MATHEMATICAL CENTRE TRACTS

10

APPLICATIONS OF DISTRIBUTIONS IN MATHEMATICAL PHYSICS

BY

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PREFACE

to the first edition

In the years between 1945 and 1949 L. Schwartz developed the theory of distributions by giving a synthesis, a generalization and a rigorous foundation of the work of many authors, who had already used the concept of a distribution in a more or less cryptic way.
Among these there are also mathematicians and physicists who were led to the use of distributions in connection with their investigations in applied mathematics or theoretical physics; here we mention in particular the names of Heaviside, Dirac, Leray, Sobolev, Courant and Hilbert. After the publication of Schwartz's monograph "Théorie des Distributions" a large number of papers and books about the theory of distributions and its applications to partial differential equations appeared of which the latter are of special importance for the applied mathematician.
Therefore considering the history of the theory of distributions it may be expected that this theory can be apllied successfully to problems in mathematical and theoretical physics.
However, the great advantages of the use of distributions are not always exploited as it should be. In fact this is the case in several rather recent articles in which the mathematical derivations can be simplified considerably or made more rigorous by using distributions. Examples may be found for instance in fluid- and quantum mechanics.

The aim of this tract is to demonstrate the value of the theory of distributions for problems in mathematical and theoretical physics; for this purpose we have chosen illustrative applications to problems from quite different branches of mathematical physics.

Before discussing these applications a review of the theory of distributions is given in an introductory chapter. All concepts and theorems, used later on, are treated here and no a priori knowledge of distributions is assumed.

The first application, given in chapter II, concerns the well-known problem of the diffraction of a cylindrical pulse by a semi-infinite screen.

The second one, treated in chapter III, is taken from theoretical aerodynamics. The boundary value problems, occurring in the theory of supersonic flow around thin wings, are usually solved in a rather complicated way. Using distributions the theory becomes much simpler; in fact all boundary value problems may be formulated in terms of one single integral equation which can easily be inverted.

In this connection the attention of the reader may be drawn also to the work by P. Germain and R. Sauer (cf. chapter III).

The next two chapters IV and V are devoted to applications in modern theoretical physics. While the problems in the two preceding chapters may also be treated, at least in principle, in a classical way, actually this is no longer possible for the problems of the chapters IV and V.

The investigations in chapter IV concern the derivation of Lorentz-invariant Green's functions for the so-called Klein-Gordon equation which is of fundamental importance in quantum field theory. Distributions concentrated on surfaces in four dimensional space play an important role in this theory. This problem has been studied in recent years also by other authors using methods different from the one given here; we mention the work by P.D. Methée, L. Gårding and J. Lavoine. In the last application, given in chapter V, we deal with the regularization of divergent convolution integrals as they occur in quantum electrodynamics. A general method is developed for dealing with these divergencies. Although it is not claimed that the investigations of this chapter yield new results in field theory, a unifying method has been found for defining the above-mentioned divergent integrals, which includes as special cases the devices used by e.g. N.N. Bogoliubov and O. Parasiuk, A.J. Achieser and W.B. Berestezki, and H. Bremermann.

The author expresses his gratitude to the Board of Directors of the "Stichting Mathematisch Centrum" for giving him the opportunity to carry out the investigations, presented in this treatise, and for publishing this study in the series "Mathematical Centre Tracts", edited by this institute.

The investigations in chapter III and V were made by the author during his stay from 1962 to 1963 as visiting lecturer at the University of California at Berkeley; he is obliged to the National Science Foundation, which supported the research during this time under the grants N.S.F. No G.P. 2 and No 25224.

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He also takes this opportunity to extend thanks to Prof.dr. R.T. Seeley of Brandeis University for inspiring conversations on the theory of distributions in general and to Prof.dr. H. Bremermann of the University of California at Berkeley for exchanging ideas on the multiplication of causal functions.

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PREFACE

to the second edition

The second edition differs from the first one in so far that the first and the fourth chapter have undergone some alterations.
In chapter I some incomplete and inaccurate statements have been detected and these are corrected.
A basic formula used in chapter IV can be proved in a much more elegant way than has been done in the previous edition. This results in a clearer and shorter exposition of the derivation of the elementary solution of the Klein-Gordon equation.

Amsterdam, March 1969

E.M. de J.

CONTENTS

Chapter I

THE THEORY OF DISTRIBUTIONS

1. Introduction

In 1927, in a paper on the physical interpretation of quantum dynamics [1], the great physicist P.A.M. Dirac introduced a "function" $\delta(x)$ which was postulated to have the following properties: $\delta(x)$ should be zero everywhere except at the point $x=0$ at which it is infinite and such that

$$\int_{-\infty}^{+\infty} \delta(x)dx = 1. \tag{1.1}$$

Unfortunately, it can easily be shown that no function exists which is zero almost everywhere while its integral from $-\infty$ to $+\infty$ does not vanish.

Dirac however treated his so called delta function as if it were a well behaved even differentiable function.

Putting

$$x\,\delta(x) = 0, \tag{1.2}$$

he obtained by formal differentiation the formula

$$x\frac{d}{dx}\,\delta(x) = -\,\delta(x). \tag{1.3}$$

Moreover it follows from (1.1) that

$$\int_{-\infty}^{x} \delta(\xi)d\xi = \Theta(x), \tag{1.4}$$

where $\Theta(x)$ is the Heaviside unit-step function defined as

$$\Theta(x) = \begin{cases} 1 \text{ for } x > 0, \\ 0 \text{ for } x \leq 0. \end{cases} \tag{1.5}$$

Differentiating one gets again formally

$$\frac{d\Theta(x)}{dx} = \delta(x). \tag{1.6}$$

Dirac also required that the delta function should have the sifting property

$$\int_{-\infty}^{+\infty} f(x)\,\delta(a-x)\,dx = f(a), \tag{1.7}$$

where f(x) is a continuous function, defined in a neighbourhood of x=a.

When f(x) is m times differentiable, integration by parts yields

$$\int_{-\infty}^{+\infty} f(x)\,\delta^{(m)}(a-x)\,dx = f^{(m)}(a). \tag{1.8}$$

Since the delta function was introduced by Dirac it has found many applications in applied mathematics, where one often deals with "delta-like" distributions of mass, force, electrical charge or other physical quantities.

Formulae related to those given above are applied, and despite the fact that the derivations are by no means mathematically correct, they lead to correct results.

In this situation one expects that some proper justification may be possible of the delta function and the processes yielding for instance formulae such as (1.2) - (1.8). This justification is given in what is called nowadays the "theory of distributions". The name refers to distributions of certain physical quantities.

The theory of distributions had already in the thirties its roots in the work of Bochner, Leray, Sobolev, Courant, Hilbert and many others [2] - [5] .

In particular, the work of Sobolev [4] should be mentioned. He investigated functionals of the type

$$\langle f,\phi\rangle = \int_{-\infty}^{+\infty} f(x)\ \phi(x)\,dx, \tag{1.9}$$

where f(x) is locally integrable and $\phi(x)$ is a function with bounded support and continuously differentiable a certain number of times. The "generalized" derivative of f(x), which may be a discontinuous function, is defined by the functional

$$(1.10) \qquad \langle \frac{df}{dx} , \phi(x) \rangle = -\int_{-\infty}^{+\infty} f(x) \frac{d\phi}{dx} dx = - \langle f, \frac{d\phi}{dx} \rangle .$$

Using this definition the formula (1.6) is readily established where differentiation must be taken in generalized sense.
Moreover, Sobolev showed also that any locally integrable function can be considered to be a so called weak limit of infinitely differentiable functions; this means that any locally integrable function f(x) satisfies for all functions $\phi(x)$ the relation

$$(1.11) \qquad \langle f,\phi \rangle = \lim_{m \to \infty} \langle f_m,\phi \rangle \quad ,$$

where the sequence of functions $f_m(x)$ is a sequence of C^∞ functions.

In the years between 1945 and 1949 L. Schwartz [6] developed the theory of distributions by giving a synthesis, a generalization, and a foundation of the work of many mathematicians who had already used the concept of distribution in a more or less hidden way.
Schwartz introduced a distribution as a continuous linear functional on a suitable space of so called test functions. The foundation of the theory arises from the consideration of the space of test functions as a topological vector space, whereas the space of distributions is the dual of this space. The operations to be performed on distributions are defined by transposition to the test functions, cf. formula (1.10).

After the publications by Schwartz many papers and books about the theory of distributions appeared; among these are the well known books by Gelfand and Shilov in which a clearly written introduction to the theory is presented [7] . In these textbooks a distribution is also defined as a continuous linear functional on a suitable space of test functions.

It can be proved that every distribution may be obtained as the weak limit of a sequence of infinitely differentiable functions. This theorem may also be taken as the starting point for defining distributions. In the same way as irrational numbers may be defined as fundamental sequences of rational numbers (Cantor), distributions can be obtained as sequences of infinitely differentiable functions. This has been done by J. Mikusinski and he has described in ref. [8] a theory of distributions equivalent to that of L. Schwartz. In this connection we mention also the works by J. Mikusinski and R. Sikorski [9], J. Korevaar [10] , G. Temple [11] and M.J. Lighthill [12] .

Another important theorem in the theory of distributions states that every distribution can be considered as the generalized derivative of a continuous function. Therefore a distribution in n variables may also be defined by a continuous function and a set of n non-negative integers denoting the order of differentiation with respect to each variable. This process for defining distributions is given by J. Mikusinski and R. Sikorski [9] and by S. e Silva [13] .

Another important approach to the definition of distributions is due to H. Tillmann, H.Bremermann, L. Durand and H.A. Lauwerier [14], [15], [16] ; these authors have shown that a distribution, defined for example on the real axis, can be extended to functions holomorphic in the upper and lower complex half planes, such that the "jump" across the real axis again represents the distribution. In this way, distributions may be defined by a pair of functions holomorphic in the upper and lower complex half planes.

It may be remarked finally, that there already exists an extensive literature on applications of distributions to partial differential equations; we only mention here the books by Hörmander, Friedman and Gelfand-Shilov [17] - [19] .

In this chapter, a review is given of the theory of distributions, also called generalized functions; they are defined here as continuous linear functionals. We emphasize mainly the aspect of the calculus, needed by the applied mathematician, the physicist or the engineer.

The most important definitions and theorems are presented, but the proofs are mostly rather concise and they are even omitted in a few cases; otherwise, this introductory chapter would become too lengthy. However, the reader will always be referred to literature in which he can find more details and additional material; besides, the text is illustrated by many examples.

For the sake of completeness some remarks are made concerning the topological foundation of the theory in the last section of this chapter; this section may be omitted by anyone who is interested mainly in the practical side of the theory.

The following subjects are treated in consecutive sections: definitions of test functions and distributions; various operations to be performed on distributions; regularization of functions of one variable with an algebraic singularity; the convolution of two distributions; the Fourier transformation; distributions concentrated on surfaces in n-dimensional Euclidean space; regularization of functions of several independent variables with an algebraic singularity; applications to partial differential equations and, finally, some remarks on the topological foundation of the theory of distributions.

2. Test functions and Distributions

2.1. The spaces D and S and their duals D' and S'

Let us consider a family Φ of infinitely continuously differentiable complex valued functions $\phi(x) = \phi(x_1, x_2, \ldots, x_n)$, defined in every point of the n-dimensional space R_n.

We suppose, that the family Φ is a linear space and that there can be introduced in Φ a rule, which defines the convergence to zero of a sequence of functions $\phi_m(x)$ belonging to Φ, m=1,2,... . The functions $\phi(x)$ are called test functions.

A distribution may be defined as a continuous linear functional on Φ . This means, that a distributions, say f, assigns to any function $\phi(x) \in \Phi$ a complex number, denoted by $\langle f, \phi \rangle$, with the properties:

1.

(2.1) $$\langle f, a_1\phi_1 + a_2\phi_2\rangle = a_1\langle f,\phi_1\rangle + a_2\langle f,\phi_2\rangle,$$

valid for any ϕ_1 and ϕ_2; a_1 and a_2 are arbitrary real or complex numbers.

2.

(2.2) $$\lim_{m\to\infty} \langle f,\phi_m\rangle = 0,$$

for any sequence $\{\phi_m\}$, converging to zero.

The set of all distributions, which can be defined on the space Φ, is called the dual space of Φ and it is denoted by Φ'. Any linear combination of two distributions f_1 and f_2 is defined by the rule

(2.3) $$\langle a_1 f_1 + a_2 f_2,\phi\rangle = \bar{a}_1\langle f_1,\phi\rangle + \bar{a}_2\langle f_2,\phi\rangle,$$

where a_1 and a_2 are arbitrary complex numbers and $\bar{a}_1,\bar{a}_2$ denote their complex conjugates.
Therefore the space Φ' is a linear space.

One can also introduce in the space Φ' the concept of convergence. A sequence of distributions $f_m \in \Phi'$, $m=1,2,\ldots$, is said to converge to a distribution $f \in \Phi'$, when the following relation holds for every test function $\phi(x) \in \Phi$

(2.4) $$\lim_{m\to\infty} \langle f_m,\phi\rangle = \langle f,\phi\rangle.$$

This type of convergence is called weak convergence.

The properties of the space Φ' depend of course on the properties of the space Φ.
We give two important examples of spaces of test functions and distributions, which will appear to be very useful in later considerations.

1. <u>The spaces D and D'</u>

The space D consists of all complex valued infinitely continuously differentiable (C^∞) functions $\phi(x) = \phi(x_1,x_2,\ldots,x_n)$, defined in every point of the space R_n and vanishing outside a bounded subset of R_n.

The closure of the bounded region in which $\phi(x)$ does not vanish, is called the support of the test function.
A sequence $\{\phi_m(x)\}$, m=1,2,..., converges in D to zero, when the supports of the functions $\phi_m(x)$ all lie within the same bounded set of R_n and when the $\phi_m(x)$ and also all their derivatives converge uniformly to zero with respect to x. The space of all distributions, which can be defined on D, is denoted by D'.

Examples

An example of a test function belonging to the space D is given by

$$(2.5) \qquad \phi(x;a) = \begin{cases} \exp\left[-\frac{a^2}{a^2-r^2}\right] & \text{for } r < a \\ 0 & \text{for } r \geq a \end{cases},$$

where $r = |x| = \sqrt{x_1^2+x_2^2+\ldots+x_n^2}$ and a some real number.
The sequence of functions $\frac{1}{m}\phi(x;a)$ (m=1,2,...) converges in D to zero, in contrast to the sequence $\frac{1}{m}\phi(\frac{x}{m};a)$ which does not converge to zero in D.
Consider a real or complex valued locally Lebesgue integrable function $f(x) = f(x_1,x_2,\ldots,x_n)$, defined on R_n; by means of this function we form the functional $<f,\phi>$ defined by

$$(2.6) \qquad <f,\phi> = \int_{-\infty}^{+\infty} \overline{f(x)}\ \phi(x)\,dx,$$

where the integration should be performed over the support of the test function $\phi(x) \in D$ and $\overline{f(x)}$ denotes the complex conjugate of $f(x)$.
It is clear, that $<f,\phi>$ is a continuous linear functional on D and hence this functional defines a distribution, belonging to D'.
The set of all values $<f,\phi>$, where ϕ may be any element of D, defines the function f(x) almost everywhere in R_n.
Hence every locally integrable function may be identified with a distribution belonging to D' and so the distributions in D' are a generalization of the locally Lebesgue integrable functions. Another example of a distribution belonging to D' is the distribution defined by

(2.7) $$<f,\phi> = \phi(0).$$

This distribution is called the delta function of Dirac.
Instead of the symbol $<f,\phi>$ we write also:

(2.8) $$<f,\phi> = \int_{-\infty}^{+\infty} \delta(x)\ \phi(x)dx = \phi(0,0,\ldots,0) = \phi(0).$$

It may be remarked, that the integral appearing in (2.8) is only a symbolic notation, which has nothing to do with an integral as defined in the sense of Riemann or Lebesgue.
This can be shown by taking the test function $\phi(x;a)$ (2.5) with $a \to 0$.

2. The spaces S and S'

The space S consists of all complex valued C^∞ functions $\phi(x) = \phi(x_1,x_2,\ldots,x_n)$ defined in R_n with the property that $\phi(x)$ together with all its derivatives decrease for $|x| \to \infty$ stronger than any negative power of $|x|$.
This may be expressed by the following inequalities which hold for every testfunction $\phi(x) \in S$:

(2.9) $$|x^k D^q \phi(x)| < C_{kq},$$

where $k = (k_1,k_2,\ldots,k_n)$ and $q = (q_1,q_2,\ldots,q_n)$ are n-tuples of non-negative integers and C_{kq} is a constant, depending on k,q and ϕ.
x^k and $D^q\phi(x)$ are short notations for the expressions:

(2.10) $$x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n} \text{ and } D^q \phi(x) = \frac{\partial^q \phi(x)}{\partial x_1^{q_1} \partial x_2^{q_2} \ldots \partial x_n^{q_n}}.$$

A sequence $\{\phi_m\}$, m=1,2,..., is said to converge in S to zero, if the functions $\phi_m(x)$ and all their derivatives converge to zero uniformly with respect to x in every bounded region of R_n and if, moreover, the numbers C_{kq}, occurring in (2.9), can be chosen as independent of m, i.e.

(2.11) $$\left|x^k D^q \phi_m(x)\right| < C_{kq},$$

for all values of m.
The space of all distributions, which can be defined on S, is denoted by S'.

Examples

An example of a test function belonging to S is the function $\phi(x) = \exp\left[-r^2\right]$.
The sequence of functions $\phi_m(x) = \frac{1}{m}\exp\left[-r^2\right]$, m=1,2,..., converges in S to zero in contrast to the sequence $\{\phi_m(\frac{x}{m})\}$, which does not converge to zero, because the relations (2.11) are not satisfied by the latter.
Consider a locally Lebesgue integrable function f(x) of finite algebraic growth at infinity, i.e. one which does not increase at infinity stronger than any positive power of $|x|$. Due to the inequalities (2.9) we can again form the functional $<f,\phi>$, defined by

$$(2.12) \qquad <f,\phi> = \int_{-\infty}^{+\infty} \overline{f(x)}\ \phi(x)dx,$$

where the integration is performed over the whole space R_n.
It is obvious, that (2.12) defines a continuous linear functional on S and hence this functional defines a distribution, belonging to S'. The set of all values $<f,\phi>$, where ϕ may be any element of S, defines the function f(x) almost everywhere in R_n.
Therefore every locally integrable function of finite algebraic growth at infinity may be identified with a distribution belonging to S' and so the distributions of S' are a generalization of the functions of this class.

The delta function of Dirac as defined by (2.7) is also a distribution in S'.

It is clear that every test function in the space D also belongs to the space S; the space D is even dense in S. This can be proved easily as follows. Take the C^∞ function e(x), which equals 1 for $r \leq 1$ and which is identically zero for $r \geq 2$. When $\phi(x) \in S$, then the functions $\phi_m(x) = e(\frac{x}{m})\ \phi(x)$ (m=1,2,...) are test functions belonging to D, with the property that the $\phi_m(x)$ converge to $\phi(x)$ in S and hence D is

dense in S.
It follows from the definitions of convergence in D and in S, that a sequence $\{\phi_m\}$, converging in D to the function $\phi \in D$, also converges to ϕ in S.
Therefore every continuous linear functional on S is a priori a continuous linear functional on D and hence $S' \subset D'$. However, not every distribution in D' is a distribution in S'.
It will appear in the next section, that the function exp $[r^2]$ is an example of a distribution in D' but not in S'.
From the definition of weak convergence it follows immediately, that a sequence of distributions f_m, converging in S' to a distribution f, converges also in D' to the distribution f.

Therefore convergence in D implies convergence in S and convergence in S' implies convergence in D'.

2.2. The space Z and its dual Z'

In the beginning of this section we introduced the general space Φ of C^∞ functions, defined on R_n. It is also possible to take a space Ψ of entire complex functions $\psi(z) = \psi(z_1, z_2, \ldots, z_n)$, defined in every point of the n-dimensional complex space C_n; $z_p = x_p + iy_p$, $p=1,2,\ldots,n$.
In the same way as before distributions, i.e. continuous linear functionals, may be defined on the space Ψ ; the space of all distributions, which can be defined on Ψ , is denoted by Ψ'. We give the following example.

The spaces Z and Z'

The space Z consists of all entire complex functions $\psi(z) = \psi(z_1, z_2, \ldots, z_n)$, defined on C_n and satisfying the following inequalities:

$$|z^k \psi(z)| < C_k \exp [a_1|y_1| + a_2|y_2| + \ldots + a_n|y_n|], \tag{2.13}$$

where k may be any n-tuple of non-negative integers $(k_1, k_2, \ldots, k_n)$, $z^k = z_1^{k_1} z_2^{k_2} \ldots z_n^{k_n}$ and C_k and a_p $(p=1,2,\ldots,n)$ are positive numbers

depending respectively on ψ and k and ψ only.
A sequence $\{\psi_m(z)\}$ is said to converge to zero, when the functions $\psi_m(x)$ converge to zero uniformly with respect to x in every bounded region of the space R_n and when moreover for each $\psi_m(z)$ inequalities of the type (2.13) hold, where the constants C_k and a_p do not depend on the index m of the function $\psi_m(z)$.
The space of all continuous linear functionals, defined on Z, is denoted by Z'.

It may be remarked already, that every test function of Z is the Fourier transform of a test function of D; the spaces Z and Z' play an important role in the theory of the Fourier transformation, see section 6.

2.3. Local and global properties of distributions

We shall consider in the sequel mainly distributions belonging to D' or S'. Many definitions and theorems are completely analogous for both cases and therefore we shall not always make an explicit distinction when it is not necessary to do so.
A distribution $\langle f,\phi\rangle$, which can be written in the form

$$\langle f,\phi\rangle = \int_{-\infty}^{+\infty} \overline{f(x)}\ \phi(x)dx, \tag{2.14}$$

where the integral is a Lebesgue integral is called a regular distribution; all other distributions are called singular. E.g. the δ-function of Dirac is a singular distribution.
Two distributions f_1 and f_2 are said to be equal, if for every test function ϕ the following relation holds:

$$\langle f_1,\phi\rangle = \langle f_2,\phi\rangle. \tag{2.15}$$

Two locally Lebesgue integrable functions, which are equal almost everywhere, define the same distribution.
A distribution f is equal to zero in a neighbourhood U of a point x_0, if $\langle f,\phi\rangle = 0$ for any test function vanishing outside U.

A distribution f is equal to zero in a domain Ω of R_n, if f is zero in some neighbourhood of every point of Ω.
By restricting a distribution to test functions which support in a neighbourhood of a certain point we obtain information about the local behaviour of the distribution at that point.
Also conversely, the global behaviour of a distribution is determined by its local behaviour. This follows from the lemma of the decomposition of the unity.
Decomposition of the unity. Let there be given a locally finite covering of R_n by bounded neighbourhoods U_1, U_2, ..., U_n, ..., i.e. the whole space R_n is covered by the union of all neighbourhoods U_i, while every point of R_n is covered by only a finite number of neighbourhoods.
It is possible to construct C^∞ functions $\alpha_m(x)$ with the properties:

a) $0 \leq \alpha_m(x) \leq 1$

b) $\alpha_m(x) \equiv 0$ outside U_m $(m=1,2,\dots)$

c) $\sum_{m=1}^{\infty} \alpha_m(x) \equiv 1$.

Proof: see [6], Vol.I, Ch.I, §2, p.22 or [7], Vol.I, Ch.I, App.1, p.143.

Let us assume, that the local behaviour of the distribution f is known in every point of R_n; this means, that every point $x \in R_n$ has a neighbourhood V(x) with the property, that the values of $\langle f,\phi \rangle$ are known for every ϕ with support in V(x). The system of neighbourhoods V(x) covers the whole space R_n and according to the theorem of Heine-Borel one can take from this system a countable set of neighbourhoods $U_1, U_2, \dots, U_m \dots$, such that every sphere $|x| \leq \rho$ is covered by only a finite number of the neighbourhoods $U_1, U_2, \dots, U_m, \dots$.
Applying the lemma, it is easily shown, that every test function belonging to D may be written in the form

$$\phi(x) = \lim_{M\to\infty} \sum_{m=1}^{M} \phi_m(x) = \sum_{m=1}^{\infty} \phi_m(x), \tag{2.16}$$

where $\phi_m(x) = \alpha_m(x)\phi(x)$ and where the limit is taken in the sense of the convergence as defined in D; the functions $\phi_m(x)$ have their support within $\bar{U}_m$ and in the sum of (2.16) only a finite number of terms occurs. Hence for every distribution $f \in D'$ or S' and for every test function $\phi \in D$ we may write

$$(2.17) \qquad \langle f,\phi\rangle = \sum_{m=1}^{\infty} \langle f,\phi_m\rangle$$

and it follows that every distribution $f \in D'$ is determined by its local behaviour in the neighbourhoods U_m ("recollement des morceaux"). Because D is dense in S, the values of $\langle f,\phi\rangle$ with $f \in S'$ and $\phi \in S'$ are determined by the values of $\langle f,\phi\rangle$ with $\phi \in D$ and hence also every distribution out of S' is defined by its local behaviour.
In particular it follows that a distribution which is zero in a certain neighbourhood of every point $x \in R_n$ is also the zero distribution, i.e.

$$(2.18) \qquad \langle f,\phi\rangle = 0$$

for each ϕ.
A point x_0 is said to be an essential point of a distribution when there does not exist a neighbourhood of x_0 in which the distribution is equal to zero. The collection of all essential points is called the support of the distribution; e.g. the support of the δ-function of Dirac is a single point.

3. Operations on distributions

3.1. Operations on distributions in D' or S'

In this section several operations, defined in a well-known way for functions, are generalized for distributions.
The definitions are chosen in such a way as to preserve their classical meaning in case the operations are applied to distributions, which are at the same time also ordinary functions.

Definitions

a) Distributions are added according to the rule:

(3.1) $$\langle f_1 + f_2, \phi \rangle = \langle f_1, \phi \rangle + \langle f_2, \phi \rangle .$$

b) Distributions can be multiplied by infinitely continuously differentiable functions a(x); in the case of S', a(x) should be of finite algebraic growth at infinity; we have the rule:

(3.2) $$\langle af, \phi \rangle = \langle f, \bar{a}\phi \rangle ,$$

where $\bar{a}$ is the complex conjugate of a.

c) For every set of n real numbers $h = (h_1, h_2, \ldots, h_n)$ a translation of the distribution $f(x) = f(x_1, x_2, \ldots, x_n)$ is defined by

$$\langle f(x_1 - h_1, x_2 - h_2, \ldots, x_n - h_n),\ \phi(x_1, x_2, \ldots, x_n) \rangle =$$

$$= \langle f(x_1, x_2, \ldots, x_n),\ \phi(x_1 + h_1, \ldots, x_n + h_n) \rangle ,$$

or

(3.3) $$\langle f(x-h),\ \phi(x) \rangle = \langle f(x),\ \phi(x+h) \rangle .$$

d) The reflection of a distribution f(x) is denoted by f(−x) and it satisfies the relation:

(3.4) $$\langle f(-x),\ \phi(x) \rangle = \langle f(x),\ \phi(-x) \rangle .$$

e) The similarity transformation is defined by

$$\langle f(\frac{x_1}{\alpha}, \frac{x_2}{\alpha}, \ldots, \frac{x_n}{\alpha}),\ \phi(x_1, x_2, \ldots, x_n) \rangle =$$

$$= |\alpha|^n \langle f(x_1, x_2, \ldots, x_n),\ \phi(\alpha x_1, \alpha x_2, \ldots, \alpha x_n) \rangle ,$$

or

(3.5) $$\langle f(\frac{x}{\alpha}),\ \phi(x) \rangle = |\alpha|^n \langle f(x),\ \phi(\alpha x) \rangle$$

where α is some real number.

A distribution is said to be homogeneous of degree λ, if

$$f(\alpha x) = \alpha^{\lambda} f(x) \text{ for each } \alpha > 0,$$

or what amounts to the same, if

$$(3.6) \qquad \langle f(x),\ \phi(\tfrac{x}{\alpha})\rangle = \alpha^{\lambda+n}\langle f(x),\ \phi(x)\rangle .$$

Hence $\delta(x) = \delta(x_1,x_2,\ldots,x_n)$ is homogeneous of degree $-n$.

f) A general linear transformation A of the independent variables $x_1,x_2,\ldots,x_n$ is applied to distributions according to the formula:

$$(3.7) \qquad \langle f(Ax),\ \phi(x)\rangle = \frac{1}{|A|}\langle f(x),\ \phi(A^{-1}x)\rangle,$$

where A^{-1} is the inverse transformation and $|A|$ the absolute value of the determinant of the transformation.

g) A sequence of distributions f_m, m=1,2,... converges to the distribution f, if

$$(3.8) \qquad \lim_{m\to\infty} \langle f_m,\phi\rangle = \langle f,\phi\rangle \text{ for each test function } \phi .$$

f is called the distributional or the weak limit of the sequence of the distributions f_m. It is remarked, that when a sequence of distributions $f_m \in D'$ or S' has the property, that the numbers $\langle f_m,\phi\rangle$ converge for every $\phi \in D$ resp. S, then the limit, say $f(\phi)$, is also a distribution belonging to D' resp. S'. This fact is important enough to state it in a theorem.

<u>Theorem 1</u>. The spaces D' and S' are complete with respect to weak convergence.

<u>Proof</u>: This theorem is usually proved by aid of the topological structure underlying the spaces D and S and their duals D' and S'. However for the case of D' a more elementary proof understandable for any reader, not familiar with the theory of topological vector spaces, is given by Gelfand and Shilov in [7], Vol.I, Appendix, p.354.

Examples (1 independent variable)

(3.9) $$\lim_{\varepsilon\to+0} \frac{1}{\pi} \frac{\varepsilon}{x^2+\varepsilon^2} = \delta(x)$$

(3.10) $$\lim_{t\to+0} \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) = \delta(x)$$

(3.11) $$\lim_{m\to\infty} \frac{1}{\pi} \frac{\sin mx}{m} = \delta(x) .$$

h) The differentiation of distributions is defined as follows:

(3.12) $$\langle \frac{\partial f(x)}{\partial x_i}, \phi(x)\rangle = - \langle f(x), \frac{\partial \phi}{\partial x_i} \rangle .$$

This definition of the so called distributional derivative has some immediate consequences which are stated in the following two theorems.

Theorem 2. Every distribution is infinitely differentiable.
For distributions in more variables one has always the relation

(3.13) $$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} .$$

Proof: Follows immediately from the definition (3.12).

Theorem 3. The operations of differentiation and passing to the limit may always be interchanged, i.e. $\lim_{m\to\infty} f_m = f$ implies

(3.14) $$\lim_{m\to\infty} \langle \frac{\partial f_m}{\partial x_i}, \phi \rangle = \langle \frac{\partial f}{\partial x_i}, \phi \rangle .$$

Proof: $\lim_{m\to\infty} \langle \frac{\partial f_m}{\partial x_i}, \phi \rangle = - \lim_{m\to\infty} \langle f_m, \frac{\partial \phi}{\partial x_i} \rangle = - \langle f, \frac{\partial \phi}{\partial x_i} \rangle = \langle \frac{\partial f}{\partial x_i}, \phi \rangle .$

We give now some examples of the application of the distributional derivative to distributions of one independent variable; the resulting formulae are easily verified and this is left to the reader.

<u>Examples</u>

1.

$$\frac{d\theta(x)}{dx} = \delta(x), \tag{3.15}$$

where $\theta(x)$ is the unit-step function of Heaviside (see (1.5)).

2.

$$\frac{d}{dx} \log |x| = \frac{1}{x}, \tag{3.16}$$

where the distribution $\frac{1}{x}$ is defined as the Cauchy principal value of

$$\int_{-\infty}^{+\infty} \frac{\phi(x)}{x}\, dx.$$

3. The distribution x^{-n} $(n=2,3,\ldots)$ is defined by the recursive relation

$$x^{-n} = -\frac{1}{n-1} \frac{d}{dx} x^{-n+1}, \tag{3.17}$$

and hence

$$x^{-n} = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} \frac{1}{x}. \tag{3.18}$$

4. Defining

$$\log(x \pm i0) = \lim_{y \to \pm 0} \log(x+iy), \tag{3.19}$$

and

$$\frac{1}{x \pm i0} = \lim_{y \to \pm 0} \frac{1}{x + iy} \tag{3.20}$$

one finds by means of the distributional derivative of $\log(x \pm i0)$ the result

$$\frac{d}{dx} \log(x \pm i0) = \frac{1}{x \pm i0} = \frac{1}{x} \mp i\pi\delta(x). \tag{3.21}$$

3.2. Distributions and continuous functions

The concept of the distributional derivative is not only a very useful tool in distribution calculus, it is also of essential importance for the relation between distributions and continuous functions. This relation reveals the true nature of distributions and it will be given in the next theorem.

Theorem 4. Every distribution belonging to D' is in every domain Ω of R_n with compact closure $\bar{\Omega}$ equal to some distributional derivative of a continuous function with support in an arbitrary neighbourhood of $\bar{\Omega}$. This may be expressed in a shorter way by saying that every distribution out of D' is locally equal to a distributional derivative of a continuous function. Hence for every $f \in D'$ and all $\phi \in D$ with support in Ω, there exist a continuous function F and a n-tuple p of non negative integers $(p_1, p_2, \ldots, p_n)$ such that

$$<f,\phi> = \int_{\Omega} F(x) \, D^p\phi(x)\,dx, \tag{3.22}$$

with

$$D^p = \frac{\partial^{|p|}}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_n^{p_n}} \text{ and } |p| = \sum_{i=1}^{n} p_i.$$

Proof: A rather simple proof is given by Schwartz and Gelfand-Shilov using the well-known representation theorem of Riess, valid for continuous linear functionals defined on the space of functions continuous in some bounded interval. Since some knowledge of the topological foundations of the theory of distributions is needed, we postpone the proof to section 10.3 of this chapter. See also [6], Vol.I, Ch.III, §6, p.82 and [7], Vol.II, Ch.II, §4, p.92-97.

In the case of distributions in S' the formulation of the theorem should be a little bit modified. Then it runs as follows:

Theorem 4^{bis}. Every distribution belonging to S' is a distributional derivative of a continuous function of finite algebraic growth at infinity.

Proof: See [6], Vol.II, Ch.VII, §4, p.95 and [7], Vol.II, Ch.II, §4, p.92-97.

A formula analogous to (3.22) holds of course again.
Due to the validity of the latter theorem the distributions of S' are called "tempered" distributions. For instance, the function e^{r^2} is not a distribution belonging to S', but it does belong to D'. With the aid of the theorems 4 and 4^{bis} one may finally prove the following statement:

Theorem 5. A distribution belonging to S' or D', which has its support in a single point x_0, is a finite linear combination of $\delta(x-x_0)$ and some of its derivatives.

Proof: We give the proof only for the case of one independent variable x. For more independent variables the demonstration runs along more or less the same lines.

The distribution, say f(x), is the derivative of a certain order, say p+1, of a continuous function F(x). Hence

$$f(x) = \frac{d^{p+1}}{dx^{p+1}} F(x).$$

Since f(x) vanishes for $x > x_0$ and $x < x_0$, F(x) must be in these intervals a polynomial of at most degree p; assume $F(x) = \sum_{q=0}^{p} a_q(x-x_0)^q$ for $x > x_0$ and $F(x) = \sum_{q=0}^{p} b_q(x-x_0)^q$ for $x < x_0$, while $a_0 = b_0$.
Differentiating (p+1) times yields the required result

$$f(x) = \sum_{q=0}^{p-1} c_q \delta^{(q)}(x - x_0), \tag{3.23}$$

with $c_q = (a_{p-q}-b_{p-q})(p-q)!$

3.3. Operations on distributions of Z'

Apart from some modifications operations on distributions belonging to Z' can be defined in an analogous way as for distributions belonging to D' or S'.
The operations of addition, translation, reflection, similarity transformation, taking the limit of a sequence of distributions and differentiation are defined as in the formulae (3.1), (3.3) - (3.5), (3.8) and (3.12). The multiplication by a function $h(z) = h(z_1, z_2, \ldots, z_n)$ is again defined as

$$(3.24) \qquad \langle h(z)g(z), \psi(z)\rangle = \langle g(z), \overline{h(z)}\, \psi(z)\rangle .$$

However, this definition implies that $\overline{h(z)}\, \psi(z)$ should be again a test function out of Z, otherwise the right hand side of (3.24) is meaningless.
Therefore h(z) should be an analytic function, satisfying an equality of the form

$$(3.25) \qquad |h(z)| < C \exp\left[b_1|y_1| + b_2|y_2| + \ldots + b_n|y_n|\right] \cdot$$
$$\cdot (1+|z_1|)^{q_1}(1+|z_2|)^{q_2}\ldots(1+|z_n|)^{q_n},$$

where C, b_i and q_i are arbitrary real constants.

4. Regularization of functions with algebraic singularities

Consider a function f(x) which has a non integrable singularity in only a finite set of isolated points. A regularization of this function f(x) is a distribution with the property, that for test functions $\phi(x)$ with support not containing any of the singular points, it is defined by the integral

$$(4.1) \qquad \int_{-\infty}^{+\infty} f(x)\, \phi(x)dx.$$

A regularization is, in case it exists, uniquely determined apart from a linear combination of δ-functions and their derivatives concentrated at the singular points of $f(x)$.

In this section a particular regularization will be given of functions $f(x)$ of one independent variable and with an algebraic singularity; e.g. $f(x) = x^{\lambda}$ with λ complex and $\operatorname{Re} \lambda \leq -1$.

Let us take the function of one variable x_+^{λ}, defined as

$$x_+^{\lambda} = \begin{cases} x^{\lambda} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0, \end{cases} \tag{4.2}$$

where λ is a complex parameter.
This function defines for $\operatorname{Re} \lambda > -1$ a regular distribution, viz.

$$\langle x_+^{\lambda}, \phi(x) \rangle = \int_0^{\infty} x^{\lambda} \phi(x)\, dx. \tag{4.3}$$

The distribution x_+^{λ} is defined for values of λ with $\operatorname{Re} \lambda \leq -1$ by analytical continuation of (4.3) with respect to λ.
Hence for $\operatorname{Re} \lambda > -2$ and $\lambda \neq -1$ one obtains the formula

$$\langle x_+^{\lambda}, \phi(x) \rangle = \int_0^{1} x^{\lambda} \{\phi(x) - \phi(0)\}\, dx + \int_1^{\infty} x^{\lambda} \phi(x)\, dx + \frac{\phi(0)}{\lambda+1}, \tag{4.4}$$

and more generally for $\operatorname{Re} \lambda > -n-1$ and $\lambda \neq -1, -2, \ldots, -n$.

$$\langle x_+^{\lambda}, \phi(x) \rangle = \int_0^{1} x^{\lambda} \Big\{\phi(x) - \phi(0) - x\,\phi'(0) \ldots - \frac{x^{n-1}}{(n-1)!} \phi^{(n-1)}(0)\Big\}\, dx + \int_1^{\infty} x^{\lambda} \phi(x)\, dx + \sum_{k=1}^{n} \frac{\phi^{(k-1)}(0)}{(k-1)!\,(\lambda+k)}. \tag{4.5}$$

In the strip $-n-1 < \operatorname{Re} \lambda < -n$ we have the result

$$(4.6)\quad \langle x_+^\lambda, \phi(x)\rangle = \int_0^\infty x^\lambda \{\phi(x) - \phi(0) - x\phi'(0) - \ldots - \frac{x^{n-1}}{(n-1)!}\phi^{(n-1)}(0)\}\, dx.$$

The function $\langle x_+^\lambda, \phi\rangle$ is an analytic function of the complex variable λ, having simple poles in the points $\lambda=-k$ $(k=1,2,\ldots)$ with residue $\frac{\phi^{(k-1)}(0)}{(k-1)!}$. This may be expressed by saying that the distribution x_+^λ has for $\lambda=-k$ a simple pole with residue

$$(4.7)\qquad \frac{(-1)^{k-1}}{(k-1)!}\delta^{(k-1)}(x),\ k=1,2,\ldots .$$

It follows from the principle of analytic continuation that

$$(4.8)\qquad \frac{dx_+^\lambda}{dx} = \lambda x_+^{\lambda-1},\ \lambda \neq -k\ (k = 1,\ 2,\ \ldots).$$

In the same way the distributions x_-^λ may be defined; this distribution corresponds for Re $\lambda>-1$ with the function

$$(4.9)\qquad x_-^\lambda = \begin{cases} |x|^\lambda & \text{for } x<0 \\ 0 & \text{for } x \geqslant 0. \end{cases}$$

Moreover, we have also the relation

$$(4.10)\qquad \langle x_-^\lambda,\ \phi(x)\rangle = \langle x_+^\lambda,\ \phi(-x)\rangle .$$

By means of this relation the properties of x_-^λ are easily derived from those of x_+^λ.

From the distributions x_+^λ and x_-^λ one can form the new distributions $|x|^\lambda$ and $|x|^\lambda$ sign x, defined as

$$(4.11)\qquad |x|^\lambda = x_+^\lambda + x_-^\lambda$$

$$(4.12)\qquad |x|^\lambda \text{ sign } x = x_+^\lambda - x_-^\lambda .$$

The distribution $|x|^\lambda$ has simple poles for $\lambda= -(2k+1)$ $(k=0,1,2,\ldots)$ with residues $2\,\frac{\delta^{(2k)}(x)}{(2k)!}$; the poles of x_+^λ and x_-^λ cancel each other in the points $\lambda=-2n$, $(n=1,2,\ldots)$. Hence the distribution $|x|^\lambda$ has a meaning for values of λ equal to $-2n$ and then it is written as x^{-2n}. The distribution $|x|^\lambda$ sign x has simple poles for $\lambda=-2k$ $(k=1,2,\ldots)$ with residues $-2\,\frac{\delta^{(2k-1)}(x)}{(2k-1)!}$; the poles of x_+^λ and x_-^λ cancel now each other in the points $\lambda=-(2n+1)$, $(n=0,1,2,\ldots)$ and so the distribution

$|x|^{\lambda}$ sign x has a meaning for values of λ equal to $-(2n+1)$; in this case it is written as $x^{-(2n+1)}$.

Therefore the distribution x^{-n} is defined for all integer values of n; moreover we have the relation

$$\frac{d}{dx} x^{-n} = -nx^{-n-1}, \tag{4.13}$$

which is in agreement with formula (3.17) of the foregoing section. The distributions $x_+^{\lambda}, x_-^{\lambda}, |x|^{\lambda}$ and $|x|^{\lambda}$ sign x may be normalized by considering the distributions

$$\frac{x_+^{\lambda}}{\lambda!}, \frac{x_-^{\lambda}}{\lambda!}, \frac{|x|^{\lambda}}{(\frac{\lambda-1}{2})!} \text{ and } \frac{|x|^{\lambda}\text{sign } x}{(\frac{\lambda}{2})!}. \tag{4.14}$$

These normalized distributions have the property that their "functional values" (e.g. $<\frac{x_+^{\lambda}}{\lambda!}, \phi(x)>$ are entire functions of the complex variable λ.

For instance we have the formulae

$$\left.\frac{x_+^{\lambda}}{\lambda!}\right|_{\lambda=-n} = \delta^{(n-1)}(x) \text{ and } \left.\frac{x_-^{\lambda}}{\lambda!}\right|_{\lambda=-n} = (-1)^{n-1}\delta^{(n-1)}(x). \tag{4.15}$$

Other important combinations of the distributions x_+^{λ} and x_-^{λ} are the distributions

$$(x \pm i0)^{\lambda} = \lim_{y \to \pm 0} (x^2+y^2)^{\frac{\lambda}{2}} e^{i\lambda \arg(x+iy)} = x_+^{\lambda} + e^{\pm i\lambda\pi} x_-^{\lambda} \tag{4.16}$$

with $-\pi < \arg(x+iy) \leq \pi$.

By expanding $<x_+^{\lambda},\phi>$ and $<e^{\pm i\lambda\pi} x_-^{\lambda},\phi>$ in a Laurent-series in the neighbourhood of the pole $\lambda=-n$ (n=1,2,...), one finds again that the poles occurring in both terms of the right hand side of (4.16) cancel each other. In this way one obtains the result

$$(x \pm i0)^{-n} = x^{-n} \mp \frac{i\pi(-1)^{n-1}}{(n-1)!} \delta^{(n-1)}(x). \tag{4.17}$$

The method of analytical continuation, which has been used in this section for defining integrals of functions with non integrable algebraic singularities, plays an important role in the calculus of distributions; it was already discovered by J. Hadamard, when he introduced the concept of the "finite" part of an integral ([20], Book III, Ch.I, section 2). The principle of analytical continuation will also be applied in order to define distributions which correspond to functions of several independent variables with algebraic singularities (see section 8 and Chapter IV, section 3).

5. The convolution of distributions

Let g(x) be some distribution belonging to D' or S'. To any test-function $\phi(x) \in$ D or S one may apply the operator g∗, which is called the convolution of g and ϕ and which is defined by

$$\psi(x) = g(x) * \phi(x) = \langle g(\xi), \phi(x+\xi)\rangle \quad (5.1)$$

The distribution g(x) is called a convolutor, if g∗ is a continuous linear operator in D respectively S. This means that $\phi \in$ D or S implies $g * \phi \in$ D resp. S and $\phi_m \to 0$ in D or S implies $g * \phi_m \to 0$ in D resp. S. The adjoint operator in D' or S' defines the convolution in D' or S'. Hence the convolution of a distribution f(x) with the convolutor g(x) is given by

$$\langle f * g, \phi\rangle = \langle f, \bar{g} * \phi\rangle, \quad (5.2)$$

or more explicitly

$$\langle f(x) * g(x), \phi(x)\rangle = \langle f(x), \langle g(\xi), \phi(x+\xi)\rangle\rangle . \quad (5.3)$$

It follows immediately, that, when g is a convolutor, also $\frac{\partial g}{\partial x_i}$ is a convolutor and one has the relation

$$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i} . \quad (5.4)$$

Theorem 6. Every distribution with bounded support belonging to D' or S' is a convolutor.

Proof: According to theorem 4 and 4^{bis} we may write the function $\psi(x) = g(x) * \phi(x)$ in the form

$$\psi(x) = \langle g(\xi), \phi(x+\xi)\rangle = \int_V G(\xi)\phi^{(p)}(x+\xi)d\xi, \tag{5.5}$$

where $G(\xi)$ is a continuous function, V a bounded region of R_n, containing the support of $g(x)$, and $\phi^{(p)}$ a derivative of order p of the test function $\phi(x)$.

From this equation it follows, that the operation g^* is a continuous linear mapping of D (or S) into itself. The proof of this statement for the case of D is simple; for the case of S it is a little bit complicated and the reader is referred to lit. [27], Ch.II, §2.2.

Example

$\delta(x)$ is a convolutor and has the relation

$$f(x) * \delta(x) = f(x). \tag{5.6}$$

It is possible to define the convolution product of two distributions which are not both convolutors in the strict sense as given above. The following conditions are each sufficient for defining a convolution product of distributions f and g belonging to D'.

1. The support of f or g is bounded.
2. The intersection of the supports of $f(x)$ and $g(\xi-x)$ is bounded for each finite ξ.

The convolution may again be defined by formula (5.2); however the C^∞ function $g(x) * \phi(x) = \langle g(\xi), \phi(x+\xi)\rangle = \langle g(\xi-x), \phi(\xi)\rangle$ need not to be a test function out of D, but due to the conditions 1 or 2 the intersection of the supports of $g(x) * \phi(x)$ and the distribution $f(x)$ is bounded and hence $g(x) * \phi(x)$ can be made a test function with the aid of a suitable cut-off factor.

In the case that f and g belong to S' the first condition is again sufficient for defining $f * g$, but the second one is no longer sufficient in general; however, there are important examples for which the second condition suffices to define the convolution $f * g$. In view of later applications (section 8.2 and section 9) we consider the example that f and g are distributions belonging to S' and concentrated respectively in the "forward" cones $x_1 \geq a\sqrt{x_2^2 + \ldots + x_n^2}$ and $x_1 \geq b\sqrt{x_2^2 + \ldots + x_n^2}$ with $a \geq 0$ and $b \geq 0$, but a and b not simultaneously zero (we take $b > 0$).

It is clear that the intersection of f(x) and g(ξ-x) is bounded for each finite ξ.

Using theorem 4^{bis} we have

$$f(x) = D^p F(x) \text{ and } g(x) = D^q G(x),$$

where F(x) and G(x) are continuous functions of finite algebraic growth at infinity and D^p and D^q denote generalized differentiation of order $|p|$ respectively $|q|$ (confer formula (3.22)).

Therefore we obtain formally:

$$\langle f(x) * g(x), \phi(x)\rangle = \langle f(x), \langle g(\xi), \phi(x+\xi)\rangle\rangle =$$

$$= (-1)^{|p|} \int_{-\infty}^{+\infty} \overline{F(x)}\; D_x^p \langle g(\xi), \phi(x+\xi)\rangle dx =$$

$$= (-1)^{|p|} \int_{-\infty}^{+\infty} \overline{F(x)} \langle g(\xi), D_x^p\phi(x+\xi)\rangle dx =$$

$$= (-1)^{|p|+|q|} \int_{-\infty}^{+\infty} \overline{F(x)} \left\{\int_{-\infty}^{+\infty} \overline{G(\xi)}\; D_\xi^q\, D_x^p\phi(x+\xi)d\xi\right\}dx,$$

or according to Fubini's theorem:

$$(5.7)\quad \langle f(x) * g(x), \phi(x)\rangle = (-1)^{|p|+|q|} \int_{-\infty}^{+\infty} D^{p+q}\phi(\xi) \left\{\int_{-\infty}^{+\infty} \overline{F(x)}\;\overline{G(\xi-x)}\; dx\right\} d\xi .$$

The region of integration of the inner integral is certainly contained within the "backward" cone $\xi_1 - x_1 \geq b\sqrt{(\xi_2-x_2)^2 + \ldots + (\xi_n-x_n)^2}$ with $x_1 \geq 0$; hence it is also contained in a sphere with centre ξ and radius $\xi_1 \sqrt{1 + \frac{1}{b^2}}$. Because moreover F(x) and G(x) are continuous functions of finite algebraic growth for $|x| \to \infty$, it follows that the

inner integral of (5.7) will also be of finite algebraic growth for $|\xi| \to \infty$. Finally due to the strong decrease of $\phi(\xi)$ at infinity the convolution product $\langle f(x) * g(x), \phi(x)\rangle$ has a meaning for all $\phi(x) \in S$ and it may be represented by the right hand side of (5.7).
For all convolution products in case they exist the following useful formulae hold:

(5.8) $$f * g = g * f$$

(5.9) $$(f * g) * h = f * (g * h)$$

(5.10) $$\frac{\partial}{\partial x_i}(f * g) = f * \frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} * g.$$

The proof is given by writing the convolution products in the form of (5.7); (5.8) and (5.10) are shown to be valid by interchanging F and G and (5.9) by applying Fubini's theorem.

The convolution operator in the strict sense is not only a continuous operator in the space of test functions, but also in the space of distributions. This is expressed in the following theorem.

Theorem 7. If a sequence of distributions f_m converges in D' or S' to the distribution f, then also the sequence $f_m * g$ converges in D' resp. S' to the distribution $f * g$, if one of the following conditions is satisfied:

1. The distributions f_m have a uniformly bounded support.
2. The distribution g has a bounded support.

Proof: $\lim_{m\to\infty} \langle f_m * g, \phi\rangle = \lim_{m\to\infty} \langle f_m, g * \phi\rangle = \langle f, g * \phi\rangle = \langle f * g, \phi\rangle$.

This theorem has an important consequence, which will be given in the next theorem.

Theorem 8. Each distribution out of D' or S' may be obtained as the limit of a sequence of test functions belonging to D.

Proof: Consider the sequence of test functions $\phi_m(x) \in D$, defined by the formula

$$\phi_m(x) = \begin{cases} \frac{1}{K_m} \exp\left(\frac{m^2r^2}{m^2r^2-1}\right) & \text{for } r = \sqrt{x_1^2+\ldots+x_n^2} < \frac{1}{m} \\ 0 & \text{for } r \geq \frac{1}{m} \end{cases}$$

with $K_m = \int_{r<m^{-1}} \exp\left(\frac{m^2r^2}{m^2r^2-1}\right)dx$; $m=1,2,\ldots$.

It is obvious that the sequence $\{\phi_m(x)\}$ has the distributional limit $\delta(x)$. Therefore, according to theorem 7, every distribution $f(x) \in D'$ or S' satisfies the relation

$$f(x) = \lim_{m\to\infty} \phi_m(x) * f(x). \tag{5.11}$$

The functions $\psi_m(x) = \phi_m(x) * f(x)$ are C^∞ functions; introducing finally the infinitely differentiable cut-off factor $e(x)$ which is identically 1 for $r \leq 1$ and which vanishes for $r \geq 2$, the functions $e(\frac{x}{m})\ \psi_m(x)$ (with $m=1,2,\ldots$) constitute a sequence of test functions out of D', which meets the requirements of the theorem.

It may be remarked that Sobolev constructed already in 1936 the same sequence of functions $\phi_m(x)$ in order to obtain a sequence of smooth functions converging weakly to an arbitrary integrable function [4].

<u>Example</u>

Let us consider the integral equation of Abel, viz.:

$$g(x) = \frac{1}{(-\alpha)!}\int_0^x (x-\xi)^{-\alpha}\ f(\xi)d\xi, \tag{5.12}$$

with $x > 0$ and $0 < \alpha < 1$.

As is well known, the solution of this equation is given by the expression:

$$f(x) = \frac{1}{(\alpha-1)!}\int_0^x (x-\xi)^{\alpha-1}\ \frac{dg(\xi)}{d\xi}\ d\xi. \tag{5.13}$$

([21], p.229).

Considering the functions f and g as distributions, the solution (5.13) follows very easily from (5.12) and also the restriction $0 < \alpha < 1$ can be released. For this purpose we introduce the distribution of one independent variable $\Phi_\lambda(x)$, defined as:

$$\Phi_\lambda(x) = \frac{x_+^{\lambda-1}}{(\lambda-1)!} \quad , \tag{5.14}$$

where λ is an arbitrary complex number (see section 4).
These distributions enjoy the following properties:

$$\Phi_\lambda * \Phi_\mu = \Phi_{\lambda+\mu}, \tag{5.15}$$

$$\frac{d}{dx}\Phi_\lambda = \Phi_{\lambda-1}, \tag{5.16}$$

$$\Phi_{-n} = \delta^{(n)}(x),\ n=0,1,2,\ldots\ . \tag{5.17}$$

The relations (5.16) and (5.17) follow from the theory of section 4; the relation (5.15) may be verified by proving it first for $\mathrm{Re}\,\lambda > 0$ and $\mathrm{Re}\,\mu > 0$ and applying consecutively the principle of analytical continuation.

Abeld integral equation, in generalized form, may now be written as:

$$g(x) = f(x) * \Phi_{1-\alpha}, \tag{5.18}$$

where α may be any arbitrary complex number.
Taking the convolution of both sides of the equation (5.18) with the distribution $\Phi_{\alpha-1}$, we obtain immediately

$$f(x) = g(x) * \Phi_{\alpha-1} = \frac{dg}{dx} * \Phi_\alpha, \tag{5.19}$$

which is a generalization of the solution (5.13).
It will appear in section 10, that the distributions Φ_λ with the properties (5.15) - (5.17) can be generalized to distributions in more variables.

6. The Fourier transformation

6.1. General theory

Let $\phi(x)$ be a test function of D and vanishing for $|x_i| \geq a_i$, $i=1,2,\ldots,n$.
Its Fourier transform is defined by

$$(6.1) \qquad \psi(s) = F[\phi(x)] = \int_{-a}^{+a} e^{is\xi}\, \phi(\xi)d\xi,$$

with $s\xi = s_1\xi_1 + s_2\xi_2 + \ldots + s_n\xi_n$ and $s = \sigma+i\tau$.
After some considerations, involving only classical analysis, it follows from (6.1), that $\psi(s)$ belongs to the space Z and, moreover, when $\phi(x) \to 0$ in D then also $\psi(s) \to 0$ in Z. (Confer lit. [7], Vol.I, Ch.II, §1.1 - §1.3).
Conversely, we have the inverse transformation

$$(6.2) \qquad \phi(x) = F^{-1}[\psi(s)] = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} e^{-i\sigma x}\psi(\sigma)d\sigma,$$

with $\sigma x = \sigma_1x_1 + \ldots + \sigma_nx_n$; it follows that $\psi(s) \to 0$ in Z implies $\phi(x) \to 0$ in D. Hence the Fourier transformation is a continuous linear 1-1 mapping of D onto Z and conversely.

Let $\phi(x)$ now belong to the space S and consider its Fourier transform

$$(6.3) \qquad \psi(\sigma) = F[\phi(x)] = \int_{-\infty}^{+\infty} e^{i\sigma x}\phi(x)dx,$$

with $\sigma x = \sigma_1x_1 + \sigma_2x_2 + \ldots + \sigma_nx_n$; $\sigma_1,\sigma_2,\ldots,\sigma_n$ are real.
The inverse transformation is given by

$$(6.4) \qquad \phi(x) = F^{-1}[\psi(\sigma)] = \frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} e^{-i\sigma x}\psi(\sigma)d\sigma.$$

Using the definition (6.3) and (6.4) it is not difficult to show, that the Fourier transformation and its inverse are also continuous linear 1-1 mappings of S onto itself (confer lit. [7], Ch.II, §1.6).

The Fourier transform of a test function ϕ will also be denoted by the symbol $\tilde{\phi}$.

The Fourier transform $F[f]$ or $\tilde{f}$ of a distribution f D' is defined by Parseval's equality, viz.

$$(6.5) \qquad \langle f(x), \phi(x)\rangle = \frac{1}{(2\pi)^n} \langle \tilde{f}(s), \tilde{\phi}(s)\rangle,$$

where $\tilde{\phi}(s) \in Z$ and $\tilde{f}(s) \in Z'$.
The Fourier transform of distributions of S' are defined in the same way:

$$(6.6) \qquad \langle f(x), \phi(x)\rangle = \frac{1}{(2\pi)^n} \langle \tilde{f}(\sigma), \tilde{\phi}(\sigma)\rangle,$$

where $\tilde{\phi}(\sigma)$ and $\tilde{f}(\sigma)$ belong again to S respectively S'.
It is clear, that these definitions are in accordance with the classical theory in case, that f(x) is an absolutely integrable function. Therefore the Fourier transformation, as defined by (6.5) or (6.6) for distributions, generalizes the operator of the Fourier transformation for functions which are no longer absolutely integrable.

Let $\{f_m(x)\}$ be a sequence, which converges in D' or S' to the distribution f. It follows from (6.5) and (6.6), that now also the sequence $\{\tilde{f}_m\}$ converges in Z' or respectively S' to the distribution $\tilde{f}$. Also the converse is true, namely $\tilde{f}_m \to \tilde{f}$ implies $f_m \to f$. Hence we have the following important theorem:

<u>Theorem 9</u>. The Fourier transformation is a continuous linear 1-1 mapping of the spaces D, S, D' and S' onto respectively the spaces Z, S, Z' and S'; the same is true for the inverse transformation.

For all test functions belonging to D we have the following well-known relations

$$(6.7) \qquad F\left[P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)\phi(x)\right] = P(-is_1,-is_2,\ldots,-is_n)F[\phi(x)]$$

$$(6.8) \qquad F[P(ix_1,ix_2,\ldots,ix_n)\phi(x)] = P\left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \ldots, \frac{\partial}{\partial s_n}\right)F[\phi(x)],$$

where $P(x_1, x_2, \ldots, x_n)$ is an arbitrary polynomial in n independent variables. By transposition we obtain for distributions of D' the same results, viz.

(6.9) $$F\left[P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right)f(x)\right] = P(-is_1, -is_2, \ldots, -is_n)F\left[f(x)\right]$$

(6.10) $$F\left[P(ix_1, ix_2, \ldots, ix_n)f(x)\right] = P\left(\frac{\partial}{\partial s_1}, \frac{\partial}{\partial s_2}, \ldots, \frac{\partial}{\partial s_n}\right)F\left[f(x)\right].$$

For test functions and distributions of S and S' one has of course the same formulae, if one replaces the complex variables s_i by real variables, say σ_i.

Other useful formulae which can be derived easily from the definitions are

(6.11) $$F\left[f(x-a)\right] = e^{+ia\cdot s} \cdot \tilde{f}(s)$$

(6.12) $$F\left[e^{ia\cdot x}f(x)\right] = \tilde{f}(s + a)$$

(6.13) $$F\left[f(Ax)\right] = \frac{1}{|\mathrm{Det}\ A|}\tilde{f}((A^T)^{-1}s),$$

where A is a non singular linear transformation of the independent variables $x_1, x_2, \ldots, x_n$ and A^T the transpose of A. $a\cdot s$ and $a\cdot x$ denote resp. $a_1s_1 + a_2s_2 + \ldots + a_ns_n$ and $a_1x_1 + a_2x_2 + \ldots + a_nx_n$.

Examples

Using the definition (6.5) and the rules (6.9) and (6.10) - (6.12) one verifies easily the following formulae

$$F\left[\delta(x)\right] = 1 \quad , \quad F\left[\delta(x-a)\right] = e^{ia\cdot s}$$

$$F\left[1\right] = (2\pi)^n\ \delta(s) \quad , \quad F\left[e^{ia\cdot x}\right] = (2\pi)^n\ \delta(s+a)$$

$$F\left[P(x)\right] = (2\pi)^n P\left(-i\frac{d}{ds}\right)\delta(s) \quad , \quad F\left[\cos(a\cdot x)\right] = \frac{(2\pi)^n}{2}\{\delta(s+a)+\delta(s-a)\}$$

$$F\left[P\left(\frac{d}{dx}\right)\delta(x)\right] = P(-is) \quad , \quad F\left[\sin(a\cdot x)\right] = \frac{(2\pi)^n}{2i}\{\delta(s+a)-\delta(s-a)\}.$$

6.2. The structure of distributions belonging to Z'

The distributions in D' and Z' are related to each other by Parseval's rule (6.5). Hence it follows, that the properties of the distributions in Z' are completely determined by those in D'. In this section the structure of the distributions in Z' will be derived from the fact, that every distribution in D' is the derivative of a continuous function.

Every distribution $g(s) \in Z'$ may be represented as follows

$$(6.15) \qquad \langle g(s), \psi(s)\rangle = (2\pi)^n \langle f(x), \phi(x)\rangle = (2\pi)^n \int_V \overline{F(x)} D^m \phi(x)dx,$$

where $f(x) = F^{-1}[g(s)]$, $\phi(x) = F^{-1}[\psi(s)]$ and $f(x) = (-1)^m D^m F(x)$ with $F(x)$ continuous; V is a finite region in R_n containing the support of $\phi(x)$. Instead of $D^m \phi(x) = \frac{\partial^m}{\partial x_1^{m_1},\dots,\partial x_n^{m_n}} \phi(x)$ we may write

$$(6.16) \qquad D^m \phi(x) = \frac{1}{(2\pi)^n} \int_{R_n} (-i\sigma)^m \psi(\sigma) e^{-i\sigma x} d\sigma ,$$

where $\sigma^m = \sigma_1^{m_1} \sigma_2^{m_2} \dots \sigma_n^{m_n}$, $\sigma_1, \sigma_2, \dots, \sigma_n$ are real.

Substitution of (6.16) into (6.15) and application of Fubini's theorem yields

$$(6.17) \qquad \langle g(s), \psi(s)\rangle = \int_{R_n} (-i\sigma)^m \psi(\sigma)d\sigma \int_V e^{-i\sigma x} \overline{F(x)}dx.$$

The function $G(\sigma) = \int_V e^{i\sigma x} F(x)dx$ is a bounded analytic function of σ, which can be continued analytically in the whole complex n-dimensional space C_n.

Hence we obtain the result

$$(6.18) \qquad \langle g(s), \psi(s)\rangle = \int_{R_n} \overline{H(\sigma)} \psi(\sigma)d\sigma ,$$

where $H(\sigma) = (i\sigma)^m G(\sigma)$ is an analytic function of σ, which is of finite algebraic growth at infinity and which can be continued analytically in the whole space C_n.

Thus we have the theorem

Theorem 10. Every distribution in the space Z' may be written in the form:

$$(6.19) \qquad \langle g(s), \psi(s)\rangle = \int_{R_n} \overline{H(\sigma)} \psi(\sigma)d\sigma = \langle H(\sigma), \psi(\sigma)\rangle ,$$

where $H(s) = H(\sigma+i\tau)$ is an entire complex function of s and $H(\sigma)$ is of finite algebraic growth for $|\sigma| \to \infty$.

6.3. The Fourier transform of distributions with compact support

The Fourier transform $\tilde{f}(\sigma)$ of a distribution $f(x) \in S'$ with bounded support V may be represented as follows:

$$(6.20) \qquad <\tilde{f}(\sigma), \tilde{\phi}(\sigma)> = (2\pi)^n <f(x), \phi(x)> =$$

$$= \qquad = (2\pi)^n \int_V \overline{F(x)}\, D^m \phi(x)dx,$$

where F(x) is a continuous function.
Substituting again

$$D^m \phi(x) = \frac{1}{(2\pi)^n} \int_{R_n} (-i\sigma)^m \tilde{\phi}(\sigma)e^{-i\sigma x} d\sigma ,$$

we obtain

$$(6.21) \qquad \tilde{f}(\sigma) = F\left[f(x)\right] = (+i\sigma)^m \int_V e^{i\sigma x} F(x)dx.$$

We introduce now a test function $\phi(x) = \left[e^{i\sigma x}\right]$, defined as

$\left[e^{i\sigma x}\right] = e^{i\sigma x}$ for $x \in V$, and
$\left[e^{i\sigma x}\right] = 0$ for $x \notin V+\varepsilon$,

where $V+\varepsilon$ is an arbitrary neighbourhood of V.
With this convention (6.21) may be written in the form

$$(6.22) \qquad F\left[f(x)\right] = <\overline{F(x)}, D^m \left[e^{i\sigma x}\right]> = <\overline{f(x)}, \left[e^{i\sigma x}\right]> .$$

Hence we may state the theorem:

Theorem 11. A distribution f(x), belonging to S' and having bounded support, has a Fourier transform $\tilde{f}(\sigma)$, which may be written as:

$$\tilde{f}(\sigma) = <\overline{f(x)}, \left[e^{i\sigma x}\right]> .$$

This Fourier transform is an analytical function of σ, which is of finite algebraic growth for $|\sigma| \to \infty$ and which can be continued analytically in C_n. The analytical continuation is the entire function:

(6.23) $$\tilde{f}(s) = (+is)^m \int_V e^{isx} F(x)dx,$$

with $s = \sigma + i\tau$.

The result (6.23) is of course also valid for distributions with bounded support, which belong to D'.

6.4. Multiplication and convolution

One of the very useful properties of the Fourier transformation is, that in the case of two square integrable functions convolution is transformed into multiplication and vice versa.

This property holds also in the case of two distributions, for which the convolution or the multiplication may be performed.

A function f(x) is called a multiplicator in S', if for every test function $\phi(x) \in S$ the function $\overline{f(x)}\ \phi(x)$ also belongs to S, while $\overline{f}\cdot\phi_m \to 0$ in S, whenever $\phi_m \to 0$ in S. A multiplicator in S' is a C^∞ function of finite algebraic growth at infinity.

Consider a distribution $f(x) \in S'$ with compact support; according to theorem 6 the distribution f(x) is a convolutor and according to theorem 11 its Fourier transform $\tilde{f}(\sigma)$ is a multiplicator. Moreover, by applying Parseval's rule one obtains

$$<f(\xi),\ \phi(\xi+x)> = \frac{1}{(2\pi)^n} <\tilde{f}(\sigma),\ e^{-ix\sigma}\ \tilde{\phi}(\sigma)> =$$

$$\frac{1}{(2\pi)^n}\int_{-\infty}^{+\infty} e^{-ix\sigma}\ \overline{\tilde{f}(\sigma)}\ \tilde{\phi}(\sigma)d\sigma = F^{-1}\left[\ \overline{\tilde{f}(\sigma)}\cdot\tilde{\phi}(\sigma)\right] ,$$

or

(6.24) $$F\left[f(x) * \phi(x)\right] = \overline{\tilde{f}(\sigma)}\cdot\tilde{\phi}(\sigma).$$

Hence for every distribution $g(x) \in S'$, we have the relation

$$<g(x) * f(x),\ \phi(x)> = <g(x),\ f(x) * \phi(x)> =$$

$$= \frac{1}{(2\pi)^n} <\tilde{g}(\sigma),\ \overline{\tilde{f}(\sigma)} \cdot \tilde{\phi}(\sigma)> = \frac{1}{(2\pi)^n} <\tilde{g}(\sigma)\ \tilde{f}(\sigma),\ \tilde{\phi}(\sigma)> \text{ , or}$$

(6.25) $$F\left[g(x) * f(x)\right] = \tilde{g}(\sigma) \cdot \tilde{f}(\sigma).$$

This formula shows, that the Fourier transformation transforms convolution into multiplication, if the convolutor is a distribution of bounded support.

Conversely, let us consider now a function f(x), which is a multiplicator in S'; it follows, that $\overline{f(x)} \cdot \phi(x)$ is a test function in S, whenever $\phi(x) \in S$. Therefore, we may write

$$F\left[\overline{f(x)}\ \phi(x)\right] = \int_{-\infty}^{+\infty} e^{ix\sigma}\ \overline{f(x)}\ \phi(x)\,dx = <f(x),\ e^{ix\sigma}\phi(x)> =$$

$$= \frac{1}{(2\pi)^n} <\tilde{f}(\sigma'),\ \tilde{\phi}(\sigma+\sigma')> \text{ , or}$$

(6.26) $$F\left[\overline{f(x)}\ \phi(x)\right] = \frac{1}{(2\pi)^n}\ \tilde{f}(\sigma) * \tilde{\phi}(\sigma).$$

Because the Fourier transformation is a linear continuous 1-1 mapping of S onto itself it follows now immediately from (6.26) that $\tilde{f}(\sigma)$ is a convolutor in S'.

Hence for every distribution $g(x) \in S'$ we get the relation

$$<g(x)\ f(x),\ \phi(x)> = <g(x),\ \overline{f(x)}\ \phi(x)> =$$

$$= \frac{1}{(2\pi)^n} <\tilde{g}(\sigma),\ \frac{1}{(2\pi)^n}\ \tilde{f}(\sigma) * \tilde{\phi}(\sigma)> =$$

$$= \frac{1}{(2\pi)^{2n}} <\tilde{g}(\sigma) * \tilde{f}(\sigma),\ \tilde{\phi}(\sigma)> \text{ , or}$$

(6.27) $$F\left[g(x)\ f(x)\right] = \frac{1}{(2\pi)^n}\ \tilde{g}(\sigma) * \tilde{f}(\sigma).$$

This formula shows, that the Fourier transformation transforms always multiplication into convolution.

Summarizing, we have obtained the theorem

Theorem 12. If f(x) is a convolutor of bounded support in S', then its Fourier transform is a multiplicator in S' and we have the relation

$$F\left[f * g\right] = F\left[f\right] \cdot F\left[g\right] .$$

If f(x) is a multiplicator in S', then its Fourier transform is a convolutor in S' and we have the relation

$$F\left[f \cdot g\right] = \frac{1}{(2\pi)^n} F\left[f\right] * F\left[g\right] .$$

In general not all distributions are multiplicators and so in general we cannot multiply distributions [22] . This is reflected in the space of the Fourier transforms; not all distributions are convolutors and so we cannot always form the convolution product of two arbitrary distributions.
However, in practical calculations, such as e.g. in electrodynamics, multiplications of distributions occur; the formal Fourier transforms of these products give rise to the appearance of convolution integrals which diverge (see chapter V).
For a more profound treatment of the connection of convolution and multiplication the reader is referred to [6] , Vol.II, ch. VII, §5, p.99 and §8, p. 124.

6.5. Some examples of Fourier transforms

In section 4 we have treated distributions corresponding with functions with an algebraic singularity. Their Fourier transforms are given in this section; the results are equally valid, whether they are considered as distributions in D' or S'. The proofs are omitted; they may be found in [7] , Vol.I, ch.II, §2, p.167-171.

$$F\left[\frac{x_+^\lambda}{\lambda!}\right] = i\, e^{i\lambda\frac{\pi}{2}} (\sigma+i0)^{-\lambda-1},$$

$$F\left[\frac{x_-^\lambda}{\lambda!}\right] = -i\, e^{-i\lambda\frac{\pi}{2}} (\sigma-i0)^{-\lambda-1},$$

$$F\left[\theta(x)\right] = \frac{i}{\sigma} + \pi\delta(\sigma),$$

$$F\left[x^{-m}\right] = \frac{i^m \pi}{(m-1)!}\, \sigma^{m-1}\, \mathrm{sign}.\sigma \;,\; m=1,2,\ldots .$$

7. Distributions on surfaces

Distributions concentrated on hypersurfaces in n-dimensional space are very important for the applications, as will appear later in sections 8 and 9 of this chapter and in chapter IV. We introduce these distributions using an elegant method due to R.T. Seeley [23] .

The distribution $\Theta(P)$ is defined as

$$(7.1) \qquad \langle\Theta(P),\ \phi(x)\rangle = \int_{P\geq 0} \phi(x)dx,$$

where $P(x_1,x_2,\dots,x_n) = 0$ is some surface in R_n and P is a C^∞ function with $\nabla P = (\frac{\partial P}{\partial x_1}, \frac{\partial P}{\partial x_2}, \dots, \frac{\partial P}{\partial x_n})$ nowhere zero on $\{P=0\}$.

The distribution $\delta(P)$ is now introduced with the aid of the distribution $\Theta(P)$, viz.

$$(7.2) \qquad \langle\delta(P),\ \phi(x)\rangle = \lim_{c\to 0} \frac{1}{c} \langle\Theta(P+c) - \Theta(P),\ \phi(x)\rangle =$$

$$\lim_{c\to 0} \frac{1}{c} \int_{-c\leq P<0} \phi(x_1,x_2,\dots,x_n)dx_1dx_2\dots dx_n .$$

The existence of this limit presents no difficulty, since we may write for the latter integral (see fig.1)

$$(7.3) \qquad \langle\delta(P),\ \phi(x)\rangle = \lim_{c\to 0} \frac{1}{c} \int_{P=0} \phi\cdot c \frac{d\sigma}{|\nabla P|} = \int_{P=0} \phi(x_1,x_2,\dots,x_n) \frac{d\sigma}{|\nabla P|},$$

where $d\sigma$ is the surface measure on $\{P=0\}$ and $|\nabla P| = \sqrt{(\nabla P,\nabla P)} \neq 0$; $dx_1dx_2\dots dx_n = \gamma d\sigma = c\frac{d\sigma}{|\nabla P|}$.

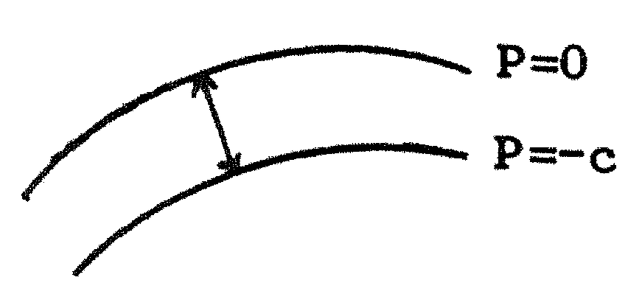

fig.1

Examples

1. Consider the distribution $\delta(x_o^2-r^2)$, where $r^2 = x_1^2+x_2^2+x_3^2$ and x_o is some parameter. x_1,x_2,x_3 may be regarded as space coordinates and x_o as a parameter, denoting the time. Hence $\delta(x_o^2-r^2)$ is concentrated on a sphere in R_3. Using polar coordinates $x_i = r\omega_i$ (i=1,2,3), we obtain $d\sigma = r^2d\Omega$ with $d\Omega$ the surface element of the unit sphere in R_3 and $|\nabla P| = 2r$.

Substitution into (9.3) yields

$$(7.4)\quad \langle\delta(x_o^2-r^2),\ \phi(x_1,x_2,x_3)\rangle = \tfrac{1}{2}\int_{r=|x_o|}\phi(r\omega_1,r\omega_2,r\omega_3)\,\frac{r^2 d\Omega}{r} =$$

$$= \tfrac{1}{2}|x_o|\int_\Omega \phi(|x_o|\omega_1,|x_o|\omega_2,|x_o|\omega_3)\,d\Omega\ .$$

2. Consider the 4-dimensional distribution $\delta(x^2-m^2)$ where $x^2 = x_o^2-x_1^2-x_2^2-x_3^2$ and m is some constant. x_1,x_2,x_3 may be regarded as space coordinates and x_o as time coordinate. This distribution is concentrated on a hyperboloid in R_4. Instead of x_1,x_2,x_3 we use again spherical coordinates $x_i = r\omega_i$ (i=1,2,3) with $r = \sqrt{x_1^2+x_2^2+x_3^2}$; hence $dx_1dx_2dx_3 = r^2 dr\ d\Omega$.

Instead of x_o we take the new coordinate $P = x_o^2-r^2-m^2$ and thus $dx_o dx_1 dx_2 dx_3 = \pm\tfrac{1}{2}(P+r^2+m^2)^{-\frac{1}{2}}r^2 dr\ dP\ d\Omega$. The distribution $\Theta(P)$ can now be written as

$$\langle\Theta(P),\ \phi(x)\rangle = \tfrac{1}{2}\int_{P\geq 0}(P+r^2+m^2)^{-\frac{1}{2}}r^2\,\phi\,dr\ dP\ d\Omega\ .$$

Hence

$$\frac{1}{c}\langle\Theta(P+c)-\Theta(P),\ \phi(x)\rangle = \frac{1}{2c}\int_{-c\leq P<0}(P+r^2+m^2)^{-\frac{1}{2}}r^2\,\phi\,dr\ dP\ d\Omega\ ;$$

after taking the limit with $c\to 0$ we obtain finally:

$$\langle\delta(x^2-m^2),\ \phi(x)\rangle = \tfrac{1}{2}\int_{x^2-m^2=0}(r^2+m^2)^{-\frac{1}{2}}r^2\,\phi\,dr\ d\Omega =$$

$$= \tfrac{1}{2}\int_0^\infty\int_\Omega (r^2+m^2)^{-\frac{1}{2}}r^2\ \phi(r\omega_1,r\omega_2,r\omega_3,\ +\sqrt{r^2+m^2})\,dr\ d\Omega\ +$$

$$+\ \tfrac{1}{2}\int_0^\infty\int_\Omega (r^2+m^2)^{-\frac{1}{2}}r^2\ \phi(r\omega_1,r\omega_2,r\omega_3,\ -\sqrt{r^2+m^2})\,dr\ d\Omega\ .$$

Performing the integration with respect to $d\Omega$ we get

$$(7.5)\quad \langle\delta(x^2-m^2),\ \phi(x)\rangle = \tfrac{1}{2}\int_0^\infty (r^2+m^2)^{-\frac{1}{2}}r^2\ \overline{\phi}(r,+\sqrt{r^2+m^2})\,dr\ +$$

$$+\ \tfrac{1}{2}\int_0^\infty (r^2+m^2)^{-\frac{1}{2}}r^2\ \overline{\phi}(r,-\sqrt{r^2+m^2})\,dr,$$

where $\overline{\phi}(r,x_o) = \int_\Omega \phi(x_1,x_2,x_3,x_o)\,d\Omega$;

i.e. $\overline{\phi}(r,x_o)$ is, apart from a constant, the mean value of ϕ on a sphere with radius r in (x_1,x_2,x_3)-space.
If in (7.5) we take the limit for $m \to 0$, we obtain a distribution concentrated on the "light cone" $x_o^2 = r^2$, viz.

$$(7.6) \qquad <\delta(x^2),\ \phi(x)> = \tfrac{1}{2}\int_0^\infty r\ \overline{\Phi}(r,+r)\,dr + \tfrac{1}{2}\int_0^\infty r\ \overline{\phi}(r,-r)\,dr.$$

The distributions $\delta_+(x^2-m^2)$ and $\delta_-(x^2-m^2)$, which are concentrated only on the upper respectively the lower sheet of the hyperboloid $x^2-m^2=0$, are given by the formulae

$$(7.7) \qquad <\delta_+(x^2-m^2),\ \phi(x)> = \tfrac{1}{2}\int_0^\infty (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \overline{\Phi}(r,+\sqrt{r^2+m^2})\,dr,$$

and

$$(7.8) \qquad <\delta_-(x^2-m^2),\ \phi(x)> = \tfrac{1}{2}\int_0^\infty (r^2+m^2)^{-\frac{1}{2}}\ r^2\ \overline{\Phi}(r,-\sqrt{r^2+m^2})\,dr.$$

The distribution $\delta(x^2)$ can be splitted in the same way into two distributions $\delta_+(x^2)$ and $\delta_-(x^2)$, which are concentrated on the forward respectively backward "light cone"; they are given by the expressions

$$(7.9) \qquad <\delta_+(x^2),\ \Phi(x)> = \tfrac{1}{2}\int_0^\infty r\ \overline{\Phi}(r,+r)\,dr,$$

and

$$(7.10) \qquad <\delta_-(x^2),\ \Phi(x)> = \tfrac{1}{2}\int_0^\infty r\ \overline{\Phi}(r,-r)\,dr.$$

The derivatives $\delta^{(k)}(P)$ of the distribution $\delta(P)$ with $|\nabla P| \neq 0$ on $\{P = 0\}$ are defined by the rule

$$(7.11) \qquad \delta^{(k+1)}(P) = \lim_{c \to 0} \frac{1}{c}\left[\delta^{(k)}(P+c)-\delta^{(k)}(P)\right], \quad k=0,1,2,\ldots .$$

<u>Example</u>

We calculate the first derivative of the three-dimensional distribution $\delta(x_o^2-r^2)$, where x_o is considered as a parameter.

$$\langle\delta^{(1)}(x_o^2-r^2),\phi\rangle =$$

(7.12)
$$\lim_{c\to 0} \tfrac{1}{2}\int_\Omega \frac{1}{c}\{\sqrt{x_o^2+c}\cdot\phi(\sqrt{x_o^2+c}\cdot\omega_1,\sqrt{x_o^2+c}\cdot\omega_2,\sqrt{x_o^2+c}\cdot\omega_3) -$$
$$- |x_o|\cdot\phi(|x_o|\omega_1,|x_o|\omega_2,|x_o|\omega_3\}d\Omega =$$
$$= \tfrac{1}{2}\int_\Omega \Big[\frac{1}{2r}\frac{\partial}{\partial r}\{r\cdot\phi(r\omega_1,r\omega_2,r\omega_3)\}\Big]_{r=|x_o|} d\Omega .$$

The derivative of $\delta^{(k)}\{P(x_1,x_2,\ldots,x_n)\}$ with respect to one of the independent variables x_i is given by the chain rule

(7.13)
$$\frac{\partial}{\partial x_i}\delta^{(k)}(P) = \delta^{(k+1)}(P)\frac{\partial P}{\partial x_i}, \quad |\nabla P| \neq 0 \text{ on } \{P\} = 0.$$

It follows from (7.3) that $P\,\delta(P) = 0$; repeated differentiation of this equation with respect to P yields the useful formula

(7.14)
$$P\,\delta^{(k)}(P) = -k\,\delta^{(k-1)}(P),\quad k=0,1,2,\ldots .$$

8. Regularization of functions of several independent variables with an algebraic singularity

8.1. The distributions r^λ

In section 4 we studied the regularization of functions of one variable with an algebraic singularity; this led to the introduction of the distributions x_+^λ, x_-^λ, $|x|^\lambda$ etc.

In this section we consider a generalization in so far as we deal now with the regularization of functions of more variables.

A first example is the function r^λ with $r = \sqrt{x_1^2+x_2^2+\ldots+x_n^2}$. The regularization of this function is given by the distribution r^λ, defined for Re $\lambda > -n$ by the integral

(8.1)
$$\langle r^\lambda,\phi\rangle = \int_{R_n} r^\lambda\,\phi(x)dx.$$

For values of λ with $\mathrm{Re}\lambda \leq -n$ the distribution r^λ is defined by the analytical continuation of $\langle r^\lambda, \phi \rangle$ with respect to λ. For this purpose we write (8.1) in the form

$$\langle r^\lambda, \phi \rangle = \Omega_n \int_0^\infty r^{\lambda+n-1} S_\phi(r)\,dr, \tag{8.2}$$

where $S_\phi(r)$ is the mean value of the function $\phi(x)$ on the surface of a sphere in R_n with center in the origin and radius r; Ω_n is the area of the unit sphere in R_n.
The analytical continuation of $\langle r^\lambda, \phi \rangle$ into the region $\mathrm{Re}\lambda \leq -n$ proceeds now in more or less the same way as in section 4. We omit the details of the calculations; for this the reader is referred to [7] , Vol.I, ch.I, §3.9, p.78.

The result is that the distribution $\langle r^\lambda, \phi \rangle$ can be defined in the whole complex λ-plane with the exception of the points $\lambda = -n-2k$ $(k=0,1,2,\ldots)$, where $\langle r^\lambda, \phi \rangle$ has simple poles with residues

$$\Omega_n \frac{S_\phi^{(2k)}(0)}{(2k)!} . \tag{8.3}$$

The distributions r^λ may be normalized by introducing

$$R^\lambda = \frac{2r^\lambda}{\Omega_n \left(\frac{\lambda+n-2}{2} \right)!} . \tag{8.4}$$

The functional $\langle R^\lambda, \phi \rangle$ is an entire function of λ; using (8.3) we obtain the formula

$$R^{-n} = \delta(x). \tag{8.5}$$

By repeated application of the Laplace operator Δ, it can easily be verified, that

$$2^k(\lambda+2)(\lambda+4)\ldots(\lambda+2k)R = \Delta^k R^{\lambda+2k}, \quad k = 1, 2, \ldots \tag{8.6}$$

and hence

$$(8.7) \qquad R^{-n-2k} \approx \frac{(-1)^k \, \Delta^k \delta(x)}{(+2)^k n(n+2)\dots(n+2k-2)} \quad , \; k = 1,\, 2,\, \dots .$$

8.2. <u>The distributions of Marcel Riesz</u>

Consider in R_n the hyperbolic distance

$$(8.8) \qquad \rho = \sqrt{x_1^2 - x_2^2 \dots - x_n^2}$$

with $x_1 \geq \sqrt{x_2^2 + \dots + x_n^2}$.

As to the applications it may be illustrative to consider in this section x_1 as the time coordinate and $x_2, \dots, x_n$ as the space coordinates. The distribution ρ^λ with $\mathrm{Re}\lambda > -2$ is defined with the aid of the integral

$$(8.9) \qquad \langle \rho^\lambda, \phi \rangle = \int_{\rho \geq 0} \rho^\lambda \phi(x)\, dx .$$

For values of λ with Re $\lambda \leq -2$ the distribution ρ^λ is again defined by the method of analytical continuation.

This case is a little bit more complicated than that of the preceding section, since ρ^λ with $\mathrm{Re}\lambda < 0$ is singular on the whole forward "light cone" $x_1 = \sqrt{x_2^2 + \dots + x_n^2}$.

We restrict our treatment by giving here again only the results of the calculations, for which the reader is referred to [6] , Vol.I, ch.II, §3 , p.49, [6], Vol. II, ch. VI, §5, p.32, [7], Vol. I, ch. III, §2, p.236 and especially [27], Ch. II, §4.3. The functional $\langle \rho^\lambda, \phi \rangle$, considered as function of λ, is analytic in λ for all complex values of λ with the exception of the following values:

1. $\lambda = -2k$ $(k=1,2,\dots)$,
2. $\lambda = -n-2k$ $(k=0,1,2,\dots)$.

For n even, it is possible that λ belongs to both sequences; in that case $\lambda=-n-2k$ is a pole of the second order of $\langle \rho^\lambda, \phi \rangle$; in the other cases the points $\lambda=-2k$ and $\lambda=-n-2k$ are simple poles of $\langle \rho^\lambda, \phi \rangle$.

a) for n even with $\lambda > -n$ and for n odd the residue of $\langle \rho^\lambda, \phi \rangle$ in the simple poles $\lambda = -2k$ equals

(8.10) $$\operatorname*{Res}_{\lambda=-2k} \langle \rho^\lambda, \phi \rangle = \frac{-1}{2^{k-1}(k-1)!(n-2)(n-4)\dots(n-2k)} \cdot$$
$$\langle \Box^k[\theta(x_1 - \sqrt{x_2^2 + \dots + x_n^2})], \phi \rangle, \quad k = 1, 2, \dots$$

where $\Box$ denotes the wave operator

$$\Box = -\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \dots + \frac{\partial^2}{\partial x_n^2} .$$

$\theta(x_1 - \sqrt{x_2^2 + \dots + x_n^2})$ equals one inside the "forward" light cone and is identically zero outside this cone; hence it is clear that $\Box^k[\theta(x_1 - \sqrt{x_2^2 + \dots + x_n^2})]$ is a distribution concentrated on the forward light cone.

b) For n odd the residue of $\langle \rho^\lambda, \phi \rangle$ in the simple poles $\lambda = -n-2k$ is

(8.11) $$\operatorname*{Res}_{\lambda=-n-2k} \langle \rho^\lambda, \phi \rangle = \frac{(-1)^{\frac{n+2k-1}{2}} \pi^{\frac{n}{2}}}{2^{2k} k! (\frac{n+2k-2}{2})!} \langle \Box^k \delta(x), \phi \rangle, \quad k = 0, 1, 2, \dots .$$

c) For n even with $\lambda \le -n$ the singularities $\lambda = -n-2k$ belong to both sequences 1 and 2. In this case $\langle \rho^\lambda, \phi \rangle$ has a pole of the second order in $\lambda = -n-2k$ $(k=0,1,2,\dots)$. The coefficient of the first term of the Laurent expansion of $\langle \rho^\lambda, \phi \rangle$ in the neighbourhood of $\lambda = -n-2k$ equals

(8.12) $$\frac{2(-1)^{\frac{n+2k-2}{2}} \pi^{\frac{n}{2}-1}}{2^{2k} k! (\frac{n+2k-2}{2})!} \langle \Box^k \delta(x), \phi \rangle, \quad k = 0, 1, 2, \dots .$$

The distributions ρ^λ may be normalized by the introduction of suitable factors. Putting

(8.13) $$Z_\mu = \frac{1}{\pi^{\frac{1}{2}n-1} \, 2^{\mu-1} (\frac{\mu-2}{2})! (\frac{\mu-n}{2})!} \rho^{\mu-n},$$

the functional $\langle Z_\mu, \phi \rangle$ becomes an entire function of the complex variable μ. These distributions Z_μ have been treated already by M. Riesz; they enjoy remarkable properties and play an important role in wave theory [24].

For $(\mu-n)$ not singular the support of Z_μ is the closure of the interior of the forward cone $x_1 = +\sqrt{x_2^2+\ldots+x_n^2}$; for $\mu = -2k$ $(k=0,1,2,\ldots)$ its support is the origin and for $\mu=n-2k$ $(k=1,2,\ldots)$ with $\mu\neq 0,-2,-4,\ldots$ its support is the surface of the forward light cone.
For non negative integral values of k we have the relation

(8.14) $$Z_{-2k} = (-1)^k \Box^k \delta(x).$$

Moreover, one has for all values of μ

(8.15) $$\Box Z_\mu = -Z_{\mu-2} \text{ and } \Box^k Z_\mu = (-1)^k Z_{\mu-2k},$$

and hence in particular

(8.16) $$Z_o = \delta(x),\ \Box Z_2 = -\delta(x) \text{ and } \Box^k Z_{2k} = (-1)^k \delta(x).$$

The convolution property, as given by equation (5.13) for the distributions $\Phi_\lambda(x)$, is also valid for the distributions Z_μ, i.e.

(8.17) $$Z_\mu * Z_\nu = Z_{\mu+\nu}.$$

This relation can be verified by proving (8.17) for Re $\mu,\nu > n$ and applying consecutively the principle of analytical continuation. The formulae (8.14) - (8.17) are of fundamental importance for the theory of the wave equation.

9. Some applications of distribution theory to partial differential equations

The applications of distribution theory to partial differential equations are only illustrated in this section by giving some examples.

Consider a partial differential equation

(9.1) $$L\, f(x) = g(x),$$

where L is an arbitrary linear differential operator with constant coefficients, which is applied to a distribution f(x) of n independent variables.
The relevant differentiations are taken in the distributional

sense; the right hand side $g(x)$ may be a distribution.
A solution of this inhomogeneous equation is obtained with the aid of a so called elementary solution $E(x)$, which satisfies the equation

$$L\,E(x) = \delta(x). \tag{9.2}$$

Assuming that $g(x)$ is such a distribution that $E(x) * g(x)$ exists, then a solution of (9.1) is given by

$$f(x) = E(x) * g(x), \tag{9.3}$$

where $E(x)$ is an arbitrary solution of (9.2).

This technique to obtain solutions of partial differential equations has many applications in applied mathematics.
In particular, let us consider as a rather general example a linear partial differential equation with constant coefficients of order m; viz.

$$L\,f(x) = \left(\frac{\partial^m}{\partial x_1^{\,m}} + \sum_{\substack{|p| \leq m \\ p_1 < m}} a_p\, D^p\right) f(x) = 0, \tag{9.4}$$

with $p = (p_1, p_2, \ldots, p_n)$, $D^p = \dfrac{\partial^{|p|}}{\partial x_1^{\,p_1}\, \partial x_2^{\,p_2} \ldots \partial x_n^{\,p_n}}$ and $|p| = \sum_{i=1}^{n} p_i$.

The function $f(x)$ is supposed to be differentiable up to the order m outside the surface $x_1 = 0$ and to satisfy (9.4) for $x_1 \neq 0$. We assume further, that the unknown function $f(x)$ and its derivatives with respect to x_1 up to the order m-1 have prescribed jumps across the surface $x_1 = 0$. These jumps, taken in the positive direction of x_1, are denoted by

$$\begin{aligned} \Delta f(0, x_2, \ldots, x_n) &= f^0(x_2, \ldots, x_n), \\ \Delta \frac{\partial f}{\partial x_1}(0, x_2, \ldots, x_n) &= f^1(x_2, \ldots, x_n), \\ &\;\;\vdots \\ \Delta \frac{\partial f^{m-1} f}{\partial x_1^{\,m-1}}(0, x_2, \ldots, x_n) &= f^{m-1}(x_2, \ldots, x_n). \end{aligned} \tag{9.5}$$

The functions $f^{\nu}(x_2,\ldots,x_n)$, $(\nu=0,1,\ldots,m-1)$ are submitted to certain conditions of differentiability, otherwise the function $f(x)$ cannot be m times differentiable outside the plane $x_1 = 0$.
It is rather difficult to give these conditions of differentiability for the general case considered here.
Therefore we shall not enter into a discussion of these conditions, but we assume that the functions f^{ν} are sufficiently smooth, such that the differentiability of $f(x)$ will not be violated (see also example 3 of this section).

When the function $f(x)$ is considered as a distribution $F(x)$ belonging to D' or S', we may write instead of the classical derivatives $\frac{\partial f}{\partial x_1}$, $\frac{\partial^2 f}{\partial x_1^2}$, etc.

$$\frac{\partial f}{\partial x_1} = \frac{\partial F}{\partial x_1} - \delta(x_1) f^{o}(x_2,\ldots,x_n),$$

$$\frac{\partial^2 f}{\partial x_1^{\,2}} = \frac{\partial^2 F}{\partial x_1^{\,2}} - \delta(x_1)\, f^{1}(x_2,\ldots,x_n) - \delta^{(1)}(x_1)\, f^{o}(x_2,\ldots,x_n),$$

(9.6) -

$$\frac{\partial^m f}{\partial x_1^{\,m}} = \frac{\partial^m F}{\partial x_1^{\,m}} - \delta(x_1)\, f^{m-1}(x_2,\ldots,x_n) - \delta^{(1)}(x_1)\, f^{m-2}(x_2,\ldots,x_n) \; \ldots\ldots\ldots - \delta^{(m-1)}(x_1)\, f^{o}(x_2,\ldots,x_n),$$

where $\frac{\partial F}{\partial x_1}$, $\frac{\partial^2 F}{\partial x_1^{\,2}}$ etc. denote now distributional derivatives.
Substituting (9.6) into (9.4) one obtains for the distribution $F(x)$ the differential equation

$$\text{(9.7)} \qquad L\,F(x) = \left(\frac{\partial^m}{\partial x_1^{\,m}} + \sum_{\substack{|p| \leq m \\ p_1 < m}} a_p\, D^p \right) F(x) = H(x),$$

where $H(x)$ is the distribution

(9.8) $$H(x) = \sum_{\nu=1}^{m} \delta^{(m-\nu)}(x_1)\, f^{\nu-1}(x_2,\ldots,x_n) + \sum_{\substack{|p|\le m \\ p_1<m}} a_p \frac{\partial^{|p|-p_1}}{\partial x_2^{p_2}\ldots\partial x_n^{p_n}} \sum_{\nu=1}^{p_1} \delta^{(p_1-\nu)}(x_1)\, f^{\nu-1}(x_2,\ldots,x_n).$$

The distribution H(x) represents a layer of poles, dipoles and multipoles, concentrated at the plane $x_1 = 0$. The distribution F(x) satisfies now the differential equation (9.7) in the whole space R_n.
According to the beginning of this section, a general solution of equation (9.7) is given by

(9.9) $$F(x) = H(x) * E(x),$$

where E(x) is an elementary solution of the equation

$$L\, E(x) = \delta(x).$$

The distribution F(x) coincides outside the plane $x_1 = 0$ with the function f(x), and therefore a solution satisfying (9.4) and (9.5) is given by:

(9.10) $$f(x) = H(x) * E(x), \quad x_1 \neq 0.$$

The distribution E(x) corresponds with the so-called Green's function of classical analysis.

This procedure may be generalized to the case that the functions $f^{\nu}(x_2,\ldots,x_n)$ are no longer smooth functions; instead it may even be assumed that they are distributions.
The distribution $F(x) = H(x) * E(x)$ again satisfies outside the plane $x_1 = 0$ the differential equation (9.4), where the differentiations are now of course taken in the distributional sense. In general, F(x) will not coincide for $x_1 \neq 0$ with an m times differentiable function (see also example 3 of this section).

When we take instead of the coordinate x_1 the time variable t, the above sketched technique may be applied for solving initial value problems. Assuming f(x,t) identically equal to zero for $t < 0$, the jumps at t=0 are given by the initial conditions, to which f(x,t) is subject. As to the uniqueness of the solution of initial value problems the reader is referred to the second example which we give below.

Other important applications are found in potential theory, when the potential or one of its normal derivatives should have a jump across a certain surface, which is a layer of poles, dipoles or multipoles. It is obvious, that the determination of elementary solutions is of the utmost importance in the theory of partial differential equations.

Examples

1. The Laplace operator

The elementary solution of the Laplace operator in R_n with $n > 2$ follows immediately from the formulae (8.4) - (8.6).
We have the relation

$$(9.11) \qquad \Delta \left[- \frac{1}{\Omega_n (n-2)} r^{-n+2} \right] = \delta(x),$$

with $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}$.
The Newtonian potential in R_3 due to a mass distribution $\mu(x_1, x_2, x_3)$ satisfies the differential equation

$$(9.12) \qquad \Delta f(x) = \mu(x),$$

and hence

$$(9.13) \qquad f(x) = \frac{-1}{4\pi r} * \mu(x).$$

2. The wave operator

According to formula (8.16), viz.

$$(8.16) \qquad \Box^k (-1)^k Z_{2k} = \delta(x),$$

the distribution $(-1)^k Z_{2k}$ is an elementary solution of the k-times iterated wave equation; this elementary solution is the only one, which vanishes for $x_1 < 0$. This follows from the fact, that the equation

$$(9.14) \qquad \Box^m f(x) = g(x),$$

with the distributions f(x) and g(x) identically equal to zero for $x_1 < 0$, has always a unique solution. This unique solution is

(9.15)
$$f(x) = (-1)^m Z_{2m} * g(x).$$

This follows easily from the following implications

$$f(x) = (-1)^m Z_{2m} * g(x) \Longrightarrow \Box^m f(x) = \Box^m (-1)^m Z_{2m} * g(x) = g(x)$$

$$\Box^m f(x) = g(x) \Longrightarrow (-1)^m Z_{2m} * \Box^m f(x) = \Box^m (-1)^m Z_{2m} * f(x) =$$

$$= f(x) = (-1)^m Z_{2m} * g(x).$$

It has been shown in section 5 that the convolution product $(-1)^m Z_{2m} * g(x)$ always exists, whenever $g(x)$ is concentrated in the region $x_1 \geq 0$. When we have an initial value problem for the iterated wave equation $\Box^m f(x) = 0$, the initial data give to the right hand side of (9.14) a contribution, which is concentrated at the surface $x_1 = 0$ (confer (9.8)). Hence, every initial value problem for the iterated wave equation has always a unique solution.

3. The vibrating string

The differential equation for the vibrating string is

(9.16)
$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} .$$

Suppose the initial conditions are

$$u(x,0) = u_o(x) \text{ and } \frac{\partial u}{\partial t}(x,0) = u_1(x).$$

According to (8.13) and (8.16) the elementary solution is:

$$Z_2(x,t) = \tfrac{1}{2} \text{ for } t > |x| ,$$

$$Z_2(x,t) = 0 \text{ for } t < |x| .$$

Applying (9.8) and (9.10) we obtain the solution

(9.17)
$$u(x,t) = Z_2(x,t) * \{u_o(x)\ \delta'(t) + u_1(x)\ \delta(t)\} .$$

In the case that $u_1(x)$ is an integrable function, this expression reduces to

(9.18)
$$u(x,t) = \frac{u_o(x+t)+u_o(x-t)}{2} + \tfrac{1}{2} \int_{x-t}^{x+t} u_1(\xi)d\xi \ , \ t \geq 0$$

which is nothing else as the well-known formula of d'Alembert.

When it is assumed, that $u_0(x)$ is two times and $u_1(x)$ is one time differentiable in classical sense, the solution $u(x,t)$ is a twice differentiable function, satisfying (9.16) in classical sense. However, when $u_o(x)$ and $u_1(x)$ are no longer subject to these conditions, the solution (9.17) is not twice differentiable and it satisfies (9.16), for $t > 0$, only in the distributional sense.
If we take for instance for $u_o(x)$ a function with a discontinuity at $x=a$, the solution $u(x,t)$ has a jump across the characteristics $x \pm t=a$. It may also be assumed, that $u_o(x)$ and $u_1(x)$ are distributions. When e.g. $u_o = \delta(x)$ and $u_1 \equiv 0$, one obtains $u(x,t) = \frac{\delta(x+t)+\delta(x-t)}{2}$; taking $u_o \equiv 0$ and $u_1 = \delta(x)$, the solution becomes $u(x,t) = Z_2(x,t)$.

4. Other applications

In chapter III we shall consider functions $f(x,y,z)$, which have prescribed jumps across the plane $z=0$ and which are solutions of the equation $-\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$.
In chapter IV we shall investigate extensively the elementary solutions of a generalized wave equation, namely the equation of Klein-Gordon.

10. The topological foundations of the theory of distributions

This section is devoted to a concise description of the topological foundation of the theory of distributions. This foundation has been given by Schwartz and Gelfand - Shilov in different ways, which are both presented here for comparison.

10.1. The topological foundation as given by Schwartz

a. The topology of the space D

We consider first the subspace $D_K \subset D$, consisting of all infinitely differentiable functions with fixed bounded support K. A topology is introduced in this space by the following system of zero-neighbourhoods.

$V(m;\varepsilon;K)$ is the set of all functions belonging to D_K with the property that all its derivatives of order $p \leq m$ are absolutely bounded by ε, i.e.

(10.1) $$|D^p \phi(x)| = \left| \frac{\partial^p}{\partial x_1^{p_1} \partial x_2^{p_2} \ldots \partial x_n^{p_n}} \phi(x_1, x_2, \ldots, x_n) \right| < \varepsilon .$$

A system of norms, which defines the same topology, is given by

(10.2) $$\|\phi\|_p = \sup_{x \in R_n} |D^p \phi(x)| .$$

D_K is a locally convex topological space with a countable basis of zero-neighbourhoods; it is very easy to show that D_K is complete with respect to the topology defined above. Therefore D_K is a socalled Fréchet-space, i.e. a locally convex complete topological vector space with a countable basis of zero-neighbourhoods.

Consider now a sequence of neighbourhoods in R_n, viz.

(10.3) $$\Omega = \{ \Omega_o = \phi , \Omega_1, \Omega_2, \ldots, \Omega_\nu, \ldots \},$$

where Ω_ν is the open sphere $|x| = \sqrt{x_1^2 + \ldots + x_n^2} < \nu$.

Let $\{\varepsilon\} = \{\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_\nu, \ldots\}$ be a sequence of decreasing positive numbers, approaching zero for $\nu \to \infty$ and $\{m\} = \{m_0, m_1, \ldots, m_\nu, \ldots\}$ a sequence of increasing non-negative integers approaching ∞ for $\nu \to \infty$.

A system of zero-neighbourhoods in D is given by the neighbourhoods $V(\{m\};\{\varepsilon\})$, which are defined as the sets of all functions $\phi \in D$ satisfying for every ν and $x \notin \Omega_\nu$ the relation

(10.4) $$|D^p \phi(x)| \leq \varepsilon_\nu, \text{ if } p \leq m_\nu .$$

A system of norms, defining the same topology, is given by

(10.5) $$\|\phi\|_{\{m\};\{\varepsilon\}} = \sup_\nu \left\{ \sup_{|x| \geq \nu, p \leq m_\nu} \left(\frac{|D^p \phi(x)|}{\varepsilon_\nu} \right) \right\}.$$

The neighbourhoods $V(\{m\};\{\varepsilon\})$ consist of all functions $\phi \in D$ with

(10.6) $$\|\phi\|_{\{m\};\{\varepsilon\}} \leq 1.$$

It follows, that the space D is a locally convex topological space with a non-countable basis of zero-neighbourhoods.
It is not difficult to show that the definition of convergence with respect to this topology coincides with that introduced in section 1 of this chapter. Moreover, the space D is complete with respect to this convergence.
If we consider only test functions belonging to D_K, the topology in D induces in the subspace D_K the same topology as was defined above by the neighbourhoods $V(m;\varepsilon;K)$.
D is the so-called "inductive" limit of the spaces D_K.
In every topological vector space the concept of bounded subsets is very important. The bounded subsets of D may be identified with the sets $B(\{M\},K)$, consisting of all $\phi \in D_K$, which satisfy the relations

(10.7) $$|D^p \phi(x)| \leq M_m \quad \text{for } p \leq m,\ m=0,1,2,\dots$$

where M_m is an element of a sequence $\{M\}$ of increasing positive numbers $M_0, M_1, M_2, \dots$.
It can be proved that every bounded set B of D is relatively compact and vice versa; this means, B is bounded if and only if every family of open sets covering $\overline{B}$ contains a finite subfamily, which covers also $\overline{B}$.
The space D is an example of a so-called Montel-space.

b. <u>The topology of the space D'</u>

A topology is defined in the dual space D', i.e. the space of distributions, with the aid of the bounded sets B of D. A neighbourhood $V(B,\varepsilon)$ of the zero-distribution is the set of all distributions f with the property, that $|\langle f,\phi\rangle| \leq \varepsilon$ for all $\phi \in B$, where B is an arbitrary bounded set of D.
Hence D' is a locally convex topological space with a non-countable basis of zero-neighbourhoods.
A system of norms is introduced by

(10.8) $$\|f\|_B = \sup_{\phi \in B} |\langle f,\phi\rangle|.$$

A set B' in D' is bounded, if for any bounded subset B of D

(10.9)
$$\sup_{f\in B',\phi\in B} |\langle f,\phi\rangle| < \infty .$$

The properties of the space D' can now be established by applying directly the theorems of the theory of the topological vector spaces [25] .

Some important consequences are the following. When a sequence of distributions $f_m \in D'$ has the property, that for every $\phi \in D$ the sequence $\{\langle f_m,\phi\rangle\}$ converges to say $f(\phi)$, then the functional $f(\phi)$ is also a distribution belonging to D'; the space D' is complete in the weak sense.

Every bounded set B' of D' is relatively compact and vice versa; the space D' is again a Montel-space.

c. The topology of the spaces S and S'

The topology, introduced in the space S, is simpler than that in D. A system of zero-neighbourhoods is defined by the sets $V(m;k;\varepsilon)$, where m and k are non-negative integers and $\varepsilon > 0$; $\phi(x) \in V(m;k;\varepsilon)$, if $\phi \in S$ and if

(10.10)
$$|(1+|x|^2)^k D^p\phi(x)| \le \varepsilon \quad \text{for } p \le m.$$

Hence S is a locally convex topological vector space with countable basis. A sequence of norms is given by

(10.11)
$$\|\phi\|_m = \sup_{\substack{|k|\le m;\, p\le m\\ x\in R_n}} |x^k D^p \phi(x)| ,$$

with $x^k = x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}$ and $|k| = \sum_{i=1}^{n} k_i$.

The bounded sets B of S may be identified with the sets of test functions $\phi(x) \in S$, which satisfy relations of the type

(10.12)
$$|D^p \phi(x)| \le k_p\, k(x),$$

where k(x) is a continuous function, decreasing for $|x| \to \infty$ stronger

than any negative power of $|x|$, while k_p is an element of a sequence of positive constants $k_o, k_1, k_2, \ldots$.
It can again be shown that S is complete and that bounded sets are relatively compact and vice versa.
A topology is introduced in the dual space S' by means of the bounded sets B of S; this can be done in the same way as in the case of D'.
The properties of S' follow again from the theory of the topological vector spaces. S' is complete in the weak sense and bounded sets are again relatively compact.
For a detailed treatment of the topological foundation of the theory of distributions, as given by Schwartz, the reader is referred to [6] , Vol.I, ch.III, p.63-103 and [6] , Vol.II, ch.VII, §3 - §4, p.89-99.

10.2. The topological foundation as given by Gelfand and Shilov

Let us consider a general linear space X with elements x, for which there can be defined an increasing sequence of norms

(10.13) $$\|x\|_o \leq \|x\|_1 \leq \cdots \leq \|x\|_m \leq \cdots \quad .$$

Moreover it will be assumed, that each pair of norms of the sequence are coordinated; this means, that whenever $\{x_n\}$ is a Cauchy sequence in X with respect to two norms and $\{x_n\}$ is convergent to zero in one of the norms, then it is also convergent to zero in the other norm. The space X is called a sequentially normed space.
A topology is introduced by a system of zero-neighbourhoods $V(m;\varepsilon)$, which consist of the elements x with

(10.14) $$\|x\|_o < \varepsilon, \ \|x\|_1 < \varepsilon, \ldots, \|x\|_m < \varepsilon .$$

Denoting the completion of the space X with respect to the m^{th} norm by X_m, one obtains a decreasing sequence of Banach spaces

(10.15) $$X_o \supset X_1 \ldots \supset X_m \supset \ldots \supset X.$$

It can be proved, that the space X is complete, if and only if X coincides with the intersection of all the Banach spaces X_m.
Together with the Banach space X_m one may consider also its conjugate

X'_m, consisting of all continuous linear functionals, which can be defined on X_m. In this way an increasing sequence of Banach spaces X'_m is obtained, viz.

(10.16) $$X'_o \subset X'_1 \subset \ldots \subset X'_m \subset \ldots \subset X',$$

while

(10.17) $$X' = \bigcup_{m=1}^{\infty} X'_m.$$

If a functional f belongs to X', it certainly belongs to some X'_m and consequently also to X'_{m+p} with $p \geq 1$.
In $X'_m, X'_{m+1}, \ldots$ it has the following norms

(10.18) $$\|f\|_m = \sup_{\|x\|_m=1} |<f,x>| \ , \ \|f\|_{m+1} = \sup_{\|x\|_{m+1}=1} |<f,x>|, \ldots,$$

and hence

(10.19) $$\|f\|_m \geq \|f\|_{m+1} \geq \ldots .$$

The space X' is the union of an increasing sequence of complete normed spaces with norms becoming weaker and weaker.
The subset B of a sequentially normed space is called bounded, if and only if

(10.20) $$\|x\|_m < C_m \quad (m=0,1,2,\ldots)$$

for all $x \in B$.
The topology in the space X' is again defined with the aid of the bounded subsets of X; the neighbourhoods $V(B;\epsilon)$ are defined as the sets of those $f \in X'$, for which

(10.21) $$\sup_{x \in B} |<f,x>| < \varepsilon .$$

The subset $B' \subset X'$ is bounded, if for every bounded subset B of X

(10.22) $$\sup_{f \in B'; x \in B} |<f,x>| < \infty .$$

Gelfand and Shilov have introduced a large class of spaces of test functions, which may be considered as sequentially normed spaces. Moreover, these authors have given also a sufficient condition for these spaces of test functions in order to have the property, that all their bounded subsets are compact. If this property is fulfilled for a sequentially normed space of test functions, the space is called perfect.

The spaces D(a) and S are very special examples from this general class of spaces of test functions; D(a) is the space of all test functions belonging to D with support $|x| < a$.

The sequence of norms to be defined in the space D(a) is given by

$$\|\phi\|_m = \sup_{|x| \leqslant a;\, p \leqslant m} |D^p \phi(x)| \quad , \; m=0,1,2,\dots \; . \tag{10.23}$$

The space D(a) coincides with the intersection of all the completions of D(a) with respect to the norms $\|\cdot\|_m$ (m=0,1,2,...) and therefore D(a) is complete. The space D(a) is also perfect.

The sequence of norms to be defined in the space S is given by

$$\|\phi\|_m = \sup_{\substack{|k| \leqslant m;\, p \leqslant m \\ x \in R_n}} |x^k D^p \phi(x)| \quad , \; m=0,1,2,\dots \; . \tag{10.24}$$

It can again be shown, that the space S is complete and perfect.

The properties of the dual spaces D'(a) and S' are established with the aid of the theory of linear topological vector spaces.

The spaces D'(a) and S' are complete with respect to weak convergence, i.e. if $\langle f_m, \phi \rangle$ converges for every test function out of D(a) or S to a limit $f(\phi)$, then $f(\phi)$ is again a linear continuous functional belonging to D'(a) respectively S'.

Finally, let there be given an increasing sequence of linear topological spaces

$$X^{(1)} \subset X^{(2)} \subset \dots \subset X^{(m)} \subset \dots \; . \tag{10.25}$$

It is assumed, that with each inclusion convergence is preserved.

The union of all spacex $X^{(m)}$ is indicated by $X^{(\omega)}$.
The space $X^{(\omega)}$ is not considered as a topological space, but only the following type of convergence is introduced. The sequence $\{x_n\}$ is said to converge to the limit x, if all elements x_n and x belong to some subspace $X^{(m)}$ and if $x_n \to x$ in the topology of $X^{(m)}$.
All continuous linear functionals on $X^{(\omega)}$ form the dual space $X^{(\omega)'}$.
The sequence of functionals $f_n \in X^{(\omega)'}$ is said to converge to the functional f, if for each $x \in X^{(\omega)}$

(10.26) $$\lim_{n \to \infty} \langle f_n, x \rangle = \langle f, x \rangle \,, \qquad \text{(weak convergence)}.$$

This principle of forming the union of topological vector spaces is applied to the spaces D(a) with a = 1,2,3,... .
In this way Gelfand and Shilov define the space of test functions D and D' is obtained as the dual of the union of the spaces D(a). The space D' is complete with respect to the weak convergence as defined in (10.26).

In contrast to the theory of L. Schwartz, the space D as defined by Gelfand and Shilov is not considered as a topological vector space. In this respect the presentation of the topological foundation of the theory of distributions as given by Schwartz is to be preferred to that presented by Gelfand and Shilov. On the other hand the theory as given by Gelfand and Shilov in literature [7],Vol. II, ch. I and II, p. 1-102, is easier to understand by those who are not too well familiar with the theory of topological vector spaces.

10.3. Distributions and continuous functions

In the present chapter we have used several times the important property that every distribution may be considered as a distributional derivative of a continuous function; therefore, we shall give here a short proof of this theorem for the case of distributions of one variable and belonging to the space D': confer theorem 4.
Consider a distribution $f(x) \in D'$; hence it is also a priori a distribution belonging to D'(a), where a may be any real number. D'(a) is the union of

all the spaces $D'_m(a)$, which are the conjugates of the spaces $D_m(a)$, while $D_m(a)$ is the completion of $D(a)$ with respect to the norm $\|\phi\|_m$. This means, that there exists a certain integer p, such that f(x) is a continuous linear functional on $D_p(a)$.

The space $D_p(a)$ consists of the functions $\phi(x)$, which have continuous derivatives up to the order p and which vanish outside (-a,+a).

To any $\phi(x) \in D_p(a)$ we may associate the continuous function $\psi(x) = \phi^{(p)}(x)$. This defines a continuous one to one mapping of $D_p(a)$ on a subspace of the space C(a), consisting of all functions continuous in the interval (-a,+a). Therefore f(x) is equivalent to a continuous linear functional g(x), defined on a subspace of C(a), with

$$\langle f,\phi\rangle = \langle g,\psi\rangle \quad .$$

According to the theorem of Hahn-Banach (see [26] , ch.III, §21, p. 106) this functional can be extended to the whole space C(a).

Using the representation theorem of Riesz (see [26] , ch.III, §22, p. 112), it is clear, that there exists a function of bounded variation $\mu(x)$, such that

$$\langle g,\psi\rangle = \int_{-a}^{+a} \psi(x)\, d\,\mu(x),$$

or

$$\langle f,\phi\rangle = \int_{-a}^{+a} \phi^{(p)}(x)\, d\,\mu(x).$$

Integration by parts yields finally

$$\langle f,\phi\rangle = \int_{-a}^{+a} F(x)\, \phi^{(p+2)}(x)\, dx, \tag{10.27}$$

where F(x) is a continuous function.

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Chapter II

THE DIFFRACTION OF A CYLINDRICAL PULSE BY A REFLECTING HALF-PLANE

1. Introduction

In this chapter we consider a special problem from the theory of diffraction.
In the case of the diffraction of acoustic or seismic disturbances the state of the medium may be described by a so-called wave or potential function. This potential is defined as the function the gradient of which yields the displacement vector of the medium.
In the case of the diffraction of light it is not allowed to use such a potential function, since an electromagnetic field, owing to the property of polarization, cannot be represented by a single scalar potential. Instead of the potential function one may take now one of the components of the electric or magnetic vector.

In order to fix our ideas we restrict our attention to the case of the diffraction of acoustic or seismic disturbances. The treatment of electromagnetic disturbances is formally quite the same; only the various mathematical symbols represent other physical quantities.

The wave function satisfies the wave equation and a boundary condition at the surface of some obstacle which causes the diffraction. An analytical solution of this boundary value problem has been found for only relatively simple geometrical configurations such as a half-plane and in two dimensions a wedge.

The case of harmonic plane waves disturbed by an absorbing or reflecting half-plane was solved in 1896 by Sommerfeld, who introduced the concept of Riemann spaces in diffraction theory, [1] .
Carslaw too used this concept in order to give the solution for the diffraction of harmonic cylindrical waves expanding from a line source

parallel to the edge of the half-plane, [2].
The method of Sommerfeld has been applied also by Cagniard for the case of spherical waves of rather general shape, [3] .
Turner and Lauwerier have discussed the diffraction of a cylindrical pulse expanding again from a line parallel to the edge of the half-plane, [4] , [5] . Turner applies consecutively the transformations of Laplace and Kantorovich-Lebedev [6] , with the aid of which the wave equation is reduced to a simple ordinary differential equation of the second order. Lauwerier too uses the Laplace transformation and the transformed boundary value problem is reduced to a Hilbert problem.

As to the application of the Laplace transformation, used by Turner and Lauwerier, it may be remarked, that this method is more or less a detour for finding the wave function.
The boundary value problem is transformed and after obtaining the solution of this transformed problem one has to determine the inverse transform of this solution. In this process many calculations concerning special functions occur, which however do not appear in the final solution; cf. [4] and [5] .
In this chapter we present another way, very simple and more direct, for obtaining the wave function corresponding to the diffraction of a cylindrical pulse expanding from a line parallel to the edge of a reflecting half-plane. The case of an absorbing half-plane can be treated quite similarly.

The boundary value problem is formulated in section 2.
In section 3 a tentative solution is given, which is based mainly on physical considerations and which satisfies the boundary condition at the half-plane. Finally it is shown in section 4, that this solution is actually a solution of the wave equation. This will be done by calculating directly $\Box f$ with the aid of the theory of distributions, where f is the proposed tentative solution and $\Box$ the wave operator.

For additional literature on diffraction theory the reader is referred to the textbooks by Baker and Copson and by Friendlander, [7] , [8].

The author is indebted to Prof. Lauwerier for suggesting the method, presented in this chapter.
The use of the theory of distributions may show, that this theory can be applied successfully in solving diffraction problems.

2. Formulation of the diffraction problem

We consider a pulse, producing a cylindrical disturbance which expands from a line parallel to the straight edge of a semi-infinite reflecting or rigid screen.
Introducing Euclidean rectangular space coordinates x,y and z and the time t, we suppose that the edge of the screen coincides with the z-axis, while the screen itself lies in the plane y=0 with $x \leq 0$, see fig. 1. The pulse is assumed to start at time t=0 from the line l passing through the point $(x_o,y_o,0)$ and parallel to the z-axis. The problem to determine the corresponding wave function is obviously two dimensional and therefore the wave function may be written as f(x,y,t). The pulse is represented by the delta-function $\delta(x-x_o,y-y_o,t)$ and the wave function f(x,y,t) satisfies the following differential equation

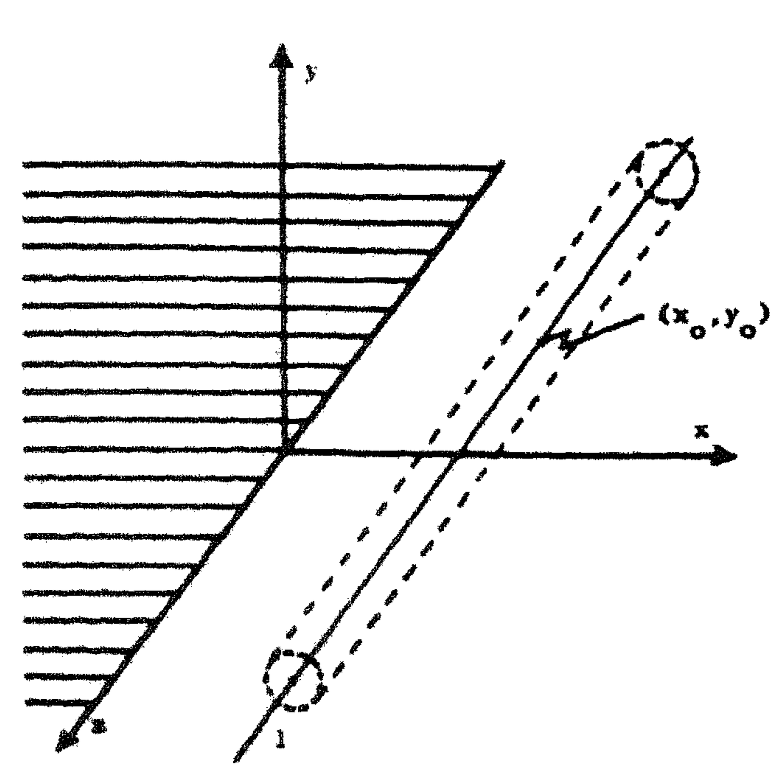

figure 1

(2.1) $$\Box f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial t^2} = -2\pi\ \delta(x-x_o,y-y_o,t),$$

where the propagation velocity of the disturbances has been put equal to 1.
The function f(x,y,t) is of course zero for values of $t<0$, while the y-component of the displacement vector vanishes at the screen, due to the condition of reflection. This means that f(x,y,t) is subject to the additional conditions

(2.2) $$f(x,y,t) \equiv 0 \quad \text{for } t<0,$$

and

$$\frac{\partial f}{\partial y}(x,y,t) = 0 \quad \text{for } y=0 \text{ and } x<0. \tag{2.3}$$

As to the solution of this boundary value problem it should be remarked that due to the appearance of the delta function in the right hand side of (2.1), the function $f(x,y,t)$ cannot be twice differentiable everywhere in the (x,y,t)-space. Actually, there are certain characteristic surfaces, across which $f(x,y,t)$ and its derivatives are discontinuous. Therefore, the differentiations occurring in (2.1) are to be taken in distributional sense and the wave function is to be considered essentially as a distribution in the neighbourhood of these characteristic surfaces.
The relation (2.3) makes only sense for those points of the screen, in the neighbourhood of which $f(x,y,t)$ may be considered as a function with a continuous derivative with respect to y; therefore, the condition (2.3) should be restricted to the points of the screen, which do not lie on the above-mentioned characteristic surfaces.

3. Tentative solution of the boundary value problem

The potential due to a pulse, starting at time $t=0$ and expanding from the point (x_o,y_o) into a region free from obstacles, is given by the generalized function Z_2, except for a factor 2π defined by (8.13), Ch.I, viz.

$$Z_2 = \Theta(t-R)\,(t^2-R^2)^{-\frac{1}{2}}, \tag{3.1}$$

where

$$R = \{(x-x_o)^2 + (y-y_o)^2\}^{\frac{1}{2}}. \tag{3.2}$$

R denotes the distance between the observer at the point (x,y) and the source at the point (x_o,y_o); see figure 2, where we present the situation in the (x,y) plane at some time t.
Introducing polar coordinates

$$x=r\cos\phi\,,\; y=r\sin\phi\,;\; x_o=r_o\cos\phi_o,\; y_o=r_o\sin\phi_o, \tag{3.3}$$

with $-\pi < \phi \leq +\pi$, $-\pi < \phi_o \leq +\pi$, we may write (3.2) also in the form

$$R = \{r^2 + r_o^2 - 2rr_o \cos(\phi-\phi_o)\}^{\frac{1}{2}}. \tag{3.4}$$

The distribution Z_2 has been treated in chapter I, section 8.2, and according to formula (8.16) it satisfies the wave equation (2.1). Because the reflection at the screen has to be taken into account, it is useful to consider also the potential due to a pulse located in $(x_o,-y_o)$. This potential is given by the generalized function

$$Z_2' = \Theta(t-R')(t^2-R'^2)^{-\frac{1}{2}}, \tag{3.5}$$

where

$$R' = \{(x-x_o)^2+(y+y_o)^2\}^{\frac{1}{2}} = \{r^2+r_o^2-2rr_o \cos(\phi+\phi_o)\}^{\frac{1}{2}}. \tag{3.6}$$

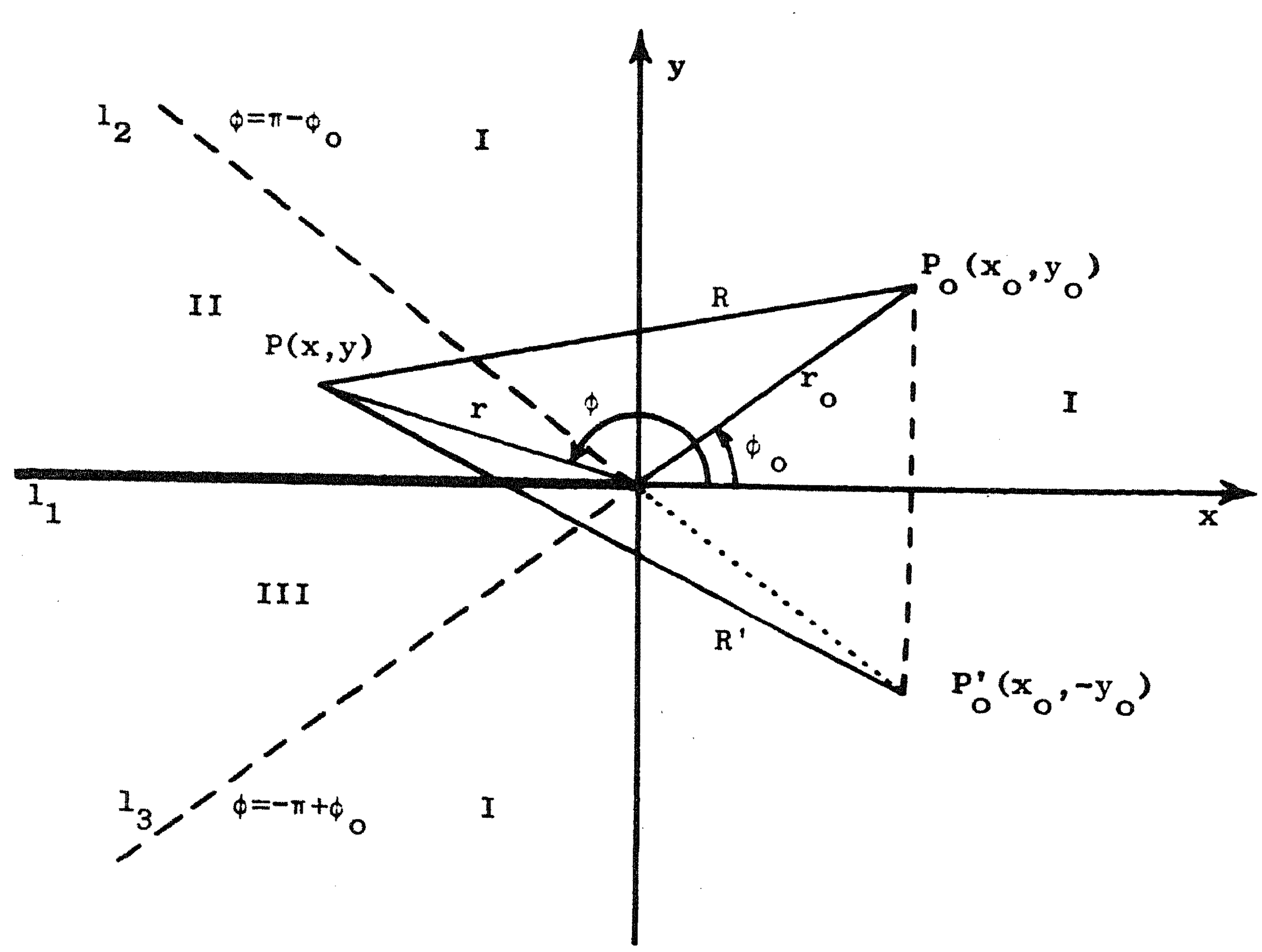

figure 2

We divide the (x,y)-plane into three regions, denoted by I, II and III, see fig.2.

The first region I, where $-\pi+\phi_o<\phi<\pi-\phi_o$, is bounded by the "shadow" lines l_2 and l_3; the second region II, where $\pi-\phi_o<\phi<\pi$, by the "shadow" line l_2 and the screen l_1 and finally the third region III, where $-\pi<\phi<-\pi+\phi_o$, by the screen l_1 and the "shadow" line l_3. The pulse at (x_o,y_o) is observed only in the regions I and II, while the reflected one at $(x_o,-y_o)$ is observed only in the region II.

Therefore, the direct pulse gives to the potential f(x,y,t) in the regions I and II a contribution which is equal to Z_2, whereas the reflected pulse gives to the potential f(x,y,t) in the region II still another contribution, which is equal to Z_2'. Neither the direct pulse nor the reflected one can give in the region III any contribution to the potential f(x,y,t).

However, the solution f(x,y,t) should be continuous almost everywhere across the "shadow" lines l_2 and l_3. In order to satisfy this condition we introduce an "edge effect" with the property that the total potential due to the direct pulse, the reflected pulse and this edge effect is continuous almost everywhere in the (x,y)-plane.

Observing that the edge of the screen disturbs the wave, expanding from the point (x_o,y_o), only after the time $t=r_o$ and that its region of influence is bounded by the cone $r=t-r_o$, it is to be expected that the potential due to the edge of the screen contains terms of the form

$$\Theta(t-r-r_o)(t^2-R^2)^{-\frac{1}{2}} \quad \text{and} \quad \Theta(t-r-r_o)(t^2-R'^2)^{-\frac{1}{2}}. \tag{3.7}$$

These functions are infinitely differentiable at all points inside and outside the cone $r=t-r_o$; however, they have a finite jump across the surface of this cone, except along the generators with respectively $R = r+r_o$ and $R' = r+r_o$, where this jump is infinite.

So we try the following solution of our boundary value problem:

I. In the region I, where $-\pi+\phi_o \leq \phi \leq \pi-\phi_o$

$$f(x,y,t) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}} + \Theta(t-r-r_o)\{A_1(t^2-R^2)^{-\frac{1}{2}} + B_1(t^2-R'^2)^{-\frac{1}{2}}\}. \tag{3.8}$$

The potential consists of two parts, namely the direct disturbance due to the pulse at (x_o,y_o) and a diffraction term due to the edge of the screen.

II. In the region II, where $\pi - \phi_o \leq \phi < +\pi$

$$(3.9)\quad f(x,y,t) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}} + \Theta(t-R')(t^2-R'^2)^{-\frac{1}{2}} + \Theta(t-r-r_o)\,\{A_2(t^2-R^2)^{-\frac{1}{2}} + B_2(t^2-R'^2)^{-\frac{1}{2}}\}\ .$$

The potential consists of three parts, namely the direct disturbance due to the pulse at (x_o,y_o), the disturbance due to the reflection at the screen and a diffraction term due to the edge of the screen.

III. In the region III, where $-\pi < \phi \leq -\pi + \phi_o$

$$(3.10)\qquad f(x,y,t) = \Theta(t-r-r_o)\,\{A_3(t^2-R^2)^{-\frac{1}{2}} + B_3(t^2-R'^2)^{-\frac{1}{2}}\}.$$

The potential is now only due to diffraction.
The coefficients A_i and B_i (i=1,2,3) are assumed to be constant and independent of the location of the pulse. They are determined in such a way that the boundary condition at the screen and the condition of continuity across the "shadow" lines are satisfied.
The first condition yields

$$(3.11)\qquad A_2 = B_2 \quad\text{and}\quad A_3 = B_3,$$

while the second one gives the simple equations

$$(3.12)\qquad A_1 = A_2 \quad\text{and}\quad B_1 = B_2 + 1,$$

$$(3.13)\qquad A_1 + 1 = A_3 \quad\text{and}\quad B_1 = B_3.$$

These six equations for A_i and B_i are not independent and we need still another condition to find the coefficients A_i and B_i.
Taking the pulse in a point of the positive x-axis, the regions II and III disappear and the solution of the diffraction problem is simply

$$f(x,y,t) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}}.$$

Hence it follows, that we have also the relation

(3.14) $$A_1 + B_1 = 0.$$

Solving finally (3.11) - (3.14) we obtain

(3.15) $$A_1 = A_2 = -\tfrac{1}{2},\ A_3 = +\tfrac{1}{2},\ B_1 = B_3 = +\tfrac{1}{2} \text{ and } B_2 = -\tfrac{1}{2}.$$

Thus we have found the tentative solution

I.

(3.16) $$f(x,y,t) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}} - \tfrac{1}{2}\Theta(t-r-r_o)\{(t^2-R^2)^{-\frac{1}{2}} - (t^2-R'^2)^{-\frac{1}{2}}\},$$

valid for $-\pi + \phi_o \leq \phi \leq +\pi - \phi_o$.

II.

(3.17) $$f(x,y,t) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}} + \Theta(t-R')(t^2-R'^2)^{-\frac{1}{2}} -$$

$$- \tfrac{1}{2}\Theta(t-r-r_o)\{(t^2-R^2)^{-\frac{1}{2}} + (t^2-R'^2)^{-\frac{1}{2}}\},$$

valid for $\pi - \phi_o \leq \phi < \pi$.

III.

(3.18) $$f(x,y,t) = \tfrac{1}{2}\Theta(t-r-r_o)\{(t^2-R^2)^{-\frac{1}{2}} + (t^2-R'^2)^{-\frac{1}{2}}\},$$

valid for $-\pi < \phi \leq -\pi + \phi_o$.

We note that we have in particular for $t > r+r_o$

(3.19) $$f(x,y,t) = \tfrac{1}{2}\{(t^2-R^2)^{-\frac{1}{2}} + (t^2-R'^2)^{-\frac{1}{2}}\}.$$

It is clear, that this proposed solution is twice differentiable almost everywhere outside the screen. The only exceptions are the characteristic conical surfaces $t=R$ with $-\pi + \phi_o \leq \phi < \pi$, $t=R'$ with $\pi - \phi_o \leq \phi < \pi$ and $t=r+r_o$ with $-\pi < \phi < +\pi$.

It can easily be verified that the solution (3.16) - (3.18), which vanishes for $t < 0$, satisfies the differential equation (2.1) and the boundary condition (2.3) in respectively all points of the (x,y,t)-

space and the screen, which do not lie on the above-mentioned characteristic surfaces.
In the next section we shall show, that the proposed solution satisfies the differential equation (2.1) in the distributional sense everywhere outside the screen.

The whole situation is illustrated by figure 3, where the conical regions of influence belonging to the pulse, the reflected pulse and the edge of the screen are shown.

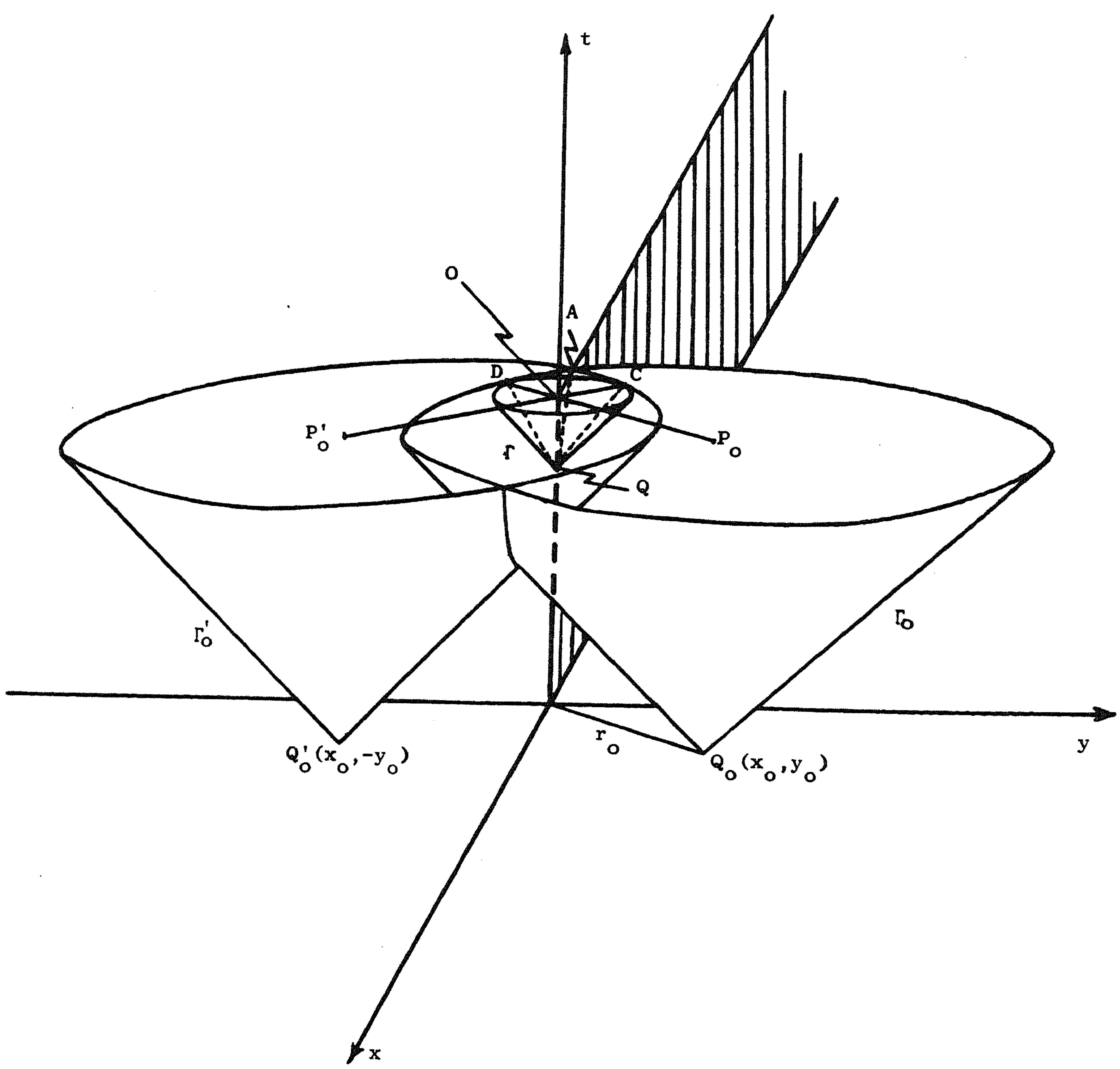

figure 3

These conical regions are denoted by respectively Γ_o, Γ'_o and Γ. The cone Γ has its vertex in the common point of intersection of the t-axis and the cones Γ_o and Γ'_o; the t-coordinate of this point Q is $t=r_o$. Further, the cone Γ is tangent to Γ_o and Γ'_o and the common generators pass through respectively the points D and C.
Finally a cross section of this figure with a plane t = constant is presented in figure 4.

The influence of the direct pulse is confined to the parts of the regions I and II which lie within the circle with centre (r_o, ϕ_o) and radius t.
The influence of the reflected pulse is confined to the part of region II which lies within the circle with centre $(r_o, -\phi_o)$ and radius t.
Finally, the influence of the edge effect, i.e. the diffraction term, is confined to the whole interior of the circle with the edge O of the screen as centre and with radius $t-r_o$.

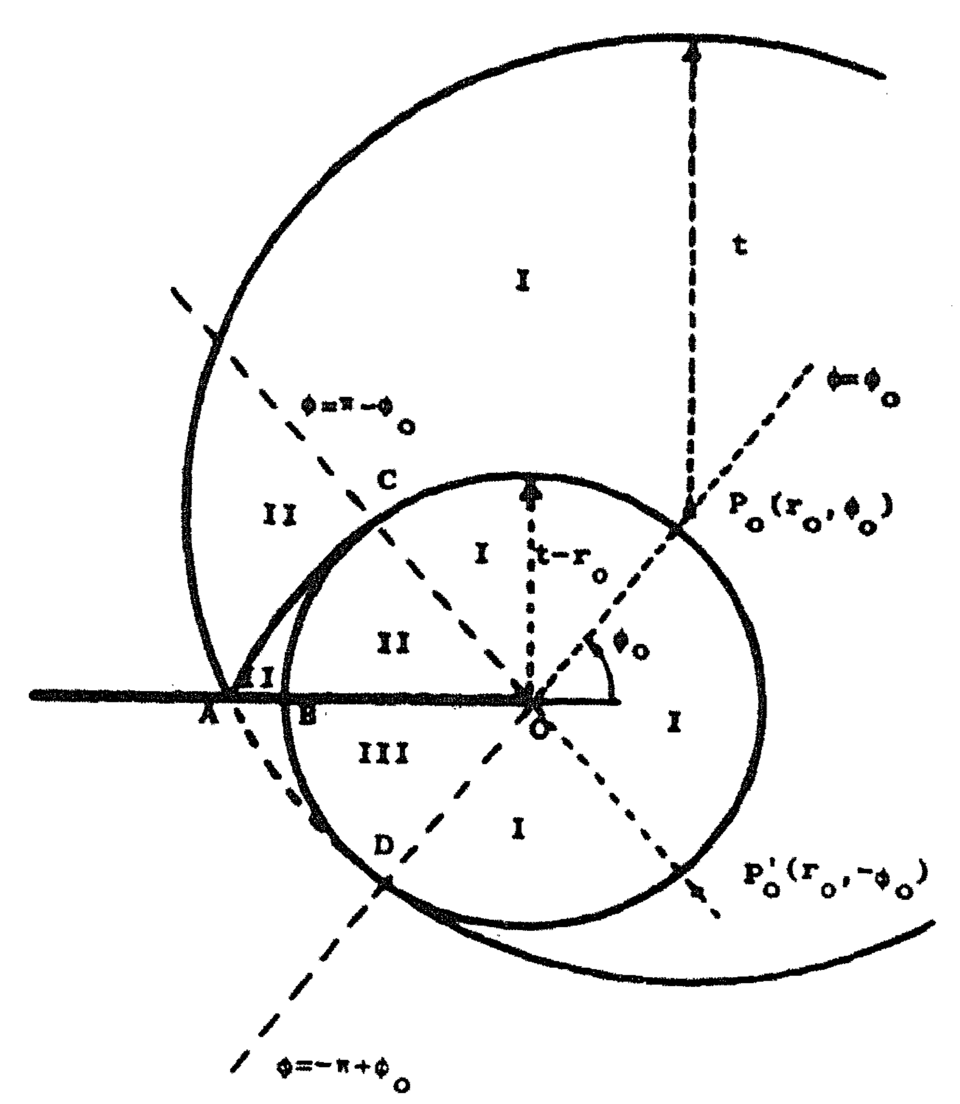

figure 4

4. <u>The verification of the proposed solution</u>

Introducing the functions

(4.1) $g_1(t,R) = \Theta(t-R)(t^2-R^2)^{-\frac{1}{2}}$,

(4.2) $g_2(t,r,R,R') = \frac{1}{2}\Theta(t-r-r_o)\{(t^2-R^2)^{-\frac{1}{2}}-(t^2-R'^2)^{-\frac{1}{2}}\}$,

(4.3) $g_3(t,r,R) = \{\Theta(t-R)- \Theta(t-r-r_o)\}\{\Theta(\pi+\phi)-\Theta(\pi+\phi-\phi_o)\}(t^2-R^2)^{-\frac{1}{2}}$,

(4.4) $g_4(t,r,R') = \{\Theta(t-R')- \Theta(t-r-r_o)\}\{\Theta(-\pi+\phi+\phi_o)-\Theta(-\pi+\phi)\}(t^2-R'^2)^{-\frac{1}{2}}$,

we may express our proposed solution (3.16) - (3.18) into one single formula, viz.

(4.5) $f(x,y,t) = g_1(t,R) - g_2(t,r,R,R') - g_3(t,r,R) + g_4(t,r,R')$.

The function $f(x,y,t)$ and also the functions g_1, g_2, g_3 and g_4 will be considered in this section as distributions belonging to D'.

The support of $g_1(t,R)$ is the closure of the interior of the cone $t=R$ and that of $g_2(t,r,R,R')$ in the closure of the interior of the cone $t-r_o=r$.
The support of the distribution $g_3(t,r,R)$ is bounded by the screen and the characteristic surfaces $t=R$ and $t-r_o=r$, while the support of $g_4(t,r,R')$ is bounded by the screen and the characteristic surfaces $t=R'$ and $t-r_o=r$.
The intersection of the support of the distribution g_1 resp. g_2 with the (x,y)-plane at some time t is the region bounded by a circle with (x_o,y_o) resp. (0,0) as centre and with radius t resp. $t-r_o$, see fig.4.
The intersection of the support of g_3 with the (x,y)-plane is the triangular region ABD and that of g_4 is the triangular region ABC, see fig.4.

The distributions g_i (i=1,2,3,4) have the property that they are integrable functions, finite in the interior of their support, but infinite in certain points of the boundary of their support. In fact $g_1(t,R)$ is infinite in all points of the boundary of its support, while $g_2(t,r,R,R')$ becomes infinite along the generators CQ and DQ, see figure 3.
The function $g_3(t,r,R)$ respectively $g_4(t,r,R')$ is infinite in all points of the boundary which is formed by the cone $t=R$ respectively $t=R'$.

According to chapter I, formula (8.16), the distribution $g_1(t,R)$ satisfies the differential equation (2.1). Hence there remains to show that the other terms of (4.5) satisfy outside the screen the homogeneous equation

$$\Box h = \frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} - \frac{\partial^2 h}{\partial t^2} = 0, \tag{4.6}$$

where the differentiations are taken in distributional sense. We shall prove (4.6) for the terms containing R; the reasoning for the terms containing R' is completely the same.

We introduce now the distribution

$$g = \Theta(G)(t^2-R^2)^{-\frac{1}{2}}, \tag{4.7}$$

where G is either the region bounded by the cone $t-r_o=r$ or the region bounded by the screen and the cones t=R and $t-r_o=r$. $\Theta(G) = 1$ for $(x,y,t) \in \bar{G}$ and $\Theta(G) = 0$ elsewhere; $\bar{G}$ is the region G, including its boundary.

Because we have to prove the validity of (4.6) in every (x,y,t)-neighbourhood not intersecting the screen, we shall show that

$$\langle \Box g, \phi \rangle = 0, \tag{4.8}$$

for every test function ϕ belonging to D and having a support, which does not contain points with $x < 0$ and $y=0$.
When the support S of ϕ does not intersect the boundary of G the validity of (4.8) follows immediately from differentiation in the classical sense. Therefore, we assume that S intersects the boundary of G.

Instead of the distribution g we consider the distribution $g^{(\alpha)}$, defined as

$$g^{(\alpha)} = \Theta(G)(t^2-R^2)^{\alpha} . \tag{4.9}$$

The functional $\langle g^{(\alpha)},\phi\rangle$ is an analytic function of α for $\mathrm{Re}\,\alpha > -1$. For values of α with $\mathrm{Re}\,\alpha \leq -1$ the distribution $g^{(\alpha)}$ is defined by analytic continuation of

$$\langle g^{(\alpha)},\phi\rangle = \iiint\limits_G g^{(\alpha)} \cdot \phi(x,y,t)\,dx\,dy\,dt, \quad \mathrm{Re}\,\alpha > -1. \tag{4.10}$$

The distributions g and $g^{(\alpha)}$ are connected with each other by the relation

$$\langle g,\phi\rangle = \lim_{\alpha \to -\frac{1}{2}} \langle g^{(\alpha)},\phi \rangle . \tag{4.11}$$

Since the operations of taking the limit and of differentiation may be interchanged (chapter I, theorem 3), it follows from (4.11) that

$$(4.12) \qquad \langle \Box g, \phi \rangle = \lim_{\alpha \to -\frac{1}{2}} \langle \Box g^{(\alpha)}, \phi \rangle \ .$$

Hence we have to prove that

$$(4.13) \qquad \lim_{\alpha \to -\frac{1}{2}} \langle \Box g^{(\alpha)}, \phi \rangle = 0.$$

A simple calculation yields for Re $\alpha > 2$

$$(4.14) \qquad \Box (t^2-R^2)^\alpha = -2\alpha \ (2\alpha+1)(t^2-R^2)^{\alpha-1}.$$

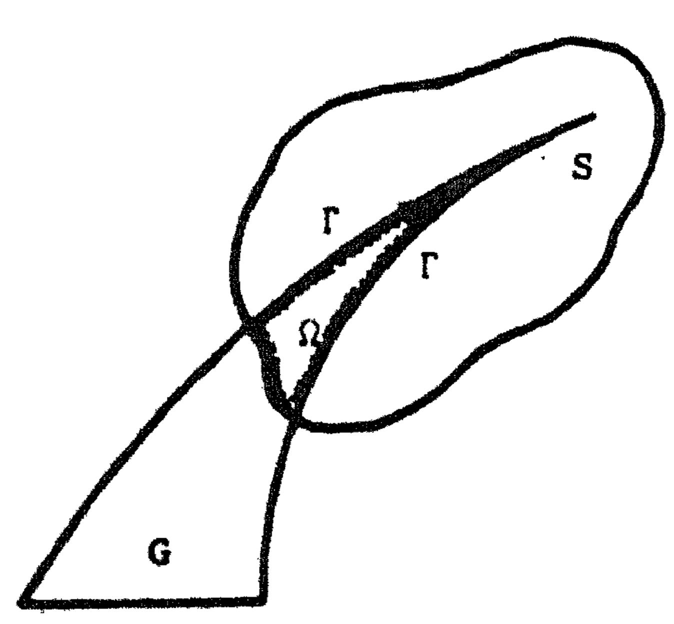

figure 5

As the intersection Ω of the region $\bar{G}$ and the support S of ϕ is bounded on the one hand by characteristic surfaces and on the other hand by a part of the boundary of S (see fig.5), we have also on Ω for Re $\alpha > 2$

$$(4.15) \qquad \Box \Theta(G)(t^2-R^2)^\alpha = \Box g^{(\alpha)} = -2\alpha(2\alpha+1)\Theta(G)(t^2-R^2)^{\alpha-1}.$$

The factor $\Theta(G)$ does not yield a contribution to the right hand side of (4.15), because the differential operator is an internal differentiation along the boundary Γ (see fig.5).

Thus we have for Re $\alpha > 2$ the relation

$$(4.16) \qquad \langle \Box g^{(\alpha)}, \phi \rangle = \langle g^{(\alpha)}, \Box \phi \rangle = -2\alpha(2\alpha+1) \langle g^{(\alpha-1)}, \phi \rangle \ .$$

Since this relation is certainly valid for Re $\alpha > 2$, it is also valid for its analytic continuation with respect to α .

Therefore there remains to show

$$(4.17) \qquad \lim_{\alpha \to -\frac{1}{2}} \left[-2\alpha(2\alpha+1) \langle g^{(\alpha-1)}, \phi \rangle \right] = 0.$$

The functional $\langle g^{(\alpha-1)}, \phi \rangle = \langle \Theta(G)(t^2-R^2)^{\alpha-1}, \phi \rangle$ is an analytic function of α for Re $\alpha > 0$.

For Re $\alpha > 0$ we may write

$$(4.18)\quad \begin{aligned} <g^{(\alpha-1)},\phi> &= \iiint_{\Omega} (t^2-R^2)^{\alpha-1}\,\phi(x,y,t)\,dx\,dy\,dt = \\ &\iiint_{\Omega} (t^2-R^2)^{\alpha-1}\{\phi(x,y,t)-\phi(x,y,R)\}\,dx\,dy\,dt\, + \\ &+ \iiint_{\Omega} (t^2-R^2)^{\alpha-1}\phi(x,y,R)\,dx\,dy\,dt. \end{aligned}$$

The first term of this expression is an analytic function of α for Re $\alpha>-1$. Using instead of the coordinate t the coordinate $\mu=t-R$, we can write the second term of (4.18) in the form

$$(4.19)\quad \int_{\mu_1}^{\mu_2} \mu^{\alpha-1}d\mu \iint_{S(\mu)} (\mu+2R)^{\alpha-1}\phi(x,y,R)\,dx\,dy,$$

where μ_1 and μ_2 are some real numbers with $\mu_2 > \mu_1 \geq 0$ and where $S(\mu)$ is the area formed by the intersection of Ω and the cone $\mu=t-R$. μ_1 is actually equal to zero, if the boundary of Ω contains one or more points of the cone $t=R$.

Putting

$$(4.20)\quad \iint_{S(\mu)} (\mu+2R)^{\alpha-1}\phi(x,y,R)\,dx\,dy = \Phi(\mu,\alpha),$$

we remark that $\Phi(\mu,\alpha)$ is an analytic function of α for all values of α. ($\mu+2R = t+R$ is positive.)

Substituting (4.20) into (4.19), we obtain

$$(4.21)\quad \begin{aligned} \iiint_{\Omega} (t^2-R^2)^{\alpha-1}\phi(x,y,R)\,dx\,dy\,dt &= \int_{\mu_1}^{\mu_2} \mu^{\alpha-1}\,\Phi(\mu,\alpha)\,d\mu = \\ \int_{\mu_1}^{\mu_2} \mu^{\alpha-1}\,\{\Phi(\mu,\alpha) - \Phi(0,\alpha)\}\,d\mu &+ \frac{\Phi(0,\alpha)}{\alpha}\,(\mu_2^{\alpha} - \mu_1^{\alpha}), \end{aligned}$$

valid for Re $\alpha > 0$.

The first term of this expression is again an analytic function of α for Re $\alpha > -1$; the second term is analytic for all values of α whenever $\mu_1 > 0$, but this term has a simple pole at $\alpha=0$ with residue $\Phi(0,0)$ if $\mu_1 = 0$.

Summarizing, we have now obtained the result that $<g^{(\alpha-1)},\phi>$ is an analytic function of α for Re $\alpha > -1$ with the possible exception of $\alpha=0$, where it may have a simple pole.

Hence

(4.16) $$<\Box g^{(\alpha)},\phi> \; = \; <-2\alpha(2\alpha+1)g^{(\alpha-1)},\phi>$$

is an analytic function of α for Re $\alpha > -1$.

Taking finally the limit for $\alpha \to -\frac{1}{2}$, we get

(4.22) $$<\Box g,\varphi> \; = \lim_{\alpha \to -\frac{1}{2}} \; <\Box g^{(\alpha)},\phi> \; = 0.$$

Thus we have proved, that outside the screen g satisfies the homogeneous differential equation (4.6) in the distributional sense. Therefore the proposed solution (4.5) satisfies the differential equation (2.1) in the distributional sense and we may conclude that (4.5) satisfies the boundary value problem, which has been posed in section 2.

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Chapter III

SUPERSONIC WING THEORY

1. Introduction

As is well-known the behaviour of a steady supersonic irrotational non-viscous flow around bodies may be described with the aid of a so-called velocity potential, the gradient of which yields the velocity vector of the flow. In the case of a very thin nearly flat wing, moving with constant speed at a small angle of incidence, the velocity potential satisfies approximately a linear hyperbolic differential equation in three space variables, which is formally the same as the wave equation.
Since the elementary solutions of this equation are singular at the so-called characteristic Mach cone, it is expected that in the derivations expressions will occur, which can not be defined in a satisfactory way in terms of classical analysis.

In papers and textbooks, such as e.g. the encyclopedic work "General theory of high speed aero-dynamics", edited by W.R. Sears [1] , the difficulties are overcome by using the concept of the "finite part" of an integral, as introduced by Hadamard [2] . However, the calculations are complicated, tedious and lengthy.

The theory can be developed in a much shorter and more elegant way by employing the theory of distributions.

Applying this theory we present in this chapter a rather complete and systematic treatment of linearized steady supersonic wing theory. It appears that the classical theory, as presented by Heaslet and Lomax in their detailed contribution to reference [1] , can be greatly simplified.

The mathematical theory of linearized steady supersonic flow around wings amounts essentially to solving boundary value problems

for the velocity potential. This potential has to satisfy the above-mentioned linear hyperbolic differential equation and it is subject to certain boundary conditions at the surface of the wing. A large variety of boundary value problems is obtained according to the planform of the wing surface and the prescribed boundary conditions. However, all the results of this chapter are derived from a single basic formula which relates the velocity potential at the plane of the wing to the velocity component normal to this plane.
In this way a very simple and unified theory of the supersonic flow around wings is obtained.

Already in 1954 Sauer emphasized in [3] that it is worthwhile to introduce the theory of distributions in supersonic aerodynamics. Actually, in reference [4] he solved one of the problems treated in this chapter by using distribution techniques.
In this connection, the work of Dorfner [5] should also be mentioned. Dorfner uses distributions for establishing the basic general equations, but his calculations for deriving expressions for the potential of the flow around actual wings again run more or less along classical lines; cf. [5] , chapter II, §6.[1)]

In section 2 we give a general outline of the theory of supersonic flow around wings. The boundary conditions are represented by a layer of poles and dipoles, concentrated at the plane of the wing. The differential equation for the velocity potential is taken in the sense of distributions and contains an extra term due to this layer; cf. chapter I, section 9.
The general solution of this differential equation is given in section 3.

In sections 4,5 and 6 we deal with the problem of calculating the pressure distribution on wings with given geometrical properties such as planform, thickness, camber and angle of incidence.
An interesting case is the calculation of the influence of wing tips on the lift distribution.

The inverse problem of calculating the geometry of the wing surface from a given pressure distribution has been solved by Sauer with

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the aid of distributions [4] .
A simplified version of this solution is given in section 7.

Finally, in section 8, we consider so-called "mixed" problems. This means that the pressure distribution is given on only one part of the wing surface, while the geometry of the wing is prescribed on the remaining part. The problem is to find an expression for the pressure distribution valid at the whole surface of the wing. These problems occur in lift cancellation technique, which is a useful tool for calculating the lift distribution on airfoils of rather general planform.

For additional literature about the theory of supersonic flow around wings the reader is referred to [1] , [6] and [7] .

2. Supersonic flow around wings; the differential equation for the velocity potential

Consider a uniform steady supersonic flow which is disturbed by a very thin nearly flat wing, set at small incidence to the oncoming stream.
Introducing Euclidean rectangular coordinates (x,y,z), we take the positive x-axis in the direction of the undisturbed flow, while the wing lies approximately in the plane z=0. The origin of the coordinate system is chosen in the most forward point of the wing; see figure 1. Assuming that the resulting flow is irrotational, a perturbation velocity potential f(x,y,z) may be defined. This function has the property, that its gradient yields the perturbation velocity vector; i.e.

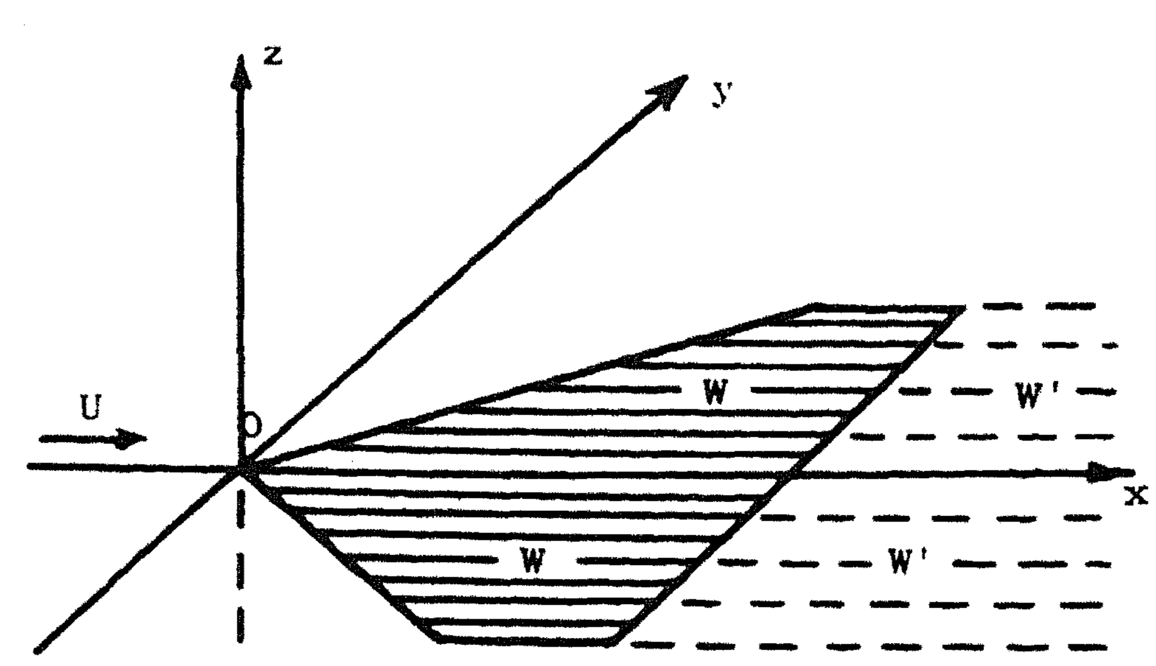

figure 1

(2.1) $$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = (u,v,w),$$

where u,v and w are the components of the perturbation velocity vector in respectively the x-,y- and z-direction.

Because the wing has been taken as very thin, nearly flat and at small incidence to the oncoming stream, u,v and w may be assumed small in comparison with the velocity U of the undisturbed flow. Therefore, the relevant equations can be linearized; this means that terms non-linear in u,v or w will be deleted.
Then it follows from the equations of motion and the equation of continuity, that f(x,y,z) satisfies the linearized differential equation

$$(2.2) \quad \left[(1-M^2)\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] f(x,y,z) = 0,$$

where M is the so-called Mach number, defined as U/a_∞ with a_∞ as the velocity of sound at infinity (cf. lit. [1] , [6] and [7]). In the case of supersonic flow U is larger than a_∞ and hence the differential equation (2.2) is of hyperbolic type. Taking another scale for the x-coordinate, one obtains the normal form

$$(2.3) \qquad -\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

This equation is formally the same as the wave equation, where the role of the time is played by the space coordinate x.

The function f(x,y,z) is subject to several conditions, resulting from mathematical and physical considerations.

1. The function f(x,y,z) is twice continuously differentiable outside the wing and its wake, with the exception of certain characteristic surfaces across which the external derivatives of f(x,y,z) may have a jump. These characteristic surfaces are circular cones with the axis in the direction of the x-coordinate - so-called Mach cones - or envelopes of these cones.
One distinguishes between a forward and a backward Mach cone. The forward Mach cone with vertex (x_o,y_o,z_o) is the conical surface, characterized by the equation

$$(2.4) \qquad x-x_o = -\sqrt{(y-y_o)^2 + (z-z_o)^2},$$

while the backward Mach cone is given by the equation

$$x-x_o = + \sqrt{(y-y_o)^2 + (z-z_o)^2}. \tag{2.5}$$

2. Because the wing has no influence on the flow in upstream direction, f(x,y,z) vanishes identically in all points lying upstream of the characteristic surface Ω , which is the boundary of the region of influence of the wing. f(x,y,z) and its internal derivatives vanish at Ω, while the external derivatives may have a jump across Ω.
As to the shapeof this characteristic surface Ω there are several possibilities according to the form of the wing.
An edge of a wing is called subsonic respectively supersonic in a point, if the component of the stream velocity normal to the edge at this point is respectively smaller or larger than the speed of sound. If an edge is subsonic in a certain point, say P, then all points of the edge in a sufficiently small neighbourhood of P lie within the Mach cone with P as vertex. If an edge is supersonic in P, then all points of a neighbourhood of P lie outside the Mach cone with P as vertex.
An edge is shortly called subsonic or supersonic, if it is respectively subsonic or supersonic in all its points.
In the case of a subsonic leading edge, the surface Ω coincides with the backward Mach cone with vertex at the nose of the wing; if, however, the leading edge is supersonic, the surface Ω consists partly of the envelope of all Mach cones with vertex at the leading edge of the wing.

3. Because the direction of the velocity in points at the wing surface is tangent to this surface, the derivatives of the function f(x,y,z) satisfy a boundary condition at this surface.
The wing has been assumed very thin and at small incidence to the oncoming stream, while its surface is nearly plane. Therefore this boundary condition may be prescribed at the projection W of the wing surface onto the plane z=0 (see fig.1). W is called the planform of the wing.

If the upper and lower surface of the wing is given by respectively the equations

(2.6) $$z = S_+(x,y), \quad \text{and} \quad z = S_-(x,y),$$

the boundary condition may be written in linear approximation as

(2.7) $$\frac{\partial f}{\partial z}(x,y,+0) = U \frac{\partial S_+}{\partial x} \quad \text{for } (x,y) \in W, \quad \text{and} \quad \frac{\partial f}{\partial z}(x,y,-0) = U \frac{\partial S_-}{\partial x} \quad \text{for } (x,y) \in W.$$

(cf. lit. [1] , [6] and [7]).

It follows from (2.7), that the function $\frac{\partial f}{\partial z}$, representing the vertical velocity component, is in general discontinuous across the plane z=0 in all points, lying within the region W.
However, the function $\frac{\partial f}{\partial z}$ is continuous in all points of the plane z=0, which lie outside the region W.

4. The pressure p(x,y,z) follows immediately from integrating Euler's equations of motion; this gives within the linear approximation

(2.8) $$p-p_\infty = -\frac{\rho_\infty U}{\sqrt{M^2-1}} \frac{\partial f}{\partial x} ,$$

where ρ_∞ is the density and p_∞ the pressure in the undisturbed flow.
If we consider a lifting wing, there is a pressure difference between the upper and lower surface of the wing. Hence, in linearized wing theory $\frac{\partial f}{\partial x}$ is discontinuous across the plane z=0 in all points lying within the region W, while $\frac{\partial f}{\partial x}$ is continuous across this plane in all points outside W.
It follows that the velocity potential f(x,y,z) itself is discontinuous in all points of the plane z=0, which lie in W or in the projection W' of the wake onto the plane z=0 (see fig.1).

Considering a general situation, we put

$$\lim_{z \to +0} f(x,y,z) = f_+(x,y),$$

(2.9)

$$\lim_{z \to -0} f(x,y,z) = f_-(x,y),$$

and

$$\lim_{z \to +0} \frac{\partial f}{\partial z}(x,y,z) = w_+(x,y),$$

(2.10)

$$\lim_{z \to -0} \frac{\partial f}{\partial z}(x,y,z) = w_-(x,y).$$

If the ordinary derivatives in the differential equation (2.3) are replaced by the distributional derivatives, one obtains the differential equation

(2.11) $$-\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = (w_+ - w_-)\cdot\delta(z) + (f_+ - f_-)\cdot\delta'(z).$$

This equation should now be taken in distributional sense and it is valid everywhere in the (x,y,z)-space. The function $w_+ - w_-$ vanishes outside W and the function $f_+ - f_-$ vanishes outside W+W'.

A useful property of thin wing theory is the fact that the geometry of the wing can be assumed to consist of two parts, namely a nonlifting or symmetric part and a lifting or anti-symmetric part. Besides the equations (2.6) for the upper and lower surface of the wing we consider the surfaces

(2.12) $$z = S_t(x,y) = \tfrac{1}{2}\{S_+(x,y) - S_-(x,y)\} ,$$

and

(2.13) $$z = S_c(x,y) = \tfrac{1}{2}\{S_+(x,y) + S_-(x,y)\} .$$

Due to the linearity of the differential equation the problem of the calculation of the velocity potential for an arbitrary wing with upper and lower surface $z = S_+(x,y)$ resp. $z = S_-(x,y)$ may be reduced to the calculation of the velocity potential $f_t(x,y,z)$ for a wing with upper and lower surface $z = S_t(x,y)$ resp. $z = -S_t(x,y)$ and to the calculation of the velocity potential $f_c(x,y,z)$ for an infinitely thin cambered wing, for which the upper and lower surfaces coincide with the surface $z = S_c(x,y)$.

The velocity potential f(x,y,z) of the original wing is then obtained as the sum of the potentials $f_t(x,y,z)$ and $f_c(x,y,z)$.

(2.14) $$f(x,y,z) = f_t(x,y,z) + f_c(x,y,z).$$

The first term of the right-hand side is symmetric with respect to z and this part of the potential accounts for the thickness distribution of the wing. The second term is anti-symmetric with respect to z and this part accounts for the angle of incidence and the camber of the wing. The lift distribution of the wing is only due to the second term $f_c(x,y,z)$.

The differential equation (2.11) becomes now for the case of the non-lifting wing with symmetric cross-section

(2.15) $$-\frac{\partial^2 f_t}{\partial x^2} + \frac{\partial^2 f_t}{\partial y^2} + \frac{\partial^2 f_t}{\partial z^2} = 2w_+ \cdot \delta(z),$$

where

(2.16) $$w_+(x,y) = \lim_{z \to +0} \frac{\partial f_t}{\partial z} .$$

The function $w_+(x,y)$ vanishes at all points of the plane $z=0$, which lie outside the region W.

For the case of the infinitely thin lifting airfoil we obtain the equation

(2.17) $$-\frac{\partial^2 f_c}{\partial x^2} + \frac{\partial^2 f_c}{\partial y^2} + \frac{\partial^2 f_c}{\partial z^2} = 2f_+ \cdot \delta'(z),$$

where

(2.18) $$f_+(x,y) = \lim_{z \to +0} f_c(x,y,z).$$

The functions $f_c(x,y,z)$ and $\frac{\partial f_c}{\partial x}(x,y,z)$ are odd with respect to the coordinate z.

Therefore, it follows from (2.8) that the lift distribution $\Delta p(x,y)$ at the wing surface is given by the expression

(2.19) $$\Delta p(x,y) = -\frac{2\rho_\infty U}{\sqrt{M^2-1}} \lim_{z \to +0} \frac{\partial f_c}{\partial x}(x,y,z).$$

Putting

(2.20) $$\lim_{z \to +0} \frac{\partial f_c}{\partial x}(x,y,z) = u_+(x,y)$$

it is clear that the lift distribution at the wing is proportional to the velocity component $u_+(x,y)$.

Since there cannot be a pressure jump at the plane z=0 outside the planform of the wing, we have always for lifting wings the boundary condition that $u_+(x,y) \equiv 0$ outside W.

The function $f_+(x,y)$ vanishes outside the region W+W'; it is independent of the coordinate x in the wake W'.

The cases of the nonlifting wing with thickness and the lifting airfoil without thickness may be treated by using the same differential equation.

For this purpose it is remarked that the calculations may be restricted to the upper half-space $z > 0$; the results for the lower half-space follow easily by considering the symmetry with respect to the plane z=0.

We introduce the velocity potential $f_+(x,y,z)$, which is defined as

$$f_+(x,y,z) = \begin{cases} f_t(x,y,z) \\ \text{or} \\ f_c(x,y,z) \end{cases} \quad \text{for } z > 0, \tag{2.21}$$

and

$$f_+(x,y,z) \equiv 0 \qquad \text{for } z < 0.$$

Substituting this into the differential equation (2.3) and replacing ordinary derivatives by distributional derivatives, we obtain the differential equation

$$-\frac{\partial^2 f_+}{\partial x^2} + \frac{\partial^2 f_+}{\partial y^2} + \frac{\partial^2 f_+}{\partial z^2} = w_+ \cdot \delta(z) + f_+ \cdot \delta'(z), \tag{2.22}$$

with

$$w_+(x,y) = \lim_{z \to +0} \frac{\partial f_+}{\partial z}(x,y,z),$$

and

$$f_+(x,y) = \lim_{z \to +0} f_+(x,y,z).$$

In the case of a nonlifting wing with symmetric cross-section $w_+(x,y)$ vanishes outside W and $f_+(x,y)$ vanishes outside the region of influence of the wing. In the case of an infinitely thin lifting airfoil $u_+(x,y) = \frac{\partial f_\pm}{\partial x}(x,y)$ vanishes outside W, while $w_+(x,y)$ vanishes outside the region of influence of the wing.

As has been shown in chapter I, section 8.2, the differential equations (2.15), (2.17) and (2.22) determine $f_t(x,y,z)$, $f_c(x,y,z)$ and $f_+(x,y,z)$ uniquely in terms of their boundary values $w_+(x,y)$ and $f_+(x,y)$.

3. The general solution of the differential equation

The differential equations (2.15), (2.17) and (2.20) are solved with the aid of the elementary solution $E(x,y,z)$, satisfying the equation

$$(3.1) \qquad \left(-\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right) E(x,y,z) = \delta(x,y,z),$$

with

$$(3.2) \qquad E(x,y,z) \equiv 0 \quad \text{for } x < 0.$$

According to the equations (8.13) and (8.16) of chapter I, section 8.2, the distribution $E(x,y,z)$ is given by the expression

$$(3.3) \qquad E(x,y,z) = \begin{cases} -\frac{1}{2\pi}\,(x^2-y^2-z^2)^{-\frac{1}{2}} & \text{for } x > \sqrt{y^2+z^2} \\ 0 & \text{for } x < \sqrt{y^2+z^2}. \end{cases}$$

The support of the distribution E is the closure of the interior of the backward Mach cone with vertex at the origin.
Using the theory of chapter I, section 9, it follows from (2.15) that for a wing, with cross-section symmetric with respect to the plane z=0, the velocity potential is given by the equation

$$(3.4) \qquad f_t(x,y,z) = +2w_+(x,y)\ \delta(z) * E(x,y,z).$$

This equation represents a potential due to a distribution of sources in the region W of the plane z=0.
Further, it follows from (2.17) that for a lifting wing without thickness the velocity potential is given by

$$(3.5) \qquad \begin{aligned} f_c(x,y,z) &= 2f_+(x,y)\ \delta'(z) * E(x,y,z) \\ &= 2f_+(x,y)\ \delta(x) * \frac{\partial E}{\partial z}(x,y,z). \end{aligned}$$

This equation represents a potential due to a distribution of dipoles in the region W+W' of the plane z=0.
In the same way we obtain finally with the aid of (2.22) for the potential $f_+(x,y,z)$ the expression

$$(3.6)\qquad f_+(x,y,z) = w_+(x,y)\,\delta(z) * E(x,y,z) + f_+(x,y)\delta'(z) * E(x,y,z)$$
$$= w_+(x,y)\delta(z) * E(x,y,z) + $$
$$+ \frac{\partial}{\partial z}\left[f_+(x,y)\delta(z) * E(x,y,z)\right] .$$

Since w_+, f_+ and E vanish for $x < 0$ and since the support of E is the closure of the interior of the backward Mach cone with vertex at the origin, the existence of the convolution products, appearing in (3.4), (3.5) and (3.6), is assured.
As $f_+(x,y,z) \equiv 0$ for $z < 0$, we have the relation

$$w_+(x,y).\delta(-z) * E(x,y,-z) - \frac{\partial}{\partial z}\left[f_+(x,y).\delta(-z) * E(x,y,-z)\right] \equiv 0$$

for $z > 0$.
Because $\delta(z)$ and $E(x,y,z)$ are even in z, we obtain the following interesting relation between the potential $f_+(x,y)$ and the "downwash" $w_+(x,y)$ in the plane z=+0, namely

$$(3.7)\qquad \frac{\partial}{\partial z}\left[f_+(x,y).\delta(z) * E(x,y,z)\right] = w_+(x,y)\cdot\delta(z) * E(x,y,z).$$

Substitution of (3.7) into (3.6) yields immediately the important result

$$(3.8)\qquad f_+(x,y,z) = 2w_+(x,y)\delta(z) * E(x,y,z).$$

Assuming $w_+(x,y)$ integrable and applying the definition of convolution, as given by formula (5.3) of chapter I, section 5, it can easily be shown that this relation may also be written as

$$(3.9)\qquad f_+(x,y,z) = -\frac{1}{\pi}\iint_A w_+(\xi,\eta)\{(x-\xi)^2-(y-\eta)^2-z^2\}^{-\frac{1}{2}}d\xi\, d\eta\ ,$$

where A is the region of the (ξ,η)-plane lying within the forward

Mach cone from the point (x,y,z); see figure 2. Letting z approach +0, we obtain for the potential at the upper side of the (x,y)-plane the expression

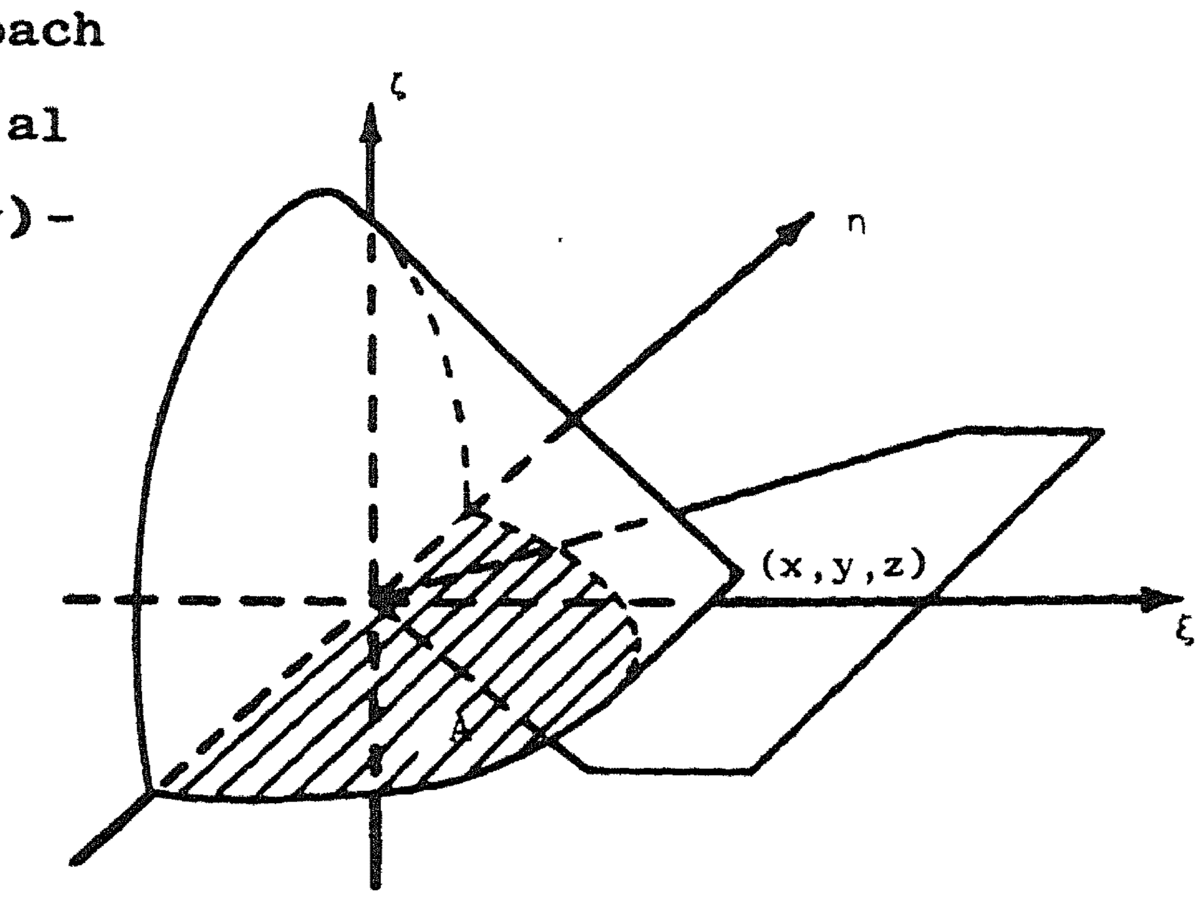

figure 2

$$(3.10) \qquad f_+(x,y) = -\frac{1}{\pi}\iint_B w_+(\xi,\eta)\{(x-\xi)^2-(y-\eta)^2\}^{-\frac{1}{2}}d\xi\, d\eta \ ,$$

where B is the triangular region bounded by the half-lines $\eta-y=\xi-x$ and $\eta-y=-\xi+x$ with $\xi < x$; see figure 3.

It is useful to introduce at this stage the characteristic coordinates

$$(3.11) \qquad \begin{aligned} r &= x-y \\ s &= x+y \ . \end{aligned}$$

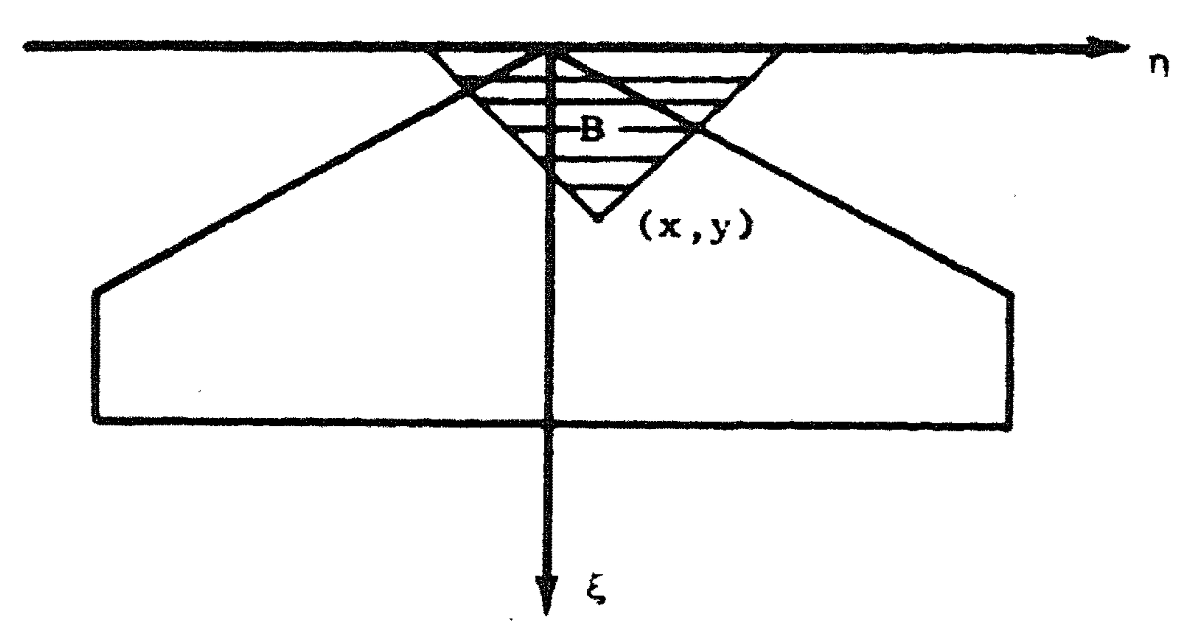

figure 3

Using these coordinates, we can write (3.10) in the form

$$(3.12) \qquad f_+(r,s) = -\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(r)\cdot \Phi_{\frac{1}{2}}(s),$$

where we have applied the distribution $\Phi_\lambda(x)$, introduced in chapter I, section 5. In view of its frequent use, we repeat here its definition and most important properties.

$$(3.13)\qquad \Phi_\lambda(x) = \frac{x_+^{\lambda-1}}{(\lambda-1)!}, \quad \Phi_o(x) = \delta(x),$$

$$(3.14)\qquad \Phi_\lambda(x) * \Phi_\mu(x) = \Phi_{\lambda+\mu}(x), \quad \frac{d}{dx}\Phi_\lambda(x) = \Phi_{\lambda-1}(x).$$

If the downwash $w_+(r,s)$ is given within the triangle B, we have obtained by aid of (3.12) an expression for the potential at the upper surface of the wing. However, in most cases of practical interest the data are not so simple; nevertheless also in these cases the equation (3.12) can be used as the basic formula, from which we start our calculations for deriving an expression for the potential, the pressure distribution or another physical quantity we are interested in.

Due to the properties of the distribution $\Phi_\lambda(x)$, stated in (3.13) and (3.14), the equation (3.12) can easily be inverted by taking the convolution of both sides with the product $\Phi_{-\frac{1}{2}}(r)\cdot\Phi_{-\frac{1}{2}}(s)$. The result is

$$(3.15)\qquad w_+(r,s) = -2f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\cdot\Phi_{-\frac{1}{2}}(s).$$

This formula expresses the downwash at the plane of the wing in the potential at this plane.
In the literature on supersonic wing theory the inversion of the equations (3.12) and (3.15) gives rise to difficulties. One relies on the theory of integral equations with singular kernels (cf. [1] , section D.12); this introduces lengthy calculations, which are now avoided with the aid of the distributions $\Phi_\lambda(x)$.

4. Nonlifting wings with symmetrical sections; lifting airfoils with supersonic leading edge

In this section we treat nonlifting wings with a given thickness distribution and symmetric with respect to the plane z=0; as to the planform of these wings no restrictions need be made. In this case the function w_+ is given in the whole plane z=+0. It is determined in the region W by the thickness distribution of the wing, while it vanishes outside W.

The same situation occurs, if one considers an infinitely thin lifting airfoil with supersonic leading edge and with prescribed surface slope. Moreover, it is assumed in this case, that the airfoil extends to infinity, so that no wake is present.

The velocity potential $f_+(r,s)$ in points at the upper surface of the wing is given by formula (3.10). Using characteristic coordinates, we obtain the result

$$(4.1) \qquad f_+(r,s) = -\frac{1}{2\pi}\iint_B w_+(\rho,\sigma)(r-\rho)^{-\frac{1}{2}}(s-\sigma)^{-\frac{1}{2}}d\rho \; d\sigma \; ,$$

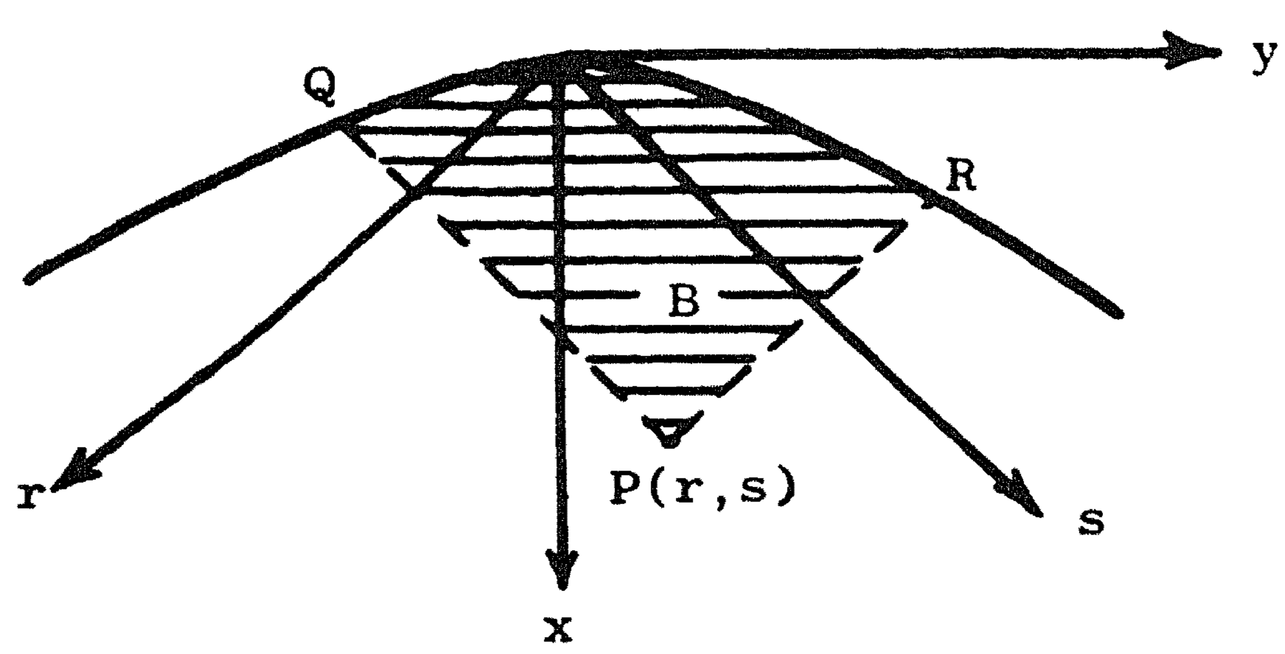

figure 4

where the area B is the part of the region W, enclosed within the forward Mach cone with vertex at the point (r,s,0); see figure 3.

If the airfoil with supersonic leading edge has a trailing edge, then the formula (4.1) is valid for all points at the upper wing surface, which lie outside the region of influence of this trailing edge.

The pressure distribution at the wing can easily be obtained by applying the differentiation $\frac{\partial}{\partial x} = \frac{\partial}{\partial r} + \frac{\partial}{\partial s}$ to the right-hand side of equation (4.1); cf. equation (2.8).

This differentiation will be carried out for the case of an infinitely thin airfoil with supersonic leading edge.

This edge is denoted by the equation $r = l(s)$; since the edge is supersonic, $l(s)$ is a monotonic function of s and therefore the edge may also be represented by the inverse function $s = m(r)$.

Applying $(\frac{\partial}{\partial r} + \frac{\partial}{\partial s})$ to (3.12), we obtain

$$(4.2) \qquad u_+(r,s) = -\tfrac{1}{2}(\frac{\partial}{\partial r} + \frac{\partial}{\partial s})w_+(r,s) * \Phi_{\frac{1}{2}}(r)\cdot \Phi_{\frac{1}{2}}(s).$$

The differentiation is taken in the distributional sense; this means, that we have to take into account the jump of the function $w_+(r,s)$ across the edge of the wing.
Assuming that $l(s)$ and $w_+(l(s),s)$ are infinitely differentiable, we may write

$$(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})w_+(r,s) = (\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\{w_+(r,s)+\Theta(r-l(s))\cdot w_+(l(s),s)\} + \tag{4.3}$$
$$+ w_+(l(s),s)\cdot(1-\frac{dl}{ds})\cdot \delta(r-l(s)),$$

where the differentiation in the right-hand side may now be taken in the ordinary sense.
The distribution $\Theta(r-l(s))$ and $\delta(r-l(s))$ are defined as

$$<\Theta(r-l(s)),\ \phi(r,s)> = \iint_{r>l(s)} \phi(r,s)\ dr\ ds,$$

and

$$<\delta(r-l(s)),\phi(r,s)> = \int_{-\infty}^{+\infty}\phi(l(s),s)\ ds,$$

where $\phi(r,s)$ is some test function belonging to D or S; cf. chapter I, section 7.
Substitution of equation (4.3) into equation (4.2) gives the result

$$u_+(r,s) = -\frac{1}{2\pi}\iint_B\{(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\sigma})w_+(\rho,\sigma)\}\cdot(r-\rho)^{-\frac{1}{2}}(s-\sigma)^{-\frac{1}{2}}d\rho\ d\sigma + \tag{4.4}$$
$$+ T(r,s),$$

with

$$T(r,s) = -\tfrac{1}{2}w_+(l(s),s)\cdot(1-\frac{dl}{ds})\cdot\ \delta(r-l(s)) * \Phi_{\frac{1}{2}}(r)\ \Phi_{\frac{1}{2}}(s) =$$
$$= -\tfrac{1}{2}\left[w_+(l(s),s)\cdot(1-\frac{dl}{ds})\cdot\ \delta(r-l(s)) * \Phi_{\frac{1}{2}}(r)\cdot\delta(s)\right]$$
$$* \Phi_{\frac{1}{2}}(s)\delta(r).$$

This term is reduced as follows.
The convolution product between the brackets vanishes for $r<l(s)$, while it can be written for $r>l(s)$ in the form

$$\frac{1}{\sqrt{\pi}}\, w_+(l(s),s)\cdot(1-\frac{dl}{ds})\int_0^r \delta(\rho-l(s))\cdot(r-\rho)^{-\frac{1}{2}}d\rho =$$

$$\frac{1}{\sqrt{\pi}}\,(1-\frac{dl}{ds})w_+(l(s),s)\cdot(r-l(s))^{-\frac{1}{2}}.$$

Hence we obtain for the term T(r,s) the expression

$$T(r,s) = -\frac{1}{2\pi}\int_{m(r)}^{s}(1-\frac{dl(\sigma)}{d\sigma})\cdot w_+(l(\sigma),\sigma)\cdot(r-l(\sigma))^{-\frac{1}{2}}(s-\sigma)^{-\frac{1}{2}}d\sigma \tag{4.5}$$

This integral is an integral along the leading edge from Q to R (see fig.4).

For the case of thin wings with cross-section symmetric with respect to z=0, the velocity component $u_+(r,s)$ may be obtained in more or less the same way.

5. Infinitely thin airfoils with tips

5.1. Tip effects

In the previous section we have treated all nonlifting wings with given thickness distribution and also infinitely thin lifting airfoils of given shape, but with the restriction that the leading edge should be supersonic.

Therefore, we are now led to the discussion of infinitely thin airfoils for which the leading edge is no longer supersonic.

In this section we discuss airfoils for which the edge is partly supersonic and partly subsonic; see figure 5.

Suppose $s=l_1(r)$ and $s=l_2(r)$ are the equations for respectively the supersonic and the subsonic part of the leading edge.

In the sequel it will always be assumed that wing planforms are bounded by curves, given by monotonic functions s=l(r), so that the edges may also be represented by r=m(s), where r=m(s) is the inverse of the function s=l(r).

In the part of the wing with $s < 0$ (or $y < -x$) the forward Mach cone from a point P' at the wing surface cuts from the airfoil a triangular region A'P'B', in which the downwash w_+ is given by the slope of the

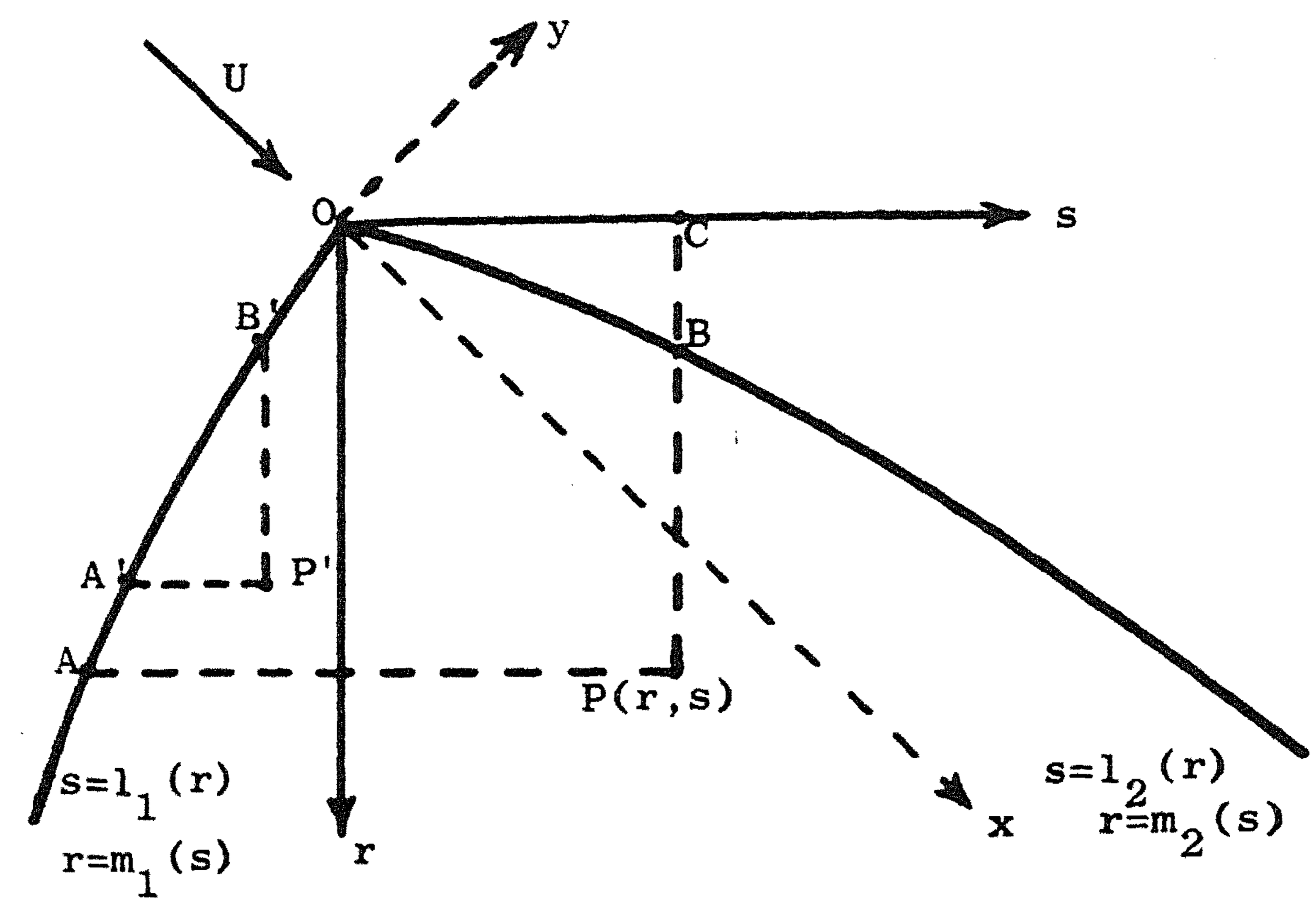

figure 5

wing and we may apply again formula (4.4).

In the part of the wing with $s > 0$ (or $y > -x$) this is no longer true. Application of formula (3.10) gives rise to a convolution integral over the area OAPBCO (see figure 5). However, the function $w_+(r,s)$ is now given by the slope of the wing only over the region OAPBO, while it is unknown in the region OBCO. In the latter region the function

$$u_+(r,s) = (\frac{\partial}{\partial r} + \frac{\partial}{\partial s})\ f_+(r,s) \tag{5.1}$$

vanishes.

We have now the problem to determine the function $u_+(r,s)$ in the region $s > 0$, while $w_+(r,s)$ is given for $l_1(r) < s < l_2(r)$ and $u_+(r,s) \equiv 0$ for $s > l_2(r)$.

Problems of this kind are important for calculating the influence of wing tips on the lift distribution at the surface of the wing.

Two different cases are distinguished, viz. the "raked-in" tip with $l_2(r) < r$ and $\frac{dl_2}{dr} < 1$ and the "raked-out" tip with $l_2(r) > r$ and $\frac{dl_2}{dr} > 1$.

In the first case the edge $s=l_2(r)$ is a subsonic trailing edge and therefore the Kutta condition should be satisfied at $s=l_2(r)$.

This condition states that the flow at both sides of the wing is tangential to the wing surface at the edge and that there is no flow around this edge. This means that the downwash distribution $w_+(r,s)$ is continuous across $s=l_2(r)$, while the lift distribution vanishes at

$s=l_2(r)$, i.e. $u_+(r,l_2(r)) \equiv 0$. In the second case the edge is a subsonic leading edge. There is now a flow around this edge from the lower side to the upper side of the wing surface. Since the edge is infinitely sharp and since the flow is assumed tangential to the surface of the airfoil, the lift distribution must become infinite at the edge; i.e. $u_+(r,s)$ becomes infinite at $s=l_2(r)$.

These two cases will be treated in the next two sections 5.2 and 5.3.

5.2. The raked-in tip

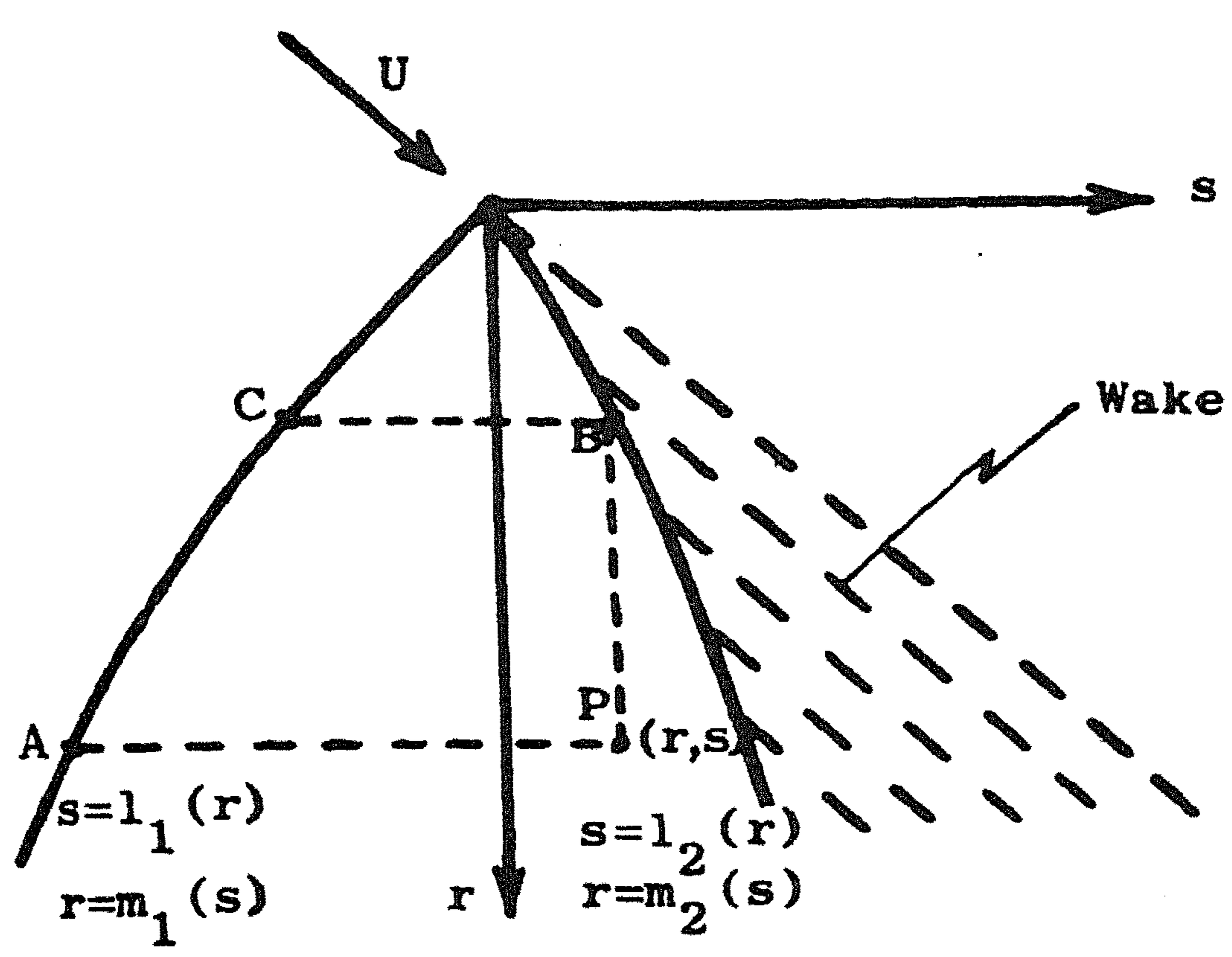

figure 6

Using the basic formula (3.12), we can carry out the following reduction.

$$f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s) = -\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(r)\Phi_{\frac{1}{2}}(s) * \Phi_{-\frac{1}{2}}(r)\delta(s) =$$
$$= -\tfrac{1}{2}w_+(r,s) * \{\Phi_{\frac{1}{2}}(r) * \Phi_{-\frac{1}{2}}(r)\} \cdot$$
$$\cdot \{\Phi_{\frac{1}{2}}(s) * \delta(s)\} =$$
$$= -\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(s)\cdot\delta(r).$$

Hence

$$f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s) = -\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(s)\delta(r). \tag{5.2}$$

After differentiating with respect to x we obtain

$$u_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s) = -\tfrac{1}{2}\left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) \cdot \int_{l_1(r)}^{s} w_+(r,\sigma)\, \Phi_{\frac{1}{2}}(s-\sigma)\,d\sigma . \tag{5.3}$$

Due to the Kutta condition the function $w_+(r,s)$ is continuous across the line $s=l_2(r)$. If moreover the given downwash distribution is assumed to be bounded at the wing surface, the right-hand side of equation (5.3) is locally integrable in the neighbourhood of $s=l_2(r)$ $(r=m_2(s))$. Because $u_+(r,s)$ vanishes for $r<m_2(s)$, $s>0$, the right-hand side of equation (5.3) is also zero for $r<m_2(s)$, $s>0$.
Hence after taking the convolution of both sides of (5.3) with $\Phi_{+\frac{1}{2}}(r)\delta(s)$, we obtain

$$u_+(r,s) = -\tfrac{1}{2}\int_{m_2(s)}^{r} \Phi_{\frac{1}{2}}(r-\rho)\ \{(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial s})\int_{l_1(\rho)}^{s} w_+(\rho,\sigma)\Phi_{\frac{1}{2}}(s-\sigma)d\sigma\}\,d\rho =$$

$$(5.4)\qquad = -\frac{1}{2\pi}\int_{m_2(s)}^{r} (r-\rho)^{-\frac{1}{2}}\ \{(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial s})\int_{l_1(\rho)}^{s} w_+(\rho,\sigma)(s-\sigma)^{-\frac{1}{2}}d\sigma\}\,d\rho,$$

valid for $0<s<l_2(r)$.
The integral is an integral over the area APBC and the lift in P does not depend on the slope of the airfoil in the region OBC. This fact is usually stated as Evvard's rule [8] .

5.3. The raked-out tip

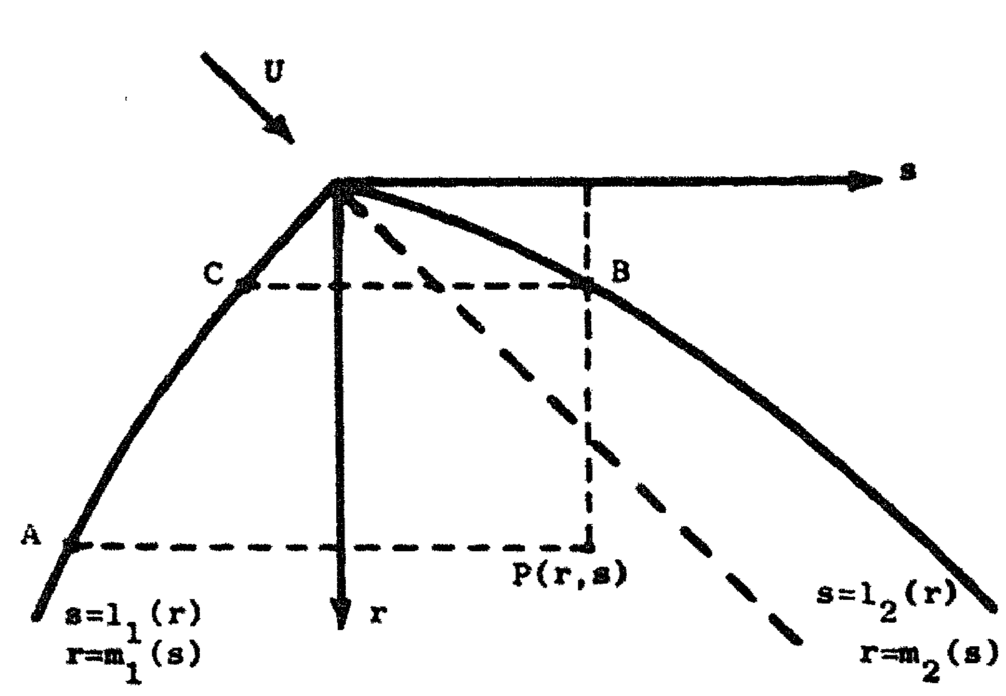

figure 7

Because $u_+(r,s)$ vanishes for $r<m_2(s)$, $s>0$ we have also $f_+(r,s)\equiv 0$ for $r<m_2(s)$, $s>0$; see figure 7. Hence it follows from equation (5.2), that

$$\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(s)\ \delta(r)\equiv 0$$

for $r<m_2(s)$, $s>0$. This means

$$\tfrac{1}{2}\int_{l_1(r)}^{s} w_+(r,\sigma)\ \Phi_{\frac{1}{2}}(s-\sigma)d\sigma \equiv 0$$

for $s>l_2(r)$.
On the other hand we have in general

$$\tfrac{1}{2}\int_{l_1(r)}^{l_2(r)} w_+(r,\sigma)\ \Phi_{\frac{1}{2}}(l_2(r)-\sigma)d\sigma \neq 0$$

and therefore

$$\tfrac{1}{2}\int_{l_1(r)}^{s} w_+(r,\sigma)\ \Phi_{\frac{1}{2}}(s-\sigma)d\sigma$$

is a discontinuous function of s along the leading edge $s=l_2(r)$. The differentiation $(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})$ (compare 5.3) will now introduce a "line" distribution concentrated at the edge $s=l_2(r)$.
We introduce the function

$$F(r) = \int_{l_1(r)}^{l_2(r)} w_+(r,\sigma)\ \Phi_{\frac{1}{2}}(l_2(r)-\sigma)d\sigma\ . \tag{5.5}$$

If we assume the slope and the edges of the wing surface infinitely differentiable, the function F(r) is a C^∞ function for $r>0$.
Applying equation (5.3) we obtain

$$u_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s) =$$

$$-\tfrac{1}{2}(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\left[\int_{l_1(r)}^{s} w_+(r,\sigma)\ \Phi_{\frac{1}{2}}(s-\sigma)d\sigma + \Theta(s-l_2(r))\cdot F(r)\right] + \tag{5.6}$$

$$+\tfrac{1}{2}\delta(s-l_2(r))\ (1-\frac{dl_2}{dr})\cdot F(r) + \tfrac{1}{2}\,\Theta(s-l_2(r))\cdot\frac{dF}{dr}\ ,$$

where the differentiation $(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})$ may now be taken in the ordinary sense.
The distributions $\Theta(s-l_2(r))$ and $\delta(s-l_2(r))$ are defined as

$$<\Theta(s-l_2(r)),\ \phi(r,s)> = \iint_{s>l_2(r)} \phi(r,s)dr\ ds, \tag{5.7}$$

and

$$<\delta(s-l_2(r)),\ \phi(r,s)> = \int_{-\infty}^{\infty}\phi(r,l_2(r))dr, \tag{5.8}$$

where ϕ is some test function belonging to D or S and having its support in the region $r>0$; cf. chapter I, section 7. The left-hand side and hence also the right-hand side of (5.6) vanishes for $r<m_2(s)$, $s>0$; taking the convolution of both sides of equation (5.6) with $\Phi_{\frac{1}{2}}(r)\ \delta(s)$, we obtain for $0<s<l_2(r)$ the result

$$u_+(r,s) =$$

$$(5.9)\quad -\frac{1}{2\pi}\int\limits_{m_2(s)}^{r} \Phi_{\frac{1}{2}}(r-\rho)d\rho \left(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial s}\right) \int\limits_{l_1(\rho)}^{s} w_+(\rho,\sigma)\, \Phi_{\frac{1}{2}}(s-\sigma)d\sigma + T(r,s) =$$

$$-\frac{1}{2\pi}\int\limits_{m_2(s)}^{r} (r-\rho)^{-\frac{1}{2}}d\rho \left(\frac{\partial}{\partial\rho}+\frac{\partial}{\partial s}\right) \int\limits_{l_1(\rho)}^{s} w_+(\rho,\sigma)(s-\sigma)^{-\frac{1}{2}}d\sigma + T(r,s),$$

where the function $T(r,s)$ is given by the expression

$$(5.10)\qquad T(r,s) = \tfrac{1}{2}\delta(s-l_2(r)) \,.\, \left(1-\frac{dl_2}{dr}\right) \cdot F(r) * \Phi_{\frac{1}{2}}(r)\ \delta(s).$$

Apart from the extra term $T(r,s)$, we have obtained the same result as for the "raked-in" tip, given by equation (5.4).
The term $T(r,s)$ will be reduced as follows.
Instead of the distribution $\delta(s-l_2(r))$ we may write

$$\delta(r-m_2(s))\ \frac{dm_2}{ds}\ ,$$

and we obtain for $r > m_2(s)$, $s > 0$

$$T(r,s) = \tfrac{1}{2}\delta(r-m_2(s)).\left(\frac{dm_2}{ds}-1\right) \cdot F(m_2(s)) * \Phi_{\frac{1}{2}}(r)\delta(s) =$$

$$\frac{1}{2\sqrt{\pi}}\left(\frac{dm_2}{ds}-1\right) \cdot F(m_2(s)) \int\limits_{m_2(s)-0}^{r} \delta(\rho-m_2(s))(r-\rho)^{-\frac{1}{2}}d\rho =$$

$$\frac{1}{2\sqrt{\pi}}\left(\frac{dm_2}{ds}-1\right)\{r-m_2(s)\}^{-\frac{1}{2}}\ F(m_2(s)).$$

Substituting the expression (5.5), we finally get

$$(5.11)\qquad T(r,s) = \frac{1}{2\pi}\left(\frac{dm_2}{ds}-1\right)\{r-m_2(s)\}^{-\frac{1}{2}} \cdot \int\limits_{l_1(m_2(s))}^{s} w_+(m_2(s),\sigma)\cdot(s-\sigma)^{-\frac{1}{2}}d\sigma\ .$$

The integral in the right-hand side is an integral along the line CB. If the point P at the wing approaches the point B at the edge of the wing, the velocity component $u_+(r,s)$ behaves as $\{r-m_2(s)\}^{-\frac{1}{2}}$, as was expected.

6. Airfoils with subsonic leading edge

Having treated the airfoil with partly supersonic and partly subsonic leading edge, it would be interesting to deal now with the case of a completely subsonic leading edge.
However, this case is essentially much more difficult than that of the preceding section. This is due to the fact that there is a mutual influence of the downwash fields at both sides of the wing; see figure 8. Actually, the problem to determine the lift distribution of an arbitrary airfoil with subsonic leading edge has not been solved up till now in a satisfactory way. A solution has been constructed only for the special case of a triangular wing with straight edges, while the downwash distribution at the wing surface is of the form

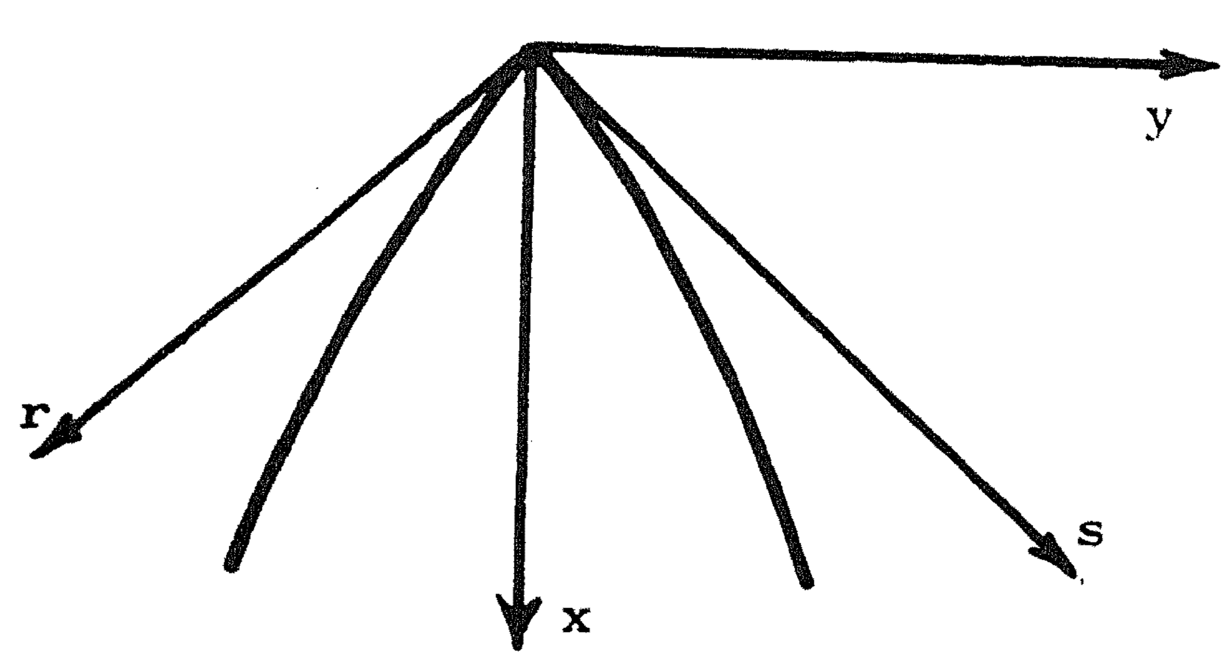

figure 8

(6.1) $$w_+(x,y) = x^n g(\frac{y}{x}), \quad n=0,1,2,\dots,$$

where g is a sufficiently differentiable function [5] .
In this connection we refer the reader also to the theory of conical flows; see for example reference [9] .
Since the difficulties for obtaining a general solution are outside the scope of distribution theory, we shall not deal with this problem in this general treatment.

7. Wings with prescribed pressure distribution

We consider in this section the case that instead of the downwash the pressure distribution is given at the wing surface, and the geometry of the wing (i.e. thickness distribution or camber and angle of incidence) is to be determined.
For infinitely thin wings this problem is tantamount to the problem of the determination of the downwash distribution $w_+(r,s)$ from the

velocity component $u_+(r,s)$, which is prescribed at the wing surface W and which vanishes outside W.
In the language of distributions this is a very simple problem.

We first introduce the function $V(x,y,z)$, defined as:

$$(7.1) \qquad V(x,y,z) = \begin{cases} +\frac{1}{2\pi} \operatorname{Arcosh}\left[\frac{x}{+(y^2+z^2)^{\frac{1}{2}}}\right] & \text{for } x > +(y^2+z^2)^{\frac{1}{2}}, \\ 0 & \text{for } x < +(y^2+z^2)^{\frac{1}{2}}. \end{cases}$$

This function is known as Volterra's solution of the wave equation [10]; the coordinate x plays again the role of the time.
One can easily verify that the function $V(x,y,z)$ and the elementary solution $E(x,y,z)$, given by equation (3.2), are connected with each other by the relation

$$(7.2) \qquad \frac{\partial V}{\partial x} = E(x,y,z).$$

Taking z=0 and using again characteristic coordinates (r,s), we obtain a function, denoted by $\Psi(r,s)$ and given by the expression

$$(7.3) \qquad V(x,y,0) = \Psi(r,s) = \frac{1}{2\pi}\Theta(r)\ \Theta(s)\ \operatorname{Arcosh}\left|\frac{r+s}{r-s}\right| .$$

Moreover, we have the relation

$$(7.4) \qquad \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s}\right) \Psi(r,s) = \tfrac{1}{2}\ \Phi_{\frac{1}{2}}(r)\ \Phi_{\frac{1}{2}}(s).$$

In order to calculate the slope of the surface of an airfoil from its prescribed lift distribution we have to determine the downwash w_+ at the wing surface from the values of u_+, given in the plane z=+0.
We start again from our basic formula (3.11), viz.

$$(3.12) \qquad f_+(r,s) = -\tfrac{1}{2}w_+(r,s) * \Phi_{\frac{1}{2}}(r)\cdot\Phi_{\frac{1}{2}}(s)$$

or in equivalent form

$$(7.5) \qquad w_+(r,s) = -2f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\cdot\ \Phi_{-\frac{1}{2}}(s).$$

Instead of $\Phi_{-\frac{1}{2}}(r)\cdot\ \Phi_{-\frac{1}{2}}(s)$ we may write

$$(7.6)\qquad \Phi_{-\frac{1}{2}}(r)\ \Phi_{-\frac{1}{2}}(s) = \tfrac{1}{4}\{(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})^2-(\frac{\partial}{\partial r}-\frac{\partial}{\partial s})^2\}\Phi_{\frac{1}{2}}(r)\cdot\Phi_{\frac{1}{2}}(s).$$

Substituting (7.6) into (7.5), using

$$u_+(r,s) = (\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\ f_+(r,s)$$

and applying the relation (7.4), we obtain the result

$$(7.7)\qquad \begin{aligned} w_+(r,s) &= -\tfrac{1}{2}(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\ u_+(r,s) * \Phi_{\frac{1}{2}}(r)\ \Phi_{\frac{1}{2}}(s) \\ &+ (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ u_+(r,s) * (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ \Psi(r,s). \end{aligned}$$

Equation (7.7) yields the downwash at the plane of the airfoil in terms of the function $u_+(r,s)$, which is proportional to the lift distribution at the airfoil.

Reducing (7.7) to an expression with ordinary integrals we should be careful in integrating across the edge of the wing, since $u_+(r,s)$ may be discontinuous at this edge and the differentiations give rise to the appearance of δ-functions concentrated along the edge of the airfoil. When $u_+(r,s)$ is infinite at the edge (e.g. in the case of subsonic leading edge), the calculations become even more complicated. We can avoid these complications by writing

$$(7.8)\qquad \begin{aligned} w_+(r,s) &= -\tfrac{1}{2}(\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\ \left[u_+(r,s) * \Phi_{\frac{1}{2}}(r)\ \Phi_{\frac{1}{2}}(s)\right] + \\ &+ (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ \left[u_+(r,s) * (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ \Psi(r,s)\right]. \end{aligned}$$

Substituting

$$(7.9)\qquad (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ \Psi(r,s) = \frac{1}{2\pi}\ (rs)^{-\frac{1}{2}}\ \frac{s+r}{s-r}$$

we obtain finally the result

$$(7.10)\qquad \begin{aligned} w_+(r,s) &= -\frac{1}{2\pi}\ (\frac{\partial}{\partial r}+\frac{\partial}{\partial s})\ \iint_{PAOB} u_+(r-\rho,s-\sigma)\rho^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}d\rho\ d\sigma \\ &+ \frac{1}{2\pi}\ (\frac{\partial}{\partial r}-\frac{\partial}{\partial s})\ \iint_{PAOB} u_+(r-\rho,s-\sigma)\rho^{-\frac{1}{2}}\sigma^{-\frac{1}{2}}\ \frac{\sigma+\rho}{\sigma-\rho}\ d\rho\ d\sigma\ , \end{aligned}$$

where the second integral should be considered in the sense of the

Cauchy principal value. As to the region PAOB see figure 9. It is to be noted that the equation (7.10) is valid for infinitely thin airfoils with arbitrary leading edge, while the prescribed lift distribution is only restricted by the requirement, that the integrals in the right-hand side of (7.10) exist.

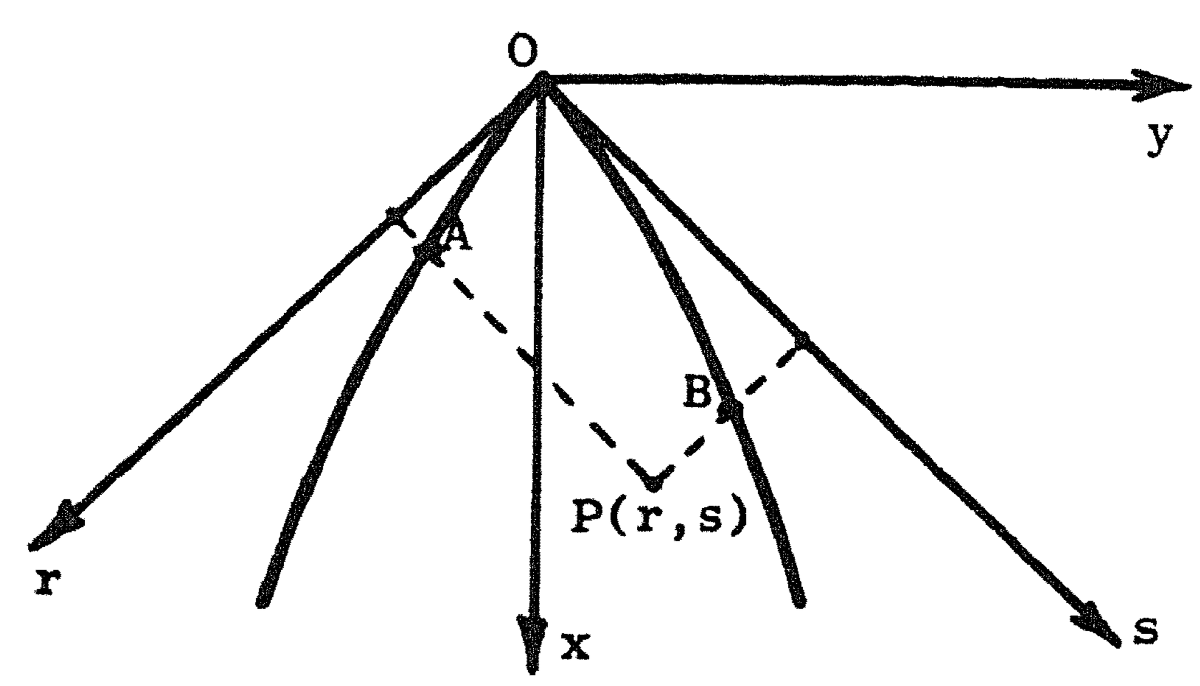

figure 9

The formula (7.10) may also be applied for calculating the thickness distribution of a nonlifting wing, symmetric with respect to the plane z=0, while its leading edge should be supersonic.

However, if its leading edge is subsonic, the given function $u_+(r,s)$ must be subject to an extra condition, since the function $w_+(r,s)$ vanishes identically outside the wing surface. Moreover, the solution $w_+(r,s)$ is no longer uniquely determined, because in this case the equation

$$w_+(r,s) * \Phi_{\frac{1}{2}}(r)\ \Phi_{\frac{1}{2}}(s) = 0$$

has a nonvanishing solution (cf. [1] , D 13).

8. Mixed problems

Another class of important problems are the so-called mixed problems. They occur in lift cancellation technique, which is a useful tool for the calculation of the lift distribution on infinitely thin airfoils of rather general planform.

Consider an airfoil at the surface of which the downwash and the lift distribution are already known; let us assume $u_+(r,s) = u^*(r,s)$ and $w_+(r,s) = w^*(r,s)$, where u^* and w^* are given functions of r and s. The leading edges of the airfoil are given by the equations $s=l_o(r)$ $(r=m_o(s))$ and $s=l_1(r)$ $(r=m_1(s))$; see figure 10[a]. The edges $s=l_o(r)$ and $s=l_1(r)$ are supposed to be respectively supersonic and subsonic.

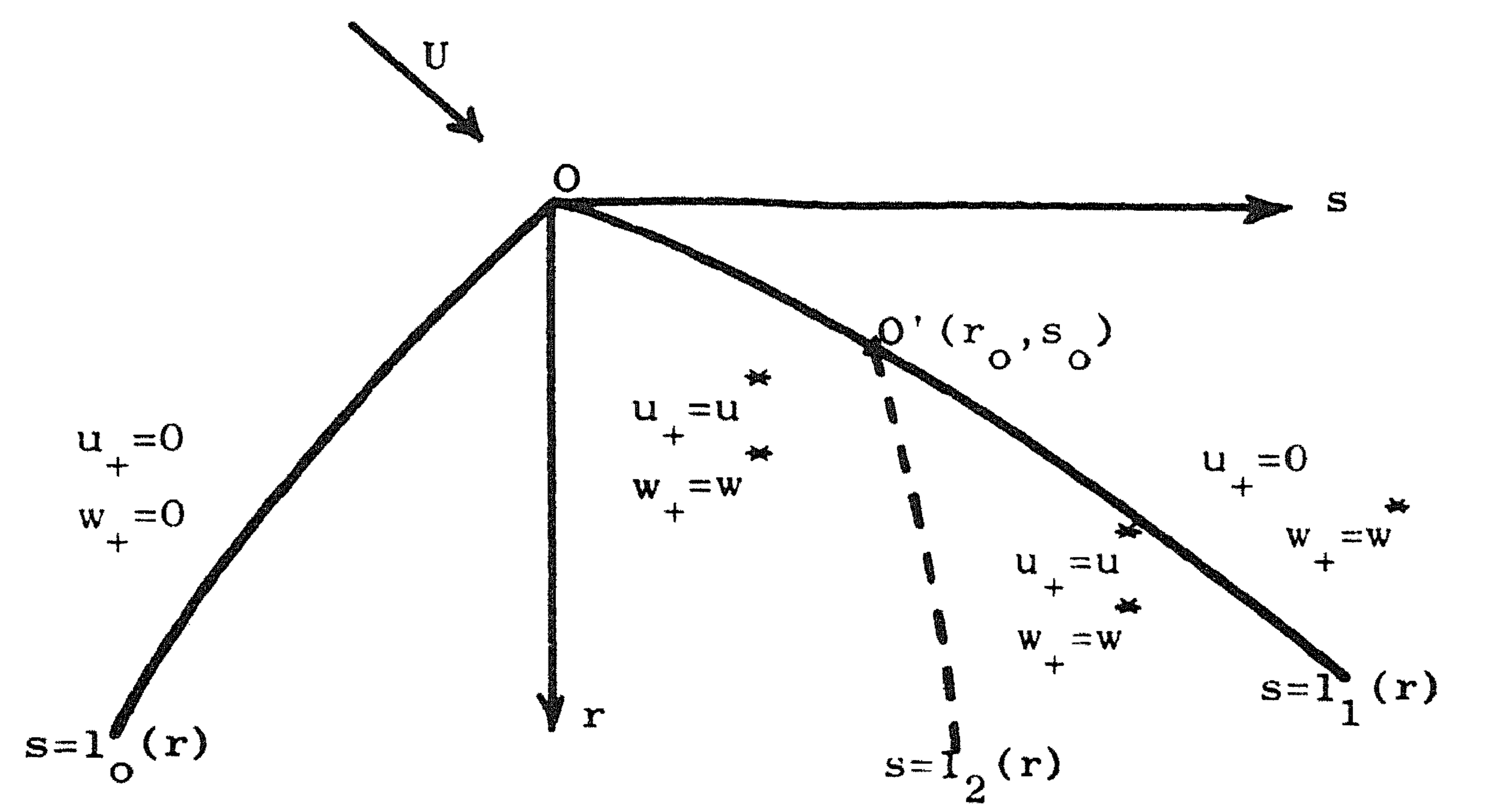

figure 10[a]

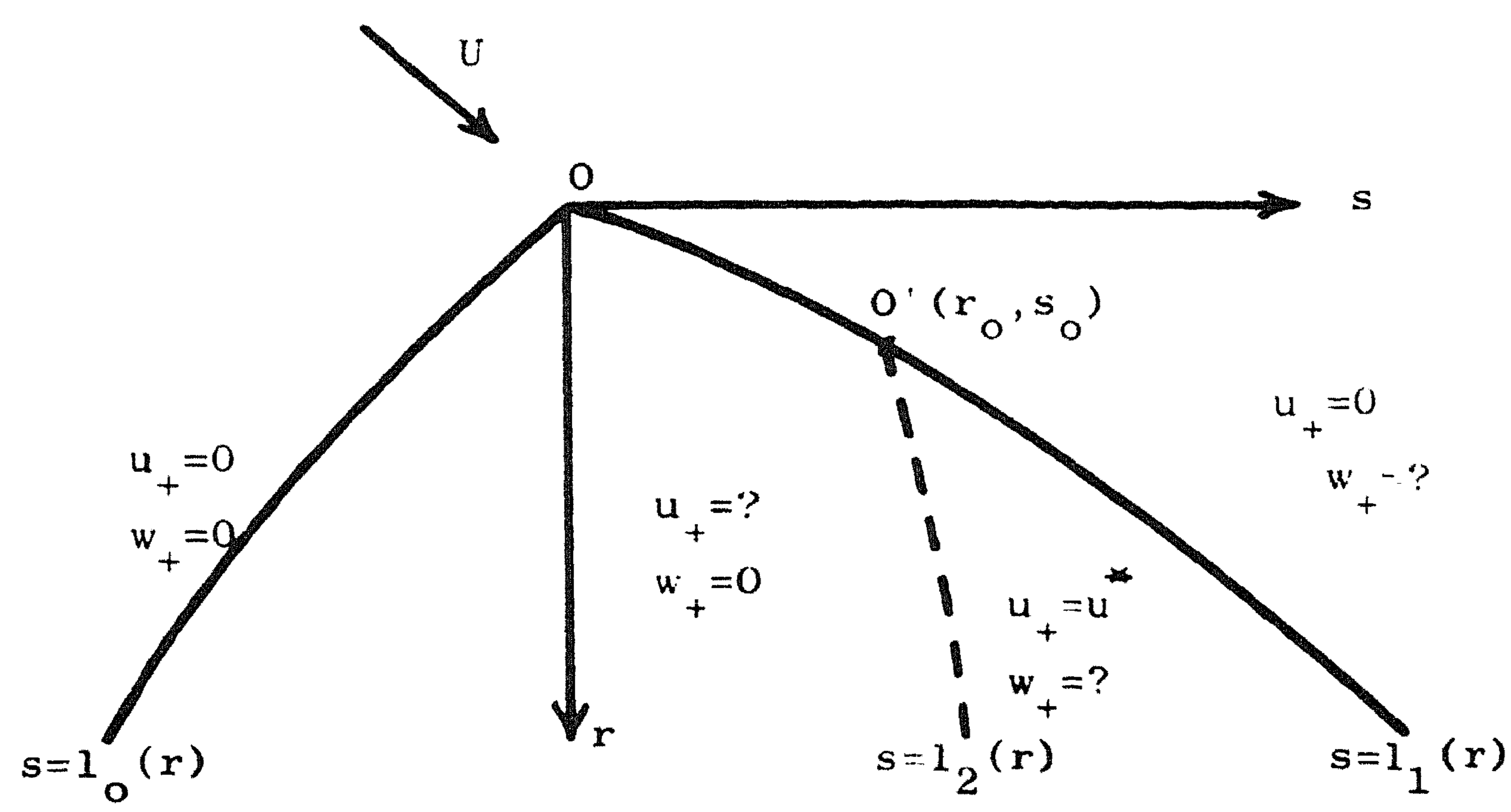

figure 10[b]

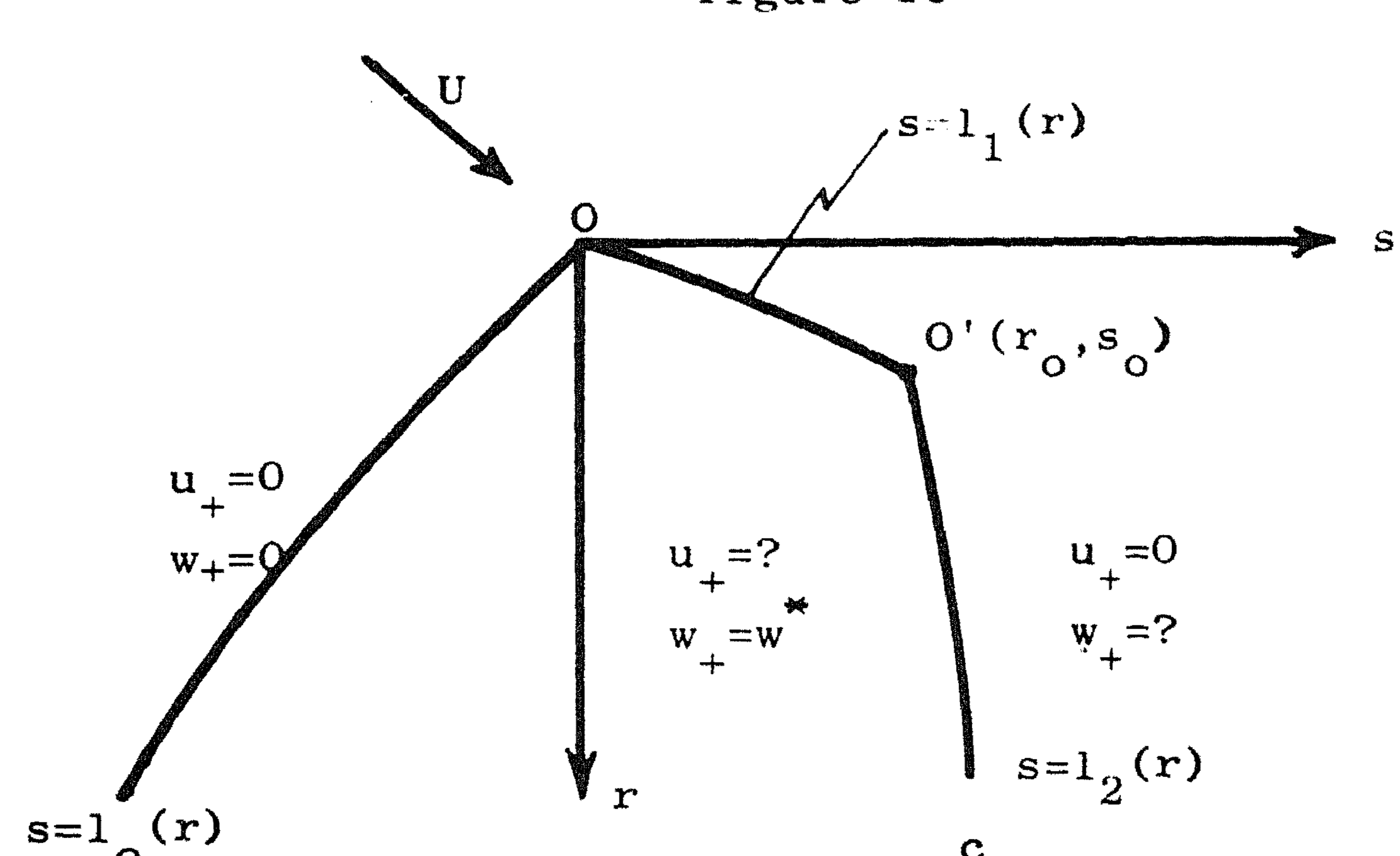

figure 10[c]

Now, consider a second airfoil, travelling at the same speed and having the same planform as the first one, but its surface is constructed in such a way, that to the left of a certain line $s=l_2(r)$ (or $r=m_2(s)$) the downwash distribution $w_+(r,s)$ equals zero and that to the right of this line the lift distribution is the same as that on the first wing. However, the lift distribution is now unknown at points to the left of $s=l_2(r)$, while the downwash distribution is unknown at points to the right of $s=l_2(r)$; see figure 10^b. If the boundary values for the second wing are subtracted from those of the first, there result the boundary values for a third wing with planform bounded by the lines $s=l_o(r)$, $s=l_1(r)$ and $s=l_2(r)$ and with a downwash distribution, which coincides at the surface of the wing with that prescribed on the first wing.
Therefore the influence of cutting off an edge of a given wing can be determined by solving the second boundary value problem, which is an example of a mixed boundary value problem.

We have to distinguish here again between the two cases, whether the new edge is "raked-in" or "raked-out".
In the first case the new edge is a trailing edge and therefore the Kutta condition, $u_+=0$, should be satisfied at the edge $s=l_2(r)$. Hence $u_+(r,s)$ must be continuous across the line $s=l_2(r)$ in the boundary value problem, corresponding with figure 10^b.
In the second case, the new edge is part of the leading edge and $u_+(r,s)$ behaves at the wing surface as $\{l_2(r)-s\}^{-\frac{1}{2}}$ for (r,s) in the neighbourhood of $s=l_2(r)$. It follows now that,in the boundary value problem of figure 10^b, $u_+(r,s)$ is discontinuous across the line $s=l_2(r)$.
The case of the raked-in edge will be treated first.

The origin of the coordinate system is translated to the point of intersection (r_o,s_o) of the leading edge $s=l_1(r)$ and the new edge $s=l_2(r)$; one obtains the coordinate system (r',s'), in which the edges are given by $s' = l_i'(r')$, $i=0,1,2$. However, in order to maintain a simple notation, we shall omit the primes in the sequel.
The influence of the new edge $s=l_2(r)$ is only restricted to the region

$s > 0$ and for calculating the influence on the lift distribution we need only to determine the function $u_+(r,s)$ in the region $0 < s < l_2(r)$; see figure 11.

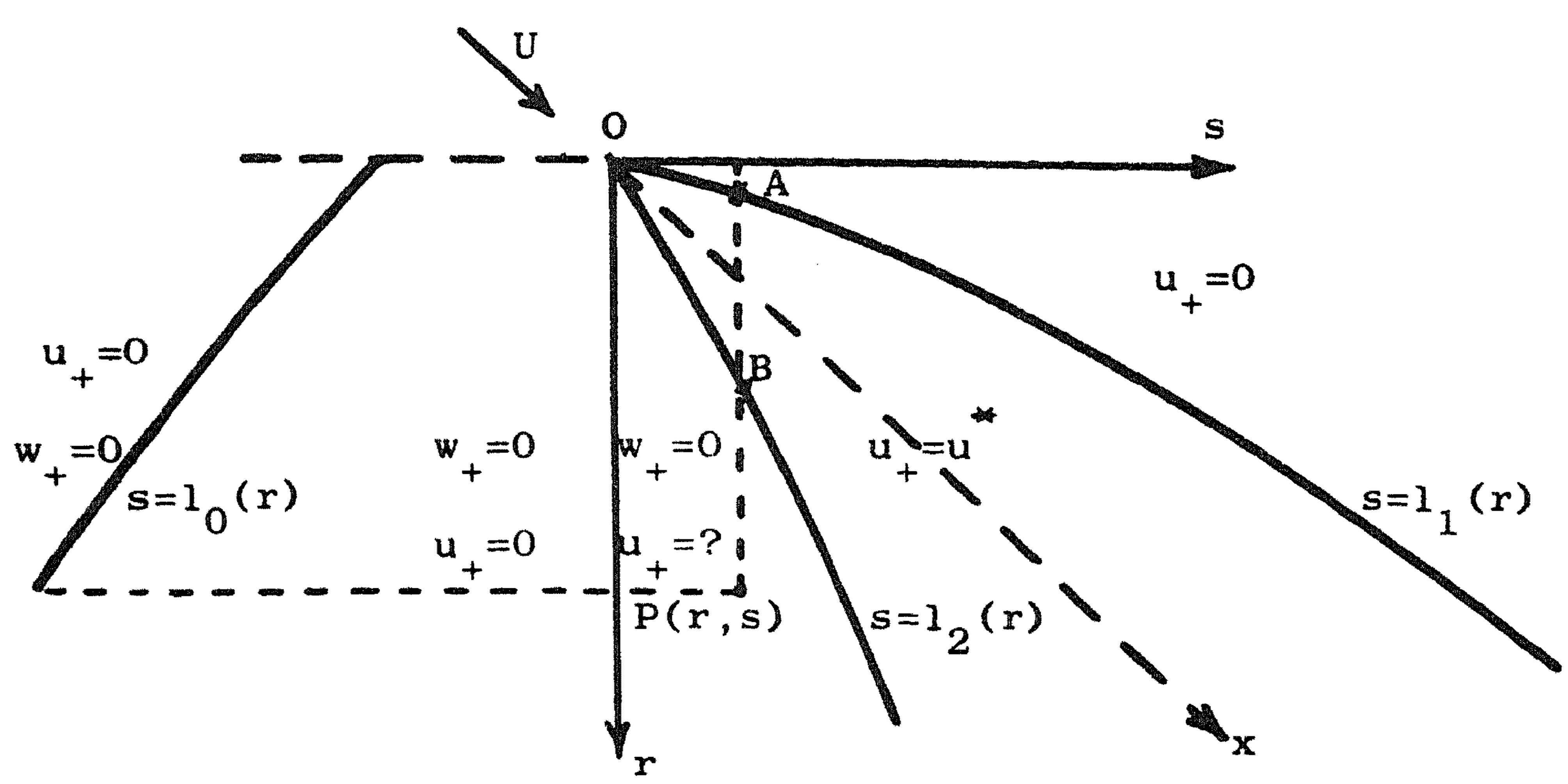

figure 11

According to formula (3.12) we may write

(8.1) $$w_+(r,s) * \Phi_{\frac{1}{2}}(s)\ \delta(r) = -2f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s).$$

Since $w_+(r,s) \equiv 0$ for $s < l_2(r)$, it follows immediately that

(8.2) $$f_+(r,s) * \Phi_{-\frac{1}{2}}(r)\cdot\delta(s) \equiv 0,$$

for $0 < s < l_2(r)$.

Applying the operator $(\frac{\partial}{\partial r} + \frac{\partial}{\partial s})$ to the left-hand side of (8.2) we obtain

(8.3) $$u_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s) \equiv 0,$$

for $0 < s < l_2(r)$.

Because $u_+(r,s)$ is continuous across the line $r=m_2(s)$ and because $u_+(r,s)$ vanishes for $r < m_1(s)$, we may write

(8.4) $$u_+(r,s) = \Theta(r-m_2(s))u_+(r,s) + \{\Theta(m_2(s)-r) - \Theta(m_1(s)-r)\}\cdot$$
$$\cdot\, u^*(r,s).$$

We introduce now the distribution $q(r,s)$, defined as

$$q(r,s) = \Theta(r-m_2(s))\; u_+(r,s) * \Phi_{-\frac{1}{2}}(r)\delta(s), \quad s>0. \tag{8.5}$$

Obviously, this distribution vanishes for $r<m_2(s)$, $s>0$; substitution of (8.4) into (8.3) gives for $r>m_2(s)$, $s>0$ the result

$$q(r,s) = \frac{+1}{2\sqrt{\pi}} \int_{m_1(s)}^{m_2(s)} u^*(\rho,s)(r-\rho)^{-3/2} d\rho\,. \tag{8.6}$$

Thus the distribution $q(r,s)$ is determined everywhere in the half-plane $s>0$ with the exception of the line $r=m_2(s)$.
Hence $q(r,s)$ is known apart from a finite linear combination of the distribution $\delta(r-m_2(s))$ and its derivatives.
This linear combination is denoted by

$$d(r,s) = \sum_{i=0}^{p} a_i(s)\, \delta^{(i)}(r-m_2(s)), \tag{8.7}$$

where the coefficients $a_i(s)$ are infinitely differentiable functions of s. Taking the convolution of $q(r,s)$ with $\Phi_{\frac{1}{2}}(r)\delta(s)$ we obtain the result

$$u_+(r,s) = \frac{1}{2\pi} \int_{m_2(s)}^{r} (r-\rho)^{-\frac{1}{2}} d\rho \int_{m_1(s)}^{m_2(s)} u^*(\rho',s)(\rho-\rho')^{-3/2} d\rho' + $$
$$+ \sum_{i=0}^{p} a_i(s) \frac{\partial^i}{\partial r^i} \{r-m_2(s)\}^{-\frac{1}{2}},$$

valid for $r>m_2(s)$, $s>0$.
This formula can be simplified by interchanging the order of integration and calculating consecutively the inner integral. Then we get

$$u_+(r,s) = \frac{1}{\pi} \{r-m_2(s)\}^{\frac{1}{2}} \int_{m_1(s)}^{m_2(s)} \frac{u^*(\rho,s)}{r-\rho} \{m_2(s)-\rho\}^{-\frac{1}{2}} d\rho + $$
$$+ \sum_{i=0}^{p} a_i(s) \frac{\partial^i}{\partial r^i} \{r-m_2(s)\}^{-\frac{1}{2}}, \tag{8.8}$$

valid for $r>m_2(s)$, $s>0$.
It is not difficult to show that the first term of the right-hand side of (8.8) tends to $u_+^*(m_2(s),s)$ for $r\to m_2(s)$.

In the case that $u_+(r,s)$ is continuous across the line $r=m_2(s)$, all coefficients $a_i(s)$ are necessarily equal to zero.
Hence the final result for the "raked-in" tip becomes

$$(8.9) \qquad u_+(r,s) = \frac{1}{\pi}\{r-m_2(s)\}^{\frac{1}{2}} \int_{m_1(s)}^{m_2(s)} \frac{u^*(\rho,s)}{r-\rho} \{m_2(s)-\rho\}^{-\frac{1}{2}} d\rho ,$$

valid for $0 < s < l_2(r)$.

In the case of the "raked-out" tip the function $u_+(r,s)$ is not continuous across the line $r=m_2(s)$, in contrast with the potential $f_+(r,s)$ which is certainly continuous across this line.
The values of $f_+(r,s)$ for $l_2(r) < s < l_1(r)$ can be obtained by integrating $u^*(r,s)$ in the direction of the x-coordinate. This integration is possible, since $s=l_2(r)$ is part of the leading edge of the airfoil. Denoting these values of $f_+(r,s)$ by $f^*(r,s)$, we can now calculate $f_+(r,s)$ in the same way as $u_+(r,s)$ in the case of the "raked-in" edge by replacing u_+ by f_+, u^* by f^* and omitting the differentiation $(\frac{\partial}{\partial r} + \frac{\partial}{\partial s})$.
We obtain for the case of the "raked-out" edge

$$(8.10) \qquad f_+(r,s) = \frac{1}{\pi}\{r-m_2(s)\}^{\frac{1}{2}} \int_{m_1(s)}^{m_2(s)} \frac{f^*(\rho,s)}{r-\rho} \{m_2(s)-\rho\}^{-\frac{1}{2}} d\rho ,$$

valid for $0 < s < l_2(r)$.
Applying the differentiation $\frac{\partial}{\partial r} + \frac{\partial}{\partial s}$, we get finally the result

$$(8.11) \qquad u_+(r,s) = \frac{1}{\pi}(\frac{\partial}{\partial r} + \frac{\partial}{\partial s})\Big[\{r-m_2(s)\}^{\frac{1}{2}} \cdot$$

$$\cdot \int_{m_1(s)}^{m_2(s)} \frac{f^*(\rho,s)}{r-\rho} \{m_2(s)-\rho\}^{-\frac{1}{2}} d\rho\Big] ,$$

valid for $0 < s < l_2(r)$.

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CHAPTER IV

THE LORENTZ-INVARIANT SOLUTIONS OF THE KLEIN-GORDON EQUATION

1. Introduction

The solutions, invariant under all proper Lorentz transformations, of the homogeneous and inhomogeneous Klein-Gordon equation (see e.g. [1], [2] and [3])

(1.1) $$(\Box - m^2)\ f(x) = 0,$$

(1.2) $$(\Box - m^2)\ g(x) = -\ \delta(x),$$

play an important role in relativistic quantum field theory. The function f(x) or g(x) is the wave function connected with a particle of mass m, x denotes the coordinates of a point (x_0, x_1, x_2, x_3) in R_4; x_1, x_2, x_3 are space coordinates and x_0 is the time coordinate. The symbol $\Box$ stands for the differential operator

(1.3) $$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_0^2}$$

and $\delta(x)$ denotes the four dimensional Dirac function concentrated in the origin of the coordinate system.
In the case of m = 0, the equations (1.1) and (1.2) reduce to the ordinary wave equations in three dimensional space and the functions f(x) and g(x) are wave functions connected with a photon.

In textbooks on field theory the solutions of (1.1) and (1.2) are usually obtained in a rather formal way, see e.g. [1], [2]. They are determined by applying a Fourier transformation to (1.1) and (1.2). The Fourier transform $\hat{g}(k)$ of e.g. g(x) satisfies the equation

(1.4) $$(k^2 - m^2)\ \hat{g}(k) = -\ 1,$$

with $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$.
The general Lorentz-invariant solution of (1.4) is of the form

$$(1.5) \qquad \hat{g}(k) = \frac{1}{m^2 - k^2} + c_+\delta_+(m^2 - k^2) + c_-\delta_-(m^2 - k^2),$$

where $\delta_+(m^2 - k^2)$ and $\delta_-(m^2 - k^2)$ are Dirac functions concentrated on respectively the upper and lower sheets of the hyperboloid $m^2 - k^2 = 0$ and c_+ and c_- are arbitrary constants.
The inverse transforms of $\delta_+(m^2 - k^2)$ and $\delta_-(m^2 - k^2)$ are obtained by purely formal calculations; for example, divergent integrals are converted into convergent integrals by merely interchanging the operations of differentiation and integration (see [1, §15.1], [2, §15.b]). It is obvious that this rather formal procedure cannot claim sufficiently mathematical rigor. The difficulties stem essentially from the fact that the Dirac-functions $\delta_\pm(m^2 - k^2)$ are not functions in the classical sense; they are generalized functions or distributions and they should be treated as such. To obtain the Lorentz-invariant solutions of (1.1) and (1.2) in a rigorous way one needs essentially the theory of distributions and the calculations have to be performed within the frame work of this theory.
A proper derivation of solutions of (1.1) and (1.2) has been given by several authors, a.o. L. Schwartz [4, Vol. II, pp. 33-36], P.D. Methée [5] - [6], J. Lavoine [7], and the present author [8] - [9].
L. Schwartz has determined the solution of (1 2) which vanishes for $x_0 < 0$. By using a convolution algebra it is not difficult to show that this solution is uniquely determined. The equation (1.2) is written in the form

$$(\Box - m^2)\delta(x) * g(x) = -\delta(x)$$

and its solution as

$$g(x) = -\left[(\Box - m^2)\delta(x)\right]^{*(-1)}.$$

A formal expansion yields

(1.6) $$g(x) = - \sum_{k=0}^{\infty} m^{2k} \Box^{*(1-k)}$$

with
$$\Box^{*(+1)} = \Box\delta(x)$$
$$\Box^{*(0)} = \delta(x)$$
$$\Box^{*(-1)} = - Z_2(x)$$
$$\Box^{*(p)} * \Box^{*(q)} = \Box^{*(p+q)} \quad \text{(p and q integer)}.$$

$Z_2(x)$ is the Riesz-distribution given in Chapter I, section 8.2. In particular one has

(1.7) $$\Box^{*(-k-1)} = (-1)^{k+1} Z_{2k+2} = \frac{(-1)^{k+1}}{\pi \cdot 2^{2k+1}(k)!(k-1)!} \sqrt{x_0^2-x_1^2-x_2^2-x_3^2}^{\,2k-2}$$

Substitution of (1.7) into (1.6) yields a convergent series and so the formal expansion is justified and after some elementary calculations the result becomes

(1.8) $$g(x) = \frac{1}{2\pi} \delta_+(x_0^2 - x_1^2 - x_2^2 - x_3^2) - \frac{m}{4\pi} \theta(x_0^2 - x_1^2 - x_2^2 - x_3^2) \frac{\mathcal{J}_1(m\sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2})}{\sqrt{x_0^2 - x_1^2 - x_2^2 - x_3^2}} \quad x_0>0 \text{ and } m>0$$

with $\mathcal{J}_1$ as the Bessel function.
As to the meaning of the distribution $\delta_+(x_0^2 - x_1^2 - x_2^2 - x_3^2)$ the reader is referred to Chapter I, section 7. For the derivation of formula (1.8) see also [10], Chapter II.

Methée [5] applies a mapping of the n-dimensional space R_n on the line R by aid of the transformation

(1.9) $$u = x_0^2 - \sum_{i=1}^{n-1} x_i^2 .$$

Lorentz-invariant solutions of the Klein-Gordon equation in n dimensions are derived by means of pairs of distributions defined on the space D of test functions. Asymptotic expansions of distributions concentrated on the hyperboloid u = ε, in the neighbourhood of ε = 0, play an important role in this theory.

A simplified version of this theory is due to J.E. Roos and L. Gårding [11]. These authors use the following transformations for the test functions $\phi(x)$:

(1.10) $$(M\phi)(\tau) = \langle\delta(\tau - x^2), \phi(x)\rangle$$

(1.11) $$(M_1\phi)(\tau) = \langle\delta(\tau - x^2)\text{sign } x_0, \phi(x)\rangle$$

with $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

Linear homeomorphisms are established between the spaces of even and uneven Lorentz-invariant distributions and the duals of the spaces of the functions $(M\phi)(\tau)$ respectively $(M_1\phi)(\tau)$; hence the necessary calculations can be performed in these dual spaces.

Lavoine [7] and the present author [8 - 9] follow the physicists and they solve the equations (1.1) and (1.2) by applying Fourier transformation. Whereas Methée uses distributions defined on the space D of test functions with compact support, Lavoine and the present author consider only tempered distributions. This results in the fact that the solutions obtained in this chapter and also in [7], contrary to those obtained by Methée in [5], do not have terms which increase exponentially at infinity; however terms of this kind are usually disregarded by the physicists.

2. Outline of the method

A proper Lorentz-transformation is a linear transformation of the space R_4 leaving invariant the quadratic form

$$x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$$

and not interchanging the forward and backward light cone; moreover its determinant equals +1. In the sequel we shall always write Lorentz transformation instead of proper Lorentz transformation. In order to obtain the solutions, invariant under all such transformations, of the Klein-Gordon equations (1.1) and(1.2) we follow the physicists by applying to (1.1) and (1.2) a slightly modified Fourier transformation F^*. This transformation applied to a function of the class $L[-\infty,+\infty]$ is defined by

$$(2.1) \qquad F^*[f(x)] = \hat{f}(k) = \int_{-\infty}^{+\infty} e^{ik\cdot x} f(x)dx,$$

with $k\cdot x = k_0x_0 - k_1x_1 - k_2x_2 - k_3x_3$.

It is not difficult to show that for every Lorentz-transformation Λ and for every $f(x) \in L[-\infty,+\infty]$ the following relation holds:

$$(2.2) \qquad F^*[f(\Lambda x)] = \hat{f}(\Lambda k).$$

Using the transformation rules for distributions and (2.2) we obtain for every tempered distribution $f(x) \in S'$:

$$\langle F^*[f(\Lambda x)], \hat{\phi}(k)\rangle = (2\pi)^4\langle f(\Lambda x), \phi(x)\rangle =$$

$$(2\pi)^4\langle f(x), \phi(\Lambda^{-1}x)\rangle = \langle\hat{f}(k), \hat{\phi}(\Lambda^{-1}k)\rangle = \langle\hat{f}(\Lambda k), \hat{\phi}(k)\rangle.$$

Hence the relation (2.2) is also valid for distributions. It follows that if the distribution $f(x)$ is invariant under a particular Lorentz transformation Λ then also its Fourier transform $\hat{f}(k)$ is invariant under this transformation. The same is of course also true for the inverse transform F^{*-1}.

If one uses the usual Fourier transformation F, defined by

$$(2.3) \qquad F[f(x)] = \tilde{f}(k) = \int_{-\infty}^{+\infty} e^{i(x_0k_0+x_1k_1+x_2k_2+x_3k_3)} f(x)dx$$

it is again not difficult to show that we have for any Lorentz-transformation Λ and for all tempered distributions $f(x)$ the formula:

$$F[f(\Lambda x)] = \hat{f}(G\Lambda Gk),$$

with G being the matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

It follows that if the disyribution f(x) is invariant under a particular Lorentz transformation Λ then its Fourier transform $\hat{f}(k)$ is invariant under the particular Lorentz-transformation $G\Lambda G$.
Hence if f(x) is a distribution invariant under all Lorentz-transformations Λ, then also its Fourier transforms $F^{*}[f(x)]$ and $F[f(x)]$ are invariant under all transformations Λ and so Fourier transformation preserves Lorentz-invariance; the same applies for the inverse transformations F^{*-1} and F^{-1}.

The connection between the Fourier transform F^{*} and the Fourier transform F is given by the formula

(2.4) $$F\,F^{*}[f(x_1,x_2,x_3,x_0)] = (2\pi)^4\, f(x_1,x_2,x_3,-x_0),$$

which is equally valid for integrable functions as well distributions. We now apply F^{*} to (1.1) and (1.2) and we get:

(2.5) $$(m^2 - k^2)\,\hat{f}(k) = 0,$$

and

(2.6) $$(m^2 - k^2)\,\hat{g}(k) = +1.$$

Since F^{*} and F^{*-1} preserve Lorentz-invariance we have only to determine the Lorentz-invariant solutions of (2.5) and (2.6) and to transform the results again to configuration space.

The general Lorentz-invariant solutions of (2.5) and (2.6) are readily obtained. A particular Lorentz-invariant solution of (2.6) is given by

(2.7) $$\hat{g}_p(k) = \frac{1}{m^2 - k^2},$$

where the distribution $(m^2 - k^2)^{-1}$ is defined as the Cauchy principal value:

(2.8) $$< \frac{1}{m^2 - k^2}, \phi(k)> = \lim_{\varepsilon \to +0} \int_{|m^2-k^2|>\varepsilon} \frac{\hat{\phi}(k)}{m^2 - k^2} dk.$$

The general Lorentz-invariant solution $\hat{f}(k)$ of the homogeneous equation (2.5) is clearly concentrated in the hyperboloid $(m^2 - k^2) = 0$; it is zero outside the surface of the hyperboloid; for surface concentrated distributions see section 7 of Chapter I. Because the only properly Lorentz invariant subsets of the hyperboloid $m^2 - k^2 = 0$ are its upper and lower sheet the general Lorentz invariant solution of (2.5) consists of one invariant distribution concentrated on the upper sheet and one invariant distribution concentrated on the lower sheet. In the same way as for distributions in one independent variable one can show that a distribution concentrated on a surface $P = 0$ is a linear combination of $\delta(P)$ and its derivatives $\delta^{(k)}(P)$. Because of the relation (2.5) and the formula (7.14) of Chapter I the only distributions which are to be considered for a Lorentz-invariant solution of (2.5) are $\delta_+(m^2 - k^2)$ and $\delta_-(m^2 - k^2)$; these distributions have been defined explicitly in Chapter I, section 7, formulae (7.7) and (7.8). Hence we obtain

(2.9) $$\hat{f}(k) = c_+\delta_+(m^2 - k^2) + c_-\delta_-(m^2 - k^2),$$

and

(2.10) $$\hat{g}(k) = (m^2 - k^2)^{-1} + \hat{f}(k).$$

Due to the required Lorentz-invariance the coefficients c_+ and c_- must be constants. For the sake of completeness we recall here the expressions for $<\delta_\pm(m^2 - k^2), \hat{\phi}(k)>$ (see [I (7.7)] and [I (7.8)])

$$(2.11)\begin{cases} \langle \delta_{\pm}(m^2 - k^2), \hat{\phi}(k)\rangle = \\ = \frac{1}{2}\int_0^\infty \int_\Omega (\kappa^2 + m^2)^{-\frac{1}{2}}\kappa^2 \hat{\phi}(\kappa\omega_1, \kappa\omega_2, \kappa\omega_3, \pm\sqrt{\kappa^2 + m^2})\, d\kappa\, d\Omega \\ = \frac{1}{2}\int_0^\infty (\kappa^2 + m^2)^{-\frac{1}{2}}\kappa^2 \bar{\hat{\phi}}(\kappa, \pm\sqrt{\kappa^2 + m^2})\, d\kappa, \end{cases}$$

and

$$\delta(m^2 - k^2) = \delta_+(m^2 - k^2) + \delta_-(m^2 - k^2).$$

$\kappa^2 = k_1^2 + k_2^2 + k_3^2$, Ω denotes the unit sphere in (k_1, k_2, k_3)-space and $d\Omega$ its surface measure.

$\bar{\hat{\phi}}(\kappa, k_0)$ is apart from a constant the mean value of $\hat{\phi}(k_1, k_2, k_3, k_0)$ on a sphere in (k_1, k_2, k_3)-space with centre in the origin and κ as radius.

All that remains now is to determine the inverse Fourier transforms of $(m^2 - k^2)^{-1}$ and $\delta_{\pm}(m^2 - k^2)$. According to (2.4) we have to calculate the usual Fourier transforms of these distributions and to change consecutively x_0 to $-x_0$ and to divide by $(2\pi)^4$.

In the determination of the Fourier transforms of (2.9) and (2.10) we use the following important formula:

$$(2.12)\quad \begin{aligned} \frac{1}{m^2 - k^2 \pm i0} &\overset{\text{def}}{=} \lim_{\varepsilon \to +0} \frac{1}{(m^2 - k^2) \pm i\varepsilon(k_1^2 + k_2^2 + k_3^2 + k_0^2)} \\ &= \frac{1}{m^2 - k^2} \mp i\pi\delta(m^2 - k^2). \end{aligned}$$

This formula being a generalization of the well-known one dimensional equation

$$(2.13)\qquad \lim_{\varepsilon \to +0} \frac{1}{x \pm i\varepsilon} = \frac{1}{x} \mp i\pi\delta(x), \text{ (conf. Ch. I, (3.21))}$$

will be proved in section 3 of this chapter.

Gelfand and Shilov have already given the Fourier transforms of $(m^2 - k^2 \pm i0)^{-1}$ (see [12], Chapter III, §2.8). Using this result we obtain, with the aid of (2.12), $F[(m^2 - k^2)^{-1}]$ and $F[\delta(m^2 - k^2)]$ by addition and subtraction.

The only final problem is to determine $F[\delta_+(m^2 - k^2)]$ and $F[\delta_-(m^2 - k^2)]$. This will be done in section 4, but we give here already a sketch of the method that will be used. Let us suppose for the moment that

$$F[\delta(m^2 - k^2)] = X(x_1, x_2, x_3, x_0),$$

then it will be shown that it is possible to make the Hilbert splitting

$$X(x_1, x_2, x_3, x_0) = X_1(x_1, x_2, x_3, x_0) + X_2(x_1, x_2, x_3, x_0),$$

where X_1 and X_2 can be continued analytically into the upper respectively lower half of the complex $(x_0 + iy_0)$-plane. According to a theorem that the distributional limits $g(x_1, x_2, x_3, x_0 \pm i0)$ of a function $g(x_1, x_2, x_3, x_0 + iy_0)$ holomorphic in the upper or lower half plane, $y_0 > 0$ respectively $y_0 < 0$, are Fourier transforms of distributions concentrated in the region $k_0 \geq 0$ respectively $k_0 \leq 0$, one finds after some reasoning that

$$F[\delta_+(m^2 - k^2)] = X_1(x_1, x_2, x_3, x_0)$$

and

$$F[\delta_-(m^2 - k^2)] = X_2(x_1, x_2, x_3, x_0).$$

Finally we present in section 5.1 the following Lorentz invariant solutions of the homogeneous Klein-Gordon equation (1.1): the Pauli-Jordan function $\Delta(x)$, its positive and negative frequency parts $\Delta^+(x)$ and $\Delta^-(x)$ and the distribution $\Delta^{(1)}(x) = \Delta^+(x) - \Delta^-(x)$.

In section 5.2 we derive the following invariant solutions of the inhomogeneous equation (1.2): the advanced and retarted Green's functions $\Delta_A(x)$ and $\Delta_R(x)$, vanishing for $x_0 > 0$ respectively $x_0 < 0$

and the causal Green's functions $\Delta_C(x)$ and $\Delta_{AC}(x)$.
This chapter is concluded by treating the Cauchy problems for the Klein-Gordon equation.

Remarks

1. Lavoine [7] has derived the Lorentz-invariant solutions of the equations (1.1) and (1.2) by applying also the Fourier-transformation. The inverse transformation of the distributions $\delta_+(k^2 - m^2)$, $\delta_-(k^2 - m^2)$ and $(k^2 - m^2)^{-1}$ are obtained by calculating the inverse transforms of $[(\exp - \beta/k_0)] \cdot \delta_{\pm}(k^2 - m^2)$, $(\exp - \beta|k_0|) \cdot \delta_-(k^2 - m^2)$ and $(\exp - \beta\sqrt{k_1^2 + k_2^2 + k_3^2 + m^2}) \cdot (k^2 - m^2)^{-1}$ and by taking consecutively the limit for $\beta \to +0$.

2. Methée [5] has derived the Lorentz-invariant solutions of the equations (1.1) and (1.2) without using the Fourier transformation; in ref. [13] he compares a.o. the results of [5] with the formulae (2.9) and (2.10) of this section and he obtains in this way the Fourier transforms of $(m^2 - k^2)^{-1}$, $\delta_+(m^2 - k^2)$ and $\delta_-(m^2 - k^2)$.

3. The Fourier transforms of $(m^2 - k^2)^{-1}$ and $\delta(m^2 - k^2)$.

In order to obtain the Fourier transforms of $(m^2 - k^2)^{-1}$ and $\delta(m^2 - k^2)$ we shall use the formula

$$(2.12) \quad \frac{1}{m^2 - k^2 \pm i0} \overset{\text{def}}{=} \lim_{\varepsilon \to +0} \frac{1}{(m^2 - k^2) \pm i\varepsilon(k_1^2 + k_2^2 + k_3^2 + k_0^2)}$$

$$= \frac{1}{m^2 - k^2} \mp i\pi\delta(m^2 - k^2),$$

where $(m^2 - k^2)^{-1}$ is taken in the sense of the Cauchy principal value. The proof of this formula is analogous to that of

$$(2.13) \quad \frac{1}{x \pm i0} = \frac{1}{x} \mp i\pi\delta(x), \text{ (see Ch. I, (3.21))}.$$

The relation (2.13) is shown to be valid by writing instead of $\frac{1}{x \pm i0}$ the expression $\frac{d}{dx}\log(x \pm i0) = \frac{d}{dx}[\log|x| \pm i\pi\theta(-x)]$.

The proof of (2.12) consists essentially in the replacement of the differential operator $\frac{d}{dx}$ by another suitable differential operator.

Proof of (2.12). We introduce the differential operator

(3.1) $$L_\varepsilon = \frac{1}{1 \pm i\varepsilon}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) - \frac{1}{1 \mp i\varepsilon}\frac{\partial^2}{\partial x_0^2},$$

with $\lim_{\varepsilon \to +0} L_\varepsilon = \Box$.

Putting

(3.2) $$m^2 - x^2 \pm i\varepsilon(x_1^2 + x_2^2 + x_3^2 + x_0^2) = m^2 - x^2 \pm i\varepsilon(x,x) = \mathcal{P}_\varepsilon,$$

with $-\pi < \arg \mathcal{P}_\varepsilon \leq +\pi$, we obtain for all values of λ (both ε and m not zero) the relation

$$L_\varepsilon\left[\mathcal{P}_\varepsilon^{\lambda+1}\right] = 4(\lambda + 1)(\lambda + 2)\mathcal{P}_\varepsilon^{\lambda} - 4m^2\lambda(\lambda + 1)\mathcal{P}_\varepsilon^{\lambda-1},$$

or for $\lambda \neq -1$,

$$\mathcal{P}_\varepsilon^{-\lambda} L_\varepsilon\left[\frac{\mathcal{P}_\varepsilon^{\lambda+1}}{\lambda + 1}\right] = \mathcal{P}_\varepsilon^{-\lambda} L_\varepsilon\left[\frac{\exp[(\lambda + 1)\log \mathcal{P}_\varepsilon]}{\lambda + 1}\right]$$

$$= 4(\lambda + 2) - 4m^2\lambda\mathcal{P}_\varepsilon^{-1}.$$

After expansing the exponential function into a power series and taking the limit for $\lambda \to -1$ we get the identity

(3.3) $$\mathcal{P}_\varepsilon\, L_\varepsilon\left[\log \mathcal{P}_\varepsilon\right] = 4m^2\, \mathcal{P}_\varepsilon^{-1} + 4,$$

and thus

(3.4) $$\lim_{\varepsilon \to +0} \mathcal{P}_\varepsilon^{-1} = \lim_{\varepsilon \to +0} \left\{\frac{\mathcal{P}_\varepsilon}{4m^2}\, L_\varepsilon\left[\log \mathcal{P}_\varepsilon\right]\right\} - \frac{1}{m^2}.$$

The right hand side of (3.4) may now be reduced with the aid of the theory of Chapter I, section 7, as follows:

$$\lim_{\varepsilon \to +0} \mathcal{P}_\varepsilon^{-1} = \frac{m^2 - x^2}{4m^2} \Box[\log |m^2 - x^2|]$$

$$+ \frac{m^2 - x^2}{4m^2} \Box[\pm i\pi\theta(x^2 - m^2)] - \frac{1}{m^2}$$

$$(3.5) \qquad = \frac{m^2 - x^2}{4m^2} \Box[\log |m^2 - x^2|] - \frac{1}{m^2} \mp i\pi\delta(m^2 - x^2).$$

The distribution $\langle \frac{m^2 - x^2}{4m^2} \Box[\log |m^2 - x^2|], \phi(x) \rangle$ may be written as

$$\frac{1}{4m^2} \langle \log |m^2 - x^2|, \Box[(m^2 - x^2)\phi(x)] \rangle =$$

$$\lim_{\delta \to 0} \frac{1}{4m^2} \int_{|m^2-x^2|>\delta} \{\log |m^2 - x^2|\} \Box[(m^2 - x^2)\phi(x)]\, dx.$$

Applying Green's theorem to the right hand side of this equation we get:

$$\lim_{\delta \to 0} \frac{1}{4m^2} \int_{|m^2-x^2|>\delta} \{\log |m^2 - x^2|\} \Box[(m^2 - x^2)\phi(x)]\, dx =$$

$$\lim_{\delta \to 0} \frac{1}{4m^2} \int_{|m^2-x^2|>\delta} \{\Box \log |m^2 - x^2|\} (m^2 - x^2)\phi(x)\, dx$$

$$+ \lim_{\delta \to 0} \frac{1}{4m^2} \int_{|m^2-x^2|=\delta} \{\log |m^2 - x^2|\} \cdot \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_0}\right)$$

$$[(m^2 - x^2)\phi(x)] \cdot \vec{n}\, d\sigma$$

$$- \lim_{\delta \to 0} \frac{1}{4m^2} \int_{|m^2-x^2|=\delta} (m^2 - x^2)\phi(x) \cdot \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, -\frac{\partial}{\partial x_0}\right)$$

$$[\log |m^2 - x^2|] \cdot \vec{n}\, d\sigma,$$

where $\vec{n}$ is the normal on the surface $|m^2 - x^2| = \delta$ and $d\sigma$ its surface measure. Observing that the integrals over the surfaces $m^2 - x^2 = \pm\delta$ cancel each other in the limit $\delta \to 0$ we obtain

$$<\frac{m^2 - x^2}{4m^2}\,\Box\left[\log\,|m^2 - x^2|\right],\ \phi(x)> =$$

$$\lim_{\delta\to 0}\frac{1}{4m^2}\int_{|m^2-x^2|>\delta}\{\Box\log\,|m^2 - x^2|\}\cdot(m^2 - x^2)\phi(x)\,dx.$$

Calculating finally $(m^2 - x^2)\,\Box\,\log\,|m^2 - x^2|$ we get

$$<\frac{m^2 - x^2}{4m^2}\,\Box\left[\log\,|m^2 - x^2|\right],\ \phi(x)> =$$

$$\lim_{\delta\to 0}\int_{|m^2-x^2|>\delta}\frac{\phi(x)}{m^2 - x^2}\,dx + \int_{R_4}\frac{\phi(x)\,dx}{m^2}\ .$$

Substituting this result into the right hand side of (3.5) we have

$$(3.6)\qquad \lim_{\varepsilon\to+0}\mathcal{P}_{\varepsilon}^{-1} = (m^2 - x^2 \pm i0)^{-1} = (m^2 - x^2)^{-1} \mp i\pi\delta(m^2 - x^2),$$

where $(m^2 - x^2)^{-1}$ should be taken in the sense of the Cauchy principal value.

The result (3.6) is also valid for $m = 0$ and the proof is rather similar to the one given above (see [14]).

As to the definition of the distribution $\delta(x^2)$ the reader is referred to Chapter I, section 7, formula (7.6).

Moreover for $m \neq 0$ the formula (3.6) may be generalized to an arbitrary number of variables and to arbitrary negative integer values of the exponent (see [14]).

Following the method of Gelfand and Shilov [12], Chapter III, §2.8, we determine now the Fourier transforms of $\{(m^2 - k^2) \pm i0\}^{-1}$, from the results of which those of $(m^2 - k^2)^{-1}$ and $\delta(m^2 - k^2)$ immediately follow by aid of the formula (3.6) (or (2.12)).

Let D be a positive definite quadratic form in k. The Fourier transform of $(m^2 + D)^{\lambda}$ with Re $\lambda < -2$ is

$$(3.7)\qquad F\left[(m^2 + D)^{\lambda}\right] = \int_{-\infty}^{+\infty}(m^2 + D)^{\lambda}\,e^{i(x,k)}\,dk,$$

with $(x,k) = x_1k_1 + x_2k_2 + x_3k_3 + x_0k_0$.
We write D in the form

$$D = (k, Gk) = \sum_{r,s=0}^{3} g_{rs}\, k_r\, k_s, \tag{3.8}$$

where G denotes the matrix of the coefficients g_{rs}.
Making a transformation $k = Tk'$ to principal axes the quadratic form D can be written as

$$D = (k, Gk) = (k', T^{*}GTk') = (k', k')$$

where the matrix T^{*} is the adjoint of the matrix T.
$T^{*}GT = 1$ and $|T|^2 = 1/|G|$, where $|T|$ and $|G|$ are the determinants of the matrices T resp. G.

Hence

$$F\left[(m^2 + D)^{\lambda}\right] = \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \{m^2 + (k',k')\}^{\lambda}\, e^{i(T^{*}x,k')}\, dk'$$

$$= \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \{m^2 + (k',k')\}^{\lambda}\, e^{i(x',k')}\, dk'$$

with $x' = T^{*}x$.
It is clear that $F\left[(m^2 + D)^{\lambda}\right]$ depends only on the length $|x'|$ of the fourvector x'. Taking $x' = (|x'|, 0, 0, 0)$ we obtain

$$F\left[(m^2 + D)^{\lambda}\right] = \frac{1}{\sqrt{|G|}} \int_{-\infty}^{+\infty} \{m^2 + (k',k')\}^{\lambda}\, e^{i|x'|k_1'}\, dk'. \tag{3.9}$$

To reduce further the right hand side of (3.9) we introduce spherical coordinates

$k_1' = \kappa \cos \phi_1$, $k_2' = \kappa \sin \phi_1 \cos \phi_2$, $k_3' = \kappa \sin \phi_1 \sin \phi_2 \cos \phi_3$

and $k_0' = \kappa \sin \phi_1 \sin \phi_2 \sin \phi_3$.

$\kappa^2 = (k',k') = k_1'^2 + k_2'^2 + k_3'^2 + k_0'^2$ and $dk' = \kappa^3 \sin^2\phi_1\, d\kappa\, d\phi_1\, d\Omega$, where $d\Omega$ is the surface element of the unit sphere in R_3.

Performing the integration with respect to $d\Omega$ we obtain

$$F\left[(m^2 + D)^\lambda\right] = \frac{4\pi}{\sqrt{|G|}} \int_0^\infty \int_0^\pi (m^2 + \kappa^2)^\lambda \, e^{i|x'|\kappa \cos \phi_1} \cdot \kappa^3 \sin^2\phi_1 \, d\phi_1 d\kappa .$$

Using the following integral relations for Bessel functions

$$\int_0^\pi e^{i|x'|\kappa \cos \phi_1} \cdot \sin^2\phi_1 \, d\phi_1 = \frac{\pi}{\kappa|x'|} J_1(\kappa|x'|),$$

and

$$\int_0^\infty \kappa^2 \, J_1(\kappa|x'|)(m^2 + \kappa^2)^\lambda d\kappa = \left(\frac{2}{|x'|}\right)^{\lambda+1} \frac{m^{2+\lambda}}{\Gamma(-\lambda)} K_{2+\lambda}(m|x'|),$$

where J_1 is the Bessel function of the first kind and $K_{2+\lambda}$ the modified Bessel-function, (see lit. [15], section 7.12, formula (9) and section 7.14.2 formula (51), we get the result

$$F\left[(m^2 + D)^\lambda\right] = \frac{2\pi^2}{\sqrt{|G|}} \left(\frac{2m}{|x'|}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)} K_{\lambda+2}(m|x'|). \tag{3.10}$$

Since $x' = T^* x$ we have

$$|x'|^2 = (x',x') = (x,TT^* x) = (x,G^{-1}x) = \sum_{r,s=0}^{3} g^{rs} x_r x_s .$$

Hence $|x'|^2$ is a positive definite quadratic form in x of which the coefficients form a matrix which is the inverse of the matrix of the coefficients of the positive quadratic form D.

Putting

$$|x'|^2 = \sum_{r,s=0}^{3} g^{rs} \, x_r \, x_s = E, \tag{3.11}$$

we obtain finally

$$(3.12)\qquad F\left[(m^2 + D)^\lambda\right] = \frac{2\pi^2}{\sqrt{|G|}}\left(\frac{2m}{E^{\frac{1}{2}}}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)}\, K_{\lambda+2}(mE^{\frac{1}{2}}).$$

With the aid of the principle of analytic continuation one proves easily that this result obtained for Re $\lambda < -2$ is also valid for all other complex values of λ.

We have derived formula (3.12) under the assumption that D is a positive definite quadratic form.
The left and right hand side of (3.12) is also an analytic function of the coefficients g_{rs} belonging to the quadratic form D and this is true for those ranges of g_{rs} where D is positive definite.
We continue now the left and right hand side of (3.12) analytically with respect to the coefficients g_{rs} into those ranges of complex values of g_{rs} where the so obtained new quadratic form $\mathcal{D}$ has either a positive or negative definite imaginary part. The complex $\mathcal{D}$-plane has a cut along the negative real axis.
The function $(m^2 + \mathcal{D})^\lambda$ is defined as

$$(3.13)\qquad (m^2 + \mathcal{D})^\lambda = |m^2 + \mathcal{D}|^\lambda \exp\cdot\left[i\lambda \arg(m^2 + \mathcal{D})\right],$$

with $-\pi < \arg(m^2 + \mathcal{D}) < +\pi$.
Using again the principle of analytic continuation we have for the complex quadratic forms $\mathcal{D}$ and all values of λ the formula

$$(3.14)\qquad F\left[(m^2 + \mathcal{D})^\lambda\right] = \frac{2\pi^2}{\sqrt{|\mathcal{G}|}}\left(\frac{2m}{\mathcal{E}^{\frac{1}{2}}}\right)^{\lambda+2} \frac{1}{\Gamma(-\lambda)}\, K_{\lambda+2}(m\mathcal{E}^{\frac{1}{2}}),$$

where $|\mathcal{G}|$ is the determinant of the quadratic form $\mathcal{D}$. $\mathcal{E}$ is a quadratic form in x whose coefficients form a matrix which is the inverse of the matrix of the coefficients of $\mathcal{D}$.
When $\mathcal{D}$ has a positive or negative definite imaginary part then $\mathcal{E}$ has a negative respectively positive definite imaginary part; when $\mathcal{D}$ itself is positive or negative definite then $\mathcal{E}$ is also positive respectively negative definite. Hence the complex $\mathcal{E}$-plane has also a cut along the negative real axis.

Taking for λ the value $\lambda = -1$, and for $\mathfrak{D}$ the quadratic form

$$\mathfrak{D} = k_1^2 + k_2^2 + k_3^2 - k_0^2 \pm i\varepsilon(k_1^2 + k_2^2 + k_3^2 + k_0^2)$$

$$= -k^2 \pm i\varepsilon(k,k),$$

and considering finally the limit for $\varepsilon \to +0$, one gets the result of Gelfand and Shilov:

$$F[(m^2 - k^2 \pm i0)^{-1}] = \mp 4\pi^2 im \frac{K_1\{m(-x^2 \mp i0)^{\frac{1}{2}}\}}{(-x^2 \mp i0)^{\frac{1}{2}}} \quad , \tag{3.15}$$

with $-x^2 = x_1^2 + x_2^2 + x_3^2 - x_0^2$. It follows now immediately from the result (2.12) that

$$F\left[\frac{1}{m^2 - k^2}\right] = 2\pi^2 im\left[\frac{K_1\{m(-x^2 + i0)^{\frac{1}{2}}\}}{(-x^2 + i0)^{\frac{1}{2}}} - \frac{K_1\{m(-x^2 - i0)^{\frac{1}{2}}\}}{(-x^2 - i0)^{\frac{1}{2}}}\right] \tag{3.16}$$

and

$$F[\delta(m^2 - k^2)] = 2\pi m\left[\frac{K_1\{m(-x^2 + i0)^{\frac{1}{2}}\}}{(-x^2 + i0)^{\frac{1}{2}}} + \frac{K_1\{m(-x^2 - i0)^{\frac{1}{2}}\}}{(-x^2 - i0)^{\frac{1}{2}}}\right]. \tag{3.17}$$

Using the relations ([15], §7.2.1)

$$K_1(ze^{i\pi/2}) = -\frac{1}{2}\pi H_1^{(2)}(z) = -\frac{1}{2}\pi J_1(z) + \frac{1}{2}\pi iY_1(z),$$

$$K_1(ze^{-i\pi/2}) = -\frac{1}{2}\pi H_1^{(1)}(z) = -\frac{1}{2}\pi J_1(z) - \frac{1}{2}\pi iY_1(z),$$

where $H_1^{(i)}$ is the Hankel and Y_1 the Neumann function, we get

$$F\left[\frac{1}{m^2 - k^2}\right] = 0 \text{ for } x^2 < 0,$$

i.e., outside the lightcone, and

$$F\left[\frac{1}{m^2 - k^2}\right] = -2\pi^3 m \frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \quad \text{for } x^2 > 0,$$

i.e., inside the lightcone.

However, for x^2 in a neighbourhood of $x^2 = 0$, we obtain from (3.16)

$$F\left[\frac{1}{m^2 - k^2}\right] = 2\pi^2 i\left[\frac{1}{-x^2 + i0} + \frac{1}{x^2 + i0}\right] + O(\log |x^2|)$$

$$= 4\pi^3\delta(x^2) + O(\log |x^2|).$$

Summarizing these results we may write

$$(3.18)\qquad F\left[\frac{1}{m^2 - k^2}\right] = -2\pi^3 m\theta(x^2)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} + 4\pi^3\delta(x^2).$$

In the same way we can reduce (3.17):

$$F[\delta(m^2 - k^2)] = 4\pi m\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} \quad \text{for } x^2 < 0,$$

i.e., outside the lightcone, and

$$F[\delta(m^2 - k^2)] = i\pi^2 m\frac{H_1^{(2)}(m\sqrt{x^2}) - H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} \quad \text{for } x^2 > 0,$$

i.e., inside the lightcone. The square roots are to be taken positive. Further, it follows from (3.17) that, for small values of x^2,

$$F[\delta(m^2 - k^2)] = 2\pi\left[\frac{1}{-x^2 + i0} - \frac{1}{x^2 + i0}\right] + O(\log |x^2|)$$

$$= -\frac{4\pi}{x^2} + O(\log |x^2|),$$

where $1/x^2$ should be interpreted in the sense of Cauchy.

Finally, also these results may be summarized in one single formula, viz.,

$$(3.19)\qquad F[\delta(m^2 - k^2)] = 4\pi m\theta(-x^2)\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} + i\pi^2 m\theta(x^2)\frac{H_1^{(2)}(m\sqrt{x^2}) - H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}}.$$

The results (3.18) and (3.19) yield the Fourier transforms of $(m^2 - k^2)^{-1}$ and $\delta(m^2 - k^2)$; there remains now to determine the Fourier transforms of the distributions $\delta_{\pm}(m^2 - k^2)$; this will be done in the next section.

4. The Fourier transform of $\delta_{\pm}(m^2 - k^2)$.

In order to determine $F[\delta_{\pm}(m^2 - k^2)]$ we need first a lemma.

Lemma. Let

$$g(u_1, u_2, \ldots, u_n, w_0) = g(u_1, u_2, \ldots, u_n, u_0 + iv_0)$$

be holomorphic in the upper half-plane $v_0 > 0$ for any set of real values of $(u_1, u_2, \ldots, u_n)$. The function $g(u_1, \ldots, u_n, w_0)$ can be majorized in any region $v_0 > \delta > 0$ as

$$|g(u_1, \ldots, u_n, u_0 + iv_0)| < C_\delta \prod_{i=1}^{n} (1 + u_i^2)^{p_i} |u_0 + iv_0|^{p_0},$$

where the p_i, $i = 0, \ldots, n$, are positive integers independent of δ. If $\lim_{v_0 \to +0} g(u_1, u_2, \ldots, u_n, u_0 + iv_0)$ exists in the distributional sense on the space S of test functions, then

$$\lim_{v_0 \to +0} g(u_1, \ldots, u_n, u_0 + iv_0)$$

is the (n+1)-dimensional Fourier transform of a distribution $f_+(x_1, x_2, \ldots, x_n, x_0)$ belonging to S' and vanishing for $x_0 < 0$.

An analogous result holds of course for functions $g(u_1, \ldots, u_n, w_0)$ holomorphic in the lower half-plane $v_0 < 0$. Mutatis mutandis one obtains that

$$\lim_{v_0 \to -0} g(u_1, \ldots, u_n, u_0 + iv_0)$$

is the (n+1)-dimensional Fourier transform of a distribution $f_-(x_1, \ldots, x_n, x_0)$ belonging to S' and vanishing for $x_0 > 0$.

Remark. H.A. Lauwerier [16] has shown that a function $g(u + iv)$, holomorphic in the upper half-plane $v > 0$ and bounded by a polynomial uniformly in every half-plane $v \geq \delta > 0$, possesses for $v \to +0$ a distributional limit $g(u) \in Z'$ which is the Fourier transform of a distribution $f_+(x) \in D'$ vanishing for $x < 0$.

E.J. Beltrami and M.R. Wohlers [17] have proved the same theorem for distributions $f(x) \in S'$; however, in the latter case one needs an extra condition, namely, the existence of the distributional limit of $g(u + iv)$ in S', which is no longer a consequence of the data as stated in the theorem of Lauwerier. In our case we deal with distributions in more independent variables, and therefore we have to make a modification of the result of Beltrami and Wohlers.

Proof of the lemma. According to the assumptions there exist positive integers $p_1, p_2, \ldots, p_n, p_0$ such that

$$h(u_1, \ldots, u_n, u_0 + iv_0)$$

$$= \{\prod_{i=1}^{n} (1 + u_i^2)^{-p_i - 1}\}(u_0 + iv_0)^{-p_0 - 2} g(u_1, \ldots, u_n, u_0 + iv_0)$$

is absolutely integrable over the whole space $R_{n+1}(u_1, \ldots, u_n, u_0)$ for any value of $v_0 > 0$. We consider now the integral

$$\frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp[-i\{x_1 u_1 + \ldots + x_n u_n + x_0(u_0 + iv_0)\}]$$

$$\cdot\, h(u_1, \ldots, u_n, u_0 + iv_0) du_1 \ldots du_n du_0 \tag{4.1}$$

$$= \frac{1}{(2\pi)^{n+1}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \exp[-i(x_1 u_1 + \ldots + x_n u_n)] du_1 \ldots du_n$$

$$\cdot \int_{-\infty}^{+\infty} \exp[-ix_0(u_0 + iv_0)] h(u_1, \ldots, u_n, u_0 + iv_0) du_0 .$$

Since h is holomorphic in the upper half-plane $v_0 > 0$, this integral is independent of v_0; hence the left-hand side of (4.1) is a function depending only on $x_1, \ldots, x_n, x_0$; taking $x_0 < 0$ and $v_0 \to +\infty$, it appears that this function, denoted by $f_+^*(x_1, \ldots, x_n, x_0)$, must vanish for $x_0 < 0$. It follows now immediately from (4.1) that

$$F[e^{-v_0 x_0} f_+^*(x_1, \ldots, x_n, x_0)] = h(u_1, \ldots, u_n, u_0 + iv_0).$$

Introducing the operator

$$D = \left\{ \prod_{i=1}^{n} (1 - \frac{\partial^2}{\partial x_i^2})^{p_i+1} \right\} (i \frac{\partial}{\partial x_0} + iv_0)^{p_0+2} ,$$

we obtain

$$F[De^{-v_0 x_0} f_+^*(x_1, \ldots, x_n, x_0)] = g(u_1, \ldots, u_n, u_0 + iv_0).$$

Taking finally the limit for $v_0 \to +0$, which by supposition exists on S, we obtain due to the continuity of the Fourier transformation

$$\lim_{v_0 \to +0} g(u_1, u_2, \ldots, u_n, u_0 + iv_0) =$$

$$= F[\lim_{v_0 \to +0} \{De^{-v_0 x_0} f_+^*(x_1, \ldots, x_n, x_0)\}] .$$

Finally, because f_+^* vanishes for $x_0 < 0$, we obtain the result that

$$\lim_{v_0 \to +0} g(u_1, \ldots, u_n, u_0 + iv_0)$$

is the Fourier transform of a distribution vanishing for $x_0 < 0$. In the case that $g(u_1, \ldots, u_n, u_0 + iv_0)$ is holomorphic for $v_0 < 0$ one shows quite similarly that $g(u_1, \ldots, u_n, u_0 - i0)$ is the Fourier transform of a distribution vanishing for $x_0 > 0$.

We consider now the function

$$g(x_1, x_2, x_3, x_0) = 2\pi m \frac{K_1\{m\sqrt{-x^2}\}}{\sqrt{-x^2}} \tag{4.2}$$

with $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2 < 0$ as a function of x_0, while x_1, x_2 and x_3 are supposed to have fixed values. The square root is taken positive. Putting

$$\sqrt{x_1^2 + x_2^2 + x_3^2} = r,$$

and introducing a complex $z_0 = x_0 + iy_0$ plane with cuts along the real axis, viz., $-\infty < x_0 < -r$ and $+r < x_0 < +\infty$, we can continue the function g analytically into the whole z_0-plane. Using the formulas

$$(4.3) \qquad \begin{aligned} K_1(ze^{i\pi/2}) &= -\frac{1}{2}\pi H_1^{(2)}(z), \\ K_1(ze^{-i\pi/2}) &= -\frac{1}{2}\pi H_1^{(1)}(z), \end{aligned}$$

and the well-known expansions for K_1, $H_1^{(1)}$ and $H_2^{(2)}$ for small and large arguments [15], one finds after some simple considerations that the analytic continuation of $g(x_1, x_2, x_3, x_0)$ is uniformly bounded in any region $y_0 > \delta > 0$ and $y_0 < -\delta < 0$. Applying again (4.3) we obtain the following limits:

$$(4.4) \qquad \begin{aligned} \lim_{y_0 \to +0} g(x_1, x_2, x_3, x_0 + iy_0) &= \frac{-i\pi^2 m H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} && \text{for } x_0 > r, \\ \lim_{y_0 \to +0} g(x_1, x_2, x_3, x_0 + iy_0) &= \frac{2\pi m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} && \text{for } |x_0| < r, \\ \lim_{y_0 \to +0} g(x_1, x_2, x_3, x_0 + iy_0) &= \frac{+i\pi^2 m H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} && \text{for } x_0 < -r, \end{aligned}$$

and

$$\lim_{y_0\to -0} g(x_1,x_2,x_3,x_0+iy_0) = \frac{+i\pi^2 m H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} \qquad \text{for } x_0 > r,$$

$$(4.5)\quad \lim_{y_0\to -0} g(x_1,x_2,x_3,x_0+iy_0) = \frac{2\pi m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} \qquad \text{for } |x_0| < r,$$

$$\lim_{y_0\to -0} g(x_1,x_2,x_3,x_0+iy_0) = \frac{-i\pi^2 m H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} \qquad \text{for } x_0 < -r.$$

The square roots should of course again be taken as positive. We investigate now the limits of $g(x_1, x_2, x_3, x_0 + iy_0)$ for x_0 in the neighbourhood of $x_0 = \pm r$.

For small values of $x_0 \mp r$ and y_0, the function $g(x_1,x_2,x_3,x_0 + iy_0)$ may be written as

$$g(x_1,x_2,x_3,x_0+iy_0) = \frac{2\pi}{r^2-(x_0+iy_0)^2} + O\{\log|r^2-(x_0+iy_0)^2|\},$$

and we have

$$(4.6)\quad \lim_{y_0\to\pm 0} g(x_1,x_2,x_3,x_0+iy_0) = \lim_{y_0\to\pm 0} \frac{2\pi}{r^2-(x_0+iy_0)^2} + O\{\log|x^2|\}.$$

The limit of the first term is reduced as follows:

$$(4.7)\quad \lim_{y_0\to\pm 0} \frac{2\pi}{r^2-(x_0+iy_0)^2} = 2\pi \lim_{y_0\to\pm 0} \frac{\partial}{\partial x_0}\left[\frac{1}{2r}\log\frac{(x_0+iy_0)+r}{(x_0+iy_0)-r}\right]$$

$$= 2\pi \frac{\partial}{\partial x_0} \lim_{y_0\to\pm 0} \left[\frac{1}{2r}\log\frac{(x_0+iy_0)+r}{(x_0+iy_0)-r}\right]$$

$$= 2\pi \frac{\partial}{\partial x_0}\left[\frac{1}{2r}\log\left|\frac{r+x_0}{r-x_0}\right| \mp \frac{i\pi}{2r}\theta(-x^2)\right]$$

$$= \frac{-2\pi}{x^2} \pm 2i\pi^2 \frac{x_0}{r}\delta(x^2)$$

$$= \frac{-2\pi}{x^2} \pm 2i\pi^2 \{\delta_+(x^2) - \delta_-(x^2)\},$$

where $1/x^2$ should be taken in the sense of Cauchy. The distributions $\delta_{\pm}(x^2)$ have been defined in Chapter I, section 7, formulae (7.9) and (7.10).

Inserting (4.7) into (4.6) we get for values of x_0 in the neighbourhood of $x_0 = \pm r$,

(4.8)
$$\lim_{y_0 \to \pm 0} g(x_1, x_2, x_3, x_0 + iy_0)$$
$$= \pm 2i\pi^2\{\delta_+(x^2) - \delta_-(x^2)\} - \frac{2\pi}{x^2} + O\{\log |x^2|\}.$$

The last two terms of this expression are already contained in (4.4) and (4.5); summarizing the results (4.4), (4.5) and (4.8) we obtain:

(4.9)
$$\lim_{y_0 \to +0} g(x_1, x_2, x_3, x_0 + iy_0)$$
$$= \theta(x^2)\left[\theta(x_0)\frac{-i\pi^2 m H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} + \theta(-x_0)\frac{i\pi^2 m H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}}\right]$$
$$+ \theta(-x^2)\frac{2\pi m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} + 2i\pi^2\{\delta_+(x^2) - \delta_-(x^2)\}$$

and

(4.10)
$$\lim_{y_0 \to -0} g(x_1, x_2, x_3, x_0 + iy_0)$$
$$= \theta(x^2)\left[\theta(x_0)\frac{i\pi^2 m H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} + \theta(-x_0)\frac{-i\pi^2 m H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}}\right]$$
$$+ \theta(-x^2)\frac{2\pi m K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} - 2i\pi^2\{\delta_+(x^2) - \delta_-(x^2)\}.$$

The function $g(x_1, x_2, x_3, x_0 + iy_0)$ satisfies all conditions of the lemma, and hence it follows that

(4.11)
$$g(x_1, x_2, x_3, x_0 \pm i0) = F\left[f_{\pm}(k_1, k_2, k_3, k_0)\right],$$

where f_+ and f_- are distributions vanishing for, respectively, $k_0 < 0$ and $k_0 > 0$.

Moreover, it follows from the result (3.19) that

$$g(x_1,x_2,x_3,x_0 + i0) + g(x_1,x_2,x_3,x_0 - i0) = F[\delta(m^2 - k^2)],$$

and hence

$$f_+(k) + f_-(k) = \delta(m^2 - k^2) = \delta_+(m^2 - k^2) + \delta_-(m^2 - k^2).$$

Because $\delta_+(m^2 - k^2)$ is concentrated on the upper sheet of the hyperboloid $m^2 - k^2 = 0$ and $\delta_-(m^2 - k^2)$ on its lower sheet, it follows further that

$$\begin{aligned} f_+(k) &= \delta_+(m^2 - k^2) + h(k), \\ f_-(k) &= \delta_-(m^2 - k^2) - h(k), \end{aligned} \tag{4.12}$$

where $h(k)$ is a distribution concentrated on the coordinate plane $k_0 = 0$.

The distributions $g(x_1,x_2,x_3,x_0 \pm i0)$ are invariant under all proper Lorentz transformations and since the Fourier transformation F preserves this Lorentz-invariance the same is true for the distributions $f_\pm(k_1,k_2,k_3,k_0)$. It follows now from (4.12) that the distribution $\overline{h}(k)$ is also properly Lorentz-invariant. A distribution which is properly Lorentz-invariant and concentrated on the plane $k_0 = 0$ must be concentrated in the origin. Gårding and Lions [11] showed that any Lorentz-invariant distribution concentrated in the origin has the form

$$R[\Box]\ \delta(k_1,k_2,k_3,k_0), \tag{4.13}$$

where R is an arbitrary polynomial in $\Box = \frac{\partial^2}{\partial k_1^2} + \frac{\partial^2}{\partial k_2^2} + \frac{\partial^2}{\partial k_3^2} - \frac{\partial^2}{\partial k_0^2}$.

Hence $h(k)$ has the form (4.13).

Using formula (2.11) we get

$$\langle \delta_+(m^2 - k^2),\ \hat{\phi}(k_1,k_2,k_3,-k_0)\rangle = \langle \delta_-(m^2 - k^2),\ \hat{\phi}(k_1,k_2,k_3,+k_0)\rangle. \tag{4.14}$$

Moreover it follows from (4.9), (4.10) and a formula, analogous to (4.14) with $m = 0$, that

$$g(x_1,x_2,x_3,-x_0 + i0) = g(x_1,x_2,x_3,+x_0 - i0)$$

and consequently

$$f_+(k_1,k_2,k_3,-k_0) = f_-(k_1,k_2,k_3,+k_0). \tag{4.15}$$

Substituting finally the results (4.14) and (4.15) into (4.12) we obtain

$$h(k_1,k_2,k_3,-k_0) = - h(k_1,k_2,k_3,+k_0)$$

and hence $h(k)$ is odd in k_0. Since moreover $h(k)$ is of the form (4.13) it follows that $h(k) \equiv 0$. Hence

$$\begin{aligned} f_+(k) &= \delta_+(m^2 - k^2) \\ f_-(k) &= \delta_-(m^2 - k^2). \end{aligned} \tag{4.16}$$

Combination of (4.9), (4.10), (4.11) and (4.16) yields finally the results

$$\begin{aligned} F[\delta_+(m^2 - k^2)] = {} & 2i\pi^2\{\delta_+(x^2) - \delta_-(x^2)\} + \\ & + 2\pi m\theta(-x^2) \frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} \\ & - i\pi^2 m\theta(x^2)\left[\theta(x_0) \frac{H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} - \theta(-x_0) \frac{H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}}\right] \end{aligned} \tag{4.17}$$

$$(4.18)\quad F[\delta_-(m^2-k^2)] = -2i\pi^2\{\delta_+(x^2)-\delta_-(x^2)\} + 2\pi m\theta(-x^2)\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} + i\pi^2 m\theta(x^2)\left[\theta(x_0)\frac{H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} - \theta(-x_0)\frac{H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}}\right].$$

5. The Lorentz-invariant solutions of the Klein-Gordon equation

5.1. The solutions of the homogeneous equation

It has been shown in section 2 (formula (2.9)) that the general Lorentz-invariant solution of the homogeneous Klein-Gordon equation can be written as:

$$(5.1)\quad f(x_1,x_2,x_3,x_0) = c_+F^{*-1}[\delta_+(m^2-k^2)] + c_-F^{*-1}[\delta_-(m^2-k^2)].$$

Putting

$$(5.2)\qquad \Delta^+(x) = -2\pi i\ F^{*-1}[\delta_+(m^2-k^2)],$$

and

$$(5.3)\qquad \Delta^-(x) = +2\pi i\ F^{*-1}[\delta_-(m^2-k^2)],$$

we may also write

$$(5.4)\qquad f(x_1,x_2,x_3,x_0) = c_1\ \Delta^+(x) + c_2\ \Delta^-(x).$$

According to equation (2.4) the distributions $\Delta^{\pm}(x)$ satisfy the relation

$$(5.5)\qquad \Delta^{\pm}(x_1,x_2,x_3,-x_0) = \mp\frac{i}{(2\pi)^3}\,F[\delta_{\pm}(m^2-k^2)].$$

Setting

$$\phi(x_1,x_2,x_3,x_0) = F[\tilde{\tilde{\phi}}(k_1,k_2,k_3,k_0)] =$$
$$= \int_{R_4} e^{i(k_1x_1+k_2x_2+k_3x_3+k_0x_0)} \tilde{\tilde{\phi}}(k_1,k_2,k_3,k_0)\,dk$$

we may write

$$<\Delta^{\pm}(x_1,x_2,x_3,x_0),\ \phi(x_1,x_2,x_3,x_0)> = <\Delta^{\pm}(x_1,x_2,x_3,-x_0),\ \phi(x_1,x_2,x_3,-x_0)>$$

$$= \pm \frac{i}{(2\pi)^3} <F[\delta_{\pm}(m^2 - k^2)],\ F[\tilde{\tilde{\phi}}(k_1,k_2,k_3,-k_0)]>$$

$$= \pm\, i(2\pi) <\delta_{\pm}(m^2 - k^2),\ \tilde{\tilde{\phi}}(k_1,k_2,k_3,-k_0)>$$

$$= \pm\, i(2\pi) <\delta_{\mp}(m^2 - k^2),\ \tilde{\tilde{\phi}}(k_1,k_2,k_3,+k_0)>$$

$$= \frac{\pm i}{(2\pi)^3} <F[\delta_{\mp}(m^2 - k^2)],\ F[\tilde{\tilde{\phi}}(k_1,k_2,k_3,+k_0)]>$$

$$= <\frac{\mp i}{(2\pi)^3} F[\delta_{\mp}(m^2 - k^2)],\ \phi(x_1,x_2,x_3,x_0)>.$$

Hence

$$\Delta^{\pm}(x) = \mp \frac{i}{(2\pi)^3} F[\delta_{\mp}(m^2 - k^2)].$$

Applying finally the results (4.17) and (4.18) of the foregoing section we obtain:

$$(5.6)\quad \Delta^{+}(x) = -\frac{1}{4\pi}\{\delta_{+}(x^2) - \delta_{-}(x^2)\} - \frac{i}{4\pi^2} m\theta(-x^2)\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} + $$
$$+ \frac{m}{8\pi}\theta(x^2)\left[\theta(x_0)\frac{H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} - \theta(-x_0)\frac{H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}}\right]$$

and

$$(5.7)\quad \Delta^-(x) = -\frac{1}{4\pi}\{\delta_+(x^2) - \delta_-(x^2)\} + \frac{im}{4\pi^2}\,\theta(-x^2)\,\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}$$

$$+ \frac{m}{8\pi}\,\theta(x^2)\left[\theta(x_0)\,\frac{H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} - \theta(-x_0)\,\frac{H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}}\right].$$

Other important solutions of the homogeneous differential equation are the solutions $\Delta(x)$ and $\Delta^{(1)}(x)$, which are respectively odd and even in x_0.

They are defined as:

$$(5.8)\qquad \Delta(x) = \Delta^+(x) + \Delta^-(x) \text{ and } \Delta^{(1)}(x) = i\{\Delta^+(x) - \Delta^-(x)\}.$$

With the aid of the well known relations ([15], §7.2.1)

$$H_1^{(1)}(z) + H_1^{(2)}(z) = 2J_1(z)$$

(5.9) and

$$H_1^{(1)}(z) - H_1^{(2)}(z) = 2iY_1(z)$$

it follows immediately from the formulae (5.6) and (5.7), that

$$(5.10)\quad \Delta(x) = -\frac{1}{2\pi}\{\delta_+(x^2) - \delta_-(x^2)\} +$$

$$+ \theta(x^2)\,\frac{m}{4\pi}\left[\theta(x_0)\,\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} - \theta(-x_0)\,\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}}\right]$$

and

$$(5.11)\quad \Delta^{(1)}(x) = \theta(-x^2)\frac{m}{2\pi^2}\,\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}} + \theta(+x^2)\,\frac{m}{4\pi}\,\frac{Y_1(m\sqrt{x^2})}{\sqrt{x^2}}\,.$$

The distribution $\Delta(x)$, which is called the Pauli-Jordan propagator function, has the important property that it vanishes according to (5.10) outside the lightcone. It will become clear in section (5.3), why it is called a propagator function.

5.2. The solutions of the inhomogeneous equation

We have shown in section 2 (formula (2.10)) that the general Lorentz-invariant solution of the inhomogeneous Klein-Gordon equation (1.2) is given by

$$(5.12) \qquad g(x_1,x_2,x_3,x_0) = F^{*-1}\left[(m^2 - k^2)^{-1}\right] + c_+F^{*-1}\left[\delta_+(m^2 - k^2)\right] + c_-F^{*-1}\left[\delta_-(m^2 - k^2)\right].$$

Special solutions are obtained by taking the following values of c_+ and c_-.

1e. $c_+ = c_- = 0$.

The solution g(x) is now denoted by $\bar{\Delta}(x)$ and we get according to (5.12) and (2.4)

$$\bar{\Delta}(x_1,x_2,x_3,-x_0) = \frac{1}{(2\pi)^4} F\left[\frac{1}{(m^2 - k^2)}\right].$$

Using the result (3.18) we obtain

$$\bar{\Delta}(x_1,x_2,x_3,-x_0) = -\frac{m}{8\pi}\,\theta(x^2)\,\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} + \frac{1}{4\pi}\,\delta(x^2),$$

and since this distribution is even in x_0 we have the result

$$(5.13) \quad \bar{\Delta}(x_1,x_2,x_3,x_0) = -\frac{m}{8\pi}\,\theta(x^2)\,\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} + \frac{1}{4\pi}\,\delta(x^2).$$

This distribution has its support in the closure of the forward and backward light cone.

2e. $c_+ = c_- = \pi i$.

We denote g(x) by $\Delta_C(x)$ and we obtain

$$(5.14) \qquad \Delta_C(x) = F^{*-1}\left[(m^2 - k^2)^{-1} + i\pi\delta(m^2 - k^2)\right] = F^{*-1}\left[(m^2 - k^2 - i0)^{-1}\right].$$

Using (5.2), (5.3), (5.8) we get

$$\Delta_C(x) = \bar{\Delta}(x) - \frac{1}{2}\{\Delta^+(x) - \Delta^-(x)\} = \bar{\Delta}(x) + \frac{i}{2}\Delta^{(1)}(x). \tag{5.15}$$

With the aid of the formulae (5.13) and (5.11) the result becomes

$$\Delta_C(x) = \frac{1}{4\pi}\delta(x^2) - \frac{m}{8\pi}\theta(x^2)\frac{H_1^{(2)}(m\sqrt{x^2})}{\sqrt{x^2}} + \frac{im}{(2\pi)^2}\theta(-x^2)\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}. \tag{5.16}$$

3e. $c_+ = c_- = -i\pi$.

The distribution g(x) is now denoted by $\Delta_{AC}(x)$ and we obtain

$$\Delta_{AC}(x) = F^{*-1}\left[(m^2 - k^2)^{-1} - i\pi\delta(m^2 - k^2)\right] = F^{*-1}\left[(m^2 - k^2 + i0)^{-1}\right]. \tag{5.17}$$

In the same way as in the foregoing case we may write

$$\Delta_{AC}(x) = \bar{\Delta}(x) + \frac{1}{2}\{\Delta^+(x) - \Delta^-(x)\} = \bar{\Delta}(x) - \frac{i}{2}\Delta^{(1)}(x). \tag{5.18}$$

Using again the results (5.13) and (5.11) we get

$$\Delta_{AC}(x) = \frac{1}{4\pi}\delta(x^2) - \frac{m}{8\pi}\theta(x^2)\frac{H_1^{(1)}(m\sqrt{x^2})}{\sqrt{x^2}} - \frac{im}{(2\pi)^2}\theta(-x^2)\frac{K_1(m\sqrt{-x^2})}{\sqrt{-x^2}}. \tag{5.19}$$

4e. $c_+ = -c_- = +i\pi$.

The distribution g(x) is denoted by $\Delta_R(x)$ and we obtain

$$\Delta_R(x) = F^{*-1}\left[(m^2 - k^2)^{-1} + i\pi\delta_+(m^2 - k^2) - i\pi\delta_-(m^2 - k^2)\right] \tag{5.20}$$

or by aid of (5.2), (5.3) and (5.8)

$$\Delta_R(x) = \bar{\Delta}(x) - \frac{1}{2}\{\Delta^+(x) + \Delta^-(x)\} = \bar{\Delta}(x) - \frac{1}{2}\Delta(x). \tag{5.21}$$

Using now the results (5.13) and (5.10) the result becomes

$$\begin{aligned} \Delta_R(x) &= \frac{1}{2\pi}\delta_+(x^2) - \frac{m}{4\pi}\theta(x^2)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} \quad &\text{for } x_0 \geq 0 \\ \Delta_R(x) &\equiv 0 \quad &\text{for } x_0 < 0. \end{aligned} \tag{5.22}$$

5e. $c_+ = - c_- = -i\pi$.

The distribution g(x) is denoted by $\Delta_A(x)$ and we obtain

(5.23) $$\Delta_A(x) = F^{*-1}\left[(m^2 - k^2)^{-1} - i\pi\delta_+(m^2 - k^2) + i\pi\delta_-(m^2 - k^2)\right],$$

or by aid of (5.2), (5.3) and (5.8)

(5.24) $$\Delta_A(x) = \bar{\Delta}(x) + \frac{1}{2}\Delta(x).$$

Using again the results (5.13) and (5.10) we get the result

(5.25) $$\begin{aligned} \Delta_A(x) &\equiv 0 && \text{for } x_0 > 0 \\ \Delta_A(x) &= \frac{1}{2\pi}\delta_-(x^2) - \frac{m}{4\pi}\theta(x^2)\frac{J_1(m\sqrt{x^2})}{\sqrt{x^2}} && \text{for } x_0 \leq 0. \end{aligned}$$

From these results it follows immediately that $\bar{\Delta}(x)$ has its support in the closure of the forward and backward light cone, $\Delta_R(x)$ in the closure of the forward light cone, $\Delta_A(x)$ in the closure of the backward light cone, whereas the supports of $\Delta_C(x)$ and $\Delta_{AC}(x)$ have no boundaries at all.

$\Delta_R(x)$ is called the retarded Green's function, $\Delta_A(x)$ the advanced Green's function, $\Delta_C(x)$ the causal Green's function and finally $\Delta_{AC}(x)$ the anticausal Green's function.

The following relations are easily verified:

(5.26) $$\Delta_C(x) = \begin{cases} -\Delta^+(x) & \text{for } x_0 > 0 \\ +\Delta^-(x) & \text{for } x_0 < 0 \end{cases} \qquad \Delta_{AC}(x) = \begin{cases} -\Delta^-(x) & \text{for } x_0 > 0 \\ +\Delta^+(x) & \text{for } x_0 < 0. \end{cases}$$

(5.27) $$\Delta_R(x) = \begin{cases} -\Delta(x) & \text{for } x_0 > 0 \\ 0 & \text{for } x_0 < 0 \end{cases} \qquad \Delta_A(x) = \begin{cases} 0 & \text{for } x_0 > 0 \\ \Delta(x) & \text{for } x_0 < 0. \end{cases}$$

(5.28) $$2\bar{\Delta}(x) = \Delta_R(x) + \Delta_A(x) = \Delta_C(x) + \Delta_{AC}(x).$$

(5.29) $$i\Delta^{(1)}(x) = \Delta^-(x) - \Delta^+(x) = \Delta_C(x) - \Delta_{AC}(x).$$

From the formulae (5.14), (5.17), (5.20) and (5.23) useful and interesting relations follow for the positive and negative frequency parts ($k_0 > 0$ resp. $k_0 < 0$) of the distributions $\Delta_C(x)$, $\Delta_{AC}(x)$, $\Delta_R(x)$ and $\Delta_A(x)$. Denoting them by $\Delta_C^+(x)$, $\Delta_C^-(x)$ etc., we have

$$\Delta_C^+(x) = \Delta_R^+(x) , \qquad \Delta_C^-(x) = \Delta_A^-(x)$$

(5.30)

$$\Delta_{AC}^+(x) = \Delta_A^+(x), \qquad \Delta_{AC}^-(x) = \Delta_R^-(x).$$

We consider now the inhomogeneous Klein-Gordon equation

(5.31) $$(\Box - m^2)\phi(x) = - j(x),$$

where $j(x)$ is an arbitrary distribution with bounded support. The Lorentz invariant solution complying with conditions at infinity can be represented in several ways, e.g.

(5.32) $$\phi(x) = \phi_\alpha(x) + \Delta_\alpha(x) * j(x),$$

where we may write instead of α: R, A, C, AC; the convolution product is taken with respect to all variables x_1, x_2, x_3, and x_0. Taking $\alpha = R$ and x_0 sufficiently small we find that $\phi_R(x)$ must be the potential of the incoming field; on the other hand, taking $\alpha = A$ and x_0 sufficiently large we obtain that ϕ_A must be the potential of the outgoing field.

We consider now the cases $\alpha = C$ and $\alpha = AC$; taking Fourier transforms, splitting into positive and negative frequency parts and using (5.30) we get

(5.33) $$\hat{\phi} = \hat{\phi}_R^+ + \hat{\phi}_R^- + \hat{\Delta}_R^+ \cdot \hat{j} + \hat{\Delta}_R^- \cdot \hat{j}.$$

(5.34) $$\hat{\phi} = \hat{\phi}_A^+ + \hat{\phi}_A^- + \hat{\Delta}_A^+ \cdot \hat{j} + \hat{\Delta}_A^- \cdot \hat{j}.$$

(5.35) $$\hat{\phi} = \hat{\phi}_C^+ + \hat{\phi}_C^- + \hat{\Delta}_C^+ \cdot \hat{j} + \hat{\Delta}_C^- \cdot \hat{j} = \hat{\phi}_C^+ + \hat{\phi}_C^- + \hat{\Delta}_R^+ \cdot \hat{j} + \hat{\Delta}_A^- \cdot \hat{j}.$$

(5.36) $$\hat{\phi} = \hat{\phi}_{AC}^+ + \hat{\phi}_{AC}^- + \hat{\Delta}_{AC}^+ \cdot \hat{j} + \hat{\Delta}_{AC}^- \cdot \hat{j} = \hat{\phi}_{AC}^+ + \hat{\phi}_{AC}^- + \hat{\Delta}_A^+ \cdot \hat{j} + \hat{\Delta}_R^- \cdot \hat{j}.$$

From these equations we derive easily:

$$\hat{\phi}_C^+ + \hat{\phi}_C^- + \hat{\Delta}_A^- \cdot \hat{j} = \hat{\phi}_R^+ + \hat{\phi}_R^- + \hat{\Delta}_R^- \cdot \hat{j}.$$

$$\hat{\phi}_C^+ + \hat{\phi}_C^- + \hat{\Delta}_R^+ \cdot \hat{j} = \hat{\phi}_A^+ + \hat{\phi}_A^- + \hat{\Delta}_A^+ \cdot \hat{j}.$$

$$\hat{\phi}_{AC}^+ + \hat{\phi}_{AC}^- + \hat{\Delta}_A^+ \cdot \hat{j} = \hat{\phi}_R^+ + \hat{\phi}_R^- + \hat{\Delta}_R^+ \cdot \hat{j}.$$

$$\hat{\phi}_{AC}^+ + \hat{\phi}_{AC}^- + \hat{\Delta}_R^- \cdot \hat{j} = \hat{\phi}_A^+ + \hat{\phi}_A^- + \hat{\Delta}_A^- \cdot \hat{j}.$$

Equating finally in each equation positive and negative frequency parts we obtain:

(5.37) $$\hat{\phi}_C^+ = \hat{\phi}_R^+ , \quad \hat{\phi}_C^- = \hat{\phi}_A^- , \quad \hat{\phi}_{AC}^+ = \hat{\phi}_A^+ , \quad \hat{\phi}_{AC}^- = \hat{\phi}_R^- .$$

Hence $\phi_C(x)$ is the sum of the positive frequency part of the potential of the incoming field and the negative frequency part of the potential of the outgoing field; $\phi_{AC}(x)$ is the sum of the positive frequency part of the potential of the outgoing field and the negative frequency part of the potential of the incoming field.

For these and many other interesting physical considerations the reader is also referred to refs. [3] and [18].

5.3. Cauchy problems for the Klein-Gordon equation

We consider the following classical Cauchy problem: the function $\phi(x)$ satisfies for $x_0 > 0$ the differential equation:

(5.38) $$(\Box - m^2)\phi(x) = 0, \quad x_0 > 0,$$

$\phi(x) \equiv 0$ for $x_0 < 0$, while we have for $x_0 = 0$

(5.39)
$$\lim_{x_0 \to +0} \phi(x_1,x_2,x_3,x_0) = \phi_0(x_1,x_2,x_3),$$

$$\lim_{x_0 \to +0} \frac{\partial \phi}{\partial x_0}(x_1,x_2,x_3,x_0) = \phi_1(x_1,x_2,x_3),$$

where ϕ_0 and ϕ_1 are functions defined on R_3 and of finite algebraic growth at infinity. We suppose further ϕ_0 and ϕ_1 sufficiently regular in order that this Cauchy problem has a twice differentiable solution (differentiable in classical sense).

According to Chapter I, section 9, we obtain the solution of this Cauchy problem by considering the differential equation

$$(5.40) \qquad (\Box - m^2)\phi(x) = -\phi_0(x_1,x_2,x_3)\delta'(x_0) - \phi_1(x_1,x_2,x_3)\delta(x_0),$$

where the differentiations are now taken in distributional sense; for $x_0 < 0$ one has still the condition $\phi(x) \equiv 0$.

Similarly as in Chapter I, section 9, Example 2, for the wave equation one can show that there exists a unique solution of (5.40) with the condition $\phi(x) \equiv 0$ for $x_0 < 0$. This unique solution is given by (5.32) with $\alpha = R$, $\phi_R \equiv 0$ and $j(x) = \phi_0(x_1,x_2,x_3)\delta'(x_0) + \phi_1(x_1,x_2,x_3)\delta(x_0)$; hence

$$\phi(x) = \Delta_R(x) * \{\phi_0(x_1,x_2,x_3)\delta'(x_0) + \phi_1(x_1,x_2,x_3)\delta(x_0)\}$$

or

$$(5.41) \qquad \phi(x) = \phi_0(x_1,x_2,x_3) * \frac{\partial}{\partial x_0}\Delta_R(x) + \phi_1(x_1,x_2,x_3) * \Delta_R(x).$$

In the latter equation the convolution should be taken with respect to the three space variables x_1, x_2 and x_3 only. In the same way one may treat the Cauchy problem:

$$(5.42) \qquad (\Box - m^2)\phi(x) = 0, \quad x_0 < 0,$$

$\phi(x) \equiv 0$ for $x_0 > 0$ and for $x_0 = 0$ one has the conditions

$$(5.43) \qquad \begin{aligned} &\lim_{x_0 \to -0} \phi(x_1,x_2,x_3,x_0) = \phi_0(x_1,x_2,x_3) \\ &\lim_{x_0 \to -0} \frac{\partial \phi}{\partial x_0}(x_1,x_2,x_3,x_0) = \phi_1(x_1,x_2,x_3), \end{aligned}$$

where ϕ_0 and ϕ_1 are again submitted to the conditions stated above. The solution of this problem is given by the solution of the partial differential equation

$$(5.44) \qquad (\Box - m^2)\phi(x) = + \phi_0(x_1,x_2,x_3)\delta'(x_0) + \phi_1(x_1,x_2,x_3)\delta(x_0),$$

where the differentiations should now be taken in distributional sense, while $\phi(x)$ is submitted to the condition $\phi(x) \equiv 0$ for $x_0 > 0$. Using now (5.32) with $\alpha = A$, $\phi_A \equiv 0$ and $j(x) = - \phi_0(x_1,x_2,x_3)\delta'(x_0) - \phi_1(x_1,x_2,x_3)\delta(x_0)$ we obtain for the unique solution of the Cauchy problem (5.42), (5.43):

$$\phi(x) = - \Delta_A(x) * \{\phi_0(x_1,x_2,x_3)\delta'(x_0) + \phi_1(x_1,x_2,x_3)\delta(x_0)\}$$

or

$$(5.45) \qquad \phi(x) = - \phi_0(x_1,x_2,x_3) * \frac{\partial}{\partial x_0} \Delta_A(x) - \phi_1(x_1,x_2,x_3) * \Delta_A(x).$$

In the latter equation the convolution is again taken with respect to the space variables x_1, x_2 and x_3 only.
Finally we consider the following problem: the function $\phi(x)$ satisfies for $x_0 > 0$ and for $x_0 < 0$ the Klein-Gordon equation

$$(5.46) \qquad (\Box - m^2)\phi(x) = 0,$$

while the function $\phi(x)$ for $x_0 = 0$ is submitted to the conditions

$$\lim_{x_0 \to 0} \phi(x_1,x_2,x_3,x_0) = \phi_0(x_1,x_2,x_3)$$

$$\lim_{x_0 \to 0} \frac{\partial}{\partial x_0} \phi(x_1,x_2,x_3,x_0) = \phi_1(x_1,x_2,x_3),$$

where ϕ_0 and ϕ_1 satisfy the same conditions as before.
Using now the results (5.41) and (5.45) we obtain immediately for $x_0 \neq 0$

$$\phi(x) = \phi_0(x_1,x_2,x_3) * \frac{\partial}{\partial x_0} \{\Delta_R(x) - \Delta_A(x)\} + \phi_1(x_1,x_2,x_3) * \{\Delta_R(x) - \Delta_A(x)\}$$

or by aid of (5.22), (5.25) and (5.10)

$$(5.47) \quad \phi(x) = \phi_0(x_1,x_2,x_3) * \frac{\partial}{\partial x_0}\Delta(x) + \phi_1(x_1,x_2,x_3) * \Delta(x),$$

where $\Delta(x)$ is the Pauli-Jordan propagator function.
Hence the Pauli-Jordan function determines the propagation of the values of solutions of the homogeneous Klein-Gordon equation as function of place and time.
In order that the solution (5.47) is invariant under proper Lorentz-transformations one should submit the functions $\phi_0(x_1,x_2,x_3)$ and $\phi_1(x_1,x_2,x_3)$ to some extra conditions; for this the reader is referred to a paper by Mustapha Raïs [19].

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Chapter V

DIVERGENT CONVOLUTION INTEGRALS IN ELECTRODYNAMICS

1. Introduction

Consider a scattering process at the beginning and at the end of which there are only particles that are widely separated from each other and may be considered to be free. If the probability amplitude of the initial state is denoted by $\Phi(-\infty)$ and that of the final state by $\Phi(+\infty)$, the scattering process is described by an operator S, the S-matrix, which is defined by the relation

$$\Phi(+\infty) = S\,\Phi(-\infty); \tag{1.1}$$

see e.g. [1] , chapter III, [2] , chapter IV, [3] , chapter VII. The elements of the S-matrix contain products of the so-called causal function $\Delta_c(x)$ and its derivatives. The function $\Delta_c(x)$ has been treated in the preceding chapter, where it appeared to be essentially a distribution.
However, as has been shown by L. Schwartz [4] , distributions cannot in general be multiplied with each other and therefore the elements of the S-matrix are not well defined.

By application of the modified Fourier transformation, the distribution $\Delta_c(x)$ and its derivatives are transformed according to formula (6.25) of chapter IV into distributional limits of rational functions, viz.:

$$\begin{aligned} &F^{*}\left[P\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_o}\right) \Delta_c(x)\right] = \\ &\lim_{\varepsilon \to +0} \left[P(ik_1, ik_2, ik_3, -ik_o) \frac{-1}{k^2-m^2+i\varepsilon(k,k)}\right], \end{aligned} \tag{1.2}$$

with

$$k^2 = k_o^2 - k_1^2 - k_2^2 - k_3^2, \tag{1.3}$$

(1.4) $$(k,k) = k_o^2+k_1^2+k_2^2+k_3^2,$$

and where $P(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_o})$ is some Lorentz-invariant polynomial in $\frac{\partial}{\partial x_i}$. [1] The products containing $\Delta_C(x)$ and (or) its derivatives are transformed formally into distributional limits of convolution integrals, which turn out to be divergent.
The impossibility of defining products of causal functions is reflected in the "momentum space" by the fact that the formal Fourier transforms of these products are divergent convolution integrals.

In the literature on quantum electrodynamics much attention has been paid to the meaning of these divergent convolution integrals and several devices have been developed for assigning to them a well-defined finite value. We mention here the method used by Bogoliubov and Parasiuk in [1] Ch.IV, [5] and [6] , that used by Achieser and Berestezki in [2] Ch.VII and a rather recent method developed by Bremermann in [7] .

Bogoliubov and Parasiuk define a divergent convolution product, say $f(k) * g(k)$ as the weak limit of a convergent convolution product on an appropriate space of test functions. The convergent convolution product consists of specially chosen factors, say $f_M(k)$ and $g_M(k)$, depending on a fictive mass M, such that $f_M(k)$ and $g_M(k)$ converge for $M \to \infty$ weakly to $f(k)$ and $g(k)$, while $f_M(k) * g_M(k)$ is convergent. Finally $f(k) * g(k)$ is defined as the weak limit of $f_M(k) * g_M(k)$ for $M \to \infty$.
The replacement of $f(k), g(k)$ and $f(k) * g(k)$ by respectively $f_M(k)$, $g_M(k)$ and $f_M(k) * g_M(k)$ is called the regularization of $f(k), g(k)$ and $f(k) * g(k)$.
However, this procedure is rather unsatisfactory since

i) $f_M(k)$ and $g_M(k)$ are chosen in a very special way.

ii) $f(k)$ and $g(k)$ are regularized with the aid of the same parameter M.

Achieser and Berestezki define the divergent convolution integrals also by means of a special limiting procedure. The integration is performed over a finite volume V of the four dimensional space. If the limit is taken for $V \to \infty$, the result is of course infinite, but it turns out that the divergencies are only contained in a certain polynomial in k. Disregarding this polynomial one obtains the so-called "regular" part, which is taken as the definition of the divergent convolution integral.

Finally Bremermann defines the divergent convolution integrals by the requirement that they should satisfy certain sets of differential equations.

In this chapter we confine our investigations to the case of divergent convolution products containing only two factors. They are introduced in a very natural way as a functional defined on an appropriate space of test functions. The definition does not involve any limiting procedure as in the definitions used by Bogoliubov-Parasiuk and Achieser-Berestezki.

It appears that the results of the above-mentioned methods are in agreement with the definition introduced in this chapter. Actually they are very special cases of the theory given here. For example the restrictions i) and ii) are quite irrelevant. Apart from presenting a theory for dealing with divergent convolution integrals we establish in this way also the equivalence of the above-mentioned methods.

It may be remarked that Bogoliubov and Parasiuk have investigated in [5] and [6] the much more general problem of defining the product of causal functions containing an arbitrary number of factors. In fact, these authors have considered expressions such as

$$\prod_{1} D_C^{\,1}(x_r - x_s), \tag{1.5}$$

where $D_C^{\,1}$ is the causal function or one of its derivatives; the product is taken over all directed line segments occurring in an arbitrary graph connecting the points $x_1, x_2, \ldots, x_n$ in the four-dimensional space of position and time.

It may be, that the method given here can be extended to this much more general case. This is a suggestion for further research.

In section 2 we introduce the convolution product of bounded rational functions in n variables. Consecutively, several useful properties of this product will be derived in section 3.

In the next section 4 a summary is given of the methods, mentioned above, which provide a well-defined meaning of the divergent convolution integrals. It is shown that they all are special cases of the definition of the convolution as given in section 2.

The chapter is concluded by section 5, in which there are made some additional remarks.

2. The convolution of bounded rational functions

2.1. The boundedness of a convolution product

We consider the class A of functions $f(k) = f(k_1,k_2,\ldots,k_n)$ which are defined and uniformly bounded in the whole n-dimensional space R_n; $k=(k_1,k_2,\ldots,k_n)$ denotes a point of this n-dimensional space. For every function f(k), belonging to A, there exists a non-negative number $m(f) \geq 0$ with the property that

$$\{1+|k|\}^{m(f)}\,|f(k)| \tag{2.1}$$

is uniformly bounded in the whole space R_n;

$$|k| = \sqrt{k_1^{\,2}+k_2^{\,2}\ldots+k_n^{\,2}}.$$

It is clear, that there exists in this case a continuous set of values m(f) such that (2.1) is uniformly bounded in R_n.
The upper bound of all these values m(f) will be denoted by $\bar{m}(f)$ and we call $\bar{m}(f)$ the index of the function f(k). E.g. the function

$$\frac{\log(1+|k|)}{1+|k|}$$

has index 1. Hence all functions of the class A have non-negative index.

Taking two functions f(k) and g(k), both belonging to A, one can form the convolution of f(k) and g(k), viz.

$$(2.2) \qquad F(p) = \int_{-\infty}^{+\infty} f(k)g(p-k)dk = \int_{-\infty}^{+\infty} f(p-k)g(k)dk,$$

where the integration is performed over the whole space R_n.
The integral converges absolutely for all finite values of p, if the condition

$$(2.3) \qquad \overline{m}(f) + \overline{m}(g) > n,$$

is satisfied.
The following lemma will be very useful in the development of the theory.

Lemma

If f(k) and g(k) belong to A, $\overline{m}(f) \leq n+\nu$, $\overline{m}(g) \leq n+\nu$ with $\nu \geq 0$ and if

$$(2.4) \qquad \overline{m}(f) + \overline{m}(g) > n+\nu,$$

then F(p) belongs also to A and the index of the function F(p) is at least equal to $\overline{m}(f) + \overline{m}(g)-n-\nu$.

Proof:

Due to the assumptions of the Lemma, there exists a positive number m(f) and a constant B(f), independent of k, such that

$$|f(k)| \leq \frac{B(f)}{\{1+|k|\}^{m(f)}},$$

valid for all values of k; we may choose m(f) arbitrarely close to $\overline{m}(f)$.
Therefore f(k) belongs to the class $L^{q_1}(-\infty,+\infty)$ with

$$q_1 = \frac{n+\nu+\delta_1}{\overline{m}(f)} > 1,$$

where δ_1 is some quantity larger than zero.
We choose m(f) such that

$$\frac{m(f)}{\overline{m}(f)} (n+\nu+\delta_1) = n+\nu+\tfrac{1}{2}\delta_1.$$

Hence

$$|f(k)|^{q_1} \leq \{B(f)\}^{q_1} \cdot (1+|k|)^{-(n+\nu+\frac{1}{2}\delta_1)} . \tag{2.5}$$

In the same way, the function $g(k)$ belongs to the class $L^{q_2}(-\infty,+\infty)$, with

$$q_2 = \frac{n+\nu+\delta_2}{\overline{m}(g)} > 1 \text{ and } \delta_2 > 0.$$

Moreover $m(g)$ is chosen, such that

$$|g(k)|^{q_2} \leq \{B(g)\}^{q_2}(1+|k|)^{-(n+\nu+\frac{1}{2}\delta_2)} . \tag{2.6}$$

Finally we put $\delta_1 = \delta_2 = \delta$ and we take such a value for δ that $\frac{1}{q_1} + \frac{1}{q_2} > 1$; due to the relation (2.4) this is always possible.

From Hölder's inequality for three functions, viz.

$$\left| \int_{-\infty}^{+\infty} \phi\psi\chi \, dk \right| \leq \left(\int_{-\infty}^{+\infty} |\phi|^{\frac{1}{\alpha}} dk\right)^{\alpha} \left(\int_{-\infty}^{+\infty} |\psi|^{\frac{1}{\beta}} dk\right)^{\beta} \left(\int_{-\infty}^{+\infty} |\chi|^{\frac{1}{\gamma}} dk\right)^{\gamma} ,$$

where $\alpha+\beta+\gamma=1$, $\alpha > 0$, $\beta > 0$ and $\gamma > 0$, one derives easily Young's inequality

$$\left| \int_{-\infty}^{+\infty} f(k)g(k)\,dk \right| \leq \left\{\int_{-\infty}^{+\infty} |f|^{q_1} |g|^{q_2} dk\right\}^{\frac{1}{q_1}+\frac{1}{q_2}-1} \cdot \tag{2.7}$$
$$\cdot \left\{\int_{-\infty}^{+\infty} |f|^{q_1} dk\right\}^{1-\frac{1}{q_2}} \cdot \left\{\int_{-\infty}^{+\infty} |g|^{q_2} dk\right\}^{1-\frac{1}{q_1}},$$

by setting $|\phi|^{\frac{1}{\alpha}} = |\psi|^{\frac{1}{\beta}} |\chi|^{\frac{1}{\gamma}}, |\psi|^{1+\frac{\alpha}{\beta}} = |f|$, $|\chi|^{1+\frac{\alpha}{\gamma}} = |g|$ and $\gamma = 1 - \frac{1}{q_1}$ and $\beta = 1 - \frac{1}{q_2}$.

Applying Young's inequality (2.7) to the function $F(p)$ we obtain

$$|F(p)| \leq \left\{\int_{-\infty}^{+\infty} |f(k)|^{q_1} \cdot |g(p-k)|^{q_2} dk\right\}^{\frac{1}{q_1}+\frac{1}{q_2}-1} \cdot$$
$$\cdot \left\{\int_{-\infty}^{+\infty} |f(k)|^{q_1} dk\right\}^{1-\frac{1}{q_2}} \cdot \left\{\int_{-\infty}^{+\infty} |g(k)|^{q_2} dk\right\}^{1-\frac{1}{q_1}} .$$

Since $f(k)$ and $g(k)$ belong to $L^{q_1}(-\infty,+\infty)$ respectively $L^{q_2}(-\infty,+\infty)$, there exists a constant C_1, independent of p such that

$$|F(p)| \le C_1 \left\{ \int_{-\infty}^{+\infty} |f(k)|^{q_1} |g(p-k)|^{q_2} dk \right\}^{\frac{1}{q_1} + \frac{1}{q_2} - 1} . \tag{2.8}$$

In virtue of (2.5) and (2.6) we can now make the following estimate, valid for all values μ with $0 \le \mu \le n+\nu$

$$|p|^{\mu} \int_{-\infty}^{+\infty} |f(k)|^{q_1} \; |g(p-k)|^{q_2} dk \le$$

$$C_2 \int_{-\infty}^{+\infty} |p|^{\mu} \; (1+ \; |k|)^{-n-\nu-\frac{\delta}{2}} (1+|p-k|)^{-n-\nu-\frac{\delta}{2}} dk \le$$

$$C_2 \int_{-\infty}^{+\infty} (|k| \; +|p-k|)^{\mu} (1+|k|)^{-n-\nu-\frac{\delta}{2}} (1+|p-k|)^{-n-\nu-\frac{\delta}{2}} dk \le$$

$$C_2 \; 2^{\mu} \int_{-\infty}^{+\infty} (|k|^{\mu} \; + \; |p-k|^{\mu}) (1+|k|)^{-n-\nu-\frac{\delta}{2}} (1+|p-k|)^{-n-\nu-\frac{\delta}{2}} dk < C_3 ,$$

where C_2 and C_3 are constants independent of p.
Hence it follows, that there exists a uniform constant C_4, such that

$$(1+|p|)^{n+\nu} \int_{-\infty}^{+\infty} |f(k)|^{q_1} \cdot \; |g(p-k)|^{q_2} dk < C_4 . \tag{2.9}$$

Substituting (2.9) into (2.8) we obtain the result

$$\begin{aligned} |F(p)| &< C(1+|p|)^{-(n+\nu)\left(\frac{1}{q_1} + \frac{1}{q_2} - 1\right)} = \\ & C(1+|p|)^{n+\nu - \frac{n+\nu}{n+\nu+\delta} \{\bar{m}(f) \; + \; \bar{m}(g)\}} , \end{aligned} \tag{2.10}$$

where C is again a constant.
In virtue of the relation (2.4), we can make δ arbitrarily small without violating the condition $\frac{1}{q_1} + \frac{1}{q_2} > 1$. Moreover, since $\bar{m}(f) + \bar{m}(g) > n+\nu$, it is clear that F(p) belongs to the class A and has index

$$\bar{m}(F) \ge \bar{m}(f) + \bar{m}(g) - (n+\nu) , \qquad \text{q.e.d.} \tag{2.11}$$

The following corollaries follow immediately from the lemma.

Corollaries:

1. If f(k) and g(k) belong to the class A, $\bar{m}(f) \leq n$, $\bar{m}(g) \leq n$ and $\bar{m}(f) + \bar{m}(g) > n$, then F(p) belongs also to A and its index is at least equal to $\bar{m}(f) + \bar{m}(g) - n$.

2. If f(k) and g(k) belong to the class A, $\bar{m}(f) > 0$, $\bar{m}(g) > 0$ and $\bar{m}(f)$ or $\bar{m}(g) > n$, then F(p) belongs also to A and its index is at least equal to min $[\bar{m}(f), \bar{m}(g)]$.

It is to be noticed, that the conditions $\bar{m}(f) > 0$, $\bar{m}(g) > 0$ in the second corollary may be omitted. Suppose e.g. $\bar{m}(f) = 0$, then $\bar{m}(g) > n$. Because f(k) is uniformly bounded in R_n, we can make the estimate

$$|F(p)| \leq \int_{-\infty}^{+\infty} |f(p-k)| \cdot |g(k)| dk < C \int_{-\infty}^{+\infty} |g(k)| dk.$$

The third member of this relation is a finite number, not necessarily zero. Therefore F(p) is uniformly bounded in R_n and its index is at least zero.

It follows that we may replace the second corollary by the slightly more general statement

2^{bis}. If f(k) and g(k) belong to the class A, $\bar{m}(f)$ or $\bar{m}(g) > n$, then F(p) belongs also to A and its index is at least equal to min $[\bar{m}(f), \bar{m}(g)]$.

2.2. Spaces of test functions C(q,r,n) and their dual spaces C'(q,r,n)

The functions f(k), g(k) and F(p), belonging to the class A, will be considered as continuous linear functionals on certain spaces of test functions.

For this purpose the following space C(q,r,n) of test functions is introduced; compare also ref. [1] and [5] .

The space C(q,r,n) consists of all functions $\phi(k) = \phi(k_1, k_2, \ldots, k_n)$, which are defined in R_n and which are continuous together with all their derivatives up to the q^{th} order inclusive. Moreover all the products

$$(2.12)\qquad |k_1^{r_1} k_2^{r_2} \ldots k_n^{r_n} \frac{\partial^p \phi(k_1,k_2,\ldots,k_n)}{\partial k_1^{p_1} \partial k_2^{p_2} \ldots \partial k_n^{p_n}}|,$$

are uniformly bounded for all values of $k_1,k_2,\ldots,k_n$; $p_1,p_2,\ldots,p_n$ are non-negative integers with $\sum_{i=1}^{n} p_i = p \leq q$ and $r,r_1,r_2,\ldots,r_n$ are non-negative numbers (not necessarily integers), such that $\sum_{j=1}^{n} r_i \leq r$.

This space is clearly a linear space; we introduce in this space a norm by defining the norm of the element $\phi(k)$ as

$$(2.13)\qquad ||\phi|| = \sup_{\substack{r_1\ldots r_n \\ p_1\ldots p_n}} | k_1^{r_1} k_2^{r_2} \ldots k_n^{r_n} \frac{\partial^p \phi(k_1,k_2,\ldots,k_n)}{\partial k_1^{p_1} \partial k_2^{p_2} \ldots \partial k_n^{p_n}} |,$$

where the supremum is taken over all values of $k_1,k_2,\ldots,k_n$ and over all combinations of r_i and p_i with $\sum_{i=1}^{n} p_i = p \leq q$ and $\sum_{i=1}^{n} r_i \leq r$.

The topology is introduced in the usual way; a neighbourhood of the zero-element is the set of all test functions ϕ with $||\phi||<\varepsilon$. It can easily be proved that the space C(q,r,n) is complete with respect to the norm (2.13) and hence C(q,r,n) is a Banach-space.

It is clear that we have for $q_1 \geq q_2$ and $r_1 \geq r_2$ the inclusion $C(q_1,r_1,n) \subset C(q_2,r_2,n)$ and that $||\phi||_{C(q_1,r_1,n)} \geq ||\phi||_{C(q_2,r_2,n)}$. Therefore convergence in the space $C(q_1,r_1,n)$ implies also convergence in the space $C(q_2,r_2,n)$.

The dual space consisting of all continuous linear functionals on C(q,r,n) is denoted by C'(q,r,n); the application of $f(k) \in C'(q,r,n)$ to $\phi(k) \in C(q,r,n)$ is written as

$$(2.14)\qquad \langle f(k),\ \phi(k)\rangle\ .$$

If f(k) is a locally Lebesgue-integrable function, such that the integral

$$\int_{-\infty}^{+\infty} f(k)\ \phi(k)dk$$

exists in the sense of Lebesgue, then f(k) considered as a continuous linear functional on C(q,r,n) is defined as

(2.15) $$<f(k),\ \phi(k)> = \int_{-\infty}^{+\infty} f(k)\ \phi(k)dk.$$

In the dual space C'(q,r,n) one can introduce again a norm. The norm of the continuous linear functional f(k) is defined as

(2.16) $$\|f(k)\| = \lim \sup_{\|\phi(k)\| = 1} |<f(k),\ \phi(k)>|.$$

A sequence of functionals $f_n(k) \in C'(q,r,n)$ is said to converge weakly to a functional $f(k) \in C'(q,r,n)$, if

(2.17) $$\lim_{n \to \infty} <f_n, \phi> = <f, \phi> \quad ,$$

for all functions $\phi \in C(q,r,n)$.

A sequence of functionals $f_n(k) \in C'(q,r,n)$ is said to converge strongly to a functional $f(k) \in C'(q,r,n)$, if

(2.18) $$\lim_{n \to \infty} \|f_n - f\| = 0.$$

This means that f_n converges strongly to f, if f_n converges in norm to f.

It is a well-known fact that C'(q,r,n) is complete with respect to the strong convergence and thus C'(q,r,n) is also a Banach space; see e.g. [8] , §19.

In the sequel we shall also need the derivative of a functional on a Banach space. This concept cannot be introduced as easily as in the case of functionals, defined e.g. on the space S of test functions, which are infinitely many times differentiable and which fall off at infinity stronger than any negative power of $|k|$; see chapter I, section 3.

In this case the functional is a distribution, say f(k), and its derivative is defined by the relation

(2.19) $$<\frac{\partial f}{\partial k_i}\ ,\ \phi> = -<f,\ \frac{\partial \phi}{\partial k_i}> .$$

This definition is meaningful, since $\frac{\partial \phi}{\partial k_i}$ belongs to S, whenever $\phi(k)$ belongs to S.

However, if $\phi(k)$ belongs to the Banach space $C(q,r,n)$, then $\frac{\partial\phi}{\partial k_i}$ belongs no longer to $C(q,r,n)$, but $\frac{\partial\phi}{\partial k_i}$ belongs to the space $C(q-1,r,n)$; q is assumed to be larger than or equal to one. We modify now our definition of derivative in the following way. If $f(k)$ is a functional, defined on the space of test functions $\psi(k)$, which can be written in the form $\psi(k) = \frac{\partial\phi}{\partial k_i}$, where $\phi(k)$ belongs to $C(q,r,n)$ with $q \geq 1$, then the "functional" derivative $\frac{\partial f}{\partial k_i}$ is defined on the space $C(q,r,n)$ by the relation

(2.20) $$< \frac{\partial f}{\partial k_i}, \phi> = - <f, \frac{\partial\phi}{\partial k_i}>.$$

Hence, if $f(k)$ is a functional on the space $C(q-1,r,n)$, then its derivative $\frac{\partial f}{\partial k_i}$ can certainly be defined on the space $C(q,r,n)$, which is a subspace of $C(q-1,r,n)$.

2.3. The convolution of bounded rational functions

In this section we define the convolution of bounded rational functions, say $f(k)$ and $g(k)$, as a functional on some space of test functions. This will be done in such a way that the new definition generalizes the classical concept of convolution which has only a well defined meaning if $\overline{m}(f) + \overline{m}(g) > n$.

We confine our treatment to bounded rational functions, because the functions appearing in the applications to electrodynamics are of this type and because it will be seen in the sequel that the functions $f(k)$ and $g(k)$ are required to be subject to the condition that differentiation raises the index by one.

The analysis will be facilitated by assuming that the indices of the functions $f(k)$ and $g(k)$ are positive. This is due to the fact that the first corollary of the lemma applies only for functions with positive index. If the index happens to be zero one obtains analogous results, but the proofs need only be a little bit modified; this is left to the reader.

Let $f(k)$ be a rational bounded function with positive index; hence its index $\overline{m}(f)$ is a positive integer and we assume $1 \leq \overline{m}(f) \leq n$.

Therefore f(k) can be considered as a functional on the space $C(q,n-\bar{m}(f)+\delta,n)$, where q may be any non-negative integer and δ an arbitrary positive number.

The functional $\langle f(k),\ \phi(k)\rangle$ is defined as

$$\langle f(k),\ \phi(k)\rangle = \int_{-\infty}^{+\infty} f(k)\ \phi(k)dk, \tag{2.21}$$

where the integration is performed over the whole space R_n and where $\phi(k)$ belongs to $C(q,n-\bar{m}(f)+\delta,n)$.
In the same way g(k) with index $\bar{m}(g)$, $1 \leq \bar{m}(g) \leq n$, may be considered as a functional on the space $C(q,n-\bar{m}(g)+\delta,n)$.

Let us consider now the convolution F(p) of the functions f(k) and g(k), defined by

$$F(p) = \int_{-\infty}^{+\infty} f(k)g(p-k)dk = \int_{-\infty}^{+\infty} f(p-k)g(k)dk. \tag{2.2}$$

For the moment we assume $\bar{m}(f) + \bar{m}(g) \geq n+1$ and hence the integral (2.2) converges absolutely.
According to the first corollary of the lemma of section (2.1) the function F(p) belongs to the class A and the index of F(p) is at least equal to $\bar{m}(f) + \bar{m}(g)-n$. Therefore F(p) may be considered as a continuous linear functional on the space $C(q,2n-\bar{m}(f)-\bar{m}(g)+\delta,n)$, where again δ may be any positive number. Because $C(q,r_1,n) \subset C(q,r_2,n)$ for $r_1 \geq r_2$, we may assume without loss of generality that $0 < \delta < 2$.

Hence for any test function $\phi(p_1,p_2,\ldots,p_n)$ belonging to $C(q,2n-\bar{m}(f)-\bar{m}(g)+\delta,n)$ we obtain the functional

$$\langle F(p),\ \phi(p)\rangle = \int_{-\infty}^{+\infty} F(p)\phi(p)dp = \int_{-\infty}^{+\infty} \phi(p)dp \int_{-\infty}^{+\infty} f(k)g(p-k)dk = \int_{-\infty}^{+\infty} \phi(p)dp \int_{-\infty}^{+\infty} f(p-k)g(k)dk. \tag{2.22}$$

In virtue of the absolute convergence of the integrals we may apply Fubini's theorem and we obtain

$$\begin{aligned} \langle F(p),\phi(p)\rangle &= \int_{-\infty}^{+\infty} f(k)dk \int_{-\infty}^{+\infty} g(p)\phi(p+k)dp = \\ &= \int_{-\infty}^{+\infty} g(k)dk \int_{-\infty}^{+\infty} f(p)\phi(p+k)dp. \end{aligned} \tag{2.23}$$

Again with the aid of the corollaries of the lemma of section (2.1), the function

$$\psi(k) = \int_{-\infty}^{+\infty} g(p)\phi(p+k)\,dp = \int_{-\infty}^{+\infty} \phi(k-p)g(-p)\,dp, \qquad (2.24)$$

has the property that it belongs to the class A and $\psi(k)$ has an index which is at least $\bar{m}(\phi) + \bar{m}(g)-n$ or $\bar{m}(g)$, according as $\bar{m}(\phi) \leq n$ or $\bar{m}(\phi)>n$. In case ϕ has an index smaller than or equal to n, the index of ϕ is a at least $2n-\bar{m}(f)-\bar{m}(g)+ \delta$ and hence $\psi(k)$ has an index which is at least $n-\bar{m}(f)+\delta$.

In case ϕ has an index larger than n, $\psi(k)$ has an index which is at least $\bar{m}(g)$ and this number is larger than or equal to $n+1-\bar{m}(f)$.

From the fact that $\phi(k)$ is q times continuously differentiable, it follows by a rather simple argument, that $\psi(k)$ is also q times continuously differentiable.

Hence the function $\psi(k)$ belongs certainly to the space $C(q,n-\bar{m}(f)+\frac{1}{2}\delta,n)$ with $0<\frac{1}{2}\delta<1$.

Therefore it is allowed to write instead of (2.23)

$$\langle F(p),\phi(p)\rangle = \langle f(k),\ \psi(k)\rangle\ . \qquad (2.25)$$

Moreover, because the space $C(q,2n-\bar{m}(f)-\bar{m}(g)+\delta,n)$ is included in the space $C(q,n-\bar{m}(g)+\delta,n)$, we may also write instead of $\psi(k)$

$$\psi(k) = \langle g(p),\ \phi(p+k)\rangle\ . \qquad (2.26)$$

Substitution of (2.26) onto (2.25) yields finally

$$\langle F(p),\ \phi(p)\rangle = \langle f(k),\ \langle g(p),\ \phi(p+k)\rangle\rangle\ . \qquad (2.27)$$

Interchanging the role of f(k) and g(k) we obtain in the same way

$$\langle F(p),\ \phi(p)\rangle = \langle g(k),\ \langle f(p),\ \phi(p+k)\rangle\rangle\ . \qquad (2.28)$$

The formula (2.27) or (2.28) is now considered as the <u>definition</u> of the convolution of the functionals f(k) and g(k). The functional given by the right-hand side of (2.27) is denoted by $f(k) * g(k)$ and similarly that of the right-hand side of (2.28) by $g(k) * f(k)$.

Hence we obtain the formulae

$$\langle f(k) * g(k),\ \phi(k)\rangle = \langle f(k),\ \langle g(p),\ \phi(p+k)\rangle\rangle \ , \tag{2.29}$$

$$\langle g(k) * f(k),\ \phi(k)\rangle = \langle g(k),\ \langle f(p),\ \phi(p+k)\rangle\rangle \ . \tag{2.30}$$

From (2.27) and (2.28) it follows immediately that the operation of convolution is commutative and therefore we have the identity

$$f(k) * g(k) = g(k) * f(k). \tag{2.31}$$

Our definition of convolution is meaningful as long as $\langle g(p),\ \phi(p+k)\rangle \in C(q,n-\bar{m}(f)+\mu_1,n)$ and $\langle f(p),\ \phi(p+k)\rangle \in C(q,n-\bar{m}(g)+\mu_2,n)$, where μ_1 and μ_2 are some quantities larger than zero. This is guaranteed, if the conditions $1 \leq \bar{m}(f) \leq n$, $1 \leq \bar{m}(g) \leq n$ and $\bar{m}(f) + \bar{m}(g) \geq n+1$ are satisfied. Summing up, one arrives at the following theorem

Theorem 1

If f and g are bounded rational functions of $k(k_1,k_2,\ldots,k_n)$ with index $\bar{m}(f)$ respectively $\bar{m}(g)$, such that $1 \leq \bar{m}(f) \leq n$, $1 \leq \bar{m}(g) \leq n$ and $\bar{m}(f) + \bar{m}(g) \geq n+1$, then $f(k)$ is a continuous linear functional on $C(q,n-\bar{m}(f)+\delta,n)$, $g(k)$ is a continuous linear functional on $C(q,n-\bar{m}(g)+\delta,n)$ and $f(k) * g(k)$ is a continuous linear functional on $C(q,2n-\bar{m}(f)-\bar{m}(g)+\delta,n)$. q may be any non-negative integer and δ any arbitrary positive number. Moreover

$$f(k) * g(k) = g(k) * f(k).$$

The above stated theorem gives conditions for which the classical convolution of two bounded rational functions with indices between 1 and n can be expressed as a continuous linear functional on a certain Banach space of test functions. We wish now to extend the concept of the convolution of bounded rational functions with positive index to the case where the classical integral definition ceases to have any meaning owing to the divergency of the integral.

Therefore suppose that $f(k)$ and $g(k)$ are bounded rational functions with indices $\bar{m}(f) \geq 1$ and $\bar{m}(g) \geq 1$, while $\bar{m}(f) + \bar{m}(g) \leq n$. Assume that $\bar{m}(f) + \bar{m}(g) = n-s$ with $0 \leq s \leq n-2$.
It is clear that the convolution

(2.2)
$$F(p) = \int_{-\infty}^{+\infty} f(k)g(p-k)dk$$

does not exist, since the integral diverges.

Instead of the convolution of f(k) and g(k) we consider now the convolution of the functions f(k) and $g^{(s+1)}(k)$, where $g^{(s+1)}(k)$ denotes any of the derivatives of g(k) of the order s+1.
Since g(k) is a bounded rational function with index $\bar{m}(g)$, $g^{(s+1)}(k)$ is also a bounded rational function, but its index is raised by (s+1), i.e. $\bar{m}(g^{(s+1)}) = \bar{m}(g)+s+1$.
Therefore $\bar{m}(f) + \bar{m}(g^{(s+1)}) = n+1$ and we can apply Theorem 1. Hence $f(k) * g^{(s+1)}(k)$ is a functional on the space $C(q,n-1+\delta,n)$ and we have according to (2.29) for any $\phi(k) \in C(q,n-1+\delta,n)$ the relation

(2.32)
$$\langle f(k) * g^{(s+1)}(k), \phi(k)\rangle = \langle f(k), \langle g^{(s+1)}(p), \phi(p+k)\rangle\rangle .$$

We take now for q the value s+1, i.e. we use the space of test functions $\phi(k)$ with the property that $\phi(k)$ and its derivatives up to the order (s+1) inclusive are continuous.
By means of integration by parts it is clear that

(2.33)
$$\langle g^{(s+1)}(p), \phi(p+k)\rangle = (-1)^{s+1} \langle g(p), \phi^{(s+1)}(p+k)\rangle .$$

Substituting this result into (2.32) we obtain

$$\langle f(k), \langle g(p), \phi^{(s+1)}(p+k)\rangle\rangle = (-1)^{s+1} \langle f(k) * g^{(s+1)}(k), \phi(k)\rangle ,$$

r according to the definition (2.29) of convolution

(2.34)
$$\langle f(k) * g(k), \phi^{(s+1)}(k)\rangle = (-1)^{s+1} \langle f(k) * g^{(s+1)}(k), \phi(k)\rangle .$$

Hence the convolution $f(k) * g(k)$ exists as a functional on the space of test functions $\psi(k)$ which can be written as $\phi(k) = \phi^{(s+1)}(k)$, where $\phi(k)$ belongs to the Banach space $C(s+1,n-1+\delta,n)$ with δ arbitrary positive.
Because $\psi(k)$ belongs to $C(0,n-1+\delta,n)$, it is clear that $f(k) * g(k)$ is only defined as a functional on a linear subspace of $C(0,n-1+\delta,n)$.

The properties of our convolution product $f(k) * g(k)$ are determined by the right-hand side of (2.34).
If we take for δ a positive number smaller than or equal to one, the convolution $f(k) * g(k)$ is uniquely defined on the space of test functions $\psi(k) = \phi^{(s+1)}(k)$ with $\phi(k) \in C(s+1, n-1+\delta, n)$.
However, if we restrict our convolution product to a smaller space by choosing δ larger than one, the convolution $f(k) * g(k)$ is no longer uniquely defined; it is now only defined apart from a polynomial. The maximal degree of this polynomial is determined by the chosen value of δ.
For $1 \leq t < \delta \leq t+1 \leq s+2$ with t integer the degree of the polynomial is at most $t-1$ and for $\delta > s+2$ the degree is at most s.
The coefficients have to be determined from extra conditions to be imposed on the convolution of $f(k)$ and $g(k)$; e.g. conditions resulting from physical considerations.

The same arguments can be used after interchanging $f(k)$ and $g(k)$; therefore we have also the relation

$$(2.35) \qquad \langle g(k) * f(k), \phi^{(s+1)}(k)\rangle = (-1)^{s+1} \langle g(k) * f^{(s+1)}(k), \phi(k)\rangle .$$

Instead of the right-hand side of equation (2.35) we may write

$$(-1)^{s+1} \langle g(k) * f^{(s+1)}(k), \phi(k)\rangle = (-1)^{s+1} \int_{-\infty}^{+\infty} \phi(p)dp \int_{-\infty}^{+\infty} g(k)\, f^{(s+1)}(p-k)dk.$$

By means of integration by parts the inner integral may also be written as

$$\int_{-\infty}^{+\infty} g^{(s+1)}(k)\, f(p-k)dk.$$

Hence we obtain the relation

$$(-1)^{s+1} \langle g(k) * f^{(s+1)}(k), \phi(k)\rangle = (-1)^{s+1} \int_{-\infty}^{+\infty} \phi(p)dp \int_{-\infty}^{+\infty} g^{(s+1)}(k)\, f(p-k)\,dk = (-1)^{s+1} \langle g^{(s+1)}(k) * f(k), \phi(k)\rangle .$$

Inserting this result into (2.35) gives finally

$$<g(k) * f(k), \phi^{(s+1)}(k)> = (-1)^{s+1} <g^{(s+1)}(k) * f(k), \phi(k)> =$$

$$(-1)^{s+1} <f(k) * g^{(s+1)}(k), \phi(k)> = <f(k) * g(k), \phi^{(s+1)}(k)> .$$

It follows that apart from a polynomial of at most degree s, the commutative property $f(k) * g(k) = g(k) * f(k)$ is again valid.

Summarizing our results we obtain finally the following theorem.

Theorem 2

If f(k) and g(k) are bounded rational functions with indices $\bar{m}(f)$ and $\bar{m}(g)$ such that $1 \leq \bar{m}(f)$, $1 \leq \bar{m}(g)$, $\bar{m}(f) + \bar{m}(g) = n-s$ and $0 \leq s \leq n-2$, then the convolution $f(k) * g(k)$ is defined on the space of test functions $\psi(k)$ with the property that $\psi(k) = \phi^{(s+1)}(k)$, where $\phi(k) \in C(s+1, n-1+\delta, n)$ and δ is an arbitrary positive number. The convolution $f(k) * g(k)$ is defined uniquely, apart from a polynomial of at most degree s, by the relation

$$(2.34) \quad <f(k) * g(k), \phi^{(s+1)}(k)> = (-1)^{s+1} <f(k) * g^{(s+1)}(k), \phi(k)> .$$

Moreover the convolution satisfies the property

$$(2.36) \qquad f(k) * g(k) = g(k) * f(k) + P(k),$$

where P(k) is an arbitrary polynomial of at most degree s.

If f(k) and g(k) have indices $\bar{m}(f)$ respectively $\bar{m}(g)$ with $1 \leq \bar{m}(f) \leq n$, $1 \leq \bar{m}(g) \leq n$, but $\bar{m}(f) + \bar{m}(g) \geq n+1$, then the convolution $f(k) * g(k)$ is defined by equation (2.29) on the space of test functions $C(q, 2n-\bar{m}(f)-\bar{m}(g)+\delta, n)$, where q is an arbitrary non-negative integer. Because in this case $2n-\bar{m}(f)-\bar{m}(g)+\delta \leq n-1+\delta$, every test function belonging to $C(q, n-1+\delta, n)$ belongs also to $C(q, 2n-\bar{m}(f)-\bar{m}(g)+\delta, n)$ and hence $f(k) * g(k)$ is also defined on $C(0, n-1+\delta, n)$. According to the equations (2.29) and (2.30), we have for every $\phi(k) \in C(s+1, n-1+\delta, n)$ the relation

$$(2.37) \quad <f(k) * g(k), \phi^{(s+1)}(k)> = <f(k), <g(p), \phi^{(s+1)}(p+k)> =$$

$$= <g(k), <f(p), \phi^{(s+1)}(p+k)> .$$

Integration by parts yields immediately

$$
(2.38)\quad
\begin{aligned}
<f(k) * g(k), \phi^{(s+1)}(k)> &= (-1)^{s+1} <f(k), <g^{(s+1)}(p), \phi(p+k)>> \\
&= (-1)^{s+1} <g(k), <f^{(s+1)}(p), \phi(p+k)>> \\
&= (-1)^{s+1} <f(k) * g^{(s+1)}(k), \phi(k)> \\
&= (-1)^{s+1} < g(k) * f^{(s+1)}(k), \phi(k)> .
\end{aligned}
$$

Thus in the case of $\overline{m}(f) + \overline{m}(g) \geq n+1$ the convolution of f(k) and g(k) satisfies also the relations (2.34) and (2.35).

Therefore the definition of the convolution of two bounded rational functions with $\overline{m}(f) + \overline{m}(g) \leq n$, given by the formulae (2.34) and (2.35), is really a generalization of the definition given by the formulae (2.29) and (2.30), which are only valid for bounded rational functions with $\overline{m}(f) + \overline{m}(g) > n$.

Remark

The diverging convolution F(p) has been defined as a continuous linear functional on the space of test functions $\psi(k)$, which can be written as $\psi(k) = \phi^{(s+1)}(k)$, where $\phi(k) \in C(s+1, n-1+\delta, n)$ with δ arbitrary positive.

This space is only a linear subspace of the Banach space $C(0, n-1+\delta, n)$, but in virtue of the theorem of Hahn-Banach, [8] , §21, the functional F(p) can be extended to the whole Banach space $C(0, n-1+\delta, n)$. As to this extension the reader is referred to the paper [5] by Bogoliubov and Parasiuk.

3. Properties of the convolution

In the previous section we have shown that it is possible to define a convolution product of bounded rational functions f(k) and g(k) for which the sum of the indices needs not to be larger than the dimension n of the space R_n. This convolution product has several interesting and useful properties which will be investigated in this section.

We confine our investigation to the most important case, in which we have the relation $\bar{m}(f) + \bar{m}(g) \leq n$; the other case $\bar{m}(f) + \bar{m}(g) > n$ can be treated in the same way.

3.1. The differential equation for $f(k) * g(k)$

In this section it is shown that the convolution $f(k) * g(k)$ is the solution of a system of differential equations of the order (s+1). According to Theorem 2 the convolution $f(k) * g(k)$ is defined on the space of test functions $\psi(k) = \phi^{(s+1)}(k)$, where $\phi(k) \in C(s+1),n-1+\delta,n)$; moreover, we have the equation

$$(2.34) \qquad \langle f(k) * g(k), \phi^{(s+1)}(k)\rangle = (-1)^{s+1} \langle f(k) * g^{(s+1)}(k), \phi(k)\rangle .$$

The derivative of order (s+1) of the convolution $f(k) * g(k)$ is defined on the space $C(s+1,n-1+\delta,n)$ by means of equation (2.20) of section 2.2. The result becomes

$$(3.1) \qquad \langle (f(k) * g(k))^{(s+1)}, \phi(k)\rangle = (-1)^{s+1} \langle f(k) * g(k), \phi^{(s+1)}(k)\rangle .$$

Combining (2.34) and (3.1) we obtain for every $\phi(k) \in C(s+1,n-1+\delta,n)$

$$(3.2) \qquad \langle (f(k) * g(k))^{(s+1)}, \phi(k)\rangle = \langle f(k) * g^{(s+1)}(k), \phi(k)\rangle ,$$

or

$$(3.3) \qquad (f(k) * g(k))^{(s+1)} = f(k) * g^{(s+1)}(k).$$

From this result it is again evident, that $f(k) * g(k)$ can only be determined apart from a polynomial of degree s. Hence we have obtained the following theorem.

Theorem 3

If $f(k)$ and $g(k)$ are bounded rational functions with index $\bar{m}(f)$ respectively $\bar{m}(g)$, such that $\bar{m}(f) \geq 1$, $\bar{m}(g) \geq 1$, $\bar{m}(f) + \bar{m}(g) = n-s$ and $0 \leq s \leq n-2$, then the convolution $f(k) * g(k)$ is a solution of the set of differential equations

$$(f(k) * g(k))^{(s+1)} = f(k) * g^{(s+1)}(k),$$

defined on the space $C(s+1,n-1+\delta,n)$, where δ is an arbitrary positive number.

3.2. Limit properties of the convolution

Our convolution product has some limit properties which will turn out to be of great value for the applications of the theory. These limit properties are given in the theorems 4,5 and 6 of this section.

Theorem 4

Let $f(k)$ and $g(k)$ be bounded rational functions with index $\overline{m}(f)$ respectively $\overline{m}(g)$, such that $\overline{m}(f) \geq 1$, $\overline{m}(g) \geq 1$, $\overline{m}(f) + \overline{m}(g) = n-s$ and $0 \leq s \leq n-2$.
Let $\{f_M(k)\}$ be a set of bounded rational functions, depending on k and the real parameter M; the index $\overline{m}(f_M)$ is independent of M and it satisfies the relation $n \geq \overline{m}(f_M) \geq \overline{m}(f)$.
Moreover, the set $\{f_M(k)\}$ converges for $M \to \infty$ weakly on the space $C(0,n-\overline{m}(f)+\delta_1,n)$ to the function $f(k)$; δ_1 is some positive number with $0 < \delta_1 < 1$.
In the same way $\{g_N(k)\}$ is a set of bounded rational functions, depending on k and the real parameter N; the index $\overline{m}(g_N)$ is independent of N and $n \geq \overline{m}(g_N) \geq \overline{m}(g)$. The set $\{g_N(k)\}$ converges for $N \to \infty$ weakly on the space $C(0,n-\overline{m}(g)+\delta_2,n)$ to the function $g(k)$; δ_2 is some positive number with $0 < \delta_2 < 1$.

Under these assumptions the convolution $f_M(k) * g_N(k)$ converges for M and $N \to \infty$ weakly to the convolution $f(k) * g(k)$. The limit has to be taken as a repeated limit on the space of test functions $\psi(k)$ which can be written as $\psi(k) = \phi^{(s+1)}(k)$, while $\psi(k) \in C(s+1,n-1+\delta_3,n)$ and δ_3 may be any positive number larger than $\max(\delta_1,\delta_2)$. The limit is independent of the order of the repeated limit; i.e.

$$(3.4)\quad \begin{aligned} &\lim_{M\to\infty}\lim_{N\to\infty} \langle f_M(k) * g_N(k),\ \psi(k)\rangle = \\ &\lim_{N\to\infty}\lim_{M\to\infty} \langle f_M(k) * g_N(k),\ \psi(k)\rangle = \langle f(k) * g(k),\ \psi(k)\rangle . \end{aligned}$$

Proof:

For any $\phi(k) \in C(s+1,n-1+\delta_3,n)$ we have the equation

$$(3.5)\quad \begin{aligned} &\langle f_M(k) * g_N(k),\ \phi^{(s+1)}(k)\rangle = \langle f_M(k),\ \langle g_N(p),\ \phi^{(s+1)}(k+p)\rangle\rangle = \\ &(-1)^{s+1} \langle f_M(k),\ \langle g_N^{(s+1)}(p),\ \phi(k+p)\rangle\rangle . \end{aligned}$$

The function $<g_N^{(s+1)}(p),\ \phi(k+p)>$ is continuous in k and using the corollaries of the lemma of section 2.1, we see that it belongs to the class A. Its index is either at least $\bar{m}(g_N)+s+1+\bar{m}(\phi)-n$ or at least $\min\{\bar{m}(g_N)+s+1,\ \bar{m}(\phi)\}$, according as $\bar{m}(g_N)+s+1$ and $\bar{m}(\phi)$ are both $\le n$ or not.

In the first case the index of $<g_N^{(s+1)}(p),\ \phi(k+p)>$ is larger than or equal to $\bar{m}(g)+s+1+n-1+\delta_3-n = n-\bar{m}(f)+\delta_3$; in the second case the index is larger than or equal to $\min\{\bar{m}(g_N)+s+1,\ \bar{m}(\phi)\} \ge \min\{n-\bar{m}(f)+1,\ n-\bar{m}(f)+\delta_3\}$. Hence $<g_N^{(s+1)}(p),\ \phi(k+p)>$ belongs certainly to the Banach space $C(0, n-\bar{m}(f)+\delta_1, n)$. Therefore we can take the limit for $M\to\infty$ in the left- and right-hand side of equation (3.5) and we obtain

$$\lim_{M\to\infty} <f_M(k) * g_N(k),\ \phi^{(s+1)}(k)> = (-1)^{s+1} <f(k),\ <g_N^{(s+1)}(p),\ \phi(k+p)>> . \tag{3.6}$$

The right-hand side of this equation can be written as

$$\begin{aligned}(-1)^{s+1} <f(k),\ <g_N^{(s+1)}(p),\ \phi(k+p)>> &= <f(k) * g_N(k),\ \phi^{(s+1)}(k)> = \\ <g_N(k) * f(k),\ \phi^{(s+1)}(k)> &= (-1)^{s+1} <g_N(k),\ <f^{(s+1)}(p),\ \phi(k+p)>> .\end{aligned} \tag{3.7}$$

Inserting (3.7) into (3.6) one gets

$$\lim_{M\to\infty} <f_M(k) * g_N(k),\ \phi^{(s+1)}(k)> = (-1)^{s+1} <g_N(k),\ <f^{(s+1)}(p),\ \phi(k+p)>> . \tag{3.8}$$

The function $<f^{(s+1)}(p),\ \phi(k+p)>$ is continuous in k and again according to the corollaries of the lemma of section 2.1 it belongs to the class A. If $\bar{m}(f)+s+1$ and $\bar{m}(\phi)$ are both $\le n$, its index is at least $\bar{m}(f)+s+1 + \bar{m}(\phi)-n$; if $\bar{m}(f)+s+1$ or $\bar{m}(\phi)$ is larger than n, then its index is at least $\min\{\bar{m}(f)+s+1,\ \bar{m}(\phi)\}$.

In the first case the index of $<f^{(s+1)}(p),\ \phi(k+p)>$ is larger than or equal to $n-\bar{m}(g)+\delta_3$; in the second case the index is larger than or equal to $\min\{n-\bar{m}(g)+1, n-1+\delta_3\} \ge \min\{n-\bar{m}(g)+1, n-\bar{m}(g)+\delta_3\}$. Therefore in both cases $<f^{(s+1)}(p),\ \phi(k+p)>$ belongs certainly to the

Banach space $C(0,n-\bar{m}(g)+\delta_2,n)$. Hence we can now take the limit for $N \to \infty$ in the left- and right-hand side of equation (3.8) and we obtain the result

$$\lim_{N \to \infty} \lim_{M \to \infty} \langle f_M(k) * g_N(k),\ \phi^{(s+1)}(k)\rangle =$$

$$(-1)^{s+1} \langle g(k),\ \langle f^{(s+1)}(p),\ \phi(k+p)\rangle\rangle ,$$

or in virtue of the definition (2.35)

$$(3.9) \quad \lim_{N \to \infty} \lim_{M \to \infty} \langle f_M(k) * g_N(k),\ \phi^{(s+1)}(k)\rangle = \langle f(k) * g(k),\ \phi^{(s+1)}(k)\rangle .$$

Interchanging the role of $f(k)$ and $g(k)$ we could have taken also first the limit with respect to N and thereafter with respect to M. The arguments would be quite analogous and because our convolution product is commutative, we would have obtained the same result.

Hence the order of the limits in (3.9) is not essential and we have the result

$$(3.10) \quad \lim_{M \to \infty} \lim_{N \to \infty} \langle f_M(k) * g_N(k),\ \psi(k)\rangle =$$

$$\lim_{N \to \infty} \lim_{M \to \infty} \langle f_M(k) * g_N(k),\ \psi(k)\rangle = \langle f(k) * g(k),\ \psi(k)\rangle ,$$

valid for all test functions $\psi(k) = \phi^{(s+1)}(k)$ with $\phi(k) \in C(s+1,n-1+\delta_3,n)$ q.e.d.

It may be remarked, that if the double limit, namely $\lim_{\substack{M \to \infty \\ N \to \infty}} f_M(k) * g_N(k)$, exists in the weak sense on the space of test functions $\psi(k)$, then it is equal to $\lim_{M \to \infty} \lim_{N \to \infty} f_M(k) * g_N(k)$. Hence, in virtue of the theorem, the weak double limit is also equal to $f(k) * g(k)$, provided that this double limit exists. Therefore the weak limit of $f_M(k) * g_N(k)$ for M and N going to infinity is always unique and it is independent of the way in which the limit has been taken, provided that the double limit exists.

In the following theorem we give a sufficient condition for the existence of the weak double limit.

Theorem 5

Let $f(k), g(k), f_M(k)$ and $g_N(k)$ satisfy the conditions of theorem 4. If $\phi(k) \in C(s+1, n-1+\delta_3, n)$ and if for M or $N \to \infty$ the function $<f_M(p)-f(p),\ \phi^{(s+1)}(k+p)>$ or $<g_N(p)-g(p),\ \phi^{(s+1)}(k+p)>$ converges to zero in the topology of $C(0, n-\bar{m}(g)+\delta_2, n)$ respectively $C(0, n-\bar{m}(f)+\delta_1, n)$, then the weak double limit $\lim\limits_{\substack{M \to \infty \\ N \to \infty}} f_M(k) * g_N(k)$ exists on the space of test functions $\psi(k)$, which can be written as $\psi(k) = \phi^{(s+1)}(k)$.

Proof:

With the aid of the corollaries of the lemma it can easily be shown, in a similar way as before, that $<f_M(p)-f(p),\ \phi^{(s+1)}(k+p)>$ and $<g_N(p)-g(p),\ \phi^{(s+1)}(k+p)>$ belong to the space $C(0, n-\bar{m}(g)+\delta_2, n)$ respectively $C(0, n-\bar{m}(f)+\delta_1, n)$. Without loss of generality we may assume that $<g_N(p) - g(p),\ \phi^{(s+1)}(k+p)>$ converges for $N \to \infty$ to zero in the topology of the space $C(0, n-\bar{m}(f)+\delta_1, n)$. In the other case where $<f_M(p)-f(p),\ \phi^{(s+1)}(k+p)>$ converges for $M \to \infty$ to zero in the topology of the space $C(0, n-\bar{m}(g)+\delta_2, n)$, we can use the same argument after interchanging again the role of $f(k)$ and $g(k)$.

For any $\phi(k) \in C(s+1, n-1+\delta_3, n)$ we have the relation

$$<f_M(k) * g_N(k), \phi^{(s+1)}(k)> = <f_M(k),\ <g_N(p), \phi^{(s+1)}(k+p)>> =$$

$$<f_M(k),\ <g(p), \phi^{(s+1)}(k+p)>> + <f_M(k),\ <g_N(p)-g(p), \phi^{(s+1)}(k+p)>> \quad . \tag{3.11}$$

The existence of both terms in the right-hand side of (3.11) can easily be proved by means of the corollaries of the lemma.
Taking the double limit in both sides of (3.11) we obtain

$$\begin{aligned} &\lim_{\substack{M \to \infty \\ N \to \infty}} < f_M(k) * g_N(k), \phi^{(s+1)}(k)> = <f(k) * g(k), \phi^{(s+1)}(k+p)> + \\ &\lim_{\substack{M \to \infty \\ N \to \infty}} <f_M(k),\ \ g_N(p)-g(p), \phi^{(s+1)}(k+p)>> \quad . \end{aligned} \tag{3.12}$$

We consider now the remainder term in the right-hand side of (3.12), viz.

$$<f_M(k),\ <g_N(p)-g(p), \phi^{(s+1)}(k+p)>> \quad .$$

Because the function $<g_N(p)-g(p),\phi^{(s+1)}(k+p)>$ converges to zero in the topology of $C(0,n-\bar{m}(f)+\delta_1,n)$, we can find for every ε a number N^*, such that for $N > N^*$

(3.13) $$\| < g_N(p)-g(p),\phi^{(s+1)}(k+p)> \| < \varepsilon \text{ for } N > N^*.$$

Since $\{f_M(k)\}$ is a weakly convergent set of continuous linear functionals on the Banach space $C(0,n-\bar{m}(f)+\delta_1,n)$, the set $\{f_M(k)\}$ is uniformly bounded according to the principle of uniform boundedness of Banach-Steinhaus (see e.g. [8] , §19). Hence there exists a number, say C, such that

(3.14) $$\| f_M \| < C,$$

valid for all values of M.

Using the relation

$$|<f_M(k), <g_N(p)-g(p),\phi^{(s+1)}(k+p)>>| \leq$$
$$\| f_M \| \cdot \| < g_N(p)-g(p),\phi^{(s+1)}(k+p)> \| ,$$

we obtain for $N > N^*$

$$|<f_M(k), <g_N(p)-g(p),\phi^{(s+1)}(k+p)>> | < \varepsilon . C,$$

and hence

$\lim\limits_{\substack{M \to \infty \\ N \to \infty}} <f_M(k), <g_N(p)-g(p),\phi^{(s+1)}(k+p)>>$ exists and is equal to zero. It follows finally from (3.12) that also $\lim\limits_{\substack{M \to \infty \\ N \to \infty}} <f_M(k) * g_N(k), \phi^{(s+1)}(k)>$ exists and that it is equal to $<f(k) * g(k),\phi^{(s+1)}(k)>$.
q.e.d.

In the last two theorems we have considered the bounded rational functions f(k) and g(k) as weak limits of other bounded rational functions $f_M(k)$ respectively $g_N(k)$. However, it is not necessary to assume that $f_M(k)$ and $g_N(k)$ are always rational functions.

We assume now that f(k) is the weak limit of a set of functions $f_M(k)$, which are only supposed to be functionals on an appropriate Banach space. Under certain conditions it is not difficult to prove that we have also in this case the property that $\lim\limits_{M \to \infty} f_M(k) * g(k) =$

$= f(k) * g(k)$; in fact, we have the following theorem.

Theorem 6

Let $f(k)$ and $g(k)$ be bounded rational functions with indices $\bar{m}(f) \geq 1$, $\bar{m}(g) \geq 1$, $\bar{m}(f) + \bar{m}(g) = n-s$ and $0 \leq s \leq n-2$. Let $\{f_M(k)\}$ be a set of functions depending on k and the real parameter M; these functions may be considered as continuous linear functionals on $C(0,n-\bar{m}(f)+\delta_1,n)$, where δ_1 is some positive number with $0< \delta_1 < 1$. If, moreover, the set $\{f_M(k)\}$ converges for $M \to \infty$ weakly on the space $C(0,n-\bar{m}(f)+\delta_1,n)$ to the function $f(k)$, then the convolution $f_M(k) * g(k)$ converges for $M \to \infty$ weakly to $f(k) * g(k)$ on the space of test functions $\psi(k)$, which can be written as $\psi(k) = \phi^{(s+1)}(k)$, while $\phi(k)$ belongs to the space $C(s+1,n-1+\delta_2,n)$ with $\delta_2 > \delta_1$.

Proof

The convolution of the functions $f_M(k)$ and $g(k)$ can be defined by the equation

$$(3.15) \quad \langle f_M(k) * g(k), \phi^{(s+1)}(k)\rangle = (-1)^{s+1} \langle f_M(k), \langle g^{(s+1)}(p), \phi(k+p)\rangle\rangle ,$$

which has a well-defined meaning for all test functions $\phi(k) \in C(s+1,n-1+\delta_2,n)$. If $\phi(k) \in C(s+1,n-1+\delta_2,n)$, the function $\langle g^{(s+1)}(p), \phi(k+p)\rangle$ belongs certainly to the space $C(0,n-\bar{m}(f)+\delta_1,n)$; since the function $f_M(k)$ may be considered as a functional on this space, the right-hand side of (3.15) is meaningful and therefore (3.15) yields a definition for the convolution $f_M(k) * g(k)$.

Due to the assumptions of the theorem we can take in the left- and right-hand side of (3.15) the limit for $M \to \infty$ and we obtain the result

$$(3.16) \quad \lim_{M \to \infty} \langle f_M(k) * g(k), \phi^{(s+1)}(k)\rangle = (-1)^{s+1} \langle f(k), \langle g^{(s+1)}(p), \phi(k+p)\rangle\rangle = \langle f(k) * g(k), \phi^{(s+1)}(k)\rangle \ .$$

q.e.d.

4. Application of the theory to divergent convolution integrals in electrodynamics

4.1. Divergent integrals in electrodynamics

As has been pointed out already in the introduction, the determination of the S-matrix involves the calculation of products, containing as factors the so-called causal function $\Delta_C(x)$ and its derivatives. The causal function $\Delta_C(x)$ has been treated in Chapter IV and it has been shown that this "function" is in fact a distribution on the space S. However, this distribution may also be obtained as a functional on the Banach space C(q,r,4), provided q and r are chosen large enough. In this connection it may be remarked that S is the intersection of all spaces C(q,r,4) with q=1,2,... and r=1,2,... .

In order to give an illustrative example of the application of the theory of divergent convolution integrals, we consider interaction processes which involve the multiplication of the distributions

(4.1)
$$\Delta_C(x) = \frac{1}{4\pi}\,\delta(R^2) - \frac{m}{8\pi}\,\theta(R^2)\,\frac{H_1^{(2)}(mR)}{R} + i\,\frac{m}{4\pi^2}\,\theta(-R^2)\,\frac{K_1(m\sqrt{-R^2})}{\sqrt{-R^2}},$$

and

(4.2)
$$S_C(x) = (i\hat{\partial}+m)\Delta_C(x),$$

where m denotes the mass of the particle under consideration, $R^2 = x_o^2-x_1^2-x_2^2-x_3^2$, $H_1^{(2)}$ is the Hankel function of the second kind and K_1 the modified Bessel function; see formula (6.26) of Chapter IV. Furthermore $\hat{\partial}$ is a short notation for the operator

(4.3)
$$\hat{\partial} = \sum_{n=0}^{3} \gamma^n \frac{\partial}{\partial x_n},$$

where γ^n denote the Dirac matrices; confer [1] , §6, §7, §14 and §24. According to formula (6.25) of Chapter IV the causal functions (4.1) and (4.2) can be written in "momentum" representation as follows

(4.4)
$$\Delta_C(x) = \lim_{\varepsilon\to+0}\left[-\frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\frac{e^{-ik.x}}{k^2-m^2+i\varepsilon(k,k)}\,dk\right],$$

and

(4.5) $$S_C(x) = \lim_{\varepsilon \to +0} \left[-\frac{1}{(2\pi)^4} \int_{-\infty}^{+\infty} \frac{(m+\hat{k})e^{-ik.x}}{k^2-m^2+i\varepsilon(k,k)} dk \right],$$

with $k.x = k_o x_o - k_1 x_1 - k_2 x_2 - k_3 x_3$, $k^2 = k_o^2 - k_1^2 - k_2^2 - k_3^2$, $(k,k) = k_o^2 + k_1^2 + k_2^2 + k_3^2$ and $\hat{k} = \gamma^o k_o - \gamma^1 k_1 - \gamma^2 k_2 - \gamma^3 k_3$; the limit for $\varepsilon \to +0$ should be taken in the distributional sense. Multiplication of these causal functions with each other leads in the momentum representation formally to convolution products of the bounded rational functions

(4.6) $$\tilde{\Delta}_C(k;\varepsilon) = \frac{-1}{k^2-m^2+i\varepsilon(k,k)},$$

and

(4.7) $$\tilde{S}_C(k;\varepsilon) = -\frac{\hat{k}+m}{k^2-m^2+i\varepsilon(k,k)}.$$

It should be remarked that, in contrast with the notation of Chapter IV, the symbols $\tilde{\Delta}_C$ and $\tilde{S}_C$ represent "modified" Fourier transforms. The limit for $\varepsilon \to +0$ is taken at the end of the procedure. It is evident that convolution integrals containing as factors $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$ are divergent. These divergent integrals may now be defined by means of the theory of section 2, i.e. by considering them as functionals on an appropriate Banach space.

Several physicists and mathematicians have invented other methods to define these divergent integrals; in particular we mention the methods used by Bogoliubov and Parasiuk in [1] , [5] and [6] , by Achieser and Berestezki in [2] and by Bremermann in [7] . These methods will be reviewed shortly in the next sections and it will be shown that they are all special cases of the theory developed in this chapter.

4.2. The method of Bogoliubov and Parasiuk

We consider a convolution integral of the following kind

(4.8) $$F(p) = \int_{-\infty}^{+\infty} f(k)g(p-k)dk,$$

where f(k) and g(k) are bounded rational functions in n independent variables vanishing at infinity.

The indices of f(k) and g(k) have the property that

(4.9) $$\bar{m}(f)+\bar{m}(g) = n-s, \text{ with } 0 \leq s \leq n-2.$$

The integration is performed over the whole n-dimensional space R_n and hence (4.8) diverges. Nevertheless, the convolution F(p) can be defined in virtue of Theorem 2 as a functional on a certain space of test functions.

Instead of f(k) and g(k) we take the bounded rational functions $f_M(k)$ and $g_N(k)$, which depend on a real parameter M respectively N.

The functions $f_M(k)$ and $g_N(k)$ are assumed to have the following properties

1. The indices $\bar{m}(f_M)$ and $\bar{m}(g_N)$ are independent of M respectively N, while

(4.10) $$\bar{m}(f_M) + \bar{m}(g_N) > n.$$

2. The functions $f_M(k)$ and $g_N(k)$ converge for M, $N \to \infty$ weakly to the functions f(k) and g(k) on the spaces $C(0,n-\bar{m}(f)+\delta_1,n)$ respectively $C(0,n-\bar{m}(g)+\delta_2,n)$, where δ_1 and δ_2 are arbitrary numbers with $0 < \delta_1 < 1$ and $0 < \delta_2 < 1$.

Since $\bar{m}(f_M) + \bar{m}(g_N) > n$, the convolution

(4.11) $$F_{M,N}(p) = \int_{-\infty}^{+\infty} f_M(k) g_N(p-k)\, dk$$

exists. In virtue of theorem 4 the convolution $F_{M,N}(p)$ converges for M and N going to infinity weakly to the convolution F(p).

More precisely, we have the relation

(4.12) $$\lim_{M \to \infty} \lim_{N \to \infty} F_{M,N}(p) = \lim_{N \to \infty} \lim_{M \to \infty} F_{M,N}(p) = F(p),$$

where the repeated limits should be taken on the space of test functions $\psi(k) = \phi^{(s+1)}(k)$ with $\phi(k) \in C(s+1,n-1+\delta_3,n)$ and $\delta_3 > \max(\delta_1,\delta_2)$. If moreover $f_M(k)$ <u>or</u> $g_N(k)$ satisfies the condition

3^a $\langle f_M(p) - f(p),\ \phi^{(s+1)}(k+p)\rangle$ converges for $M \to \infty$ to zero in the topology of $C(0,n-\bar{m}(g)+\delta_2,n)$,

or

3^b $\langle g_N(p)-g(p),\phi^{(s+1)}(k+p)\rangle$ converges for $N \to \infty$ to zero in the topology of $C(0,n-\bar{m}(f)+\delta_1,n)$,

then, in virtue of theorem 5, also the double limit of $F_{M,N}(p)$ exists on the above-mentioned space of test functions $\psi(k)$.

Further, we have the relation

(4.13)
$$\lim_{\substack{M \to \infty \\ N \to \infty}} F_{M,N}(p) = F(p).$$

It follows that we can determine our divergent integral $F(p)$ by calculating the convergent integral $F_{M,N}(p)$ and by taking consecutively the weak limit for $M,N \to \infty$. The ultimate result is defined apart from a polynomial of at most degree s.

The functions $f_M(k)$ and $g_N(k)$ are called the regularizations of $f(k)$ and $g(k)$.

This method is essentially the procedure by which Bogoliubov and Parasiuk define in [1] , [5] and [6] the divergent convolution integrals occurring in electrodynamics.

Let us consider for example the so-called electron self-energy, which leads to a convolution product containing as factors $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$; cf. [1] , §24; [2] , §47; [3], Ch.9, § 4.

The indices of $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(\tilde{k};\varepsilon)$ equal respectively 2 and 1; the convolution of these functions diverges since the sum of the indices is 3 while the dimension of the space is 4; hence s=1.

Bogoliubov and Parasiuk use for the regularization of $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$ the regularization of Pauli-Villars [9] .

In its most simple form this regularization yields for $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$ the expressions

(4.14)
$$\text{Reg. } \tilde{\Delta}_C(k;\varepsilon) = \frac{1}{k^2-m^2+i\varepsilon(k,k)} - \frac{1}{k^2-M^2+i\varepsilon(k,k)},$$

and

(4.15) $$\text{Reg.}\tilde{S}_C(k;\varepsilon) = \text{Reg}\left[(\hat{k}+m)\tilde{P}_C(k;\varepsilon)\right] = (\hat{k}+m)\text{Reg }\tilde{D}_C(k;\varepsilon) =$$

$$(\hat{k}+m)\left[\frac{1}{k^2-m^2+i\varepsilon(k,k)} \quad \frac{1}{k^2-N^2+i\varepsilon(k,k)}\right].$$

The index of Reg. $\tilde{\Delta}_C(k;\varepsilon)$ equals 4, while that of Reg. $\tilde{S}_C(k;\varepsilon)$ equals 3.

It can be shown that $\tilde{\Delta}_C(k;\varepsilon)$, $\tilde{S}_C(k;\varepsilon)$, Reg. $\tilde{\Delta}_C(k;\varepsilon)$ and Reg. $\tilde{S}_C(k;\varepsilon)$ satisfy the conditions of theorem 4 and 5; see [10] .
Therefore the convolution of $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$ may be determined by calculating the converging convolution of Reg. $\tilde{\Delta}_C(k;\varepsilon)$ and Reg. $\tilde{S}_C(k;\varepsilon)$ and by taking consecutively the limit for $M,N \to \infty$.
In virtue of theorem 5 N may be taken equal to M.
If the calculations are carried out, such as has been done in [1] , §24, then it appears that the part of the convolution of Reg. $\tilde{\Delta}_C(k;\varepsilon)$ and Reg. $\tilde{S}_C(k;\varepsilon)$, which actually diverges for $M \to \infty$, is contained only in a term linear in p. This term is however immaterial, because the convolution product is considered as a linear functional on the space of test functions $\psi(p)$ which, due to the fact that s=1, can be written as

$$\psi(p) = \frac{\partial^2}{\partial p_\mu \partial p_\nu}\phi(p),$$

where $\phi(p) \in C(2,3+\delta,4)$ with δ some number between 0 and 1.
It may be remarked that in this case it is not necessary to regularize both $\tilde{\Delta}_C(k;\varepsilon)$ and $\tilde{S}_C(k;\varepsilon)$; it is already sufficient to regularize only one of these factors and the result will of course be the same.

Another example of a diverging convolution integral is furnished by the photon self-energy; cf. [1] , §24, [2] , §47, [3] ,Ch.9, § 5. In this case both factors are of the type $\tilde{S}_C(k;\varepsilon)$ and s=2. The diverging part of the regularized convolution integral is now contained only in a term which is quadratic in p.

4.3. The method of Achieser and Berestezki

Consider again the divergent convolution product

$$F(p) = f(k) * g(k),$$

where f(k) and g(k) are bounded rational functions, vanishing at infinity, while

$$\bar{m}(f) + \bar{m}(g) = n-s,$$

with $0 \leq s \leq n-2$.

Instead of F(p) we consider the integral

$$F_V(p) = \int f(k)g(p-k)\,dk, \tag{4.16}$$

where the integration is now performed over a finite volume V of the n-dimensional space R_n.

The integral $F_V(p)$ can be calculated and it can be written as

$$F_V(p) = f_V(k) * g(k), \tag{4.17}$$

where

$$\begin{aligned} f_V(k) &\equiv f(k) \text{ for } k \text{ inside } V, \\ f_V(k) &\equiv 0 \text{ for } k \text{ outside } V. \end{aligned} \tag{4.18}$$

It is clear that f(k) is the weak limit of $f_V(k)$ for $V \to \infty$ on the space of test functions $C(0, n-\bar{m}(f)+\delta_1, n)$, where δ_1 may be any number with $0 < \delta_1 < 1$. According to theorem 6, the original convolution $F(p) = f(k) * g(k)$ is the weak limit of $F_V(p) = f_V(k) * g(k)$ for $V \to \infty$ on the space of test functions $\psi(k)$ which can be written as $\psi(k) = \phi^{(s+1)}(k)$, where $\phi(k) \in C(s+1, n-1+\delta_2, n)$ and $\delta_2 > \delta_1$. Therefore we can obtain the divergent convolution $F(p) = f(k) * g(k)$ by calculating $f_V(k) * g(k)$ and taking consecutively the weak limit for $V \to \infty$.

For the actual calculation of F(p) we consider the Taylor expansion of $F_V(p)$, viz.

$$\begin{aligned} F_V(p) = F_V(0) &+ \sum_{i=1}^{n} \frac{\partial}{\partial p_i} F_V(p) \Big|_{p=0} p_i + \ldots\ldots\ldots \\ \ldots &+ \sum_{i_1,i_2,\ldots,i_s=1}^{n} \frac{\partial^s}{\partial p_{i_1} \partial p_{i_2} \ldots \partial p_{i_s}} F_V(p) \Big|_{p=0} p_{i_1} p_{i_2} \ldots p_{i_s} + \\ &+ \sum_{i_1,i_2,\ldots,i_{s+1}=1} \frac{\partial^{s+1}}{\partial p_{i_1} \partial p_{i_2} \ldots \partial p_{i_{s+1}}} F_V(\theta p) p_{i_1} \ldots p_{i_{s+1}}, \end{aligned} \tag{4.19}$$

with $0 < \theta < 1$.

Denoting the last term of the right-hand side of (4.19) by $\overline{F}_V(p)$, we may write

$$F_V(p) = \overline{F}_V(p) + P(p),$$

where P(p) is the polynomial of degree s, obtained by taking the first (s+1) terms of the Taylor expansion of $F_V(p)$.

Since the differentiations of $F_V(\theta p)$ with respect to p_i can be carried out after the integration symbol, it is clear that $\overline{F}_V(p)$ converges in the ordinary classical sense for V going to infinity.

Hence it follows that the divergence of our original convolution integral is contained only in a polynomial P(p) of at most degree s.

If we take now the weak limit of $F_V(p)$ on the space of test functions $\psi(k)$, with $\psi(k) = \phi^{(s+1)}(k)$ and $\phi(k) \in C(s+1, n-1+\delta_2, n)$, the polynomial P(p) is completely irrelevant and the divergence is removed.

Thus one obtains finally

$$F(p) = \lim_{V\to\infty} F_V(p) = \lim_{V\to\infty} \overline{F}_V(p). \tag{4.20}$$

This procedure is essentially the method used by Achieser and Berestezki in [2] , Ch.VII, §47.1; it appears again, that also this method is a special case of the theory developed in the sections 2 and 3.

For actual computations concerning the electron and photon self-energy the reader is referred to [2] , Ch. VII, §47, 1,2,3.

4.4. The method of Bremermann

Bremermann has proposed in [11] to define divergent convolution integrals of the kind

$$F(p) = \int_{-\infty}^{+\infty} f(k)g(p-k)\,dk, \tag{4.21}$$

where f and g are bounded rational functions, as the solution of the set of differential equations

$$\frac{\partial^{s+1}}{\partial p_{i_1} \partial p_{i_2} \cdots \partial p_{i_{s+1}}} F(p) = \int_{-\infty}^{+\infty} f(k) \frac{\partial^{s+1}}{\partial p_{i_1} \partial p_{i_2} \cdots \partial p_{i_{s+1}}} \cdot g(p-k)\,dk. \tag{4.22}$$

The right-hand side of (4.22) is supposed to be a convergent integral and s is chosen as small as possible.

In virtue of theorem 3, formula (3.3), the convolution $F(p) = f(k) * g(k)$ as defined in theorem 2 satisfies on the Banach space $C(s+1, n-1+\delta, n)$ the set of differential equations (4.22). Hence, also Bremermann's method is included in the general theory of the preceding sections.

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