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MATHEMATICAL CENTRE TRACTS 102

REFLEXIVE AND SUPERREFLEXIVE BANACH SPACES

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PREFACE

This book has developed from lectures on reflexive and superreflexive Banach spaces given at the University of Amsterdam. A Banach space X is called reflexive iff the canonical embedding of X into its second conjugate space X^{**} is surjective. Although clear and simple, in some respects this definition is not very satisfactory. For instance, checking the reflexivity of a given space against this definition requires the computation of the first and second conjugate spaces, a generally difficult task. One would like to avoid this and to characterize reflexive spaces intrinsically, i.e. without reference to their conjugates. Developing such characterizations is what a large part of Chapter I of this book is about. In the process we shall gain a better insight into the structure of reflexive spaces. In particular it will become apparent that reflexivity is a property intimately connected with the geometry of the unit ball. Roughly, what makes a space reflexive is the absence of certain large (i.e. infinite-dimensional) flat areas in the unit ball, away from the origin. Thus reflexive spaces can be thought of as spaces whose unit balls possess a certain degree of infinite-dimensional rotundity. This is just one of various types of characterizations developed in Chapter I, but it is of particular importance in connection with superreflexivity, the subject of Chapter II.

It is a classical result that every uniformly convex space is reflexive. However, not every reflexive space is uniformly convexifiable (i.e. can be given an equivalent uniformly convex norm). The question of characterizing the uniformly convexifiable spaces has long been open. It is now known that the latter coincide with the so-called superreflexive spaces, which form a proper subclass of the reflexive ones. It was R.C. JAMES ([54]) who first introduced the notion of superreflexivity, expecting it to be equivalent to uniform convexifiability. As P. ENFLO ([33]) has shown, this idea was sound. In terms of the geometry of the unit ball, a space fails to be superreflexive iff its unit ball satisfies certain finite-dimensional flatness conditions analogous to the infinite-dimensional ones characterizing non-reflexive spaces.

Further geometric properties of superreflexive spaces are taken up in the latter part of Chapter II. In particular we relate superreflexivity to a geometric parameter called the girth ([90]) of the unit ball, defined as the infimum of the lengths of all closed centrally symmetric curves on the unit sphere. A Banach space is superreflexive iff its girth is larger than 4 ([59]). If the girth of the unit ball of X is 4 and is achieved, i.e. if there exists a so-called "girth curve" of length exactly 4, then X is called flat ([42]). Some results on flatness are proved and a characterization of superreflexivity in terms of flatness is given. We also discuss other super-properties equivalent to superreflexivity. The final section of Chapter II is devoted to connections with ℓ^p spaces. It was long hoped that the spaces ℓ^p $(1 \leq p < \infty)$ and c_0 would turn out to be the fundamental building stones in Banach space theory, in the sense that every Banach space would contain one of them isomorphically. This hope was crushed by B.S. TSIRELSON ([101]), however, who constructed a reflexive space containing no ℓ^p. Notwithstanding some positive results connecting superreflexive spaces with ℓ^p spaces ([55], [38]), even superreflexive spaces need not contain any ℓ^p, as T. FIGIEL & W.B. JOHNSON ([35]) have shown by a modification of Tsirelson's example.

The selection of material for this book has been determined to some extent by personal taste and prejudice (and, undoubtedly, lack of knowledge), but mainly by our desire to concentrate on the geometry of the subject. The emphasis is on general theory. Examples and counterexamples are given only to illustrate the various notions introduced and to indicate the scope of the theorems proved. We make no pretence of being complete and the specialist may find his favorite topic missing. This is a price we gladly paid for our wish to keep both the size of this volume down and the exposition detailed enough for the beginning graduate student to be able to proceed without undue hardship. A preliminary section of prerequisites reduces the presupposed knowledge to a minimum. By providing full details of proof throughout the text we hope to have made this beautiful subject accessible to a large audience.

Rather than scattering them throughout the text we have collected biographical references at the end of each section. Notation is for the most part standard. Let us mention here a few uses which may not be so common.

sp A, sp{x,y,...}: linear hull of a subset A, respectively {x,y,...}, of a linear space, i.e. the set of all (finite) linear combinations formed with elements of A, respectively {x,y,...}.

$[A] = \overline{\text{sp}}\, A$, $[x_n]_{n=1}^{\infty}$: closed linear hull of a subset A, respectively $\{x_n\colon n \in \mathbb{N}\}$, of a normed linear space, i.e. the closure of sp A, respectively $sp\{x_1,x_2,\ldots\}$.

$\bar{A}^{\mathcal{T}}$: closure of a subset A of a space equipped with a topology $\mathcal{T}$.

I am indebted to Mr. A.J. Pach for the work he did in connection with this book. Not only did he write a preliminary (Dutch) version of part of the present text while I was lecturing on it, but he also critically read a large part of the final manuscript, pointing out errors and suggesting many improvements. Needless to say, all remaining mistakes are mine. Finally I would like to thank Mrs. C. Klein Velderman for her excellent typing and the Mathematical Center for accepting this volume in their series.

D. van Dulst

August 1978.

0. PREREQUISITES

For the convenience of the reader we review in this section some basic facts frequently used later on in these notes and which the reader should be familiar with. We shall presuppose only very little knowledge of Banach spaces, roughly the contents of Chapter I of [44]. All other results recalled in this section will be provided with (short) proofs.

In the study of Banach spaces also topologies other than that defined by the norm are important tools, notably the weak and weak* topologies. These can best be understood against the background of topological vector spaces which we now briefly discuss. Let X be a vector space. (We always consider vector spaces over the reals, for simplicity.) A subset $A \subset X$ is called *convex* if $\lambda x+(1-\lambda)y \in A$ whenever $x,y \in A$ and $0 \leq \lambda \leq 1$, and *balanced* (or *circled*) if $\lambda A \subset A$ for all $|\lambda| \leq 1$. $\bigcup_{|\lambda|\leq 1} \lambda A$ is the smallest balanced set containing A and is called the *balanced hull* of A. Likewise the *convex hull*, denoted by co A, is the smallest convex set containing A and consists of all elements of the form $\lambda_1 x_1 + \dots + \lambda_n x_n$, with $n \in \mathbb{N}$, $\lambda_1,\dots,\lambda_n \geq 0$, $\sum_{i=1}^n \lambda_i = 1$. A set which is both balanced and convex is called *absolutely convex*. The smallest absolutely convex set containing a given subset $A \subset X$ is called the *absolutely convex hull* of A. It is easily verified that it equals the convex hull of the balanced hull of A and consists of all elements $\lambda_1 x_1 + \dots + \lambda_n x_n$ with $n \in \mathbb{N}$, $\lambda_1,\dots,\lambda_n \in \mathbb{R}$ and $\sum_{i=1}^n |\lambda_i| \leq 1$. $A \subset X$ is called *absorbing* if for every $x \in X$ there exists a $\lambda_0 \geq 0$ such that $x \in \lambda A$ whenever $|\lambda| \geq \lambda_0$. If A is balanced, then it is absorbing whenever $\bigcup_{n\in\mathbb{N}} nA = X$. A function p on X is a *seminorm* on X if

(i) $0 \leq p(x) < \infty$ for all $x \in X$,

(ii) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$ and $x \in X$,

(iii) $p(x+y) \leq p(x)+p(y)$ for all $x,y \in X$.

Thus a seminorm differs from a norm in that it may be 0 on non-zero elements.

DEFINITION 0.1. A *topological vector space* (t.v.s.) X is a vector space equipped with a topology for which the maps

(i) $(x,y) \to x+y$ from $X \times X$ into X, and

(ii) $(\lambda,x) \to \lambda x$ from $\mathbb{R} \times X$ into X

are continuous ($X \times X$ and $\mathbb{R} \times X$ have the product topologies).

In particular, translations in a t.v.s. are homeomorphisms, so the topology of a t.v.s. is completely determined by a base $\mathcal{U}$ of neighborhoods U for the zero element. It easily follows from the continuity of the algebraic operations that all 0-neighborhoods are absorbing and that the closed balanced 0-neighborhoods form a basis. If X has also a 0-neighborhood base consisting of convex sets (which does not follow from the definition of a t.v.s.) then X is said to be *locally convex*. A locally convex topological vector space is briefly called a *locally convex space* (l.c.s.). A l.c.s. is easily seen to possess a 0-neighborhood base consisting of closed absolutely convex sets. Namely, take any 0-neighborhood base $\mathcal{U}$ and form the closed absolutely convex hulls of all $U \in \mathcal{U}$ (the *closed absolutely convex hull* of a set A is by definition the smallest closed absolutely convex set containing A and is obviously the closure of its absolutely convex hull).

If A is an absolutely convex absorbing set in a vector space X (no topology for the moment), then the *gauge* or *Minkowski functional* p_A is the function on X, defined as follows:

$$p_A(x) := \inf\{\lambda \geq 0: x \in \lambda A\} \qquad (x \in X).$$

p_A is easily seen to be a seminorm and, moreover,

$$\{x \in X: p_A(x) < 1\} \subset A \subset \{x \in X: p_A(x) \leq 1\}.$$

Conversely, if p is a seminorm on X then $\{x \in X: p(x) < 1\}$ and $\{x \in X: p(x) \leq 1\}$ are absolutely convex and absorbing and $p = p_A$ for an absolutely convex absorbing set A iff

$$\{x \in X: p(x) < 1\} \subset A \subset \{x \in X: p(x) \leq 1\}.$$

Supposing now that X is a l.c.s., it is obvious from these observations that there is a 1-1 correspondence between the closed absolutely convex 0-neighborhoods U and the continuous (i.e. continuous at 0) seminorms p on X, namely

$$U = \{x \in X: p(x) \leq 1\} \leftrightarrow p = p_U.$$

A linear form f on X is continuous iff f is continuous at 0, i.e. iff $|f(x)| \leq 1$ for all $x \in U$, where U is some closed absolutely convex 0-neighhorhood, i.e. iff $|f(x)| \leq p(x)$ $(x \in X)$, for some continuous seminorm on X (take $p = p_U$). Two topological vector spaces X and Y are called (topologically) *isomorphic* (notation: $X \simeq Y$) if there exists a linear homeomorphism of X onto Y. Every finite-dimensional t.v.s. X is easily shown to be isomorphic to $\mathbb{R}^n$, where n = dim X. Whenever we wish to emphasize the topology $\mathcal{T}$ of a t.v.s. X, we write it as $(X,\mathcal{T})$.

The *dual space* X^* of a t.v.s. X is the set of all continuous linear forms on X. X^* is a vector space with the usual definitions of addition and scalar multiplication. The hyperplanes in X are the kernels of the linear forms on X and they are either closed or dense, according as the corresponding linear form is continuous or discontinuous. In the case of a l.c.s. X the Hahn-Banach theorem guarantees that X^* contains "sufficiently many" elements. We state it in two forms, a geometric and an analytic one, which can be derived from one another. One should observe that in the absence of local convexity, the only non-empty open convex subset of X may be X, in which case the Hahn-Banach theorem is empty.

<u>THEOREM 0.2</u> (Hahn-Banach).
<u>geometric form</u>: *Let* X *be a t.v.s.,* $A \neq \phi$ *an open convex subset of* X *and* M *a linear manifold in* X *such that* $M \cap A = \phi$. *Then there exists a (closed) affine hyperplane* H *in* X *such that* $H \cap A = \phi$ *and* $H \supset M$.
<u>analytic form</u>: *Let* X *be a vector space,* p *a seminorm on* X *and* M *a linear subspace of* X. *If* f *is a linear form on* M *satisfying* $|f(x)| \leq p(x)$ *for all* $x \in M$, *then* f *can be extended to a linear form on* X *satisfying the same inequality on* X, *i.e. there exists a linear form* g *on* X *such that* $f(x) = g(x)$ *for all* $x \in M$ *and* $|g(x)| \leq p(x)$ *for all* $x \in X$.

In particular, taking for X a l.c.s. and for p a suitable continuous seminorm on X, one sees that every continuous linear form on a linear subspace $M \subset X$ can be extended continuously to X. In the case of normed linear spaces, the extension can be made with preservation of the norm (take $p(x) = \|f\|\|x\|$ $(x \in X)$, $\|f\|$ denoting the norm of f on M).

<u>DEFINITION 0.3</u>. Let X and Y be two vector spaces and let $\langle\cdot,\cdot\rangle$ be a bilinear form on $X \times Y$, i.e. for fixed $x \in X$ (respectively $y \in Y$) $y \to \langle x,y\rangle$

(respectively, $x \to \langle x,y\rangle$) is a linear form on Y (respectively, X). Then the pair X,Y with this bilinear form is called a *dual pair* (denoted by $\langle X,Y\rangle$) provided the following conditions hold:

(i) $\langle x,y\rangle = 0$ for all $x \in X \Rightarrow y = 0$,

(ii) $\langle x,y\rangle = 0$ for all $y \in Y \Rightarrow x = 0$.

An example of a dual pair is a l.c.s. X together with its dual X^* and the "canonical" bilinear form $\langle \cdot,\cdot \rangle$ on $X \times X^*$ defined by

$$\langle x,x^*\rangle := x^*(x) \qquad (x \in X,\ x^* \in X^*).$$

Indeed, (i) is trivially satisfied and (ii) holds by the Hahn-Banach theorem: given $x \neq 0$, define a continuous linear form x^* on sp{x} (= the linear span of x) by $x^*(\alpha x) = \alpha$ ($\alpha \in \mathbb{R}$) and extend it continuously to X. Then $x^* \in X^*$ and $\langle x,x^*\rangle = 1 \neq 0$.

Now let $\langle X,Y\rangle$ be a given dual pair. We shall define Hausdorff locally convex topologies $\sigma(X,Y)$ and $\sigma(Y,X)$ on X and Y respectively and derive some properties of these topologies. $\sigma(X,Y)$ will be determined completely once we have described a neighborhood base for an arbitrary fixed element $x \in X$. By definition such a neighborhood base consists of all sets of the form

$$V(x;y_1,\dots,y_n;\varepsilon) := \{x' \in X\colon |\langle x-x',y_i\rangle| \le \varepsilon,\ i = 1,\dots,n\},$$

where $n \in \mathbb{N}$, $y_1,\dots,y_n \in Y$ and $\varepsilon > 0$ are arbitrary. It is easily checked that these sets are convex and define a topology on X for which the algebraic operations are jointly continuous. Thus X with the topology $\sigma(X,Y)$ is a l.c.s. Furthermore,

I. <u>$\sigma(X,Y)$ is a Hausdorff topology.</u>

<u>PROOF</u>. Let $x_1,x_2 \in X$ with $x_1 \neq x_2$ be arbitrary. Using (ii) in Definition 0.3, choose $y \in Y$ such that $\langle x_1-x_2,y\rangle \neq 0$ and $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{2}|\langle x_1-x_2,y\rangle|$. Then $V(x_1;y;\varepsilon) \cap V(x_2;y;\varepsilon) = \phi$. □

II. <u>$(X,\sigma(X,Y))^*$ can be identified with Y.</u>

<u>PROOF</u>. Let us denote $(X,\sigma(X,Y))^*$ by X^*. For each $y \in Y$ we define a linear form $\phi(y)$ on X by

$$\phi(y)(x) = \langle x,y\rangle \qquad (x \in X,\ y \in Y).$$

Clearly, each $\phi(y)$ is continuous for $\sigma(X,Y)$, so ϕ maps Y into X^*.

Furthermore, ϕ is linear and 1-1 (by (i) in Definition 0.3), so it remains to show that ϕ is surjective. Let $x^* \in X^*$ be arbitrary. Then $x^*(0) = 0$ and by continuity there is a 0-neighborhood $V(0;y_1,\ldots,y_n;\varepsilon)$ on which $|x^*|$ is bounded by 1, i.e.

$$x \in X, \quad |\langle x,y_i\rangle| \le \varepsilon \ (i=1,\ldots,n) \Rightarrow |x^*(x)| \le 1.$$

It follows in particular that $\ker x^* \supset \bigcap_{i=1}^{n} \ker \phi(y_i)$. It is now an easy exercise (use induction on n) to show that $x^* = \lambda_1\phi(y_1) + \ldots + \lambda_n\phi(y_n) = \phi(\lambda_1 y_1 + \ldots + \lambda_n y_n)$ for some $\lambda_1,\ldots,\lambda_n \in \mathbb{R}$. □

Thus we may and do identify Y with X^* (i.e. we suppress the ϕ notation).

III. <u>$\sigma(X,Y)$ is the coarsest topology on X for which the elements of Y are continuous.</u>

<u>PROOF</u>. We have already noted that the elements of Y are continuous for $\sigma(X,Y)$. Conversely, let $\mathcal{T}$ be any topology on X for which the elements of Y are continuous. Let $V(x;y_1,\ldots,y_n;\varepsilon)$ be an arbitrary basic $\sigma(X,Y)$-neighborhood of an arbitrary $x \in X$. By assumption, for every $i \in \{1,\ldots,n\}$ there exists a $\mathcal{T}$-open O_i such that $x \in O_i$ and $|\langle x'-x,y_i\rangle| \le \varepsilon$ whenever $x' \in O_i$. Then $O := \bigcap_{i=1}^{n} O_i$ is $\mathcal{T}$-open and $x \in O \subset V(x;y_1,\ldots,y_n;\varepsilon)$, proving that $\mathcal{T}$ is finer than $\sigma(X,Y)$. □

By the symmetry of X and Y in the definition of a dual pair, it is obvious how $\sigma(Y,X)$ should be defined. Analogous properties hold for $\sigma(Y,X)$ (simply interchange X and Y everywhere).

Now let X be a l.c.s. with topology $\mathcal{T}$ and let us denote $(X,\mathcal{T})^*$ by X^*. We have seen above that $\langle X,X^*\rangle$ is a dual pair (with the canonical bilinear form $\langle x,x^*\rangle = x^*(x)$), so that $\sigma(X,X^*)$ and $\sigma(X^*,X)$ are defined. In particular we have now two topologies on X, namely $\mathcal{T}$ and $\sigma(X,X^*)$. By II both give rise to the same dual space, namely X^*, and by III $\sigma(X,X^*)$ is coarser than $\mathcal{T}$. In general $\sigma(X,X^*)$ is strictly coarser than $\mathcal{T}$. As an example, let us consider a normed linear space X (so that $\mathcal{T}$ is the norm topology).

<u>PROPOSITION 0.4</u>. *Let* X *be a normed linear space and* X^* *its dual. Then* $\sigma(X,X^*)$ *coincides with the norm topology iff* $\dim X < \infty$.

<u>PROOF</u>. We have already remarked that finite-dimensional topological vector spaces of a fixed dimension n are all isomorphic to $\mathbb{R}^n$, so that sufficiency

is clear. Suppose now that dim X = ∞ and let $V = V(0;x_1^*,\dots,x_n^*;\varepsilon)$ be an arbitrary basic 0-neighborhood for $\sigma(X,X^*)$. Then V contains the linear subspace $L := \bigcap_{i=1}^{n} \ker x_i^*$. Since dim X = ∞, $L \neq \{0\}$ and, therefore, unbounded (in norm). Thus the unit ball of X cannot contain V, proving that $\sigma(X,X^*)$ is strictly coarser than the norm topology. □

Let X be a vector space and let T_1 and T_2 be two locally convex topologies on X (i.e. (X,T_1) and (X,T_2) are l.c.s.). T_1 and T_2 are called *compatible* iff they yield the same dual spaces (i.e. $(X,T_1)^* = (X,T_2)^*$).

PROPOSITION 0.5. *Let* X *be a vector space and let* T_1 *and* T_2 *be compatible locally convex topologies on* X. *Then for any convex set* $A \subset X$ *(so in particular for every linear subspace of* X*) the* T_1*-closure of* A *coincides with the* T_2*-closure.*

PROOF. Let X^* denote the joint dual $(X,T_1)^* = (X,T_2)^*$ and let $A \subset X$ be convex. It obviously suffices to show that the $\sigma(X,X^*)$-closure equals the T_1-closure of A. Let us denote these by $\tilde{A}$ and $\bar{A}$, respectively. Since $\sigma(X,X^*)$ is coarser than T_1 (by III), $\tilde{A} \supset \bar{A}$. To prove the other inclusion, let $x \notin \bar{A}$ and let O be an open convex T_1-neighborhood of x such that $O \cap \bar{A} = \phi$. Then $\bar{A} - O$ is a T_1-open convex set (note that $\bar{A}$ is again convex) not containing 0. By the Hahn-Banach theorem (geometric form with M = {0}), there exists a T_1-closed hyperplane H (through 0) with $H \cap (\bar{A} - O) = \phi$. Hence there exists an $x^* \in X^*$ with $x^* > 0$ on $\bar{A} - O$. It follows, since O is open, that $\beta := \inf_{x' \in A} \langle x',x^*\rangle > \langle x,x^*\rangle$. Thus, choosing $0 < \varepsilon < \beta - \langle x,x^*\rangle$, the $\sigma(X,X^*)$-neighborhood $V(x;x^*;\varepsilon)$ is disjoint with A, proving that $x \notin \tilde{A}$. □

A similar argument proves the so called bipolar theorem. We need some definitions first. Let $\langle X,Y\rangle$ be a dual pair and let $A \subset X$ be arbitrary. Then the *polar (set)* A^0 of A is $A^0 := \{y \in Y: |\langle x,y\rangle| \le 1 \text{ for all } x \in A\}$. Similarly, interchanging X and Y, the polar $B^0 \subset X$ of a subset $B \subset Y$ is defined. In particular $A^{00} := (A^0)^0$ is a subset of X, whenever $A \subset X$. A^{00} is called the *bipolar (set)* of A with respect to the dual pair $\langle X,Y\rangle$. It is not difficult to show that polar sets are always absolutely convex, $\sigma(X,Y)$ (or $\sigma(Y,X)$)-closed and that $A \subset A^{00}$ for all $A \subset X$. The bipolar theorem asserts that A^{00} is (for any $A \subset X$) the $\sigma(X,Y)$-closed absolutely convex hull of A.

PROPOSITION 0.6 (bipolar theorem). *Let* $\langle X,Y\rangle$ *be a dual pair and let* $A \subset X$ *be arbitrary. Then* A^{00} *is the* $\sigma(X,Y)$*-closed absolutely convex hull of* A.

PROOF. We have already observed that A^{00} is $\sigma(X,Y)$-closed and absolutely convex. Let A_1 be any $\sigma(X,Y)$-closed absolutely convex subset of X containing A and let $x \in X \backslash A_1$ be arbitrary. By the Hahn-Banach theorem (cf. the proof of Proposition 0.5) there exists a $\sigma(X,Y)$-continuous linear form, i.e. an element of Y such that $\langle x,y\rangle > 1$ and $\langle x',y\rangle < 1$ for all $x' \in A_1$. Since A_1 is balanced, $|\langle x',y\rangle| < 1$ for all $x' \in A_1$, i.e. $y \in A_1^0 \subset A^0$. Thus $x \notin A^{00}$, since $\langle x,y\rangle > 1$. This shows that $A^{00} \subset A_1$ and therefore concludes the proof since A_1 was an arbitrary $\sigma(X,Y)$-closed absolutely convex set containing A. □

Let $\langle X,Y\rangle$ be a dual pair and let $V \subset Y$ be a linear subspace. Occasionally we shall have to consider the locally convex topology $\sigma(X,V)$. A $\sigma(X,V)$-neighborhood base for an element $x \in X$ is given by the sets $V(x;y_1,\ldots,y_n;\varepsilon)$ with $y_1,\ldots,y_n \in V$. $\sigma(X,V)$ is not Hausdorff in general. It is iff for every $x \in X$, $x \neq 0$, there exists a $y \in V$ such that $\langle x,y\rangle \neq 0$, as one immediately sees. Since $(Y,\sigma(Y,X))^* = X$, this condition is equivalent to V being $\sigma(Y,X)$-dense in Y, by the Hahn-Banach theorem.

We now turn to Banach spaces. If X is a Banach space, let X^* and $X^{**} = (X^*)^*$ denote its dual (or conjugate) and bidual, respectively. Elements of X, X^*, and X^{**} are written as x, x^*, x^{**}, respectively. We define a map π_X of X into X^{**} as follows (using the notation $\langle\cdot,\cdot\rangle$ for the canonical duality of the pairs $\langle X,X^*\rangle$ and $\langle X^*,X^{**}\rangle$):

$$(0.1) \qquad \langle x^*,\pi_X(x)\rangle = \langle x,x^*\rangle \qquad (x \in X, \quad x^* \in X^*).$$

π_X is clearly linear. We show that it is an isometry. Indeed, for any $x_0 \in X$ we have

$$\|\pi_X(x_0)\| = \sup_{\|x^*\|\le 1} |\langle x^*,\pi_X(x_0)\rangle| = \sup_{\|x^*\|\le 1} |\langle x_0,x^*\rangle| \le \|x_0\|.$$

On the other hand, by the Hahn-Banach theorem there exists an $x_0^* \in X^*$ satisfying $\langle x_0,x_0^*\rangle = \|x_0\|$ and $\|x_0^*\| = 1$, so that the last inequality is an equality.

π_X is called the canonical embedding of X into X^{**}. We often write π for π_X if no confusion is likely. Even more often we identify X with πX and simply regard X as a subspace of X^{**}.

DEFINITION 0.7. A Banach space X is called *reflexive* iff πX is surjective, i.e. $\pi_X X = X^{**}$.

COROLLARY 0.8. *A Banach space* X *is reflexive iff* $\sigma(X^*,X) = \sigma(X^*,X^{**})$.

PROOF. ⇒: immediate from (0.1).
⇐: We know that the locally convex topologies $\sigma(X^*,X)$ and $\sigma(X^*,X^{**})$ on X^* yield duals X and X^{**}, respectively. On the other hand, since $\sigma(X^*,X)$ and $\sigma(X^*,X^{**})$ are equal, so are their duals. Thus for every $x^{**} \in X^{**}$ there exists an $x \in X$ such that

$$\langle x^*,x^{**}\rangle = \langle x,x^*\rangle \quad \text{for all } x^* \in X^*.$$

Since $\langle x,x^*\rangle = \langle x^*,\pi_X(x)\rangle$ this means precisely that $\pi_X X = X^{**}$. □

For a Banach space X $\sigma(X,X^*)$ is called the *weak* (w) *topology* on X, while $\sigma(X^*,X)$ is called the *weak** (w^*) *topology* on X^*. Thus on a dual Banach space X^* we have, apart from the norm topology, two generally distinct topologies, the weak topology $\sigma(X^*,X^{**})$ and the weak* topology $\sigma(X^*,X)$, which should not be confused. Equality occurs precisely when X is reflexive, by Corollary 0.8.

We now want to characterize reflexive spaces by the weak compactness of their unit balls. The essential step is Alaoglu's theorem.

PROPOSITION 0.9 (Alaoglu's theorem). *The unit ball* B_{X^*} *of a dual Banach space* X^* *is weak***-compact.*

PROOF. For every $x \in X$ put $\mathbb{R}_x := \mathbb{R}$, with the usual topology. Consider the product $\prod_{x\in X} \mathbb{R}_x$ with the product topology and denote its elements by $\alpha = (\alpha_x)_{x\in X}$. We define a map

$$\phi\colon X^* \to \prod_{x\in X} \mathbb{R}_x$$

by

$$\phi(x^*) = (\langle x,x^*\rangle)_{x\in X} \qquad (x^* \in X^*).$$

Then ϕ is a homeomorphism (into) for the weak* topology $\sigma(X^*,X)$ on X^*. This is an immediate consequence of the definitions of $\sigma(X^*,X)$ and of the product topology. Hence it remains to be shown that $\phi(B_{X^*})$ is compact. Note first that $\phi(B_{X^*}) \subset \prod_{x\in X} [-\|x\|,\|x\|]$, since for every $x^* \in B_{X^*}$ and every $x \in X$, $|\langle x,x^*\rangle| \le \|x\|\|x^*\| \le \|x\|$. $\prod_{x\in X} [-\|x\|,\|x\|]$ being compact by the Tychonoff theorem, it suffices to prove that $\phi(B_{X^*})$ is closed in $\prod_{x\in X} \mathbb{R}_x$. Let $x,y \in X$ and $\lambda,\mu \in \mathbb{R}$ be arbitrary and consider the map $\alpha \to \alpha_{\lambda x+\mu y} - \lambda\alpha_x - \mu\alpha_y$ ($\alpha \in \prod_{x\in X} \mathbb{R}_x$). Clearly it vanishes on $\phi(B_{X^*})$ and therefore, by continuity, on

$\overline{\phi(B_{X^*})}$ as well. Also, since $\phi(B_{X^*}) \subset \prod_{x\in X} [-\|x\|,\|x\|]$ and the latter set is closed, we have $\overline{\phi(B_{X^*})} \subset \prod_{x\in X} [-\|x\|,\|x\|]$. Combining both facts we conclude that for any $\alpha = (\alpha_x)_{x\in X} \in \overline{\phi(B_{X^*})}$ the map $x \to \alpha_x$ $(x \in X)$ is a continuous linear form with norm ≤ 1, meaning that $\alpha \in \phi(B_{X^*})$. This proves that $\phi(B_{X^*})$ is closed. □

PROPOSITION 0.10. *Let* X *be a Banach space and let us identify* X *with the subspace* πX *of* X^{**}. *Then the* $\sigma(X^{**},X^*)$*-closure of* B_X *is* $B_{X^{**}}$. *In particular* X *is* $\sigma(X^{**},X^*)$*-dense in* X^{**}.

PROOF. Observe that B_{X^*} is the polar of B_X with respect to the dual pair $\langle X,X^*\rangle$ (or $\langle X^{**},X^*\rangle$ if we wish). Similarly $B_{X^{**}}$ is the polar of B_{X^*} with respect to $\langle X^{**},X^*\rangle$. Hence $B_{X^{**}}$ is the bipolar of B_X with respect to $\langle X^{**},X^*\rangle$. B_X being absolutely convex, the $\sigma(X^{**},X^*)$ density of B_X in $B_{X^{**}}$ is now an immediate consequence of the bipolar theorem. The last statement is obvious, since $X^{**} = \bigcup_{n\in\mathbb{N}} nB_{X^{**}}$. □

PROPOSITION 0.11. X *is reflexive iff* B_X *is weakly compact*.

PROOF. By Alaoglu's theorem $B_{X^{**}}$ is $\sigma(X^{**},X^*)$-compact. It is obvious, moreover, that $\sigma(X^{**},X^*)_{|X}$ (= the topology induced by $\sigma(X^{**},X^*)$ on X) coincides with $\sigma(X,X^*)$. Thus if $X = X^{**}$ then B_X (= $B_{X^{**}}$) is $\sigma(X,X^*)$-compact. Conversely, since B_X is $\sigma(X^{**},X^*)$-dense in $B_{X^{**}}$, $\sigma(X,X^*)$-compactness of B_X implies $B_{X^{**}} = B_X$, so $X^{**} = X$. □

Two Banach spaces X and Y are called (topologically) *isomorphic* (notation $X \simeq Y$) iff there exists a (topological) isomorphism (i.e. a linear homeomorphism) of X onto Y. If there exists a linear isometry of X onto Y then X and Y are called *isometric* (notation $X \cong Y$). Let T be an isomorphism of X onto Y. Then $\|T^{-1}\|^{-1} B_Y \subset TB_X \subset \|T\| B_Y$. Also (see below) T is a weak homeomorphism, i.e. a homeomorphism for the weak topologies $\sigma(X,X^*)$ and $\sigma(Y,Y^*)$. Hence Proposition 0.11 implies, since B_X and B_Y are weakly closed by Proposition 0.5,

COROLLARY 0.12. *Let* X *and* Y *be isomorphic Banach spaces. Then* X *is reflexive iff* Y *is reflexive*.

PROPOSITION 0.13. *Let* X *be a Banach space. Then* X *is reflexive iff* X^* *is reflexive*.

PROOF. Suppose that X^* is reflexive and assume for contradiction that $\pi_X X \subsetneq X^{**}$. By the Hahn-Banach theorem there exists an $x^{***} \in X^{***}$ such that $x^{***} \neq 0$ and $x^{***}_{|\pi_X X} = 0$. By assumption $X^{***} = \pi_{X^*}X^*$, so there exists an $x^* \in X^*$ with $\pi_{X^*}(x^*) = x^{***}$, i.e. $\langle x^{**},x^{***}\rangle = \langle x^*,x^{**}\rangle$ for all $x^{**} \in X^{**}$. In particular $\langle\pi_X(x),x^{***}\rangle = \langle x^*,\pi_X(x)\rangle = \langle x,x^*\rangle$ for all $x \in X$. Since $x^{***}_{|\pi_X X} = 0$, it follows that $x^* = 0$, contradicting $\pi_{X^*}(x^*) = x^{***} \neq 0$. Conversely, suppose that X is reflexive. By Alaoglu's theorem B_{X^*} is $\sigma(X^*,X)$-compact, and therefore $\sigma(X^*,X^{**})$-compact by Corollary 0.8. Now apply Proposition 0.11 to X^*. □

To conclude this discussion of the weak and weak* topologies, we prove two simple facts which will be of use later.

PROPOSITION 0.14. *Let X be a l.c.s. with dual X^* and let $Y \subset X$ be a linear subspace. Then $\sigma(Y,Y^*) = \sigma(X,X^*)_{|Y}$.*

PROOF. Observe that Y is a l.c.s. in its own right, so that its dual Y^* and $\sigma(Y,Y^*)$ are defined. The equality of $\sigma(Y,Y^*)$ and $\sigma(X,X^*)_{|Y}$ follows from $\{x^*_{|Y}: x^* \in X^*\} = Y^*$ and this last equality is an immediate consequence of the Hahn-Banach theorem. □

Unlike norm-topologies, weak and weak* topologies are in general not metrizable. We have, however,

PROPOSITION 0.15. *Let X be a separable Banach space. Then B_{X^*} with the w^* topology is metrizable.*

PROOF. Let us first recall the following simple topological fact: If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two compact Hausdorff topologies on a set T which are comparable (i.e. $\mathcal{T}_1$ finer than $\mathcal{T}_2$ or conversely), then $\mathcal{T}_1 = \mathcal{T}_2$. Now let $\{x_n\}$ be a dense sequence in the unit sphere $S_X := \{x \in X: \|x\| = 1\}$. We define a metric d on X^* as follows

$$d(x^*,y^*) = \sum_{n=1}^{\infty} 2^{-n} |\langle x_n,x^*-y^*\rangle| \qquad (x^*,y^* \in X^*).$$

We leave to the reader the easy proof that d is indeed a metric and show that the identity map $(B_{X^*},\sigma(X^*,X)) \xrightarrow{I} (B_{X^*},d)$ is continuous. Let $x_0^* \in B_{X^*}$ and $\varepsilon > 0$ be arbitrary. Choose $n_0 \in \mathbb{N}$ so that $\sum_{n=n_0+1}^{\infty} 2^{-n+1} < \frac{\varepsilon}{2}$ and put $V = V(x_0^*;x_1,\ldots,x_{n_0};\frac{\varepsilon}{2})$. Then for any $x^* \in V \cap B_{X^*}$ we have $d(x^*,x_0^*) =$

$\sum_{n=1}^{\infty} 2^{-n}|<x_n,x^*-x_0^*>| = \sum_{n=1}^{n_0} 2^{-n}|<x_n,x^*-x_0^*>| + \sum_{n=n_0+1}^{\infty} 2^{-n}|<x_n,x^*-x_0^*>| \leq \sum_{n=1}^{n_0} 2^{-n}\frac{\varepsilon}{2} + \sum_{n=n_0+1}^{\infty} 2^{-n}\cdot 2 < \varepsilon$. This proves the continuity of I. Since B_{X^*} is $\sigma(X^*,X)$-compact by Alaoglu's theorem, the observation made at the beginning of the proof shows that I is a homeomorphism. Thus $(B_{X^*},\sigma(X^*,X))$ is metrizable. □

Let X and Y be Banach spaces and $T: X \to Y$ a bounded linear operator. The *adjoint operator* $T^*: Y^* \to X^*$ is defined by the formula

$$<x,T^*y^*> = <Tx,y^*> \qquad (x \in X,\ y^* \in Y^*).$$

(Note that for every $y^* \in Y^*$ the map $x \to <Tx,y^*>$ is continuous and linear in x, so that there exists an element $T^*y^* \in X^*$ satisfying this formula.) T^* is clearly linear. Also T^* is bounded and $\|T^*\| = \|T\|$. Indeed,

$$\|T^*\| = \sup_{\|y^*\|\leq 1} \|T^*y^*\| = \sup_{\|x\|,\|y^*\|\leq 1} |<x,T^*y^*>| =$$

$$= \sup_{\|x\|,\|y^*\|\leq 1} |<Tx,y^*>| \leq \sup_{\|x\|,\|y^*\|\leq 1} \|T\|\,\|x\|\,\|y^*\| = \|T\|.$$

To prove the reverse inequality, choose $\varepsilon > 0$ arbitrarily, and $x_0 \in X$ with $\|x_0\| = 1$ so that $\|Tx_0\| \geq \|T\|-\varepsilon$. Then, using the Hahn-Banach theorem, choose $y_0^* \in Y^*$ so that $<Tx_0,y_0^*> = \|Tx_0\|$ and $\|y_0^*\| = 1$. It now follows that $\|T^*\| \geq \|T^*y_0^*\| \geq <x_0,T^*y_0^*> = <Tx_0,y_0^*> = \|Tx_0\| > \|T\|-\varepsilon$ and, therefore, $\|T^*\| \geq \|T\|$ since $\varepsilon > 0$ was arbitrary.

We list now some properties of bounded linear operators $T: X \to Y$ and their adjoints which will be used repeatedly without further reference.

I. <u>T is weakly continuous, i.e. continuous for the topologies $\sigma(X,X^*)$ and $\sigma(Y,Y^*)$.</u>

<u>PROOF</u>. Let $x \in X$ and a basic $\sigma(Y,Y^*)$-neighborhood $V := V(Tx;y_1^*,\ldots,y_n^*;\varepsilon)$ for Tx be given arbitrarily. Then $U := V(x;T^*y_1^*,\ldots,T^*y_n^*;\varepsilon)$ is a $\sigma(X,X^*)$-neighborhood of x satisfying $TU \subset V$. □

In particular, since $T^*: Y^* \to X^*$ is bounded, it is weakly continuous, i.e. continuous for $\sigma(Y^*,Y^{**})$ and $\sigma(X^*,X^{**})$. Also

II. <u>T^* is weak* continuous, i.e. continuous for $\sigma(Y^*,Y)$ and $\sigma(X^*,X)$.</u>

<u>PROOF</u>. $y^* \in Y^*$ and a basic w^*-neighborhood $V := V(T^*y^*;x_1,\ldots,x_n;\varepsilon)$ being

arbitrary, $U := V(y^*;Tx_1,\dots,Tx_n;\varepsilon)$ is a weak*-neighborhood of y^* satisfying $T^*U \subset V$. □

In particular $T^{**}: X^{**} \to Y^{**}$ is both w- and w*-continuous. Identifying X and Y with the subspaces $\pi_X X$ and $\pi_Y Y$ of X^{**} and Y^{**}, respectively, we have

III. $T^{**}|_X = T$ and T^{**} is the unique w*-continuous extension of $T: X \to Y$ to an operator from X^{**} into Y^{**}.

PROOF. Let $x \in X$ be given. Then for all $y^* \in Y^*$ we have $\langle y^*,T^{**}\pi_X(x)\rangle = \langle T^*y^*,\pi_X(x)\rangle = \langle x,T^*y^*\rangle = \langle Tx,y^*\rangle = \langle y^*,\pi_Y Tx\rangle$. Hence $T^{**}\pi_X = \pi_Y T$ and therefore, suppressing the identification maps π_X and π_Y, $T^{**}|_X = T$. As to the second statement in III, we know already that T^{**} is w*-continuous and $T^{**}|_X = T$, so that only uniqueness remains to be proved. But this is immediate since X is $\sigma(X^{**},X^*)$-dense in X^{**} (Proposition 0.10). □

Let $\langle X,Y\rangle$ be a dual pair. For linear subspaces $L \subset X$ the polar L^0 of L in Y is customarily denoted by $L^\perp$ and is called the *annihilator* of L. Since L is linear, we have

$$L^\perp = \{y \in Y: \langle x,y\rangle = 0 \text{ for all } x \in L\}.$$

Clearly $L^\perp$ is also a linear subspace (of Y) and (as a polar set) $\sigma(Y,X)$-closed.

Now let X be a Banach space and $L \subset X$ a linear subspace. Then (without further reference to dual pairs) $L^\perp$ denotes the annihilator of L in X^*. Thus $L^{\perp\perp} := (L^\perp)^\perp$ is a subspace of X^{**}. The annihilator of a subspace $M \subset X^*$ with respect to the dual pair $\langle X,X^*\rangle$ will be denoted by $M^\top$. Clearly $L^{\perp\top} = L^{\perp\perp} \cap X$ (X is identified, as usual, with $\pi_X X$). By the bipolar theorem $L^{\perp\top}$ is the $\sigma(X,X^*)$-closure of L, which equals the norm-closure (by Proposition 0.5). Similarly, $L^{\perp\perp}$ is the $\sigma(X^{**},X^*)$-closure of L.

Using these notations we now identify the duals of subspaces and quotients of Banach spaces.

PROPOSITION 0.16. *Let* Y *be a closed subspace of a Banach space* X. *Then*
(i) $Y^* \cong X^*/Y^\perp$,
(ii) $(X/Y)^* \cong Y^\perp$.

PROOF. (i): Let $I: Y \to X$ be the identity embedding. Then, for all $x^* \in X^*$

and $y \in Y$,

$$\langle y, I^* x^* \rangle = \langle Iy, x^* \rangle = \langle y, x^* \rangle,$$

so that $I^* x^* = x^*_{|Y}$. In other words, I^* is nothing but the restriction map. Clearly $\ker I^* = Y^{\perp}$, and by the Hahn-Banach theorem I^* is surjective. Thus I^* induces a linear bijection T from $X^*/Y^{\perp}$ onto Y^*. We claim that T is an isometry. Indeed, given any $y_0^* \in Y^*$ there exists by the Hahn-Banach theorem an $x_0^* \in X^*$ with $x^*_{0|Y} = y_0^*$ and $\|x_0^*\| = \|y_0^*\|$. Therefore we have

$$\|T^{-1} y_0^*\| = \inf_{x^*_{|Y} = y_0^*} \|x^*\| = \|x_0^*\| = \|y_0^*\|.$$

(ii): Let $Q: X \to X/Y$ be the quotient map and $Q^*: (X/Y)^* \to X^*$ its adjoint. It is immediate from the definition of the quotient norm that $Q(\text{int } B_X) = \text{int } B_{X/Y}$. Hence $\text{int } B_{X/Y} \subset QB_X \subset B_{X/Y}$. We claim that Q^* is an isometry (into). Indeed, for all $z^* \in (X/Y)^*$ we have

$$\|Q^* z^*\| = \sup_{\|x\| \le 1} |\langle x, Q^* z^* \rangle| = \sup_{\|x\| \le 1} |\langle Qx, z^* \rangle| =$$

$$= \sup_{z \in QB_X} |\langle z, z^* \rangle| = \sup_{z \in B_{X/Y}} |\langle z, z^* \rangle| = \|z^*\|.$$

It remains to be shown that $Q^*((X/Y)^*) = Y^{\perp}$. $Q^*((X/Y)^*) \subset Y^{\perp}$ is clear, since for all $z^* \in (X/Y)^*$ and $y \in Y$ we have $\langle y, Q^* z^* \rangle = \langle Qy, z^* \rangle = \langle 0, z^* \rangle = 0$. To prove the reverse inclusion, let $x^* \in Y^{\perp}$ be arbitrary. Define $z^* \in (X/Y)^*$ by $\langle Qx, z^* \rangle = \langle x, x^* \rangle$ $(x \in X)$. (z^* is well defined since $x^* \in Y^{\perp}$, and z^* is continuous). Then $Q^* z^* = x^*$, since for all $x \in X$ we have $\langle x, Q^* z^* \rangle = \langle Qx, z^* \rangle = \langle x, x^* \rangle$. □

As examples of Banach spaces we shall often use the classical sequence spaces c_0, and ℓ^p $(1 \le p \le \infty)$. We recall the definitions here. c_0 is the space of real sequences $x = \{\xi_n\}$ with $\lim_{n \to \infty} \xi_n = 0$ and with norm $\|x\| = \sup_{n \in \mathbb{N}} |\xi_n|$. For any $1 \le p < \infty$, ℓ^p is the space of real sequences $x = \{\xi_n\}$ with $\sum_{n=1}^{\infty} |\xi_n|^p < \infty$, and norm $\|x\| = (\sum_{n=1}^{\infty} |\xi_n|^p)^{1/p}$. Finally, ℓ^∞ is the space of all bounded real sequences $x = \{\xi_n\}$ with norm $\|x\| = \sup_{n \in \mathbb{N}} |\xi_n|$. The dual of c_0 can be identified with ℓ^1, that of ℓ^1 with ℓ^∞ and that of ℓ^p $(1 < p < \infty)$ with ℓ^q, where $\frac{1}{p} + \frac{1}{q} = 1$. With these identifications, the canonical duality becomes, in each case, $\langle \{\xi_n\}, \{\eta_n\} \rangle = \sum_{n=1}^{\infty} \xi_n \eta_n$. In particular

ℓ^p $(1 < p < \infty)$ is reflexive. For each $n \in \mathbb{N}$, ℓ_n^p $(1 \le p \le \infty)$ and $(c_0)_n$ denote the n-dimensional space $\mathbb{R}^n$ with the ℓ^p- and c_0-norm, respectively. For any compact topological space K, C(K) denotes the Banach space of all continuous real functions x on K with norm $\|x\| = \sup_{t \in K} |x(t)|$.

ℓ^∞ and C([0,1]) share an important property.

<u>PROPOSITION 0.17</u>. *Every separable Banach space* X *can be isometrically embedded in* ℓ^∞ *as well as in* C([0,1]).

<u>PROOF</u>. a) Let X be separable and $\{x_n\}$ a dense sequence in its unit sphere S_X. For each $n \in \mathbb{N}$ use the Hahn-Banach theorem to select an $x_n^* \in X^*$ with $\langle x_n, x_n^* \rangle = \|x_n^*\| = 1$. Now define the map $T: X \to \ell^\infty$ by $Tx = \{\langle x, x_n^* \rangle\}$ $(x \in X)$. Clearly T is linear. To show that T is an isometry, let $x \in X$ with $\|x\| = 1$ be given and let $\varepsilon > 0$ be arbitrary. Choose $n_0 \in \mathbb{N}$ so that $\|x - x_{n_0}\| < \varepsilon$. Then $\|Tx\| \ge |\langle x, x_{n_0}^* \rangle| \ge |\langle x_{n_0}, x_{n_0}^* \rangle| - |\langle x - x_{n_0}, x_{n_0}^* \rangle| > 1 - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $\|Tx\| \ge 1 = \|x\|$. Obviously $\|Tx\| \le \|x\|$ so $\|Tx\| = \|x\|$ for all $x \in X$.

b) Again let X be a separable Banach space. We know already that B_{X^*}, with the topology $\sigma(X^*, X)$ is a compact metric space (Propositions 0.9 and 0.15). Let us denote it by K. Then clearly the map $T: X \to C(K)$ defined by $(Tx)(x^*) = \langle x, x^* \rangle$ $(x \in X,\ x^* \in K)$ is a linear isometry. It therefore remains only to exhibit a linear isometry of C(K) into C([0,1]). It is a well known topological fact ([66], page 166) that every compact metric space, in particular K, is the continuous image of the Cantor discontinuum Δ. So let $\phi: \Delta \to K$ be a continuous surjection. Now define $R: C(K) \to C(\Delta)$ by $(Rf)(t) = f(\phi(t))$ $(t \in \Delta,\ f \in C(K))$. Clearly R is a linear isometry. Finally, viewing Δ as a subset of [0,1], every $f \in C(\Delta)$ can be extended to a function $Sf \in C([0,1])$ by defining Sf linearly in all components of $[0,1] \setminus \Delta$. It is simple to check that the so defined map $S: C(\Delta) \to C([0,1])$ is a linear isometry. Now SRT is the desired isometry of X into C([0,1]). □

Apart from taking subspaces and quotients there are other ways of forming new Banach spaces from given ones. One of these we now discuss.

<u>DEFINITION 0.18</u>. Let $\{X_n\}$ be a sequence of Banach spaces. Then, for any $1 \le p < \infty$, the ℓ^p-sum $(\sum_{n=1}^\infty \oplus X_n)_{\ell^p}$ is the Banach space consisting of all sequences $x = \{x_n\}$ such that $x_n \in X_n$ $(n = 1,2,\ldots)$ and $\sum_{n=1}^\infty \|x_n\|^p < \infty$, with the norm $\|x\| = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p}$. Similarly the c_0-sum $(\sum_{n=1}^\infty \oplus X_n)_{c_0}$ (resp. the ℓ^∞-sum $(\sum_{n=1}^\infty \oplus X_n)_{\ell^\infty}$) is defined as the Banach space of all sequences

$x = \{x_n\}$ with $x_n \in X_n$ $(n = 1,2,\dots)$ and $\lim_{n\to\infty} \|x_n\| = 0$ (resp. $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$), with the norm $\|x\| = \sup_{n\in\mathbb{N}} \|x_n\|$.

We leave it to the reader to verify that the objects defined above are indeed Banach spaces. The proofs are completely analogous to those for the case $X_n = \mathbb{R}$ $(n = 1,2,\dots)$, i.e. X one of the classical sequence spaces c_0, ℓ^p $(1 \le p \le \infty)$.

We now identify the duals of c_0- and ℓ^p-sums.

<u>PROPOSITION 0.19</u>. *Let* $\{X_n\}$ *be a sequence of Banach spaces and let* $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell^p}$ $(1 < p < \infty)$. *Then* $X^* \cong (\sum_{n=1}^{\infty} \oplus X_n^*)_{\ell^q}$, *with* $\frac{1}{p} + \frac{1}{q} = 1$. *Analogously,* $((\sum_{n=1}^{\infty} \oplus X_n)_{c_0})^* \cong (\sum_{n=1}^{\infty} \oplus X_n^*)_{\ell^1}$ *and* $((\sum_{n=1}^{\infty} \oplus X_n)_{\ell^1})^* \cong (\sum_{n=1}^{\infty} \oplus X_n^*)_{\ell^\infty}$.

<u>PROOF</u>. We carry out the proof for the case $1 < p < \infty$ and leave it to the reader to modify it so as to fit the other cases. For every $n \in \mathbb{N}$ let I_n denote the linear isometry of X_n onto the subspace $\{0\} \oplus \{0\} \oplus \dots \oplus \{0\} \oplus X_n \oplus \{0\} \dots$ of $X = (\sum_{n=1}^{\infty} \oplus X_n)_{\ell^p}$ defined by

$$x_n \xrightarrow{I_n} (0,\dots,0,x_n,0,\dots) \qquad (x_n \in X_n).$$

Then I_n^* is an isometry of $(I_n X_n)^*$ onto X_n^*. For every $x^* \in X^*$ and every $n \in \mathbb{N}$ write $x_n^* := I_n^*(x^*|_{I_n X_n})$ and define a map $T\colon X^* \to \prod_{n=1}^{\infty} X_n^*$ by $Tx^* = \{x_n^*\}$ $(x^* \in X^*)$. Clearly T is linear and we intend to show that T is an isometry onto $(\sum_{n=1}^{\infty} \oplus X_n^*)_{\ell^q}$, $\frac{1}{p} + \frac{1}{q} = 1$. First let us note that for every $x^* \in X^*$ and every $x = \{x_n\} \in X$ we have

$$(0.2)\qquad \begin{aligned} \langle x,x^*\rangle &= \langle \sum_{n=1}^{\infty} I_n x_n, x^*\rangle = \sum_{n=1}^{\infty} \langle I_n x_n, x^*|_{I_n X_n}\rangle = \\ &= \sum_{n=1}^{\infty} \langle x_n, I_n^*\left(x^*|_{I_n X_n}\right)\rangle = \sum_{n=1}^{\infty} \langle x_n, x_n^*\rangle. \end{aligned}$$

We claim that

$$(0.3)\qquad \left(\sum_{n=1}^{\infty} \|x_n^*\|^q\right)^{1/q} \le \|x^*\| \qquad \text{for every } x^* \in X^*,$$

implying, in particular, that

$$(0.4)\qquad TX^* \subset \left(\sum_{n=1}^{\infty} \oplus X_n^*\right)_{\ell^q}.$$

Fix $x^* \in X^*$. It suffices to show that for an arbitrary $k \in \mathbb{N}$ $(\sum_{n=1}^k \|x_n^*\|^q)^{1/q} \le \|x^*\|$. Putting $\lambda_n := \|x_n^*\|$ $(n = 1,\dots,k)$ and using the duality of ℓ_k^p and ℓ_k^q, choose $\mu_1,\dots,\mu_k \ge 0$ such that $(\sum_{n=1}^k \mu_n^p)^{1/p} = 1$ and $\sum_{n=1}^k \lambda_n\mu_n = (\sum_{n=1}^k \lambda_n^q)^{1/q}$. Now let $\varepsilon > 0$ be arbitrary and select for each $n \in \{1,\dots,k\}$ an $x_n \in X_n$ such that $\|x_n\| = \mu_n$ and $<x_n,x_n^*> \, > \|x_n^*\|\|x_n\| - \frac{\varepsilon}{k}$. Then $x := (x_1,\dots,x_k,0,0,\dots) \in X$ and $\|x\| = (\sum_{n=1}^k \mu_n^p)^{1/p} = 1$. Using (0.2) we obtain $\|x^*\| \ge <x,x^*> = \sum_{n=1}^k <x_n,x_n^*> \ge \sum_{n=1}^k (\|x_n^*\|\|x_n\| - \frac{\varepsilon}{k}) = (\sum_{n=1}^k \lambda_n\mu_n) - \varepsilon = (\sum_{n=1}^k \lambda_n^q)^{1/q} - \varepsilon = (\sum_{n=1}^k \|x_n^*\|^q)^{1/q} - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves (0.3) and therefore (0.4). On the other hand, by (0.2) and Hölder's inequality,

$$
\begin{aligned}
\|x^*\| &= \sup_{\|x\| = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p} \le 1} |<x,x^*>| \le \\
&\le \sup_{(\sum_{n=1}^\infty \|x_n\|^p)^{1/p} \le 1} \sum_{n=1}^\infty |<x_n,x_n^*>| \le \qquad (0.5)\\
&\le \sup_{(\sum_{n=1}^\infty \|x_n\|^p)^{1/p} \le 1} \sum_{n=1}^\infty \|x_n\|\|x_n^*\| \le \left(\sum_{n=1}^\infty \|x_n^*\|^q\right)^{1/q}.
\end{aligned}
$$

Together, (0.3) and (0.5) show that T is an isometry of X^* into $(\sum_{n=1}^\infty \oplus X_n^*)_{\ell^q}$. But T is also surjective, since every $\{x_n^*\} \in (\sum_{n=1}^\infty \oplus X_n^*)_{\ell^q}$ is the T-image of the element $x^* \in X^*$ defined by $<x,x^*> = \sum_{n=1}^\infty <x_n,x_n^*>$ $(x = \{x_n\} \in (\sum_{n=1} \oplus X_n)_{\ell^p})$. □

<u>REMARK 0.20</u>. If $1 < p < \infty$, then $X^{**} = ((\sum_{n=1}^\infty \oplus X_n)_{\ell^p})^{**} \cong (\sum_{n=1}^\infty \oplus X_n^{**})_{\ell^p}$ (apply Proposition 0.19 twice). Some reflection shows that under the identification of X^{**} with $(\sum_{n=1}^\infty \oplus X_n^{**})_{\ell^p}$, $\pi_X X$ corresponds to $(\sum_{n=1}^\infty \oplus \pi_{X_n} X_n)_{\ell^p}$. Hence $X = (\sum_{n=1}^\infty \oplus X_n)_{\ell^p}$ is reflexive iff each X_n is. Finally, let us consider the degenerate case of finitely many Banach spaces $X_1,\dots,X_k$. Then $(\sum_{n=1}^k \oplus X_n)_{\ell^p}$ $(1 \le p \le \infty)$ and $(\sum_{n=1}^k \oplus X_n)_{c_0}$ all coincide as vector spaces with the product X^k. Furthermore, all norms are equivalent and generate the product topology on X^k. Whenever we consider a product of finitely many Banach spaces without explicitly mentioning the norm, we shall have in mind any of the above equivalent norms.

We end this summary by recalling some miscellaneous results.

<u>PROPOSITION 0.21</u> (Banach-Steinhaus theorem). *Let X be a Banach space and Y a normed linear space. If $\{T_\alpha\}_{\alpha\in A}$ is a set of bounded linear operators*

from X *into* Y *such that* $\{T_\alpha x\}_{\alpha\in A}$ *is bounded in* Y *for every* $x \in X$, *then* $\sup_{\alpha\in A} \|T_\alpha\| < \infty$.

COROLLARY 0.22. *If* X *is a Banach space and* $A \subset X$, *then* A *is norm bounded if* $\{\langle x,x^*\rangle: x \in A\}$ *is bounded for each* $x^* \in X^*$.

PROOF. Apply Proposition 0.21 with X and Y replaced by X^* and $\mathbb{R}$ and to the subset $\pi A \subset X^{**}$. □

A linear operator T from a Banach space X into a Banach space Y is called *compact* iff $\overline{TB_X}$ is compact. Trivial examples are finite rank operators, i.e. linear operators with finite-dimensional range. The compact operators form a closed subset of the Banach space B(X,Y) of all bounded linear operators from X into Y, with the operator norm. In particular, limits of finite rank operators are compact. The converse does not hold in general. It does for most concrete Banach spaces.

Let X be a Banach space with dual X^* and let $\|\cdot\|_1$ be a norm on X^* equivalent to the given dual norm $\|\ \|$ on X^*. Then $\|\ \|_1$ is a dual norm (i.e. there exists a norm $\|\ \|_1$ on X such that $\|x^*\|_1 = \sup\{|\langle x,x^*\rangle|: \|x\|_1 \le 1\}$ for all $x^* \in X^*$) iff $B := \{x^* \in X^*: \|x^*\|_1 \le 1\}$ is w^*-closed. Indeed, we have observed earlier that a dual unit ball is a polar set and as such w^*-closed (even w^*-compact), so that the condition is necessary. Conversely, if B is w^*-closed then, by the bipolar theorem, $B = B^{00}$ (with respect to $\langle X,X^*\rangle$). Clearly the gauge of $B^0 \subset X$ is a norm whose dual is $\|\ \|_1$. Another way to express the condition that B is w^*-closed is to say that $\|\ \|_1$ is w^*-*lower semicontinuous*. By definition a real function f on a topological space T is lower semicontinuous iff $\{t \in T: f(t) \le a\}$ is closed for every $a \in \mathbb{R}$. Finite sums as well as arbitrary suprema of lower semicontinuous functions are again lower semicontinuous.

PROPOSITION 0.23 (closed graph theorem). *Let* X *and* Y *be Banach spaces and let* $T: X \to Y$ *be a closed linear operator (i.e. the graph* $\{(x,Tx): x \in X\}$ *of* T *is closed in* $X\times Y$*). Then* T *is bounded.*

A vector space X is said to be the *algebraic direct sum* of subspaces Y and Z, notation $X = Y \oplus Z$, iff $X = Y + Z$ and $Y \cap Z = \{0\}$. Suppose that X is a Banach space and that $X = Y \oplus Z$. Then by the closed graph theorem it follows that whenever Y and Z are closed, the projections from X onto Y (resp. Z) with kernel Z (resp. Y) are bounded (and conversely). Equivalently, this means that X is isomorphic to the product space $Y \times Z$ via the map $(y,z) \longrightarrow y+z$ $(y \in Y,\ z \in Z)$. In this case X is called the *topological direct sum* of Y and Z.

PROPOSITION 0.24 (open mapping theorem). *Let* X *and* Y *be Banach spaces and* $T\colon X \to Y$ *a bounded linear map onto* Y. *Then* T *is open (i.e.* TO *is open in* Y *for every open* $O \subset X$*) and therefore* T *induces an isomorphism from* $X/\ker T$ *onto* Y.

Our next goal is to prove the Krein-Milman theorem.

DEFINITION 0.25. Let X be a vector space and $C \subset X$. A subset $B \subset C$ is called *C-extremal* if $x,y \in B$ whenever $x,y \in C$ and $\lambda x + (1-\lambda)y \in B$ for some $0 < \lambda < 1$. If a one-point subset $\{x_0\} \subset C$ is C-extremal, then it is called an *extreme point* of C. Thus x_0 is an extreme point of C iff $x_0 = \lambda x + (1-\lambda)y$, $0 < \lambda < 1$, $x,y \in C$ imply $x_0 = x = y$.

Observe that if $A \subset B \subset C \subset X$ and A is B-extremal and B is C-extremal, then A is C-extremal.

LEMMA 0.26. *Let* X *be a l.c.s. and* C *a compact subset of* X. *Then* C *has an extreme point.*

PROOF. Consider the collection $\mathcal{B}$ of all non-empty compact C-extremal subsets of C, partially ordered by inclusion. Observe that $\mathcal{B} \neq \phi$, since trivially $C \in \mathcal{B}$. If $\{B_\alpha\}_{\alpha\in I}$ is a chain in $\mathcal{B}$ (i.e. a totally ordered subcollection of $\mathcal{B}$), then, by compactness, $B := \bigcup_{\alpha\in I} B_\alpha \neq \phi$. It is also readily verified that B is C-extremal and therefore is a lower bound for $\{B_\alpha\}_{\alpha\in I}$ in $\mathcal{B}$. Consequently by Zorn's lemma $\mathcal{B}$ has a minimal element, say B_0. It will now suffice to show that B_0 consists of one point. Suppose for contradiction that $x_1,x_2 \in B_0$, $x_1 \neq x_2$. Use the Hahn-Banach theorem to select an $x^* \in X^*$ such that $\langle x_1,x^*\rangle \neq \langle x_2,x^*\rangle$. Since B_0 is compact, x^* attains its infimum over B_0, say α. Thus $B_1 := \{x \in B_0 : \langle x,x^*\rangle = \alpha\}$ is non-empty, compact and also $B_1 \subsetneqq B_0$, since x_1 and x_2 do not both belong to B_1. Finally, B_1 is C-extremal, contradicting the minimality of B_0. Indeed, suppose that $x,y \in B_0$ and $\lambda x + (1-\lambda)y \in B_1$, for some $0 < \lambda < 1$. Then $\lambda\langle x,x^*\rangle + (1-\lambda)\langle y,x^*\rangle = \alpha$, implying that $\langle x,x^*\rangle = \langle y,x^*\rangle = \alpha$, i.e. $x,y \in B_1$. Thus B_1 is B_0-extremal. Since B_0 is C-extremal, so is B_1, by the observation immediately preceeding the lemma. □

PROPOSITION 0.27 (Krein-Milman theorem). *Let* X *be a l.c.s. and let* $B \subset C \subset X$ *with* C *compact and convex. Then the following are equivalent:*

(i) $\overline{co}\, B = C$,

(ii) $\text{ext}\, C \subset \bar{B}$.

($\overline{co}\, B$ denotes the closure of the convex hull of B and ext C the set of extreme points of C.) In particular, $C = \overline{co}\, \text{ext}\, C$ and $\overline{\text{ext}\, C}$ is the smallest closed subset B of C satisfying $\overline{co}\, B = C$.

PROOF. (ii) ⇒ (i): Suppose $\overline{co}\, B \subsetneq C$ and choose $x_0 \in C \setminus \overline{co}\, B$. By the Hahn-Banach theorem there exists an $x^* \in X^*$ such that $\langle x_0, x^* \rangle > \max_{x \in \overline{co}\, B} \langle x, x^* \rangle$. Put $\alpha := \max\{\langle x, x^* \rangle : x \in C\}$. As in the preceeding proof it follows that $A := \{x \in C: \langle x, x^* \rangle = \alpha\}$ is non-empty, compact and C-extremal. By the choice of x^*, $A \cap \overline{co}\, B = \phi$, so $A \cap \bar{B} = \phi$. By lemma 0.26 A has an extreme point, say x_1. Since A is C-extremal, x_1 is also an extreme point of C, while $x_1 \notin \bar{B}$. This contradicts (ii).

(i) ⇒ (ii): Let $x \in \text{ext}\, C$ be arbitrary and let V be any closed absolutely convex 0-neighborhood in X. The compactness of $\bar{B}$ implies the existence of finitely many $x_1, \ldots, x_n \in B$ such that $B \subset \bigcup_{i=1}^{n} (x_i + V)$. Put $K_i := \overline{co}((x_i+V) \cap B)$ $(i = 1, \ldots, n)$. Then $K_1, \ldots, K_n$ are convex and compact, and $B \subset \bigcup_{i=1}^{n} K_i \subset C$. Hence

$$(0.6) \qquad C = \overline{co}\, B = \overline{co}\, (\bigcup_{i=1}^{n} K_i) = co\Big(\bigcup_{i=1}^{n} K_i\Big).$$

(Observe that $co(\bigcup_{i=1}^{n} K_i)$ is compact as the image of the compact set $\{(\lambda_1, \ldots, \lambda_n, y_1, \ldots, y_n): y_i \in K_i,\ \lambda_i \geq 0\ (i = 1, \ldots, n),\ \sum_{i=1}^{n} \lambda_i = 1\} \subset \mathbb{R}^n \times K_1 \times \ldots \times K_n$ under the continuous map $(\lambda_1, \ldots, \lambda_n, y_1, \ldots, y_n) \to \lambda_1 y_1 + \ldots + \lambda_n y_n$). (0.6) implies that $x = \lambda_1 y_1 + \ldots + \lambda_n y_n$ for some $y_i \in K_i$, $\lambda_i \geq 0$ $(i = 1, \ldots, n)$, with $\sum_{i=1}^{n} \lambda_i = 1$. At least one λ_i is non-zero, say λ_{i_0}. If $\lambda_{i_0} \neq 1$ we write $x = \lambda_{i_0} y_{i_0} + (1-\lambda_{i_0})(\sum_{i \neq i_0} \frac{\lambda_i}{1-\lambda_{i_0}} y_i)$ and infer from $x \in \text{ext}\, C$ that $x = y_{i_0}$. The same conclusion holds if $\lambda_{i_0} = 1$. Hence, since $y_{i_0} \in K_{i_0}$ and V is closed and convex, $x \in K_{i_0} \subset x_{i_0} + V$. V being also balanced, we have $x_{i_0} \in x + V$ and therefore, since $x_{i_0} \in B$, $(x+V) \cap B \neq \phi$. Since the collection of all closed absolutely convex 0-neighborhoods forms a base, and V was arbitrarily chosen in this collection, it follows that $x \in \bar{B}$. Thus $\text{ext}\, C \subset \bar{B}$, since $x \in \text{ext}\, C$ was arbitrary. The last statements of the proposition are obvious consequences of the equivalence (i) ⟺ (ii). □

Finally we mention without proof (cf. [87]) a special case of the Krein-Šmulian theorem. We shall use it only twice.

<u>PROPOSITION 0.28</u> (Krein-Šmulian theorem). *Let* X *be a Banach space and let* V *be a linear subspace of* X^*. Then V *is* $\sigma(X^*,X)$*-closed if (and only if)* $B_{X^*} \cap V = B_V$ *is* $\sigma(X^*,X)$*-closed.*

CHAPTER I

1. THE EBERLEIN-ŠMULIAN THEOREM

After a preliminary discussion of the notions of compactness, sequential compactness, countable compactness and the implications between them, we come to the main result in this section, the famous Eberlein-Šmulian theorem. It states that in every topological space homeomorphic to a subset of a Banach space with its relative weak topology, the above three notions coincide.

DEFINITION 1.1. Let T be a Hausdorff topological space.

(a) T is *compact* iff any of the following equivalent conditions is satisfied:

- (i) each open cover of T has a finite subcover,
- (ii) each family of closed subsets of T with the finite intersection property has a non-empty intersection,
- (iii) each net in T has a limit (or cluster) point,
- (iv) each net in T has a convergent subnet.

(b) T is *countably compact* iff any of the following equivalent conditions holds:

- (i) each countable open cover of T has a finite subcover,
- (ii) each countable family of closed subsets of T with the finite intersection property has a non-empty intersection,
- (iii) each sequence in T has a limit point.

(c) T is *sequentially compact* iff

- (i) each sequence in T has a convergent subsequence.

For the proof of these equivalences we refer to [66].

REMARK 1.2. Evidently the following implications always hold:

$$\text{compact} \Rightarrow \text{countably compact}$$

and

$$\text{sequentially compact} \Rightarrow \text{countably compact}.$$

No other implications are true in general, as we shall presently see.

The next two lemmas deal with special cases in which some or all of these notions coincide.

LEMMA 1.3. *A Hausdorff space* T *satisfying the first axiom of countability is sequentially compact iff it is countably compact.*

PROOF. Assuming T to be countably compact, let $\{t_n\}$be any sequence in T. Then $\{t_n\}$ has a limit point t. If $\{U_n\}$ is a decreasing neighborhood base for t, then we may choose a subsequence $\{n_k\} \subset \mathbb{N}$ such that $t_{n_k} \in U_k$ $(k = 1,2,\ldots)$. Obviously $\lim_{k\to\infty} t_{n_k} = t$. Thus T is sequentially compact. □

LEMMA 1.4. *Let* T *be a metric space. Then*
T *compact* $\Leftrightarrow$ T *countably compact* $\Leftrightarrow$ T *sequentially compact.*

PROOF. Since a metric space satisfies the first axiom of countability, it suffices, by Remark 1.2 and Lemma 1.3, to show: T countably compact $\Rightarrow$ T compact. So let T be countably compact. We prove compactness in three steps.

(i) For every $\varepsilon > 0$ T has a finite ε-net (i.e. given $\varepsilon > 0$, there exist finitely many points $t_1,\ldots,t_n \in T$ so that $T = \bigcup_{i=1}^{n} U_\varepsilon(t_i)$, where $U_\varepsilon(t) = \{t' \in T\colon d(t,t') < \varepsilon\}$, d the metric of T). Indeed, if not, then for some $\varepsilon > 0$ there exists an infinite sequence $\{t_n\}$ in T with $d(t_i,t_j) \geq \varepsilon$ whenever $i \neq j$. Such a sequence can have no limit point.

(ii) For every open cover $\{O_\alpha\}_{\alpha\in A}$ of T there exists an $\varepsilon > 0$ such that for each $t \in T$ there is an $\alpha = \alpha(t) \in A$ satisfying $U_\varepsilon(t) \subset O_\alpha$. Indeed, if not, then for some open cover $\{O_\alpha\}_{\alpha\in A}$ of T there exists for every $n \in \mathbb{N}$ a $t_n \in T$ such that $U_{1/n}(t_n) \not\subset O_\alpha$ for all $\alpha \in A$. Let t be a limit point of $\{t_n\}$ and suppose that $t \in O_{\alpha_0}$. Let $\varepsilon > 0$ satisfy $U_\varepsilon(t) \subset O_{\alpha_0}$. By the definition of a limit point there is an $n_0 > \frac{2}{\varepsilon}$ with $t_{n_0} \in U_{\varepsilon/2}(t)$. Thus $U_{1/n_0}(t_{n_0}) \subset U_{\varepsilon/2}(t_{n_0}) \subset U_\varepsilon(t) \subset O_{\alpha_0}$, a contradiction.

(iii) Finally, to show compactness, let $\{O_\alpha\}_{\alpha\in A}$ be any open cover of T and let us take $\varepsilon > 0$ as in (ii). By (i) there exists an ε-net $\{t_1,\ldots,t_n\}$ for T. By the choice of ε, for each $i \in \{1,\ldots,n\}$ there exists an O_{α_i} with $U_\varepsilon(t_i) \subset O_{\alpha_i}$. Hence $\{O_{\alpha_i}\}_{i=1}^{n}$ is a finite subcover of $\{O_\alpha\}_{\alpha\in A}$. □

We now give two examples showing that the two implications in Remark 1.2 are the only ones that hold in general.

EXAMPLE 1. Let A be an uncountable set and put $T := [0,1]^A$, with the product topology. Elements of T are denoted by $t = \{t_\alpha\}_{\alpha\in A}$. Let us also consider the subset $S := \{t = \{t_\alpha\}_{\alpha\in A} \in T\colon \{\alpha \in A\colon t_\alpha \neq 0\} \text{ countable}\}$. Then $S \subsetneq \bar{S} = T$. Thus, since T is Hausdorff (and, of course, compact by the Tychonoff theorem), S is not compact. However, S is sequentially compact and therefore also countably compact. Indeed, let $\{t^{(n)}\}_{n=1}^{\infty}$ be a sequence in S, $t^{(n)} = \{t_\alpha^{(n)}\}_{\alpha\in A}$ $(n = 1,2,\dots)$. Putting $A_n := \{\alpha \in A\colon t_\alpha^{(n)} \neq 0\}$ $(n = 1,2,\dots)$, it is clear that $t_\alpha^{(n)} = 0$ for all n and for all α outside the countable set $B := \bigcup_{n=1}^{\infty} A_n$. A simple diagonal argument now yields a subsequence $\{t^{(n_k)}\}$ of $\{t^{(n)}\}$ converging to some $t = \{t_\alpha\} \in T$. Obviously $t_\alpha = 0$ for $\alpha \notin B$, so that $t \in S$.

EXAMPLE 2. Let T be as in Example 1, with the special choice $A = [0,1]$. Again T (which we regard as the space of all functions $f\colon [0,1] \to [0,1]$) is compact, hence countably compact, but not sequentially compact. To see this, consider the functions $f_n \in T$ $(n = 1,2,\dots)$ defined as follows: for fixed n, f_n increases linearly from 0 to 1 in each interval $[\frac{k}{10^n}, \frac{k+1}{10^n})$, $k = 0,\dots,10^n-1$, and $f_n(1) = 0$. In other words, if $x = 0,x_1x_2\dots x_n\dots$ is the decimal expansion of $x \in [0,1)$ (allowing no infinite repetition of 9's) then

(1.1) $\quad f_n(x)$ has decimal expansion $0,x_{n+1}x_{n+2}\dots\dots$.

We claim that the sequence $\{f_n\}$ has no convergent subsequence. Indeed, suppose for contradiction that $\{f_{n_k}\}$ converges. This means that $\{f_{n_k}(x)\}$ converges for every $x \in [0,1]$. Now let x be any point in $[0,1]$ whose decimal expansion satisfies e.g. the relations

$$x_{n_k+1} = \begin{cases} 1 & \text{if } k \text{ even} \\ 3 & \text{if } k \text{ odd.} \end{cases}$$

Then, by (1.1), $|f_{n_{k+1}}(x) - f_{n_k}(x)| \geq \frac{1}{10}$ for all $k \in \mathbb{N}$, contradicting the convergence of $\{f_{n_k}\}$.

REMARK 1.5. Since T in Example 2 is countably compact, the sequence $\{f_n\}$ does have a limit point (infinitely many, in fact). It is therefore a mistake to think of a limit point of a sequence as a limit of a convergent subsequence, unless the first axiom of countability holds (cf. the proof of Lemma 1.3). A further word of caution seems appropriate here: Since every sequence is in particular a net, the sequence $\{f_n\}$ in Example 2, by

Definition 1.1(a)(iv), has a convergent subnet. Thus a subnet of a sequence need not be a subsequence.

Besides for metric spaces the equivalences of Lemma 1.4 also hold for every topological space which is homeomorphic to a subset of a Banach space with its relative weak topology. One should note that in general the weak topology of a Banach space is not metrizable and does not satisfy the first axiom of countability, not even when restricted to weakly compact sets (consider e.g. the weakly compact unit ball of a non-separable Hilbert space). This makes the above mentioned fact all the more surprising. In order to be able to deal also with non-closed sets, we give the following

DEFINITION 1.6. Let T be a Hausdorff topological space and let $A \subset T$. Then

(i) A is *relatively compact* iff $\bar{A}$ is compact,

(ii) A is *relatively countably compact* (in T) iff every sequence in A has a limit point in T,

(iii) A is *relatively sequentially compact* (in T) iff every sequence in A has a subsequence that converges in T.

REMARK 1.7. One should not confuse "A relatively sequentially (countably) compact" with "$\bar{A}$ sequentially (countably) compact", since these may be different statements. E.g. the set S in Example 1 is (relatively) sequentially compact, but $\bar{S} = T$ is not sequentially compact (cf. Example 2).

We are now ready for the statement of the Eberlein-Šmulian theorem.

THEOREM 1.8. *Let* A *be a subset of a Banach space* X. *Then the following are equivalent.*

(a) A *is relatively* $\sigma(X,X^*)$*-compact,*

(b) A *is relatively sequentially* $\sigma(X,X^*)$*-compact,*

(c) A *is relatively countably* $\sigma(X,X^*)$*-compact.*

Moreover, these equivalences also hold with "relatively" deleted everywhere.

Our proof will follow R. WHITLEY's ([105]). To keep it relatively short, we isolate some technicalities first in the next lemmas. X will denote a Banach space throughout the rest of this section.

LEMMA 1.9. *Let* X *be separable. Then* X^* *contains a countable set* $\{x_n^*\}_{n=1}^{\infty}$ *which is total (i.e.* $\langle x,x_n^*\rangle = 0$ *for all* $n \in \mathbb{N}$ *implies* $x = 0$*).*

PROOF. Let $\{x_n\}$ be a dense sequence in the unit sphere S_X of X. Using the Hahn-Banach theorem, select for each $n \in \mathbb{N}$ an $x_n^* \in X^*$ with $\langle x_n, x_n^*\rangle = \|x_n^*\| = 1$. We claim that $\{x_n^*\}$ is total. For, if $x \in S_X$ is arbitrary, there exists an $n \in \mathbb{N}$ with $\|x-x_n\| < 1$. Then $|\langle x,x_n^*\rangle| \geq |\langle x_n,x_n^*\rangle| - |\langle x-x_n,x_n^*\rangle| \geq 1 - \|x_n^*\|\|x-x_n\| > 0$, proving that $\{x_n^*\}$ is total. □

LEMMA 1.10. *Let F be a finite-dimensional linear subspace of* X^* *and let* $\varepsilon > 0$ *be arbitrary. Then there exist finitely many points* $x_1,\ldots,x_n \in S_X$ *such that*

$$\max_{i=1,\ldots,n} |\langle x_i,x^*\rangle| > (1-\varepsilon)\|x^*\| \qquad \textit{for all } x^* \in F.$$

PROOF. Since S_F is compact it has a finite $\frac{\varepsilon}{2}$-net $\{x_1^*,\ldots,x_n^*\}$. Now choose, for each $i \in \{1,\ldots,n\}$, an $x_i \in S_X$ such that $|\langle x_i,x_i^*\rangle| > 1 - \frac{\varepsilon}{2}$. These $x_1,\ldots,x_n$ will do, as we now show. Let $x^* \in F$ be given. We may assume without loss of generality that $\|x^*\| = 1$. Let $i_0 \in \{1,\ldots,n\}$ be selected so that $\|x^*-x_{i_0}^*\| < \frac{\varepsilon}{2}$. Then $|\langle x_{i_0},x^*\rangle| \geq |\langle x_{i_0},x_{i_0}^*\rangle| - |\langle x_{i_0},x^*-x_{i_0}^*\rangle| > 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1-\varepsilon$. □

LEMMA 1.11. *Let* X^* *contain a countable total subset (this is the case whenever X is separable, by Lemma* 1.9, *but also e.g. for* $X = \ell^\infty$*) and let* $A \subset X$ *be* $\sigma(X,X^*)$*-compact. Then* $\sigma(X,X^*)_{|A}$ *is metrizable.*

PROOF. Observe first that for every $x^* \in X^*$ the set $\{\langle x,x^*\rangle : x \in A\}$ is compact, whence bounded in $\mathbb{R}$, so that A is norm bounded by the Banach-Steinhaus theorem (see Corollary 0.22). Let $\{x_n^*\} \subset X^*$ be a total sequence. We may assume that $\|x_n^*\| = 1$ $(n = 1,2,\ldots)$. From this point on the proof runs as that of Proposition 0.15:

$$d(x,y) := \sum_{n=1}^{\infty} 2^{-n}|\langle x-y,x_n^*\rangle| \qquad (x,y \in A)$$

defines a metric on A and the identity map $(A,\sigma(X,X^*)) \to (A,d)$ is continuous, whence a homeomorphism. □

PROOF OF THEOREM 1.8. (a) ⇒ (b): Let $A \subset X$ be relatively $\sigma(X,X^*)$-compact and let $\{x_n\}$ be any sequence in A. Since $Y := [x_n]$ is separable and $\sigma(X,X^*)$-closed (Proposition 0.5) and $\sigma(Y,Y^*) = \sigma(X,X^*)_{|Y}$ (Proposition 0.14), Lemma 1.11 implies that the $\sigma(X,X^*)$-compact set $\overline{\{x_n : n \in \mathbb{N}\}}^{\sigma(X,X^*)}$ is metrizable. Therefore, by Lemma 1.4, $\{x_n\}$ has a weakly convergent subsequence.

(b) ⇒ (c): trivial.

(c) ⇒ (a): Let $A \subset X$ be relatively countably $\sigma(X,X^*)$-compact. We observe first that A is norm bounded, by the Banach-Steinhaus theorem, so that $\overline{\pi A}^{\sigma(X^{**},X^*)}$ is $\sigma(X^{**},X^*)$-compact by Alaoglu's theorem. Throughout the rest of the proof we suppress the π-notation and identify X with πX. Since $\sigma(X^{**},X^*)|_X = \sigma(X,X^*)$ the proof will therefore be finished once we have shown that $\bar{A}^{\sigma(X^{**},X^*)} \subset X$. Let $x_0^{**} \in \bar{A}^{\sigma(X^{**},X^*)}$. In order to show that $x_0^{**} \in X$, we shall construct a sequence $\{x_n\}$ in A such a way that its only possible $\sigma(X^{**},X^*)$-limit point is x_0^{**}. Since by assumption $\{x_n\}$ has a $\sigma(X,X^*)$-limit point, i.e. a $\sigma(X^{**},X^*)$-limit point which belongs to X, we must have $x_0^{**} \in X$.

We choose inductively an increasing sequence $1 = n_1 < n_2 < \ldots < n_k < \ldots$ in $\mathbb{N}$, a sequence $\{x_k\}_{k=1}^{\infty}$ in A and a sequence $\{x_n^*\}_{n=1}^{\infty}$ in S_{X^*} so that the following conditions are satisfied:

(i) $\max\{|\langle x_n^*, x_0^{**}-x_k\rangle| : n=1,\ldots,n_k\} < \frac{1}{k}$ $(k=1,2,\ldots)$;

(ii) $\max\{|\langle x_n^*, x^{**}\rangle| : n_k < n \le n_{k+1}\} > \frac{1}{2}\|x^{**}\|$ for all $x^{**} \in sp\{x_0^{**},x_1,\ldots,x_k\}$ $(k=1,2,\ldots)$.

This is done as follows. We begin by choosing $x_1^* \in S_{X^*}$ arbitrarily. Since $x_0^{**} \in \bar{A}^{\sigma(X^{**},X^*)}$, there exists an $x_1 \in A$ such that $|\langle x_1^*, x_0^{**}-x_1\rangle| < 1$. Using Lemma 1.10 we can now choose finitely many elements $x_2^*,\ldots,x_{n_2}^* \in S_{X^*}$ such that $\max\{|\langle x_n^*,x^{**}\rangle| : 1 < n \le n_2\} > \frac{1}{2}\|x^{**}\|$ for all x^{**} in the finite-dimensional space $sp\{x_0^{**},x_1\}$. Thus with these choices (i) and (ii) hold for k = 1. Suppose now that $1 = n_1 < n_2 < \ldots < n_{k_0+1}$, $x_1,\ldots,x_{k_0} \in A$ and $x_1^*,\ldots,x_{n_{k_0+1}}^* \in S_{X^*}$ have been chosen so that (i) and (ii) hold for $k = 1,\ldots,k_0$. Since $x_0^{**} \in \bar{A}^{\sigma(X^{**},X^*)}$, there exists an $x_{k_0+1} \in A$ such that $\max\{|\langle x_n^*,x_0^{**}-x_{k_0+1}\rangle| : n=1,\ldots,n_{k_0+1}\} < \frac{1}{k_0+1}$. By Lemma 1.10 we can now select finitely many $x_{n_{k_0+1}}^*,\ldots,x_{n_{k_0+2}}^* \in S_{X^*}$ so that $\max\{|\langle x_n^*,x^{**}\rangle| : n_{k_0+1} < n \le n_{k_0+2}\} > \frac{1}{2}\|x^{**}\|$ for all $x^{**} \in sp\{x_0^{**},x_1,\ldots,x_{k_0+1}\}$. This completes the inductive definition of the three sequences.

By assumption $\{x_k\}$ has a $\sigma(X,X^*)$-limit point $x_0 \in X$. To show that $x_0^{**} = x_0$, let us observe that, clearly, $x_0 \in \overline{sp}\{x_1,x_2,\ldots\}$, so that $x_0^{**}-x_0 \in \overline{sp}\{x_0^{**},x_1,x_2,\ldots\}$. In particular, (ii) then implies that

$$\sup\{|\langle x_n^*,x_0^{**}-x_0\rangle| : n \in \mathbb{N}\} \ge \frac{1}{2}\|x_0^{**}-x_0\|.$$

Hence $x_0^{**} = x_0$ will be proved once we have shown that

$$\langle x_m^*,x_0^{**}-x_0\rangle = 0 \quad \text{for all } m \in \mathbb{N},$$

or, equivalently, that

$$|\langle x_m^*, x_0^{**} - x_0 \rangle| < \frac{2}{k} \text{ whenever } n_k \geq m.$$

To prove this last inequality, fix m and k so that $n_k \geq m$. By the triangle inequality we have for all $n \in \mathbb{N}$

$$(1.2) \qquad |\langle x_m^*, x_0^{**} - x_0 \rangle| \leq |\langle x_m^*, x_0^{**} - x_n \rangle| + |\langle x_n - x_0, x_m^* \rangle|.$$

By (i) $|\langle x_m^*, x_0^{**} - x_n \rangle| < \frac{1}{n} < \frac{1}{k}$ for all $n > k$, since $m \leq n_k < n_n$ for all these n. Also, x_0 being a weak limit point of $\{x_k\}$, there exists an $n > k$ such that $|\langle x_n - x_0, x_m^* \rangle| < \frac{1}{k}$. Substituting this x_n in (1.2) yields $|\langle x_m^*, x_0^{**} - x_0 \rangle| < \frac{2}{k}$ and therefore the desired conclusion $x_0^{**} = x_0$.

A trivial modification of the above proof shows that one may delete the word "relatively" everywhere in the statement of Theorem 1.8. Indeed, one only needs to observe that if A is countably $\sigma(X,X^*)$-compact, the above proof of (c) ⇒ (a) shows that $\bar{A}^{\sigma(X^{**},X^*)} = A$, so that A is $\sigma(X,X^*)$-compact. □

<u>REMARK 1.12</u>. It follows in particular that $\bar{A}^{\sigma(X,X^*)}$ is sequentially (countably) $\sigma(X,X^*)$-compact if (and only if) A is relatively sequentially (countably) $\sigma(X,X^*)$-compact, a fact which is not true for general topological spaces (cf. Remark 1.7).

To conclude this section, we note the following useful consequence of the Eberlein-Šmulian theorem.

<u>COROLLARY 1.13</u>. X is reflexive iff all its separable closed linear subspaces are reflexive.

<u>PROOF</u>. For the "if" part, observe that the assumption implies sequential $\sigma(X,X^*)$-compactness and therefore, by the Eberlein-Šmulian theorem, $\sigma(X,X^*)$-compactness of B_X, since every sequence in B_X is contained in a closed (and therefore weakly closed, by Proposition 0.5) separable linear subspace Y, and since $\sigma(Y,Y^*) = \sigma(X,X^*)_{|Y}$ (Proposition 0.14). The "only if" part is a trivial consequence of Propositions 0.5, 0.11 and 0.14. □

<u>REMARK 1.14</u>. Corollary 1.13 can be strengthened. It suffices, for reflexivity of X, that all its (separable) closed linear subspaces <u>with a basis</u> are reflexive. This result was first proved by A. PELCZYNSKI ([80]) in answer to a question raised by I. SINGER ([95]). We shall encounter a different proof of this fact in Section 10 (cf. Remark 10.2).

NOTES. Examples 1 and 2 are taken from [69]. Theorem 1.8 goes back to [32] and [99]. Our presentation follows [105]. Another proof of the Eberlein-Šmulian theorem is due to A. PELCZYNSKI ([81]). Many interesting and deep results have been proved in recent years about Eberlein compacts, i.e. topological spaces homeomorphic to weakly compact subsets of Banach spaces with their relative weak topology. To mention a few: if T is an Eberlein compact and $S \subset T$, then the closure of S coincides with its sequential closure; any separable $S \subset T$ is metrizable, and T has a dense set of G_δ-points. Also a purely topological characterization of Eberlein compacts is known ([1],[18],[86]).

2. SUBREFLEXIVITY

If X is reflexive, then by the weak compactness of B_X there exists for every $x^* \in X^*$ an $x \in B_X$ such that $\langle x,x^*\rangle = \|x^*\|$. In other words every $x^* \in X^*$ attains its supremum $\|x^*\|$ on B_X. It is a deep result, due to R.C. JAMES ([53]) and to be proved in a later section, that this property characterizes reflexive spaces. Thus for a non-relexive X the elements $x^* \in X^*$ attaining their sup on B_X form a proper subset of X^*. E.g. let X be the non-reflexive space c_0 and let us identify X^* with ℓ^1. An element $x^* = \{\eta_n\} \in \ell^1$ takes the value $\|x^*\| = \sum_{n=1}^{\infty} |\eta_n|$ on B_{c_0} iff $\eta_n = 0$ for all but finitely many n, since $x = \{\xi_n\} \in c_0$ with $\|x\| = \sup_{n\in\mathbb{N}} |\xi_n| \le 1$ satisfies $\langle x,x^*\rangle = \sum_{n=1}^{\infty} \eta_n\xi_n = \sum_{n=1}^{\infty} |\eta_n|$ iff $\xi_n = \frac{|\eta_n|}{\eta_n}$ whenever $\eta_n \neq 0$. In this example the (proper) subset of X^* of all x^* attaining their sup on B_X is (norm) dense in X^*. Hence c_0 is subreflexive in the sense of the following

<u>DEFINITION 2.1</u>. A normed linear space X is called *subreflexive* iff $\{x^* \in X^* \mid \exists x \in B_X: \langle x,x^*\rangle = \|x^*\|\}$ is dense in X^*.

The main result in this section is

<u>THEOREM 2.2</u>. *Every Banach space is subreflexive.*

For the proof we need the lemma below. It is a precise statement of the following intuitively obvious fact: two elements $x^*,y^* \in S_{X^*}$ whose kernels are almost parallel must be either almost equal or almost antipodal.

<u>LEMMA 2.3</u>. *Let* X *be a normed linear space,* $\varepsilon > 0$ *and* $x^*,y^* \in S_{X^*}$. *Suppose that* $|\langle x,y^*\rangle| \le \frac{\varepsilon}{2}$ *whenever* $x \in (\ker x^*) \cap B_X$, *i.e.* $\|y^*|_{\ker x^*}\| \le \frac{\varepsilon}{2}$. *Then either* $\|x^*+y^*\| \le \varepsilon$ *or* $\|x^*-y^*\| \le \varepsilon$.

<u>PROOF</u>. Let z^* be a Hahn-Banach extension of $y^*|_{\ker x^*}$ to X, so $y^*-z^* = 0$ on $\ker x^*$ and $\|z^*\| \le \frac{\varepsilon}{2}$. Then $y^*-z^* = \alpha x^*$ for some $\alpha \in \mathbb{R}$. Furthermore, $|1-|\alpha|| = |\|y^*\| - \|y^*-z^*\|| \le \|z^*\| \le \frac{\varepsilon}{2}$. Thus we have, if $\alpha \ge 0$,

$\|x^*-y^*\| = \|(1-\alpha)x^*-z^*\| \le |1-\alpha| + \|z^*\| = \big|1-|\alpha|\big| + \|z^*\| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,$

and, if $\alpha \le 0$,

$\|x^*+y^*\| = \|(1+\alpha)x^*+z^*\| \le |1+\alpha| + \|z^*\| = \big|1-|\alpha|\big| + \|z^*\| \le \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$ □

Theorem 2.2 is a special case of the following slightly more general result.

THEOREM 2.4. *Let A be a bounded closed convex set in a Banach space X. Then* $\{x^* \in X^* \mid \exists x \in A: \langle x,x^*\rangle = \sup\{\langle y,x^*\rangle : y \in A\}$ *is dense in* X^*.

PROOF. It clearly suffices to show that $M := \{x^* \in S_{X^*} \mid \exists x \in A: \langle x,x^*\rangle = \sup\{\langle y,x^*\rangle : y \in A\}\}$ is dense in S_{X^*}. So let $x^* \in S_{X^*}$ and $0 < \varepsilon < 1$ be arbitrary. Our aim is to produce a $y^* \in M$ satisfying $\|x^*-y^*\| \le \varepsilon$. Fix $K > 1 + \frac{2}{\varepsilon}$. We partially order A by means of the closed convex cone

(2.1) $\qquad C(x^*,K) := \{x \in X: \|x\| \le K\langle x,x^*\rangle\},$

i.e.

(2.2) $\qquad x \le y \iff \|y-x\| \le K\{\langle y,x^*\rangle - \langle x,x^*\rangle\}.$

a) We first use Zorn's lemma to show that A has a maximal element. To this end, let W be a chain in A. (2.2) implies that $\{\langle x,x^*\rangle : x \in W\}$ is a (bounded) monotone net in $\mathbb{R}$ and therefore converges to its supremum. Again using (2.2), we infer that W is a Cauchy net (in norm) and thus, by the completeness of X, W converges to some $y \in A$. Since x^* and the norm are both continuous, it follows that y is an upper bound for W in A. Now Zorn's lemma yields the existence of a maximal element $z \in A$. In geometric terms, this means that

(2.3) $\qquad A \cap \{z + C(x^*,K)\} = \{z\}.$

Since int $\underline{C(x^*,K) \supset}$ $\{x \in X: \|x\| < K\langle x,x^*\rangle\} \ne \phi$, it is easily checked that $C(x^*,K) = \operatorname{int} C(x^*,K)$. Thus the Hahn-Banach theorem (applied to the open convex set $A - \operatorname{int} C(x^*,K)$ and the linear manifold $\{z\}$) enables us to separate $z + C(x^*,K)$ from A by an element $y^* \in S_{X^*}$ satisfying

(2.4) $\qquad \sup\{\langle x,y^*\rangle : x \in A\} = \langle z,y^*\rangle = \inf\{\langle x,y^*\rangle : x \in z + C(x^*,K)\}.$

It follows from (2.4) that $y^* \in M$ and that

(2.5) $\qquad \langle x,y^*\rangle \ge 0 \quad$ for all $x \in C(x^*,K)$.

b) The next step is to show that our choice of $K > 1+\frac{2}{\varepsilon}$ makes the cone $C(x^*,K)$ large enough so that (2.5) implies $\|y^*|_{\ker x^*}\| \le \frac{\varepsilon}{2}$. First let us choose $x \in S_X$ so that $1 + \frac{2}{\varepsilon} < K\langle x,x^*\rangle$. For every $v \in (\ker x^*) \cap B_X$ we have $\|x \pm \frac{2}{\varepsilon}v\| \le 1+\frac{2}{\varepsilon} < K\langle x,x^*\rangle = K\langle x \pm \frac{2}{\varepsilon}v,x^*\rangle$, so that $x \pm \frac{2}{\varepsilon}v \in C(x^*,K)$. Hence (2.5) yields $\langle x \pm \frac{2}{\varepsilon}v,y^*\rangle \ge 0$, i.e. $|\langle v,y^*\rangle| \le \frac{\varepsilon}{2}\langle x,y^*\rangle \le \frac{\varepsilon}{2}$. This proves that $\|y^*|_{\ker x^*}\| \le \frac{\varepsilon}{2}$. Lemma 2.3 now shows that either $\|x^*+y^*\| \le \varepsilon$ or $\|x^*-y^*\| \le \varepsilon$.

c) It remains to rule out the case $\|x^*+y^*\| \le \varepsilon$. To this end, note that since ε and K^{-1} were chosen < 1, we can pick a $w \in S_X$ satisfying $\langle x,w^*\rangle >$ $> \max(\varepsilon,K^{-1})$. In particular $\|w\| \le K\langle w,x^*\rangle$, i.e. $w \in C(x^*,K)$. Thus, by (2.5), $\langle w,y^*\rangle \ge 0$ and therefore $\varepsilon < \langle w,x^*\rangle \le \langle w,x^*+y^*\rangle \le \|x^*+y^*\|$. This completes the proof. □

The remainder of this section is devoted to two examples. The first one shows that completeness of X is essential for subreflexivity and the second, in conjunction with the first, that subreflexivity is not an isomorphic invariant.

<u>EXAMPLE 1</u>. Let X be ℓ^2, with the following norm (which is equivalent to the ℓ^2-norm):

$$\|x\| = |\xi_1| + \left(\sum_{n=2}^{\infty} |\xi_n|^2\right)^{\frac{1}{2}} \qquad (x = \{\xi_n\}).$$

Then, by Proposition 0.19, X^* is ℓ^2 with the norm

$$\|x^*\| = \max\left(|\eta_1|, \left(\sum_{n=2}^{\infty} |\eta_n|^2\right)^{\frac{1}{2}}\right) \qquad (x^* = \{\eta_n\}).$$

Let $\{e_n\}$ (respectively, $\{e_n^*\}$) denote the sequence of unit vectors in X (respectively X^*) and let I be a Hamel base for X containing $\{e_n\}$. The linear form f on X is defined by $f(x) = 1$ for $x \in I$. Clearly f is not bounded, so $Y := \ker f$ is dense in X. Thus Y^* can be identified with X^*. We claim that Y is not subreflexive. In fact $\{y^* \in S_{Y^*}: \|y^*-e_1^*\| < 1\}$ contains no element that attains its sup on B_Y. To see this, note that for an element $y^* = \{\eta_n\} \in Y^* = X^*$ it follows from $\|y^*\| = \|\{\eta_n\}\| = 1$ and $\|y^*-e_1^*\| < 1$ that $\eta_1 = 1$ and $(\sum_{n=2}^{\infty} |\eta_n|^2)^{\frac{1}{2}} < 1$. Hence $\langle x,y^*\rangle < 1$ for all $x \in B_X\backslash\{e_1\}$. It remains to observe that $e_1 \notin Y$.

<u>EXAMPLE 2</u>. Since the norm $\|x\| = |\xi_1| + (\sum_{n=2}^{\infty}|\xi_n|^2)^{\frac{1}{2}}$ $(x = \{\xi_n\} \in \ell^2)$ is equivalent to the Hilbertnorm, the non-subreflexive space Y above is isomorphic to a dense subspace of the Hilbertspace ℓ^2. We claim that any

dense subspace X of ℓ^2 is subreflexive. Thus subreflexivity is not an isomorphic invariant. X^* can again be identified with ℓ^2. Let $x^* \in S_{X^*}$ be arbitrary. Then there exists a sequence $\{x_n\} \subset S_X$ such that $\lim_{n\to\infty} \langle x_n, x^* \rangle = 1$. By the Hahn-Banach theorem, for each $n \in \mathbb{N}$ there exists an $x_n^* \in S_{X^*}$ such that $\langle x_n, x_n^* \rangle = 1$. Then

$$2 \geq \|x^*+x_n^*\| \geq |\langle x_n, x^* \rangle + \langle x_n, x_n^* \rangle| = |\langle x_n, x^* \rangle + 1| \to 2, \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} \|x^*+x_n^*\| = 2$. In view of the parallelogram law

$$\|x^*+x_n^*\|^2 + \|x^*-x_n^*\|^2 = 2\|x^*\|^2 + 2\|x_n^*\|^2,$$

it follows that $\lim_{n\to\infty} \|x^*-x_n^*\| = 0$. This proves our claim, since each x_n^* attains its sup on B_X.

<u>NOTES</u>. The main result in this section is due to E. BISHOP & R.R. PHELPS ([10]). The present proof owes some streamlining to J. DIESTEL ([27]). The examples 1 and 2 appear in [84]. Given two Banach spaces X and Y, let B(X,Y) denote the Banach space of all bounded linear operators from X into Y, and let P(X,Y) be the subset of B(X,Y) consisting of all the operators which attain their norm on B_X, i.e. all those T for which there exists an $x \in B_X$ with $\|Tx\| = \|T\|$. Theorem 2.3 says that P(X,Y) is dense in B(X,Y) if dim Y = 1. One may ask more generally for which X and Y P(X,Y) is dense in B(X,Y). For a discussion of this question see [70].

3. LOCAL REFLEXIVITY

In view of the result of R.C. James quoted at the beginning of section 2, it is natural to regard subreflexivity as a weakened version of the property of reflexivity. By Theorem 2.2 it holds for every Banach space. In this section we consider another weakening of reflexivity which holds for every Banach space. The result we have in mind is known as the principle of local reflexivity. It will be of great use to us further on in these notes. It is valid for arbitrary Banach spaces X and says roughly that, although X^{**} may be much larger than X, it has essentially the same finite-dimensional subspaces. More precisely:

THEOREM 3.1 (principle of local reflexivity).
Let X *be a Banach space (identified with* πX*) and let* G *be a finite-dimensional subspace of* X^{**}*. Then, given* $\varepsilon > 0$ *there exists a* 1-1 *linear operator* $T: G \to X$ *satisfying*

(i) $Tx = x$ *for all* $x \in X \cap G$,

(ii) $\|T\|\,\|T^{-1}\| < 1+\varepsilon$.

If, in addition, a finite-dimensional subspace $F \subset X^*$ *is given,* T *may be chosen so that also*

(iii) $\langle Tx^{**}, x^*\rangle = \langle x^*, x^{**}\rangle$ *for all* $x^{**} \in G$ *and all* $x^* \in F$.

REMARK 3.2. Assuming that $X \cap G \neq \{0\}$, (i) and (ii) imply that $(1+\varepsilon)^{-1}\|x\| \leq \|Tx\| < (1+\varepsilon)\|x\|$ for all $x \in G$. Hence the theorem says that every finite-dimensional subspace $G \subset X^{**}$ can be "almost isometrically" mapped (by T) into X, subject to the further conditions that $T_{|X\cap G} = id_{X\cap G}$ and that Tx^{**} and x^{**} coincide on F for every $x^{**} \in G$.

We shall first show that a T satisfying (i) and (ii) exists and deal with (iii) later. We begin with a few lemmas.

LEMMA 3.3. *Let* $C_0, C_1, \ldots, C_n$ *be* n+1 *open convex sets in a normed linear* *space* X, *with* $\bigcap_{i=0}^{n} C_i = \phi$. *Then there exists a continuous linear map* $T: X \to \mathbb{R}^n$ *satisfying* $\bigcap_{i=0}^{n} TC_i = \phi$.

PROOF. We set $X_i = X$ $(i = 1,\dots,n)$ and consider in the product space $\prod_{i=1}^{n} X_i$ the convex open subset

$$C := \{x_0-x_1,x_0-x_2,\dots,x_0-x_n): x_i \in C_i,\ i = 0,1,\dots,n\}.$$

The assumption $\bigcap_{i=0}^{n} C_i = \phi$ translates into $(0,0,\dots,0) \notin C$. Hence by the Hahn-Banach theorem there exists an element $x^* = (x_1^*,\dots,x_n^*) \in (\prod_{i=1}^{n} X_i)^* = \prod_{i=1}^{n} X_i^*$ such that $x^* > 0$ on C, i.e. $\sum_{i=1}^{n} \langle x_0-x_i,x_i^*\rangle > 0$ for all choices of $x_i \in C_i$, $i = 0,1,\dots,n$. Now we define a continuous linear $T: X \to \mathbb{R}^n$ by $Tx = (\langle x,x_1^*\rangle,\dots,\langle x,x_n^*\rangle)$ $(x \in X)$ and claim that $\bigcap_{i=0}^{n} TC_i = \phi$. Indeed, suppose $\alpha = (\alpha_1,\dots,\alpha_n) \in \bigcap_{i=0}^{n} TC_i$. Then there exist elements $x_i \in C_i$ $(i = 0,\dots,n)$ such that $(\alpha_1,\dots,\alpha_n) = (\langle x_i,x_1^*\rangle,\dots,\langle x_i,x_n^*\rangle)$ for all $i = 0,\dots,n$. In particular $\langle x_0,x_i^*\rangle = \langle x_i,x_i^*\rangle$ $(i = 1,\dots,n)$, so $\sum_{i=1}^{n} \langle x_0-x_i,x_i^*\rangle = 0$, a contradiction. □

LEMMA 3.4. *Let* C *be an open convex set in a normed linear space* X. *Then* $C = \text{int}\,\bar{C}$.

PROOF. The inclusion $C \subset \text{int}\,\bar{C}$ being trivial, let $x \in \text{int}\,\bar{C}$ be given. If $y \in C$, $y \neq x$ is arbitrary, then x is an interior point of a segment $[y,z]$ with $z \in \bar{C}$, since $\text{int}\,\bar{C}$ is open. Choose a sequence $\{z_n\} \subset C$ with $\lim_{n\to\infty} z_n = z$. Then, for some $0 < \lambda < 1$,

$$x = \lambda y + (1-\lambda)z = \lambda\Big(y + \frac{1-\lambda}{\lambda}(z-z_n)\Big) + (1-\lambda)z_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\lim_{n\to\infty} z_n = z$ the assumption that C is open implies that $y + \frac{1-\lambda}{\lambda}(z-z_n) \in C$ for large n. Hence $x \in C$, since C is convex. □

LEMMA 3.5. *Let* Y *be a Banach space,* $K \subset Y$ *an open convex subset,* Z *a finite-dimensional Banach space and* $S: Y \to Z$ *a bounded linear surjection. Then* $S^{**}(\underline{K}) = S(K)$, *where* $\underline{K}$ *denotes the (norm) interior of the* $\sigma(Y^{**},Y^*)$*-closure of* K.

PROOF. Since S^{**} is the unique w^*-continuous extension of S to a map from Y^{**} to $Z^{**} = Z$, and the w^*- and the norm topologies coincide on the finite-dimensional space Z, we have

$$S^{**}(\underline{K}) \subset S^{**}(\bar{K}^{\sigma(Y^{**},Y^*)}) \subset \overline{S^{**}(K)} = \overline{S(K)}. \tag{3.1}$$

Also, it follows from the equality $\bar{B}_Y^{\sigma(Y^{**},Y^*)} = B_{Y^{**}}$ (Proposition 0.10) that the interior of any set in Y is contained in the interior of its w^*-closure in Y^{**}. In particular $K \subset \underline{K}$, since K is open. Therefore

(3.2) $\qquad S(K) \subset S^{**}(\underline{K})$.

Combining (3.1) and (3.2) yields

(3.3) $\qquad S(K) \subset S^{**}(\underline{K}) \subset \overline{S(K)}$.

Now by the open mapping theorem (Proposition 0.24) S and S^{**} are open, so that both S(K) and $S^{**}(\underline{K})$ are open. Applying Lemma 3.4 to the open convex set S(K) yields $S(K) = \operatorname{int} \overline{S(K)}$. Hence, since $S^{**}(\underline{K})$ is open, we have, by (3.3), $S(K) = S^{**}(\underline{K})$. □

We now come to the final preparatory lemma. It states the intuitively obvious fact that the unit sphere of a finite-dimensional space can be approximated by a polyhedron.

<u>LEMMA 3.6</u>. *Let G be a finite-dimensional Banach space,* $0 < \delta < 1$, *and let* $\{y_1,\dots,y_m\}$ *be a* δ *-net for* S_G. *Then the set* $C := \{\sum_{i=1}^m \alpha_i y_i : \alpha_i \in \mathbb{R}\ (i = 1,\dots,m),\ \sum_{i=1}^m |\alpha_i| \le 1\}$ *satisfies*

$$(1-\delta)B_G \subset C \subset B_G.$$

<u>PROOF</u>. The second inclusion is obvious. Suppose the first is false. Then there exists a $y_0 \notin C$ satisfying $\|y_0\| \le 1-\delta$. C is absolutely convex and compact as the image of the absolutely convex and compact set $\{(\alpha_1,\dots,\alpha_m) : \sum_{i=1}^m |\alpha_i| \le 1\} \subset \mathbb{R}^m$ under the continuous linear map $(\alpha_1,\dots,\alpha_m) \to \sum_{i=1}^m \alpha_i y_i$. Therefore the Hahn-Banach theorem yields a $y^* \in G^*$ satisfying

(3.4) $\qquad \langle y_0,y^*\rangle > \max_{y\in C} |\langle y,y^*\rangle|$.

We may assume $\|y^*\| = 1$. Then

(3.5) $\qquad \langle y_0,y^*\rangle \le \|y^*\|\|y_0\| \le 1-\delta$.

On the other hand, for every $y \in S_G$ there exists an $i_0 \in \{1,\dots,m\}$ with $\|y-y_{i_0}\| < \delta$, so that

$$|\langle y,y^*\rangle| \le |\langle y_{i_0},y^*\rangle| + |\langle y-y_{i_0},y^*\rangle| \le |\langle y_{i_0},y^*\rangle| + \|y^*\|\|y-y_{i_0}\| < |\langle y_{i_0},y^*\rangle| + \delta.$$

Therefore

$$(3.6)\qquad \|y^*\| = \max_{\|y\|=1} |\langle y,y^*\rangle| \le \max_{i=1,\dots,m} |\langle y_i,y^*\rangle| + \delta .$$

Thus (3.4) and (3.6) yield the following inequality contradicting (3.5):

$$\langle y_0,y^*\rangle > \max_{y\in C}|\langle y,y^*\rangle| \ge \max_{i=1,\dots,m} |\langle y_i,y^*\rangle| \ge \|y^*\|-\delta = 1-\delta . \qquad \square$$

PROOF OF THEOREM 3.1.

Step 1. Let $\varepsilon > 0$ be arbitrary. Choose $0 < \delta < 1$ such that

$$(3.7)\qquad 0 < \frac{1+\delta}{(1-\delta)(1-2\delta) - \delta(1+\delta)} < 1+\varepsilon .$$

Compactness allows us to choose a δ-net $\{y_1^{**},\dots,y_m^{**}\}$ for S_G. We now claim that a linear map $T: G \to X$ will satisfy (ii) whenever

$$(3.8)\qquad 1-2\delta < \|Ty_i^{**}\| < 1+\delta \qquad (i = 1,\dots,m).$$

Indeed, suppose that (3.8) holds. By Lemma 3.6 any $y^{**} \in S_G$ can be written as $y^{**} = \sum_{i=1}^m \alpha_i y_i^{**}$ with $\sum_{i=1}^m |\alpha_i| \le \frac{1}{1-\delta}$. Hence, by (3.8),

$$\|Ty^{**}\| \le \sum_{i=1}^m |\alpha_i|\ \|Ty_i^{**}\| \le (1+\delta) \sum_{i=1}^m |\alpha_i| \le \frac{1+\delta}{1-\delta},$$

so

$$(3.9)\qquad \|T\| \le \frac{1+\delta}{1-\delta}.$$

On the other hand there exists an $i_0 \in \{1,\dots,m\}$ such that $\|y^{**}-y_{i_0}^{**}\| < \delta$, so that it follows from (3.8) and (3.9) that

$$\|Ty^{**}\| \ge \|Ty_{i_0}^{**}\| - \|T(y^{**}-y_{i_0}^{**})\| > (1-2\delta) - \|T\|\,\|y^{**}-y_{i_0}^{**}\| \ge$$

$$\ge (1-2\delta) - \left(\frac{1+\delta}{1-\delta}\right)\delta = \frac{(1-\delta)(1-2\delta) - \delta(1+\delta)}{1-\delta}.$$

Thus

$$(3.10)\qquad \|T^{-1}\| \le \frac{1-\delta}{(1-\delta)(1-2\delta) - \delta(1+\delta)}.$$

Together with (3.9) and (3.7), this yields (ii).

Step 2. We now make an attempt to define $T: G \to X$ so that it satisfies (i) and (3.8). Let $k = \dim G/G\cap X$ and let $z_1^{**},\dots,z_k^{**} \in G$ be such that $G = \mathrm{sp}\big(\{z_1^{**},\dots,z_k^{**}\} \cup (G\cap X)\big)$. Then the elements $y_1^{**},\dots,y_m^{**}$ (chosen in step 1) can be uniquely written as

$$(3.11)\qquad y_i^{**} = \sum_{j=1}^{k} \alpha_{ij} z_j^{**} + b_i \qquad (i = 1,\dots,m),$$

with $\alpha_{ij} \in \mathbb{R}$ and $b_i \in G\cap X$. Clearly a linear map $T: G \to X$ satisfying (i) can be fully defined by arbitrarily prescribing the images $Tz_1^{**},\dots,Tz_k^{**} \in X$. We denote these (yet to be determined) images by $x_1,\dots,x_k$, respectively. Thus, the problem of satisfying (3.8) is to pick $x_1,\dots,x_k \in X$ in such a way that the inequalities

$$(3.12)\qquad 1-2\delta < (\|Ty_i^{**}\| =)\ \|\sum_{j=1}^{k} \alpha_{ij}x_j + b_i\| < 1+\delta \qquad (i = 1,\dots,m)$$

hold. For this let us first select $x_1^*,\dots,x_m^* \in S_{X^*}$ so that

$$(3.13)\qquad |\langle x_i^*, y_i^{**}\rangle > 1-\delta \qquad (i = 1,\dots,m)$$

and then consider in X^k ($= \prod_{j=1}^{k} X_j$ with $X_j = X$, $j = 1,\dots,k$) the sets

$$C_i := \{(x_1,\dots,x_k): |\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij}x_j - \sum_{j=1}^{k} \alpha_{ij}z_j^{**}\rangle| < \delta\} \ (i = 1,\dots,m).$$

Suppose for the moment that $\bigcap_{i=1}^{m} (K_i\cap C_i) \neq \phi$. Let $(x_1,\dots,x_k) \in \bigcap_{i=1}^{m} (K_i\cap C_i)$. We claim that with this choice of $x_1,\dots,x_k$ the inequalities (3.12) hold. This is obvious for the inequalities on the right. For those on the left, observe that, for all $i \in \{1,\dots,m\}$

$$\|\sum_{j=1}^{k} \alpha_{ij}x_j + b_i\| \ge |\langle \sum_{j=1}^{k} \alpha_{ij}x_j + b_i, x_i^*\rangle| \ge |\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij} z_j^{**} + b_i\rangle|$$

$$- |\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij}x_j - \sum_{j=1}^{k} \alpha_{ij}z_j^{**}\rangle| =$$

$$|\langle x_i^*, y_i^{**}\rangle| - |\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij}x_j - \sum_{j=1}^{k} \alpha_{ij}z_j^{**}\rangle| > (1-\delta) - \delta = 1 - 2\delta.$$

<u>Step 3</u>. It remains to be shown that $\bigcap_{i=1}^{m} (K_i\cap C_i) \neq \phi$. Suppose for contradiction that $\bigcap_{i=1}^{m} (K_i\cap C_i) = \phi$. Since the K_i and C_i are open and convex we may use Lemma 3.3 to obtain a continuous linear map $S: X^k \to \mathbb{R}^{2m-1}$ such that

(3.14) $\bigcap_{i=1}^{m} \big(S(K_i) \cap S(C_i)\big) = \phi$.

We now introduce the following auxiliary subsets of $(X^{**})^k$.

$$K_i^{**} := \{(x_1^{**},\dots,x_k^{**}) : \| \sum_{j=1}^{k} \alpha_{ij} x_j^{**} + b_i \| < 1+\delta\} \qquad (i = 1,\dots,m),$$

and

$$C_i^{**} := \{x_1^{**},\dots,x_k^{**}) : |\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij} x_j^{**} - \sum_{j=1}^{k} \alpha_{ij} z_j^{**}\rangle| < \delta\} \quad (i = 1,\dots,m).$$

At this point the reader should realize (cf. Remark 0.20) that

(3.15) $\begin{cases} (X^k)^{**} \text{ and } (X^{**})^k \text{ may be identified and that the } w^*\text{-topology} \\ \big((X^k)^{**},(X^k)^*\big) \text{ corresponds under this identification with} \\ \text{the product of the } w^*\text{-topologies } \sigma(X^{**},X^*). \end{cases}$

Let us suppose for the moment that we have proved the inclusions

(3.16) $K_i^{**} \subset \underline{K}_i$ and $C_i^{**} \subset \underline{C}_i$ $(i = 1,\dots,m)$,

where $\underline{K}_i$ and $\underline{C}_i$ are defined as in Lemma 3.5, but this time with respect to the Banach space X^k. Since obviously $K_i \subset K_i^{**}$, $C_i \subset C_i^{**}$ $(i = 1,\dots,m)$ and since by Lemma 3.5 $S(K_i) = S^{**}(\underline{K}_i)$, $S(C_i) = S^{**}(\underline{C}_i)$ $(i = 1,\dots,m)$, it follows, using (3.16), that

(3.17) $S^{**}(K_i^{**}) = S(K_i)$, $S^{**}(C_i^{**}) = S(C_i)$ $(i = 1,\dots,m)$.

Thus, by (3.14) $\bigcap_{i=1}^{m} \big(S^{**}(K_i^{**}) \cap S^{**}(C_i^{**})\big) = \phi$, and therefore in particular $\bigcap_{i=1}^{m} (K_i^{**} \cap C_i^{**}) = \phi$, contradicting the obvious fact that $(z_1^{**},\dots,z_k^{**}) \in$ $\in \bigcap_{i=1}^{m} (K_i^{**} \cap C_i^{**})$.

<u>Step 4</u>. We now prove (3.16). Since all C_i^{**} and K_i^{**} are open it suffices to show that C_i^{**} (respectively K_i^{**}) is contained in the w^*-closure of C_i (respectively K_i) $(i = 1,\dots,m)$. First we deal with the C_i. Fix i and $(x_1^{**},\dots,x_k^{**}) \in C_i^{**}$. For every $j \in \{1,\dots,k\}$ let $\{x_{j,\alpha}\}_{\alpha\in A}$ be a net in X that w^*-converges to x_j^{**}. (Observe that we are allowed to use the same index set A for all j, namely a $\sigma(X^{**},X^*)$-0-neighborhood base.) Then the net $\{(x_{1,\alpha},\dots,x_{k,\alpha})\}_{\alpha\in A}$ in X^k w^*-converges to $(x_1^{**},\dots,x_k^{**})$, by (3.15). It is now obvious that the inequality

$$|\langle x_i^*, \sum_{j=1}^{k} \alpha_{ij} x_{j,\alpha} - \sum_{j=1}^{k} \alpha_{ij} z_j^{**}\rangle| < \delta$$

holds, i.e. $(x_{1,\alpha},\dots,x_{k,\alpha}) \in C_i$, for sufficiently large α.

In the case of the K_i we must manoeuvre a bit more carefully: the argument analogous to the one given above for C_i yields only that $\sum_{j=1}^{k} \alpha_{ij}x_{j,\alpha} + b_i \to \sum_{j=1}^{k} \alpha_{ij}x_j^{**} + b_i$ weak*, but allows no conclusion at all about $\|\sum_{j=1}^{k} \alpha_{ij}x_{j,\alpha} + b_i\|$. Fix i and $(x_1^{**},\dots,x_k^{**}) \in K_i^{**}$. Let us observe first that the case $\alpha_{i,j} = 0$ for $j = 1,\dots,k$ is trivial, so that we may assume $\alpha_{i,j_0} \neq 0$ for some $j_0 \in \{1,\dots,k\}$. We now choose nets $\{x_\alpha\}_{\alpha\in A}$ and $\{x_{j,\alpha}\}_{\alpha\in A}$ $(j \in \{1,\dots,k\}\setminus\{j_0\})$ in X such that

$$w^* - \lim_\alpha x_\alpha = \sum_{j=1}^{k} \alpha_{ij}x_j^{**} + b_i$$

$$w^* - \lim_\alpha x_{j,\alpha} = x_j^{**} \quad (j \in \{1,\dots,k\}\setminus\{j_0\})$$

and such that, in addition, $\|x_\alpha\| < 1+\delta$ for all $\alpha \in A$ (use Proposition 0.10). Then we define, for $\alpha \in A$,

$$x_{j_0,\alpha} = \frac{1}{\alpha_{ij_0}}\Big(x_\alpha - \big(\sum_{j\neq j_0} \alpha_{ij}x_{j,\alpha} + b_i\big)\Big).$$

It follows that also $w^* - \lim_\alpha x_{j_0,\alpha} = x_{j_0}^{**}$ and therefore, by (3.15), $w^*\text{-}\lim_\alpha (x_{1,\alpha},\dots,x_{k,\alpha}) = (x_1^{**},\dots,x_k^{**})$. Moreover, $\|\sum_{j=1}^{k}\alpha_{ij}x_{j,\alpha} + b_i\| = \|x_\alpha\| < 1+\delta$, so that $(x_{1,\alpha},\dots,x_{k,\alpha}) \in K_i$ for all $\alpha \in A$. Hence $K_i^{**} \subset w^*\text{-cl}\,K_i$ and therefore $K_i^{**} \subset \underline{K}_i$. This completes the proof of the first part of Theorem 3.1, concerning (i) and (ii).

At this point we interrupt the proof for two more lemmas.

<u>LEMMA 3.7</u>. *Let X be a Banach space, $K \subset X$ open and convex, and let $L \subset X$ be a closed linear subspace of X with codim $L < \infty$. Then $\underline{K} \cap \tilde{L} \subset \widetilde{K\cap L}$, where $\underline{K}$ is defined as in Lemma 3.5 and $\sim$ denotes $\sigma(X^{**},X^*)$-closure.*

<u>PROOF</u>. Fix $x_0^{**} \in \underline{K} \cap \tilde{L}$. We shall construct a w^*-convergent net in $K \cap L$ with limit x_0^{**}. For the index set of this net we take the collection $\mathcal{H}$ of all finite-dimensional subspaces of X^* containing $L^\perp$, ordered by inclusion. Given $H \in \mathcal{H}$, let us consider the restriction map $x^{**} \to x^{**}|_H$ of X^{**} onto H^*. Since it is clearly w^*-continuous and dim $H^* < \infty$, it is legitimate to write it as T^{**}, where $T: X \to H^*$ is the restriction of T^{**} to X. By Lemma 3.5 $T^{**}(\underline{K}) = T(K)$, so in particular there exists an element $x_H \in K$ with $T^{**}(x_0^{**}) = T(x_H)$. The last equality means that x_0^{**} coincides with x_H on H, and therefore on $L^\perp \subset H$. Since $x_0^{**} \in \tilde{L} = L^{\perp\perp}$, it follows that $x_H = 0$ on $L^\perp$,

i.e. $x_H \in L^{\perp\perp} \cap X = L$. In this way we find for every $H \in \mathcal{H}$ an $x_H \in K \cap L$ so that $x_H = x_0^{**}$ on H. Evidently the net $\{x_H\}_{H \in \mathcal{H}}$ w^*-converges to x_0^{**} and the proof is complete. □

<u>LEMMA 3.8</u>. *Let* X *be a Banach space,* $K_1,\ldots,K_n \subset X$ *open and convex, and* $L \subset X$ *a closed affine subspace with* codim $L < \infty$. *Then we have*

$$\text{(3.18)} \qquad \tilde{L} \cap \underline{K}_1 \cap \ldots \cap \underline{K}_n \neq \phi \;\Rightarrow\; L \cap K_1 \cap \ldots \cap K_n \neq \phi .$$

<u>PROOF</u>. Applying a translation if necessary, we may assume that $o \in L$, i.e. that L is a linear subspace.

a) Let us observe that (3.18) holds in the special case that $L = X$. Indeed, in that case (3.18) reads, since $\tilde{L} = \tilde{X} = X^{**}$,

$$\text{(3.19)} \qquad \underline{K}_1 \cap \ldots \cap \underline{K}_n \neq \phi \;\Rightarrow\; K_1 \cap \ldots \cap K_n \neq \phi$$

and (3.19) is easily proved by combining Lemmas 3.3 and 3.5.

b) We now prove (3.18) for $L \neq X$. Let $M_i := K_i \cap L$ $(i = 1,\ldots,n)$ and let $\underline{\underline{M}}_i$ be the (norm) interior in L^{**} of the $\sigma(L^{**},L^*)$-closure of M_i. Then (3.19) applied to the Banach space L and the convex open sets $M_1,\ldots,M_n \subset L$ yields the implication

$$\text{(3.20)} \qquad \underline{\underline{M}}_1 \cap \ldots \cap \underline{\underline{M}}_n \neq \phi \;\Rightarrow\; M_1 \cap \ldots \cap M_n \neq \phi .$$

Next we recall that L^{**} may be identified with $\tilde{L}$ (Proposition 0.16, and $\tilde{L} = L^{\perp\perp}$) and that with this identification, $\sigma(L^{**},L^*)$ coincides with $\sigma(X^{**},X^*)_{|\tilde{L}}$. Hence, since $\tilde{L}$ is $\sigma(X^{**},X^*)$-closed, the $\sigma(L^{**},L^*)$-closure of M_i equals $\tilde{M}_i$, the $\sigma(X^{**},X^*)$-closure of M_i $(i = 1,\ldots,n)$. Since also by the preceding Lemma

$$\underline{K}_i \cap \tilde{L} \subset \widetilde{K_i \cap L} = \tilde{M}_i \qquad (i = 1,\ldots,n),$$

and $\underline{K}_i \cap \tilde{L}$ is norm open in $\tilde{L}$, we get

$$\text{(3.21)} \qquad \underline{K}_i \cap \tilde{L} \subset \underline{\underline{M}}_i \qquad\qquad (i = 1,\ldots,n).$$

It now remains to combine (3.20) and (3.21):

$$\begin{aligned} &\tilde{L} \cap \underline{K}_1 \cap \ldots \cap \underline{K}_n = (\underline{K}_1 \cap \tilde{L}) \cap \ldots \cap (\underline{K}_n \cap \tilde{L}) \neq \phi \\ &\Rightarrow \underline{\underline{M}}_1 \cap \ldots \cap \underline{\underline{M}}_n \neq \phi \\ &\Rightarrow M_1 \cap \ldots \cap M_n = L \cap K_1 \cap \ldots \cap K_n \neq \phi . \end{aligned}$$

□

We are now prepared to finish the proof of Theorem 3.1.

<u>END OF THE PROOF OF THEOREM 3.1</u>. Let $z_1^{**},\dots,z_k^{**}$ be as in step 2 and put $L := \big((z_1^{**},\dots,z_k^{**}) + (F^{\perp})^k\big) \cap X^k$. F being finite-dimensional, there exists for every $j \in \{1,\dots,k\}$ a $y_j \in X$ with $y_j|_F = z_j^{**}|_F$. Hence

$$(3.22) \qquad L = (y_1,\dots,y_k) + (F^{\perp} \cap X)^k$$

and L is therefore a closed affine subspace of X^k with codim $L < \infty$. Moreover, it is easily checked that $(F^{\perp}\cap X)^{\perp} = F$, so that $\widetilde{F^{\perp}\cap X} = F^{\perp}$. Therefore, using (3.15) again,

$$(3.23) \qquad \tilde{L} = (y_1,\dots,y_k) + (F^{\perp})^k = (z_1^{**},\dots,z_k^{**}) + (F^{\perp})^k.$$

Suppose now that it has been proved that $L \cap \big(\bigcap_{i=1}^{m} (K_i \cap C_i)\big) \neq \phi$. Taking $(x_1,\dots,x_k) \in L \cap \big(\bigcap_{i=1}^{m} (K_i \cap C_i)\big)$, we know from step 2 that $Tz_1^{**} = x_1,\dots,Tz_k^{**} = x_k$ defines a linear map satisfying (i) and (ii). Also (iii) holds, for this choice of $x_1,\dots,x_k$ and T. Indeed, since $(x_1,\dots,x_k) \in L$ we have $(Tz_1^{**},\dots,Tz_k^{**}) - (z_1^{**},\dots,z_k^{**}) \in (F^{\perp})^k$, i.e. $Tz_j^{**}-z_j^{**} \in F^{\perp}$ for $j = 1,\dots,k$, and therefore also $Tx^{**}-x^{**} \in F^{\perp}$ for all $x^{**} \in G$, which proves (iii). Finally, to see that $L \cap \big(\bigcap_{i=1}^{m} (K_i \cap C_i)\big) \neq \phi$, observe that $(z_1^{**},\dots,z_k^{**}) \in \tilde{L} \cap \big(\bigcap_{i=1}^{m} (K_i^{**} \cap C_i^{**})\big)$. Since $K_i^{**} \subset \underline{K}_i$ and $C_i^{**} \subset \underline{C}_i$ $(i = 1,\dots,m)$ (by step 4), we therefore have $\tilde{L} \cap \big(\bigcap_{i=1}^{m} (\underline{K}_i \cap \underline{C}_i)\big) \neq \phi$. Hence, by Lemma 3.8, $L \cap \big(\bigcap_{i=1}^{m} (K_i \cap C_i)\big) \neq \phi$. □

<u>NOTES</u>. Theorem 3.1 is due to J. LINDENSTRAUSS & H.P. ROSENTHAL ([73]), except for the last part concerning (iii), which was obtained later in [62]. Lemma 3.3 is due to V. KLEE ([67]), but the present proof is taken from [73]. For a different proof of Theorem 3.1 we refer to [25].

4. A RENORMING OF NON-REFLEXIVE SPACES

If one defines on a reflexive space $(X, \|\cdot\|)$ a new equivalent norm $|||\cdot|||$, then also $(X, |||\ |||)$ is reflexive (Corollary 0.12). In particular for any choice of equivalent norm $(X, |||\ |||)$ is isometric to a conjugate Banach space. It may be asked whether this last property characterizes reflexive spaces. Phrased differently, the question reads: Does there exist on every non-reflexive space $(X, \|\ \|)$ an equivalent norm $|||\ |||$ such that $(X, |||\cdot|||)$ is not isometric to a conjugate Banach space? Since for every conjugate Banach space X (identified with πX) there exists a projection P from X^{**} onto X with $\|P\| = 1$ (if $X = Y^*$, take $P = \pi_Y^*$), the following result implies a positive answer to the question above: For every non-reflexive space $(X, \|\ \|)$ there exists an equivalent norm $|||\ |||$ such that $(X, |||\ |||)$ admits no projection P from X^{**} onto X with $|||P||| = 1$. In this connection let us point out that some spaces which are not isometric to conjugate Banach spaces are nevertheless the range of norm 1 projections from their biduals, e.g. $L^1[0,1]$, or any other non-dual AL-space (see [88]). The proof of the above renorming result will occupy us for the rest of this section. An important tool will be the notion of the characteristic of a linear subspace of a conjugate Banach space.

Let X be a Banach space and $V \subset X^*$ a linear subspace. As before, for any subset A of X^*, $\tilde{A}$ will denote its w^*-closure. We define $\approx\!\!V := \bigcup_{\alpha \geq 0} \alpha \tilde{B}_V$, i.e. $\approx\!\!V$ is formed by adjoining to V all w^*-limit points of bounded subsets of V. Clearly $\approx\!\!V$ is again a linear subspace and $V \subset \approx\!\!V \subset \tilde{V}$. If X is separable $\approx\!\!V$ is nothing but the w^*-sequential closure of V, since in this case the w^*-topology is metrizable on bounded sets (Proposition 0.15) and since w^*-convergent sequences are always norm bounded (Banach-Steinhaus theorem). Furthermore, we note the following restatement of the Krein-Šmulian theorem (Proposition 0.28):

(4.1) $\qquad V = \tilde{V} \iff V = \approx\!\!V.$

A more careful look at the operation $V \to \approx{V}$ shows that it can occur (even for separable X) that

$$(4.2) \qquad \tilde{V} = X^* \quad \text{and} \quad \approx{V} \neq X^*.$$

An example will be given later. Suppose V_0 is a fixed linear subspace of X^* satisfying (4.2). Substituting $\approx{V}_0$ for V in (4.1) shows that $(\approx{V}_0)^{\approx} \neq \approx{V}_0$, so that $V \to \approx{V}$ is not a legitimate closure operation. To round off these preliminary observations let us point out the following equivalence:

$$(4.3) \qquad \approx{V} = X^* \Longleftrightarrow \exists\alpha > 0 \ \tilde{B}_V \supset \alpha B_{X^*}.$$

Indeed, $\approx{V} = X^*$ means $\bigcup_{n=1}^{\infty} n\tilde{B}_V = X^*$, so by Baire's theorem $\tilde{B}_V$ contains an interior point. Hence $o \in \text{int } \tilde{B}_V$. The converse implication is trivial.

We are ready now to introduce the characteristic of V.

DEFINITION 4.1. Let X be a Banach space and $V \subset X^*$ a linear subspace. Then the number

$$(4.4) \qquad r(V) := \max\{\alpha \geq 0: \tilde{B}_V \supset \alpha B_{X^*}\}$$

is called the *characteristic* of V.

REMARK 4.2. Evidently $0 \leq r(V) \leq 1$ for every $V \subset X^*$. If $\tilde{V} \neq X^*$, then $r(V) = 0$. However, the converse does not hold, as is seen by combining (4.2) and (4.3).

We now derive some useful expressions for r(V).

LEMMA 4.3. *Let X be a Banach space and $V \subset X^*$ a linear subspace. Then*

$$(4.5) \qquad r(V) = \inf_{x \in S_X} \sup_{x^* \in B_V} |\langle x, x^* \rangle|.$$

PROOF. This is a simple application of the bipolar theorem applied to the dual pair $\langle X, X^* \rangle$. Recall that for an absorbing absolutely convex set A, p_A denotes its gauge. Since $\sup_{x^* \in B_V} |\langle x, x^* \rangle| = p_{(B_V)^0}(x)$, the right member of (4.5) equals

$$\inf_{x \in S_X} p_{(B_V)^0}(x) = \sup\{\alpha \geq 0: \alpha (B_V)^0 \subset B_X\} =$$

$$= \sup\{\alpha \geq 0: B_V^{00} \supset \alpha B_{X^*}\} =$$

$$= \sup\{\alpha \geq 0: \tilde{B}_V \supset \alpha B_{X^*}\} = r(V). \qquad \square$$

LEMMA 4.4. *Let* X *be a Banach space and* $V \subset X^*$ *a linear subspace. Then*

$$r(V) = \left[\sup\{\|x\| : x \in \bar{B}_X^{\sigma(X,V)}\}\right]^{-1}. \tag{4.6}$$

PROOF. We first assume that V is $\sigma(X^*,X)$-dense in X^*. Then both $\langle X,X^*\rangle$ and $\langle X,V\rangle$ are dual pairs. By the bipolar theorem applied to $\langle X,X^*\rangle$ we have the equivalence

$$\alpha B_{X^*} \subset \bar{B}_V^{\sigma(X^*,X)} \iff \frac{1}{\alpha}B_X \supset B_V^0 \quad \text{for all } \alpha \geq 0. \tag{4.7}$$

On the other hand the bipolar theorem applied to $\langle X,V\rangle$ yields

$$B_V^0 = B_X^{00} = \bar{B}_X^{\sigma(X,V)}. \tag{4.8}$$

Thus, combining (4.7) and (4.8),

$$\alpha B_{X^*} \subset \bar{B}_V^{\sigma(X^*,X)} \iff B_X \supset \alpha\bar{B}_X^{\sigma(X,V)} \quad \text{for all } \alpha \geq 0.$$

(4.6) now follows from this and the definition of $r(V)$. In the case that V is not w^*-dense in X^* we have $r(V) = 0$. Also, there exists in this case an $x_0 \in V^\top$, $x_0 \neq 0$. Clearly $sp\{x_0\} \subset \bar{B}_X^{\sigma(X,V)}$, so that $\sup\{\|x\| : x \in \bar{B}_X^{\sigma(X,V)}\} = \infty$. Hence (4.6) holds also in this case. □

The next lemma connects $r(V)$ with the norm of a certain projection.

LEMMA 4.5. *Let* X *be a Banach space and* $V \subset X^*$ *a linear subspace. Then*

$$r(V) = \inf\{\|x+x^{**}\| : x \in S_X,\ x^{**} \in V^\perp\}. \tag{4.9}$$

In particular, if V *is* w^**-dense in* X^* *(equivalently,* $V^\perp \cap X = \{0\}$*) then* $r(V) = \frac{1}{\|P\|}$*, where* P *is the projection from* $X \oplus V^\perp$ *onto* X *with kernel* $V^\perp$.

PROOF. We prove (4.9) first. Let $x \in S_X$ and $x^{**} \in V^\perp$ be arbitrary. Then, using (4.5), we have

$$\|x+x^{**}\| = \sup_{x^* \in B_{X^*}} |\langle x^*, x+x^{**}\rangle| \geq \sup_{x^* \in B_V} |\langle x^*, x+x^{**}\rangle| =$$

$$= \sup_{x^* \in B_V} |\langle x, x^*\rangle| \geq r(V).$$

On the other hand if $\varepsilon > 0$ is arbitrary, (4.5) implies the existence of an $x \in S_X$ such that

$$\sup_{x^* \in B_V} |\langle x, x^* \rangle| < r(V)+\varepsilon.$$

This means that the restriction of x to V has norm $< r(V)+\varepsilon$. This restriction therefore has a Hahn-Banach extension $y^{**} \in X^{**}$ with $\|y^{**}\| < r(V)+\varepsilon$. Since $y^{**}-x \in V^{\perp}$, we have $y^{**} = x+x^{**}$ for some $x^{**} \in V^{\perp}$. Thus (4.9) is proved.

Let us now suppose that $\tilde{V} = X^*$, i.e. $V^{\perp} \cap X = \{0\}$. P being the projection defined above, we have, by (4.9),

$$\|P\| = \sup\left\{\frac{\|x\|}{\|x+x^{**}\|}: 0 \neq x \in X,\ x^{**} \in V^{\perp}\right\} =$$

$$= \left[\inf\left\{\frac{\|x+x^{**}\|}{\|x\|}: 0 \neq x \in X,\ x^{**} \in V^{\perp}\right\}\right]^{-1} = r(V)^{-1}. \qquad \square$$

For completeness we now give the example promised earlier of a V satisfying (4.2).

<u>EXAMPLE</u>. Let $X = c_0$. We shall construct a linear subspace $V \subset X^* = \ell^1$ which is w^*-dense, but satisfies $r(V) = 0$, i.e. $\tilde{V} = X^*$ and $\tilde{\tilde{V}} \neq X^*$. We partition $\mathbb{N}$ into infinitely many infinite sequences $(n_1^1, n_2^1, n_3^1, \ldots), (n_1^2, n_2^2, n_3^2, \ldots), \ldots, (n_1^k, n_2^k, \ldots), \ldots$ and define a linear subspace $W \subset X^{**} = \ell^\infty$ as follows:

$$W := \{x^{**} = \{\zeta_n\} \in \ell^\infty: \zeta_{n_i^k} = k^{-1}\zeta_{n_1^k} \quad \text{for all } i = 2,3,\ldots \text{ and } k = 1,2,\ldots\}.$$

It is easily checked that W is w^*-closed and that $W \cap c_0 = \{0\}$. Hence $W = V^{\perp}$, for some linear subspace $V \subset \ell^1$ ($V := W^{\top}$ will do, by the bipolar theorem). From $V^{\perp} \cap c_0 = \{0\}$ it follows that V is w^*-dense in ℓ^1. We claim that $r(V) = 0$. To prove this it suffices, by Lemma 4.5, to produce for every $k \in \mathbb{N}$ an $x \in S_{c_0}$ and an $x^{**} \in V^{\perp} = W$ with $\|x+x^{**}\| \leq \frac{1}{k}$. Fix $k \in \mathbb{N}$. We choose $x = \{\xi_n\} \in c_0$ such that $\xi_{n_1^k} = -1$ and $\xi_n = 0$ for all $n \neq n_1^k$, and $x^{**} = \{\xi_n\} \in W$ such that $\zeta_{n_1^k} = 1$, $\zeta_{n_i^k} = \frac{1}{k}$ $(i = 2,3,\ldots)$ and $\zeta_{n_i^\ell} = 0$ for all $\ell \neq k$ and all $i \in \mathbb{N}$. Then $\|x+x^{**}\| = \frac{1}{k}$ and the proof is complete.

<u>REMARK 4.6</u>. Examples like the above abound. In fact we shall see in Theorem 7.21 that for every Banach space X the existence of a w^*-dense linear subspace $V \subset X^*$ with $r(V) = 0$ is equivalent to non-quasireflexivity of X (X is

called quasireflexive iff $\dim X^{**}/\pi X < \infty$).

We now return to the question discussed at the beginning of this section. First we wish to prove a renorming theorem which we shall need again in section 8. The main result (Theorem 4.8) will be a corollary of this.

PROPOSITION 4.7. *Let* X *be a Banach space and* W *a separable linear subspace of* X^*. *Then there exists an equivalent norm* $|||\ |||$ *on* X *such that, whenever* $\{x_\alpha^*\}$ *is a net in* X^* *satisfying* $w^*\text{-}\lim_\alpha x_\alpha^* = x^*$ *and* $\lim_\alpha |||x_\alpha^*||| = |||x^*|||$ *for some* $x^* \in W$, *then we have* $\lim_\alpha x_\alpha^* = x^*$ *in norm.*

PROOF. Let $F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots$ be an increasing sequence of finite-dimensional subspaces of W such that $\overline{\bigcup_{n=1}^{\infty} F_n} = W$. We define a norm $|||\cdot|||$ on X^* as follows:

$$|||x^*||| := \|x^*\| + \sum_{n=1}^{\infty} \frac{1}{2^n} \operatorname{dist}(x^*, F_n) \qquad (x^* \in X^*).$$

(By dist we mean the distance with respect to $\|\cdot\|$.) Checking that $|||\cdot|||$ is a norm is no problem. Also clear is that $\|x^*\| \le |||x^*||| \le 2\|x^*\|$ for all $x^* \in X^*$, so that $\|\cdot\|$ and $|||\cdot|||$ are equivalent on X^*. We need to know that $|||\cdot|||$ is the dual of some norm $|||\cdot|||$ on X (the latter is then evidently also equivalent to $\|\cdot\|$ on X). Necessary and sufficient for this is that $|||\cdot|||$ is w^* lower semicontinuous. This condition is satisfied here, however, since each of the functions $x^* \to \operatorname{dist}(x^*, F_n)$ is easily seen to be w^*-lower semicontinuous.

Now suppose that $x^* \in W$ and a net $\{x_\alpha^*\} \subset X^*$ are given so that $w^*\text{-}\lim_\alpha x_\alpha^* = x^*$ and $\lim_\alpha |||x_\alpha^*||| = |||x^*|||$. We show by contradiction that $\lim_\alpha \|x_\alpha^* - x^*\| = 0$ (equivalently, $\lim_\alpha |||x_\alpha^* - x^*||| = 0$). If not, there exists a subnet $\{x_{\alpha'}^*\}$ and $\varepsilon > 0$ so that $\|x_{\alpha'}^* - x^*\| \ge 2\varepsilon$ for all α'. Note that by w^* lower semicontinuity we have

$$\liminf_{\alpha'} \|x_{\alpha'}^*\| \ge \|x^*\| \quad \text{and} \quad \liminf_{\alpha'} \operatorname{dist}(x_{\alpha'}^*, F_n) \ge \operatorname{dist}(x^*, F_n),$$

for each $n \in \mathbb{N}$. Thus the definition of $|||\cdot|||$, together with $\lim_{\alpha'} |||x_{\alpha'}^*||| = |||x^*|||$ implies that, for each $n \in \mathbb{N}$, $\lim_{\alpha'} \operatorname{dist}(x_{\alpha'}^*, F_n) = \operatorname{dist}(x^*, F_n)$. Since $x^* \in W = \overline{\bigcup_{n=1}^{\infty} F_n}$, we have $\lim_{n\to\infty} \operatorname{dist}(x^*, F_n) = 0$. Thus there exist an $n_0 \in \mathbb{N}$ and an α_0' such that for each $\alpha' \ge \alpha_0'$, there is a $y_{\alpha'}^* \in F_{n_0}$ satisfying $\|x_{\alpha'}^* - y_{\alpha'}^*\| < \varepsilon$. Passing to a subnet if necessary, using the finite-dimensionality of F_{n_0}, we may suppose that $\{y_{\alpha'}^*\}$ converges to some $y^* \in F_{n_0}$.

Thus $\|x^*_{\alpha'}-y^*\| < \varepsilon$ for large enough α'. Taking the w^*-limit we obtain $\|x^*-y^*\| \leq \varepsilon$. Combining the last two inequalities yields $\|x^*_{\alpha'}-x^*\| < 2\varepsilon$ for large α', contradicting the choice of $\{x^*_{\alpha'}\}$ and ε. □

THEOREM 4.8. *Let X be a non-reflexive Banach space. Then there exists a norm* $|||\cdot|||$ *on X, equivalent to the given norm, such that there exists no projection P from* X^{**} *onto X with norm* $|||P||| = 1$.

PROOF. Since X is not reflexive, neither is X^* (Proposition 0.13). Hence B_{X^*} is not weakly compact (Proposition 0.11). By the Eberlein-Šmulian theorem B_{X^*} therefore contains a sequence $\{x^*_n\}$ which fails to have a weak limit point. By Alaoglu's theorem $\{x^*_n\}$ does have a w^*-limit point, say x^*. Now let W be the (separable) linear subspace of X^* spanned by $x^*, x^*_1, x^*_2, \ldots$. We claim that $X^{**} \neq X + W^{\perp}$. Indeed, $X^{**} = X + W^{\perp}$ would imply the equality of $\sigma(X^*,X^{**})_{|W}$ and $\sigma(X^*,X)_{|W}$ and this is precluded by the choice of W: $x^* \in W$ is a $\sigma(X^*,X)$ - but not a $\sigma(X^*,X^{**})$-limit point of $\{x^*_n\} \subset W$.

Now let $|||\cdot|||$ be any norm satisfying Proposition 4.7, with respect to this W. We claim that $r_{|||\cdot|||}(V) < 1$ for every closed linear subspace $V \subset X^*$ with $V \not\supset W$. ($r_{|||\cdot|||}(V)$ of course denotes $r(V)$ with respect to the norm $|||\cdot|||$.) Indeed, let $V \not\supset W$ and pick $x^* \in W\backslash V$, $|||x^*||| = 1$. Supposing that $r_{|||\cdot|||}(V) = 1$, there exists a net $\{x^*_\alpha\} \subset V$ such that $w^*\text{-}\lim_\alpha x^*_\alpha = x^*$ and $|||x^*_\alpha||| \leq 1$ for all α. By w^* lower semicontinuity it follows that $\lim|||x^*_\alpha||| = 1 = |||x^*|||$. Hence Proposition 4.7 implies $\lim_\alpha\|x^*_\alpha-x^*\| = 0$, so $x^* \in V$, a contradiction.

Finally, suppose that P is a projection from X^{**} onto X with $|||P||| = 1$. Since $X^{**} \neq X + W^{\perp}$, there exists an $x^{**} \in (\ker P)\backslash W^{\perp}$. Taking $V := \ker x^{**} \subset X^*$, we have $V \not\supset W$, since $x^{**} \in V^{\perp}\backslash W^{\perp}$. Hence $r_{|||\cdot|||}(V) < 1$, by the above. On the other hand, however, by Lemma 4.5 we have, since $V^{\perp} = \mathrm{sp}\{x^{**}\}$,

$$r_{|||\cdot|||}(V) = \frac{1}{|||P_{|X\oplus \mathrm{sp}\{x^{**}\}}|||} \geq \frac{1}{|||P|||} = 1,$$

a contradiction which finishes the proof. □

COROLLARY 4.9. *Every non-reflexive Banach space admits an equivalent norm for which it is not isometric to any conjugate Banach space.*

PROOF. Any norm $|||\ |||$ as in Theorem 4.8 does the job. Indeed, it suffices to observe that for any conjugate Banach space $X = Y^*$, π^*_Y is a norm one projection from X^{**} onto X. □

<u>NOTES</u>. The phenomenon expressed in (4.2) was first noticed by S. MAZURKIEWICZ ([77]), in response to a question of S. Banach. J. DIXMIER defined and systematically studied characteristics in [29]. Among other things he proved the Lemmas 4.3, 4.4 and 4.5. The example is also taken from [29]. The result mentioned in Remark 4.6 was proved by W.J. DAVIS & J. LINDENSTRAUSS ([23]). Proposition 4.7 was first proved by M.I. KADEC ([63]) and V. KLEE ([68]) in the case of a Banach space X with separable dual $X^* = W$. The present proof was given by W.J. DAVIS & W.B. JOHNSON ([20]), who actually proved a slightly stronger result. Finally, Theorem 4.8 is due to D. VAN DULST & I. SINGER ([31]). Corollary 4.9 was already shown in [20].

5. ELEMENTARY FACTS ABOUT BASES AND BASIC SEQUENCES

Later we shall wish to characterize reflexivity in terms of bases and basic sequences. In this section we collect some facts needed for this and other purposes.

DEFINITION 5.1. A sequence $\{x_n\}_{n=1}^{\infty}$ in a Banach space X is called a *(Schauder) basis* for X iff for every $x \in X$ there exists a unique sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$ (where the series converges in norm).

Evidently the existence of a basis implies separability. The converse is not true: P. ENFLO ([34]) has constructed a separable Banach space which fails to have a basis (and does not even satisfy the so-called approximation property), thereby solving a famous problem. In fact it is now known ([76]) that the classical sequence spaces c_0 and ℓ^p $(2 < p < \infty)$ all contain subspaces without bases. Most concretely defined spaces have bases. Trivial examples are c_0 and ℓ^p $(1 \le p < \infty)$. For each of these the unit vectors $e_n = (\underbrace{0,0,\dots,0}_{n-1},1,0,\dots)$ $(n \in \mathbb{N})$ form a basis, usually referred to as the *standard basis*.

A somewhat less trivial example of a Banach space with a basis is $C([0,1])$. Let $t_1 := 0$, $t_2 := 1$, $t_3, t_4, \dots$ be a sequence of distinct points which form a dense subset of $[0,1]$. For each $n > 2$ let I_n be the open interval of the partition of $[0,1]$ determined by $t_1,\dots,t_{n-1}$ which contains t_n. We now define $\{x_n\} \subset C([0,1])$ as follows:

$$(5.1)\qquad \begin{cases} x_1(t) = 1 \quad \text{and} \quad x_2(t) = t \qquad \text{for all } t \in [0,1], \\ x_n(t) = \begin{cases} 0 & \text{if } t \notin I_n \\ 1 & \text{if } t = t_n. \end{cases} \text{ and extended linearly to } [0,1] \ (n = 3,4,\dots) \end{cases}$$

We postpone the proof that $\{x_n\}$ is a basis for $C([0,1])$ until later.

Let $\{x_n\}$ be a basis for X. Then for each fixed $n \in \mathbb{N}$ the n-th coefficient α_n in the expansion $x = \sum_{n=1}^{\infty} \alpha_n x_n$ clearly depends linearly on x. A first important fact is that this linear form is also continuous.

PROPOSITION 5.2. *Let $\{x_n\}$ be a basis for a Banach space X. For each $n \in \mathbb{N}$ the linear form x_n^* defined by*

$$\langle x, x_n^* \rangle := \alpha_n \qquad (x = \sum_{n=1}^{\infty} \alpha_n x_n \in X)$$

is bounded on X (and therefore the notation x_n^* is justified). Moreover, there exists a constant $C > 0$ such that

$$\|x_n\| \|x_n^*\| \leq C \quad \textit{for all } n \in \mathbb{N}.$$

In fact (with a notation to be introduced later) $C = 2\nu_{\{x_n\}}$ will do. $\{x_n^*\}$ is called the sequence of coefficient functionals of the basis $\{x_n\}$.

PROOF. We introduce on X a new norm as follows:

$$\|x\|_1 := \sup_{n \in \mathbb{N}} \|\sum_{i=1}^{n} \alpha_i x_i\| \qquad (x = \sum_{i=1}^{\infty} \alpha_i x_i).$$

It is trivial to check the norm properties (for the proof of $\|x\|_1 = 0 \Rightarrow x = 0$ observe that $x_n \neq 0$ for all $n \in \mathbb{N}$, by the uniqueness of the expansions $x = \sum_{n=1}^{\infty} \alpha_n x_n$). Also clearly $\|x\| \leq \|x\|_1$ for all $x \in X$. Let us suppose for the moment that we have shown $(X, \| \ \|_1)$ to be complete. Then by the open mapping theorem there exists a constant $C > 0$ such that

$$\|x\| \leq \|x\|_1 \leq \tfrac{1}{2} C \|x\| \qquad \text{for all } x \in X.$$

In particular for every $n \in \mathbb{N}$ and all $x \in X$ we have

$$|\langle x, x_n^* \rangle| = \frac{\|\langle x, x_n^* \rangle x_n\|}{\|x_n\|} \leq \frac{\|\sum_{i=1}^{n} \langle x, x_i^* \rangle x_i\| + \|\sum_{i=1}^{n-1} \langle x, x_i^* \rangle x_i\|}{\|x_n\|} \leq$$

$$\leq \frac{2\|x\|_1}{\|x_n\|} \leq \frac{C\|x\|}{\|x_n\|}.$$

Thus $\|x_n^*\| \|x_n\| \leq C$ for all $n \in \mathbb{N}$ and the proof is finished. (The statement that $2\nu_{\{x_n\}}$ may be taken for C will become clear right after the definition of $\nu_{\{x_n\}}$, see Remark 5.8.)

Verification of the completeness of $(X, \| \ \|_1)$ is an easy exercise the details of which we leave to the reader: If $\{x^{(k)}\}_{k=1}^{\infty}$ is a Cauchy sequence for $\| \ \|_1$, where $x^{(k)} = \sum_{i=1}^{\infty} \alpha_i^{(k)} x_i$ $(k = 1,2,\ldots)$, then for each fixed $i \in \mathbb{N}$ the sequence $\{\alpha_i^{(k)}\}_{k=1}^{\infty}$ is Cauchy and thus convergent to some $\alpha_i \in \mathbb{R}$. The series $\sum_{i=1}^{\infty} \alpha_i x_i$ is then seen to converge and its sum x is the limit of $\{x^{(k)}\}$. □

We now want to derive a more geometric criterion for deciding whether or not a given sequence is a basis. First we introduce the notion of a biorthogonal system.

DEFINITION 5.3. A *biorthogonal system* in a Banach space X is a pair $\{x_n\},\{x_n^*\}$ of sequences $\{x_n\} \subset X$, $\{x_n^*\} \subset X^*$ satisfying the orthogonality relations

$$\langle x_i, x_j^* \rangle = \delta_{ij} \qquad (i,j = 1,2,\ldots).$$

It is called *complete* if $[x_n] = X$.

Note that if a biorthogonal system $\{x_n\},\{x_n^*\}$ for X is complete, the x_n^* are uniquely determined by the x_n. Obviously, if $\{x_n\}$ is a basis for X, then $\{x_n\}$ and its sequence of coefficient functionals $\{x_n^*\}$ form a complete biorthogonal system. Furthermore, to every biorthogonal system $\{x_n\},\{x_n^*\}$, in particular to every basis $\{x_n\}$, there is associated a sequence of bounded projections $P_n\colon X \to X$ of finite rank, defined by

$$P_n(x) = \sum_{i=1}^{n} \langle x, x_i^* \rangle x_i \qquad (n \in \mathbb{N}).$$

Note that $P_n P_m = P_m P_n = P_{\min(n,m)}$ for all $n,m \in \mathbb{N}$.

The next proposition gives a necessary and sufficient condition for a complete biorthogonal system to be a basis (together with its coefficient functionals). We shall see later (Example 2) that this is not always the case.

PROPOSITION 5.4. *Let $\{x_n\},\{x_n^*\}$ be a complete biorthogonal system for a Banach space X and let $\{P_n\}$ be the associated sequence of projections. Then the following are equivalent:*

(i) *$\{x_n\}$ is a basis for X,*

(ii) *the sequence $\{P_n\}$ is uniformly bounded,*

(iii) *$\lim_{n\to\infty} P_n(x) = x$ for all $x \in X$.*

PROOF. (i) ⇒ (iii): If $\{x_n\}$ is a basis for X, then by uniqueness $\{x_n^*\}$ must be the sequence of its coefficient functionals. Hence $x = \sum_{i=1}^{\infty} \langle x, x_i^* \rangle x_i = \lim_{n\to\infty} P_n(x)$ for all $x \in X$.

(iii) ⇒ (ii): Apply the Banach-Steinhaus theorem.

(ii) ⇒ (i): Assume that $\{P_n\}$ is uniformly bounded. Let $x = \sum_{i=1}^{m} \alpha_i x_i = \sum_{i=1}^{m} \langle x, x_i^* \rangle x_i$ be an arbitrary element of $sp\{x_n\}$. Then $x = P_n x$ for all $n \geq m$, so $x = \lim_{n\to\infty} P_n x$ holds on $sp\{x_n\}$. Now the fact that $X = [x_n]$ together with the uniform boundedness of $\{P_n\}$ implies that

$$x = \lim_{n\to\infty} P_n(x) = \sum_{i=1}^{\infty} \langle x, x_i^* \rangle x_i \quad \text{for all } x \in X.$$

Since this expansion is obviously unique, we have shown that $\{x_n\}$ is a basis for X (with $\{x_n^*\}$ as its sequence of coefficient functionals). □

The next question that needs to be answered is the following: given a sequence $\{x_n\} \subset X$ such that $[x_n] = X$, when does there exist a sequence $\{x_n^*\} \subset X^*$ such that $\{x_n\},\{x_n^*\}$ forms a (complete) biorthogonal system? A necessary condition is clearly that $\{x_n\}$ be linearly independent, but this does not suffice, as we shall see below (Example 1).

PROPOSITION 5.5. *Let $\{x_n\}$ be a sequence of non-zero elements in a Banach space X such that $[x_n] = X$. Then the following are equivalent:*

(i) $x_n \notin [x_1,\dots,x_{n-1},x_{n+1},\dots]$ *for all* $n \in \mathbb{N}$.

(ii) *There exists a sequence $\{x_n^*\} \subset X^*$ such that $\{x_n\},\{x_n^*\}$ is a (complete) biorthogonal system.*

(iii) *For every $n \in \mathbb{N}$ there exists a constant $C_n > 0$ such that* $\|\sum_{i=1}^{n} \alpha_i x_i\| \leq C_n \|\sum_{i=1}^{n+m} \alpha_i x_i\|$ *for all* $m \in \mathbb{N}$ *and all* $\alpha_1,\dots,\alpha_{n+m} \in \mathbb{R}$.

PROOF. (i) ⟺ (ii): This is an obvious consequence of the Hahn-Banach theorem.

(ii) ⇒ (iii): For every choice of $n,m \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_{n+m} \in \mathbb{R}$ we have $\|\sum_{i=1}^{n} \alpha_i x_i\| = \|P_n(\sum_{i=1}^{n+m} \alpha_i x_i)\| \leq \|P_n\| \|\sum_{i=1}^{n+m} \alpha_i x_i\|$, where P_n is the n-th projection associated to the biorthogonal system $\{x_n\},\{x_n^*\}$. Take $C_n = \|P_n\|$ $(n = 1,2,\dots)$.

(iii) ⇒ (ii): Let us observe first that (iii) implies linear independence of the sequence $\{x_n\}$. Thus for every $n \in \mathbb{N}$ there is a well defined linear form x_n^* on $sp\{x_n\}$ satisfying

$$\langle x_m, x_n^* \rangle = \delta_{mn} \quad (m,n = 1,2,\dots).$$

It now suffices to show that each x_n^* is bounded on $sp\{x_n\}$, so that it can be extended uniquely to an element of X^* (this will justify the somewhat premature notation x_n^*). Fix $n \in \mathbb{N}$ and let $x = \sum_{i=1}^k \alpha_i x_i \in sp\{x_n\}$ be arbitrary. We may assume $k \geq n$ since otherwise $\langle x, x_n^* \rangle = 0$. Then by (iii) we have

$$|\langle x, x_n^* \rangle| = \frac{\| \langle x, x_n^* \rangle x_n \|}{\| x_n \|} = \frac{\| \alpha_n x_n \|}{\| x_n \|} \leq \frac{\| \sum_{i=1}^n \alpha_i x_i \| + \| \sum_{i=1}^{n-1} \alpha_i x_i \|}{\| x_n \|} \leq$$

$$\leq \frac{(C_n + C_{n-1})}{\| x_n \|} \| x \|,$$

so that

$$\| x_n^* \| \leq \frac{C_n + C_{n-1}}{\| x_n \|} < \infty. \qquad \square$$

Now that we know (i), (ii) and (iii) to be equivalent we should observe that (iii) amounts to the statement $\| P_n \| \leq C_n$ $(n \in \mathbb{N})$, where P_n is the n-th projection associated to the (unique) biorthogonal system $\{x_n\}, \{x_n^*\}$.

Combining the two preceding propositions and the above observation yields the following characterization of bases. Note that (ii) below is a purely geometric and intrinsic criterion.

<u>THEOREM 5.6</u>. *Let* $\{x_n\}$ *be a sequence of non-zero elements in a Banach space* X *such that* $[x_n] = X$. *Then the following are equivalent:*

(i) $\{x_n\}$ *is a basis for* X.

(ii) *There exists a constant* $C \geq 1$ *such that* $\| \sum_{i=1}^n \alpha_i x_i \| \leq C \| \sum_{i=1}^{n+m} \alpha_i x_i \|$ *for all* $n,m \in \mathbb{N}$ *and all* $\alpha_1, \dots, \alpha_{n+m} \in \mathbb{R}$.

(iii) *There exists a (unique) sequence* $\{x_n^*\} \subset X^*$ *such that* $\{x_n\}, \{x_n^*\}$ *is a (complete) biorthogonal system and the associated sequence of projections* $\{P_n\}$ *is uniformly bounded.*

(iv) *There exists a (unique) sequence* $\{x_n^*\} \subset X^*$ *such that* $\{x_n\}, \{x_n^*\}$ *is a (complete) biorthogonal system and* $x = \sum_{n=1}^\infty \langle x, x_n^* \rangle x_n$ *for every* $x \in X$.

As an application we check that (5.1) defines a basis for $C([0,1])$. Let $x \in C([0,1])$ and $\varepsilon > 0$ be arbitrary. Using the density of $\{t_n\}$ in $[0,1]$, choose $n \in \mathbb{N}$ so that the oscillation of x is less than ε on every interval of the partition of $[0,1]$ determined by $t_1, \dots, t_n$. Now consider the function $y = \sum_{i=1}^n \beta_i x_i$, where the coefficients β_i are inductively defined by

$$\beta_i = x(t_i) - \sum_{j=1}^{i-1} \beta_j x_j(t_i).$$

Since $x_i(t_j) = 0$ whenever $j < i$, by (5.1), we find, for all $1 \le j \le n$,

$$y(t_j) = \sum_{i=1}^{n} \beta_i x_i(t_j) = \beta_j x_j(t_j) + \sum_{i=1}^{j-1} \beta_i x_i(t_j) =$$

$$= \beta_j + (x(t_j) - \beta_j) = x(t_j).$$

Since y is clearly piecewise linear, it follows now from the choice of n that $\|x-y\| < \varepsilon$. This proves that $[x_n] = C([0,1])$, x and ε being arbitrary. It remains to check (ii) in Theorem 5.6. Let $n > 1$ and $\alpha_1,\ldots,\alpha_{n+1} \in \mathbb{R}$ be arbitrary and compare the functions $\sum_{i=1}^{n} \alpha_i x_i$ and $\sum_{i=1}^{n+1} \alpha_i x_i$. They coincide outside I_{n+1}, while $\sum_{i=1}^{n} \alpha_i x_i$ is linear on I_{n+1}. Obviously therefore $\|\sum_{i=1}^{n} \alpha_i x_i\| \le \|\sum_{i=1}^{n+1} \alpha_i x_i\|$. The same inequality is trivially satisfied for $n = 1$. We conclude that (ii) holds with $C = 1$.

$\{x_n\}$ has norm 1 in the sense of the following

DEFINITION 5.7. If $\{x_n\}$ is a basis for X then the smallest C for which (ii) in Theorem 5.6 holds, i.e. $\sup_{n\in\mathbb{N}}\|P_n\|$, is called the *norm of the basis* $\{x_n\}$, (or the *basis constant*) and is denoted by $\nu_{\{x_n\}}$.

REMARK 5.8. A look back at the proof of Proposition 5.2 now shows that if $\{x_n\}$ is a basis for X, with coefficient functionals $\{x_n^*\}$, then $\|x_n\|\|x_n^*\| \le 2\nu_{\{x_n\}}$ for all $n \in \mathbb{N}$.

EXAMPLE 1. Let $\{x_n\}$ be a basis for a Banach space X, with coefficient functionals $\{x_n^*\}$. Let $x \in X$ be such that $\langle x,x_n^*\rangle \ne 0$ for all $n \in \mathbb{N}$, e.g. take $x = \sum_{n=1}^{\infty} \frac{1}{2^n\|x_n\|} x_n$. Then the sequence $\{x,x_1,x_2,\ldots\}$ is linearly independent, but clearly does not satisfy the equivalent conditions of Proposition 5.5.

EXAMPLE 2. Let $\{e_n\}$ be the standard basis for ℓ^1, with coefficient functionals denoted by $\{e_n^*\}$. Let

$$x_n := \sum_{i=1}^{n} e_i, \qquad x_n^* := e_n^* - e_{n+1}^* \qquad (n = 1,2,\ldots).$$

Then $\{x_n\},\{x_n^*\}$ is evidently a complete biorthogonal system, but $\{x_n\}$ fails to be a basis for ℓ^1. This is most easily proved by observing that $\|x_n\|\|x_n^*\| = n$, so that Proposition 5.2 is not satisfied.

Our next two results show that, with some care, new bases can be constructed from old ones.

PROPOSITION 5.9. *Let* $\{x_n\}$ *be a basis for a Banach space* X *with* $0 < \inf_{n\in\mathbb{N}} \|x_n\| \le \sup_{n\in\mathbb{N}} \|x_n\| < \infty$ *and let* $\{\alpha_n\}$ *be a sequence of non-zero real numbers. Putting* $y_n := \sum_{i=1}^n \alpha_i x_i$ $(n = 1,2,\ldots)$, *we have that* $\{y_n\}$ *is a basis for* X *iff the sequence* $\{\|y_n\|/|\alpha_{n+1}|\}$ *is bounded.*

PROOF. Let $\{x_n^*\}$ be the sequence of coefficient functionals. Putting

$$y_n^* := \frac{1}{\alpha_n} x_n^* - \frac{1}{\alpha_{n+1}} x_{n+1}^* \quad (n = 1,2,\ldots), \tag{5.2}$$

one easily verifies that $\{y_n\},\{y_n^*\}$ is a complete biorthogonal system. Moreover, we claim that for all $x \in X$ and $n \in \mathbb{N}$,

$$\sum_{i=1}^n \langle x,y_i^*\rangle y_i = \sum_{i=1}^n \langle x,x_i^*\rangle x_i - \frac{\langle x,x_{n+1}^*\rangle}{\alpha_{n+1}} y_n. \tag{5.3}$$

Indeed, for any $x = \sum_{i=1}^\infty \beta_i x_i \in X$ and any $n \in \mathbb{N}$, we have, by (5.2)

$$\sum_{i=1}^n \langle x,y_i^*\rangle y_i = \sum_{i=1}^n \left[\frac{1}{\alpha_i} \langle x,x_i^*\rangle - \frac{1}{\alpha_{i+1}} \langle x,x_{i+1}^*\rangle\right] y_i =$$

$$= \sum_{i=1}^n \left[\frac{\beta_i}{\alpha_i} - \frac{\beta_{i+1}}{\alpha_{i+1}}\right] y_i = \sum_{i=1}^n \left[\frac{\beta_i}{\alpha_i} - \frac{\beta_{i+1}}{\alpha_{i+1}}\right]\left(\sum_{k=1}^i \alpha_k x_k\right) =$$

$$= \sum_{k=1}^n \left[\sum_{i=k}^n \left(\frac{\beta_i}{\alpha_i} - \frac{\beta_{i+1}}{\alpha_{i+1}}\right)\right]\alpha_k x_k = \sum_{k=1}^n \left(\frac{\beta_k}{\alpha_k} - \frac{\beta_{n+1}}{\alpha_{n+1}}\right)\alpha_k x_k =$$

$$= \sum_{k=1}^n \beta_k x_k - \frac{\beta_{n+1}}{\alpha_{n+1}} \sum_{k=1}^n \alpha_k x_k = \sum_{k=1}^n \langle x,x_k^*\rangle x_k - \frac{\langle x,x_{n+1}^*\rangle}{\alpha_{n+1}} y_n.$$

Now let us assume that the sequence of numbers $\gamma_n := \|y_n\|/|\alpha_{n+1}|$ is unbounded. Then, using that $\sup_{n\in\mathbb{N}} \|x_n\| < \infty$, we can extract a subsequence $\{\gamma_{n_k}\}$ such that $\lim_{k\to\infty} \gamma_{n_k} = \infty$ and $\sum_{k=1}^\infty 1/\sqrt{\gamma_{n_k}}\, x_{n_k+1}$ converges, say to z. Then $\langle z,x_{n_k+1}^*\rangle = 1/\sqrt{\gamma_{n_k}}$ $(k = 1,2,\ldots)$ and $\langle z,x_i^*\rangle = 0$ for $i \notin \{n_k+1 : k \in \mathbb{N}\}$. Hence

$$\lim_{k\to\infty} \left\| \frac{\langle z,x_{n_k+1}^*\rangle}{|\alpha_{n_k+1}|} y_{n_k} \right\| = \lim_{k\to\infty} \sqrt{\gamma_{n_k}} = \infty. \tag{5.4}$$

On the other hand

$$(5.5) \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \langle z, x_i^* \rangle x_i = z.$$

(5.3), (5.4) and (5.5) now imply that the series $\sum_{i=1}^{\infty} \langle z, y_i^* \rangle y_i$ diverges, so that $\{y_n\}$ fails to be a basis for X, by Theorem 5.6 (iv).

For the converse, suppose that $\{\|y_n\|/|\alpha_{n+1}|\}$ is bounded. Observe first that the convergence of $\sum_{n=1}^{\infty} \langle x, x_n^* \rangle x_n$ together with $\inf_{n\in\mathbb{N}} \|x_n\| > 0$ implies that $\lim_{n\to\infty} \langle x, x_n^* \rangle = 0$ for every $x \in X$. Now the convergence of $\sum_{i=1}^{\infty} \langle x, x_i^* \rangle x_i$, the boundedness of $\{\|y_n\|/|\alpha_{n+1}|\}$ and $\lim_{n\to\infty} \langle x, x_{n+1}^* \rangle = 0$, imply, by (5.3), that $\sum_{i=1}^{\infty} \langle x, y_i^* \rangle y_i$ converges to x for every $x \in X$. Thus $\{y_n\}$ is a basis for X, again by Theorem 5.6 (iv). □

<u>PROPOSITION 5.10</u>. *Let $\{x_n\}$ be a basis for a Banach space X, let $0 = m_0 < m_1 < \ldots < m_n < \ldots$ be an increasing subsequence of $\mathbb{N}$, and let $\{y_n\}$ be a sequence in X such that for each $n \in \mathbb{N}$ $\{y_i\}_{i=m_{n-1}+1}^{m_n}$ is a basis for $[x_i]_{i=m_{n-1}+1}^{m_n}$. Suppose furthermore that there exists an $M > 0$ such that*

$$\nu_{\{y_i\}_{i=m_{n-1}+1}^{m_n}} \le M$$

for all $n \in \mathbb{N}$. Then $\{y_n\}$ is a basis for X.

<u>PROOF</u>. Since clearly $[y_n] = X$ it suffices to verify (ii) in Theorem 5.6. Let $\ell, k \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_{\ell+k} \in \mathbb{R}$ be arbitrary. We must show that $\|\sum_{i=1}^{\ell} \alpha_i y_i\| \le C\|\sum_{i=1}^{\ell+k} \alpha_i y_i\|$ for some constant C independent of these choices. Choose n and q so that

$$m_{n-1}+1 \le \ell \le m_n \quad \text{and} \quad m_{q-1}+1 \le \ell+k \le m_q.$$

Then $n \le q$ and we assume that $n < q$, leaving the (simpler) case $n = q$ to the reader. Since for each $j \in \mathbb{N}$ $\{x_i\}_{i=m_{j-1}+1}^{m_j}$ and $\{y_i\}_{i=m_{j-1}+1}^{m_j}$ are bases for the same subspace $[x_i]_{i=m_{j-1}+1}^{m_j}$, we can write

$$\sum_{i=m_{j-1}+1}^{m_j} \alpha_i y_i = \sum_{i=m_{j-1}+1}^{m_j} \beta_i x_i \qquad (j = 1, \ldots, q-1)$$

and

$$\sum_{i=m_{q-1}+1}^{\ell+k} \alpha_i y_i = \sum_{i=m_{q-1}+1}^{m_q} \beta_i x_i.$$

Then we have, by the definition of the norm of a basis,

$$\begin{aligned}
\Big\| \sum_{i=1}^{\ell} \alpha_i y_i \Big\| &\le \Big\| \sum_{i=1}^{m_{n-1}} \alpha_i y_i \Big\| + \Big\| \sum_{i=m_{n-1}+1}^{\ell} \alpha_i y_i \Big\| \le \\
&\le \Big\| \sum_{i=1}^{m_{n-1}} \beta_i x_i \Big\| + M \Big\| \sum_{i=m_{n-1}+1}^{m_n} \alpha_i y_i \Big\| = \\
&= \Big\| \sum_{i=1}^{m_{n-1}} \beta_i x_i \Big\| + M \Big\| \sum_{i=m_{n-1}+1}^{m_n} \beta_i x_i \Big\| \le \\
&\le \Big\| \sum_{i=1}^{m_{n-1}} \beta_i x_i \Big\| + M \left(\Big\| \sum_{i=1}^{m_{n-1}} \beta_i x_i \Big\| + \Big\| \sum_{i=1}^{m_n} \beta_i x_i \Big\| \right) \le \\
&\le \nu_{\{x_n\}} \Big\| \sum_{i=1}^{m_q} \beta_i x_i \Big\| + 2M\nu_{\{x_n\}} \Big\| \sum_{i=1}^{m_q} \beta_i x_i \Big\| = \\
&= (2M+1)\nu_{\{x_n\}} \Big\| \sum_{i=1}^{\ell+k} \alpha_i y_i \Big\| .
\end{aligned}$$

Thus (ii) in Theorem 5.6 holds with $C = (2M+1)\nu_{\{x_n\}}$. □

<u>DEFINITION 5.11</u>. A sequence $\{x_n\}$ in a Banach space is called a *basic sequence* iff it is a basis for its closed linear span $[x_n]$.

It is obvious that (ii) in Theorem 5.6 characterizes basic sequences among all non-zero sequences (simply replace X by $[x_n]$). The norm of a basic sequence (which is by definition the norm of the basis $\{x_n\}$ for $[x_n]$) is again the smallest C satisfying (ii) in Theorem 5.6. A subsequence of a basis is always a basic sequence. One might think that, given a basic sequence in a separable Banach space X, there exists a basis for X of which it is a subsequence. However, since there exists a separable space without a basis and since (as we shall soon see) every Banach space contains basic sequences, this is not true. Even if X has a basis, it may contain a basic sequence which cannot be extended to a basis for X ([98]).

The existence of basic sequences will be a consequence of the following simple

<u>LEMMA 5.12</u>. *Let* X *be a Banach space with* $\dim X = \infty$. *Then, given any* $\varepsilon > 0$ *and any subspace* $N \subset X$ *with* $\dim N < \infty$, *there exists a subspace* $M \subset X$ *with* $\operatorname{codim} M < \infty$ *such that* $\operatorname{dist}(S_N, M) > 1-\varepsilon$.

PROOF. Let $\{x_1,\dots,x_k\}$ be an ε-net for S_N and choose for every $i \in \{1,\dots,k\}$ an $x_i^* \in X^*$ such that $\|x_i^*\| = \langle x_i, x_i^*\rangle = 1$. We claim that $M := \bigcap_{i=1}^{k} \ker x_i^*$ satisfies the requirement. Indeed, let $x \in S_N$ and $y \in M$ be arbitrary. Pick $i_0 \in \{1,\dots,k\}$ so that $\|x-x_{i_0}\| < \varepsilon$. Then

$$\|x+y\| \geq \|x_{i_0}+y\| - \|x-x_{i_0}\| \geq \langle x_{i_0}+y, x_{i_0}^*\rangle - \|x-x_{i_0}\| =$$

$$= \langle x_{i_0}, x_{i_0}^*\rangle - \|x-x_{i_0}\| > 1-\varepsilon. \qquad \square$$

PROPOSITION 5.13. *Let* X *be an infinite-dimensional Banach space and let* $\varepsilon > 0$. *Then* X *contains a basic sequence with norm* $< \frac{1}{1-\varepsilon}$.

PROOF. We define the required basic sequence inductively. Let $x_1 \neq 0$ be arbitrary. Then, putting $N_1 := [x_1]$, let M_1 with $\operatorname{codim} M_1 < \infty$ be such that Lemma 5.12 is satisfied for N_1, M_1 and ε. Pick $x_2 \in M_1\setminus\{0\}$. Now let us suppose that for some $n > 1$, $x_1,\dots,x_n$ and $M_1,\dots,M_{n-1}$ have been defined so that $\operatorname{codim} M_i < \infty$ $(i = 1,\dots,n-1)$, $x_{i+1} \in \bigcap_{j=1}^{i} M_j\setminus\{0\}$ $(i = 1,\dots,n-1)$ and $\operatorname{dist}(S_{N_i}, M_i) > 1-\varepsilon$ $(i = 1,\dots,n-1)$, where $N_i := [x_j]_{j=1}^{i}$. Then, putting $N_n := [x_j]_{j=1}^{n}$, we determine M_n as in Lemma 5.12 so that $\operatorname{codim} M_n < \infty$ and $\operatorname{dist}(S_{N_n}, M_n) > 1-\varepsilon$ and thereafter pick $x_{n+1} \in \bigcap_{j=1}^{n} M_j\setminus\{0\}$. This completes the inductive definition of $\{x_n\}$. We claim that (ii) in Theorem 5.6 now holds for $\{x_n\}$ with $C = \frac{1}{1-\varepsilon}$. To see this, let $n,m \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_{n+m} \in \mathbb{R}$ be arbitrary. We may assume without restricting the generality that $\|\sum_{i=1}^{n} \alpha_i x_i\| = 1$. Then, since $x_{n+1},\dots,x_{n+m}$ are chosen in M_n and $\sum_{i=1}^{n} \alpha_i x_i \in S_{N_n}$, we have

$$\|\sum_{i=1}^{n+m} \alpha_i x_i\| \geq \operatorname{dist}(S_{N_n}, M_n) > 1-\varepsilon = (1-\varepsilon)\|\sum_{i=1}^{n} \alpha_i x_i\|. \qquad \square$$

Although basic sequences (even in a Banach space with a basis) cannot in general be extended to bases, there exists a certain special type of basic sequences (called block basic sequences) which always do admit such extensions.

LEMMA 5.14. *Let* $\{x_n\}$ *be a basis for a Banach space* X, $0 = m_0 < m_1 < \dots < m_n < \dots$ *an increasing subsequence of* $\mathbb{N}$ *and* $\{\alpha_n\} \subset \mathbb{R}$. *Then the sequence* $\{y_n\}$ *defined by*

$$y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \qquad (n = 1,2,\dots)$$

is a basic sequence provided $y_n \neq 0$ *for all* $n \in \mathbb{N}$. *Moreover,* $\nu_{\{y_n\}} \leq \nu_{\{x_n\}}$.

PROOF. For every $n,k \in \mathbb{N}$ and $\gamma_1,\dots,\gamma_{n+k} \in \mathbb{R}$ we have

$$\|\sum_{j=1}^{n} \gamma_j y_j\| = \|\sum_{j=1}^{n} \sum_{i=m_{j-1}+1}^{m_j} \gamma_j \alpha_i x_i\| \leq \nu_{\{x_n\}} \|\sum_{j=1}^{n+k} \sum_{i=m_{j-1}+1}^{m_j} \gamma_j \alpha_i x_i\| =$$

$$= \nu_{\{x_n\}} \|\sum_{j=1}^{n+k} \gamma_j y_j\|.$$

Hence $\{y_n\}$ is basic and $\nu_{\{y_n\}} \leq \nu_{\{x_n\}}$, by Theorem 5.6 (ii). □

DEFINITION 5.15. Let $\{x_n\}$ be a basis for a Banach space X. Any sequence $\{y_n\}$ as in Lemma 5.14 is called a *block basic sequence* (with respect to the basis $\{x_n\}$).

Before showing now that any block basic sequence can be extended to a basis, we make a simple observation.

LEMMA 5.16. *Let* Y *and* Z *be two hyperplanes in a finite-dimensional Banach space* X. *Then there exists an isomorphism* $T: Y \to Z$ *satisfying*

$$\tfrac{1}{3}\|y\| \leq \|Ty\| \leq 3\|y\| \quad \textit{for all } y \in Y.$$

PROOF. Unless $Y = Z$ (in which case the lemma is trivial), $Y \cap Z$ has codimension 1 in Y as well as in Z. Let us write $Y \cap Z = \{y \in Y: \langle y,y^*\rangle = 0\}$, with $y^* \in Y^*$, $\|y^*\| = 1$. Then by compactness there exists a $y_0 \in Y$ such that $\langle y_0,y^*\rangle = \|y_0\| = 1$. It follows that $P(y) = y - \langle y,y^*\rangle y_0$ $(y \in Y)$ defines a projection from Y onto $Y \cap Z$ with $\|I_Y - P\| = 1$, so $\|P\| \leq 2$. Similarly, there exists a projection Q from Z onto $Y \cap Z$ with $\|I_Z - Q\| = 1$, so $\|Q\| \leq 2$. Choose $z_0 \in \ker Q$, $\|z_0\| = 1$. We claim that the formula

$$T(x + \alpha y_0) = x + \alpha z_0 \quad (x \in Y \cap Z,\ \alpha \in \mathbb{R})$$

defines the desired isomorphism. Indeed, for all $x \in Y \cap Z$ and $\alpha \in \mathbb{R}$,

$$\|T(x+\alpha y_0)\| = \|x+\alpha z_0\| \leq \|x\|+|\alpha| = \|x\|+\|\alpha y_0\| =$$

$$= \|P(x+\alpha y_0)\| + \|(I_Y-P)(x+\alpha y_0)\| \leq 3\|x+\alpha y_0\|,$$

and analogously

$$\|T^{-1}(x+\alpha z_0)\| \leq 3\|x+\alpha z_0\|.$$ □

PROPOSITION 5.17. *Let* $\{x_n\}$ *be a basis for a Banach space* X *and let* $\{y_n\}$ *with*

$$y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \neq 0 \quad (n \in \mathbb{N},\ 0 = m_0 < m_1 < \dots)$$

be a block basic sequence with respect to $\{x_n\}$. *Then there exists a basis* $\{z_n\}$ *for* X *such that*

$$(5.6) \qquad z_{m_n} = y_n \qquad (n = 1,2,\dots)$$

and

$$(5.7) \qquad [z_i]_{i=m_{n-1}+1}^{m_n} = [x_i]_{i=m_{n-1}+1}^{m_n} \qquad (n = 1,2,\dots).$$

PROOF. Put $W_n := [x_i]_{i=m_{n-1}+1}^{m_n}$ $(n = 1,2,\dots)$. Then $y_n \in W_n$ $(n \in \mathbb{N})$ and we can select for every n a $w_n^* \in W_n^*$ satisfying $\langle y_n, w_n^*\rangle = \|w_n^*\|\|y_n\| = 1$. Evidently

$$P_n(x) = x - \langle x, w_n^*\rangle y_n \qquad (x \in W_n)$$

defines a projection P_n of W_n onto a hyperplane $Z_n \subset W_n$ with kernel spanned by y_n and with $\|P_n\| \le 2$. Since both Z_n and $Y_n := [x_i]_{i=m_{n-1}+1}^{m_n-1}$ are hyperplanes in W_n, by Lemma 5.16 there exists an isomorphism $T_n\colon Y_n \to Z_n$ satisfying

$$(5.8) \qquad \tfrac{1}{3}\|x\| \le \|T_n x\| \le 3\|x\| \quad \text{for all } x \in Y_n.$$

We now define $\{z_n\}$ by

$$z_i = \begin{cases} T_n(x_i) & \text{if } m_{n-1}+1 \le i \le m_n - 1 \\ y_n & \text{if } i = m_n \end{cases} \qquad (i = 1,2,\dots).$$

It is obvious that (5.6) and (5.7) are satisfied. To prove that $\{z_n\}$ is a basis we use Proposition 5.10. Note first that, for all choices of $n \in \mathbb{N}$ and $\beta_{m_{n-1}+1},\dots,\beta_{m_n} \in \mathbb{R}$

$$(5.9) \qquad P_n\Bigl(\sum_{i=m_{n-1}+1}^{m_n} \beta_i z_i\Bigr) = \sum_{i=m_{n-1}+1}^{m_n-1} \beta_i z_i,$$

since $z_{m_n} = y_n \in \ker P_n$ and since $z_{m_{n-1}+1},\dots,z_{m_n-1} \in Z_n = \text{range } P_n$. Now

(5.8), (5.9) and $\|P_n\| \leq 2$ $(n = 1,2,\ldots)$ imply that for all choices of $n \in \mathbb{N}$, $m_{n-1}+1 \leq k < \ell \leq m_n$ and $\beta_{m_{n-1}+1},\ldots,\beta_\ell \in \mathbb{R}$, we have, taking $\beta_{\ell+1} = \ldots = \beta_{m_n} = 0$,

$$\begin{aligned}
\Big\|\sum_{i=m_{n-1}+1}^{k} \beta_i z_i\Big\| &= \Big\|\sum_{i=m_{n-1}+1}^{k} \beta_i T_n(x_i)\Big\| \leq \\
&\leq 3\Big\|\sum_{i=m_{n-1}+1}^{k} \beta_i x_i\Big\| \leq 3\nu_{\{x_j\}}\Big\|\sum_{i=m_{n-1}+1}^{m_n-1} \beta_i x_i\Big\| \leq \\
&\leq 9\nu_{\{x_j\}}\Big\|\sum_{i=m_{n-1}+1}^{m_n-1} \beta_i T_n(x_i)\Big\| = 9\nu_{\{x_j\}}\Big\|P_n\Big(\sum_{i=m_{n-1}+1}^{m_n} \beta_i z_i\Big)\Big\| \leq \\
&\leq 18\nu_{\{x_j\}}\Big\|\sum_{i=m_{n-1}+1}^{m_n} \beta_i z_i\Big\| = 18\nu_{\{x_j\}}\Big\|\sum_{i=m_{n-1}+1}^{\ell} \beta_i z_i\Big\|.
\end{aligned}$$

Thus the condition of Proposition 5.10 is satisfied with

$$M = 18\nu_{\{x_j\}},$$

so that $\{z_n\}$ is a basis for X. □

<u>NOTES</u>. Most of the material of this section is well known. The notion of a basis goes back to J. SCHAUDER ([93]), who also constructed the basis for C([0,1]) given here. S. BANACH (cf. [5]) was already aware of Theorem 5.6 and Proposition 5.13. Proposition 5.9 is due to B.R. GELBAUM ([36]). Explicit proofs of Propositions 5.10 and 5.17 and Lemma 5.16 were given by M. ZIPPIN ([106]). A general reference for questions concerning bases is [97], from which some of our proofs were taken.

6. CHARACTERIZATIONS OF REFLEXIVITY IN TERMS OF BASES AND BASIC SEQUENCES

Before proceeding to the main results of this section we introduce shrinking and boundedly complete bases. First we have a look at the coefficient functionals of a basis and at the subspace spanned by them.

PROPOSITION 6.1. *Let* $\{x_n\}$ *be a basis for a Banach space* X *and let* $\{x_n^*\}$ *be its sequence of coefficient functionals. Then* $\{x_n^*\}$ *is a basic sequence and* $\nu_{\{x_n^*\}} \le \nu_{\{x_n\}}$. *The sequence of coefficient functionals associated to the basis* $\{x_n^*\}$ *for* $[x_n^*]$ *is*

$$\{\pi(x_n)\big|_{[x_n^*]}\}.$$

Hence

$$x^* = \sum_{n=1}^{\infty} \langle x^*, \pi(x_n)\rangle x_n^* = \sum_{n=1}^{\infty} \langle x_n, x^*\rangle x_n^* \quad \textit{for all } x^* \in [x_n^*].$$

PROOF. Let $\{P_n\}$ be the sequence of projections associated to the basis $\{x_n\}$. By the biorthogonality of $\{x_n\},\{x_n^*\}$ we have, for all $x \in X$, $n,m \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_{n+m} \in \mathbb{R}$,

$$\langle x, P_n^*(\sum_{i=1}^{n+m} \alpha_i x_i^*)\rangle = \langle P_n x, \sum_{i=1}^{n+m} \alpha_i x_i^*\rangle = \langle \sum_{j=1}^{n} \langle x, x_j^*\rangle x_j, \sum_{i=1}^{n+m} \alpha_i x_i^*\rangle =$$

$$= \langle x, \sum_{i=1}^{n} \alpha_i x_i^*\rangle.$$

Thus

$$P_n^*(\sum_{i=1}^{n+m} \alpha_i x_i^*) = \sum_{i=1}^{n} \alpha_i x_i^*$$

and therefore

$$\|\sum_{i=1}^{n} \alpha_i x_i^*\| \le \|P_n^*\| \|\sum_{i=1}^{n+m} \alpha_i x_i^*\| \qquad (n,m \in \mathbb{N}, \alpha_1,\dots,\alpha_{n+m} \in \mathbb{R}).$$

By Theorem 5.6 it follows that $\{x_n^*\}$ is a basic sequence with norm

$$\nu_{\{x_n^*\}} \le \sup_{n\in\mathbb{N}} \|P_n^*\| = \sup_{n\in\mathbb{N}} \|P_n\| = \nu_{\{x_n\}}.$$

The last statement in the proposition is clear since $\langle x_j^*, \pi x_i\rangle = \langle x_i, x_j^*\rangle = \delta_{ij}$ $(i = 1,2,\ldots)$. □

REMARK 6.2. Observe that the above proof shows that the n-th projection associated to the basis $\{x_n^*\}$ for $[x_n^*]$ is

$$P_n^*\big|_{[x_n^*]}.$$

In particular, since $\dim R(P_n^*) = n$, it follows that

$$R(P_n^*) = R(P_n^*\big|_{[x_n^*]}) = [x_i^*]_{i=1}^n.$$

PROPOSITION 6.3. *Let* $\{x_n\}$ *be a basis for a Banach space* X *and let* $V := [x_n^*] \subset X^*$. *Then* $r(V) \ge \dfrac{1}{\nu\{x_n\}} > 0$.

PROOF. Let $x \in S_X$ and $\varepsilon > 0$ be arbitrary. Then since $x = \lim_{n\to\infty} P_n x$, there exists an n_0 such that $\|x - P_{n_0}x\| < \varepsilon$. Thus

$$1 = \|x\| = \sup_{x^*\in B_{X^*}} |\langle x,x^*\rangle| \le \sup_{x^*\in B_{X^*}} |\langle P_{n_0}x,x^*\rangle| + \varepsilon =$$

$$= \sup_{x^*\in B_{X^*}} |\langle x,P_{n_0}^* x^*\rangle| + \varepsilon \le \|P_{n_0}^*\| \sup_{x^*\in B_V} |\langle x,x^*\rangle| + \varepsilon.$$

Hence

$$\sup_{x^*\in B_V} |\langle x,x^*\rangle| \ge \frac{1-\varepsilon}{\sup_{n\in\mathbb{N}}\|P_n\|} = \frac{1-\varepsilon}{\nu\{x_n\}}.$$

Since $\varepsilon > 0$ and $x \in S_X$ were arbitrary, Lemma 4.3 now shows that $r(V) \ge \dfrac{1}{\nu_{\{x_n\}}}$. □

DEFINITION 6.4. A basis $\{x_n\}$ for a Banach space X is called *shrinking* iff $[x_n^*] = X^*$ and *boundedly complete* iff for every sequence $\{\alpha_n\} \subset \mathbb{R}$ $\sum_{i=1}^\infty \alpha_i x_i$ converges whenever $\{\sum_{i=1}^n \alpha_i x_i\}_{n=1}^\infty$ is bounded. A basic sequence $\{x_n\}$ is called shrinking (respectively, boundedly complete) iff $\{x_n\}$ is a shrinking (respectively, boundedly complete) basis for $[x_n]$.

The following simple lemma gives an equivalent formulation of shrinkingness.

LEMMA 6.5. *A basis* $\{x_n\}$ *for a Banach space* X *is shrinking iff* $\lim_{n\to\infty}\|x^*\|_n = 0$ *for every* $x^* \in X^*$, *where*

$$\|x^*\|_n := \|x^*|_{[x_i]_{i=n+1}^{\infty}}\|.$$

PROOF. $\|x^*\|_n$ is by definition the norm of x^* restricted to $[x_i]_{i=n+1}^{\infty}$. Since

$$([x_i]_{i=n+1}^{\infty})^* \cong X^*/([x_i]_{i=n+1}^{\infty})^{\perp}$$

and since clearly $([x_i]_{i=n+1}^{\infty})^{\perp} = [x_i^*]_{i=1}^{n}$, we have $\|x^*\|_n = \text{dist}(x^*,[x_i^*]_{i=1}^{n})$. Hence $\lim_{n\to\infty}\|x^*\|_n = 0$ for all $x^* \in X^*$ is equivalent to $[x_n^*] = X^*$. □

DEFINITION 6.6. A sequence $\{x_n\}$ in a topological vector space X is called a *basis* for X iff every $x \in X$ admits a unique expansion $x = \sum_{i=1}^{\infty} \alpha_i x_i$. It is called a *Schauder basis* if, moreover, the associated coefficient functionals are continuous.

In case X is a Banach space every basis for X is a Schauder basis, as we have seen in Proposition 5.2. The proof depended on the closed graph theorem and for general topological vector spaces it does not hold. (See [97] for examples.)

We now establish a duality between bases and w^*-Schauder bases.

PROPOSITION 6.7. *Let* $\{x_n^*\}$ *be a sequence in a dual Banach space* X^*. *Then the following are equivalent:*

(i) $\{x_n^*\}$ *is a* w^**-Schauder basis for* X^* *(i.e. a Schauder basis for the t.v.s.* $(X^*,\sigma(X^*,X))$*);*

(ii) X *has a basis* $\{x_n\}$ *which has* $\{x_n^*\}$ *for its sequence of coefficient functionals.*

PROOF. (i) ⇒ (ii): Assume that $\{x_n^*\}$ is a w^*-Schauder basis for X^*. By definition the coefficient functionals are w^*-continuous, so we may identify them with a sequence $\{x_n\}$ in X. Then $\{x_n\},\{x_n^*\}$ is a biorthogonal system and for every $x^* \in X^*$ we have $x^* = \sum_{i=1}^{\infty} \langle x_i,x^*\rangle x_i^*$, where the convergence is w^*. This means that

$$(6.1) \qquad \langle x,x^*\rangle = \sum_{i=1}^{\infty} \langle x_i,x^*\rangle\langle x,x_i^*\rangle \qquad \text{for all } x \in X,\ x^* \in X^*.$$

It follows now from the Banach-Steinhaus theorem applied twice that the projections $P_n := \sum_{i=1}^n \langle\cdot,x_i^*\rangle x_i$ $(n = 1,2,\dots)$ are uniformly bounded. By Theorem 5.6 it remains only to show that $[x_n] = X$. If not, then $\langle x_n,x_0^*\rangle = 0$ $(n = 1,2,\dots)$ for some $x_0^* \neq 0$ in X^*, contradicting (6.1).

(ii) $\Rightarrow$ (i): Let $\{x_n\}$ be a basis for X, with coefficient functionals $\{x_n^*\}$. Then for arbitrary $x \in X$ and $x^* \in X^*$ we have $x = \sum_{i=1}^\infty \langle x,x_i^*\rangle x_i$, so $\langle x,x^*\rangle = \sum_{i=1}^\infty \langle x,x_i^*\rangle\langle x_i,x^*\rangle$. Hence every $x^* \in X^*$ has the expansion $x^* = \sum_{i=1}^\infty \langle x_i,x^*\rangle x_i^*$, where the convergence is w^*. This expansion is unique. Indeed, if $\sum_{i=1}^\infty \alpha_i x_i^* = 0$, i.e. $\sum_{i=1}^\infty \alpha_i\langle x,x_i^*\rangle = 0$ for all $x \in X$, then substituting $x = x_i$ and using the biorthogonality of $\{x_n\},\{x_n^*\}$ yields $\alpha_i = 0$ $(i = 1,2,\dots)$. Finally, it is obvious that the coefficient functionals are $\{\pi(x_i)\}_{i=1}^\infty$, and these are w^*-continuous. □

We now prove a duality between shrinking and boundedly complete bases.

PROPOSITION 6.8. *Let $\{x_n\}$ be a basis for a Banach space X, $\{x_n^*\}$ its sequence of coefficient functionals, and put $V := [x_n^*]$. Then*

(i) *$\{x_n\}$ is boundedly complete iff $\{x_n^*\}$ is a shrinking basic sequence (i.e. a shrinking basis for V);*

(ii) *$\{x_n\}$ is shrinking iff $\{x_n^*\}$ is a boundedly complete basic sequence (i.e. a boundedly complete basis for V).*

PROOF. Let us observe first that, since $r(V) > 0$ by Proposition 6.3, the canonical map $T\colon X \to V^*$ defined by $Tx = \pi(x)_{|V}$ $(x \in X)$ is an isomorphic embedding. Indeed, by Lemma 4.3 $r(V)\|x\| \le \|Tx\| \le \|x\|$ for all $x \in X$.

Now, for the proof of (ii), suppose first that $\{x_n\}$ is shrinking. Then $V = X^*$, i.e. $\{x_n^*\}$ is a basis for X^*, with coefficient functionals $\{\pi x_n\}$. Let a sequence $\{\alpha_n\} \subset \mathbb{R}$ be given such that $\{\sum_{i=1}^n \alpha_i x_i^*\}_{n=1}^\infty$ is bounded. Then by Alaoglu's theorem $\{\sum_{i=1}^n \alpha_i x_i^*\}_{n=1}^\infty$ has a w^*-limit point $x^* \in X^*$. Since $\langle x_i,\sum_{j=1}^n \alpha_j x_j^*\rangle = \alpha_i$ whenever $n \ge i$, we have $\langle x^*,\pi x_i\rangle = \langle x_i,x^*\rangle = \alpha_i$ for all $i \in \mathbb{N}$ and it follows that $x^* = \sum_{i=1}^\infty \alpha_i x_i^*$. Thus $\sum_{i=1}^\infty \alpha_i x_i^*$ converges in norm. This proves that $\{x_n^*\}$ is boundedly complete.

Conversely, suppose that $\{x_n^*\}$ is a boundedly complete basis for V. By Proposition 6.7, $\{x_n^*\}$ is a w^*-Schauder basis for X^*. We must show that $V = X^*$. Let $x^* \in X^*$ be given arbitrarily. Then $x^* = w^* \lim_{n\to\infty} \sum_{i=1}^n \langle x_i,x^*\rangle x_i^*$. Thus by the Banach-Steinhaus theorem $\{\sum_{i=1}^n \langle x_i,x^*\rangle x_i^*\}_{n=1}^\infty$ is bounded. Since $\{x_n^*\}$ is boundedly complete, the series $\sum_{i=1}^{\infty} \langle x_i, x^* \rangle x_i^*$ is norm convergent, obviously to x^*. Hence $x^* \in V$. This completes the proof that $\{x_n\}$ is shrinking.

To prove (i), observe that, since $\{x_n^*\}$ is a basis for V and $\{Tx_n\}$ its sequence of coefficient functionals, it follows from (ii) that $\{x_n^*\}$ is shrinking iff $\{Tx_n\}$ is boundedly complete, and therefore, since T is an isomorphism, iff $\{x_n\}$ is boundedly complete. □

We now come to the first main theorem of this section, which characterizes reflexive spaces among those with basis.

THEOREM 6.9. *Let X be a Banach space with a basis $\{x_n\}$. Then X is reflexive iff $\{x_n\}$ is shrinking and boundedly complete. Hence all bases for a reflexive space are shrinking and boundedly complete.*

PROOF. Suppose that X is reflexive. Again put $V := [x_n^*]$, where $\{x_n^*\}$ is the sequence of coefficient functionals of $\{x_n\}$. We know already that V is w^* dense in X^* (even $r(V) > 0$). By reflexivity the w^* and the norm topology on X^* are compatible, so $V = X^*$ (Proposition 0.5), i.e. $\{x_n\}$ is shrinking. To show that $\{x_n\}$ is boundedly complete it suffices to observe that $\{x_n^*\}$ is a basis for the reflexive space $V = X^*$ and therefore, by the above, shrinking. Proposition 6.8 (i) now shows that $\{x_n\}$ is boundedly complete.

Let us now suppose, conversely, that $\{x_n\}$ is shrinking and boundedly complete. Let $x^{**} \in X^{**}$. We claim that $x^{**} = \sum_{i=1}^{\infty} \langle x_i^*, x^{**} \rangle \pi x_i$, in the w^* sense. Indeed, since $\{x_n\}$ is shrinking, $\{x_n^*\}$ is a basis for X^* and therefore every $x^* \in X^*$ can be written as $x^* = \sum_{i=1}^{\infty} \langle x_i, x^* \rangle x_i^*$. Hence $\langle x^*, x^{**} \rangle =$ $= \sum_{i=1}^{\infty} \langle x_i, x^* \rangle \langle x_i^*, x^{**} \rangle = \sum_{i=1}^{\infty} \langle x^*, \pi x_i \rangle \langle x_i^*, x^{**} \rangle$, proving our claim. It follows, by the Banach-Steinhaus theorem, that the sequence $\{\sum_{i=1}^{n} \langle x_i^*, x^{**} \rangle \pi x_i\}_{n=1}^{\infty}$ is bounded. The assumption that $\{x_n\}$ is boundedly complete now implies that $\sum_{i=1}^{\infty} \langle x_i^*, x^{**} \rangle \pi x_i$ norm converges, obviously to x^{**}. Hence $x^{**} \in \pi X$ and we have shown that X is reflexive. □

REMARK 6.10. The standard basis $\{e_n\}$ for ℓ^1 is boundedly complete, but not shrinking (since $(\ell^1)^* = \ell^\infty$ is non-separable). On the other hand, the standard basis $\{e_n\}$ for c_0 is shrinking, since its coefficient functionals form the standard basis for ℓ^1, but not boundedly complete, for $\{\sum_{i=1}^{n} e_i\}$ is bounded but not convergent. Hence shrinkingness or boundedly completeness alone do not imply reflexivity. The next result shows, however, that the requirement that all bases for a space are either all shrinking or all boundedly complete, is enough to guarantee reflexivity.

THEOREM 6.11. *Let X be a Banach space with a basis. Then X is reflexive if (and only if) either* (i) *or* (ii) *below is satisfied.*

(i) *All bases for X are shrinking.*

(ii) *All bases for X are boundedly complete.*

PROOF. In view of Theorem 6.9 it suffices to prove the equivalence of (i) and (ii).

(i) ⇒ (ii): Suppose $\{x_n\}$ is a non-boundedly complete basis for X. Then for some sequence $\{\alpha_n\} \subset \mathbb{R}$ and some constant M we have

$$\| \sum_{i=1}^{n} \alpha_i x_i \| \le M \quad (n = 1,2,\ldots), \tag{6.2}$$

while $\{\sum_{i=1}^{n} \alpha_i x_i\}$ is not a Cauchy sequence. Thus there exists an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ $\|\sum_{i=n}^{n+k} \alpha_i x_i\| > \varepsilon$ for some $k \in \mathbb{N}$. In particular there is a sequence $0 = m_0 < m_1 < \ldots < m_n < \ldots$ in $\mathbb{N}$ such that

$$\varepsilon < \| \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \| \le 2M \quad \text{for all } n = 1,2,\ldots . \tag{6.3}$$

Now by Proposition 5.17 there exists a basis $\{z_n\}$ for X with

$$z_{m_n} = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \quad (n = 1,2,\ldots). \tag{6.4}$$

Let $\{z_n^*\}$ be its sequence of coefficient functionals. We now define

$$\begin{aligned} u_i &= \begin{cases} z_i & \text{if } i \notin \{m_1, m_2, \ldots\} \\ \sum_{j=1}^{n} z_{m_j} & \text{if } i = m_n \ (n = 1,2,\ldots) \end{cases} \\ u_i^* &= \begin{cases} z_i^* & \text{if } i \notin \{m_1, m_2, \ldots\} \\ z_{m_n}^* - z_{m_{n+1}}^* & \text{if } i = m_n \ (n = 1,2,\ldots), \end{cases} \quad (i = 1,2,\ldots). \end{aligned} \tag{6.5}$$

Evidently $\{u_n\},\{u_n^*\}$ is a complete biorthogonal system for X. We shall show that $\{u_n\}$ is a non-shrinking basis for X, contradicting the assumption (i). For $\{u_n\}$ to be a basis, it suffices by Proposition 5.4 that the projections $U_n := \sum_{i=1}^{n} \langle \cdot, u_i^* \rangle u_i$ $(n = 1,2,..)$ be uniformly bounded. Pick $n \in \mathbb{N}$ and suppose $m_k \le n < m_{k+1}$. Then for every $x \in X$ we have, by (6.5), a change of summation, and (6.2),

$$\|U_n(x)\| = \Big\| \sum_{\substack{i=1\\ i\neq m_j}}^{n} \langle x,z_i^*\rangle z_i + \sum_{j=1}^{k} [\langle x,z_{m_j}^*\rangle - \langle x,z_{m_{j+1}}^*\rangle] \sum_{i=1}^{j} z_{m_i} \Big\|$$

$$\le \Big\| \sum_{i=1}^{n} \langle x,z_i^*\rangle z_i\Big\| + \Big\| - \sum_{j=1}^{k} \langle x,z_{m_j}^*\rangle z_{m_j} + \sum_{j=1}^{k} [\langle x,z_{m_j}^*\rangle - \langle x,z_{m_{j+1}}^*\rangle] \sum_{i=1}^{j} z_{m_i}\Big\|$$

$$= \Big\| \sum_{i=1}^{n} \langle x,z_i^*\rangle z_i\Big\| + \Big\| - \sum_{j=1}^{k} \langle x,z_{m_j}^*\rangle z_{m_j} + \sum_{j=1}^{k} [\langle x,z_{m_j}^*\rangle - \langle x,z_{m_{k+1}}^*\rangle] z_{m_j}\Big\|$$

$$\le \nu_{\{z_n\}}\|x\| + \Big\|\langle x,z_{m_{k+1}}^*\rangle \sum_{j=1}^{k} z_{m_j}\Big\| \le (\nu_{\{z_n\}} + \|z_{m_{k+1}}^*\| M)\|x\| .$$

Since by Proposition 5.2 $\|z_n^*\|\|z_n\| \le 2\nu_{\{z_n\}}$ for all $n \in \mathbb{N}$, and since also $\|z_{m_n}\| > \varepsilon$ for all $n \in \mathbb{N}$, it follows that $\{z_{m_n}^*\}$ is uniformly bounded. Hence $\{U_n\}$ is uniformly bounded and we have proved that $\{u_n\}$ is a basis for X. Furthermore, we have

$$\langle u_{m_n}, z_{m_1}^*\rangle = \langle \sum_{j=1}^{n} z_{m_j}, z_{m_1}^*\rangle = 1 \quad \text{for all } n \in \mathbb{N}$$

and it follows therefore from (6.2) and Lemma 6.5 that $\{u_n\}$ is not shrinking.

(ii) ⇒ (i): Suppose $\{x_n\}$ is a non-shrinking basis for X. Lemma 6.5 then yields the existence of an $x^* \in X^*$, $\|x^*\| = 1$ and an $\varepsilon > 0$ such that $\|x^*\|_n \ge \varepsilon$ for all $n \in \mathbb{N}$. Thus there exists for this x^* a sequence $\{y_n\}$ of the form

$$(6.6) \qquad y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \qquad (n = 1,2,\ldots),$$

where $0 = m_0 < m_1 < \ldots < m_n < \ldots$, such that

$$(6.7) \qquad 1 \le \|y_n\| \le \frac{2}{\varepsilon} \qquad (n = 1,2,\ldots)$$

and

$$(6.8) \qquad \langle y_n, x^*\rangle = 1 \qquad (n = 1,2,\ldots).$$

Let us put $W_n := [x_i]_{i=m_{n-1}+1}^{m_n}$ and $Z_n := \{x \in W_n\colon \langle x,x^*\rangle = 0\}$ $(n = 1,2,\ldots)$. Then the formula $P_n(x) = x - \langle x,x^*\rangle y_n$ $(x \in W_n)$ defines, for each n, a bounded projection from W_n onto Z_n. In fact the sequence $\{P_n\}$ is uniformly bounded: $\|P_n\| \le 1 + \|x^*\|\|y_n\| \le 1 + \|x^*\| \frac{2}{\varepsilon}$. Now the same argument that was used in the proof of Proposition 5.17 shows that there exists a basis $\{z_n\}$ for X satisfying

$$(6.9) \qquad 1 \le \|z_n\| \le \frac{2}{\varepsilon} \qquad (n = 1,2,\dots)$$

$$(6.10) \qquad z_{m_n} = y_n \qquad (n = 1,2,\dots)$$

$$(6.11) \qquad \langle z_i, x^* \rangle = 0 \qquad \text{for every } i \notin \{m_1, m_2, \dots\}.$$

As usual $\{z_n^*\}$ will denote its sequence of coefficient functionals. Let us now define

$$(6.12) \qquad u_i = \begin{cases} z_i & \text{if } i \notin \{m_2, m_3, \dots\} \\ z_{m_n} - z_{m_{n-1}} & \text{if } i = m_n \quad (n = 2,3,\dots) \end{cases}$$

$$u_i^* = \begin{cases} z_i^* & \text{if } i \notin \{m_1, m_2, \dots\} \\ x^* & \text{if } i = m_1 \\ x^* - \sum_{j=1}^{n-1} z_{m_j}^* & \text{if } i = m_n \quad (n = 2,3,\dots) \end{cases} \quad (i = 1,2,\dots).$$

Using (6.8), (6.10) and (6.11), it is easy to check that this system $\{u_n\},\{u_n^*\}$ is biorthogonal and, moreover, complete. We shall show that $\{u_n\}$ is a non-boundedly complete basis for X, contradicting the assumption (ii). For $\{u_n\}$ to be a basis, it suffices by Proposition 5.4 that the projections $Q_n := \sum_{i=1}^n \langle \cdot, u_i^* \rangle u_i$ $(n = 1,2,\dots)$ are uniformly bounded. Pick $n \in \mathbb{N}$ and suppose for definiteness that $m_k \le n < m_{k+1}$. Then, using (6.12), a change of summation and (6.9), it follows that for every $x \in X$,

$$\|Q_n x\| = \Big\| \sum_{\substack{i=1 \\ i \notin \{m_1,\dots,m_k\}}}^{n} \langle x, z_i^* \rangle z_i + \langle x, x^* \rangle z_{m_1} + \sum_{i=2}^{k} \Big[\langle x, x^* \rangle - \sum_{j=1}^{i-1} \langle x, z_{m_j}^* \rangle \Big] (z_{m_i} - z_{m_{i-1}}) \Big\|$$

$$(6.13) \qquad = \Big\| \sum_{\substack{i=1 \\ i \notin \{m_1,\dots,m_k\}}}^{n} \langle x, z_i^* \rangle z_i + \langle x, x^* \rangle z_{m_1} + \langle x, x^* \rangle (z_{m_k} - z_{m_1}) - \Big[\sum_{j=1}^{k-1} \langle x, z_{m_j}^* \rangle \Big] z_{m_k} + \sum_{j=1}^{k-1} \langle x, z_{m_j}^* \rangle z_{m_j} \Big\| =$$

$$= \Big\| \sum_{i=1}^{n} \langle x, z_i^* \rangle z_i + \langle x, x^* \rangle z_{m_k} - \sum_{j=1}^{k} \langle x, z_{m_j}^* \rangle z_{m_k} \Big\| \le$$

$$\le \Big[\nu_{\{z_n\}} + \|x^*\| \frac{2}{\varepsilon} + \Big\| \sum_{j=1}^{k} z_{m_j}^* \Big\| \frac{2}{\varepsilon} \Big] \|x\|.$$

By (6.8), (6.10) and (6.11) we have for every $x = \sum_{i=1}^{\infty} \alpha_i z_i \in X$

$$\lim_{n\to\infty} \sum_{j=1}^{n} \langle x, z_{m_j}^* \rangle = \lim_{n\to\infty} \sum_{j=1}^{n} \alpha_{m_j} = \sum_{j=1}^{\infty} \alpha_{m_j} = \langle x, x^* \rangle .$$

Thus the sequence $\{\sum_{j=1}^{n} z_{m_j}^*\}$ is w^*-convergent and therefore norm bounded. Together with (6.13) this shows that $\{Q_n\}$ is uniformly bounded, so that $\{u_n\}$ is a basis for X. However, $\{u_n\}$ is not boundedly complete, since by the second inequality in (6.9),

$$\| \sum_{k=2}^{n} u_{m_k} \| = \| \sum_{k=2}^{n} (z_{m_k} - z_{m_{k-1}}) \| = \| z_{m_n} - z_{m_1} \| \le \frac{4}{\varepsilon} \quad (n = 2,3,\ldots),$$

whereas $\sum_{k=2}^{\infty} u_{m_k}$ diverges, since, by the first inequality in (6.9),

$$\| u_{m_k} \| = \| z_{m_k} - z_{m_{k-1}} \| \ge \nu_{\{z_n\}}^{-1} \| z_{m_{k-1}} \| \ge \nu_{\{z_n\}}^{-1} > 0 \quad (k = 2,3,\ldots). \ \square$$

Borrowing now from section 10 the fact that every non-reflexive space X has a non-reflexive subspace with a basis, we get from Theorems 6.9 and 6.11 the following characterization of reflexive spaces among general Banach spaces.

THEOREM 6.12. *Let* X *be any Banach space (possibly without basis, or even non-separable). Then the following are equivalent:*

(i) X *reflexive;*

(ii) *all basic sequences in* X *are shrinking;*

(iii) *all basic sequences in* X *are boundedly complete.*

* * *

We now wish to consider Banach spaces with unconditional bases and to prove the second main theorem of this section, stating that such a space is reflexive iff it does not isomorphically contain either c_0 or ℓ^1. Generalizations of this theorem will be mentioned in the Notes at the end of this section. We begin with a discussion of unconditional convergence of series in Banach spaces.

PROPOSITION 6.13. *Let* X *be a Banach space and* $\{x_n\} \subset X$. *Then the following are equivalent:*

(i) $\sum_{i=1}^{\infty} x_i$ *converges unconditionally, i.e.* $\sum_{i=1}^{\infty} x_{\sigma(i)}$ *converges for every permutation* σ *of* $\mathbb{N}$.

(ii) $\lim_{F\in\mathcal{F}} \sum_{i\in F} x_i$ *exists, where* $\mathcal{F}$ *is the net of all finite subsets* $F \subset \mathbb{N}$, *ordered by inclusion.*

(iii) $\sum_{i=1}^{\infty} x_{n_i}$ *converges for every subsequence* $\{n_i\} \subset \mathbb{N}$.

(iv) $\sum_{i=1}^{\infty} \varepsilon_i x_i$ *converges for all choices of* $\varepsilon_i = \pm 1$ $(i = 1,2,\ldots)$.

(v) $\sum_{i=1}^{\infty} \alpha_i x_i$ *converges for every bounded sequence* $\{\alpha_i\} \subset \mathbb{R}$.

PROOF. (i) ⇒ (ii): Suppose that $\sum_{i=1}^{\infty} x_i$ converges unconditionally. Put $x = \sum_{i=1}^{\infty} x_i$. We claim that $\lim_{F\in\mathcal{F}} \sum_{i\in F} x_i = x$. If not, then for some $\varepsilon > 0$ there exists for every $F \in \mathcal{F}$ an $F' \in \mathcal{F}$ such that $F \subset F'$ and

$$(6.14)\qquad \|x - \sum_{i\in F'} x_i\| \geq \varepsilon.$$

Also, since $x = \sum_{i=1}^{\infty} x_i$, there exists an $n_0 \in \mathbb{N}$ such that

$$(6.15)\qquad \|x - \sum_{i=1}^{n} x_i\| < \frac{\varepsilon}{2} \quad \text{for all } n \geq n_0.$$

Combining (6.14) and (6.15) we can construct an increasing sequence $\{F_n\} \subset \mathcal{F}$ satisfying the following two properties:

(a) $F_n = \{i \in \mathbb{N}\colon i \leq k\}$ for some $k = k(n) \geq n_0$ whenever n is odd;

(b) $\|x - \sum_{i\in F_n} x_i\| \geq \varepsilon$ whenever n is even.

It follows from (a) and (b) that for every $n \in \mathbb{N}$

$$(6.16)\qquad \|\sum_{i\in F_{n+1}} x_i - \sum_{i\in F_n} x_i\| \geq \left|\|x - \sum_{i\in F_{n+1}} x_i\| - \|x - \sum_{i\in F_n} x_i\|\right| > \frac{\varepsilon}{2}.$$

Now let σ be the permutation of $\mathbb{N}$ obtained by first enumerating (in arbitrary fashion) the elements of F_1, then those of $F_2\backslash F_1$, then of $F_3\backslash F_2$, etc. It is obvious from (6.16) that $\sum_{i=1}^{\infty} x_{\sigma(i)}$ is not Cauchy, contradicting (i).

(ii) ⇒ (i): Assume (ii), put $x = \lim_{F\in\mathcal{F}} \sum_{i\in F} x_i$ and let σ be any permutation of $\mathbb{N}$. If $\varepsilon > 0$ is given arbitrarily, there exists an $F_0 \in \mathcal{F}$ such that $\|x - \sum_{i\in F} x_i\| < \varepsilon$ for all $F \in \mathcal{F}$, $F \supset F_0$. Choosing $n_0 \in \mathbb{N}$ such that $F_0 \subset \{\sigma(1),\ldots,\sigma(n_0)\}$, we then have $\|x - \sum_{i=1}^{n} x_{\sigma(i)}\| < \varepsilon$ whenever $n \geq n_0$. Thus $\sum_{i=1}^{\infty} x_{\sigma(i)} = x$.

(ii) ⇒ (v): We show first that (ii) implies

$$(6.17)\qquad \lim_{n\to\infty} \sup_{x^*\in B_{X^*}} \sum_{i=n}^{\infty} |\langle x_i, x^*\rangle| = 0.$$

Again let $x := \lim_{F \in \mathcal{F}} \sum_{i \in F} x_i$. Let $\varepsilon > 0$ be given and let $F_0 \in \mathcal{F}$ be such that $\|x - \sum_{i \in F} x_i\| < \varepsilon$ for all $F \in \mathcal{F}$, $F \supset F_0$. Then

$$(6.18) \qquad \|\sum_{i \in F} x_i\| < 2\varepsilon \quad \text{for all } F \in \mathcal{F}, \quad F \cap F_0 = \phi.$$

Now let $n_0 := \max\{i: i \in F_0\}$. Then we have for all $x^* \in B_{X^*}$ and for all $n,k \in \mathbb{N}$ with $n > n_0$,

$$(6.19) \qquad \sum_{i=n}^{n+k} |\langle x_i, x^* \rangle| = |\langle \sum_{i=n}^{n+k}{}' \, x_i, x^* \rangle| + |\langle \sum_{i=n}^{n+k}{}'' \, x_i, x^* \rangle|,$$

where $\sum'$ (respectively $\sum''$) denotes the restriction of the sum to those indices i for which $\langle x_i, x^* \rangle \geq 0$ (respectively $\langle x_i, x^* \rangle < 0$). Hence (6.18) and (6.19) yield

$$(6.20) \qquad \sum_{i=n}^{n+k} |\langle x_i, x^* \rangle| \leq \|x^*\| \|\sum_{i=n}^{n+k}{}' \, x_i\| + \|x^*\| \|\sum_{i=n}^{n+k}{}'' \, x_i\| \leq 4\varepsilon.$$

This proves (6.17).

Now we proceed to the proof of (v). Let $\varepsilon > 0$ be arbitrary and let $n_0 \in \mathbb{N}$ be as above. Then for any bounded sequence $\{\alpha_i\} \subset \mathbb{R}$ and all $k \in \mathbb{N}$ we have for suitable $x^* \in S_{X^*}$, by (6.20),

$$\|\sum_{i=n}^{n+k} \alpha_i x_i\| = |\langle \sum_{i=n}^{n+k} \alpha_i x_i, x^* \rangle| \leq \sup|\alpha_i| \sum_{i=n}^{n+k} |\langle x_i, x^* \rangle| \leq$$
$$\leq 4\varepsilon \sup|\alpha_i|,$$

whenever $n \geq n_0$. Thus $\sum_{i=1}^{\infty} \alpha_i x_i$ is Cauchy and (v) is proved.

(v) ⇒ (iv): trivial.

(iv) ⇒ (iii): Assume that (iv) holds and that $\{n_i\} \subset \mathbb{N}$ is a given subsequence. We define sequences $\{\varepsilon_i\}$ and $\{\varepsilon_i'\}$ by

$$\varepsilon_i = 1 \ (i = 1,2,\ldots), \qquad \varepsilon_i' = \begin{cases} 1 & \text{if } i \in \{n_1, n_2, \ldots\} \\ -1 & \text{if } i \notin \{n_1, n_2, \ldots\}. \end{cases}$$

Then, clearly, for all $k, \ell \in \mathbb{N}$,

$$\sum_{i=k}^{k+\ell} x_{n_i} = \frac{1}{2}\left(\sum_{i=n_k}^{n_{k+\ell}} \varepsilon_i x_i + \sum_{i=n_k}^{n_{k+\ell}} \varepsilon_i' x_i \right)$$

and it follows from the assumption that $\sum_{i=1}^{\infty} x_{n_i}$ converges.

(iii) ⇒ (ii): Assume that (iii) holds and that $\lim_{F \in \mathcal{F}} \sum_{i \in F} x_i$ does not exist. Then there exists an $\varepsilon > 0$ with the property that for every $F \in \mathcal{F}$ there exists an $F' \in \mathcal{F}$ with $F \cap F' = \phi$ and $\|\sum_{i \in F'} x_i\| > \varepsilon$. Therefore we can choose a sequence $\{F_n\}$ of disjoint elements of $\mathcal{F}$ satisfying

(a) $\max F_n < \min F_{n+1}$ $(n = 1,2,\ldots)$;

(b) $\|\sum_{i \in F_n} x_i\| > \varepsilon$ $(n = 1,2,\ldots)$.

Let $\{n_i\}$ be the subsequence of $\mathbb{N}$ obtained by enumerating first (in their natural order) the elements in F_1, then those of F_2, etc. Then evidently $\sum_{i=1}^{\infty} x_{n_i}$ is not Cauchy, contradicting (iii). □

<u>REMARK 6.14</u>. Implicitly the proof shows that $\sum_{i=1}^{\infty} x_{\sigma(i)} = \sum_{i=1}^{\infty} x_i$ for every permutation σ of $\mathbb{N}$, if $\sum_{i=1}^{\infty} x_i$ converges unconditionally.

<u>DEFINITION 6.15</u>. A basis $\{x_n\}$ for a Banach space X is called *unconditional* if for every $x \in X$ the expansion $x = \sum_{i=1}^{\infty} \langle x, x_i^* \rangle x_i$ converges unconditionally. A basic sequence $\{x_n\}$ is called *unconditional* if it is an unconditional basis for $[x_n]$.

<u>EXAMPLES</u>. It is easy to verify that the standard bases $\{e_n\}$ in c_0, ℓ^p $(1 \le p < \infty)$ are unconditional. We shall see later that not every space with a basis has an unconditional basis. Examples are $C([0,1])$ and $L^1[0,1]$. Furthermore, it is known that every space with a basis has a conditional (i.e. not unconditional) basis ([B2]). We give a simple example of such a conditional basis for c_0.

Let $x_n := \sum_{i=1}^{n} e_i$ $(n = 1,2,\ldots)$. To check that $\{x_n\}$ is a basis for c_0, it suffices to show, by Theorem 5.6, that for all $n,k \in \mathbb{N}$ and all $\alpha_1,\ldots,\alpha_{n+k} \in \mathbb{R}$ we have $\|\sum_{i=1}^{n} \alpha_i x_i\| \le 2\|\sum_{i=1}^{n+k} \alpha_i x_i\|$. But this is obvious, since

$$\|\sum_{i=1}^{n} \alpha_i x_i\| = \sup_{j=1,\ldots,n} |\sum_{i=j}^{n} \alpha_i|,$$

$$\|\sum_{i=1}^{n+k} \alpha_i x_i\| = \sup_{j=1,\ldots,n+k} |\sum_{i=j}^{n+k} \alpha_i|,$$

and for every $j \in \{1,\ldots,n\}$

$$|\sum_{i=j}^{n} \alpha_i| \le |\sum_{i=j}^{n+k} \alpha_i| + |\sum_{i=n+1}^{n+k} \alpha_i| \le 2\|\sum_{i=1}^{n+k} \alpha_i x_i\|.$$

(Alternatively, use Proposition 5.9.)

To show that the basis $\{x_n\}$ is conditional, note that for arbitrary $n,k \in \mathbb{N}$ and $\alpha_n,\dots,\alpha_{n+k} \in \mathbb{R}$,

$$\|\sum_{i=n}^{n+k} \alpha_i x_i\| = \sup_{j=n,\dots,n+k} |\sum_{i=j}^{n+k} \alpha_i| .$$

Thus $\sum_{i=1}^{\infty} \alpha_i x_i$ converges iff $\sum_{i=1}^{\infty} \alpha_i$ converges. Hence for a conditionally convergent series $\sum_{i=1}^{\infty} \alpha_i$, $\sum_{i=1}^{\infty} \alpha_i x_i$ converges, but not unconditionally.

We now want to prove a result that gives some geometric insight into what distinguishes conditional and unconditional bases (cf. Theorem 5.6).

<u>PROPOSITION 6.16</u>. *Let $\{x_n\}$ be a sequence of non-zero elements in a Banach space X such that $[x_n] = X$ and let $\mathcal{F}$ be as in Proposition 6.13. Then the following are equivalent:*

(i) *$\{x_n\}$ is an unconditional basis for X.*

(ii) *For every permutation σ of $\mathbb{N}$ $\{x_{\sigma(i)}\}$ is a basis for X.*

(iii) *There exists a (unique) sequence $\{x_n^*\} \subset X^*$ such that $\{x_n\},\{x_n^*\}$ is a (complete) biorthogonal system and the associated net of projections $\{P_F\}_{F\in\mathcal{F}}$ defined by*

$$P_F(x) = \sum_{i\in F} \langle x,x_i^*\rangle x_i \qquad (x \in X)$$

is uniformly bounded.

(iv) *There exists a constant $C > 0$ such that for every pair $F, F' \in \mathcal{F}$ with $F \subset F'$ and all $\alpha_i \in \mathbb{R}$ $(i \in F')$ we have*

$$\|\sum_{i\in F} \alpha_i x_i\| \le C\|\sum_{i\in F'} \alpha_i x_i\| .$$

<u>PROOF</u>. (i) ⇒ (ii): Let $\{x_n\}$ be an unconditional basis for X, $\{x_n^*\}$ its sequence of coefficient functionals, and let σ be a permutation of $\mathbb{N}$. Then, by the definition of unconditionality (see also Remark 6.14)
$x = \sum_{i=1}^{\infty} \langle x,x_{\sigma(i)}^*\rangle x_{\sigma(i)}$ for every $x \in X$. Hence $\{x_{\sigma(i)}\}$ is a basis for X.

(ii) ⇒ (i): The assumption (ii) implies in particular that $\{x_n\}$ is a basis for X. Let $\{x_n^*\}$ be the sequence of coefficient functionals. Since clearly $\{x_{\sigma(i)}^*\}$ is the sequence of coefficient functionals associated to the basis $\{x_{\sigma(i)}\}$ (for every permutation σ of $\mathbb{N}$), we have $x = \sum_{i=1}^{\infty} \langle x,x_{\sigma(i)}^*\rangle x_{\sigma(i)}$ for every $x \in X$ and every σ. Thus $\sum_{i=1}^{\infty} \langle x,x_i^*\rangle x_i$ converges unconditionally for every $x \in X$, proving that $\{x_n\}$ is unconditional.

(i) ⇒ (iii): Only the assertion that $\{P_F\}_{F\in\mathcal{F}}$ is uniformly bounded needs proof. Since by Proposition 6.13 (ii) for every $x \in X$ $\lim_{F\in\mathcal{F}} P_F(x)$ exists, it follows easily that $\{P_F(x): F \in \mathcal{F}\}$ is bounded for every $x \in X$. Thus $\{P_F\}_{F\in\mathcal{F}}$ is uniformly bounded by the Banach-Steinhaus theorem.

(iii) ⇒ (ii): Let σ be any permutation of $\mathbb{N}$. Clearly the assumption implies that $\{x_{\sigma(i)}\},\{x^*_{\sigma(i)}\}$ is a complete biorthogonal system. Also, since for all $x \in X$ $\sum_{i=1}^n <x,x^*_{\sigma(i)}>x_{\sigma(i)} = P_F(x)$ with $F = \{\sigma(1),\ldots,\sigma(n)\}$, the sequence of projections associated with this system is uniformly bounded. Hence, by (iii) ⇒ (i) of Theorem 5.6, $\{x_{\sigma(i)}\}$ is a basis for X.

We have now proved the equivalence of (i), (ii) and (iii). Since we know already that (iv) implies that $\{x_n\}$ is a basis (Theorem 5.6), it is clear that (iv) is nothing but a restatement of (iii). □

Observe that (iv) above is a purely geometric and intrinsic characterization of unconditional bases. It should be compared with (ii) in Theorem 5.6.

Let $\{x_n\}$ be an unconditional basis for X. Consider the bilinear map $\phi: \ell^\infty \times X \to X$ defined by

$$\phi(\{\alpha_n\},x) = \sum_{i=1}^\infty \alpha_i <x,x_i^*>x_i \qquad (\{\alpha_n\} \in \ell^\infty, \quad x \in X).$$

Note that the series on the right converges by Proposition 6.13 (v), so that ϕ is well defined. For every fixed $\{\alpha_n\} \in \ell^\infty$ the map

$$x = \sum_{i=1}^\infty <x,x_i^*>x_i \longrightarrow \sum_{i=1}^\infty \alpha_i <x,x_i^*>x_i \qquad (x \in X)$$

is evidently closed and therefore bounded by the closed graph theorem. Similarly, for every fixed $x \in X$ the map

$$\{\alpha_n\} \longrightarrow \sum_{i=1}^\infty \alpha_i <x,x_i^*>x_i \qquad (\{\alpha_n\} \in \ell^\infty)$$

is closed, whence bounded. (The boundedness is also a direct consequence of (6.17).) Combining both these observations, it follows that the set of bounded operators

$$x = \sum_{i=1}^\infty <x,x_i^*>x_i \longrightarrow \sum_{i=1}^\infty \alpha_i <x,x_i^*>x_i \qquad (x \in X),$$

with $\{\alpha_n\}$ varying over B_{ℓ^∞}, is pointwise bounded. Hence, by the Banach-Steinhaus theorem, it is bounded in norm. This means that ϕ is bounded, i.e.

there exists a constant $C < \infty$ such that

$$(6.21)\qquad \|\sum_{i=1}^{\infty} \alpha_i \langle x, x_i^* \rangle x_i\| \le C \quad, \text{for all } x \in B_X \text{ and } \{\alpha_n\} \in B_{\ell^\infty}.$$

DEFINITION 6.17. Let $\{x_n\}$ be an unconditional basis for X. The smallest number C satisfying (6.21) is called the *unconditional norm* (or *unconditional basis constant*) of $\{x_n\}$ and is denoted by $\nu^u_{\{x_n\}}$.

In a space with an unconditional basis $\{x_n\}$ we can introduce a new norm by

$$\|x\|_1 := \sup\{\|\sum_{i=1}^{\infty} \alpha_i \langle x, x_i^* \rangle x_i\| : \{\alpha_n\} \in B_{\ell^\infty}\}.$$

By (6.21), $\|\ \|_1$ is equivalent to $\|\ \|$. Clearly $\nu^u_{\{x_n\}} = 1$, with respect to $\|\ \|_1$, or equivalently,

$$(6.22)\qquad \|\sum_{i=1}^{\infty} \beta_i x_i\|_1 \le \|\sum_{i=1}^{\infty} \alpha_i x_i\|_1, \quad \text{whenever } |\beta_i| \le |\alpha_i| \quad (i = 1,2,\ldots),$$

and the series converge. In particular $\|P_F\|_1 = 1$ for all $F \in \mathcal{F}$.

We are now prepared for the second main result in this section, stating that a space with an unconditional basis is reflexive iff it does not contain either c_0 or ℓ^1 isomorphically. We divide its proof in two parts.

PROPOSITION 6.18. *Let X be a Banach space with an unconditional basis* $\{x_n\}$. *If X does not contain* c_0 *isomorphically, then* $\{x_n\}$ *is boundedly complete.*

PROOF. Passing to an equivalent norm if necessary, we may suppose that $\nu^u_{\{x_n\}} = 1$. Let us assume for contradiction that $\{x_n\}$ is not boundedly complete. Then there exists a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that $\{\sum_{i=1}^n \alpha_i x_i\}_{n=1}^\infty$ is bounded in norm by 1 and diverges. Hence there is an $\varepsilon > 0$ and a subsequence $\{n_k\} \subset \mathbb{N}$ such that

$$(6.23)\qquad \|\sum_{i=n_k+1}^{n_{k+1}} \alpha_i x_i\| \ge \varepsilon \qquad (k = 1,2,\ldots).$$

Thus, putting $y_k := \sum_{i=n_k+1}^{n_{k+1}} \alpha_i x_i$ $(k = 1,2,\ldots)$, we have $\|y_k\| \ge \varepsilon$ $(k=1,2,\ldots)$ and, since $\nu^u_{\{x_n\}} = 1$,

$$(6.24)\qquad \|\sum_{k=1}^{n} y_k\| \le 1 \quad (n = 1,2,\ldots)$$

$\{y_k\}$ is a block basic sequence with respect to $\{x_n\}$ and therefore a basis for $Y := [y_k]$. We shall show that $Y \simeq c_0$, contradicting the assumption. For this it is enough to prove the following inequalities:

$$(6.25) \qquad \sup_{i \in \mathbb{N}} |\lambda_i| \geq \|x\| \geq \varepsilon \sup_{i \in \mathbb{N}} |\lambda_i| \qquad (x = \sum_{i=1}^{\infty} \lambda_i y_i \in Y).$$

Indeed, if (6.25) holds, the map $Tx := \{\lambda_i\}$ $(x = \sum_{i=1}^{\infty} \lambda_i y_i \in Y)$ is the desired isomorphism from Y onto c_0. By (6.23) and (6.22) we have, for every $k \in \mathbb{N}$,

$$\varepsilon|\lambda_k| \leq \|\lambda_k y_k\| \leq \|\sum_{i=1}^{\infty} \lambda_i y_i\| = \|x\|,$$

proving the second part of (6.25). Also, by (6.22) and (6.24), we have for every $n \in \mathbb{N}$

$$\|\sum_{i=1}^{n} \lambda_i y_i\| \leq \sup|\lambda_i| \|\sum_{i=1}^{n} y_i\| \leq \sup_{i \in \mathbb{N}} |\lambda_i|,$$

so that

$$\|x\| \sup_{i \in \mathbb{N}} |\lambda_i|. \qquad \square$$

PROPOSITION 6.19. *Let X be a Banach space with an unconditional basis $\{x_n\}$. If X does not contain ℓ^1 isomorphically, then $\{x_n\}$ is shrinking.*

PROOF. We may assume again that $\nu^u_{\{x_n\}} = 1$. Let us suppose for contradiction that $\{x_n\}$ is not shrinking. Then there exists by Lemma 6.5 an $x^* \in X^*$ satisfying $\lim_{n\to\infty} \|x^*\|_n = 1$. Hence there exists an increasing sequence $\{n_k\} \subset \mathbb{N}$ and elements $y_k \in [x_i]_{i=n_k+1}^{n_{k+1}}$ satisfying

$$(6.26) \qquad \|y_k\| = 1 \quad \text{and} \quad \langle y_k, x^* \rangle > \tfrac{1}{2} \qquad (k = 1,2,\ldots).$$

Note that $\{y_k\}$, as a block basic sequence with respect to the unconditional basis $\{x_n\}$, is an unconditional basic sequence. This is most easily seen by checking that Proposition 6.16 (iv) holds for $\{y_k\}$ (cf. the proof of Lemma 5.14). We claim that $Y := [y_k]$ is isomorphic to ℓ^1, contradicting the assumption. For this it suffices to prove the existence of a constant $M > 0$ such that the following inequalities hold:

$$(6.27) \qquad \|x\| \leq \sum_{i=1}^{\infty} |\alpha_i| \leq M\|x\| \qquad (x = \sum_{i=1}^{\infty} \alpha_i y_i \in Y).$$

Indeed, $Tx = \{\alpha_i\}$ $(x = \sum_{i=1}^{\infty} \alpha_i y_i \in Y)$ is then an isomorphism from Y onto ℓ^1. The first inequality is trivial, since $\|y_k\| = 1$ $(k = 1,2,\ldots)$. To prove the second one, let $\alpha_i^+ = \max(\alpha_i, 0)$, $\alpha_i^- = \max(-\alpha_i, 0)$ $(i = 1,2,\ldots)$. Any $x = \sum_{i=1}^{\infty} \alpha_i y_i$ can be written as $x = \sum_{i=1}^{\infty} \alpha_i^+ y_i - \sum_{i=1}^{\infty} \alpha_i^- y_i$. Let us observe that both series on the right converge and that

$$(6.28) \qquad \Big\|\sum_{i=1}^{\infty} \alpha_i^+ y_i\Big\| \le \|x\|, \qquad \Big\|\sum_{i=1}^{\infty} \alpha_i^- y_i\Big\| \le \|x\|,$$

since $\nu^u_{\{x_n\}} = 1$. Furthermore, by (6.26),

$$(6.29) \qquad \|x^*\| \Big\|\sum_{i=1}^{\infty} \alpha_i^+ y_i\Big\| \ge \Big\langle \sum_{i=1}^{\infty} \alpha_i^+ y_i, x^* \Big\rangle \ge \tfrac{1}{2} \sum_{i=1}^{\infty} \alpha_i^+$$

and

$$(6.30) \qquad \|x^*\| \Big\|\sum_{i=1}^{\infty} \alpha_i^- y_i\Big\| \ge \Big\langle \sum_{i=1}^{\infty} \alpha_i^- y_i, x^* \Big\rangle \ge \tfrac{1}{2} \sum_{i=1}^{\infty} \alpha_i^-.$$

Adding (6.29) and (6.30) and using (6.28), we obtain

$$\sum_{i=1}^{\infty} |\alpha_i| = \sum_{i=1}^{\infty} \alpha_i^+ + \sum_{i=1}^{\infty} \alpha_i^- \le$$

$$\le 2\|x^*\| \left\{ \Big\|\sum_{i=1}^{\infty} \alpha_i^+ y_i\Big\| + \Big\|\sum_{i=1}^{\infty} \alpha_i^- y_i\Big\| \right\} \le$$

$$\le 4\|x^*\| \|x\|.$$

Thus (6.27) is proved and we are done. □

Combining the two preceding propositions we arrive at

THEOREM 6.20. *Let* X *be a Banach space with an unconditional basis. Then the following are equivalent:*

(i) X *is reflexive;*

(ii) X *does not contain either* c_0 *or* ℓ^1 *isomorphically;*

(iii) X^{**} *is separable.*

PROOF. (ii) ⇒ (i): This is a consequence of the Propositions 6.18 and 6.19 and Theorem 6.9.

(i) ⇒ (iii): trivial.

(iii) ⇒ (ii): Suppose that X^{**} is separable. Observe that if Y is a subspace of X then Y^{**} is embedded in X^{**} in a canonical way (as $Y^{\perp\perp}$). Suppose $Y \subset X$

is isomorphic to c_0 (respectively ℓ^1). Then Y^{**} is isomorphic to ℓ^∞ (respectively $(\ell^\infty)^*$). Since both these spaces are non-separable, this contradicts the separability of X^{**}. □

* * *

In connection with Theorem 6.20 we wish to introduce the famous James space J. It is a counterexample to many conjectures, and in particular it shows that Theorem 6.20 is no longer valid when the assumption of unconditionality is dropped completely (it may be replaced by weaker assumptions): J^{**} is separable, so J contains neither c_0 nor ℓ^1 isomorphically (by the proof of (iii) ⇒ (ii) in Theorem 6.20), but J is not reflexive. Moreover, J has a shrinking basis, and $\dim J^{**}/\pi J = 1$. Although non-reflexive, J is isometric to J^{**}. Of course, this isometry cannot be the canonical map π_J. All these facts we shall prove below.

In a later section we shall be concerned with the question of what consequences can be deduced from the assumption X^{**} separable for a general Banach space X. It will turn out that it implies that both X and X^* are "somewhat reflexive", meaning that each closed infinite-dimensional subspace of X (and of X^*) contains an infinite-dimensional reflexive subspace.

<u>The James space J</u>

By J we denote the linear space of all sequences $\{\alpha_n\} \subset \mathbb{R}$ satisfying

$$(6.31)\qquad \lim_{n\to\infty} \alpha_n = 0$$

and

$$(6.32)\qquad \sup\left\{\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2\right\}^{\frac{1}{2}} < \infty,$$

where the supremum is taken over all $n \in \mathbb{N}$ and all finite increasing sequences $k_1 < k_2 < \ldots < k_{n+1}$ in $\mathbb{N}$. The norm $\|\{\alpha_n\}\|$ of $\{\alpha_n\} \in J$ is defined to be this supremum. It is easy to check the norm properties. We only show here how the triangle inequality follows from the one in ℓ^2. If $x = \{\alpha_n\}$ and $y = \{\beta_n\}$ are in J, and $\varepsilon > 0$ is arbitrary, then there exists a finite increasing sequence $k_1 < k_2 < \ldots < k_{n+1}$ in $\mathbb{N}$ such that

$$\|x+y\| < \left\{\sum_{i=1}^{n} [(\alpha_{k_i}+\beta_{k_i}) - (\alpha_{k_i}+\beta_{k_{i+1}})]^2 + [(\alpha_{k_{n+1}}+\beta_{k_{n+1}}) - (\alpha_{k_1}+\beta_{k_1})]^2\right\}^{\frac{1}{2}} + \varepsilon$$

$$\leq \left\{\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2\right\}^{\frac{1}{2}} + \left\{\sum_{i=1}^{n} (\beta_{k_i}-\beta_{k_{i+1}})^2 + (\beta_{k_{n+1}}-\beta_{k_1})^2\right\}^{\frac{1}{2}}+\varepsilon$$

$$\leq \|x\| + \|y\| + \varepsilon.$$

Thus $\|x+y\| \leq \|x\| + \|y\|$, since $\varepsilon > 0$ was arbitrary.

The completeness of J can be proved directly in the usual way. There is no need to check it here, since it will follow later from the fact (proved below) that J is a closed subspace of J^{**}. Thus J is a Banach space.

We now derive several properties of J.

PROPERTY I. The sequence $\{e_n\}$, where $e_n = (\underbrace{0,\ldots,0}_{n-1},1,0,\ldots)$ $(n \in \mathbb{N})$ is a monotone basis for J (i.e. $\nu_{\{e_n\}} = 1$).

PROOF. In the first place we have for all $n,m \in \mathbb{N}$ and all $\alpha_1,\ldots,\alpha_{n+m} \in \mathbb{R}$,

$$\|\sum_{i=1}^{n} \alpha_i e_i\| = \|(\alpha_1,\ldots,\alpha_n,0,\ldots)\| \leq \|(\alpha_1,\ldots,\alpha_n,\ldots,\alpha_{n+m},0,0\ldots)\|$$

$$= \|\sum_{i=1}^{n+m} \alpha_i e_i\|,$$

as one easily verifies. Hence $\{e_n\}$ is a monotone basic sequence and it remains to be shown that $[e_n] = J$. Let $x = \{\alpha_n\} \in J$ be arbitrary. We claim that $x = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i e_i$. Suppose not. Then, for some $\varepsilon > 0$ there are arbitrarily large $n \in \mathbb{N}$ with the property that $\|x - \sum_{i=1}^{n} \alpha_i e_i\| = \|(0,\ldots,0,\alpha_{n+1},\alpha_{n+2},\ldots)\| > \varepsilon$. Hence there exist infinitely many finite increasing sequences $k_1^{(j)} < k_2^{(j)} < \ldots < k_{n_j+1}^{(j)}$ $(j = 1,2,\ldots)$ in $\mathbb{N}$ such that $k_{n_j+1}^{(j)} < k_1^{(j+1)}$ for all $j \in \mathbb{N}$, with the property that

$$\sum_{j=1}^{n_j} (\alpha_{k_i^{(j)}}-\alpha_{k_{i+1}^{(j)}})^2 + (\alpha_{k_{n_j+1}^{(j)}}-\alpha_{k_1^{(j)}})^2 > \varepsilon^2 \quad (j = 1,2,\ldots).$$

Combining this with the fact that $\lim_{n\to\infty} \alpha_n = 0$, we obtain

$$\sum_{j=1}^{\infty} \sum_{i=1}^{n_j} (\alpha_{k_i^{(j)}}-\alpha_{k_{i+1}^{(j)}})^2 = \infty,$$

contradicting $\|\{\alpha_n\}\| < \infty$. □

<u>PROPERTY II</u>. The basis $\{e_n\}$ is shrinking.

<u>PROOF</u>. We use Lemma 6.5 and show that $\lim_{n\to\infty} \|x^*\|_n = 0$, for every $x^* \in J^*$. Suppose this is false for some $x^* \in J^*$. Then for some $\varepsilon > 0$ there exists a block basic sequence $\{y_n\}$ with respect to $\{e_n\}$, of the form $y_n = \sum_{i=m_{n-1}+1}^{m_n} \beta_i e_i$ $(n = 1,2,\dots)$ with $0 = m_0 < m_1 < \dots < m_n < \dots$, such that

$$(6.33) \qquad \|y_n\| = 1 \qquad (n = 1,2,\dots)$$

and

$$(6.34) \qquad \langle y_n, x^*\rangle > \varepsilon \qquad (n = 1,2,\dots).$$

We consider the series $\sum_{n=1}^{\infty} \frac{1}{n} y_n$ and show that it converges. This will finish the proof, since (6.34) then yields the contradiction $\langle\sum_{n=1}^{\infty} \frac{1}{n} y_n, x^*\rangle \geq \sum_{n=1}^{\infty} \frac{1}{n} \varepsilon = \infty$. Observe that checking the convergence of $\sum_{n=1}^{\infty} \frac{1}{n} y_n$ is equivalent to showing that $\{\alpha_i\} \in J$, where

$$(6.35) \qquad \alpha_i := \frac{1}{n} \beta_i \quad \text{whenever} \quad m_{n-1}+1 \leq i \leq m_n \qquad (n = 1,2,\dots).$$

Clearly $\lim_{i\to\infty} \alpha_i = 0$, since it easily follows from (6.33) that

$$(6.36) \qquad |\beta_i| \leq \frac{1}{\sqrt{2}} \qquad (i = 1,2,\dots).$$

Hence the problem is to show that

$$(6.37) \qquad \sup\left\{\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2\right\} < \infty.$$

Let us consider an arbitrary sum of the form $\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2$, $k_1 < \dots < k_{n+1}$. For each term $(\alpha_{k_i}-\alpha_{k_{i+1}})^2$ we have that either there exists an $\ell \in \mathbb{N}$ such that $m_{\ell-1}+1 \leq k_i < k_{i+1} \leq m_\ell$ or no such ℓ exists. The terms of the second type add up to at most $\frac{1}{2}\sum_{\ell=1}^{\infty} (\frac{1}{\ell} + \frac{1}{\ell+1})^2 < \infty$, in view of (6.36), and those of the first type to at most $\sum_{\ell=1}^{\infty} \frac{1}{\ell^2} < \infty$, since for $m_{\ell-1}+1 \leq k_p < k_{p+1} < \dots < k_q < k_{q+1} \leq m_\ell$ we have, by (6.33),

$$\sum_{i=p}^{q} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 = \frac{1}{\ell^2} \sum_{i=p}^{q} (\beta_{k_i}-\beta_{k_{i+1}})^2 \leq \frac{1}{\ell^2} \|y_\ell\|^2 \leq \frac{1}{\ell^2}.$$

Since also $\{\alpha_n\}$ is bounded, it is now clear that the sup in (6.37) is finite and we are done. □

PROPERTY III. $\dim J^{**}/\pi J = 1$.

This will follow from the preceding two properties and the following

PROPOSITION 6.21. *Let X be a Banach space with a monotone shrinking basis* $\{e_n\}$, *and let* $\{e_n^*\}$ *be the sequence of coefficient functionals. Let A be the linear space of all sequences* $\{\alpha_n\} \subset \mathbb{R}$ *satisfying* $\sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \alpha_i e_i\| < \infty$, *equipped with the norm* $\|\{\alpha_n\}\| := \sup_{n\in\mathbb{N}} \|\sum_{i=1}^n \alpha_i e_i\|$. *Then the linear map* $\phi: X^{**} \to A$ *defined by*

$$\phi(x^{**}) = \{<e_i^*, x^{**}>\}_{i=1}^{\infty} \qquad (x^{**} \in X^{**})$$

is an isometry onto A.

PROOF. Since $\{e_n\}$ is shrinking, $\{e_n^*\}$ is a basis for X^*, so its sequence of coefficient functionals $\{\pi e_n\}$ is a w^*-Schauder basis for X^{**}, by Proposition 6.7. If $\{P_n\}$ denotes the sequence of projections associated to the basis $\{e_n\}$, we have

$$(6.38) \qquad x^{**} = w^*\text{-}\lim_{n\to\infty} P_n^{**} x^{**} = w^*\text{-}\lim_{n\to\infty} \sum_{i=1}^n <e_i^*, x^{**}>\pi e_i \qquad (x^{**} \in X^{**}).$$

By the monotonicity of $\{e_n\}$, $\|P_n\| = \|P_n^{**}\| = 1$ $(n = 1,2,\dots)$, so

$$(6.39) \qquad \|\sum_{i=1}^n <e_i^*, x^{**}>\pi e_i\| \le \|\sum_{i=1}^{n+1} <e_i^*, x^{**}>\pi e_i\| \le \|x^{**}\| \qquad (x^{**} \in X^{**},\ n \in \mathbb{N}).$$

Combining (6.38) and (6.39) with the w^*-lower semi-continuity of the (dual) norm on X^{**}, we obtain

$$\|x^{**}\| = \lim_{n\to\infty} \|\sum_{i=1}^n <e_i^*, x^{**}>e_i\| = \sup_{n\in\mathbb{N}} \|\sum_{i=1}^n <e_i^*, x^{**}>e_i\| \qquad (x^{**} \in X^{**}).$$

This proves that ϕ is isometric. To show surjectivity, let $\{\alpha_n\} \in A$, i.e. $\{\sum_{i=1}^n \alpha_i e_i\}_{n=1}^{\infty}$ bounded. If $x^{**} \in X^{**}$ is a w^*-limit point of $\{\sum_{i=1}^n \alpha_i \pi e_i\}$, it follows, since $<e_m^*, \sum_{i=1}^n \alpha_i \pi e_i> = \alpha_m$ whenever $n \ge m$, that $<e_i^*, x^{**}> = \alpha_i$ for all $i \in \mathbb{N}$, showing that $\phi(x^{**}) = \{\alpha_n\}$. This completes the proof. □

We now apply this result to J with the monotone shrinking basis $\{e_n\}$ defined above. ϕ is clearly the identity on J. In addition, we get that ϕJ^{**} is the space of sequences $\{\alpha_n\} \subset \mathbb{R}$ with norm

$$\|\{\alpha_n\}\| = \lim_{m\to\infty}\|\sum_{i=1}^{m} \alpha_i e_i\| = \sup_{m\in\mathbb{N}} \|\sum_{i=1}^{m} \alpha_i e_i\| =$$

(6.40)

$$= \sup_{m\in\mathbb{N}} \left[\sup\left\{\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2\right\}^{\frac{1}{2}}\right] < \infty,$$

where the inner sup is taken over all $n \in \mathbb{N}$ and all $k_1 < k_2 < \ldots < k_{n+1}$, with the understanding that α_{k_j} is read as 0 whenever $k_j > m$. $J = \phi J$ consists of all null sequences $\{\alpha_n\}$ for which (6.40) holds. Observe next that $\|\{\alpha_n\}\| < \infty$ implies that $\lim_{n\to\infty} \alpha_n$ exists, so that ϕJ^{**} consists of convergent sequences. Finally we have by (6.40) that $(1,1,1,\ldots) \in \phi J^{**}$, since $\|\sum_{i=1}^{n} e_i\| = \sqrt{2}$ for all $n \in \mathbb{N}$. Set $x_0^{**} = \phi^{-1}((1,1,1,\ldots))$. All these facts combined yield $J^{**} = \pi J \oplus [x_0^{**}]$, completing the proof of Property III. □

<u>REMARK 6.22</u>. Observe that for elements $\{\alpha_n\} \in \phi J^{**}$ the expressions (6.32) and (6.40) do not coincide: For the sequence $(1,1,\ldots) = \phi(x_0^{**}) \in \phi J^{**}$ (6.32) gives 0, while (6.40) yields $\sqrt{2} = \|x_0^{**}\|$. Of course (6.32) and (6.40) are equal on J, since J consists of null sequences.

Property III says that πJ is a hyperplane in the Banach space J^{**}. It cannot be dense since it follows easily from (6.40) that $d(x_0^{**},\pi J) = \sqrt{2}$. Hence πJ is closed in J^{**} and therefore complete.

<u>PROPERTY IV</u>. J^{**} is linearly isometric to J.

<u>PROOF</u>. From now on, for every sequence $\{\alpha_n\} \subset \mathbb{R}$ $\|\{\alpha_n\}\|$ will denote the expression in (6.40), whether finite of infinite. As we have seen, we may identify J and J^{**} with, respectively, the null sequences and the convergent sequences $\{\alpha_n\}$ satisfying $\|\{\alpha_n\}\| < \infty$. Now consider the linear map T from real sequences to real sequences defined by

$$T((\alpha_1,\alpha_2,\ldots,\alpha_n,\ldots)) = (\alpha_2-\alpha_1,\alpha_3-\alpha_1,\ldots,\alpha_n-\alpha_1,\ldots).$$

Evidently T is a bijection from the null sequences onto the convergent sequences. In view of the above identification it will therefore be clear that T establishes a linear isometry from J onto J^{**}, once we have proved that $\|T(\{\alpha_n\})\| = \|\{\alpha_n\}\|$ for all null sequences $\{\alpha_n\}$. Observe that for null sequences $\{\alpha_n\}$ the expressions for $\|\{\alpha_n\}\|$ in (6.32) and (6.40) are equal. Now given any null sequence $\{\alpha_n\}$, consider any sum of the form $\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2$. If $k_1 > 1$, we can write it as

$$\sum_{i=1}^{n}[(\alpha_{k_i}-\alpha_1) - (\alpha_{k_{i+1}}-\alpha_1)]^2 + [(\alpha_{k_{n+1}}-\alpha_1) - (\alpha_{k_1}-\alpha_1)]^2,$$

showing that it is $\leq \|T(\{\alpha_n\})\|$. If $k_1 = 1$, we have

$$\sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2 = (\alpha_1-\alpha_{k_2})^2 + \sum_{i=2}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 +$$

$$+ (\alpha_{k_{n+1}}-\alpha_1)^2 =$$

$$= \sum_{i=2}^{n}[(\alpha_{k_i}-\alpha_1) - (\alpha_{k_{i+1}}-\alpha_1)]^2 + [(\alpha_{k_{n+1}}-\alpha_1) - 0]^2 + [0 - (\alpha_{k_2}-\alpha_1)]^2 \leq$$

$$\leq \|(\alpha_2-\alpha_1,\ldots,\alpha_{k_{n+1}}-\alpha_1,0,0,\ldots)\|^2 \leq \|T(\{\alpha_n\})\|^2,$$

where the last inequality follows from (6.40). Hence, both cases combined yield $\|\{\alpha_n\}\| \leq \|T(\{\alpha_n\})\|$. For the proof of the reverse inequality, consider a sum of the form

$$\sum_{i=1}^{n}\left\{(\alpha_{k_i}-\alpha_1) - (\alpha_{k_{i+1}}-\alpha_1)\right\}^2 + \left\{(\alpha_{k_{n+1}}-\alpha_1) - (\alpha_{k_1}-\alpha_1)\right\}^2,$$

with $2 \leq k_1 < \ldots < k_{n+1}$, and where for some $m \in \mathbb{N}$ $(\alpha_{k_j}-\alpha_1)$ is to be read as 0 whenever $k_j > m$. If $k_{n+1} \leq m$, this sum equals

$$\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2 \leq \|\{\alpha_n\}\|^2.$$

If for some $i_0 \in \{1,\ldots,n\}$ we have $k_{i_0} \leq m < k_{i_0+1}$, then the sum equals

$$\sum_{i=1}^{i_0-1}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{i_0}}-\alpha_1)^2 + (\alpha_{k_1}-\alpha_1)^2 \leq \|\{\alpha_n\}\|^2.$$

This proves that $\|T(\{\alpha_n\})\| \leq \|\{\alpha_n\}\|$, and hence $\|T(\{\alpha_n\})\| = \|\{\alpha_n\}\|$. □

J is often equipped with other equivalent norms, such as

$$\|\{\alpha_n\}\|_1 := \sup\left\{\sum_{i=1}^{n}(\alpha_{k_{2i-1}}-\alpha_{k_{2i}})^2 + \alpha_{k_{2n+1}}^2\right\}^{\frac{1}{2}},$$

where the sup is taken over all $n \in \mathbb{N}$ and $k_1 < k_2 < \ldots < k_{2n+1}$ in $\mathbb{N}$, or

$$\|\{\alpha_n\}\|_2 := \sup\left\{\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2\right\}^{\frac{1}{2}},$$

where the sup is taken over all $n \in \mathbb{N}$ and $k_1 < k_2 < \ldots < k_{n+1}$ in $\mathbb{N}$. To show that $\| \; \|_1$ and $\| \; \|$ are equivalent, let us consider any sum of the form $\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2$. If $(\alpha_{k_{n+1}}-\alpha_{k_1})^2 \le \sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2$, then

$$\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2 \le 2\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 =$$

$$= 2\sum_{\substack{i=1 \\ i \text{ even}}}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 + 2\sum_{\substack{i=1 \\ i \text{ odd}}}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 \le 4\|\{\alpha_n\}\|_1^2.$$

If not, then

$$\sum_{i=1}^{n}(\alpha_{k_i}-\alpha_{k_{i+1}})^2 + (\alpha_{k_{n+1}}-\alpha_{k_1})^2 \le 2(\alpha_{k_{n+1}}-\alpha_{k_1})^2 \le 2\|\{\alpha_n\}\|_1^2.$$

Hence, combining both cases we obtain

(6.41) $\quad \|\{\alpha_n\}\| \le 2\|\{\alpha_n\}\|_1 \qquad (\{\alpha_n\} \in J).$

Now consider any sum of the form $\sum_{i=1}^{n}(\alpha_{k_{2i-1}}-\alpha_{k_{2i}})^2 + \alpha_{k_{2n+1}}^2$. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \alpha_n = 0$, we have for a sufficiently large choice of $k_{2n+2} \in \mathbb{N}$,

$$\sum_{i=1}^{n} (\alpha_{k_{2i-1}}-\alpha_{k_{2i}})^2 + \alpha_{k_{2n+1}}^2 \le \sum_{i=1}^{n+1} (\alpha_{k_{2i-1}}-\alpha_{k_{2i}})^2 + \varepsilon$$

$$\le \|\{\alpha_n\}\|^2 + \varepsilon.$$

$\varepsilon > 0$ being arbitrary, this proves that

(6.42) $\quad \|\{\alpha_n\}\|_1 \le \|\{\alpha_n\}\| \qquad (\{\alpha_n\} \in J).$

(6.41) and (6.42) show the equivalence of $\| \; \|$ and $\| \; \|_1$. The equivalence of $\| \; \|$ and $\| \; \|_2$ is even easier and we leave its proof to the reader. One obtains

$$2^{-\frac{1}{2}}\|\{\alpha_n\}\| \le \|\{\alpha_n\}\|_2 \le \|\{\alpha_n\}\| \quad (\{\alpha_n\} \in J).$$

PROPERTY V. J is the closure of the sum of two closed subspaces J_1 and J_2 with $J_1 \cap J_2 = \{0\}$, each of which is isomorphic to ℓ^2.

PROOF. It is convenient here to use the equivalent norm $\| \ \|_2$. Take $J_1 := [e_{2n}]_{n=1}^{\infty}$ and $J_2 := [e_{2n-1}]_{n=1}^{\infty}$. Clearly $J_1 \cap J_2 = \{0\}$ and $\overline{J_1+J_2} = J$, since $\{e_n\}$ is a basis for J. To show that $J_1 \simeq \ell^2$, let $\{\alpha_n\} \in J_1$ be arbitrary, i.e. $\alpha_{2n-1} = 0$ for all $n \in \mathbb{N}$. Then

$$(6.43) \qquad \sum_{i=1}^{n} \alpha_{2i}^2 = \sum_{i=1}^{n} (\alpha_{2i-1}-\alpha_{2i})^2 \le \|\{\alpha_n\}\|_2^2 \qquad (n = 1,2,\ldots).$$

On the other hand, for every $n \in \mathbb{N}$ and $k_1 < k_2 < \ldots < k_{n+1}$ in $\mathbb{N}$ we have, since $(\alpha-\beta)^2 \le 2(\alpha^2+\beta^2)$ for all $\alpha,\beta \in \mathbb{R}$,

$$(6.44) \qquad \sum_{i=1}^{n} (\alpha_{k_i}-\alpha_{k_{i+1}})^2 \le 4 \sum_{i=1}^{\infty} \alpha_{2i}^2.$$

Combining (6.43) and (6.44) we get

$$\left(\sum_{i=1}^{\infty} \alpha_{2i}^2\right)^{\frac{1}{2}} \le \|\{\alpha_n\}\|_2 \le 2\left(\sum_{i=1}^{\infty} \alpha_{2i}^2\right)^{\frac{1}{2}},$$

so

$$J_1 \simeq \ell^2.$$

The proof that $J_2 \simeq \ell^2$ is similar. □

In section 9 we shall prove the following related, but much stronger result: Every closed infinite-dimensional subspace of J contains an infinite dimensional subspace isomorphic to ℓ^2.

REMARK 6.23. It follows from Proposition 5.9 that the sequence $\{x_n\}$ with $x_n := \sum_{i=1}^{n} e_i$ $(n = 1,2,\ldots)$ is also a basis for J. Let $x = \sum_{i=1}^{\infty} \alpha_i e_i = \sum_{j=1}^{\infty} \beta_j x_j$ be an arbitrary element of J. Then

$$(6.45) \qquad \alpha_i = \langle x,e_i^*\rangle = \sum_{j=1}^{\infty} \beta_j \langle x_j,e_i^*\rangle = \sum_{j=i}^{\infty} \beta_j \qquad (i = 1,2,\ldots),$$

and therefore

$$(6.46) \qquad \|x\|_2 = \sup\left\{ \sum_{i=1}^{n} \left(\sum_{j=k_i}^{k_{i+1}-1} \beta_j\right)^2 \right\}^{\frac{1}{2}},$$

where the sup is taken over all $n \in \mathbb{N}$ and $k_1 < k_2 < \ldots < k_{n+1}$ in $\mathbb{N}$. Thus the space J can be alternatively defined as the space of all sequences $\{\beta_n\} \subset \mathbb{R}$ for which

$$(6.47) \qquad \sup_{\substack{n\in \mathbb{N} \\ k_1<\dots<k_{n+1}}} \left\{\sum_{i=1}^{n}\left(\sum_{j=k_i}^{k_{i+1}-1} \beta_j\right)^2\right\}^{\frac{1}{2}}$$

is finite, taking this expression as the norm of $\{\beta_n\}$. (Note that the finiteness of (6.47) implies that $\lim_{i\to\infty} \sum_{j=i}^{\infty} \beta_j = 0$, so, by (6.45), $\lim_{i\to\infty} \alpha_i = 0$.) Furthermore, it is obvious from the definitions involved, that $\{x_n\}$ is a boundedly complete basis for J. Thus J has a boundedly complete basis $\{x_n\}$ and a shrinking basis $\{e_n\}$ without being reflexive.

NOTES. Theorem 6.9 is due to R.C. JAMES ([47]) and Theorem 6.11 to M. ZIPPIN ([106]). A. PELCZYNSKI first proved that every non-reflexive space has a (non-reflexive) subspace with a non-shrinking basis in [80], where also Theorem 6.12 appears. [95] contains further characterizations of reflexivity in terms of basic sequences. Apart from the preliminary material on unconditional bases all the rest of this section, in particular the main Theorem 6.20, is due to R.C. JAMES ([47],[48]). Other authors have subsequently generalized Theorem 6.20. C. BESSAGA & A. PELCZYNSKI ([8]) have shown that it suffices to assume that X can be isomorphically embedded in a space with an unconditional basis. More recently, L. TZAFRIRI ([103]) has proved that Theorem 6.20 holds for any closed subspace X of a σ-Dedekind complete Banach lattice with σ-order continuous norm. This result actually includes the previous one of C. Bessaga and A. Pelczynski. For details and the definitions involved we refer to [102], [103] and [75]. A related paper is [60], where a finite-dimensional version of Theorem 6.20 is proved for spaces having "local unconditional structure".

7. QUASI-REFLEXIVITY

In Section 6 we have seen an example of a Banach space X with the property that $\dim X^{**}/\pi X = 1$. It follows that for every $n \in \mathbb{N}$ there exists a Banach space X with $\dim X^{**}/\pi X = n$. E.g. $X = J^n$ has this property.

DEFINITION 7.1. A Banach space X with $\dim X^{**}/\pi X = n < \infty$ is called *quasi-reflexive* (*of order* n).

In this section we study quasi-reflexive spaces in general and characterize them in two ways: first by compactness of the unit ball in a suitable "weak" topology, and secondly by the non-existence of w^*-dense subspaces V of the dual space with characteristic $r(V) = 0$. Before giving these results we must prove some theorems characterizing spaces isomorphic (respectively, isometric) to dual or bidual spaces. At this point the reader should recall the definition of the characteristic $r(V)$ of a subspace $V \subset X^*$, and the various expressions for it derived in section 4, since these will be used repeatedly below.

Let X be a Banach space and $V \subset X^*$ a linear subspace. Following J. DIXMIER ([29]), we call V *minimal* if V is closed and w^*-dense, and there exists no proper subspace $W \subset V$ with both these properties. The following proposition describes minimal subspaces $V \subset X^*$ in a different way.

PROPOSITION 7.2. *Let* X *be a Banach space and* $V \subset X^*$ *a closed* w^**-dense subspace. Then* V *is minimal iff* $X^{**} = X \oplus V^{\perp}$.

PROOF. Observe first that the w^*-density of V is equivalent to $V^{\perp} \cap X = \{0\}$. Suppose that $X^{**} = X \oplus V^{\perp}$ and assume for contradiction that $W \subsetneq V$, W closed and w^*-dense. Then there exists an $x^{**} \in W^{\perp}\setminus V^{\perp}$. Writing $x^{**} = x + y^{**}$, $x \in X$ and $y^{**} \in V^{\perp}$, we then have $x \neq 0$. On the other hand $x^{**}|_W = y^{**}|_W = 0$, so $x|_W = 0$, contradicting the w^*-density of W.

Conversely, let $V \subset X^*$ be minimal and let $x^{**} \in X^{**}\setminus V^{\perp}$ be arbitrary.

$W := V \cap \ker x^{**}$ is a closed hyperplane in V, and therefore cannot be w^*-dense in X^*, by the minimality of V. Hence there exists an $x \in X$, $x \neq 0$, $x_{|W} = 0$. Now $x_{|W} = x^{**}{}_{|W} = 0$ and $x_{|V} \neq 0$ (since V is w^*-dense), so $x^{**}{}_{|V} = \lambda x_{|V}$, for some $\lambda \in \mathbb{R}$. But this means that $\lambda x - x^{**} \in V^\perp$, and therefore $x^{**} = \lambda x + y^{**}$, with $y^{**} \in V^\perp$. This proves that $X^{**} = X \oplus V^\perp$, since $x^{**} \in X^{**}$ was arbitrary. □

COROLLARY 7.3. *Let X be a Banach space and $V \subset X^*$ a minimal subspace. Then* $r(V) > 0$.

PROOF. By the previous proposition $X^{**} = X \oplus V^\perp$, so by the closed graph theorem the projection P from X^{**} onto X with kernel $V^\perp$ is bounded. The conclusion now follows from Lemma 4.5. □

A description of minimal subspaces $V \subset X^*$ in terms of compactness is given in

PROPOSITION 7.4. *Let X be a Banach space and $V \subset X^*$ a closed w^*-dense subspace. Then V is minimal iff B_X is relatively $\sigma(X,V)$-compact. In this case $\bar{B}_X^{\sigma(X,V)}$ is bounded.*

PROOF. The last assertion follows from Corollary 7.3 and Lemma 4.4.

Let us assume first that V is minimal. Then $X^{**} = X \oplus V^\perp$, by Proposition 7.2, so $X \simeq X^{**}/V^\perp$, while $X^{**}/V^\perp \cong V^*$. Thus X is isomorphic to V^* by the map T defined by $Tx = (\pi x)_{|V}$ $(x \in X)$. T is clearly also an isomorphism for the topologies $\sigma(X,V)$ and $\sigma(V^*,V)$. By Alaoglu's theorem the image TB_X is relatively $\sigma(V^*,V)$-compact, so B_X is relatively $\sigma(X,V)$-compact.

Conversely, let B_X be relatively $\sigma(X,V)$-compact. Suppose that $W \subset V$ is a closed w^*-dense subspace of X^*. We show that $W = V$. B_X being relatively $\sigma(X,V)$-compact, it follows that $\sigma(X,V)$ and $\sigma(X,W)$ coincide on B_X (cf. the observation made at the beginning of the proof of Proposition 0.15). Let $x^* \in V$ be arbitrary. Then x^* is continuous on B_X for the topology $\sigma(X,W)$, i.e. given $\varepsilon > 0$ there exist $x_1^*,\dots,x_n^* \in W$ such that

$$\sup_{i=1,\dots,n} |\langle x,x_i^*\rangle| \le 1, \quad \|x\| \le 1 \Rightarrow |\langle x,x^*\rangle| \le \varepsilon.$$

In particular, putting $N := \bigcap_{i=1}^n \ker x_i^*$, we have $\|x^*{}_{|N}\| \le \varepsilon$. Let $y^* \in X^*$ be a Hahn-Banach extension of $x^*{}_{|N}$, i.e. $y^*{}_{|N} = x^*{}_{|N}$ and $\|y^*\| \le \varepsilon$. Since $\ker(x^*-y^*) \supset N = \bigcap_{i=1}^n \ker x_i^*$, we have $x^*-y^* \in [x_i^*]_{i=1}^n \subset W$. Hence $\operatorname{dist}(x^*,W) \le \|y^*\| \le \varepsilon$. $\varepsilon > 0$ and $x^* \in V$ being arbitrary, this proves that $V = W$. □

COROLLARY 7.5. *Let* X *be a Banach space and* $V \subset X^*$ *a closed* w^**-dense subspace. Then the following are equivalent:*

(i) V *minimal and* $r(V) = 1$;

(ii) B_X *is* $\sigma(X,V)$*-compact.*

PROOF. Proposition 7.4 and Lemma 4.4. □

We are now going to characterize Banach spaces which are isometric to dual spaces.

THEOREM 7.6. *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is isometric to a dual space.*

(ii) X^* *contains a minimal subspace* V *with* $r(V) = 1$.

(iii) *There exists a projection* P *from* X^{**} *onto* X *with* $\|P\| = 1$ *and* $\ker P$ w^**-closed.*

(iv) X^* *contains a closed* w^**-dense subspace* V *such that* B_X *is* $\sigma(X,V)$*-compact.*

PROOF. (i) ⇒ (iv): Suppose $X \cong Y^*$. Then $V := Y$ is a closed w^*-dense subspace of Y^{**}, and B_{Y^*} is $\sigma(Y^*,Y)$-compact by Alaoglu's theorem.

(iv) ⟺ (ii): Corollary 7.5.

(ii) ⇒ (i): Let $V \subset X^*$ be minimal with $r(V) = 1$. Then by Proposition 7.2 $X^{**} = X \oplus V^{\perp}$. Also Lemma 4.5 implies that $\|x\| \le \|x+x^{**}\|$ for all $x \in X$ and $x^{**} \in V^{\perp}$, since $r(V) = 1$. It follows now that $X \cong X^{**}/V^{\perp}$, and since generally $X^{**}/V^{\perp} \cong V^*$, we obtain that X is isometric to the dual space V^*.

(i) ⇒ (iii): Let $X \cong Y^*$. Then, as is easily checked, $(\pi_Y)^*$ is a projection from Y^{***} onto its subspace Y^*, obviously with norm 1 and with w^*-closed kernel.

(iii) ⇒ (ii): Let us put $V := (\ker P)^{\top}$, where P is a projection from X^{**} onto X as postulated in (iii). Then $V^{\perp} = \ker P$, since $\ker P$ is w^*-closed and therefore we have $X^{**} = X \oplus V^{\perp}$. Proposition 7.2 now yields that V is minimal and furthermore, by Lemma 4.5, $r(V) = 1$ since $\|P\| = 1$. □

Before giving an isomorphic version of the above theorem let us observe the following. If $V \subset X^*$ is a closed subspace with $r(V) > 0$, then there exists an equivalent norm on X for which $r(V) = 1$. Indeed, we know from Lemma 4.4 that $\bar{B}_X^{\sigma(X,V)}$ is bounded, and obviously closed and absolutely convex. Hence the gauge of this set defines an equivalent norm $\|\ \|_1$ on X.

Clearly for this new norm the unit ball is $\sigma(X,V)$-closed and therefore, by Lemma 4.4, $r(V) = 1$. Hence, from an isomorphic point of view $r(V)$ has only two distinguishable values, namely 0 and 1. Taking this into account, the following result is clear.

THEOREM 7.7. *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is isomorphic to a dual space;*

(ii) X^* *contains a minimal subspace* V*;*

(iii) *there exists a bounded projection* P *from* X^{**} *onto* X *with* $\ker P$ w^**-closed;*

(iv) X^* *contains a closed* w^**-dense subspace* V *such that* B_X *is relatively* $\sigma(X,V)$*-compact.*

The next result is proved along the same lines and characterizes bidual spaces.

THEOREM 7.8. *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is isometric to a bidual space;*

(ii) $X^* = W \oplus V$, *where* V *is a closed* w^**-dense subspace of* X^* *such that* B_X *is* $\sigma(X,V)$*-compact,* W *a* w^**-closed subspace of* X^**, and the projection* P *from* X^* *onto* V *with kernel* W *has norm* 1.

PROOF. (i) $\Rightarrow$ (ii): Let $X \cong Y^{**}$. Then $Y^{***} = Y^* \oplus \ker P$, where P is the norm one projection from Y^{***} onto Y^* with w^*-closed kernel introduced in the proof of (i) $\Rightarrow$ (iii) of Theorem 7.6. Clearly $V := Y^*$ and $W := \ker P$ satisfy the properties of (ii).

(ii) $\Rightarrow$ (i): Let V, W and P be as in (ii). We claim that X is isometric to $(W^{\top})^{**}$. Indeed, since W is w^*-closed and $\|P\| = 1$, we have $(W^{\top})^* \cong X^*/(W^{\top})^{\perp} = X^*/W \cong V$. Moreover, by Corollary 7.5 and the proof of (ii) $\Rightarrow$ (i) in Theorem 7.6, $V^* \cong X$. Hence $X \cong (W^{\top})^{**}$. $\square$

An isomorphic version is

THEOREM 7.9. *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is isomorphic to a bidual space;*

(ii) $X^* = W \oplus V$, *where* V *is a closed* w^**-dense subspace of* X^* *such that* B_X *is relatively* $\sigma(X,V)$*-compact and* W *is a* w^**-closed subspace of* X^*.

PROOF. (i) $\Rightarrow$ (ii): Clear from the previous theorem.

(ii) ⇒ (i): Argue as in (ii) ⇒ (i) above, replacing isometries by isomorphisms. (Note that P is bounded by the closed graph theorem.) □

<u>COROLLARY 7.10</u>. *Let* X *be a Banach space. Then the following are equivalent:*

(i) $X \simeq X^{**}$ *and* X *not reflexive;*

(ii) X^* *contains a closed* w^**-dense subspace* $V \neq X^*$ *such that* B_X *is relatively* $\sigma(X,V)$*-compact and* $V \simeq X^*$.

<u>PROOF</u>. Obvious from Theorem 7.7. □

The next result characterizing quasi-reflexive spaces is now almost self-evident.

<u>THEOREM 7.11</u>. *Let* X *be a Banach space. Then the following are equivalent:*

(i) X *is quasi-reflexive of order* n*;*

(ii) $X^* = W \oplus V$, *with* W *and* V *as in Theorem* 7.9 (ii), *and* dim W = n.

<u>PROOF</u>. Assuming (i), $X^{**} = X \oplus N$, dim N = n. Let $V := N^{\top}$. Then $V^{\perp} = (N^{\top})^{\perp} = N$, so $X^{**} = X \oplus V^{\perp}$. Hence V is minimal, by Proposition 7.2. Thus by Proposition 7.4 B_X is relatively $\sigma(X,V)$-compact. Evidently $\dim X^*/V = n$, so any complement W of V in X^* will satisfy the requirements.

For the converse, let us suppose that (ii) holds. Then, using Propositions 7.4 and 7.2, we see that $X^{**} = X \oplus V^{\perp}$. Since clearly $\dim V^{\perp} = \dim W = n$, we have (i). □

It will be convenient from now on to sometimes use the following notation for any Banach space X: $H(X) := X^{**}/\pi X$.

The known fact that X is reflexive iff X^* is reflexive (Proposition 0.13) is generalized in the following

<u>PROPOSITION 7.12</u>. *Let* X *be a Banach space. Then* X *is quasi-reflexive of order* n *iff* X^* *is quasi-reflexive of order* n.

<u>PROOF</u>. We have already seen repeatedly that $P := \pi_{X^*}(\pi_X)^*$ projects X^{***} onto $\pi_{X^*}X^*$. Since $\ker P = (\pi_X X)^{\perp}$, we therefore have $X^{***} = \pi_{X^*}X^* \oplus (\pi_X X)^{\perp}$. Thus $H(X^*) \simeq (\pi_X X)^{\perp}$. On the other hand $(\pi_X X)^{\perp} \cong H(X)^*$, so $H(X^*) \simeq H(X)^*$. The proposition is now obvious. □

It follows that all higher conjugates of X are quasi-reflexive of order n if X is. Also, combining Theorems 7.11 and 7.7 we obtain

PROPOSITION 7.13. *Let* X *be quasi-reflexive of order* n. *Then for every* $j \in \mathbb{N}$ *there exists a Banach space* X_j, *quasi-reflexive of order* n, *such that the* j-th *conjugate of* X_j *is isomorphic to* X.

PROOF. From Theorems 7.7 and 7.11 we obtain that $X \simeq X_1^*$, for some X_1. By the previous proposition X_1 is quasi-reflexive of order n. Repreating the argument, $X_1 \simeq X_2^*$ for some X_2, again quasi-reflexive of order n. Thus $X \simeq X_2^{**}$. Etc. □

REMARK 7.14. The reader should observe that Theorem 3.5 in [16] is incorrect: A quasi-reflexive space of order $n \geq 1$ is not always isometric to a conjugate space (see Corollary 4.9).

The next result shows that subspaces and quotients of quasi-reflexive spaces are again quasi-reflexive and that the order of quasi-reflexivity behaves additively.

THEOREM 7.15. *Let* X *be a Banach space and* $Y \subset X$ *a closed subspace. Then* $\pi X + Y^{\perp\perp}$ *is closed in* X^{**}. *Furthermore*

$$(7.1) \qquad H(Y) \simeq (\pi X + Y^{\perp\perp})/\pi X$$

and

$$(7.2) \qquad H(X/Y) \simeq X^{**}/(\pi X + Y^{\perp\perp}),$$

where $\pi = \pi_X$. *Hence*

$$(7.3) \qquad \dim H(X) = \dim H(Y) + \dim H(X/Y),$$

and therefore X *is quasi-reflexive iff both* Y *and* X/Y *are.*

PROOF. Let us first observe that the second adjoint of the identity embedding $i: Y \to X$ is an isometry from Y^{**} onto the subspace $Y^{\perp\perp}$ of X^{**}. Therefore we may and shall identify Y^{**} with the subspace $Y^{\perp\perp} \subset X^{**}$. Since i^{**} extends i (see Section 0 Property III p.12), we have $i^{**}\pi_Y = \pi i$. In particular $i^{**}\pi_Y Y = \pi i Y$. Hence, suppressing the identity map i in the notation from now on, we may identify $Y^{**}/\pi_Y(Y) = H(Y)$ with $Y^{\perp\perp}/\pi Y$. Secondly, let us note that $Y^{\perp\perp} \cap \pi X = \pi Y$, by the Hahn-Banach theorem.

Now let $Q: X \to X/Y$ be the quotient map and $\pi_1: X/Y \to (X/Y)^{**}$ the canonical embedding. Again we have, since Q^{**} extends Q, that

$$(7.4) \qquad Q^{**}\pi = \pi_1 Q.$$

It follows from (7.4) that $Q^{**}\pi X = \pi_1 QX = \pi_1(X/Y)$. Thus, since $\ker Q^{**} = Y^{\perp\perp}$,

$$\pi X + Y^{\perp\perp} = (Q^{**})^{-1}\pi_1(X/Y). \tag{7.5}$$

Consequently $\pi X + Y^{\perp\perp}$ is closed in X^{**}.

For the proof of (7.1) we consider the quotient map $X^{**} \to X^{**}/\pi X$, restrict it to $Y^{\perp\perp}$ and replace $X^{**}/\pi X$ by its range. Let us call the resulting map A. Thus

$$Y^{\perp\perp} \xrightarrow{A} (\pi X+Y^{\perp\perp})/\pi X.$$

Since $\ker A = Y^{\perp\perp} \cap \pi X = \pi Y$, A defines an isomorphism of $Y^{\perp\perp}/\pi Y$ onto $(\pi X+Y^{\perp\perp})/\pi X$. Identifying $Y^{\perp\perp}/\pi Y$ with $H(Y)$ we obtain (7.1).

For the proof of (7.2), let $Q_1\colon (X/Y)^{**} \to H(X/Y)$ be the quotient map and consider $Q_1Q^{**}\colon X^{**} \to H(X/Y)$. Since both Q_1 and Q^{**} are surjective, so is Q_1Q^{**}. Furthermore, $\ker Q_1Q^{**} = Q^{**-1}\pi_1(X/Y) = \pi X + Y^{\perp\perp}$, by (7.5). Hence Q_1Q^{**} defines an isomorphism of $X^{**}/(\pi X+Y^{\perp\perp})$ onto $H(X/Y)$, proving (7.2). (7.3) follows directly from (7.1) and (7.2). □

We finally note the following obvious consequence of Theorem 7.15.

<u>COROLLARY 7.16</u>. *Let* X *be a Banach space and* $Y \subset X$ *a closed subspace. Then* Y *is reflexive iff* $Y^{\perp\perp} \subset \pi X$ *and* X/Y *is reflexive iff* $\pi X + Y^{\perp\perp} = X^{**}$. *If* X/Y *is reflexive then* $H(Y) \simeq H(X)$ *and if* Y *is reflexive then* $H(X/Y) \simeq H(X)$.

* * *

We now start working towards the final goal in this section, which is to prove that X is quasi-reflexive iff there exists no closed w^*-dense subspace $V \subset X^*$ with $r(V) = 0$. One half of this equivalence is trivial now. Indeed, let X be quasi-reflexive and $V \subset X^*$ closed and w^*-dense. Then $V^\perp \cap X = \{0\}$ and so by quasi-reflexivity $\dim V^\perp < \infty$. It follows now that the projection P from $X \oplus V^\perp$ onto X with kernel $V^\perp$ is bounded, and therefore $r(V) > 0$ by Lemma 4.5. The converse is a little more involved. We begin with a criterion for the existence of subspaces $V \subset X^*$ with $r(V) = 0$. Subsequent results will show that the conditions imposed can be fulfilled in any non-quasi-reflexive space.

PROPOSITION 7.17. *Let* X *be a Banach space and let* Y *and* Z *be infinite-dimensional closed subspaces of* X *and* X^{**}*, respectively, such that* $Z \cap (Y^{**}+X) = \{0\}$ *(*Y^{**} *and* X *are identified with the subspaces* $Y^{\perp\perp}$ *and* πX *of* X^{**}*, respectively). Then* X^* *contains a closed* w^**-dense subspace* V *with* $r(V) = 0$.

PROOF. We may assume without loss of generality that Y is separable. Let $\{y_n^*\}$ be a sequence in Y^* such that $\|y_n^*\| = 1$ $(n = 1,2,\dots)$ and $[y_n^*]$ is w^*-dense in Y^*. (E.g. take $\{y_n\}$ dense in S_Y and define y_n^* so that $\|y_n^*\| = \langle y_n,y_n^*\rangle = 1$ $(n = 1,2,\dots)$.) Now let $\{z_n\}$ be a basic sequence in Z with $\|z_n\| = 1$ $(n = 1,2,\dots)$ (Proposition 5.13). We now choose $\varepsilon_n > 0$ $(n=1,2,\dots)$ so that $\sum_{n=1}^{\infty} \varepsilon_n \le \frac{1}{2}$ and define a linear map $T\colon Y^{**} \to Z$ by

$$Ty^{**} = \sum_{n=1}^{\infty} \varepsilon_n \langle y_n^*,y^{**}\rangle z_n.$$

Then $\|T\| \le \frac{1}{2}$ and it is easily seen that $T_{|Y}$ is injective, since $[y_n^*]$ is w^*-dense in Y^* and $\{z_n\}$ is a basic sequence. We define W to be the subspace $W := \{y^{**}+Ty^{**}\colon y^{**} \in Y^{**}\}$ of X^{**} and claim that B_W is w^*-closed in X^{**}. Indeed, let $\{y_\alpha^{**}+Ty_\alpha^{**}\}$ be a net in B_W that converges to some $x^{**} \in X^{**}$ for $\sigma(X^{**},X^*)$. Since B_W is clearly $\sigma(X^{**},X^*)$-closed in W, it suffices to show that $x^{**} \in W$. Note first that $\{y_\alpha^{**}\}$ is bounded since $\|T\| \le \frac{1}{2}$ implies, for all α,

$$1 \ge \|y_\alpha^{**}+Ty_\alpha^{**}\| \ge \|y_\alpha^{**}\| - \|Ty_\alpha^{**}\| \ge \tfrac{1}{2}\|y_\alpha^{**}\|.$$

By the $\sigma(Y^{**},Y^*)$-compactness of $B_{Y^{**}}$ and the fact that $\sigma(X^{**},X^*)$ coincides on Y^{**} with $\sigma(Y^{**},Y^*)$, there exists a subnet $\{y_{\alpha'}^{**}\}$ of $\{y_\alpha^{**}\}$ that converges to some $y^{**} \in Y^{**}$ for $\sigma(X^{**},X^*)$. It follows now immediately from the definition of T and the boundedness of $\{y_{\alpha'}^{**}\}$ that $\{Ty_{\alpha'}^{**}\}$ converges to Ty^{**} in norm. Hence $x^{**} = y^{**}+Ty^{**} \in W$ and our claim has been proved. The Krein-Šmulian theorem (Proposition 0.28) now implies that W is $\sigma(X^{**},X^*)$ closed. We show finally that $V := W^{\top} \subset X^*$ is w^*-dense and $r(V) = 0$.

For the first assertion, since $V^{\perp} = W$ by the w^*-closedness of W, it suffices to show that $W \cap X = \{0\}$. Suppose that $y^{**}+Ty^{**} =: x \in X$ for some $y^{**} \in Y^{**}$. Then $Ty^{**} = -y^{**}+x \in Z \cap (Y^{**}+X)$, so $Ty^{**} = -y^{**}+x = 0$ by assumption. Hence $y^{**} \in X \cap Y^{**} = Y$ (see the proof of Theorem 7.15 for this last equality). But now, since $T_{|Y}$ is injective and $Ty^{**} = 0$, it follows that $y^{**} = 0$. Hence $x = y^{**}+Ty^{**} = 0$ and $W \cap X = \{0\}$ is proved.

Finally, to see that $r(V) = 0$, let $\varepsilon > 0$ be arbitrary and choose

$y \in Y$ so that $\|y\| = 1$ and $\|Ty\| \le \varepsilon$. (Take $y \in \bigcap_{i=1}^{n} \ker y_i^*$ for sufficiently large n.) Then for every $x^* \in V$ we have $\langle x^*, y+Ty\rangle = 0$, so $|\langle y,x^*\rangle| = |\langle x^*,Ty\rangle| \le \varepsilon\|x^*\|$. Thus $\sup\{|\langle y,x^*\rangle|: x^* \in B_V\} \le \varepsilon$. Lemma 4.3 now says that $r(V) \le \varepsilon$. Hence $r(V) = 0$, since $\varepsilon > 0$ was arbitrary. □

For the proof of the announced theorem we need two auxiliary results.

PROPOSITION 7.18. *Let X be a non-reflexive Banach space. Then there exists an infinite-dimensional closed subspace* $Y \subset X$ *such that* X/Y *is not reflexive.*

PROOF. By Theorem 6.12 there exists in X a basic sequence $\{x_n\}$ which fails to be boundedly complete. Thus for some sequence $\{\alpha_n\} \subset \mathbb{R}$ we have that $\{\sum_{i=1}^{n} \alpha_i x_i\}_{n=1}^{\infty}$ is bounded but divergent. It follows that there exist a $\delta > 0$ and a sequence $0 = m_0 < m_1 < \dots < m_n < \dots$ in $\mathbb{N}$ such that for the block basic sequence $\{y_n\}$, $y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i$ $(n = 1,2,\dots)$ we have $\{\sum_{i=1}^{n} y_i\}_{n=1}^{\infty}$ bounded and $\|y_n\| \ge \delta$ $(n = 1,2,\dots)$. Now put $Y := [y_{2n-1}]$ and let $Q: X \to X/Y$ be the quotient map. We claim that $\{Qy_{2n}\}$ is a basic sequence in X/Y and that

$$\|Qy_{2n}\| \ge \frac{\delta}{2\nu_{\{y_n\}}} \qquad (n = 1,2,\dots),$$

while $\{\sum_{i=1}^{n} Qy_{2i}\}_{n=1}^{\infty}$ is bounded. Once this has been proved, we are done, since a basic sequence with these properties fails to be boundedly complete, and therefore X/Y is not reflexive, by Theorem 6.12.

We first show that $\{Qy_{2n}\}$ is basic. For this let $\ell, k \in \mathbb{N}$ and $\beta_1,\dots,\beta_{\ell+k} \in \mathbb{R}$ be given arbitrarily. We prove that $\|\sum_{i=1}^{\ell} \beta_i Qy_{2i}\| \le \nu_{\{y_n\}}\|\sum_{i=1}^{\ell+k} \beta_i Qy_{2i}\|$. Indeed, let $\varepsilon > 0$ be arbitrary and let $z = \sum_{i=1}^{\infty} \lambda_i y_{2i-1} \in Y$ be such that

$$\|\sum_{i=1}^{\ell+k} \beta_i Qy_{2i}\| > \|\sum_{i=1}^{\ell+k} \beta_i y_{2i} + z\| - \varepsilon.$$

Then

$$\|\sum_{i=1}^{\ell} \beta_i Qy_{2i}\| \le \|\sum_{i=1}^{\ell} \beta_i y_{2i} + \sum_{i=1}^{\ell} \lambda_i y_{2i-1}\| \le \nu_{\{y_n\}}\|\sum_{i=1}^{\ell+k} \beta_i y_{2i} + z\|$$

$$\le \nu_{\{y_n\}}\Big(\|\sum_{i=1}^{\ell+k} \beta_i Qy_{2i}\| + \varepsilon\Big).$$

Hence

$$\|\sum_{i=1}^{\ell} \beta_i Qy_{2i}\| \le \nu_{\{y_n\}}\|\sum_{i=1}^{\ell+k} \beta_i Qy_{2i}\|,$$

since $\varepsilon > 0$ was arbitrary.

Secondly, denoting by $\{y_n^*\}$ the sequence of coefficient functionals of the basis $\{y_n\}$ for $[y_n]$, we have for every $z \in [y_{2n-1}]$ and every $n \in \mathbb{N}$,

$$\|y_{2n}+z\| \|y_{2n}^*\| \geq \langle y_{2n}+z, y_{2n}^*\rangle = \langle y_{2n}, y_{2n}^*\rangle = 1,$$

so, by Proposition 5.2,

$$\|y_{2n}+z\| \geq \frac{1}{\|y_{2n}^*\|} \geq \frac{\|y_{2n}\|}{2\nu_{\{y_n\}}} \geq \frac{\delta}{2\nu_{\{y_n\}}}.$$

Thus

$$\|Qy_{2n}\| \geq \frac{\delta}{2\nu_{\{y_n\}}} \qquad (n = 1,2,\ldots).$$

Finally, $\{\sum_{i=1}^n Qy_{2i}\}$ is bounded, since $\sum_{i=1}^n Qy_{2i} = Q(\sum_{i=1}^{2n} y_i)$ $(n = 1,2,\ldots)$ and $\{\sum_{i=1}^{2n} y_i\}$ is bounded. □

Our next auxiliary result is a perturbation lemma for basic sequences. It says that if one changes the elements of a basic sequence slightly, the resulting sequence is still basic. With a view toward later applications we state it in a somewhat stronger form than is needed here. First we give a

DEFINITION 7.19. Two basic sequences $\{x_n\}$ and $\{y_n\}$ in Banach spaces X and Y, respectively, are called *equivalent* provided, for every sequence $\{\alpha_n\} \subset \mathbb{R}$, $\sum_{n=1}^\infty \alpha_n x_n$ converges iff $\sum_{n=1}^\infty \alpha_n y_n$ converges. Or, what is the same (by an application of the closed graph theorem), $\{x_n\}$ and $\{y_n\}$ are equivalent iff there is a (unique) isomorphism T from $[x_n]$ onto $[y_n]$ satisfying $Tx_n = y_n$ $(n = 1,2,\ldots)$. If $\|T\|\|T^{-1}\| \leq \alpha$, then $\{x_n\}$ and $\{y_n\}$ are called *α-equivalent*.

PROPOSITION 7.20. *Let $\{x_n\}$ be a basic sequence in a Banach space X and let $\{x_n^*\} \subset [x_n]^*$ denote its sequence of coefficient functionals. Then any sequence $\{y_n\} \subset X$ satisfying $\sum_{n=1}^\infty \|x_n-y_n\|\|x_n^*\| =: \delta < 1$ is again a basic sequence, and, moreover, $\{y_n\}$ is $\frac{1+\delta}{1-\delta}$-equivalent to $\{x_n\}$.*

PROOF. For any $x := \sum_{i=1}^n \alpha_i x_i \in sp\{x_n\}$ we have

$$\|\sum_{i=1}^n \alpha_i y_i\| - \|\sum_{i=1}^n \langle x, x_i^*\rangle (x_i - y_i)\| \leq \|\sum_{i=1}^n \alpha_i x_i\| \leq$$

$$\leq \| \sum_{i=1}^{n} \alpha_i y_i \| + \| \sum_{i=1}^{n} \langle x, x_i^* \rangle (x_i - y_i) \| .$$

Hence, since $\sum_{i=1}^{\infty} \|x_i^*\| \|x_i - y_i\| = \delta < 1$, it follows that

$$\| \sum_{i=1}^{n} \alpha_i y_i \| - \delta \|x\| \leq \|x\| \leq \| \sum_{i=1}^{n} \alpha_i y_i \| + \delta \|x\| .$$

This shows that the map $T: sp\{x_n\} \to sp\{y_n\}$ defined by $Tx_n := y_n$ $(n = 1,2,\ldots)$ satisfies $\|T\| \|T^{-1}\| \leq \frac{1+\delta}{1-\delta}$. The same estimate then holds for the unique extension $T: [x_n] \to [y_n]$. Since an isomorphic image of a basic sequence is clearly again a basic sequence, $\{y_n\}$ is therefore basic and $\frac{1+\delta}{1-\delta}$ - equivalent to $\{x_n\}$. □

We are now prepared to prove the main result.

THEOREM 7.21. *Let* X *be a Banach space. Then* X *is quasi-reflexive iff there exists no closed* w^**-dense subspace* $V \subset X^*$ *with* $r(V) = 0$.

PROOF. The necessity of the condition has been explained prior to Proposition 7.17. For sufficiency we assume that $\dim X^{**}/X = \infty$ and show the existence of a closed w^*-dense $V \subset X^*$ with $r(V) = 0$. It suffices to show that there exist subspaces $Y \subset X$ and $Z \subset X^{**}$ as in Proposition 7.17. First we construct a subspace $Y \subset X$ such that $\dim X^{**}/(X+Y^{**}) = \infty$ and worry about Z later. If X has an infinite-dimensional reflexive subspace Y, then $X^{**}/(X+Y^{**}) = X^{**}/X$ (cf. Corollary 7.16) and we are done. If not, then we can find, using Proposition 7.18, a chain of infinite-dimensional closed subspaces

$$X = X_1 \supset X_2 \supset X_3 \supset \ldots \supset X_k \supset \ldots$$

such that X_k/X_{k+1} is not reflexive $(k = 1,2,\ldots)$. Next, for each $k \in \mathbb{N}$ let us choose $y_k \in X_k \setminus X_{k+1}$. We claim that $Y := [y_k]$ does the job. To see this, observe first that

$$Y = [y_i]_{i=1}^{k-1} + (X_k \cap Y) \qquad (k = 1,2,\ldots),$$

so that

$$Y^{**} = [y_i]_{i=1}^{k-1} + (X_k \cap Y)^{**}$$

and therefore

$$X+Y^{**} = X + (X_k \cap Y)^{**} \subset X + X_k^{**} \qquad (k = 1,2,\ldots).$$

Since we have proved in Theorem 7.15 that $X^{**}/(X+X_k^{**}) \simeq H(X/X_k)$, it now suffices to show that $\dim H(X/X_k) \geq k-1$ $(k = 2,3,\ldots)$. This is done by induction. Clearly $\dim H(X/X_2) \geq 1$ and, more generally, $\dim H(X_k/X_{k+1}) \geq 1$ for all $k \in \mathbb{N}$. Suppose now that $\dim H(X/X_k) \geq k-1$ has been proved for some fixed $k \geq 2$. Since

$$X/X_{k+1} \Big/ X_k/X_{k+1} \simeq X/X_k,$$

we then have, by (7.3)

$$\dim H(X/X_{k+1}) = \dim H(X_k/X_{k+1}) + \dim H(X/X_k) \geq 1 + (k-1) = k.$$

Finally, for the existence of Z it suffices to prove the following general statement: Let X be any Banach space and $Y \subset X$ a closed infinite-dimensional subspace with $\dim X/Y = \infty$. Then there exists a closed infinite-dimensional subspace $Z \subset X$ such that $Y \cap Z = \{0\}$. The proof is an application of the perturbation lemma Proposition 7.20. Let $\{y_n\}$ and $\{\hat{z}_n\}$ be basic sequences in Y and X/Y, respectively, and let $Q: X \to X/Y$ be the quotient map. Define $z_n^* := Q^*\hat{z}_n^* \in X^*$ $(n = 1,2,\ldots)$, where $\{\hat{z}_n^*\} \subset (X/Y)^*$ is a sequence of Hahn-Banach extensions of the coefficient functionals associated to $\{\hat{z}_n\}$. Clearly then $\langle z_m, z_n^*\rangle = \delta_{mn}$ $(m,n = 1,2,\ldots)$ and $z_n^*|_Y = 0$ $(n = 1,2,\ldots)$. By Proposition 7.20 $\{y_n+\varepsilon_n z_n\}$ is a basic sequence for sufficiently small choices of the $\varepsilon_n > 0$. Then $Z := [y_n+\varepsilon_n z_n]$ is a closed infinite-dimensional subspace of X satisfying $Z \cap Y = \{0\}$. Indeed, any $x \in Y \cap Z$ is of the form $x = \sum_{n=1}^{\infty} \lambda_n (y_n+\varepsilon_n z_n)$. We have $\langle x, z_n^*\rangle = \varepsilon_n\lambda_n = 0$ $(n = 1,2,\ldots)$, by the choice of z_n^*. Thus $\lambda_n = 0$ $(n = 1,2,\ldots)$ and therefore $x = 0$. □

<u>NOTES</u>. The first example of a quasi-reflexive Banach space was the space J constructed by R.C. JAMES ([47]). Quasi-reflexive spaces in general were studied by P. CIVIN & B. YOOD ([16]). The material up to Theorem 7.7 is taken from [29]. The subsequent results up to Corollary 7.16 are from [16] and the remaining ones are due to W.J. DAVIS & J. LINDENSTRAUSS ([23]), with the exception of Proposition 7.20 which was proved by C. BESSAGA & A. PELCZYNSKI ([7]). Further results on quasi-reflexive spaces can be found in [21], [96] and [58]. E.g. in [21] the following analogue appears of a result we have already seen for non-reflexive spaces: a Banach space is non-quasi-reflexive iff it has a non-quasi-reflexive subspace with a basis.

The paper [58] shows that even after almost thirty years, the space J has not yet yielded all of its secrets: It is shown that J^* is not isomorphic to any subspace of J, but it is as yet unknown whether J can be isomorphically embedded into J^*.

8. SOMEWHAT REFLEXIVE SPACES

We have seen in section 6 that for a Banach space X which has an unconditional basis (more generally, which is isomorphic to a closed subspace of a σ-Dedekind complete Banach lattice with σ-order continuous norm), separability of X^{**} implies reflexivity of X. One reason why the James space is so remarkable is that for J this implication is false. Although J is not reflexive, we have seen (Property V in section 6) that it contains reflexive subspaces (even subspaces isomorphic to ℓ^2). The main result in this section will be that a much stronger statement is generally true: any Banach space with separable bidual is somewhat reflexive, in the sense of the following

DEFINITION 8.1. A Banach space X is called *somewhat reflexive* if every closed infinite-dimensional subspace $Y \subset X$ contains an infinite-dimensional reflexive subspace Z.

The proof is not easy. We first introduce the notion of w^*-basic sequences and then prove two existence theorems for special basic sequences: If X is a Banach space with separable dual X^*, then X contains a shrinking basic sequence (Corollary 8.9), and any w^* null sequence in X^* which is bounded away from 0 contains a subsequence which is a boundedly complete w^* basic sequence (Proposition 8.12). The main Theorem 8.13 and its Corollary 8.14 will be easy consequences of this. The first of these existence theorems (Corollary 8.9) can be proved directly in an elementary way via Proposition 5.13. We do not take the shortest route here. Instead we first prove Proposition 8.6, since it will be needed later anyway, and use it to derive Proposition 8.8, which is much stronger than its Corollary 8.9. The second existence theorem (Proposition 8.12) is deeper. Among other things the proof uses a renorming result of M.I. Kadec and V. Klee, which is a special case of Porposition 4.7.

The reader should recall at this point the duality between bases for X and w^* Schauder bases for X^* established in Proposition 6.7.

DEFINITION 8.2. Let X be a Banach space. A sequence $\{x_n^*\}$ in X^* is called a w^*-*basic sequence* iff it is a w^* -Schauder basis for $\widetilde{[x_n^*]}$ ($=\sigma(X^*,X)$-closure of $[x_n^*]$).

REMARK 8.3. An equivalent definition is the following: A sequence $\{x_n^*\} \subset X^*$ is a w^*-basic sequence provided there exists in X a sequence $\{x_n\}$ with the following properties:

(i) The system $\{x_n\}$, $\{x_n^*\}$ is biorthogonal.

(ii) For every $x^* \in \widetilde{[x_n^*]}$ we have $x^* = \sum_{n=1}^{\infty} \langle x_n, x^* \rangle x_n^*$, where the convergence is weak*.

Indeed, if $\{x_n^*\}$ is a w^*-Schauder basis for $\widetilde{[x_n^*]}$, then its w^*-continuous coefficient functionals can be extended w^*-continuously to X^*. This defines a sequence $\{x_n\} \subset X$ which clearly satisfies (i) and (ii). The converse is obvious. It should be observed that the above sequence $\{x_n\}$ is not uniquely determined, unless $\widetilde{[x_n^*]} = X^*$.

PROPOSITION 8.4. *A sequence $\{x_n^*\}$ in a dual Banach space X^* is a w^*-basic sequence iff $\{Q^{*-1}x_n^*\}$ is a w^*-Schauder basis for $(X/[x_n^*]^\top)^*$, where $Q: X \to X/[x_n^*]^\top$ is the quotient map.*

PROOF. Consider $Q^*: (X/[x_n^*]^\top)^* \to X^*$. Q^* is not only an isometric embedding, but also an isomorphic embedding for the respective w^*-topologies. Since $Q^*(X/[x_n^*]^\top)^* = [x_n^*]^{\top\perp} = \widetilde{[x_n^*]}$, the proposition is now obvious.

We summarize some facts about w^*-basic sequences in the next

PROPOSITION 8.5. *Let $\{x_n^*\}$ be a sequence in a dual Banach space X^*, and let $Q: X \to X/[x_n^*]^\top$ be the quotient map. Then*

(i) *$\{x_n^*\}$ is a w^*-basic sequence iff $X/[x_n^*]^\top$ has a basis $\{y_n\}$, the coefficient functionals $\{y_n^*\}$ of which satisfy $Q^*y_n^* = x_n^*$ (n = 1,2,...). In particular every w^*-basic sequence is a basic sequence.*

(ii) *The following are equivalent:*

(a) *$\{x_n^*\}$ is a boundedly complete w^*-basic sequence (i.e. a w^*-basic sequence which is a boundedly complete basic sequence),*

(b) *$\{x_n^*\}$ is a w^*-basic sequence and $\widetilde{[x_n^*]} = [x_n^*]$,*

(c) $X/[x_n^*]^\top$ *has a shrinking basis* $\{y_n\}$ *the coefficient functionals* $\{y_n^*\}$ *of which satisfy* $Q^* y_n^* = x_n^*$ $(n = 1,2,\ldots)$.

(iii) *Equivalent are*:

(a) $\{x_n^*\}$ *is a shrinking* w^**-basic sequence (i.e. a* w^**-basic sequence which is a shrinking basic sequence)*,

(b) $X/[x_n^*]^\top$ *has a boundedly complete basis* $\{y_n\}$ *the coefficient functionals* $\{y_n^*\}$ *of which satisfy* $Q^* y_n^* = x_n^*$ $(n = 1,2,\ldots)$.

PROOF (i): For the first statement, combine Propositions 8.4 and 6.7. The last statement is a consequence of the fact that Q^* is an isometry and the coefficient functionals of any basis form a basic sequence, by Proposition 6.1.

(iii): Since Q^* is isometric, $\{x_n^*\}$ is a shrinking basic sequence iff $\{Q^{*-1}x_n^*\}$ is a shrinking basic sequence. Thus the equivalence of (a) and (b) in (iii) follows from (i) and from the duality between shrinking and boundedly complete basic sequences established in Proposition 6.8.

(ii): The equivalence (a) $\Leftrightarrow$ (c) is similar to (a) $\Leftrightarrow$ (b) in (iii). It remains to be shown that (b) $\Leftrightarrow$ (c). Since Q^* is an isometry from $(X/[x_n^*]^\top)^*$ onto $\widetilde{[x_n^*]}$ which is also a w^*-isomorphism, $\widetilde{[x_n^*]} = [x_n^*]$ iff $(X/[x_n^*]^\top)^* = [Q^{*-1}x_n^*]$. Hence (b) $\Leftrightarrow$ (c) follows from (i) and the definition of a shrinking basis. □

The proof of the announced main result of this section will be a combination of two existence theorems, one for shrinking basic sequences and another for boundedly complete w^* basic sequences. We start deriving these now. First we prove a criterion for a sequence to contain a basic subsequence.

PROPOSITION 8.6. *Let* X *be a Banach space with a basis* $\{x_n\}$, *and let* $\{x_n^*\}$ *be its sequence of coefficient functionals. If* $\{y_n\} \subset X$ *is a sequence such that*

(i) $\limsup_{n\to\infty} \|y_n\| > 0$, *and*

(ii) $\lim_{n\to\infty} \langle y_n, x_i^*\rangle = 0$ *for all* $i \in \mathbb{N}$,

then, for every $\varepsilon > 0$ $\{y_n\}$ *contains a basic subsequence which is* $(1+\varepsilon)$-*equivalent to a block basic sequence with respect to* $\{x_n\}$.

PROOF. Put $C := \nu_{\{x_n\}}$. Passing to a subsequence if necessary, we may assume, by (i), that

$$(8.1) \qquad \inf \|y_n\| =: \eta > 0.$$

Let $0 < \delta \leq 1$ be arbitrary. Using (ii), we can select inductively two subsequences $\{p_n\}$ and $\{q_n\}$ of $\mathbb{N}$ satisfying

$$(8.2) \qquad \Big\| \sum_{i=q_n+1}^{\infty} \langle y_{p_n}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^{n+2}} \qquad (n = 1,2,\ldots)$$

and

$$(8.3) \qquad \Big\| \sum_{i=1}^{q_n} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^{n+2}} \qquad (n = 1,2,\ldots).$$

Indeed, put $p_1 = 1$. Since $y_{p_1} = \sum_{i=1}^{\infty} \langle y_{p_1}, x_i^* \rangle \, x_i$, there exists a $q_1 \in \mathbb{N}$ such that

$$\Big\| \sum_{i=q_1+1}^{\infty} \langle y_{p_1}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^3}.$$

Now suppose that for some $n \geq 1$ $p_1 < \ldots < p_n$ and $q_1 < \ldots < q_n$ have been selected so that (8.2) and (8.3) hold. Then, by (ii) there exists a $p_{n+1} > p_n$ in $\mathbb{N}$ such that

$$\Big\| \sum_{i=1}^{q_n} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^{n+2}}.$$

Since $y_{p_{n+1}} = \sum_{i=1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i$, we can then find a $q_{n+1} > q_n$ in $\mathbb{N}$ with

$$\Big\| \sum_{i=q_{n+1}+1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^{n+3}}.$$

This completes the inductive definition of the sequences $\{p_n\}$ and $\{q_n\}$. We now put

$$(8.4) \qquad z_n := \sum_{i=q_n+1}^{q_{n+1}} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \qquad (n = 1,2,\ldots).$$

By (8.1), (8.3), (8.4), (8.2) and $C \geq 1$, $\delta \leq 1$, we have, for all $n \in \mathbb{N}$,

$$\eta \leq \| y_{p_{n+1}} \| = \Big\| \sum_{i=1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| \leq \Big\| \sum_{i=1}^{q_n} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| + \Big\| \sum_{i=q_n+1}^{q_{n+1}} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| + \Big\| \sum_{i=q_{n+1}+1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle \, x_i \Big\| \leq \frac{\eta\delta}{4C2^{n+2}} + \| z_n \| + \frac{\eta\delta}{4C2^{n+3}} \leq \frac{\eta}{2} + \| z_n \|.$$ Hence

$$(8.5) \qquad \| z_n \| \geq \frac{\eta}{2} \qquad (n = 1,2,\ldots).$$

Let us observe that $\{z_n\}$ is a block basic sequence with respect to $\{x_n\}$. Let $\{z_n^*\} \subset [z_n]^*$ be its sequence of coefficient functionals. Since $\nu_{\{z_n\}} \leq \nu_{\{x_n\}} = C$, by Lemma 5.14, we have by Proposition 5.2 that $1 \leq \|z_n\| \|z_n^*\| \leq 2C$ $(n = 1,2,\ldots)$, so that (8.5) implies

$$(8.6) \qquad \|z_n^*\| \leq \frac{4C}{\eta} \qquad (n = 1,2,\ldots).$$

On the other hand it follows from (8.4), (8.3) and (8.2) that for all $n \in \mathbb{N}$

$$(8.7) \qquad \| y_{p_{n+1}} - z_n \| = \| \sum_{i=1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle x_i - \sum_{i=q_n+1}^{q_{n+1}} \langle y_{p_{n+1}}, x_i^* \rangle x_i \| \leq$$

$$\| \sum_{i=1}^{q_n} \langle y_{p_{n+1}}, x_i^* \rangle x_i \| + \| \sum_{i=q_{n+1}+1}^{\infty} \langle y_{p_{n+1}}, x_i^* \rangle x_i \| \leq$$

$$\frac{\eta\delta}{4C2^{n+2}} + \frac{\eta\delta}{4C2^{n+3}} < \frac{\eta\delta}{4C2^{n+1}} .$$

Finally, (8.6) and (8.7) imply that

$$(8.8) \qquad \sum_{n=1}^{\infty} \| z_n^* \| \, \| z_n - y_{p_{n+1}} \| < \frac{4C}{\eta} \sum_{n=1}^{\infty} \frac{\eta\delta}{4C2^{n+1}} = \frac{\delta}{2} ,$$

so that by Proposition 7.20 $\{y_{p_{n+1}}\}$ is a basic sequence $\frac{1+\frac{\delta}{2}}{1-\frac{\delta}{2}}$-equivalent to the block basic sequence $\{z_n\}$ with respect to $\{x_n\}$. Since $\delta > 0$ was arbitrary, this completes the proof. □

COROLLARY 8.7. *Let* X *be a Banach space and let* $\{y_n\}$ *be a sequence in* X *such that*

(i) $\limsup_{n\to\infty} \|y_n\| > 0$, *and*

(ii) $\lim_{n\to\infty} y_n = 0$, *weakly*.

Then $\{y_n\}$ *has a basic subsequence.*

PROOF. There is no loss of generality in assuming that X is separable. X can be isometrically embedded in C([0,1]), by Prop. 0.17, and C([0,1]) has a basis $\{x_n\}$ (see section 5). Identifying X with a subspace of C([0,1]), it is now clear that the conditions of Proposition 8.6 are satisfied. □

The following result strengthens the conclusion of Corollary 8.7 in the case that X^* is separable.

PROPOSITION 8.8. *Suppose that* X *is a Banach space with separable dual* X^* *and that* $\{x_n\} \subset X$ *satisfies*

(i) $\limsup_{n\to\infty} \|x_n\| > 0$, *and*

(ii) $\lim_{n\to\infty} x_n = 0$, *weakly*.

Then $\{x_n\}$ *contains a shrinking basic subsequence.*

PROOF. Passing to a subsequence and normalizing, if necessary, we may assume that $\|x_n\| = 1$ $(n = 1,2,\dots)$, and also, by Corollary 8.7, that $\{x_n\}$ is basic. Let $\{x_n^*\} \subset [x_n]^*$ be the sequence of coefficient functionals. By Proposition 5.2, $1 \le \|x_n^*\| \le 2\nu_{\{x_n\}} =: C$. Therefore by Proposition 7.20 any sequence $\{y_n\} \subset X$ satisfying

$$(8.9) \qquad \sum_{n=1}^{\infty} \|x_n - y_n\| < \frac{1}{C}$$

is a basic sequence equivalent to $\{x_n\}$. Similarly, for any subsequence $\{n_i\} \subset \mathbb{N}$ and any choice of y_i such that

$$(8.10) \qquad \sum_{i=1}^{\infty} \|x_{n_i} - y_i\| < \frac{1}{C} ,$$

$\{y_i\}$ is a basic sequence equivalent to $\{x_{n_i}\}$. We shall now determine $\{n_i\} \subset \mathbb{N}$ and $\{y_i\}$ so that (8.10) holds and, in addition, $\{y_i\}$ is shrinking. Then, by the equivalence of $\{y_i\}$ and $\{x_{n_i}\}$, $\{x_{n_i}\}$ is the shrinking basic subsequence we were looking for.

Using the separability of X^*, let $\{z_n^*\} \subset X^*$ be dense, with $z_1^* = 0$. We now choose inductively a subsequence $\{n_i\} \subset \mathbb{N}$ and a sequence $\{y_i\} \subset X$ such that for all $i \in \mathbb{N}$

(iii) $y_i \in ([z_j^*]_{j=1}^i)^\top$, and

(iv) $\|y_i - x_{n_i}\| < \dfrac{1}{C2^i}$.

Let $y_1 = x_1$, $n_1 = 1$. Then (iii) and (iv) hold for $i = 1$, since $z_1^* = 0$. Suppose that for some $i \ge 2$ we have chosen $y_1,\dots,y_{i-1} \in X$ and $n_1 < n_2 < \dots < n_{i-1}$ in $\mathbb{N}$ such that (iii) and (iv) hold. For all $x \in X$ and $i \in \mathbb{N}$ we have $\mathrm{dist}(x,([z_j^*]_{j=1}^i)^\top) = \|Qx\|$, where $Q: X \to X/([z_j^*]_{j=1}^i)^\top$ denotes the quotient map. Since $(X/([z_j^*]_{j=1}^i)^\top)^* \cong [z_j^*]_{j=1}^i$, we have therefore

$$\mathrm{dist}(x,([z_j^*]_{j=1}^i)^\top) = \sup\{|\langle x,x^*\rangle| : x^* \in [z_j^*]_{j=1}^i, \|x^*\| \le 1\}.$$

It is now easily verified, using (ii) and the compactness of $B_{[z_j^*]_{j=1}^i}$, that $\lim_{n\to\infty} \mathrm{dist}(x_n,([z_j^*]_{j=1}^i)^T) = 0$ $(i = 1,2,\ldots)$. Hence we can select an $n_i > n_{i-1}$ in $\mathbb{N}$ and a $y_i \in ([z_j^*]_{j=1}^i)^T$ such that $\|x_{n_i}-y_i\| < \frac{1}{C2^i}$. This completes the inductive definition. It remains to be shown that $\{y_i\}$ is shrinking. Let $y^* \in [y_n]^*$ be arbitrary and let $x^* \in X^*$ be chosen so that $x^*|_{[y_n]} = y^*$. Since $\{z_n^*\}$ is dense in X^*, there exists a subsequence $\{z_{m_i}^*\}$ of $\{z_n^*\}$ such that

$$(8.11) \qquad \lim_{i\to\infty} z_{m_i}^* = x^* .$$

By (iii) we have

$$(8.12) \qquad z_{m_i}^*|_{[y_j]_{j=m_i}^\infty} = 0 \quad \text{for all } i \in \mathbb{N} .$$

Therefore, by (8.11) and (8.12),

$$\|x^*|_{[y_j]_{j=m_i}^\infty}\| \le \|(x^*-z_{m_i}^*)|_{[y_j]_{j=m_i}^\infty}\| + \|z_{m_i}^*|_{[y_j]_{j=n}^\infty}\| \le \|x^*-z_{m_i}^*\| \to 0$$

as $i \to \infty$. Hence $\lim_{n\to\infty} \|y^*|_{[y_j]_{j=n}^\infty}\| = \lim_{n\to\infty} \|x^*|_{[y_j]_{j=n}^\infty}\| = 0.$

By Lemma 6.5 this means that $\{y_i\}$ is shrinking. □

<u>COROLLARY 8.9</u>. *Let* X *be a Banach space with separable dual. Then* X *contains a shrinking basic sequence.*

<u>PROOF</u>. Let $\{x_n^*\}$ be a dense sequence in X^*. Choose a sequence $\{x_n\}$ in X such that

$$\|x_n\| = 1 \quad \text{and} \quad x_n \in \bigcap_{i=1}^n \ker x_i^* \qquad (n = 1,2,\ldots).$$

Then clearly $\lim_{n\to\infty} \langle x_n,x_i^*\rangle = 0$ for all $i \in \mathbb{N}$, so by the density of $\{x_n^*\}$ and the boundedness of $\{x_n\}$, we have $\lim_{n\to\infty} x_n = 0$, weakly. It now suffices to apply Proposition 8.8 to $\{x_n\}$. □

We now come to an existence theorem for w^* basic sequences which, together with Corollary 8.9, will lead to the main result.

PROPOSITION 8.10. *Let X be a separable Banach space and let* $\{x_n^*\} \subset X^*$ *be a sequence satisfying*

(i) $\limsup_{n\to\infty} \|x_n^*\| > 0$, *and*

(ii) $w^*\text{-}\lim_{n\to\infty} x_n^* = 0$.

Then $\{x_n^*\}$ *has a subsequence* $\{x_{n_k}^*\}$ *which is* w^* *basic. Moreover, this subsequence can be chosen in such a way that* $\lim_{n\to\infty} \|S_n\| = 1$, *where* $\{S_n\}$ *is the sequence of projections associated to the* w^* *basic sequence* $\{x_{n_k}^*\}$.
($S_n : \widetilde{[x_{n_k}^*]} \to \widetilde{[x_{n_k}^*]}$ *is defined by* $S_n x^* = \sum_{k=1}^n \langle x_k, x^*\rangle x_{n_k}^*$, *where* $\{x_k\} \subset X$ *is such that* $\{x_k\}$, $\{x_{n_k}^*\}$ *is biorthogonal, cf. Remark 8.3).*

PROOF. Passing to a subsequence if necessary we may assume, by (i), that $\inf\|x_n^*\| > 0$, and hence, by normalization, that $\|x_n^*\| = 1$ ($n = 1,2,\ldots$). (Observe that this normalization does not affect S_n.) We now describe the construction of $\{x_{n_k}^*\}$ first and then show that it satisfies the requirements. Choose a sequence $\{\varepsilon_n\}$, $0 < \varepsilon_n < 1$ ($n = 1,2,\ldots$) such that

$$\sum_{n=1}^{\infty} \varepsilon_n < \infty, \quad \text{so} \quad \prod_{n=1}^{\infty} \frac{1}{1-\varepsilon_n} < \infty.$$

We now determine inductively a subsequence $\{n_k\} \subset \mathbb{N}$ and an increasing sequence of finite subsets $\{F_k\}$ of S_X such that $[\bigcup_{k=1}^{\infty} F_k] = X$ and such that the following conditions are satisfied for each $k \in \mathbb{N}$:

(a) For every $v^* \in ([x_{n_i}^*]_{i=1}^k)^*$ with $\|v^*\| = 1$ there exists an $x \in F_k$ such that

$$(8.13) \qquad |\langle x,x^*\rangle - \langle x^*,v^*\rangle| \le \frac{\varepsilon_k}{3}\|x^*\| \quad \text{for all } x^* \in [x_{n_i}^*]_{i=1}^k$$

$$\text{(b)} \qquad |\langle x,x_{n_{k+1}}^*\rangle| < \frac{\varepsilon_k}{3} \quad \text{for all } x \in F_k.$$

This is done as follows. First by the separability of X we can choose an increasing sequence $\{F_k'\}$ of finite subsets of S_X such that $[\bigcup_{k=1}^{\infty} F_k'] = X$. Choose $n_1 = 1$. The existence of F_1 is included in the general induction step below, putting $F_0 = \emptyset$. Suppose now that $n_1 < \ldots < n_k$ in $\mathbb{N}$ and finite subsets $F_1 \subset F_2 \subset \ldots \subset F_{k-1}$ in S_X have been determined in such a way that $F_i' \subset F_i$ ($i = 1,\ldots,k-1$) and (a) and (b) hold. We indicate how F_k and n_{k+1} are selected. For the choice of F_k, let $\{v_1^*,\ldots,v_n^*\}$ be a δ-net of $S_{([x_{n_i}^*]_{i=1}^k)^*}$, $\delta > 0$. Extend these elements to unit vectors $x_1^{**},\ldots,x_n^{**}$ of X^{**}. Then apply local reflexivity to find a map T from $[x_i^{**}]_{i=1}^n$ into X such that $1 \le \|T\|\|T^{-1}\| < 1+\delta$ and

$$\langle Tx_i^{**}, x^* \rangle = \langle x^*, x_i^{**} \rangle = \langle x^*, v_i^* \rangle \text{ for all } x^* \in [x_{n_i}^*]_{i=1}^k \quad (i = 1,\dots,n),$$

and put

$$x_i := \frac{Tx_i^{**}}{\|Tx_i^{**}\|} \qquad (i = 1,\dots,n).$$

It should be clear now that if δ is chosen sufficiently small, then for every $v^* \in ([x_{n_i}^*]_{i=1}^k)^*$ with $\|v^*\| = 1$ there exists an $i \in \{1,\dots,n\}$ such that (8.13) holds with $x = x_i$. Therefore $F_k := F_k' \cup F_{k-1} \cup \{x_1,\dots,x_n\}$ will do. The choice of n_{k+1} such that (b) holds is no problem, since we have (ii). Finally, since $[\bigcup_{k=1}^\infty F_k'] = X$ and $F_k' \subset F_k$ $(k = 1,2,\dots)$, we have $[\bigcup_{k=1}^\infty F_k] = X$.

We now check in several steps that $\{x_{n_k}^*\}$ is as desired.

<u>STEP 1</u>. $\{x_{n_k}^*\}$ is a basic sequence.

<u>PROOF</u>. Let $k \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_k \in \mathbb{R}$ be such that $\|\sum_{i=1}^k \alpha_i x_{n_i}^*\| = 1$, but otherwise arbitrary. Choose a $v^* \in ([x_{n_i}^*]_{i=1}^k)^*$ such that

$$\langle \sum_{i=1}^k \alpha_i x_{n_i}^*, v^* \rangle = \|v^*\| = 1.$$

By (a) there exists an $x \in F_k$ so that (8.13) holds for this v^*. Then

$$|\langle x, \sum_{i=1}^k \alpha_i x_{n_i}^* \rangle| \geq 1 - \frac{\varepsilon_k}{3}. \tag{8.14}$$

Now for every choice of $\alpha_{k+1} \in \mathbb{R}$ we have, by (8.14) and (b),

$$\|\sum_{i=1}^{k+1} \alpha_i x_{n_i}^*\| \geq \begin{cases} |\langle x, \sum_{i=1}^k \alpha_i x_{n_i}^* \rangle + \langle x, \alpha_{k+1} x_{n_{k+1}}^* \rangle| \geq 1 - \frac{\varepsilon_k}{3} - \frac{2\varepsilon_k}{3} = 1 - \varepsilon_k \\ \qquad \text{if } |\alpha_{k+1}| \leq 2 \\ \|\alpha_{k+1} x_{n_{k+1}}^*\| - \|\sum_{i=1}^k \alpha_i x_{n_i}^*\| > 2 - 1 = 1 \quad \text{if } |\alpha_{k+1}| > 2. \end{cases}$$

Thus it follows that

$$\|\sum_{i=1}^k \alpha_i x_{n_i}^*\| \leq \frac{1}{1-\varepsilon_k} \|\sum_{i=1}^{k+1} \alpha_i x_{n_i}^*\| \text{ for all } k \in \mathbb{N} \text{ and } \alpha_1,\dots,\alpha_{k+1} \in \mathbb{R},$$

and therefore, by iteration,

$$\left\|\sum_{i=1}^{k}\alpha_i x^*_{n_i}\right\| \le \left(\prod_{i=k}^{k+\ell-1}\frac{1}{1-\varepsilon_i}\right)\left\|\sum_{i=1}^{k+\ell}\alpha_i x^*_{n_i}\right\| \text{ for all } k,\ell\in\mathbb{N} \text{ and } \alpha_1,\dots,\alpha_{k+\ell}\in\mathbb{R}.$$

This proves that $\{x^*_{n_k}\}$ is basic and, moreover, that for the projections Q_k associated to this basic sequence we have

$$\|Q_k\| \le \prod_{i=k}^{\infty}\frac{1}{1-\varepsilon_i} \qquad (k = 1,2,\dots),$$

so

(8.15) $$\lim_{n\to\infty}\|Q_n\| = 1.$$

Now let $\{v^*_k\}\subset[x^*_{n_k}]^*$ be the sequence of coefficient functionals associated to the basis $\{x^*_{n_k}\}$ for $[x^*_{n_k}]$. Then $\{v^*_k\}$ is a basis for $[v^*_k]$ and the projections P_k associated to this basis look as follows

$$P_k v^* = \sum_{i=1}^{k} \langle x^*_{n_i}, v^*\rangle\, v^*_i \qquad (v^*\in[v^*_k], \qquad k = 1,2,\dots).$$

Since the P_k are the restrictions of the Q^*_k to $[v^*_k]$, it follows from (8.15) that

(8.16) $$\lim_{k\to\infty}\|P_k\| = 1.$$

We now define the linear map $A\colon X \to [x^*_{n_k}]^*$ by

$$\langle x^*, Ax\rangle = \langle x, x^*\rangle \qquad (x\in X, x^*\in[x^*_{n_k}]).$$

It is immediate that $\|A\| \le 1$. We now proceed to show that $AX = [v^*_k]$.

<u>STEP 2</u>. $AX \subset [v^*_k]$.

<u>PROOF</u>. Since $X = [\bigcup_{k=1}^{\infty} F_k]$, it suffices to show that $AF_k \subset [v^*_k]$ for each $k\in\mathbb{N}$. Let $x\in F_k$, for some $k\in\mathbb{N}$, and consider the series $\sum_{i=1}^{\infty}\langle x, x^*_{n_i}\rangle\, v^*_i$. Observe that it converges, by (b), the choice of the ε_k and by the boundedness of $\{v^*_k\}$ ($\|v^*_k\|\|x^*_{n_k}\| \le 2\nu_{\{x^*_{n_k}\}}$, by Proposition 5.2,

and $\|x^*_{n_k}\| = 1$ $(k = 1,2,\ldots))$. Hence it defines an element of $[v^*_k]$. Furthermore, for every $i \in \mathbb{N}$

$$<x^*_{n_i}, \sum_{j=1}^{\infty} <x,x^*_{n_j}>v^*_j> = <x,x^*_{n_i}> = <x^*_{n_i},Ax>,$$

which implies that

$$Ax = \sum_{i=1}^{\infty} <x,x^*_{n_i}> v^*_i \in [v^*_k]. \tag{8.17}$$

STEP 3. For every $v^*_0 \in [v^*_k]$ with $\|v^*_0\| = 1$ and for every $\varepsilon > 0$ there exists an $x_0 \in X$ with $\|x_0\| = 1$ such that $\|Ax_0 - v^*_0\| < 4\varepsilon$.

PROOF. We may assume that $\varepsilon < 1$. Choose $N \in \mathbb{N}$ so that

$$\sum_{j=n}^{\infty} \varepsilon_j < \frac{4\varepsilon}{C} \quad \text{and} \quad \|Q_n\| < 1 + \varepsilon \quad \text{for } n > N, \tag{8.18}$$

where $C := \max(4, \sup_{k\in\mathbb{N}} \|v^*_k\|)$. Now fix $n > N$. For convenience we define for elements $v^* \in [v^*_k]_{k=1}^n$

$$\|v^*\|_1 := \|v^*|_{[x^*_{n_k}]_{k=1}^n}\|.$$

We then have, for all $v^* \in [v^*_k]_{k=1}^n$,

$$\|v^*\|_1 \le \|v^*\| \le \|Q_n\| \|v^*\|_1 \le 2\|v^*\|_1. \tag{8.19}$$

The first of these inequalities is clear and the last one follows from (8.18). To see the middle one, let $v^* = \sum_{k=1}^n \alpha_k v^*_k \in [v^*_k]_{k=1}^n$ be given. Then, clearly, for every $x^* \in [x^*_{n_k}]$ we have $<x^*,v^*> = <Q_n x^*,v^*>$. Hence, since $Q_n x^* \in [x^*_{n_k}]_{k=1}^n$,

$$\|v^*\| = \sup_{\substack{\|x^*\|\le 1\\ x^*\in[x^*_{n_k}]}} |<x^*,v^*>| = \sup_{\substack{\|x^*\|\le 1\\ x^*\in[x^*_{n_k}]}} |<Q_n x^*,v^*>| \le$$

$$\|Q_n\| \sup_{\substack{\|x^*\|\le 1\\ x^*\in[x^*_{n_k}]_{k=1}^n}} |<x^*,v^*>| = \|Q_n\| \|v^*\|_1.$$

This proves (8.19).

Now let $w^* \in [v_k^*]_{k=1}^n$ with $\|w^*\| = 1$ be given and put

$$v^* = \frac{w^*}{\|w^*\|_1} .$$

Then $v^* = \sum_{k=1}^n \langle x_{n_k}^*, v^* \rangle v_k^*$ and $\|v^*\|_1 = 1$. Hence, by (a), there exists an $x_0 \in F_n$ such that (8.13) holds with $k = n$ and $x = x_0$. Each $x^* \in [x_{n_k}^*]_{k=1}^n$ is of the form $x^* = \sum_{k=1}^n \langle x^*, v_k^* \rangle x_{n_k}^*$, so $\langle x_0, x^* \rangle = \sum_{k=1}^n \langle x^*, v_k^* \rangle \langle x_0, x_{n_k}^* \rangle$. Thus (8.13) says that

$$(8.20) \qquad \Big\| \sum_{k=1}^{n} \langle x_0, x_{n_k}^* \rangle v_k^* - v^* \Big\|_1 < \frac{\varepsilon_n}{3} .$$

Therefore, by (8.19) and (8.18),

$$(8.21) \qquad \Big\| \sum_{k=1}^{n} \langle x_0, x_{n_k}^* \rangle v_k^* - v^* \Big\| < \frac{2\varepsilon_n}{3} < \frac{2}{3}\varepsilon .$$

Also, since $\sup_{k\in\mathbb{N}} \|v_k^*\| \le C$, and by (b) and (8.18),

$$(8.22) \qquad \Big\| \sum_{k=n+1}^{\infty} \langle x_0, x_{n_k}^* \rangle v_k^* \Big\| \le C \sum_{k=n+1}^{\infty} |\langle x_0, x_{n_k}^* \rangle| \le C \sum_{k=n+1}^{\infty} \frac{\varepsilon_k}{3} < C\frac{4\varepsilon}{3C} = \frac{4}{3}\varepsilon .$$

Since we have shown in (8.17) that $Ax_0 = \sum_{k=1}^\infty \langle x_0, x_{n_k}^* \rangle v_k^*$,

(8.21) and (8.22) yield

$$(8.23) \qquad \|Ax_0 - v^*\| = \Big\| \sum_{k=1}^{\infty} \langle x_0, x_{n_k}^* \rangle v_k^* - v^* \Big\| \le \Big\| \sum_{k=1}^{n} \langle x_0, x_{n_k}^* \rangle v_k^* - v^* \Big\| + \Big\| \sum_{k=n+1}^{\infty} \langle x_0, x_{n_k}^* \rangle v_k^* \Big\| < \frac{2}{3}\varepsilon + \frac{4}{3}\varepsilon = 2\varepsilon .$$

Finally, by (8.19) and (8.18), $1 = \|w^*\| \le \|Q_n\| \|w^*\|_1 \le (1+\varepsilon)\|w^*\|_1$, so

$$\frac{1}{\|w^*\|_1} \le 1 + \varepsilon .$$

Hence

$$(8.24) \qquad \|v^* - w^*\| = \Big\| \frac{w^*}{\|w^*\|_1} - w^* \Big\| = \|w^*\| \left(\frac{1}{\|w^*\|_1} - 1 \right) \le \varepsilon .$$

Therefore, by (8.23) and (8.24),

$$(8.25)\qquad \|Ax_0-w^*\| \le \|Ax_0-v^*\| + \|v^*-w^*\| \le 3\varepsilon.$$

To conclude the proof, let us assume that $v_0^* \in [v_k^*]$ with $\|v_0^*\| = 1$ is given. Then we can select an $n > N$ and a $w^* \in [v_k^*]_{k=1}^n$ with $\|w^*\| = 1$ such that $\|v_0^*-w^*\| < \varepsilon$. For this w^* we now determine an $x_0 \in X$, $\|x_0\| = 1$ such that (8.25) holds. Then $\|Ax_0-v_0^*\| \le \|Ax_0-w^*\| + \|w^*-v_0^*\| < 4\varepsilon$ and the proof is finished.

STEP 4. $AX = [v_k^*]$, and moreover, A maps the open unit ball of X onto the open unit ball of $[v_k^*]$.

PROOF. Steps 2 and 3 and the fact that $\|A\| \le 1$ imply that A maps the unit ball of X onto a dense subset of the unit ball of $[v_k^*]$. The usual proof of the open mapping theorem then shows that the open ball of X is mapped onto the open ball of $[v_k^*]$. In particular $AX = [v_k^*]$.

STEP 5. $\{x_{n_k}^*\}$ is a w^*-basic sequence and $\lim_{n\to\infty} \|S_n\| = 1$.

PROOF. Since $\ker A = [x_{n_k}^*]^\top$, step 4 implies that A defines an isometry $\hat{A}$ of $X/[x_{n_k}^*]^\top$ onto $[v_k^*]$. $\{v_k^*\}$ being a basis for $[v_k^*]$, it follows that $\{\hat{A}^{-1}v_k^*\}$ is a basis for $X/[x_{n_k}^*]^\top$. If, as before, $Q\colon X \to X/[x_{n_k}^*]^\top$ denotes the quotient map, then it is evident that $\{Q^{*-1}x_{n_k}^*\}$ is the sequence of coefficient functionals of the basis $\{\hat{A}^{-1}v_k^*\}$. By Proposition 8.5 (i) this means that $\{x_{n_k}^*\}$ is a w^*-basic sequence. Finally, since Q^* and $\hat{A}$ are isometries, it follows that $\|S_n\| = \|P_n^*\|$ $(n = 1,2,\ldots)$ and therefore, by (8.16), $\lim_{n\to\infty} \|S_n\| = 1$. □

It is not without purpose that we have taken the trouble to construct the w^*-basic sequence in Proposition 8.10 in such a way that $\lim_{n\to\infty} \|S_n\| = 1$ for the associated projections S_n. This extra property will be instrumental in showing that the w^*-basic sequence can be chosen to be boundedly complete in case X^* is separable. The proof of this depends on the following renorming theorem, which is a special case of Proposition 4.7.

PROPOSITION 8.11. *Let X be a Banach space with a separable dual. Then X has an equivalent norm $|||\cdot|||$ with the property that, for this new norm, w^* sequential convergence coincides with norm convergence on the unit sphere of X. More precisely: if $\{x_n^*\}$ is a sequence in X^* such that $\{x_n^*\}$ w^* converges to x^* and, in addition, $\lim_{n\to\infty} |||x_n^*||| = |||x^*|||$, then $\{x_n^*\}$ converges* *to* x^* *in norm.*

PROPOSITION 8.12. *Let* X *be a Banach space with separable dual and let* $\{x_n^*\} \subset X^*$ *be a sequence satisfying*

(i) $\limsup_{n\to\infty} \|x_n^*\| > 0$, *and*

(ii) $w^*\text{-}\lim_{n\to\infty} x_n^* = 0$.

Then $\{x_n^*\}$ *contains a boundedly complete* w^* *basic subsequence.*

PROOF. We may assume, by passing to an equivalent norm, that $\|\cdot\|$ has the property of Proposition 8.11. Since X^* is separable, so is X. Thus by Proposition 8.10 $\{x_n^*\}$ has a w^* basic subsequence $\{x_{n_k}^*\}$ with the property that

$$\lim_{n\to\infty} \|S_n\| = 1. \tag{8.34}$$

The projections $S_n: \widetilde{[x_{n_k}^*]} \to \widetilde{[x_{n_k}^*]}$ are defined by

$$S_n x^* = \sum_{k=1}^{n} \langle x_k, x^* \rangle x_{n_k}^* \qquad (x^* \in \widetilde{[x_{n_k}^*]}),$$

where $\{x_k\} \subset X$ is such that $\{x_k\}$, $\{x_{n_k}^*\}$ is biorthogonal. Let $x^* \in \widetilde{[x_{n_k}^*]}$ be arbitrary. Then, by Remark 8.3,

$$w^*\text{-}\lim_{n\to\infty} S_n x^* = x^*. \tag{8.35}$$

Since $\lim_{n\to\infty} \|S_n\| = 1$ we have $\limsup_{n\to\infty} \|S_n x^*\| \le \|x^*\|$, while on the other hand the w^* lower semi continuity of $\|\ \|$ on X^* implies $\liminf_{n\to\infty} \|S_n x^*\| \ge \|x^*\|$. Thus $\lim_{n\to\infty} \|S_n x^*\| = \|x^*\|$. Together with (8.35) this implies, by Proposition 8.11, that $\lim_{n\to\infty} S_n x^* = x^*$, in norm, so that $x^* \in [x_{n_k}^*]$. We have now shown that $\widetilde{[x_{n_k}^*]} = [x_{n_k}^*]$, which, by Proposition 8.5 (ii), means precisely that $\{x_{n_k}^*\}$ is boundedly complete. □

It remains to combine Corollary 8.9 and Proposition 8.12 to prove the main result.

THEOREM 8.13. *Let* X *be a Banach space with separable dual* X^* *and let* $Y \subset X^*$ *be an infinite-dimensional closed subspace with separable dual* Y^*. *Then* Y *is somewhat reflexive.*

PROOF. Let $Z \subset Y$ be any closed infinite-dimensional subspace. Since Y^* is separable, so is Z^*. By Corollary 8.9, therefore, we can pick in Z a shrinking basic sequence $\{z_n\}$ which we may assume to be normalized. Let $\{z_n^*\} \subset [z_n]^*$ be its sequence of coefficient functionals. Then $[z_n^*] = [z_n]^*$ and $\lim_{n\to\infty} \langle z_n, z_k^* \rangle = 0$ for all $k \in \mathbb{N}$. The boundedness of $\{z_n\}$ now implies that $\lim z_n = 0$, for $\sigma(Z,Z^*)$. If we define $x_n^* = Iz_n$ $(n = 1,2,\dots)$, where $I: Z \to X^*$ is the identity, then clearly $w^*\text{-}\lim_{n\to\infty} x_n^* = 0$, and of course $\{x_n^*\}$ is shrinking. Application of Proposition 8.12 now yields a subsequence $\{x_{n_k}^*\}$ which is a boundedly complete w^*-basic sequence. As a subsequence of a shrinking basic sequence, $\{x_{n_k}^*\}$ is also shrinking, as is easily verified using Lemma 6.5. It follows now from Theorem 6.9 that $[x_{n_k}^*]$ is reflexive. We have now proved that any closed $Z \subset Y$ with $\dim Z = \infty$ contains a subspace $[z_{n_k}]$ which is reflexive, i.e. Y is somewhat reflexive. □

COROLLARY 8.14. *Let X be a Banach space such that X^{**} is separable. Then both X and X^* are somewhat reflexive.*

PROOF. Applying Theorem 8.13 with $Y = X^*$ immediately shows that X^* is somewhat reflexive. Now let $Y \subset X$ be an arbitrary closed subspace with $\dim Y = \infty$. Then Y^* is separable since X^* is. Identifying Y with the subspace $\pi_X Y$ of X^{**}, Theorem 8.13 applied to X^* instead of X shows that Y is somewhat reflexive. In particular Y has a reflexive subspace Z with $\dim Z = \infty$. Thus X is somewhat reflexive, as was to be proved. □

NOTES. Proposition 8.6 and its Corollary 8.7 are due to C. BESSAGA and A. PELCZYNSKI ([7]). All other results in this section come from [61]. An exception is Proposition 8.11 which is due to M.I. KADEC ([63]) and V. KLEE ([68]) in this form. The existence of shrinking basic sequences in Banach spaces with separable dual (which is here a consequence of Proposition 8.8) was first proved in [26]. Precursors of the main result of this section appeared in [45]. E.g. it was shown there that quasi-reflexive spaces are somewhat reflexive. Also an example of a non-quasi-reflexive somewhat reflexive space was given.

9. A SEPARABLE SOMEWHAT REFLEXIVE SPACE WITH NON-SEPARABLE DUAL

It seems reasonable to conjecture, as a kind of converse of Theorem 8.13, that a somewhat reflexive separable (dual) space should have separable dual. This section is devoted to a counterexample to this conjecture due to R.C. JAMES. It is modeled on the James space and is known as the James tree space. It is a separable dual space with non-separable dual and has the property that every closed infinite-dimensional subspace has a subspace isomorphic to ℓ^2. There are several other conjectures to which the James tree space is a counterexample. E.g. it refutes the idea long held that every separable space with non-separable dual must contain ℓ^1.

We now describe the space in question. Let us define a subset T of $\mathbb{N} \times \mathbb{N}$ as follows:

$$T := \{(n,i): n = 1,2,\ldots;\ 1 \le i \le 2^n\}.$$

We partially order T by putting $(n,i) \le (m,j)$ iff $n \le m$ and there exist integers $i = i_n, i_{n+1}, \ldots, i_m = j$ with $i_\ell \in \{2i_{\ell-1}-1, 2i_{\ell-1}\}$ ($\ell = n+1, \ldots, m$). A set of the form $\{(n,i_n),(n+1,i_{n+1}),\ldots,(n+k,i_{n+k})\}$ with $i_\ell \in \{2i_{\ell-1}-1, 2i_{\ell-1}\}$ ($\ell = n+1,\ldots,n+k$) is called *segment* ($n = 1,2,\ldots;\ k = 0,1,\ldots$). For each $n \in \mathbb{N}$ the points (n,i), $1 \le i \le 2^n$, are called *branch points of order* n. A *branch of order* n or an n-*branch* is an infinite segment starting with a branch point of order n, i.e. a set of the form $\{(n,i_n),(n+1,i_{n+1}),\ldots\}$ with $i_\ell \in \{2i_{\ell-1}-1, 2i_{\ell-1}\}$ ($\ell = n+1,n+2,\ldots$). A 1-branch is simply called a *branch*.

Now let X (the James tree space) be the set of all real functions x on T such that

$$(9.1) \qquad \|x\| := \sup\left(\sum_{j=1}^{k}\left(\sum_{(n,i)\in S_j} x(n,i)\right)^2\right)^{\frac{1}{2}} < \infty,$$

where the sup is taken over all $k \in \mathbb{N}$ and all sets of pairwise disjoint segments $S_1,\ldots,S_k$. It is not hard to prove that $(X,\|\cdot\|)$ is a Banach space.

Note that $x \in X$ implies that for every $\varepsilon > 0$ $|x(n,i)| < \varepsilon$ except for finitely many $(n,i) \in T$.

We derive now several properties of X.

<u>PROPERTY I</u>. X is isometric to a separable dual space.

<u>PROOF</u>. For each $(n,i) \in T$ let $e_{n,i}$ be the characteristic function of $\{(n,i)\}$. It is straightforward that these elements, enumerated in lexicographic order

$$e_{1,1}, e_{1,2}, e_{2,1}, e_{2,2}, e_{2,3}, e_{2,4}, \ldots, e_{n,1}, \ldots, e_{n,2^n}, e_{n+1,1}, \ldots$$

form a boundedly complete monotone basis for X. Let $\{e^*_{n,i}\} \subset X^*$ be its sequence of coefficient functionals and put $V := [e^*_{n,i}]$. By Proposition 6.3 $r(V) = 1$ and therefore by Lemma 4.3 the map $A: X \to V^*$ defined by $\langle x^*, Ax\rangle = \langle x, x^*\rangle$ $(x \in X,\ x^* \in V)$ is isometric. It follows from Proposition 6.8 that $\{e^*_{n,i}\}$ is a shrinking basis for V, so that $[Ae_{n,i}] = V^*$. Thus A is onto and we have proved that $X \cong V^*$. (In fact this proof shows that any Banach space with a boundedly complete basis is isomorphic to a dual space, and isometric if this basis is monotone.) □

<u>PROPERTY II</u>. X^* is non-separable.

<u>PROOF</u>. Let B be any branch of T. We define an element $x^*_B \in X^*$ by

$$\langle x, x^*_B\rangle = \sum_{(n,i)\in B} x(n,i) \qquad (x \in X).$$

(Here the summation is in the order that B inherits from T.) Observe first that this series converges for every $x \in X$, by (9.1). Suppose that $x \in X$, $\|x\| = 1$. Then for any initial segment S of B we have $(\sum_{(n,i)\in S} x(n,i))^2 \le 1$, so $|\sum_{(n,i)\in S} x(n,i)| \le 1$ and therefore $|\langle x, x^*_B\rangle| \le 1$. Hence $\|x^*_B\| \le 1$. Also $\langle e_{n,i}, x^*_B\rangle = \|e_{n,i}\| = 1$ whenever $(n,i) \in B$, so $\|x^*_B\| = 1$ for all B. On the other hand, if B_1 and B_2 are two different branches of T then, choosing $(n,i) \in B_1\backslash B_2$ and $(m,j) \in B_2\backslash B_1$, we have $\langle e_{n,i}-e_{m,j}, x^*_{B_1}-x^*_{B_2}\rangle = 2$. Since $\|e_{n,i}-e_{m,j}\| = \sqrt{2}$, it follows that $\|x^*_{B_1}-x^*_{B_2}\| \ge \sqrt{2}$. Thus X^* is non-separable, since the number of branches is uncountable (branches are in 1-1 correspondence with dyadic expansions of real numbers in $[0,1]$). □

We now come to the deepest property of X, namely that every closed infinite-dimensional subspace of X contains a subspace isomorphic to ℓ^2. We shall in fact prove the stronger statement that ℓ^2 is contained "uniformly" in every closed infinite-dimensional subspace of X:

PROPERTY III. Let $\theta > \sqrt{2}$. Then for every closed infinite-dimensional subspace $Y \subset X$ there exists a closed infinite-dimensional subspace $Z \subset Y$ and an inner product norm $|||\cdot|||$ on Z such that

$$(9.2) \qquad |||x||| \le \|x\| \le \theta|||x||| \qquad \text{for all } x \in Z.$$

The proof of this will occupy us for the rest of this section. Let us denote the dense subspace $sp\{e_{n,i}\}$ of X by E. For the proof of PROPERTY III it suffices to show

THEOREM 9.1. *Let* $\theta > \sqrt{2}$. *Then for every infinite-dimensional (non-closed) linear subspace* F *of* E *there exists an infinite-dimensional subspace* G *of* F *and an inner product norm* $|||\cdot|||$ *on* G *such that*

$$(9.3) \qquad |||x||| \le \|x\| \le \theta|||x||| \qquad \text{for all } x \in G.$$

Indeed, suppose Theorem 9.1 has been proved. Let $\theta > \theta' > \sqrt{2}$. Given a closed infinite-dimensional subspace Y of X, choose a basic sequence $\{y_n\}$ in Y (Proposition 5.13). Let $\{\varepsilon_n\}$ be a sequence of positive numbers. For each $n \in \mathbb{N}$ we select an $e_n \in E$ such that $\|y_n - e_n\| < \varepsilon_n$. We know from Proposition 7.20 that for sufficiently small choices of the ε_n, $\{e_n\}$ is basic and $sp\{e_n\}$ is isomorphic to $sp\{y_n\}$. By assumption $sp\{e_n\}$ has a subspace G with $\dim G = \infty$ on which an inner product norm exists such that

$$|||x||| \le \|x\| \le \theta'|||x||| \qquad \text{for all } x \in G.$$

It is also clear from Proposition 7.20 that for suitably small choices of the ε_n there exists an isomorphism $T: sp\{e_n\} \to sp\{y_n\}$ with

$$\|x\| \le \|Tx\| \le \frac{\theta}{\theta'}\|x\| \quad (x \in sp\{e_n\}).$$

It now follows that on $Z := \overline{TG}$ the unique inner product norm defined by $|||Tx||| := |||x|||$ $(x \in G)$, satisfies (9.2).

Before starting the proof of Theorem 9.1 we derive some simple estimates needed later.

LEMMA 9.2.

(i) *For all* $a,b,c \in \mathbb{R}$ *we have*

$$(a+b+c)^2 \le 2a^2+4b^2+4c^2.$$

(ii) *For all* $a,b,c \in \mathbb{R}$ *and* $\varepsilon > 0$ *we have*

$$(a+b+c)^2 \le (2+\varepsilon)(a^2+b^2) + (1+\frac{2}{\varepsilon})c^2.$$

(iii) *For all* $n \in \mathbb{N}$ *and all* $a_1,\ldots,a_n \in \mathbb{R}$ *we have*

$$\left(\sum_{k=1}^{n} a_k\right)^2 \le \sum_{k=1}^{n} 2^k a_k^2.$$

PROOF. (i): For fixed b and c the function

$$f(x) := 2x^2+4b^2+4c^2-(x+b+c)^2 = x^2-(2b+2c)x+(3b^2+3c^2-2bc)$$

is minimal for $x = b+c$ and $f(b+c) = 2(b-c)^2 \ge 0$.

(ii): We may assume without loss of generality that $a,b,c \ge 0$ and $a+b > 0$ and $c > 0$. Consider for fixed a,b,c the function $f(\varepsilon) = (2+\varepsilon)(a^2+b^2) + (1+\frac{2}{\varepsilon})c^2$. It is minimal for

$$\varepsilon = \frac{c\sqrt{2}}{\sqrt{a^2+b^2}} \quad \text{and} \quad f\left(\frac{c\sqrt{2}}{\sqrt{a^2+b^2}}\right) = (c+\sqrt{2}\sqrt{a^2+b^2})^2 \ge (a+b+c)^2.$$

(iii): By the Cauchy-Schwartz inequality

$$\left(\sum_{k=1}^{n} a_k\right)^2 = \left[\sum_{k=1}^{n} 2^{-\frac{1}{2}k}(2^{\frac{1}{2}k}a_k)\right]^2 \le$$

$$\le \sum_{k=1}^{n} 2^{-k} \cdot \sum_{k=1}^{n} 2^k a_k^2 \le \sum_{k=1}^{n} 2^k a_k^2. \qquad \square$$

PROOF OF THEOREM 9.1. Let $F \subset E$ be an infinite-dimensional subspace. For every $k \in \mathbb{N}$ let us put $F_k := F \cap sp\{e_{n,i}: n \ge k\}$ and let us define a seminorm $\|\cdot\|_k$ on E by

$$(9.4) \qquad \|x\|_k := \sup\left\{\sum_{j=1}^{2^k}\Big(\sum_{(n,i)\in B_j} x(n,i)\Big)^2\right\}^{\frac{1}{2}} \qquad (x \in E),$$

where the sup is taken over all sets of pairwise disjoint k-branches $\{B_1,\ldots,B_{2^k}\}$. For every $k \in \mathbb{N}$, put

$$(9.5) \qquad \lambda_k := \inf\{\|y\|_k: y \in F_k \text{ and } \|y\| = 1\}.$$

Since $F_{k+1} \subset F_k$ $(k = 1,2,\ldots)$ and since it is easily verified that $\|x\|_{k+1} \ge \|x\|_k$ for all $x \in F_{k+1}$, it follows that $\{\lambda_k\}$ is a non-decreasing sequence. Also, as is easily checked, $\lambda_k \le 1$ for all k. Let us put

$\lambda := \lim \lambda_k$. We now show that

(9.6) $\qquad \lambda = 0, \quad$ i.e. $\quad \lambda_k = 0 \quad$ for all $k \in \mathbb{N}$.

For contradiction, suppose $\lambda > 0$. Choose an integer $N > 8$ and an $\varepsilon > 0$ so small that

(9.7) $\qquad 8\lambda^2 + 10N^2\varepsilon < N(\lambda^2-\varepsilon)$.

Since $\{\lambda_k\}$ is non-decreasing, there exists a $K \in \mathbb{N}$ so that

(9.8) $\qquad \lambda_k^2 = \inf\{\|y\|_k^2 : y \in F_k \text{ and } \|y\| = 1\} > \lambda^2-\varepsilon$,

whenever $k \geq K$. Furthermore, the definition of λ_k and λ enables us to choose an increasing sequence of integers $K = m(1) < m(2) < \ldots < m(k) < \ldots$, and a sequence $\{y^k\}_{k=1}^{\infty}$ in F such that, for each k, $\|y^k\| = 1$ and $y^k(n,i) = 0$ whenever $n < m(k)$ or $n \geq m(k+1)$, and

(9.9) $\qquad \|y^k\|_{m(k)}^2 < \lambda^2+\varepsilon$.

Since, by (9.8), for each $k \in \mathbb{N}$ we have $\|y^k\|_K^2 > \lambda^2-\varepsilon$, there exist 2^K K-branches $B_1^k,\ldots,B_{2^K}^k$ with initial points $(K,1),\ldots,(K,2^K)$ respectively, such that

(9.10) $$\sum_{i=1}^{2^K}\Bigl(\sum_{(n,j)\in B_i^k} y^k(n,j)\Bigr)^2 > \lambda^2-\varepsilon.$$

For each $i \in \{1,\ldots,2^K\}$ let $(m(k),p_i^k)$ be the unique branch point of B_i^k of order $m(k)$ and let $B^+(m(k),p_i^k)$ denote the unique $m(k)$-branch contained in B_i^k. Then since the support of y^k consists only of branch points with orders in the interval $[m(k),m(k+1))$, we can write (9.10) as

(9.11) $$\sum_{i=1}^{2^K}\Bigl(\sum_{(n,j)\in B^+(m(k),p_i^k)} y^k(n,j)\Bigr)^2 > \lambda^2-\varepsilon \qquad (k = 1,2,\ldots).$$

Next, for each $k \in \mathbb{N}$ and $i \in \{1,\ldots,2^K\}$ let $B^-(m(k),p_i^k)$ be the unique segment contained in B_i^k with initial point (K,i) and endpoint $(m(k),p_i^k)$. For fixed $i \in \{1,\ldots,2^K\}$ let us consider the sequence $\{B^-(m(k),p_i^k)\}_{k=1}^{\infty}$. There are now two possibilities. In the first place, there may be a subsequence $\{k_n\} \subset \mathbb{N}$ such that $\{B^-(m(k_n),p_i^{k_n})\}_{n=1}^{\infty}$ is totally ordered by inclusion. In the contrary case there exists a subsequence $\{k_n\} \subset \mathbb{N}$ such that no two elements of $\{B^-(m(k_n),p_i^{k_n})\}_{n=1}^{\infty}$ are related by inclusion. Indeed, in case the first alternative does not occur, it suffices to choose a subsequence of maximal (with respect to inclusion) elements of $\{B^-(m(k),p_i^k)\}_{k=1}^{\infty}$. Repeating this procedure of taking subsequences 2^K times (namely, for each i) and suppressing this in the notation, we may therefore assume that for each $i \in \{1,\ldots,2^K\}$ the sequence $\{B^-(m(k),p_i^k)\}_{k=1}^{\infty}$ is either totally ordered by inclusion or that none of its elements is contained in any other.

For the branchpoints $(m(k),p_i^k)$ $(k \in \mathbb{N};\ i = 1,\ldots,2^K)$ we have now achieved the following situation. The set $\{1,\ldots,2^K\}$ is the union of two disjoint subsets I_1 and I_2 such that

$$(9.12)\qquad \begin{cases} \text{for each } i \in I_1 \text{ there exists a (unique) K-branch } B_0(i) \\ \text{containing all } (m(k),p_i^k)\ (k \in \mathbb{N}) \\ \text{and} \\ \text{for each } i \in I_2 \text{ no K-branch contains more than one} \\ (m(k),p_i^k). \end{cases}$$

Furthermore, since for each $i \in I_1$ the sequence $\{\sum_{(n,j)\in B_0(i)} \cdot$ $\cdot\, y^k(n,j)\}_{k=1}^{\infty}$ is bounded, we may assume, by passing to a suitable subsequence if necessary, that for every $i \in I_1$

$$(9.13)\qquad \Big|\sum_{(n,j)\in B_0(i)} y^k(n,j) - \sum_{(n,j)\in B_0(i)} y^{k'}(n,j)\Big| < 2^{-K/2}\varepsilon^{\frac{1}{2}} \quad \text{for all } k,k' \in \mathbb{N}.$$

Next let us observe that also for each $i \in \{1,\ldots,2^K\}$ the sequence $\{\mu_{i,k}\}_{k=1}^{\infty}$ with

$$\mu_{i,k} := \sup\Big[\Big\{\sum_{(n,j)\in B} y^k(n,j)\Big\}^2 : (K,i) \in B,\ B \text{ a K-branch}\Big] \quad (i = 1,\ldots,2^K;\ k = 1,2,\ldots)$$

is bounded. Considering Cauchy subsequences, it follows in particular that there exist integers $1 \le k_1 < k_2 < \ldots < k_N$ (N defined as in (9.7)) such that for each $i \in \{1,\ldots,2^K\}$ we have $\mu_{i,k_\ell} < \mu_{i,k_1} + 2^{-K}\varepsilon$ $(\ell = 1,\ldots,N)$, i.e.

$$(9.14)\qquad \begin{cases} \sup\Big[\Big\{\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big\}^2 : (K,i) \in B,\ B \text{ a K-branch},\ \ell = 1,\ldots,N\Big] \\ < \sup\Big[\Big\{\sum_{(n,j)\in B} y^{k_1}(n,j)\Big\}^2 : (K,i) \in B,\ B \text{ a K-branch}\Big] + 2^{-K}\varepsilon\ . \end{cases}$$

Let B now be an arbitrary fixed K-branch and suppose that $(K,i) \in B$.

Let us consider

$$(9.15)\qquad \Big[\sum_{(n,j)\in B}\Big\{\sum_{\ell=1}^{N}(-1)^{\ell}\, y^{k_\ell}(n,j)\Big\}\Big]^2 = \Big[\sum_{\ell=1}^{N}(-1)^{\ell}\Big\{\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big\}\Big]^2 .$$

For each $\ell \in \{1,\dots,N\}$ we put

$$\left.\begin{aligned}\rho_B^{k_\ell} &:= \Big|\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big|\\ \sigma_B^{k_\ell} &:= 0\end{aligned}\right\}\quad \text{if } (m(k_\ell),p_i^{k_\ell}) \in B$$

$$\left.\begin{aligned}\rho_B^{k_\ell} &:= 0\\ \sigma_B^{k_\ell} &:= \Big|\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big|\end{aligned}\right\}\quad \text{if } (m(k_\ell),p_i^{k_\ell}) \notin B.$$

Then by the alternative (9.12) we have the following.

(i) If $i \in I_1$, then there exists an $M = M(B) \in \{0,1,\dots,N\}$ such that $(m(k_\ell),p_i^{k_\ell}) \in B$ for $1 \le \ell \le M$ and $(m(k_\ell),p_i^{k_\ell}) \notin B$ for $M < \ell \le N$, i.e. M is the largest number (if $M \neq 0$) such that B coincides with $B_0(i)$ up to $(m(k_M),p_i^{k_M})$. Hence, by (9.13), we have in this case

$$(9.16)\qquad \Big|\sum_{\ell=1}^{N}(-1)^{\ell}\Big\{\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big\}\Big| \le \big([\tfrac{M}{2}]2^{-K/2}\varepsilon^{\frac12}\big) + \big(\rho_B^{k_{M-1}}+\rho_B^{k_M}\big) + \Big(\sum_{\ell=1}^{N}\sigma_B^{k_\ell}\Big),$$

with the convention $\rho_B^{k_{-1}} = \rho_B^{k_0} = 0$.

(ii) If $i \in I_2$, then either there exists exactly one $M = M(B) \in \{1,\dots,N\}$ such that $(m(k_M),p_i^{k_M}) \in B$, or there exists no such M, in which case we put $M = M(B) = 0$. Now we have

$$\Big|\sum_{\ell=1}^{N}(-1)^{\ell}\Big\{\sum_{(n,j)\in B} y^{k_\ell}(n,j)\Big\}\Big| \le \rho_B^{k_M} + \Big(\sum_{\ell=1}^{N}\sigma_B^{k_\ell}\Big),$$

again with the convention $\rho_B^{k_0} = 0$.

Combining (i) and (ii) we see that (9.16) holds in both cases, if we take $M = M(B)$ to be the largest integer $\le N$ for which $(m(k_M),p_i^{k_M}) \in B$, if any, and $M = M(B) = 0$ otherwise. Using now Lemma 9.2(i), (9.16) yields that the expression (9.15) is bounded above by

$$(9.17)\qquad 2[\rho_B^{k_{M-1}}+\rho_B^{k_M}]^2 + 4[[\tfrac{M}{2}]2^{-K/2}\varepsilon^{1/2}]^2 + 4\Big[\sum_{\ell=1}^{N} \sigma_B^{k_\ell}\Big]^2 .$$

To find an upper estimate for $\|\sum_{\ell=1}^N (-1)^\ell y^{k_\ell}\|_K^2$ we must sum the expressions (9.15) or (9.17) over any 2^K disjoint K-branches. By (9.14) and (9.9), the sum of $[\rho_B^{k_{M(B)-1}}+\rho_B^{k_{M(B)}}]^2$ over any 2^K pairwise disjoint K-branches is at most $4\|y^{k_1}\|_K^2 + 4\varepsilon < 4\lambda^2 + 8\varepsilon$. Also, by (9.9) and (9.11), for a fixed ℓ the sum of $(\sigma_B^{k_\ell})^2$ over 2^K pairwise disjoint K-branches is less than 2ε. Putting these observations together and using the triangle inequality in ℓ^2, the assumption $N > 8$, and (9.7), we find for any 2^K pairwise disjoint K-branches $B(1),\dots,B(2^K)$:

$$\sum_{k=1}^{2^K}\Big[\sum_{(n,j)\in B(k)}\Big\{\sum_{\ell=1}^{N}(-1)^\ell\, y^{k_\ell}(n,j)\Big\}\Big]^2 \le 8\lambda^2 + 16\varepsilon + N^2\varepsilon +$$

$$+ 4\Big[\sum_{\ell=1}^{N}\Big\{\sum_{k=1}^{2^K}(\sigma_{B(k)}^{k_\ell})^2\Big\}^{1/2}\Big]^2 \le 8\lambda^2 + 16\varepsilon + N^2\varepsilon + 4(N\sqrt{2\varepsilon})^2 \le$$

$$\le 8\lambda^2 + 10N^2\varepsilon < N(\lambda^2-\varepsilon).$$

Hence $\|\sum_{\ell=1}^N (-1)^\ell y^{k_\ell}\|_K^2 < N(\lambda^2-\varepsilon)$. On the other hand we clearly have, since $\|y^{k_\ell}\| = 1$ for $\ell = 1,\dots,N$, that $\|\sum_{\ell=1}^N (-1)^\ell y^{k_\ell}\|^2 \ge N$. Thus

$$\left\|\frac{\sum_{\ell=1}^N (-1)^\ell y^{k_\ell}}{\|\sum_{\ell=1}^N (-1)^\ell y^{k_\ell}\|}\right\|_K^2 < \lambda^2 - \varepsilon,$$

which contradicts (9.8). This completes the proof of (9.6).

We are now ready to describe G and to finish the proof. Let $\varepsilon > 0$ be arbitrary. Since $\lambda = 0$, i.e. $\lambda_k = 0$ for all $k \in \mathbb{N}$, we can choose an increasing sequence $\{n(k)\}_{k=1}^\infty$ of integers and a sequence $\{y^k\} \subset F$ such that $\|y^k\| = 1$, $y^k(n,j) = 0$ whenever $n < n(k)$ or $n \ge n(k+1)$ and

$$(9.18)\qquad \|y^k\|_{n(k)}^2 < 2^{-k}\varepsilon^2 \qquad (k = 1,2,\dots).$$

We claim that $G := \mathrm{sp}\{y^k\}$ satisfies the requirement in Theorem 9.1. Indeed, let $p \in \mathbb{N}$ and $a_1,\dots,a_p \in \mathbb{R}$ be given arbitrarily. Then clearly, since $\|y^k\| = 1$ for all $k \in \mathbb{N}$, $\|\sum_{k=1}^p a_k y^k\|^2 \ge \sum_{k=1}^p a_k^2$. We now establish an upper estimate for $\|\sum_{k=1}^p a_k y^k\|^2$. For simplicity we put $\sum_{k=1}^p a_k y^k =: x$. Since x finite support, there exist pairwise disjoint segments $S_1,\dots,S_m$ such that

$$\|x\|^2 = \sum_{i=1}^{m} \Big(\sum_{(n,j)\in S_i} x(n,j) \Big)^2 .$$

We may assume that all points of S_i $(i = 1,\dots,m)$ have orders in the interval $[n(1),n(p+1))$. For each $i \in \{1,\dots,m\}$ and $\ell \in \{1,\dots,p\}$ let $S_{i,\ell}$ be the intersection of S_i with the set of branchpoints with orders in $[n(\ell),n(\ell+1))$. Now let $i \in \{1,\dots,m\}$ be fixed. We wish to consider $[\sum_{(n,j)\in S_i} x(n,j)]^2$. We first determine $\ell_1(i),\ell_2(i) \in \{1,\dots,p\}$ so that $S_{i,\ell_1(i)}$ and $S_{i,\ell_2(i)}$ are the first and the last non-empty subsegment among the $S_{i,1},\dots,S_{i,p}$, respectively. (If just one of $S_{i,1},\dots,S_{i,p}$ is non-empty, call this subsegment $S_{i,\ell_1(i)}$ and put $S_{i,\ell_2(i)} := \phi$.) Next we split the sum $\sum_{(n,j)\in S_i} x(n,j)$ into three parts:

$$\sum_{(n,j)\in S_i} x(n,j) = \sum_{(n,j)\in S_{i,\ell_1(i)}} x(n,j) + \sum_{(n,j)\in S_{i,\ell_2(i)}} x(n,j) + \\ + \sum_{\substack{\ell=1\\ \ell\neq\ell_1(i),\ell_2(i)}}^{p} \sum_{(n,j)\in S_{i,\ell}} x(n,j).$$

Applying Lemma 9.2 (ii) and (iii) successively, we get

$$\Big[\sum_{(n,j)\in S_i} x(n,j)\Big]^2 \le (2+\varepsilon)\Big[\Big\{\sum_{(n,j)\in S_{i,\ell_1(i)}} x(n,j)\Big\}^2 + \\ + \Big\{\sum_{(n,j)\in S_{i,\ell_2(i)}} x(n,j)\Big\}^2\Big] + (1+\tfrac{2}{\varepsilon}) \sum_{\substack{\ell=1\\ \ell\neq\ell_1(i),\ell_2(i)}}^{p} \cdot \\ \cdot 2^{\ell}\Big\{\sum_{(n,j)\in S_{i,\ell}} x(n,j)\Big\}^2 =$$

$$= (2+\varepsilon)a^2_{\ell_1(i)}\Big\{\sum_{(n,j)\in S_{i,\ell_1(i)}} y^{\ell_1(i)}(n,j)\Big\}^2 + \\ + (2+\varepsilon)a^2_{\ell_2(i)}\Big\{\sum_{(n,j)\in S_{i,\ell_2(i)}} y^{\ell_2(i)}(n,j)\Big\}^2 + \\ + (1+\tfrac{2}{\varepsilon}) \sum_{\substack{\ell=1\\ \ell\neq\ell_1(i),\ell_2(i)}}^{p} 2^{\ell} a^2_{\ell}\Big\{\sum_{(n,j)\in S_{i,\ell}} y^{\ell}(n,j)\Big\}^2 .$$

Summing this last expression over all $i = 1,\dots,m$, performing a change of summation, recalling that $\|y^k\| = 1$ $(k = 1,2,\dots)$, and using (9.18), we obtain

$$\|x\|^2 = \|\sum_{k=1}^{p} a_k y^k\|^2 =$$

$$(2+\varepsilon)\sum_{k=1}^{p} a_k^2\Big[\sum_{\substack{i=1\\ \ell_1(i)=k}}^{m}\Big\{\sum_{(n,j)\in S_{i,k}} y^k(n,j)\Big\}^2 + \sum_{\substack{i=1\\ \ell_2(i)=k}}^{m}\Big\{\sum_{(n,j)\in S_{i,k}} y^k(n,j)\Big\}^2\Big] +$$

$$+ (1+\tfrac{2}{\varepsilon})\sum_{k=1}^{p} 2^k a_k^2 \sum_{\substack{i=1\\ \ell_1(i),\ell_2(i)\neq k}}^{m}\Big\{\sum_{(n,j)\in S_{i,k}} y^k(n,j)\Big\}^2 \le$$

$$\le (2+\varepsilon)\sum_{k=1}^{p} a_k^2\|y^k\|^2 + (1+\tfrac{2}{\varepsilon})\sum_{k=1}^{p} 2^k a_k^2\|y^k\|^2_{n(k)}$$

$$\le (2+\varepsilon)\sum_{k=1}^{p} a_k^2 + (\varepsilon^2+2\varepsilon)\sum_{k=1}^{p} a_k^2 = (2+3\varepsilon+\varepsilon^2)\sum_{k=1}^{p} a_k^2.$$

If we choose $\varepsilon > 0$ so small that $2+3\varepsilon+\varepsilon^2 < \theta^2$, then this proves that

$$\Big(\sum_{k=1}^{p} a_k^2\Big)^{\frac12} \le \|\sum_{k=1}^{p} a_k y^k\| \le \theta\Big(\sum_{k=1}^{p} a_k^2\Big)^{\frac12},$$

for all $p \in \mathbb{N}$ and all $a_1,\dots,a_p \in \mathbb{R}$, i.e. (9.3). □

<u>REMARK 9.3</u>. Let B be any branch of T and let X_B be the closed subspace of X spanned by all $e_{n,j}$ with $(n,j) \in B$. Let the projection P_B be defined on X by

$$P_B\, x = \sum_{(n,j)\in B} x(n,j)e_{n,j} \qquad (x \in X).$$

Then it is immediate that P_B has range X_B and norm 1. Also, for any branch B the subspace X_B is isometric to the classical James space equipped with the norm $\|\cdot\|_2$ of formula (6.46). Thus the James tree space contains many complemented copies of the classical James space and it follows in particular from Theorem 9.1 that every closed infinite-dimensional subspace of J contains a subspace isomorphic to ℓ^2, a result we have mentioned earlier (immediately preceding Remark 6.23).

NOTES. The contents of this section are from [56]. For a detailed study of the James tree space and the continuous analogue of J, the James function space, we refer to [74]. In this paper also the relevance of these spaces to questions other than reflexivity is discussed. The fact mentioned in Remark 9.3, that every closed infinite-dimensional subspace of J contains ℓ^2 isomorphically, was first proved in [45]. As we observed earlier, the James tree space is also a counterexample to the conjecture that every separable space with non-separable dual must contain ℓ^1 isomorphically. Another such counterexample with curious properties was recently constructed by J. HAGLER ([39]). For yet another (flat) space of this kind, see the Notes to Section 18.

10. SOME GEOMETRIC PROPERTIES EQUIVALENT TO NON-REFLEXIVITY

Several properties of Banach spaces are introduced in this section which will turn out to be equivalent to non-reflexivity. We number them $P_2^\infty, P_3^\infty, P_4^\infty$. P_1^∞ will be defined later. The superindex ∞ serves to distinguish these properties from their finite-dimensional versions, to be discussed at length in the next chapter. All of $P_2^\infty, P_3^\infty, P_4^\infty$ are of a geometric nature and require, in some form or other, the existence of infinite-dimensional flat areas in the unit ball away from the origin. Their equivalence with non-reflexivity suggests the idea that a space is reflexive iff its unit ball possesses a certain degree of infinite-dimensional rotundity.

<u>DEFINITION 10.1</u>. Let X be a Banach space.

X has P_2^∞ iff $\exists \varepsilon > 0\ \exists \{x_n\} \subset B_X$: $\forall k \in \mathbb{N}$ $\operatorname{dist}(\operatorname{co}\{x_1,\ldots,x_k\}, \operatorname{co}\{x_{k+1},\ldots\}) \geq \varepsilon$.

X has P_3^∞ iff $\exists \varepsilon > 0\ \exists \{x_n\} \subset B_X$: $[\operatorname{dist}(\operatorname{co}\{x_1,x_2,\ldots\},\{0\}) \geq \varepsilon$ and $\forall k,n \in \mathbb{N}\ \forall \alpha_1,\ldots,\alpha_{k+n} \in \mathbb{R}: \|\sum_{i=1}^{k+n} \alpha_i x_i\| \geq \frac{\varepsilon}{2}\|\sum_{i=1}^{k} \alpha_i x_i\|]$.

X has P_4^∞ iff $\exists \varepsilon > 0\ \exists \{x_n\} \subset B_X\ \exists \{x_n^*\} \subset B_{X^*}$: $\forall k,i \in \mathbb{N}$: $[k \leq i \Rightarrow \langle x_i, x_k^*\rangle \geq \varepsilon,\ k > i \Rightarrow \langle x_i, x_k^*\rangle = 0]$.

<u>REMARK 10.2</u>. For obvious reasons P_2^∞ is known as the infinite flatness property, P_3^∞ as the infinite basic sequence property (note that the second requirement in P_3^∞ means that $\{x_n\}$ is a basic sequence with norm $\leq \frac{2}{\varepsilon}$), and P_4^∞ as the infinite triangular matrix property. Assuming for the moment that we have proved P_3^∞ to be equivalent to non-reflexivity, it is clear that every non-reflexive Banach space has a non-reflexive subspace with a basis. Indeed, if $\{x_n\} \subset X$ is as postulated in P_3^∞, then the subspace $[x_n] \subset X$ obviously also satisfies P_3^∞ and has a basis. This result has been used earlier to prove Theorem 6.12.

<u>EXAMPLES</u>. (a) ℓ^1 satisfies P_2^∞, P_3^∞ and P_4^∞. Indeed, if $\{e_n\}$ denotes the standard basis for ℓ^1, then P_2^∞ holds with $x_n = e_n$ $(n = 1,2,\ldots)$ and $\varepsilon = 2$ and P_3^∞ holds with $x_n = e_n$ $(n = 1,2,\ldots)$ and $\varepsilon = 1$. For P_4^∞, take $x_n = e_n$ and

$x_n^* = (\underbrace{0,\dots,0}_{n-1},1,1,1,\dots) \in \ell^\infty \cong (\ell^1)^*$ $(n = 1,2,\dots)$. Then P_4^∞ holds with $\varepsilon = 1$.

(b) c_0 also satisfies P_2^∞, P_3^∞ and P_4^∞ with $\varepsilon = 1$ and $x_n = \sum_{i=1}^n e_i$ $(n=1,2,\dots)$. (Observe that $\{x_n\}$ is a basis for c_0 by Proposition 5.9.) For the proof of P_2^∞, let $k \in \mathbb{N}$ be arbitrary and let $x = \{\xi_n\} \in \mathrm{co}\{x_1,\dots,x_k\}$ and $y = \{\eta_n\} \in \mathrm{co}\{x_{k+1},x_{k+2},\dots\}$ be given. Then $\xi_{k+1} = 0$, $\eta_{k+1} = 1$ and therefore $\|x-y\| \geq |\xi_{k+1}-\eta_{k+1}| = 1$. Hence $\mathrm{dist}(\mathrm{co}\{x_1,\dots,x_k\},\mathrm{co}\{x_{k+1},\dots\}) \geq 1$. For the proof of P_3^∞, observe that for all $x = \{\xi_n\} \in \mathrm{co}\{x_1,x_2,\dots\}$ we have $\xi_1 = 1$ so that $\mathrm{dist}(\mathrm{co}\{x_1,x_2,\dots\},\{0\}) \geq 1$. Also the basic sequence $\{x_n\}$ has norm ≤ 2 which means that the second part of P_3^∞ is satisfied. Indeed, for all $k,n \in \mathbb{N}$ and all $\alpha_1,\dots,\alpha_{k+n} \in \mathbb{R}$ we have

$$\Big\|\sum_{i=1}^k \alpha_i x_i\Big\| = \sup_{\ell=1,\dots,n} \Big|\sum_{i=\ell}^k \alpha_i\Big| \leq$$

$$\leq \sup_{\ell=1,\dots,k+n} \Big|\sum_{i=\ell}^{k+n} \alpha_i\Big| + \Big|\sum_{i=k+1}^{k+n} \alpha_i\Big| \leq$$

$$\leq 2 \sup_{\ell=1,\dots,k+n} \Big|\sum_{i=\ell}^{k+n} \alpha_i\Big| = 2\Big\|\sum_{i=1}^{k+n} \alpha_i x_i\Big\|.$$

Finally, P_4^∞ is satisfied by $\{x_n\}$ and $\{x_n^*\}$ with $\varepsilon = 1$ if we take $x_n^* = (\underbrace{0,\dots,0}_{n-1},1,0,\dots) \in \ell^1 \cong (c_0)^*$.

We now show that the spaces satisfying any one (equivalently, all) of $P_2^\infty, P_3^\infty, P_4^\infty$ are precisely the non-reflexive ones.

THEOREM 10.3. *Let X be a Banach space. Then the following are equivalent:*

(i) X *is non-reflexive;*

(ii) X *satisfies* P_2^∞;

(iii) X *satisfies* P_3^∞;

(iv) X *satisfies* P_4^∞.

PROOF. We prove the cycles (i) ⇒ (iv) ⇒ (ii) ⇒ (i) and (i) ⟺ (iii).

(ii) ⇒ (i): For contradiction assume that X is reflexive and satisfies P_2^∞. Let $\{x_n\} \subset B_X$ and $\varepsilon > 0$ be as postulated in P_2^∞. By the weak (countable) compactness of B_X the sequence $\{x_n\}$ has a weak limit point x. In particular x belongs to the weak closure of $\mathrm{co}\{x_1,x_2,\dots\}$. Therefore, by Proposition 0.5, there exists a $k \in \mathbb{N}$ and a convex combination $x' = \sum_{i=1}^k \lambda_i x_i$ $(\lambda_i \geq 0$, $\sum_{i=1}^k \lambda_i = 1)$ such that $\|x-x'\| < \frac{\varepsilon}{2}$. Similarly $x \in \overline{\mathrm{co}}\{x_{k+1},x_{k+2},\dots\}$, so there exists an $\ell \in \mathbb{N}$ and a convex combination $x'' = \sum_{i=k+1}^{k+\ell} \lambda_i x_i$ with $\|x-x''\| < \frac{\varepsilon}{2}$. Thus $\|x'-x''\| < \varepsilon$, contradicting $\mathrm{dist}(\mathrm{co}\{x_1,\dots,x_k\},\mathrm{co}\{x_{k+1},\dots\}) \geq \varepsilon$.

(iii) $\Rightarrow$ (i): Again assume that X is reflexive and that it satisfies P_3^∞, for some sequence $\{x_n\} \subset B_X$ and some $\varepsilon > 0$. As before, $\{x_n\}$ has a weak limit point, say x, which belongs to $\overline{\mathrm{co}}\{x_1,x_2,\dots\}$. It is impossible that $x = 0$, since this would imply $\mathrm{dist}(\mathrm{co}\{x_1,x_2,\dots\},\{0\}) = 0$. Thus $\|x\| =: \delta > 0$. As in the previous case, for every $\eta > 0$ there exist $k \in \mathbb{N}$ and elements $x' \in \mathrm{co}\{x_1,\dots,x_k\}$, $x'' \in \mathrm{co}\{x_{k+1},\dots\}$ such that $\|x-x'\| < \eta\delta$ and $\|x-x''\| < \eta\delta$. Thus $\|x'-x''\| < 2\eta\delta$. Writing

$$(10.1) \qquad x' = \sum_{i=1}^{k} \lambda_i x_i, \quad x'' = \sum_{i=k+1}^{k+n} \lambda_i x_i \quad (\lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = \sum_{i=k+1}^{k+n} \lambda_i = 1),$$

it follows from the second part of P_3^∞ that

$$(10.2) \qquad \|x'-x''\| = \|\sum_{i=1}^{k} \lambda_i x_i - \sum_{i=k+1}^{k+n} \lambda_i x_i\| \geq \frac{\varepsilon}{2}\|\sum_{i=1}^{k} \lambda_i x_i\| =$$

$$= \frac{\varepsilon}{2}\|x'\| \geq \frac{\varepsilon}{2}(\|x\|-\|x-x'\|) > \frac{\varepsilon}{2}(\delta-\eta\delta).$$

(10.1) and (10.2) are contradictory for sufficiently small $\eta > 0$.

(i) $\Rightarrow$ (iv): Let $0 < \varepsilon < 1$ be arbitrary. Then, assuming that X is non-reflexive, there exists an $x^{**} \in X^{**}\setminus\pi X$ with $\|x^{**}\| < 1$ and $0 < \varepsilon < \mathrm{dist}(\{x^{**}\},\pi X)$. Indeed, let $x^{***} \in X^{***}$ be such that $x^{***}|_{\pi X} = 0$ and $\|x^{***}\| = 1$. Then any $x^{**} \in X^{**}$ such that $\|x^{**}\| < 1$ and $\langle x^{**},x^{***}\rangle > \varepsilon$ will do, since $\|x^{**}-\pi(x)\| \geq |\langle x^{**}-\pi(x),x^{***}\rangle| = \langle x^{**},x^{***}\rangle > \varepsilon$ for all $x \in X$. In particular $\|x^{**}\| > \varepsilon$, so there exists an $x_1^* \in B_{X^*}$ with $\langle x_1^*,x^{**}\rangle > \varepsilon$. Applying Theorem 3.1 with $G := \mathrm{sp}\{x^{**}\}$ and $F := \mathrm{sp}\{x_1^*\}$, we find an $x_1 \in X$ with $\|x_1\| < 1$ such that $\langle x_1,x_1^*\rangle = \langle x_1^*,x^{**}\rangle > \varepsilon$. Having now constructed x_1 and x_1^*, we proceed with induction. Suppose that, for some fixed $n \geq 1$ we have found $x_1,\dots,x_n \in B_X$ and $x_1^*,\dots,x_n^* \in B_{X^*}$ satisfying

$$(10.3) \qquad \begin{aligned} &\langle x_i,x_k^*\rangle > \varepsilon && \text{for all } 1 \leq k \leq i \leq n \\ &\langle x_i,x_k^*\rangle = 0 && \text{for all } 1 \leq i < k \leq n \\ &\langle x_k^*,x^{**}\rangle > \varepsilon && \text{for all } 1 \leq k \leq n. \end{aligned}$$

We now select an $x^{***} \in X^{***}$ with $\|x^{***}\| < 1$ such that $x^{***}|_{\pi X} = 0$ (so in particular $\langle\pi(x_i),x^{***}\rangle = 0$ for $i = 1,\dots,n$) and $\langle x^{**},x^{***}\rangle > \varepsilon$. This is possible by the Hahn-Banach theorem, since $\operatorname{dist}(\{x^{**}\},\pi X) > \varepsilon$. Now Theorem 3.1 applied to X^* with $G := sp\{x^{***}\}$ and $F := sp\{x^{**},\pi(x_1),\dots,\pi(x_n)\} \subset X^{**}$ yields an $x^*_{n+1} \in X^*$ with $\|x^*_{n+1}\| < 1$, $\langle x_i,x^*_{n+1}\rangle = \langle\pi(x_i),x^{***}\rangle = 0$ for $i = 1,\dots,n$, and $\langle x^*_{n+1},x^{**}\rangle = \langle x^{**},x^{***}\rangle > \varepsilon$. Finally, one more application of local reflexivity, this time to X with $G := sp\{x^{**}\}$ and $F := sp\{x^*_1,\dots,x^*_{n+1}\}$ yields an $x_{n+1} \in X$ with $\|x_{n+1}\| < 1$ and $\langle x_{n+1},x^*_k\rangle = \langle x^*_k,x^{**}\rangle > \varepsilon$ for $k = 1,\dots,n+1$. The system $x_1,\dots,x_{n+1} \in B_X$, $x^*_1,\dots,x^*_{n+1} \in B_{X^*}$ now satisfies (10.3) with n replaced by n+1, so this completes the inductive definition of the sequences $\{x_n\} \subset B_X$, $\{x^*_n\} \subset B_{X^*}$ satisfying P^∞_4.

(iv) ⇒ (ii): Let $\varepsilon > 0$, $\{x_n\} \subset B_X$ and $\{x^*_n\} \subset B_{X^*}$ be as postulated in P^∞_4. Let $k \in \mathbb{N}$ be arbitrary. Then for every choice of $n \in \mathbb{N}$ and $\lambda_1,\dots,\lambda_{k+n} \geq 0$ with $\sum_{i=1}^k \lambda_i = \sum_{i=k+1}^{k+n} \lambda_i = 1$ we have

$$\Big\|\sum_{i=1}^{k}\lambda_i x_i - \sum_{i=k+1}^{k+n}\lambda_i x_i\Big\| \geq \Big|\Big\langle \sum_{i=1}^{k}\lambda_i x_i - \sum_{i=k+1}^{k+n}\lambda_i x_i, x^*_{k+1}\Big\rangle\Big| =$$

$$= \Big|\sum_{i=k+1}^{k+n}\lambda_i\langle x_i,x^*_{k+1}\rangle\Big| \geq \varepsilon \sum_{i=k+1}^{k+n}\lambda_i = \varepsilon.$$

Thus $\operatorname{dist}(co\{x_1,\dots,x_k\},co\{x_{k+1},\dots\}) \geq \varepsilon$ for all $k \in \mathbb{N}$, i.e. P^∞_2 holds.

(i) ⇒ (iii): Let $\frac{2}{3} < \varepsilon_0 < 1$ be arbitrary. We have seen already in the proof of (i) ⇒ (iv) that there exists an $x^{**}_0 \in X^{**}\setminus\pi X$ such that $\|x^{**}_0\| < 1$ and $\operatorname{dist}(\{x^{**}_0\},\pi X) > \varepsilon_0$. In particular $\|x^{**}_0\| > \varepsilon_0$, so there exists an $x^*_0 \in X^*$ with $\|x^*_0\| < 1$ and $\langle x^*_0,x^{**}_0\rangle > \varepsilon_0$. We now define inductively a sequence $\{x_n\} \subset B_X$ and an increasing sequence $\{H_n\}$ of finite subsets of B_{X^*} such that the following conditions are satisfied:

(a) $\langle x_i,x^*\rangle = 0$ for all $k,i \in \mathbb{N}$ with $k < i$ and for all $x^* \in H_k$;

(b) $\sup_{x^*\in H_k}|\langle x,x^*\rangle| \geq (\frac{3}{2}\varepsilon_0-1)\ \|x\|$ for all $k \in \mathbb{N}$ and all $x \in sp\{x_1,\dots,x_k\}$ i.e. $[H_k]$ is a $(\frac{3}{2}\varepsilon_0-1)$-norming subspace for $sp\{x_1,\dots,x_k\}$;

(c) $\langle x_i,x^*_0\rangle > \varepsilon_0$ for all $i \in \mathbb{N}$;

(d) $\langle x^*,x^{**}_0\rangle = 0$ for all $x^* \in \bigcup_{k=1}^\infty H_k$.

For convenience we begin with the proof of the induction step, as it will become clear that the choice of x_1 and H_1 is included in it. So suppose that for some $k \geq 2$ elements $x_1,\dots,x_{k-1} \in B_X$ and finite subsets $H_1 \subset H_2 \subset \dots \subset H_{k-1}$ of B_{X^*} have been defined which satisfy (a)...(d). Since $\|x^{**}_0\| < 1$, $\langle x^*_0,x^{**}_0\rangle > \varepsilon_0$ and $\langle x^*,x^{**}_0\rangle = 0$ for all $x^* \in H_{k-1}$, Theorem 3.1 applied with $G := sp\{x^{**}_0\}$ and $F := sp\{H_{k-1} \cup \{x^*_0\}\}$ yields an $x_k \in X$

such that $\|x_k\| < 1$, $\langle x_k, x_0^*\rangle = \langle x_0^*, x_0^{**}\rangle > \varepsilon_0$ and $\langle x_k, x^*\rangle = \langle x^*, x_0^{**}\rangle = 0$ for all $x^* \in H_{k-1}$.

Defining H_k is a bit more complicated. By compactness we can select a $(1-\varepsilon_0)$-net $\{y_1,\dots,y_m\}$ for $S_{sp\{x_1,\dots,x_k\}}$. We now define for each fixed $\ell \in \{1,\dots,m\}$ a linear form ϕ_ℓ on $sp\{x_0^{**}, \pi y_\ell\}$ by

$$\phi_\ell(\alpha x_0^{**} + \beta\pi y_\ell) = \beta \quad (\alpha,\beta \in \mathbb{R}).$$

We claim that $\|\phi_\ell\| < \gamma_0$, where $\gamma_0 := \frac{2}{\varepsilon_0}$. Indeed, for any $\alpha,\beta \in \mathbb{R}$ we have

$$|\phi_\ell(\alpha x_0^{**} + \beta\pi y_\ell)| = |\beta| = \|\beta\pi y_\ell\| \le \|\alpha x_0^{**} + \beta\pi y_\ell\| + \|\alpha x_0^{**}\|.$$

Assuming $\alpha \neq 0$, as we may, we have

$$\|\alpha x_0^{**}\| < |\alpha| \le \frac{|\alpha|\operatorname{dist}(\{x_0^{**}\},\pi X)}{\varepsilon_0} \le \frac{|\alpha|}{\varepsilon_0}\,\|x_0^{**} + \frac{\beta}{\alpha}\pi y_\ell\| = \frac{\|\alpha x_0^{**} + \beta\pi y_\ell\|}{\varepsilon_0}.$$

So, putting these inequalities together, we find

$$|\phi_\ell(\alpha x_0^{**} + \beta\pi y_\ell)| \le \left(1 + \frac{1}{\varepsilon_0}\right)\|\alpha x_0^{**} + \beta\pi y_\ell\|,$$

i.e.

$$\|\phi_\ell\| \le 1 + \frac{1}{\varepsilon_0} < \gamma_0.$$

Each ϕ_ℓ $(\ell = 1,\dots,m)$ can be extended, by the Hahn-Banach theorem, to an element $x_\ell^{***} \in X^{***}$, with

$$\|x_\ell^{***}\| < \gamma_0, \quad \langle x_0^{**}, x_\ell^{***}\rangle = 0, \quad \langle \pi y_\ell, x_\ell^{***}\rangle = 1 \quad (\ell = 1,\dots,m).$$

By local reflexivity of X^*, applied with

$$G := sp\{x_1^{***},\dots,x_m^{***}\} \quad \text{and} \quad F := sp\{x_0^{**}, \pi y_1,\dots,\pi y_m\}$$

we obtain $y_1^*,\dots,y_m^* \in X^*$ with

$$\|y_\ell^*\| < \gamma_0, \quad \langle y_\ell^*, x_0^{**}\rangle = \langle x_0^{**}, x_\ell^{***}\rangle = 0, \; \langle y_\ell, y_\ell^*\rangle = \langle \pi(y_\ell), x_\ell^{***}\rangle = 1$$

$$(\ell = 1,\dots,m).$$

Now define

$$H_k := H_{k-1} \cup \{\frac{1}{\gamma_0} y_1^*,\dots,\frac{1}{\gamma_0} y_m^*\}.$$

It is evident that the so constructed x_k and H_k satisfy (a), (c) and (d).

We show that (b) holds. Let $x \in sp\{x_1,\dots,x_k\}$ with $\|x\| = 1$ be arbitrary. Then for some $\ell \in \{1,\dots,m\}$ we have $\|x-y_\ell\| < \varepsilon_0$, so

$$|<x,\frac{1}{\gamma_0} y_\ell^*>| \ge |<y_\ell,\frac{1}{\gamma_0} y_\ell^*>| - |<x-y_\ell,\frac{1}{\gamma_0} y_\ell^*>| \ge$$

$$\ge \frac{1}{\gamma_0} - \|\frac{1}{\gamma_0} y_\ell^*\|\|x-y_\ell\| \ge \frac{1}{\gamma_0} - (1-\varepsilon_0) = \frac{3}{2}\varepsilon_0 - 1.$$

This proves (b).

The reader should observe now that the definition of x_1 and H_1 (in this order) is included in the above, if one reads $H_0 = \phi$. Thus the inductive definition of the sequences $\{x_n\}$ and $\{H_n\}$ satisfying (a)...(d) is completed. It remains to be proved that P_3^∞ is satisfied with this $\{x_n\}$ and for some $\varepsilon > 0$. In the first place (c) immediately implies that $\text{dist}(\text{co}\{x_1,x_2,\dots\},\{0\}) \ge \varepsilon_0$, since for any convex combination $\sum_{i=1}^k \lambda_i x_i$ we have

$$\|\sum_{i=1}^k \lambda_i x_i\| \ge <\sum_{i=1}^k \lambda_i x_i, x_0^*> \ge \varepsilon_0 \sum_{i=1}^k \lambda_i = \varepsilon_0.$$

Finally, let $k,n \in \mathbb{N}$ and $\alpha_1,\dots,\alpha_{k+n} \in \mathbb{R}$ be arbitrary. Then, by (a) and (b),

$$(\tfrac{3}{2}\varepsilon_0 - 1)\|\sum_{i=1}^k \alpha_i x_i\| \le \sup_{x^* \in H_k} |<\sum_{i=1}^k \alpha_i x_i, x^*>| = \sup_{x^* \in H_k} |<\sum_{i=1}^{k+n} \alpha_i x_i, x^*>|$$

$$\le \sup_{x^* \in H_k} \|x^*\|\|\sum_{i=1}^{k+n} \alpha_i x_i\| \le \|\sum_{i=1}^{k+n} \alpha_i x_i\|.$$

Thus P_3^∞ holds with any ε satisfying $0 < \varepsilon \le \min(\varepsilon_0, 3\varepsilon_0 - 2) = 3\varepsilon_0 - 2$.

REMARK 10.4. It is clear from the proofs of (i) ⇒ (iv) ⇒ (ii) and (i) ⇒ (iii) that in a non-reflexive space the properties P_2^∞, P_3^∞ and P_4^∞ are satisfied with any choice of $0 < \varepsilon < 1$. Thus, replacing in Definition 10.1 "$\exists \varepsilon > 0$" by "$\forall 0 < \varepsilon < 1$" wherever it occurs, leads to equivalent properties.

* * *

Recall that by the subreflexivity theorem of E. Bishop and R.R. Phelps, for any Banach space X the subset of X^* consisting of all those elements which attain their sup on B_X is dense in X^*. Obviously for reflexive X this set equals X^*, by the weak compactness of B_X. We now show that the converse is true: if every $x^* \in X^*$ attains its sup on B_X, then X is reflexive. We give the proof only for the separable case. Essentially the same method of proof also works for non-separable spaces.

We begin with the following technical

LEMMA 10.5. *Let X be a Banach space. Let a number ε with $0 < \varepsilon < 1$, a sequence $\{x_n^*\} \subset B_{X^*}$ and a sequence $\{\lambda_n\}$ of positive numbers be given such that*

$$(10.4) \qquad \mathrm{dist}(\mathrm{co}\{x_1^*,x_2^*,\ldots\},\{0\}) \geq \varepsilon \quad \textit{and} \quad \sum_{n=1}^{\infty} \lambda_n = 1.$$

Then there exists a sequence $\{y_n^*\} \subset X^*$ such that
(i) $y_n^* \in \mathrm{co}\{x_n^*,x_{n+1}^*,\ldots\}$ $(n = 1,2,\ldots)$, *and*
(ii) $\|\sum_{i=1}^n \lambda_i y_i^*\| < \|\sum_{i=1}^\infty \lambda_i y_i^*\|(1 - \varepsilon \sum_{i=n+1}^\infty \lambda_i)$ $(n = 1,2,\ldots)$.

PROOF. We choose numbers ε_n so that

$$(10.5) \qquad 0 < \varepsilon_n < (1-\varepsilon)\Big(\sum_{i=n+1}^{\infty} \lambda_i\Big)\Big(\sum_{i=n}^{\infty} \lambda_i\Big) \qquad (n = 1,2,\ldots).$$

Then

$$(10.6) \qquad \sum_{n=1}^{\infty} \frac{\lambda_n \varepsilon_n}{(\sum_{i=n+1}^\infty \lambda_i)(\sum_{i=n}^\infty \lambda_i)} < 1-\varepsilon \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 0.$$

The sequence $\{y_n^*\}$ is now selected as follows. Put

$$\alpha_1 := \inf\{\|y^*\| : y^* \in \mathrm{co}\{x_1^*,x_2^*,\ldots\}\}$$

and pick $y_1^* \in \mathrm{co}\{x_1^*,x_2^*,\ldots\}$ so that

$$\|y_1^*\| < \alpha_1(1+\varepsilon_1).$$

Then define

$$\alpha_2 := \inf\{\|\lambda_1 y_1^* + (\sum_{i=2}^{\infty} \lambda_i) y^*\| : y^* \in \mathrm{co}\{x_2^*,x_3^*,\ldots\}\}.$$

Clearly $\varepsilon \leq \alpha_1 \leq \alpha_2 \leq 1$. Select $y_2^* \in \mathrm{co}\{x_2^*,x_3^*,\ldots\}$ so that

$$\|\lambda_1 y_1^* + (\sum_{i=2}^{\infty} \lambda_i^*) y_2^*\| < \alpha_2(1+\varepsilon_2).$$

Proceeding by induction we so define a sequence $\{y_n^*\} \subset X^*$ satisfying

$$(10.7) \qquad y_n^* \in \mathrm{co}\{x_n^*,x_{n+1}^*,\ldots\} \qquad (n = 1,2,\ldots)$$

and (with $\sum_{i=1}^{0} \lambda_i y_i^* := 0$),

$$\|\sum_{i=1}^{n-1} \lambda_i y_i^* + (\sum_{i=n}^{\infty} \lambda_i) y_n^*\| < \alpha_n(1+\varepsilon_n) \qquad (n = 1,2,\dots), \tag{10.8}$$

where

$$\alpha_n := \inf\{\|\sum_{i=1}^{n-1} \lambda_i y_i^* + (\sum_{i=n}^{\infty} \lambda_i) y^*\| : y^* \in \mathrm{co}\{x_n^*, x_{n+1}^*, \dots\}\} \qquad (n = 1,2,\dots). \tag{10.9}$$

From (10.4), (10.9) and $\|x_n^*\| \le 1$ $(n = 1,2,\dots)$ it follows that

$$\varepsilon \le \alpha_1 \le \alpha_2 \le \dots \le \alpha_n \le \alpha_{n+1} \le \dots \le 1, \tag{10.10}$$

so that $\alpha := \lim_{n\to\infty} \alpha_n$ exists and satisfies $\varepsilon \le \alpha \le 1$. Moreover, (10.8) and (10.9) imply that

$$\alpha_n \le \|\sum_{i=1}^{n-1} \lambda_i y_i^* + (\sum_{i=n}^{\infty} \lambda_i) y_n^*\| < \alpha_n(1+\varepsilon_n) \qquad (n = 1,2,\dots).$$

Hence

$$\left|\|\sum_{i=1}^{n-1} \lambda_i y_i^*\| - \alpha_n\right| \le \alpha_n\varepsilon_n + \|(\sum_{i=n}^{\infty} \lambda_i) y_n^*\| \le \varepsilon_n + \sum_{i=n}^{\infty} \lambda_i \to 0 \text{ as } n \to \infty$$

and therefore

$$\|\sum_{i=1}^{\infty} \lambda_i y_i^*\| = \alpha. \tag{10.11}$$

Before proving (ii) we first show that for all $n \in \mathbb{N}$,

$$\|\sum_{i=1}^{n} \lambda_i y_i^*\| < (\sum_{i=n+1}^{\infty} \lambda_i)\left[\frac{\lambda_n \alpha_n (1+\varepsilon_n)}{(\sum_{i=n+1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \frac{1}{\sum_{i=n}^{\infty}\lambda_i} \cdot \|\sum_{i=1}^{n-1} \lambda_i y_i^*\|\right]. \tag{10.12}$$

Indeed, for every $n \in \mathbb{N}$ we have by the triangle inequality and (10.8)

$$\|\sum_{i=1}^{n} \lambda_i y_i^*\| = \left\|\frac{\lambda_n + \sum_{i=n+1}^{\infty}\lambda_i}{\sum_{i=n}^{\infty}\lambda_i} \sum_{i=1}^{n-1} \lambda_i y_i^* + \frac{\sum_{i=n}^{\infty}\lambda_i}{\sum_{i=n}^{\infty}\lambda_i} \lambda_n y_n^*\right\| \le$$

$$\leq \frac{\lambda_n}{\sum_{i=n}^{\infty}\lambda_i} \|\sum_{i=1}^{n-1}\lambda_i y_i^* + (\sum_{i=n}^{\infty}\lambda_i)y_n^*\| + \frac{\sum_{i=n+1}^{\infty}\lambda_i}{\sum_{i=n}^{\infty}\lambda_i} \|\sum_{i=1}^{n-1}\lambda_i y_i^*\| <$$

$$< \frac{\lambda_n}{\sum_{i=n}^{\infty}\lambda_i} (\alpha_n(1+\varepsilon_n)) + \frac{\sum_{i=n+1}^{\infty}\lambda_i}{\sum_{i=n}^{\infty}\lambda_i} \|\sum_{i=1}^{n-1}\lambda_i y_i^*\| =$$

$$= \sum_{i=n+1}^{\infty}\lambda_i\left[\frac{\lambda_n\alpha_n(1+\varepsilon_n)}{(\sum_{i=n+1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \frac{1}{(\sum_{i=n}^{\infty}\lambda_i)}\|\sum_{i=1}^{n-1}\lambda_i y_i^*\|\right].$$

We now replace in (10.12) the expression $\|\sum_{i=1}^{n-1}\lambda_i y_i^*\|$ by the right side of the inequality that results from (10.12) by substituting n-1 for n. This yields, for $n \geq 2$,

$$\|\sum_{i=1}^{n}\lambda_i y_i^*\| < (\sum_{i=n+1}^{\infty}\lambda_i)\left[\frac{\lambda_n\alpha_n(1+\varepsilon_n)}{(\sum_{i=n+1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \right.$$

$$+ \frac{1}{(\sum_{i=n}^{\infty}\lambda_i)}\sum_{i=n}^{\infty}\lambda_i\left\{\frac{\lambda_{n-1}\alpha_{n-1}(1+\varepsilon_{n-1})}{(\sum_{i=n}^{\infty}\lambda_i)(\sum_{i=n-1}^{\infty}\lambda_i)} + \right.$$

$$\left.\left. + \frac{1}{(\sum_{i=n-1}^{\infty}\lambda_i)}\left\|\sum_{i=1}^{n-2}\lambda_i y_i^*\right\|\right\}\right] =$$

$$= (\sum_{i=n+1}^{\infty}\lambda_i)\left[\frac{\lambda_n\alpha_n(1+\varepsilon_n)}{(\sum_{i=n+1}^{\infty}\lambda_i)(\sum_{i=n}^{\infty}\lambda_i)} + \frac{\lambda_{n-1}\alpha_{n-1}(1+\varepsilon_{n-1})}{(\sum_{i=n}^{\infty}\lambda_i)(\sum_{i=n-1}^{\infty}\lambda_i)} + \right.$$

$$\left. + \frac{1}{(\sum_{i=n-1}^{\infty}\lambda_i)}\left\|\sum_{i=1}^{n-2}\lambda_i y_i^*\right\|\right].$$

Repeating this argument we finally obtain, by (10.6) and (10.11),

$$\|\sum_{i=1}^{n}\lambda_i y_i^*\| < (\sum_{i=n+1}^{\infty}\lambda_i)\sum_{k=1}^{n}\frac{\lambda_k\alpha_k(1+\varepsilon_k)}{(\sum_{i=k+1}^{\infty}\lambda_i)(\sum_{i=k}^{\infty}\lambda_i)} \leq$$

$$\leq \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\sum_{k=1}^{n}\frac{\lambda_k(1+\varepsilon_k)}{(\sum_{i=k+1}^{\infty}\lambda_i)(\sum_{i=k}^{\infty}\lambda_i)} \leq$$

$$\leq \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left[\left\{\sum_{k=1}^{n}\frac{\lambda_k}{(\sum_{i=k+1}^{\infty}\lambda_i)(\sum_{i=k}^{\infty}\lambda_i)}\right\} + (1-\varepsilon)\right] =$$

$$= \alpha(\sum_{i=n+1}^{\infty}\lambda_i)\left[\left\{\sum_{k=1}^{n}\left(\frac{1}{\sum_{i=k+1}^{\infty}\lambda_i} - \frac{1}{\sum_{i=k}^{\infty}\lambda_i}\right)\right\} + 1 - \varepsilon\right] =$$

$$= \alpha\Big(\sum_{i=n+1}^{\infty}\lambda_i\Big)\Big[\frac{1}{\sum_{i=n+1}^{\infty}\lambda_i} - \frac{1}{\sum_{i=1}^{\infty}\lambda_i} + 1 - \varepsilon\Big] = \alpha\Big(\sum_{i=n+1}^{\infty}\lambda_i\Big)\Big(\frac{1}{\sum_{i=n+1}^{\infty}\lambda_i} - \varepsilon\Big) =$$

$$= \alpha\Big(1 - \varepsilon\sum_{i=n+1}^{\infty}\lambda_i\Big) = \Big\|\sum_{i=1}^{\infty}\lambda_i y_i^*\Big\|\Big(1 - \varepsilon\sum_{i=n+1}^{\infty}\lambda_i\Big) \quad (n = 1,2,\ldots). \ \square$$

In the following theorem the equivalence that interests us is (i) $\Leftrightarrow$ (iv). It says that X is reflexive iff every $x^* \in X^*$ attains its sup on B_X. The properties (ii) and (iii) are merely intermediate steps in the proof of (i) $\Rightarrow$ (iv).

THEOREM 10.6. *Let X be a separable Banach space. Then the following are equivalent:*

(i) *X is non-reflexive.*

(ii) *For every* $0 < \varepsilon < 1$ *there exists a sequence* $\{x_n^*\} \subset B_{X^*}$ *satisfying*

(a) $w^*\text{-}\lim_{n\to\infty} x_n^* = 0$,

(b) $\operatorname{dist}(\operatorname{co}\{x_1^*, x_2^*, \ldots\}, \{0\}) \geq \varepsilon$.

(iii) *For every* $0 < \varepsilon < 1$ *and for every sequence* $\{\lambda_n\}$ *of positive numbers with* $\sum_{n=1}^{\infty}\lambda_n = 1$ *there exists a sequence* $\{y_n^*\} \subset B_{X^*}$ *satisfying*

(c) $w^*\text{-}\lim_{n\to\infty} y_n^* = 0$,

(d) $\|\sum_{i=1}^{n}\lambda_i y_i^*\| < \|\sum_{i=1}^{\infty}\lambda_i y_i^*\|(1 - \varepsilon\sum_{i=n+1}^{\infty}\lambda_i)$ $(n = 1,2,\ldots)$.

(iv) *There exists an* $x^* \in X^*$ *such that* $\{x \in B_X : \langle x, x^*\rangle = \|x^*\|\} = \phi$.

PROOF. (i) $\Rightarrow$ (ii): Let X be non-reflexive and let $0 < \varepsilon < 1$ be given. There exist (see the proof of Theorem 10.3) elements $x^{**} \in X^{**}\setminus\pi X$ and $x^{***} \in X^{***}$ satisfying

$$\varepsilon < \|x^{***}\| < 1, \quad x^{***}|_{\pi X} = 0, \quad \langle x^{**}, x^{***}\rangle > \varepsilon \quad \text{and} \quad \|x^{**}\| < 1.$$

Let $\{x_n\}$ be a sequence dense in X. We now select a sequence $\{x_n^*\} \subset X^*$ such that the following hold for every $n \in \mathbb{N}$:

(1) $\|x_n^*\| < 1$,

(2) $\langle x_n^*, x^{**}\rangle > \varepsilon$,

(3) $\langle x_i, x_n^*\rangle = 0$ for all $i = 1,\ldots,n$.

This can be done inductively with a (by now standard) application of Theorem 3.1 to X^* with $G := \operatorname{sp}\{x^{***}\}$ and $F := \operatorname{sp}\{x^{**}, \pi x_1, \ldots, \pi x_n\}$. It remains to be verified that this sequence $\{x_n^*\}$ satisfies (a) and (b). Obviously (a) follows from (3), (1) and the density of $\{x_n\}$. (b) is a simple consequence of (2), since for any convex combination $\sum_{i=1}^{n} \lambda_i x_i^*$ we have

$$\| \sum_{i=1}^{n} \lambda_i x_i^* \| \geq < \sum_{i=1}^{n} \lambda_i x_i^*, x^{**} > > \varepsilon \sum_{i=1}^{n} \lambda_i = \varepsilon .$$

(ii) ⇒ (iii): Let ε with $0 < \varepsilon < 1$ and a sequence $\{\lambda_n\}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ be given arbitrarily. Then choose a sequence $\{x_n^*\} \subset B_{X^*}$ satisfying (a) and (b), for this ε. Having now an ε, $0 < \varepsilon < 1$, a sequence of positive numbers $\{\lambda_n\}$, $\sum_{n=1}^{\infty} \lambda_n = 1$, and a sequence $\{x_n^*\} \subset B_{X^*}$, with $\text{dist}(\text{co}\{x_1^*, x_2^*, \ldots\}, \{0\}) \geq \varepsilon$, we now use Lemma 10.5 to find a sequence $\{y_n^*\} \subset X^*$ satisfying (i) and (ii) in Lemma 10.5 for this triple $\varepsilon, \{\lambda_n\}, \{x_n^*\}$. Now (c) follows from (a) and (i) in Lemma 10.5, and (d) is the same as (ii) in Lemma 10.5.

(iii) ⇒ (iv): Let $0 < \varepsilon < 1$ and a sequence $\{\lambda_n\}$ of positive numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ be given arbitrarily. Select $\{y_n^*\} \subset B_{X^*}$ so that (c) and (d) hold. We claim that $x^* := \sum_{n=1}^{\infty} \lambda_n y_n^*$ satisfies (iv). Indeed, if $x \in B_X$ is arbitrary, then $\lim_{n\to\infty} <x, y_n^*> = 0$, by (c). Hence there exists an $n_0 \in \mathbb{N}$ such that $<x, y_n^*> < \varepsilon \| x^* \|$ whenever $n \geq n_0$. Then, using (d),

$$<x, x^*> = \sum_{n=1}^{\infty} \lambda_n <x, y_n^*> < \sum_{n=1}^{n_0} \lambda_n <x, y_n^*> + (\sum_{n=n_0+1}^{\infty} \lambda_n) \varepsilon \| x^* \| \leq$$

$$\leq \| \sum_{n=1}^{n_0} \lambda_n y_n^* \| + (\sum_{n=n_0+1}^{\infty} \lambda_n) \varepsilon \| x^* \| <$$

$$< \| x^* \| (1 - \varepsilon \sum_{n=n_0+1}^{\infty} \lambda_n) + (\sum_{n=n_0+1}^{\infty} \lambda_n) \varepsilon \| x^* \| = \| x^* \| .$$

Hence $\{x \in B_X : <x, x^*> = \| x^* \| \} = \phi$, since $x \in B_X$ was arbitrary.

(iv) ⇒ (i): This is trivial if one knows that the unit ball of a reflexive space is weakly compact. Also without this knowledge the proof is trivial, however: Suppose $x^* \in X^*$ is as in (iv). By the Hahn-Banach theorem there exists an $x^{**} \in X^{**}$ with $\| x^{**} \| = 1$ and $<x^*, x^{**}> = \| x^* \|$. The assumption (iv) means that $x^{**} \in X^{**} \setminus \pi X$. Thus X is non-reflexive. □

NOTES. All results in this section are due to R.C. JAMES. The properties P_2^{∞}, P_3^{∞} and P_4^{∞} were introduced and studied in [54] in the context of the search for an isomorphic analogue of uniform convexity. (This subject will be dealt with in the next chapter.) Theorem 10.6 has a rather long history. It was first proved in [49] for separable spaces, and later in [51] for the non-separable case as well. The more general result that a bounded weakly closed subset K of a complete locally convex topological vector space E is weakly compact iff each continuous linear form on E attains its sup on K, first appeared in [50] . The proofs of these results given in the papers cited above were difficult. Considerably simpler proofs (essentially the ones reproduced here) were given by R.C. JAMES in [53]. Related papers are [85] and [94].

11. THE INFINITE TREE PROPERTY

We now introduce the infinite tree property, labelled P_1^∞. It is not equivalent to non-reflexivity but implies it. The proof of this last statement is the main goal in this section.

DEFINITION 11.1. Let X be a Banach space, $\varepsilon > 0$ and $n \in \mathbb{N}$. An *ε-tree of length* n, or an *(n,ε)-tree* in X is a subset

$$T = \{x_{k,i}: k = 1,\dots,n;\ i = 1,\dots,2^k\}$$

of X satisfying the following relations:

$$x_{k,i} = \tfrac{1}{2}(x_{k+1,2i-1}+x_{k+1,2i}) \qquad (k = 1,\dots,n-1;\ i = 1,\dots,2^k), \tag{11.1}$$

$$\|x_{k,2i-1}-x_{k,2i}\| \geq \varepsilon \qquad (k = 1,\dots,n;\ i = 1,\dots,2^{k-1}) \tag{11.2}$$

An *ε-tree of infinite length,* or an *(∞,ε)-tree* in X is a subset

$$T = \{x_{k,i}: k = 1,2,\dots;\ i = 1,\dots,2^k\}$$

of X satisfying (11.1) and (11.2) for all $k \in \mathbb{N}$ and $i \in \{1,\dots,2^k\}$.

On should view an ε-tree as the result of a branching process starting with two points $x_{1,1}$ and $x_{1,2}$ having distance $\geq \varepsilon$. Each of these points is then written as the midpoint of a segment $[x_{2,1},x_{2,2}]$ (respectively $[x_{2,3},x_{2,4}]$) having length at least ε. Etc.

An alternative inductive definition of an (n,ε)-tree is

DEFINITION 11.1*. A $(1,\varepsilon)$-tree in X is a pair of points $\{x_1,x_2\}$ with $\|x_1-x_2\| \geq \varepsilon$. Supposing that we have defined an $(n-1,\varepsilon)$-tree for some $n \geq 2$, then an (n,ε)-tree in X is a subset $T = \{x_1,x_2,\dots,x_{2^n}\}$ of X satisfying

$$\|x_{2i-1}-x_{2i}\| \geq \varepsilon \quad (i = 1,\ldots,2^{n-1})$$

and such that

$$T' := \{\tfrac{1}{2}(x_1+x_2), \tfrac{1}{2}(x_3+x_4),\ldots,\tfrac{1}{2}(x_{2^n-1}+x_{2^n})\}$$

is an $(n-1,\varepsilon)$-tree. T' is called the *derived tree* of T.

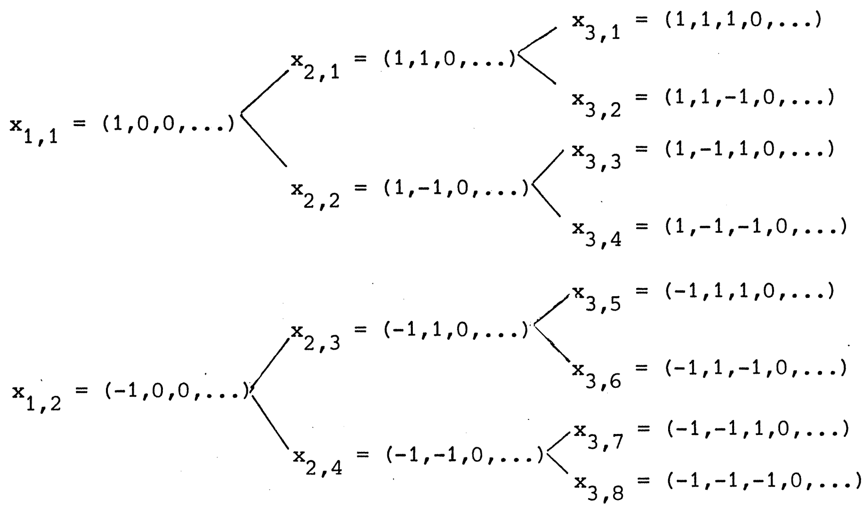
REMARK 11.2. One should observe that formally at least two different objects are involved here: An (n,ε)-tree in the sense of Definition 11.1* consists of the "endpoints" of an (n,ε)-tree in the sense of Definition 11.1. We shall use both definitions, whichever is the more convenient. The difference in indexation precludes confusion. Obviously an (∞,ε)-tree cannot be defined as in Definition 11.1*, since it has no endpoints.

DEFINITION 11.3. Let T be an ε-tree in a Banach space X (of finite or infinite length). By the *norm of* T we mean the number $\|T\| := \sup\{\|x\| : x \in T\}$. (Note that it makes no difference, in the case of finite trees, whether one uses Definition 11.1 or Definition 11.1*.)

In every Banach space X and for every $\varepsilon > 0$ there exist in X ε-trees of arbitrary (finite or infinite) length, since there is obviously no limit to the branching process. In general, however, for fixed $\varepsilon > 0$, the infimum of the norms of all (n,ε)-trees in X will increase indefinitely with n. X will be said to have the infinite tree property if, for some $\varepsilon > 0$, there exists in X an (∞,ε)-tree with finite norm. The main result in this section will be that reflexive spaces do not have the infinite tree property. The proper feeling here is that the unit ball of a reflexive space is too rotund to contain (∞,ε)-trees.

DEFINITION 11.4. A Banach space X has the property P_1^∞ (= *infinite tree property*) if for some $\varepsilon > 0$ there exists an (∞,ε)-tree T in B_X, i.e. with norm $\|T\| \leq 1$.

EXAMPLE. c_0 has P_1^∞. Indeed, the following branching process, which the reader will undoubtedly be able to continue, leads to an $(\infty,2)$-tree with norm 1:



<u>THEOREM 11.5</u>. *A reflexive Banach space* X *does not have* P_1^∞.

The proof rests on the following Proposition, which holds also in more general situations (cf. Remark 11.7).

<u>PROPOSITION 11.6</u>. *Let* $K \neq \phi$ *be a convex separable and weakly compact subset of a Banach space* X. *Then there exists for every* $\varepsilon > 0$ *a closed convex subset* $C \subset K$ *such that* $C \neq K$ *and diam* $(K \backslash C) \leq \varepsilon$.

<u>PROOF</u>. Let $\varepsilon > 0$ be arbitrary. By the Krein-Milman theorem ext $K \neq \phi$. Let D be the weak closure of ext K and let $U = \frac{\varepsilon}{4} B_X$. Since K is separable, there exists a sequence $\{k_n\} \subset K$ such that $K \subset \bigcup_{n=1}^{\infty} (k_n+U)$. In particular $D \subset \bigcup_{n=1}^{\infty} (k_n+U)$. By Baire's theorem for compact Hausdorff spaces, applied to D with the weak topology, at least one of the sets $(k_n+U) \cap D$, say $(k_{n_0}+U) \cap D$ has an interior point relative to the weak topology on D. Hence there exists a weakly open subset $W \subset X$ such that

$$(k_{n_0}+U) \cap D \supset W \cap D \neq \phi .$$

We now define

$$K_1 := \overline{\text{co}}\ (D\backslash W) \quad \text{and} \quad K_2 := \overline{\text{co}}\ (D \cap W) .$$

Since ext $K \subset D \subset K_1 \cup K_2 \subset K$, the Krein-Milman theorem implies that

$$K = \overline{co}\ (K_1 \cup K_2) = co\ (K_1 \cup K_2).$$

Moreover, $K_1 \neq K$. Indeed, $K = K_1 = \overline{co}\ (D \setminus W)$ would imply, again by the Krein-Milman theorem, that $D \setminus W$ contains ext K, contradicting $D \cap W \neq \phi$. Also, since $K_2 \subset k_{n_0} + U$ and diam $U = \frac{\varepsilon}{2}$,

$$\text{diam } K_2 \le \frac{\varepsilon}{2}.$$

Now let $r \in (0,1]$ be arbitrary and consider the map $f_r : K_1 \times K_2 \times [r,1] \to K$ defined by

$$f_r(x_1, x_2, \lambda) = \lambda x_1 + (1-\lambda) x_2 \qquad (x_1 \in K_1, x_2 \in K_2, \lambda \in [r,1]).$$

Since f_r is continuous for the product of the weak topologies on K_1 and K_2 and the usual topology on $[r,1]$, to the weak topology on K, the image C_r of f_r is weakly compact. It is also convex. Indeed, for any two triples (x_1, x_2, λ) and (x_1', x_2', λ') in $K_1 \times K_2 \times [r,1]$ and any $\mu \in (0,1)$ we have

$$\mu(\lambda x_1 + (1-\lambda)x_2) + (1-\mu)(\lambda' x_1' + (1-\lambda')x_2') =$$

$$= \underbrace{(\mu\lambda + (1-\mu)\lambda')}_{=:\alpha \in [r,1]} \underbrace{\frac{\mu\lambda x_1 + (1-\mu)\lambda' x_1'}{\mu\lambda + (1-\mu)\lambda'}}_{\in K_1} +$$

$$+ \underbrace{(\mu(1-\lambda) + (1-\mu)(1-\lambda'))}_{=1-\alpha} \underbrace{\frac{\mu(1-\lambda)x_2 + (1-\mu)(1-\lambda')x_2'}{\mu(1-\lambda) + (1-\mu)(1-\lambda')}}_{\in K_2} \in C_r .$$

We observe next that $C_r \neq K$. Indeed, suppose $C_r = K$. Then in particular every $x \in$ ext K is of the form $x = \lambda x_1 + (1-\lambda)x_2$ for some $x_1 \in K_1$, $x_2 \in K_2$ and $\lambda \in [r,1]$. Hence $x = x_1 \in K$, so that ext $K \subset K_1$. But then, by the Krein-Milman theorem, $K = K_1$, contradicting $K_1 \neq K$.

Finally we estimate diam $(K \setminus C_r)$. It follows from $K = co\ (K_1 \cup K_2)$ and the definition of C_r that every $y \in K \setminus C_r$ is of the form

$$y = \lambda x_1 + (1-\lambda)x_2 \quad \text{with} \quad x_1 \in K_1,\ x_2 \in K_2 \quad \text{and} \quad \lambda \in [0,r),$$

so $\|y-x_2\| = \|\lambda x_1-\lambda x_2\| \le \lambda$ diam $K \le r$ diam K. Hence for every pair $y = \lambda x_1 + (1-\lambda)x_2$, $y' = \lambda' x_1' + (1-\lambda')x_2' \in K\backslash C_r$ we have, since diam $K_2 \le \frac{\varepsilon}{2}$,

$$\|y-y'\| \le \|y-x_2\| + \|x_2-x_2'\| + \|y'-x_2'\| \le 2r \text{ diam } K + \frac{\varepsilon}{2}\,.$$

Thus diam $(K\backslash C_r) \le 2r$ diam $K + \frac{\varepsilon}{2}$ and for the choice $r = \frac{\varepsilon}{4\text{diam}K}$ the closed convex set $C := C_r$ satisfies the requirement. □

PROOF OF THEOREM 11.5. Let us assume that X is reflexive and that $T = \{x_{k,i}: k \in \mathbb{N},\ i = 1,\dots,2^k\}$ is an (∞,ε)-tree in B_X, for some $\varepsilon > 0$. Then by reflexivity the convex set $K := \overline{co}\ T \subset B_X$ is weakly compact, and also separable, since T is countable. Thus there exists, by Proposition 11.6, a closed convex set $C \subsetneqq K$, with diam $(K\backslash C) \le \frac{\varepsilon}{3}$. Since $K = \overline{co}\ T$ and $C \ne K$, we have $T\backslash C \ne \phi$. Suppose $x_{k,i} \in T\backslash C$. The equality $x_{k,i} = \frac{1}{2}(x_{k+1,2i-1}+x_{k+1,2i})$ and the convexity of C imply that either $x_{k+1,2i-1}$ or $x_{k+1,2i}$, say $x_{k+1,2i}$, does not belong to C. But then

$$\text{diam } (K\backslash C) \ge \|x_{k,i}-x_{k+1,2i}\| = \tfrac{1}{2}\,\|x_{k+1,2i-1}-x_{k+1,2i}\| \ge \frac{\varepsilon}{2}\,,$$

a contradiction. □

REMARK 11.7. The proof of Theorem 11.5 shows that the infinite tree property does not hold in any Banach space X with the property that for every closed convex bounded set $K \subset X$ and for every $\varepsilon > 0$ there exists a closed convex $C \subsetneqq K$ with diam $(K\backslash C) \le \varepsilon$. This last property (the so called "dentability" of closed convex bounded sets) is known to be equivalent to the Radon-Nikodym (R.N.) property, i.e. to the validity of the Radon-Nikodym theorem for X-valued measures. The R.N. property holds not only for reflexive spaces, but e.g. also for separable dual spaces. Therefore also separable dual spaces do not have P_1^∞. An example is $\ell^1 = (c_0)^*$. In particular, since we have seen that c_0 has P_1^∞, P_1^∞ is not preserved under duality.

NOTES. The infinite tree property was introduced by R.C. JAMES in [54], which also contains Theorem 11.5. Lemma 11.6 is due to I. NAMIOKA and E. ASPLUND ([3]). For more information about dentability, the R.N. property and related matters, the reader should consult [27], Chapter 6 and its references or the recently published monograph [28].

CHAPTER II

12. UNIFORM CONVEXITY AND UNIFORM SMOOTHNESS

We shall be interested later in the problem which Banach spaces admit an equivalent uniformly convex norm. As a preparation we discuss in this section uniform convexity and the dual notion of uniform smoothness. In particular we show that uniform convexity implies reflexivity. Furthermore we develop an averaging procedure to show that if a Banach space X admits an equivalent uniformly convex norm and also an equivalent uniformly smooth norm, then there exists an equivalent norm on X with both these properties. Combining this with a fact to be proved later, namely that X admits an equivalent uniformly convex norm iff it admits an equivalent uniformly smooth norm, it will follow that every "uniformly convexifiable" space admits an equivalent norm which is both uniformly convex and uniformly smooth.

Of course it would be more natural to start with a discussion of the weakest convexity and differentiability properties and then to gradually strengthen them. Since all this material is classical and can be found in detail in e.g. [24] or [27], we feel justified in omitting it. So we restrict ourselves to what is strictly needed to make this account self-contained.

DEFINITION 12.1. A Banach space X is called *uniformly convex* (or *uniformly rotund*) iff for every sequence of pairs $x_n, y_n \in B_X$, $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$ implies $\lim_{n\to\infty} \|x_n-y_n\| = 0$. In words: if the midpoints of a sequence of segments in B_X approach the unit sphere, then their lengths converge to 0. Or, equivalently, for every $0 < \varepsilon \le 2$ the number

$$\delta(\varepsilon) := \inf\{1 - \|\tfrac{x+y}{2}\| : x,y \in B_X,\ \|x-y\| \ge \varepsilon\}$$

is positive. For any Banach space X the function $\delta = \delta_X : (0,2] \to [0,1]$ defined above is called the *modulus of convexity* of X. Thus X is uniformly convex iff $\delta_X(\varepsilon) > 0$ for every $\varepsilon > 0$.

REMARK 12.2. Sometimes a slightly different definition of the modulus of convexity is used:

$$\delta'(\varepsilon) := \inf\{1 - \|\tfrac{x+y}{2}\| : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\} \quad (0 < \varepsilon \leq 2).$$

Clearly $\delta(\varepsilon) \leq \delta'(\varepsilon)$ for all $\varepsilon > 0$. It is also obvious that $\delta(\varepsilon)$ is strictly positive for positive ε if δ' has this property. Indeed, suppose for some $\varepsilon > 0$ there exist $x_n, y_n \in B_X$ with $\|x_n - y_n\| \geq \varepsilon$ $(n = 1,2,\dots)$ and $\lim_{n\to\infty} \|x_n + y_n\| = 2$. Then $\lim_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|y_n\| = 1$ and it follows that

$$\left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| > \frac{\varepsilon}{2} \text{ for large } n,$$

while

$$\lim_{n\to\infty} \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| = 2,$$

contradicting $\delta'(\frac{\varepsilon}{2}) > 0$. Thus uniform convexity can be defined equivalently by requiring that $\delta'(\varepsilon) > 0$ for all $\varepsilon > 0$.

Well-known examples of uniformly convex spaces are the spaces ℓ^p $(1 < p < \infty)$ for which the modulus of convexity can be computed explicitly. We have already seen in Section 10 that reflexivity of a space X means the non-existence of certain infinite-dimensional flat areas in B_X close to S_X. Hence it is not surprising (and actually contained implicitly in Theorem 10.3 and Remark 10.4, as will be shown in Proposition 16.1) that a uniformly convex space is reflexive. We now give a direct and simple proof of this.

PROPOSITION 12.3. *A uniformly convex Banach space* X *is reflexive.*

PROOF. Let $x^{**} \in S_{X^{**}}$ be arbitrary. Since B_X is $\sigma(X^{**},X^*)$-dense in $B_{X^{**}}$ (Proposition 0.10), there exists a net $\{x_\alpha\}_{\alpha\in A} \subset B_X$ such that $w^* \lim_\alpha x_\alpha = x^{**}$. Consider the net $\{\frac{1}{2}(x_\alpha + x_\beta)\}_{(\alpha,\beta)\in A\times A}$, where $A \times A$ is partially ordered by

$$(\alpha,\beta) \geq (\alpha',\beta') \Longleftrightarrow \alpha \geq \alpha' \quad\text{and}\quad \beta \geq \beta'.$$

Clearly $\{\frac{1}{2}(x_\alpha + x_\beta)\}$ w^* converges to x^{**}, while $\|\frac{1}{2}(x_\alpha + x_\beta)\| \leq 1$ for all (α,β). Hence, by the w^*-lower semi continuity of $\|\cdot\|$ on X^{**}, $\lim_{(\alpha,\beta)} \|\frac{1}{2}(x_\alpha + x_\beta)\| = \|x^{**}\| = 1$. By the uniform convexity of X it follows that $\lim_{(\alpha,\beta)} \|x_\alpha - x_\beta\| = 0$, i.e. $\{x_\alpha\}$ is a Cauchy net. Then $\{x_\alpha\}$ converges in norm, obviously to x^{**}. Hence $x^{**} \in X$ and the reflexivity of X is proved. □

The notion dual to uniform convexity is uniform smoothness.

DEFINITION 12.4. A Banach space X is called *uniformly smooth* iff the norm is differentiable in the following strong sense:

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} =: D(x,y)$$

exists, uniformly for all $x,y \in S_X$.

REMARK 12.5. For $x,y \in S_X$ and $t \neq 0$ let us denote $\frac{\|x+ty\| - \|x\|}{t}$ by $\Delta(x,y,t)$. We show that $\Delta(x,y,\cdot)$ is a non-decreasing function ($x,y \in S_X$ fixed). For $0 < t_1 < t_2$ we have

$$t_2\|x+t_1y\| = \|t_2x+t_2t_1y\| \leq t_1\|x+t_2y\| + (t_2-t_1)\|x\|,$$

so

$$t_2(\|x+t_1y\| - \|x\|) \leq t_1(\|x+t_2y\| - \|x\|),$$

i.e.

$$\Delta(x,y,t_1) = \frac{\|x+t_1y\| - \|x\|}{t_1} \leq \frac{\|x+t_2y\| - \|x\|}{t_2} = \Delta(x,y,t_2).$$

Hence $\Delta(x,y,\cdot)$ is non-decreasing for positive t. Since $\Delta(x,y,-t) = -\Delta(x,-y,t)$ for any $t \neq 0$, $\Delta(x,y,\cdot)$ is also non-decreasing for negative t. Furthermore, for any $t > 0$,

$$2\|x\| \leq \|x+ty\| + \|x-ty\|,$$

so

$$\|x+ty\| - \|x\| \geq -(\|x-ty\| - \|x\|)$$

and therefore

$$\Delta(x,y,t) = \frac{\|x+ty\| - \|x\|}{t} \geq \frac{\|x-ty\| - \|x\|}{-t} = \Delta(x,y,-t).$$

Together these facts prove that $\Delta(x,y,\cdot)$ is non-decreasing for all t.

It follows from this that uniform smoothness is equivalent to

$$\text{(12.1)} \qquad \lim_{t\downarrow 0} \left[\left\{\frac{\|x+ty\|-\|x\|}{t}\right\} - \left\{\frac{\|x-ty\|-\|x\|}{-t}\right\}\right] = 0,$$

uniformly for $x,y \in S_X$. Equivalently, replacing ty by y, this means

$$\text{(12.2)} \qquad \lim_{y\to 0} \frac{\|x+y\|+\|x-y\|-2}{\|y\|} = 0$$

uniformly for $x \in S_X$. Expressed differently again, (12.2) is equivalent to

(12.3) $\qquad \forall\varepsilon > 0 \quad \exists\delta > 0\colon x \in S_X,\ \|y\| < \delta \ \Rightarrow\ \|x+y\|+\|x-y\| < 2+\varepsilon\|y\|.$

For any Banach space X we define, for all $\tau > 0$,

$$\rho(\tau) := \sup\left\{\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} - 1\colon x \in S_X,\ \|y\| = \tau\right\}.$$

The function $\rho = \rho_X\colon [0,\infty) \to [0,\infty)$ is called the *modulus of smoothness* of X and it is clear from (12.2) that X is uniformly smooth iff $\lim_{\tau\downarrow 0} \frac{\rho(\tau)}{\tau} = 0$.

We now prove the duality of uniform convexity and uniform smoothness.

PROPOSITION 12.6. *Let X be a Banach space. Then*

(i) X *is uniformly smooth iff* X^* *is uniformly convex;*

(ii) X *is uniformly convex iff* X^* *is uniformly smooth.*

PROOF. a) Let us assume that X is uniformly convex. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

(12.4) $\qquad \|x\|,\|y\| \le 1,\ \|x-y\| \ge \varepsilon \ \Rightarrow\ 2 - \|x+y\| \ge \delta.$

We must prove that X^* is uniformly smooth, i.e., by (12.3), that for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

(12.5) $\qquad \|x^*\| = 1,\ \|y^*\| < \eta \ \Rightarrow\ \|x^*+y^*\|+\|x^*-y^*\| < 2+\varepsilon\|y^*\|.$

Let $\varepsilon > 0$ be fixed. Choose $\delta > 0$ such that (12.4) holds and let $\eta > 0$ be so small that $\frac{2-2\eta}{1+\eta} > 2-\delta$. We claim that with this η (12.5) holds. Indeed, suppose $x^*,y^* \in X^*$ satisfy $\|x^*\| = 1$, $\|y^*\| < \eta$. Choose $x,y \in S_X$ such that

$$\|x^*+y^*\| = \langle x,x^*+y^*\rangle, \qquad \|x^*-y^*\| = \langle y,x^*-y^*\rangle.$$

(These elements exist by reflexivity.) Then on the one hand we have

$$\begin{aligned}\|x^*+y^*\| + \|x^*-y^*\| &= \langle x,x^*+y^*\rangle + \langle y,x^*-y^*\rangle \le \\ &\le \|x+y\| + \|y^*\|\,\|x-y\| \le 2+\|y^*\|\,\|x-y\|,\end{aligned} \tag{12.6}$$

and on the other hand

$$\begin{aligned}\|x+y\| &\ge \frac{\langle x+y,x^*+y^*\rangle}{1+\eta} \ge \frac{\langle x,x^*+y^*\rangle + \langle y,x^*-y^*\rangle + 2\langle y,y^*\rangle}{1+\eta} \ge \\ &\ge \frac{\|x^*+y^*\| + \|x^*-y^*\| - 2\eta}{1+\eta} \ge \frac{2-2\eta}{1+\eta} > 2-\delta.\end{aligned} \tag{12.7}$$

It follows from (12.7) and (12.4) that $\|x-y\| < \varepsilon$. Thus the right side of (12.5) holds, by (12.6).

b) Let us assume now that X is uniformly smooth. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\text{(12.8)} \qquad \|x\| = 1,\ \|y\| < \delta \ \Rightarrow\ \|x+y\| + \|x-y\| < 2 + \tfrac{\varepsilon}{2}\|y\| .$$

We must show that X^* is uniformly convex, i.e. that for every $\varepsilon > 0$ there exists an $\eta > 0$ such that

$$\text{(12.9)} \qquad \|x^*\|,\|y^*\| \le 1,\ \|x^*-y^*\| \ge \varepsilon \Rightarrow 2 - \|x^*+y^*\| \ge \eta .$$

Fix $\varepsilon > 0$ and suppose that $x^*,y^* \in X^*$ satisfy $\|x^*\|,\|y^*\| \le 1$, $\|x^*-y^*\| \ge \varepsilon$. Choose $\delta > 0$ such that (12.8) holds. Then there exists an $x_0 \in X$ with $\|x_0\| = \frac{\delta}{2}$ such that $\langle x_0,x^*-y^*\rangle > \frac{\varepsilon\delta}{3}$. It follows now, using (12.8), that for every $x \in S_X$,

$$\langle x,x^*+y^*\rangle = \langle x+x_0,x^*\rangle + \langle x-x_0,y^*\rangle - \langle x_0,x^*-y^*\rangle \le$$
$$\le \|x+x_0\| + \|x-x_0\| - \frac{\varepsilon\delta}{3} < 2 + \frac{\varepsilon\delta}{4} - \frac{\varepsilon\delta}{3} = 2 - \frac{\varepsilon\delta}{12}.$$

Hence, with $\eta := \frac{\varepsilon\delta}{12} > 0$, the right side of (12.9) holds.

c) We have now shown necessity in both (i) and (ii). Combining this with Proposition 12.3 and the fact that X is reflexive iff X^* is reflexive (Proposition 0.13), sufficiency in both (i) and (ii) follows immediately. □

Our next topic is an averaging procedure devised by E. Asplund for the purpose of combining good properties of two equivalent norms into one single norm.

On a Banach space X let us consider the function

$$\text{(12.10)} \qquad f(x) = \tfrac{1}{2}\|x\|^2 \quad (x \in X).$$

Then f is convex and homogeneous of degree 2, i.e. $f(tx) = t^2 f(x)$ for all $t \in \mathbb{R}$ and $x \in X$. The last statement is clear. For the proof of the convexity of f, let $x,y \in X$, $0 < \lambda < 1$ be given. Then

$$f(\lambda x + (1-\lambda)y) = \tfrac{1}{2}\|\lambda x + (1-\lambda)y\|^2 \le \tfrac{1}{2}(\lambda\|x\| + (1-\lambda)\|y\|)^2 =$$
$$= \tfrac{1}{2}\lambda^2\|x\|^2 + \tfrac{1}{2}(1-\lambda)^2\|y\|^2 + \lambda(1-\lambda)\|x\|\|y\| \le \tfrac{1}{2}\lambda^2\|x\|^2 + \tfrac{1}{2}(1-\lambda)^2\|y\|^2 +$$
$$+ \lambda(1-\lambda)[\tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2] = \tfrac{1}{2}\lambda\|x\|^2 + \tfrac{1}{2}(1-\lambda)\|y\|^2 = \lambda f(x) + (1-\lambda)f(y).$$

Observe that any convex function which is homogeneous of degree 2 vanishes at 0 and is non-negative, and that the particular function f defined by (12.10) has the additional property that it vanishes only at 0. Conversely, let f be a convex function on a vector space X which is homogeneous of degree 2 and vanishes only at 0. (Hence $f(x) > 0$ whenever $x \neq 0$.) Then (12.10) defines a norm on X. Indeed, all norm properties are evident except perhaps the triangle inequality. Let $x, y \in X$ and assume, as we may, that $\|x\| \neq 0$, $\|y\| \neq 0$. Then, putting $t := \sqrt{f(y)/f(x)}$, we have

$$\begin{aligned} \|x+y\|^2 &= 2f(x+y) = 2f\Big(\frac{1}{1+t}(1+t)x + \frac{t}{1+t}\,\frac{1+t}{t}\,y\Big) \leq \\ &\leq \frac{2}{1+t}\,f((1+t)x) + 2\frac{t}{1+t}\,f\big(\frac{1+t}{t}\,y\big) = \\ &= 2(1+t)f(x) + 2\,\frac{1+t}{t}\,f(y) = 2f(x) + 2f(y) + 4\sqrt{f(x)}\sqrt{f(y)} \\ &= \big(\sqrt{2f(x)} + \sqrt{2f(y)}\big)^2 = (\|x\| + \|y\|)^2. \end{aligned}$$

Thus we have shown that, given a vector space X, there is a 1-1 correspondence between norms on X and convex functions on X which are homogeneous of degree 2 and vanish exactly at the origin. This correspondence is given by (12.10).

We now establish a formula relating the two functions corresponding to a given norm on X and to its dual norm, respectively.

PROPOSITION 12.7. *Let* X *be a Banach space and let* f *and* f^* *be the convex functions on* X *and* X^* *corresponding to the norm on* X *and its dual norm on* X^**, respectively. Then*

$$(12.11) \qquad f^*(x^*) = \sup\{\langle x, x^*\rangle - f(x) : x \in X\} \qquad (x^* \in X^*).$$

PROOF. Formula (12.11) means

$$(12.12) \qquad \tfrac{1}{2}\|x^*\|^2 = \sup\{\langle x, x^*\rangle - \tfrac{1}{2}\|x\|^2 : x \in X\}.$$

For arbitrary $x \in X$ and $x^* \in X^*$ we have

$$\langle x, x^*\rangle \leq \|x\|\,\|x^*\| \leq \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|x^*\|^2,$$

so

$$\tfrac{1}{2}\|x^*\|^2 \geq \sup\{\langle x, x^*\rangle - \tfrac{1}{2}\|x\|^2 : x \in X\},$$

which proves one half of (12.12). For the other half we may assume $x^* \neq 0$. Let $\varepsilon > 0$ be arbitrary and choose $x_0 \in X$ with $\|x_0\| = 1$ and $<x_0,x^*> > \|x^*\| - \varepsilon$. Putting $x = \|x^*\| x_0$, we have, since $\|x\| = \|x^*\|$,

$$<x,x^*> = \|x^*\| <x_0,x^*> \geq \|x^*\| (\|x^*\| - \varepsilon) \geq (\|x^*\| - \varepsilon)(\|x\| - \varepsilon) =$$

$$= \tfrac{1}{2}(\|x^*\| - \varepsilon)^2 + \tfrac{1}{2}(\|x\| - \varepsilon)^2.$$

Hence, letting $\varepsilon \to 0$, it follows that $\sup\{<x,x^*> - \frac{1}{2}\|x\|^2 : x \in X\} \geq \frac{1}{2}\|x^*\|^2$. □

We now want to express the uniform convexity of a Banach space by a condition on the function f corresponding to its norm (as in (12.10)). Let us agree to call this function f uniformly convex iff

$$(12.13) \qquad \inf\{f(x) - 2f(\tfrac{x+y}{2}) + f(y) : \|x\|, \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\}$$

is positive for every $0 < \varepsilon \leq 2$.

Observe that in (12.13) $\|\cdot\|$ and f are supposed to be related by (12.10). However, it is clear that replacing $\|\ \|$ in (12.13) by an equivalent norm, but keeping f fixed, the new expression is positive for every $\varepsilon > 0$ iff the original one is.

PROPOSITION 12.8. *Let X be a Banach space and let f correspond to the norm $\|\ \|$ of X as in* (12.10). *Then X is uniformly convex iff f is uniformly convex.*

PROOF. Suppose that f is uniformly convex and let $0 < \varepsilon \leq 2$ be arbitrary. Expressing (12.13) in terms of the norm, we get

$$(12.14) \qquad \inf\{\tfrac{1}{2}\|x\|^2 - \|\tfrac{x+y}{2}\|^2 + \tfrac{1}{2}\|y\|^2 : \|x\|, \|y\| \leq 1,\ \|x-y\| \geq \varepsilon\} > 0.$$

Taking x and y so that $\|x\| = \|y\| = 1$ (and $\|x-y\| \geq \varepsilon$), it follows that $\inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\}$ is positive, i.e. X is uniformly convex (cf. Remark 12.2).

For the converse let us assume that X is uniformly convex and let us fix $0 < \varepsilon \leq 2$. Then there exists a $\delta > 0$ such that

$$(12.15) \qquad u,v \in X,\quad \|u\| = \|v\| = 1,\quad \|u-v\| \geq \tfrac{\varepsilon}{2} \Rightarrow 1 - \|\tfrac{u+v}{2}\| > \delta.$$

We are going to show that the inf in (12.14) is positive. Let $x,y \in X$ with $\|x\|, \|y\| \leq 1$, $\|x-y\| \geq \varepsilon$. We distinguish three cases.

I: $|\|x\|-\|y\|| \geq \frac{\varepsilon}{2}$, II: $0 \leq \|x\|-\|y\| \leq \frac{\varepsilon}{2}$, III: $0 < \|y\|-\|x\| \leq \frac{\varepsilon}{2}$.

Put $u := \frac{x}{\|x\|}$, $v := \frac{y}{\|y\|}$.

Case I: $\frac{1}{2}\|x\|^2 - \|\frac{x+y}{2}\|^2 + \frac{1}{2}\|y\|^2 \geq \frac{1}{2}\|x\|^2 - \frac{1}{4}(\|x\|+\|y\|)^2 + \frac{1}{2}\|y\|^2 = \frac{1}{4}(\|x\|-\|y\|)^2 \geq \frac{\varepsilon^2}{16}$.

Case II: Let us put $k := \frac{\|x\|}{\|y\|} \geq 1$ (note that $y \neq 0$) and observe that

$$\|y\|(u+v+(k-1)u) = \|y\|\left(\frac{x}{\|x\|} + \frac{y}{\|y\|} + \frac{\|x\|-\|y\|}{\|y\|}\frac{x}{\|x\|}\right) = x+y.$$

Using this, and also that $\left(\frac{1+k^2}{2}\right)^{\frac{1}{2}} \geq \frac{k+1}{2}$, we obtain

$$\frac{1}{2}\|x\|^2 - \|\frac{x+y}{2}\|^2 + \frac{1}{2}\|y\|^2 = \frac{1}{2}\|y\|^2\left(\frac{\|x\|}{\|y\|}\right)^2 - \left(\|y\|\left\|\frac{u+v+(k-1)u}{2}\right\|\right)^2 + \frac{1}{2}\|y\|^2$$

$$\geq \|y\|^2\left(\frac{1}{2}k^2 - \left(\left\|\frac{u+v}{2}\right\| + \frac{k-1}{2}\right)^2 + \frac{1}{2}\right) = \|y\|^2\left(\frac{1+k^2}{2} - \left(\left\|\frac{u+v}{2}\right\| + \frac{k-1}{2}\right)^2\right) =$$

$$= \|y\|^2\left[\left(\frac{1+k^2}{2}\right)^{\frac{1}{2}} + \frac{k-1}{2} + \left\|\frac{u+v}{2}\right\|\right]\left[\left(\frac{1+k^2}{2}\right)^{\frac{1}{2}} - \frac{k-1}{2} - \left\|\frac{u+v}{2}\right\|\right] \geq$$

$$\geq \|y\|^2\, k\left(1 - \left\|\frac{u+v}{2}\right\|\right) = \|x\|\,\|y\|\left(1 - \left\|\frac{u+v}{2}\right\|\right).$$

Case III: $\frac{1}{2}\|x\|^2 - \left\|\frac{x+y}{2}\right\|^2 + \frac{1}{2}\|y\|^2 \geq \|x\|\,\|y\|\left(1 - \left\|\frac{u+v}{2}\right\|\right)$
(as in case II, by interchanging x and y).

In case I we are done. In cases II and III we must estimate $\|x\|\,\|y\|(1 - \|\frac{u+v}{2}\|)$. First note that in both these cases

(12.16) $$\|u-v\| = \frac{1}{\|y\|}\left\|x-y - \frac{\|x\|-\|y\|}{\|x\|}x\right\| \geq \frac{1}{\|y\|}(\|x-y\| - |\|x\|-\|y\||) \geq \frac{\varepsilon}{2\|y\|} \geq \frac{\varepsilon}{2}.$$

Furthermore, $\|y\| \geq \|x-y\|-\|x\| \geq \varepsilon-\|x\| \geq \varepsilon-(\|y\|+\frac{\varepsilon}{2})$, so that

(12.17) $\|y\| \geq \frac{\varepsilon}{4}$ and, similarly, $\|x\| \geq \frac{\varepsilon}{4}$.

(12.16) combined with (12.15) yields $1 - \|\frac{u+v}{2}\| > \delta$, and therefore, using (12.17), we obtain in both cases II and III that

$$\frac{1}{2}\|x\|^2 - \|\frac{x+y}{2}\|^2 + \frac{1}{2}\|y\|^2 \geq \frac{\varepsilon^2}{16}\delta > 0. \qquad \square$$

Let f_0 and g_0 be two convex functions on a vector space X which are homogeneous of degree 2 and vanish precisely at the origin. Let us assume they are equivalent in the sense that there exists a positive C such that

(12.18) $$g_0 \leq f_0 \leq (1+C)g_0.$$

We now average f_0 and g_0 in two different ways:

$$f_1 = \tfrac{1}{2}(f_0 + g_0)$$

$$g_1(x) = \inf\{\tfrac{1}{2}(f_0(x+y) + g_0(x-y)) : y \in X\} \quad (x \in X).$$

We claim that both these averages f_1 and g_1 are again convex and homogeneous of degree 2, vanish exactly at 0 and satisfy

$$g_0 \le g_1 \le f_1 \le f_0, \tag{12.19}$$

$$f_1 \le (1 + 2^{-1}C)g_1. \tag{12.20}$$

Indeed, since $g_0 \le f_0$ and g_0 is convex, we get for every $x \in X$,

$$g_1(x) = \inf\left\{\frac{f_0(x+y) + g_0(x-y)}{2} : y \in X\right\} \ge \inf\left\{\frac{g_0(x+y) + g_0(x-y)}{2} : \ y \in X\right\}$$

$$\ge g_0(x).$$

Hence $g_0 \le g_1$. Also, since $f_0 \le (1+C)g_0$, we have

$$f_1 = \frac{f_0 + g_0}{2} \le \frac{(2+C)}{2}g_0 \le (1 + 2^{-1}C)g_1.$$

The remaining inequalities in (12.19) are clear, as is the homogeneity of f_1 and g_1 and the convexity of f_1. We prove now the convexity of g_1. Let $x_1, x_2 \in X$ and $0 \le \lambda \le 1$ be given. Let us choose, for an arbitrary $\varepsilon > 0$, y_1 and y_2 in X such that

$$\tfrac{1}{2}f_0(x_1+y_1) + \tfrac{1}{2}g_0(x_1-y_1) \le g_1(x_1) + \varepsilon$$

and

$$\tfrac{1}{2}f_0(x_2+y_2) + \tfrac{1}{2}g_0(x_2-y_2) \le g_1(x_2) + \varepsilon.$$

Then, by the convexity of f_0 and g_0, we have

$$\lambda g_1(x_1) + (1-\lambda)g_1(x_2) \ge \lambda[\tfrac{1}{2}f_0(x_1+y_1) + \tfrac{1}{2}g_0(x_1-y_1)] +$$

$$+ (1-\lambda)[\tfrac{1}{2}f_0(x_2+y_2) + \tfrac{1}{2}g_0(x_2-y_2)] - \varepsilon \ge \tfrac{1}{2}f_0\big(\lambda(x_1+y_1) + (1-\lambda)(x_2-y_2)\big) +$$

$$+ \tfrac{1}{2}g_0\big(\lambda(x_1-y_1) + (1-\lambda)(x_2-y_2)\big) - \varepsilon = \tfrac{1}{2}f_0\big((\lambda x_1+(1-\lambda)x_2)+(\lambda y_1+(1-\lambda)y_2)\big) +$$

$$+ \tfrac{1}{2}g_0\big((\lambda x_1+(1-\lambda)x_2) - (\lambda y_1+(1-\lambda)y_2)\big) - \varepsilon \ge g_1\big(\lambda x_1+(1-\lambda)x_2\big) - \varepsilon.$$

Thus g_1 is convex, since $\varepsilon > 0$ was arbitrary.

We now iterate this procedure and define for every $n = 0,1,2,\dots$

$$(12.21)\qquad \begin{aligned} f_{n+1} &= \tfrac{1}{2}(f_n + g_n) \\ g_{n+1}(x) &= \inf\{\tfrac{1}{2}(f_n(x+y) + g_n(x-y)) : y \in X\} \qquad (x \in X). \end{aligned}$$

In this way we get sequences $\{f_n\}$ and $\{g_n\}$ of functions, satisfying the relations

$$(12.22)\qquad g_n \le g_{n+1} \le f_{n+1} \le f_n,$$

$$(12.23)\qquad f_n \le (1 + 2^{-n}C)g_n \qquad (n = 0,1,2,..).$$

It follows that both sequences converge to a limit function h, which is obviously again convex, homogeneous of degree 2, and vanishes exactly at 0, and that

$$(12.24)\qquad (1 + 2^{-n}C)^{-1}h \le g_n \le h \le f_n \le (1 + 2^{-n}C)h \qquad (n = 0,1,\dots).$$

The estimates (12.23) and (12.24) can and must be improved for subsequent applications.

LEMMA 12.9.

$$(12.25)\qquad g_n \le f_n \le (1 + 4^{-n}C)g_n,$$

and consequently

$$(12.26)\qquad (1 + 4^{-n}C)^{-1}h \le g_n \le h \le f_n \le (1 + 4^{-n}C)h \qquad (n = 0,1,\dots).$$

PROOF. By induction on n. For $n = 0$ (12.25) is nothing but (12.18). Suppose that (12.25) has been proved for some n. Let us put $a := 1 + 4^{-n}C/2$. We have $f_{n+1} \le f_n$ and $f_{n+1} = \tfrac{1}{2}(f_n+g_n)$, so by the induction hypothesis, $f_{n+1} \le ag_n$. Thus, using homogeneity and convexity of the involved functions, we have for all $x,y \in X$,

$$\begin{aligned} &\tfrac{1}{2}(f_n(x+y) + g_n(x-y)) \ge \tfrac{1}{2}\left(\frac{1}{a^2}f_{n+1}(ax+ay) + \frac{1}{a}f_{n+1}(x-y)\right) = \\ &= \tfrac{1}{2}\frac{1+a}{a^2}\left(\frac{1}{1+a}f_{n+1}(ax+ay) + \frac{a}{1+a}f_{n+1}(x-y)\right) \ge \frac{1+a}{2a^2}f_{n+1}\left(\frac{2ax}{1+a}\right) = \\ &= \frac{2}{1+a}f_{n+1}(x). \end{aligned}$$

Taking the inf over all $y \in X$ yields

$$f_{n+1} \le (1+4^{-(n+1)}C)g_{n+1},$$

completing the proof. □

We show now that the limit function h inherits uniform convexity from f_0.

PROPOSITION 12.10. *If f_0 is uniformly convex, then so is h.*

PROOF. The inductive definition of the f_n shows that $f_n = \frac{f_0}{2^n} + h_n$, where h_n is the convex function $\sum_{k=0}^{n-1} \frac{g_k}{2^{n-k}}$. By Lemma 12.9 we have, for all n,

$$0 \le f_n - h \le f_n - g_n \le 4^{-n}Cg_n \le 4^{-n}Cf_0,$$

so

$$\left(\frac{1}{2^n} - \frac{C}{4^n}\right)f_0 + h_n \le h \le \frac{1}{2^n} f_0 + h_n. \tag{12.27}$$

It follows from (12.27) and the convexity of h_n that, for all $x,y \in X$ and all $n = 0,1,\ldots,$

$$\begin{aligned} h(x) - 2h(\tfrac{x+y}{2}) + h(y) &\ge \left[\frac{1}{2^n}f_0(x) - \frac{C}{4^n}f_0(x) + h_n(x)\right] - \\ &- 2\left[\frac{1}{2^n}f_0(\tfrac{x+y}{2}) + h_n(\tfrac{x+y}{2})\right] + \left[\frac{1}{2^n}f_0(y) - \frac{C}{4^n}f_0(y) + h_n(y)\right] \ge \\ &\ge \frac{1}{2^n}\left[f_0(x) - 2f_0(\tfrac{x+y}{2}) + f_0(y) - \frac{C}{2^n}(f_0(x)+f_0(y))\right]. \end{aligned} \tag{12.28}$$

Let us fix $\varepsilon > 0$. By assumption

$$\inf\{f_0(x) - 2f_0(\tfrac{x+y}{2}) + f_0(y) : \|x\|,\|y\| \le 1,\ \|x-y\| \ge \varepsilon\}$$

is positive, where $\|\ \|$ is related to f_0 by (12.10). Hence, taking n sufficiently large it follows from (12.28) that

$$\inf\{h(x) - 2h(\tfrac{x+y}{2}) + h(y) : \|x\|,\|y\| \le 1,\ \|x-y\| \ge \varepsilon\} > 0. \tag{12.29}$$

This holds for every $\varepsilon > 0$. Clearly then, since the norm related to h by (12.10) is equivalent to $\|\ \|$, h is uniformly convex (see the observation immediately preceding Proposition 12.8). □

As a final step before we can apply this averaging procedure, we need to know how the "conjugate" functions f_n^*, g_n^* and h^* are related. It is immediate from (12.26) and the fact that conjugate functions correspond to dual norms, that on X^* (= the dual of X for any of the equivalent norms corresponding to the equivalent functions f_n, g_n, h)

$$(12.30)\qquad (1+4^{-n}C)^{-1}h^* = ((1+4^{-n}C)h)^* \le f_n^* \le h^* \le g_n^* \le ((1+4^{-n}C)^{-1}h)^* = = (1+4^{-n}C)h^*.$$

In particular, both $\{f_n^*\}$ and $\{g_n^*\}$ converge to h^*. More interesting, and deeper, is the fact that the relations (12.21) are inverted upon passing to conjugate functions.

PROPOSITION 12.11. *For all* $n = 0,1,\ldots$ *and all* $x^* \in X^*$ *we have*

$$(12.31)\qquad f_{n+1}^*(x^*) = \inf\{\tfrac{1}{2}(f_n^*(x^*+y^*) + g_n^*(x^*-y^*)): y^* \in X^*\},$$

$$(12.32)\qquad g_{n+1}^* = \tfrac{1}{2}(f_n^* + g_n^*).$$

PROOF. Fix n. We know that $f_{n+1} = \frac{1}{2}(f_n+g_n)$. Let $\| \ \|_1$ and $\| \ \|_2$ denote the norms corresponding to f_n and g_n, respectively. Then the norm $||| \ |||$ corresponding to f_{n+1} satisfies

$$\tfrac{1}{2}|||x|||^2 = \tfrac{1}{4}(\|x\|_1^2 + \|x\|_2^2) \qquad (x \in X).$$

With these notations (and using the same symbol for a norm and its dual) (12.31) means

$$(12.33)\qquad \tfrac{1}{2}|||x^*|||^2 = \inf\{\tfrac{1}{4}\|x^*+y^*\|_1^2 + \tfrac{1}{4}\|x^*-y^*\|_2^2: \ y^* \in X^*\}.$$

To prove this we introduce the product space $Z := X\times X$ equipped with the norm $\|(x,y)\| := \sqrt{\|x\|_1^2 + \|y\|_2^2}$ $(x,y \in X)$ and consider the isometric embedding $I: (X, ||| \ |||) \to Z$ defined by

$$Ix = (\frac{1}{\sqrt{2}}x, \frac{1}{\sqrt{2}}x) \qquad (x \in X).$$

The adjoint $I^*: Z^* = X^*\times X^* \to (X, |||\cdot|||)^*$ evidently satisfies

$$I^*(x^*,y^*) = \frac{1}{\sqrt{2}}x^* + \frac{1}{\sqrt{2}}y^*.$$

Also it is clear that $\ker I^* = (\operatorname{im} I)^\perp = \{(y^*,-y^*): y^* \in X^*\}$. Since I^* defines an isometry of $Z^*/\ker I^*$ onto $(X, ||| \ |||)^*$, we have, for every $x^* \in X^*$,

$$\|\!| x^{*} |\!\| = \inf\{\| (\tfrac{1}{2}\sqrt{2}\; x^{*}, \tfrac{1}{2}\sqrt{2}\; x^{*}) + (y^{*}, -y^{*})\| : y^{*} \in X^{*}\} =$$

$$= \inf\{(\|\tfrac{1}{\sqrt{2}}\; x^{*}+y^{*}\|_1^2 + \|\tfrac{1}{\sqrt{2}}\; x^{*}-y^{*}\|_2^2)^{\frac{1}{2}} : y^{*} \in X^{*}\} =$$

$$= \inf\{(\tfrac{1}{2}\|x^{*}+y^{*}\|_1^2 + \tfrac{1}{2}\|x^{*}-y^{*}\|_2^2)^{\frac{1}{2}} : y^{*} \in X^{*}\}.$$

Hence

$$\tfrac{1}{2}\|\!| x^{*} |\!\|^2 = \inf\{\tfrac{1}{4}\|x^{*}+y^{*}\|_1^2 + \tfrac{1}{4}\|x^{*}-y^{*}\|_2^2 : y^{*} \in X^{*}\},$$

i.e. (12.33). Finally, (12.32) follows similarly by a dual argument. □

We are now prepared to show how these methods can be applied to obtain from two equivalent norms a new norm (by mixing the two) which combines the differentiability and convexity properties each of them may have. We restrict attention here to uniform smoothness and uniform convexity.

THEOREM 12.12. *Let X be a Banach space with two equivalent norms* $\|\ \|_1$ *and* $\|\ \|_2$ *and suppose that* $\|\ \|_1$ *is uniformly convex and* $\|\ \|_2$ *is uniformly smooth. Then there exists on X a third norm* $\|\ \|_3$*, equivalent to* $\|\ \|_1$ *and* $\|\ \|_2$*, which is both uniformly convex and uniformly smooth.*

PROOF. Let f_0 and g_0 be the convex functions corresponding to $\|\ \|_1$ and $\|\ \|_2$, respectively. We may assume that $g_0 \le f_0 \le (1+C)g_0$, for some $C > 0$. Let h be the function that results from the averaging procedure described above. By Proposition 12.8 f_0 is uniformly convex, and so is g_0^{*}, by Proposition 12.8 and 12.6 (i). Now Proposition 12.10 implies that h is uniformly convex, and by the same result applied to the conjugate functions, it follows, using Proposition 12.11 as well, that also h^{*} is uniformly convex. Hence the norm $\|\ \|_3$ corresponding to h and its dual (which corresponds to h^{*}) are both uniformly convex. Thus, by the other half of Proposition 12.6 (i), $\|\ \|_3$ is uniformly convex and uniformly smooth. □

NOTES. The material on uniform convexity and uniform smoothness is classical. Uniformly convex spaces were introduced by J.A. CLARKSON ([17]), who also proved the uniform convexity of the L^p-spaces $(1 < p < \infty)$. The precise value of the modulus of convexity for L^p-spaces was calculated by O. HANNER ([40]). For a more complete discussion of differentiability and convexity of norms we refer to [69], [27] and [24]. M.M. DAY's monograph contains an encyclopedic account of what is known in this area. Proposition 12.3 is independently due to D.P. MILMAN ([78]) and B.J. PETTIS ([83]). The proof given here was taken from [75]. For still other proofs of this result see [27]. The averaging procedure described in the second half of this section (which we have narrowed down here to suit our specific needs) is due to E. ASPLUND ([2]).

13. THE PROBLEM OF UNIFORM CONVEXIFIABILITY

Which Banach spaces admit an equivalent uniformly convex norm, or, shortly, are uniformly convexifiable? The problem of characterizing this class of spaces has long been open. In this section we do a bit of reconnoitering in the hope of getting some feeling for the direction in which the solution should be sought. The latter will be given in subsequent sections.

Obviously, by Proposition 12.3, the class of uniformly convexifiable spaces is a subclass of that of the reflexive spaces. We shall see later that it is a proper subclass. We prove a preliminary result here that provides some insight into this class. For this we need the notion of finite representability.

DEFINITION 13.1. Let X and Y be Banach spaces. We say that Y is *finitely representable* in X, notation $Y \prec X$, if for every $\varepsilon > 0$ and for every finite-dimensional subspace $F \subset Y$ there exists an isomorphism T from F into X satisfying

$$(13.1) \qquad (1-\varepsilon)\|y\| \le \|Ty\| \le (1+\varepsilon)\|y\| \quad \text{for all } y \in F.$$

Roughly speaking, $Y \prec X$ means that of every finite-dimensional subspace of Y there exists an "almost isometric" copy in X. Clearly the relation $\prec$ is reflexive and transitive. We shall see below that it is not antisymmetric: it occurs that $Y \neq X$, while $X \prec Y$ and $Y \prec X$.

The following result is a kind of stability property of the class of uniformly convexifiable spaces. The condition that Y is separable is not essential (as will become clear later), but we need it in the present proof.

PROPOSITION 13.2. *Let X and Y be Banach spaces such that X is uniformly convexifiable, Y separable, and $Y \prec X$. Then Y is uniformly convexifiable.*

PROOF. With $\|\cdot\|$ we denote the given norms on X and Y. Let $\|\cdot\|_0$ be an equivalent uniformly convex norm on X. Thus, for some c, C > 0,

$$c\|x\| \le \|x\|_0 \le C\|x\| \quad \text{for all } x \in X. \tag{13.2}$$

Since Y is separable there exists a dense sequence $\{y_k\}$ in Y. Putting $F_n := [y_k]_{k=1}^n$ (n = 1,2,...), we have $Y = \overline{\bigcup_{n=1}^{\infty} F_n}$. Now let $0 < \varepsilon < 1$ be arbitrary. The assumption $Y \prec X$ implies, for every $n \in \mathbb{N}$, the existence of an isomorphism $T_n\colon F_n \to Y$ such that

$$(1-\varepsilon)\|y\| \le \|T_n y\| \le (1+\varepsilon)\|y\| \quad \text{for all } y \in F_n. \tag{13.3}$$

It follows from (13.2) and (13.3) that, with $C' := C(1+\varepsilon)$ and $c' := c(1-\varepsilon)$,

$$c'\|y\| \le \|T_n y\|_0 \le C'\|y\| \quad \text{for all } y \in F_n. \tag{13.4}$$

Hence, for every $n \in \mathbb{N}$, we can define an equivalent uniformly convex norm $\|\cdot\|_n$ on F_n by

$$\|y\|_n = \|T_n y\|_0 \quad (y \in F_n). \tag{13.5}$$

Then, by (13.4), we have

$$c'\|y\| \le \|y\|_n \le C'\|y\| \quad \text{for all } y \in F_n, \tag{13.6}$$

where it is important to observe that the constants c' and C' are independent of n. We now construct the desired equivalent uniformly convex norm on Y by a limit process. For a fixed $k \in \mathbb{N}$ the sequence $\{\|y_k\|_n\}_{n=1}^{\infty}$ is defined for $n \ge k$ and bounded by (13.6). Hence, by a diagonal procedure, we can find a subsequence $\{n_\ell\}$ of $\mathbb{N}$ such that

$$\{\|y_k\|_{n_\ell}\}_{\ell=1}^{\infty} \tag{13.7}$$

converges for every $k \in \mathbb{N}$. Clearly, by the density of $\{y_k\}$ in Y and by (13.6), it follows that $\{\|y\|_{n_\ell}\}_{\ell=1}^{\infty}$ converges for every $y \in \bigcup_{n=1}^{\infty} F_n$. Let us define

$$|||y||| = \lim_{\ell\to\infty} \|y\|_{n_\ell}, \quad \text{for all } y \in \bigcup_{n=1}^{\infty} F_n. \tag{13.8}$$

By (13.6) $|||\cdot|||$ is a norm on $\bigcup_{n=1}^{\infty} F_n$ which is equivalent to the given norm $\|\ \|$:

$$c'\|y\| \le |||y||| \le C'\|y\| \quad \text{for all } y \in \bigcup_{n=1}^{\infty} F_n. \tag{13.9}$$

Therefore it can be extended uniquely to an equivalent norm $|||\ |||$ on Y. Finally we show that $|||\ |||$ is uniformly convex. Since for every $n \in \mathbb{N}$, $(F_n, \|\cdot\|_n)$ is isometric to a subspace of $(X, \|\ \|_0)$ (via T_n), it is obvious that for the respective moduli of convexity we have

$$(13.10) \qquad \delta_{(F_n,\|\cdot\|_n)}(\varepsilon) \leq \delta_{(X,\|\cdot\|_0)}(\varepsilon) > 0 \quad \text{for all } 0 < \varepsilon \leq 2 \text{ and all } n \in \mathbb{N}.$$

Let us fix $0 < \varepsilon \leq 2$. Clearly it suffices to prove that

$$(13.11) \qquad \inf\{1 - |||\tfrac{x+y}{2}|||:\ x,y \in Y,\ |||x||| < 1,\ |||y||| < 1,\ |||x-y||| > \varepsilon\} > 0.$$

By the density of $\bigcup_{n=1}^{\infty} F_n$ in Y, this number equals

$$\inf\{1 - |||\tfrac{x+y}{2}|||:\ x,y \in \bigcup_{n=1}^{\infty} F_n,\ |||x||| < 1,\ |||y||| < 1,\ |||x-y||| > \varepsilon\}.$$

Fixing $x,y \in \bigcup_{n=1}^{\infty} F_n$ with $|||x||| < 1$, $|||y||| < 1$ and $|||x-y||| > \varepsilon$, it follows from (13.8) that for sufficiently large ℓ we have

$$\|x\|_{n_\ell} < 1,\ \|y\|_{n_\ell} < 1,\ \|x-y\|_{n_\ell} > \varepsilon.$$

Hence, by (13.10),

$$1 - |||\tfrac{x+y}{2}||| = 1 - \lim_{\ell\to\infty}\|\tfrac{x+y}{2}\|_{n_\ell} \geq \delta_{(X,\|\ \|_0)}(\varepsilon) > 0.$$

Thus (13.11) holds and the proof is complete. □

<u>REMARK 13.3</u>. An examination of the proof shows that we have not fully used the assumption $Y \prec X$. In fact we need only the existence of two positive constants d and D such that for every finite-dimensional subspace F of Y there exists an isomorphism T of F into X satisfying

$$d\|y\| \leq \|Ty\| \leq D\|y\| \quad \text{for all } y \in Y.$$

R.C. James has expressed this condition by saying that Y is *crudely finitely representable* in Y.

We now give some examples of finite representability.

<u>EXAMPLE 1</u>. $X^{**} \prec X$ for every Banach space X. This is in fact a much weaker property than the already proved principle of local reflexivity (Theorem 3.1).

<u>EXAMPLE 2</u>. $X \prec c_0$ for every Banach space X. Indeed, let X, a finite-dimensional subspace $F \subset X$ and $\varepsilon > 0$ be given. By compactness S_{F^*} has a finite ε-net $\{x_1^*,\dots,x_n^*\}$. We now define $T: F \to c_0$ by

$$Tx = (\langle x,x_1^*\rangle,\dots,\langle x,x_n^*\rangle,0,0,\dots) \qquad (x \in F).$$

Clearly $\|T\| \le 1$ and it remains to be shown that $(1-\varepsilon)\|x\| \le \|Tx\|$ for all $x \in F$. Suppose $\|x\| = 1$. Then there exists an $x^* \in S_{F^*}$ such that $\langle x,x^*\rangle = 1$. Let x_i^* $(1 \le i \le n)$ be such that $\|x^*-x_i^*\| < \varepsilon$. Then $\langle x,x_i^*\rangle = \langle x,x^*\rangle - \langle x,x^*-x_i^*\rangle \ge 1-\varepsilon$, so $\|Tx\| \ge 1-\varepsilon = (1-\varepsilon)\|x\|$.

<u>EXAMPLE 3</u>. $X \prec (\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ for every Banach space X. Indeed, observe that the above proof that $X \prec c_0$ in fact shows the following: given any finite-dimensional Banach space F and any $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ (namely, the cardinality of an ε-net $\{x_1^*,\dots,x_n^*\}$ for S_{F^*}) and an isomorphism T_n from F into $(c_0)_n$ (defined by $T_nx = (\langle x,x_1^*\rangle,\dots,\langle x,x_n^*\rangle)$ $(x \in F)$) satisfying $(1-\varepsilon)\|x\| \le \|T_nx\| \le 1$ for all $x \in X$. It now suffices to note that the space $(\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ contains each $(c_0)_n$ isometrically.

Combining the Examples 2 and 3, we see that

$$c_0 \prec \Big(\sum_{n=1}^{\infty} \oplus (c_0)_n\Big)_{\ell^2} \quad \text{and} \quad \Big(\sum_{n=1}^{\infty} \oplus (c_0)_n\Big)_{\ell^2} \prec c_0.$$

Roughly, this means that c_0 and $(\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ have the "the same" finite-dimensional subspaces, in the sense that any finite-dimensional subspace of one of them can be embedded as "nicely" as we wish into the other. Nevertheless c_0 is non-reflexive (even "very much" so, since its bidual ℓ^∞ is non-separable), whereas $(\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ is reflexive (see Remark 0.20). This illustrates clearly that information concerning the finite-dimensional subspaces does not suffice to decide the question of reflexivity. Evidently it is the way in which the finite-dimensional subspaces are put together that is crucial. These remarks also explain why all the characterizations of reflexivity that we have seen so far (notably those in section 10) are essentially infinite-dimensional, i.e. involve infinite-dimensional subspaces.

The situation is different for uniform convexifiability. First of all let us note that uniform convexity is a property of a finite-dimensional (even 2-dimensional) nature. Indeed, if we know all 2-dimensional subspaces F of a given Banach space X, we can decide the question of uniform convexity since obviously

$$\delta_X(\varepsilon) = \inf\{\delta_F(\varepsilon) : F \subset X, \dim F = 2\} \quad (0 < \varepsilon \le 2).$$

It is not at all obvious that knowledge of all finite-dimensional subspaces of X is enough to decide on uniform convexifiability. Some hope that this is so can be derived from Proposition 13.2, at least in the case of separable spaces: in fact in Proposition 13.2 the only information we have about Y concerns its finite-dimensional subspaces, and the conclusion is that Y is uniformly convexifiable.

The above remarks and the fact that uniformly convexifiable spaces are reflexive, make it plausible that finite-dimensional versions P_2, P_3, P_4 of the properties $P_2^\infty, P_3^\infty, P_4^\infty$ (which characterize non-reflexivity) might characterize the class of spaces which fail to be uniformly convexifiable. Also a finite-dimensional version P_1 of P_1^∞ should be considered. We shall show in the next few sections that indeed uniform convexifiability is equivalent to the negation of each of these (yet to be defined) properties P_1, P_2, P_3, P_4.

NOTES. Almost all work on the problem of uniform convexifiability was done by R.C. JAMES. He introduced finite representability, super-reflexivity and the properties P_1, P_2, P_3, P_4 in the course of his investigations. The paper [54] shows how far he got. The missing link was provided by P. ENFLO ([33]). Proposition 13.2 is a variant of Lemma 1.1 in [52].

14. THE FINITE TREE PROPERTY

We prove in this section that a Banach space is uniformly convexifiable iff it does not have the finite tree property.

DEFINITION 14.1. A Banach space X is said to have the property P_1 (= the *finite tree property*) iff

$$\exists\varepsilon > 0 \quad \forall n \in \mathbb{N} \quad B_X \text{ contains an } (n,\varepsilon)\text{-tree.}$$

REMARK 14.2. Clearly $P_1^\infty \Rightarrow P_1$, but the converse is not true. In fact we have already seen (although not proved) that ℓ^1 does not have the infinite tree property (Remark 11.7). However, ℓ^1 has the finite tree property, since it is easily verified that for every $n \in \mathbb{N}$ $T = \{e_1,\ldots,e_{2^n}\}$ is an $(n,2)$-tree in B_{ℓ^1} ($\{e_n\}$ denotes the standard basis for ℓ^1).

REMARK 14.3. P_1 is an isomorphic invariant (as is P_1^∞). Indeed, if A is an isomorphism from a Banach space X onto a Banach space Y and T is an ε-tree (finite or infinite) in B_X, then it is easily checked that $\frac{A}{\|A\|}T$ is an $\frac{\varepsilon}{\|A\|\|A^{-1}\|}$- tree in B_Y.

PROPOSITION 14.4. *A uniformly convexifiable Banach space* X *does not have* P_1.

PROOF. By the preceding remark we may assume that X is uniformly convex. Let us suppose that $T = \{x_1,\ldots,x_{2^n}\}$ is an (n,ε)-tree in B_X, for some $n \in \mathbb{N}$ and some $\varepsilon > 0$. We shall establish an upper bound for n in terms of ε and the modulus of convexity $\delta_X(\cdot)$, thus proving that P_1 fails to hold. Recall that $\delta(\varepsilon) = \inf\{1 - \|\frac{x+y}{2}\|: \|x\|,\|y\| \le 1,\ \|x-y\| \ge \varepsilon\}$ $(0 < \varepsilon \le 2)$. From this it follows that we have the following implication for all $x,y \in X$:

$$(14.1) \qquad \|x-y\| \ge \varepsilon \max(\|x\|,\|y\|) \Rightarrow \max(\|x\|,\|y\|) \ge (1-\delta(\varepsilon))^{-1}\|\tfrac{1}{2}(x+y)\|.$$

Indeed, putting $M := \max(\|x\|,\|y\|)$, the left member of (14.1) says $\|\frac{x}{M}-\frac{y}{M}\| \ge \varepsilon$.

Since $\|\frac{x}{M}\|$, $\|\frac{y}{M}\| \le 1$, the definition of $\delta(\cdot)$ implies $1 - \|\frac{x+y}{2M}\| > \delta(\varepsilon)$, which is the right member of (14.1). Now (14.1) has the following consequence for the derived trees (cf. Definition 11.1*) $T' = \{\frac{1}{2}(x_1+x_2), \frac{1}{2}(x_3+x_4), \ldots, \frac{1}{2}(x_{2^n-1}+x_{2^n})\}$, $T'' := (T')', \ldots, T^{(n)} := (T^{(n-1)})'$:

$$(14.2) \qquad 1 \ge \|T\| \ge (1-\delta(\varepsilon))^{-1}\|T'\| \ge (1-\delta(\varepsilon))^{-2}\|T''\| \ge \ldots$$
$$\ge (1-\delta(\varepsilon))^{-(n-1)}\|T^{(n-1)}\|.$$

$T^{(n-1)}$ consists of two points with distance $\ge \varepsilon$. Therefore $\|T^{(n-1)}\| \ge \frac{\varepsilon}{2}$, so that (14.2) implies

$$1 \ge (1-\delta(\varepsilon))^{-(n-1)}\,\frac{\varepsilon}{2}.$$

Since $\delta(\varepsilon) > 0$, we see that for fixed ε, n is bounded above and we are done.□

We have now proved one half of the main

<u>THEOREM 14.5</u>. *A Banach space is uniformly convexifiable iff it does not have the finite tree property.*

The proof of the other half will be accomplished in a series of lemmas. We need the following inductive

<u>DEFINITION 14.6</u>. Let X be a Banach space and let $x \in X$ and $\varepsilon > 0$.

(i) The single element x is a $(0,\varepsilon)$- *partition* of x.

(ii) An ordered 2^n-tuple $P = (x_1,\ldots,x_{2^n})$ ($n \in \mathbb{N}$) is called an (n,ε)-*partition* of x iff

$$x = \sum_{i=1}^{2^n} x_i, \quad \|x_{2j-1}\| = \|x_{2j}\|, \quad \left\|\frac{x_{2j-1}}{\|x_{2j-1}\|} - \frac{x_{2j}}{\|x_{2j}\|}\right\| \ge \varepsilon \qquad (j = 1,\ldots,2^{n-1})$$

and $P' := (x_1+x_2, x_3+x_4, \ldots, x_{2^n-1}+x_{2^n})$ is an $(n-1,\varepsilon)$-partition of x. P' is called the *derived partition* of P.

<u>LEMMA 14.7</u>. *If X is a Banach space which does not have P_1 then for every $\varepsilon > 0$ there exist an $n \in \mathbb{N}$ and a $\delta > 0$, such that for every $x \in X$ and every (n,ε)-partition $P = (x_1,\ldots,x_{2^n})$ of x,*

$$\sum_{i=1}^{2^n} \|x_i\| \ge (1+\delta)\|x\|.$$

<u>PROOF</u>. Let $\varepsilon > 0$ be given. Choose $\delta' > 0$ arbitrarily. Since by assumption

X does not have P_1, there exists an $n \in \mathbb{N}$ such that B_X contains no $(n,\frac{\varepsilon}{2(1+\delta')})$-tree, i.e.

(14.3) there exists no $(n,\frac{\varepsilon}{2})$-tree with norm $\le 1+\delta'$.

We claim that n and $\delta := 2^{-n}\delta'$ satisfy the requirements. To see this, let $P = (x_1,\ldots,x_{2^n})$ be an (n,ε)-partition for some $x \in X$. Without loss of generality we may assume that $\|x\| = 1$. Let $P^{(0)} := P$, $P^{(1)} := P'$, $P^{(2)} := (P')',\ldots,P^{(n)} := (P^{(n-1)})'$ be the derived partitions and let us write $P^{(k)} = (x_1^{(k)},x_2^{(k)},\ldots,x_{2^{n-k}}^{(k)})$ $(k = 0,\ldots,n)$. Then

$$(14.4) \qquad \|x_j^{(k)}\| \ge 2^{-(n-k)} \qquad (k = 0,\ldots,n;\ \ j = 1,\ldots,2^{n-k}),$$

as is easily seen by induction, using the triangle inequality, the assumption $\|x\| = 1$, and $\|x_{2j-1}^{(k)}\| = \|x_{2j}^{(k)}\|$. We now claim that $T := \{2^n x_1, 2^n x_2,\ldots$ $\ldots 2^n x_{2^n}\}$ is an $(n,\frac{\varepsilon}{2})$-tree and that all elements of all $T^{(k)}$ $(k = 0,\ldots,n-1)$ have norm ≥ 1. The assertion about the norms clearly follows from (14.4) and the obvious relation $T^{(k)} = 2^{n-k}P^{(k)}$ $(k = 0,\ldots,n-1)$. To see that T is an $\frac{\varepsilon}{2}$-tree, let $k \in \{0,\ldots,n-1\}$ and let $z_{2j-1}^{(k)} := 2^{n-k}x_{2j-1}^{(k)}$ and $z_{2j}^{(k)} := 2^{n-k}x_{2j}^{(k)}$ be two adjoining points of $T^{(k)}$. By Definition 14.6 we have

$$\left\| \frac{x_{2j-1}^{(k)}}{\|x_{2j-1}^{(k)}\|} - \frac{x_{2j}^{(k)}}{\|x_{2j}^{(k)}\|} \right\| = \left\| \frac{z_{2j-1}^{(k)}}{\|z_{2j-1}^{(k)}\|} - \frac{z_{2j}^{(k)}}{\|z_{2j}^{(k)}\|} \right\| \ge \varepsilon.$$

Since also $\|z_{2j-1}^{(k)}\|,\|z_{2j}^{(k)}\| \ge 1$, as we have already seen, it easily follows that $\|z_{2j-1}^{(k)} - z_{2j}^{(k)}\| \ge \frac{\varepsilon}{2}$, proving our claim. Now by (14.3) at least one of the points $2^n x_i$ $(i = 1,\ldots,2^n)$ has norm $> 1+\delta'$. On the other hand all points $2^n x_i$ have norm ≥ 1. Thus we have, dividing by 2^n,

$$\sum_{i=1}^{2^n} \|x_i\| \ge 2^{-n}(1+\delta') + (2^n-1)2^{-n} = 1 + 2^{-n}\delta' =$$

$$= 1 + \delta = (1+\delta)\|x\|. \qquad \square$$

<u>DEFINITION 14.8</u>. A real function $|\cdot|$ on a vector space X is called an *ecart* if

(i) $|x| \ge 0$ for all $x \in X$ and $|x| = 0$ iff $x = 0$;

(ii) $|\lambda x| = |\lambda||x|$ for all $\lambda \in \mathbb{R}$ and $x \in X$.

What we are trying to do is to define an equivalent uniformly convex norm on a Banach space X which does not have P_1. A first step towards this goal is taken in the next lemma, where an ecart is defined with a property reminiscent of uniform convexity.

LEMMA 14.9. *Suppose that* X *is a Banach space which does not have* P_1. *Let* $\varepsilon > 0$ *be arbitrary and let* $n \in \mathbb{N}$ *and* $\delta > 0$ *be determined as in Lemma* 14.7. *We assume, as we clearly may, that* $0 < \delta < \frac{1}{8}$. *Then there exists an ecart* $|\cdot|$ *on* X *and a* $\delta_1 > 0$ *such that*

(a) $(1-\delta)\|x\| \le |x| \le (1-\frac{\delta}{3})\|x\|$ *for all* $x \in X$;

(b) $\|x\| = \|y\| = 1,\ \|x-y\| \ge \varepsilon \Rightarrow |x+y| < |x|+|y| - \delta_1 \quad (x,y \in X)$.

PROOF. We define, for every $x \in X$,

$$(14.5)\qquad |x| := \inf\left\{\frac{\sum_{i=1}^{2^m} \|x_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^m})} : \begin{array}{l} 0 \le m \le n \text{ and } (x_1,\ldots,x_{2^m}) \\ \text{an } (m,\varepsilon)\text{-partition of } x \end{array}\right\}.$$

Then $|\cdot|$ is an ecart satisfying (a). Indeed, taking $m = 0$ in (14.5), we obtain $|x| \le \frac{\|x\|}{1+\delta/2} < (1-\frac{\delta}{3})\|x\|$ (since $\delta < 1$). Also, by the triangle inequality, for every (m,ε)-partition $(x_1,\ldots,x_{2^m})$ of x,

$$\frac{\sum_{i=1}^{2^m} \|x_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^m})} \ge \frac{\|x\|}{1+\frac{\delta}{2}(1+\ldots+\frac{1}{4^m})} \ge \frac{\|x\|}{1+\delta} > (1-\delta)\|x\|.$$

This proves (a). The fact that $|\cdot|$ is an ecart is now trivial.

For the proof of (b) let us first observe the following. If $(x_1,\ldots,x_{2^n})$ is an (n,ε)-partition of x, then by Lemma 14.7 and the choice of n and δ,

$$\frac{\sum_{i=1}^{2^n} \|x_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^m})} \ge \frac{(1+\delta)\|x\|}{1+\delta} = \|x\|.$$

Also $\|x\| > |x|$ if $x \ne 0$, by (a). Thus in (14.5) we may take the infimum over all $m \in \{0,1,\ldots,n-1\}$, without changing $|\cdot|$. Now let $x,y \in X$ be such that $\|x\| = \|y\| = 1$ and $\|x-y\| \ge \varepsilon$. By the above observation, there exist an (m,ε)-partition $P_1 = (x_1,\ldots,x_{2^m})$ of x and a (k,ε)-partition $P_2 = (y_1,\ldots,y_{2^k})$ of y, with $0 \le m < n$, $0 \le k < n$ such that

$$(14.6)\qquad |x| > \frac{\sum_{i=1}^{2^m} \|x_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^m})} - \gamma;\quad \frac{\sum_{i=1}^{2^k} \|y_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^k})} - \gamma,\ \ldots$$

for some $0 < \gamma < \frac{\delta}{4^{2n+1}}$. For reasons of symmetry we may assume $0 \le m \le k < n$.

Let $P_2^{(k-m)} = (w_1, w_2, \dots, w_{2^m})$. Then, since

$$(14.7) \qquad \sum_{i=1}^{2^k} \|y_i\| \geq \sum_{i=1}^{2^m} \|w_i\|$$

and

$$(14.8) \qquad 1+\tfrac{1}{4}+\dots+\frac{1}{4^k} \leq 1+\tfrac{1}{4}+\dots+\frac{1}{4^m}+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}},$$

it follows from (14.6) that

$$(14.9) \qquad |y| > \frac{\sum_{i=1}^{2^m} \|w_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})} - \gamma.$$

Also, since $(x_1, x_2, \dots, x_{2^m}, w_1, w_2, \dots, w_{2^m})$ is an $(m+1,\varepsilon)$-partition of $x+y$, and $m+1 \leq n$, we have by the definition of $|\cdot|$,

$$(14.10) \qquad |x+y| \leq \frac{\sum_{i=1}^{2^m} \|x_i\| + \sum_{i=1}^{2^m} \|w_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}})}.$$

Together (14.6), (14.9) and (14.10) yield

$$(14.11) \qquad \begin{aligned} |x|+|y| - |x+y| &> \frac{\sum_{i=1}^{2^m} \|x_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^m})} - \gamma + \\ &+ \frac{\sum_{i=1}^{2^m} \|w_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})} - \gamma - \frac{\sum_{i=1}^{2^m} \|x_i\| + \sum_{i=1}^{2^m} \|w_i\|}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}})} = \\ &= \sum_{i=1}^{2^m} \|x_i\| \left(\frac{1}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^m})} - \frac{1}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}})} \right) - \\ &- \sum_{i=1}^{2^m} \|w_i\| \left(\frac{1}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}})} - \frac{1}{1+\frac{\delta}{2}(1+\frac{1}{4}+\dots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})} \right) - 2\gamma. \end{aligned}$$

Observe that, since $\|y\| = 1$, we have by (14.9), (a) and $\gamma < \frac{\delta}{4^{2n+1}}$,

$$(14.12) \qquad \begin{aligned} \sum_{i=1}^{2^m} \|w_i\| &\leq (|y|+\gamma)\left[1+\frac{\delta}{2}(1+\tfrac{1}{4}+\dots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})\right] \leq \\ &\leq (1-\frac{\delta}{3}+\frac{\delta}{4^{2n+1}})\left[1+\frac{\delta}{2}(1+\tfrac{1}{4}+\dots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})\right] = \\ &= (1-\frac{\delta}{3}+\frac{\delta}{4^{2n+1}})(1+\tfrac{2}{3}\delta) < 1+\delta. \end{aligned}$$

Using also that $\sum_{i=1}^{2^m} \|x_i\| \geq \|x\| = 1$, it follows from (14.11) and (14.12) that

$$|x|+|y| - |x+y| > \frac{\frac{\delta}{2}\frac{1}{4^{m+1}}}{[1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^m})][1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^{m+1}})]} -$$

$$- (1+\delta)\frac{\frac{\delta}{2}\frac{1}{3.4^{m+1}}}{[1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^{m+1}})][1+\frac{\delta}{2}(1+\frac{1}{4}+\ldots+\frac{1}{4^{m+1}}+\frac{1}{3.4^{m+1}})]} - 2\gamma.$$

Since all factors in the denominators are between 1 and $1+\delta$, and since

$$\gamma < \frac{\delta}{4^{2n+1}}, \quad \delta < \frac{1}{8},$$

we finally obtain

$$|x|+|y| - |x+y| > \frac{\delta}{2}\frac{1}{4^{m+1}}\frac{1}{(1+\delta)^2} - (1+\delta)\frac{\delta}{2}\frac{1}{3.4^{m+1}} - \frac{2\delta}{4^{2n+1}} >$$

$$> \frac{\delta}{2}\frac{1}{4^{m+1}}((\frac{8}{9})^2 - \frac{9}{8}\cdot\frac{1}{3} - \frac{4}{4^{2n-m}}) > \frac{\delta}{2}\frac{1}{4^{m+1}}((\frac{8}{9})^2 - \frac{3}{8} - \frac{1}{4^n}) >$$

$$> \frac{\delta}{2}\frac{1}{4^{m+1}}\frac{1}{16} = \frac{\delta}{2}\frac{1}{4^{m+3}} \geq \frac{\delta}{2}\frac{1}{4^{n+2}}.$$

Hence with $\delta_1 =: \frac{\delta}{2}\frac{1}{4^{n+2}}$ (b) holds. □

In the next and final lemma before the proof of Theorem 14.5, we improve on Lemma 14.9 by constructing out of the ecart $|\cdot|$ a norm, while essentially retaining the properties (a) and (b).

<u>LEMMA 14.10</u>. *Suppose that* X *is a Banach space which does not have* P_1. *Let* $\varepsilon > 0$ *be arbitrary and let* $n \in \mathbb{N}$ *and* $\delta > 0$ *be determined as in Lemma* 14.7. *We may and do assume that* $0 < \delta < \frac{\varepsilon}{1+\varepsilon}$ *and* $\delta < \frac{1}{8}$. *Let* $\delta_1 > 0$ *be as in Lemma* 14.9. *Then there exists a norm* $\|\ \|_{5\varepsilon}$ *on* X *such that*

(α) $(1-\delta)\|x\| \leq \|x\|_{5\varepsilon} \leq (1-\frac{\delta}{3})\|x\|$ *for all* $x \in X$;

(β) $\|x\| = \|y\| = 1,\ \|x-y\| \geq 5\varepsilon \Rightarrow \|x+y\|_{5\varepsilon} \leq \|x\|_{5\varepsilon}+\|y\|_{5\varepsilon} - \varepsilon\delta_1$ $(x,y \in X)$.

<u>PROOF</u>. Let $|\cdot|$ be the ecart of Lemma 14.9 and put $B := \{x \in X\colon |x| \leq 1\}$ (while B_X continues to denote $\{x \in X\colon \|x\| \leq 1\}$). The norm $\|\cdot\|_{5\varepsilon}$ is defined as follows:

$$(14.13) \qquad \|x\|_{5\varepsilon} = \inf\left\{\sum_{i=1}^{m} |x_i| : m \in \mathbb{N} \text{ and } x = \sum_{i=1}^{m} x_i\right\} \quad (x \in X).$$

It is easy to check that (14.13) is the gauge of co B: If $0 < \|x\|_{5\varepsilon} < 1$, then $x = \sum_{i=1}^m x_i$ with $0 < \sum_{i=1}^m |x_i| < 1$, so $x = \sum_{i=1}^m \lambda_i y_i$, where

$$\lambda_i := \frac{|x_i|}{\sum_{i=1}^m |x_i|} \geq 0 \quad \text{and} \quad y_i := \frac{x_i}{\lambda_i} \quad (i = 1,\dots,m).$$

Since $\sum_{i=1}^m \lambda_i = 1$ and $y_i \in B$ $(i = 1,\dots,m)$, it follows that $x \in$ co B. Conversely, every $x \in$ co B is of the form $x = \sum_{i=1}^m \lambda_i x_i$ with $\lambda_i \geq 0$, $\sum_{i=1}^m \lambda_i = 1$ and $x_i \in B$ $(i = 1,\dots,m)$, so $\|x\|_{5\varepsilon} \leq \sum_{i=1}^m |\lambda_i x_i| \leq \sum_{i=1}^m \lambda_i = 1$.

Formula (a) in Lemma 14.9 can be written equivalently as $\frac{1}{1-\delta/3} B_X \subset B \subset \frac{1}{1-\delta} B_X$. It follows immediately that $\frac{1}{1-\delta/3} B_X \subset \text{co } B \subset \frac{1}{1-\delta} B_X$ and this is equivalent to (α).

For the proof of (β), let $x,y \in X$ be given with $\|x\| = \|y\| = 1$ and $\|x-y\| \geq 5\varepsilon$. Choose γ such that $0 < \gamma < \min\{\delta(\frac{1}{3} - \delta),\ \delta_1 \varepsilon/4\}$. By (14.13) there exist representations $x = \sum_{i=1}^k x_i$ and $y = \sum_{i=1}^m y_i$ such that

$$(14.14) \qquad \|x\|_{5\varepsilon} > \sum_{i=1}^{k} |x_i| - \gamma \quad \text{and} \quad \|y\|_{5\varepsilon} > \sum_{i=1}^{m} |y_i| - \gamma.$$

We may assume without loss of generality that $m \leq k$ and, furthermore, that $\|x_i\| = \|y_i\|$ for $i = 1,\dots,m$. Indeed, if $\|x_1\| < \|y_1\|$, we write $y_1 = \lambda y_1 + (1-\lambda) y_1$ with $\lambda = \frac{\|x_1\|}{\|y_1\|}$. Then $\|x_1\| = \|\lambda y_1\|$. Since $|y_1| = |\lambda y_1| + |(1-\lambda)y_1|$, we can replace y_1 in (14.14) by the two elements λy_1 and $(1-\lambda)y_1$. We continue now with x_2 and $(1-\lambda)y_1$ and split either one if the norms differ. We proceed in this manner until either the x_i or the y_i are exhausted and then renumber them. Furthermore, we have

$$(14.15) \qquad 1 \leq \sum_{i=1}^{k} \|x_i\| < 1+\delta \quad \text{and} \quad 1 \leq \sum_{i=1}^{m} \|y_i\| < 1+\delta.$$

Indeed, $\sum_{i=1}^k \|x_i\| \geq \|\sum_{i=1}^k x_i\| = \|x\| = 1$ and by (a), (14.14), (α), and $\gamma < \delta(\frac{1}{3} - \delta)$,

$$\sum_{i=1}^{k} \|x_i\| \leq \frac{1}{1-\delta} \sum_{i=1}^{k} |x_i| \leq \frac{1}{1-\delta}(\|x\|_{5\varepsilon}+\gamma) \leq \frac{1}{1-\delta}((1-\frac{\delta}{3})\|x\| + \gamma) <$$
$$< \frac{1}{1-\delta}(1 - \frac{\delta}{3} + \frac{\delta}{3} - \delta^2) = 1 + \delta.$$

The proof for $\sum_{i=1}^m \|y_i\|$ is the same. Since $\|x_i\| = \|y_i\|$ $(i = 1,\dots,m)$ it follows from (14.15) that

$$(14.16) \qquad \sum_{i=m+1}^{k} \|x_i\| < \delta.$$

We now split $\{1,\dots,m\}$ in two parts:

$$J := \{i: 1 \le i \le m \text{ and } \|x_i-y_i\| < \varepsilon \|x_i\|\}$$

$$J' := \{i: 1 \le i \le m \text{ and } \|x_i-y_i\| \ge \varepsilon \|x_i\|\}.$$

Then we have, using (14.16),

$$5\varepsilon \le \|x-y\| \le \sum_{i\in J} \|x_i-y_i\| + \sum_{i\in J'} \|x_i-y_i\| + \sum_{i=m+1}^{k} \|x_i\| \le$$

$$\le \varepsilon \sum_{i\in J} \|x_i\| + \sum_{i\in J'} \|x_i-y_i\| + \delta,$$

and therefore, by (14.15) and the assumption $\delta < \frac{\varepsilon}{1+\varepsilon}$,

$$(14.17) \qquad \sum_{i\in J'} \|x_i-y_i\| \ge 5\varepsilon - \varepsilon \sum_{i=1}^{k} \|x_i\| - \delta > 5\varepsilon - \varepsilon(1+\delta)-\delta > 3\varepsilon.$$

Since

$$x+y = \sum_{i\in J'} (x_i+y_i) + \sum_{\substack{i=1 \\ i\notin J'}}^{k} x_i + \sum_{\substack{i=1 \\ i\notin J'}}^{m} y_i,$$

we have

$$(14.18) \qquad \|x+y\|_{5\varepsilon} \le \sum_{i\in J'} |x_i+y_i| + \sum_{\substack{i=1 \\ i\notin J'}}^{k} |x_i| + \sum_{\substack{i=1 \\ i\notin J'}}^{m} |y_i|.$$

Together, (14.14) and (14.18) yield

$$\|x\|_{5\varepsilon} + \|y\|_{5\varepsilon} - \|x+y\|_{5\varepsilon} \ge \sum_{i=1}^{k} |x_i| - \gamma + \sum_{i=1}^{m} |y_i| - \gamma -$$

$$(14.19) \qquad - \sum_{i\in J'} |x_i+y_i| - \sum_{\substack{i=1 \\ i\notin J'}}^{k} |x_i| - \sum_{\substack{i=1 \\ i\notin J'}}^{m} |y_i| =$$

$$= \sum_{i\in J'} (|x_i|+|y_i|-|x_i+y_i|) - 2\gamma.$$

Observe that for $i \in J'$ we have, since $\|x_i\| = \|y_i\|$,

$$\left\|\frac{x_i}{\|x_i\|}\right\| = \left\|\frac{y_i}{\|y_i\|}\right\| = 1 \quad \text{and} \quad \left\|\frac{x_i}{\|x_i\|} - \frac{y_i}{\|y_i\|}\right\| = \frac{\|x_i-y_i\|}{\|x_i\|} \geq \varepsilon,$$

so that, by (b),

$$\left|\frac{x_i}{\|x_i\|} + \frac{y_i}{\|y_i\|}\right| < \left|\frac{x_i}{\|x_i\|}\right| + \left|\frac{y_i}{\|y_i\|}\right| - \delta_1,$$

i.e.

(14.20) $\quad |x_i|+|y_i| - |x_i+y_i| > \delta_1\|x_i\| \quad (i \in J')$.

Thus, combining (14.19) and (14.20),

(14.21) $\quad \|x\|_{5\varepsilon} + \|y\|_{5\varepsilon} - \|x+y\|_{5\varepsilon} \geq \delta_1 \sum_{i\in J'} \|x_i\| - 2\gamma.$

Furthermore, (14.17) implies that

$$3\varepsilon < \sum_{i\in J'} \|x_i-y_i\| \leq \sum_{i\in J'} \|x_i\| + \sum_{i\in J'} \|y_i\| = 2 \sum_{i\in J'} \|x_i\|,$$

so

(14.22) $\quad \sum_{i\in J'} \|x_i\| > \frac{3\varepsilon}{2}.$

(14.21), (14.22) and the fact that $\gamma < \frac{\delta_1\varepsilon}{4}$ now yield the desired conclusion

$$\|x\|_{5\varepsilon} + \|y\|_{5\varepsilon} - \|x+y\|_{5\varepsilon} > \tfrac{3}{2}\delta_1\varepsilon - 2\gamma > \delta_1\varepsilon. \qquad \square$$

PROOF OF THEOREM 14.5. Suppose X does not possess P_1. Using the notations of the preceding lemmas, put

(14.23) $\quad |||x||| := \sum_{n=1}^{\infty} 2^{-n}\|x\|_{2^{-n}} \quad (x \in X)$.

We claim that $|||\ |||$ is an equivalent uniformly convex norm on X. By (α) we we have for all $n \in \mathbb{N}$

$$\tfrac{1}{2}\|x\| \leq \|x\|_{2^{-n}} \leq \|x\| \quad (x \in X).$$

Therefore

(14.24) $\quad \tfrac{1}{2}\|x\| \leq |||x||| \leq \|x\| \quad (x \in X)$,

which proves the equivalence of $\|\cdot\|$ and $|||\cdot|||$.

Next we prove that for every $\varepsilon > 0$ there exists a $\delta' = \delta'(\varepsilon) > 0$ such that

(14.25) $\quad \|x\| = \|y\| = 1, \ \|x-y\| \geq \varepsilon \Rightarrow |||x+y||| \leq |||x||| + |||y||| - \delta'(\varepsilon).$

Indeed, if $\varepsilon > 0$ is given arbitrarily, pick $n_0 \in \mathbb{N}$ so that $2^{-n_0} < \varepsilon$. Suppose that $\|x\| = \|y\| = 1$, $\|x-y\| \geq \varepsilon$. Then by (β) (with $\varepsilon = \frac{1}{5}\,2^{-n_0}$)

$$\|x+y\|_{2^{-n_0}} \leq \|x\|_{2^{-n_0}} + \|y\|_{2^{-n_0}} - \frac{1}{5}2^{-n_0}\,\delta_1.$$

Thus, by (14.23), and the triangle inequality,

$$|||x+y||| \leq |||x||| + |||y||| - \frac{1}{5}2^{-2n_0}\,\delta_1,$$

and therefore $\delta'(\varepsilon) := \frac{1}{5}2^{-2n_0}\delta_1$ satisfies (14.25).

Finally we show that $|||\ |||$ is uniformly convex. We prove in fact that for $\delta'(\cdot)$ defined by (14.25), we have, for all $\eta > 0$:

(14.26) $\quad |||x||| = |||y||| = 1, \ |||x-y||| \geq \eta \Rightarrow |||x+y||| \leq 2 - \delta'(\frac{\eta}{4}).$

Let $x,y \in X$ with $|||x||| = |||y||| = 1$, $|||x-y||| \geq \eta$ be given. It is not difficult to show that

(14.27) $\quad |||x-ty||| \geq \frac{\eta}{2} \quad$ for all $t \in \mathbb{R}$.

Put $\alpha := \frac{1}{\|x\|}$, $\beta := \frac{1}{\|y\|}$. By (14.24), $\frac{1}{2} \leq \alpha, \beta \leq 1$ and therefore, by (14.27), we have

$$\|\alpha x-\beta y\| \geq |||\alpha x-\beta y||| = \alpha|||x - \frac{\beta}{\alpha}y||| \geq \alpha\frac{\eta}{2} \geq \frac{\eta}{4}.$$

Applying (14.25) to αx, βy, and $\varepsilon = \frac{\eta}{4}$ yields $|||\alpha x+\beta y||| \leq \alpha|||x||| + \beta|||y||| - \delta'(\frac{\eta}{4})$, and this in turn implies $|||x+y||| \leq |||\alpha x+\beta y||| + (1-\alpha)|||x||| + (1-\beta)|||y||| \leq$ $\leq \alpha|||x||| + \beta|||y||| - \delta'(\frac{\eta}{4}) + (1-\alpha)|||x||| + (1-\beta)|||y||| = |||x||| + |||y||| - \delta'(\frac{\eta}{4}) =$ $= 2 - \delta'(\frac{\eta}{4})$. □

<u>NOTES</u>. R.C. JAMES ([54]) introduced P_1 and proved Proposition 14.4. The other implication in Theorem 14.5 is due to P. ENFLO ([33]).

15. SUPERREFLEXIVITY

In this section we introduce finite versions P_2, P_3, P_4 of the properties $P_2^\infty, P_3^\infty, P_4^\infty$, respectively, and show that all P_i $(i = 1,\dots,4)$ are mutually equivalent. Hence, by Theorem 14.5, the negation of each of the P_i is equivalent to uniform convexifiability. The proof of the equivalence of the P_i follows a general pattern and is accomplished via the notion of superreflexivity. We now give the necessary definitions.

DEFINITION 15.1. Let X be a Banach space.

X has P_2 iff $\exists \varepsilon > 0\ \ \forall n \in \mathbb{N}\ \exists\{x_1,\dots,x_n\} \subset B_X$: $\forall k \in \{1,\dots,n-1\}$ $\operatorname{dist}(\operatorname{co}\{x_1,\dots,x_k\},\operatorname{co}\{x_{k+1},\dots,x_n\}) \geq \varepsilon$.

X has P_3 iff $\exists \varepsilon > 0\ \ \forall n \in \mathbb{N}\ \exists\{x_1,\dots,x_n\} \subset B_X$: $\operatorname{dist}(\operatorname{co}\{x_1,\dots,x_n\},\{0\}) \geq \varepsilon$ and $\forall k \in \{1,\dots,n\}$ $\forall \alpha_1,\dots,\alpha_n \in \mathbb{R}$ $\|\sum_{i=1}^n \alpha_i x_i\| \geq \frac{\varepsilon}{2}\|\sum_{i=1}^k \alpha_i x_i\|$].

X has P_4 iff $\exists \varepsilon > 0\ \ \forall n \in \mathbb{N}\ \exists\{x_1,\dots,x_n\} \subset B_X$ $\exists\{x_1^*,\dots,x_n^*\} \subset B_{X^*}$ $\forall k,i \in \{1,\dots,n\}$: $[k \leq i \Rightarrow \langle x_i,x_k^*\rangle \geq \varepsilon,\ k > i \Rightarrow \langle x_i,x_k^*\rangle = 0]$.

It is evident that P_i^∞ implies P_i $(i = 1,\dots,4)$. For convenience we also introduce here the following notations.

DEFINITION 15.2. Let X be a Banach space, $\varepsilon > 0$ and $n \in \mathbb{N}$.

(a) $\{x_1,\dots,x_{2^n}\} \subset B_X$ satisfies $P_1^n(\varepsilon)$ iff $\{x_1,\dots,x_{2^n}\}$ is an (n,ε)-tree.

(b) $\{x_1,\dots,x_n\} \subset B_X$ satisfies $P_2^n(\varepsilon)$ iff $\forall k \in \{1,\dots,n-1\}$ $\operatorname{dist}(\operatorname{co}\{x_1,\dots,x_k\},\operatorname{co}\{x_{k+1},\dots,x_n\}) \geq \varepsilon$.

(c) $\{x_1,\dots,x_n\} \subset B_X$ satisfies $P_3^n(\varepsilon)$ iff $\operatorname{dist}(\operatorname{co}\{x_1,\dots,x_n\},\{0\}) \geq \varepsilon$ and $\forall k \in \{1,\dots,n\}$ $\forall \alpha_1,\dots,\alpha_n \in \mathbb{R}$ $\|\sum_{i=1}^n \alpha_i x_i\| \geq \frac{\varepsilon}{2}\|\sum_{i=1}^k \alpha_i x_i\|$.

(d) $\{x_1,\dots,x_n\} \subset B_X$ and $\{x_1^*,\dots,x_n^*\} \subset B_{X^*}$ satisfy $P_4^n(\varepsilon)$ iff $\forall k,i \in \{1,\dots,n\}$: $[k \leq i \Rightarrow \langle x_i,x_k^*\rangle \geq \varepsilon,\ k < i \Rightarrow \langle x_i,x_k^*\rangle = 0]$.

Similar notations will be used for $n = \infty$.

<u>DEFINITION 15.3</u>. A Banach space X is called *superreflexive* iff $Y \prec X$ implies that Y is reflexive.

<u>REMARK 15.4</u>. It is trivial that every superreflexive Banach space is reflexive (take Y = X in the above definition). The converse is not true. In fact we have seen in Section 13, Example 3, that every Banach space is finitely representable in the reflexive space $(\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$. Thus $(\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ is definitely not superreflexive.

We now state the main result

<u>THEOREM 15.5</u>. *For every Banach space* X *the following are equivalent:*

(i) X *has* P_1;

(ii) X *has* P_2;

(iii) X *has* P_3;

(iv) X *has* P_4;

(v) X *is not superreflexive;*

(vi) X *is not uniformly convexifiable.*

<u>PROOF</u>. By Theorem 14.5,(i) $\Leftrightarrow$ (vi). It therefore suffices to show that for each i = 1,...,4 we have

(15.1) X has $P_i \Leftrightarrow$ X not superreflexive.

The implication from right to left in (15.1) is simple. Indeed, let us fix $i \in \{1,\dots,4\}$ and let us assume that X is not superreflexive. Then there exists a non-reflexive Y such that $Y \prec X$ and Y has P_i (if i = 1 this follows from Theorem 14.5 and if $i \in \{2,3,4\}$ this is a consequence of Theorem 10.3 and the trivial implication $P_i^{\infty} \Rightarrow P_i$). It follows now from the definition of finite representability that also X has P_i. For if $0 < \varepsilon < 1$ is as in Definition 15.1 or 14.1 for P_i and if $n \in \mathbb{N}$ is arbitrary, then there exist systems $\{y_1,\dots,y_{2^n}\} \subset B_Y$ (if i = 1), $\{y_1,\dots,y_n\} \subset B_Y$ (if i = 2,3), and $\{y_1,\dots,y_n\} \subset B_Y$, $\{y_1^*,\dots,y_n^*\} \subset B_{Y^*}$ (if i = 4) satisfying $P_i^n(\varepsilon)$. In each case the elements y_j span a finite-dimensional subspace $F \subset Y$ and therefore we can determine a "good" (i.e. almost isometric) isomorphism $T: F \to X$. More precisely, given any ε' with $0 < \varepsilon' < \varepsilon$ a choice for T is possible so that the systems $\{Ty_1,\dots,Ty_{2^n}\}$ (if i = 1), $\{Ty_1,\dots,Ty_n\}$ (if i = 2,3) and $\{Ty_1,\dots,Ty_n\},\{x_1^*,\dots,x_n^*\}$ (if i = 4) satisfy $P_i^n(\varepsilon')$, where in the case i = 4 the x_j^* (j = 1,...,n) are extensions with preservation of norm to X of the elements $(T^*)^{-1}y_j^*$, T being regarded as a map from $[y_1,\dots,y_n]$ onto

$[Ty_1,\dots,Ty_n]$. (Possibly the x_j^* must be multiplied with suitable constants so that $\|x_j^*\| \le 1$, $j = 1,\dots,n$.)

The implication from left to right in (15.1) follows from the proposition below, since for each $i = 1,\dots,4$ P_i^∞ implies non-reflexivity.

PROPOSITION 15.6. *If a Banach space* X *has* P_i *for some* $i \in \{1,\dots,4\}$, *then there exists a Banach space* Y *such that* $Y \prec X$ *and* Y *satisfies* P_i^∞.

PROOF. Essentially the proof is the same in all cases and we shall give it simultaneously for all $i = 1,\dots,4$. If for technical reasons it becomes necessary in some step of the proof to distinguish between the four cases, we shall indicate this by labelling the step with a subindex i ($i = 1,\dots,4$).

(a_1) If X has P_1 then there exists an $\varepsilon_0 > 0$ such that for each $n \in \mathbb{N}$ B_X contains an (n,ε_0)-tree $T_n = \{x_{k,i}^n : k = 1,\dots,n;\ i = 1,\dots,2^k\}$. Renumbering the elements of T_n in lexicographic order, we write

$$T_n = \{x_1^n, x_2^n, \dots, x^n_{\sum_{k=1}^n 2^k}\} \quad (n = 1,2,\dots).$$

($a_{2,3}$) There exists an $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there is a system $\{x_1^n,\dots,x_n^n\} \subset B_X$ satisfying $P_i^n(\varepsilon_0)$ ($i = 2,3$).

(a_4) There exists an $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ there exist $\{x_1^n,\dots,x_n^n\} \subset B_X$, $x_1^{n*},\dots,x_n^{n*}\} \subset B_{X^*}$ satisfying $P_4^n\ (\varepsilon_0)$.

(b) Let Y be the linear space of all real sequences with finitely many non-zero elements. Using the notation $e_n = (\underbrace{0,\dots,0}_{n-1},1,0,\dots)$ $(n \in \mathbb{N})$, each element of Y can be written as $\sum_{j=1}^r \alpha_j e_j$ $(r \in \mathbb{N};\ \alpha_1,\dots,\alpha_r \in \mathbb{R})$.

($c_{1,2,3}$) For each finite set $\{\alpha_1,\dots,\alpha_r\} \subset \mathbb{R}$ the sequence $\{\|\sum_{j=1}^r \alpha_j x_j^n\|\}_{n=r}^\infty$ is bounded. Hence, by a diagonal procedure, we can find a subsequence $\{n_\ell\} \subset \mathbb{N}$ such that $\lim_{\ell\to\infty} \|\sum_{j=1}^r \alpha_j x_j^{n_\ell}\|$ exists for all $r \in \mathbb{N}$ and all $\alpha_1,\dots,\alpha_r \in \mathbb{Q}$. (Observe that $x_j^{n_\ell}$ is defined for $\ell \ge j$.)

(c_4) For each finite set $\{\alpha_1,\dots,\alpha_r\} \subset \mathbb{R}$ and for each pair $i,k \in \mathbb{N}$ the sequences $\{\|\sum_{j=1}^r \alpha_j x_j^n\|\}_{n=r}^\infty$ and $\{\langle x_i^n, x_k^{n*}\rangle\}_{n=\max(i,k)}^\infty$ are bounded. Again by a diagonal procedure, there exists a subsequence $\{n_\ell\} \subset \mathbb{N}$ such that $\lim_{\ell\to\infty} \|\sum_{j=1}^r \alpha_j x_j^{n_\ell}\|$ and $\lim_{\ell\to\infty} \langle x_i^{n_\ell}, x_k^{n_\ell *}\rangle$ exist for all finite sets $\{\alpha_1,\dots,\alpha_r\} \subset \mathbb{Q}$ and all pairs $i,k \in \mathbb{N}$.

(d) Observe that by the density of $\mathbb{Q}$ in $\mathbb{R}$ the limits $\lim_{\ell\to\infty} \|\sum_{j=1}^r \alpha_j x_j^{n_\ell}\|$ also exist for all finite sets $\{\alpha_1,\dots,\alpha_r\} \subset \mathbb{R}$. We now define on Y a semi-norm as follows:

$$(15.2) \qquad \Big\|\sum_{j=1}^r \alpha_j e_j\Big\| := \lim_{\ell\to\infty} \Big\|\sum_{j=1}^r \alpha_j x_j^{n_\ell}\Big\| \quad (r \in \mathbb{N},\ \alpha_1,\dots,\alpha_r \in \mathbb{R}).$$

Let us put $\hat{Y} := Y/N$, where $N = \{y \in Y: \|y\| = 0\}$. We shall denote elements $y+N \in \hat{Y}$ by $\hat{y}$. Then $\|\cdot\|$ induces a norm on $\hat{Y}$ (denoted by the same symbol) and trivially

(15.3) $\qquad \|\hat{y}\| = \|y\| \quad$ for every $y \in Y$.

We claim that the completion of $\hat{Y}$ is the space we are looking for.

(e) In this step we show that $\hat{Y} \prec X$. Since $\{e_m\}$ spans Y, $\{\hat{e}_m\}$ contains a linearly independent subsequence $\{\hat{e}_{m_k}\}$ that spans $\hat{Y}$. Now let $\varepsilon > 0$ be arbitrary and let F be any finite-dimensional subspace of $\hat{Y}$. Enlarging F if necessary, we may assume that $F = [\hat{e}_{m_1},\dots,\hat{e}_{m_r}]$. By compactness, the set $K := \{(\alpha_1,\dots,\alpha_r) \in \mathbb{R}^r: \|\sum_{j=1}^r \alpha_j \hat{e}_{m_j}\| = 1\}$, equipped with the norm $\|(\alpha_1,\dots,\alpha_r)\| = \sum_{j=1}^r |\alpha_j|$, has a finite ε-net $\{\alpha^{(i)} = (\alpha_1^{(i)},\dots,\alpha_r^{(i)})\}_{i=1}^p$. By (15.2) we can determine an $\ell_0 \in \mathbb{N}$ such that for all $i = 1,\dots,p$

(15.4) $$\left| \left\| \sum_{j=1}^r \alpha_j^{(i)} \hat{e}_{m_j} \right\| - \left\| \sum_{j=1}^r \alpha_j^{(i)} x_{m_j}^{n_{\ell_0}} \right\| \right| < \varepsilon.$$

(Note that $\|\sum_{j=1}^r \alpha_j^{(i)} \hat{e}_{m_j}\| = \|\sum_{j=1}^r \alpha_j^{(i)} e_{m_j}\|$ by (15.3).) Now we define a linear map $T: F \to X$ by

(15.5) $$T\left(\sum_{j=1}^r \alpha_j \hat{e}_{m_j}\right) = \sum_{j=1}^r \alpha_j x_{m_j}^{n_{\ell_0}} \quad (\alpha_1,\dots,\alpha_r \in \mathbb{R})$$

and claim that T satisfies

(15.6) $$(1-2\varepsilon)\|\hat{y}\| \le \|T\hat{y}\| \le (1+2\varepsilon)\|\hat{y}\| \quad \text{for all } \hat{y} \in F.$$

Indeed, let $\sum_{j=1}^r \alpha_j \hat{e}_{m_j} \in F$ be given such that $\|\sum_{j=1}^r \alpha_j \hat{e}_{m_j}\| = 1$. Since $\{\alpha^{(1)},\dots,\alpha^{(p)}\}$ is an ε-net for K, there exists an $i_0 \in \{1,\dots,p\}$ such that

(15.7) $$\sum_{j=1}^r |\alpha_j - \alpha_j^{(i_0)}| < \varepsilon.$$

Then, since $\|x_{m_j}^{n_{\ell_0}}\| \le 1$, $\|\hat{e}_{m_j}\| \le 1$ $(j = 1,\dots,r)$ and $\alpha^{(i_0)} \in K$, we have by (15.5), (15.7) and (15.4),

$$\left\|T\left(\sum_{j=1}^r \alpha_j \hat{e}_{m_j}\right)\right\| = \left\|\sum_{j=1}^r \alpha_j x_{m_j}^{n_{\ell_0}}\right\| \le \left\|\sum_{j=1}^r \alpha_j^{(i_0)} x_{m_j}^{n_{\ell_0}}\right\| + \varepsilon \le$$

$$\leq \|\sum_{j=1}^{r} \alpha_j^{(i_0)} \hat{e}_{m_j}\| + 2\varepsilon = 1 + 2\varepsilon,$$

and

$$\|T(\sum_{j=1}^{r} \alpha_j \hat{e}_{m_j})\| = \|\sum_{j=1}^{r} \alpha_j x_{m_j}^{n_{\ell_0}}\| \geq \|\sum_{j=1}^{r} \alpha_j^{(i_0)} x_{m_j}^{n_{\ell_0}}\| - \varepsilon \geq$$

$$\geq \|\sum_{j=1}^{r} \alpha_j^{(i_0)} \hat{e}_{m_j}\| - 2\varepsilon = 1 - 2\varepsilon.$$

This proves (15.6) and completes the proof that $\hat{Y} \prec X$.

(f_1) We show that $\hat{Y}$ has P_1^∞. In fact the sequence $\{\hat{e}_m\} \subset B_{\hat{Y}}$ forms an (∞, ε_0)-tree when its elements are relabelled as follows: $\hat{e}_{1,1} := \hat{e}_1$, $\hat{e}_{1,2} := \hat{e}_2$, $\hat{e}_{2,1} := \hat{e}_3$, $\hat{e}_{2,2} := \hat{e}_4$, $\hat{e}_{2,3} := \hat{e}_5$, $\hat{e}_{2,4} := \hat{e}_6$, $\hat{e}_{3,1} := \hat{e}_7$, etc. Indeed, for every $k \in \mathbb{N}$ the relations

$$\|x_{k,i}^{n_\ell} - \tfrac{1}{2}(x_{k+1,2i-1}^{n_\ell} + x_{k+1,2i}^{n_\ell})\| = 0 \qquad (i = 1,\ldots,2)$$

and

$$\|x_{k,2i-1}^{n_\ell} - x_{k,2i}^{n_\ell}\| \geq \varepsilon_0 \qquad (i = 1,\ldots,2^{k-1})$$

hold whenever $n_\ell > k$. This implies, by (15.2), that

$$\|\hat{e}_{k,i} - \tfrac{1}{2}(\hat{e}_{k+1,2i-1} + \hat{e}_{k+1,2i})\| = \lim_{\ell\to\infty}\|x_{k,i}^{n_\ell} - \tfrac{1}{2}(x_{k+1,2i-1}^{n_\ell} + x_{k+i,2i}^{n_\ell})\| = 0,$$

so

$$\hat{e}_{k,i} = \tfrac{1}{2}(\hat{e}_{k+1,2i-1} + \hat{e}_{k+1,2i}) \qquad (k = 1,2,\ldots;\ i = 1,\ldots,2^k)$$

and

$$\|\hat{e}_{k,2i-1} - \hat{e}_{k,2i}\| = \lim_{\ell\to\infty}\|x_{k,2i-1}^{n_\ell} - x_{k,2i}^{n_\ell}\| \geq \varepsilon_0$$
$$(k = 1,2,\ldots;\ i = 1,\ldots,2^{k-1}).$$

(f_2) $\hat{Y}$ has P_2^∞. Indeed, let $k,n \in \mathbb{N}$ with $k < n$ and $\lambda_1,\ldots,\lambda_{k+n} \geq 0$ with $\sum_{i=1}^k \lambda_i = \sum_{i=k+1}^{k+n} \lambda_i = 1$ be given arbitrarily. Then

$$\|\sum_{i=1}^{k} \lambda_i x_i^{n_\ell} - \sum_{i=k+1}^{k+n} \lambda_i x_i^{n_\ell}\| \geq \varepsilon_0$$

whenever $n_\ell \geq k+n$, so

$$\|\sum_{i=1}^{k} \lambda_i \hat{e}_i - \sum_{i=k+1}^{k+n} \lambda_i \hat{e}_i\| = \lim_{\ell\to\infty}\|\sum_{i=1}^{k} \lambda_i x_i^{n_\ell} - \sum_{i=k+1}^{k+n} \lambda_i x_i^{n_\ell}\| \geq \varepsilon_0.$$

(f_3) The proof that $\hat{Y}$ has P_3^∞ is similar.

(f_4) To show that $\hat{Y}$ has P_4^∞, we first define a sequence $\{e_k^*\}$ of linear forms on Y by

$$(15.8)\qquad \begin{aligned} &\langle e_i, e_k^* \rangle := \lim_{\ell\to\infty} \langle x_i^{n_\ell}, x_k^{n_\ell *} \rangle \quad (i,k \in \mathbb{N}) \text{ and} \\ &\langle \sum_{j=1}^{r} \alpha_j e_j, e_k^* \rangle := \sum_{j=1}^{r} \alpha_j \langle e_j, e_k^* \rangle \quad (k,r \in \mathbb{N},\ \alpha_1,\dots,\alpha_r \in \mathbb{R}). \end{aligned}$$

Observe that we have

$$(15.9)\qquad \begin{aligned} &|\langle \sum_{j=1}^{r} \alpha_j e_j, e_k^* \rangle| = |\lim_{\ell\to\infty} \langle \sum_{j=1}^{r} \alpha_j x_j^{n_\ell}, x_k^{n_\ell *} \rangle| \le \\ &\le \limsup_{\ell\to\infty} (\| x_k^{n_\ell *} \| \| \sum_{j=1}^{r} \alpha_j x_j^{n_\ell} \|) \le \lim_{\ell\to\infty} \| \sum_{j=1}^{r} \alpha_j x_j^{n_\ell} \| = \| \sum_{j=1}^{r} \alpha_j e_j \|, \end{aligned}$$

so that each e_k^* vanishes on N and therefore defines a unique linear form $\hat{e}_k^*$ on $\hat{Y}$ satisfying $\langle \hat{y}, \hat{e}_k^* \rangle = \langle y, e_k^* \rangle$ for all $y \in Y$. Obviously, by (15.9), all $\hat{e}_k^*$ are continuous on $\hat{Y}$ with norm ≤ 1. Now let $k,i \in \mathbb{N}$ be arbitrary. Then, since whenever $n_\ell \ge \max(k,i)$ we have

$$\langle x_i^{n_\ell}, x_k^{n_\ell *} \rangle \begin{cases} \ge \varepsilon_0 & \text{if } k \le i \\ = 0 & \text{if } k > i \end{cases},$$

it follows that

$$\langle \hat{e}_i, \hat{e}_k^* \rangle = \lim_{\ell\to\infty} \langle x_i^{n_\ell}, x_k^{n_\ell *} \rangle \begin{cases} \ge \varepsilon_0 & \text{if } k \le i \\ = 0 & \text{if } k > i. \end{cases}$$

Hence $\hat{Y}$ has P_4^∞.

(g) Evidently the completion of $\hat{Y}$ is a Banach space which also has P_i^∞. Moreover, this completion is also finitely representable in X. We omit a formal proof of this last statement since it is similar to the argument used in Section 9 to show that Theorem 9.1 implies Property III.

This completes the proof of Proposition 15.6 and thereby also that of Theorem 15.5. □

COROLLARY 15.7. *All properties P_1, P_2, P_3, P_4, superreflexivity and uniform convexifiability are self-dual, i.e. a Banach space X has any one of them iff X^* does.*

PROOF. By Theorem 15.5 it suffices to show this for one of these properties. But for P_4 self-duality is obvious: If $\{x_1,\dots,x_n\},\{x_1^*,\dots,x_n^*\}$ satisfies $P_4^n(\varepsilon)$ for X, then $\{y_1^*,\dots,y_n^*\},\{y_1^{**},\dots,y_n^{**}\}$ satisfies $P_4^n(\varepsilon)$ for X^*, if we take $y_i^* = x_{n-i}^*$ and $y_k^{**} = \pi x_{n-k}$ $(i,k = 1,\dots,n)$. □

In particular,

COROLLARY 15.8. *For a Banach space* X *the following are equivalent:*

(i) X *has an equivalent uniformly convex norm;*

(ii) X *has an equivalent uniformly smooth norm;*

(iii) X *has an equivalent norm that is both uniformly convex and uniformly smooth.*

PROOF. Combine Corollary 15.7 with Proposition 12.6 (i) to obtain the equivalence of (i) and (ii). That (i) and (ii) imply (iii) is Theorem 12.12.

NOTES. This section is entirely the work of R.C. JAMES ([54]), modulo P. ENFLO's result ([33]).

16. SQUARES IN THE UNIT SPHERE

We have seen in Remark 10.4 that for i = 2,3,4 P_i^∞ is equivalent to the same property with "$\exists\varepsilon > 0$" replaced by "$\forall 0 < \varepsilon < 1$". An analogous remark holds for the finite properties P_i (i = 2,3,4). Indeed, if X has P_i (i = 2,3,4) then X is not superreflexive (Theorem 15.5), so there exists a non-reflexive Y with $Y \prec X$. Y then has P_i^∞ with "$\forall 0 < \varepsilon < 1$" instead of "$\exists\varepsilon > 0$", so certainly it has P_i with "$\forall 0 < \varepsilon < 1$". The definition of finite representability then easily implies that also X has P_i with "$\forall 0 < \varepsilon < 1$" (cf. the proof of the easy half of Theorem 15.5). We shall see later that in the cases i = 1,2 we can even replace "$\exists\varepsilon > 0$" by "$\forall 0 < \varepsilon < 2$", without changing the property P_i. (Note that for i = 3,4 no such replacement makes sense: 1 is obviously the largest possible value of ε for which P_3 and P_4 can hold.) The proofs of these statements will follow later. In this section we deal with a preliminary result in this direction. We have shown (Proposition 12.3) that a uniformly convex space is reflexive (in fact we now know it is even superreflexive). It is not difficult to see, with the knowledge we now have, that in this result the weaker assumption "$\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 1$" suffices (Proposition 16.1). The main result in this section (Theorem 16.4) is that even the assumption "$\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$" suffices. Later we shall generalize this last result and use it to show that in P_i (i = 1,2) we may read "$\forall 0 < \varepsilon < 2$".

PROPOSITION 16.1. *Let X be a Banach space. If for some $0 < \varepsilon < 1$ the modulus of convexity $\delta(\varepsilon)$ is positive, then X is reflexive, even superreflexive.*

PROOF. Explicitly, the assumption is:

$$(16.1) \qquad \exists 0 < \varepsilon < 1 \ \exists \delta > 0 \colon [\|x\|, \|y\| \le 1, \ \left\|\frac{x+y}{2}\right\| > 1-\delta \Rightarrow \|x-y\| < \varepsilon].$$

Let us put $\eta := \max(\varepsilon, 1-\delta)$. Suppose for contradiction that X is not superreflexive. Then X has P_4 (with $\forall 0 < \varepsilon < 1$), so in particular there exist

$x_1, x_2 \in B_X$ and $x_1^*, x_2^* \in B_{X^*}$ such that $<x_i, x_k^*> > \eta$ if $k \le 1$ and $<x_i, x_k^*> = 0$ if $k > 1$ $(k,i = 1,2)$. But then

$$\left\|\frac{x_1+x_2}{2}\right\| \ge <\frac{x_1+x_2}{2}, x_1^*> = \tfrac{1}{2}<x_1, x_1^*> + \tfrac{1}{2}<x_2, x_1^*> > \eta \ge 1-\delta$$

and

$$\|x_1-x_2\| \ge |<x_1-x_2, x_2^*>| = |<x_2, x_2^*>| > \eta \ge \varepsilon,$$

contradicting (16.1). □

We now want to show that Proposition 16.1 still holds if we assume that $\delta(\varepsilon) > 0$ for some $0 < \varepsilon < 2$. First we wish to give a geometric interpretation to the assumption $\exists 0 < \varepsilon < 2 \;\; \delta(\varepsilon) > 0$.

DEFINITION 16.2. A Banach space X is called *uniformly non-square* if there exists an $\eta > 0$ such that no $x, y \in B_X$ satisfy $\|\tfrac{1}{2}(x+y)\| > 1-\eta$ and $\|\tfrac{1}{2}(x-y)\| > 1-\eta$.

The interpretation is clear: for small $\eta > 0$ the situation $\|x\|, \|y\| \le 1$, $\|\tfrac{1}{2}(x+y)\| > 1-\eta$ and $\|\tfrac{1}{2}(x-y)\| > 1-\eta$ (and therefore $\|x\| \ge \|x+y\| - \|y\| > 2-2\eta-1 = 1-2\eta$ and similarly $\|y\| > 1-2\eta$) means that the unit sphere of $sp\{x,y\}$ closely resembles a square.

LEMMA 16.3. *Let X be a Banach space. Then X is uniformly non-square iff* $\delta_X(\varepsilon) > 0$ *for some* $0 < \varepsilon < 2$.

PROOF. X uniformly non-square $\Longleftrightarrow$
$\Longleftrightarrow \exists \eta > 0 \; [x, y \in B_X, \; \|\tfrac{1}{2}(x+y)\| > 1-\eta \Rightarrow \|\tfrac{1}{2}(x-y)\| \le 1-\eta]$
$\Longleftrightarrow \exists \eta > 0 \; [x, y \in B_X, \; 1 - \|\tfrac{x+y}{2}\| < \eta \;\; \Rightarrow \|x-y\| \le 2-2\eta]$
$\Longleftrightarrow \exists \eta > 0 \;\; \delta_X(2-2\eta) \ge \eta$
$\Longleftrightarrow \exists 0 < \varepsilon < 2 \;\; \delta_X(\varepsilon) > 0$.
In the last equivalence the monotonicity of the function $\delta_X(\cdot)$ has been used. □

Thus the assumption $\delta_X(\varepsilon) > 0$ for some $0 < \varepsilon < 2$ means that there is a uniform bound to how closely the unit sphere of a 2-dimensional subspace of X can approximate a square. We are now ready for the main result.

THEOREM 16.4. *Let X be a Banach space. If* $\delta_X(\varepsilon) > 0$ *for some* $0 < \varepsilon < 2$ *then X is reflexive.*

PROOF. Suppose that X is non-reflexive. We define for every sequence

$\{x_j^*\} \subset S_{X^*}$, every $n \in \mathbb{N}$ and every finite increasing sequence $p_1 < p_2 < \dots$
$\dots < p_{2n}$ of natural numbers,

$$S(p_1,\dots,p_{2n};\{x_j^*\}) := \{x \in X: \tfrac{1}{2} \le (-1)^{i-1}\langle x,x_k^*\rangle \le 1 \quad \text{if } p_{2i-1} \le k \le p_{2i} \quad (i = 1,\dots,n)\}$$

(16.2) and

$$R(p_1,\dots,p_{2n};\{x_j^*\}) := \inf\{\|x\|: x \in S(p_1,\dots,p_{2n};\{x_j^*\})\}.$$

(We use the convention here that the inf over the empty set is ∞.) Let us observe that if $n \in \mathbb{N}$, $\{x_j^*\} \subset S_{X^*}$ and all $p_1,\dots,p_{2n}$ save one of them, say p_ℓ, are kept fixed, then $S(p_1,\dots,p_{2n};\{x_j^*\})$ is a monotone function of p_ℓ (with respect to inclusion), increasing if ℓ is odd, and decreasing if ℓ is even. Consequently $R(p_1,\dots,p_{2n};\{x_j^*\})$ is also monotone in p_ℓ when the other variables are kept fixed. It follows from this that the following definition makes sense, since all limits involved exist

$$(16.3) \qquad K(n;\{x_j^*\}) := \lim_{p_1\to\infty}\lim_{p_2\to\infty}\dots\lim_{p_{2n}\to\infty} R(p_1,\dots,p_{2n};\{x_j^*\}) \qquad (n \in \mathbb{N};\{x_j^*\} \subset S_{X^*}).$$

We now define

$$(16.4) \qquad K_n := \inf\{K(n;\{x_j^*\}): \{x_j^*\} \subset S_{X^*}\} \quad (n = 1,2,\dots).$$

As a first step in the proof we show that

$$(16.5) \qquad K_n \le 2n \quad (n = 1,2,\dots).$$

Indeed, let us fix $n \in \mathbb{N}$ and choose r such that $\max(\frac{7}{8}, 1 - \frac{1}{8n}) < r < 1$. Since X is assumed to be non-reflexive, it has P_4^∞ (with $\forall 0 < \varepsilon < 1$) so there exist sequences $\{x_n\} \subset B_X$, $\{x_n^*\} \subset B_{X^*}$ such that

$$(16.6) \qquad \langle x_i,x_k^*\rangle > r \text{ if } k \le i \quad\text{and}\quad \langle x_i,x_k^*\rangle = 0 \text{ if } k > i.$$

Obviously we may assume that $\{x_n^*\} \subset S_{X^*}$. Now let natural numbers $p_1,\dots,p_{2n}$ be given and define

$$(16.7) \qquad w := \sum_{i=1}^{n} (-1)^{i-1}(-x_{p_{2i-1}-1} + x_{p_{2i}}).$$

Using the notation

$$A_{k,i} := -\langle x_{p_{2i-1}-1}, x_k^*\rangle + \langle x_{p_{2i}}, x_k^*\rangle \qquad (k \in \mathbb{N},\ i \in \{1,\dots,n\}),$$

we have

$$\langle w, x_k^*\rangle = \sum_{i=1}^{n} (-1)^{i-1} A_{k,i} \qquad (k = 1,2,\dots).$$

It follows from (16.6) that

$$\left.\begin{array}{ll} A_{k,i} = 0 & \text{if } k > p_{2i} \\ A_{k,i} = \langle x_{p_{2i}}, x_k^*\rangle & \text{if } p_{2i-1} \le k \le p_{2i} \\ |A_{k,i}| < 1-r & \text{if } k < p_{2i-1} \end{array}\right\} \quad (k \in \mathbb{N},\ i \in \{1,\dots,n\}).$$

Hence, for $i \in \{1,\dots,n\}$ and $p_{2i-1} \le k \le p_{2i}$ we have

(16.8) $\qquad (-1)^{i-1}\langle w, x_k^*\rangle + \langle x_{p_{2i}}, x_k^*\rangle + E_k, \quad \text{with} \quad |E_k| < n(1-r).$

Since $r > \max(\frac{7}{8}, 1 - \frac{1}{8n})$, we also have

(16.9) $\qquad 1 \ge \langle x_{p_{2i}}, x_k^*\rangle > r > \frac{7}{8}$

and

(16.10) $\qquad |E_k| < n(1-r) < n(1 - (1 - \frac{1}{8n})) = \frac{1}{8}.$

(16.8), (16.9) and (16.10) imply that, for $i \in \{1,\dots,n\}$ and $p_{2i-1} \le k \le p_{2i}$,

$$\frac{9}{8} > \langle x_{p_{2i}}, x_k^*\rangle + |E_k| \ge (-1)^{i-1}\langle w, x_k^*\rangle \ge \langle x_{p_{2i}}, x_k^*\rangle - |E_k| > \frac{3}{4},$$

so that

$$1 > (-1)^{i-1} \langle \tfrac{8}{9}w, x_k^*\rangle > \tfrac{1}{2}.$$

Thus $\frac{8}{9}w \in S(p_1,\dots,p_{2n};\{x_j^*\})$. Since, by (16.7), $\|\frac{8}{9}w\| \le \|w\| \le 2n$, we have now shown that the set $S(p_1,\dots,p_{2n};\{x_j^*\})$ contains an element of norm $\le 2n$. This means that for the sequence $\{x_j^*\}$ introduced above (dependent on n, via r, but not on $p_1,\dots,p_{2n}$), we have $K(n;\{x_j^*\}) \le 2n$. Consequently (16.5) holds, since $n \in \mathbb{N}$ was arbitrary.

Let us observe next that

(16.11) $\qquad K_n \ge \frac{1}{2} \quad (n = 1,2,\dots),$

since $|\langle x, x_k^*\rangle| \ge \frac{1}{2}$ implies $\|x\| \ge \frac{1}{2}$ (cf. (16.2)). Also

(16.12) $\{K_n\}$ is non-decreasing.

To see this it suffices to show that, for every choice of $n \in \mathbb{N}$ and $\{x_j^*\} \subset S_{X^*}$ we have

$$K(n;\{x_j^*\}) \le K(n+1,\{x_j^*\}).$$

This last inequality is immediate, since for every choice of natural numbers $p_1 < \dots < p_{2n+2}$,

$$S(p_1,\dots,p_{2n};\{x_j^*\}) \supset S(p_1,\dots,p_{2n+2};\{x_j^*\}),$$

and consequently,

$$R(p_1,\dots,p_{2n};\{x_j^*\}) \le R(p_1,\dots,p_{2n+2};\{x_j^*\}).$$

To complete the proof we shall show, for every $\delta > 0$, the existence of a pair $x,y \in B_X$ satisfying

(16.13) $\|\frac{1}{2}(x+y)\| > 1-\delta$ and $\|\frac{1}{2}(x-y)\| > 1-\delta$.

By Lemma 16.3, this contradicts the assumption and we are done. So let $\delta > 0$ be arbitrary and choose r so that $1-\delta < r < 1$. By (16.11) and (16.12) there exists an $\varepsilon > 0$ and an $N \in \mathbb{N}$ such that

(16.14) $\dfrac{K_n - \varepsilon}{K_n+2\varepsilon} > r > 1-\delta$ for all $n \ge N$.

We claim that there even exists an $m \ge N$ such that

(16.15) $\dfrac{K_{m-1}-\varepsilon}{K_m + 2\varepsilon} > 1-\delta.$

This is simple if $\lim_{n\to\infty} K_n < \infty$, since then $\lim_{n\to\infty}(K_n - K_{n-1}) = 0$. In the general case, where possibly $\lim_{n\to\infty} K_n = \infty$, (16.15) easily follows from (16.14) once we have proved that

(16.16) $\liminf\limits_{n\to\infty} \dfrac{K_n}{K_{n-1}} = 1.$

To see (16.16), suppose $\liminf_{n\to\infty} \frac{K_n}{K_{n-1}} > \alpha > 1$. Then there exists an $n_0 \in \mathbb{N}$ such that $K_n > \alpha^{n-n_0} K_{n_0}$ for all $n \ge n_0$ and this clearly contradicts (16.5). (This is the only place where (16.5) is used.)

Having determined $\varepsilon > 0$ and $m \in \mathbb{N}$ so that (16.15) holds, we keep them fixed throughout the rest of the proof. By (16.4) there exists a sequence $\{x_j^*\} \subset S_{X^*}$ such that

$$K(m;\{x_j^*\}) < K_m + \varepsilon. \tag{16.17}$$

Also this sequence $\{x_j^*\}$ is kept fixed from now on. We now choose distinct natural numbers $p_1,\ldots,p_{2m}, q_1,\ldots,q_{2m}$ so that

$$p_1, q_1, p_2, p_3, q_2, q_3, \ldots, p_{2m-2}, p_{2m-1}, q_{2m-2}, q_{2m-1}, p_{2m}, q_{2m} \tag{16.18}$$

represents not only the order of choice but also the order of magnitude of these numbers. The choice is made in such a way that the following conditions are satisfied:

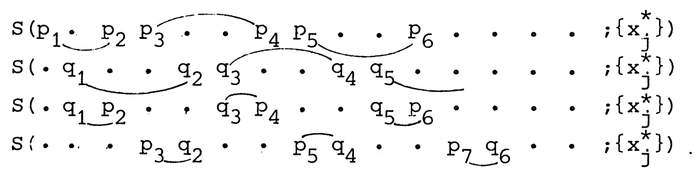
(a) $\exists u \in S(p_1,\ldots,p_{2m};\{x_j^*\}) : \|u\| < K(m;\{x_j^*\}) + \varepsilon$, i.e.
$R(p_1,\ldots,p_{2m};\{x_j^*\}) < K(m;\{x_j^*\}) + \varepsilon$,
$\exists v \in S(q_1,\ldots,q_{2m};\{x_j^*\}) : \|v\| < K(m;\{x_j^*\}) + \varepsilon$, i.e.
$R(q_1,\ldots,q_{2m};\{x_j^*\}) < K(m;\{x_j^*\}) + \varepsilon$.

(b) $\forall x \in S(q_1,p_2,q_3,p_4,q_5,p_6,\ldots,q_{2m-1},p_{2m};\{x_j^*\}) : K(m;\{x_j^*\}) - \varepsilon \le \|x\|$, i.e.
$R(q_1,p_2,q_3,p_4,q_5,p_6,\ldots,q_{2m-1},p_{2m};\{x_j^*\}) \ge K(m;\{x_j^*\}) - \varepsilon$.

(c) $\forall x \in S(p_3,q_2,p_5,q_4,p_7,q_6,\ldots,p_{2m-1},q_{2m-2};\{x_j^*\}) : K(m-1;\{x_j^*\}) - \varepsilon \le \|x\|$, i.e.
$R(p_3,q_2,p_5,q_4,p_7,q_6,\ldots,p_{2m-1},q_{2m-2}\{x_j^*\}) \ge K(m-1;\{x_j^*\}) - \varepsilon$.

It follows from the definition of $K(m;\{x_j^*\})$ and $K(m-1;\{x_j^*\})$ that this choice is possible. (Note that in each of the sets $S(\ldots\ \ldots\ ;\{x_j^*\})$ occurring in (a), (b) and (c) the integers appear in the same order as in (16.18).) The following scheme is an attempt to visualize the situation:

(16.19)



The symbols $\smile$ and $\frown$ have the following meaning: If x is an element of any one of the above four sets then by (16.2) we have either $\frac{1}{2} \le \langle x, x_k^* \rangle \le 1$ or $\frac{1}{2} \le -\langle x, x_k^* \rangle \le 1$, depending on the interval in which k lies. We have indicated with $\smile$ the intervals of the k's for which any element in the set satisfies $\frac{1}{2} \le \langle x, x_k^* \rangle \le 1$ and with $\frown$ the others.

It is obvious from (16.19) that (a) implies

$$\tfrac{1}{2}(u+v) \in S(q_1,p_2,q_3,p_4,q_5,p_6,\ldots,q_{2m-1},p_{2m};\{x_j^*\}) \tag{16.20}$$

and

$$(16.21)\qquad \tfrac{1}{2}(v-u) \in S(p_3,q_2,p_5,q_4,p_7,q_6,\dots,p_{2m-1},q_{2m-2};\{x_j^*\}).$$

From (16.20) and (b) it follows now that

$$(16.22)\qquad \|\tfrac{1}{2}(u+v)\| \geq K(m;\{x_j^*\}) - \varepsilon,$$

and from (16.21) and (c) that

$$(16.23)\qquad \|\tfrac{1}{2}(u-v)\| \geq K(m-1;\{x_j^*\}) - \varepsilon.$$

We claim that

$$x := \frac{u}{K_m+2\varepsilon} \quad\text{and}\quad y := \frac{v}{K_m+2\varepsilon}$$

satisfy (16.13). Indeed, by (a) and (16.17),

$$(16.24)\qquad \|x\| = \frac{\|u\|}{K_m+2\varepsilon} < \frac{K(m;\{x_j^*\}) + \varepsilon}{K_m + 2\varepsilon} < \frac{K_m + 2\varepsilon}{K_m + 2\varepsilon} = 1,$$

and similarly,

$$(16.25)\qquad \|y\| < 1.$$

Also, by (16.22), (16.4) and (16.14),

$$(16.26)\qquad \|\tfrac{1}{2}(x+y)\| = \frac{\|\tfrac{1}{2}(u+v)\|}{K_m + 2\varepsilon} \geq \frac{K(m;\{x_j^*\}) - \varepsilon}{K_m + 2\varepsilon} \geq \frac{K_m - \varepsilon}{K_m + 2\varepsilon} > 1-\delta,$$

and, by (16.23), (16.4) and (16.15)

$$(16.27)\qquad \|\tfrac{1}{2}(x-y)\| = \frac{\|\tfrac{1}{2}(u-v)\|}{K_m + 2\varepsilon} \geq \frac{K(m-1;\{x_j^*\}) - \varepsilon}{K_m + 2\varepsilon} \geq \frac{K_{m-1} - \varepsilon}{K_m + 2\varepsilon} > 1-\delta.$$

This completes the proof. □

The conclusion can even be strengthened to superreflexivity, as we now show.

COROLLARY 16.5. *Let* X *be a Banach space and suppose that* $\delta_X(\varepsilon) > 0$ *for some* $0 < \varepsilon < 2$. *Then* X *is superreflexive.*

PROOF. Suppose not. Then there exists a non-reflexive Y such that $Y \prec X$. By Theorem 16.4, $\delta_Y(\eta) = 0$ for all $0 < \eta < 2$. Hence, since the number $\delta_X(\varepsilon)$ is positive, there exist $y_1, y_2 \in Y$ satisfying

$$\|y_1\|,\|y_2\| < 1, \quad \|y_1-y_2\| > \varepsilon \quad \text{and} \quad 1 - \left\|\frac{y_1+y_2}{2}\right\| < \delta_X(\varepsilon).$$

A suitable isomorphism $T\colon sp\{y_1,y_2\} \to X$ now yields elements $x_1 := Ty_1$, $x_2 := Ty_2$ in X satisfying

$$\|x_1\|,\|x_2\| < 1, \quad \|x_1-x_2\| > \varepsilon \quad \text{and} \quad 1 - \left\|\frac{x_1+x_2}{2}\right\| < \delta_X(\varepsilon),$$

contradicting the definition of δ_X. □

In conjunction with Theorem 15.5, Corollary 16.5 leads to

COROLLARY 16.6. *A Banach space* X *is superreflexive iff* X *has an equivalent norm* $|||\cdot|||$ *satisfying* $\delta_{(X,|||\ |||)}(\varepsilon) > 0$ *for some* $0 < \varepsilon < 2$, *i.e. such that* $(X,|||\ |||)$ *is uniformly non-square.*

NOTES. Theorem 16.4 was proved by R.C. JAMES in [52]. Corollary 16.6 is also due to R.C. JAMES (see [54]), modulo P. ENFLO's result ([33]). The main result Theorem 16.4 can be restated as follows: $\ell_2^1 \prec X$ for every non-reflexive space. The so-called ℓ_n^1-problem is a direct generalization of this: If X is non-reflexive, is it true that $\ell_n^1 \prec X$ for all $n \in \mathbb{N}$? This question was raised in [52] by R.C. JAMES. In the same paper he proved that the answer is positive for all non-reflexive spaces having an unconditional basis. However, he has shown recently ([57]) that in general the answer is negative, by the construction of a non-reflexive space X which is uniformly non-octahedral, i.e. $\ell_3^1 \not\prec X$. There are some partial positive results for special classes of non-reflexive spaces. E.g. if $X^{**}/_{\pi X}$ is non-reflexive, then $\ell_3^1 \prec X$. For the proof of this and similar statements for arbitrary $n \in \mathbb{N}$ we refer to [22].

17. GIRTHS OF UNIT SPHERES AND FLAT BANACH SPACES

We introduce in this section a geometric parameter for Banach spaces called the girth of the unit ball. It is the infimum of the lengths of all closed curves γ in S_X which are centrally symmetric (i.e. $x \in \gamma$ iff $-x \in \gamma$). In the next section we shall connect girth and superreflexivity. It will be shown that a space is superreflexive iff the girth of its unit ball is larger than 4. If there exists on S_X a closed centrally symmetric curve of length exactly 4 (i.e. if the girth is 4 and is a minimum), then X is called flat. The reason for the term "flat" will become clear in the course of this section. Among other things we shall show that flat spaces have non-separable duals which are also flat. In particular flat spaces are non-reflexive. More strongly, it can be shown that they cannot be isomorphically embedded into any separable conjugate space. This last result follows from a characterization of flat spaces as those spaces which satisfy the so-called infinite supported tree property. This in turn will lead to a result connecting flatness and superreflexivity: X is superreflexive iff every space Y with $Y \prec X$ fails to be flat.

In this section we lay the groundwork with some preliminary results on girth and flatness. Connections with superreflexivity will be discussed in the next section.

DEFINITION 17.1. Let X be a Banach space and let $g\colon [a,b] \to X$ be a continuous map from a segment $[a,b] \subset \mathbb{R}$ into X. The image $\gamma := g([a,b])$ is called a *curve* in X. $g(a)$ and $g(b)$ are called the *initial* and the *endpoint* of γ, respectively, and g is called a *representation* or *parametrization* of γ. If a curve γ has a representation $g\colon [a,b] \to X$ which is injective and therefore a homeomorphism, then it is called *simple*. The *length* $\ell(\gamma)$ of a simple curve with injective representation g is defined by

$$\ell(\gamma) := \sup\{ \sum_{i=1}^{n} \| g(t_i) - g(t_{i-1}) \| \},$$

where the sup is taken over all $n \in \mathbb{N}$ and all partitions $a = t_0 < t_1 < \dots$ $\dots < t_n = b$ of $[a,b]$. Observe that $\ell(\gamma)$ does not depend on the representation g, as long as g is injective. If $\ell(\gamma) < \infty$, then γ is called *rectifiable*.

REMARK 17.2. We shall work only with simple curves in these notes. Of course a simple curve has numerous different parametrizations. We single out a special one, called the *standard representation* (in terms of arc length), as follows. Let $g\colon [a,b] \to X$ be an injective representation of a simple curve γ. Let us consider the function $h\colon [a,b] \to [0,\ell(\gamma)]$ defined by $h(t) = \ell(g([a,t]))$, $a \le t \le b$. Since g is injective, h is strictly increasing. It is also continuous. In fact the continuity from the left is obvious and the continuity from the right follows from the formula

$$\ell(\gamma) = \ell(g([a,t])) + \ell(g([t,b])) \qquad (a \le t \le b)$$

and the obvious right continuity of $t \to \ell(g([t,b]))$. Thus h is a homeomorphism. Now the representation

$$g_\gamma := gh^{-1}\colon [0,\ell(\gamma)] \to X$$

is called the *standard representation* (*in terms of arc length*) of γ. It is evidently characterized by the property that

$$(17.1) \qquad \ell(g_\gamma([\alpha,\beta])) = \beta - \alpha \qquad \text{for all } 0 \le \alpha \le \beta \le \ell(\gamma).$$

DEFINITION 17.3. Let X be a Banach space with $\dim X \ge 2$. For every $x \in S_X$ let $m(x)$ be the infimum of the lengths of all simple curves on S_X with initial point x and endpoint $-x$. Put

$$m(X) := \inf\{m(x)\colon x \in S_X\}.$$

The number $2m(X)$ is called the *girth* of B_X (or of X). Clearly $2m(X)$ can be defined alternatively as the infimum of the lengths of all curves $\gamma = g([a,b])$ on S_X which are closed (i.e. $g(a) = g(b)$ in the only multiple point) and centrally symmetric (i.e. $x \in \gamma$ iff $-x \in \gamma$). This better explains the term "girth".

EXAMPLES. (a) Let X be $\mathbb{R}^2$ with a parallelogram as its unit sphere. It is not difficult to see that $m(x) = 4$ for all $x \in S_X$, so that the girth of X is 8. In fact there are only two simple curves on S_X connecting x and $-x$ and the length of each of them is one half of the perimeter of the parallelogram.

(b) If X is $\mathbb{R}^2$ with an affinely regular hexagon as its unit sphere, then it is easily seen that $m(X) = 3$ and therefore the girth of X is 6.

REMARK 17.4. It is possible to show that for every 2-dimensional Banach space X we have $3 \le m(X) \le 4$ and that $m(X) = 3$ iff $_X$ is an affinely regular hexagon and $m(X) = 4$ iff S_X is a parallelogram. A consequence is that for every Banach space X (of finite or infinite dimension ≥ 2) we have $2 \le m(X) \le 4$. In fact $2 \le m(X)$ is a trivial consequence of the triangle inequality and $m(X) \le 4$ follows because $m(X) \le m(Y)$ for every subspace $Y \subset X$, and $m(Y) \le 4$ if $\dim Y = 2$, by the above.

DEFINITION 17.5. A Banach space X is called *flat* if there exists an $x \in S_X$ and a simple curve γ on S_X with initial point x and endpoint $-x$ and with length $\ell(\gamma) = 2$. Such a curve is called a *girth curve.*

It seems to contradict our geometric intuition that such a curve should ever exist. In fact no such thing is possible in a finite-dimensional space as we shall soon see. But even for infinite-dimensional spaces the phenomenon of flatness may be quite surprising. Here is an example.

EXAMPLE. Let X be $C([0,1])$ and let us define for each $s \in [0,2]$ a function $x_s \in X$ by

$$x_s(t) = \begin{cases} 2t + (1-s) & \text{if } 0 \le t \le \frac{s}{2} \\ -2t + (1+s) & \text{if } \frac{s}{2} < t \le 1. \end{cases}$$

One easily checks that $\|x_s\| = 1$ for all $s \in [0,2]$, $\|x_{s_1}-x_{s_2}\| = |s_1-s_2|$ for all $s_1,s_2 \in [0,2]$, and $x_0 = -x_2$. Thus $g: [0,2] \to X$ defined by $g(s) = x_s$ $(0 \le s \le 2)$ is the standard representation of a simple curve on S_X with length 2, joining the antipodal points x_0 and x_2. Therefore $C([0,1])$ is flat.

We now investigate flatness more thoroughly. Let X be a flat Banach space and let $g: [0,2] \to X$ be the standard representation of a girth curve. Then, by (17.1), for any $0 \le s < t \le 2$,

$$2 = \|g(0)-g(2)\| \le \|g(0)-g(s)\| + \|g(s)-g(t)\| + \|g(t)-g(2)\|$$

$$\le s + (t-s) + (2-t) = 2$$

and therefore

$$\|g(s)-g(t)\| = |s-t| \quad \text{for all } s,t \in [0,2]. \tag{17.2}$$

For each $t \in [0,2]$ let $f^*(t) \in X^*$ be a support functional at $g(t)$, i.e.

$$(17.3) \qquad \langle g(t), f^*(t)\rangle = \|f^*(t)\| = 1.$$

By (17.3), (17.2) and $g(0) = -g(2)$ we have the following inequalities for all $s,t \in [0,2]$:

$$(17.4) \qquad |\langle g(s), f^*(t)\rangle - 1| = |\langle g(s)-g(t), f^*(t)\rangle| \le \|g(s)-g(t)\| = |s-t|,$$

and

$$|\langle g(s), f^*(t)\rangle + 1| = |\langle g(s)+g(t), f^*(t)\rangle| \le \|g(s)+g(t)\| \le$$

$$(17.5) \qquad \le \begin{cases} \|g(s)-g(0)\| + \|g(2)-g(t)\| = s + (2-t) \\ \|g(s)-g(2)\| + \|g(0)-g(t)\| = (2-s)+t. \end{cases}$$

It follows from (17.4) and (17.5) that

$$(17.6) \qquad \langle g(s), f^*(t)\rangle = 1 - |s-t| \quad \text{for all } s,t \in [0,2].$$

This equality (17.6) will be useful in the proof of the next theorem characterizing flat Banach spaces as those possessing a certain special tree property which we now define.

DEFINITION 17.6. A Banach space X is said to have the *infinite supported tree property* (*ISTP*) iff there exists a subset $T = \{x_{k,i}: k = 1,2,\ldots; i = 1,\ldots,2^k\}$ of S_X and a subset $\{x^*_{k,i}: k = 0,1,\ldots; i = 1,\ldots,2^k\}$ of S_{X^*} such that

$$(17.7) \qquad x_{k,i} = \tfrac{1}{2}(x_{k+1,2i-1} + x_{k+1,2i}) \quad (k = 1,2,\ldots; i = 1,\ldots,2^k)$$

and such that

$$(17.8) \qquad \langle x_{k_1,i_1}, x^*_{k_2,i_2}\rangle = \begin{cases} -1 & \text{if } \dfrac{i_1}{2^{k_1}} \le \dfrac{2i_2-1}{2^{k_2+1}} \\[2ex] +1 & \text{if } \dfrac{2i_2-1}{2^{k_2+1}} \le \dfrac{i_1-1}{2^{k_1}} \end{cases}$$

$$(k_1 = 1,2,\ldots;\ k_2 = 0,1,\ldots;\ i_1 = 1,\ldots,2^{k_1};\ i_2 = 1,\ldots,2^{k_2}).$$

(17.8) means that T is "supported" by two hyperplanes in the following sense: each point x_{k_1,i_1} either lies in one of the two hyperplanes $\{x: \langle x, x^*_{k_2,i_2}\rangle = 1\}$, $\{x: \langle x, x^*_{k_2,i_2}\rangle = -1\}$ (namely, whenever $k_1 > k_2$), or, by (17.7), is a finite convex combination of points that do.

Note furthermore that (17.7) and (17.8) imply that T is a $(\infty,2)$-tree, since, for any $k+1 \in \mathbb{N}$ and $i \in \{1,\dots,2^k\}$, $\langle x_{k+1,2i-1}, x^*_{k,i}\rangle$, $\langle x_{k+1,2i}, x^*_{k,i}\rangle = +1$, so that $\|x_{k+1,2i-1} - x_{k+1,2i}\| = 2$.

<u>THEOREM 17.7</u>. *A Banach space X is flat iff it has the infinite supported tree property.*

<u>PROOF</u>. Suppose first that X is flat. Let $g\colon [0,2] \to X$ be the standard representation of a girth curve and let $f^*\colon [0,2] \to X^*$ be as in (17.3), so that (17.6) holds. Put

$$x_{k,i} := 2^k\left[g\left(\frac{i-1}{2^k}\right) - g\left(\frac{i}{2^k}\right)\right] \qquad (k = 1,2,\dots;\ i = 1,\dots,2^k). \tag{17.9}$$

It is immediate from (17.2) that $\|x_{k,i}\| = 1$. Also (17.9) implies that

$$x_{k,i} = 2^k\left[g\left(\frac{2(i-1)}{2^{k+1}}\right) - g\left(\frac{2i-1}{2^{k+1}}\right)\right] + 2^k\left[g\left(\frac{2i-1}{2^{k+1}}\right) - g\left(\frac{2i}{2^{k+1}}\right)\right] =$$

$$= \tfrac{1}{2}(x_{k+1,2i-1} + x_{k+1,2i}).$$

Finally, let us define

$$x^*_{k,i} := f^*\left(\frac{2i-1}{2^{k+1}}\right) \qquad (k = 0,1,\dots;\ i = 1,\dots,2^k).$$

Then (17.6) yields

$$\langle x_{k_1,i_1}, x^*_{k_2,i_2}\rangle = \left\langle 2^{k_1}\left[g\left(\frac{i_1-1}{2^{k_1}}\right) - g\left(\frac{i_1}{2^{k_1}}\right)\right], f^*\left(\frac{2i_2-1}{2^{k_2+1}}\right)\right\rangle =$$

$$= \begin{cases} -1 & \text{if } \dfrac{i_1}{2^{k_1}} \le \dfrac{2i_2-1}{2^{k_2+1}} \\[2ex] +1 & \text{if } \dfrac{2i_2-1}{2^{k_2+1}} \le \dfrac{i_1-1}{2^{k_1}}\ . \end{cases}$$

We have thus proved that X has the ISTP.

Conversely, suppose X has the ISTP. Let $\{x_{k,i}\colon k = 1,2,\dots;\ i = 1,\dots,2^k\} \subset S_X$ and $\{x^*_{k,i}\colon k = 0,1,\dots;\ i = 1,\dots,2^k\} \subset S_{X^*}$ be as in Definition 17.6. Repeated application of (17.7) shows that, for all $p \ge k$,

$$x_{k,i} = \frac{1}{2^{p-k}} \sum_{j=2^{p-k}(i-1)+1}^{2^{p-k}\cdot i} x_{p,j} \qquad (k = 1,2,\dots;\ i = 1,\dots,2^k). \tag{17.10}$$

We now define $g: \{j/2^k: k = 0,1,\ldots; j = 0,\ldots,2^{k+1}\} \to X$ as follows:

$$(17.11) \qquad g\left(\frac{j}{2^k}\right) = -\frac{1}{2^{k+1}}\sum_{i=1}^{j} x_{k+1,i} + \frac{1}{2^{k+1}}\sum_{i=j+1}^{2^{k+1}} x_{k+1,i},$$

where, of course, one of the sums may be empty. It follows from (17.10) that g is well defined, meaning that the value $g\left(\frac{j}{2^k}\right)$ does not depend on the representation of $\frac{j}{2^k}$. Also clearly $g(0) = -g(2)$, and obviously $\|g\left(\frac{j}{2^k}\right)\| \le 1$. We show now that in fact $\|g\left(\frac{j}{2^k}\right)\| = 1$.

Indeed, if we fix $k \in \mathbb{N}$ and $j \in \{1,\ldots,2^{k+1}\}$, then for every $\ell \in \mathbb{N}$, by (17.10) and (17.11),

$$g\left(\frac{j}{2^k}\right) = g\left(\frac{j\cdot 2^\ell}{2^{k+\ell}}\right) = -\frac{1}{2^{k+\ell+1}}\sum_{i=1}^{j\cdot 2^\ell} x_{k+\ell+1,i} + \frac{1}{2^{k+\ell+1}}\sum_{i=j2^\ell+1}^{2^{k+\ell+1}} x_{k+\ell+1,i}.$$

(17.8) implies that

$$\langle x_{k+\ell+1,i}, x^*_{k+\ell+1,j\cdot 2^\ell}\rangle = \begin{cases} -1 & \text{if } i \in \{1,\ldots,j\cdot 2^\ell-1\} \\ +1 & \text{if } i \in \{j\cdot 2^\ell+1,\ldots,2^{k+\ell+1}\}. \end{cases}$$

Thus $\langle g\left(\frac{j}{2^k}\right), x^*_{k+\ell+1,j\cdot 2^\ell}\rangle \ge 1 - \frac{2}{2^{k+\ell+1}}$. Consequently, since $\|x^*_{k+\ell+1,j\cdot 2^\ell}\| = 1$, $\|g\left(\frac{j}{2^k}\right)\| \le 1$ and since $\ell \in \mathbb{N}$ was arbitrary, we have $\|g\left(\frac{j}{2^k}\right)\| = 1$ for $j = 1,\ldots,2^{k+1}$, and, of course, also for $j = 0$ because $\|g(0)\| = \|g(2)\|$.

We now estimate $\|g\left(\frac{j_1}{2^k}\right)-g\left(\frac{j_2}{2^k}\right)\|$ for $0 \le j_1 < j_2 \le 2^{k+1}$, using (17.11):

$$\left\|g\left(\frac{j_1}{2^k}\right) - g\left(\frac{j_2}{2^k}\right)\right\| \le \frac{2}{2^{k+1}}\left\|\sum_{i=j_1+1}^{j_2} x_{k+1,i}\right\| \le \frac{j_2-j_1}{2^k}.$$

This shows that g is Lipschitz continuous with constant 1 on its domain of definition $\{\frac{j}{2^k}: k = 0,1,\ldots; j = 0,\ldots,2^{k+1}\}$. The unique continuous extension to $[0,2]$ of g is then again Lipschitz continuous with constant 1 and satisfies $\|g(t)\| = 1$ for all $t \in [0,2]$. Since, moreover, $g(0) = -g(2)$, $g: [0,2] \to X$ is the standard representation of a girth curve. This completes the proof that X is flat. □

<u>REMARK 17.8</u>. The following weaker property is called the infinite supported tree property by R.E. HARRELL & L.A. KARLOVITZ ([43]): There exists a $\delta > 0$ and subsets $\{x_{k,i}: k = 1,2,\ldots; i = 1,\ldots,2^k\}$ of B_X, and $\{x^*_{k,i}: k = 0,1,\ldots; i = 1,\ldots,2^k\}$ of B_{X^*}, such that for all appropriate indices

$$x_{k,i} = \tfrac{1}{2}(x_{k+1,2i-1} + x_{k+1,2i})$$

and

$$\langle x_{k_1,i_1}, x^*_{k_2,i_2}\rangle = \begin{cases} -\delta & \text{if } \frac{i_1}{2^{k_1}} \le \frac{2i_2-1}{2^{k_2+1}} \\ \delta & \text{if } \frac{2i_1-1}{2^{k_1+1}} \le \frac{i_2-1}{2^{k_2}} . \end{cases}$$

A slight modification of the above proof will show that this property characterizes those Banach spaces that are isomorphic to a flat Banach space. We shall not need this here and therefore omit the details. One consequence we would like to mention though. We have already remarked that separable dual spaces admit no bounded (∞,ε)-trees (Remark 11.7). Since the ISTP as formulated in this remark implies that the set $\{x_{k,i}: k = 1,2,\dots;$ $i = 1,\dots,2^k\} \subset B_X$ is a $(\infty,2\delta)$-tree (compare the observation directly preceeding Theorem 17.7), it follows that a flat Banach space cannot be isomorphically embedded into a separable dual space. In particular a flat space is not reflexive. This last fact is also a simple consequence of Proposition 17.11 below.

Our next result is that flatness is preserved under duality.

PROPOSITION 17.9. *If a Banach space* X *is flat then so is its dual* X^*.

PROOF. Let $g: [0,2] \to X$ be the standard representation of a girth curve and let $f^*: [0,2] \to X^*$ be as in (17.3). For every $s \in [0,2]$ and $n \in \mathbb{N}$ we define

$$(17.12) \qquad g^*_n(s) = \frac{1}{2n}\left(-\sum_{i=1}^{[ns]} f^*(\tfrac{i}{n}) + \sum_{i=[ns]+1}^{2n} f^*(\tfrac{i}{n})\right),$$

where $[ns]$ denotes the largest integer $\le ns$. Clearly $\|g^*_n(s)\| \le 1$ and $g^*_n(0) = -g^*_n(2)$. Using the w^*-compactness of B_{X^*}, let $g^*(s)$ be a w^* limit point of $\{g^*_n(s)\}_{n=1}^{\infty}$, for each $s \in [0,2]$. Then evidently $g^*(0) = -g^*(2)$ and $\|g^*(s)\| \le 1$. We show now that in fact $\|g^*(s)\| = 1$ $(0 \le s \le 2)$. To see this, let $s \in [0,2)$ be arbitrary and let $0 < \varepsilon \le 2-s$. Then, putting $x_\varepsilon := \frac{1}{\varepsilon}(g(s+\varepsilon) - g(s))$, we have $\|x_\varepsilon\| = 1$, by (17.2). Also it follows from (17.6) that

$$\langle x_\varepsilon, f^*(\tfrac{i}{n})\rangle = \begin{cases} -1 & \text{if } \frac{i}{n} \le s \\ 1 & \text{if } \frac{i}{n} \ge s+\varepsilon \\ \ge -1 & \text{otherwise.} \end{cases}$$

Therefore

$$\langle x_\varepsilon, g_n^*(s)\rangle = \frac{1}{2n}\left(-\sum_{i=1}^{[ns]}(-1) + \sum_{i=[ns]+1}^{[n(s+\varepsilon)]}\langle x_\varepsilon, f^*(\tfrac{i}{n})\rangle + \sum_{i=[n(s+\varepsilon)]+1}^{2n}(1)\right)$$

$$\geq 1 - \left(\frac{[n(s+\varepsilon)]-[ns]}{n}\right) \geq 1 - \frac{n\varepsilon+1}{n} = 1-\varepsilon-\frac{1}{n} \quad (n = 1,2,\ldots).$$

Hence $\langle x_\varepsilon, g^*(s)\rangle \geq 1-\varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small and $\|g^*(s)\| \leq 1$, $\|x_\varepsilon\| = 1$, it follows that $\|g^*(s)\| = 1$ $(0 \leq s < 2)$. Since $g^*(0) = -g^*(2)$, we now have $\|g^*(s)\| = 1$ for all $s \in [0,2]$.

Finally we estimate $\|g^*(s_1) - g^*(s_2)\|$ for $0 \leq s_1 < s_2 \leq 2$, using (17.12). For all $n \in \mathbb{N}$

$$\|g_n^*(s_1) - g_n^*(s_2)\| = \frac{2}{2n}\left\|\sum_{i=[ns_1]+1}^{[ns_2]} f^*(\tfrac{i}{n})\right\| \leq \frac{1}{n}([ns_2]-[ns_1]) \leq s_2-s_1+\frac{1}{n}.$$

Since $g^*(s_1) - g^*(s_2)$ is a w^* limit point of $\{g_n^*(s_1)-g_n^*(s_2)\}$, it follows that $\|g^*(s_1)-g^*(s_2)\| \leq s_2-s_1$. Together with $g^*(0) = -g^*(2)$ and $\|g^*(s)\| = 1$ $(0 \leq s \leq 2)$ this shows that $g^*\colon [0,2] \to X^*$ is the standard representation of a girth curve. Thus X^* is flat. □

<u>REMARK 17.10</u>. There is no converse to Proposition 17.9. Indeed, $\ell^\infty = (\ell^1)^*$ is flat (since it contains the flat space $C([0,1])$ isometrically, see Proposition 0.17), but ℓ^1 is not flat. In fact we have even seen that ℓ^1, as a separable dual space, does not have the infinite tree property (Remark 11.7), let alone the infinite supported tree property.

We have observed in Remark 17.8 that a flat Banach space cannot be embedded into a separable dual space. Together with Proposition 17.9 this implies that the dual of a flat space is non-separable. This can also be seen directly in a simple way.

<u>PROPOSITION 17.11</u>. *Let X be a flat Banach space. Then* X^* *is nonseparable.*

<u>PROOF</u>. Let $g\colon [0,2] \to X$ be the standard representation of a girth curve and let $f^*\colon [0,2] \to X^*$ be as in (17.3). It suffices to show that $\|f^*(t_1)-f^*(t_2)\| = 2$ for all $0 \leq t_1 < t_2 \leq 2$. Since $\|f^*(t)\| = 1$ for all $t \in [0,2]$, we have $\|f^*(t_1)-f^*(t_2)\| \leq 2$. Equality follows by taking $x := \frac{1}{t_2-t_1}\{g(t_2)-g(t_1)\}$ and observing that, by (17.2), $\|x\| = 1$ and, by (17.6), $\langle x, f^*(t_1)-f^*(t_2)\rangle = -2$. □

<u>COROLLARY 17.12</u>. *A reflexive Banach space X is not flat.*

PROOF. Suppose that γ is a girth on S_X. Application of Proposition 17.11 to the flat separable reflexive space $Y := \overline{sp}\,\gamma$ yields a contradiction. □

The following result justifies the term "flat". It says essentially that every point of a girth curve is contained in a large flat area in the unit sphere. Before stating it we introduce some notation.

Let $g: [0,2] \to X$ be the standard representation of a girth curve γ on S_X. For every $s \in (0,2]$ and $h \in (0,s]$ we define the *difference quotient* $\Delta_g(s,h)$ by

$$\Delta_g(s,h) = h^{-1}(g(s-h) - g(s)).$$

Furthermore, for each $s \in [0,2]$ we define the *chord set* of g, denoted by $\chi(g,s)$, by

$$\chi(g,s) := \overline{co}(\{-\Delta_g(t,h): 0 < h \le t \le s\} \cup \{\Delta_g(t,h): 0 < h \le t-s \le 2-s\}).$$

If $s = 0$ (respectively, $s = 2$), the first (respectively, the second) set on the right is empty. In words, $\chi(g,s)$ is formed by taking all chord vectors of γ whose initial and endpoints both lie on one side of $g(s)$ and which point towards $g(s)$, normalizing them and forming their closed convex hull. Finally, $f^*: [0,2] \to X^*$ is as in (17.3).

PROPOSITION 17.13. *Let X be a flat Banach space and let $g: [0,2] \to X$ be the standard representation of a girth curve γ on S_X. Then for each $s \in [0,2]$*

$$(17.13) \qquad g(s) \in \chi(g,s) \subset \{x: \langle x,f^*(s)\rangle = 1\} \cap S_X.$$

Moreover, for every $y \in \chi(g,s)$,

$$(17.14) \qquad \sup_{x\in\chi(g,s)} \|y-x\| = \sup_{x,z\in\chi(g,s)} \|x-z\| = 2,$$

i.e., diam $\chi(g,s) = 2$ *and each point of* $\chi(g,s)$ *is a diametral.*

Finally, if $X = \overline{sp}\,\gamma$, then

$$(17.15) \qquad \text{closed affine hull } \chi(g,s) = \{x: \langle x,f^*(s)\rangle = 1\}.$$

PROOF. It is immediate that, for $0 < s < 2$,

$$\frac{2-s}{2}\,\Delta_g(2,2-s) + \frac{s}{2}(-\Delta_g(s,s)) = g(s),$$

so that $g(s) \in \chi(g,s)$. Similarly

$$g(0) = \tfrac{1}{2}(g(0) - g(2)) = \Delta_g(2,2) \in \chi(g,0)$$

and

$$g(2) = -\tfrac{1}{2}(g(0) - g(2)) = -\Delta_g(2,2) \in \chi(g,2).$$

Since also, by (17.6),

$$\langle \Delta_g(t,h), f^*(s)\rangle = \begin{cases} -1 & \text{if } t \le s \\ +1 & \text{if } s \le t-h, \end{cases} \tag{17.16}$$

it follows that $\chi(g,s) \subset \{x\colon \langle x, f^*(s)\rangle = 1\}$. Therefore, since $\|f^*(s)\| = 1$, we must have $\|x\| = 1$ whenever $x \in \chi(g,s)$. This finishes the proof of (17.13).

To prove (17.14), let us first observe that, whenever $0 < h' < h \le s$,

$$\Delta_g(s,h) = (h-h')h^{-1}\Delta_g(s-h',h-h') + h'h^{-1}\Delta_g(s,h'). \tag{17.17}$$

Using (17.16) and (17.17), we obtain, whenever $0 < h' < h \le t \le s \le 2$,

$$\begin{aligned} \langle -\Delta_g(t,h), f^*(s-h')\rangle &= (h-h')h^{-1}\langle -\Delta_g(t-h',h-h'), f^*(s-h')\rangle + \\ &\quad + h'h^{-1}\langle -\Delta_g(t,h'), f^*(s-h')\rangle = \\ &= (h-h')h^{-1} - h'h^{-1}A, \end{aligned} \tag{17.18}$$

with $|A| \le 1$, and, whenever $0 < h' < h \le t-s \le 2-s$,

$$\begin{aligned} \langle \Delta_g(t,h), f^*(s-h')\rangle &= (h-h')h^{-1}\langle \Delta_g(t-h',h-h'), f^*(s-h')\rangle + \\ &\quad + h'h^{-1}\langle \Delta_g(t,h'), f^*(s-h')\rangle = \\ &= (h-h')h^{-1} + h'h^{-1}. \end{aligned} \tag{17.19}$$

Hence, for fixed $s \in (0,2]$ and any $y \in \{-\Delta_g(t,h)\colon 0 < h \le t \le s\} \cup \{\Delta_g(t,h)\colon 0 < h \le t-s \le 2-s\}$, we obtain from (17.18) and (17.19),

$$\lim_{h'\to 0} \langle y, f^*(s-h')\rangle = 1.$$

Obviously the same then holds for all $y \in \chi(g,s)$, since f^* is a bounded function. Thus

$$\lim_{h'\to 0} \langle y, f^*(s-h')\rangle = 1 \quad \text{for every } s \in (0,2] \text{ and } y \in \chi(g,s). \tag{17.20}$$

Hence, if $s \in (0,2]$ and $y \in \chi(g,s)$, by (17.16) and (17.20),

$$\begin{aligned} \|y - (-\Delta_g(s,h')\| &\ge \langle y, f^*(s-h')\rangle + \langle \Delta_g(s,h'), f^*(s-h')\rangle = \\ &= \langle y, f^*(s-h')\rangle + 1 \to 2 \quad \text{as } h' \to 0. \end{aligned}$$

Since $-\Delta_g(s,h') \in \chi(g,s)$ for $0 < h' \leq s$, this shows that $\sup\{\|y-x\|: x \in \chi(g,s)\} \geq 2$. On the other hand, $\chi(g,s) \subset B_X$, so that $\sup\{\|y-x\|: x \in \chi(g,s)\} \leq 2$. This proves (17.14) for $0 < s \leq 2$. Since also $\chi(g,0) = -\chi(g,2)$, this finishes the proof of (17.14).

Finally, assuming that $X = \overline{\text{sp}}\ \gamma$, it follows, using $g(0) = -g(2)$, that

$$X = \overline{\text{sp}}\{\Delta_g(t,h) : 0 < h \leq t \leq 2\}. \quad \text{Fix } s \in [0,2].$$

It is easily verified by means of (17.17) that any $\Delta_g(t,h)$ $(0 < h \leq t \leq 2)$ and therefore any $x \in \text{sp}\{\Delta_g(t,h) : 0 < h \leq t \leq 2\}$ can be written as a linear combination of elements from the set

$$A := \{-\Delta_g(t,h) : 0 < h \leq t \leq s\} \cup \{\Delta_g(t,h) : 0 < h \leq t-s \leq 2-s\}.$$

Thus sp A is dense in X. Since, by (17.16), $f^*(s)$ is identically 1 on A, it follows that an element $x \in \text{sp}\ A$ satisfies $\langle x, f^*(s)\rangle = 1$ iff $x \in$ affine hull A. By the continuity of $f^*(s)$ and the density of sp A, (17.15) now follows. □

NOTES. The concept of girth was introduced by J.J. SCHÄFFER in [90], and studied thereafter in several other papers. The monograph [91] contains a systematic account of the present knowledge about girth and other geometric parameters in Banach spaces. The statements made in Remark 17.4 are proved in [90]. Flat Banach spaces were first defined and studied by R.E. HARRELL & L.A. KARLOVITZ ([41],[42]). Flatness of C([0,1]) was observed in [41]. Theorem 17.7 is a modification of the main result in [43]. Propositions 17.9 and 17.11 are due to L.A. KARLOVITZ ([65]) and Proposition 17.13 to R.E. HARRELL & L.A. KARLOVITZ ([42]).

18. SUPERREFLEXIVITY, GIRTH AND FLATNESS

We prove two main results in this section. The first characterizes superreflexivity in terms of girth. It says that a space is superreflexive iff the girth of its unit sphere is larger than 4. The second connects superreflexivity (or rather its negation) with flatness: a space X is not superreflexive iff there exists a flat space Y which is finitely representable in X.

We have seen (Theorem 16.4) that a non-reflexive Banach space X fails to be uniformly non-square (i.e. $\delta_X(\varepsilon) = 0$ for all $0 < \varepsilon \le 2$). The principal ingredient of the proof of the first main theorem is a strengthening of the result just quoted. We need a definition first.

<u>DEFINITION 18.1</u>. Let X be a Banach space and let $n \in \mathbb{N}$ and $\rho \in (0,1)$ be arbitrary. We say that X has the *property* $J_{n,\rho}$ iff there exist $x_1,\ldots,x_n \in B_X$ such that for all $j = 0,1,\ldots,n$,

$$(18.1) \qquad \left\|-\sum_{k=1}^{j} x_k + \sum_{k=j+1}^{n} x_k\right\| > \rho n,$$

where for $j = 0$ (respectively, $j = n$) the first (respectively, second) sum is interpreted as 0. If $J_{n,\rho}$ holds for all $\rho \in (0,1)$, we say that X has the *property* J_n. Similarly, X is said to have the *property* J iff it satisfies J_n for all $n \in \mathbb{N}$.

<u>EXAMPLES</u>. ℓ^1 has the property J, since for every $n \in \mathbb{N}$ and $j \in \{0,1,\ldots,n\}$ we have $\|-\sum_{k=1}^{j} e_k + \sum_{k=j+1}^{n} e_k\| = n$. Also c_0 has J. Indeed, if for fixed but arbitrary $n \in \mathbb{N}$ we define the vectors $x_1,\ldots,x_n$ by

$$x_k = (\underbrace{1,\ldots,1}_{k},\underbrace{-1,\ldots,-1}_{n-k},0,0,\ldots) \qquad (k = 1,\ldots,n),$$

then it is easy to see that $\|-\sum_{k=1}^{j} x_k + \sum_{k=j+1}^{n} x_k\| = n$ for all $j \in \{0,1,\ldots,n\}$.

One should observe that J_2 is nothing but the negation of "uniformly non-square", so that Theorem 16.4 can be reformulated by saying that a non-reflexive Banach space satisfies J_2. Using the method of proof of Theorem 16.4 we shall now generalize this as follows.

PROPOSITION 18.2. *Every non-reflexive Banach space satisfies* J.

PROOF. Suppose X is non-reflexive and let us fix $m \in \mathbb{N}$ and $\rho \in (0,1)$ arbitrarily. It suffices to show that $J_{m,\rho}$ holds. Using the notations introduced in the proof of Theorem 16.4, we may write (16.16) as

$$\limsup_{n\to\infty} \frac{K_{n-1}}{K_n} = 1. \tag{18.2}$$

Choose $n_0 \in \mathbb{N}$ such that $K_{n_0-1}/K_{n_0} > \rho$. By (16.4) there exists a sequence $\{x_j^*\} \subset S_{X^*}$ satisfying

$$\frac{K_{n_0-1}}{K(n_0;\{x_j^*\})} > \rho,$$

so certainly

$$\frac{K(n_0-1;\{x_j^*\})}{K(n_0;\{x_j^*\})} > \rho. \tag{18.3}$$

We now pick $\tau > 1$ such that

$$\frac{K(n_0-1;\{x_j^*\})}{K(n_0;\{x_j^*\})} > \tau^2\rho. \tag{18.4}$$

$n_0, \{x_j^*\}$ and τ are kept fixed throughout the rest of this proof. Next we select $2n_0m$ distinct natural numbers p_ℓ^k ($k = 1,\ldots,m$; $\ell = 1,\ldots,2n_0$) so that

$$\begin{aligned} &\underbrace{p_1^1,p_1^2,p_1^3,\ldots,p_1^m}_{m},\ \underbrace{p_2^1,p_3^1,p_2^2,p_3^2,\ldots,p_2^m,p_3^m}_{2m},\ p_4^1,p_5^1,p_4^2,p_5^2,\ldots \\ &\qquad\ldots\ \underbrace{p_{2i}^1,p_{2i+1}^1,p_{2i}^2,p_{2i+1}^2,\ldots,p_{2i}^m,p_{2i+1}^m}_{2m},\ldots \\ &\ldots\ \underbrace{p_{2n_0-2}^1,p_{2n_0-1}^1,\ldots,p_{2n_0-2}^m,p_{2n_0-1}^m}_{2m},\ \underbrace{p_{2n_0}^1,p_{2n_0}^2,p_{2n_0}^3,\ldots,p_{2n_0}^m}_{m} \end{aligned} \tag{18.5}$$

represents not only the order of choice but also the order of magnitude of these numbers. We take care to choose them in such a manner that the following conditions are satisfied:

(a') $R(p_1^k,\dots,p_{2n_0}^k;\{x_j^*\}) < \tau K(n_0;\{x_j^*\})$ for all $k = 1,\dots,m$

(b') $R(p_1^m,p_2^1,p_3^m,p_4^1,\dots,p_{2i-1}^m,p_{2i}^1,\dots,p_{2n_0-3}^m,p_{2n_0-2}^1;\{x_j^*\}) \geq \tau^{-1}K(n_0-1;\{x_j^*\})$

(c') $R(p_3^k,p_2^{k+1},p_5^k,p_4^{k+1},\dots,p_{2i-1}^k,p_{2i-2}^{k+1},\dots,p_{2n_0-1}^k,p_{2n_0-2}^{k+1};\{x_j^*\}) \geq$
$\tau^{-1}K(n_0-1;\{x_j^*\})$ for all $k = 1,\dots,m-1$.

The possibility of satisfying these conditions is immediate from the definitions of $K(n_0-1,\{x_j^*\})$, $K(n_0;\{x_j^*\})$ and the fact that in the sets $R(\dots\dots;\{x_j^*\})$ of (a'), (b') and (c') the order of the p's is that of (18.5).

It is clear from (18.5) that $[p_{2i-1}^m,p_{2i}^1] \subset [p_{2i-1}^k,p_{2i}^k]$ for all $i = 1,\dots,n_0-1$ and $k = 1,\dots,m$ and this implies that for all $k = 1,\dots,m$,

$$S(p_1^k,\dots,p_{2n_0}^k;\{x_j^*\}) \subset S(p_1^m,p_2^1,p_3^m,p_4^1,\dots,p_{2n_0-3}^m,p_{2n_0-2}^1;\{x_j^*\}). \tag{18.6}$$

Similarly (18.5) yields that $[p_{2i+1}^\ell,p_{2i}^{\ell+1}] \subset [p_{2i-1}^k,p_{2i}^k]$ for all $1 \leq \ell < k \leq m$ and $i = 1,\dots,n_0-1$, so that, for all $1 \leq \ell < k \leq m$,

$$S(p_1^k,\dots,p_{2n_0}^k;\{x_j^*\}) \subset S(p_3^\ell,p_2^{\ell+1},p_5^\ell,p_4^{\ell+1},\dots,p_{2n_0-1}^\ell,p_{2n_0-2}^{\ell+1};\{x_j^*\}). \tag{18.7}$$

On the other hand it follows from (18.5) that $[p_{2i+1}^\ell,p_{2i}^{\ell+1}] \subset [p_{2i+1}^k,p_{2i+2}^k]$ for all $1 \leq k \leq \ell \leq m$ and $i = 1,\dots,n_0-1$, whence, for all $1 \leq k \leq \ell \leq m$,

$$-S(p_1^k,\dots,p_{2n_0}^k;\{x_j^*\}) \subset S(p_3^\ell,p_2^{\ell+1},p_5^\ell,p_4^{\ell+1},\dots,p_{2n_0-1}^\ell,p_{2n_0-2}^{\ell+1};\{x_j^*\}). \tag{18.8}$$

By (a') there exists, for every $k = 1,\dots,m$, an element $u_k \in S(p_1^k,\dots,p_{2n_0}^k;\{x_j^*\})$ satisfying

$$\|u_k\| \leq \tau K(n_0;\{x_j^*\}) \qquad (k = 1,\dots,m). \tag{18.9}$$

In virtue of (18.6), $u_k \in S(p_1^m,p_2^1,\dots,p_{2n_0-3}^m,p_{2n_0-2}^1;\{x_j^*\})$ for all $k=1,\dots,m$, whence, by the convexity of the latter set,

$$m^{-1} \sum_{k=1}^{m} u_k \in S(p_1^m,p_2^1,\dots,p_{2n_0-3}^m,p_{2n_0-2}^1;\{x_j^*\}).$$

Condition (b') now implies

$$m^{-1}\left\|\sum_{k=1}^{m} u_k\right\| = m^{-1}\left\|-\sum_{k=1}^{m} u_k\right\| \geq \tau^{-1}K(n_0-1;\{x_j^*\}).$$

Similarly, it follows from (18.7), (18.8) and the condition (c') that, for all $\ell = 1,\dots,m-1$

$$(18.11) \qquad m^{-1}\Big\|-\sum_{k=1}^{\ell} u_k + \sum_{k=\ell+1}^{m} u_k\Big\| \geq \tau^{-1}K(n_0-1;\{x_j^*\}).$$

Finally, let us define

$$x_k := \frac{u_k}{\tau K(n_0;\{x_j^*\})} \qquad (k = 1,\dots,m).$$

Then $\|x_k\| \leq 1$ for all $k = 1,\dots,m$, by (18.9), and it follows from (18.10), (18.11) and (18.4) that, for all $j = 0,\dots,m$

$$\Big\|-\sum_{k=1}^{j} x_k + \sum_{k=j+1}^{m} x_k\Big\| \geq m\,\frac{\tau^{-1}K(n_0-1;\{x_j^*\})}{\tau K(n_0;\{x_j^*\})} > \rho m.$$

This concludes the proof that X has $J_{m,\rho}$. □

The next step towards the main Theorem 18.6 consists in showing that J is equivalent to the property $m(X) = 2$.

<u>PROPOSITION 18.3</u>. *Let X be a Banach space and let $n \in \mathbb{N}$ and $\rho \in (0,1)$ be arbitrary. Then*

(i) $m(X) < 2\rho^{-1} \Rightarrow$ X *has* $J_{n,\rho}$;

(ii) X *has* $J_{n,\rho}$ *and* $\rho n > 1 \Rightarrow m(X) \leq 2(\rho-n^{-1})^{-1}$.

Hence X *has* J *iff* $m(X) = 2$.

<u>PROOF</u>. The last statement is an immediate consequence of (i) and (ii).

(i): Suppose $m(X) < 2\rho^{-1}$. Then there exist an $x \in S_X$ and a simple curve γ on S_X joining x to $-x$ such that $\ell := \ell(\gamma) < 2\rho^{-1}$. Let $g\colon [0,\ell] \to S_X$ be the standard representation of γ and let us put

$$x_k := g(\tfrac{k}{n}\ell) \qquad (k = 0,\dots,n)$$

and

$$y_k := \tfrac{n}{\ell}(x_k - x_{k-1}) \qquad (k = 1,\dots,n).$$

Then $x = x_0$, $-x = x_n$ and

$$(18.12) \qquad \|y_k\| = \tfrac{n}{\ell}\|g(\tfrac{k}{n}\ell) - g(\tfrac{k-1}{n}\cdot\ell)\| \leq \tfrac{n}{\ell}(\tfrac{k}{n}\cdot\ell - \tfrac{(k-1)}{n}\ell) = 1 \qquad (k = 1,\dots,n).$$

Furthermore, since $x_n = -x_0$, we have for all $j = 0,\dots,n$,

$$-\sum_{k=1}^{j} y_k + \sum_{k=j+1}^{n} y_k = -\frac{2n}{\ell}\,x_j,$$

whence

$$(18.13)\qquad \Big\| -\sum_{k=1}^{j} y_k + \sum_{k=j+1}^{n} y_k \Big\| = \frac{2n}{\ell}\|x_j\| = \frac{2n}{\ell} > \frac{2n}{2\rho^{-1}} = \rho_n .$$

(18.12) and (18.13) prove that X satisfies $J_{n,\rho}$.

(ii): Suppose that $J_{n,\rho}$ holds and that $\rho n > 1$. Let us put $\mu := (\rho n-1)^{-1}$ and let $x_1,\dots,x_n \in B_X$ satisfy (18.1). We now define $y_1,\dots,y_n$ by

$$y_j := \mu\Big(-\sum_{k=1}^{j-1} x_k + \sum_{k=j+1}^{n} x_k\Big) \quad (j = 1,\dots,n),$$

$$y_0 := -y_n,$$

and show that the polygon p with vertices $y_0,\dots,y_n$ lies outside B_X. Indeed, a point of a segment $[y_{j-1},y_j]$ $(j = 1,\dots,n)$ is of the form $\lambda y_{j-1} + (1-\lambda)y_j$ with $0 \le \lambda \le 1$ and we have on the one hand

$$(18.14)\qquad \lambda y_0 + (1-\lambda)y_1 = \lambda\Big(\mu \sum_{k=1}^{n-1} x_k\Big) + (1-\lambda)\mu \sum_{k=2}^{n} x_k = $$

$$= \mu\Big(\sum_{k=1}^{n} x_k - (1-\lambda)x_1 - \lambda x_n\Big)$$

and on the other hand, for all $j = 2,\dots,n$,

$$\lambda y_{j-1} + (1-\lambda)y_j = \mu\Big(-\lambda \sum_{k=1}^{j-2} x_k + \lambda \sum_{k=j}^{n} x_k - (1-\lambda)\sum_{k=1}^{j-1} x_k +$$

$$(18.15)\qquad + (1-\lambda)\sum_{k=j+1}^{n} x_k\Big) =$$

$$= \mu\Big(-\sum_{k=1}^{j-1} x_k + \sum_{k=j}^{n} x_k - (1-\lambda)x_j + \lambda x_{j-1}\Big).$$

By the choice of $x_1,\dots,x_n$, it follows from (18.14) and (18.15) that, for all $j = 1,\dots,n$,

$$\|\lambda y_{j-1} + (1-\lambda)y_j\| > \mu(\rho n - (1-\lambda) - \lambda) = \mu(\rho n-1) = 1.$$

Thus p lies outside B_X. Furthermore, it is readily seen that $y_1-y_0 = \mu(x_n-x_1)$ and $y_j-y_{j-1} = -\mu(x_{j-1}+x_j)$ for $j = 2,\dots,n$. Hence for the length of p we find

$$\ell(p) = \sum_{j=1}^{n} \|y_j-y_{j-1}\| \le \sum_{j=1}^{n} 2\mu = 2n\mu = 2n(\rho n-1)^{-1} = 2(\rho-n^{-1})^{-1}.$$

We have now shown that the antipodal points y_0 and $y_n = -y_0$ with norm > 1 can be joined by a curve p in $X\backslash B_X$ with $\ell(p) \le 2(\rho-n^{-1})^{-1}$. One is inclined to regard it as geometrically obvious now that $m(X) \le 2(\rho-n^{-1})^{-1}$. A proof by radial projection of p on S_X is incorrect, however, since radial projection may increase distances (cf. [89]). A correct but surprisingly awkward proof is given in the next lemma.

LEMMA 18.4. *Let* X *be a Banach space,* $x \in X$, $\|x\| \ge 1$, *and let* γ *be a simple curve in* $X\backslash \text{int } B_X$ *with initial point* x *and endpoint* -x. *Then there exists a* $y \in S_X$ *and a simple curve* γ_1 *in* S_X *with initial point* y *and endpoint* -y *such that* $\ell(\gamma_1) \le \ell(\gamma)$.

PROOF. a) Let $g: [0,\ell(\gamma)] \to X$ be the standard representation of γ. We may assume that x,-x is the only antipodal pair on γ. Putting $d := \min\{\|x\|: x \in \gamma\}$, $\frac{1}{d}\gamma$ contains a point y with $\|y\| = 1$. Suppose $y = \frac{1}{d}g(t_1)$. Then $\gamma_0 := \frac{1}{d}(g([t_1,\ell(\gamma)] \cup g([0,t_1]))$ is a curve in $X\backslash \text{int } B_X$ which can be represented as a simple curve with initial point y and endpoint -y and with $\ell(\gamma_0) = \frac{1}{d}\ell(\gamma) \le \ell(\gamma)$.

b) Let $g_0: [0,\ell(\gamma_0)] \to X$ be the standard representation of γ_0. The set $\{s \in [0,\ell(\gamma_0)]: \|g_0(s)\| > 1\}$ is open and contains neither 0 nor $\ell(\gamma_0)$ and is therefore the union of countably many disjoint open subintervals of $(0,\ell(\gamma_0))$. Let (a,b) be such an interval. Then $\|g_0(a)\| = \|g_0(b)\| = 1$. We now wish to replace the part $g_0([a,b])$ of γ_0 by a curve in S_X with the same initial and endpoint and with length $\le \ell(g_0([a,b])) = b-a$. Doing this for each one of these disjoint intervals (a,b) and pasting the replacements together yields the desired curve γ_1 in S_X connecting y and -y with length $\ell(\gamma_1) \le \ell(\gamma_0) \le \ell(\gamma)$.

c) Let us concentrate on $g_0([a,b])$. Pick $s_0 \in (a,b)$ arbitrarily and consider the function $f: (s_0,b] \to \mathbb{R}$ defined by

$$f(s) := \min\{\|x\|: x \in [g_0(s_0),g_0(s)]\}$$

([y,z] denotes the line segment joining y to z). f is continuous, $f(b) \le 1$ (since $\|g_0(b)\| = 1$) and $\lim_{s\downarrow s_0} f(s) = \|g_0(s_0)\| > 1$. Hence there exists an $s_1 \in (s_0,b]$ such that $f(s_1) = 1$. Let $x_1 \in [g_0(s_0),g_0(s_1)]$ be such that $\|x_1\| = 1$. This means that the segment $[g_0(s_0),g_0(s_1)]$ is tangent to B_X at x_1, or that $x_1 = s_1 = b$. If $s_1 < b$, we can repeat this process with s_1 instead of s_0, etc. Thus we find a finite or infinite sequence $s_0 < s_1 < s_2 < \ldots$ $\ldots \le b$ and points $x_1,x_2,\ldots$ such that

(18.16) $\quad x_i \in [g_0(s_{i-1}),g_0(s_i)]$ and $\|x_i\| =$

$$= \min\{\|x\|: x \in [g_0(s_{i-1}), g_0(s_i)]\} = 1 \quad (i = 1,2,\ldots).$$

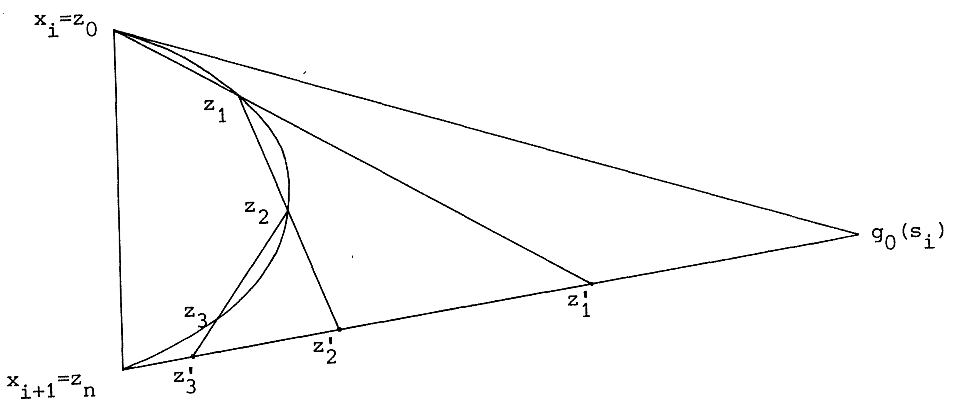
The same procedure can be followed in the other direction, yielding sequences $s_0 > s_{-1} > s_{-2} \ldots \geq a$ and $x_0, x_{-1}, x_{-2}, \ldots$ such that (18.16) holds for all $i \in \mathbb{Z}$. Observe that $\lim\limits_{n\to\infty} s_n = b$ (and similarly $\lim\limits_{n\to-\infty} s_n = a$) in the case of infinite sequences, since

$$\|g_0(s_n)\| \leq \|x_n\| + \|g_0(s_n)-x_n\| \leq 1 + \|g_0(s_n)-g_0(s_{n-1})\| \leq$$

$$\leq 1 + (s_n - s_{n-1}) \to 1$$

as $n \to \infty$. We now replace all parts $g_0([s_{i-1}, s_i])$ of $g_0([a,b])$ by the corresponding line segments $[g_0(s_{i-1}), g_0(s_i)]$. Clearly this decreases lengths, and preserves initial and endpoints.

d) The resulting curve consists of pieces $[x_i, g_0(s_i)] \cup [g_0(s_i), x_{i+1}]$ with $x_i, x_{i+1} \in S_X$ $(i \in \mathbb{Z})$ (see picture). Fix $i \in \mathbb{Z}$. The intersection of S_X with the triangle with vertices x_i, x_{i+1} and $g_0(s_i)$ is a plane curve σ_i



with initial point x_i and endpoint x_{i+1}. We claim that

(18.17) $\quad \ell(\sigma_i) \leq \|g_0(s_i)-x_i\| + \|g_0(s_i)-x_{i+1}\|.$

Once this is proved, we are done, since then $\sigma := \bigcup_{i\in\mathbb{Z}} \sigma_i$ is the desired replacement of $g_0([a,b])$ with $\ell(\sigma) \leq \ell(g_0([a,b])) = b-a$. For the proof of (18.17), let $z_0 := x_i, z_1, z_2, \ldots, z_n := x_{i+1}$ be any finite number of successive points on σ_i and let z'_j be the intersection of the line through z_{j-1} and z_j with $[g_0(s_i), x_{i+1}]$ $(j = 1,\ldots,n)$. Repeated application of the triangle inequality shows that

$$\|x_i-g_0(s_i)\| + \|g_0(s_i)-z_1'\| \geq \|z_0-z_1\| + \|z_1-z_1'\|$$

$$\|z_1-z_1'\| + \|z_1'-z_2'\| \geq \|z_1-z_2\| + \|z_2-z_2'\|$$

$$\vdots$$

$$\|z_{n-1}-z'_{n-1}\| + \|z'_{n-1}-x_{i+1}\| \geq \|z_{n-1}-z_n\|.$$

Adding these inequalities and subtracting the terms occurring on both sides, yields

$$\|x_i-g_0(s_i)\| + \|g_0(s_i)-x_{i+1}\| \geq \sum_{j=1}^{n} \|z_{j-1}-z_j\|.$$

This proves (18.17) since $\{z_1,\ldots,z_n\}$ was an arbitrary partition of σ_i. If the curve γ_1 is not simple, it can obviously be made simple be removing its loops.

Since the polygon p in the proof of Proposition 18.3 can be assumed to be simple, Lemma 18.4 is applicable with $\gamma = p$ and thus Proposition 18.3 is now completely proved. □

One more lemma is needed for the proof of Theorem 18.6.

LEMMA 18.5. *Let X be a Banach space and let* $0 < \varepsilon < 1$, $n \in \mathbb{N}$. *Suppose* $x_1,\ldots,x_n \in B_X$ *satisfy* $\|\sum_{k=1}^n x_k\| > n-\varepsilon$. *Then* $\|\sum_{k=1}^n \gamma_k x_k\| > (1-\varepsilon)\sum_{k=1}^n \gamma_k$ *for all choices of* $\gamma_1,\ldots,\gamma_n \geq 0$, *save* $\gamma_1 = \gamma_2 = \ldots = \gamma_n = 0$.

PROOF. Suppose that $\gamma_1,\ldots,\gamma_n \geq 0$, not all 0. Defining

$$\gamma_{n+k} := \gamma_k \qquad (k = 1,\ldots,n-1),$$

we have

$$(n-\varepsilon)\sum_{k=1}^{n}\gamma_k < \Big\|\sum_{k=1}^{n}x_k\Big\|\sum_{k=1}^{n}\gamma_k = \Big\|\sum_{k=1}^{n}\Big(\sum_{i=1}^{n}\gamma_i\Big)x_k\Big\| =$$

$$= \Big\|\sum_{k=1}^{n}\Big(\sum_{i=0}^{n-1}\gamma_{k+i}\Big)x_k\Big\| = \Big\|\sum_{i=0}^{n-1}\sum_{k=1}^{n}\gamma_{k+i}\,x_k\Big\| \leq \Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\| +$$

$$+ \sum_{i=1}^{n-1}\sum_{k=1}^{n}\gamma_{k+i}\|x_k\| \leq \Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\| + (n-1)\sum_{k=1}^{n}\gamma_k.$$

Thus

$$\Big\|\sum_{k=1}^{n}\gamma_k x_k\Big\| > (n-\varepsilon)\sum_{k=1}^{n}\gamma_k - (n-1)\sum_{k=1}^{n}\gamma_k =$$

$$= (1-\varepsilon)\sum_{k=1}^{n}\gamma_k. \quad \square$$

THEOREM 18.6. *Let* X *be a Banach space. Then* X *is superreflexive iff* $m(X) > 2$.

PROOF. By Proposition 18.3 the following statement is equivalent to that in the Theorem:

$$X \text{ not superreflexive} \iff X \text{ has } J.$$

Suppose first that X is not superreflexive. Then there exists a non-reflexive Y with $Y \prec X$. By Proposition 18.2 Y has J. The finite-dimensional character of J and the definition of $\prec$ imply that X has J.

Conversely, let X have J. We shall show that P_2 holds in X, so that, by Theorem 15.5, X is not superreflexive. Since $J_{n,\rho}$ holds in X for all $\rho \in (0,1)$ and $n \in \mathbb{N}$, there exists for every $n \in \mathbb{N}$ and every $\varepsilon > 0$ a system $\{x_1,\dots,x_n\} \subset B_X$ satisfying

$$\left\| -\sum_{k=1}^{j} x_k + \sum_{k=j+1}^{n} x_k \right\| > n-\varepsilon \qquad (j = 0,\dots,n)$$

(take $\rho = 1 - \frac{\varepsilon}{n}$). By the previous Lemma we then have, for all $j = 0,\dots,n$, and all $\gamma_1,\dots,\gamma_n \geq 0$, not all 0,

$$\left\| -\sum_{k=1}^{j} \gamma_k x_k + \sum_{k=j+1}^{n} \gamma_k x_k \right\| > (1-\varepsilon) \sum_{k=1}^{n} \gamma_k .$$

In particular it follows from this last inequality that

$$\text{(18.18)} \qquad \operatorname{dist}(\operatorname{co}\{x_1,\dots,x_j\},\operatorname{co}\{x_{j+1},\dots,x_n\}) > 2(1-\varepsilon) \quad \text{for all } j = 1,\dots,n-1.$$

This proves P_2. Consequently X is not superreflexive. □

COROLLARY 18.7. *The property* "$m(X) = 2$" *(and therefore also* "$m(X) > 2$"*) is an isomorphic invariant.*

PROOF. Immediate, since superreflexivity is an isomorphic invariant. □

COROLLARY 18.8. P_1 *(resp.* P_2*) is equivalent to the property obtained from it by replacing* "$\exists\varepsilon > 0$" *by* "$\forall 0 < \varepsilon < 2$".

PROOF. For P_2 this is immediate from the proof of Theorem 18.6: we have in fact proved that P_2 implies, for every $0 < \eta < 2$ and every $n \in \mathbb{N}$, the existence of $\{x_1,\dots,x_n\} \subset B_X$ satisfying $\operatorname{dist}(\operatorname{co}\{x_1,\dots,x_j\},\operatorname{co}\{x_{j+1},\dots,x_n\}) > \eta$ for all $j = 1,\dots,n-1$.

Now suppose P_1 holds. Since $P_1 = P_2$ and for P_2 the claim has been proved, there exists for every $0 < \varepsilon < 2$ and every $n \in \mathbb{N}$ a system $\{x_1,\dots,x_{2^n}\} \subset B_X$ satisfying

$$(18.19) \qquad \operatorname{dist}(\operatorname{co}\{x_1,\dots,x_k\},\operatorname{co}\{x_{k+1},\dots,x_{2^n}\}) \ge \varepsilon \quad \text{for all } k = 1,\dots,2^n-1.$$

But then $T := \{x_1,\dots,x_{2^n}\}$ is an (n,ε)-tree in B_X, since, by (18.19), $\|x_1-x_2\|, \|x_3-x_4\|,\dots,\|x_{2^n-1}-x_{2^n}\| \ge \varepsilon$, $\|\frac{1}{2}(x_1+x_2)-\frac{1}{2}(x_3+x_4)\|,\dots,\|\frac{1}{2}(x_{2^n-3}+x_{2^n-2}) - \frac{1}{2}(x_{2^n-1}+x_{2^n})\| \ge \varepsilon$, etc. Thus we have shown that P_1 with "$\forall 0 < \varepsilon < 2$" holds. □

We now come to the second main theorem of this section, connecting superreflexivity and flatness.

THEOREM 18.9. *A Banach space X is superreflexive iff every Banach space Y which is finitely representable in X, fails to be flat.*

PROOF. Since a reflexive space is not flat (Corollary 17.12), the "only if" part is trivial. It remains to be shown that if X is not superreflexive, then there exists a flat Y with $Y \prec X$. So suppose X is not superreflexive. The idea of the proof is the following. We first show, using Theorem 18.6, that X has a finite version of the ISTP. Then the argument of Proposition 15.6 yields a space Y with $Y \prec X$ and Y having the ISTP. This Y is flat by Theorem 17.7.

We claim that X has the following property which we call the *finite supported tree property* (FSTP):

For each $n \in \mathbb{N}$ there exist a subset $T = \{x^n_{k,i}: k = 1,\dots,n;\ i = 1,\dots \dots,2^k\}$ of X and a subset $\{x^{n*}_{k,i}: k=0,1,\dots,n;\ i=i,\dots,2^k\}$ of X^* such that for all the appropriate indices

$$(18.20) \qquad x^n_{k,i} = \tfrac{1}{2}(x^n_{k+1,2i-1} + x^n_{k+i,2i}),$$

and

$$(18.21) \qquad \langle x^n_{k_1,i_1}, x^{n*}_{k_2,i_2}\rangle \begin{cases} \le -1 + n^{-1} & \text{if } \dfrac{i_1}{2^{k_1}} \le \dfrac{2i_2-1}{2^{k_2+1}} \\[2ex] \ge 1 - n^{-1} & \text{if } \dfrac{2i_2-1}{2^{k_2+1}} \le \dfrac{i_1-1}{2^{k_1}} \end{cases}$$

and

$$(18.22) \qquad \|x^n_{k,i}\| \le 1 + n^{-1}, \quad \|x^{n*}_{k,i}\| = 1.$$

Indeed, since $m(X) = 2$ by Theorem 18.6, for any $\varepsilon > 0$ there exists a simple curve γ_ε on S_X joining some $x_\varepsilon \in S_X$ to its antipode $-x_\varepsilon$, with length

$\ell(\gamma_\varepsilon) \le 2(1+\varepsilon)$. Obviously, a representation $g_\varepsilon\colon [0,2] \to S_X$ for γ_ε can be chosen so that

$$(18.23) \qquad g_\varepsilon(0) = x_\varepsilon = -g_\varepsilon(2)$$

and

$$(18.24) \qquad \|g_\varepsilon(s)-g_\varepsilon(t)\| \le (1+\varepsilon)|s-t| \quad \text{for all } s,t \in [0,2].$$

Let $f_\varepsilon^*\colon [0,2] \to S_{X^*}$ be such that

$$(18.25) \qquad \langle g_\varepsilon(t), f_\varepsilon^*(t)\rangle = 1 \quad \text{for all } t \in [0,2].$$

We have now, for all $0 \le t,s \le 2$,

$$(18.26) \qquad 1-(1+\varepsilon)|s-t| \le \langle g_\varepsilon(s), f_\varepsilon^*(t)\rangle \le 1-(1+\varepsilon)|s-t| + 2\varepsilon.$$

Indeed, by (18.25) and (18.24),

$$(18.27) \qquad \begin{aligned} |\langle g_\varepsilon(s), f_\varepsilon^*(t)\rangle - 1| &= |\langle g_\varepsilon(s), f_\varepsilon^*(t)\rangle - \langle g_\varepsilon(t), f_\varepsilon^*(t)\rangle| \le \\ &\le \|g_\varepsilon(s)-g_\varepsilon(t)\| \le (1+\varepsilon)|s-t| \end{aligned}$$

and, using also (18.23),

$$(18.28) \qquad \begin{aligned} &\langle g_\varepsilon(s), f_\varepsilon^*(t)\rangle + 1 = \langle g_\varepsilon(s)+g_\varepsilon(t), f_\varepsilon^*(t)\rangle \le \\ &\le \|g_\varepsilon(s)+g_\varepsilon(t)\| \le \begin{cases} \|g_\varepsilon(s)-g_\varepsilon(0)\| + \|g_\varepsilon(2)-g_\varepsilon(t)\| \le (1+\varepsilon)[s+(2-t)] \\ \|g_\varepsilon(s)-g_\varepsilon(2)\| + \|g_\varepsilon(0)-g_\varepsilon(t)\| \le (1+\varepsilon)[(2-s)+t]. \end{cases} \end{aligned}$$

From (18.27) and (18.28) formula (18.26) readily follows.

Now, if $n \in \mathbb{N}$ is given, let us choose $\varepsilon < \frac{1}{n\cdot 2^{n+1}}$ and, with g_ε and f_ε^* defined as above, let us put

$$(18.29) \qquad x_{k,i}^n := 2^k\left[g_\varepsilon\left(\tfrac{i-1}{2^k}\right) - g_\varepsilon\left(\tfrac{i}{2^k}\right)\right] \quad (k = 1,\dots,n;\ i = 1,\dots,2^k)$$

and

$$(18.30) \qquad x_{k,i}^{n*} := f_\varepsilon^*\left(\tfrac{2i-1}{2^{k+1}}\right) \qquad (k = 0,1,\dots,n;\ i = 1,\dots,2^k).$$

We claim that with these definitions (18.20), (18.21) and (18.22) are satisfied. Indeed, for the appropriate k and i,

$$\begin{aligned} x_{k,i}^n &= 2^k\left[g_\varepsilon\left(\frac{2(i-1)}{2^{k+1}}\right) - g_\varepsilon\left(\frac{2i-1}{2^{k+1}}\right)\right] + 2^k\left[g_\varepsilon\left(\frac{2i-1}{2^{k+1}}\right) - g_\varepsilon\left(\frac{2i}{2^{k+1}}\right)\right] = \\ &= \tfrac{1}{2}(x_{k+1,2i-1}^n + x_{k+1,2i}^n), \end{aligned}$$

$\|x^n_{k,i}\| \le 2^k(1+\varepsilon)\frac{1}{2^k} \le 1+n^{-1}$ (by (18.24)), $\|x^{n*}_{k,i}\| = 1$ by definition, and finally, by (18.26),

$$<x^n_{k_1,i_1}, x^{n*}_{k_2,i_2}> = 2^{k_1}<g_\varepsilon\left(\frac{i_1-1}{2^{k_1}}\right), f^*_\varepsilon\left(\frac{2i_2-1}{2^{k_2+1}}\right)> - 2^{k_1}<g_\varepsilon\left(\frac{i_1}{2^{k_1}}\right), f^*_\varepsilon\left(\frac{2i_2-1}{2^{k_2+1}}\right)>$$

$$\begin{cases} \le 2^{k_1}\left[1-(1+\varepsilon)\left(\frac{2i_2-1}{2^{k_2+1}}-\frac{i_1-1}{2^{k_1}}\right)+2\varepsilon-1+(1+\varepsilon)\left(\frac{2i_2-1}{2^{k_2+1}}-\frac{i_1}{2^{k_1}}\right)\right] = -1-\varepsilon+\varepsilon 2^{k_1+1} \le -1+n^{-1} \\ \qquad\qquad \text{if } \frac{i_1}{2^{k_1}} \le \frac{2i_2-1}{2^{k_2+1}} \\ \ge 2^{k_1}\left[1-(1+\varepsilon)\left(\frac{i_1-1}{2^{k_1}}-\frac{2i_2-1}{2^{k_2+1}}\right)-1+(1+\varepsilon)\left(\frac{i_1}{2^{k_1}}-\frac{2i_2-1}{2^{k_2+1}}\right)-2\varepsilon\right] = 1+\varepsilon-\varepsilon 2^{k_1+1} \ge 1-n^{-1} \\ \qquad\qquad \text{if } \frac{2i_2-1}{2^{k_2+1}} \le \frac{i_1-1}{2^{k_1}}\,. \end{cases}$$

Having now verified the FSTP, we use the argument of Proposition 15.6 to show that there exists a Y having the ISTP which is finitely representable in X. This part of the proof may be sketchy since we have given complete details earlier. Let Y be the linear space spanned by a sequence of independent vectors $e_{k,i}$ ($k = 1,2,\ldots;\ i = 1,\ldots,2^k$). Applying a diagonal procedure, we may assume that for some subsequence $\{n_\ell\}$ of $\mathbb{N}$ and for every finite set A of rational (and therefore also of real) numbers $\alpha_{k,i}$, $\lim_{\ell\to\infty}\|\sum_A \alpha_{k,i}x^{n_\ell}_{k,i}\|$ exists, as well as $\lim_{\ell\to\infty}<x^{n_\ell}_{k_1,i_1}, x^{n_\ell *}_{k_2,i_2}>$, for all (k_1,i_1), (k_2,i_2). (Note that for sufficiently large ℓ all these elements are defined.) On Y we now define a seminorm by

$$(18.31)\qquad \left\|\sum_{\text{finite}} \alpha_{k,i}\, e_{k,i}\right\| := \lim_{\ell\to\infty}\left\|\sum_{\text{finite}} \alpha_{k,i}\, x^{n_\ell}_{k,i}\right\|.$$

Now consider the quotient $\hat{Y} = Y/N$, where $N = \{y \in Y\colon\ \|y\| = 0\}$. This space $\hat{Y}$ is spanned by the images $\hat{e}_{k,i}$ of the $e_{k,i}$ under the quotient map, and is finitely representable in X (see the proof of Proposition 15.6). To show that $\hat{Y}$ has the ISTP, one first defines elements $e^*_{k,i} \in Y^*$ ($k = 0,1,\ldots;\ i = 1,\ldots,2^k$) by the formulas

$$(18.32)\qquad \begin{aligned} &<e_{k_1,i_1}, e^*_{k_2,i_2}> := \lim_{\ell\to\infty} <x^{n_\ell}_{k_1,i_1}, x^{n_\ell *}_{k_2,i_2}>, \\ &<\sum_{\text{finite}} \alpha_{k_1,i_1}\, e_{k_1,i_1}, e^*_{k_2,i_2}> := \sum_{\text{finite}} \alpha_{k_1,i_1} <e_{k_1,i_1}, e^*_{k_2,i_2}>. \end{aligned}$$

It is readily seen that each $e_{k,i}^{*}$ annihilates N and therefore defines a unique $\hat{e}_{k,i}^{*} \in \hat{Y}^{*}$. Furthermore, by (18.20), (18.21), (18.22), (18.31) and (18.32),

$$\|\hat{e}_{k,i}^{*}\| \le \limsup_{\ell\to\infty} \|x_{k,i}^{n_\ell *}\| = 1,$$

$$\|\hat{e}_{k,i}\| = \lim_{\ell\to\infty}\|x_{k,i}^{n_\ell}\| \le 1,$$

$$\|\hat{e}_{k,i}-\tfrac{1}{2}\hat{e}_{k+1,2i-1}-\tfrac{1}{2}\hat{e}_{k+1,2i}\| = \lim_{\ell\to\infty}\|x_{k,i}^{n_\ell}-\tfrac{1}{2}x_{k+1,2i-1}^{n_\ell}-\tfrac{1}{2}x_{k+1,2i}^{n_\ell}\| = 0,$$

$$<\hat{e}_{k_1,i_1},\hat{e}_{k_2,i_2}^{*}> = \lim_{\ell\to\infty} <x_{k_1,i_1}^{n_\ell},x_{k_2,i_2}^{n_\ell *}> \begin{cases} \le -1 & \text{if } \dfrac{i_1}{2^{k_1}} \le \dfrac{2i_2-1}{2^{k_2+1}} \\ \ge 1 & \text{if } \dfrac{2i_2-1}{2^{k_2+1}} \le \dfrac{i_1-1}{2^{k_1}} . \end{cases}$$

It follows that all these inequalities are equalities, and therefore the elements $\{\hat{e}_{k,i}: k = 1,2,\ldots; i = 1,\ldots,2^k\}$ and $\{\hat{e}_{k,i}^{*}: k = 0,1,2,\ldots; i = 1,\ldots,2^k\}$ satisfy the ISTP for the completion $\tilde{Y}$ of $\hat{Y}$. Thus $\tilde{Y}$ is flat and, of course, also finitely representable in X. This completes the proof. □

<u>NOTES</u>. The property J was introduced and Propositions 18.2 and 18.3 were proved by J.J. SCHÄFFER & K. SUNDARESAN ([92]), while Theorem 18.6 is due to R.C. JAMES & J.J. SCHÄFFER ([59]). Corollary 18.8 was originally proved by R.C. JAMES ([54]). The technical Lemma 18.4 can be found in [91]. Theorem 18.9 is due to D. VAN DULST ([30]). An interesting aspect of Theorem 18.9 is that it shows the existence of certain unusual flat spaces. E.g. if X is the uniformly non-octahedral space constructed by R.C. James (cf. the Notes to Section 16), then by Theorem 18.9 there exists a flat space Y such that $Y \prec X$. Since $\ell_3^1 \not\prec X$, it follows that $\ell_3^1 \not\prec Y$. Also, by the result from [22] mentioned in the Notes to Section 16, $Y^{**}/\pi Y$ is reflexive. Another result connecting girth and superreflexivity was recently proved by A.J. Pach: every non-superreflexive space has an equivalent norm for which there exists an $x \in S_X$ with $m(x) = 2$ (cf. Definition 17.3).

19. OTHER SUPERPROPERTIES EQUIVALENT TO SUPERREFLEXIVITY

If P is a property of Banach spaces, a space X is said to have super P iff every Banach space Y with $Y \prec X$ has P. Let us observe that if P implies Q then super P implies super Q. In this section we discuss various properties whose corresponding superproperties are equivalent to superreflexivity. For our first result we rely on some known theorems whose proofs we do not give here. They can be found in full detail in [27] and [28].

DEFINITION 19.1. A Banach space X has the *Krein-Milman* (K.M.) *property* iff every closed bounded convex set in X is the closed convex hull of its extreme points.

It is known that the R.N. property implies the K.M. property. In particular reflexive and separable dual spaces have the K.M. property (Remark 11.7). It is an open problem whether the K.M. property and the R.N. property are equivalent. This equivalence holds for dual spaces. In fact we have

PROPOSITION 19.2. *For every Banach space* X *the following are equivalent:*
(i) X^* *has the R.N. property;*
(ii) X^* *has the K.M. property;*
(iii) *every separable subspace* $Y \subset X$ *has separable dual.*

Using this result, it is an almost immediate corollary of Theorem 18.9 that the corresponding superproperties coincide with superreflexivity.

THEOREM 19.3. *For every Banach space* X *the following are equivalent:*
(i) X *is superreflexive;*
(ii) X *is super R.N.*
(iii) X *is super K.M.*
(iv) X *is super non-flat.*

PROOF. Theorem 18.9 asserts that (i) $\Longleftrightarrow$ (iv). Also, since reflexive $\Rightarrow$ R.N. $\Rightarrow$

$\Rightarrow$ K.M., we have (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). It therefore suffices to prove (iii) $\Rightarrow$ (iv). Suppose (iv) does not hold. Then there exists a flat space Y with $Y \prec X$. By local reflexivity also $Y^{**} \prec X$. Moreover, by Proposition 17.9 Y^* is flat. Since every flat space has non-separable dual (Proposition 17.11), in particular Y^* contains a separable subspace (the closed linear span of a girth curve) with non-separable dual. Hence, by Proposition 19.2 Y^{**} does not have the K.M. property. This completes the proof of (iii) $\Rightarrow$ (iv) and of the Theorem. □

REMARK 19.4. We know that flat spaces do not have the R.N. property: By Theorem 17.7 they have the I.S.T.P., while spaces with the R.N. property do not possess bounded (∞,ε)-trees (Remark 11.7). It seems likely that every flat space also fails to have the K.M. property, but we have no proof of this. In [42] this is shown to hold under an additional hypothesis.

The remainder of this section is devoted to the study of several summability and ergodic properties of Banach spaces and a proof that all corresponding superproperties are equivalent to superreflexivity.

A real infinite matrix $A = (\alpha_{ij})$ is called a *convergence-preserving method* iff for every convergent real sequence $\{x_n\}$ the sequence $\{\sum_{j=1}^{\infty} \alpha_{ij}x_j\}_{i=1}^{\infty}$ converges. It is not difficult to see that the following are necessary and sufficient conditions for A to be convergence-preserving.

(i) $\{\sum_{j=1}^{\infty} |\alpha_{ij}|\}_{i=1}^{\infty}$ is bounded;

(ii) $\{\sum_{j=1}^{\infty} \alpha_{ij}\}_{i=1}^{\infty}$ converges, say to α;

(iii) $\{\alpha_{ij}\}_{i=1}^{\infty}$ converges for every j, say to α_j.

Also, if $A = (\alpha_{ij})$ is convergence-preserving, then it preserves limits (i.e. $\lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij}x_j = \lim_{n\to\infty} x_n$ for all convergent $\{x_n\}$), iff, in addition, $\alpha = 1$ and $\alpha_j = 0$ for all j. A well known example of a method that preserves convergence as well as limits is that of taking Cesaro sums:

$$A = (\alpha_{ij}) \quad \text{with} \quad \alpha_{ij} = \begin{cases} \frac{1}{i} & \text{for } j = 1,\dots,i \\ 0 & \text{otherwise.} \end{cases}$$

Our first goal is to characterize reflexivity in terms of a summability property. For this we need the following

DEFINITION 19.5. A real infinite matrix $A = (\alpha_{ij})$ is called an *R-matrix* iff it satisfies the following conditions:

(A) $\qquad \sum_{j=1}^{\infty} \alpha_{ij} \not\to 0 \qquad \text{if } i \to \infty,$

(B) $\qquad \lim_{i\to\infty} \alpha_{ij} = 0 \qquad \text{for all } j \in \mathbb{N}.$

Condition (A) means that $\sum_{j=1}^{\infty} \alpha_{ij}$ converges for every i and the sequence $\{\sum_{j=1}^{\infty} \alpha_{ij}\}_{i=1}^{\infty}$ either diverges or converges to a limit different from 0.

PROPOSITION 19.6. *A Banach space X is reflexive iff for every sequence* $\{x_n\} \subset B_X$ *there exists an* R-*matrix* $A = (\alpha_{ij})$ *such that* $\{\sum_{j=1}^{\infty} \alpha_{ij}x_j\}_{i=1}^{\infty}$ *converges weakly.*

PROOF. The "only if" part is trivial. Indeed, if X is reflexive then $\{x_n\}$ has a weakly convergent subsequence $\{x_{n_k}\}$. For the R-matrix it suffices to take $\alpha_{ij} = \delta_{n_i,j}$ for all i,j. The converse lies a bit deeper. Suppose X is not reflexive. Then by Theorem 6.12 X contains a non-shrinking basic sequence $\{x_n\}$. In view of Lemma 6.5 this means that for some $x^* \in X^*$ and some $\varepsilon > 0$ we have $\|x^*|_{[x_i]_{i=n+1}^{\infty}}\| > \varepsilon$ for all $n \in \mathbb{N}$. This easily implies the existence of a block basic sequence $\{y_n\}$, with $y_n = \sum_{i=m_{n-1}+1}^{m_n} \beta_i x_i$ $(0 = m_0 < m_1 < \dots)$ such that $\|y_n\| = 1$ and $\langle y_n,x^*\rangle > \varepsilon$ $(n = 1,2,\dots)$. Putting $z_n := \frac{y_n}{\langle y_n,x^*\rangle}$ $(n = 1,2,\dots)$, $\{z_n\}$ is a bounded basic sequence and $\langle z_n,x^*\rangle = 1$ for all $n \in \mathbb{N}$. Assume now that for some R-matrix (α_{ij}) we have $\lim_{i\to\infty} \sum_{j=1}^{\infty}\alpha_{ij}z_j = z$, weakly. Then $z \in [z_n]$ and therefore can be written as $z = \sum_{i=1}^{\infty} \alpha_i z_i$. Let $\{z_n^*\} \subset [z_n]^*$ be the sequence of coefficient functionals of $\{z_n\}$. The (weak) continuity of the z_n^*, together with the condition (B) implies that, for all $n \in \mathbb{N}$,

$$\alpha_n = \langle z,z_n^*\rangle = \lim_{i\to\infty} \langle \sum_{j=1}^{\infty} \alpha_{ij}z_j,z_n^*\rangle = \lim_{i\to\infty} \alpha_{in} = 0.$$

Thus z = 0 and it follows that

$$0 = \langle z,x^*\rangle = \lim_{i\to\infty} \langle \sum_{j=1}^{\infty} \alpha_{ij}z_j,x^*\rangle = \lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij},$$

contradicting (A). □

We now consider a stronger summability property.

DEFINITION 19.7. A Banach space X is said to have the *Banach-Saks* (B.S.) *property* iff every sequence $\{x_n\} \subset B_X$ has a subsequence $\{x_{n_k}\}$ whose Cesaro averages converge in norm, i.e.

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} x_{n_i}$$

exists, in norm.

The following consequence of Proposition 19.6 is immediate.

COROLLARY 19.8. *If* X *has the B.S. property, then* X *is reflexive.*

For some time it was an open problem whether all reflexive spaces have the B.S. property. The following example shows that this is not the case.

EXAMPLE. Let Γ be the family of all finite non-empty subsets γ of $\mathbb{N}$ with the property $|\gamma| \le \min \gamma$. ($|\gamma|$ denotes the cardinality of γ.) For $\gamma, \gamma' \in \Gamma$ we write $\gamma < \gamma'$ iff $\max \gamma < \min \gamma'$. Let $x = \{x_n\}$ be any real sequence. For all $\gamma \in \Gamma$ we define

$$\sigma(x,\gamma) := \sum_{n\in\gamma} |x_n|.$$

For any increasing sequence $\{\gamma_k\}$ in Γ (i.e. $\gamma_k < \gamma_{k+1}$, $k \in \mathbb{N}$) we put

$$(19.1) \qquad \sigma(x,\{\gamma_k\}) := \{\sum_{k=1}^{\infty} \sigma(x,\gamma_k)^2\}^{\frac{1}{2}}.$$

Putting

$$(19.2) \qquad \|x\| := \sup \sigma(x,\{\gamma_k\}),$$

where the sup is taken over all increasing sequences $\{\gamma_k\}$ in Γ, we define the space X as the set of all real sequences $x = \{x_n\}$ for which (19.2) is finite. We omit the easy proof that X is a Banach space with the norm defined by (19.2). Let us denote by $\{e_n\}$ the sequence of unit vectors. Obviously $\|e_n\| = 1$ for all $n \in \mathbb{N}$. We intend to show that

(19.3) $\{e_n\}$ is a boundedly complete shrinking basis for X.

Suppose for the moment that (19.3) has been proved. Then by Theorem 6.9 X is reflexive. However, X does not have the B.S. property. Indeed, suppose that for some subsequence $\{n_i\}$ of $\mathbb{N}$ the sequence $\{s_k\}$ with $s_k := \frac{1}{k}\sum_{i=1}^{k} e_{n_i}$ ($k = 1,2,\ldots$), converges in norm. Since $\{e_n\}$ is a basis for X (with coefficient functionals denoted by $\{e_n^*\}$), the limit must be 0, because $\langle \lim_{k\to\infty} s_k, e_n^*\rangle = \lim_{k\to\infty} \langle s_k, e_n^*\rangle = 0$ for all $n \in \mathbb{N}$. But on the other hand, taking $\gamma_k = \{n_{k+1},\ldots,n_{2k}\} \in \Gamma$, it follows that $\|s_{2k}\| \ge \sigma(s_{2k},\gamma_k) = \frac{1}{2}$ for all $k = 1,2,\ldots$, contradicting $\lim_{k\to\infty} s_k = 0$.

We now complete the example by showing (19.3). Since any representation $x = \sum_{n=1}^{\infty} x_n e_n$ is obviously unique, the proof that $\{e_n\}$ is a basis for X reduces to showing that for any $x = \{x_n\} \in X$ we have $\lim_{n\to\infty} \|x-\sum_{k=1}^{n} x_k e_k\| = 0$. Suppose not. Then for some $x = \{x_n\} \in X$ and some $\varepsilon > 0$, $\|x-\sum_{k=1}^{n} x_k e_k\| > \varepsilon$ for all $n = 0,1,2,\ldots$, since the sequence $\{\|x - \sum_{k=1}^{n} x_k e_k\|\}_{n=0}^{\infty}$ is clearly non-increasing ($\sum_{k=1}^{0} x_k e_k$ is to be read as 0). In particular $\|x\| > \varepsilon$, so there exists $\gamma_1 < \gamma_2 < \ldots < \gamma_{p(1)}$ in Γ satisfying

$$\sum_{k=1}^{p(1)} \sigma(x,\gamma_k)^2 > \varepsilon^2.$$

Let m be the largest element of $\gamma_{p(1)}$. Since also $\|x - \sum_{k=1}^{m} x_k e_k\| > \varepsilon$, there exist $\gamma_{p(1)+1} < \gamma_{p(1)+2} < \ldots < \gamma_{p(2)}$ in Γ with $\gamma_{p(1)} < \gamma_{p(1)+1}$ so that

$$\sum_{k=p(1)+1}^{p(2)} \sigma(x,\gamma_k)^2 > \varepsilon^2.$$

Continuing this procedure, we find that $\|x\|^2 \geq \sum_{i=1}^{\infty} \sum_{k=p(i)+1}^{p(i+1)} \sigma(x,\gamma_k)^2 \geq \sum_{i=1}^{\infty} \varepsilon^2 = \infty$, a contradiction. Thus $\{e_n\}$ is a basis for X. To show that it is boundedly complete, let $x = \{x_n\}$ be a real sequence such that

$$\sup_{n\in \mathbb{N}} \left\| \sum_{i=1}^{n} x_i e_i \right\| < \infty.$$

It is readily seen that this implies $\|x\| < \infty$, so that $x \in X$. Hence, $\{e_n\}$ being a basis for X, $x = \sum_{n=1}^{\infty} x_n e_n$, so that $\sum_{n=1}^{\infty} x_n e_n$ converges.

Finally we show that $\{e_n\}$ is shrinking. This is the trickiest part of the example and the restriction on the cardinality of the γ_k appearing in the definition of $\| \ \|$, plays an essential role here. Suppose that $\{e_n\}$ is not shrinking. Then in view of Lemma 6.5 $\|x^*|_{[e_i]_{i=n}^{\infty}}\| > \delta$ for some $x^* \in X^*$, some $\delta > 0$ and all $n \in \mathbb{N}$. This implies the existence of a strictly increasing sequence $0 = m_0 < m_1 < \ldots < m_n < \ldots$ in $\mathbb{N}$ and a block basic sequence $\{y^n\} \subset X$ such that

$$\|y^n\| = 1, \quad \langle y^n,x^*\rangle > \delta \quad \text{and} \quad y^n \in [e_i]_{i=m_{n-1}+1}^{m_n} \qquad (n = 1,2,\ldots).$$

Let us define a function F on $\mathbb{N} \cup \{0\}$ by

$$F(0) = 0, \quad F(1) = 1 \quad \text{and } F(n) = F(n-1) + m_{F(n-1)} \qquad (n = 2,3,\ldots).$$

Put

$$w^n := \frac{1}{n} \frac{1}{F(n)-F(n-1)} \sum_{m=F(n-1)+1}^{F(n)} y^m \qquad (n = 1,2,\ldots)$$

and let x be the sequence defined by

$$x_i := w_i^n \quad (m_{F(n-1)} < i \le m_{F(n)}; \quad n = 1,2,\ldots).$$

The proof will be complete once we have shown that $x \in X$, for then $x = \sum_{n=1}^{\infty} w^n$, yielding the contradiction $\langle x,x^*\rangle = \sum_{n=1}^{\infty} \langle w^n,x^*\rangle \ge \sum_{n=1}^{\infty} \frac{1}{n}\delta = \infty$.

Let $\{\gamma_k\}$ be any increasing sequence in Γ. Let us fix $n \in \mathbb{N}$ and let us put

$$A(n) := \{k\colon \min \gamma_k \in [m_{F(n-1)}+1, m_{F(n)}]\}.$$

Suppose that $\mu(n)$ is the largest element of $A(n)$ (if $A(n) \ne \emptyset$). Whenever $k \in A(n)$ and $k < \mu(n)$, we have $\sigma(x,\gamma_k) = \sigma(w^n,\gamma_k)$, so that

$$(19.4) \qquad \sum_{k\in A(n)} \sigma(x,\gamma_k)^2 \le \|w^n\|^2 + \sigma(x,\gamma_{\mu(n)})^2.$$

Observe that $\|w^n\| \le \frac{1}{n}$ $(n = 1,2,\ldots)$. To estimate the term $\sigma(x,\gamma_{\mu(n)})^2$, write $\gamma_{\mu(n)} = \gamma' \cup \gamma''$, where $\gamma' = \gamma_{\mu(n)} \cap [1,m_{F(n)}]$ and $\gamma'' = \gamma_{\mu(n)} \cap [m_{F(n)}+1,\infty)$. Then $\sigma(x,\gamma') = \sigma(w^n,\gamma') \le \frac{1}{n}$. Moreover, $\gamma'' \subset \gamma_{\mu(n)} \in \Gamma$, so $\sigma(x,\gamma'')$ is the sum of at most $m_{F(n)}$ terms, each of which has the form $\frac{1}{N}\,\frac{1}{F(N)-F(N-1)}\,\alpha$, with $0 \le \alpha \le 1$ (since $\|y^m\| \le 1$ for all $m \in \mathbb{N}$) and $N \ge n+1$. Since $F(N)-F(N-1) = m_{F(N-1)} \ge m_{F(n)}$, each term is bounded above by $\frac{1}{n}\,\frac{1}{m_{F(n)}}$. Hence $\sigma(x,\gamma'') \le \frac{1}{n}$, and therefore

$$\sigma(x,\gamma_{\mu(n)})^2 = \{\sigma(x,\gamma') + \sigma(x,\gamma'')\}^2 \le \frac{4}{n^2}.$$

Thus, by (19.4), $\sum_{k\in A(n)} \sigma(x,\gamma_k)^2 \le \frac{5}{n^2}$. The conclusion is that $\sigma(x,\{\gamma_k\})^2 \le 5\sum_{n=1}^{\infty} \frac{1}{n^2}$ for all increasing sequences $\{\gamma_k\}$ in Γ, proving that $\|x\| < \infty$, i.e. $x \in X$. □

It is a long known fact that every uniformly convex space has the B.S. property. We include a proof here for completeness.

<u>PROPOSITION 19.9.</u> *Every uniformly convex Banach space X has the B.S. property.*

<u>PROOF.</u> a) Let $\{x_n\} \subset B_X$ be given. By the reflexivity of X $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ that weakly converges, say to x. Then $\{(x_{n_k}-x)/2\}$ weakly converges to 0 and is contained in B_X. Clearly if this last sequence has a Cesaro summable subsequence, then so does $\{x_n\}$. Thus we may assume that $\{x_n\}$ is a weak null sequence.

b) We first prove that there exists a $0 < \theta < 1$ satisfying the following property: For every weak null sequence and for every $n \in \mathbb{N}$ there exists an $m > n$ such that

$$(19.5) \qquad \left\|\frac{x_n+x_m}{2}\right\| \le \theta M, \text{ where } M := \sup_{n\in\mathbb{N}} \|x_n\|.$$

Recall that the following fact has been proved earlier (see (14.1))

$$(19.6) \qquad \|x-y\| \ge \varepsilon \max(\|x\|,\|y\|) \Rightarrow (1-\delta(\varepsilon))\max(\|x\|,\|y\|) \ge \|\tfrac{1}{2}(x+y\|,$$

where $\delta(\cdot)$ is the modulus of convexity of X. Now put $\theta := \max(\frac{3}{4}, 1-\delta(\frac{1}{2})) < 1$. (Note that θ depends only on δ_X.) Let x_n be any element of a weak null sequence $\{x_n\}$. We distinguish two cases. If $\|x_n\| \le \frac{1}{2}M$, we are done, since for every $m > n$ we then have

$$\left\|\frac{x_n+x_m}{2}\right\| \le \tfrac{1}{2}\|x_n\| + \tfrac{1}{2}\|x_m\| \le \tfrac{3}{4}M \le \theta M.$$

In case $\|x_n\| > \frac{1}{2}M$, the assertion will follow from (19.6). Indeed, first note that there exists an $m > n$ such that $\|x_n-x_m\| > \frac{1}{2}M$. For $\|x_n-x_m\| \le \frac{1}{2}M$ for all $m > n$ would imply, for all $x^* \in B_{X^*}$,

$$|\langle x_n,x^*\rangle| = |\lim_{m\to\infty} \langle x_n-x_m,x^*\rangle| \le \limsup_{m\to\infty}\|x_n-x_m\| \le \tfrac{1}{2}M.$$

Hence $\|x_n\| \le \frac{1}{2}M$, contradicting the assumption. So pick $m > n$ such that $\|x_n-x_m\| > \frac{1}{2}M$. Then (19.6) yields

$$\|\tfrac{1}{2}(x_n+x_m)\| \le (1-\delta(\tfrac{1}{2}))M \le \theta M.$$

Thus (19.5) is proved.

c) Using (19.5) we now construct inductively sequences $\{x_n^{(k)}\}_{n=1}^{\infty}$, $k = 0,1,2,\ldots$ as follows. $\{x_n^{(0)}\} := \{x_n\}$. To define $\{x_n^{(1)}\}$, note first that, by (19.5), there exists a subsequence $\{k_n^{(0)}\} \subset \mathbb{N}$ satisfying

$$k_1^{(0)} = 2 \quad \text{and} \quad \left\|\frac{x^{(0)}_{k^{(0)}_{2n-1}} + x^{(0)}_{k^{(0)}_{2n}}}{2}\right\| \le \theta \quad \text{for all } n = 1,2,\ldots$$

Let us put

$$x_n^{(1)} := \frac{x^{(0)}_{k^{(0)}_{2n-1}} + x^{(0)}_{k^{(0)}_{2n}}}{2} \qquad (n = 1,2,\ldots).$$

Then

$$\sup_{n\in\mathbb{N}} \|x_n^{(1)}\| \le \theta.$$

Suppose that for some $p \in \mathbb{N}$ we have constructed a weak null sequence $\{x_n^{(p)}\}_{n=1}^{\infty}$ such that $\sup_{n\in\mathbb{N}} \|x_n^{(p)}\| \le \theta^p$. Then, by (19.5), we can select a subsequence $\{k_n^{(p)}\} \subset \mathbb{N}$ satisfying

$$k_1^{(p)} = 2 \quad \text{and} \quad \left\| \frac{x_{k_{2n-1}^{(p)}}^{(p)} + x_{k_{2n}^{(p)}}^{(p)}}{2} \right\| \le \theta^{p+1} \quad \text{for all } n = 1,2,\ldots .$$

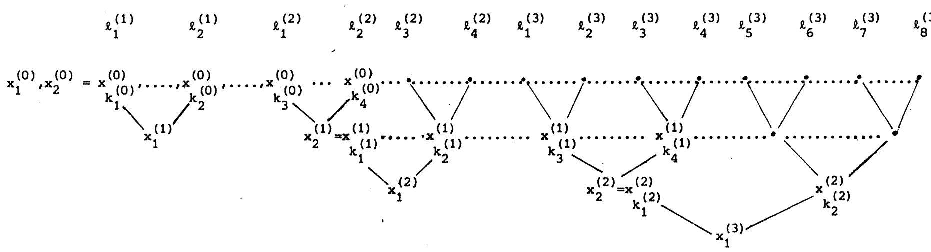
We now define

$$x_n^{(p+1)} := \tfrac{1}{2}\left(x_{k_{2n-1}^{(p)}}^{(p)} + x_{k_{2n}^{(p)}}^{(p)}\right) \qquad (n = 1,2,\ldots).$$

Thus

$$\sup_{n\in\mathbb{N}} \|x_n^{(p+1)}\| \le \theta^{p+1}.$$

This completes the induction definition of the sequence $\{x_n^{(p)}\}$, $p = 1,2,\ldots$. The picture below is an attempt to visualize what happens.



d) Two facts are clear from the construction. (Note that $k_1^{(p)} = 2$ for all p!)

I. For each $k \in \mathbb{N}$, $x_1^{(k)}$ is of the form

$$x_1^{(k)} = 2^{-k}\left(x_{\ell_1^{(k)}} + x_{\ell_2^{(k)}} + \ldots + x_{\ell_{2^k}^{(k)}}\right),$$

where $\ell_1^{(k)} < \ell_2^{(k)} < \ldots < \ell_{2^k}^{(k)}$ and where the "supports" $\{\ell_1^{(k)},\ldots,\ell_{2^k}^{(k)}\}$ of the $x_1^{(k)}$ are pairwise disjoint and ordered as follows: $\ell_{2^k}^{(k)} < \ell_1^{(k+1)2^k}$ for all $k = 1,2,\ldots$.

II. If $1 \le k \le m$ and $1 \le i \le 2^{m-k}$, then

$$2^{-k}\left(x_{\ell^{(m)}_{(i-1)2^k+1}} + \ldots + x_{\ell^{(m)}_{i\cdot 2^k}}\right)$$

is an element of the sequence $\{x_n^{(k)}\}_{n=1}^{\infty}$, and therefore its norm is $\leq \theta^k$.

e) We now enumerate the "supports" $\{\ell_1^{(k)},\ldots,\ell_{2^k}^{(k)}\}$ of the $x_1^{(k)}$ in their natural order, adding an initial element:

$$n_1 := 1,\ \ n_2 := \ell_1^1,\ \ n_3 := \ell_2^{(1)},\ n_4 := \ell_1^{(2)},\ldots,n_7 := \ell_4^{(2)},$$
$$n_8 := \ell_1^{(3)},\ldots,n_{15} := \ell_8^{(3)},\ n_{16} := \ell_1^{(4)},\ldots .$$

The claim is that the subsequence $\{x_{n_k}\}$ converges to 0 in the Cesaro sense.

Indeed, let $\ell \in \mathbb{N}$ be given and let us try to estimate $\frac{1}{\ell}\|\sum_{j=1}^{\ell} x_{n_j}\|$. Suppose $i\cdot 2^k \leq \ell < (i+1)2^k$ for some $k,i \in \mathbb{N}$. Then, by the triangle inequality and (II),

$$\|\sum_{j=1}^{\ell} x_{n_j}\| = \|\sum_{j=1}^{2^k-1} x_{n_j} + \sum_{p=2}^{i}\sum_{j=(p-1)2^k}^{p\cdot 2^k-1} x_{n_j} + \sum_{j=i\cdot 2^k}^{\ell} x_{n_j}\|$$
$$\leq \|\sum_{j=1}^{2^k-1} x_{n_j}\| + \sum_{p=2}^{i}\|\sum_{j=(p-1)2^k}^{p\cdot 2^k-1} x_{n_j}\| + \|\sum_{j=i\cdot 2^k}^{\ell} x_{n_j}\|$$
$$\leq (2^k-1) + (i-1)2^k\theta^k + 2^k.$$

Hence

$$\frac{1}{\ell}\|\sum_{j=1}^{\ell} x_{n_j}\| \leq \frac{2^{k+1}}{\ell} + \frac{(i-1)2^k\theta^k}{\ell} \leq \frac{2}{i} + \theta^k. \tag{19.6}$$

Finally, observe that for every $N_1 \in \mathbb{N}$ there exists an $N_2 \in \mathbb{N}$ such that for every $\ell \geq N_2$ there exist $i,k \geq N_1$ satisfying $i\cdot 2^k \leq \ell < (i+1)2^k$. This observation together with (19.6) shows that $\lim_{\ell\to\infty} \|\sum_{j=1}^{\ell} x_{n_j}\| = 0$. □

<u>REMARK 19.10</u>. Obviously the Banach-Saks property is an isomorphic invariant. Hence all uniformly convexifiable, i.e. all superreflexive spaces have the Banach-Saks property. These do not exhaust the class of "B.S. spaces", however, as the following example shows. Let $X = (\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$. We know from Section 15 that X is reflexive, but not superreflexive. To show that X has the B.S. property it suffices, by the proof of Proposition 19.9, to prove the existence of a number θ, $0 < \theta < 1$ satisfying (19.5). Let $\{x^{(k)}\}$ be a weak null sequence in $X = (\sum_{n=1}^{\infty} \oplus (c_0)_n)_{\ell^2}$ and let $x_n^{(k)} \in (c_0)_n$ denote the "n-th coordinate" of $x^{(k)}$ $(n = 1,2,\ldots)$. As before we put $M := \sup_{k\in\mathbb{N}}\|x^{(k)}\|$. Given $\varepsilon > 0$ and $k \in \mathbb{N}$, there exists an $N \in \mathbb{N}$ such that $\sum_{n>N}\|x_n^{(k)}\|^2 < \varepsilon^2$. Since weak convergence to 0 implies "coordinatewise" convergence to 0, for sufficiently large $\ell \in \mathbb{N}$ we have $\sum_{n\leq N}\|x_n^{(\ell)}\|^2 < \varepsilon^2$. Then, for such ℓ,

$$\tfrac{1}{2}\| x^{(k)}+x^{(\ell)}\| \le \tfrac{1}{2}\| (x_1^{(k)},\dots,x_N^{(k)},x_{N+1}^{(\ell)},\dots)\| +$$

$$+ \tfrac{1}{2}\| (x_1^{(\ell)},\dots,x_N^{(\ell)},x_{N+1}^{(k)},\dots)\|$$

$$= \tfrac{1}{2}\Big(\sum_{n\le N}\|x_n^{(k)}\|^2 + \sum_{n>N}\|x_n^{(\ell)}\|^2\Big)^{\frac{1}{2}} + \tfrac{1}{2}\Big(\sum_{n\le N}\|x_n^{(\ell)}\|^2 + \sum_{n>N}\|x_n^{(k)}\|^2\Big)^{\frac{1}{2}}$$

$$\le \tfrac{1}{2}(M^2+M^2)^{\frac{1}{2}} + \tfrac{1}{2}(\varepsilon^2+\varepsilon^2)^{\frac{1}{2}} = 2^{-\frac{1}{2}}(M+\varepsilon).$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, it follows that any $\theta > 2^{-\frac{1}{2}}$ satisfies (19.5).

We now consider two ergodic properties.

DEFINITION 19.11. a) A Banach space X is called *ergodic* (*for isometries*) iff for every linear isometry T on X the Cesaro averages $A_n = A_n(T) := \frac{1}{n}(T^0+T^1+\dots+T^{n-1})$ converge in the strong operator topology, i.e. $\{A_n x\}$ converges for every $x \in X$.

b) A Banach space X is called R-*ergodic* (*for isometries*) iff for every linear isometry T on X and for every $x \in X$ there exists an R-matrix (α_{ij}) such that $\{\sum_{j=1}^{\infty} \alpha_{ij}\, T^{j-1}x\}_{i=1}^{\infty}$ converges weakly.

REMARK 19.12. It is a classical fact that reflexive spaces are ergodic. We shall not give the proof at this point since it will be a corollary of a later result. It is known that the converse is not true: there exist examples of non-reflexive spaces (even a space isomorphic to ℓ^1) for which $\pm I$ (the identity) are the only isometries. Such spaces are trivially ergodic. Of course, every ergodic space is R-ergodic. Note also that Proposition 19.6 directly implies that a reflexive space is R-ergodic.

We wish to prove the following

THEOREM 19.13. *For every Banach space* X *the following are equivalent:*

(i) X *super B.S.;*

(ii) X *superreflexive;*

(iii) X *superergodic;*

(iv) X *super R-ergodic.*

We have the following implications:

B.S. $\Rightarrow$ reflexive $\Rightarrow$ ergodic $\Rightarrow$ R-ergodic

The first implication is Corollary 19.8 and the second follows from the (yet to be proved) ergodic theorem of Yosida-Kakutani. The same implications hold for the corresponding superproperties, i.e. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). We shall eventually prove (iv) $\Rightarrow$ (i), completing the proof of Theorem 19.13. Observe that Theorem 19.13 includes Proposition 19.9, since the latter is nothing but the implication (ii) $\Rightarrow$ (i). (Note that $Y \prec X$, X superreflexive, implies Y superreflexive.) Since the argument proving (iv) $\Rightarrow$ (i) makes no use of Proposition 19.9, it implicitly provides a new proof for it.

We first establish an auxiliary result which relates (for fixed $x \in X$ and certain matrices (α_{ij})) the behavior of $\{\sum_{j=1}^{\infty} \alpha_{ij} T^{j-1}x\}$ to that of $\{A_n(T)x\}$. A corollary will be the ergodic theorem of Yosida-Kakutani, and in particular the second implication in (19.7). Let us agree to call (α_{ij}) an *A-matrix* if it satisfies (A) in Definition 19.5. Furthermore let S denote the set of all real sequences which are eventually 0. Whenever we write an infinite series, we tacitly assume that it converges

PROPOSITION 19.14. *Let* T *be a linear operator in a Banach space* X, (α_{ij}) *an* A-*matrix and let* $x,\bar{x} \in X$ *satisfy*

$$(19.7) \qquad \sum_{j=1}^{\infty} \alpha_{ij} T^{j-1}x \longrightarrow \bar{x} \text{ weakly, as } i \to \infty.$$

Then

$$(19.8) \qquad x-\alpha\bar{x} \in \overline{(I-T)X}, \qquad \text{for some } \alpha \in \mathbb{R},$$

and this holds for $\alpha = 1$ *if* $\lim_{i\to\infty} \sum_{j=1}^{\infty} \alpha_{ij} = 1$. *If, moreover,*

$$(19.9) \qquad \sup_{n\in\mathbb{N}} \|A_n\| < \infty \quad \text{and} \quad \frac{T^n}{n} \to 0 \text{ strongly},$$

then

$$\lim_{n\to\infty} A_n(x-\alpha\bar{x}) = 0.$$

PROOF. a) We first prove the proposition under the additional assumption that $\{\alpha_{ij}\}_{j=1}^{\infty} \in S$ and $\sum_{j=1}^{\infty} \alpha_{ij} = 1$ for all i. Let us define the map $\phi: S \to X$ by

$$\phi(a) = \sum_{j=1}^{\infty} a_j T^{j-1}x \quad (a = \{a_j\} \in S).$$

Suppose that for some $n \in \mathbb{N}$ and $b_1,\dots,b_n \in \mathbb{R}$ we have $\sum_{j=1}^{n} b_j = 0$. Then

$$\sum_{j=1}^{n} b_j\, T^{j-1} = b_1(I-T) + (b_1+b_2)(T-T^2) + (b_1+b_2+b_3)(T^2-T^3)\dots$$
$$\dots + (b_1+\dots+b_n)(T^{n-1}-T^n)$$
$$= P(T)(I-T) = (I-T)P(T),$$

where $P(T)$ is a polynomial in T. It follows that $\phi(b) \in (I-T)X$, whenever $b = \{b_j\} \in S$ and $\sum_{j=1}^{\infty} b_j = 0$. By the assumption that $\sum_{j=1}^{\infty}\alpha_{ij} = 1$ and $\{\alpha_{ij}\}_{j=1}^{\infty} \in S$ for all i, this remark can be applied to the sequence $\{b_j\}$ defined by

$$b_1 = a_{i1}-1,\quad b_j = a_{ij} \quad \text{for } j = 2,3,\dots,\quad i \text{ fixed}.$$

Therefore for each $i \in \mathbb{N}$ there exists a $y_i \in X$ such that $\sum_{j=1}^{\infty} a_{ij}\, T^{j-1}x-x = y_i - Ty_i$. Thus, by (19.7), $\bar{x} - x$ belongs to the closure (= weak closure) of $(I-T)X$.

b) In the general case we may assume, by (A), that there exists a $\beta > 0$ such that $\sum_{j=1}^{\infty} \alpha_{ij} > \beta$ for all i. (If necessary, replace (α_{ij}) by $(\alpha_{k_i,j})$, $\{k_i\} \subset \mathbb{N}$ a subsequence and change signs.) Using the convergence (for each i) of the series $\sum_{j=1}^{\infty} \alpha_{ij}$ (implicit in (A)) and $\sum_{j=1}^{\infty} \alpha_{ij}\, T^{j-1}x$ (implicit in (19.7)), we determine a subsequence $\{k_i\} \subset \mathbb{N}$ such that

$$\text{(19.10)} \qquad \Big|\sum_{j>k_i} \alpha_{ij}\Big| < \frac{1}{i} \quad \text{and} \quad \Big\|\sum_{j>k_i} \alpha_{ij}\, T^{j-1}x\Big\| < \frac{1}{i} \qquad (i = 1,2,\dots).$$

Put

$$\beta_{ij} := \begin{cases} \alpha_{ij} & \text{if } j \le k_i \\ 0 & \text{if } j > k_i \end{cases} \qquad (i,j = 1,2,\dots).$$

Then (β_{ij}) is clearly an A-matrix again, $\{\beta_{ij}\}_{j=1}^{\infty} \in S$ for all $i \in \mathbb{N}$, and $\beta \le \liminf_{i\to\infty} \sum_{j=1}^{\infty} \beta_{ij} =: \beta_0 \le \infty$. Replacing (β_{ij}) by $(\beta_{k_i,j})$, $\{k_i\} \subset \mathbb{N}$ a subsequence, we may assume that this liminf is a limit. Finally, we put

$$\gamma_{ij} := \frac{\beta_{ij}}{\sum_{j=1}^{\infty} \beta_{ij}} \qquad (i,j = 1,2,\dots).$$

Then (γ_{ij}) is again an A-matrix and, moreover, satisfies the assumptions of part a) of this proof, namely, $\{\gamma_{ij}\}_{j=1}^{\infty} \in S$ and $\sum_{j=1}^{\infty} \gamma_{ij} = 1$ for all $i \in \mathbb{N}$. Furthermore, (19.7) and the second part of (19.10) imply that $\sum_{j=1}^{\infty} \beta_{ij}\, T^{j-1}x \to \bar{x}$ weakly, and therefore $\sum_{j=1}^{\infty} \gamma_{ij}\, T^{j-1}x \to \alpha\bar{x}$ weakly, with $\alpha = 1/\beta_0$. By part

a) of the proof we have $x - \alpha\bar{x} \in \overline{(I-T)X}$. Thus (19.8) has now been proved in the general case.

c) Assume now that, in addition, (19.9) holds. For every $y \in X$ we have $A_n(T)(I-T)y = \frac{1}{n}(I-T^n)y \to 0$, since $\lim_{n\to\infty} \frac{T^n}{n} y = 0$ by assumption. It follows now that $\lim_{n\to\infty} A_n(x-\alpha\bar{x}) = 0$, by approximating $x - \alpha\bar{x}$ with elements $(I-T)y$, and using the boundedness of $\{A_n\}$. □

<u>COROLLARY 19.15</u> (Ergodic theorem of Yosida-Kakutani).
Let T be a bounded linear operator in a Banach space X and suppose that

$$\sup_{n\in\mathbb{N}} \|A_n(T)\| < \infty, \quad \frac{1}{n} T^n \to 0 \text{ strongly, as } n \to \infty. \tag{19.11}$$

Then, for every $x \in X$, $\{A_nx\}$ converges whenever $\{A_nx\}$ has a weakly convergent subsequence. In particular every reflexive Banach space is ergodic (for isometries).

<u>PROOF</u>. Suppose that, for some $x \in X$, $\{A_{n_i}x\}$ is a subsequence of $\{A_nx\}$ which converges weakly, say to $\bar{x}$. Then $\{(I-T)A_{n_i}x\} = \{\frac{1}{n_i}(I-T^{n_i})x\}$ converges to 0 in norm, by the second half of (19.11), but also to $(I-T)\bar{x}$ weakly. Therefore $T\bar{x} = \bar{x}$. Writing $A_{n_i}x = \sum_{j=1}^{\infty} \alpha_{ij} T^{j-1}x$ with

$$\alpha_{ij} = \begin{cases} 0 & \text{for } j > n_i \\ \frac{1}{n_i} & \text{for } j = 1,\dots,n_i \end{cases} \qquad (i = 1,2,\dots),$$

(α_{ij}) is an A-matrix satisfying $\sum_{j=1}^{\infty} \alpha_{ij} = 1$ for all $i \in \mathbb{N}$. Thus, by Proposition 19.14 and the already proved fact $\bar{x} = T\bar{x}$, $A_n(x-\bar{x}) = A_xx-\bar{x} \to 0$, as $n \to \infty$. □

We now sketch a rough outline of the proof of (iv) ⇒ (i) in Theorem (19.13). Let X be super R-ergodic and let Y with $Y \prec X$ be arbitrary. We would like to show that Y is B.S., so let $\{x_n\}$ be any sequence in B_Y. Suppose we can extract a subsequence $\{e_n\}$ from $\{x_n\}$ such that the shift operator T defined on $F := [e_n]$ by $Te_n = e_{n+1}$ $(n = 1,2,\dots)$ is an isometry and equals the identity on $\bigcap_{j=0}^{\infty} T^jF$. Then we are done. Indeed, F is R-ergodic, so for some R-matrix (α_{ij}) we now have that $\{\sum_{j=1}^{\infty} \alpha_{ij} T^{j-1}e_1\} = \{\sum_{j=1}^{\infty} \alpha_{ij} e_j\}$ converges weakly, say to e. It is readily deduced from condition (B) in Definition 19.5 that $e \in \bigcap_{j=0}^{\infty} T^jF$, so that $Te = e$. Proposition 19.14 then implies that $\lim_{n\to\infty} A_n(e_1-\alpha e) = 0$, for some $\alpha \in \mathbb{R}$. Hence $\lim_{n\to\infty} A_ne_1 = \lim_{n\to\infty} \frac{e_1+\dots+e_n}{n} = \alpha e$ exists.

Unfortunately such a subsequence $\{e_n\}$ of $\{x_n\}$ is not easily found, and its existence can certainly not be deduced from the R-ergodicity of Y alone, since even a reflexive space need not have the B.S. property (cf. the Example). Hence we must refine our approach. What we shall do is the following. We extract from $\{x_n\}$ a subsequence $\{e_n\}$ of a special nature (see Proposition 19.17), introduce on $sp\{e_n\}$ a new norm $|\cdot|$ and denote by F the completion of $sp\{e_n\}$ for $|\cdot|$. All this is done in such a way that the following holds.

(1°) $F \prec Y$ (and hence $F \prec X$, so F is R-ergodic).

(2°) The shift operator T defined on F by $Te_n = e_{n+1}$ $(n = 1,2,\ldots)$ is an isometry and equals the identity on $\bigcap_{j=0}^{\infty} T^jF$.

(Hence by the argument given above and applied to the R-ergodic space F, $\lim_{n\to\infty} \frac{e_1+\ldots+e_n}{n}$ exists in F.)

(3°) The existence in F of $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} e_j$ implies that $\{e_n\}$ has a subsequence whose Cesaro sums converge in Y.

Before we can define the right subsequence $\{e_n\}$ we must prove the following combinatorial result which is known as Ramsey's theorem. The proof of it was kindly shown to us by H.W. Lenstra. If V is a set and $k \in \mathbb{N}$ then $P_k(V)$ denotes the collection of all subsets of V consisting of k elements.

PROPOSITION 19.16. *Let* V *be an infinite set and let* $k \in \mathbb{N}$. *Suppose that* $P_k(V) = A \cup B$, *with* $A \cap B = \phi$. *Then there exists an infinite set* $W \subset V$ *such that either* $P_k(W) \subset A$ *or* $P_k(W) \subset B$.

PROOF. We use induction on k. For k = 1 the proposition is evidently true. So suppose $k > 1$ and assume the result has been proved for $k-1$. Pick $v_1 \in V$ arbitrarily. Then $P_{k-1}(V\backslash\{v_1\}) = A_1 \cup B_1$ with $A_1 \cap B_1 = \phi$, where A_1 and B_1 are defined as follows: $E \in P_{k-1}(V\backslash\{v_1\})$ belongs to A_1 iff $E \cup \{v_1\} \in A$ and to B_1 iff $E \cup \{v_1\} \in B$. By the induction hypothesis there exists an infinite set $W_1 \subset V\backslash\{v_1\}$ such that either $P_{k-1}(W_1) \subset A_1$ or $P_{k-1}(W_1) \subset B_1$. Having chosen W_1, we agree to call v_1 of type A (respectively, of type B) in the first (respectively, second) case. Now choose $v_2 \in W_1$ arbitrarily. Again there is a natural partition $P_{k-1}(W_1\backslash\{v_2\}) = A_2 \cup B_2$ defined by $E \in A_2$ iff $E \cup \{v_2\} \in A$ and $E \in B_2$ iff $E \cup \{v_2\} \in B$ $(E \in P_{k-1}(W_1\backslash\{v_2\}))$. By the induction hypothesis we can determine an infinite subset W_2 of $W_1\backslash\{v_2\}$ such that either $P_{k-1}(W_2) \subset A_2$ or $P_{k-1}(W_2) \subset B_2$. In the first (second) case we call v_2 of type A (of type B). Continuing this procedure we define a sequence $\{v_n\} \subset V$, and a sequence $V := W_0, W_1, W_2, \ldots$ of infinite subsets of V satisfying the following conditions for each $n \in \mathbb{N}$:

(i) $W_n \subsetneqq W_{n-1}$;

(ii) $v_n \in W_{n-1} \backslash W_n$;

(iii) $E \in P_{k-1}(W_n)$ implies $E \cup \{v_n\} \in A$ (respectively, $E \cup \{v_n\} \in B$) iff v_n if of type A (respectively, of type B).

It follows in particular from this that any set $\{v_{n_1}, v_{n_2}, \ldots, v_{n_k}\}$ with $n_1 < n_2 < \ldots < n_k$ belongs to A iff v_{n_1} if of type A and to B iff v_{n_1} if of type B. One at least of the sets $\{v_n \mid v_n \text{ is of type A}\}$, $\{v_n \mid v_n \text{ is of type B}\}$ is infinite and can be taken as the desired set W. □

This combinatorial result is useful in the proof of

<u>PROPOSITION 19.17</u>. *Let X be a Banach space and* $\{x_n\} \subset X$ *a bounded sequence. Then* $\{x_n\}$ *contains a subsequence* $\{e_n\}$ *satisfying the following property: For each* $a \in S$ *and each* $\varepsilon > 0$ *there exists a* $\nu = \nu(a,\varepsilon) \in \mathbb{N}$ *with*

$$\left| \left\| \sum_{i=1}^{\infty} a_i e_{n_i} \right\| - L(a) \right| \le \varepsilon$$

whenever $\nu \le n_1 < n_2 < \ldots$.

<u>PROOF</u>. Let us assume for simplicity, as we clearly may, that $\|x_n\| \le 1$ for all $n \in \mathbb{N}$. Let $a = (a_1, \ldots, a_k, 0, 0, \ldots)$ be a fixed element of S with rational coordinates and consider the function $\psi: P_k(\mathbb{N}) \to \mathbb{R}$ defined by

$$\psi(E) = \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\|,$$

where $\{n_1, n_2, \ldots, n_k\}$ is the set E enumerated in increasing order. Clearly ψ is bounded, say by α. Let us write

$$P_k(\mathbb{N}) = A \cup B,$$

with $A := \{E: 0 \le \psi(E) < \frac{\alpha}{2}\}$ and $B := \{E: \frac{\alpha}{2} \le \psi(E) \le \alpha\}$. By the previous result $\mathbb{N}$ contains an infinite subset $\mathbb{N}_1$ such that either $P_k(\mathbb{N}_1) \subset A$ or $P_k(\mathbb{N}_1) \subset B$. Put differently, this means that there exists a subsequence $\{i_n^{(1)}\}$ of $\mathbb{N}$ such that we have

$$\text{either } 0 \le \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| < \frac{\alpha}{2} \quad \text{or} \quad \frac{\alpha}{2} \le \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \le \alpha$$

for all $n_1 < \ldots < n_k$ taken from $\{i_n^{(1)}\}$. Suppose we are in the first case. Continuing the procedure, we can extract a subsequence $\{i_n^{(2)}\}$ of $\{i_n^{(1)}\}$ such that

$$\text{either } 0 \le \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| < \frac{\alpha}{4} \quad \text{or} \quad \frac{\alpha}{4} \le \left\| \sum_{i=1}^{k} a_i x_{n_i} \right\| \le \frac{\alpha}{2},$$

for all $n_1 < \ldots < n_k$ taken from $\{i_n^{(2)}\}$. Inductively, we so obtain a nested sequence of closed subintervals of $[0,\alpha]$ having exactly one point, $L(a)$, in common, and a diagonal seuquence $\{i_n^{(n)}\}$ such that, for every $\varepsilon > 0$, $\left| \|\sum_{i=1}^k a_i x_{n_i}\| - L(a) \right| < \varepsilon$ for any choice of $n_1 < \ldots < n_k$ sufficiently far in $\{i_n^{(n)}\}$. Since the set of $a \in S$ with rational coefficients is countable, another diagonal argument produces an infinite sequence, which we call $\{i_n\}$, such that for every $\varepsilon > 0$ and for every $a \in S$ with rational coefficients,

$$\left| \left\| \sum_{i=1}^{\infty} a_i x_{n_i} \right\| - L(a) \right| < \varepsilon,$$

whenever $n_1 < n_2 < \ldots$ are chosen sufficiently far in $\{i_n\}$. We claim that $\{e_n\} := \{x_{i_n}\}$ satisfies the requirements. For this it remains to consider the case of an arbitrary not necessarily rational $a = (a_1,\ldots,a_k,0,\ldots)$ $\in S$. Let $\varepsilon > 0$ be given and choose a rational $a' = (a_1',\ldots,a_k',0,\ldots) \in S$ such that $\|a-a'\|_1 := \sum_{i=1}^k |a_i - a_i'| < \frac{\varepsilon}{2}$. Then on the one hand

$$\left| \left\| \sum_{i=1}^{k} a_i e_{n_i} \right\| - \left\| \sum_{i=1}^{k} a_i' e_{n_i} \right\| \right| \le \left\| \sum_{i=1}^{k} (a_i - a_i') e_{n_i} \right\| \le \|a-a'\|_1 < \frac{\varepsilon}{2}$$

for all $n_1 < \ldots < n_k$ in $\mathbb{N}$, and on the other, there exists a $\nu \in \mathbb{N}$ such that $\left| \|\sum_{i=1}^k a_i' e_{n_i}\| - L(a') \right| < \frac{\varepsilon}{2}$ whenever $\nu \le n_1 < \ldots < n_k$ and $n_1,\ldots,n_k$ are taken from $\{i_n\}$. It follows now that $\lim_{n_1\to\infty} \|\sum_{i=1}^k a_i e_{n_i}\| =: L(a)$ exists, uniformly for all $n_1 < \ldots < n_k$ taken from $\{i_n\}$. □

We return now to the setting outlined before Proposition 19.16. Let X be super R-ergodic, $Y \prec X$ and $\{x_n\}$ a sequence in B_Y. We are interested in extracting a subsequence from $\{x_n\}$ whose Cesaro averages converge. We may assume that $\{x_n\}$ contains no convergent subsequence since such a subsequence would also be Cesaro summable. Now let $\{e_n\}$ be a subsequence of $\{x_n\}$ satisfying Proposition 19.17. We define a function $|\cdot|$ on $\mathrm{sp}\{e_n\}$ as follows:

$$(19.12) \qquad \left| \sum_{i=1}^{\infty} a_i e_i \right| := L(a) \qquad (a = \{a_i\} \in S).$$

<u>PROPOSITION 19.18.</u> $|\cdot|$ *is a norm on* $\mathrm{sp}\{e_n\}$ *which is invariant under spreading, i.e.*

$$(19.13) \qquad \left| \sum_{i=1}^{k} a_i e_{n_i} \right| = \left| \sum_{i=1}^{k} a_i e_{m_i} \right| \quad \textit{for all } k \in \mathbb{N},\ a_1,\ldots,a_k \in \mathbb{R},$$

$n_1 < n_2 < \dots < n_k$, $m_1 < m_2 < \dots < m_k$.

PROOF. The seminorm properties of $|\cdot|$ are obvious and (19.13) follows immediately from (19.12) and the definition of $L(\cdot)$ given in the statement of Proposition 19.17. Suppose $|\cdot|$ fails to be a norm, i.e. $|\sum_{i=1}^{\infty} a_i e_i| = 0$ for some $a = \{a_i\} \in S$, $a \neq 0$. Let a_{i_0} be the first non-zero coefficient of a. Since $|\sum_{i=1}^{\infty} a_i e_i| = 0$ there exists for each $\varepsilon > 0$ a $\nu \in \mathbb{N}$ such that, whenever $\nu \le n < p < n_1 < n_2 < \dots$, we have

$$\| a_{i_0} e_n + a_{i_0+1} e_{n_1} + a_{i_0+2} e_{n_2} + \dots \| \le \frac{\varepsilon}{2} |a_{i_0}|$$

and

$$\| a_{i_0} e_p + a_{i_0+1} e_{n_1} + a_{i_p+2} e_{n_2} + \dots \| \le \frac{\varepsilon}{2} |a_{i_0}|.$$

Thus $\|e_n - e_p\| \le \varepsilon$ whenever $\nu \le n < p$, so that $\{e_n\}$ is a Cauchy sequence in Y. This contradicts our assumption that $\{x_n\}$ contains no convergent subsequences. □

Now let F be the completion of $sp\{e_n\}$ with respect to the norm $|\cdot|$ and put $F_n := [e_i]_{i=n}^{\infty}$ $(n = 1,2,\dots)$, $F_\infty := \bigcap_{n=1}^{\infty} F_n$. It follows in particular from (19.13) that the shift operator T on F defined by $Te_n = e_{n+1}$ $(n = 1,2,\dots)$ is an isometry. We claim that T is the identity precisely on F_∞.

LEMMA 19.19. *For every* $y \in F$ *we have* $y \in F_\infty$ *iff* $Ty = y$.

PROOF. Let $y \in F_\infty$ and let $\varepsilon > 0$ be arbitrary. Choose a $y_1 \in sp\{e_n\}$ such that $|y - y_1| \le \frac{\varepsilon}{4}$. Suppose $y_1 \in sp\{e_1,\dots,e_n\}$. Since $y \in F_{n+1}$, there also exists a $y_2 \in sp\{e_{n+1}, e_{n+2},\dots\}$ with $|y - y_2| \le \frac{\varepsilon}{4}$. Thus $|y_1 - y_2| \le \frac{\varepsilon}{2}$. Since $|\cdot|$ is invariant under spreading, we have $|y_1 - Ty_2| = |y_1 - y_2| \le \frac{\varepsilon}{2}$. Furthermore, T being an isometry, $|Ty - Ty_2| = |y - y_2| \le \frac{\varepsilon}{4}$. Therefore $|y - Ty| \le |y - y_1| + |y_1 - Ty_2| + |Ty_2 - Ty| \le \varepsilon$, so $y = Ty$ since $\varepsilon > 0$ was arbitrary.

Conversely, $Ty = y$ implies $y = T^n y \in F_n$ for all $n \in \mathbb{N}$. □

PROPOSITION 19.20. $F \prec Y$.

PROOF. It clearly suffices to show that for every $k \in \mathbb{N}$ and every $\varepsilon > 0$ there exists an isomorphism $A: [e_i]_{i=1}^{k} \to Y$ such that $\|A\|\|A^{-1}\| < 1+\varepsilon$. So fix $k \in \mathbb{N}$, $\varepsilon > 0$. By compactness there exist, for every $\delta > 0$, finitely many $a^{(1)},\dots,a^{(n)} \in S$ with $a_i^{(j)} = 0$ for $i > k$ $(j = 1,\dots,n)$ which form a δ-net for $\{(a_1,\dots,a_k,0,0,\dots): |\sum_{i=1}^{k} a_i e_i| = 1\}$ with respect to the ℓ^1-norm. By the definition of $|\cdot|$ there exists a $\nu \in \mathbb{N}$ such that

$$\Big|\,|\sum_{i=1}^{k} a_i^{(j)} e_i| - \|\sum_{i=1}^{k} a_i^{(j)} e_{n_i}\|\,\Big| < \delta \quad \text{for all } j = 1,\dots,n,$$

whenever $\nu \le n_1 < n_2 < \dots < n_k$. Now define $A\colon [e_i]_{i=1}^{k} \to Y$ by

$$A(\sum_{i=1}^{k} a_i e_i) = \sum_{i=1}^{k} a_i e_{\nu+i}.$$

Suppose $|\sum_{i=1}^{k} a_i e_i| = 1$. Choose $a^{(j)}$ $(1 \le j \le n)$ so that $\sum_{i=1}^{k} |a_i - a_i^{(j)}| < \delta$. Then

$$\|A(\sum_{i=1}^{k} a_i e_i)\| \begin{cases} \le \|\sum_{i=1}^{k} a_i^{(j)} e_{\nu+i}\| + \|\sum_{i=1}^{k} (a_i - a_i^{(j)}) e_{\nu+i}\| \\ \ge \|\sum_{i=1}^{k} a_i^{(j)} e_{\nu+i}\| - \|\sum_{i=1}^{k} (a_i - a_i^{(j)}) e_{\nu+i}\| \end{cases}$$

$$\begin{cases} \le |\sum_{i=1}^{k} a_i^{(j)} e_i| + \delta + \sum_{i=1}^{k} |a_i - a_i^{(j)}| \le |\sum_{i=1}^{k} a_i^{(j)} e_i| + 2\delta = 1+2\delta \\ \le |\sum_{i=1}^{k} a_i^{(j)} e_i| - \delta - \sum_{i=1}^{k} |a_i - a_i^{(j)}| \ge |\sum_{i=1}^{k} a_i^{(j)} e_i| - 2\delta = 1-2\delta. \end{cases}$$

Thus $\|A\| \le 1+2\delta$ and $\|A^{-1}\| \le \frac{1}{1-2\delta}$. A sufficiently small choice of $\delta > 0$ then ensures that $\|A\|\|A^{-1}\| < 1+\varepsilon$. □

We have now defined $\{e_n\}$, $|\cdot|$ and F so that (1°) and (2°) are satisfied. It remains to show (3°) to complete the proof of Theorem 19.13.

PROPOSITION 19.21. *Suppose* $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e_i$ *exists in* F. *Then* $\{e_n\}$ *has a subsequence which is Cesaro summable in* Y.

PROOF. Suppose $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} e_i =: e$ in F. For each $k \in \mathbb{N}$ $\mathrm{dist}(\frac{1}{n}\sum_{i=1}^{n} e_i, F_k) \le |\frac{1}{n}\sum_{i=1}^{k-1} e_i| \to 0$ as $n \to \infty$, so that $e \in \bigcap_{k=1}^{\infty} F_k = F_\infty$. Hence $Te = e$, by Lemma 19.19. Since T is an isometry, $|\frac{1}{n}\sum_{i=1}^{n} e_i - e| = |T^r(\frac{1}{n}\sum_{i=1}^{n} e_i) - T^r e| = |\frac{1}{n}\sum_{i=1}^{n} e_{i+r} - e|$ for every $r \in \mathbb{N}$, so that $\{\frac{1}{n}\sum_{i=1}^{n} e_{i+r}\}_{n=1}^{\infty}$ converges in F to e, uniformly in r. In particular, for every $\varepsilon > 0$ there exists a $P(\varepsilon) \in \mathbb{N}$ such that

$$(19.14) \qquad |\frac{1}{p}\sum_{i\le p} e_i - \frac{1}{q}\sum_{i=p+1}^{p+q} e_i| < \varepsilon \quad \text{if } p,q \ge P(\varepsilon).$$

This means, by the definition of $|\cdot|$, that for every $\varepsilon > 0$ there exists a $P(\varepsilon) \in \mathbb{N}$ such that for every choice of $p,q \ge P(\varepsilon)$ there exists an

$N = N(\varepsilon,p,q) \in \mathbb{N}$ such that

(19.15) $$\|\frac{1}{p}\sum_{i\le p} e_{n_i} - \frac{1}{q}\sum_{i=p+1}^{p+q} e_{n_i}\| < \varepsilon$$

whenever $N < n_1 < n_2 < \dots < n_{p+q}$. The number $N(\varepsilon,p,q)$ depends on ε, p and q and is defined whenever $p,q \ge P(\varepsilon)$. We now put

$$P(2^{-n}) =: P_n \qquad (n = 0,1,\dots)$$

and assume, as we clearly may, that

(19.16) $$P_n > nP_{n-1} \qquad (n = 1,2,\dots).$$

Furthermore, let us put $\nu(n) := N(2^{-n},P_n,P_{n+1})$ and let us define a sequence $\{r_n\} \subset \mathbb{N}$ such that

(19.17) $$r_{n+1} \ge r_n + P_n$$

and

(19.18) $$r_n \ge \nu(n) \qquad (n = 1,2,\dots).$$

Putting

(19.19) $$\xi_n := \frac{1}{P_n}\sum_{i\le P_n} e_{r_n+i} \qquad (n = 1,2,\dots),$$

it follows from (19.15) (using also (19.17) and (19.18), that

(19.20) $$\|\xi_n-\xi_{n+1}\| < 2^{-n}.$$

Thus $\{\xi_n\}$ is a Cauchy sequence in Y and therefore converges, say to ξ. We now consider the subsequence of $\{e_n-\xi\}$, whose terms have indices (see (19.17))

$$r_1+1,r_1+2,\dots,r_1+P_1,r_2+1,\dots,r_2+P_2,\dots,r_i+1,\dots,r_i+P_i,\dots$$

and call this subsequence $\{y_n\}$.

Before proceeding, let us make the statement $\lim_{n\to\infty} \xi_n = \xi$ more precise as follows:

(19.21) $$\|\frac{1}{P_n}\sum_{i=1}^{P_n} e_{m_i}-\xi\| = \|\frac{1}{P_n}\sum_{i=1}^{P_n}(e_{m_i}-\xi)\| \le 2^{-n+3}$$

for all $n \in \mathbb{N}$ and all $m_1 < m_2 < \dots < m_{P_n}$ satisfying $m_1 > r_n$. For the proof of (19.21), apply (19.15) with $\varepsilon = 2^{-n+1}$, $p = P_{n-1}$, $q = P_n$. This yields, taking into account also (19.17) and (19.18),

$$(19.22)\qquad \left\| \frac{1}{P_{n-1}} \sum_{i=1}^{P_{n-1}} e_{r_{n-1}+i} - \frac{1}{P_n} \sum_{i=1}^{P_n} e_{m_i} \right\| = \left\| \xi_{n-1} - \frac{1}{P_n} \sum_{i=1}^{P_n} e_{m_i} \right\| \le 2^{-n+1},$$

whenever $r_n < m_1 < m_2 < \ldots < m_{P_n}$. Also

$$(19.23)\qquad \|\xi_{n-1}-\xi\| \le \sum_{i=n-1}^{\infty} \|\xi_i-\xi_{i+1}\| \le 2^{-n+2}.$$

Now (19.21) follows from (19.22) and (19.23).

To finish the proof, let us consider an arbitrary finite subsequence $y_{i_1}, y_{i_2}, \ldots, y_{i_n}$ of $\{y_n\}$ and let us write $z_j := y_{i_j}$ $(j = 1,\ldots,n)$. Let k be the integer satisfying

$$P_1 + P_2 + \ldots + P_k \le n < P_1 + P_2 + \ldots + P_{k+1}$$

(assume that n is large enough so that k is well defined), and put $m := n - \sum_{i=1}^{k} P_i$. Let $d_{k-1}, d_k, Q_{k-1}, Q_k$ be non-negative integers such that

$$m = d_k P_k + Q_k, \qquad Q_k < P_k$$

$$Q_k = d_{k-1}P_{k-1} + Q_{k-1},\ Q_{k-1} < P_{k-1}.$$

It now follows from (19.21) and the definition of the y_n's that

$$\sum_{j=1}^{n} y_{i_j} = \sum_{j=1}^{n} z_j = (z_1+\ldots+z_{P_1}) + (z_{P_1+1}+\ldots+z_{P_1+P_2}) + \ldots$$

$$\ldots + (z_{P_1+\ldots+P_{k-1}+1} + \ldots + z_{P_1+\ldots+P_k}) + (z_{P_1+\ldots+P_k+1} + \ldots + z_n)$$

is majorized in norm by

$$P_1 2^{-1+3} + P_2 2^{-2+3} + \ldots + P_k 2^{-k+3} + d_k P_k 2^{-2+3} +$$
$$+ d_{k-1} P_{k-1} 2^{-(k-1)+3} + 2Q_{k-1}.$$

Dividing by n, one obtains

$$(19.24)\qquad \left\| \frac{1}{n} \sum_{j=1}^{n} y_{i_j} \right\| \le 8 \frac{\sum_{j=1}^{k} P_j 2^{-j}}{\sum_{j=1}^{k} P_j} + 2^{3-k} + 2^{4-k} + 2\frac{Q_{k-1}}{P_k}.$$

Note finally that $n \to \infty$ implies $k \to \infty$ and that, by the assumption that $P_{n+1} > nP_n$, $\lim_{k\to\infty} \frac{Q_{k-1}}{P_k} = 0$. Thus (19.24) implies in particular $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} y_i = 0$, i.e. $\{e_n\}$ has a subsequence that Cesaro converges to ξ in Y. This completes the proof. □

REMARK 19.22. Observe that the proof shows not only that $\{\frac{1}{n}\sum_{i=1}^{n} y_i\}$ converges to 0, but (cf. (19.24)) that $\{\frac{1}{n}\sum_{j=1}^{n} y_{i_j}\}_{n=1}^{\infty}$ converges to 0 uniformly for all subsequences $\{y_{i_n}\}$ of $\{y_n\}$. It has been shown, however, by P. Erdős and M. Magidor that this uniformity can be realized in any B.S. space and thus we have not really proved anything stronger than the B.S. property for Y.

NOTES. In [71] J. LINDENSTRAUSS showed that ℓ^1 has the K.M. property. Shortly thereafter C. BESSAGA & A. PELCZYNSKI ([9]) observed that essentially the same proof works for all separable dual spaces. Proposition 19.2 is the result of work by C. STEGALL ([100]) and R.E. HUFF & P. MORRIS ([46]). A complete proof can be found in [27]. As to Theorem 19.3, the equivalence of (i) and (ii) is already implicit in the work of R.C. JAMES ([54]). (i) ⇒ (iii) was also known. The present proof is from [30]. Proposition 19.6 can be found in [11]. Related results are due to T. NISHIURA & D. WATERMAN ([79],[104]). It is shown in [11] that Proposition 19.6 can be amended to include these latter results. The Banach-Saks property goes back to 1930, when S. BANACH & S. SAKS proved that the spaces $L_p[0,1]$ $(1 < p < \infty)$ have this property ([6]). The example following Corollary 19.8 is due to A. BERNSTEIN II ([4]) and Proposition 19.9 to S. KAKUTANI ([64]). The example given in Remark 19.10 comes from [79]. The one concerning ℓ^1 mentioned in Remark 19.12 is a result of W.J. DAVIS ([19]). Finally, Theorem 19.13 is due to A. BRUNEL & L. SUCHESTON ([12],[13],[14]). In [14] yet another ergodic superproperty is shown to be equivalent to superreflexivity.

20. CONNECTIONS WITH ℓ^p-SPACES

For a long time the hope existed that the spaces c_0, ℓ^p $(1 \le p < \infty)$ would turn out to be the fundamental building stones for Banach spaces, in the sense that every infinite-dimensional Banach space would contain one of them isomorphically. This conjecture was put to rest by B.S. TSIRELSON ([101]) with his construction of a reflexive space containing no ℓ^p. At approximately the same time some theorems were proved ([38],[55]) relating superreflexive spaces to ℓ^p-spaces. These results suggested that maybe at least superreflexive spaces would always contain some ℓ^p. However, recently T. FIGIEL & W.B. JOHNSON ([35]) have modified Tsirelson's example to produce a uniformly convex space containing no ℓ^p.

This section is devoted to a presentation of the above mentioned results. We begin with a positive one.

THEOREM 20.1. *A Banach space* X *is superreflexive iff for every* $\nu \ge 1$ *there exist numbers* A,B,r,s *with* $A,B > 0$, $1 < r, s < \infty$, *such that for every normalized basic sequence* $\{x_n\}$ *in* X *with* $\nu_{\{x_n\}} \le \nu$ *and for every* $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n]$ *we have*

$$(20.1) \qquad A\Big(\sum_{n=1}^{\infty} |\alpha_n|^s\Big)^{\frac{1}{s}} \le \|x\| \le B\Big(\sum_{n=1}^{\infty} |\alpha_n|^r\Big)^{\frac{1}{r}}.$$

The proof will be split up in several propositions and lemmas. Since a superreflexive space has an equivalent uniformly convex and uniformly smooth norm (cf. Corollary 15.8), it clearly suffices to prove necessity for uniformly convex and uniformly smooth spaces. We first concentrate on uniformly convex spaces, deriving for these the second inequality in (20.1), and then obtain the first inequality by a duality argument, using the duality of uniform smoothness and uniform convexity.

LEMMA 20.2. *Let* X *be a Banach space and let* $x,y \in X$ *be given with* $\|x\| = \|y\| = 1$, $x \neq y$. *Suppose that* ε *satisfies*

$$0 < \varepsilon \le \|x-y\| \quad \textit{and that} \quad \lambda := 2(1-\delta_X(\varepsilon)) < 2.$$

Then for every r *with* $\lambda^r < 2$ *there exists an* $\eta = \eta(r,\varepsilon)$ *such that*

$$(20.2) \qquad \|x+\alpha y\|^r < 1+\alpha^r, \quad \textit{whenever } |1-\alpha| < \eta.$$

PROOF. By the definition of $\delta_X(\cdot)$ we have

$$\|x+y\| \le \lambda, \quad \text{whence } \|x+y\|^r \le \lambda^r < 2 = 1+1^r,$$

showing that (20.2) holds for $\alpha = 1$. The existence of $\eta(r,\varepsilon)$ satisfying (20.2) follows by the continuity in α of the functions $\|x+\alpha y\|^r$ and $1+\alpha^r$. Some reflection shows that, as indicated by the notation $\eta = \eta(r,\varepsilon)$, η may be chosen so that it depends only on ε and r, but not on x and y (as long as $\|x-y\| \ge \varepsilon$). □

LEMMA 20.3. *Let* $n \in \mathbb{N}$, $\varepsilon > 0$ *and real sequences* $\{\xi_i\}_{i=1}^n$, $\{\eta_i\}_{i=1}^n$ *be given such that*

$$\xi_1 < \eta_1,\ \xi_n > \eta_n,\ |\xi_{i+1}-\xi_i| < \varepsilon,\ |\eta_{i+1}-\eta_i| < \varepsilon \quad (i = i,\dots,n-1).$$

Then for some i_0 $(1 \le i_0 \le n)$ *we have* $|\xi_{i_0}-\eta_{i_0}| < \varepsilon$.

PROOF. Assume that no such i_0 exists. Let j $(j \le n-1)$ be the largest index such that $\xi_j < \eta_j$. Then $\xi_{j+1} \ge \eta_{j+1}$. By assumption we have $\xi_j \le \eta_j-\varepsilon$, $\xi_{j+1} \ge \eta_{j+1}+\varepsilon > \eta_j$. Thus $\xi_{j+1} > \xi_j+\varepsilon$, a contradiction. □

PROPOSITION 20.4. *Let* X *be a uniformly convex Banach space. Let* $0 < \varepsilon \le 1$ *be given and let* λ *be defined by* $\lambda := 2(1-\delta_X(\varepsilon))$, *so that* $\lambda < 2$. *Then for every* $r \in \mathbb{R}$ *satisfying* $\lambda^r < 2$ *there exists a constant* $B = B(r,\varepsilon)$ *with the property that for every normalized basic sequence* $\{x_n\}$ *in* X *with* $\nu_{\{x_n\}} \le \frac{1}{\varepsilon}$ *and for any* $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n]$ *we have*

$$\|x\| \le B\Big(\sum_{n=1}^{\infty} |\alpha_n|^r\Big)^{1/r}.$$

PROOF. We show that $B = \frac{2}{\eta}$ does the job, where $\eta = \eta(r,\varepsilon)$ is as in Lemma 20.2. Let $\{x_n\}$ be a given normalized basic sequence with $\nu_{\{x_n\}} \le \frac{1}{\varepsilon}$. It clearly suffices to show that

$$(20.3) \qquad \Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \le \frac{2}{\eta}\Big(\sum_{i=1}^{n} |\alpha_i|^r\Big)^{1/r}, \quad \text{for all } n \in \mathbb{N} \text{ and all } \alpha_1,\dots,\alpha_n \in \mathbb{R}.$$

The proof is by induction on the number k of non-zero terms in $\sum_{i=1}^{n} \alpha_i x_i$. If

$k = 1$, (20.3) clearly holds. (Note that $\eta \le 1$.) So suppose (20.3) holds for some $k \ge 1$ and let us consider a sum with k+1 non-zero terms. For simplicity of notation we may assume (without loss of generality) that it is of the form $\sum_{i=1}^{k+1} \alpha_i x_i$. Put $y_{k+1} := \sum_{i=1}^{k+1}\alpha_i x_i$. If $|\alpha_{i_0}| \ge \frac{\eta}{2}\|y_{k+1}\|$ for some $i_0 \in \{1,\dots,k+1\}$, then we are done, since in that case $\|y_{k+1}\| \le \frac{2}{\eta}|\alpha_{i_0}| \le \frac{2}{\eta}(\sum_{i=1}^{k+1}|\alpha_i|^r)^{1/r}$. Therefore we may assume that $|\alpha_i| < \frac{\eta}{2}\|y_{k+1}\|$ for $i = 1,\dots,k+1$. Let us define

$$y_\ell := \sum_{i=1}^{\ell} \alpha_i x_i, \quad z_\ell := \sum_{i=\ell+1}^{k+1} \alpha_i x_i \qquad (\ell = 0,1,\dots,k+1),$$

where empty sums are interpreted as 0. We then have

$$\|y_0\| < \|z_0\|, \qquad \|y_{k+1}\| > \|z_{k+1}\|,$$

$$\big|\|y_{i+1}\| - \|y_i\|\big| < \tfrac{\eta}{2}\|y_{k+1}\|, \qquad \big|\|z_{i+1}\| - \|z_i\|\big| < \tfrac{\eta}{2}\|y_{k+1}\| \,(i = 1,\dots,k).$$

Thus, by Lemma 20.3, for some ℓ_0, $0 \le \ell_0 \le k+1$,

$$\big|\|y_{\ell_0}\| - \|z_{\ell_0}\|\big| < \tfrac{\eta}{2}\|y_{k+1}\|. \tag{20.4}$$

Clearly $0 < \ell_0 < k+1$ (because $\eta \le 1$). We may suppose that $\|y_{\ell_0}\| \ge \|z_{\ell_0}\|$ (otherwise interchange $\|y_{\ell_0}\|$ and $\|z_{\ell_0}\|$ in the following argument) and furthermore, for reasons of homogeneity, that $\|y_{\ell_0}\| = 1$. Since $y_{k+1} = y_{\ell_0}+z_{\ell_0}$, it follows now that $\|y_{k+1}\| \le 2$ and so (20.4) yields

$$\big|1 - \|z_{\ell_0}\|\big| < \eta. \tag{20.5}$$

Putting $\tilde z_{\ell_0} := \frac{z_{\ell_0}}{\|z_{\ell_0}\|}$, we have $z_{\ell_0} = \alpha\tilde z_{\ell_0}$ with $\alpha := \|z_{\ell_0}\|$ satisfying, by (20.5), $|1-\alpha| < \eta$. Next we observe that $\|y_{\ell_0}-\tilde z_{\ell_0}\| \ge \frac{1}{\|P_{\ell_0}\|}\|y_{\ell_0}\| \ge \varepsilon$, since $\nu_{\{x_n\}} \le \frac{1}{\varepsilon}$ (P_n denotes the natural projection from $[x_n]$ onto $[x_i]_{i=1}^n$). Applying Lemma 20.2 and using the induction hypothesis, we obtain

$$\|y_{k+1}\|^r = \|y_{\ell_0}+z_{\ell_0}\|^r = \|y_{\ell_0}+\alpha\tilde z_{\ell_0}\|^r < \|y_{\ell_0}\|^r + \|z_{\ell_0}\|^r \le$$
$$\le (\tfrac{2}{\eta})^r \sum_{i=1}^{\ell_0} |\alpha_i|^r + (\tfrac{2}{\eta})^r \sum_{i=\ell_0+1}^{k+1} |\alpha_i|^r = (\tfrac{2}{\eta})^r \sum_{i=1}^{k+1} |\alpha_i|^r,$$

i.e.

$$\|y_{k+1}\| < \tfrac{2}{\eta}\Big(\sum_{i=1}^{k+1} |\alpha_i|^r\Big)^{1/r}. \qquad \square$$

<u>PROPOSITION 20.5</u>. *Let X be a uniformly smooth Banach space (i.e. X^* uniformly convex). Let $0 < \varepsilon \le 1$ be given and let λ be defined by* $\lambda := (1-\delta_{X^*}(\varepsilon))$,

so that $\lambda < 2$. *Then for every* $r > 1$ *satisfying* $\lambda^r < 2$ *there exists a constant* $A = A(r,\varepsilon)$ *with the property that for every normalized basic sequence* $\{x_n\}$ *in* X *with* $\nu_{\{x_n\}} \le \frac{1}{\varepsilon}$ *and for any* $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n]$ *we have*

$$A\Big(\sum_{n=1}^{\infty} |\alpha_n|^s\Big)^{1/s} \le \|x\|, \quad \text{where } \frac{1}{s} + \frac{1}{r} = 1.$$

PROOF. Let $r \in \mathbb{R}$ satisfy $r > 1$, $\lambda^r < 2$ and let $\{x_n\}$ be any normalized basic sequence in X with $\nu_{\{x_n\}} \le \frac{1}{\varepsilon}$. Put $Y := [x_n]$ and let $\{x_n^*\} \subset Y^*$ be the basis for Y^* spanned by the coefficient functionals. Then $1 \le \|x_n\|\|x_n^*\| = \|x_n^*\| \le 2\nu_{\{x_n\}} \le \frac{2}{\varepsilon}$, by Proposition 5.2 and also $\nu_{\{x_n^*\}} \le \nu_{\{x_n\}} \le \frac{1}{\varepsilon}$ (Proposition 6.1). Furthermore $Y^* \cong X^*/Y^{\perp}$ and Y^* is therefore uniformly convex and, moreover, $\delta_{Y^*} \ge \delta_{X^*}$, as is easy to check. Hence $\lambda' := 2(1-\delta_{Y^*}(\varepsilon)) \le 2(1-\delta_{X^*}(\varepsilon)) =: \lambda < 2$, so that $\lambda'^r < 2$. Applying Proposition 20.4 to Y^* and the seminormalized basis $\{x_n^*\}$, we obtain, for every $x^* = \sum_{n=1}^{\infty} \alpha_n x_n^* \in Y^*$,

$$\|x^*\| \le B'\Big(\sum_{n=1}^{\infty} |\alpha_i|^r\Big)^{1/r}. \tag{20.6}$$

Here $B' = B'(r,\varepsilon)$ is the constant $B(r,\varepsilon)$ defined as in Proposition 20.4, multiplied by a scalar (depending on ε) to account for the fact that $\{x_n^*\}$ is only *semi*normalized (i.e. $0 \le \inf_{n\in\mathbb{N}} \|x_n^*\| \le \sup_{n\in\mathbb{N}} \|x_n^*\| < \infty$). We now finish the proof by a simple duality argument. Let $n \in \mathbb{N}$, $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in \mathbb{R}$ be arbitrary. Put $x := \sum_{i=1}^{n} \alpha_i x_i$, $x^* = \sum_{i=1}^{n} \beta_i x_i^*$. Then

$$\frac{|\langle x,x^*\rangle|}{\|x\|} \le \|x^*\| \le B'\Big(\sum_{i=1}^{n} |\beta_i|^r\Big)^{1/r},$$

whence

$$\|x\| \ge \frac{1}{B'}\,\frac{|\langle x,x^*\rangle|}{(\sum_{i=1}^{n}|\beta_i|^r)^{1/r}} = \frac{1}{B'}\,\frac{|\sum_{i=1}^{n}\alpha_i\beta_i|}{(\sum_{i=1}^{n}|\beta_i|^r)^{1/r}}.$$

Putting $\beta_i = |\alpha_i|^{1/r-1}$ sign α_i $(i = 1,\dots,n)$, we obtain

$$\|x\| \ge \frac{1}{B'}\,\frac{\sum_{i=1}^{n}|\alpha_i|^{r/r-1}}{(\sum_{i=1}^{n}|\alpha_i|^{r/r-1})^{1/r}} = \frac{1}{B'}\Big(\sum_{i=1}^{n}|\alpha_i|^s\Big)^{1/s}.$$

Thus $A := \frac{1}{B'}$ satisfies the requirement. □

PROOF OF THEOREM 20.1. The necessity is an immediate consequence of Propositions 20.4 and 20.5. Indeed, let X be superreflexive. Then X has an equivalent norm $\|\ \|_1$ which is uniformly convex and uniformly smooth. Let $\nu \ge 1$ be be given. Then, putting $\varepsilon = \frac{1}{\nu}$ and applying both propositions, we find that for suitable constants A,B,r and s the assertion of Theorem 20.1 holds for $(X, \| \ \|_1)$. Obviously the validity of it is not affected by isomorphisms, although the constants A,B,r and s may change.

Conversely, suppose that X is not superreflexive. Then $P_3^n(\varepsilon)$ is satisfied (Theorem 15.5), for some $0 < \varepsilon < 1$ and all $n \in \mathbb{N}$. Suppose for contradiction that $B > 0$ and $r > 1$ are such that for every normalized basic sequence $\{x_n\}$ in X with $\nu_{\{x_n\}} \le \frac{2}{\varepsilon}$ we have, for all $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n]$,

$$(20.7) \qquad \|x\| \le B\Big(\sum_{n=1}^{\infty} |\alpha_n|^r\Big)^{1/r} .$$

Choose $n \in \mathbb{N}$ so that $n^{1-\frac{1}{r}} > \frac{B}{\varepsilon}$. Then, by $P_3^n(\varepsilon)$, there exist $x_1,\dots,x_n \in B_X$ such that

$$(20.8) \qquad \operatorname{dist}(\operatorname{co}\{x_1,\dots,x_n\},\{0\}) \ge \varepsilon$$

and

$$(20.9) \qquad \Big\| \sum_{i=1}^{n} \alpha_i x_i \Big\| \ge \frac{\varepsilon}{2} \Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| \qquad \text{for all } k = 1,\dots,n \text{ and all } \alpha_1,\dots,\alpha_n \in \mathbb{R}.$$

The last condition means that $\{x_i\}_{i=1}^n$ is a (finite) basic sequence with $\nu_{\{x_i\}_{i=1}^n} \le \frac{2}{\varepsilon}$. Therefore, by our assumption (20.7),

$$\Big\| \sum_{i=1}^{n} x_i \Big\| = \Big\| \sum_{i=1}^{n} \|x_i\| \frac{x_i}{\|x_i\|} \Big\| \le B\Big(\sum_{i=1}^{n} \|x_i\|^r\Big)^{1/r} \le B\, n^{1/r}.$$

On the other hand $\|\sum_{i=1}^n x_i\| = n \, \|\frac{1}{n} \sum_{i=1}^n x_i - 0\| \ge n\varepsilon$, by (20.8), so $\varepsilon \le Bn^{\frac{1}{r}-1}$, contradicting the choice of n. □

<u>REMARK 20.6.</u> In the proof of the sufficiency we have used much less than available. In fact it suffices, for the proof of superreflexivity, to know that for some $\nu > 2$ there exist $B > 0$ and $r > 1$ such that for every normalized basic sequence $\{x_n\}$ with $\nu_{\{x_n\}} \le \nu$ we have $\|x\| \le B(\sum_{n=1}^{\infty} |\alpha_n|^r)^{1/r}$, whenever $x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n]$. (Note that if X is not superreflexive, then $P_3^n(\varepsilon)$ holds for every $0 < \varepsilon < 1$, see the beginning of Section 16.)

In Theorem 20.1 the constants A,B,r and s are so determined that (20.1) holds for all normalized basic sequences with *norms bounded by* ν. It is natural to ask whether A,B,r and s can be so chosen that (20.1) holds for all normalized basic sequences, *without any restriction on the norms*. We shall show by an example that this is not possible. Even if one allows the constants A and B to depend on the basic sequence and takes X to be the Hilbert space, it is impossible to determine r and s uniformly for all normalized basic sequences.

THEOREM 20.7. *Given any r,s with* $1 < r,s < \infty$, *there exists a (seminormalized) basis* $\{x_n\}$ *for the Hilbert space* H *such that for any positive numbers* A *and* B *there exist finite sequences* $\alpha_1,\dots,\alpha_n \in \mathbb{R}$ *and* $\beta_1,\dots,\beta_n \in \mathbb{R}$ *such that*

$$\|\sum_{i=1}^{n} \alpha_i x_i\| < A(\sum_{i=1}^{n} |\alpha_i|^s)^{1/s} \quad \text{and} \quad \|\sum_{i=1}^{n} \beta_i x_i\| > B(\sum_{i=1}^{n} |\beta_i|^r)^{1/r} .$$

The proof will be accomplished in several steps.

PROPOSITION 20.8. *Let* $\{x_n\}$ *be a sequence in a Banach space* x *and suppose the following conditions hold:*

(i) $\|\sum_{i=1}^{n} x_i\| \geq Kn^r$ *for some* $K,\ r > 0$ *and for all* $n \in \mathbb{N}$.

(ii) $\|\sum_{i=1}^{n} \alpha_i x_i\| \leq \|\sum_{i=1}^{n} \beta_i x_i\|$ *for every* $n \in \mathbb{N}$ *and all* $\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n \in \mathbb{R}$ *satisfying* $0 \leq \alpha_i \leq \beta_i$ $(i = 1,\dots,n)$.

Let $\{\alpha_n\}$ *be a non-increasing sequence of positive numbers and suppose that* $\{\alpha_n\} \notin \ell^p$ *for some* $p > \frac{1}{r}$. *Then* $\sum_{n=1}^{\infty} \alpha_n x_n$ *diverges.*

PROOF. Choose ρ such that $p > \rho > \frac{1}{r}$. Then, since $\sum_{n=1}^{\infty} \alpha_n^p = \infty$, there exists a subsequence $\{n_k\} \subset \mathbb{N}$ such that $\alpha_{n_k} \geq \frac{1}{n_k^{1/\rho}}$ $(k = 1,2,\dots)$. Using first (ii) and then (i), we obtain for every $k \in \mathbb{N}$

$$\|\sum_{i=1}^{n_k} \alpha_i x_i\| \geq \alpha_{n_k} \|\sum_{i=1}^{n_k} x_i\| \geq \frac{1}{n_k^{1/\rho}} Kn_k^r = Kn_k^{r-\frac{1}{\rho}}.$$

Hence $\sum_{n=1}^{\infty} \alpha_n x_n$ diverges. □

PROPOSITION 20.9. *Let* $\{x_n\}$ *be a sequence in a Banach space* X *and suppose*

(iii) $\|\sum_{i=1}^{n} x_i\| \leq Kn^r$ *for some* $K,\ r > 0$ *and for all* $n \in \mathbb{N}$.

Let $\{\alpha_n\}$ *be a non-increasing sequence of positive numbers and suppose that* $\{\alpha_n\} \in \ell^p$ *for some* $0 < p < \frac{1}{r}$. *Then* $\sum_{n=1}^{\infty} \alpha_n x_n$ *converges.*

PROOF. We claim that $\lim_{n\to\infty} \alpha_n n^{1/p} = 0$. Suppose not. Then there exists a constant $A > 0$ and a subsequence $\{n_k\} \subset \mathbb{N}$ such that $\alpha_{n_k} n_k^{1/p} \geq A$ $(k=1,2,\dots)$. Obviously we may assume that $n_{k+1} \geq 2n_k$ for all $k \in \mathbb{N}$. Using the monotonicity of $\{\alpha_n\}$, we get for all $k \in \mathbb{N}$,

$$\sum_{i=n_k}^{n_{k+1}} \alpha_i^p \geq \alpha_{n_{k+1}}^p (n_{k+1} - n_k + 1) \geq \tfrac{1}{2} n_{k+1} \alpha_{n_{k+1}}^p \geq \tfrac{1}{2} A^p,$$

contradicting $\{\alpha_n\} \in \ell^p$. Thus

(20.10) $\quad \lim_{n\to\infty} \alpha_n n^{1/p} = 0.$

In particular, for some $C > 0$ we have

$$\alpha_n \leq Cn^{-1/p} \quad (n = 1,2,\ldots).$$

Writing $S_n := \sum_{i=1}^n x_i$ $(n = 1,2,\ldots)$, and using (iii), (20.11) and the fact that $(n+1)^r - n^r \leq Bn^{r-1}$ for some constant B and all $n \in \mathbb{N}$, we obtain for all $n \leq m$,

$$\begin{aligned}
\left\|\sum_{i=n}^{m} \alpha_i x_i\right\| &= \left\|\alpha_m S_m - \alpha_n S_{n-1} - \sum_{i=n}^{m-1} (\alpha_{i+1} - \alpha_i) S_i\right\| \leq \\
&\leq \alpha_m \|S_m\| + \alpha_n \|S_{n-1}\| - \sum_{i=n}^{m-1} (\alpha_{i+1} - \alpha_i) \|S_i\| \leq \\
&\leq \alpha_m K m^r + \alpha_n K (n-1)^r - K \sum_{i=n}^{m-1} (\alpha_{i+1} - \alpha_i) i^r = \\
&= K[\alpha_m m^r + \alpha_n (n-1)^r - \alpha_m m^r + \alpha_n (n-1)^r + \sum_{i=n}^{m} \alpha_i (i^r - (i-1)^r)] \leq \\
&\leq K[2\alpha_n (n-1)^r + B \sum_{i=n}^{m} \alpha_i (i-1)^{r-1}] \leq \\
&\leq K[2\alpha_n n^r + BC \sum_{i=n}^{m} i^{-\frac{1}{p}+r-1}].
\end{aligned}$$

Since $r < \frac{1}{p}$, (20.10) implies $\lim_{n\to\infty} \alpha_n n^r = 0$. Also $-\frac{1}{p}+r-1 < -1$, so it follows that $\sum_{n=1}^\infty \alpha_n x_n$ converges. □

To obtain the desired example we shall apply the preceding propositions to the particular sequences $\{|t|^\alpha \cos nt\}$ in $L^2[-\pi,\pi]$, $0 < |\alpha| < \frac{1}{2}$. We take for granted here the fact that these sequences are basic and seminormalized (cf. [97], p.351). Some technicalities must be disposed of first.

<u>LEMMA 20.10</u>. *For each* $\alpha < 1$ *there exist constants* $A,B > 0$ *such that, for all* $n \in \mathbb{N}$,

(20.12) $$An^{\alpha+1} \leq \int_0^\pi t^{-\alpha}\left[1 + \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right]^2 dt \leq Bn^{\alpha+1}.$$

PROOF. Since $\frac{2}{\pi}t \le \sin t$ for $t \in [0,\frac{\pi}{2}]$ and $\sin t \le t$ for $t \in [0,\infty)$, we have

$$\frac{2}{\pi}(n+\tfrac{1}{2})t \le \sin(n+\tfrac{1}{2})t \le (n+\tfrac{1}{2})t \qquad \text{for } t \in [0,\pi/2n+1]$$

and

$$\frac{2}{\pi}\frac{t}{2} \le \sin\frac{t}{2} \le \frac{t}{2} \qquad \text{for } t \in [0,\pi].$$

Consequently, for every $n \in \mathbb{N}$

$$K_1 n^2 \le \left[1 + \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right]^2 \le K_2 n^2 \quad \text{for } t \in [0,\pi/2n+1],$$

with constants $K_1, K_2 > 0$ independent of n. Hence, for some constants K_3, K_4 and all $n \in \mathbb{N}$,

$$K_3 n^{1+\alpha} \le \int_0^{\pi/2n+1} t^{-\alpha}\left[1+\frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right]^2 dt \le K_4 n^{1+\alpha}. \tag{20.13}$$

Furthermore, for some $K_5 > 0$ and for all $n \in \mathbb{N}$,

$$\left[1 + \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right]^2 \le \left[1 + \frac{1}{\frac{2}{\pi}\frac{t}{2}}\right]^2 \le \frac{K_5}{t^2} \quad \text{for } t \in [\tfrac{\pi}{2n+1},\pi].$$

Therefore, for some $K_6 > 0$, and all $n \in \mathbb{N}$,

$$\int_{\pi/2n+1}^{\pi} t^{-\alpha}\left[1 + \frac{\sin(n+\frac{1}{2})t}{\sin\frac{t}{2}}\right]^2 dt \le K_5 \int_{\pi/2n+1}^{\pi} t^{-\alpha-2}\, dt \le K_6 n^{1+\alpha}. \tag{20.14}$$

Clearly (20.12) follows from (20.13) and (20.14). □

LEMMA 20.11. *If* $x_n(t) := |t|^{-\alpha}\cos nt \in L^2[-\pi,\pi]$, *with* $\alpha < \frac{1}{2}$ $(n = 0,1,\ldots)$, *then there exist* $C_1, C_2 > 0$ *such that for all* $n = 0,1,2,\ldots,$

$$C_1 n^{\frac{1}{2}+\alpha} \le \left\|\sum_{k=0}^{n} x_k\right\| \le C_2 n^{\frac{1}{2}+\alpha}. \tag{20.15}$$

PROOF. $\|\sum_{k=0}^n x_k\|^2 = \int_{-\pi}^{\pi} |t|^{-2\alpha}(\sum_{k=0}^n \cos kt)^2\, dt = 2\int_0^{\pi} t^{-2\alpha}[\frac{1}{2} + \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{t}{2}}]^2\, dt,$ so that (20.15) follows from (20.12). □

LEMMA 20.12. *If* $0 < \alpha < 1$, *then*

$$\int_{-\pi}^{\pi} |t|^{-\alpha}\cos nt\, dt > 0 \quad \text{for all } n = 0,1,2,\ldots .$$

PROOF. Partial integration shows that, for $n > 0$,

$$\int_{-\pi}^{\pi} |t|^{-\alpha} \cos nt\, dt = 2\int_0^{\pi} t^{-\alpha} \cos nt\, dt = 2\left[t^{-\alpha}\frac{\sin nt}{n}\Big|_0^{\pi} + \frac{\alpha}{n}\int_0^{\pi}\frac{\sin nt}{t^{\alpha+1}}dt\right]$$

$$= \frac{2\alpha}{n}\int_0^{\pi}\frac{\sin nt}{t^{\alpha+1}}\, dt.$$

The last expression is positive since $1/t^{\alpha+1}$ decreases. □

LEMMA 20.13. *Let* $0 < \alpha < \frac{1}{2}$ *and let* x_n $(n = 0,1,\ldots)$ *be as in Lemma* 20.11. *Suppose that* $0 \le \alpha_n \le \beta_n$ $(n = 0,1,\ldots)$ *and that the series* $\sum_{n=0}^{\infty} \alpha_n x_n$ *and* $\sum_{n=0}^{\infty} \beta_n x_n$ *converge in* $L^2[-\pi,\pi]$, *say to* x *and* y, *respectively. Then* $\|x\| \le \|y\|$.

PROOF.

$$\|x\|^2 = \sum_{k,\ell=0}^{\infty} \alpha_k\alpha_\ell \int_{-\pi}^{\pi} |t|^{-2\alpha} \cos kt \cos \ell t\, dt =$$

$$= \tfrac{1}{2}\sum_{k,\ell=0}^{\infty} \alpha_k\alpha_\ell \int_{-\pi}^{\pi} [\cos(k+\ell)t + \cos(k-\ell)t]|t|^{-2\alpha}\, dt,$$

and similarly

$$\|y\|^2 = \tfrac{1}{2}\sum_{k,\ell=0}^{\infty} \beta_k\beta_\ell \int_{-\pi}^{\pi} [\cos(k+\ell)t + \cos(k-\ell)t]|t|^{-2\alpha}\, dt.$$

Since the integrals are positive, by Lemma 20.12, the conclusion is immediate. □

COROLLARY 20.14. *Let* $\{\alpha_n\}$ *be a non-increasing sequence of positive numbers with limit* 0 *and let* $p > 1$ *be arbitrary. Let* x_n $(n = 0,1,2,\ldots)$ *be as in Lemma* 20.11.

(a) *If* $\{\alpha_n\} \notin \ell^p$, *then* $\sum_{n=0}^{\infty} \alpha_n x_n$ *diverges whenever* $\frac{1}{2} > \alpha > \max(\frac{1}{p}-\frac{1}{2},0)$.

(b) *If* $\{\alpha_n\} \in \ell^p$, *then* $\sum_{n=0}^{\infty} \alpha_n x_n$ *converges whenever* $-\frac{1}{2} < \alpha < \frac{1}{p}-\frac{1}{2}$.

PROOF. Observe that in case (a) Proposition 20.8 and in case (b) Proposition 20.9 are applicable since the assumptions hold (with $r = \frac{1}{2}+\alpha$), by Lemmas 20.11 and 20.13.

PROOF OF THEOREM 20.7. Let $1 < r, s < \infty$ be given. Choose α such that $\frac{1}{2} > \alpha > \max(\frac{1}{r} - \frac{1}{2}, 0)$ and consider the sequence $\{x_n\} = \{|t|^{-\alpha} \cos nt\}$. Fix r' so that $\alpha > \frac{1}{r'} - \frac{1}{2} > \frac{1}{r} - \frac{1}{2}$ and put $\beta_n := n^{-1/r'}$ $(n = 1,2,\ldots)$. Since $\{\beta_n\} \notin \ell^{r'}$, it follows from Corollary 20.14 (a) that $\sum_{n=1}^{\infty} \beta_n x_n$ diverges. On the other hand $\{\beta_n\} \in \ell^r$ since $r > r'$.

Next let $\alpha' \neq 0$ be such that $-\frac{1}{2} < \alpha' < \frac{1}{s} - \frac{1}{2}$ and consider $\{x_n'\} := \{|t|^{-\alpha'} \cos nt\}$. Fix s' so that $\alpha' < \frac{1}{s'} - \frac{1}{2} < \frac{1}{s} - \frac{1}{2}$ and put $\alpha_n := n^{-1/s}$. Since $\{\alpha_n\} \in \ell^{s'}$, Corollary 20.14 (b) implies that $\sum_{n=1}^{\infty} \alpha_n x_n'$ converges. On the other hand $\{\alpha_n\} \notin \ell^s$.

Finally put $H := (H_1 \oplus H_2)_{\ell^2}$ where H_1 and H_2 are the closed linear subspaces of $L^2[-\pi,\pi]$ spanned by $\{x_n\}$ and $\{x_n'\}$, respectively. It is easily verified that the sequence $x_1, x_1', x_2, x_2', \ldots, x_n, x_n', \ldots$ is a seminormalized basis for H and from what we have already proved it is obvious that this basis satisfies the requirements of Theorem 20.7. □

Another natural question is suggested by Theorem 20.1: what happens if we dispense with the uniformity condition altogether and simply require that for any normalized basic sequence $\{x_n\}$ there exist constants $A,B > 0$, $1 < r, s < \infty$ (dependent on $\{x_n\}$) such that (20.1) holds? Does this still imply superreflexivity? It certainly implies reflexivity as the following result shows.

PROPOSITION 20.15. *Let X be a Banach space and suppose that for any normalized basic sequence $\{x_n\}$ there exist constants $A,B > 0$, and $1 < r, s < \infty$ such that*

$$(20.16) \qquad A\Big(\sum_{n=1}^{\infty} |\alpha_n|^s\Big)^{1/s} \le \|x\| \le B\Big(\sum_{n=1}^{\infty} |\alpha_n|^r\Big)^{1/r}$$

whenever

$$x = \sum_{n=1}^{\infty} \alpha_n x_n \in [x_n].$$

Then X is reflexive.

PROOF. By Theorem 6.12 it suffices to show that an arbitrary normalized basic sequence $\{x_n\}$ is boundedly complete. Suppose not. Then there exists an $\varepsilon > 0$, an increasing sequence of integers $0 = m_0 < m_1 < \ldots \quad m_n < \ldots$ and a real sequence $\{\alpha_n\}$ such that for the block basic sequence $\{y_n\}$, with

$$y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i x_i \qquad (n = 1,2,\ldots),$$

we have

$$\|y_n\| \geq \varepsilon \quad (n = 1,2,\ldots) \quad \text{and} \quad \{\sum_{k=1}^{n} y_k\} \text{ bounded.}$$

By assumption there exist $A > 0$, $1 < s < \infty$ such that the first inequality in (20.16) holds for the normalized basic sequence $\{\frac{y_n}{\|y_n\|}\}$. Thus, for each $n \in \mathbb{N}$

$$An^{1/s}\varepsilon \leq A(\sum_{k=1}^{n} \|y_k\|^s)^{1/s} \leq \left\|\sum_{k=1}^{n} \|y_k\| \frac{y_k}{\|y_k\|}\right\| = \left\|\sum_{k=1}^{n} y_k\right\|,$$

contradicting the boundedness of $\{\sum_{k=1}^{n} y_k\}$.

REMARK 20.16. As in the proof of the "if" part of Theorem 20.1, we have used only the first inequality of (20.16) to prove reflexivity. If we assume only the second inequality of (20.16) then it follows similarly that each (normalized) basic sequence is shrinking, so that X is reflexive by Theorem 6.12.

REMARK 20.17. The problem whether the assumptions of Proposition 20.15 imply superreflexivity is open as far as we know. The answer is yes if X is isomorphic to a subspace of a Banach space with an unconditional basis, or more generally, if X has local unconditional structure. This follows from the results of [60].

We now proceed to the promised example of a reflexive space which contains no ℓ^p. First let us give a

DEFINITION 20.18. A sequence $\{x_n\}$ is called an *unconditionally monotone basis* for a Banach space X iff it is an unconditional basis with unconditional norm $\nu^u_{\{x_n\}} = 1$, i.e. for all convergent series $\sum_{n=1}^{\infty} \alpha_n x_n$ and $\sum_{n=1}^{\infty} \beta_n x_n$ we have

$$(20.17) \qquad \|\sum_{n=1}^{\infty} \alpha_n x_n\| \leq \|\sum_{n=1}^{\infty} \beta_n x_n\| \quad \text{whenever } |\alpha_n| \leq |\beta_n| \quad (n = 1,2,\ldots).$$

Recall that (20.17) implies in particular that the associated projections P_F ($F \in \mathcal{F}$, the collection of finite subsets of $\mathbb{N}$) all have norm 1.

Let X be the space of all real sequences which are eventually 0. If $x \in X$ and $F \in \mathcal{F}$, then Fx will be short for $P_F x$, i.e. the sequence whose n-th coordinate is that of x or 0 according as n is in F or in $\mathbb{N}\setminus F$. We shall consider *unconditionally monotone* norms on X, by which we mean norms with the property that the unit vectors $\{e_n\}$ form an unconditionally monotone basis for (the completion of) $(X, \| \ \|)$. Thus $\|Fx\| \leq \|x\|$ for any such norm $(F \in \mathcal{F})$. For $F_1, F_2 \in \mathcal{F}$ we write $F_1 < F_2$ iff $\max F_1 < \min F_2$. A finite sequence $\{F_i\}_{i=1}^k \subset \mathcal{F}$ is called *admissible* provided $F_i < F_{i+1}$ $(i = 1,\dots,k-1)$ and $\{k\} < F_1$. We now define inductively a sequence of norms $\| \ \|_n$ on X as follows:

$$\|x\|_0 := \|x\|_{c_0}$$

$$(20.18) \qquad \|x\|_{n+1} := \max\Big[\|x\|_n, \ \tfrac{1}{2} \max\{\sum_{i=1}^k \|F_i x\|_n : \ k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^k \text{ is admissible}\}\Big] \qquad (n = 0,1,2,\dots) .$$

It is routine to verify that these are indeed norms. Moreover, all these norms are unconditionally monotone (use induction) and form a non-decreasing sequence. It is also easily checked that $\|e_k\|_n = 1$ for all $k,n \in \mathbb{N}$. In particular it follows that for each $x \in X$ the non-decreasing sequence $\{\|x\|_n\}$ is bounded, so that we can define the limit norm by

$$(20.19) \qquad \|x\| := \lim_{n\to\infty} \|x\|_n \qquad (x \in X).$$

Observe that $\|\cdot\|$ is again unconditionally monotone.

LEMMA 20.19. $\|\cdot\|$ *is the unique norm on* X *satisfying*

$$(20.20) \qquad \|x\| = \max[\|x\|_{c_0}, \tfrac{1}{2}\max\{\sum_{i=1}^k \|F_i x\| : k \in \mathbb{N} \textit{ and } \{F_i\}_{i=1}^k \textit{ admissible}\}].$$

PROOF. Let us first check the validity of (20.20). It is an immediate consequence of (20.18), (20.19) and the increasing nature of $\{\|\cdot\|_n\}$ that for every $x \in X$ we have

$$\|x\| \geq \max[\|x\|_{c_0}, \tfrac{1}{2}\max\{\sum_{i=1}^k \|F_i x\| : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^k \text{ admissible}\}].$$

Let us assume for contradiction that for some $x \in X$ we have

$$\|x\| > \max[\|x\|_{c_0}, \tfrac{1}{2}\max\{\sum_{i=1}^k \|F_i x\| : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^k \text{ admissible}\}].$$

Then for all sufficiently large n we have

$$\|x\|_n > \tfrac{1}{2}\max\{\sum_{i=1}^{k}\|F_i x\| : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^{k} \text{ admissible}\},$$

and consequently, by (20.18), $\|x\|_n = \|x\|_{n+1}$. Let n_0 be the smallest index such that $\|x\|_{n_0} = \|x\|_{n_0+1} = \|x\|_{n_0+2} = \ldots = \|x\|$. Note that $n_0 \neq 0$, since by the assumption $\|x\|_0 = \|x\|_{c_0} < \|x\|$. Now, using (20.18) again, we arrive at a contradiction:

$$\|x\| = \|x\|_{n_0} = \max[\|x\|_{n_0-1}, \tfrac{1}{2}\max\{\sum_{i=1}^{k}\|F_i x\|_{n_0-1} : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^{k} \text{ admissible}\}] \leq \max[\|x\|_{n_0-1}, \tfrac{1}{2}\max\{\sum_{i=1}^{k}\|F_i x\| : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^{k} \text{ admissible}\}] < \|x\|,$$

where the last inequality follows from the choice of n_0 and from the assumption.

Suppose that $|||\cdot|||$ is another norm on X satisfying (20.20), i.e.

$$|||x||| = \max[\|x\|_{c_0}, \tfrac{1}{2}\max\{\sum_{i=1}^{k}|||F_i x||| : k \in \mathbb{N} \text{ and } \{F_i\}_{i=1}^{k} \text{ admissible}\}]. \tag{20.21}$$

First of all $\|e_n\| = |||e_n||| = 1$ for all $n \in \mathbb{N}$, so $\|\ \|$ and $|||\ |||$ coincide on all x with supp x (= {$n \in \mathbb{N}$: n-th coordinate of x non-zero}) consisting of at most one element. Observe next that in (20.20) and (20.21) we may obviously assume that, for a given $x \in X$, the max is taken over all admissible $\{F_i\}_{i=1}^{k}$ such that $\bigcup_{i=1}^{k} F_i \subset \operatorname{supp} x$ and $k \geq 2$. Induction on the cardinality of supp x now shows that $|||\cdot||| = \|\cdot\|$ on X. □

Let T be X with the norm defined by (20.20) and $\tilde{T}$ its completion. As we have already observed $\{e_n\}$ is an unconditionally monotone basis for $\tilde{T}$. Furthermore,

<u>THEOREM 20.20</u>. *$\tilde{T}$ is reflexive and contains no infinite-dimensional super-reflexive subspaces. In particular $\tilde{T}$ contains none of the spaces c_0, ℓ^1, ℓ^p $(1 < p < \infty)$ isomorphically.*

We need two auxiliary results for the proof of Theorem 20.20. The first one gives some more information on the norm of T and the second one (needed to prove that $\tilde{T}$ does not contain ℓ^1) asserts roughly, that if ℓ^1 is isomorphically contained in any Banach space, then it is "almost isometrically" contained in it.

PROPOSITION 20.21. *For every* $\alpha > 1$ *there exists a* $\beta < 2$ *(in fact* $\beta = \frac{1}{2}(3+\alpha^{-1})$ *will do) such that*

$$\|x_0 + m^{-1} \sum_{i=1}^{m} x_i\| \le \beta \max_{0\le i\le m} \|x_i\|, \tag{20.22}$$

whenever $k,m \in \mathbb{N}$ *and* $x_0,\dots,x_m \in T$ *satisfy*

$$\operatorname{supp} x_0 \subseteq [1,k] < \operatorname{supp} x_1 < \dots < \operatorname{supp} x_m \text{ and } m \ge \alpha k. \tag{20.23}$$

PROOF. Let $k,m \in \mathbb{N}$ and $x_0,\dots,x_m \in T$ satisfy (20.23). Since

$$\|x_0 + m^{-1} \sum_{i=1}^{m} x_i\|_{c_0} \le \max_{0\le i\le m} \|x_i\|_{c_0} \le \max_{0\le i\le m} \|x_i\|,$$

it suffices, in view of (20.20), to show that, for $\beta = \frac{1}{2}(3+\alpha^{-1})$,

$$\sum_{j=1}^{n} \|F_j(x_0 + m^{-1} \sum_{i=1}^{m} x_i)\| \le 2\beta \max_{0\le i\le m} \|x_i\| \tag{20.24}$$

whenever $\{F_j\}_{j=1}^n$ is admissible. (20.24) holds with any $\beta \ge 1$ if $n \ge k$. Indeed, in this case $\operatorname{supp} x_0 < F_1$, by the admissibility of $\{F_j\}_{j=1}^n$, so that $F_j x_0 = 0$ for all $j = 1,\dots,n$. Hence

$$\sum_{j=1}^{n} \|F_j(x_0 + m^{-1} \sum_{i=1}^{m} x_i)\| = \sum_{j=1}^{n} \|F_j(m^{-1} \sum_{i=1}^{m} x_i)\|$$

$$\le 2\|m^{-1} \sum_{i=1}^{m} x_i\| \le 2 \max_{0\le i\le m} \|x_i\|.$$

So assume that $n < k$. Let us put

$$A := \{i \in \{1,\dots,m\}: \|F_j x_i\| \ne 0 \text{ for at least two values of } j\}$$

$$B := \{i \in \{1,\dots,m\}: \|F_j x_i\| \ne 0 \text{ for at most one value of } j\}.$$

For each $i \in A$ pick two F_j's such that $F_j x_i \neq 0$. Obviously the cardinality of the set of F_j's so chosen is at least $|A| + 1$. Therefore $|A| \leq n-1$. Using this, (20.20), the fact that at most one term of $\sum_{j=1}^{n} \|F_j x_i\|$ is non-zero whenever $i \in B$, and the assumptions $n < k$, $m \geq \alpha k$, we obtain

$$\sum_{j=1}^{n} \|F_j(x_0 + m^{-1} \sum_{i=1}^{m} x_i\| \leq \sum_{j=1}^{n} \|F_j x_0\| +$$

$$+ m^{-1}(\sum_{i\in A} \sum_{j=1}^{n} \|F_j x_i\| + \sum_{i\in B} \sum_{j=1}^{n} \|F_j x_i\|)$$

$$\leq 2\|x_0\| + m^{-1} (2 \sum_{i\in A} \|x_i\| + \sum_{i\in B} \|x_i\|)$$

$$\leq 2\|x_0\| + m^{-1}[2(n-1) + m-n+1] \max_{1\leq i\leq m} \|x_i\|$$

$$\leq 2\|x_0\| + m^{-1}(m-1+k) \max_{1\leq i\leq m} \|x_i\| \leq 2\|x_0\| + (1+m^{-1}k) \max_{1\leq i\leq m} \|x_i\|$$

$$\leq (3+\alpha^{-1}) \max_{0\leq i\leq m} \|x_i\|.$$

Thus (20.24) holds with $\beta = \frac{1}{2}(3+\alpha^{-1})$ and the proof is complete. □

PROPOSITION 20.22. *Suppose that a Banach space* X *contains a subspace isomorphic to* ℓ^1. *Then for any* $0 < \varepsilon < 1$ *there exists a sequence* $\{z_n\} \subset S_X$ *such that for all* $k \in \mathbb{N}$ *and* $\alpha_1,\ldots,\alpha_k \in \mathbb{R}$,

$$(1-\varepsilon) \sum_{n=1}^{k} |\alpha_n| \leq \|\sum_{n=1}^{k} \alpha_n z_n\| \leq \sum_{n=1}^{k} |\alpha_n|. \tag{20.25}$$

Thus $\{z_n\}$ *is* $\frac{1}{1-\varepsilon}$*-equivalent to the standard basis of* ℓ^1.

PROOF. Let Y be a subspace of X isomorphic to ℓ^1. Let $\{x_n\}$ be the sequence in Y which corresponds to the standard basis of ℓ^1 under an isomorphism. Then there are constants $M_1, M_2 > 0$ such that for all $k \in \mathbb{N}$ and $\alpha_1,\ldots,\alpha_k \in \mathbb{R}$

$$M_1 \sum_{n=1}^{k} |\alpha_n| \leq \|\sum_{n=1}^{k} \alpha_n x_n\| \leq M_2 \sum_{n=1}^{k} |\alpha_n|.$$

For each $m \in \mathbb{N}$ set

$$K_m := \inf\{\|\sum_{n=m}^{k} \alpha_n x_n\| : k \in \mathbb{N} \text{ and } \sum_{n=m}^{k} |\alpha_n| = 1\}.$$

Then $\{K_m\}$ is a non-decreasing sequence bounded below and above by M_1 and M_2, respectively. Hence $M_1 \le K := \lim_{m\to\infty} K_m \le M_2$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $1-\varepsilon \le \frac{1-\delta}{1+\delta}$ and $m_0 \in \mathbb{N}$ such that $K_{m_0} \ge (1-\delta)K$. Inductively, using the definition of the K_m, we now select an increasing sequence $m_0 < m_1 < m_2 < \ldots$ in $\mathbb{N}$ and a (block basic) sequence $\{y_n\}$ of the form

$$(20.26) \qquad y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i^n x_i \quad \text{with} \quad \sum_{i=m_{n-1}+1}^{m_n} |\alpha_i^n| = 1 \text{ and } \|y_n\| \le (1+\delta)K \qquad (n = 1,2,\ldots).$$

Putting $z_n := \frac{y_n}{\|y_n\|}$ $(n = 1,2,\ldots)$, we have $\{z_n\} \subset S_X$ and we claim that this sequence satisfies the requirements. Indeed, the second inequality in (20.25) is obvious. For the first one, observe that for all $k \in \mathbb{N}$ and all $\alpha_1,\ldots,\alpha_k \in \mathbb{R}$ we have, by the choice of m_0 and (20.26),

$$\Big\|\sum_{n=1}^{k} \alpha_n y_n\Big\| = \Big\|\sum_{n=1}^{k} \sum_{i=m_{n-1}+1}^{m_n} \alpha_n \alpha_i^n x_i\Big\| \ge K_{m_0} \sum_{n=1}^{k} \sum_{i=m_{n-1}+1}^{m_n} |\alpha_n \alpha_i^n| \ge$$
$$\ge (1-\delta)K \sum_{n=1}^{k} |\alpha_n|,$$

whence

$$\Big\|\sum_{n=1}^{k} \alpha_n z_n\Big\| = \Big\|\sum_{n=1}^{\infty} \frac{\alpha_n}{\|y_n\|} y_n\Big\| \ge \frac{(1-\delta)K}{(1+\delta)K} \sum_{n=1}^{k} |\alpha_n| \ge (1-\varepsilon) \sum_{n=1}^{k} |\alpha_n|. \qquad \square$$

One more remark should be made before the proof of Theorem 20.20.

<u>REMARK 20.23</u>. If $\{e_n\}$ denotes the standard basis of ℓ^1 or c_0, then any normalized block basic sequence $\{y_n\}$ with respect to $\{e_n\}$ (so in particular every subsequence of $\{e_n\}$) is 1-equivalent to $\{e_n\}$. Indeed, let

$$y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i e_i \qquad (n = 1,2,\ldots)$$

with

$$\|y_n\| = \sum_{i=m_{n-1}+1}^{m_n} |\alpha_i| \ (\text{respectively}, \sup_{m_{n-1}<i\le m_n} |\alpha_i|) = 1 \ (n=1,2,\ldots).$$

Then

$$\Big\|\sum_{n=1}^{k} \beta_n y_n\Big\| = \sum_{n=1}^{k} |\beta_n| \ (\text{respectively}, \sup_{1\le n\le k} |\beta_n|) \text{ for all } k \in \mathbb{N} \text{ and all } \beta_1,\ldots,\beta_k \in \mathbb{R}.$$

We are now prepared for the

<u>PROOF OF THEOREM 20.20</u>. a) Let $\{x_i\}_{i=1}^k \subset T$ be given such that $\{\text{supp } x_i\}_{i=1}^k$ is increasing, $\|x_i\| = 1$ $(i = 1,\dots,k)$ and $\{k\} < \text{supp } x_1$. Then $\{\text{supp } x_i\}_{i=1}^k$ is admissible and it follows from (20.20) that for all $\alpha_1,\dots,\alpha_k \in \mathbb{R}$,

$$\|\sum_{i=1}^k \alpha_i x_i\| \geq \tfrac{1}{2} \sum_{i=1}^k |\alpha_i|,$$

thus $\{x_i\}_{i=1}^k$ is 2-equivalent to the standard basis for ℓ_k^1. This observation implies that any normalized block basic sequence with respect to the standard basis $\{e_n\}$ of T has, for each k, subsequences of length k which are 2-equivalent to the standard basis for ℓ_k^1.

b) Suppose now that Y is an infinite-dimensional superreflexive subspace of $\tilde{T}$. Select a normalized basic sequence $\{y_n\}$ in Y. Then the coefficient functionals $\{y_n^*\}$ form a basis for Y^*, so that every $y^* \in Y^*$ can be written as $y^* = \sum_{n=1}^\infty \langle y_n, y^* \rangle y_n^*$. It follows immediately that $\lim_{n\to\infty} \langle y_n, y^* \rangle = 0$ for every $y^* \in Y^*$, since $\|y_n^*\| \geq |\langle y_n, y_n^* \rangle| = 1$ for all $n \in \mathbb{N}$. (This argument in fact shows, more generally, that any bounded basic sequence in a reflexive space is a weak null sequence.) In particular $\lim_{n\to\infty} \langle y_n, e_k^* \rangle = 0$ for all $k \in \mathbb{N}$ ($\{e_n^*\}$ is the sequence of coefficient functionals associated to $\{e_n\}$), so that the assumptions of Proposition 8.6 hold with X and $\{x_n\}$ replaced by $\tilde{T}$ and $\{e_n\}$, respectively. Hence there exists for every $\varepsilon > 0$ a subsequence $\{y_{p_n}\}$ which is $(1+\varepsilon)$-equivalent to a block basic sequence with respect to $\{e_n\}$ ($\varepsilon > 0$ arbitrary). A bit more care in the proof of Proposition 8.6 shows that this latter block basic sequence may be assumed normalized. By the observation made under a), $\{y_{p_n}\}$ therefore contains, for any $k \in \mathbb{N}$, subsequences of length k which are $2(1+\varepsilon)$-equivalent to the standard basis of ℓ_k^1. Since the standard basis for $\ell_{2^k}^1$ forms a $(k,2)$-tree (Remark 14.2), it follows that Y has P_1, contradicting its superreflexivity. Thus $\tilde{T}$ contains no non-trivial (i.e. infinite-dimensional) superreflexive subspaces, in particular no ℓ^p for $1 < p < \infty$.

c) An analogous argument will show that $\tilde{T}$ contains no subspace isomorphic to c_0. Indeed, suppose $Y \subset \tilde{T}$ is isomorphic to c_0. Let $\{y_n\}$ be the normalization of the basis for Y corresponding to the standard basis of c_0 under an isomorphism. Since the standard basis of c_0 trivially forms a weak null sequence, so does $\{y_n\}$. Again Proposition 8.6 and part a) of the present proof now show, for every $\varepsilon > 0$, the existence of a subsequence $\{y_{p_n}\}$ which contains, for any $k \in \mathbb{N}$, subsequences of length k which are $2(1+\varepsilon)$-equivalent to the standard basis of ℓ_k^1. This is obviously impossible, since any such subsequence is also K-equivalent to the standard basis of $(c_0)_k$ for some K independent of k, by Remark 20.24.

d) Since $\tilde{T}$ has an unconditional basis $\{e_n\}$, it suffices (by Theorem 6.20) for the proof of reflexivity, to show that $\tilde{T}$ does not contain ℓ^1 isomorphically. Suppose it does and let $\varepsilon > 0$ be arbitrary. Then, by Proposition 20.22, $S_{\tilde{T}}$ contains a sequence $\{z_n\}$ which is $(1+\varepsilon)$-equivalent to the standard basis of ℓ^1. It is not difficult to see that there exists a normalized block basic sequence $\{y_n\}$ with respect to $\{z_n\}$ which satisfies $\lim_{n\to\infty} \langle y_n, e_k^*\rangle = 0$ for each k. Indeed, suppose that for some $k \in \mathbb{N}$ $y_n = \sum_{i=m_{n-1}+1}^{m_n} \alpha_i z_i$, $\|y_n\| = 1$ $(n = 1,\dots,k)$ have been chosen so that $|\langle y_n, e_\ell^*\rangle| \le 1/n$ for all $1 \le \ell \le n \le k$. Now pick y_{k+1} in $[z_i]_{i=m_n+1}^{\infty} \cap (\bigcap_{i=1}^{k+1} \ker e_i^*)$ with $\|y_{k+1}\| = 1$, truncate and normalize this y_{k+1} in such a careful way that $|\langle y_{k+1}, e_\ell^*\rangle| \le \frac{1}{k+1}$ for $\ell = 1,\dots,k+1$. It is not hard to see, using Remark 20.23, that the so defined normalized sequence $\{y_n\}$ is $(1+\varepsilon)^3$-equivalent to the normalization of the corresponding block basic sequence of the standard basis of ℓ^1. The latter, in turn, is 1-equivalent to the standard basis of ℓ^1, by Remark 20.23. The upshot of all this is that we have now found a normalized basic sequence $\{y_n\}$, $(1+\varepsilon)^3$-equivalent to the standard basis of ℓ^1 and satisfying the assumption of Proposition 8.6 (with $\tilde{T}$ and $\{e_n\}$ instead of X and $\{x_n\}$). Hence application of Proposition 8.6 yields a normalized block basic sequence with respect to $\{e_n\}$, say $\{x_n\}$, which is $(1+\varepsilon)^4$-equivalent to the standard basis of ℓ^1. Now, finally, let us apply Proposition 20.21 to $\{x_n\}$. Suppose $\operatorname{supp} x_1 \subset [1,k] < \operatorname{supp} x_2$. Then, taking $\alpha = 2$, $m = 2k$, $\beta = \frac{1}{2}(3+\alpha^{-1}) = \frac{7}{4}$, we obtain

$$\|x_1 + (2k)^{-1} \sum_{i=2}^{2k+1} x_i\| \le \frac{7}{4}.$$

For sufficiently small $\varepsilon > 0$ this contradicts the $(1+\varepsilon)^4$-equivalence of $\{x_n\}$ to the standard ℓ^1-basis, since

$$1 + \sum_{i=2}^{2k+1} (2k)^{-1} = 2. \qquad \square$$

The Tsirelson space $\tilde{T}$ thus disproves the old conjecture that every Banach space must contain an isomorph of c_0 or of some ℓ^p, $1 \le p < \infty$. Retreating a little, it is quite natural to ask whether every Banach space must contain either a reflexive subspace (infinite-dimensional, of course) or an isomorph of c_0 or ℓ^1. As far as we know this question is still open today. Another positive result one might still hope for, would be that every superreflexive space contains some ℓ^p, $1 < p < \infty$, isomorphically. In particular Theorem 20.1, which shows a close connection between superreflexive and ℓ^p spaces, is suggestive in this direction. However, recently T. Figiel and W.B. Johnson have shown that this is not true. By an ingenious convexification procedure they have modified the norm of the Tsirelson space so that it becomes uniformly convex, while at the same time preserving the property that the space contains no ℓ^p. The details are rather complicated and we do not give them here. Let us at least state the result formally.

THEOREM 20.24. *There exists a uniformly convex space containing no isomorph of any* ℓ^p, $1 < p < \infty$.

NOTES. The "only if" part of Theorem 20.1 is due to V.I. & N.I. GURARIĬ ([38]) for spaces which have uniformly convex and uniformly smooth norm. R.C. JAMES ([55]) derived an even stronger result for superreflexive spaces (which at the time were not yet known to be uniformly convexifiable). He also proved the "if" part, modulo ENFLO's result ([33]). For the "only if" part we have followed [38], since the proofs there seem more natural. The example of Theorem 20.7 is due to N.I. GURARIĬ ([37]), while Proposition 20.15 is from [55]. The original paper of B.S. TSIRELSON giving a reflexive space containing no ℓ^p is [101]. Our account follows that of T. FIGIEL & W.B. JOHNSON ([35]) to whom the entire second half of this section is due. An exception is Proposition 20.22 which was proved by R.C. JAMES in [52].

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