1,2;\ n \in \mathbb{N}_o)$, then $\{v_n^{(i)}\} \in R$ $(i = 1,2)$, and hence, by the Markov-chain interpretation, also $\{v_n\} \in R$, where $v_n :=$ $:= v_n^{(1)} v_n^{(2)}$ $(n \in \mathbb{N}_o)$. But since the $\{b_n\}$ and $\{f_n\}$, that $\{v_n\}$ is associated with, are not tractable, we cannot go back to R_α.
Using the relation between R_α and R_o from theorem 4.2.6, we can say a little more, but the results are only formal and rather obscure. We give one example, where we make use of theorem 4.1.7(iii).

THEOREM 4.2.10. If $\{u_n\}$ and $\{v_n\} \in R_\alpha$, and if $\{u_n(\alpha)\}$ and $\{v_n(\alpha)\}$ are defined by

$$u_n = \sum_{k=0}^{n} \alpha^k u_k u_{n-k}(\alpha)\ ,\quad v_n = \sum_{k=0}^{n} \alpha^k v_k v_{n-k}(\alpha) \qquad (n \in \mathbb{N}_o)\ ,$$

then the sequence $\{w_n\}_o^\infty$, defined by

$$w_o = 1,\ w_n = \sum_{k=0}^{n} \alpha^k w_k u_{n-k}(\alpha)\, v_{n-k}(\alpha) \qquad (n \in \mathbb{N})\ ,$$

is also in R_α.

Not able to prove inequalities for $\{u_n\} \in R_\alpha$ using a probabilistic interpretation, we adapt the analytic proof of the first inequality in theorem

4.1.7(i). To this end we need the following lemma from De Bruijn & Erdős (1951).

LEMMA 4.2.11. If $\{v_n\}_0^\infty$ is a sequence of real numbers with $v_0 = 1$ and satisfying

$$(4.2.15) \qquad v_{n+1} = \sum_{k=0}^{n} r_{k,n} v_{n-k} \qquad (n \in \mathbb{N}_0) ,$$

where $r_{k,n} \geq 0$ ($k \in \mathbb{N}_0$; $n = k,k+1,\ldots$) and $r_{k,n}$ is nondecreasing in n for all $k \in \mathbb{N}_0$, then for $n \in \mathbb{N}_0$

$$(4.2.16) \qquad v_{n+k} \geq v_n v_k \qquad (k \in \mathbb{N}_0) .$$

PROOF. We use mathematical induction with respect to n. For $n = 0$ (4.2.16) is trivial. Now suppose that (4.2.16) holds for $n = 1,\ldots,N$, then for all $k \in \mathbb{N}_0$

$$v_{N+1+k} = \sum_{\ell=0}^{N+k} r_{\ell,N+k} v_{N+k-\ell} \geq \sum_{\ell=0}^{N} r_{\ell,N+k} v_{N-\ell} v_k \geq$$

$$\geq v_k \sum_{\ell=0}^{N} r_{\ell,N} v_{N-\ell} = v_{N+1} v_k ,$$

and (4.2.16) is proved for $n = N+1$. □

Before stating the generalized inequality, we introduce the following notation:

$$(4.2.17) \qquad h_n(\alpha) := \prod_{k=1}^{n} \frac{1-\alpha^k}{1-\alpha} \qquad (n \in \mathbb{N}_0;\ 0 \leq \alpha < 1) .$$

THEOREM 4.2.12. If $0 \leq \alpha < 1$ and $\{u_n\} \in R_\alpha$, then the following inequality holds:

$$(4.2.18) \qquad \frac{h_{n+k}(\alpha)}{h_n(\alpha) h_k(\alpha)} u_{n+k} \geq u_n u_k \qquad (n,k \in \mathbb{N}_0) .$$

PROOF. Let $\{u_n\} \in R_\alpha$. Define $v_n := h_n(\alpha) u_n$ ($n \in \mathbb{N}_0$), then $v_0 = 1$ and by (4.2.2)

$$v_{n+1} = h_n(\alpha)\frac{1-\alpha^{n+1}}{1-\alpha}\, u_{n+1} = \sum_{k=0}^{n} r_k(\alpha)\frac{h_n(\alpha)}{h_{n-k}(\alpha)}\, v_{n-k} \qquad (n \in \mathbb{N}_o) ,$$

i.e. $\{v_n\}_o^\infty$ satisfies (4.2.15) with $r_{k,n}$ given by

$$r_{k,n} = r_k(\alpha)h_n(\alpha)/h_{n-k}(\alpha) = r_k(\alpha)\prod_{j=0}^{k-1}\frac{1-\alpha^{n-j}}{1-\alpha} \ (k \in \mathbb{N}_o;\ n=k,k+1,..).$$

Obviously, $r_{k,n}$ is nonnegative and nondecreasing in n, from which by lemma 4.2.11 it follows that (4.2.16), and hence (4.2.18), holds. □

COROLLARY 4.2.13. If $0 \le \alpha < 1$ and $\{u_n\} \in R_\alpha$, then u_n can be bounded as follows:

$$(4.2.19) \qquad \frac{u_1^n}{h_n(\alpha)} \le u_n \le \frac{(1-\alpha)^{-n}}{h_n(\alpha)} \qquad (n \in \mathbb{N}_o) .$$

PROOF. The first part of the inequality follows by iteration of (4.2.18), while the second part is a reformulation of (4.2.3). □

We note that both inequalities in (4.2.19) become equalities if $u_n = \tilde{u}_n$ (cf. remark 4.2.3). Further, theorem 4.2.12 implies the following result about the zeros of $\{u_n\} \in R_\alpha$. It can also be obtained from theorem 1.5.7 and the fact that $R_\alpha \subset R_1$, where R_1 is an extension of C_1 to be defined in the next section.

COROLLARY 4.2.14. If $0 \le \alpha < 1$ and $\{u_n\} \in R_\alpha$, then the following implication holds:

$$[u_n > 0 \text{ and } u_k > 0] \Rightarrow u_{n+k} > 0 \qquad (n,k \in \mathbb{N}_o) .$$

Consequently, if $u_1 > 0$ then $u_n > 0$ for all $n \in \mathbb{N}_o$.

The upper bound for u_n, given in (4.2.19), is independent of $\{u_n\} \in R_\alpha$. Now, using the relation between R_α and R, given in theorem 4.2.7, and observing that $v_n \le 1$ $(n \in \mathbb{N}_o)$ if $\{v_n\} \in R$, we see that if $\{u_n\} \in R_\alpha$ then

$$(4.2.20) \qquad u_n \le U(\alpha) \qquad (n \in \mathbb{N}_o) ,$$

i.e. we have an upper bound for u_n, independent of $n \in \mathbb{N}_o$.

Furthermore, theorem 4.2.7 can be used to generalize the renewal theorem for R_o (theorem 4.1.2(i) and (ii) with $b_n = \delta_{o,n}$) to the R_α's. The first part has already been generalized in theorem 4.2.5, the second part will be considered after the following lemma concerning the period of $\{u_n\} \in R_\alpha$.

LEMMA 4.2.15. Let $0 < \alpha < 1$, let $\{u_n\} \in R_\alpha$, and suppose that the $r_n(\alpha)$'s, corresponding to $\{u_n\}$, are not all zero. Then the period d of $\{u_n\}$ is equal to that of the sequence $\{r_{n-1}(\alpha)\}_1^\infty$, and

$$(4.2.21)\qquad \#(\{n \in \mathbb{N} \mid u_{nd} = 0\}) < \infty .$$

PROOF. Let $\{u_n\} \in R_\alpha$ with period d, and let δ be the period of $\{r_{n-1}(\alpha)\}_1^\infty$. Since by (4.2.2) the following inequality holds:

$$\frac{1-\alpha^{n+1}}{1-\alpha}\, u_{n+1} \geq r_n(\alpha) \qquad (n \in \mathbb{N}_o) ,$$

we have

$$\{n \in \mathbb{N} \mid r_{n-1}(\alpha) > 0\} \subset \{n \in \mathbb{N} \mid u_n > 0\} ,$$

and hence $\delta \geq d$. As in the proof of theorem 2.3.4, by iteration of (4.2.4) we obtain the following expression for U in terms of R_α:

$$(4.2.22)\qquad U(z) = \prod_{k=0}^{\infty} \{1 - (1-\alpha)\alpha^k z R_\alpha(\alpha^k z)\}^{-1} .$$

Now, as $r_{n-1}(\alpha) = 0$ unless δ divides n ($n \in \mathbb{N}$), we see that each factor in the right-hand side of (4.2.22), and hence U, is a power series in z^δ, i.e.

$$\{n \in \mathbb{N} \mid u_n > 0\} \subset \{k\delta \mid k \in \mathbb{N}\} ,$$

so that $d \geq \delta$. It follows that $d = \delta$. To prove (4.2.21) we apply theorem 4.2.6, according to which the sequence $\{u_n(\alpha)\}_o^\infty$, with gf $U_\alpha(z) := U(z)/U(\alpha z)$, is in R_o. The gf F of $\{f_n\}_1^\infty$, that $\{u_n(\alpha)\}$ is associated with, satisfies

$$F(z) = 1 - 1/U_\alpha(z) = 1 - U(\alpha z)/U(z) = (1-\alpha)zR_\alpha(z) ,$$

i.e. $f_n = (1-\alpha)r_{n-1}(\alpha)$ ($n \in \mathbb{N}$). Now by theorem 4.1.14 it follows that the period of $\{u_n(\alpha)\}$ is equal to d, and since by (4.2.9) we have

$$u_n \geq u_n(\alpha) \qquad (n \in \mathbb{N}_o) ,$$

we obtain (4.2.21) from the corresponding result for $\{u_n(\alpha)\}$ (cf. (4.1.17)).$\square$

THEOREM 4.2.16. Let $0 < \alpha < 1$ and let $\{u_n\} \in R_\alpha$. Suppose that the $r_n(\alpha)$'s, corresponding to $\{u_n\}$, are such that $r(\alpha) = 1/(1-\alpha)$, and let $\{u_n\}$ have period d. Then $u_\infty := \lim_{n\to\infty} u_{nd}$ exists, and u_∞ is given by

$$(4.2.23) \quad u_\infty = (1-\alpha)^{-1}U(\alpha)\frac{d}{\mu} ,$$

where

$$(4.2.24) \quad \mu := \sum_{n=0}^{\infty} (n+1)r_n(\alpha) \qquad (\leq \infty) .$$

PROOF. Let $\{u_n\} \in R_\alpha$. By the preceding lemma we have

$$u_n = r_{n-1}(\alpha) = 0 \text{ if } n \notin \{\ell d \mid \ell \in \mathbb{N}_0\} .$$

Now, for the sequence $\{w_n\}_0^\infty$, with $w_n := u_{nd}$ $(n \in \mathbb{N}_0)$, it follows that for $n \in \mathbb{N}_0$

$$\frac{1-(\alpha^d)^{n+1}}{1-\alpha^d} w_{n+1} = \frac{1-\alpha}{1-\alpha^d}\frac{1-\alpha^{(n+1)d}}{1-\alpha} u_{(n+1)d} =$$

$$= \frac{1-\alpha}{1-\alpha^d}\sum_{\ell=1}^{n+1} r_{\ell d-1}(\alpha)u_{(n+1-\ell)d} = \sum_{k=0}^{n}\frac{1-\alpha}{1-\alpha^d} r_{(k+1)d-1}(\alpha)w_{n-k} .$$

Hence $\{w_n\} \in R_{\alpha^d}$ and, if the $r_n(\alpha^d)$'s, corresponding to $\{w_n\}$, are denoted by $\bar{r}_n(\alpha^d)$ $(n \in \mathbb{N}_0)$, then

$$\bar{r}_n(\alpha^d) = \frac{1-\alpha}{1-\alpha^d} r_{(n+1)d-1}(\alpha) \qquad (n \in \mathbb{N}_0) .$$

Now, according to theorem 4.2.7(i), the sequence $\{v_n\}_0^\infty$, with $v_n := w_n/W(\alpha^d)$ $(n \in \mathbb{N}_0)$, is a delayed renewal sequence, associated with

$$b_n = \alpha^{nd}v_n \ (n \in \mathbb{N}_0) \text{ and } f_n = (1-\alpha^d)\bar{r}_{n-1}(\alpha^d) \ (n \in \mathbb{N}) .$$

Since $\{\bar{r}_{n-1}(\alpha^d)\}_1^\infty$, and hence $\{f_n\}_1^\infty$, is aperiodic, and as

$$f = \sum_{n=1}^{\infty} f_n = (1-\alpha)\sum_{n=1}^{\infty} r_{nd-1}(\alpha) = (1-\alpha)r(\alpha) = 1 ,$$

by the renewal theorem (theorem 4.1.2) it follows that $v_\infty := \lim_{n\to\infty} v_n$ exists, and that $v_\infty = b/\mu_1$, where $\mu_1 := \Sigma nf_n$. Observing that in our case $b = 1$,

$W(\alpha^d) = U(\alpha)$, and

$$\mu_1 = \sum_{n=1}^{\infty} nf_n = (1-\alpha) \sum_{n=1}^{\infty} nr_{nd-1}(\alpha) = (1-\alpha)d^{-1}\mu ,$$

with μ given by (4.2.24), we see that $u_\infty := \lim_{n\to\infty} u_{nd}$ exists, and is equal to the expression in (4.2.23). □

If the $r_n(\alpha)$'s, corresponding to $\{u_n\} \in R_\alpha$, are such that either $r(\alpha) < 1/(1-\alpha)$ or $r(\alpha) = 1/(1-\alpha)$ and $\mu = \Sigma(n+1)r_n(\alpha) = \infty$, then $u_n \to 0$ as $n \to \infty$. In order to know how fast u_n may tend to zero, we generalize the results of Kingman (1972) and DeBruijn & Erdős (1951) for R_o (cf. theorem 4.1.14 and remark 4.1.17) to R_α. It turns out that, as in the case $\alpha = 0$, $\{u_n\} \in R_\alpha$ tends to zero not faster than exponentially.

THEOREM 4.2.17. Let $0 < \alpha < 1$ and let $\{u_n\} \in R_\alpha$. Suppose that the $r_n(\alpha)$'s, corresponding to $\{u_n\}$, are not all zero, and that $\{u_n\}$ has period d. Finally, let ρ denote the radius of convergence of the gf of $\{u_n\}_o^\infty$. Then

$$\theta := \lim_{n\to\infty} u_{nd}^{1/(nd)}$$

exists in $(0,1]$, ρ is finite, and

$$(4.2.25) \qquad \theta^{-1} = \rho = \sup\{x \geq 0 \mid (1-\alpha)xR_\alpha(x) \leq 1\} =: \gamma_\alpha .$$

Furthermore (cf. (4.2.17))

$$(4.2.26) \qquad u_n \leq \theta^n \frac{(1-\alpha)^{-n}}{h_n(\alpha)} \qquad (n \in \mathbb{N}_o) ,$$

so that also $\{\rho^n u_n\}_o^\infty \in R_\alpha$.

PROOF. Let $\{u_n\} \in R_\alpha$. We introduce the sequence $\{v_n\}$ by

$$v_n := -\log\{h_{nd}(\alpha)u_{nd}\} ,$$

which is well defined for n sufficiently large, as then $u_{nd} > 0$ (cf. (4.2.21)). By inequality (4.2.18) it follows that $\{v_n\}$ is subadditive, i.e.

$$v_{n+k} \leq v_n + v_k ,$$

and hence (cf. Pólya & Szegö (1970), I. Abschn., Kap. 3) $\lim_{n\to\infty} v_n/n$ exists in

$[-\infty,\infty)$, and is equal to $\ell := \inf v_n/n$. It follows that

$$\lim_{n\to\infty} \{h_{nd}(\alpha)u_{nd}\}^{1/(nd)} = \exp[-\lim_{n\to\infty} v_n/(nd)] = \exp[-\ell/d] ,$$

from which, observing that for $0 < \alpha < 1$

$$\log h_n(\alpha) \sim -n \log(1-\alpha) \qquad (n \to \infty) ,$$

we see that

$$\theta := \lim_{n\to\infty} u_{nd}^{1/(nd)} = (1-\alpha)\exp[-\ell/d] .$$

As $\ell < \infty$, we have $\theta > 0$. On the other hand, in view of (4.2.19) we can write

$$\ell = \inf \frac{-1}{n} \log\{h_{nd}(\alpha)u_{nd}\} \geq \inf \frac{-1}{n} \log(1-\alpha)^{-nd} = d \log(1-\alpha) \quad ,$$

and hence $\theta \leq 1$. Further, as $v_n/n \geq \ell$ for all n, we have

$$\{h_{nd}(\alpha)u_{nd}\}^{1/(nd)} \leq \exp[-\ell/d] = \theta(1-\alpha)^{-1} ,$$

which yields (4.2.26). In view of lemma 4.2.2 we may now conclude that also $\{\theta^{-n}u_n\}_0^\infty \in R_\alpha$.
Finally, we turn to relation (4.2.25). As in the proof of corollary 4.1.15, it can be shown that $\rho = \theta^{-1}$. As $\theta > 0$, ρ is finite. From (4.2.2) it is seen that

(4.2.27) $\quad u_{n+1} \geq (1-\alpha)r_n(\alpha) \qquad (n \in \mathbb{N}_0) ,$

and hence the radius of convergence of R_α is at least ρ. Now by (4.2.4) it follows that

$$(1-\alpha)R_\alpha(x) = 1 - U(\alpha x)/U(x) \leq 1 \qquad (0 \leq x < \rho) ,$$

and hence $\gamma_\alpha \geq \rho$. On the other hand, using mathematical induction, we prove that for $k \in \mathbb{N}_0$

(4.2.28) $\quad u_k \leq \gamma_\alpha^{-k} \prod_{\ell=1}^{k} (1-\alpha^\ell)^{-1}:$

suppose that (4.2.28) holds for $k = 1,\ldots,n$, then we can write

$$\gamma_\alpha^{n+1} u_{n+1} = \gamma_\alpha^{n+1} \frac{1-\alpha}{1-\alpha^{n+1}} \sum_{k=0}^{n} r_k(\alpha)u_{n-k} \leq$$

$$\leq \gamma_\alpha^{n+1} \frac{1-\alpha}{1-\alpha^{n+1}} \sum_{k=0}^{n} r_k(\alpha)\gamma_\alpha^{-(n-k)} \prod_{\ell=1}^{n-k} (1-\alpha^\ell)^{-1} \leq$$

$$\leq (1-\alpha)\gamma_\alpha R_\alpha(\gamma_\alpha) \prod_{\ell=1}^{n+1} (1-\alpha^\ell)^{-1} \leq \prod_{\ell=1}^{n+1} (1-\alpha^\ell)^{-1} .$$

It follows that there exists $M > 0$ such that $u_n \leq M\gamma_\alpha^{-n}$ for all $n \in \mathbb{N}_o$, and hence $\rho \geq \gamma_\alpha$. Since we already showed that $\gamma_\alpha \geq \rho$, we get (4.2.25). □

REMARK 4.2.18. If $\{u_n\} \in \mathcal{R}_\alpha$ with $u_1 > 0$, then $d = 1$, and (4.2.19) yields a sharper lower bound for $\theta = \lim_{n\to\infty} u_n^{1/n}$ than the value zero given in theorem 4.2.17:

(4.2.29) $\quad \theta \geq (1-\alpha)u_1 \quad .$

The first part of theorem 4.2.17 can be reformulated as follows:

(4.2.30) $\quad \lim_{n\to\infty} \{\rho^{nd}u_{nd}\}^{1/(nd)} = 1 \quad .$

Now, from the fact that $\{\rho^n u_n\} \in \mathcal{R}_\alpha$ (cf. the last part of theorem 4.2.17), we can obtain some more information about the asymptotic behaviour of $\{\rho^{nd}u_{nd}\}_o^\infty$. In fact, applying theorems 4.2.5 and 4.2.16 to $\{\rho^n u_n\}$, we get the following result, which, of course, is only interesting if $\rho > 1$.

THEOREM 4.2.19. Consider the situation from the preceding theorem, and let σ_α denote the radius of convergence of R_α. Then γ_α satisfies

$$(1-\alpha)\gamma_\alpha R_\alpha(\gamma_\alpha) \leq 1 \ ,$$

while

(i) $(1-\alpha)\gamma_\alpha R_\alpha(\gamma_\alpha) < 1$ iff $U(\rho) < \infty$, in which case $\lim_{n\to\infty} \rho^n u_n = 0$;

(ii) If $(1-\alpha)\gamma_\alpha R_\alpha(\gamma_\alpha) = 1$, or, equivalently, if either $\sigma_\alpha = \infty$ or $\sigma_\alpha < \infty$ and $(1-\alpha)\sigma_\alpha R_\alpha(\sigma_\alpha) \geq 1$, then

(4.2.31) $\quad \lim_{n\to\infty} \rho^{nd}u_{nd} = (1-\alpha)^{-1}U(\alpha\rho)\frac{d}{\mu} \ ,$

where

(4.2.32) $\quad \mu := \sum_{n=0}^{\infty} (n+1)r_n(\alpha)\rho^{n+1} \qquad (\leq \infty) \ .$

To conclude the discussion of the R_α's, we give a sufficient condition in terms of the $r_n(\alpha)$'s for $\{u_n\} \in R_\alpha$ to be *unimodal*. This is suggested by theorem 3.3.8, which gives, in fact, the same result for the extension R_1 of C_1, to be defined in the next section.

THEOREM 4.2.20. Let $0 < \alpha < 1$ and let $\{u_n\} \in R_\alpha$. Suppose that the sequence $\{r_n(\alpha)\}$, corresponding to $\{u_n\}$, is such that $\{r_n(\alpha)/\alpha^n\}_0^\infty$ is nonincreasing. Then $\{u_n\}_0^\infty$ is unimodal, i.e. $\{u_n - u_{n-1}\}_0^\infty$, with $u_{-1} := 0$, changes sign at most once. Furthermore, $\{u_n\}$ is nonincreasing iff $r_o(\alpha) \le 1$.

PROOF. Let $\{u_n\} \in R_\alpha$ and define the sequence $\{d_n\}_0^\infty$ by

$$d_n := u_n - u_{n-1} \qquad (n \in \mathbb{N}_o) .$$

Replacing in (4.2.2) n by $n-1$, we can write

$$\frac{1-\alpha^{n+1}}{1-\alpha} u_n = \alpha \frac{1-\alpha^n}{1-\alpha} u_n + u_n = \alpha \sum_{k=0}^{n-1} r_k(\alpha) u_{n-1-k} + u_n .$$

From this and (4.2.2) we obtain by subtraction

$$\text{(4.2.33)} \quad \frac{1-\alpha^{n+1}}{1-\alpha} d_{n+1} = (r_o(\alpha) - 1) u_n + \sum_{k=0}^{n-1} \{r_{k+1}(\alpha) - \alpha r_k(\alpha)\} u_{n-1-k} \quad (n \in \mathbb{N}_o).$$

Now, if the sequence $\{r_n(\alpha)/\alpha^n\}$ is nonincreasing then

$$r_{n+1}(\alpha) \le \alpha r_n(\alpha) \qquad (n \in \mathbb{N}_o) ,$$

and hence by (4.2.33) it follows that $d_n \le 0$ $(n \in \mathbb{N})$, i.e. $\{u_n\}_0^\infty$ is nonincreasing, iff $r_o(\alpha) \le 1$. Finally, if $r_o(\alpha) > 1$, then, relation (4.2.33) being very similar to (3.3.15), we obtain the unimodality of $\{u_n\}$ along the lines of the proof of theorem 3.3.8. □

REMARK 4.2.21. Neither the condition that $r(\alpha) \le 1/(1-\alpha)$ nor the nonnegativity of the $r_n(\alpha)$'s is essential in the preceding theorem. If, however, the $r_n(\alpha)$'s are nonnegative, then $\{r_n(\alpha)/\alpha^n\}$ nonincreasing implies that the radius of convergence σ_α of R_α satisfies $\sigma_\alpha \ge 1/\alpha$. Hence, unimodality of sequences $\{u_n\} \in R_\alpha$, for which $\sigma_\alpha < 1/\alpha$, cannot be proved by theorem 4.2.20.

REMARK 4.2.22. Though R_o contains many unimodal sequences, e.g. the bounded Kaluza sequences (cf. theorem 4.1.18), there exists no obvious analogue of theorem 4.2.20 for R_o. However, as for $\{u_n\} \in R_o$ we have $u_n \le 1 = u_o$, a unimodal renewal sequence is necessarily nonincreasing. Hence (cf. (4.1.5)) a necessary and sufficient condition in terms of $\{f_n\}_1^\infty$ for $\{u_n\} \in R_o$ to be unimodal is that $\{\sum_{k=1}^{n} f_n^{*k}\}_1^\infty$ is nonincreasing, but this condition is not very useful.

Finally, we mention an analogue of theorem 4.1.22 for C_α. Consider a delayed, transient renewal process $\{S_n\}_o^\infty$, associated with $\{b_n\}_o^\infty$ and $\{f_n\}_1^\infty$ such that $b = 1$ (and so $f < 1$). Let $\{v_n\}_o^\infty$ denote the corresponding delayed renewal sequence (cf. theorem 4.1.5). Then, as in the case of a pure, transient renewal process (cf. the end of section 1), it can be shown that the rv

$$N := \#(\{n \in \mathbb{N} \mid S_n < \infty\})$$

has a geometric distribution with parameter f, and that the distribution $\{p_n\}_o^\infty$ of the duration $D = S_N$ is given by

$$p_n = (1-f)v_n \qquad (n \in \mathbb{N}_o) .$$

Now in view of the relation between R and R_α (cf. theorem 4.2.7 and remark 4.2.8) and the relation between R_α and C_α (cf. lemma 4.2.4), we easily obtain the following characterization of C_α for $0 < \alpha < 1$.

THEOREM 4.2.23. For $0 < \alpha < 1$ a lattice distribution $\{p_n\}_o^\infty$ is in C_α iff there exists a delayed, transient renewal process $\{S_n\}$, associated with $\{b_n\}$ and $\{f_n\}$ satisfying $b = 1$, $b_o > 0$ and

$$(1-\alpha^n)b_n = \sum_{k=1}^{n} \alpha^k f_k b_{n-k} \qquad (n \in \mathbb{N}) ,$$

such that $\{p_n\}$ is the probability distribution of the duration of $\{S_n\}$.

4.3. *An extension of C_1; the class R_1*

In this section we briefly consider an extension R_1 of C_1. The class C_1 is in many respects different from the classes C_α for $0 \le \alpha < 1$ (cf. chapter 2). As a consequence, it is not clear how to define R_1 analogous to R_α for $\alpha < 1$, i.e. how to characterize the boundedness of the R_1-sequences in terms of the $r_n(1)$'s. Therefore we define R_1 as follows.

DEFINITION 4.3.1. A sequence $\{u_n\}_o^\infty$ with $u_o = 1$ is said to be in the class R_1 if it is bounded and if there exist nonnegative quantities $r_n(1)$ $(n \in \mathbb{N}_o)$ such that

$$(4.3.1)\qquad (n+1)u_{n+1} = \sum_{k=0}^{n} r_k(1)u_{n-k} \qquad (n \in \mathbb{N}_o) .$$

M.L.J. Hautus suggested the following generalization of our result on the boundedness of sequences $\{u_n\}$ satisfying (4.3.1) with $r_n(1) \geq 0$ $(n \in \mathbb{N}_o)$; in fact, we proved the case $\varepsilon_n = 0$ $(n \in \mathbb{N}_o)$.

LEMMA 4.3.2. Let $\{u_n\}_o^\infty$ be a sequence with $u_o = 1$ and satisfying (4.3.1) with nonnegative $r_n(1)$'s $(n \in \mathbb{N}_o)$. Then $\{u_n\}$ is bounded if there exist $N \in \mathbb{N}$ and a sequence $\{\varepsilon_n\}_o^\infty$ satisfying

$$(4.3.2)\qquad \varepsilon_n \geq 0 \ (n \in \mathbb{N}_o), \ \sum_{n=0}^{\infty} \varepsilon_n < \infty ,$$

such that

$$(4.3.3)\qquad \frac{1}{n+1}\sum_{k=0}^{n} r_k(1) \leq 1 + \varepsilon_n \qquad (n \geq N) .$$

PROOF. Define $v_n := \max\{u_o, u_1, \ldots, u_n\}$. Because of (4.3.3) we have for all $n \geq N$

$$u_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} r_k(1)u_{n-k} \leq (1+\varepsilon_n)v_n .$$

Since $v_{n+1} = \max\{v_n, u_{n+1}\}$, $\{v_n\}_o^\infty$ is now seen to satisfy

$$v_{n+1} \leq (1+\varepsilon_n)v_n \qquad (n \geq N) ,$$

or also

$$v_{n+1} \leq v_N \prod_{k=N}^{n} (1+\varepsilon_k) \qquad (n \geq N) .$$

As $\Sigma\varepsilon_n < \infty$, it follows that $\{v_n\}$, and hence $\{u_n\}$, is bounded. □

REMARK 4.3.3. If $r_n(1) \leq 1$ for all $n \in \mathbb{N}_o$, or if there exist $K \in \mathbb{N}$ and $\gamma < 1$ such that $r_n(1) \leq \gamma$ for $n \geq K$, then (4.3.3) holds and so $\{u_n\}$ is bounded. However, if, for instance, $r_n(1) = 1 + \varepsilon$ for all $n \in \mathbb{N}_o$ $(\varepsilon > 0)$, then it follows that (use (4.3.5))

$$U(z) = (1-z)^{-1-\varepsilon} ,$$

and hence

$$u_n = \binom{-1-\varepsilon}{n}(-1)^n = \binom{n+\varepsilon}{n} = \prod_{k=1}^{n} (1+\varepsilon/k) \qquad (n \in \mathbb{N}_0) ,$$

which tends to ∞ as $n \to \infty$. Thus it turns out that condition (4.3.3) is not too bad.

The gf's U and R_1 of $\{u_n\} \in \mathcal{R}_1$ and $\{r_n(1)\}$, corresponding to $\{u_n\}$, exist for $|z| < 1$, because $\{u_n\}$ is bounded and $\{r_n(1)\}$ satisfies

$$(4.3.4) \qquad r_n(1) \le (n+1)u_{n+1} \qquad (n \in \mathbb{N}_0) .$$

From (4.3.1) it follows that

$$(4.3.5) \qquad U'(z) = U(z)R_1(z) \qquad (|z| < 1) ,$$

from which by integration and analytic continuation of the result we obtain the following representation for $U \in \mathcal{R}_1$:

$$(4.3.6) \qquad U(z) = \exp\Big[\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} z^{n+1}\Big] \qquad (|z| < 1) .$$

Letting $z \uparrow 1$ we get the following analogue of theorems 4.1.2(i) and 4.2.5.

THEOREM 4.3.4. Let $\{u_n\} \in \mathcal{R}_1$. Then $u := \sum_{n=0}^{\infty} u_n < \infty$ iff $\sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} < \infty$, in which case

$$(4.3.7) \qquad \sum_{n=0}^{\infty} \frac{r_n(1)}{n+1} = \log u .$$

It is not clear how to get an analogue of theorems 4.1.2(ii) and 4.2.16, because the conditions $f = 1$ and $r(\alpha) = 1/(1-\alpha)$ have no obvious analogue for $\alpha = 1$. If, however, for a sequence $\{u_n\} \in \mathcal{R}_1$ we suppose that $u_\infty := \lim_{n\to\infty} u_n$ exists, then, using (4.3.6), we can obtain this limit as follows:

$$u_\infty = \lim_{x\uparrow 1} (1-x)U(x) = \lim_{x\uparrow 1} \exp\Big[\log(1-x) + \sum_{k=0}^{\infty} \frac{r_k(1)}{k+1} x^{k+1}\Big] =$$

$$= \exp\Big[\lim_{x\uparrow 1} \sum_{k=0}^{\infty} \frac{r_k(1)-1}{k+1} x^{k+1}\Big] .$$

If in addition the $r_n(1)$'s are such that

$$(4.3.8) \qquad \sum_{k=0}^{\infty} \frac{r_k(1) - 1}{k + 1} \text{ converges },$$

then by Abel's theorem it follows that

$$(4.3.9) \qquad u_\infty = \exp\Big[\sum_{k=0}^{\infty} \frac{r_k(1) - 1}{k + 1}\Big] .$$

Quite recently, however, Hawkes & Jenkins (1978) proved that the absolute convergence of the series in (4.3.8) is also sufficient for the existence of $u_\infty = \lim_{n\to\infty} u_n$. In fact, they have the following results.

THEOREM 4.3.5. Let $\{u_n\}_0^\infty$ be an aperiodic (cf. lemma 4.3.9) sequence with $u_0 = 1$ and satisfying (4.3.1) with $r_n(1) \geq 0$ $(n \in \mathbb{N}_0)$.
(i) If $\gamma := \lim_{n\to\infty} r_n(1)$ exists in $(0,\infty)$, then $\{u_n\}$ satisfies

$$(4.3.10) \qquad u_n \sim \frac{n^{\gamma-1}}{\Gamma(\gamma)} \exp\Big[\sum_{k=0}^{\infty} \frac{r_k(1) - \gamma}{k + 1}\Big(1 - \frac{1}{n}\Big)^{k+1}\Big] \qquad (n \to \infty) .$$

(ii) If $\sum_{k=0}^{\infty} \frac{|r_k(1) - \gamma|}{k + 1} < \infty$ for some $\gamma \geq 1$, then

$$(4.3.11) \qquad u_n \sim \frac{n^{\gamma-1}}{\Gamma(\gamma)} \exp\Big[\sum_{k=0}^{\infty} \frac{r_k(1) - \gamma}{k + 1}\Big] \qquad (n \to \infty) ;$$

when $\gamma = 1$, then $\{u_n\}$ is also of bounded variation, i.e.

$$(4.3.12) \qquad \sum_{n=0}^{\infty} |u_{n+1} - u_n| < \infty .$$

PROOF. For the sake of completeness we give a proof of the result that is most interesting for us, i.e. part (ii) with $\gamma = 1$.
Define $d_n := u_n - u_{n-1}$ $(n \in \mathbb{N}_0)$, with $u_{-1} := 0$. Then from (4.3.1) we obtain by subtraction

$$(4.3.13) \qquad (n+1)d_{n+1} = \sum_{k=0}^{n} (r_k(1) - 1)d_{n-k} \qquad (n \in \mathbb{N}_0) .$$

Now, let $\{v_n\}_0^\infty$ be the sequence with $v_0 = 1$ that satisfies (4.3.1) with $r_n(1)$ replaced by $\bar{r}_n(1) := |r_n(1) - 1|$ $(n \in \mathbb{N}_0)$. Then from (4.3.13) it follows by induction that $|d_n| \leq v_n$ $(n \in \mathbb{N}_0)$, and as by theorem 4.3.4 we have

$\Sigma v_n < \infty$, we conclude that $\{u_n\}$ is of bounded variation. Hence $\{u_n\}$ has a limit u_∞, which is necessarily equal to the expression in (4.3.9). □

REMARK 4.3.6. As noted by Hawkes & Jenkins, theorem 4.3.5(i) includes the renewal theorem: if $\{u_n\}$ is an aperiodic renewal sequence, associated with $\{f_n\}$ such that $\mu := \Sigma nf_n < \infty$, then the $r_n(1)$'s, corresponding to $\{u_n\}$, satisfy $\lim_{n\to\infty} r_n(1) = 1$ (cf. Port (1964)), and

$$(4.3.14)\qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{r_k(1) - 1}{k + 1}(1 - \frac{1}{n})^{k+1} = \frac{1}{\mu} .$$

The limit theorems 4.1.14 for R_0 and 4.2.17 for R_α $(0 < \alpha < 1)$ do have an analogue for R_1. As in the case $0 \le \alpha < 1$, we need an inequality and an observation concerning the period of $\{u_n\} \in R_1$. Since for the function $h_n(\alpha)$, defined in (4.2.17), we have

$$h_n(1) := \lim_{\alpha\uparrow 1} h_n(\alpha) = n! \qquad (n \in \mathbb{N}_0) ,$$

and as the discussion in the proof of theorem 4.2.12 also holds for $\alpha = 1$, we get the following inequality for $\{u_n\} \in R_1$.

THEOREM 4.3.7. If $\{u_n\}_0^\infty \in R_1$, then

$$(4.3.15)\qquad \binom{n+k}{n} u_{n+k} \ge u_n u_k \qquad (n,k \in \mathbb{N}_0) .$$

COROLLARY 4.3.8. If $\{u_n\} \in R_1$, then the following implication and inequality hold:

$$(4.3.16)\qquad [u_n > 0 \text{ and } u_k > 0] \Rightarrow u_{n+k} > 0 \qquad (n,k \in \mathbb{N}_0) ,$$

$$(4.3.17)\qquad u_n \ge u_1^n/n! \qquad (n \in \mathbb{N}_0) .$$

We note that the inequalities in (4.3.15) and (4.3.17) become equalities for $\{\tilde{u}_n\} \in R_1$, corresponding to $\tilde{r}_0(1) = \mu > 0$, $\tilde{r}_n(1) = 0$ $(n \in \mathbb{N})$, i.e.

$$(4.3.18)\qquad \tilde{u}_n = \mu^n/n! \qquad (n \in \mathbb{N}_0) ,$$

i.e. the R_1-analogue of the Poisson distribution (cf. $\{\tilde{u}_n\} \in R_\alpha$ in remark 4.2.3).

The period of $\{u_n\} \in R_1$ has properties similar to those in the case $0 \le \alpha < 1$ (cf. theorem 4.1.14 and lemma 4.2.15; see also Wright (1967)).

LEMMA 4.3.9. Let $\{u_n\} \in R_1$ and suppose that the $r_n(1)$'s, corresponding to $\{u_n\}$, are not all zero. Then the period d of $\{u_n\}$ is equal to that of the sequence $\{r_{n-1}(1)\}_1^\infty$, and

$$\#(\{n \in \mathbb{N} \mid u_{nd} = 0\}) < \infty . \tag{4.3.19}$$

PROOF. Let $\{u_n\} \in R_1$ with period d, and let δ be the period of $\{r_{n-1}(1)\}_1^\infty$. In view of (4.3.4) we have

$$\{n \in \mathbb{N} \mid r_{n-1}(1) > 0\} \subset \{n \in \mathbb{N} \mid u_n > 0\} ,$$

and hence $\delta \ge d$. Since $r_{n-1}(1) = 0$ unless δ divides n $(n \in \mathbb{N})$, from (4.3.6) it is seen that U is a power series in z^δ, i.e.

$$\{n \in \mathbb{N} \mid u_n > 0\} \subset \{k\delta \mid k \in \mathbb{N}\} ,$$

so that $d \ge \delta$. It follows that $d = \delta$. To prove (4.3.19) we introduce the set $A = \{k_1,k_2,\ldots\}$ as the set of $k \in \mathbb{N}$ for which $u_{kd} > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\gcd A = \gcd\{k_1,\ldots,k_N\} = 1 .$$

It is well known that there now exists $K \in \mathbb{N}$ such that every $k \ge K$ $(k \in \mathbb{N})$ can be written as

$$k = \sum_{j=1}^{N} m_j k_j ,$$

where $m_j \in \mathbb{N}$ $(j = 1,\ldots,N)$. By (4.3.15) it follows that for $k \ge K$

$$\frac{(kd)!}{\prod_{j=1}^{N} \{(k_j d)!\}^{m_j}} u_{kd} \ge \prod_{j=1}^{N} u_{k_j d}^{m_j} , \tag{4.3.20}$$

which is positive, as $u_{k_j d} > 0$ for $j = 1,\ldots,N$. Hence $u_{kd} > 0$ for $k \ge K$. □

COROLLARY 4.3.10. If $\{u_n\} \in R_1$, and if $\{u_n\}$ has period d, then also $\{u_{nd}\}_{n=0}^\infty \in R_1$.

PROOF. In view of the preceding lemma we have

$$u_n = r_{n-1}(1) = 0 \text{ if } n \notin \{\ell d \mid \ell \in \mathbb{N}_0\} .$$

Define $w_n := u_{nd}$ $(n \in \mathbb{N}_0)$, then it follows that for $n \in \mathbb{N}_0$

$$(n+1)w_{n+1} = (n+1)u_{(n+1)d} = \frac{1}{d}\sum_{j=1}^{n+1} r_{jd-1}(1)u_{(n+1-j)d} =$$

$$= \sum_{k=0}^{n} \frac{1}{d}\, r_{(k+1)d-1}(1)w_{n-k} .$$

Hence $\{w_n\} \in R_1$, with $r_n(1)$, corresponding to $\{w_n\}$ and denoted by $r_n^{(d)}(1)$, given by

$$r_n^{(d)}(1) = \frac{1}{d}\, r_{(n+1)d-1}(1) \qquad (n \in \mathbb{N}_0) .$$ □

REMARK 4.3.11. Theorem 4.1.7(ii) yields an analogue of corollary 4.3.10 for R_0. In the case $0 < \alpha < 1$, however, we have (cf. the proof of theorem 4.2.16):

(4.3.21) $\{u_n\} \in R_\alpha \Rightarrow \{u_{nd}\} \in R_{\alpha^d}$.

We are now ready to state the following result on the asymptotic behaviour of $\{u_n\} \in R_1$. By corollary 4.3.10 we can, and will, confine ourselves to aperiodic sequences $\{u_n\}$.

THEOREM 4.3.12. Let $\{u_n\}$ be an aperiodic sequence in R_1. Then

(4.3.22) $$\ell := \lim_{n\to\infty} \frac{-\log u_n - n\log n}{n}$$

exists in $[-\infty,\infty)$, while

(i) $\ell > -\infty$ iff there exists $\gamma > 0$ such that $u_n \le \gamma^n/n!$ $(n \in \mathbb{N}_0)$, in which case

(4.3.23) $$\lim_{n\to\infty} \frac{-\log u_n}{n\log n} = 1 \text{ and } \frac{-\log u_n}{n\log n} - 1 = \frac{\ell + o(1)}{\log n} \qquad (n \to \infty) ;$$

(ii) If $\ell = -\infty$, then

(4.3.24) $$\limsup_{n\to\infty} \frac{-\log u_n}{n\log n} \le 1 .$$

PROOF. Let $\{u_n\} \in R_1$ be aperiodic. Then, in view of (4.3.19), for n sufficiently large we can define v_n as follows:

$$v_n := -\log\{n!u_n\} .$$

On account of theorem 4.3.7, $\{v_n\}$ is subadditive, i.e. $v_{n+k} \leq v_n + v_k$, from which (cf. Pólya & Szegö (1970), I. Abschn., Kap. 3) it follows that $\lim_{n\to\infty} \frac{1}{n} v_n$ exists in $[-\infty,\infty)$ and is equal to $\ell' := \inf \frac{1}{n} v_n$. As $n \log n - \log n! \sim n \ (n \to \infty)$, it is seen that also

$$\ell := \lim_{n\to\infty} \frac{-\log u_n - n \log n}{n} = \lim_{n\to\infty} \{\frac{1}{n} v_n - \frac{1}{n}(n \log n - \log n!)\} =$$

$$= \ell' - 1$$

exists in $[-\infty,\infty)$. Furthermore, if $\ell > -\infty$, then we have for n sufficiently large

$$-\frac{1}{n} \log\{n!u_n\} \geq \ell' = \ell + 1 ,$$

and hence $u_n \leq \gamma^n/n!$ $(n \in \mathbb{N}_o)$ for some $\gamma > 0$. It is easily seen that the converse is also true, and that in this case (4.3.23) follows from (4.3.22). Finally, if $\ell = -\infty$, then for all sufficiently large n we have $-\log u_n - n \log n < 0$, and hence (4.3.24). □

It can be proved that for an aperiodic $\{u_n\} \in R_1$ the limit $\lim_{n\to\infty} -\log u_n/\{n \log n\}$ always exists (cf. (4.3.23) and (4.3.24)), and its value can be calculated. This has been done by Steutel & Wolfe (1977) for $\{u_n\} \in C_1$ with $u_1 > 0$. It is easily shown, however, that their proof also holds for a general aperiodic $\{u_n\} \in C_1$. Finally, as for every $\{u_n\} \in R_1$ there exists $\gamma > 0$ such that $\{u_n\gamma^n/U(\gamma)\} \in C_1$, we get the following result.

THEOREM 4.3.13. Let $\{u_n\}$ be an aperiodic sequence in R_1. Then

$$\text{(4.3.25)} \quad \lim_{n\to\infty} \frac{-\log u_n}{n \log n} = \frac{1}{N} ,$$

where N is the smallest integer (possibly infinite) such that $r_n(1) = 0$ for all $n \geq N$.

Combining this result with theorem 4.3.12, we obtain the following corollaries (note that $N = 1$ (cf. theorem 4.3.13) iff $u_n = \tilde{u}_n$, given by (4.3.18)).

COROLLARY 4.3.14. Let $\{u_n\}$ be an aperiodic sequence in $\mathcal{R}_1$. Then the limit ℓ in (4.3.22) is finite iff there exists $\mu > 0$ such that

$$u_n = \mu^n/n! \qquad (n \in \mathbb{N}_0) \ ,$$

in which case necessarily $\ell = -\log \mu - 1$.

COROLLARY 4.3.15. Let $\{u_n\}$ be an aperiodic sequence in $\mathcal{R}_1$. If $u_n \leq \gamma^n/n!$ $(n \in \mathbb{N}_0)$ for some $\gamma > 0$, then there exists $\mu > 0$ such that $u_n = \mu^n/n!$ $(n \in \mathbb{N}_0)$.

As $\mathcal{C}_\alpha \subset \mathcal{C}_1$, we have also

$$(4.3.26) \quad \mathcal{R}_\alpha \subset \mathcal{R}_1 \qquad (0 \leq \alpha < 1) \ .$$

Now, if $0 \leq \alpha < 1$ and if $\{u_n\}$ is an aperiodic sequence in $\mathcal{R}_\alpha$, then by theorem 4.2.17

$$\lim_{n\to\infty} \frac{-\log u_n}{n} = -\log \theta \in [0,\infty) \ .$$

It follows that the limit in (4.3.25) is zero, i.e. we have the following result.

THEOREM 4.3.16. Let $0 \leq \alpha < 1$ and let $\{u_n\}$ be an aperiodic sequence in $\mathcal{R}_\alpha$. Then the $r_n(1)$'s, corresponding to $\{u_n\}$, satisfy

$$\forall_{N\in\mathbb{N}} \ \exists_{n\geq N} \ r_n(1) > 0 \ .$$

CHAPTER 5

CLASSIFICATION OF THE INFINITELY DIVISIBLE DISTRIBUTIONS ON $[0,\infty)$

The first aim of this chapter is to extend the classification of $\mathcal{C}_1$, defined by the classes $\mathcal{C}_\alpha$ (cf. chapter 2), to all inf div distributions on $[0,\infty)$. Here the difficulty arises that no analogue of $\mathcal{C}_0$ is known for general df's on $[0,\infty)$, but generalizing a functional equation, by means of which the inf div df's on $[0,\infty)$ are characterized (cf. theorem 1.6.2), we obtain classes $\mathcal{F}_\lambda$ $(0 \le \lambda < \infty)$, which, by putting $\alpha = e^{-\lambda}$, can be considered to be extensions of the classes $\mathcal{C}_\alpha$ $(0 < \alpha \le 1)$. This is done in section 1, which also contains some preliminary results. These are necessary, because the proofs of the monotonicity of $\mathcal{F}_\lambda$ (section 2) and further properties (section 3) turn out to be more delicate than in the discrete case. This is mainly due to the necessity of considering also distributions on $[0,\infty)$ without a jump at zero, as will be apparent from the proofs, which will only be given as far as they are essentially different from those in the discrete case.
In section 4 we study the limiting class $\mathcal{F}_\infty := \lim_{\lambda\to\infty} \mathcal{F}_\lambda$. In many respects, this class turns out to be the analogue of $\mathcal{C}_0$ for df's on $[0,\infty)$. For instance, the df's F in $\mathcal{F}_\infty$ with $F(0) > 0$ correspond with the compound geometric df's on $[0,\infty)$, and $\mathcal{F}_\infty$ contains the log-convex, and hence the comp mon, densities on $(0,\infty)$ (cf. the end of section 1.6). Furthermore, we show that the PLST's of df's in $\mathcal{F}_\infty$ have a canonical representation that is very similar to that of the LT's of continuous analogues of the renewal sequences (class $\mathcal{R}_0$), the standard p-functions (class $\mathcal{P}$) of Kingman (1972). The resulting relation between $\mathcal{P}$ and (a part of) $\mathcal{F}_\infty$ is considered in section 5. Many properties of $\mathcal{P}$, often easily obtained by the probabilistic interpretation of a p-function, can be translated for the corresponding densities in $\mathcal{F}_\infty$. Also, the df's in $\mathcal{F}_\infty$ have interesting relations with the renewal functions (cf. Smith (1958)) and the potential kernels (cf. Berg & Forst (1975) and Hawkes (1977)). However, we shall not investigate these relations in detail in this monograph.
In section 6 we show that two different interpolations between $\mathcal{F}_\infty$ and $\mathcal{F}_0$ give both rise to classes of df's that are limits of compound negative-binomial df's on $[0,\infty)$ (cf. section 2.5). These classes also define a classification of $\mathcal{F}_0$.
Finally, in section 7 we mention some further generalizations. Specifically, we briefly discuss the classification of inf div distributions on $\mathbb{R}$ and on $[0,\infty)^2$ by means of functional equations.

5.1. *The classes F_λ; preliminaries*

In chapter 2 the classification of C_1, defined by the classes C_α $(0 \le \alpha \le 1)$, was achieved by generalizing the recurrence relations by means of which C_1 and C_0 are characterized, i.e. the relations for a lattice distribution $\{p_n\}_0^\infty$ given by

$$(5.1.1) \qquad (n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(1) \qquad (n \in \mathbb{N}_0)$$

and

$$(5.1.2) \qquad p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(0) \qquad (n \in \mathbb{N}_0) \;,$$

respectively. Now we want to give a similar classification of the class of all inf div distributions on $[0,\infty)$. The starting point for this is the analogue of (5.1.1) for densities (cf. corollary 1.6.3): a pdf f on $(0,\infty)$ is inf div iff there exists a right-continuous, nondecreasing function K_0 such that

$$(5.1.3) \qquad xf(x) = \int_{[0,x]} f(x-y)\,dK_0(y) \qquad (\text{almost all } x > 0) \;.$$

For pdf's there is no obvious analogue of (5.1.2); this would be

$$(5.1.4) \qquad f(x) = \int_{[0,x]} f(x-y)\,dK(y) \qquad (\text{almost all } x > 0) \;,$$

with a right-continuous, nondecreasing function K. However, (5.1.4) is satisfied by all pdf's if K is the unit-step function at zero and by none for any other K. To overcome this difficulty we first generalize the classes C_α for $\alpha > 0$. The set of (absolutely continuous) distributions in the intersection of the resulting classes will then be considered as the analogue of C_0 for (absolutely continuous) distributions on $[0,\infty)$.
Proceeding as in the discrete case we replace the factor x in the left-hand side of (5.1.3) by the function

$$(5.1.5) \qquad c(x;\lambda) := (1-e^{-\lambda x})/(1-e^{-\lambda}) \qquad (0 < \lambda < \infty;\; x \ge 0) \;,$$

with $c(x;0) := x$ $(x \ge 0)$. This function is obtained from $c_n(\alpha) = (1-\alpha^{n+1})/(1-\alpha)$ in (2.3.1) by replacing α with $e^{-\lambda}$; in fact, we have

$$(5.1.6) \qquad c_n(\alpha) = c(n+1;-\log \alpha) \qquad (n \in \mathbb{N}_0;\; 0 < \alpha \le 1) \;,$$

and, as in the discrete case, we define for $\lambda > 0$

$$c_\lambda := \lim_{x\to\infty} c(x;\lambda) = (1-e^{-\lambda})^{-1} .$$

Considering, more generally, df's rather than pdf's, we are led to the following analogues of the classes $\mathcal{C}_\alpha$ for $0 < \alpha \le 1$.

DEFINITION 5.1.1. For $0 \le \lambda < \infty$ a df F on $[0,\infty)$ is said to be in the class $\mathcal{F}_\lambda$ if there exists a right-continuous, nondecreasing function K_λ, vanishing on $(-\infty,0)$, such that

$$(5.1.7) \qquad \int_{[0,x]} c(y;\lambda)\,dF(y) = \int_{[0,x]} F(x-y)\,dK_\lambda(y) \qquad (x \ge 0) .$$

If $F \in \mathcal{F}_\lambda$, then we shall also say that $\hat{F} \in \mathcal{F}_\lambda$; similarly, $f \in \mathcal{F}_\lambda$ if $F \in \mathcal{F}_\lambda$ has density f. As we shall see from its LST, the function K_λ in definition 5.1.1 is uniquely determined by $F \in \mathcal{F}_\lambda$; it will be called the K_λ-*function* of $F \in \mathcal{F}_\lambda$.

It is convenient to introduce the following (disjoint) subclasses of $\mathcal{F}_\lambda$ for $0 \le \lambda < \infty$:

$$(5.1.8) \qquad \begin{aligned} \mathcal{F}_\lambda^+ &:= \{F \in \mathcal{F}_\lambda \mid F(0) > 0\} , \\ \mathcal{F}_\lambda' &:= \{F \in \mathcal{F}_\lambda \mid F \text{ is absolutely continuous}\} . \end{aligned}$$

The classes $\mathcal{F}_\lambda^+$ contain the classes $\mathcal{C}_\alpha$; in fact, for $0 < \alpha \le 1$ we have

$$(5.1.9) \qquad \mathcal{C}_\alpha = \{F \in \mathcal{F}^+_{-\log\alpha} \mid F \text{ is concentrated on } \mathbb{N}_0\} .$$

This easily follows by showing that if F is the df corresponding to a distribution $\{p_n\}_0^\infty$ on $\mathbb{N}_0$ with $p_0 > 0$ and if $\lambda = -\log\alpha$, then the recurrence relations (2.3.1) and the functional equation (5.1.7) are equivalent: use (5.1.6) and the following relation between the K_λ-function of F and the $r_n(\alpha)$'s corresponding to $\{p_n\}$ (cf. theorem 1.7.7):

$$K_\lambda(x) = \sum_{n=1}^{\infty} r_{n-1}(\alpha)\,1_{[n,\infty)}(x) \qquad (x \in \mathbb{R}) .$$

The classes $\mathcal{F}_\lambda'$ can be characterized as follows (cf. corollary 1.6.3).

THEOREM 5.1.2. For $0 \le \lambda < \infty$ a pdf f on $(0,\infty)$ is in $\mathcal{F}_\lambda'$ iff there exists a right-continuous, nondecreasing function K_λ such that

$$(5.1.10)\quad c(x;\lambda)f(x) = \int_{[0,x]} f(x-y)\,dK_\lambda(y) \qquad \text{(almost all } x > 0)\ .$$

PROOF. If F is a df on $[0,\infty)$ with pdf f, then for all $x > 0$ we have, on the one hand

$$\int_{[0,x]} c(y;\lambda)\,dF(y) = \int_{[0,x]} c(y;\lambda)f(y)\,dy\ ,$$

and on the other hand (using Fubini's theorem)

$$\int_{[0,x]} F(x-y)\,dK_\lambda(y) = \int_{[0,x]}\int_{[0,x-y]} f(u)\,du\ dK_\lambda(y) =$$

$$= \int_{[0,x]}\int_{[y,x]} f(v-y)\,dv\ dK_\lambda(y) = \int_{[0,x]}\int_{[0,v]} f(v-y)\,dK_\lambda(y)\,dv\ .$$

Now, in view of (5.1.7), the theorem easily follows. □

Taking $\lambda = 0$ in definition 5.1.1, from theorem 1.6.2 one sees that F_0 is the class of *all* inf div df's on $[0,\infty)$. Furthermore, on account of theorem 1.6.6 it follows that F_0^+ is the class of all compound Poisson distributions on $[0,\infty)$. To obtain the compound geometric distributions on $[0,\infty)$, we formally let $\lambda \to \infty$ in (5.1.7). As

$$c(x;\infty) := \lim_{\lambda\to\infty} c(x;\lambda) = \begin{cases} 0 & \text{if } x = 0\ , \\ 1 & \text{if } x > 0\ , \end{cases}$$

we get

$$(5.1.11)\quad F(x) - F(0) = \int_{[0,x]} F(x-y)\,dK_\infty(y) \qquad (x \geq 0)\ .$$

If $F(0) = 0$, the same difficulty arises as in (5.1.4) for pdf's, but if $F(0) > 0$, (5.1.11) makes sense; the df's F on $[0,\infty)$, with $F(0) > 0$ and satisfying (5.1.11) with a right-continuous, nondecreasing function K_∞, correspond to the compound geometric distributions on $[0,\infty)$ (cf. theorem 1.6.7). This class of distributions we denote by F_∞^+ (cf. (5.1.8)), since it turns out to be the subset of df's F with $F(0) > 0$ in a class F_∞ to be defined later. Thus we have the following definition:

$$(5.1.12)\quad F_\infty^+ := \{\text{df } F \text{ on } [0,\infty) \mid F \text{ is compound geometric}\}\ .$$

We require that the function K_∞ in (5.1.11) vanishes on $(-\infty,0)$. It is then uniquely determined by $F \in \mathcal{F}_\infty^+$, and will be called the K_∞-*function* of $F \in \mathcal{F}_\infty^+$.

In order to obtain a characterization of $\mathcal{F}_\lambda$ $(0 < \lambda < \infty)$ and $\mathcal{F}_\infty^+$, analogous to that of $\mathcal{F}_0$ given in the first part of theorem 1.6.1, we take LST's in (5.1.7) and (5.1.11) to obtain

$$(5.1.13)\qquad c_\lambda\{\hat{F}(\tau) - \hat{F}(\tau+\lambda)\} = \hat{F}(\tau)\hat{K}_\lambda(\tau) \qquad (\tau > 0)\ ,$$

and

$$(5.1.14)\qquad \hat{F}(\tau) - F(0) = \hat{F}(\tau)\hat{K}_\infty(\tau) \qquad (\tau > 0)\ .$$

Now, for an arbitrary df F on $[0,\infty)$, we define the φ_λ-function of F as the solution of (5.1.13) or (5.1.14) for $\hat{K}_\lambda$, i.e. we have the following definition.

DEFINITION 5.1.3. For an arbitrary df F on $[0,\infty)$ the φ_λ-*function* of F is defined by

$$\varphi_\lambda(\tau) := c_\lambda\{1 - \hat{F}(\tau+\lambda)/\hat{F}(\tau)\} \qquad \text{if } 0 < \lambda < \infty\ ,$$

$$\varphi_0(\tau) := \lim_{\lambda\downarrow 0} \varphi_\lambda(\tau) = -\hat{F}'(\tau)/\hat{F}(\tau) = -\frac{d}{d\tau}\log \hat{F}(\tau)\ ,$$

and, if $F(0) > 0$,

$$\varphi_\infty(\tau) := \lim_{\lambda\to\infty} \varphi_\lambda(\tau) = 1 - F(0)/\hat{F}(\tau)\ .$$

We note that for $0 \le \lambda < \infty$ the φ_λ-function of a df $F \in \mathcal{F}_\lambda$ coincides with the LST of the K_λ-function of F, i.e. $\varphi_\lambda = \hat{K}_\lambda$. Similarly $\varphi_\infty = \hat{K}_\infty$, if $F \in \mathcal{F}_\infty^+$. By Bernstein's theorem (theorem 1.3.7) we immediately obtain the following characterization of $\mathcal{F}_\lambda$ and $\mathcal{F}_\infty^+$ (cf. lemma 2.3.3).

LEMMA 5.1.4.
(i) For $0 \le \lambda < \infty$ a df F on $[0,\infty)$ is in $\mathcal{F}_\lambda$ iff its φ_λ-function is comp mon.
(ii) A df F on $[0,\infty)$ with $F(0) > 0$ is in $\mathcal{F}_\infty^+$ iff its φ_∞-function is comp mon.

We shall need the limiting behaviour for $\tau \to \infty$ of the φ_λ-functions of a df F. For $\lambda > 0$ this is obtained from the following lemma, which does not seem to be generally known (cf. Van Harn (1977)). The quantity $\ell(F)$ is the left extremity of F (cf. section 1.2).

LEMMA 5.1.5. If $a > 0$ and if F is a df on $[0,\infty)$, then the function $\hat{F}(\tau+a)/\hat{F}(\tau)$ is nondecreasing, and satisfies

$$(5.1.15)\qquad \lim_{\tau\to\infty} \hat{F}(\tau+a)/\hat{F}(\tau) = \exp[-a\ell(F)]\ .$$

PROOF. Let $a > 0$ and let F be a df on $[0,\infty)$. As by Bernstein's theorem $\hat{F}$ is comp mon, it is log-convex (use Schwarz's inequality; see also theorem 1.6.11), i.e. $\log \hat{F}(\tau)$ is convex. It follows that the φ_o-function φ_o of F is nonincreasing, and as

$$(5.1.16)\qquad \frac{d}{d\tau}\{\hat{F}(\tau+a)/\hat{F}(\tau)\} = \{\varphi_o(\tau) - \varphi_o(\tau+a)\}\hat{F}(\tau+a)/\hat{F}(\tau)\ ,$$

it is seen that the function $\hat{F}(\tau+a)/\hat{F}(\tau)$, which takes values in $(0,1]$, is nondecreasing, and hence has a limit in $(0,1]$ as $\tau\to\infty$.
The proof of (5.1.15) we give here, is due to W. Vervaat (see also remark 5.1.7). First, take $\ell(F) = 0$. Then for (5.1.15) it is sufficient to prove that

$$(5.1.17)\qquad \liminf_{\tau\to\infty} \hat{F}(\tau+a)/\hat{F}(\tau) \ge 1\ .$$

For every $\tau > 0$ and $\varepsilon > 0$

$$\hat{F}(\tau+a) \ge e^{-a\varepsilon}\int_{[0,\varepsilon]} e^{-\tau x}dF(x) \ge e^{-a\varepsilon}\{\hat{F}(\tau) - e^{-\tau\varepsilon}\}\ .$$

Since

$$0 \le e^{-\tau\varepsilon}\hat{F}(\tau)^{-1} \le e^{-\tau\varepsilon}\{\int_{[0,\varepsilon/2]} e^{-\tau x}dF(x)\}^{-1} \le e^{-\tau\varepsilon/2}F(\varepsilon/2)^{-1}\ ,$$

it follows that for all $\varepsilon > 0$

$$\liminf_{\tau\to\infty} \hat{F}(\tau+a)/\hat{F}(\tau) \ge e^{-a\varepsilon}\ ,$$

which yields (5.1.17), and hence (5.1.15) with $\ell(F) = 0$.
Finally, using this result for the df $G(x) := F(x+\ell(F))$, we easily obtain (5.1.15) for an arbitrary df F on $[0,\infty)$. □

LEMMA 5.1.6. If $0 \le \lambda \le \infty$ and if F is a df on $[0,\infty)$ (with $F(0) > 0$ if $\lambda = \infty$), then the φ_λ-function of F satisfies (cf. (5.1.5))

$$(5.1.18)\qquad \lim_{\tau\to\infty} \varphi_\lambda(\tau) = c(\ell(F);\lambda)\ .$$

PROOF. The result follows from (5.1.15) if $0 < \lambda < \infty$, and is trivial if $\lambda = \infty$. To prove (5.1.18) for $\lambda = 0$, we may restrict ourselves to df's F with $\ell(F) = 0$ (cf. the proof of lemma 5.1.5). Then using the identity

$$\hat{F}(\tau) = \tau \int_{(0,\infty)} e^{-\tau x} F(x)\,dx \qquad (\tau > 0) \;,$$

we obtain for $\varepsilon > 0$

$$\begin{aligned}\varphi_o(\tau) &= -\hat{F}'(\tau)/\hat{F}(\tau) \le \Big\{ \int_{(0,\infty)} e^{-\tau x} F(x)\,dx \Big\}^{-1} \int_{(0,\infty)} x e^{-\tau x} F(x)\,dx \le \\ &\le \varepsilon + \Big\{ F(\varepsilon/2) \int_{(\varepsilon/2,\infty)} e^{-\tau x}\,dx \Big\}^{-1} \int_{(\varepsilon,\infty)} x e^{-\tau x}\,dx = \\ &= \varepsilon + F(\varepsilon/2)^{-1}\{\varepsilon + 1/\tau\}\exp[-\tau\varepsilon/2] \;,\end{aligned}$$

which is less than 2ε for τ sufficiently large. Hence $\lim_{\tau\to\infty} \varphi_o(\tau) = 0$. □

REMARK 5.1.7. Since for all $\tau > 0$ there exists $\theta(\tau) \in (0,1)$ such that

$$\log\{\hat{F}(\tau + a)/\hat{F}(\tau)\} = -a\varphi_o(\tau + \theta(\tau)a) \;,$$

(5.1.15) also immediately follows from (5.1.18) with $\lambda = 0$.

In (1.6.2) and lemma 1.6.4 some properties of the K_o-function (i.e. the canonical function) of an inf div df on $[0,\infty)$ are given. The K_λ-function of a df in $\mathcal{F}_\lambda$ has analogous properties. In the following lemma we prove two of them by considering the LST $\hat{K}_\lambda$ of K_λ, but they can also be obtained from the functional equations (5.1.7) and (5.1.11). A third property of K_λ will be given in lemma 5.2.2.

LEMMA 5.1.8. If $0 < \lambda \le \infty$ and if $F \in \mathcal{F}_\lambda$ (with $F(0) > 0$ if $\lambda = \infty$), then the K_λ-function of F satisfies

(5.1.19) $\quad K_\lambda(0) = c(\ell(F);\lambda)$

and

(5.1.20) $\quad K_\lambda(\infty) = \displaystyle\int_{[0,\infty)} dK_\lambda(x) = c_\lambda\{1 - \hat{F}(\lambda)\} < c_\lambda \;.$

PROOF. If $F \in \mathcal{F}_\lambda$, then $\hat{K}_\lambda$ coincides with the φ_λ-function of F, for which we have lemma 5.1.6. Now from (1.3.5) it follows that

$$K_\lambda(0) = \lim_{\tau\to\infty} \hat{K}_\lambda(\tau) = c(\ell(F);\lambda) ,$$

and from (1.3.6)

$$K_\lambda(\infty) = \lim_{\tau\downarrow 0} \hat{K}_\lambda(\tau) = \lim_{\tau\downarrow 0} c_\lambda\{1-\hat{F}(\tau+\lambda)/\hat{F}(\tau)\} = c_\lambda\{1-\hat{F}(\lambda)\} . \qquad \square$$

Finally, we state a property of $\mathcal{F}_o$ that we need in the next section to prove the monotonicity of $\mathcal{F}_\lambda$.

THEOREM 5.1.9. If $F \in \mathcal{F}_o$ and $a > 0$, then the function φ, defined by

$$\varphi(\tau) := \hat{F}(a)\hat{F}(\tau)/\hat{F}(\tau+a) \qquad (\tau \ge 0) ,$$

is a PLST in $\mathcal{F}_o^+$.

PROOF. Let $F \in \mathcal{F}_o$ with canonical function K_o. In view of theorem 1.6.1 we calculate

$$-\frac{d}{d\tau} \log \varphi(\tau) = \hat{K}_o(\tau) - \hat{K}_o(\tau+a) ,$$

which is a comp mon function. As furthermore $\varphi(0) = 1$, it follows that φ is the PLST of a df F_a in $\mathcal{F}_o$, for which by lemma 5.1.5 we have

$$F_a(0) = \lim_{\tau\to\infty} \varphi(\tau) = \hat{F}(a)\exp[a\ell(F)] > 0 . \qquad \square$$

5.2. *The monotonicity of $\mathcal{F}_\lambda$, absolute continuity*

In this section we show that the classes $\mathcal{F}_\lambda$ define a classification of the inf div df's on $[0,\infty)$, i.e. we show that $\mathcal{F}_\lambda$ depends monotonically on λ. To this end it is convenient to consider only df's F for which

(5.2.1) $\quad \ell(F) = 0$;

this is not an essential restriction (cf. theorem 5.3.2(i)). Now, for instance, it follows that the φ_λ-function of F satisfies $\lim_{\tau\to\infty} \varphi_\lambda(\tau) = 0$ (cf. lemma 5.1.6), and, if $F \in \mathcal{F}_\lambda$, that $K_\lambda(0) = 0$ (cf. (5.1.19)).
First we prove that all distributions in the classes $\mathcal{F}_\lambda$ with $0 < \lambda < \infty$ are inf div.

THEOREM 5.2.1. For all $\lambda \in (0,\infty)$ the following inclusion holds: $\mathcal{F}_\lambda \subset \mathcal{F}_0$.

PROOF. We use the method of proof of theorem 2.3.4. It will then be clear that by iteration of (5.1.13) $\hat{F}$ can be written as

$$\hat{F}(\tau) = \frac{\hat{F}(\tau+n\lambda)}{\hat{F}(n\lambda)} \prod_{k=0}^{n-1} \frac{1 - c_\lambda^{-1}\hat{K}_\lambda(k\lambda)}{1 - c_\lambda^{-1}\hat{K}_\lambda(\tau+k\lambda)} \qquad (\tau \geq 0;\ n \in \mathbb{N}) .$$

Now using the fact that $\hat{F}(\tau+n\lambda)/\hat{F}(n\lambda)$ tends to 1 as $n \to \infty$ (cf. lemma 5.1.5), we obtain the following expression for $\hat{F}$ in its K_λ-function:

$$(5.2.2) \qquad \hat{F}(\tau) = \prod_{k=0}^{\infty} \frac{1 - c_\lambda^{-1}\hat{K}_\lambda(k\lambda)}{1 - c_\lambda^{-1}\hat{K}_\lambda(\tau+k\lambda)} \qquad (\tau \geq 0) ,$$

from which, as in the discrete case, it follows that F is inf div. □

From (5.2.2) we obtain a characterization of $\mathcal{F}_\lambda^+$ for $0 < \lambda < \infty$ in terms of the K_λ-function. It can be considered as an analogue of lemma 1.6.4(ii), where $\mathcal{F}_0^+$ is characterized similarly.

LEMMA 5.2.2. Let $0 < \lambda < \infty$ and let $F \in \mathcal{F}_\lambda$. Then $F \in \mathcal{F}_\lambda^+$, i.e. $F(0) > 0$, iff the K_λ-function of F satisfies

$$(5.2.3) \qquad \int_{(0,\infty)} c(x;\lambda)^{-1} dK_\lambda(x) < \infty ,$$

in which case

$$(5.2.4) \qquad F(0) = \prod_{k=0}^{\infty} \{1 - c_\lambda^{-1}\hat{K}_\lambda(k\lambda)\} .$$

PROOF. Let $F \in \mathcal{F}_\lambda$. Then (5.2.2) holds, from which, taking $\tau = n\lambda$, we see that

$$(5.2.5) \qquad \hat{F}(n\lambda) = \prod_{k=0}^{n-1} \{1 - c_\lambda^{-1}\hat{K}_\lambda(k\lambda)\} \qquad (n \in \mathbb{N}) .$$

As $F(0) = \lim_{n\to\infty} \hat{F}(n\lambda)$, from (5.2.5) it follows that $F(0) > 0$ iff the infinite product in (5.2.4) converges, or, equivalently, iff

$$(5.2.6) \qquad \sum_{k=0}^{\infty} \hat{K}_\lambda(k\lambda) < \infty .$$

Since by (5.2.1) we have $K_\lambda(0) = 0$, we can write, using Fubini's theorem,

$$c_\lambda^{-1} \sum_{k=0}^{\infty} \hat{K}_\lambda(k\lambda) = c_\lambda^{-1} \int_{(0,\infty)} \sum_{k=0}^{\infty} \{e^{-\lambda x}\}^k dK_\lambda(x) = \int_{(0,\infty)} c(x;\lambda)^{-1} dK_\lambda(x),$$

and the lemma is proved. □

For the classes $\mathcal{F}_o$, $\mathcal{F}_o^+$ and $\mathcal{F}_\infty^+$ we have already representation theorems (cf. theorems 1.6.1, 1.6.6 and 1.6.7, respectively). Now, from (5.2.2) we easily obtain the following representation for PLST's in $\mathcal{F}_\lambda$.

THEOREM 5.2.3. For $0 < \lambda < \infty$ a df F (with $\ell(F) = 0$) is in $\mathcal{F}_\lambda$ iff $\hat{F}$ has the form

$$(5.2.7) \qquad \hat{F}(\tau) = \prod_{k=0}^{\infty} \frac{1 - p\hat{G}(k\lambda)}{1 - p\hat{G}(\tau + k\lambda)} \qquad (\tau \geq 0),$$

where $0 \leq p < 1$ and G is a df with $G(0) = 0$. The representation (p,G) is unique.

PROOF. Apply the method of proof of theorem 2.3.5, and use the properties of the K_λ-function of F from lemma 5.1.8. □

Next we turn to the general monotonicity property. Once it has been proved for $\mathcal{F}_\cdot$, it also follows for $\mathcal{F}'_\cdot$, $\mathcal{F}^+_\cdot$ and $\mathcal{C}_\cdot$, and for every other set of classes, obtained from the $\mathcal{F}_\lambda$'s by intersecting them with a set of df's that does not depend on λ. We think it useful to give a full proof, although the first part is analogous to the proof of theorem 2.3.7.

THEOREM 5.2.4. For all $\lambda \in [0,\infty)$ and $\mu \in [0,\infty)$ the following inclusion holds:

$$\mathcal{F}_\lambda \subset \mathcal{F}_\mu \qquad \text{if } \lambda \geq \mu .$$

PROOF. The theorem has been already proved in the case $\lambda \in [0,\infty)$, $\mu = 0$. So, in view of lemma 5.1.4(i), we have to show that, if $0 < \mu \leq \lambda < \infty$ and if the φ_λ-function of a df F on $[0,\infty)$ is comp mon, then its φ_μ-function is comp mon. Now, according to definition 5.1.3, the φ_μ-function of F satisfies

$$(5.2.8)\qquad \varphi_\mu(\tau) - \varphi_\mu(\tau+\lambda) = c_\mu\{\hat{F}(\tau+\lambda+\mu)/\hat{F}(\tau+\lambda) - \hat{F}(\tau+\mu)/\hat{F}(\tau)\}\ .$$

If the right-hand side of (5.2.8) is divided by $c_\mu\hat{F}(\tau+\mu)$, it becomes symmetric in λ and μ, and so

$$(5.2.9)\qquad c_\lambda\{\varphi_\mu(\tau) - \varphi_\mu(\tau+\lambda)\} = \frac{\hat{F}(\tau+\mu)}{\hat{F}(\tau+\lambda)}\, c_\mu\{\varphi_\lambda(\tau) - \varphi_\lambda(\tau+\mu)\}\ .$$

If φ_λ is comp mon, then $\varphi_\lambda(\tau) - \varphi_\lambda(\tau+\mu)$ is comp mon too. Further, as $F \in F_\lambda \subset F_o$ (theorem 5.2.1) and as $\mu \le \lambda$, from theorem 5.1.9 it follows that the function $\hat{F}(\tau)/\hat{F}(\tau+\lambda-\mu)$, and hence $\hat{F}(\tau+\mu)/\hat{F}(\tau+\lambda)$, is comp mon. Thus, from (5.2.9) we conclude that $\varphi_\mu(\tau) - \varphi_\mu(\tau+\lambda)$ is a comp mon function. Now, using the fact that $\lim_{n\to\infty} \varphi_\mu(\tau+n\lambda) = 0$ (cf. lemma 5.1.6), for φ_μ we can write

$$\varphi_\mu(\tau) = \sum_{k=0}^{\infty} \{\varphi_\mu(\tau+k\lambda) - \varphi_\mu(\tau+k\lambda+\lambda)\}\ .$$

It follows that φ_μ is the limit of a sequence of sums of comp mon functions. Hence φ_μ is comp mon, and the theorem is proved. □

By letting $\lambda \to \infty$ in (5.2.9) we see that if φ_∞ is comp mon, then φ_μ is comp mon for all $\mu \in (0,\infty)$, so (cf. lemma 5.1.4(ii)) we have $F_\infty^+ \subset \bigcap_{\lambda<\infty} F_\lambda^+$. On the other hand $\varphi_\infty = \lim_{\lambda\to\infty} \varphi_\lambda$ is comp mon, if all φ_λ are, and so

$$(5.2.10)\qquad F_\infty^+ = \bigcap_{\lambda<\infty} F_\lambda^+\ .$$

This allows us to define the classes F_∞ and F_∞' as follows (cf. (5.1.8) and (5.1.12)):

$$(5.2.11)\qquad \begin{aligned} F_\infty &:= \bigcap_{\lambda<\infty} F_\lambda = \lim_{\lambda\to\infty} F_\lambda\ , \\ F_\infty' &:= \{F \in F_\infty \mid F \text{ is absolutely continuous}\}\ . \end{aligned}$$

Of course, theorem 5.2.4 can now be supplemented with the case $\lambda = \infty$, $\mu \in [0,\infty]$.

If $F \in F_\lambda$, then theorem 5.2.4 ensures the existence of the K_μ-functions of F for all $\mu \in [0,\lambda]$. From (5.2.9) we obtain the following properties of these functions.

THEOREM 5.2.5. Let $0 < \lambda \le \infty$ and let $F \in \mathcal{F}_\lambda$ (with $F(0) > 0$ if $\lambda = \infty$) with an absolutely continuous K_λ-function (density k_λ). Then for all $\mu \in [0,\lambda)$ the K_μ-function of F is also absolutely continuous (density k_μ), and the following inequality holds:

$$(5.2.12)\quad k_\mu(x)/c(x;\mu) \ge k_\lambda(x)/c(x;\lambda) \qquad (x > 0)\ .$$

PROOF. First take $\mu \in (0,\lambda)$. As we saw in the proof of theorem 5.2.4, $\hat{F}(\tau+\mu)/\hat{F}(\tau+\lambda)$ is comp mon, so by Bernstein's theorem there exists a right-continuous, nondecreasing function $K_{\lambda,\mu}$, zero for negative arguments, such that $\hat{F}(\tau+\mu)/\hat{F}(\tau+\lambda) = \hat{K}_{\lambda,\mu}(\tau)$. As $\varphi_\lambda = \hat{K}_\lambda$ and $\varphi_\mu = \hat{K}_\mu$, from (5.2.9) it follows that for all $x > 0$

$$(5.2.13)\quad \int_{[0,x]} c(y;\lambda)\,dK_\mu(y) = \int_{[0,x]} \int_{[0,x-y]} c(u;\mu)\,dK_\lambda(u)\,dK_{\lambda,\mu}(y)\ .$$

If K_λ has a density k_λ, then, substituting $u + y = v$ in the right-hand side of (5.2.13) and then changing the order of integration, and noting that by lemma 5.1.5

$$K_{\lambda,\mu}(0) = \lim_{\tau\to\infty} \hat{F}(\tau+\mu)/\hat{F}(\tau+\lambda) = 1\ ,$$

we obtain the absolute continuity of K_μ, with density k_μ given by

$$(5.2.14)\quad k_\mu(x) = c(x;\lambda)^{-1}\{c(x;\mu)k_\lambda(x) + \int_{(0,x]} c(x-y;\mu)k_\lambda(x-y)\,dK_{\lambda,\mu}(y)\}\ .$$

For the case $\mu = 0$, we let $\mu \downarrow 0$ in (5.2.9) to obtain

$$(5.2.15)\quad c_\lambda\{\hat{K}_o(\tau) - \hat{K}_o(\tau+\lambda)\} = \frac{\hat{F}(\tau)}{\hat{F}(\tau+\lambda)}\frac{d}{d\tau}[-\hat{K}_\lambda(\tau)]\ ,$$

which by (5.1.13) is equivalent to

$$(5.2.16)\quad \int_{[0,x]} c(y;\lambda)\,dK_o(y) = \int_{[0,x]} y\,dK_\lambda(y) + \int_{[0,x]} K_\lambda(x-y)(1-e^{-\lambda y})\,dK_o(y)\,.$$

It follows that K_o is absolutely continuous, with density k_o given by

$$(5.2.17)\quad k_o(x) = c(x;\lambda)^{-1}\{xk_\lambda(x) + \int_{[0,x]} k_\lambda(x-y)(1-e^{-\lambda y})\,dK_o(y)\}\ .$$

Finally, (5.2.12) is obtained from (5.2.14) and (5.2.17). $\square$

REMARK 5.2.6. In case $\mu = 0$ in the preceding theorem we can reverse matters. Suppose that the K_o-function of $F \in \mathcal{F}_\lambda$ has a density k_o, then from (5.2.16) it follows that the K_λ-function of F satisfies

$$(5.2.18)\qquad \int_{[0,x]} y\, dK_\lambda(y) = \int_{[0,x]} \{c(y;\lambda)k_o(y) - \int_{[0,y]} (1-e^{-\lambda(y-z)})k_o(y-z)\,dK_\lambda(z)\}dy\ .$$

Hence, as $K_\lambda(0) = 0$, K_λ is absolutely continuous and has a density k_λ, that satisfies (5.2.17).
Thus we have proved that if $F \in \mathcal{F}_\lambda$, then the absolute continuity of the K_λ-function and that of the K_o-function of F are *equivalent*. Furthermore, in view of theorem 5.2.4 it now follows more generally that if $F \in \mathcal{F}_\lambda$ and if $0 \le \mu < \lambda$, then the K_λ-function of F is absolutely continuous *iff* the K_μ-function of F is absolutely continuous.

The absolute continuity of the K_λ-function of $F \in \mathcal{F}_\lambda$ is also sufficient for the absolute continuity of $F(x) - F(0)$. This observation generalizes a theorem by Tucker (1962) or Fisz & Varadarajan (1963) (our theorem 1.7.10), if restricted to the half-line. In our case the proof is very simple.

THEOREM 5.2.7. If $0 \le \lambda \le \infty$ and if $F \in \mathcal{F}_\lambda$ (with $F(0) > 0$ if $\lambda = \infty$) has an absolutely continuous K_λ-function, then $F(x) - F(0)$ is absolutely continuous.

PROOF. If K_λ has a density k_λ, then the right-hand side of (5.1.7) or (5.1.11) is absolutely continuous with density u given by

$$u(x) = \int_{[0,x]} k_\lambda(x-y)\,dF(y) \qquad (x > 0)\ .$$

As $c(x;\lambda) > 0$ for all $x > 0$, from (5.1.7) or (5.1.11) it now follows that

$$F(x) - F(0) = \int_{(0,x]} dF(y) = \int_{(0,x]} c(y;\lambda)^{-1}u(y)\,dy\ ,$$

and hence $F(x) - F(0)$ is absolutely continuous with density f_o given by

$$(5.2.19)\qquad f_o(x) = c(x;\lambda)^{-1} \int_{[0,x]} k_\lambda(x-y)\,dF(y) \qquad (x > 0)\ . \qquad \square$$

COROLLARY 5.2.8. Let $0 \leq \lambda < \infty$ and let $F \in \mathcal{F}_\lambda$. If the K_λ-function of F is absolutely continuous with

$$(5.2.20) \qquad \int_{(0,\infty)} c(x;\lambda)^{-1} dK_\lambda(x) = \infty ,$$

then F is absolutely continuous, i.e. $F \in \mathcal{F}'_\lambda$.

PROOF. Combine theorem 5.2.7 and lemma 5.2.2. □

REMARK 5.2.9. The K_o-function of a df $F \in \mathcal{F}'_o$ satisfies (5.2.20) with $\lambda = 0$, but it is not necessarily absolutely continuous: if K_o has an absolutely continuous part, i.e. $K_o = K_o^{(1)} + K_o^{(2)}$ with $K_o^{(i)}$ nondecreasing $(i = 1,2)$ and $K_o^{(1)} \not\equiv 0$ absolutely continuous, then from (1.6.3) it is seen that F has an absolutely continuous component, and hence is itself absolutely continuous. By remark 5.2.6 it follows that also the K_λ-function of a df $F \in \mathcal{F}'_\lambda$ may not be expected to be always absolutely continuous.

5.3. *Further properties of the $\mathcal{F}_\lambda$'s, examples*

When considering properties and examples of df's in $\mathcal{F}_\lambda$, we shall frequently use the characterization of $\mathcal{F}_\lambda$ given by lemma 5.1.4. The φ_λ-function of a df that depends on some parameter ν, will then be denoted by $\varphi_\lambda^{(\nu)}$, and we shall use the same notation without further comment in several different situations.

We start with some properties of the classes $\mathcal{F}_\lambda$. The first of them is well known for $\mathcal{F}_o$ (cf. theorem 1.4.3).

THEOREM 5.3.1. For $0 \leq \lambda \leq \infty$ the class $\mathcal{F}_\lambda$ is closed under weak convergence, i.e. a df F on $[0,\infty)$, for which there exist $F_n \in \mathcal{F}_\lambda$ $(n \in \mathbb{N})$ such that $\hat{F}(\tau) = \lim_{n\to\infty} \hat{F}_n(\tau)$ $(\tau \geq 0)$, is again in $\mathcal{F}_\lambda$.

PROOF. By the definition of $\mathcal{F}_\infty$ it is sufficient to consider the case $0 < \lambda < \infty$. The φ_λ-functions $\varphi_\lambda^{(n)}$ of F_n are comp mon, and as $\varphi_\lambda^{(n)} \to \varphi_\lambda$ if $\hat{F}_n \to \hat{F}$ $(n \to \infty)$, it follows that φ_λ is comp mon too. So $F \in \mathcal{F}_\lambda$. □

It turns out that every $\mathcal{F}_\lambda$ $(0 \leq \lambda \leq \infty)$ is closed under translations, but *only $\mathcal{F}_o$ and $\mathcal{F}_\infty$ are closed under scale transformations.*

THEOREM 5.3.2. If $0 \le \lambda \le \infty$ and $a > 0$, then

(i) A df F on $[0,\infty)$ is in $\mathcal{F}_\lambda$ iff the df $F_a(x) := F(x-a)$ is in $\mathcal{F}_\lambda$;
(ii) A df F on $[0,\infty)$ is in $\mathcal{F}_\lambda$ iff the df $F_a(x) := F(ax)$ is in $\mathcal{F}_{a\lambda}$.

PROOF. The theorem is known for $\lambda = 0$ and follows for $\lambda = \infty$ as soon as it has been proved for finite λ's. So let $0 < \lambda < \infty$ and $a > 0$. In case (i) we have

$$\hat{F}_a(\tau) = e^{-a\tau}\hat{F}(\tau) ,$$

by which one easily obtains the following relation between the φ_λ-functions of F_a and F:

$$(5.3.1) \qquad \varphi_\lambda^{(a)}(\tau) = c(a;\lambda) + e^{-a\lambda}\varphi_\lambda(\tau) .$$

As $\lim_{\tau\to\infty} \varphi_\lambda(\tau) = c(\ell(F);\lambda) \ge 0$ (cf. lemma 5.1.6), it follows that φ_λ is comp mon iff $\varphi_\lambda^{(a)}$ is comp mon. Hence (i) is proved. In case (ii) $\hat{F}$ and $\hat{F}_a$ are related by

$$\hat{F}_a(\tau) = \hat{F}(\tau/a) ,$$

from which it is easily seen that

$$(5.3.2) \qquad \varphi_{a\lambda}^{(a)}(\tau) = \{c_{a\lambda}/c_\lambda\}\varphi_\lambda(\tau/a) .$$

It follows that φ_λ is comp mon iff $\varphi_{a\lambda}^{(a)}$ is comp mon, and (ii) is proved. □

In view of part (ii) of the preceding theorem, for many purposes, such as asymptotic behaviour and properties of moments, it is sufficient to consider, apart from $\mathcal{F}_o$ and $\mathcal{F}_\infty$, only, say, the class $\mathcal{F}_1$ in stead of all classes $\mathcal{F}_\lambda$ for $0 < \lambda < \infty$. Still, the monotonicity of $\mathcal{F}_\lambda$ is an interesting property, and an explicit definition of the $\mathcal{F}_\lambda$'s is needed to define the class $\mathcal{F}_\infty$. Furthermore, we note that, if we want to consider specifically *lattice* distributions, i.e. distributions on the fixed lattice $\mathbb{N}_o$, then the transformation $F_a(x) = F(ax)$ is not possible (unless $a^{-1} \in \mathbb{N}$): the classes $\mathcal{C}_\alpha$ $(0 \le \alpha \le 1)$ are essentially distinct.

In the following theorem we state some properties of the $\mathcal{F}_\lambda$'s for $0 \le \lambda \le \infty$, which are well known or trivial for $\lambda = 0$.

THEOREM 5.3.3. For $0 \le \lambda \le \infty$ the following properties hold:

(i) If $F \in \mathcal{F}_\lambda$ and $0 \le \nu < \infty$, then $\hat{F}(\tau+\nu)/\hat{F}(\nu) \in \mathcal{F}_\lambda$.

(ii) If $F \in \mathcal{F}_\lambda$ and $0 \le \nu \le 1$, then $\hat{F}^\nu \in \mathcal{F}_\lambda$.

(iii) If $F \in \mathcal{F}_\lambda$ and $0 \le \nu < \infty$, then $\hat{F}(\nu)\hat{F}(\tau)/\hat{F}(\tau+\nu) \in \mathcal{F}_\lambda$.

(iv) If $F \in \mathcal{F}_\lambda$, $n \in \mathbb{N}$ and $0 \le \nu \le \lambda/n$, then $\prod_{k=0}^{n-1} \hat{F}(\tau+k\nu)/\hat{F}(k\nu) \in \mathcal{F}_\nu$.

PROOF. It is sufficient to consider the case $0 < \lambda < \infty$. The proof is then analogous to that of theorem 2.4.2 in the discrete case (use lemma 5.1.4).□

The case $\lambda = \infty$ in (iv) can be used to construct examples of df's in $\mathcal{F}_\lambda$; we state it as a corollary.

COROLLARY 5.3.4. If $F \in \mathcal{F}_\infty$ and if $0 \le \lambda < \infty$, then for all $n \in \mathbb{N}$

$$(5.3.3) \qquad \prod_{k=0}^{n-1} \hat{F}(\tau+k\lambda)/\hat{F}(k\lambda) \in \mathcal{F}_\lambda \ .$$

We note that if $F(0) = 0$ then in (5.3.3) we cannot take $n = \infty$. In fact, the infinite product converges iff the series

$$\sum_{k=0}^{\infty} \{1 - \hat{F}(\tau+k\lambda)/\hat{F}(k\lambda)\} = c_\tau^{-1} \sum_{k=0}^{\infty} \hat{K}_\tau(k\lambda)$$

is convergent, but in view of lemma 5.2.2 and (5.2.6) this can only be the case if $F(0) > 0$. If $F \in \mathcal{F}_\infty^+$, however, then from theorem 5.2.3 it is seen that the infinite product yields a PLST in $\mathcal{F}_\lambda$ indeed. This also follows from the following characterization of $\mathcal{F}_\lambda$ in terms of $\mathcal{F}_\infty^+$ (cf. theorem 2.4.5).

THEOREM 5.3.5. Let F be a df on $[0,\infty)$, let $0 < \lambda < \infty$ and define the function ψ_λ by

$$(5.3.4) \qquad \psi_\lambda(\tau) := \hat{F}(\lambda)\hat{F}(\tau)/\hat{F}(\tau+\lambda) \qquad (\tau \ge 0) \ .$$

Then $F \in \mathcal{F}_\lambda$ iff ψ_λ is a PLST in $\mathcal{F}_\infty^+$.

PROOF. If $F \in \mathcal{F}_\lambda$, then according to theorem 5.1.9 ψ_λ is a PLST $\hat{F}_\lambda$, say, with $F_\lambda(0) = \hat{F}(\lambda)\ \exp[\lambda \ell(F)] > 0$. In this case the φ_∞-function of F_λ is given by

$$(5.3.5) \qquad \varphi_\infty^{(\lambda)}(\tau) = 1 - e^{\lambda\ell(F)} + e^{\lambda\ell(F)} c_\lambda^{-1} \varphi_\lambda(\tau) \ ,$$

and as φ_λ and $\varphi_\infty^{(\lambda)}$ are both nonnegative functions, it follows that φ_λ is comp mon iff $\varphi_\infty^{(\lambda)}$ is. Applying lemma 5.1.4 now proves the theorem. □

COROLLARY 5.3.6. A df F on $[0,\infty)$ is in $\mathcal{F}_\infty^+$ iff $\prod_{k=0}^{\infty} \hat{F}(\tau + k\lambda)/\hat{F}(k\lambda)$ is a PLST in $\mathcal{F}_\lambda$.

Using theorems 5.2.4 and 5.3.5 we can improve part (iii) of theorem 5.3.3 in the following way.

COROLLARY 5.3.7. If $0 \le \lambda \le \infty$ and if $F \in \mathcal{F}_\lambda$, then for all $\nu \in [0,\infty)$ the function $\hat{F}(\nu)\hat{F}(\tau)/\hat{F}(\tau+\nu)$ is a PLST $\hat{F}_\nu$, say, while $F_\nu \in \mathcal{F}_\lambda^+$ if $\nu > \lambda$, and $F_\nu \in \mathcal{F}_\infty^+$ if $0 \le \nu \le \lambda$.

As a last property of the $\mathcal{F}_\lambda$'s we prove that $\bigcup_{\lambda>0} \mathcal{F}_\lambda$ is dense in $\mathcal{F}_o$ in the sense of weak convergence.

THEOREM 5.3.8. If $F \in \mathcal{F}_o$, then there exists a decreasing sequence $\{\lambda_n\}$ with $\lambda_n \to 0$ and there are $F_n \in \mathcal{F}_{\lambda_n}$ $(n \in \mathbb{N})$ such that

$$\hat{F}(\tau) = \lim_{n\to\infty} \hat{F}_n(\tau) \qquad (\tau \ge 0) .$$

PROOF. As $\mathcal{F}_o^+$ coincides with the set of compound Poisson distributions on $[0,\infty)$ (cf. theorem 1.6.6), for $F \in \mathcal{F}_o^+$ we can give a proof along the same lines as in theorem 2.4.4 for the discrete case. For an arbitrary $F \in \mathcal{F}_o$ this proof and the proof of De Finetti's theorem (theorem 1.4.15) suggest the following choice: take $\lambda_n = n^{-2}$, and F_n such that for $n \ge 2$

$$(5.3.6) \qquad \hat{F}_n(\tau) = \prod_{k=0}^{n-1} \frac{1 - n^{-\frac{1}{2}}\hat{G}_n(k/n^2)}{1 - n^{-\frac{1}{2}}\hat{G}_n(\tau + k/n^2)} \qquad (\tau \ge 0) ,$$

where G_n is defined by

$$\hat{G}_n(\tau) := \hat{F}(\tau)^{1/n^{\frac{1}{2}}} \qquad (\tau \ge 0) ,$$

which because of the inf div of F is indeed a PLST. Using in corollary 5.3.4 a compound-geometric-$(n^{-\frac{1}{2}}, G_n)$ distribution, we see that $F_n \in \mathcal{F}_{\lambda_n}$. Next we rewrite $\hat{F}_n$ as follows:

$$(5.3.7)\qquad \hat{F}_n(\tau) = \{1 - n^{-\frac{1}{2}}(1 - \hat{G}_n(\tau))\}^n \prod_{k=0}^{n-1} \{1 + g_{k,n}(\tau)\} ,$$

where

$$g_{k,n}(\tau) := \frac{\hat{G}_n(\tau + k/n^2) - \hat{G}_n(k/n^2) + 1 - \hat{G}_n(\tau)}{n^{\frac{1}{2}} + \hat{G}_n(\tau) - 1} \qquad (n \in \mathbb{N};\ k = 0,1,\dots,n-1;\ \tau \geq 0) .$$

Observing that for all $\tau \geq 0$ and all $k \in \{0,1,\dots,n-1\}$ we have

$$\hat{G}_n(\tau) - \hat{G}_n(\tau + k/n^2) \leq \hat{G}_n(\tau) - \hat{G}_n(\tau + 1/n) =$$

$$= \int_{[0,\infty)} e^{-\tau x}\{1 - e^{-x/n}\} dG_n(x) \leq 1 - \hat{G}_n(1/n) ,$$

for $|g_{k,n}(\tau)|$ we obtain an upperbound, independent of k and τ:

$$(5.3.8)\qquad |g_{k,n}(\tau)| \leq 2(n^{\frac{1}{2}} - 1)^{-1}\{1 - \hat{G}_n(1/n)\} \quad (n \in \mathbb{N};\ k = 0,1,\dots,n-1;\ \tau \geq 0) .$$

Now define $\delta_n := 1 - \hat{F}(1/n)$, then $\delta_n \in (0,1)$ and

$$n^{\frac{1}{2}}\{1 - \hat{G}_n(1/n)\} = n^{\frac{1}{2}}\{1 - (1 - \delta_n)^{1/n^{\frac{1}{2}}}\} = -n^{\frac{1}{2}} \sum_{k=1}^{\infty} \binom{1/n^{\frac{1}{2}}}{k} (-\delta_n)^k =$$

$$= \delta_n \sum_{k=1}^{\infty} \frac{1}{k}\binom{1/n^{\frac{1}{2}}-1}{k-1} (-\delta_n)^{k-1} \leq \delta_n \sum_{\ell=0}^{\infty} \binom{1/n^{\frac{1}{2}}-1}{\ell} (-\delta_n)^{\ell} = \delta_n(1-\delta_n)^{1/n^{\frac{1}{2}}-1},$$

which, as $\delta_n = o(1)$ $(n \to \infty)$, tends to zero as $n \to \infty$. It follows that

$$1 - \hat{G}_n(1/n) = o(n^{-\frac{1}{2}}) \qquad (n \to \infty) ,$$

and hence by (5.3.8)

$$(5.3.9)\qquad \forall_{k \in \{0,1,\dots,n-1\}}\ \forall_{\tau \geq 0}\ |g_{k,n}(\tau)| \leq \varepsilon_n, \text{ with } \varepsilon_n = o(\tfrac{1}{n}) \qquad (n \to \infty) .$$

We can write now the following inequalities:

$$(1 - \varepsilon_n)^n \leq \prod_{k=0}^{n-1} \{1 + g_{k,n}(\tau)\} \leq (1 + \varepsilon_n)^n ,$$

from which by the asymptotic behaviour of ε_n it is easily seen that

$$(5.3.10)\qquad \lim_{n\to\infty} \prod_{k=0}^{n-1} \{1 + g_{k,n}(\tau)\} = 1 \qquad (\tau \geq 0) .$$

Turning to the first factor in the right-hand side of (5.3.7) we write

$$\{1 - n^{-\frac{1}{2}}(1 - \hat{G}_n(\tau))\}^n = \exp[n \log(1 - c_n)] ,$$

with $c_n := n^{-\frac{1}{2}}(1 - \hat{G}_n(\tau))$. Using the fact that

$$-x - x^2 \le \log(1 - x) \le -x \quad \text{for } 0 \le x \le \tfrac{1}{2} ,$$

and observing that $nc_n \to -\log \hat{F}(\tau)$ and $nc_n^2 \to 0$ as $n \to \infty$, we see that

$$(5.3.11) \quad \lim_{n\to\infty} \{1 - n^{-\frac{1}{2}}(1 - \hat{G}_n(\tau))\}^n = \hat{F}(\tau) \qquad (\tau \ge 0) .$$

Combining this result with (5.3.10), from (5.3.7) we conclude that $\lim_{n\to\infty} \hat{F}_n(\tau) = \hat{F}(\tau)$. □

To construct examples of df's in F_λ we need, apart from lemma 1.3.8, the following lemma's on comp mon functions. They can easily be proved by use of Bernstein's theorem and partial fraction expansions.

LEMMA 5.3.9. For $\mu_3 > 0$ the function φ, defined by

$$\varphi(\tau) := \frac{\mu_1 + \mu_2\tau}{\mu_3 + \tau} \qquad (\tau > 0) ,$$

is comp mon iff $\mu_2 \ge 0$ and $\mu_1 \ge \mu_2\mu_3$.

LEMMA 5.3.10. For $\mu_3 > 0$ and $\mu_4 > 0$ the function φ, defined by

$$\varphi(\tau) := \frac{\mu_1 + \mu_2\tau}{(\mu_3 + \tau)(\mu_4 + \tau)} \qquad (\tau > 0) ,$$

is comp mon iff $\mu_2 \ge 0$ and $\mu_1 \ge \mu_2 \min(\mu_3,\mu_4)$.

LEMMA 5.3.11. For $\mu_3 > 0$ and $\mu_4 > 0$ the function φ, defined by

$$\varphi(\tau) := \frac{(\mu_1 + \tau)(\mu_2 + \tau)}{(\mu_3 + \tau)(\mu_4 + \tau)} \qquad (\tau > 0) ,$$

is comp mon iff $\min(\mu_1,\mu_2) \ge \min(\mu_3,\mu_4)$, $\mu_1\mu_2 \ge \mu_3\mu_4$ and $\mu_1 + \mu_2 \ge \mu_3 + \mu_4$.

Now we mention some simple examples of distributions in F_λ. In the next section some more examples for F_∞ will be given.

1. Consider the *exponential distribution* with parameter $\mu > 0$; its pdf and PLST is given by

$$f(x) = \mu e^{-\mu x} \quad (x > 0), \text{ and } \hat{F}(\tau) = \frac{\mu}{\mu + \tau}.$$

Calculating the φ_λ-function of F, we obtain

$$(5.3.12) \quad \frac{\mu}{\mu + \tau} \in F'_\infty \qquad (\mu > 0).$$

In fact, the K_λ-function of F has a density k_λ given by

$$k_\lambda(x) = \lambda c_\lambda e^{-(\lambda+\mu)x} \qquad (x > 0;\ 0 \le \lambda < \infty).$$

2. The *gamma distribution* with parameters $\mu > 0$ and $\nu > 0$ has pdf and PLST given by

$$f(x) = \frac{\mu^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\mu x} \ (x > 0), \text{ and } \hat{F}(\tau) = \{\frac{\mu}{\mu + \tau}\}^\nu.$$

In view of theorem 5.3.3(ii), from example 1 we conclude that

$$(5.3.13) \quad \{\frac{\mu}{\mu + \tau}\}^\nu \in F'_\infty \qquad (\mu > 0;\ 0 < \nu \le 1),$$

but, as for the discrete analogue of the gamma distribution (cf. example 2, p. 56), we have

$$(5.3.14) \quad \{\frac{\mu}{\mu + \tau}\}^\nu \notin \bigcup_{\lambda>0} F_\lambda \qquad (\mu > 0;\ \nu > 1).$$

To show this we put $\nu = 1 + \varepsilon$ with $\varepsilon > 0$, and we calculate the φ_λ-function of F:

$$\varphi_\lambda(\tau) = c_\lambda\{1 - (\frac{\mu + \tau}{\mu + \lambda + \tau})^{1+\varepsilon}\} = -\sum_{k=1}^{\infty} \binom{1+\varepsilon}{k} (-\lambda)^k (\mu + \lambda + \tau)^{-k},$$

which can be written as the LT of a function k_λ that satisfies

$$\lambda^{-1} e^{(\lambda+\mu)x} k_\lambda(x) = \sum_{k=1}^{\infty} \binom{1+\varepsilon}{k} \frac{(-\lambda x)^{k-1}}{(k-1)!} = 1 + \varepsilon - \varepsilon(1 + \varepsilon) \sum_{k=1}^{\infty} \binom{k-1-\varepsilon}{k-1} \frac{(\lambda x)^k}{k(k+1)!}.$$

As for $0 < \varepsilon \le 1$ this is less than $(1 + \varepsilon)\{1 - \varepsilon\lambda x/2\}$, which tends to $-\infty$ as $x \to \infty$, (5.3.14) is proved for $1 < \nu \le 2$. Finally, from this and theorem 5.3.3(ii) it is easily seen that (5.3.14) also holds for $\nu > 2$.

3. In view of example 1 we can take $\hat{F}(\tau) = \mu/(\mu+\tau)$ in corollary 5.3.4 to obtain

$$(5.3.15)\qquad \prod_{k=0}^{n-1} \frac{\mu+k\lambda}{\mu+k\lambda+\tau} \in F'_\lambda \qquad (\mu > 0;\ n \in \mathbb{N};\ 0 \le \lambda < \infty)\ .$$

4. For a product of two exponential PLST's, which, obviously, can be given the form (5.3.15) with $n = 2$, we have

$$(5.3.16)\qquad \frac{\mu}{\mu+\tau}\,\frac{\mu+\nu}{\mu+\nu+\tau} \in F'_\lambda \Leftrightarrow \lambda \le \nu \qquad (\mu > 0;\ \nu \ge 0)\ .$$

In fact, the φ_λ-function can easily be seen to be the LT of the function k_λ given by

$$k_\lambda(x) = \lambda c_\lambda \nu^{-1}\{(\nu-\lambda) + (\nu+\lambda)e^{-\nu x}\}e^{-(\mu+\lambda)x} \qquad (x > 0)\ ,$$

from which it follows that φ_λ is comp mon, or, equivalently, $k_\lambda(x) \ge 0$ for all $x > 0$, iff $\lambda \le \nu$.

5. A quotient of two exponential PLST's with parameters $\mu > 0$ and $\nu > 0$, respectively, is again a PLST iff $\mu \le \nu$ (cf. lemma 5.3.9). We then have

$$(5.3.17)\qquad \frac{\mu}{\mu+\tau} \Big/ \frac{\nu}{\nu+\tau} \in F^+_\infty \qquad (0 < \mu \le \nu)\ .$$

In fact, the K_∞-function has a density k_∞ given by

$$k_\infty(x) = (\nu-\mu)e^{-\nu x} \qquad (x > 0)\ .$$

6. Taking F in corollaries 5.3.4 and 5.3.6 compound geometric, we get

$$(5.3.18)\qquad \prod_{k=0}^{n-1} \frac{1-p\hat{G}(k\lambda)}{1-p\hat{G}(\tau+k\lambda)} \in F_\lambda \quad (n \in \mathbb{N} \cup \{\infty\};\ 0 \le p < 1;\ G \text{ is df on } [0,\infty))\ .$$

7. Choose in (5.3.18) $\hat{G}(\tau) = \mu/(\mu+\tau)$, then it follows that

$$(5.3.19)\qquad \prod_{k=0}^{n-1} \frac{\mu_1+k\lambda}{\mu_1+k\lambda+\tau} \Big/ \frac{\mu_2+k\lambda}{\mu_2+k\lambda+\tau} \in F_\lambda \qquad (n \in \mathbb{N} \cup \{\infty\};\ 0 < \mu_1 \le \mu_2)\ .$$

Note that for $n = 1$ we get example 5.

8. Finally, we want to show that for positive μ_1, μ_2 and μ_3

$$(5.3.20)\qquad \frac{\mu_1}{\mu_1+\tau}\,\frac{\mu_3}{\mu_3+\tau}\,/\{\frac{\mu_2}{\mu_2+\tau}\}^2 \in F_\infty^+ \Leftrightarrow \tfrac{1}{2}(\mu_1+\mu_3) \le \mu_2 \le \max(\mu_1,\mu_3)\ .$$

First we note that according to lemma 5.3.11 for the function in (5.3.20) to be a PLST it is necessary and sufficient that $\mu_2 \geq \frac{1}{2}(\mu_1+\mu_3)$. Calculating the φ_∞-function in that case, we find

$$\varphi_\infty(\tau) = (\mu_2+\tau)^{-2}\{(\mu_2^2-\mu_1\mu_3) + (2\mu_2-\mu_1-\mu_3)\tau\}\ ,$$

from which by lemma 5.3.10 it follows that φ_∞ is comp mon iff $\mu_2^2-\mu_1\mu_3 \geq$ $\geq (2\mu_2-\mu_1-\mu_3)\mu_2$, i.e. $(\mu_3-\mu_2)(\mu_2-\mu_1) \geq 0$, or $\mu_2 \leq \max(\mu_1,\mu_3)$.

5.4. *The class* F_∞

In this section we study the class $F_\infty := \lim_{\lambda\to\infty} F_\lambda$ in more detail, and we do so mainly for the following two reasons. In the first place, F_∞ seems to be the analogue of C_0 for distributions on $[0,\infty)$. In fact, similarly to the way F_∞ is defined, C_0 can be obtained as the limit of the discrete analogues of the F_λ's, the classes C_α, for $\alpha \downarrow 0$. Furthermore, it will turn out that F_∞ has properties, very similar to those of C_0, and F_∞ contains the compound geometric distributions on $[0,\infty)$ just as F_0 contains the compound Poisson ones. Secondly we study F_∞ because of its interesting relations with other classes of functions occurring in probability theory, such as the standard p-functions, the renewal densities and the potential kernels. These relations will be discussed briefly in the next section.

Our first aim is to look for some basic properties of F_∞ as available for F_0 (cf. theorems 1.6.1 and 1.6.2). The relation between C_0 and R_0 (cf. section 4.1) suggests the existence of a relation between F_∞ and the class of continuous analogues of the renewal sequences, the (standard) p-functions (cf. section 5). By a method used by Kingman (1972) for p-functions, we can derive a canonical representation for the PLST's in F_∞. The starting point for this is the observation that F_∞^+ is dense in $\{F \in F_\infty \mid \ell(F) = 0\}$ in the sense of weak convergence (this can be considered as an analogue of De Finetti's theorem (theorem 1.4.15) for F_∞): if $F \in F_\infty$ with $\ell(F) = 0$, then by (5.1.13) and lemma 5.1.5 we can write

$$(5.4.1)\qquad \hat{F}(\tau) = \lim_{\lambda\to\infty} \frac{\hat{F}(\tau+\lambda)}{1-c_\lambda^{-1}\hat{K}_\lambda(\tau)} = \lim_{\lambda\to\infty} \frac{\hat{F}(\lambda)}{1-c_\lambda^{-1}\hat{K}_\lambda(\tau)} = \lim_{n\to\infty} \frac{1-p_n}{1-p_n\hat{G}_n(\tau)}\ ,$$

where for $n \in \mathbb{N}$ $p_n := 1 - \hat{F}(n) \in (0,1)$ and $\hat{G}_n(\tau) := c_n^{-1}\hat{K}_n(\tau)/\{1-\hat{F}(n)\}$ is a PLST (cf. (5.1.20)). Using a different approach, however, we can derive the canonical representation much easier. We shall do so now, and start with a characterization of F_∞ by means of a comp mon function. To this end we give the following definition.

DEFINITION 5.4.1. The ψ_∞-*function* of a df F on $[0,\infty)$ is defined by

$$\psi_\infty(\tau) := \frac{d}{d\tau}\hat{F}(\tau)^{-1} = -\hat{F}'(\tau)/\hat{F}(\tau)^2 \qquad (\tau > 0) .$$

We note the following relations with the φ_0-function and, if $F(0) > 0$, with the φ_∞-function of F (cf. definition 5.1.3):

$$(5.4.2) \qquad \psi_\infty(\tau) = \varphi_0(\tau)/\hat{F}(\tau) \text{ and } \psi_\infty(\tau) = -\varphi_\infty'(\tau)/F(0) .$$

As $\varphi_\infty(\tau) \geq 0$ $(\tau > 0)$, the latter relation shows that (cf. lemma 5.1.4(ii)) a df F with $F(0) > 0$ is in F_∞^+ iff the ψ_∞-function of F is comp mon. This characterization of F_∞^+ can be extended to F_∞ in the following way (cf. lemma 2.5.6).

THEOREM 5.4.2. If F is a df on $[0,\infty)$, then $F \in F_\infty$ with $\ell(F) = 0$ iff the ψ_∞-function of F is comp mon.

PROOF. Let $F \in F_\infty$ with $\ell(F) = 0$. Then by lemma 5.1.4(i) for all $\lambda < \infty$ the φ_λ-function of F is comp mon, and is related to the ψ_∞-function of F as follows:

$$(5.4.3) \qquad -\frac{d}{d\tau}\varphi_\lambda(\tau) = c_\lambda\hat{F}(\tau+\lambda)\{\psi_\infty(\tau) - \varphi_0(\tau+\lambda)/\hat{F}(\tau)\} .$$

Division by $\hat{F}(\lambda)$, and use of the fact that $c_\lambda \to 1$, $\hat{F}(\tau+\lambda)/\hat{F}(\lambda) \to 1$ (cf. lemma 5.1.5), and $\varphi_0(\tau+\lambda) \to 0$ (cf. lemma 5.1.6) as $\lambda \to \infty$, show that

$$(5.4.4) \qquad \psi_\infty(\tau) = \lim_{\lambda\to\infty} -\frac{d}{d\tau}\varphi_\lambda(\tau)/\hat{F}(\lambda) .$$

It follows that ψ_∞ is comp mon, since it is the limit of a sequence of comp mon functions.

Conversely, let the ψ_∞-function of a df F on $[0,\infty)$ be comp mon, and let $0 < \lambda < \infty$. As $\hat{F}$ is nonincreasing, the function

$$(5.4.5) \qquad \varphi_\lambda(\tau)/\hat{F}(\tau+\lambda) = c_\lambda\{\hat{F}(\tau+\lambda)^{-1} - \hat{F}(\tau)^{-1}\}$$

is nonnegative. Furthermore, it satisfies

$$(5.4.6)\qquad -\frac{d}{d\tau}[\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda)] = c_\lambda\{\psi_\infty(\tau) - \psi_\infty(\tau+\lambda)\}\ ,$$

from which it follows that $\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda)$, and hence $\varphi_\lambda(\tau)$, is comp mon. Thus we have proved that $F \in F_\infty$. In order to show that $\ell(F) = 0$, we note that $\lim_{\tau\to\infty}\psi_\infty(\tau)$ exists in $[0,\infty)$. Hence by lemma 5.1.6 and (5.4.2), $\ell(F)$ can be obtained as follows:

$$\ell(F) = \lim_{\tau\to\infty}\varphi_o(\tau) = \lim_{\tau\to\infty}\psi_\infty(\tau)\hat{F}(\tau) = F(0)\lim_{\tau\to\infty}\psi_\infty(\tau)\ ,$$

from which it is seen that $\ell(F) = 0$ if $F(0) = 0$. As trivially $\ell(F) = 0$ if $F(0) > 0$, the theorem is proved. □

Using Bernstein's theorem, from theorem 5.4.2 we immediately obtain a characterization of F_∞ by a functional equation (cf. lemma 2.5.8, where a similar result is given for C_o (= H_1; cf. theorem 2.5.11)).

COROLLARY 5.4.3. If F is a df on $[0,\infty)$, then $F \in F_\infty$ with $\ell(F) = 0$ iff there exists a right-continuous, nondecreasing function L such that

$$(5.4.7)\qquad \int_{[0,x]} y\, dF(y) = \int_{[0,x]} F^{*2}(x-y)\,dL(y) \qquad (x \geq 0)\ .$$

Relation (5.4.6) gives rise to a characterization of F_∞ in terms of the φ_λ-function for a *fixed* $\lambda \in [0,\infty)$. It shows in a sense which part of F_λ consists of F_∞-distributions.

THEOREM 5.4.4. If F is a df on $[0,\infty)$, then $F \in F_\infty$ with $\ell(F) = 0$ iff for some, and then for all, $\lambda \in [0,\infty)$ the function $\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda)$ is comp mon.

PROOF. First we note that if $F \in F_\infty$ with $\ell(F) = 0$, then by theorem 5.4.2 the ψ_∞-function of F is comp mon, and hence (cf. the second part of the proof of theorem 5.4.2) $\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda)$ is comp mon for all $\lambda \in [0,\infty)$. From (5.4.2) it now follows that for $\lambda = 0$ the theorem reduces to theorem 5.4.2. Therefore we take a fixed $\lambda \in (0,\infty)$ and suppose $\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda)$ to be comp mon. Then the limit

$$(5.4.8)\qquad a_\lambda := \lim_{\tau\to\infty}\varphi_\lambda(\tau)/\hat{F}(\tau+\lambda) = c_\lambda \lim_{\tau\to\infty}\frac{\hat{F}(\tau) - \hat{F}(\tau+\lambda)}{\hat{F}(\tau)\hat{F}(\tau+\lambda)}$$

exists in $[0,\infty)$. For all $\tau > 0$ there exists $\theta_\tau \in (0,1)$ such that

$$\hat{F}(\tau) - \hat{F}(\tau+\lambda) = -\lambda\hat{F}'(\tau+\theta_\tau\lambda) ,$$

and hence, as $\hat{F}'$ is nondecreasing,

$$-\lambda\hat{F}'(\tau+\lambda) \le \hat{F}(\tau) - \hat{F}(\tau+\lambda) \le -\lambda\hat{F}'(\tau) ,$$

or for $\tau > \lambda$

$$\hat{F}(\tau) - \hat{F}(\tau+\lambda) \le -\lambda\hat{F}'(\tau) \le \hat{F}(\tau-\lambda) - \hat{F}(\tau) .$$

Now, dividing by $\hat{F}(\tau)^2$, and using (5.4.8) and lemma 5.1.5, we see that ψ_∞ satisfies

$$(5.4.9) \qquad \{\lambda c_\lambda\}^{-1} a_\lambda e^{-\lambda\ell(F)} \le \liminf_{\tau\to\infty} \psi_\infty(\tau) \le \limsup_{\tau\to\infty} \psi_\infty(\tau) \le \{\lambda c_\lambda\}^{-1} a_\lambda e^{\lambda\ell(F)} .$$

If F would be such that $\ell(F) > 0$ (and so $F(0) = 0$), ψ_∞ would satisfy (cf. lemma 5.1.6)

$$\lim_{\tau\to\infty} \psi_\infty(\tau) = \lim_{\tau\to\infty} \varphi_o(\tau)/\hat{F}(\tau) = \infty ,$$

which contradicts (5.4.9). It follows that $\ell(F) = 0$, and hence

$$(5.4.10) \qquad \lim_{\tau\to\infty} \psi_\infty(\tau) = \{\lambda c_\lambda\}^{-1} a_\lambda .$$

Now ψ_∞ can be written as

$$\psi_\infty(\tau) = \{\lambda c_\lambda\}^{-1} a_\lambda + \sum_{k=0}^{\infty} \{\psi_\infty(\tau+k\lambda) - \psi_\infty(\tau+k\lambda+\lambda)\} ,$$

which by (5.4.6) implies the comp mon of ψ_∞. Hence by theorem 5.4.2: $F \in \mathcal{F}_\infty$.□

Next, from theorem 5.4.2 we derive a representation theorem for $\mathcal{F}_\infty$ that can be considered as an analogue of theorem 2.4.8, where a relation between $\mathcal{C}_o$ and $\mathcal{C}_1$ is given.

THEOREM 5.4.5. A function φ on $[0,\infty)$ is the PLST of a df $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ iff there exists a df $H \in \mathcal{F}_o$ such that φ has the form

$$(5.4.11) \qquad \varphi(\tau) = \{1 - \log \hat{H}(\tau)\}^{-1} \qquad (\tau \ge 0) .$$

PROOF. Let $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$. Then the ψ_∞-function ψ_∞ of F is comp mon. If we define the positive function ψ on $[0,\infty)$ by

(5.4.12) $\psi(\tau) := \exp[1 - \hat{F}(\tau)^{-1}] \qquad (\tau \geq 0)$,

then it is seen that $\psi(0) = 1$ and that the function

$$-\frac{d}{d\tau} \log \psi(\tau) = \frac{d}{d\tau} \hat{F}(\tau)^{-1} = \psi_\infty(\tau)$$

is comp mon. Now by theorem 1.6.1 it follows that there exists a df $H \in \mathcal{F}_0$ such that $\psi = \hat{H}$, and hence by (5.4.12) $\hat{F}$ can be represented in the form (5.4.11).

Conversely, suppose that a function φ on $[0,\infty)$ has the form (5.4.11) with $H \in \mathcal{F}_0$. Then the φ_0-function of H

$$\varphi_0^{(H)}(\tau) = -\frac{d}{d\tau} \log \hat{H}(\tau)$$

is comp mon, and as $-\log \hat{H}(\tau)$ is nonnegative, by lemma 1.3.8(vi) it follows that φ is comp mon. Also $\varphi(0) = 1$, and hence by Bernstein's theorem φ is the PLST of a df F on $[0,\infty)$, for which the ψ_∞-function

$$\psi_\infty(\tau) = \frac{d}{d\tau} \hat{F}(\tau)^{-1} = \varphi_0^{(H)}(\tau)$$

is comp mon. In view of theorem 5.4.2 we conclude that $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$.□

COROLLARY 5.4.6. If $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$, then

(5.4.13) $\hat{H}(\tau) = \exp[1 - \hat{F}(\tau)^{-1}] \qquad (\tau \geq 0)$

is the PLST of a df H in $\mathcal{F}_0$. The K_0-function $K_0^{(H)}$ of H has the ψ_∞-function of F as its LST, and is therefore related to the K_0-function K_0 of F as follows:

(5.4.14) $F * K_0^{(H)} = K_0$.

By letting $\tau \to \infty$ in (5.4.13), it is seen that $F(0) > 0$ iff $H(0) > 0$. Hence from the preceding theorem we obtain the following relation between $\mathcal{F}_0^+$ and $\mathcal{F}_\infty^+$, which can also be established by use of the representations (1.6.12) and (1.6.13) for $\mathcal{F}_0^+$ and $\mathcal{F}_\infty^+$, respectively (cf. the proof of theorem 2.4.8).

COROLLARY 5.4.7. A function φ on $[0,\infty)$ is the PLST of a df $F \in \mathcal{F}_\infty^+$ iff there exists a df $H \in \mathcal{F}_0^+$ such that (5.4.11) holds.

We want to make the representation for PLST's in F_∞, given by theorem 5.4.5, more explicit. We do so in the following theorem by introducing canonical quantities in such a way that several properties of F_∞ are easily expressible in terms of them, and such that there exists a clear resemblance to the canonical representation for the LT of a standard p-function (cf. section 5).

THEOREM 5.4.8. A function φ on $[0,\infty)$ is the PLST of a df $F \in F_\infty$ with $\ell(F) = 0$ iff there exist $\gamma \geq 0$ and a right-continuous, nondecreasing function N on $(0,\infty)$ with $N(\infty) = 0$ and satisfying

$$(5.4.15)\qquad \int_{(0,1]} x\, dN(x) < \infty ,$$

such that φ has the form

$$(5.4.16)\qquad \varphi(\tau) = \{1 + \gamma\tau + \int_{(0,\infty)} (1 - e^{-\tau x})\, dN(x)\}^{-1} \qquad (\tau \geq 0) ;$$

the representation (γ,N) for $\varphi = \hat{F}$ is unique.

PROOF. The theorem is an immediate consequence of the preceding theorem, and theorem 1.7.1, where a representation (γ,N) is given for the chf of an inf div df with finite left extremity; using this representation for the df $H \in F_0$ in (5.4.11), we get (5.4.16). □

It is useful to state the following corollary, which will be clear from corollary 1.7.2.

COROLLARY 5.4.9. If $F \in F_\infty$ with $\ell(F) = 0$, then between the representation (γ,N) in (5.4.16) for $\hat{F}$ and the K_0-function $K_0^{(H)}$ of the df H, defined by (5.4.13), the following relations hold:

$$(5.4.17)\qquad \gamma = K_0^{(H)}(0) \text{ and } N(x) = -\int_{(x,\infty)} \frac{1}{y}\, dK_0^{(H)}(y) \qquad (x > 0) ,$$

and, conversely,

$$(5.4.18)\qquad K_0^{(H)}(x) = \gamma + \int_{(0,x]} y\, dN(y) \qquad (x \geq 0) .$$

The representation (γ,N), given in (5.4.16), for the PLST of a df $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ will be called the *canonical representation* of $F \in \mathcal{F}_\infty$. It has several simple properties, which we need in the sequel. They are summarized in the following lemma.

LEMMA 5.4.10. Let $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ and canonical representation (γ,N). Then

(i) N is integrable on every finite interval, or, equivalently,

$$(5.4.19) \qquad -\int_{(0,x]} N(y)\,dy < \infty \qquad (x > 0) \ .$$

(ii) N is integrable on $(0,\infty)$ iff $\mu := \int_{[0,\infty)} x\, dF(x)$ is finite, in which case

$$(5.4.20) \qquad \mu = \gamma - \int_{(0,\infty)} N(x)\,dx = \gamma + \int_{(0,\infty)} x\, dN(x) \ .$$

(iii) The LT of N exists on $(0,\infty)$ and can be expressed in $\hat{F}$ as follows:

$$(5.4.21) \qquad \int_{(0,\infty)} e^{-\tau x} N(x)\,dx = \gamma + \tau^{-1} - \{\tau\hat{F}(\tau)\}^{-1} \qquad (\tau > 0) \ .$$

(iv) γ can be obtained from $\hat{F}$ by

$$(5.4.22) \qquad \gamma = \lim_{\tau\to\infty} \{\tau\hat{F}(\tau)\}^{-1} \ .$$

(v) The K_0-function of F is related to (γ,N) as follows:

$$(5.4.23) \qquad K_0(x) = \gamma F(x) + \int_{(0,x]} F(x-y)\,y\, dN(y) \qquad (x \geq 0) \ .$$

(vi) $F \in \mathcal{F}_\infty^+$ iff $\gamma = 0$ and N is bounded, i.e. $-N(0+) < \infty$.

PROOF.

(i) For all $x > 0$, using Fubini's theorem, we can write

$$(5.4.24) \qquad -\int_{(0,x]} N(y)\,dy = \int_{(0,x]} y\, dN(y) - xN(x) \ ,$$

which by (5.4.15) is finite.

(ii) By Fubini's theorem it follows that

$$(5.4.25) \qquad -\int_{(0,\infty)} N(x)\,dx = \int_{(0,\infty)} x\,dN(x) ,$$

where the integrals may be both infinite. Further, by (5.4.14) and (5.4.18) we can write

$$\mu = \int_{(0,\infty)} x dF(x) = \lim_{\tau\downarrow 0} -\hat{F}'(\tau)/\hat{F}(\tau)^2 = \lim_{\tau\downarrow 0} \hat{K}_o^{(H)}(\tau) = \gamma + \int_{(0,\infty)} x dN(x) ,$$

from which the assertion in (ii) immediately follows.

(iii) From (5.4.19) it is seen that the LT of N exists on $(0,\infty)$. We calculate it from (5.4.16) with $\varphi = \hat{F}$, and obtain

$$-\int_{(0,\infty)} e^{-\tau x} N(x)\,dx = \int_{(0,\infty)} e^{-\tau x} \int_{(x,\infty)} dN(y)\,dx =$$

$$= \tau^{-1} \int_{(0,\infty)} (1 - e^{-\tau y})\,dN(y) = \tau^{-1}\{\hat{F}(\tau)^{-1} - 1 - \gamma\tau\} .$$

(iv) This follows immediately from (iii).

(v) The assertion is a consequence of the relations (5.4.14) and (5.4.18).

(vi) If H is the df in F_o defined by (5.4.13), then according to corollary 5.4.7 we have $F \in F_\infty^+$ iff $H \in F_o^+$. Now, by lemma 1.6.4 it is seen that $H \in F_o^+$ iff its K_o-function $K_o^{(H)}$ satisfies

$$K_o^{(H)}(0) = 0 \text{ and } \int_{(0,\infty)} \frac{1}{x}\, dK_o^{(H)}(x) < \infty ,$$

which in view of (5.4.17) is equivalent to the condition that $\gamma = 0$ and N is bounded. □

REMARK 5.4.11. From the canonical representation (5.4.16) for $\hat{F}$ it is also easily verified that if $\gamma = 0$ and N is bounded, then F is compound-geometric-(p,G), with

$$(5.4.26) \qquad p = -N(0+)/\{1 - N(0+)\} \text{ and } G(x) = 1 - N(x)/N(0+) \qquad (x > 0) .$$

The K_∞-function of F can then be expressed in (γ,N) as follows:

$$(5.4.27) \qquad K_\infty(x) = \{N(x) - N(0+)\}/\{1 - N(0+)\} \qquad (x > 0) .$$

Thus we see that a df F in F_∞ with $\ell(F) = 0$ and for which $\gamma > 0$ or N is unbounded, is necessarily continuous at zero, and hence is *continuous every-*

where (cf. corollary 1.7.6). A further investigation of such distributions is simplified by the existence of a functional equation in terms of γ and N, by means of which $\mathcal{F}_\infty$ can be characterized. In fact, rewriting (5.4.21) as

$$\hat{F}(\tau)\{\gamma + \int_{(0,\infty)} e^{-\tau x}\{1 - N(x)\}dx\} = \tau^{-1} ,$$

and using the uniqueness theorem for LST's, we get the following result.

THEOREM 5.4.12. If F is a df on $[0,\infty)$, then $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ iff there exist $\gamma \geq 0$ and a function N satisfying the conditions for N in theorem 5.4.8 such that

$$x - \gamma F(x) = \int_{(0,x]} F(x - y)\{1 - N(y)\}dy \qquad (x \geq 0) . \tag{5.4.28}$$

The quantities γ and N are unique, and give the canonical representation of F.

We can subdivide $\mathcal{F}_\infty$ in four subclasses, characterized by the following four possibilities for the canonical representation (γ,N): $\gamma = 0$ or $\gamma > 0$ and N is bounded or not. The subclass $\mathcal{F}_\infty^+ = \{F \in \mathcal{F}_\infty$ with $\ell(F) = 0 \mid \gamma = 0$, N bounded$\}$ is well known. Now, using theorem 5.4.12, we can completely analyze the two classes of df's, for which $\gamma > 0$. To this end we state the following theorem.

THEOREM 5.4.13. If $F \in \mathcal{F}_\infty$, with $\ell(F) = 0$, has a canonical representation (γ,N) with $\gamma > 0$, then F is absolutely continuous and has a continuous density f which satisfies

$$1 - \gamma f(x) = \int_{(0,x)} f(x-y)\{1 - N(y)\}dy \qquad (x > 0) . \tag{5.4.29}$$

Furthermore, this density f has the following properties:

(i) $f(0+) = \lim_{x \downarrow 0} f(x)$ exists in $(0,\infty)$, and it satisfies

$$f(0+) = 1/\gamma > f(x) \qquad (x > 0) ; \tag{5.4.30}$$

(ii) $\lim_{x\to\infty} f(x) = 0$;

(iii) $-f_+'(0) := \lim_{x\downarrow 0} \frac{1}{x}\{f(0+) - f(x)\}$ exists in $(0,\infty]$, and is given by

$$(5.4.31)\qquad -f_+'(0) = \{1 - N(0+)\}/\gamma^2 \qquad (\leq \infty)\ .$$

PROOF. Let $F \in F_\infty$ with $\ell(F) = 0$. Then F satisfies the functional equation (5.4.28), the right-hand side of which can be written as (cf. (5.4.19))

$$\int_{[0,x]} \int_{(0,x-y]} \{1 - N(z)\}dz dF(y) = \int_{[0,x]} \int_{(y,x]} \{1 - N(u-y)\}du dF(y) =$$

$$= \int_{(0,x]} \int_{[0,u)} \{1 - N(u-y)\}dF(y)du\ .$$

It follows that

$$(5.4.32)\qquad \gamma F(x) = \int_{(0,x]} \{1 - w(u)\}du \qquad (x \geq 0)\ ,$$

where the function w is defined on $(0,\infty)$ by

$$(5.4.33)\qquad w(x) := \int_{[0,x)} \{1 - N(x-y)\}dF(y) \qquad (x > 0)\ .$$

Now, suppose that $\gamma > 0$. Then from (5.4.32) it is seen that F is absolutely continuous; the function f, defined by

$$f(x) := \{1 - w(x)\}/\gamma \qquad (x > 0)\ ,$$

is a density of F, and, because of (5.4.33), it satisfies (5.4.29). From this relation we obtain, using the monotonicity of N,

$$(5.4.34)\qquad \gamma f(x) \leq 1 - \{1 - N(x)\}F(x) \qquad (x > 0)\ .$$

It follows that $f(x) < 1/\gamma$ for $x > 0$, and, if we let $x \to \infty$, that (ii) holds. Now we can estimate as follows:

$$0 \leq \int_{(0,x)} f(x-y)\{1 - N(y)\}dy \leq \gamma^{-1}\{x - \int_{(0,x)} N(y)dy\}\ ,$$

which, because of (5.4.19), tends to zero as $x \downarrow 0$. From (5.4.29) it is now seen that $1 - \gamma f(x)$ tends to zero as $x \downarrow 0$, and (i) is proved.
To prove the continuity of f, or, equivalently, of w, we use the following well known property:

$$(5.4.35)\qquad [\forall_{x>0} \int_{(0,x]} |f(y)|dy < \infty] \Rightarrow [\forall_{a>0} \lim_{h\to 0} \int_{(0,a]} |f(x+h) - f(x)|dx = 0].$$

Let $x > 0$ and $h > 0$, then for all $x_o < x$ we have

$$|w(x+h) - w(x)| \le \int_{(0,x)} |f(x+h-y) - f(x-y)|\{1-N(y)\}dy +$$

$$+ \int_{[x,x+h)} f(x+h-y)\{1-N(y)\}dy \le 2\gamma^{-1} \int_{(0,x_o)} \{1-N(y)\}dy +$$

$$+ \{1-N(x_o)\} \int_{(0,x-x_o]} |f(z+h) - f(z)|dz + \{1-N(x)\} \int_{(0,h]} f(z)dz ,$$

which by (5.4.19) and (5.4.35) becomes arbitrarily small, if we choose x_o sufficiently small. Proceeding for $h < 0$ in a similar way, we conclude that w, and hence f, is continuous.

Finally, we prove (iii); since N is nondecreasing, we can write for $x > 0$

$$\{1-N(x)\}F(x) \le \int_{(0,x)} f(x-y)\{1-N(y)\}dy \le \{1-N(0+)\}F(x) \qquad (\le \infty),$$

from which by use of

$$\lim_{x \downarrow 0} F(x)/x = f(0+)$$

it follows that

$$\lim_{x\downarrow 0} \frac{1}{x}\{f(0+) - f(x)\} = \gamma^{-1} \lim_{x\downarrow 0} \frac{1}{x}\{1 - \gamma f(x)\} =$$

$$= \gamma^{-1} \lim_{x\downarrow 0} \frac{1}{x} \int_{(0,x)} f(x-y)\{1-N(y)\}dy = \{1-N(0+)\}/\gamma^2 \qquad (\le \infty) \;. \Box$$

From (5.4.32) we immediately obtain a characterization of $\mathcal{F}'_\infty$ and of $\{F \in \mathcal{F}_\infty$ with $\ell(F) = 0 \mid \gamma = 0\}$ by a functional equation.

COROLLARY 5.4.14. A pdf f on $(0,\infty)$ is the density of a df $F \in \mathcal{F}'_\infty$ with $\ell(F) = 0$ iff there exist $\gamma \ge 0$ and a function N satisfying the conditions for N in theorem 5.4.8 such that (5.4.29) holds for almost all $x > 0$.

COROLLARY 5.4.15. If F is a df on $[0,\infty)$, then $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ and with $\gamma = 0$ in its canonical representation (5.4.16) iff there exists a function N satisfying the conditions for N in theorem 5.4.8 such that

$$\int_{[0,x)} \{1-N(x-y)\}dF(y) = 1 \qquad \text{(almost all } x > 0) \;. \tag{5.4.36}$$

The subdivision of F_∞ in four subclasses, mentioned just before theorem 5.4.13, can be characterized in terms of properties of the df's in F_∞ themselves; this is easily verified from the preceding theorem and the following lemma.

LEMMA 5.4.16. If a df $F \in F_\infty$, with $\ell(F) = 0$ and canonical representation (γ,N), is absolutely continuous and has a density f, for which $f(0+)$ exists in $[0,\infty]$, then necessarily $f(0+) \in (0,\infty]$, and

(i) $f(0+) \in (0,\infty)$ iff $\gamma > 0$, in which case $f(0+) = 1/\gamma$;

(ii) $f(0+) = \infty$ iff $\gamma = 0$, in which case N is unbounded.

PROOF. If a df F on $[0,\infty)$ has a density f, for which $f(0+)$ exists in $[0,\infty]$, then, as is easily verified, $f(0+)$ can be obtained as follows:

$$f(0+) = \lim_{\tau\to\infty} \tau\hat{F}(\tau) \ .$$

Now, if $F \in F_\infty'$ with $\ell(F) = 0$, then in view of lemma 5.4.10(iv) it is seen that $f(0+)$ cannot be zero, and that $f(0+) < \infty$ iff $\gamma > 0$, in which case $f(0+) = 1/\gamma$. Finally, if $f(0+) = \infty$, then $\gamma = 0$ and hence N is unbounded, as otherwise by lemma 5.4.10(vi) we would have $F \in F_\infty^+$.

THEOREM 5.4.17. Let F be in F_∞ with $\ell(F) = 0$ and canonical representation (γ,N). Then the following four cases can be distinguished:

(i) $\gamma = 0$ and N is bounded iff F is compound geometric ($F \in F_\infty^+$);

(ii) $\gamma > 0$ and N is bounded iff F is absolutely continuous and has a density f, for which $f(0+)$ and $-f_+'(0)$ exist in $(0,\infty)$;

(iii) $\gamma > 0$ and N is unbounded iff F is absolutely continuous and has a density f, for which $f(0+)$ exists in $(0,\infty)$ and $-f_+'(0) = \infty$;

(iv) $\gamma = 0$ and N is unbounded iff F is continuous and F is either not absolutely continuous, or absolutely continuous such that no density of F has a finite limit as $x \downarrow 0$.

The following lemma provides a method to construct examples of distributions in each of the four subclasses of F_∞ from given inf div distributions on $[0,\infty)$.

LEMMA 5.4.18. If $H \in F_0$, then $\{1 - \log \hat{H}(\tau)\}^{-1}$ is the PLST $\hat{F}$ of a df $F \in F_\infty$ with $\ell(F) = 0$, and the canonical representation (γ,N) of F satisfies

(i) $\gamma > 0$ iff $\ell(H) > 0$;

(ii) N is bounded iff $H(\ell(H)) > 0$.

PROOF. The first part of the lemma follows from theorem 5.4.5. Further, from corollary 5.4.9 and lemma 1.6.4 it is seen that $\gamma = K_o^{(H)}(0) = \ell(H)$ and that N is bounded iff

$$\int_{(0,\infty)} \frac{1}{x}\, dK_o^{(H)}(x) < \infty ,$$

i.e. iff $H(\ell(H)) > 0$. □

Just as F_∞^+ (cf. (5.4.1)), the subclass $\{F \in F_\infty \text{ with } \ell(F) = 0 \mid \gamma > 0, N \text{ is bounded}\}$ is dense in $\{F \in F_\infty \mid \ell(F) = 0\}$ in the sense of weak convergence; in fact, if $F \in F_\infty$ with $\ell(F) = 0$ and canonical representation (γ,N), then by the monotone convergence theorem it is seen that $\hat{F}(\tau) = \lim_{n\to\infty} \hat{F}_n(\tau)$, where F_n ($n \in \mathbb{N}$) is defined as the df with PLST (5.4.16) with γ and N replaced by

$$\gamma_n := \begin{cases} \gamma, & \text{if } \gamma > 0 \\ 1/n, & \text{if } \gamma = 0 \end{cases} ; \quad N_n(x) := \begin{cases} N(x), & \text{if } N \text{ bounded or } x \geq 1/n \\ N(1/n), & \text{otherwise} . \end{cases}$$

Furthermore, this subclass of F_∞ turns out to consist of convolutions of an exponential distribution and a compound-geometric-(p,G) one, where G has the same exponential distribution as a factor. We state this in the following theorem, and note that distributions of this type also occur as first-passage time distributions in Miller (1967).

THEOREM 5.4.19. If F is a df on $[0,\infty)$, then $F \in F_\infty$ with $\ell(F) = 0$ and having a canonical representation (γ,N) with $\gamma > 0$ and N bounded iff $\hat{F}$ has the form

$$(5.4.37) \qquad \hat{F}(\tau) = \frac{\mu}{\mu+\tau}\, \frac{1-p}{1-p\{\mu/(\mu+\tau)\}\hat{G}(\tau)} ,$$

where $\mu > 0$, $0 \leq p < 1$ and G is a df on $[0,\infty)$.

PROOF. Let the PLST $\hat{F}$ of a df F on $[0,\infty)$ have the form (5.4.16) with $\gamma > 0$ and N bounded. Then, defining

$$(5.4.38) \qquad \mu := \{1-N(0+)\}/\gamma,\ p := -N(0+)/\{1-N(0+)\},\ G(x) := 1-N(x)/N(0+),$$

we see that $\hat{F}$ takes the form (5.4.37).

Conversely, the ψ_∞-function of a df F with PLST given by (5.4.37) satisfies

$$\psi_\infty(\tau) = \frac{d}{d\tau}\hat{F}(\tau)^{-1} = \frac{d}{d\tau}[1 + \{\mu(1-p)\}^{-1}\tau + \{p/(1-p)\}(1-\hat{G}(\tau))] =$$

$$= \{\mu(1-p)\}^{-1} - \{p/(1-p)\}\hat{G}'(\tau) \;,$$

which is comp mon. Hence by theorem 5.4.2 $F \in F_\infty$ with $\ell(F) = 0$, and because of corollaries 5.4.6 and 5.4.9 it follows that the canonical representation is given by

$$(5.4.39)\quad \gamma = \{\mu(1-p)\}^{-1},\; N(x) = -\{p/(1-p)\}(1-G(x)) \qquad (x > 0) \;,$$

so $\gamma > 0$ and N is bounded. □

EXAMPLE 5.4.20. In (5.4.37) take $\hat{G}(\tau) = \frac{\nu}{\nu+\tau} / \frac{\mu}{\mu+\tau}$ with $0 < \nu \le \mu$; by example 5 on p. 153, this is indeed a PLST (in F_∞^+). Then it follows that

$$\frac{\mu}{\mu+\tau}\,\frac{\nu(1-p)}{\nu(1-p)+\tau} / \frac{\nu}{\nu+\tau} \in F'_\infty \;.$$

However, we can prove a little more: if μ_1, μ_2 and μ_3 are all positive, then

$$(5.4.40)\quad \hat{F}(\tau) := \frac{\mu_1}{\mu_1+\tau}\,\frac{\mu_3}{\mu_3+\tau} / \frac{\mu_2}{\mu_2+\tau} \in F'_\infty \Leftrightarrow \min(\mu_1,\mu_3) \le \mu_2 \le \max(\mu_1,\mu_3) \,.$$

First we note that by lemma 5.3.10 $\hat{F}$ is indeed a PLST iff $\mu_2 \ge \min(\mu_1,\mu_3)$. Calculating then the ψ_∞-function of F, we get

$$\psi_\infty(\tau) = 1 + (\mu_1 - \mu_2)(\mu_2 - \mu_3)(\mu_2+\tau)^{-2} \;,$$

which is comp mon iff $(\mu_1 - \mu_2)(\mu_2 - \mu_3) \ge 0$, i.e. iff $\mu_2 \le \max(\mu_1,\mu_3)$. Finally, we note that under these conditions $\hat{F}$ is, in fact, a mixture of two exponential PLST's:

$$\hat{F}(\tau) = \{\mu_2(\mu_3 - \mu_1)\}^{-1}\{\mu_3(\mu_2 - \mu_1)\frac{\mu_1}{\mu_1+\tau} + \mu_1(\mu_3 - \mu_2)\frac{\mu_3}{\mu_3+\tau}\} \;.$$

Hence (cf. lemma 1.6.10) F has a comp mon density and therefore belongs to F'_∞, as we shall see presently.

The subclass $\{F \in F_\infty$ with $\ell(F) = 0 \mid \gamma = 0$, N unbounded$\}$ of F_∞ is still rather obscure. Apart from PLST's of the form $\{1 - \log \hat{H}(\tau)\}^{-1}$ with $H \in F_0$ and $\ell(H) = H(\ell(H)) = 0$ (cf. lemma 5.4.18), it contains all pdf's $f \in F'_\infty$, for which $f(0+) = \infty$ (cf. lemma 5.4.16(ii)). Before giving some examples, we state an analogue of corollary 5.2.8.

THEOREM 5.4.21. If $F \in F_\infty$ with $\ell(F) = 0$ and with canonical representation (γ, N) such that $\gamma = 0$ and N is unbounded and absolutely continuous, then F is absolutely continuous.

PROOF. If $\gamma = 0$ and N is absolutely continuous with density n, then from (5.4.23) it is seen that the K_o-function of F is absolutely continuous with density k_o given by

$$k_o(x) = \int_{[0,x)} (x-y)\,n(x-y)\,dF(y) \qquad (x > 0) .$$

Now, by theorem 5.2.7 it follows that $F(x) - F(0)$ is absolutely continuous. If in addition N is unbounded, then necessarily $F(0) = 0$, and hence F is absolutely continuous, with density f satisfying (cf. (5.2.19))

$$\text{(5.4.41)} \quad xf(x) = \int_{(0,x]} f^{*2}(x-y)\,yn(y)\,dy \qquad (x > 0) . \qquad \square$$

REMARK 5.4.22. Also from (5.4.23) it is seen that, contrary to the case of a general inf div df in F'_o (cf. remark 5.2.9), the K_o-function of a df $F \in F'_\infty$ with $\ell(F) = 0$ is necessarily absolutely continuous. It follows that corollary 5.2.8 gives *necessary and sufficient* conditions in terms of the K_o-function for $F \in F_\infty$ to be absolutely continuous. However, such conditions are not easily obtained in terms of the function N.

EXAMPLE 5.4.23.
(i) Consider the gamma distribution with parameters $\mu > 0$ and $\nu > 0$:

$$f(x) = \frac{\mu^\nu}{\Gamma(\nu)} x^{\nu-1} e^{-\mu x} \quad (x > 0), \text{ or } \hat{F}(\tau) = \left(\frac{\mu}{\mu+\tau}\right)^\nu .$$

By (5.3.13) we know that if $0 < \nu \le 1$ then $f \in F'_\infty$. If $\nu < 1$ then $f(0+) = \infty$, and hence in that case: $\gamma = 0$ and N unbounded. Indeed, the ψ_∞-function of f is given by

$$\psi_\infty(\tau) = \frac{\nu}{\mu}\left(\frac{\mu}{\mu+\tau}\right)^{1-\nu} ,$$

and hence (cf. corollaries 5.4.6 and 5.4.9)

$$\gamma = 0 \text{ and } N(x) = -\frac{\nu\mu^{-\nu}}{\Gamma(1-\nu)} \int_{(x,\infty)} y^{-1-\nu} e^{-\mu y} dy \qquad (x > 0) .$$

(ii) Take for H in lemma 5.4.18 a gamma distribution. Then it follows that

$$\hat{F}(\tau) := \{1 + \nu \log(1 + \rho\tau)\}^{-1} \in F'_\infty \qquad (\nu > 0;\ \rho > 0) ,$$

with $\gamma = 0$ and N unbounded. Here, F is absolutely continuous, because N is: the ψ_∞-function of F is given by

$$\psi_\infty(\tau) = \rho\nu/(1+\rho\tau) ,$$

and hence

$$N(x) = -\nu \int_{(x,\infty)} y^{-1} e^{-\rho^{-1}y} dy \qquad (x > 0) .$$

(iii) Take $0 < \alpha < 1$, and consider the function N on $(0,\infty)$, defined by

$$N(x) := (1 - x^{-\alpha}) 1_{(0,1]}(x) \qquad (x > 0) .$$

Then N is absolutely continuous and satisfies the conditions in theorem 5.4.8, so that

$$(5.4.42)\qquad \hat{F}(\tau) := \{1 + \alpha \int_{(0,1]} (1 - e^{-\tau x}) x^{-\alpha-1} dx\}^{-1} \in F'_\infty ;$$

in the canonical representation (γ,N) of F, $\gamma = 0$ and N is unbounded. The PLST (5.4.42) occurs in Feller (1971), ch. XIII (problem section) as the limiting PLST of the distribution of S_n/M_n, where $S_n := X_1 + \ldots + X_n$, $M_n := \max\{X_1,\ldots,X_n\}$ and $X_1, X_2, \ldots$ are independent rv's with common distribution belonging to the domain of attraction of the stable distribution on $[0,\infty)$ with exponent α.

After having studied the structure and basic properties of F_∞, we turn to analogues of some of the properties of C_o, mentioned in theorems 1.5.13 and 2.4.9, for F_∞.
As noted at the end of section 1.6, Goldie (1967) and Steutel (1970) prove that the classes $\mathcal{D}$ and E of comp mon and log-convex pdf's, respectively, are subclasses of F_o. We now prove that they are subclasses of F_∞, i.e.

$$(5.4.43)\qquad \mathcal{D} \subset E \subset F_\infty .$$

THEOREM 5.4.24. If f is a log-convex pdf on $(0,\infty)$, then $f \in F'_\infty$.

PROOF. Let F be a df on $[0,\infty)$ with a log-convex density f. Then for every $h > 0$ there exists $c_h > 0$ such that the sequence $\{p_n^{(h)}\}_0^\infty$, defined by

$$p_n^{(h)} := c_h h f(\tfrac{1}{2}(2n+1)h) \qquad (n \in \mathbb{N}_0) \ ,$$

satisfies $\Sigma p_n^{(h)} = 1$. As f is a log-convex function, $\{p_n^{(h)}\}$ is a log-convex sequence, and hence (cf. theorem 1.5.13) satisfies the recurrence relations (1.5.17) with nonnegative $r_n(0)$'s. Now, consider $\{p_n^{(h)}\}$ as a probability distribution on the lattice $\{0,h,2h,\ldots\}$. Then by (5.1.9) with $\alpha = 0$ it follows that $\{p_n^{(h)}\} \in \mathcal{F}_\infty^+$ for all $h > 0$. Since furthermore

$$F(x) = \lim_{h\downarrow 0} \sum_{n=0}^{[x/h]} hf(\tfrac{1}{2}(2n+1)h) = \lim_{h\downarrow 0} \sum_{nh\le x} p_n^{(h)} \qquad (x \geq 0) \ ,$$

and as $\mathcal{F}_\infty$ is closed under weak convergence (cf. theorem 5.3.1), we conclude that $F \in \mathcal{F}_\infty$. □

COROLLARY 5.4.25. If f is a comp mon pdf on $(0,\infty)$, then $f \in \mathcal{F}_\infty'$. Equivalently (cf. lemma 1.6.10), if G is a df on $(0,\infty)$, then the PLST

$$\int_{(0,\infty)} \frac{\mu}{\mu + \tau}\, dG(\mu) \in \mathcal{F}_\infty' \ . \tag{5.4.44}$$

Steutel (1970) proves the following implication:

$$F \in \mathcal{D} \Rightarrow \forall_{p\in(0,1)}\ p + (1-p)\hat{F}(\tau) \in \mathcal{F}_0^+ \ . \tag{5.4.45}$$

We can improve on this result as follows.

THEOREM 5.4.26. If $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ and if $0 < p < 1$, then the PLST

$$p + (1-p)\hat{F}(\tau) \in \mathcal{F}_\infty^+ \ .$$

PROOF. We calculate the φ_∞-function of $p + (1-p)\hat{F}$, and obtain

$$\varphi_\infty(\tau) = 1 - \frac{p + (1-p)F(0)}{p + (1-p)\hat{F}(\tau)} = (1-p)\{1 - \frac{F(0)}{\hat{F}(\tau)}\}\frac{\hat{F}(\tau)}{p + (1-p)\hat{F}(\tau)} \ . \tag{5.4.46}$$

Denoting the last factor in the right-hand side of (5.4.46) by $\varphi(\tau)$, we can write $\varphi(\tau) = \{1 - \log \hat{H}(\tau)\}^{-1}$, where according to corollary 5.4.6 and theorem 1.4.4

$$\hat{H}(\tau) := \exp[p(1 - \hat{F}(\tau)^{-1})]$$

is the PLST of a df H in F_o. Now, by theorem 5.4.5 it follows that φ is the PLST of a df in F_∞. Hence φ is comp mon, and applying lemma 5.1.4(ii) once in each direction, we conclude from (5.4.46) that $p + (1-p)\hat{F} \in F_\infty^+$. □

COROLLARY 5.4.27. If G is a df on $(0,\infty)$, and if $0 < p < 1$, then the PLST

$$p + (1-p) \int_{(0,\infty)} \frac{\mu}{\mu+\tau}\, dG(\mu) \in F_\infty^+ . \tag{5.4.47}$$

REMARK 5.4.28. Starting with continuous df's F in theorem 5.4.26, we cannot generate *all* compound geometric PLST's. In fact, in view of the result to be given in corollary 5.4.30, it will be clear that for $p \in (0,1)$ a PLST $\hat{F}$ has the form $1-p+p\hat{H}(\tau)$, where H is a continuous df in F_∞ with $\ell(H) = 0$, iff F is compound-geometric-(p,G) with $G \in F_\infty$, $\ell(G) = 0$.

The function φ from the proof of theorem 5.4.26 and the tot-dec(1) pgf's in section 3.4 suggest another closure property of F_∞.

THEOREM 5.4.29. Let F be a df on $[0,\infty)$, and define for $\mu > 0$ the function ψ_μ by

$$\psi_\mu(\tau) := \frac{\mu}{\mu - 1 + \hat{F}(\tau)^{-1}} \qquad (\tau \geq 0) . \tag{5.4.48}$$

Then:

(i) ψ_μ is a PLST for all $\mu \in (0,1]$.

(ii) If ψ_μ is a PLST, then ψ_λ is a PLST for all $\lambda \in (0,\mu]$.

(iii) ψ_μ is a PLST for all $\mu > 0$ iff $F \in F_\infty$ with $\ell(F) = 0$, in which case for all $\mu > 0$ the df F_μ, with $\hat{F}_\mu = \psi_\mu$, is also in F_∞ with $\ell(F_\mu) = 0$.

PROOF. Since ψ_μ can be written in the form

$$\psi_\mu(\tau) = \frac{1-(1-\mu)}{1-(1-\mu)\hat{F}(\tau)}\, \hat{F}(\tau) \qquad (\tau \geq 0) ,$$

it is seen that ψ_μ is a PLST for $0 < \mu \leq 1$. Using the relation

$$\hat{F}(\tau) = \frac{\psi_\mu(\tau)}{\mu + (1-\mu)\psi_\mu(\tau)} ,$$

we obtain the following relation between ψ_λ and ψ_μ:

$$\psi_\lambda(\tau) = \frac{\lambda/\mu}{1-(1-\lambda/\mu)\psi_\mu(\tau)}\,\psi_\mu(\tau) \quad ,$$

from which (ii) follows.
The "if" part of (iii) follows from the proof of theorem 5.4.26, where we showed that if $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ then ψ_μ is a PLST $\hat{F}_\mu$ in $\mathcal{F}_\infty$ with $\ell(F_\mu) = 0$ for all $\mu > 0$.
Therefore, suppose that ψ_μ is a PLST $\hat{F}_\mu$ for all $\mu > 0$. Rewrite $\hat{F}_\mu$ as

$$\hat{F}_\mu(\tau) = \{1 + \mu^{-1}[\hat{F}(\tau)^{-1} - 1]\}^{-1} \qquad (\tau \geq 0) \quad ,$$

and let $\gamma > 0$. Then, taking $\mu_n := n/\gamma$ $(n \in \mathbb{N})$, we see that the PLST $\{\hat{F}_{\mu_n}(\tau)\}^n$ satisfies

$$\hat{H}_\gamma(\tau) := \lim_{n\to\infty} \{\hat{F}_{\mu_n}(\tau)\}^n = \exp[\gamma(1 - \hat{F}(\tau)^{-1})] \ .$$

As $\hat{H}_\gamma(0+) = 1$, it follows by the continuity theorem for LST's that $\hat{H}_\gamma$ is a PLST for all $\gamma > 0$, and hence $\hat{H}_1$ is an inf div PLST. From theorem 5.4.5 we now conclude that $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$. □

COROLLARY 5.4.30. Let F be a df on $[0,\infty)$, and let $0 < p < 1$. Then $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ iff the following PLST $\hat{G}$ is in $\mathcal{F}_\infty$ with $\ell(G) = 0$:

$$(5.4.49) \qquad \hat{G}(\tau) := \frac{1-p}{1-p\hat{F}(\tau)}\,\hat{F}(\tau) \qquad (\tau \geq 0) \ .$$

Steutel (1970) has similar results for chf's. For instance, he gives an analogue of theorem 5.4.29(i); furthermore, he obtains a result (stated in the following theorem) that immediately yields also an analogue of theorem 5.4.29(iii) for chf's.

THEOREM 5.4.31. Let ψ be a function on $\mathbb{R}$. Then the function ψ_μ, defined by

$$\psi_\mu(t) := \frac{\mu}{\mu + \psi(t)} \qquad (t \in \mathbb{R}) \quad ,$$

is a chf for all $\mu > 0$ iff $\psi(t) = -\log \tilde{H}(t)$ with H an inf div df on $\mathbb{R}$.

COROLLARY 5.4.32. Let F be a df on $\mathbb{R}$. Then the function ψ_μ, defined by

$$\psi_\mu(t) := \frac{\mu}{\mu - 1 + \tilde{F}(t)^{-1}} \qquad (t \in \mathbb{R}) \quad ,$$

is a chf for all $\mu > 0$ iff $\tilde{F}(t) = \{1 - \log \tilde{H}(t)\}^{-1}$ with H an inf div df on $\mathbb{R}$.

The preceding theorems provide examples of distributions in F_∞. We mention a few of them.

EXAMPLE 5.4.33.

(i) Because of theorem 5.4.26 we have (cf. example 5, p. 153)

$$\frac{\nu}{\nu+\tau} \Big/ \frac{\mu}{\mu+\tau} = \frac{\nu}{\mu} + \left(1-\frac{\nu}{\mu}\right)\frac{\nu}{\nu+\tau} \in F_\infty^+ \qquad (0 < \nu < \mu) .$$

(ii) Scale mixtures of gamma distributions with (fixed) second parameter $\nu \in (0,1]$ can be regarded as mixtures of exponential distributions (cf. Steutel (1970)); hence they are in F'_∞. For instance

$$\int_{(1,\infty)} \left(\frac{\mu}{\mu+\tau}\right)^\nu \mu^{-2} d\mu = \frac{(1+\tau)^{1-\nu} - 1}{(1-\nu)\tau} \in F'_\infty \qquad (0 < \nu < 1) ,$$

and hence (take $\nu = \frac{1}{2}$, and apply theorems 5.4.29(iii) and 5.4.26)

$$(5.4.50) \quad p + (1-p)\frac{r+1}{r+(1+\tau)^{\frac{1}{2}}} \in F_\infty \qquad (0 \le p < 1;\ r > -1) .$$

(iii) The density f_r of X^r, where X has a gamma distribution with parameters $\mu > 0$ and $\nu > 0$, satisfies

$$(5.4.51) \quad f_r(x) = \frac{\mu^\nu}{r\Gamma(\nu)} x^{\nu/r-1} \exp[-\mu x^{1/r}] \in F'_\infty \quad (\mu > 0;\ \nu > 0;\ r \ge \max(1,\nu)) ,$$

as it is log-convex iff $r \ge \max(1,\nu)$.

In the next theorem we state a property of F_∞ that can be viewed as an analogue of the following closure property of F_0 (cf. theorem 1.4.9):

$$(5.4.52) \quad [G \in F_0,\ H \in F_0] \Rightarrow \hat{G}(-\log \hat{H}(\tau)) \text{ is a PLST in } F_0 .$$

Here, the inf div of H ensures the function $\hat{G}(-\log \hat{H}(\tau))$ to be a PLST.

THEOREM 5.4.34. If $G \in F_\infty$ with $\ell(G) = 0$, and if $H \in F_0$, then the function

$$(5.4.53) \quad \hat{G}(-\log \hat{H}(\tau)) = \int_{[0,\infty)} \hat{H}(\tau)^x dG(x) \qquad (\tau \ge 0)$$

is a PLST in F_∞ with left extremity equal to zero.

PROOF. As $H \in F_o$, the function $\hat{G}(-\log \hat{H}(\tau))$ is a PLST. Its ψ_∞-function is given by

$$\psi_\infty(\tau) = \frac{d}{d\tau}\{\hat{G}(-\log \hat{H}(\tau))\}^{-1} =$$

$$= -\{\hat{G}(-\log \hat{H}(\tau))\}^{-2}\varphi_o^{(H)}(\tau)\frac{d}{d\sigma}\hat{G}(\sigma)\Big|_{\sigma=-\log \hat{H}(\tau)} =$$

$$= \varphi_o^{(H)}(\tau)\psi_\infty^{(G)}(-\log \hat{H}(\tau)) .$$

The φ_o-function $\varphi_o^{(H)}$ of H is comp mon, as $H \in F_o$. Since $G \in F_\infty$ with $\ell(G) = 0$, its ψ_∞-function $\psi_\infty^{(G)}$ is comp mon (cf. theorem 5.4.2), and as $-\log \hat{H}(\tau)$ is nonnegative and has a comp mon derivative, we conclude from lemma 1.3.8(vi) that $\psi_\infty^{(G)}(-\log \hat{H}(\tau))$ is comp mon. It follows that ψ_∞ is comp mon, and the theorem is proved. □

In view of theorem 5.4.5 and corollary 5.4.6, (5.4.52) and theorem 5.4.34 can be reformulated as follows.

COROLLARY 5.4.35.
(i) If $G \in F_o$ and if $H \in F_\infty$ with $\ell(H) = 0$, then $\hat{G}(\hat{H}(\tau)^{-1} - 1)$ is a PLST in F_o.
(ii) If $G \in F_\infty$ with $\ell(G) = 0$, and if $H \in F_\infty$ with $\ell(H) = 0$, then $\hat{G}(\hat{H}(\tau)^{-1}-1)$ is a PLST in F_∞.

REMARK 5.4.36. If G is the df corresponding to a lattice distribution with pgf P and $P(0) > 0$, then in (5.4.52) and theorem 5.4.34 it is not necessary to require the inf div of H; in fact, for all df's H on $[0,\infty)$

(5.4.54) $\quad \hat{G}(-\log \hat{H}(\tau)) = P(\hat{H}(\tau))$

is the PLST of a compound distribution, and hence is in F_o^+ or F_∞^+ according as $P \in C_1$ or $P \in C_o$ (cf. lemma 1.4.14). Further, in view of (5.4.54), also for a non-lattice df G on $[0,\infty)$, but with an inf div df H, $\hat{G}(-\log \tilde{H}(t))$ could be called the chf of a *compound* distribution. Since every $\hat{F} \in F_\infty$ with $\ell(F) = 0$ can be written in the form $\hat{G}(-\log \hat{H}(\tau))$ with $\hat{G}(\tau) = (1+\tau)^{-1}$ (cf. theorem 5.4.5), it then follows that the distributions in F_∞ with left extremity equal to zero can be regarded as *"compound exponential"* distributions on $[0,\infty)$. Similarly, the chf's ψ_μ, considered in theorem 5.4.31, correspond to "compound exponential" distributions on $\mathbb{R}$.

To conclude this section, we return to functional equations. For $\lambda < \infty$ the $\mathcal{F}_\lambda$'s have been defined by means of the equation

$$(5.4.55)\qquad \int_{[0,x]} c(y;\lambda)\,dF(y) = \int_{[0,x]} F(x-y)\,dK_\lambda(y) \qquad (x > 0)\ ,$$

with a nondecreasing function K_λ. Letting $\lambda \to \infty$, we get (cf. theorem 1.6.7) the characterization of the class $\mathcal{F}_\infty^+$ by

$$(5.4.56)\qquad F(x) - F(0) = \int_{[0,x]} F(x-y)\,dK_\infty(y) \qquad (x > 0)\ ,$$

with a nondecreasing function K_∞.

Now, we want to know how from (5.4.55) or (5.4.56) the functional equation can be derived, by means of which $\mathcal{F}_\infty$ has been characterized (cf. theorem 5.4.12):

$$(5.4.57)\qquad x - \gamma F(x) = \int_{(0,x]} F(x-y)\{1 - N(y)\}dy \qquad (x > 0)\ ,$$

where $\gamma \geq 0$ and N is a nondecreasing function with $N(\infty) = 0$ and $\int_{(0,1)} x\,dN(x) < \infty$.

First, let us consider the case that $F \in \mathcal{F}_\infty^+$. Then $\gamma = 0$ and N is bounded, and from remark 5.4.11 it is seen that

$$(5.4.58)\qquad 1 - K_\infty(x) = F(0)\{1 - N(x)\} \qquad (x > 0)\ .$$

Rewriting (5.4.56) as

$$F(0) = F(x) - \int_{[0,x]} F(x-y)\,dK_\infty(y) \qquad (x > 0)\ ,$$

and integrating this equation over $[0,z]$, we get

$$F(0)\,z = \int_{[0,z]} F(z-y)\{1 - K_\infty(y)\}dy \qquad (z > 0)\ ,$$

which by (5.4.58) is equivalent to (5.4.57) with $\gamma = 0$.

When considering a general $F \in \mathcal{F}_\infty$, we need the first relation in the following lemma.

LEMMA 5.4.37. Let $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$. Then the following relations hold between the canonical representation (γ,N) and the K_λ-functions of F:

$$(5.4.59)\quad \lim_{\lambda\to\infty} \frac{1}{\hat{F}(\lambda)} \int_{[0,x]} \{1 - c_\lambda^{-1} K_\lambda(y)\}dy = \gamma + \int_{(0,x]} \{1 - N(y)\}dy \ ,$$

$$(5.4.60)\quad \lim_{\lambda\to\infty} \frac{1}{\hat{F}(\lambda)} \int_{[0,x]} ydK_\lambda(y) = \gamma + \int_{(0,x]} ydN(y) \ ,$$

$$(5.4.61)\quad \lim_{\lambda\to\infty} \frac{1}{\hat{F}(\lambda)}\{1 - c_\lambda^{-1} K_\lambda(x)\} = 1 - N(x) \ .$$

PROOF. In view of (5.4.1) and (5.4.21) we have the following relation:

$$(5.4.62)\quad \lim_{\lambda\to\infty} \frac{1}{\hat{F}(\lambda)} \tau^{-1}\{1 - c_\lambda^{-1}\hat{K}_\lambda(\tau)\} = \gamma + \int_{(0,\infty)} e^{-\tau x}\{1 - N(x)\}dx \ ,$$

from which by the continuity theorem for LST's (cf. Feller (1971), ch. XIII) (5.4.59) follows.

By the canonical representation (5.4.16) for $\hat{F}$ we can rewrite (5.4.4) in the form

$$(5.4.63)\quad \lim_{\lambda\to\infty} -\frac{d}{d\tau}\hat{K}_\lambda(\tau)/\hat{F}(\lambda) = \gamma + \int_{(0,\infty)} e^{-\tau x} x dN(x) \ .$$

Applying the continuity theorem once more, we get (5.4.60).

Finally, rewriting the integrals in (5.4.59) as

$$\int_{[0,x]} \{1 - c_\lambda^{-1} K_\lambda(y)\}dy = x\{1 - c_\lambda^{-1} K_\lambda(x)\} + c_\lambda^{-1} \int_{[0,x]} ydK_\lambda(y)$$

and (cf. (5.4.24))

$$\int_{(0,x]} \{1 - N(y)\}dy = x\{1 - N(x)\} + \int_{(0,x]} ydN(y) \ ,$$

respectively, and using (5.4.59) and (5.4.60), we obtain (5.4.61). $\square$

Now, we are ready to derive (5.4.57) from (5.4.55). Rewrite (5.4.55) as

$$\int_{[0,x]} e^{-\lambda y}dF(y) = F(x) - c_\lambda^{-1} \int_{[0,x]} F(x - y)dK_\lambda(y) \ ,$$

then, integrating this equation over [0,z], we get

$$\int_{[0,z]} (z - y)e^{-\lambda y}dF(y) = \int_{[0,z]} F(z - y)\{1 - c_\lambda^{-1} K_\lambda(y)\}dy \ ,$$

or, dividing by $\hat{F}(\lambda)$ and rewriting the left-hand side,

$$z - \frac{1}{\hat{F}(\lambda)}\{z \int_{(z,\infty)} e^{-\lambda y} dF(y) + \int_{[0,z]} y e^{-\lambda y} dF(y)\} =$$

(5.4.64)
$$= \int_{[0,z]} F(z-y) \frac{1 - c_\lambda^{-1} K_\lambda(y)}{\hat{F}(\lambda)} dy \ .$$

Letting $\lambda \to \infty$, and using (5.4.59) and the fact that the absolute value of the second term in the left-hand side of (5.4.64) is less than $-\hat{F}'(\lambda)/\hat{F}(\lambda)$, and hence tends to zero (cf. lemma 5.1.6), we get the functional equation (5.4.57).

5.5. *The class F_∞ in relation to standard* p-*functions*

The relation between C_o and the class R_o of renewal sequences (cf. section 4.1) suggests the existence of relations between F_∞ and the classes of p-functions and renewal densities, both of which can be considered as continuous analogues of R_o.

Let us first give the definition and some properties of p-functions; they can be found in Kingman (1972).

DEFINITION 5.5.1. A function p on $(0,\infty)$ is said to be a p-*function* if there exists a $\{0,1\}$-valued stochastic process $\{Z(t)\}_{t>0}$ (called a *regenerative phenomenon*) such that

(5.5.1) $$P(Z(t_1) = \ldots = Z(t_n) = 1) = \prod_{k=1}^{n} p(t_k - t_{k-1}) \quad (0 = t_0 < t_1 < \ldots < t_n;\ n \in \mathbb{N}) .$$

In a similar way one can define a discrete-time regenerative phenomenon $\{Z(n)\}_{n \in \mathbb{N}_o}$. The sequence $\{u_n\}_o^\infty$ that then replaces the function p in (5.5.1), can be shown to be a renewal sequence, and conversely.

From (5.5.1) it follows that a p-function p satisfies:

(5.5.2) $\quad p(t) = P(Z(t) = 1) \qquad (t > 0)$,

and hence

(5.5.3) $\quad 0 \le p(t) \le 1 \qquad (t > 0)$.

p-functions satisfy many other inequalities; in fact, they can be characterized as follows (cf. the definition of renewal sequences).

THEOREM 5.5.2. A function p on $(0,\infty)$ is a p-function iff p satisfies

$$F(t_1,\dots,t_n;p) \geq 0, \quad \sum_{k=1}^{n} F(t_1,\dots,t_k;p) \leq 1$$

for all $n \in \mathbb{N}$ and $0 < t_1 < \dots < t_n$, where

$$F(t_1,\dots,t_n;p) := p(t_n) - \sum_{1\leq i<n} p(t_i)p(t_n - t_i) +$$

$$+ \sum\sum_{1\leq i<j<n} p(t_i)p(t_j - t_i)p(t_n - t_j) - \dots + (-1)^{n-1}p(t_1)\prod_{k=2}^{n} p(t_k - t_{k-1}) .$$

COROLLARY 5.5.3. If p is a p-function, then

(5.5.4) $\quad p(s)p(t) \leq p(s+t) \leq p(s)p(t) + 1 - p(s) \qquad (s,t > 0)$.

The following two properties are of special interest.

THEOREM 5.5.4. If p_1 and p_2 are p-functions, then so is the function p, defined by

$$p(t) := p_1(t)p_2(t) \qquad (t > 0) .$$

THEOREM 5.5.5. If p is a p-function, if $h > 0$, and if $u_n(h) := p(nh)$ $(n \in \mathbb{N})$, $u_o(h) := 1$, then

$$\{u_n(h)\}_o^\infty \in \mathcal{R}_o ,$$

in which case $\{u_n(h)\}$ is associated with $f_n(h) := F(h,2h,\dots,nh;p)$.

We shall connect $\mathcal{F}_\infty$ with the class $\mathcal{P}$ of *standard* p-functions, i.e. p-functions p with the property

(5.5.5) $\quad \lim_{t\downarrow 0} p(t) = 1$.

The class $\mathcal{R}_o$ of renewal sequences coincides with the class of diagonal transition probabilities corresponding to any state in any (discrete-time) Markov chain (cf. theorem 4.1.6(i)). There exists only a partial analogue of this result for $\mathcal{P}$.

THEOREM 5.5.6. If $\{X(t)\}_{t\geq 0}$ is a standard, continuous-time Markov chain on the countable state space S, if $a \in S$, and if

$$(5.5.6)\qquad p(t) := P(X(t) = a \mid X(0) = a) \qquad (t > 0) ,$$

then $p \in \mathcal{P}$.

The following theorem shows what functions can arise as LT's of standard p-functions.

THEOREM 5.5.7. A function φ on $(0,\infty)$ is the LT of a function $p \in \mathcal{P}$ iff there exists a measure ν on $(0,\infty]$ satisfying

$$(5.5.7)\qquad \int_{(0,\infty]} (1 - e^{-x})\,\nu(dx) < \infty ,$$

such that φ has the form

$$(5.5.8)\qquad \varphi(\tau) = \{\tau + \int_{(0,\infty]} (1 - e^{-\tau x})\,\nu(dx)\}^{-1} \qquad (\tau > 0)$$

(ν, which is unique, is called the *canonical measure* of $p \in \mathcal{P}$).

Before giving the relation between $\mathcal{F}_\infty$ and $\mathcal{P}$, we state some simple, but useful properties of $\mathcal{P}$. Let $p \in \mathcal{P}$ with canonical measure ν. Then from the inequality (5.5.4) it can be shown (put $p(0) = 1$) that p is positive and uniformly continuous on $[0,\infty)$, and that $p(\infty) := \lim_{t\to\infty} p(t)$ exists in $[0,1]$, $-p_+'(0) := \lim_{t\downarrow 0} \{1 - p(t)\}/t$ in $[0,\infty]$ and $\ell := \lim_{t\to\infty} \{-\log p(t)\}/t$ in $[0,\infty)$, while furthermore

$$(5.5.9)\qquad p(\infty) = \{1 + \int_{(0,\infty]} x\nu(dx)\}^{-1},\quad -p_+'(0) = \nu((0,\infty])$$

and

$$(5.5.10)\qquad \exp[p_+'(0)t] \le p(t) \le \exp[-\ell t] \qquad (t > 0) .$$

Finally, from (5.5.8) it follows that $\int_{(0,\infty)} p(t)dt < \infty$ iff $\nu(\{\infty\}) > 0$, in which case

$$(5.5.11)\qquad \int_{(0,\infty)} p(t)dt = \nu(\{\infty\})^{-1} .$$

THEOREM 5.5.8.

(i) Let $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$, and let $\gamma > 0$ in the canonical representation (γ,N) of F (cf. (5.4.16)). Then F has a continuous density f, for which $f(0+)$ exists in $(0,\infty)$, and

$$p(x) := f(x)/f(0+) \in \mathcal{P} .$$

(ii) Let $p \in \mathcal{P}$ with canonical measure ν, and let $\nu(\{\infty\}) > 0$. Then

$$f(x) := \nu(\{\infty\})p(x) \in \mathcal{F}'_\infty .$$

(iii) Let $p \in \mathcal{P}$, and let $a > 0$. Then

$$f(x) := e^{-ax}p(x)/\hat{P}(a) \in \mathcal{F}'_\infty .$$

PROOF. One easily obtains parts (i) and (ii) by using theorem 5.4.13 and (5.5.11), and by comparing the canonical representations (5.4.16) for $\mathcal{F}_\infty$ and (5.5.8) for $\mathcal{P}$. Here the following relation holds between (γ,N) and ν:

(5.5.12) $\quad \gamma\nu(dx) = dN(x)$ on $(0,\infty)$, $\gamma\nu(\{\infty\}) = 1$.

To prove part (iii), we define $\pi(x) := e^{-ax}p(x)$ $(x \geq 0)$. Then in view of theorem 5.5.4 we have $\pi \in \mathcal{P}$. Since by (5.5.11) the canonical measure of π has mass $\hat{P}(a)^{-1} > 0$ at ∞, we can apply part (ii), and (iii) follows. □

By means of theorem 5.5.8 it is possible to obtain properties of the class $\{F \in \mathcal{F}_\infty \mid \ell(F) = 0, \gamma > 0\}$ from those of $\mathcal{P}$ (and conversely), and one could try to extend these properties to the whole class $\mathcal{F}'_\infty$, or, if the property considered can be transfered to df's, to $\mathcal{F}_\infty$. However, we shall restrict ourselves to the following interesting analogue of theorem 5.5.4 for $\mathcal{F}_\infty$ (see also Hawkes (1977), who gives a similar result for potential kernels (cf. the end of the present section)).

THEOREM 5.5.9. Let f be the continuous density of a df $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$ and $\gamma > 0$ (i.e. $f(0+) \in (0,\infty)$) in its canonical representation (γ,N) (cf. theorem 5.4.13), and let $G \in \mathcal{F}_\infty$ with $\ell(G) = 0$. Then there exists $c > 0$ such that the function

(5.5.13) $$H(x) := c \int_{[0,x]} f(y)\,dG(y) \qquad (x \geq 0)$$

is a df in $\mathcal{F}_\infty$ with $\ell(H) = 0$.

PROOF. First we note that by the boundedness of f (cf. (5.4.30)) (5.5.13) defines a df H for a suitably chosen $c > 0$. By the remark, following lemma 5.4.18, there exist df's G_n $(n \in \mathbb{N})$ in $\mathcal{F}_\infty$ with $\ell(G_n) = 0$ and with $\gamma > 0$ in their canonical representation such that G is the weak limit of the sequence $\{G_n\}$. Since f is continuous and bounded, it follows by the extended version of Helly's second theorem that

$$(5.5.14)\quad \hat{H}(\tau) = \lim_{n\to\infty} c \int_{[0,\infty)} e^{-\tau x} f(x)\, dG_n(x) \ .$$

By theorem 5.5.8(i), each of the df's G_n has a continuous density g_n, for which $g_n(x)/g_n(0+) \in \mathcal{P}$. Applying theorem 5.5.4, we see that also $f(x)g_n(x)/\{f(0+)g_n(0+)\} \in \mathcal{P}$, and hence (cf. theorem 5.5.8(ii)) $f(x)g_n(x)$ is, save for norming, a density in $\mathcal{F}'_\infty$. By (5.5.14) it now follows that H is the weak limit of a sequence of df's in $\mathcal{F}'_\infty$, and hence (cf. theorem 5.3.1) $H \in \mathcal{F}_\infty$. □

REMARK 5.5.10. If f is the density of a df $F \in \mathcal{F}'_\infty$, for which $f(0+) = \infty$, then the integral in (5.5.13) is not necessarily convergent for every df $G \in \mathcal{F}_\infty$ with $\ell(G) = 0$: take, for instance, a gamma df with second parameter $\nu = \frac{1}{2}$ for both F and G.

The renewal sequences have a second continuous analogue, the *renewal densities*. This will be apparent from theorem 4.1.5 and the following definition and properties of (pure) renewal functions (cf. Smith (1958)).
Let $T_1, T_2, \ldots$ be nonnegative, independent rv's with common df F. Then the sequence $\{S_n\}_1^\infty$, defined by

$$S_n := \sum_{k=1}^{n} T_k \qquad (n \in \mathbb{N}) \ ,$$

is called the *renewal process* associated with F. The *renewal function* U associated with F, is defined by

$$U(x) := EN_x, \text{ with } N_x := \#(\{n \in \mathbb{N} \mid S_n \le x\}) \ .$$

Both $\{S_n\}$ and U are called *persistent* if F is proper, and *transient* if F is defective. The renewal function U is the solution of the *renewal equation* associated with F:

$$(5.5.15)\quad U(x) = F(x) + \int_{[0,x]} U(x-y)\, dF(y) \qquad (x \in \mathbb{R}) \ ,$$

and U satisfies

(5.5.16) $U(x) = \sum_{n=1}^{\infty} F^{*n}(x) \qquad (x \in \mathbb{R})$.

If F has a density f, then U has a density u, the *renewal density*, which satisfies

(5.5.17) $u(x) = f(x) + \int_{[0,x]} u(x-y)f(y)\,dy \qquad$ (almost all $x \in \mathbb{R}$) .

First, let us consider a transient renewal function U. It is associated with pF, for $p \in (0,1)$ and F a proper df on $[0,\infty)$. By (5.5.15) it follows that the LST $\hat{U}$ of U satisfies

(5.5.18) $\hat{U}(\tau) = \dfrac{p\hat{F}(\tau)}{1 - p\hat{F}(\tau)}$,

from which, using corollary 5.4.30, we obtain the following relation with $\mathcal{F}_\infty$.

THEOREM 5.5.11. Let U be a transient renewal function, associated with pF $(0 < p < 1)$. Then:

(i) $\hat{H}(\tau) := (1-p)p^{-1}\hat{U}(\tau)$ is a PLST in $\mathcal{F}_\infty$ with $\ell(H) = 0$ iff $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$;

(ii) $(1-p)\{1 + \hat{U}(\tau)\}$ is a PLST in $\mathcal{F}_\infty^+$.

Next, let us consider the persistent case. Then we have the following relations for a renewal function associated with F:

(5.5.19) $\hat{U}(\tau) = \dfrac{\hat{F}(\tau)}{1 - \hat{F}(\tau)}$ and $\hat{F}(\tau) = \dfrac{\hat{U}(\tau)}{1 + \hat{U}(\tau)} \qquad (\tau > 0)$.

Daley (1965) characterizes the renewal functions U for which μU is again a renewal function for all $\mu > 0$. His result is not only most easily formulated by means of the class $\mathcal{F}_\infty$, but it also immediately follows from theorem 5.4.29; one need only observe that if U is the renewal function associated with F and if μU is a renewal function, then (cf. (5.5.19)) μU is associated with F_μ, given by

(5.5.20) $\hat{F}_\mu(\tau) = \dfrac{\mu}{\mu - 1 + \hat{F}(\tau)^{-1}} \qquad (\tau \geq 0)$.

This proves Daley's result, which we formulate as follows.

THEOREM 5.5.12. Let U be a (persistent) renewal function, associated with F. Then:

(i) μU is a renewal function for all $\mu \in (0,1]$;

(ii) μU is a renewal function for all $\mu > 0$ iff $F \in \mathcal{F}_\infty$ with $\ell(F) = 0$, in which case the df F_μ that μU is associated with, is also in $\mathcal{F}_\infty$ with $\ell(F_\mu) = 0$.

Some authors (e.g. Feller (1971)) put $S_o = 0$ and count this as a renewal epoch, i.e. they consider $1_{[0,\infty)}(x) + U(x)$ in stead of $U(x)$. In connection with this, Runnenburg (1965) notes that for most purposes it is sufficient to consider only renewal functions associated with a *compound geometric* df on $[0,\infty)$. In fact, he gives the following rather remarkable result (cf. theorem 5.5.11(ii)), which is easily verified.

LEMMA 5.5.13. If U is the renewal function associated with the df F, then the function

$$(5.5.21) \quad V(x) := 1_{[0,\infty)}(x) + U(x) \qquad (x \in \mathbb{R})$$

is a renewal function associated with a df $G \in \mathcal{F}_\infty^+$, given by

$$(5.5.22) \quad \hat{G}(\tau) = \{2 - \hat{F}(\tau)\}^{-1} \qquad (\tau \geq 0) \ .$$

In view of theorem 5.5.12 this lemma can be generalized as follows.

THEOREM 5.5.14. If U is the renewal function associated with the df F, then for all $\mu > 0$ the function

$$V_\mu(x) := \mu\{1_{[0,\infty)}(x) + U(x)\} \qquad (x \in \mathbb{R})$$

is a renewal function associated with a df in $\mathcal{F}_\infty^+$.

If $F \in \mathcal{F}_\infty$ has canonical representation (γ,N), then by (5.5.19) the renewal function U, associated with F, satisfies

$$(5.5.23) \quad \hat{U}(\tau) = \{\gamma\tau + \int_{(0,\infty)} (1 - e^{-\tau x})\,dN(x)\}^{-1} \qquad (\tau > 0) \ .$$

If $\gamma > 0$, then by theorem 5.5.7 and (5.5.11) it follows that U has a continuous renewal density u, for which $p(x) := \gamma u(x)$ is a standard p-function with $\int_{(0,\infty)} p(x)\,dx = \infty$. Obviously, the converse is also true, i.e. (cf. theorem 5.5.12) we have the following relation between renewal densities and p-functions (see also Daley (1965) and Kingman (1972)), which, by theorem 5.5.8(iii), also yields a relation between renewal densities and F'_∞.

THEOREM 5.5.15. Let u be a continuous function on $[0,\infty)$ with $u(0) = 1$. Then μu is a (persistent) renewal density for all $\mu > 0$ iff $u \in P$ and $\int_{(0,\infty)} u(x)dx = \infty$.

Recently, Forst (1978) proved Daley's result by making use of the fact that a nondecreasing function U, vanishing on $(-\infty,0)$, is a persistent renewal function iff the measure μ, with df $1_{[0,\infty)}(x) + U(x)$ $(x \in \mathbb{R})$, is a potential kernel on $[0,\infty)$ with infinite mass. Here, a *potential kernel* on $[0,\infty)$ is a measure μ of the form

$$(5.5.24) \qquad \mu = \int_{(0,\infty)} \mu_t dt ,$$

where $(\mu_t)_{t>0}$ is a convolution semigroup of possibly defective probability measures on $[0,\infty)$ (see e.g. Berg & Forst (1975)).
The potential kernels are intimately connected with the class F_∞. This is apparent from the canonical representations. In terms of LST's, (5.5.24) can be rewritten as

$$(5.5.25) \qquad \hat{\mu}(\tau) = \int_{(0,\infty)} \{p\hat{H}(\tau)\}^t dt = \{-\log p - \log \hat{H}(\tau)\}^{-1} ,$$

where $0 < p \leq 1$ and $H \in F_0$. Note that for $p < 1$ we have, save for norming, a special case of theorem 5.4.34: take for G in (5.4.53) an exponential df. Using for $\hat{H}$ the representation (γ,N) from theorem 1.7.1, we obtain from (5.5.25) the canonical representation for the LST $\hat{\mu}$ of a potential kernel μ:

$$(5.5.26) \qquad \hat{\mu}(\tau) = \{\beta + \gamma\tau + \int_{(0,\infty)} (1 - e^{-\tau x})dN(x)\}^{-1} \qquad (\tau > 0) ,$$

where $\beta \geq 0$, $\gamma \geq 0$ and N is a nondecreasing function on $(0,\infty)$ with $N(\infty) = 0$ and $\int_{(0,1)} x dN(x) < \infty$ (not simultaneously $\beta = 0$, $\gamma = 0$, $N \equiv 0$). In fact, (5.5.26) reduces to the canonical representations (5.4.16) for F_∞ and (5.5.8) for P, if we take $\beta = 1$ and $\gamma = 1$, respectively. Furthermore, if U is the renewal function associated with pF, where $0 < p \leq 1$ and F is a (proper) df on $[0,\infty)$, then, obviously, the LST

$$1 + \hat{U}(\tau) = \{1 - p\hat{F}(\tau)\}^{-1} \qquad (\tau > 0)$$

is of the form (5.5.26) with $\gamma = 0$ and N bounded. Thus, properties of F_∞, P and renewal functions can be obtained from known properties of potential kernels (cf. Berg & Forst (1978)).
Finally, we note that the potential kernels correspond with more general renewal functions. The continuous-time analogue of the renewal process $\{S_n\}_1^\infty$ (cf. p. 181), regarded as a discrete-time process, is an increasing process $\{S_t\}_{t>0}$ that is inf div, i.e.

$$Ee^{-\tau S_t} = \{p\hat{F}(\tau)\}^t \qquad (\tau \geq 0) ,$$

where $0 < p \leq 1$ and $F \in F_0$. Now, the analogue of the renewal function U is the function V, defined by (λ is the Lebesgue measure)

$$V(x) := E\lambda(\{t > 0 \mid S_t \leq x\}) = \int_{(0,\infty)} p^t F^{*t}(x)\,dt ,$$

which shows that the potential kernels, considered by e.g. Hawkes (1977), are identical to those defined in (5.5.24) or (5.5.25).

5.6. *Other classifications*

In section 2.5 we introduced the classes C_0^u of compound negative-binomial lattice distributions with parameter $u > 0$. They define a classification of C_1, and can be characterized as follows:

$$P \in C_0^u \Leftrightarrow P^{1/u} \text{ is a pgf in } C_0 .$$

Now, for the sake of completeness, we briefly consider the analogous classes F_∞^u of distributions on $[0,\infty)$.

DEFINITION 5.6.1. For $u > 0$ a df F on $[0,\infty)$ is said to be in the class F_∞^u if $\hat{F}^{1/u}$ is a PLST in F_∞.

If F is a df on $[0,\infty)$, if $\gamma > 0$ and if $\hat{F}^\gamma$ is a PLST (of $F^{*\gamma}$), then the φ_0-function of $F^{*\gamma}$ is equal to that of F, multiplied by γ. Hence, by lemma 5.1.6 we have in this case

$$(5.6.1) \qquad \ell(F^{*\gamma}) = \gamma\ell(F) .$$

Now, by theorem 5.4.5 it follows that a PLST $\hat{F}$ in F_∞^u with $\ell(F) = 0$ has the canonical form

$$(5.6.2)\qquad \hat{F}(\tau) = \{1 - \log \hat{H}(\tau)\}^{-u} \qquad (\tau \geq 0)\ ,$$

with $H \in F_o$. Thus, the df's F in F_∞^u with $\ell(F) = 0$ can be regarded as *"compound gamma"* df's (cf. remark 5.4.36).
Since F_∞^+ is the class of compound geometric distributions on $[0,\infty)$, the class $(F_\infty^u)^+$ of df's F in F_∞^u with $F(0) > 0$ coincides with the class of *compound negative-binomial* distributions on $[0,\infty)$ with parameter u (cf. theorem 1.6.6). Also, F_∞^+ is dense in $\{F \in F_\infty \mid \ell(F) = 0\}$ in the sense of weak convergence (cf. (5.4.1)), and this result, together with (5.6.1), immediately yields the following characterization of F_∞^u (cf. theorem 1.4.15).

THEOREM 5.6.2. For $u > 0$ a df F, with $\ell(F) = 0$, is in F_∞^u iff F is the weak limit of compound negative-binomial df's with parameter u, i.e. iff $\hat{F}$ has the form

$$(5.6.3)\qquad \hat{F}(\tau) = \lim_{n\to\infty} \left\{\frac{1 - p_n}{1 - p_n \hat{G}_n(\tau)}\right\}^u \qquad (\tau \geq 0)\ ,$$

where $0 < p_n < 1$ and G_n is a df on $[0,\infty)$ $(n \in \mathbb{N})$.

The family $(F_\infty^u \mid u > 0)$ defines a classification of F_o: similar to the proof of theorem 2.5.3 it can be shown that

$$(5.6.4)\qquad u < v \Rightarrow F_\infty^u \subset F_\infty^v\ .$$

Furthermore, $\bigcup_{u>0} F_\infty^u$ is dense in F_o; in fact, if $F \in F_o$, then F is the weak limit of the sequence of df's F_n, defined by

$$(5.6.5)\qquad \hat{F}_n(\tau) := \left\{\frac{n}{n - \log \hat{F}(\tau)}\right\}^n \qquad (\tau \geq 0)\ ,$$

which (cf. (5.6.2)) is a PLST in F_∞^n $(n \in \mathbb{N})$.

Another interpolation between F_∞ and F_o is suggested by the relation (5.4.2) between the ψ_∞-function and the φ_o-function of a df F on $[0,\infty)$: we briefly indicate the analogues to the classes H_γ $(\gamma \geq 0)$ from definition 2.5.7.

DEFINITION 5.6.3. For $\gamma \geq 0$ a df F on $[0,\infty)$ is said to be in the class L_γ if the function

(5.6.6) $\quad -\hat{F}'(\tau)/\hat{F}(\tau)^{1+\gamma} \qquad (\tau > 0)$

is comp mon.

Obviously, $L_o = F_o$ and $L_1 = \{F \in F_\infty \mid \ell(F) = 0\}$ (cf. theorem 5.4.2). Now, let $\gamma > 0$ and let $F \in L_\gamma$. Then the function in (5.6.6) has a finite limit c, say, as $\tau \to \infty$, and therefore by lemma 5.1.6 it follows that

$$\ell(F) = \lim_{\tau\to\infty} -\hat{F}'(\tau)/\hat{F}(\tau) = cF(0)^\gamma ,$$

and hence

(5.6.7) $\quad F \in L_\gamma \Rightarrow \ell(F) = 0 \qquad (\gamma > 0)$.

Furthermore, if $F \in L_\gamma$, then $\hat{F}^{-\gamma}$ has a comp mon derivative; hence by lemma 1.3.8(vi) $\hat{F}^\gamma = \{\hat{F}^{-\gamma}\}^{-1}$ is a PLST, and its ψ_∞-function is comp mon. Now it easily follows that

(5.6.8) $\quad F \in L_\gamma \Leftrightarrow \hat{F}^\gamma$ is a PLST in L_1 ,

which by (5.6.7) can be reformulated as follows.

THEOREM 5.6.4. For $\gamma > 0$ the following relation holds:

$$L_\gamma = \{F \in F_\infty^{1/\gamma} \mid \ell(F) = 0\} .$$

Thus, for df's F on $[0,\infty)$ with $\ell(F) = 0$ the family $(L_\gamma \mid \gamma > 0)$ defines the same classification of F_o as $(F_\infty^u \mid u > 0)$.

5.7. *Further generalizations*

In analogy to the investigations in chapter 3 we could consider tot dec, tot fact and tot-dec(1) PLST's. We only mention the following analogue of theorem 3.2.4, and note that the analogue of theorem 3.4.5 has already been given in theorem 5.4.29(iii).

THEOREM 5.7.1. A df F on $[0,\infty)$ is inf div iff for all $\lambda > 0$, or, equivalently, for all $\lambda \in (0,\varepsilon)$ (some $\varepsilon > 0$), there exists a df F_λ on $[0,\infty)$ such that $\hat{F}$ can be written as

$$\hat{F}(\tau) = \frac{\hat{F}(\tau+\lambda)}{\hat{F}(\lambda)}\,\hat{F}_\lambda(\tau) \qquad (\tau \ge 0) .$$

In chapter 4 we studied extensions of the $\mathcal{C}_\alpha$'s, the classes $\mathcal{R}_\alpha$ of generalized renewal sequences. Similarly, one could consider the functional equations (5.1.7) for nondecreasing functions F that are not necessarily df's. This would give rise to classes of generalized p-functions, renewal functions and potential kernels.

In view of a further generalization one might ask to what extent the results of this monograph can be extended to distributions on the whole real line. Trying to extend the classification of $\mathcal{F}_0$ by the $\mathcal{F}_\lambda$'s to df's on $\mathbb{R}$, one encounters the difficulty that the PLST $\hat{F}(\tau+\lambda)/\hat{F}(\lambda)$, which plays an essential role for $\mathcal{F}_\lambda$, has no obvious analogue for general df's; $\tilde{F}(t+\lambda)/\tilde{F}(\lambda)$ is, in general, not a chf for $\lambda > 0$. Also, no functional equations are known that characterize the class of all inf div df's on $\mathbb{R}$.
Another possibility is writing an inf div chf $\tilde{F}$ as a function of chf's of df's on $[0,\infty)$, and then requiring that these df's are in $\mathcal{F}_\lambda$. Of course, every chf $\tilde{F}$ can be written as

$$(5.7.1) \qquad \tilde{F}(t) = (1-p)\tilde{F}_1(t) + p\tilde{F}_2(-t) \qquad (t \in \mathbb{R}) ,$$

where $0 \le p \le 1$, and F_1 and F_2 are df's on $[0,\infty)$. However, if F_1 and F_2 are in $\mathcal{F}_0$, then F is not necessarily inf div.
The following decomposition gives a better result.

DEFINITION 5.7.2. For $0 \le \lambda \le \infty$ a df F on $\mathbb{R}$ is said to be in the class $\mathcal{G}_\lambda$ if there exist df's F_1 and F_2 in $\mathcal{F}_\lambda$ such that

$$(5.7.2) \qquad \tilde{F}(t) = \tilde{F}_1(t)\tilde{F}_2(-t) \qquad (t \in \mathbb{R}) .$$

Obviously, we have the desired monotonicity property:

$$\mathcal{G}_\lambda \subset \mathcal{G}_\mu \text{ if } 0 \le \mu \le \lambda \le \infty ,$$

but $\mathcal{G}_0$ does not coincide with the class of *all* inf div df's on $\mathbb{R}$; in fact, by theorem 1.7.11 it is seen that an inf div df F is in $\mathcal{G}_0$ iff its Lévy representation (a,σ^2,M) satisfies

$$(5.7.3) \qquad \sigma^2 = 0, \quad \int_{(-1,1)\setminus\{0\}} |x|\,dM(x) < \infty .$$

Therefore, for $0 \le \lambda \le \infty$ we introduce the classes $\bar{\mathcal{G}}_\lambda$ of df's that are weak limits of df's in $\mathcal{G}_\lambda$. Now, observing that the compound Poisson df's on $\mathbb{R}$

are in $\mathcal{G}_o$ (cf. theorem 1.7.13 and (5.7.3)), we conclude from De Finetti's theorem (theorem 1.4.15) that $\bar{\mathcal{G}}_o$ is the class of *all* inf div df's on $\mathbb{R}$. As obviously

$$\bar{\mathcal{G}}_\lambda \subset \bar{\mathcal{G}}_\mu \text{ if } 0 \le \mu \le \lambda \le \infty ,$$

the family $(\bar{\mathcal{G}}_\lambda \mid 0 \le \lambda \le \infty)$ defines a classification of the inf div df's on $\mathbb{R}$. Unfortunately, this does not lead to a characterization of $\bar{\mathcal{G}}_\lambda$ by a single functional equation.
We note that the classification of $\mathcal{F}_o$ given by $(\mathcal{F}_\infty^u \mid u > 0)$ (cf. section 6), is easily extended to general df's on $\mathbb{R}$: consider chf's $\tilde{F}$ of the form

$$\tilde{F}(t) = \{1 - \log \tilde{H}(t)\}^{-u} \quad (t \in \mathbb{R}) ,$$

with H an inf div df on $\mathbb{R}$.

Finally, we consider the multivariate case. Generalizing the functional equations by means of which Horn & Steutel (1978) characterize the distributions of nonnegative multivariate inf div random vectors, we can obtain a classification of these distributions analogous to that of $\mathcal{F}_o$ by the $\mathcal{F}_\lambda$'s. We illustrate this by giving this characterization for distributions on $\mathbb{N}_o^2$.

THEOREM 5.7.3. Let $\{p_{k,\ell}\}$ be a probability distribution on $\mathbb{N}_o^2$ with $p_{o,o} > 0$. Then there exists a unique solution $\{a_{k,\ell}\}$ with $a_{o,o} = 0$ of the following system of recurrence relations:

$$(5.7.4) \qquad \begin{aligned} (k+1)p_{k+1,\ell} &= \sum_{i=0}^{k} \sum_{j=0}^{\ell} (i+1)a_{i+1,j}p_{k-i,\ell-j} \\ (\ell+1)p_{k,\ell+1} &= \sum_{i=0}^{k} \sum_{j=0}^{\ell} (j+1)a_{i,j+1}p_{k-i,\ell-j} \end{aligned} \qquad (k,\ell \in \mathbb{N}_o) .$$

Furthermore, $\{p_{k,\ell}\}$ is inf div iff $a_{k,\ell} \ge 0$ for all $k,\ell \in \mathbb{N}_o$, in which case

$$\sum_{k,\ell} a_{k,\ell} = -\log p_{o,o} < \infty .$$

Now, we can replace the factors $k+1$ and $\ell+1$ in (5.7.4) by

$$(1-\alpha^{k+1})/(1-\alpha) \text{ and } (1-\alpha^{\ell+1})/(1-\alpha) ,$$

or by

$$(1-\alpha^{k+1})/(1-\alpha) \text{ and } (1-\beta^{\ell+1})/(1-\beta) ,$$

respectively, to obtain classifications of the inf div distributions on $\mathbb{N}_o^2$.

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