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Introduction to option pricing in a securities market

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Preface

These lecture notes are based on the material of the papers [25], [26] and [27]. Part I is a revised version of the paper [25] by K. Dzhaparidze and M. van Zuijlen. The first draft of this paper appeared in June 1992 as the technical report No 9209 of the Department of Mathematics during my detachment at the Catholic University of Nijmegen. The opportunity to complete this work occurred a few years later with the contribution [25] to the special issues of the CWI Quarterly on *Mathematics in Finance.* With kind permission of Martien van Zuijlen, I got the opportunity to base the paper [25] and related parts of the present work on our earlier notes, and I want to thank him sincerely.

Due to the growing interest in both research and teaching, there appeared in the recent years many new textbooks – our list of references, though far from being complete, contains a number of interesting titles, e.g. [3], [4], [5], [19], [22], [33], [44], [47], [49], [52], [54], [62], [78] and [66]. All these textbooks typically require to various extend the knowledge of the elements of probability theory, in particular martingale theory and stochastic calculus.

The objective of the papers [25], [26] and [27], hence of the present lecture notes, was somewhat different. To suit the editorial policy of the CWI Quarterly that publishes expository and survey papers aimed at once at the broad auditory of mathematicians and computer scientists, an attempt is made to facilitate reading without advanced probabilistic prerequisite and to keep the presentation at a low technical level. Moreover, in the paper [25] and in the corresponding Part I of the present lecture notes, the idea is pushed forward, like in [73], Chapter 15, that essentially no probability theory is needed for the buildup of the finite theory of options pricing. In this Part I all basic notions and tools of the mathematical theory of securities markets are introduced and simply explained. A path by path approach pursued throughout this course follows certain unsophisticated algebraic considerations, in contrast with usual treatment based on the probabilistic approach in the textbooks mentioned above.

Part II is devoted to the limiting transition towards usual continuous trading models, the so-called Merton model in Chapter 4 and the Black-Scholes model in Chapter 5. Since the reader is not supposed to be familiar with advanced methods of the probability theory, the presentation is kept on the same low technical level as in the previous part with the help of certain heuristic arguments. For full rigour would require a higher level of the general theory of stochastic processes. For further reading we provide a selective list of related papers some of which may carry the reader far afield. Our attention is restricted to emphasis on phenomenological understanding the cash flow mechanism in securities markets that is shown to reveal strong similarity to the physical Brownian motion (more generally, to diffusion with drift) or to the molecular mechanism of heat flow. Our description of the economical laws governing the market resembles in some respect the seminal paper [6] by Black and Scholes.

Contents

Chapter 1

Introduction

These lecture notes are aimed at the reader, e.g. a mathematician, a computer scientist or an econometrist, who is willing to get acquainted straightly with the mathematical theory of assets trading in securities markets, without the recourse to the advanced methods of the probability theory. The course starts from a simple mathematical model of a market, a so-called *binomial model* that is a typical representative of the class of binary models treated in Part I. This model is quite often used for the introductory purposes as it brings forth usual continuous time models, Merton's model of Section 4.4 and the Black-Scholes model of Section 5.5. The suitable limiting transitions are described in Chapters 4 and 5, respectively. There will be no need of probabilistic methods throughout Part I, for every notion within the finite theory can be explained and every statement can be rigorously proved by means of certain unsophisticated algebraic considerations based on the path by path approach. A few probabilistic terms occasionally used is meant to facilitate further reading and to maintain connections to the traditional literature based on a probabilistic approach. The matters are somewhat more complicated in Part II. We get on with the previous path by path approach. It turns out soon, however, that we are bound to resort to certain arguments of heuristic nature. For rigorous considerations and refined statements should carry us far beyond the present framework towards the highly technical theory of the limit theorems for stochastic processes. Instead, in all of our market models we thoroughly investigate path by path the cash flow dynamics governed by appropriate equations, namely the backward recurrence relations of Proposition 3.3.4 in the case of binary models in Part I, the differential-difference equations of Proposition 4.4.3 in the case of Merton's model of Chapter 4 and the second order partial differential equations of Proposition 5.5.3 in the case of the Black-Scholes model of Chapter 5 (to trace an intrinsic relationship between these equations, see Proposition 4.3.9 in Chapter 4 and Proposition 5.4.3 in Chapter 5). The latter equations are well-known in thermodynamics, referred to usually as the *heat equations.* This brings us at the beginning of Chapter 5 into the area of the physical Brownian motion, diffusion and thermal conduction.

Our mathematical models for securities markets are diverse in nature. But they have always in common that only two assets (or securities) are traded over a finite period of time denoted throughout by $[0, T]$. The origin $t = 0$ is understood as the current date and $t = T$ as a terminal date. As usual, t is the time coordinate. The main difference between the models is that in binary models of Part I the new prices of the traded assets are announced at certain finite number of fixed instants, in Merton's model of Chapter 4 the price changes occur at certain random sequence of instants, while in the Black-Scholes model of Chapter 5 the price processes develop continuously within the interval $[0, T]$.

One of the assets, called the *bond*, is riskless since its future prices are currently predetermined, while another asset, called the *stock*, is risky since its price process is allowed to develop along diverse trajectories. On pricing financial derivatives, it is often desirable to choose the bond as a common unit on the basis of which the prices on the stock are expressed. In this manner the bond prices become the *numeraire* – the stock price at each instant t is divided by the bond price. The resulting process is called the *discounted* stock price process. We will reserve the symbols B and S to denote the bond and the stock prices, respectively. The graved symbols will denote discounted quantities, e.g. the discounted stock price is denoted by $\grave{S} = S/B$. As the asset prices vary in time, they get the index t so that we write either B_t, S_t and $\grave{S}_t$ or equally $B(t)$, $S(t)$ and $\grave{S}(t)$.

Imagine now an investor who is willing to invest currently an amount $v \geq 0$ by buying Ψ_0 shares of the bond and Φ_0 shares of the stock. So, if the current price of the bond and stock are fixed to $B_0 = 1$ (throughout we agree upon this simplification) and $S_0 = s > 0$, respectively, then the investment amounts to $v = \Psi_0 + \Phi_0 s$. As time advances, the investor adjusts his portfolio according to his observation on the past development of the stock price. Hence at a given instant t his portfolio depends on stock prices occurred strictly before time t, but not on S_t, for instance, or on yet unobserved future prices. To place emphasis on this property, we say that the portfolio is *predictable*, borrowing this term from stochastic calculus. At instant t let the investor own Ψ_t shares of the bond and Φ_t shares of the stock. Then the market value of his holding amounts to $\Psi_t B_t + \Phi_t S_t$. The entire process $\pi \doteq (\Psi_t, \Phi_t)_{t\in[0,T]}$ of selecting the portfolio components is called the investor's trading strategy. With each trading strategy π we associate the so-called value process $V(\pi) = \Psi B + \Phi S$ which at each instant t presents the market value of the holding (Ψ_t, Φ_t) so that $V_t(\pi) = \Psi_t B_t + \Phi_t S_t$. This gives rise to the term *value process.* Clearly, the initial endowment amounts to $v = V_0(\pi)$.

By definition, the discounted value process is $\grave{V}(\pi) = \Psi + \Phi\grave{S}$. It is said that a trading strategy π is *self-financing*, if its construction is founded only on the initial endowment so that all changes in the portfolio values are due to capital gains during the trading and no infusion or withdrawal of funds is required. The first mathematical result of our theory concerns the discounted value process $\grave{V}(\pi)$ for a self-financing strategy π. It is proved that such process

$\grave{V}(\pi)$ possesses an integral representation property, in the sense that it can always be represented as the integral with respect to the discounted stock price process $\grave{S}$, with the integrand that is the stock component Φ of the portfolio. Symbolically, this statement is displayed as follows: at each instant t

$$\grave{V}_t(\pi) = \int_0^t \Phi_u d\grave{S}_u.$$

In all our models this integral will be given an appropriate meaning. The result complementary to this is Clarck's formula telling us that in all our cases the integrand, the stock component Φ of the portfolio, can be written in a certain special way. As applied to the continuous-time trading models like in Chapter 5, this formula is well-known since [6] or [39], Formula (1.9), but under the present name it has been applied by Ocone and Karatzas [59] to portfolio optimization problems, cf also [47]–[49]. Within the context of Malliavin calculus it is used for the stochastic integral representation of a random variable, see [58], [49], Appendix E, [56], Section 1.3.3, or [57], Section 1.6. It seems worth of notice that the similar formula does exist also within the context of the finite theory of Part I as well as in the Poisson theory of Chapter 4, as it will appear in Sections 3.2.2 and 4.4.2.

In all our models, the aforementioned integral representation of the discounted value process $\grave{V}(\pi)$ has important consequences. In particular, in virtue of linearity of the transformation, $\grave{V}(\pi)$ preserves the property of the discounted stock price process that is described in Chapter 3 by equations (3.3.6), in Chapter 4 by equations (4.4.13) and in Chapter 5 by equations (5.5.5), and concerns the relationship between the increments in the time coordinate of the discounted stock price states and the increments in the space coordinate. It is then proved that under the self-financing condition on the trading strategy π the exactly same relations are satisfied by the states of the discounted value process $\grave{V}(\pi)$, see Propositions 3.3.4, 4.4.2 and 5.5.3 that describe in this manner the cash flow mechanism in the markets in question. As was already mentioned, in the latter special case of the Black-Scholes market partial differential equations appear that are accustomed in physics for modeling thermal conductance. The equations involved in all three of these assertions are *backward* in the sense that subject to fixed boundary conditions at the terminal date T, they all allow for unique solutions. In the binary case of Part I the backward recurrence procedure and the final results are presented already in Section 2.2.2, prior to the market description. Likewise, in the Poisson case the basic differential-difference equations and their solution is presented in the preliminary Section 4.2.2. Finally, Section 5.3 is devoted to the heat equations and their solution.

Suppose that an investor is willing to invest now (at $t = 0$) in the bond and the stock in order to attain at the terminal date T a certain wealth, say W_T, without infusion or withdrawal of funds. The investor determines the desired wealth W_T so as to respond to all possibilities of the stock price development. It is said that the market in question is *complete* if there exists a self-financing trading strategy which attains any desired wealth W_T with a certain initial endowment. Besides, a specific trading strategy that at the terminal date T yields the desired wealth W_T is said to be a *hedging strategy* against W_T. To convince oneself that our markets are complete one only needs to look for the solution to the equations governing the discounted value process $\grave{V}(\pi)$ and satisfying the boundary condition $\grave{V}_T(\pi) = \grave{W}_T$. The solution always exists and is unique, as we said, for any fixed $\grave{W}_T = W_T/B_T$. Moreover, in all three cases the explicit procedure can be described for constructing the hedging strategy against the desired wealth W_T, cf Sections 3.4.2, 4.4.3 and 5.5.3.

These considerations brings us in position to turn to our main task referred usually to as *options pricing* or *options valuation.* The emphasis is placed on a particular option called the *European call option* that is described as follows. Suppose that today, at time $t = 0$, we sign a contract giving us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time.* If the stock price S_T is below the exercise price at maturity, i.e. $S_T \leq K$, then the contract is worthless to us. On the other hand, if $S_T > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price S_T. Thus this option, called the *European call option*, yields a profit at maturity T equal to

$$\max\{0, S_T - K\} = (S_T - K)^+.$$

This function of the stock price S_T at maturity T, is called the *payoff function* for the European call option. Now, how much would we be willing to pay at time $t = 0$ for a ticket which gives the right to buy at maturity $t = T$ one share of stock with exercise price K? To put this in another way, what is a fair price to pay at time $t = 0$ for the ticket? The answer to come is free of the aforementioned specific form of the payoff function, hence applicable to any *contingent claim* with any fixed *payoff function* of above type. The device to be followed may be summarized as follows:

(i) construct the hedging strategy against the contingent claim in question, which duplicates the payoff;

(ii) determine the initial wealth needed for construction in (i);

(iii) equate this initial wealth to the fair price of the contingent claim.

In a complete market this leads to the unique definition of the fair price, provided of course that the payoff function of the contingent claim in question defines properly the boundary condition at the maturity T. For reasoning behind this definition, see Section 3.5. The explicit formulas can be found in Sections 3.5.2, 4.4.3 and 5.5.3.

The present introductory course is completed at the option pricing formulas. Some material for further reading and some other relevant references can be found at the beginning of each of the forthcoming chapters. The next two chapters constitute Part I and are devoted to the binary models for finite securities markets. This part is self-contained and therefore can be read independently. The objective in Chapters 4 and 5, constituting Part II, is to shed some extra light to the circumstances under which the well-known Merton and Black-Scholes models emerge, with the help of direct, path by path derivations of the necessary Poisson and Gaussian approximations. We intend to provide in this manner an easy access to much higher technicalities of the specialized literature in the field of the limit theorems of the general theory of stochastic processes.

Part I

Finite Securities Markets

Chapter 2

Binary Models

2.1 Introduction

In this preparatory chapter basic notations are settled and the primary notions are introduced. Throughout Part I we will deal with sequences in time (therefore called often processes) presenting e.g. the price development of the securities or assets traded in a market under consideration. Since the whole time stretch of trading is always finite, all these sequences consist of fixed finite number of entries. This explains the common usage of the term *finite securities markets*, see e.g. Section 2 in the seminal paper [39] devoted to the finite theory and based on the preceding paper [38]. Cf also [68], [74]–[77] and the follow-up paper [18] (the results of the latter paper can be also found in Chapter 3 of the recent book [33], devoted to the fundamental theorem of asset pricing).

One of the assets is always chosen to serve as a common unit, on the bases of which prices of other assets or financial derivatives are expressed. Such an asset is called a *numeraire*. It is usually found handy to take as a numeraire asset the bank account that earns interest at a riskless interest rate. From Section 2.3.1 onwards the riskless asset is simply called the *bond*, and the risky asset the *stock*. The risky nature of the stock, described in Section 2.2.1, means that starting from a certain fixed value, the stock price process is allowed to evolve along one of a finite number of trajectories that will be, for convenience, portrayed by drawing a price tree with several branches of consecutive offshoots (like e.g. in (2.2.14)).

For simplicity we restrict ourselves to a market where only a stock and a bond are traded, unlike in the papers mentioned above in which several stocks are allowed. For this simplifies considerably the necessary algebra, which is otherwise quite complicated, while the essential ideas are still present. Moreover, this also dictates a risky nature of the stock, as a comprehensive theory turns out to require the equality of the number of assets traded to the number of nodes allowed in the stock price tree. This is exactly the reason why we do focus exclusively on *binary* models, the models in which the stock price tree always has precisely two notes. For any deviation from this model would violate the existence or uniqueness of the solution to the system of equations basical for constructions in the next chapter. It may be already seen in Section 2.2.2 where this system of backward recurrence equations is discussed, for at each stage of the backward solution one has to have a suitable number of unknowns.

Since the paper [13] many authors use for introductory purposes the simplest binary model, the so-called *binomial* model, to let the reader to get acquainted with the basic notions of the theory and to provide simple access to more complicated situation in continuous-time models. See e.g. [3], [4], [14], [19], [33], [44], [52], [53] and [62]. By similar reasons we also place emphasis on this particular example, especially in Part II when preparing circumstances for the Poisson and Gaussian approximations. Meanwhile, in this chapter the binomial model is presented as a simplest representative of a class of the so-called *moving averages* models, cf Section 2.3.2. These are fancy mathematical models serving only to demonstrate a wide range of the possibilities within the binary framework, but basically having a little to do with reality. Nevertheless, they seem to be of certain theoretical interest like their analogies from time series analysis and of some use for the model fitting purposes (goodness-of-fit or other statistical problems are beyond the scope of present lecture notes; the interested reader may consult e.g. the recent book [8]).

The present chapter is organized as follows. In Section 2.2.1 necessary facts are summarized concerning the risky sequences in general and binary sequences in particular. As is already indicated in the general introduction (cf the similar notice in [73], Chapter 15), the presentation in Part I does not make use of probabilistic considerations so that the pathwise approach pursued throughout needs considerable adjustments. In attempt to facilitate parallel reading of the specialized literature, we provide close probabilistic references, e.g. [65], Chapter I, for elementary constructions of the sample space, or Chapter VII, in particular Definition 7.1.5 on p 450, for martingale transforms. Regarding the latter transforms and their properties mentioned in the subsections on the summation by parts and stochastic exponentials, we also refer to more sophisticated textbooks [30], [45] and [63]. As was said above, Section 2.2.2 is devoted to solving the underlying system of backward recurrence equations. Besides, the special short subsection on random walks is inserted exclusively on purpose to indicate a proper place for few probabilistic terminology that transpires time to time. A general mathematical model for a securities market, called the *binary securities market*, is defined in Section 2.3. The latter section consists of two parts. In Section 2.3.1 primary notions of asset trading are defined, such as the bond, the stock, the discounted stock and their returns. Several illustrative examples of the binary market are gathered in Section 2.3.2.

2.2 Auxiliary notions and results

2.2.1 Sequences and their transforms

Sequences with finite state space. The prime mathematical object with which we will deal throughout this chapter is a sequence of *risky* variables, say $X_0, X_1, \ldots$. This term is used in the theory of securities market to characterize the unpredictable behaviour in time of asset prices. What we mean under this is that each of the variables involved is allowed to occupy one of a certain finite number of states: the n^{th} variable X_n, say, may be in one of the J_n states. For convenience, these states will be denoted by the corresponding lower case symbols and enumerated from 1 to J_n. So, the set of states of X_n will be presented in the following form:

$$\{x_{1n}, \ldots, x_{J_n n}\}. \tag{2.2.1}$$

It is assumed that $J_0 = 1$ so that the state x_{10} at the origin is fixed. Starting from the origin x_{10} the sequence $X_0, X_1, \ldots$ is thus allowed to develop along different trajectories. If X_n is in state $x_{j_n n}$ for a certain $j_n \in \{1, \ldots, J_n\}$ then we get the following transition scheme:

$$x_{10} \longrightarrow x_{j_1 1} \longrightarrow x_{j_2 2} \longrightarrow \cdots \tag{2.2.2}$$

Hereby a sampled trajectory may be imagined as consecutive displacements along the states that occur at consecutive time instants $t = t_1, t_2, \ldots$. In theory the physical nature of these states needs no specification, and the mentioned trajectories are merely schematical, presenting consecutive state transitions. In practice, however, each of these states get certain numerical values attached via outcomes of some measurements. It is not excluded of course that the admissible sample paths, plotted against the vertical space (of measurements) axis and horizontal time axis, do overlap as the same outcome may occur at several different states. Clearly, a distinction is necessary between the states and the numerical values they take on. But since it will be always clear from the context whether the states or their numerical values are meant, we prefer not to complicate notations and to use the same symbol $x_{j_n n}$ for the state and for its numerical value at the same time. This should cause no ambiguity.

Our attention will be focused on the simplest situation in the market in which only one risky asset is traded, and the process of its price development will underlie the risky behaviour of all other sequences we will come across.

Transforms. Let $X = \{X_n\}_{n=0,1,\ldots}$ be a certain sequence as described in the previous subsection. Let us consider it as a basic sequence. Any other sequence $Y = \{Y_n\}_{n=0,1,\ldots}$ is said to be of the same type as X, if the set $\{y_{1n}, \ldots, y_{J_n n}\}$ of the possible states for Y_n consists of the same number J_n of entries as in (2.2.1) and, moreover, the fact that X_n is in a particular state $x_{j_n n}$ means that Y_n is in state $y_{j_n n}$. To the sequence X the backward shift operator is applied to obtain the new sequence $X_- = \{X_{n-1}\}_{n=1,2,\ldots}$ The difference operator in time Δ is then defined by $\Delta X = X - X_-$, i.e. $\Delta X = \{\Delta X_n\}_{n=1,2,\ldots}$ with

$$\Delta X_n = X_n - X_{n-1}$$

for any positive integer n. Obviously, if our sequence is observed to develop along the particular trajectory described by the transition scheme (2.2.2), then for the associated sequence $\Delta X_1, \Delta X_2, \ldots$ we get the transition scheme

$$x_{j_1\,1} - x_{10} \longrightarrow x_{j_2\,2} - x_{j_1\,1} \longrightarrow \cdots$$

For completeness, put $\Delta X_0 = 0$.

A sequences will be called *predictable* if it is of the same type as X_-. Clearly, if Y is of the same type as X, then Y_- is of the same type as X_-. Note that the product $Y_- X$ is of the same type as X. If $\Phi = \{\Phi_n\}_{n=1,2,\ldots}$ is a predictable sequence, then Φ_n may occupy one of J_{n-1} states, say

$$\{\phi_{1\,n-1}, \ldots, \phi_{J_{n-1}\,n-1}\}.$$

Moreover, the states of Φ_n are completely determined by the states of X_{n-1}, for Φ_n is in a particular state $\phi_{j\,n-1}$ if and only if X_{n-1} has been in state $x_{j\,n-1}$, for any $j \in \{1, \ldots, J_{n-1}\}$.

The transform of X by a certain predictable sequence Φ is defined by

$$Y_n = Y_0 + \sum_{\nu=1}^{n} \Phi_\nu \Delta X_\nu. \tag{2.2.3}$$

The summation here is carried out path by path, along the trajectories of the sequence X. In case of the transition scheme (2.2.2), for instance, the set of possible states of Y_n is $\{y_{j_n\,n}\}_{j_n=1,\ldots,J_n}$ where

$$y_{j_n\,n} = y_{j_0\,0} + \sum_{\nu=1}^{n} \phi_{j_{\nu-1}\,\nu-1}(x_{j_\nu\,\nu} - x_{j_{\nu-1}\,\nu-1}). \tag{2.2.4}$$

for $n = 1, 2, \ldots$. We thus obtain the new sequence Y of the same type as the basic sequence X. In stochastic calculus the following abbreviation is customary

$$Y_n \doteq Y_0 + \Phi \cdot X_n, \tag{2.2.5}$$

cf e.g. [65], Definition 7.1.5 on p 450. Later, in Chapters 4 and 5, we will make use of this notation for integral extensions to (2.2.3) as well. Having this in mind, we will make already in Section 3.2.2 of the present chapter use of the term *integral representation* for a representation of type (2.2.5).

Summation by parts. If Y is of the same type as X, then the notation (2.2.5) reads

$$Y_- \cdot X_n \doteq \sum_{\nu=1}^{n} Y_{\nu-1} \Delta X_\nu. \tag{2.2.6}$$

The following notation

$$[X, Y]_n = \sum_{\nu=1}^{n} \Delta Y_\nu \Delta X_\nu \tag{2.2.7}$$

is also borrowed from stochastic calculus. In these notations, we have the so-called *summation by parts formula:*

$$X_nY_n - X_0Y_0 = Y_- \cdot X_n + X_- \cdot Y_n + [X, Y]_n. \tag{2.2.8}$$

Indeed, by simple algebra we get

$$\begin{aligned}\Delta(X_nY_n) &= Y_{n-1}\Delta X_n + X_n\Delta Y_n \\ &= Y_{n-1}\Delta X_n + X_{n-1}\Delta Y_n + \Delta X_n\Delta Y_n.\end{aligned} \tag{2.2.9}$$

By summing up both sides of this equation we get (2.2.8).

Exponentials. For a fixed sequence $X = \{X_n\}_{n=0,1,\dots}$ with $X_0 = 0$, consider the linear difference equations

$$\Delta Z_n = Z_{n-1}\Delta X_n, \quad n = 1, 2, \dots, \tag{2.2.10}$$

subject to the initial condition $Z_0 = 1$. These difference equations are equivalent to

$$Z_n = 1 + Z_- \cdot X_n, \quad n = 1, 2, \dots. \tag{2.2.11}$$

They have unique solutions given by

$$Z_n = \prod_{\nu=1}^{n}(1 + \Delta X_\nu) \doteq \mathcal{E}(X)_n, \quad n = 1, 2, \dots. \tag{2.2.12}$$

Assume $\Delta X_n > -1$ for all $n = 1, 2, \dots$ to get a positive solution. The symbol $\mathcal{E}$ is borrowed from the theory of stochastic differential equations. It denotes the solution of (2.2.10) (or, equivalently, (2.2.11)) and is called the stochastic, or Doléans-Dade exponential; see for instance [30], [45] or [63]. Note the following property of the Doléans-Dade exponential

$$\mathcal{E}(X)_n\mathcal{E}(Y)_n = \mathcal{E}(X + Y + [X, Y])_n \tag{2.2.13}$$

(cf (2.2.7)) which can be verified directly or by using (2.2.9).

Difference operator in the state space. Throughout this lecture notes we deal exclusively with the so-called binary sequences. It is said that a sequence $X = \{X_n\}_{n=0,1,\dots}$ is *binary*, if the set (2.2.1) of states of X_n consists of $J_n = 2^n$ entries and the transition scheme is portrayed by the n^{th} transition

$$x_{j\,n-1} \left\langle \begin{array}{l} x_{2j\,n} \\ x_{2j-1\,n}. \end{array} \right. \tag{2.2.14}$$

The first two instances, for example, are

$$x_{10} \left\langle \begin{array}{l} x_{21} \left\langle \begin{array}{l} x_{42} \\ x_{32} \end{array} \right. \\ x_{11} \left\langle \begin{array}{l} x_{22} \\ x_{12}. \end{array} \right. \end{array} \right. \tag{2.2.15}$$

Along with the difference operator in time Δ, in this particular case we can also define the difference operator in the state space, say D. To this end, a predictable sequence $DX = \{DX_n\}_{n=1,2,\dots}$, i.e. a sequence of the same type as X_-, is completely defined by the following statement: for each $j \in \{1, \dots, 2^{n-1}\}$ the claim that

$$X_{n-1} \text{ is in state } x_{j\,n-1} \tag{2.2.16}$$

is equivalent to the claim that

$$DX_n \text{ is in state } D_j(X_n) \doteq x_{2j\,n} - x_{2j-1\,n}. \tag{2.2.17}$$

The sequence DX is called predictable by the obvious reason that the observation at instant t_{n-1} of a certain state of X_{n-1} allows for exact forecasting the state at the forthcoming instant t_n of DX_n. It is easily verified that if Ψ and Φ are certain predictable sequences, then

$$D(\Psi + \Phi X) = \Psi + \Phi\, DX. \tag{2.2.18}$$

The operator D is iterated as follows. For $n = 2, 3, \dots$, look at the two step transition scheme:

$$x_{j\,n-2} \left\langle \begin{array}{l} x_{2j\,n-1} \left\langle \begin{array}{l} x_{4j\,n} \\ x_{4j-1\,n} \end{array} \right. \\ x_{2j-1\,n-1} \left\langle \begin{array}{l} x_{4j-2\,n} \\ x_{4j-3\,n}. \end{array} \right. \end{array} \right.$$

The sequence $D^2X = \{D^2X_n\}_{n=2,3,\dots}$ is completely defined by the following statement: for $n = 2, 3, \dots$ and $j \in \{1, \dots, 2^{n-2}\}$ the claim that

$$X_{n-2} \text{ is in state } x_{j\,n-2}$$

is equivalent to the claim that

$$\begin{aligned} D^2X_n \text{ is in state } D_j^2(X_n) &\doteq D_{2j}(X_n) - D_{2j-1}(X_n) \\ &= x_{4j\,n} - x_{4j-1\,n} - x_{4j-2\,n} + x_{4j-3\,n}. \end{aligned}$$

Note that for the same sequence $Z = \Psi + \Phi X$ as in (2.2.18) we have

$$\begin{aligned} D_j^2(Z_n) &\doteq D_{2j}(Z_n) - D_{2j-1}(Z_n) \\ &= \phi_{2j\,n-1} D_{2j}(X_n) - \phi_{2j-1\,n-1} D_{2j-1}(X_n). \end{aligned} \tag{2.2.19}$$

2.2.2 Backward recurrence equations

Elementary solutions. Throughout this lecture notes we make use of the following notations. For any number x we denote by $\lceil x \rceil$ the smallest integer exceeding x and by $\lfloor x \rfloor$ the largest integer not exceeding x, which is usually called the *integer part* of x. Obviously

$$\lceil x \rceil - \lfloor x \rfloor = 1.$$

With any positive integer k we associate its n^{th} *dyadic fraction* k_n defined by

$$k_n \doteq \lceil k/2^n \rceil, \tag{2.2.20}$$

i.e. $k_0 = k$, $k_1 = \lceil k/2 \rceil$, $k_2 = \lceil k/4 \rceil$ and so forth. Obviously, this sequence, starting from k, gradually descends to 1: schematically

$$k \searrow k_1 \searrow k_2 \searrow \cdots \searrow 1. \tag{2.2.21}$$

With these notations at hand, we are now going to describe the solution to the following boundary problem that plays an important rôle in the considerations of the present chapter. Suppose that for each $n = 1, \dots, N$ the set of 2^n coefficients is given $\{p_{1n}, \dots, p_{2^n\,n}\}$. For each such n, consider the system of 2^{n-1} equations

$$\begin{aligned} x_{1\,n-1} &= p_{1n}x_{1n} + p_{2n}x_{2n} \\ x_{2\,n-1} &= p_{3n}x_{3n} + p_{4n}x_{4n} \\ &\dots \\ x_{2^{n-1}\,n-1} &= p_{2^n-1\,n}x_{2^n-1\,n} + p_{2^n\,n}x_{2^n\,n}. \end{aligned} \tag{2.2.22}$$

To solve this system of recurrence equations we need to fix a boundary condition at the terminal date $t_N = T$ by assigning a certain value to each component of the set $\{x_{1N}, \dots, x_{2^N\,N}\}$. We get the simplest case with the following boundary conditions: for a certain fixed index $k \in \{1, \dots, 2^N\}$

$$x_{jN} = \delta_{jk} \doteq \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \,. \end{cases} \tag{2.2.23}$$

The symbol δ_{jk} used here is known as *Kronecker's delta.* The system of equations (2.2.22) can now be solved for $n = N$. The solution for the preceding set $\{x_{1\,N-1}, \dots, x_{2^{N-1}\,N-1}\}$ is given by

$$x_{j\,N-1} = \delta_{jk_1}p_{kN} \tag{2.2.24}$$

with the same Kronecker symbol as above.

Having (2.2.24) at hand, we may proceed by solving (2.2.22) for $n = N-1$. The set $\{x_{1\,N-2}, \dots, x_{2^{N-2}\,N-2}\}$ is thus evaluated by

$$x_{j\,N-2} = \delta_{jk_2}p_{k_1\,N-1}p_{kN}. \tag{2.2.25}$$

Moving backwards in this way we obtain all the solutions step by step. After $n < N$ steps e.g., we arrive at the set $\{x_{1\,N-n}, \dots, x_{2^{N-n}\,N-n}\}$ with

$$x_{j\,N-n} = \delta_{jk_n} \prod_{0 \le \nu < n} p_{k_\nu\,N-\nu}. \tag{2.2.26}$$

The final N^{th} step brings us to

$$x_{10} = \prod_{0 \le \nu < N} p_{k_\nu\,N-\nu}. \tag{2.2.27}$$

Use for convenience the notations

$$P^n_{kN} \doteq \prod_{0 \le \nu < n} p_{k_\nu\, N-\nu} \tag{2.2.28}$$

for $n = 1, \ldots, N$ (obviously $P^1_{kN} = p_{kN}$) and, in particular,

$$P_{kN} \doteq P^N_{kN} = \prod_{0 \le \nu < N} p_{k_\nu\, N-\nu}. \tag{2.2.29}$$

Then the result may be summarized in the following assertion.

Proposition 2.2.1. *The unique solution of the system of backward recurrence equations (2.2.22) subject to the boundary conditions (2.2.23) is described by using the notations (2.2.28) and (2.2.29) as follows: for* $n \in \{1, \ldots, N\}$

$$x_{j\, N-n} = \delta_{jk_n} P^n_{kN}. \tag{2.2.30}$$

In particular

$$x_{10} = P_{kN}. \tag{2.2.31}$$

Random walk. The results of the previous subsection may be applied to a physical model for displacements of a particle from state to state, imparted to it by a certain external force at some instants of time. We use this example to illustrate a few basic ideas and terms from probability theory, necessary for understanding the impact of this theory upon options pricing methodology.

The initial state of the particle is supposed to be fixed to x_{10}, say. Suppose that it remains in this state during some period of time and then switches to one of the alternative states, either to x_{11} or to x_{21}. This process develops as follows: if within a certain period, say the $n-1^{th}$ one, n is a positive integer, the particle stays in state $x_{k\, n-1}$ with some $k \in \{1, \ldots, 2^{n-1}\}$, then at the end it switches either to $x_{2k-1\, n}$ or to $x_{2k\, n}$. Moreover, suppose that the external force is focused so that in the $p_{2k-1\, n}100\%$ of cases the transition $x_{k\, n-1} \searrow x_{2k-1\, n}$ occurs and in the remaining $p_{2k\, n}100\%$ of cases the transition $x_{k\, n-1} \nearrow x_{2k-1\, n}$ occurs, where $p_{2k-1\, n}$ and $p_{2k\, n}$ are given positive numbers such that

$$p_{2k-1\, n} + p_{2k\, n} = 1. \tag{2.2.32}$$

These numbers are called the *transition probabilities.* They are also interpreted as the *conditional probabilities* of corresponding events, under the condition that the particle departs from state $x_{k\, n-1}$.

In terms of Section 2.2.1, we deal with the binary sequence $X_0, X_1, \ldots$ with X_n whose admissible states are (2.2.1) where $J_n = 2^n$, and the transition scheme is portrayed by (2.2.14). In case of 2 periods, for instance, we have (2.2.15) and

$$\begin{array}{llll}
x_{10} \nearrow x_{21} \nearrow x_{42} & \text{in} & p_{21}p_{42}100\% & \text{of cases} \\
x_{10} \nearrow x_{21} \searrow x_{32} & \text{in} & p_{21}p_{32}100\% & \text{of cases} \\
x_{10} \searrow x_{11} \nearrow x_{22} & \text{in} & p_{11}p_{22}100\% & \text{of cases} \\
x_{10} \searrow x_{11} \searrow x_{12} & \text{in} & p_{11}p_{12}100\% & \text{of cases}
\end{array}$$

with the corresponding transition probabilities which sum up to 1:

$$p_{21}p_{42} + p_{21}p_{32} + p_{11}p_{22} + p_{11}p_{12} = 1 \tag{2.2.33}$$

by (2.2.32).

These notions easily extend to the general case of N periods: $P^1_{kN} = p_{kN}$ is the probability of the transition $x_{k_1\,N-1} \longrightarrow x_{kN}$, as we have already observed, $P^2_{kN} = p_{k_1\,N-1}p_{kN}$ the probability of the transition $x_{k_2\,N-2} \longrightarrow x_{kN}$ and so forth. Generally, P^n_{kN} given by (2.2.28) is the probability of the transition $x_{k_n\,N-n} \longrightarrow x_{kN}$. Using (2.2.32) repeatedly, we obtain the following extension to (2.2.33):

$$\sum_{2^n(j-1)<k\le 2^n j} P^n_{kN} = 1 \quad \text{for} \quad j = 1, \dots , 2^{N-n}. \tag{2.2.34}$$

The extreme case $n = N$ is special because each trajectory starts from a fixed point x_{10}, hence no conditioning is needed and P_{kN} for an integer $k \in \{1, \dots , 2^N\}$, given by (2.2.28), is now the (unconditional) probability that after N periods the particle will occupy the state x_{kN}. Clearly

$$\sum_{k=1}^{2^N} P_{kN} = 1. \tag{2.2.35}$$

The evolution of the particle just described is usually referred to as a *random walk* and the whole collection $\{P_{kN}\}_{k=1,\dots,2^N}$ as its *probability distribution.*

General solutions. In addition to the coefficients $\{p_{1\,n}, \dots , p_{2^n\,n}\}$, let the set of 2^N numbers $\{c_1, \dots , c_{2^N}\}$ be given. Consider the system of backward equations (2.2.22) subject to the boundary conditions

$$x_{jN} = c_j \quad \text{for} \quad j = 1, \dots , 2^N. \tag{2.2.36}$$

Obviously, these conditions reduce to (2.2.23) in the special case of $c_j = \delta_{jk}$.

Observe that the solution to the present boundary problem has the following additivity property. Let two different solutions to the system (2.2.22) be known which correspond to two different boundary conditions, say (2.2.36) with c'_j in the first case and c''_j in the second case. If these solution are $x'_{j\,N-n}$ and $x''_{j\,N-n}$, respectively, then the solution to (2.2.22) subject to new boundary conditions with $c_j = c'{}_j + c''{}_j$ is composed by $x_{j\,N-n} = x'{}_{j\,N-n} + x''{}_{j\,N-n}$.

Due to this observation, we can easily deduce from Proposition 2.2.1 the solution also to the present general case (2.2.36), for these boundary conditions may be rewritten in the form

$$x_{jN} = \sum_{k=1}^{2^N} c_k\delta_{jk} \quad \text{for} \quad j = 1, \dots , 2^N.$$

Then the problem can be solved for each summand separately. This yields the following result.

Proposition 2.2.2. *The system of recurrence equations (2.2.22) subject to the boundary conditions (2.2.36) has a unique solution: for* $n \in \{1, \ldots, N\}$

$$x_{j\,N-n} = \sum_{2^n(j-1)<k\leq 2^n j} c_k P^n_{kN} \quad \textit{for} \quad j = 1, \ldots, 2^{N-n}. \tag{2.2.37}$$

In particular

$$x_{10} = \sum_{k=1}^{2^N} c_k P_{kN}. \tag{2.2.38}$$

Proof. Due to the additivity property observed above, it follows from Proposition 2.2.1 that

$$x_{j\,N-n} = \sum_{k=1}^{2^N} \delta_{jk_n} c_k P^n_{kN}. \tag{2.2.39}$$

Thus, it remains only to verify that Kronecker's δ_{jk_n} equals to 1 when $j - 1 < k/2^n \leq j$, otherwise it equals to 0. The solution (2.2.37) is obtained. To get (2.2.38) substitute $n = N$ in (2.2.37). □

2.3 A binary model

2.3.1 A market with two securities: a bond and a stock

Securities market. Consider a *securities market* in which two *assets* (or *securities*) are traded at successive time periods marked by $0 = t_0 < t_1 < \cdots < t_N = T$. The instant $t_0 = 0$ is interpreted as the current date and $t_N = T < \infty$ as a terminal date which is considered to be fixed. It is usually said in this case that the time horizon T is finite and the trading takes place over N periods. The instants $t_0, t_1, \dots, t_N$ are called trading times, since these are the dates at which new prices are announced in the market.

Bond. One of these assets, the *bond*, has price B_n over the period $[t_n, t_{n+1})$ for $n = 0, 1, \dots, N-1$ and B_N is the price announced at the terminal date $t_N = T$. Fix for simplicity $B_0 = 1$. These prices $\{B_n\}_{n=0,1,\dots,N}$ are usually related to interest rates over the corresponding periods as follows: for $n = 1, \dots, N$

$$B_n = r_n B_{n-1} \tag{2.3.1}$$

and

$$B_n = r_1 \cdots r_n \tag{2.3.2}$$

where $r_n \geq 1$ is one plus the interest rate in the interval $[t_{n-1}, t_n)$. Since here the interest rates are predetermined, we call the bond the riskless asset. The quantity r_n itself is sometimes called the *gross return* on the bond at the trading time t_n, in contrast with $r_n - 1$ which is called the *net return* (cf e.g. [8], p 9).

Parallel to the bond price process $\{B_n\}_{n=0,1,\dots,N}$, it is found often useful to work with the so-called *return process* (or *cumulative return process*) on the bond $\{\mathcal{R}_n\}_{n=0,1,\dots,N}$, starting from the origin $\mathcal{R}_0 = 0$, that is defined for each $n \in \{1, \dots, N\}$ as the accumulation of all net returns up to the trading time t_n, i.e.

$$\mathcal{R}_n = \sum_{\nu=1}^{n} (r_\nu - 1), \tag{2.3.3}$$

With the notations of section 2.2.1 we have

$$\Delta\mathcal{R}_n = \frac{\Delta B_n}{B_{n-1}} = r_n - 1, \tag{2.3.4}$$

since $B_n / B_{n-1} = r_n$ by (2.3.1). Hence, the return process is expressed in terms of the bond prices

$$\mathcal{R}_n = \sum_{\nu=1}^{n} \frac{\Delta B_\nu}{B_{\nu-1}}$$

and the bond price process in terms of the returns

$$B_n = \prod_{\nu=1}^{n}(1 + \Delta\mathcal{R}_\nu).$$

Compare the later equation with (2.2.12) to see that the bond price process is the unique solution of the linear difference equations (2.2.11) with X substituted by $\mathcal{R}$. In terms of Section 2.3.1, the bond price process is described as the Doléans-Dade exponential of the cumulative return process, i.e. for $n = 1, \ldots, N$

$$B_n = \mathcal{E}(\mathcal{R})_n. \tag{2.3.5}$$

Stock. Unlike the bond, the second asset, the *stock*, is risky in the sense that its price, denoted by S_n at time t_n, is allowed to evolve in time along more then one trajectory. The set of all admissible trajectories is described as follows. Currently (at $t_0 = 0$) the stock price S_0 is fixed in state s_{10}, say. At the next trading time t_1 a new price is announced and S_1 will occupy either state s_{11} or s_{21}. Schematically, the transition from the state s_{10} of S_0 to two alternative states s_{21} or s_{11} of S_1 can be portrayed as follows:

$$s_{10} \begin{array}{l} \diagup \; s_{21} \\ \diagdown \; s_{11}. \end{array} \tag{2.3.6}$$

Generally, if the stock price at the trading time t_{n-1} is in state $s_{j\,n-1}$ (i.e. S_{n-1} is in state $s_{j\,n-1}$), then at the next trading time t_n it will be announced either in state $s_{2j\,n}$ or $s_{2j-1\,n}$. In other words, $s_{2j\,n}$ and $s_{2j-1\,n}$ are two alternative states of S_n, provided S_{n-1} has been in state $s_{j\,n-1}$. Schematically,

$$s_{j\,n-1} \begin{array}{l} \diagup \; s_{2j\,n} \\ \diagdown \; s_{2j-1\,n}. \end{array} \tag{2.3.7}$$

This is completely similar to the model for a particle evolution of Section 2.2.2, called the random walk. In the latter case the transition scheme has been (2.2.14), alike (2.3.7). Similarly to (2.2.15), the 2 period model, for instance, is portrayed as follows:

$$s_{10} \begin{array}{l} \diagup \; s_{21} \begin{array}{l} \diagup \; s_{42} \\ \diagdown \; s_{32} \end{array} \\ \diagdown \; s_{11} \begin{array}{l} \diagup \; s_{22} \\ \diagdown \; s_{12}. \end{array} \end{array} \tag{2.3.8}$$

We want to emphasize at this point that unlike the previous case of random walk we need no mentioning of chances of particular transitions, for this is completely superfluous within the theory to be developed.

The transition scheme (2.3.7) describes a so-called *binary* market in which the stock price is allowed to evolve along one of 2^N different trajectories: at the terminal date $t_N = T$ the stock price S_N occupies one of the states $\{s_{kN}\}_{k=1,\ldots,2^N}$. For convenience we assign to the trajectories the same index as to the states of S_N. This means that the following two statements are equivalent:

"the stock price evolves along the k^{th} trajectory of states"

and

"at the terminal date $t_N = T$ the stock price S_N is in state s_{kN}".

The stock price development along a particular trajectory is described by using the notations introduced at the beginning of Section 2.2.2. Denoting by $\lceil x \rceil$ the smallest integer exceeding x, we have defined by $k_n = \lceil k/2^n \rceil$ the n^{th} dyadic fraction of a positive integer k, cf (2.2.20). This relationship was portrayed by the scheme (2.2.21). Along the k^{th} trajectory, for any $k \in \{1, \dots, 2^N\}$, the stock price is in the states $\{s_{k_{N-n}\, n}\}_{n=0,1,\dots,N}$. In other words, at the trading time t_n the variable S_n is in state $s_{k_{N-n}\, n}$.

The returns are encountered in the present case too. At the trading time t_n with $n \in \{1, \dots, N\}$, the *gross return* on the stock is defined by

$$Z_n \doteq \frac{S_n}{S_{n-1}}. \tag{2.3.9}$$

With this notation

$$S_n = S_0 Z_1 \cdots Z_n. \tag{2.3.10}$$

In terms of Section 2.2.1, this is the Doléans-Dade exponential

$$S_n = S_0 \mathcal{E}(R)_n \tag{2.3.11}$$

where

$$R_n \doteq \sum_{\nu=1}^{n} (Z_\nu - 1). \tag{2.3.12}$$

So, the *cumulative return process* $\{R_n\}_{n=0,1,\dots,N}$ on the stock is defined, that starts from the origin $R_0 = 0$. It follows from (2.3.12) that the *net return* on the stock at the trading time t_n with $n \in \{1, \dots, N\}$ is

$$\Delta R_n = Z_n - 1. \tag{2.3.13}$$

By (2.3.9) and (2.3.13) $\Delta R_n = \Delta S_n / S_{n-1}$.

Let us denote by $\{z_{kn}\}_{k=1,\dots,2^n}$ and $\{r_{kn}\}_{k=1,\dots,2^n}$ the states of the gross return Z_n and the net return ΔR_n, respectively, which are defined as follows: if the stock price S_n is in state s_{kn}, then

$$r_{kn} = z_{kn} - 1 \quad \text{and} \quad z_{kn} = \frac{s_{kn}}{s_{k_1\, n-1}} \tag{2.3.14}$$

where $k_1 = \lceil k/2 \rceil$ as usual, cf (2.2.20). As the cumulative return R_n is the sum of all net returns up to the trading time t_n, the corresponding state of R_n is obtained by summing up these returns along the path (2.2.21). This results in $\sum_{\nu=1}^{n} r_{k_{n-\nu}\, \nu}$.

Thus we have described in terms of consecutive states the whole range of possibilities for the stock price development. Surely, the prices are announced in currency amounts and thus all admissible states s_{kn} take on the corresponding numerical values. The expressions like (2.3.14) get then a proper meaning upon substituting these numerical values and carrying out necessary operations. As was already emphasized in the first subsection of the introductory Section 2.2.1, this convention will always kept in mind, besides it will be always assumed that $s_{10} = s > 0$ and that $s_{2k\,n} > s_{2k-1\,n} > 0$. Clearly, there is no reason to think of different prices at different states, so that if we draw all possible price graphs, they may overlap. We will see this in Section 2.3.2 where special models are discussed.

Discounting. It is usually desirable to take into account the development of the stock prices relative to the bond prices. This is achieved by making the latter prices the *numeraire* that defines the *discounted* stock prices $\{\grave{S}_n\}_{n=0,1,\dots,N}$ with $\grave{S}_n = S_n/B_n$. To this new process the net returns are related in the precisely same manner as before: $\grave{R}_0 = 0$ and

$$\Delta\grave{R}_n = \frac{\Delta\grave{S}_n}{\grave{S}_{n-1}}$$

for $n = 1, \dots, N$. We thus have the usual relationship between the discounted stock price process $\{\grave{S}_n\}_{n=0,1,\dots,N}$ and the cumulative return process $\{\grave{R}_n\}_{n=0,1,\dots,N}$: for $n = 1, \dots, N$

$$\grave{S}_n = S_0\grave{Z}_1 \cdots \grave{Z}_n = S_0\mathcal{E}(\grave{R})_n$$

(cf (2.3.11) and (2.3.10)) with the discounted gross returns

$$\grave{Z}_n = 1 + \Delta\grave{R}_n = \frac{\grave{S}_n}{\grave{S}_{n-1}}.$$

The states of the discounted variables are defined by the considerations parallel to the previous subsection. In particular, if at the trading time t_n with $n = 1, \dots, N$, the stock price S_n is in state s_{kn} for some $k = 1, \dots, 2^n$, then the discounted gross return on the stock $\grave{Z}_n$ is in state

$$\grave{z}_{kn} \doteq \frac{\grave{s}_{kn}}{\grave{s}_{k_1\,n-1}} = \frac{z_{kn}}{r_n}. \tag{2.3.15}$$

Note finally that for each $n \in \{1, \dots, N\}$ the net returns $\Delta\mathcal{R}_n$, ΔR_n and $\Delta\grave{R}_n$ are related as follows:

$$\Delta\grave{R}_n = \frac{\Delta(R-\mathcal{R})_n}{1+\Delta\mathcal{R}_n} = \frac{\Delta(R-\mathcal{R})_n}{r_n}. \tag{2.3.16}$$

The difference $\Delta(R-\mathcal{R})_n = \Delta R_n - \Delta\mathcal{R}_n$ is called the *excess return*, like in [8], p 12. The relationship (2.3.16) is equivalent to

$$\Delta R_n = \Delta\mathcal{R}_n + \Delta\grave{R}_n + \Delta\mathcal{R}_n\Delta\grave{R}_n.$$

This can also be confirmed by using (2.2.13) as follows:

$$\mathcal{E}(R)_n = \frac{S_n}{S_0} = B_n \frac{\grave{S}_n}{S_0} = \mathcal{E}(\mathcal{R})_n \mathcal{E}(\grave{R})_n = \mathcal{E}(\mathcal{R} + \grave{R} + [\mathcal{R}, \grave{R}])_n.$$

2.3.2 Binomial model and moving averages

Up to now we have made frequent use of the difference operator in time Δ, introduced in the second subsection of Section 2.2.1. Beginning from the present section we are going to make use also of the difference operator in the state space D. According to the definitions in the last subsection of Section 2.2.1, the latter operator is applied to the stock price process and to the corresponding gross returns as follows.

Fix the trading time t_n with $n \in \{1, \dots, N\}$. Let $j \in \{1, \dots, 2^{n-1}\}$. The claim that

$$S_{n-1} \text{ is in state } s_{j\,n-1}$$

is then equivalent to the claim that

$$DS_n \text{ is in state } D_j(S_n) \doteq s_{2j\,n} - s_{2j-1\,n}$$

or, in virtue of (2.2.18), that

$$DZ_n \text{ is in state } D_j(Z_n) \doteq D_j(S_n)/s_{j\,n-1}.$$

We can thus write $DZ = DS/S_-$. The operator D is iterated as follows. Fix the trading time t_n with $n \in \{2, \dots, N\}$ and let $j \in \{1, \dots, 2^{n-2}\}$. The claim

$$S_{n-2} \text{ is in state } s_{j\,n-2}$$

is equivalent to the claim that

$$\begin{aligned} D^2 S_n \text{ is in state } D_k^2(S_n) &\doteq D_{2j}(S_n) - D_{2j-1}(S_n) \\ &= s_{4j\,n} - s_{4j-1\,n} - s_{4j-2\,n} + s_{4j-3\,n} \end{aligned}$$

or, in virtue of (2.2.19), that

$$\begin{aligned} D^2 Z_n \text{ is in state } D_j^2(Z_n) &\doteq D_{2j}(Z_n) - D_{2j-1}(Z_n) \\ &= \frac{D_{2j}(S_n)}{s_{2j\,n-1}} - \frac{D_{2j-1}(S_n)}{s_{2j-1\,n-1}}. \end{aligned}$$

These notions are particularly simple in the case of the so-called *binomial model* to be described next.

Binomial model. We begin with simplest case of the binomial model that occurs most often in literature. This model will serve us in Part II as a starting point for transition to more complicated models in continuous time.

Example 2.3.1. HOMOGENEOUS CASE. In this model the prices on the bond and stock are assumed to develop homogeneously in time, as the returns on both assets are free of the time index. Hence, the interest rate remains constant and the gross returns in the bond pricing formula (2.3.2) are equal $r = r_1 = \cdots = r_N$ where $r \geq 1$, so that the bond prices grow exponentially $B_n = r^n$ as n runs through $\{0, 1, \ldots, N\}$. Accordingly, the rate of net returns on the bond at any trading time t_n with $n \in \{1, \ldots, N\}$ amounts to

$$\Delta\mathcal{R}_n = \frac{\Delta B_n}{B_{n-1}} = r - 1 \geq 0,$$

cf (2.3.4). Note that the extreme case $r = 1$, when the bond prices do not vary, may be interpreted as if the stock prices in the market were discounted before their announcement.

The rate of net and gross returns on the stock defined at the trading time t_n with $n \in \{1, \ldots, N\}$ by $\Delta R_n = \Delta S_n / S_{n-1}$ and $Z_n = S_n / S_{n-1}$, respectively, remains homogeneous in time as well. For, independently of n the gross return Z_n jumps either up or down, always with the same step sizes u and d, respectively, where $u > d > 0$ (the equalities are excluded to retain the risky nature of the stock price process). In other words, all the even states $z_{2k\,n}$ takes on the value u and all the odd states $z_{2k-1\,n}$ the value d, i.e.

$$z_{2k\,n} = \frac{s_{2k\,n}}{s_{k\,n-1}} = u \quad \text{and} \quad z_{2k-1\,n} = \frac{s_{2k-1\,n}}{s_{k\,n-1}} = d \tag{2.3.17}$$

whatever $n = 1, \ldots N$ and $k = 1, \ldots, 2^{n-1}$. Thus, if the stock price at the trading time t_{n-1} is S_{n-1}, then at the end of the following period it will be either $S_{n-1}u$ or $S_{n-1}d$. So, at the trading time t_n there occurs either $S_n = S_{n-1}u$ or $S_{n-1}d$. Schematically,

$$S_{n-1}\begin{cases} S_{n-1}u \\ S_{n-1}d, \end{cases} \tag{2.3.18}$$

cf (2.3.7). For instance, in the 2 period model we have $s_{42} = su^2$, $s_{32} = s_{22} = sud$ and $s_{12} = sd^2$, hence the following transitions take place:

$$\begin{array}{ccccc} & & & & su^2 \\ & & su & \langle & \\ & & & & sud \\ s & \langle & & & \\ & & sd & \langle & \\ & & & & sd, \end{array}$$

cf (2.3.8). This is the example mentioned already above in which not all states differ in value. To extend this to the general N period model with 2^N states of S_N, for each $n \in \{0, 1, \ldots, N\}$ select those trajectories which evolve with exactly n upward and $N-n$ downward displacements. In other words, partition the set of all possible terminal states $\{s_{kN}\}_{k=1,\ldots,2^N}$ into $N+1$ disjoint subsets, say $\Pi_0, \Pi_1, \ldots, \Pi_N$, so that if a state belongs to Π_n, then the corresponding trajectory evolves with exactly n upward and $N-n$ downward displacements. As we know from combinatorics, the number of all such trajectories equals to

$\binom{N}{n}$. By the binomial formula we have $\sum_{n=0}^{N}\binom{N}{n} = 2^N$. If now the stock price S_N occupies one of $\binom{N}{n}$ states from the subset Π_n, then it is announced to be equal $su^n d^{N-n}$. Thus, not all 2^N different states at the terminal date yield different stock prices and the number of the possibilities is limited to $N+1$.

There is no need so far to confine our consideration only to the homogeneous case, at least until Example 3.3.5 where the homogeneity hypotheses will bring significant simplification in the basic formulas. Therefore consider the following extension to the previous example.

Example 2.3.2. NONHOMOGENEOUS CASE. Let the rate of returns on the bond be not necessarily homogeneous, and replace (2.3.17) with

$$z_{2k\,n} = \frac{s_{2k\,n}}{s_{k\,n-1}} = u_n \quad \text{and} \quad z_{2k-1\,n} = \frac{s_{2k-1\,n}}{s_{k\,n-1}} = d_n \tag{2.3.19}$$

for $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$. Thus the dependence on time is introduced, but the homogeneity in space is still retained. Therefore, the difference operator in the space D yields elementary results summarized in Proposition 2.3.3 below. Since the transition scheme (2.3.18) extends to

$$S_{n-1}\begin{matrix} \nearrow S_{n-1}u_n \\ \searrow S_{n-1}d_n, \end{matrix} \tag{2.3.20}$$

we simply have $DS_n = (u_n - d_n)S_{n-1}$ and the following statement holds true:

Proposition 2.3.3. *Let the underlying model be binomial. Then at each trading time t_n with $n = 1, \dots, N$ the gross return Z_n on the stock possesses the following properties:*
(i) the variable DZ_n is constant in the state space: all its 2^{n-1} states take on the same numerical value $u_n - d_n$, i.e.

$$DZ_n = u_n - d_n \tag{2.3.21}$$

whatever the states $\{s_{k\,n-1}\}_{k=1,\dots,2^{n-1}}$ of the variable S_{n-1}.
(ii) Moreover, at each trading time t_n with $n = 2, \dots, N$ the variable D^2Z_n is constant in the state space: all its 2^{n-2} states vanish, i.e.

$$D^2Z_n = 0 \tag{2.3.22}$$

whatever the states $\{s_{k\,n-2}\}_{k=1,\dots,2^{n-2}}$ of the variable S_{n-2}.

Adapted models. We are going to describe certain transformations of the binomial model that retain the properties of its returns asserted in Proposition 2.3.3. To this end, we shall introduce special notations for the gross returns in the binomial model. Let ϵ_n be an elementary variable which may occupy one of the following states $\{e_{kn}\}_{k=1,\dots,2^n}$ with

$$e_{kn} = \begin{cases} u_n & \text{if } k \text{ is even} \\ d_n & \text{if } k \text{ is odd} \end{cases}$$

where u_n and d_n are given numbers so that $u_n > d_n > 0$. In view of (2.3.19), the sequence $\epsilon_1, \ldots, \epsilon_N$ constitutes the sequence of gross returns in the binomial model. A new sequence $\{Z_n\}_{n=1,\ldots,N}$ of gross returns is obtained by the following transformation. Set $Z_1 = \epsilon_1$ and for $n = 2, \ldots, N$

$$Z_n = \epsilon_n + f_n(\epsilon_1, \ldots, \epsilon_{n-1}), \tag{2.3.23}$$

with certain functions f_n of $n-1$ arguments. By our conventions this means that for $n = 1, \ldots, N$ the states $\{z_{kn}\}_{k=1,\ldots,2^n}$ of Z_n and the states $\{e_{k\nu}\}_{k=1,\ldots,2^\nu}$ of ϵ_ν for $\nu = 1, \ldots, n$ are related as follows:

$$z_{kn} = e_{kn} + f_n(e_{k_{n-1}\,1}, \ldots, e_{k_1\,n-1}). \tag{2.3.24}$$

Once again the notations (2.2.20) and (2.2.21) are used. Clearly, this model is portrayed by the following transition scheme: for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$

$$s_{k\,n-1} \begin{matrix} \nearrow s_{k\,n-1}\, z_{2k\,n} \\ \searrow s_{k\,n-1}\, z_{2k-1\,n} \end{matrix}$$

where

$$\begin{aligned} z_{2k\,n} &= u_n + f_n(e_{k_{n-1}\,1}, \ldots, e_{k_1\,n-1}) \\ z_{2k-1\,n} &= d_n + f_n(e_{k_{n-1}\,1}, \ldots, e_{k_1\,n-1}) \end{aligned} \tag{2.3.25}$$

according to (2.3.24). It is easily verified that the result of Proposition 2.3.3 extends to the present case. We have

Proposition 2.3.4. *Let a binary model be adapted to the binomial model according to the relationship (2.3.23) between the corresponding returns. Then the resulting sequence of gross returns retains the properties (2.3.21) and (2.3.22).*

Moving averages. If the basic variables $\epsilon_1, \ldots, \epsilon_N$ are interpreted as innovations at the corresponding trading times, then the representation (2.3.23) gets the following meaning. At the trading time t_n the gross return Z_n consists of the innovation term ϵ_n plus a certain function of the past. In this manner we get a large variety of binomial models with the desired properties, as the functions of the past can vary unrestrictedly. We only intend, however, to focus in the sequel on a special case of linear functions f_n. It will be supposed, in particular, that

$$f_n(x_1, \ldots, x_{n-1}) = \alpha_1 x_{n-1} + \cdots + \alpha_{n-1} x_1$$

with some real parameters $\alpha_1, \ldots, \alpha_{N-1}$, i.e. $Z_1 = \epsilon_1$ and for $n = 2, \ldots, N$

$$Z_n = \epsilon_n + \alpha_1 \epsilon_{n-1} + \cdots + \alpha_{n-1} \epsilon_1. \tag{2.3.26}$$

Note that the expression (2.3.25) for states of the gross return Z_n reduces in the present special case to

$$\begin{aligned} z_{2k\,n} &= u_n + \alpha_1 e_{k_1\,n-1} + \cdots + \alpha_{n-1} e_{k_{n-1}\,1} \\ z_{2k-1\,n} &= d_n + \alpha_1 e_{k_1\,n-1} + \cdots + \alpha_{n-1} e_{k_{n-1}\,1}. \end{aligned} \tag{2.3.27}$$

In time series analysis such models are called *moving averages.* Consider the following special examples.

Example 2.3.5. 1^{st} ORDER MOVING AVERAGES MODEL. In (2.3.26) assume the dependence only on the nearest past and put $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_{n-1} = 0$. Fix again the current stock price $S_0 = s$ and at the first trading time t_1 assume the transition (2.3.6) with $s_{21} = su_1$ and $s_{11} = sd_1$ (like in the nonhomogeneous binomial model). Moreover, for $n = 2, \dots, N$ assume

$$s_{k\,n-1}\begin{matrix} \nearrow s_{k\,n-1}(\alpha u_{n-1} + u_n) \\ \searrow s_{k\,n-1}(\alpha u_{n-1} + d_n) \end{matrix} \quad \text{if } k \text{ is even}$$

and

$$s_{k\,n-1}\begin{matrix} \nearrow s_{k\,n-1}(\alpha d_{n-1} + u_n) \\ \searrow s_{k\,n-1}(\alpha d_{n-1} + d_n) \end{matrix} \quad \text{if } k \text{ is odd.}$$

As in the binomial case (see Example 2.3.2), it is useful to describe the present 1^{st} order moving averages model via its gross returns, since the variables $Z_n = S_n/S_{n-1}$ are quite simple: $Z_1 = \epsilon_1$ and

$$Z_n = \alpha\epsilon_{n-1} + \epsilon_n \tag{2.3.28}$$

for $n = 2, 3, \dots, N$, according to (2.3.26). This means that depending on the state of the stock price S_n at the trading time t_n, the gross return Z_n takes on only one of 4 numeric values:

$$\begin{aligned}
&\alpha u_{n-1} + u_n \text{ if } S_n \text{ is in state } s_{4j+4\,n}\\
&\alpha u_{n-1} + d_n \text{ if } S_n \text{ is in state } s_{4j+3\,n}\\
&\alpha d_{n-1} + u_n \text{ if } S_n \text{ is in state } s_{4j+2\,n}\\
&\alpha d_{n-1} + d_n \text{ if } S_n \text{ is in state } s_{4j+1\,n}
\end{aligned}$$

for all $j = 0, 1, \dots, 2^{n-2} - 1$. Note the properties (2.3.21) and (2.3.22) for the present model.

Example 2.3.6. 1^{st} ORDER AUTOREGRESSIVE MODEL. Consider another special case of the model (2.3.26) with $\alpha_k = \alpha^k$ for some parameter α. This model is called autoregressive because the sequence of gross returns $\{Z_n\}_{n=1,\dots,N}$ satisfies the following difference equations: $Z_1 = \epsilon_1$ and for $n = 2, 3, \dots, N$

$$Z_n = \alpha Z_{n-1} + \epsilon_n. \tag{2.3.29}$$

Chapter 3

Hedging and Options Valuation

3.1 Introduction

In this chapter all notions regarding trading in a market are discussed and properly formalized. The primary notions are an investor's *portfolio* and its market value at each instant $t \in [0, T]$. The entire process of selecting portfolio is called the investor's *trading strategy.* With each trading strategy we associate a so-called *value process* that represents the development in time of the market value of the investor's holding.

The first mathematical result is obtained by an elementary algebra, see Proposition 3.2.1. It takes advantage of the difference operator in space D introduced in Section 2.2.1 and tells us that the stock component of the portfolio is a special predictable process representable in the form of the ratio (3.2.9). This gives rise to the so-called Clark's formula in Section 3.2.2, the integral representation (3.2.22) of the value process for self-financing trading strategies. Recall the definition of the self-financing property. It is said that the investor's strategy is self-financing if at the current date $t = 0$ he buys some shares of the bond and stock and afterwards readjusts his portfolio using exclusively capital gains from the trading, without infusion or withdrawal of funds. This is easily formalized in the present setting of the discrete time, see Definition 3.2.2, as the increments in time of price processes have clear meaning. In the continuous time models, however, a little bit more precaution is needed. Though standard arguments like in [35], [39], [47] – [49] etc. extend straightforwardly the considerations of Section 3.2.1, as is seen in Part II.

The states of the discounted stock price process can always be arranged into the sequence of recurrence identities of the same form as we have had in Section 2.2.2, cf (2.2.22). This is just a matter of suitable notations for the coefficients p_{kn}, cf (3.3.1) – (3.3.4). In terms of the general theory of stochastic processes, this is called the martingale property of the discounted stock price process with respect to a special probability distribution generated by the transition probabilities p_{kn} (cf Section 2.2.2, the second subsection on random walks where these probabilistic terms are introduced). Due to this connection, it became customary to attribute to p_{kn} the term *risk neutral probabilities.* We also make use of this term, but with certain reservation because up to the present point nothing restricts these numbers to be negative. It will be seen in Section 3.6 closing this chapter that some additional conditions has to be required from the asset price processes to enforce strictly positive p's. It is quite remarkable that this requirement has clear economical meaning as it excludes arbitrage opportunities that are special self-financing trading strategies of making profit without any investment. Therefore since [38] and [39] a lot of work is going on refining this statement in various settings and at various degree of abstraction, see e.g. the recent books [33] and [66] where further references are provided.

There is of course nothing deep-lying about the identity (3.3.4), except that this "martingale property" of the discounted stock price process is inherited by all value processes for the self-financing trading strategies, hence we have the parallel relations (3.3.14). The probabilistic counterpart of Proposition 3.3.4 is that the integral transforms preserve the martingale property, cf e.g. [65], Theorem 7.1.1.

From Section 2.2.2 we already know how to solve the backward recurrence equations subject to any boundary condition at the terminal date T, if we want to find out how much has to be invested now at $t = 0$ in order to attain at $t = T$ a certain desired wealth. Since a unique solution always exists, we conclude in Section 3.4.1 that our binary market is *complete* in the sense that any desired wealth is attainable starting from a certain initial endowment and choosing for a suitable self-financing trading strategy. See Section 3.4.2 for the explicit construction of such strategy called the *hedging strategy* against given desired wealth.

Suppose now that in a binary market *derivative securities* or *contingent claims* are exercised in the form of financial contracts written on the stock. Their values are derived from, or contingent on the stock price development. One of such derivatives is a *call option* that is a contract that entitles its owner with the right to hold some shares of stock (or to sell shares in case of a put option, see Section 3.5, the subsection on call-put parity) by a certain date called the *expiration time* or *maturity.* For the sake of simplicity we discuss exclusively the so-called *European call option* that is only allowed to be exercised at maturity, leaving aside e.g. the *American call option* that can be exercised at any date prior to maturity, or some other exotic options, for their pricing would require much more efforts. An interested reader may consult the recent introductory books [44], [78], [79], [80] or [81]. A contingent claim provides a positive payoff function at maturity. The European call option, for instance, provides the payoff (3.5.1). Pricing a contingent claim is to determine its market value at any date prior to maturity, in particular, at the current date $t = 0$. It is explained in Section 3.5 how to handle this task and how to find the current fair price by relating the problem to the construction of the hedging strategy against the contingent claim in question, the strategy that duplicates the payoff. The initial wealth needed for the construction is then taken for the fair price of the contingent claim. See Section 3.5.2 for concrete examples of pricing the European call option, in particular the well-known Cox-Ross-Rubinstein pricing formula for the binomial model, cf [13], [14] or [53].

This chapter is organized as follows. In Section 3.2 the portfolio and value processes are treated and the self-financing strategies are characterized. In Section 3.3 the recurrence relations for the value processes are discussed. In Section 3.4 the completeness of a binary market is defined and the explicit procedure is provided for the construction of the hedging strategies. The result is used in Section 3.5 for pricing contingent claims, in particular the European options. Finally, in Section 3.6 the class of binary markets excluding arbitrage opportunities is characterized.

3.2 Portfolio and value process

3.2.1 Self-financing strategies

Portfolio. Suppose that one invests an amount $v \geq 0$ in the two assets described in Section 2.3.1. Let Ψ_0 and Φ_0 denote the numbers of shares of the bond and stock, respectively, owned by the investor at the current date $t_0 = 0$. The asset prices at the current date were fixed to $B_0 = 1$ and $S_0 = s$, so that the investment amounts to

$$v \doteq \Psi_0 + s\Phi_0. \tag{3.2.1}$$

Furthermore, let Ψ_n and Φ_n denote the number of shares of the bond and stock, respectively, owned by the investor at the consecutive trading times t_n, $n = 1, \dots, N$. The couple (Ψ_n, Φ_n) is called the investor's *portfolio* at time t_n. Observe that the components Ψ_n and Φ_n of a portfolio may become negative, which has to be interpreted as short-selling the bond or the stock. Since the investor selects his portfolio at time t_n with $n = 1, \dots, N$ on the basis of the history of the price development in the market, the number of shares Ψ_n and Φ_n of the bond and stock he owns at time t_n may depend on prices B_ν and S_ν with $\nu < n$, but not on prices not yet announced, e.g. B_n and S_n. In particular

$$(\Psi_0, \Phi_0) = (\Psi_1, \Phi_1), \tag{3.2.2}$$

which means that the currently selected portfolio is kept unchanged during the whole first period $[t_0, t_1]$. Afterwards, just after the stock price S_1 is announced at time t_1 the portfolio turns into (Ψ_2, Φ_2) and stays unchanged during the whole period $(t_1, t_2]$. The investor proceeds further in the same manner, by selecting last time his portfolio (Ψ_N, Φ_N) just after the announcement of the stock price S_{N-1} at time t_1 and keeping it until the terminal date $t_N = T$.

The entire process $\pi = (\Psi_n, \Phi_n)_{n=0,1,\dots,N}$ of selecting the portfolio components is called a *trading strategy.* With each trading strategy π we associate the process

$$V(\pi) = \{V_n(\pi)\}_{n=0,1,\dots,N}$$

by

$$V_n(\pi) \doteq \Psi_n B_n + \Phi_n S_n. \tag{3.2.3}$$

In view of (3.2.1) $V_0(\pi) = v \geq 0$. After discounting, i.e. dividing both sides by B_n, we get

$$\grave{V}_n(\pi) \doteq \Psi_n + \Phi_n \grave{S}_n. \tag{3.2.4}$$

The process $V(\pi)$ is usually called the *value process* for a trading strategy π, since $V_n(\pi)$ represents the market value of the portfolio at time t_n held just before any changes are made in the portfolio.

As was observed above, a portfolio (Ψ_n, Φ_n) depends only on the history of the price development before the trading time t_n. In terms of Section 2.2.1, this means that both components $\Psi = \{\Psi_n\}_{n=0,1,\dots,N}$ and $\Phi = \{\Phi_n\}_{n=0,1,\dots,N}$ of the trading strategy π are predictable. That is, both sequences are of the same type as S_-. So, if S_n is in state s_{kn} for some $k \in \{1, \dots, 2^n\}$, then Ψ_n and Φ_n are in the states $\psi_{k_1\, n-1}$ and $\phi_{k_1\, n-1}$, respectively. As usual $k_1 = \lceil k/2 \rceil$. According to (3.2.4), the investor's discounted wealth at time t_n is then in state

$$\grave{v}_{kn}(\pi) = \psi_{k_1\, n-1} + \phi_{k_1\, n-1}\grave{s}_{kn}. \tag{3.2.5}$$

Recall that also $DS = \{DS_n\}_{n=1,\dots,N}$ and $DV(\pi) = \{DV_n(\pi)\}_{n=1,\dots,N}$ are predictable processes and the knowledge that S_{n-1} has been in state $s_{k\, n-1}$ allows to predict the states of DS_n and $DV_n(\pi)$ that are

$$\begin{aligned} D_k(S_n) &= s_{2k\, n} - s_{2k-1\, n} \\ D_k(V_n(\pi)) &= v_{2k\, n}(\pi) - v_{2k-1\, n}(\pi), \end{aligned} \tag{3.2.6}$$

respectively. Anticipating (3.2.8) and (3.2.9), we put $DS_0 = DS_1$ and $DV_0(\pi) = DV_1(\pi)$ to meet (3.2.2). The assertion of the following proposition is an elementary consequence of (2.2.18) applied to (3.2.3), and (3.2.4) for the predictability of the portfolio components implies

$$DV(\pi) = \Phi DS \quad \text{and} \quad D\grave{V}(\pi) = \Phi D\grave{S}. \tag{3.2.7}$$

To refresh the arguments, however, we will provide the detailed proof.

Proposition 3.2.1. *The components of the portfolio (Ψ_n, Φ_n) at trading time t_n with $n = 0, 1, \dots, N$ are given by*

$$\Psi_n B_n = \frac{V_n(\pi) DS_n - S_n DV_n(\pi)}{DS_n} \tag{3.2.8}$$

and

$$\Phi_n = \frac{DV_n(\pi)}{DS_n}. \tag{3.2.9}$$

Proof. Fix $n \in \{1, \dots, N\}$. At the trading time t_{n-1} let S_{n-1} be in state $s_{k\, n-1}$ with some $k \in \{1, \dots, 2^{n-1}\}$. By (3.2.5) the investor's discounted wealth at the next trading time t_n may then be in one of the following two alternative states:

$$\begin{aligned} \grave{v}_{2k\, n}(\pi) &= \psi_{k\, n-1} + \phi_{k\, n-1}\grave{s}_{2k\, n} \\ \grave{v}_{2k-1\, n}(\pi) &= \psi_{k\, n-1} + \phi_{k\, n-1}\grave{s}_{2k-1\, n}. \end{aligned}$$

By solving the system of these two equations with respect to $\psi_{k\, n-1}$ and $\phi_{k\, n-1}$, we get

$$\psi_{k\, n-1} = \frac{\grave{v}_{2k-1\, n}(\pi)\grave{s}_{2k\, n} - \grave{v}_{2k\, n}(\pi)\grave{s}_{2k-1\, n}}{\grave{s}_{2k\, n} - \grave{s}_{2k-1\, n}} \tag{3.2.10}$$

and

$$\phi_{k\, n-1} = \frac{\grave{v}_{2k\, n}(\pi) - \grave{v}_{2k-1\, n}(\pi)}{\grave{s}_{2k\, n} - \grave{s}_{2k-1\, n}}. \tag{3.2.11}$$

In non-discounted form we have

$$\psi_{k\,n-1}B_n = \frac{v_{2k-1\,n}(\pi)s_{2k\,n} - v_{2k\,n}(\pi)s_{2k-1\,n}}{s_{2k\,n} - s_{2k-1\,n}}$$

and

$$\phi_{k\,n-1} = \frac{v_{2k\,n}(\pi) - v_{2k-1\,n}(\pi)}{s_{2k\,n} - s_{2k-1\,n}}.$$

Due to (3.2.6), this is equivalent to (3.2.8) and (3.2.9). The proof is complete. □

Self-financing. It will be shown next that the value process $V(\pi)$ is of a special structure when π belongs to the class of so-called *self-financing* strategies defined as follows:

A trading strategy π is said to be self-financing, if its construction is founded only on the initial endowment so that all changes in the portfolio values are due to capital gains during the trading and no infusion or withdrawal of funds is required.

To formalize this definition, we may argue as follows. We have already defined by (3.2.3) the market value $V_n(\pi)$ at the trading time t_n of the portfolio (Ψ_n, Φ_n). Next, the market value of the consecutive portfolio (Ψ_{n+1}, Φ_{n+1}) just after it has been selected at the trading time t_n may be evaluated by $\Psi_{n+1}B_n + \Phi_{n+1}S_n$. But this value can't differ from the previous $V_n(\pi)$, since no infusion or withdrawal of funds took place. Thus, apart from (3.2.3) the following relation holds:

$$V_n(\pi) = \Psi_{n+1}B_n + \Phi_{n+1}S_n \tag{3.2.12}$$

or, in the discounted form,

$$\grave{V}_n(\pi) = \Psi_{n+1} + \Phi_{n+1}\grave{S}_n. \tag{3.2.13}$$

These considerations allow us to equate the right hand sides in (3.2.3) and (3.2.12), as well as in (3.2.4) and (3.2.13). We get

$$\Psi_n B_n + \Phi_n S_n = \Psi_{n+1}B_n + \Phi_{n+1}S_n$$

and

$$\Psi_n + \Phi_n\grave{S}_n = \Psi_{n+1} + \Phi_{n+1}\grave{S}_n.$$

It is easily seen that these identities are equivalent to (3.2.14) and (3.2.15) below, hence we have the following formal definition.

Definition 3.2.2. *A trading strategy π is said to be self-financing, if at each trading time t_n with $n = 1, \dots, N$ the corresponding portfolio satisfies the condition*

$$B_{n-1}\Delta\Psi_n + S_{n-1}\Delta\Phi_n = 0 \tag{3.2.14}$$

or, in the discounted form,

$$\Delta\Psi_n + \grave{S}_{n-1}\Delta\Phi_n = 0. \tag{3.2.15}$$

The self-financing is a basic notion in the forthcoming sections as well. Let us therefore make it universally applicable by rewriting (3.2.14) and (3.2.15) in an integral form. To this end, we sum up both sides of these equations and use the notations

$$B_- \cdot \Psi_n \doteq \sum_{\nu=1}^{n} B_{\nu-1}\Delta\Psi_\nu \quad \text{and} \quad S_- \cdot \Phi_n \doteq \sum_{\nu=1}^{n} S_{\nu-1}\Delta\Phi_\nu$$

like in Section 2.2.1, formula (2.2.5). We thus get the following equivalent conditions

$$B_- \cdot \Psi_n + S_- \cdot \Phi_n = 0 \tag{3.2.16}$$

or, in the discounted form,

$$\Psi_n - \Psi_0 + \grave{S}_- \cdot \Phi_n = 0. \tag{3.2.17}$$

With the notations $k_1 = \lceil k/2 \rceil$ and $k_2 = \lceil k/4 \rceil$, the self-financing condition (3.2.15) means that

$$\psi_{k_1\, n-1} - \psi_{k_2\, n-1} = -\grave{s}_{k_1\, n-1}(\phi_{k_1\, n-1} - \phi_{k_2\, n-1}).$$

We occasionally will use the term *admissibility* of a trading strategy, meaning the following

Definition 3.2.3. *If the investor follows a self-financing strategy π so that his wealth remains nonnegative, i.e. $V_n(\pi) \geq 0$ at each trading time t_n with $n = 0, 1, \dots, N$, then it is said that the strategy π is admissible.*

3.2.2 Integral representation

The term *predictability* of portfolio components is borrowed from the theory of stochastic calculus, where predictable processes play the rôle of integrands in stochastic integrals. The reader familiar with this theory, as well as with the theory of continuous trading in the spirit of, e.g. [39], Section 3, could trace the analogy of the representation (3.2.19) below and the integral representation of the discounted value process for a self-financing strategy as a stochastic integral with respect to the discounted stock price process. Later we will return to this, see Chapter 4, Proposition 4.4.2, and Chapter 5, Formula (5.5.8). But meanwhile the situation is quite simple as we will only deal in the present chapter with the transform of $\grave{S}$ by Φ which is

$$\Phi \cdot \grave{S}_n \doteq \sum_{\nu=1}^{n} \Phi_\nu \Delta\grave{S}_\nu, \tag{3.2.18}$$

as was already defined in Section 2.2.1.

Proposition 3.2.4. *Let π be a self-financing strategy. Then the corresponding value process $V(\pi) = \{V_n(\pi)\}_{n=0,1,\dots,N}$ with the initial endowment $v = V_0(\pi) \geq 0$ has the following representation: for all $n = 1, \dots, N$*

$$\grave{V}_n(\pi) = v + \Phi \cdot \grave{S}_n, \tag{3.2.19}$$

cf (3.2.18).

Proof. Obviously, it suffices to prove that for $n = 1, \dots, N$

$$\Delta\grave{V}_n(\pi) = \Phi_n \Delta\grave{S}_n. \tag{3.2.20}$$

But this is easily seen since by applying (2.2.9) to (3.2.4) we obtain

$$\Delta\grave{V}_n(\pi) = \Delta\Psi_n + \Delta(\Phi_n\grave{S}_n) = \Delta\Psi_n + \Delta\Phi_n\grave{S}_{n-1} + \Phi_n\Delta\grave{S}_n,$$

which equals to $\Phi_n\Delta\grave{S}_n$ due to equation (3.2.15). □

As was mentioned in Section 2.2.1, cf (2.2.4), the summation in (3.2.19) is carried out path by path, which means that if for a fixed $n \in \{1, \dots, N\}$ we deal with S_n that occupies the state s_{kn} with some $k \in \{1, \dots, 2^n\}$, then investor's discounted wealth at time t_n is in state

$$\grave{v}_{kn}(\pi) = v + \sum_{\nu=1}^{n} \phi_{k_{n-\nu+1}\,\nu-1}(\grave{s}_{k_{n-\nu}\,\nu} - \grave{s}_{k_{n-\nu+1}\,\nu-1}). \tag{3.2.21}$$

Clark's formula. We want to emphasize further analogy with the theory of continuous trading. Within this theory the discounted value process is represented as a stochastic integral with respect to the discounted stock price process, with the integrand - the stock component of the portfolio - that is of a special form, namely given by so-called Clark's formula. See [39], Formula (1.9), or [59]. As is seen in Corollary 3.2.5 to Proposition 3.2.1 below, the analogous formula in the case of a binary market is quite elementary. It is based on the simple usage of the difference operator in the state space as defined in Section 2.2.1. For further references on the analogous case of continuous trading where certain Malliavin derivatives occur (functional derivatives in the state space), see [58], [56], Section 1.3.3, [57], Section 1.6, or [49], Appendix E. Cf also Chapter 4, Proposition 4.3.7, and Chapter 5, Remark 5.5.4. In virtue of Propositions 3.2.4 and 3.2.1 we obtain

Corollary 3.2.5. *Under the self-financing condition (3.2.14) we have for $n = 1, \dots, N$*

$$\grave{V}_n(\pi) = v + \frac{DV(\pi)}{DS} \cdot \grave{S}_n \tag{3.2.22}$$

and

$$\frac{\Delta\grave{V}_n(\pi)}{\Delta\grave{S}_n} = \frac{D\grave{V}_n(\pi)}{D\grave{S}_n}. \tag{3.2.23}$$

Proof. Substitute the integrand (3.2.9) in (3.2.19) to get (3.2.22). Apply the difference operator in time Δ to both sides of (3.2.22). We get

$$\Delta \grave{V}_n(\pi) = \frac{DV_n(\pi)}{DS_n} \Delta \grave{S}_n.$$

But this coincides with (3.2.23), since $DV/dS = D\grave{V}/D\grave{S}$ by (3.2.7). $\square$

3.3 Recurrence relations

3.3.1 Risk neutral probabilities

Fix a trading time t_n for some $n = 1, \ldots, N$ and consider the particular branch of the discounted price tree for the stock

$$\grave{s}_{k\,n-1}\begin{array}{l} \nearrow \grave{s}_{2k\,n} \\ \searrow \grave{s}_{2k-1\,n} \end{array}$$

for some $k = 1, \ldots, 2^{n-1}$. It is easy to express $\grave{s}_{k\,n-1}$ as a linear combination of the future states $\grave{s}_{2k\,n}$ and $\grave{s}_{2k-1\,n}$ by solving the equation

$$\grave{s}_{k\,n-1} = x\grave{s}_{2k\,n} + (1-x)\grave{s}_{2k-1\,n}$$

with respect to the unknown x. The solution is

$$x = \frac{\grave{s}_{k\,n-1} - \grave{s}_{2k-1\,n}}{\grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}}.$$

In order to exhibit the dependence of this solution on the time and state indices, we use throughout the following notations:

$$p_{2k\,n} = \frac{\grave{s}_{k\,n-1} - \grave{s}_{2k-1\,n}}{\grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}} \tag{3.3.1}$$

and

$$p_{2k-1\,n} = \frac{\grave{s}_{2k\,n} - \grave{s}_{k\,n-1}}{\grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}}. \tag{3.3.2}$$

So, for any trading time t_n and any state of $\grave{S}_n$, the numerical values of $p_{2k\,n}$ and $p_{2k-1\,n}$ satisfy

$$p_{2k\,n} + p_{2k-1\,n} = 1. \tag{3.3.3}$$

With these notations we get for all $n = 1, \ldots, N$ that

$$\grave{s}_{k\,n-1} = p_{2k\,n}\grave{s}_{2k\,n} + p_{2k-1\,n}\grave{s}_{2k-1\,n}, \quad k = 1, \ldots, 2^{n-1}. \tag{3.3.4}$$

Observe that negative values of p_{kn} are not excluded, since it may happen that either $\grave{s}_{k\,n-1} \leq \grave{s}_{2k-1\,n}$ or $\grave{s}_{k\,n-1} \geq \grave{s}_{2k\,n}$. Later, in Section 3.6, we will treat separately a class of markets in which both of these possibilities are excluded. We will see that this is justified by certain arguments having a clear economical meaning. Within the latter class of markets, the numerical values of p_{kn} are positive, satisfying (3.3.3). This makes possible then to give them the same interpretation as in the special subsection of Section 2.2.2 concerning random walk. Recall that p_{kn} have been interpreted as the transition probabilities. By this reason, we will permit ourself the liberty to associate the term *probability* with the numerical values of p_{kn} even beyond the situation of Section 3.6, although it makes no sense to think of negative probabilities, of course. Furthermore, following the established tradition in the present field, the numerical values of p_{kn} will be referred to as the *risk neutral probabilities.* This may be justified as follows. In terms of the states of the discounted net returns $\grave{r}_{kn}$ the equation (3.3.4) is equivalent to

$$0 = p_{2k\,n}\grave{r}_{2k\,n} + p_{2k-1\,n}\grave{r}_{2k-1\,n} \tag{3.3.5}$$

(for the proof, see Lemma 3.3.1 below) which means that the weights (3.3.1) and (3.3.2) with property (3.3.3) are chosen so as to neutralize the upward displacements in the discounted net returns on the stock by the downward displacements. The reader familiar with the theory of stochastic processes (e.g. to the extend of the introductory Section 7.1 in [65]) should notice that the discounted net returns on the stock with property (3.3.5) constitute *martingale differences* with respect to the probability distribution generated by the collection of the transition probabilities p_{kn}, and that the discounted stock price process with property (3.3.4) constitutes a *martingale.* As $p_{2k\,n}$ and $p_{2k-1\,n}$ are the conditional probabilities of the upward and downward displacements, respectively, given that the stock price S_{n-1} has been in state $s_{k\,n-1}$, the right hand sides both in (3.3.4) and (3.3.5) are understood as the corresponding *expectations.* We are not in position to enter here in more details on these notions, as this would require a probabilistic buildup and would carry us far afield.

Instead, we want to stress an additional characteristic feature of the quantities $p_{2k\,n}$ and $p_{2k-1\,n}$. To this end, rewrite (3.3.1) and (3.3.2) in the form

$$\begin{aligned} \grave{s}_{2k-1\,n} - \grave{s}_{k\,n-1} &= -p_{2k\,n}D_k(\grave{S}_n) \\ \grave{s}_{2k\,n} - \grave{s}_{k\,n-1} &= p_{2k-1\,n}D_k(\grave{S}_n) \end{aligned} \tag{3.3.6}$$

with $D_k(\grave{S}_n) = \grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}$ as usual. So we see that $-p_{2k\,n}$ and $p_{2k-1\,n}$ serve as the proportionality coefficients relating the increments in time of the discounted states on the left-hand side of the identities (3.3.6) to the corresponding stretch in space of states on the right.

We now give the proof of the identities (3.3.5) for each $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$. Note first that in terms of the states of the gross returns on the stock, both (3.3.4) and (3.3.5) are equivalent to

$$r_n = p_{2k\,n}z_{2k\,n} + p_{2k-1\,n}z_{2k-1\,n} \tag{3.3.7}$$

where r_n is one plus the interest rate as in (2.3.2) and $\{z_{kn}\}_{k=1,\dots,2^n}$ are the states of Z_n as in (2.3.14). Besides, by using

$$1 + \grave{r}_{kn} = \grave{z}_{kn} = \frac{z_{kn}}{r_n},$$

cf (2.3.15), we may rewrite (3.3.1) and (3.3.2) in the following alternative form:

$$p_{2k\,n} = \frac{r_n - z_{2k-1\,n}}{z_{2k\,n} - z_{2k-1\,n}} \tag{3.3.8}$$

and

$$p_{2k-1\,n} = \frac{z_{2k\,n} - r_n}{z_{2k\,n} - z_{2k-1\,n}}. \tag{3.3.9}$$

Lemma 3.3.1. *Under the self-financing condition (3.2.14) the states of the discounted net returns on the stock satisfy the identities (3.3.5) for all $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$.*

Proof. By (3.3.8) and (3.3.9) we get

$$p_{2k\,n} z_{2k\,n} + p_{2k-1\,n} z_{2k-1\,n} = \frac{(r_n - z_{2k-1\,n}) z_{2k\,n} + (z_{2k\,n} - r_n) z_{2k-1\,n}}{z_{2k\,n} - z_{2k-1\,n}}$$

which equals to r_n. □

Examples. In the moving averages model the states $\{z_{kn}\}_{k=1,\dots,2^n}$ of the gross return on the stock Z_n, defined by (2.3.14), are specified by (2.3.27) so that (3.3.8) and (3.3.9) yield

$$p_{2k\,n} = \frac{r_n - d_n - \sum_{\nu=1}^{n-1} \alpha_{n-\nu} e_{k_{\nu+1}\,\nu}}{u_n - d_n} \tag{3.3.10}$$

and

$$p_{2k-1\,n} = \frac{u_n - r_n + \sum_{\nu=1}^{n-1} \alpha_{n-\nu} e_{k_{\nu+1}\,\nu}}{u_n - d_n} \tag{3.3.11}$$

for each $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$. Consider two special examples.

Example 3.3.2. BINOMIAL MODEL. For each $n = 1, \dots, N$ substitute $\alpha_1 = \dots = \alpha_{n-1} = 0$ in (3.3.10) and (3.3.11). In this special case we see that $p_{2k\,n}$ and $p_{2k-1\,n}$ take on the same numerical value for all $k = 1, \dots, 2^{n-1}$, namely

$$p_{2k\,n} = \frac{r_n - d_n}{u_n - d_n} \quad \text{and} \quad p_{2k-1\,n} = \frac{u_n - r_n}{u_n - d_n}. \tag{3.3.12}$$

Example 3.3.3. 1st ORDER MOVING AVERAGES MODEL. For every $n = 1, \dots, N$ put $\alpha_1 = \alpha$ and $\alpha_2 = \dots = \alpha_{n-1} = 0$ in (3.3.10) and (3.3.11). We obtain

$$\begin{aligned} p_{2k\,n} &= \frac{r_n - d_n - \alpha e_{k\,n-1}}{u_n - d_n} \\ = p_{2k-1\,n} &= \frac{u_n - r_n + \alpha e_{k\,n-1}}{u_n - d_n} \end{aligned} \tag{3.3.13}$$

for all $k = 1, \dots, 2^{n-1}$. So, these equations yield

$$p_{4k\,n} = \frac{r_n - d_n - \alpha u_{n-1}}{u_n - d_n} \quad \text{and} \quad p_{4k-1\,n} = \frac{u_n - r_n + \alpha u_{n-1}}{u_n - d_n}$$

for even indices, and

$$p_{4k-2\,n} = \frac{r_n - d_n - \alpha d_{n-1}}{u_n - d_n} \quad \text{and} \quad p_{4k-3\,n} = \frac{u_n - r_n + \alpha d_{n-1}}{u_n - d_n}$$

for odd indices, where $k = 1, \dots, 2^{n-2}$.

3.3.2 Recurrence relations for value processes

It is well-known in the theory of stochastic processes that any transform of a martingale by a predictable integrand is again a martingale, see e.g. [65], Theorem 7.1.1. As was already seen in Section 3.2.2, Proposition 3.2.4, under the self-financing strategy the value process is a such transform of the discounted stock price process by the stock component of the portfolio. Hence (3.2.19) is a martingale transform. Let us now recall the remark in Section 3.3.1 about the relationship between the equations (3.3.4) and the martingale property of the discounted stock price process. This suggests us that the states of the discounted value process do satisfy equations of form similar to (3.3.4). This is indeed easy to confirm.

Proposition 3.3.4. *In a binary market a trading strategy π is self-financing if and only if for all $n \in \{1, \ldots, N\}$ the states $\{\grave{v}_{kn}(\pi)\}_{k=1,\ldots,2^n}$ of the discounted value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\ldots,N}$ at the trading time t_n and the states $\{\grave{v}_{k\,n-1}(\pi)\}_{k=1,\ldots,2^{n-1}}$ at the previous trading time t_{n-1} are related by the equations*

$$\grave{v}_{k\,n-1}(\pi) = p_{2k\,n}\grave{v}_{2k\,n}(\pi) + p_{2k-1\,n}\grave{v}_{2k-1\,n}(\pi) \tag{3.3.14}$$

where $k = 1, \ldots, 2^{n-1}$. The weights $p_{2k\,n}$ and $p_{2k-1\,n}$ are given by (3.3.1) and (3.3.2).

Proof. (i) Assume self-financing of the trading strategy π. Then the corresponding discounted value process satisfies (3.2.13), i.e. its states satisfy

$$\grave{v}_{k\,n-1}(\pi) = \psi_{k\,n-1} + \phi_{k\,n-1}\grave{s}_{k\,n-1} \tag{3.3.15}$$

for $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$. These equations and (3.2.10)–(3.2.11) yield

$$\begin{aligned} \grave{v}_{k\,n-1}(\pi) &= \frac{\grave{v}_{2k-1\,n}(\pi)\grave{s}_{2k\,n} - \grave{v}_{2k\,n}(\pi)\grave{s}_{2k-1\,n}}{\grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}} \\ &+ \frac{\grave{v}_{2k\,n}(\pi) - \grave{v}_{2k-1\,n}(\pi)}{\grave{s}_{2k\,n} - \grave{s}_{2k-1\,n}}\grave{s}_{k\,n-1}. \end{aligned} \tag{3.3.16}$$

It can be easily verified that the coefficients of $\grave{v}_{2k\,n}(\pi)$ and $\grave{v}_{2k-1\,n}(\pi)$ are indeed given by (3.3.1) and (3.3.2).

(ii) Conversely, (3.3.14), (3.3.1) and (3.3.2) imply (3.3.16), and hence (3.3.15). By Definition (3.2.3), (3.3.15) implies (3.2.5), i.e. the strategy in question is indeed self-financing. □

Similarly to (3.3.6), we thus have from (3.3.14) the following relationship between the increments in time and in space of the discounted states of the value process:

$$\begin{aligned} \grave{v}_{2k-1\,n}(\pi) - \grave{v}_{k\,n-1}(\pi) &= -p_{2k\,n}D_k(\grave{V}_n(\pi)) \\ \grave{v}_{2k\,n}(\pi) - \grave{v}_{k\,n-1}(\pi) &= p_{2k-1\,n}D_k(\grave{V}_n(\pi)). \end{aligned} \tag{3.3.17}$$

This tells us that the previous proportionality coefficients $-p_{2k\,n}$ and $p_{2k-1\,n}$ are preserved.

The equations (3.3.14) are *backward recurrent* in the sense that if the states of the discounted value process are given at the terminal date $t_N = T$, then working backwards one can determine the states at the previous trading times step by step, each time using the system of equations (3.3.14). In Section 2.2.2 we have already discussed the solution of this problem. Indeed the system of equations (3.3.14) takes the form (2.2.22) upon the substitution $x_{jn} = \grave{v}_{jn}(\pi)$. Thus if at the terminal date $t_N = T$ the states of the discounted value process are fixed, say

$$\grave{v}_{jN}(\pi) = c_j, \quad j = 1, \dots, 2^N, \tag{3.3.18}$$

where $\{c_1, \dots, c_{2^N}\}$ is a set of given numbers, cf (2.2.36), then the solutions are given by (2.2.37) and (2.2.38), upon the same substitution $x_{jn} = \grave{v}_{jn}(\pi)$, of course. Regarding their applications in the present setup we want to make following remarks.

Firstly, the weights in (2.2.22) are now specified by (3.3.1) and (3.3.2). Negative numerical values are not excluded. But in the strictly positive case, the numerical values of p_{kn} may be interpreted as transition probabilities, as in the subsection on random walk in Section 2.2.2. Likewise, the strictly positive weights

$$P^n_{kN} = \prod_{0 \le \nu < n} p_{k_\nu\, N-\nu} \tag{3.3.19}$$

in (2.2.28) get the meaning of transition probabilities, and the collection $\{P_{kN}\}_{k=1,}$ of weights in (2.2.29) with

$$P_{kN} = \prod_{0 \le \nu < N} p_{k_\nu\, N-\nu} \tag{3.3.20}$$

the meaning of a probability distribution.

Secondly, we will need in the sequel to specify the boundary conditions (3.3.18) as follows. Define a variable $\grave{H}_N$ which is allowed to occupy one of the 2^N states $\{\grave{h}_{kN}\}_{k=1,\dots,2^N}$ by means of a certain function H (to be called in Section 3.5.2, the *payoff function*) that maps each trajectory of the stock price development to the corresponding states of $\grave{H}_N$. We may portray this mapping from the states of the terminal stock price to that of the payoff as follows:

$$\grave{s}_{jN} \xmapsto{H} \grave{h}_{jN}, \quad j = 1, \dots, 2^N. \tag{3.3.21}$$

Consider then the system of recurrence equations

$$x_{j\,n-1} = p_{2j\,n} x_{2k\,n} + p_{2j-1\,n} x_{2j-1\,n} \tag{3.3.22}$$

for $n = 1, \dots, N$ and $j = 1, \dots, 2^{n-1}$ (cf (2.2.22), as well as (3.3.4) and (3.3.14)), subject to the boundary conditions

$$x_{jN} = \grave{h}_{jN}, \quad j = 1, \dots, 2^N. \tag{3.3.23}$$

As in Section 2.2.2, the solution to this system of equations is obtained by the procedure starting with the substitution $n = N$ in (3.3.22) that determines $\{x_{j\,N-1}\}_{j=1,\dots,2^{N-1}}$. Working backwards in this manner, after n such steps the solution $\{x_{j\,N-n}\}_{j=1,\dots,2^{N-n}}$ is obtained with

$$x_{j\,N-n} = \sum_{2^n(j-1)<k\leq 2^n j} P^n_{kN} \grave{h}_{kN}. \tag{3.3.24}$$

This procedure is terminated after N steps and yields

$$x_{10} = \sum_{k=1}^{2^N} P_{kN} \grave{h}_{kN}. \tag{3.3.25}$$

To grasp the meaning of these solutions in the present context, consider the payoff function $H(x) = x$ to reduce (3.3.21) to the simplest case $\grave{H}_N = \grave{S}_N$ and $\grave{h}_{kN} = \grave{s}_{kN}$. This specifies the boundary conditions (3.3.23) so that upon the substitution $x_{jn} = \grave{s}_{jn}$ the solutions reduce to

$$\grave{s}_{j\,N-n} = \sum_{2^n(j-1)<k\leq 2^n j} P^n_{kN} \grave{s}_{kN} \tag{3.3.26}$$

and, in particular,

$$s = \sum_{k=1}^{2^N} P_{kN} \grave{s}_{kN}. \tag{3.3.27}$$

The reader familiar with the probability theory must recognize in the latter equations yet another characterization, additional to (3.3.4), of the martingale property of the discounted stock process. It is indeed seen from (3.3.27) that the expectation of the terminal discounted stock price $\grave{S}_N$ (relative to the probability distribution $\{P_{kN}\}_{k=1,\dots,2^N}$) equals to the fixed initial value s. Likewise, the sum on the right-hand side of (3.3.26) gets the interpretation of the conditional expectation of the terminal discounted stock price $\grave{S}_N$, given that n periods before the discounted stock price $\grave{S}_{N-n}$ has been in state $\grave{s}_{k\,N-n}$. Consider now another example of the payoff function

$$\grave{s}_{jN} \xmapsto{H} \frac{\grave{s}_{jN}}{\grave{s}_{j_1\,N}} - 1, \quad j = 1,\dots,2^N.$$

In this case $\grave{h}_{kN} = \grave{r}_{kN}$ and we get, using (2.2.34) and (2.2.35) that

$$\sum_{2^n(j-1)<k\leq 2^n j} P^n_{kN} \grave{r}_{kN} = 0 \tag{3.3.28}$$

and, in particular,

$$\sum_{k=1}^{2^N} P_{kN} \grave{r}_{kN} = 0. \tag{3.3.29}$$

This is an alternative to (3.3.5) characterization of the fact that the discounted net returns constitute martingale differences.

Let us turn back to the discounted value process $\grave{V}(\pi)$ for a certain self-financing strategy π. As we know, its integral representation (3.2.19) is a martingale transform. This property of $\grave{V}(\pi)$ is characterized either by (3.3.14) or by the relations

$$\grave{v}_{j\,N-n}(\pi) = \sum_{2^n(j-1)<k\leq 2^n j} P^n_{kN}\grave{v}_{kN}(\pi) \tag{3.3.30}$$

which for $n = N$ are reduced to

$$v = \sum_{k=1}^{2^N} P_{kN}\grave{v}_{kN}(\pi). \tag{3.3.31}$$

Obviously, the relations (3.3.30) are associated with (3.3.14) in the precisely same way as the relations (3.3.26) with (3.3.4).

Examples of solution. It will be necessary for concrete applications to have at hand the explicit solutions to the system of equations (3.3.22) subject to the boundary conditions (3.3.23).

Example 3.3.5. BINOMIAL MODEL: HOMOGENEOUS CASE. For given numbers u, d and r, denote

$$p_u = \frac{r-d}{u-d} \quad \text{and} \quad p_d = \frac{u-r}{u-d}$$

which satisfy $p_u + p_d = 1$. These are the risk neutral probabilities in the present special case, cf (3.3.12). Recall the partition $\Pi_0, \Pi_1, \ldots, \Pi_N$ of the set of all possible terminal states $\{s_{kN}\}_{k=1,\ldots,2^N}$, discussed in Example 2.3.1. If the stock price at the terminal date occupies one of the state from the subset Π_n, then it amounts to $su^n d^{N-n}$. Let us consider now a special payoff function that makes distinction only of prices announced at the terminal date. Namely, let us specify the mapping (3.3.21) as follows: for all $n \in \{0, 1, \ldots, N\}$

$$\grave{h}_{jN} = r^{-N} H(su^n d^{N-n}) \quad \text{if} \quad s_{jN} \in \Pi_n \tag{3.3.32}$$

where H is a certain function of a single argument. This gives to the solutions (3.3.24) and (3.3.25) a special form that may be described in terms of the sequence of functions $\{f_n\}_{n=0,1,\ldots,N}$ with

$$f_n(x) = \sum_{j=0}^{n} \binom{n}{j} p_u^j p_d^{n-j} H(xu^j d^{n-j}).$$

Note that $f_0(x) = H(x)$. With these notations (3.3.24) and (3.3.25) are reduced to

$$x_{j\,N-n} = r^{-N} f_n(s_{j\,N-n}), \quad j = 1, \ldots, 2^{N-n}, \tag{3.3.33}$$

and

$$x_{10} = r^{-N} f_N(s). \tag{3.3.34}$$

Thus, if X_n is a variable with the set of states $\{x_{kn}\}_{k=1,\dots,2^n}$, then

$$X_{N-n} = r^{-N} f_n(S_{N-n}).$$

Example 3.3.6. BINOMIAL MODEL: NONHOMOGENEOUS CASE. For given numbers u_n, d_n and r_n with $n = 1, \dots, N$ denote

$$p_{u_n} = \frac{r_n - d_n}{u_n - d_n} \quad \text{and} \quad p_{d_n} = \frac{u_n - r_n}{u_n - d_n}$$

which satisfy $p_{u_n} + p_{d_n} = 1$ and present the risk neutral probabilities in the present special case, cf (3.3.12). Alternatively to Example 2.3.2, it turns out useful to describe the situation by means of N binary indices $\iota_1, \dots, \iota_N$ with ι_n taking on either the value d_n or u_n. For, the summation we will need to carry out in the formulas (3.3.35) and (3.3.36) below (cf also (3.3.37) and (3.3.38)) will extend over these binary indices. Moreover, with their help the set of states for the stock price S_n may be described as the product $\{s\iota_1 \cdots \iota_n\}$ which indeed yields 2^n of appropriate outcomes when the indices vary. A payoff may then be defined by means of a certain function H of a single argument so that

$$s\iota_1 \cdots \iota_N \longmapsto H(s\iota_1 \cdots \iota_N).$$

Define now the sequence of functions $\{f_n\}_{n=1,\dots,N}$ with $f_0(x) = H(x)$ and

$$f_n(x) = \sum_{\iota_{N-n+1} \cdots \iota_N} p_{\iota_{N-n+1}} \cdots p_{\iota_N} H(x\iota_{N-n+1} \cdots \iota_N) \tag{3.3.35}$$

for $n = 1, \dots, N$, where the summation extends over the n binary indices $\iota_{N-n+1}, \dots, \iota_N$. Then the solution (3.3.24) reduces to

$$x_{j\,N-n} = \frac{f_n(s_{j\,N-n})}{r_1 \cdots r_N}.$$

In particular

$$x_{10} = \sum_{\iota_1 \cdots \iota_N} \frac{p_{\iota_1} \cdots p_{\iota_N}}{r_1 \dots r_N} H(s\iota_1 \cdots \iota_N). \tag{3.3.36}$$

Note that if X_n is a variable with the set of states $\{x_{kn}\}_{k=1,\dots,2^n}$, then

$$X_{N-n} = \frac{f_n(S_{N-n})}{r_1 \cdots r_N}.$$

Example 3.3.7. 1^{st} ORDER MOVING AVERAGES MODEL. We retain here the notations of the previous example. Moreover, we denote by

$$p_{u_n|\iota_n} = \frac{r_n - d_n - \alpha\iota_n}{u_n - d_n} \quad \text{and} \quad p_{d_n|\iota_n} = \frac{u_n - r_n + \alpha\iota_n}{u_n - d_n}$$

with $p_{u_n|\iota_n} + p_{d_n|\iota_n} = 1$, the risk neutral probabilities, cf (3.3.13). For $n = 1, \ldots, N$ define the following functions of two variables

$$\begin{aligned} f_n(x,y) = & \sum_{\iota_{N-n+1}\cdots\iota_N} p_{\iota_{N-n+1}|y} p_{\iota_{N-n+2}|\iota_{N-n+1}} \cdots p_{\iota_N|\iota_{N-1}} \times \\ & H(x(\iota_{N-n+1} + \alpha y)(\iota_{N-n+2} + \alpha\iota_{N-n+1}) \cdots (\iota_N + \alpha\iota_{N-1})) \end{aligned} \tag{3.3.37}$$

where H is again a function of a single argument. Then the solution (3.3.24 may be given by

$$x_{2j\,N-n} = \frac{f_n(s_{2j\,N-n}, u_{N-n})}{r_1 \cdots r_N}$$

and

$$x_{2j-1\,N-n} = \frac{f_n(s_{2j-1\,N-n}, d_{N-n})}{r_1 \cdots r_N}$$

for $j = 1, \ldots, 2^{n-1}$. In particular

$$\begin{aligned} x_{10} &= \sum_{\iota_1\cdots\iota_N} \frac{p_{\iota_1} p_{\iota_2|\iota_1} \cdots p_{\iota_N|\iota_{N-1}}}{r_1 \cdots r_N} \times \\ & \quad H(s(\iota_1 + \alpha s)(\iota_2 + \alpha\iota_1) \cdots (\iota_N + \alpha\iota_{N-1})). \end{aligned} \tag{3.3.38}$$

Note that if X_n is a variable with the set of states $\{x_{kn}\}_{k=1,\ldots,2^n}$, then

$$X_{N-n} = \frac{f_n(S_{N-n}, \epsilon_{N-n})}{r_1 \cdots r_N}.$$

3.4 Completeness and hedging strategies

3.4.1 Completeness

In the theory of securities markets the following question arises. Suppose that an investor is willing to invest now (at $t = 0$) in the bond and the stock in order to attain at the terminal date T a certain wealth, say W_T, without infusion or withdrawal of funds. The investor determines the desired wealth W_T so as to respond to all possibilities of the stock price development. Is then this goal attainable? The answer depends on the conditions in the market in question, of course. In this connection the term *completeness* of a market is used, that is defined as follows.

Definition 3.4.1. *A market is called complete if there exists a self-financing trading strategy which attains any desired wealth W_T with a certain initial endowment.*

These strategies bear a special name.

Definition 3.4.2. *A specific trading strategy that at the terminal date T yields the desired wealth W_T is called the hedging strategy against W_T.*

In this chapter we are concerned exclusively with binary markets, so the knowledge of the conditions in a market means the knowledge of all 2^N possibilities of the stock price development up to the terminal date $t_N = T$. By evaluating each of these possibilities, the investor determines then the wealth he desires to attain at the terminal date $t_N = T$. In this way W_N becomes a variable with 2^N possible states $\{w_{kN}\}_{k=1,\ldots,2^N}$ depending on the states of the terminal stock price: if the stock price evolves along the k^{th} trajectory, i.e. S_N is in state s_{kN}, then W_N is in state w_{kN}.

It will be shown in this section that a binary market is complete in the sense of Definition 3.4.1. Moreover, for a fixed W_N the required initial endowment and the hedging self-financing strategy will be determined. To this end, fix a wealth W_N desired at the terminal date $t_N = T$, i.e. fix its states $\{w_{kN}\}_{k=1,\ldots,2^N}$. For the discounted wealth $\grave{W}_N = W_N/B_N$ with the corresponding states $\{\grave{w}_{kN}\}_{k=1,\ldots,2^N}$ where $\grave{w}_{kN} = w_{kN}/B_N$, solve the recurrence equations (3.3.22), subject to the boundary conditions

$$x_{jN} = \grave{w}_{jN}, \quad j = 1, \ldots, 2^N. \tag{3.4.1}$$

We need to apply the formula (3.3.24) with $\grave{h}_{kN}$ substituted by $\grave{w}_{kN}$. Denote the solutions by $\{\grave{w}_{kn}\}_{k=1,\ldots,2^n}$ for $n = 0, 1, \ldots, N-1$. We obtain

$$\grave{w}_{j\,N-n} = \sum_{2^n(j-1)<k\leq 2^n j} P^n_{kN}\grave{w}_{kN}. \tag{3.4.2}$$

In particular

$$w_{10} = \sum_{k=1}^{2^N} P_{kN}\grave{w}_{kN}, \tag{3.4.3}$$

cf (3.3.30) and (3.3.31). Finally, for each $n = 0, 1, \ldots, N$ denote by $\grave{W}_n$ the variable with states $\{\grave{w}_{kn}\}_{k=1,\ldots,2^n}$. We are now in a position to describe a particular trading strategy which attains the desired wealth W_N with the initial endowment (3.4.3).

3.4.2 Hedging strategy

According to Definition 3.4.2, a specific trading strategy π whose value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\ldots,N}$ is such that $\grave{V}_N(\pi) = W_N$ is the hedging strategy against the desired wealth W_N. This strategy is uniquely determined by the following procedure:

- Given the price development of the bond and stock, determine the numerical values of all transition probabilities $\{p_{kn}\}_{k=1,\ldots,2^n}$ for $n = 1, \ldots, N$ by the formulas (3.3.1) and (3.3.2).

- Given the wealth W_N, determine recurrently the numerical values of the $\{\grave{w}_{kn}\}_{k=1,\ldots,2^n}$ for $n = 0, 1, \ldots, N$ by formula (3.4.2), as described above.

- Determine, in particular, the numerical values of the probability distribution $\{P_{kN}\}_{k=1,\ldots,2^N}$ by formula (3.3.20), and then the initial endowment w_{10} by formula (3.4.3).

- Invest currently the amount w_{10} in Ψ_0 and Φ_0 shares of the bond and the stock respectively, where Ψ_0 and Φ_0 are calculated as follows. Calculate

$$\begin{aligned}\psi_{10} &= \frac{\grave{w}_{21}\grave{s}_{21} - \grave{w}_{11}\grave{s}_{11}}{\grave{s}_{21} - \grave{s}_{11}}\\ \phi_{10} &= \frac{\grave{w}_{21} - \grave{w}_{11}}{\grave{s}_{21} - \grave{s}_{11}}\end{aligned}$$

  according to the formulas (3.2.10) and (3.2.11) with $n = k = 1$ and $\grave{v}_{21}(\pi) = \grave{w}_{21}$, $\grave{v}_{11}(\pi) = \grave{w}_{11}$. According to (3.2.2), identify these numeric values of the stock and bond components Ψ_1 and Φ_1 with Ψ_0 and Φ_0, respectively.

- During the first period keep the portfolio unchanged, i.e. keep Ψ_1 shares of the bond and Φ_1 shares of the stock, in order to get the wealth $\grave{V}_1(\pi)$ determined by formula (3.2.3) with $n = 1$, which coincides with $\grave{W}_1$.

- If at the trading time t_1 the stock price s_{21} is announced, then during the second period keep ψ_{21} shares of the bond and ϕ_{21} shares of the stock. These numbers of shares are calculated by

$$\begin{aligned}\psi_{21} &= \frac{\grave{w}_{42}\grave{s}_{42} - \grave{w}_{32}\grave{s}_{32}}{\grave{s}_{42} - \grave{s}_{32}}\\ \phi_{21} &= \frac{\grave{w}_{42} - \grave{w}_{32}}{\grave{s}_{42} - \grave{s}_{32}}\end{aligned}$$

  according to the formulas (3.2.10) and (3.2.11) with $n = 2, k = 2, \grave{v}_{42}(\pi) = \grave{w}_{42}$ and $\grave{v}_{32}(\pi) = \grave{w}_{32}$.

However, if the announced stock price is s_{11}, then keep ψ_{11} shares of the bond and ϕ_{11} shares of the stock, again determined by the formulas (3.2.10) and (3.2.11) but now with $n = 2, k = 1, \grave{v}_{22}(\pi) = \grave{w}_{22}$ and $\grave{v}_{12}(\pi) = \grave{w}_{12}$, i.e.

$$\begin{aligned} \psi_{11} &= \frac{\grave{w}_{22}\grave{s}_{22} - \grave{w}_{12}\grave{s}_{12}}{\grave{s}_{22} - \grave{s}_{22}} \\ \phi_{11} &= \frac{\grave{w}_{22} - \grave{w}_{12}}{\grave{s}_{22} - \grave{s}_{12}} \end{aligned}$$

Then the wealth $\grave{V}_2(\pi)$ attained at the trading time t_2 is determined by formula (3.2.5) with $n = 2$. It coincides with $\grave{W}_2$.

- If during the forthcoming trading periods the portfolio will be held which is always determined by the same formulas (3.2.10) and (3.2.11) with $\{\grave{v}_{kn}(\pi) = \grave{w}_{kn}\}_{k=1,\dots,2^n}$ for the integers n increasing up to N, then the value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\dots,N}$ will develop in such a way that $\grave{V}_n(\pi)$ will coincide with $\grave{W}_n$ for $n = 1, \dots, N$. In particular, at the terminal date $t_N = T$ the wealth W_N will be attained.

The trading strategy just described is indeed the hedging strategy against the wealth W_N and the construction is unique. Clearly, this strategy is applicable to any desired wealth of the type W_N with 2^N possible states $\{w_{kN}\}_{k=1,\dots,2^N}$ numbered according to the states of the terminal stock price. This proves the completeness of a binary market. We have

Proposition 3.4.3. *A binary market is complete: any wealth W_N, desired at the terminal date $t_N = T$, is attainable with an initial endowment uniquely defined by (3.4.3). If for $n = 0, 1, \dots, N$ the states $\{\grave{w}_{kn}\}_{k=1,\dots,2^n}$ of $\grave{W}_n$ are obtained by (3.4.2), then the so-called hedging strategy against W_N is uniquely determined by selecting the portfolio according to (3.2.10) and (3.2.11) with $\{\grave{v}_{kn}(\pi) = \grave{w}_{kn}\}_{k=1,\dots,2^n}$ for $n = 0, 1, \dots, N$.*

If $\{P_{kN}\}_{k=1,\dots,2^N}$ is indeed a probability distribution, i.e. all the weights in the sum (3.4.3) are strictly positive, then the sum itself is positive and the possibility is excluded of attaining a positive wealth at the terminal date $t_N = T$ with a nonpositive initial endowment. Moreover, in this case the hedging strategy of the present section is not only self-financing but also admissible in the sense of Definition 3.2.3, since the corresponding value process – the solution of the above recurrence equations – cannot be negative.

Example 3.4.4. CONSTANT PORTFOLIO. It is not hard to see that a strategy of selecting a constant portfolio (a portfolio selected at the current date $t_0 = 0$ and kept unchanged over consecutive periods of trading) is the hedging strategy against some W_N, if and only if W_N is representable as a linear combination of the bond and stock prices B_N and S_N at $t_N = T$, which means that there are constants ϕ and ψ such that

$$W_N = \psi B_N + \phi S_N. \tag{3.4.4}$$

Indeed, in order to attain at $t_N = T$ the wealth W_N of the form (3.4.4), one has to invest the amount $v = \psi + \phi s$ ($s = S_0$ as usual) by buying currently, at $t_0 = 0$, ψ shares of the bond and ϕ shares of the stock. In other words, the investor has to select at $t_0 = 0$ the portfolio $(\Psi_0, \Phi_0) = (\psi, \phi)$. If this portfolio is kept unchanged, i.e. $(\Psi_n, \Phi_n) = (\psi, \phi)$ for $n = 0, 1, \ldots, N$, then the corresponding value process $V(\pi) = \{V_n(\pi)\}_{n=0,1,\ldots,N}$ is given by

$$V_n(\pi) = \psi B_n + \phi S_n \tag{3.4.5}$$

so that we also have the desired equality $V_N(\pi) = W_N$.

Note that according to the assertion of Proposition 3.4.3 the discounted value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,1,\ldots,N}$ with $\grave{V}_n(\pi) = \psi + \phi \grave{S}_n$ solves the recurrence equations (3.3.22) subject to the boundary conditions $x_{kN} = \psi + \phi \grave{s}_{kN}$, $k = 1, \ldots, 2^N$. Hence the strategy $\pi = (\Psi_n, \Phi_n)_{n=0,1,\ldots,N}$ of holding the constant portfolio $(\Psi_n, \Phi_n) = (\psi, \phi)$ is the hedging strategy against W_N as in (3.4.4).

3.5 Option pricing

3.5.1 European call option

Suppose that today, at time $t = 0$, we sign a contract giving us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time.* If the stock price S_T is below the exercise price at maturity, i.e. $S_T \leq K$, then the contract is worthless to us. On the other hand, if $S_T > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price S_T. Thus this option, called the *European call option*, yields a profit at maturity T equal to

$$\max\{0, S_T - K\} = (S_T - K)^+. \tag{3.5.1}$$

The function (3.5.1) of the stock price S_T at maturity T, is called the *payoff function* for the European call option.

Now, how much would we be willing to pay at time $t = 0$ for a ticket which gives the right to buy at maturity $t = T$ one share of stock with exercise price K? To put this in another way, what is a fair price to pay at time $t = 0$ for the ticket? This question has a direct answer only in the trivial case where the exercise price K lays outside the range of all possible values of S_T. If, for instance, the exercise price is too high (exceeding all possible values of S_T), then clearly the contract is worthless and the fair price of the ticket is 0. On the other hand, consider another extreme situation in which the exercise price K is too low; i.e. below all possible values of S_T, so that $S_T \geq K$. Obviously, the payoff function (3.5.1) then reduces to

$$S_T - K. \tag{3.5.2}$$

In order to determine the fair price to pay at time $t = 0$ for the ticket which entitles us to the payoff (3.5.2), suppose that instead of buying the ticket we act as follows. Currently, at $t = 0$, we borrow an amount $\grave{K}$ and buy one stock, where $\grave{K} = K/B_T$ is the exercise price, discounted by the bond price B_T at maturity $t = T$. In terms of Section 3.2.1, we select the special portfolio $(\Psi_n, \Phi_n) = (-\grave{K}, 1)$ by investing an amount $v = V_0(\pi) = s - \grave{K}$, where as usual $s = S_0 > 0$ is the stock price at $t = 0$. We keep consequently this portfolio unchanged over N periods of trading i.e. we hold the constant portfolio $(\Psi_n, \Phi_n) = (-\grave{K}, 1)$ for all $n = 0, 1, \ldots, N$ like in Example 3.4.4 where $\phi = 1$ and $\psi = -\grave{K}$. Our wealth at maturity $t_N = T$ will then amount to $V_T(\pi) = S_T - K$, which equals to the payoff (3.5.2); see (3.4.4) and (3.4.5) with $\phi = 1$ and $\psi = -\grave{K}$ (for convenience, we have identified $V_N(\pi)$, S_N and B_N with $V_T(\pi)$, S_T and B_T). In other words, holding the call with payoff (3.5.2) is exactly equivalent to holding the above mentioned portfolio which requires, as we have seen, the investment of the amount $v = S_0 - \grave{K}$. Any option price different from $S_0 - \grave{K}$ would enable either the option seller or the option buyer to make a sure profit without any risk or, as is sometimes said, to have "free lunch" (in Section 3.6 below we will choose for the term "an arbitrage opportunity"). It is natural, therefore, to conclude that the fair price of the equivalent call is $S_0 - \grave{K}$.

The discounted value process $\grave{V}$ for the present strategy π of keeping the constant portfolio $(-\grave{K}, 1)$ is determined by substituting in the discounted version of (3.4.5) $\phi = 1$ and $\psi = -\grave{K}$, which yields

$$\grave{V}_n(\pi) = \grave{S}_n - \grave{K}, \quad n = 0, 1, \ldots, N.$$

It solves the recurrence equations (3.3.22), subject to the boundary conditions

$$x_{jN} = \grave{s}_{jN} - \grave{K} = \frac{s_{jN} - K}{B_N}, \quad j = 1, \ldots, 2^N,$$

see Example 3.4.4. Hence the strategy of holding the portfolio $(\Psi_n, \Phi_n) = (-\grave{K}, 1)$ for all $n = 0, 1, \ldots, N$ is the *hedging strategy* against the payoff (3.5.2).

Remark 3.5.1. The fair price of a call will be denoted throughout by C. Thus if the exercise price K in (3.5.1) is too high (exceeding all possible values of S_T), then $C = 0$. On the other hand, if the exercise price K is too low (below all possible values of S_T), then $C = S_0 - \grave{K}$. This is strictly positive if at least one of the states of $\grave{S}_T$ is strictly below $S_0 = s$, so that also $\grave{K} < S_0$.

Although we have used above the "no arbitrage" principle to obtain fair option prices, we want to emphasize here that we will not need that principle in the next section where we give a formal definition of the fair price of a contingent claim.

3.5.2 Pricing a contingent claim

A contract with some fixed discounted payoff function $\grave{H}_N$ that assigns to each of 2^N trajectories of the discounted stock price process some nonnegative numerical value, is called a *contingent claim.* Let $\{\grave{h}_{kN}\}_{k=1,\ldots,2^N}$ be the states of $\grave{H}_N$, like in (3.3.21). The European call option is thus a special contingent claim with payoff (3.5.1). Consider, for example, the two period binomial model. Suppose that the contract we are going to sign yields the discounted payoff $\grave{h}_{12}$ if the gross return on the stock goes down at both of trading times, $\grave{h}_{22}$ if it first goes down and then up, $\grave{h}_{32}$ if it first goes up and then down and $\grave{h}_{42}$ if it goes up in both cases. If we deal with the European call option, then

$$\begin{aligned}
\grave{h}_{12} &= \frac{1}{r_1 r_2}(sd_1 d_2 - K)^+ & \grave{h}_{22} &= \frac{1}{r_1 r_2}(sd_1 u_2 - K)^+ \\
\grave{h}_{32} &= \frac{1}{r_1 r_2}(su_1 d_2 - K)^+ & \grave{h}_{42} &= \frac{1}{r_1 r_2}(su_1 u_2 - K)^+.
\end{aligned}$$

In contrast to the general case (3.3.21), the payoff (3.5.1) of the European call option makes no distinction between those states in which the terminal stock prices are identical, e.g. in the homogeneous model we get $\grave{h}_{22} = \grave{h}_{32}$.

The fair price of a contingent claim is defined by the same considerations as in the special case of the linear payoff (3.5.2). The procedure used in the preceding section can be described as follows:

(i) construct the hedging strategy against the contingent claim in question, which duplicates the payoff;

(ii) determine the initial wealth needed for construction in (i);

(iii) equate this initial wealth to the fair price of the contingent claim.

According to Proposition 3.4.3 the hedging strategy against the contingent claim with a payoff function $\grave{H}_N$ consists in holding the portfolio with the components (3.2.10) and (3.2.11), where $\{\grave{v}_{kn}(\pi)\}_{k=1,\dots,2^n}$ for $n = 0, 1, \dots, N$ are determined by solving the recurrence equations (3.3.22), subject to the boundary conditions (3.3.23). This strategy indeed duplicates the payoff, since $\grave{V}_N(\pi) = \grave{H}_N$. It requires the initial wealth $v = V_0(\pi)$ which is calculated according to (3.3.25):

$$v = \sum_{k=1}^{2^N} P_{kN} \grave{h}_{kN}$$

with P_{kN} and $\grave{h}_{kN}$ defined by (3.3.20) and (3.3.21), respectively. The fair price $C = C(\grave{H}_N)$ of the contingent claim with the payoff function $\grave{H}_N$ is thus defined by

$$C(\grave{H}_N) = \sum_{k=1}^{2^N} P_{kN} \grave{h}_{kN}. \tag{3.5.3}$$

The European call option (3.5.1), in particular, has a special payoff function depending only on the stock price at maturity $t_N = T$ and its fair price is

$$C = \sum_{k=1}^{2^N} P_{kN} (\grave{s}_{kN} - \grave{K})^+. \tag{3.5.4}$$

If all the weights are strictly positive summing up to 1 and $\{P_{kN}\}_{k=1,\dots,2^N}$ is a probability distribution, then we can apply (2.2.35) and (3.3.27) to get $C \geq s - \grave{K}$. In view of Remark 3.5.1, this means that the fair price of the European call option is not less then the fair price of the contingent claim with linear payoff (3.5.2).

Let us turn back to the example of two period binomial model. Using the same notations as in Example 3.3.6 we get the fair price of the European call option

$$\begin{aligned} C = \frac{p_{d_1} p_{d_2}}{r_1 r_2} (s d_1 d_2 - K)^+ \quad &+ \quad \frac{p_{d_1} p_{u_2}}{r_1 r_2} (s d_1 u_2 - K)^+ \\ + \frac{p_{u_1} p_{d_2}}{r_1 r_2} (s u_1 d_2 - K)^+ \quad &+ \quad \frac{p_{u_1} p_{u_2}}{r_1 r_2} (s u_1 u_2 - K)^+. \end{aligned} \tag{3.5.5}$$

The call-put parity. Suppose that today, at time $t = 0$, we sign a contract which gives us the right to sell, at specified time T one share of a stock at a specified price K. If the stock price S_T is above the exercise price at maturity, i.e. $S_T \geq K$, the contract is worthless. On the other hand, if $S_T < K$, we can exercise our option: we can sell one share of the stock at the fixed price K and then buy it immediatelv in the market for the price S_T. Thus this option, called thc *European put opnon*, yields the following profit at maturity T:

$$\max\{0, K - S_T\} = (K - S_T)^+.$$

This function $(K - S_T)^+$ of the stock price S_T at maturity T, is called the *payoff function* for the European put option. The fair price of a put is denoted by P, and according to our theory we have

$$P = \sum_{k=1}^{2^N} P_{kN}(\grave{K} - \grave{s}_{kN})^+.$$

Since

$$(S_T - K)^+ - (K - S_T)^+ = S_T - K,$$

we obtain by (2.2.35) and (3.3.27) the so-called call-put parity relationship:

$$C - P = \sum_{k=1}^{2^N} P_{kN}(\grave{s}_{kN} - \grave{K}) = s - \grave{K}.$$

Option strategies. We can easily use the linearity of the summation operator in (3.5.4) to evaluate options formed as certain linear combinations of contingent claims. For instance prices of *spreads* in the exercise price are obtained as linear combinations of individual option prices. Also the price of for instance a *strangle* with payoff function $|S_T - K|$ turns out to be $C + P$ since $(S_T - K)^+ + (K - S_T)^+ = |S_T - K|$.

Examples of option pricing. We conclude this section by applications of the general option pricing formula (3.5.4) to the following special models.

Example 3.5.2. BINOMIAL MODEL. Consider first the homogeneous case. According to (3.5.5) in the two period model we have

$$C = \frac{1}{r^2}\left(p_d^2(sd^2 - K)^+ + 2p_dp_u(sdu - K)^+ + p_u^2(su^2 - K)^+\right).$$

This extends straightforwardly to the N period model, since formula (3.3.25) reduces to (3.3.34) and

$$C = \frac{1}{r^N}\sum_{j=0}^{N}\binom{N}{j}p_u^j p_d^{N-j}(su^j d^{N-j} - K)^+ \tag{3.5.6}$$

which is the well-known Cox-Ross-Rubinstein option pricing formula, see [13].

As for the nonhomogeneous case, the fair price of the European call option is determined by applying formula (3.3.36). So, it equals

$$C = \sum_{\varepsilon_1 \dots \varepsilon_N} \frac{p_{\varepsilon_1} \cdots p_{\varepsilon_N}}{r_1 \dots r_N} (s\varepsilon_1 \dots \varepsilon_N - K)^+. \tag{3.5.7}$$

This is the straightforward extension to the two period formula (3.5.5).

Example 3.5.3. 1^{st} ORDER MOVING AVERAGE. In this case we have (3.3.38) so that the fair price of the European call option equals

$$\begin{aligned} C &= \sum_{\varepsilon_1 \dots \varepsilon_N} \frac{p_{\varepsilon_1} p_{\varepsilon_2|\varepsilon_1} \cdots p_{\varepsilon_N|\varepsilon_{N-1}}}{r_1 \dots r_N} \\ &\quad (s(\varepsilon_1 + \alpha s)(\varepsilon_2 + \alpha\varepsilon_1) \dots (\varepsilon_N + \alpha\varepsilon_{N-1}) - K)^+ . \end{aligned} \tag{3.5.8}$$

3.6 Markets excluding arbitrage opportunities

3.6.1 Arbitrage opportunities

It will be shown in Proposition 3.6.2 below that under natural restrictions on the asset prices development in a binary market the possibility is excluded of making a profit without any initial endowment. More precisely, the possibility is excluded of constructing a self-financing trading strategy $\pi = (\Psi_n, \Phi_n)_{n=0,1,\dots,N}$ with an initial endowment

$$v = V_0(\pi) = \Psi_0 + \Phi_0 S_0 = 0 \tag{3.6.1}$$

and with a value process which attains at t_N only states with nonnegative values, i.e. $V_N(\pi) \geq 0$, and at least one state with a strictly positive value. A strategy of selecting such a portfolio is called an *arbitrage opportunity.* Thus, an arbitrage opportunity represents a riskless plan of making a profit without any investment: there is no threat of loss, since $V_N(\pi) \geq 0$ and moreover, there is a chance of a pure gain in case the stock price develops along one of those trajectories for which $V_N(\pi)$ attains one of the states with a strictly positive value. It is, therefore, economically meaningful (and mathematically useful, as we will see below), to treat separately the security markets which exclude arbitrage opportunities. These markets also allow for a certain economic equilibrium as will be seen in the concluding part of the present course.

Remark 3.6.1. Since negative values of the components of the portfolio (Ψ_0, Φ_0) at $t = 0$ are not excluded, there is no real reason for keeping the initial endowment $v = V_0(\pi) = \Psi_0 + \Phi_0 S_0$ nonnegative, as in Section 3.2.1. We may drop this assumption and take into consideration the possibility of an arbitrary initial endowment v, not necessarily $v = V_0(\pi) \geq 0$. The equality to 0 in (3.6.1) has to be replaced then by the sign $\leq$. Note also that the trivial strategy with $(\Psi_n, \Phi_n) \equiv (0, 0)$ is not an arbitrage opportunity.

Proposition 3.6.2. *A binary market excludes arbitrage opportunities if and only if the states $\{\grave{s}_{kn}\}_{k=1,\dots,2^n}$ of the discounted stock prices $\grave{S}_n, n = 1, \dots, N$, take on values which satisfy the inequalities*

$$\grave{s}_{2k-1\,n} < \grave{s}_{k\,n-1} < \grave{s}_{2k\,n} \tag{3.6.2}$$

for $k = 1, \dots, 2^{n-1}$ and $n = 1, \dots, N$.

It is instructive to prove the assertion of Proposition 3.6.2 first in the special case of the one-period model where $N = 1$, because one can easily trace in that case the main idea behind the proof.

Lemma 3.6.3. *(i) If the transition (2.3.6) of the stock price is such that*

$$\grave{s}_{11} < s < \grave{s}_{21}, \tag{3.6.3}$$

then for every portfolio $(\Psi_0, \Phi_0) = (\Psi_1, \Phi_1)$, whose value process satisfies

$$\{\grave{v}_{21}(\pi) > 0 \text{ and } \grave{v}_{11}(\pi) \geq 0\} \tag{3.6.4}$$

or

$$\{\grave{v}_{21}(\pi) \geq 0 \text{ and } \grave{v}_{11}(\pi) > 0\},$$

we have $v = V_0(\pi) > 0$, so that there is no arbitrage opportunity.

(ii) Conversely, if (3.6.3) is violated, so that $s \leq \grave{s}_{11}$ or $s \geq \grave{s}_{21}$, then the trading strategy of selecting portfolio

$$\pi_0 = \pi_1 = (-\phi s, \phi) \tag{3.6.5}$$

with some

$$\begin{cases} \phi > 0 & \text{if } s \leq \grave{s}_{11} \\ \phi < 0 & \text{if } s \geq \grave{s}_{21} \end{cases}$$

is an arbitrage opportunity.

Proof. (i) Suppose that (3.6.3) and (3.6.4) hold. Put $\Psi_0 = \Psi_1 = \psi$ and $\Phi_0 = \Phi_1 = \phi$. If $\phi = 0$, then $V_0(\pi) = \grave{V}_1(\pi) = \psi$ is strictly positive by (3.6.4). We get the desired strict inequality also for $\phi \neq 0$, since

$$V_0(\pi) = \psi + \phi s > \begin{cases} \psi + \phi \grave{s}_{21} = \grave{v}_{21}(\pi) \geq 0 & \text{if } \phi < 0 \\ \psi + \phi \grave{s}_{11} = \grave{v}_{11}(\pi) \geq 0 & \text{if } \phi > 0. \end{cases}$$

(ii) To see that the portfolio (3.6.5) yields an arbitrage opportunity, observe that $v = V_0(\pi) = -\phi s + \phi s = 0$ and that $\grave{V}_1(\pi) = \phi(\grave{S}_1 - s)$ can be in two alternative states: either $\grave{v}_{21}(\pi) = \phi(\grave{s}_{21} - s)$ or $\grave{v}_{11}(\pi) = \phi(\grave{s}_{11} - s)$. By the definition of ϕ these states satisfy condition (3.6.4). Hence, the portfolio (3.6.5) yields an arbitrage opportunity. □

Corollary 3.6.4. *The assertion in Proposition 3.6.2 holds in the special case where $N = 1$.*

Proof. We note that the notion of self-financing is empty in case $N = 1$. (i) The sufficiency of (3.6.3) for excluding arbitrage opportunities is indeed reduced to the statement (i) that under (3.6.3) and (3.6.4) we have $v = V_0(\pi) > 0$ whatever the components $\Psi_0 = \Psi_1 = \psi$ and $\Phi_0 = \Phi_1 = \phi$ of a portfolio, hence the self-financing strategy attaining (3.6.4) cannot be an arbitrage opportunity. □

3.6.2 Proof of Proposition 3.6.2

Let us turn back to the general multi-period model of a binary market. It is useful to extend first in a separate lemma the arguments used in the course of proving assertion (i) of Lemma 3.6.3.

Lemma 3.6.5. *Let the discounted value process $\grave{V}(\pi) = \{\grave{V}_n(\pi)\}_{n=0,\ldots,N}$ of a self-financing strategy π be such that for some trading time $t_n, n \in \{1, \ldots, N\}$, we have $\grave{V}_n(\pi) \geq 0$ and at the same time at least one of the states $\{\grave{v}_{kn}(\pi)\}_{k=1,\ldots,}$*

takes on a strictly positive value. Then under condition (3.6.2) $\grave{V}_{n-1}(\pi)$ *is of the same type:* $\grave{V}_{n-1}(\pi) \geq 0$ *and at least one of the states* $\{\grave{v}_{k\,n-1}(\pi)\}_{k=1,\dots,2^{n-1}}$ *takes on a strictly positive value.*

Proof. Since the self-financing condition (3.2.14) is assumed to hold, the set of states $\{\grave{v}_{k\,n-1}(\pi)\}_{k=1,\dots,2^{n-1}}$ at the fixed trading time t_{n-1} satisfies

$$\grave{v}_{k\,n-1}(\pi) = \psi_{k\,n-1} + \phi_{k\,n-1}\grave{s}_{k\,n-1}, \tag{3.6.6}$$

with strict inequality if $\grave{v}_{2j\,n}(\pi) = \grave{v}_{2j-1\,n}(\pi) > 0$, as we already know. If $\phi_{j\,n-1} \neq 0$, then by assumption (3.6.2) and by (3.6.6)

$$\grave{v}_{j\,n-1}(\pi) > \begin{cases} \psi_{j\,n-1}+\phi_{j\,n-1}\grave{s}_{2j\,n}=\grave{v}_{2j\,n}(\pi)\geq 0 \text{ if } \phi_{j\,n-1}<0 \\ \psi_{j\,n-1}+\phi_{j\,n-1}\grave{s}_{2j-1\,n}=\grave{v}_{2j-1\,n}(\pi)\geq 0 \text{ if } \phi_{j\,n-1}>0. \end{cases}$$

The proof is complete. □

We present now the proof of Proposition 3.6.2.

Proof. (i) Suppose that condition (3.6.2) holds and suppose we have a self-financing strategy for which $\grave{V}_N(\pi) \geq 0$, with at least one strictly positive state. We will show that this strategy requires a positive investment $v = V_0(\pi) > 0$ and therefore cannot be an arbitrage opportunity. To prove this claim we apply Lemma 3.6.5. If $\grave{V}_N(\pi)$ is of the above type, then $\grave{V}_{N-1}(\pi)$ is of the same type, and so on. Hence the states of $\grave{V}_1(\pi)$ satisfy (3.6.4) and by Lemma 3.6.3, assertion (i), we have $v = V_0(\pi) > 0$.
(ii) As in the special case of $N = 1$ (cf Lemma 3.6.3, assertion (ii)), the necessity of (3.6.2) for $N > 1$ will be proved by contradiction: it will be shown that there is an arbitrage opportunity, provided (3.6.2) is violated. Suppose that for some $m \in \{1,\dots,N\}$ we have

$$\grave{s}_{j\,m-1} \leq \grave{s}_{2j-1\,m} \tag{3.6.7}$$

or

$$\grave{s}_{j\,m-1} \geq \grave{s}_{2j\,m}$$

for some $j \in \{1,\dots,2^{m-1}\}$. Consider the self-financing strategy of selecting the following portfolio. At the trading times preceding t_m we take the trivial portfolio:

$$(\Psi_n, \Phi_n) = (0,0), \quad \text{for } n < m.$$

At t_m the portfolio (Ψ_m, Φ_m) is selected according to

$$(\psi_{k\,m-1}, \phi_{k\,m-1}) = \begin{cases} (0,0) & \text{if } k \neq j \\ (-\phi s_{j\,m-1}, \phi) & \text{if } k = j \end{cases} \tag{3.6.8}$$

where $\phi > 0$ if $\grave{s}_{j\,m-1} \leq \grave{s}_{2j-1\,m}$ and $\phi < 0$ if $\grave{s}_{j\,m-1} \geq \grave{s}_{2j\,m}$.

Next, at t_{m+1} the portfolio $(\Psi_{m+1},\ \Phi_{m+1})$ is nonzero only when $k = j$ and

$$\begin{cases} (\psi_{2j\,m}, \phi_{2j\,m}) & = & (\phi(\grave{s}_{2j\,m} - \grave{s}_{j\,m-1}), 0) \\ \\ (\psi_{2j-1\,m}, \phi_{2j-1\,m}) & = & (\phi(\grave{s}_{2j-1\,m} - \grave{s}_{j\,m-1}), 0). \end{cases} \tag{3.6.9}$$

Finally, for $n > m+1$ we do not change the portfolio $(\Psi_n,\ \Phi_n)$ anymore (note that this strategy is self-financing). Choosing for this strategy, the investor is not taking any risk before and after the trading time t_{m-1}. Awaiting the stock price announcement at the trading time t_{m-1}, the investor acts only if state $s_{j\,m-1}$ occurs, by selecting the portfolio according to (3.6.8) and choosing the appropriate sign of ϕ. It will be shown that this strategy is an arbitrage opportunity. Observe that the corresponding value process evolves as follows: $V_n(\pi) = 0$ for all $n = 0, 1, \ldots, m-1$, while $\grave{V}_m(\pi) \geq 0$ is in one of the following states. Fix $k \in \{1, \ldots, 2^{m-1}\}$. If $k \neq j$, then both $\grave{v}_{2k\,m}$ and $\grave{v}_{2k-1\,m}$ vanish. If $k = j$ we have

$$\begin{aligned} \grave{v}_{2k\,m}(\pi) &= \phi(\grave{s}_{2j\,m} - \grave{s}_{j\,m-1}) \\ \grave{v}_{2k-1\,m}(\pi) &= \phi(\grave{s}_{2j-1\,m} - \grave{s}_{j\,m-1}). \end{aligned}$$

Therefore either

$$\{\grave{v}_{2j\,m}(\pi) > 0 \text{ and } \grave{v}_{2j-1\,m}(\pi) \geq 0\}$$

or

$$\{\grave{v}_{2j\,m}(\pi) \geq 0 \text{ and } \grave{v}_{2j-1\,m}(\pi) > 0\},$$

depending on whether $\grave{s}_{j\,m-1} \leq \grave{s}_{2j-1\,m}$ or $\grave{s}_{j\,m-1} \geq \grave{s}_{2j\,m}$. Hence, there is no threat of loss. Moreover, if at t_{m-1} the stock price is in state $s_{j\,m-1}$, then a pure gain is attained (unless the next state is either $s_{2j-1\,m}$ and $\grave{s}_{j\,m-1} = \grave{s}_{2j-1\,m}$, or $s_{2j\,m}$ and $\grave{s}_{j\,m-1} = \grave{s}_{2j\,m}$). Since in the subsequent trading intervals no risk is taken, the investor's wealth remains nonnegative, and thus the above strategy is indeed an arbitrage opportunity. The proof of Proposition 3.6.2 is complete. □

Note that condition (3.6.2) can be written in the following alternative form: for $n = 1, \ldots, N$

$$M < r_n < m \tag{3.6.10}$$

where $z_{kn} = s_{kn}/s_{k_1\,n-1}$, cf (2.3.14), while M and m are the maximum and the minimum, respectively, over the set $\{z_{2k-1\,n} : k = 1, \ldots, 2^{n-1}\}$.

3.6.3 No arbitrage for moving averages models

The condition of no arbitrage is simply tractable in the following

Example 3.6.6. BINOMIAL MODEL. Since $z_{2k-1\,n} = d_n$ and $z_{2k\,n} = u_n$ whatever the index k (cf Example 2.3.2), the condition (3.6.10) for no arbitrage opportunities reduces to

$$d_n < r_n < u_n \quad \text{for} \quad n = 1, \dots, N.$$

For the general moving averages model of Section 2.3.2, however, the condition of no arbitrage opportunities is quite complicated, since (3.6.10) means that $d_1 < r_1 < u_1$ and for $n = 2, \dots, N$

$$M + d_n < r_n < m + u_n$$

where the maximum $M = \max\{\alpha_{n-1} e_{k_{n-1}\,1} + \cdots + \alpha_1 e_{k_1\,n-1}\}$ and the minimum $m = \min\{\alpha_{n-1} e_{k_{n-1}\,1} + \cdots + \alpha_1 e_{k_1\,n-1}\}$ are taken over all possible values of the variables $\{e_{k_{n-\nu}\,\nu}\}_{\nu=1,\dots,n}$ (cf (2.3.24) and (2.3.26)).

Example 3.6.7. 1^{st} ORDER MOVING AVERAGES MODEL. In the present model of a market with $\alpha_1 = \alpha$ and $\alpha_2 = \cdots = \alpha_{n-1} = 0$ (cf (2.3.28)) arbitrage opportunities are excluded only if

$$|\alpha| < \frac{u_n - d_n}{u_{n-1} - d_{n-1}} \quad \text{for} \quad n = 2, \dots, N.$$

In this case the above condition of no arbitrage opportunities reduces to the following conditions:

$$\begin{cases} d_1 < r_1 < u_1 \\ \alpha u_{n-1} + d_n < r_n < \alpha d_{n-1} + u_n \quad \text{for } n = 2, \dots, N \end{cases}$$

if α is positive and

$$\begin{cases} d_1 < r_1 < u_1 \\ \alpha d_{n-1} + d_n < r_n < \alpha u_{n-1} + u_n \quad \text{for } n = 2, \dots, N \end{cases}$$

if α is negative.

Part II

Towards Continuous-Time Models

Chapter 4

Poisson Approximation

4.1 Introduction

In this chapter the material is used of Part I. Most of this material is presented in Section 4.2 in the form aimed at the limiting transition in Section 4.4 towards the Poisson model. See Section 4.4 for details on this model; for some related references see e.g. [12], [14], [16], [32], [39], [61] and [64]. The presentation in this chapter is kept at the same low technical level as in Part I. A path by path approach pursued here is based on certain unsophisticated algebraic considerations, in contrast with the usual treatment based on a probabilistic approach, namely on a martingale approach. The results obtained in this manner in Section 4.4 are of an heuristic nature, for the full rigour would require higher technical level of the general theory of stochastic processes, see e.g. [1], [2], [22], [46], [50] and [51].

As in Chapter 2, it is assumed that in a securities market two assets, called the bond and stock, are traded during the time interval $[0, T]$. New prices on both assets are announced at certain fixed trading times, say $t_0 < t_1 < \cdots < t_N$ where $t_0 = 0$ is the current date and $t_N = T$ the terminal date. Thus the whole time interval $[0, T]$ is divided in N trading periods by a grid $\{t_0, t_1, \ldots, t_N\}$. It is supposed throughout the present chapter that the number N of the trading times is very large, and possibilities are sought for approximating the option pricing formulas of Part I, Section 3.5. To this end, we let $N \to \infty$. We can expect in the limit sensible results if only the grid $\{t_0, t_1, \ldots, t_N\}$ of trading times becomes finer and finer in the sense that the mesh size of the grid tends to zero as $N \to \infty$ (the mesh size is the maximal length of the trading periods) and if the asset prices are made dependent on the index N in a certain special manner. See Section 4.2.1 for the conditions under which the Poisson approximation of the present paper is obtained. Asymptotically, the cumulative return process on the bond is assumed to increase with a constant interest rate, see (4.3.11). The asymptotics of the returns on the stock is characterized by the displacements at certain random instants, upwards with a constant amplitude or downwards with an infinitesimal amplitude, cf (4.3.17) or (4.3.18). To these displacements certain weights are assigned (called as in Part I, Section 3.3.1, *risk neutral probabilities*, cf (4.3.35) and (4.3.36)) so that under the conditions 3.1.1 and 3.1.2 the approximation (4.3.40) becomes valid. Using probabilistic terminology one may argue that the upward displacements become rare events, since the right hand side in (4.3.40) is infinitesimal, of magnitude $1/T$. This is the necessary prerequisite for the Poisson approximation of Section 4.4.

In Section 4.3 the complete description is provided of the Poisson model (or Merton's model, as it is sometimes called, cf [53]). The price processes on the bond and the stock are given by (4.4.2) and (4.4.4), respectively, see (4.4.1) and (4.4.3) for the corresponding returns. The Poisson model describes the situation when the stock price develops with sudden jumps of a constant amplitude at random instants. As usual, the self-financing strategy in the Poisson model is defined by the portfolio selection founded only on an initial endowment so that all changes in the portfolio values are due to capital gains during trading and no infusion or withdrawal of funds is allowed. It is shown that the value process of a self-financing strategy has the integral representation and, moreover, Clark's formula holds; cf the Propositions 4.3.5 and 4.3.7 in the binary case and the similar Propositions 4.4.2 and 4.3.5 in the Poisson case. As was already mentioned, the usage of this term in both of these cases stems from the analogy to the genuine Clark formula in [58] or in [59].

Next, it is shown in Proposition 4.4.3 that this value process satisfies the differential equations (4.4.29) which play the same rôle in the Poisson case as equations (4.3.46) in the binary case. In particular, they entail the completeness of a Poisson market, see Proposition 4.4.5. The hedging strategy against any desired wealth is explicitly defined by the portfolio components (4.4.32) and (4.4.33) in terms of the Poisson distribution (4.2.14) (in fact, the right hand side of (4.4.35) is a certain conditional expectation). This gives rise to the term *Poisson market.*

Finally, the option pricing formulas are presented for contingent claims (see (4.4.36) with a Poisson expectation on the right hand side) and for the European call option in particular, see Proposition 4.4.6.

The integral representations (4.3.30) and (4.4.26) mentioned above involve the Riemann-Stieltjes integrals with respect to piecewise continuous functions. Certain elementary facts concerning this kind of functions and respective integrals are gathered in the next section.

4.2 Auxiliary notions and results

4.2.1 Piecewise continuous functions

In this chapter the asset prices are supposed to evolve along piecewise continuous trajectories within a time period $[0,T]$. Therefore we will need to settle some notations and to present some common facts concerning functions of this type. For the definitions below we make use of the indicator function $I_{\mathcal{T}}$ of a set $\mathcal{T} \subset [0,T]$ which is a function of time $t \in [0,T]$ such that

$$I_{\mathcal{T}}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{T} \\ 0 & \text{otherwise.} \end{cases}$$

Let F be a function of the same argument $t \in [0,T]$, discontinuous at certain instants $T_1, \dots, T_m$ so that $0 < T_1 < \dots < T_m \leq T$ and continuous in-between. Let F be defined within the time stretch between the jumps by means of certain continuous functions $\{f_k\}_{k=0,1,\dots,m}$ so that

$$F(t) = \sum_{k=0}^{m} f_k(t) I_{[T_k, T_{k+1})}(t). \tag{4.2.1}$$

Let us set $T_0 = 0$ and $T_{m+1} \geq T$ for convenience. Note that F is a right-continuous function in the sense that by approaching an instant $t \in [0,T)$ from the right we get $\lim_{s \downarrow t} F(s) = F(t)$. With this function F another function F_- is associated by the following conventions: $F_-(0) = F(0)$ and $F_-(t) = F(t-) \doteq \lim_{s \uparrow t} F(s)$ for $t \in (0,T]$. By continuity of the components $\{f_k\}_{k=0,1,\dots,n}$ we have

$$F_-(t) = f_0(t) I_{[T_0, T_1]}(t) + \sum_{k=1}^{m} f_k(t) I_{(T_k, T_{k+1}]}(t). \tag{4.2.2}$$

Obviously, F_- is a left-continuous function. We will write alternatively $F(t)$ or F_t, $F(t-)$ or F_{t-}, for the notation with the variable as a subscript is more widely used in stochastic calculus. Next, the function ΔF of jumps of F is defined by $\Delta F = F - F_-$. In view of (4.2.1) and (4.2.2), ΔF takes on non-zero values only at the instants of discontinuity $T_1, \dots, T_m$ when

$$\Delta F(T_k) = f_k(T_k) - f_{k-1}(T_k), \quad k = 1, \dots, m. \tag{4.2.3}$$

Riemann-Stieltjes integrals. In the Propositions 4.3.5 and 4.4.2 integral representations are asserted, in terms of the Riemann-Stieltjes integrals with respect to piecewise continuous functions. The definition of such integrals is as follows (see e.g. [65], Section II.6.10, for more details).

Let H be a piecewise continuous function of the same type as F, with the representation similar to (4.2.1)

$$H(t) = \sum_{k=0}^{m} h_k(t) I_{[T_k, T_{k+1})}(t).$$

The function H_- is then defined similarly to (4.2.2). The integral of H_- with respect to F is a function G of the same type as F. Namely, at time $t \in [0,T]$ the integral

$$G(t) \doteq \int_0^t H_{u-} dF_u$$

is represented in the form

$$G(t) = \sum_{k=0}^{m} g_k(t) I_{[T_k, T_{k+1})}(t)$$

where

$$g_k(t) = g_k(T_k) + \int_{T_k}^{t} h_k(u) df_k(u)$$

for $t \in [T_k, T_{k+1})$, with $g_0(0) = 0$ and

$$g_k(T_k) = \sum_{j=0}^{k-1} \int_{T_j}^{T_{j+1}} h_j(u) df_j(u) + \sum_{j=1}^{k} h_{j-1}(T_j) \Delta F(T_j)$$

for $k = 1, \ldots, m$. To give a proper meaning to the integrals just introduced, assume all $\{f_k\}_{k=0,1,\ldots,n}$ to be of bounded variation. Though this is truly superfluous, as in the present paper only continuously differentiable functions f_k will occur, with $df_k(u)$ to be understood as $f_k'(u)du$ where f_k' is the derivative of f_k. We often will use the alternative notation

$$\int_0^t H_{u-} dF_u = H_- \cdot F_t,$$

usual in stochastic calculus. Note that for $k = 1, \ldots, m$ and $t \in [T_k, T_{k+1})$

$$g_k(t) - g_{k-1}(T_k) = h_{k-1}(T_k) \Delta F(T_k) + \int_{T_k}^{t} h_k(u) df_k(u).$$

Hence by (4.2.3)

$$\Delta(H_- \cdot F) = H_- \Delta F. \tag{4.2.4}$$

In Section 4.3 the trajectories of price development are certain piecewise constant functions. In this special case of F given by (4.2.1) with the constant components $f_k(t) \equiv f_k$, it follows from (4.2.4) that

$$H_- \cdot F_t = \sum_{u \in [0,t]} H_{u-} \Delta F_u. \tag{4.2.5}$$

Integration by parts. Integrals defined in the previous section allow for integrating by parts: for each $t \in [0,T]$

$$H_t F_t - H_0 F_0 = H_- \cdot F_t + F_- \cdot H_t + [H,F]$$

where

$$[H,F] = \sum_{u \in [0,t]} \Delta H_u \Delta F_u$$

as in Chapter 2, Formulas (2.2.7) and (2.2.8) (see [65], Section II.6.11 for more details). In the course of proving Proposition 4.3.5 we will use the following consequence of this integration by parts formula.

Proposition 4.2.1. *Let H', F' and H'', F'' be piecewise continuous functions of the above type. The function*

$$F = F'H' + F''H''$$

has integral representation

$$F - F_0 = H' \cdot F' + H'' \cdot F''$$

if and only if

$$F'_- \cdot H' + F''_- \cdot H'' = 0.$$

Exponentials. The details on the material of present section can be found in [65], Section II.6.12. See also [30], [45] or [63].

Obviously, in case of a continuous function F of bounded variation the solution of the integral equation

$$H_t = 1 + H_- \cdot F_t \tag{4.2.6}$$

is uniquely defined by

$$H_t = e^{F_t - F_0}.$$

In the another extreme case of a piecewise constant F

$$H_t = \prod_{u \in [0,t]} (1 + \Delta F_u).$$

In case of a piecewise continuous function F the above two cases are combined in the solution

$$H_t = e^{F_t - F_0} \prod_{u \in [0,t]} (1 + \Delta F_u) e^{-\Delta F_u}. \tag{4.2.7}$$

In stochastic calculus the solution to (4.2.6) is usually denoted by $H_t = \mathcal{E}(F)_t$ and called Doléans-Dade exponential (or stochastic exponential).

We apply (4.2.7) to the following special case. Let $\mathcal{N} = \{\mathcal{N}_t\}_{t\in[0,T]}$ be the so-called *counting process* with $\mathcal{N}(t)$ which counts the number of jumps observed up to time $t \in [0,T]$. By assumption $\mathcal{N}(0) = 0$. Further $\mathcal{N}(t) = 0$ if no jumps occur up to time t and $\mathcal{N}(t) = k$ if and only if $T_k \leq t$ for $k \in \{1,\ldots,m\}$. Let

$$F(t) = a(\mathcal{N}(t) - \lambda t)$$

with certain positive numbers a and λ, cf (4.4.6). Then

$$\Delta F_t = \begin{cases} a & \text{if } t \in \{T_1,\ldots,T_m\} \\ 0 & \text{otherwise.} \end{cases}$$

Hence (cf (4.4.5))

$$\mathcal{E}(F)_t = (1+a)^{\mathcal{N}(t)} e^{-a\lambda t}.$$

Difference operator in the state space. In the present chapter the asset prices will be allowed to evolve along piecewise continuous trajectories, but the discontinuities in Sections 4.3 and 4.4 are of a completely different nature.

In Section 4.3 the situation of Chapter 2 is retained in which new prices are announced at fixed trading times, denoted by $t_0, t_1, \ldots, t_N$ with $t_0 = 0$ and $t_N = T$. Since in-between the prices remain unchanged, the trajectories of their development are piecewise constant, of type (4.2.1) but with the functions in the sum on the right hand side replaced by constants and with the instants T_j fixed to be exactly $N+1$ trading times mentioned above. Moreover, in Section 4.3 the stock price processes, the value processes etc. are again risky, described by means of the same binary sequence $\{X_n\}_{n=0,1,\ldots}$ as in the last subsection of Part I, Section 2.2.1. They all will then get the form $X = \{X_t\}_{t\in[0,T]}$ say, with

$$X(t) = \sum_{n=0}^{N} X_n I_{[t_n,t_{n+1})}(t), \tag{4.2.8}$$

cf e.g. (4.3.2) below. The definition of the difference operator D in the state space of such processes is straightforward: the process $DX = \{DX_t\}_{t\in[0,T]}$ is defined so that

$$DX(t) = DX_1 I_{[t_0,t_1]}(t) + \sum_{n=1}^{N} DX_{n+1} I_{(t_n,t_{n+1}]}(t), \tag{4.2.9}$$

with the same DX_n as in Part I, Section 2.2.1. Recall that we have set $DX_0 = DX_1$. According to the definition (4.2.2), we associate with X the process $X_- = \{X(t-)\}_{t\in[0,T]}$ where

$$X(t-) = X_1 I_{[t_0,t_1]}(t) + \sum_{n=1}^{N} X_{n+1} I_{(t_n,t_{n+1}]}(t), \tag{4.2.10}$$

cf (4.2.2). Any process of the latter type, e.g. DX, is called *predictable.* For, the knowledge that $X(t-)$ is in state x_{kn} allows one to predict that $DX(t)$ will be in state $D_k(X_{n+1}) = x_{2k\,n+1} - x_{2k-1\,n+1}$, cf Part I, Formulas (2.2.16) and (2.2.17). Note that if Ψ and Φ are two predictable processes, then we have the same relation (2.2.18) as in Part I.

In Section 4.4 the source of uncertainty will be completely different. A special securities market will be treated in which some fixed net returns on the stock occur unexpectedly at certain instants. The possible trajectories of the stock price processes, the value processes etc. get then the form (4.2.1) with certain fixed functions on the right hand side and with the instants of jumps T_j. To adequately define the difference operator D in the state space of such processes, we argue as follows. Suppose that there where exactly k jumps prior to the instant t so that $F(t-)$ takes on the value $f_k(t)$. Then at time t either nothing happens or a new jump does occur. Consequently, the value of the function $F(t)$ either remains $f_k(t)$ or gets equal to $f_{k+1}(t)$. Clearly, the latter switch takes place if the $k+1^{th}$ jump does occur at instant t. By this consideration, at instant t the difference operator D is applied to the function F as follows: if $F(t-)$ has taken on the value $f_k(t)$, then $DF(t)$ takes on the value

$$D_{k+1}(F_t) \doteq f_{k+1}(t) - f_k(t). \tag{4.2.11}$$

Hence the process $DF = \{DF_t\}_{t\in[0,T]}$ is defined by

$$DF(t) = D_1(F_t)I_{[0,T_1]}(t) + \sum_{k=1}^{m} D_{k+1}(F_t)I_{(T_k,T_{k+1}]}(t) \tag{4.2.12}$$

with the states given by (4.2.11). Compare (4.2.2) and (4.2.12) to conclude that DF is a process in time of the same type as F_-. This process, as well as any process of type F_-, is called predictable. Note again that if Ψ and Φ are two predictable processes, then we have

$$D(\Psi + \Phi F) = \Psi + \Phi DF, \tag{4.2.13}$$

the same relation (2.2.18) as in Part I, Section 2.2.1.

4.2.2 Differential-difference equations

Poisson distribution. In the probability theory the *Poisson distribution* with intensity $\lambda > 0$ is defined by $\mathcal{P}_\lambda = \{p_j(\lambda)\}_{j=0,1,\dots}$ with

$$p_j(\lambda) = \frac{\lambda^j}{j!}e^{-\lambda}. \tag{4.2.14}$$

The positive numbers (4.2.14) sum up to 1, so that $\mathcal{P}_\lambda$ is a probability distribution. Note that definition (4.2.14) extends to $\lambda = 0$ as follows:

$$p_j(0) = \delta_{j0} \doteq \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.15}$$

Here we use again Kronecker's symbol, as in Part I, Section 2.2.2. The following property of the Poisson distribution is well-known.

Lemma 4.2.2. *At each $t \in (0,T]$ let the Poisson distribution $\mathcal{P}_{t\lambda} = \{p_j(t\lambda)\}_{j=0}$ be defined by (4.2.14) with the intensity $t\lambda$. This distribution satisfies the following system of differential-difference equations*

$$\frac{dp_j(t\lambda)}{dt} = -\lambda(p_j(t\lambda) - p_{j-1}(t\lambda)), \;\; j = 0,1,\ldots$$

with $p_{-1}(t\lambda) \equiv 0$ and the initial conditions (4.2.15).

Proof. This is easily verified by the direct differentiation of (4.2.14). □

For more details see, e.g. [34], vol 1, Section 17.2, or [11], Section 4.1.

Solution to the differential-difference equations. Let $t \in [0,T]$. Consider the system of differential-difference equations

$$\frac{d\grave{x}_k(t)}{dt} = -\lambda\left(\grave{x}_{k+1}(t) - \grave{x}_k(t)\right), \;\; k = 0,1,\ldots, \tag{4.2.16}$$

subject to the boundary conditions

$$\grave{x}_k(T) = \grave{h}_k(T), \;\; k = 0,1,\ldots, \tag{4.2.17}$$

with given numbers $\{\grave{h}_k(T)\}_{k=0,1,\ldots}$. The parameter $\lambda > 0$ is the same as in (4.2.14), cf also (4.4.13) and (4.4.29) below. Lemma 4.2.2 allows for the following explicit solution of the system of equations (4.2.16).

Proposition 4.2.3. *The system (4.2.16) of differential equations in the interval $t \in [0,T]$, subject to the boundary conditions (4.2.17), is satisfied by*

$$\grave{x}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T), \;\; k = 0,1,\ldots \tag{4.2.18}$$

provided that the numbers $\{\grave{h}_k(T)\}_{k=0,1,\ldots}$ allow the differentiation under the summation sign. In particular

$$\grave{x}_0(0) = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{h}_j(T). \tag{4.2.19}$$

Proof. The boundary conditions (4.2.17) are satisfied due to property (4.2.15) of the Poisson distribution. Differentiating both sides of (4.2.18) we get by Lemma 4.2.2 that

$$\begin{aligned}\frac{d\grave{x}_k(t)}{dt} = & - \lambda\{\sum_{j=1}^{\infty} p_{j-1}(\lambda(T-t))\grave{h}_{k+j}(T) \\ & - \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{h}_{k+j}(T)\} = -\lambda\left(\grave{x}_{k+1}(t) - \grave{x}_k(t)\right).\end{aligned}$$

This yields (4.2.16). The proof is complete. □

4.3 Binary market

4.3.1 Conditions on the bond and stock price processes

Consider a binary securities market in which the bond and stock are traded during the time interval $[0, T]$ which is partitioned in N trading periods by a grid $\{t_0, t_1, \ldots, t_N\}$. Unlike in Chapter 2, the prices on the bond and stock announced at the n^{th} trading time t_n with $n \in \{0, 1, \ldots, N\}$ are now denoted by B_n^N and S_n^N, respectively, in order to express the dependence on N. Moreover, the corresponding price processes $B^N = \{B_t^N\}_{t\in[0,T]}$ and $S^N = \{S_t^N\}_{t\in[0,T]}$ are defined in the entire time interval $[0, T]$ by

$$B^N(t) = \sum_{n=0}^{N} B_n^N I_{[t_n, t_{n+1})}(t) \tag{4.3.1}$$

and

$$S^N(t) = \sum_{n=0}^{N} S_n^N I_{[t_n, t_{n+1})}(t). \tag{4.3.2}$$

As in (4.2.1), an additional instant $t_{N+1} \geq T$ is introduced for convenience. Put $B^N(0) = 1$ and $S^N(0) = s$ for simplicity, where s is a certain positive number. The discounted stock price process is denoted as in Chapter 2 by $\grave{S}^N = \{\grave{S}_t^N\}_{t\in[0,T]}$ with

$$\grave{S}^N(t) = \frac{S^N(t)}{B^N(t)}. \tag{4.3.3}$$

The bond is a riskless asset and the price process B^N evolves along a prescribed piecewise constant trajectory, while the stock is a risky asset and the price process S^N is allowed to evolve along 2^N different piecewise constant trajectories. These trajectories are specified by the binary transition scheme of Part I, Section 2.3.1. They all start from the same fixed state s, the current state of the stock price

$$s = s_{10}^N > 0.$$

Further, the whole price tree is uniquely determined by two offsprings at each trading time. If at t_{n-1} with $n = 1, \ldots, N$ the stock price was in state $s_{k\,n-1}^N$, then at the consecutive trading time t_n it is announced either in state $s_{2k\,n}^N$ or $s_{2k-1\,n}^N$ with

$$s_{2k\,n}^N > s_{2k-1\,n}^N > 0.$$

Hence if $t \in [t_n, t_{n+1})$ with some $n = 0, 1, \ldots, N$ the stock price $S^N(t)$ may occupy one of the states $\{s_k^N(t)\}_{k=1,\ldots,2^n}$ with $s_k^N(t) \equiv s_{kn}^N$. During the first period $[t_0, t_1)$, for instance, the stock price stays in the current state $s > 0$. At the terminal date $t_N = T$ the stock price $S^N(T)$ may occupy one of 2^N

states $s_k^N(T)$ with some $k = 1, \ldots, 2^N$. In Part I, Section 2.3.1, we have agreed to also say that the stock price evolves along the k^{th} trajectory. In order to describe the stock price development along this particular trajectory, we specify the stock price state at each $t \in [t_n, t_{n+1})$ for $n = 1, \ldots, N$ by the identity $s^N_{k_{N-n}}(t) \equiv s^N_{k_{N-n}\,n}$ where k_n is the n^{th} dyadic fraction of k, i.e.

$$k_n = \lceil k/2^n \rceil \,, \tag{4.3.4}$$

see Part I, Section 2.2.2, for more details. Note that by definition

$$S^N(t-) = sI_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} S_n^N I_{(t_n,t_{n+1}]}(t), \tag{4.3.5}$$

cf (4.2.10).

We shall now formulate the conditions of the present chapter which restrict the behaviour of asset prices in the market, to allow for the limiting transition in Section 4.5 below, when the number of trading periods N increases unboundedly while the stretch of each trading period, say $\Delta t_n = t_n - t_{n-1}$ with $n \in \{1, \ldots, N\}$, tends to zero. For instance, think of the special case of markets where new prices are announced regularly so that the trading times are equidistant, given by $t_n = nT/N$, and the corresponding mesh is given by $\Delta t_n = T/N$ (see Section 4.5.1 below for more details on this special case). In fact, all the entries $\{t_j\}_{j=0,1,\ldots,N}$ in the N^{th} grid depend on N and one should write $\{t_j^N\}_{j=0,1,\ldots,N}$ instead, but for simplicity the upper index is always suppressed.

Our conditions will be formulated in terms of net returns on both assets. At the current date $t_0 = 0$ the net return on the bond equals to 0 by convention and at $t \in (0, T]$ to

$$\Delta\mathcal{R}^N(t) \doteq \frac{\Delta B_t^N}{B_{t-}^N} = \frac{B_t^N}{B_{t-}^N} - 1. \tag{4.3.6}$$

Obviously, this is non-zero only at $t \in \{t_1, \ldots, t_N\}$. The cumulative return process on the bond $\mathcal{R}^N = \{\mathcal{R}_t^N\}_{t\in[0,T]}$ is defined as the sum of all previous net returns:

$$\mathcal{R}^N(t) = \sum_{u\in(0,t]} \frac{\Delta B_u^N}{B_{u-}^N} = \int_0^t \frac{dB_u^N}{B_{u-}^N} \tag{4.3.7}$$

and, in terms of Section 4.2.1,

$$B^N(t) = \prod_{u\in(0,t]} (1 + \Delta\mathcal{R}_u^N) = \mathcal{E}(\mathcal{R}^N)_t. \tag{4.3.8}$$

Condition 4.3.1. *As $N \to \infty$ the increase of the return process on the bond over each trading period becomes proportional to the length of this period: for each $n = 1, \ldots, N$*

$$\frac{\mathcal{R}^N(t_n) - \mathcal{R}^N(t_{n-1})}{t_n - t_{n-1}} = r + \varrho_n^N,$$

where $r > 0$ is a positive constant, while ϱ_n^N is a negligible remainder term.

Obviously, Condition 4.3.1 means that at the trading time t_n with $n \in \{1, \dots, N\}$ the return on the bond

$$\rho_n^N \doteq \Delta\mathcal{R}^N(t_n) = \mathcal{R}^N(t_n) - \mathcal{R}^N(t_{n-1}) \tag{4.3.9}$$

is asymptotically proportional to the length of the preceding period:

$$\rho_n^N = (r + \varrho_n^N)\Delta t_n \sim r\Delta t_n. \tag{4.3.10}$$

Here and elsewhere below the sign $\sim$ indicates that the ratio of the two sides tends to unity. In view of (4.3.6)–(4.3.8), we have for each $t \in [0, T]$ that

$$\mathcal{R}^N(t) \sim rt \tag{4.3.11}$$

and

$$B^N(t) \sim e^{rt}. \tag{4.3.12}$$

Indeed, by (4.3.10)

$$\log B^N(t) \sim \sum_k \log(1 + r\Delta t_k) \sim r \sum_k \Delta t_k$$

(here $\log(1 + x) \sim x$ is used) and

$$\mathcal{R}^N(t) \sim r \sum_k \Delta t_k \sim rt$$

where the summation extends over the lengths of all past periods prior to time t. Asymptotically, this means that the cumulative return process on the bond is assumed to increase with a constant interest rate r.

The cumulative return process on the stock $R^N = \{R_t^N\}_{t\in[0,T]}$ is defined similarly as the sum of all previous net returns. At the current date $t_0 = 0$ the net return equals to 0 and at $t \in (0, T]$ to

$$\Delta R^N(t) \doteq \frac{\Delta S_t^N}{S_{t-}^N} = \frac{S_t^N}{S_{t-}^N} - 1.$$

Obviously, the latter expression is non-zero only at $t \in \{t_1, \dots, t_N\}$. So

$$R^N(t) = \int_0^t \frac{dS_u^N}{S_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta S_u^N}{S_{u-}^N}$$

and

$$S^N(t) = s\mathcal{E}(R^N)_t = s \prod_{u\in(0,t]} (1 + \Delta R_u^N), \tag{4.3.13}$$

analogously to (4.3.7) and (4.3.8). In Part I, Section 2.3.1, we also introduced the discounted cumulative return process $\grave{R}^N = \{\grave{R}_t^N\}_{t\in[0,T]}$ by

$$\grave{R}^N(t) = \int_0^t \frac{d\grave{S}_u^N}{\grave{S}_{u-}^N} = \sum_{u\in(0,t]} \frac{\Delta\grave{S}_u^N}{\grave{S}_{u-}^N},$$

with the discounted stock price process $\grave{S}^N$ defined by (4.3.3). Obviously,

$$\grave{S}^N(t) = s\mathcal{E}(\grave{R}^N)_t = s \prod_{u\in(0,t]} (1 + \Delta\grave{R}^N_u).$$

We are now going to formulate conditions on the behaviour at the trading times $t_n, n = 1, \dots, N$, of the net returns on the stock $\Delta R^N(t_n)$ in terms of their states

$$r^N_{kn} = z^N_{kn} - 1 = \frac{s^N_{kn}}{s^N_{k_1\,n-1}} - 1, \quad k = 1, \dots, 2^n, \tag{4.3.14}$$

where $k_1 = \lceil k/2 \rceil$, as in Part I, Formula (2.3.14).

Condition 4.3.2. *At the trading time t_n with some $n = 1, \dots, N$ the net return on the stock $\Delta R^N(t_n)$ is in one of the 2^n states*

$$r^N_{2k\,n} = a + \alpha^N_{2k\,n}, \quad k = 1, \dots, 2^{n-1}, \tag{4.3.15}$$

or

$$r^N_{2k-1\,n} = -(b + \beta^N_{2k-1\,n})\Delta t_n, \quad k = 1, \dots, 2^{n-1}, \tag{4.3.16}$$

where a and b are some positive constants, while $\{\alpha^N_{2k\,n}\}_{k=1,\dots,2^{n-1}}$ and $\{\beta^N_{2k-1\,n}\}$ are negligible remainder terms as $N \to \infty$. In fact b can be negative but exceeding $-r$ with $r > 0$ of Condition 4.3.1, to guarantee inequality (4.3.19) below.

Using the same sign $\sim$ as above we may express (4.3.15) and (4.3.16) in the following form

$$r^N_{kn} \sim \begin{cases} a & \text{if } k \text{ is even} \\ -b\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{4.3.17}$$

If Condition 4.3.1 holds as well, then the states $\{\grave{r}^N_{kn}\}_{k=1,\dots,2^n}$ of the discounted net return $\Delta\grave{R}^N(t_n)$ with $n \in \{1, \dots, N\}$ are approximated as follows. Due to (4.3.10) it follows from the relation (2.3.16) in Part I, Chapter 2, that the net returns on the stock $\Delta\grave{R}^N(t_n)$ is approximated by the excess return $\Delta(R^N - \mathcal{R}^N)_n$. Apply (4.3.10) and (4.3.17) to the latter expression to get

$$\grave{r}^N_{kn} \sim \begin{cases} a & \text{if } k \text{ is even} \\ -a\lambda\Delta t_n & \text{if } k \text{ is odd} \end{cases} \tag{4.3.18}$$

where

$$\lambda = \frac{r + b}{a} > 0 \tag{4.3.19}$$

is a parameter which later on will take on the rôle of the intensity of the Poisson distribution, cf (4.2.14). Since the even state indices correspond to the upward displacements, and the odd indices to the downward displacements, the asymptotic relations (4.3.17) and (4.3.18) tell us that for N sufficiently large all the upward displacements are of the same order $a > 0$. We call this parameter a the *amplitude* of the upward displacements. On the other hand, all the downward displacements are infinitesimal, of magnitude Δt_n. Look, for instance, at the condition (4.3.18) concerning the discounted net returns on the stock. The range of the parameters a and b are restricted so that the inequality (4.3.19) holds and in contrast to the upward displacements, the downward displacements are negative. It will be seen below that this is crucial for developing the theory of the present chapter. In particular, the following lemma will be referred to.

Lemma 4.3.3. *Under the conditions 4.3.1 and 4.3.2 the weights $\lambda\Delta t_n$ and $1-\lambda\Delta t_n$ neutralize the upward displacements of the discounted net returns on the stock against the downward displacements, in the sense that*

$$\lambda\Delta t_n \grave{r}^N_{2k\,n} + (1-\lambda\Delta t_n)\grave{r}^N_{2k-1\,n} \sim 0 \tag{4.3.20}$$

for all $k = 1, \ldots, 2^{n-1}$ and $n = 1, \ldots, N$.

Proof. By (4.3.18)

$$\begin{aligned}\lambda\Delta t_n \grave{r}^N_{2k\,n} + (1-\lambda\Delta t_n)\grave{r}^N_{2k-1\,n} &\sim a\lambda\Delta t_n + (1-\lambda\Delta t_n)(-a\lambda\Delta t_n)\\ &= a(\lambda\Delta t_n)^2\end{aligned}$$

which is of a lower magnitude than (4.3.40). This is equivalent to the desired relation (4.3.20), in view of our convention about the usage of $\sim$. □

The comparison of the assertions of Lemma 4.3.3 and Lemma 3.3.1 in Part I suggests us to assign the weights $\lambda\Delta t_n$ and $1-\lambda\Delta t_n$ to the upward and downward displacements, respectively, and to bring them in connection with the risk neutral probabilities. We will return to this in the next section, see formula (4.3.40). Note meanwhile that in this probabilistic interpretation the upward displacements become rare events, since the corresponding transition probabilities become infinitesimal, proportional to Δt_n. This will provide in Section 4.5 the conditions for the Poisson approximation.

Remark 4.3.4. In Condition 4.3.1 the remainder term ϱ^N_n is called negligible, because it can be suppressed in the asymptotic relation (4.3.10) (as well as in the consequent relations (4.3.11) and (4.3.12) or elsewhere below). This is achieved, in particular, if

$$\max_{n\in\{1,\ldots,N\}} |\varrho^N_n| \to 0$$

as $N \to \infty$. In Condition 4.3.2 the remainder terms, all the α's and β's, are negligible in the asymptotic relation (4.3.17) (or elsewhere below) if, for instance, it holds that

$$\max_{n\in\{1,\ldots,N\}} \max_{k\in\{1,\ldots,2^{n-1}\}} |\alpha^N_{2k\,n}| \to 0$$

$$\max_{n\in\{1,\ldots,N\}} \max_{k\in\{1,\ldots,2^{n-1}\}} |\beta^N_{2k-1\,n}| \to 0$$

as $N \to \infty$.

4.3.2 Self-financing strategies and value processes

Self-financing. In the present setup the situation of Section 3.2.1 is described as follows. One starts again with the investment that amounts to

$$v = \Psi_0 + s\Phi_0 \geq 0 \tag{4.3.21}$$

where Ψ_0 and Φ_0 are the numbers of shares of the bond and stock, respectively, owned at the current date $t_0 = 0$. Let Ψ_n^N and Φ_n^N denote the number of shares of the bond and stock, respectively, owned by the investor at the consecutive trading times t_n, $n = 1, \ldots, N$. At any time $t \in [0, T]$ investor's portfolio (Ψ_t^N, Φ_t^N) is then defined by the bond and stock components

$$\Psi^N(t) = \Psi_1^N I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \Psi_{n+1}^N I_{(t_n,t_{n+1}]}(t) \tag{4.3.22}$$

and

$$\Phi^N(t) = \Phi_1^N I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \Phi_{n+1}^N I_{(t_n,t_{n+1}]}(t). \tag{4.3.23}$$

Note that the possible trajectories of both components are piecewise constant functions of the same type as those of S_-^N, compare (4.3.22) and (4.3.23) with (4.3.5). At each $t \in (0, T]$ the dependence of the portfolio only on the past prices means that if $t \in (t_n, t_{n+1}]$ and $S^N(t-)$ is in state s_{kn}^N for some $n = 0, 1, \ldots, N-1$ and $k = 1, \ldots, 2^n$, then Ψ_t^N is in state ψ_{kn}^N, likewise Φ_t^N is in state ϕ_{kn}^N. In stochastic calculus processes of this type are called *simple predictable*, cf [63], p 43.

The process $\pi^N = (\Psi_t^N, \Phi_t^N)_{t\in[0,T]}$ is called a *self-financing trading strategy*, if the portfolio is selected so that for all $t \in [0, T]$

$$B_-^N \cdot \Psi_t^N + S_-^N \cdot \Phi_t^N = 0, \tag{4.3.24}$$

cf (3.2.16) in Part I. The notion just introduced is of universal use whenever the integrals in (4.3.24) are well-defined, which in the present special case of piecewise constant portfolio components (4.3.22) and (4.3.23) are particularly simple:

$$B_-^N \cdot \Psi_t^N = \sum_{u\in[0,t]} B_{u-}^N \Delta\Psi_u^N \tag{4.3.25}$$

and

$$S_-^N \cdot \Phi_t^N = \sum_{u\in[0,t]} S_{u-}^N \Delta\Phi_u^N, \tag{4.3.26}$$

cf (4.2.5). In view of (4.3.25) and (4.3.26) the condition (4.3.24) is equivalent to

$$B_-^N \Delta\Psi^N + S_-^N \Delta\Phi^N = 0. \tag{4.3.27}$$

Value process. Like in Part I, Formula (3.2.3), with each trading strategy π^N we associate the value process $V^N(\pi) = \Psi^N B^N + \Phi^N S^N$, which means that the process $V^N(\pi) = \{V_t^N(\pi)\}_{t\in[0,T]}$ is defined by

$$V^N(t;\pi) = \Psi_t^N B_t^N + \Phi_t^N S_t^N \tag{4.3.28}$$

so that $V^N(0;\pi) = v \geq 0$, cf (4.3.21).

According to Formula (3.2.12) in Part I, if π^N is self-financing trading strategy, then we also have $V_-^N(\pi) = \Psi^N B_-^N + \Phi^N S_-^N$. That is

$$V^N(t-;\pi) = \Psi_t^N B_{t-}^N + \Phi_t^N S_{t-}^N. \tag{4.3.29}$$

Using the integrating by parts formula of Section 4.2.1, we obtain exactly in the same manner as in Part I, Section 3.2.2, the following characterization of self-financing strategies.

Proposition 4.3.5. *A trading strategy π^N is self-financing if and only if its discounted value process $\grave{V}^N(\pi) = \{\grave{V}_t^N(\pi)\}_{t\in[0,T]}$ admits the following integral representation: at each $t \in [0,T]$*

$$\grave{V}^N(t;\pi) = v + \Phi^N \cdot \grave{S}_t^N. \tag{4.3.30}$$

Proof. Take into consideration (4.3.25)–(4.3.28) and apply the same arguments as in Part I, Proposition 3.2.4. □

Note that the integral representation (4.3.30) is valid also in general, if only the integrals are well-defined and Proposition 4.2.1 holds. This is easily seen by taking into consideration that (4.3.24) is equivalent to

$$\Psi_t^N - \Psi_0 + \grave{S}_-^N \cdot \Phi_t^N = 0,$$

cf Part I, (3.2.17). We want to emphasize that the value process for a self-financing strategy is a process of the same type as the stock price process, since it evolves along one of 2^N piecewise constant trajectories. In fact we have

$$\grave{V}^N(t;\pi) = \sum_{n=0}^{N} \grave{V}_n^N(\pi) I_{[t_n,t_{n+1})}(t), \tag{4.3.31}$$

like (4.3.2), where $\grave{V}_n^N(\pi)$ is a variable whose states are described according to formula (3.2.21) in Part I, i.e.

$$\grave{v}_{kn}^N(\pi) = v + \sum_{\nu=1}^{n} \phi_{k_{n-\nu+1}\,\nu-1}^N (\grave{s}_{k_{n-\nu}\,\nu}^N - \grave{s}_{k_{n-\nu+1}\,\nu-1}^N) \tag{4.3.32}$$

with $k = 1, \ldots, 2^n$ and k_n given by (4.3.4).

Example 4.3.6. Constant portfolio. Like in the Example 3.4.4 in Part I, suppose that one keeps the constant portfolio $(\Psi_t^N, \Phi_t^N) = (\psi, \phi)$ all the time $t \in [0,T]$, that is obviously a self-financing strategy - no infusion or withdrawal of funds is needed (both terms on the left-hand side of (4.3.27) equal to 0). The integral representation (4.3.30) gives $\grave{V}^N(t;\pi) = v + \phi(\grave{S}^N(t) - s) = \psi + \phi\grave{S}^N(t)$.

Clark's formula. The integral representation (4.3.30) will again be given the form of Clark's formula, as in Part I, Section 3.2.2. To this end we need to recall how the difference operator D acts in the state space. In view of (4.2.8) and (4.2.9) the process $DV^N(\pi)/DS^N$ is defined by

$$\frac{DV^N(t;\pi)}{DS^N(t)} = \frac{DV_1^N(\pi)}{DS_1^N} I_{[t_0,t_1]}(t) + \sum_{n=1}^{N-1} \frac{DV_{n+1}^N(\pi)}{DS_{n+1}^N} I_{(t_n,t_{n+1}]}(t). \tag{4.3.33}$$

This is in state

$$\frac{v_{2k\,n+1}^N(\pi) - v_{2k-1\,n+1}^N(\pi)}{s_{2k\,n+1}^N - s_{2k-1\,n+1}^N},$$

provided $S^N(t-)$ has been in state s_{kn}^N. The results similar to Proposition 3.2.1 and Corollary 3.2.5 in Part I are now formulated as follows:

Proposition 4.3.7. *Under the self-financing condition (4.3.24) the stock component of the portfolio is given by (4.3.33): for each $t \in [0,T]$*

$$\Phi^N(t) = \frac{DV^N(t;\pi)}{DS^N(t)}.$$

Therefore the integral representation (4.3.30) takes the form

$$\grave{V}^N(t;\pi) = v + \frac{DV^N(\pi)}{DS^N} \cdot \grave{S}_t^N. \tag{4.3.34}$$

Risk neutral probabilities. In markets excluding arbitrage opportunities the discounted stock price process evolves so that

$$\grave{s}_{2k-1\,n}^N < \grave{s}_{k\,n-1}^N < \grave{s}_{2k\,n}^N$$

for each $n = 1, \ldots, N$ and $k = 1, \ldots, 2^{n-1}$, as was shown in Part I, Section 3.6. This means that the numerical values of

$$p_{2k\,n}^N = \frac{\grave{s}_{k\,n-1}^N - \grave{s}_{2k-1\,n}^N}{\grave{s}_{2k\,n}^N - \grave{s}_{2k-1\,n}^N} \tag{4.3.35}$$

and

$$p_{2k-1\,n}^N = \frac{\grave{s}_{2k\,n}^N - \grave{s}_{k\,n-1}^N}{\grave{s}_{2k\,n}^N - \grave{s}_{2k-1\,n}^N} \tag{4.3.36}$$

are positive and satisfy

$$p_{2k\,n}^N + p_{2k-1\,n}^N = 1. \tag{4.3.37}$$

In Part I, Section 3.3.1, we have seen that every state $s_{k\,n-1}^N$ at trading time t_{n-1} is expressed as a convex combination of two alternative states $\grave{s}_{2k-1\,n}^N$ and $\grave{s}_{2k\,n}^N$ at the consecutive trading time t_n, i.e.

$$\grave{s}_{k\,n-1}^N = p_{2k\,n}^N \grave{s}_{2k\,n}^N + p_{2k-1\,n}^N \grave{s}_{2k-1\,n}^N. \tag{4.3.38}$$

According to Lemma 3.3.1 in Part I, this is expressed in terms of the discounted net returns as follows:

$$0 = p_{2k\,n}^N \hat{r}_{2k\,n}^N + p_{2k-1\,n}^N \hat{r}_{2k-1\,n}^N. \tag{4.3.39}$$

The latter relations tell us that the weights $p_{2k\,n}^N$ and $p_{2k-1\,n}^N$ are chosen so as to neutralize the upward displacements in the discounted returns by the downward displacements. This gives rise to the term *risk neutral probabilities.* Note the following consequence of (4.3.20) and (4.3.39): under the conditions 4.3.1 and 4.3.2 the risk neutral probabilities are proportional to Δt_n. Namely, for each $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$

$$p_{2k\,n}^N \sim \lambda \Delta t_n. \tag{4.3.40}$$

We have already mentioned this at the end of the previous section. We will show next that the relationship (4.3.40) can be made precise.

Lemma 4.3.8. *Under the conditions 4.3.1 and 4.3.2 we have for each $n = 1, \dots, N$ that*

$$p_{2k\,n}^N = (\lambda + \lambda_{2k\,n}^N)\Delta t_n, \quad k = 1, \dots, 2^{n-1}, \tag{4.3.41}$$

with remainder terms $\{\lambda_{2k\,n}^N\}_{k=1,\dots,2^{n-1}}$, negligible in the sense of Remark 4.3.4.

Proof. It will be shown that with the notations of the conditions 4.3.1 and 4.3.2

$$\lambda_{2k\,n}^N = \frac{\varrho_n^N + \beta_{2k-1\,n}^N}{r_{2k\,n}^N - r_{2k-1\,n}^N} - \lambda \frac{\alpha_{2k\,n}^N - r_{2k-1\,n}^N}{r_{2k\,n}^N - r_{2k-1\,n}^N}, \tag{4.3.42}$$

which is indeed negligible under these conditions (cf Remark 4.3.4). In view of the notation (4.3.9) for ρ_n^N and the usual relationship (4.3.14) between the gross and net returns, we can rewrite formula (3.3.8) in Part I in terms of net returns on both assets:

$$p_{2k\,n}^N = \frac{\rho_n^N - r_{2k-1\,n}^N}{r_{2k\,n}^N - r_{2k-1\,n}^N}. \tag{4.3.43}$$

Due to (4.3.9), (4.3.15), (4.3.16) and (4.3.19) it follows from the latter equality that

$$\frac{p_{2k\,n}^N - \lambda\Delta t_n}{\Delta t_n} = \frac{\varrho_n^N + \beta_{2k-1\,n}^N}{r_{2k\,n}^N - r_{2k-1\,n}^N} - \lambda \frac{\alpha_{2k\,n}^N + (b + \beta_{2k-1\,n}^N)\Delta t_n}{r_{2k\,n}^N - r_{2k-1\,n}^N}.$$

Compare this with (4.3.41). It is easily seen that (4.3.42) holds. □

In the sequel we will retain the notations (3.3.19) and (3.3.20) of Part I. We thus write

$$P_{kN}^n = \prod_{0 \le \nu < n} p_{k_\nu\,N-\nu}^N \tag{4.3.44}$$

for the transition probabilities and $\{P_{kN}\}_{k=1,\dots,2^N}$ for the probability distribution with

$$P_{kN} = \prod_{0 \le \nu < N} p_{k_\nu\,N-\nu}^N. \tag{4.3.45}$$

Recurrence relations for value processes. In Part I, Sections 3.3.1 and 3.3.2, the equations (4.3.38) were related to the martingale property of the discounted stock price process with respect to the probability distribution $\{P_{kN}\}_{k=1,\dots,2^N}$. It has been observed that the integral representation (4.3.30) preserves the martingale property and parallel to (4.3.38) we have that the states (4.3.32) of the value process $V^N(\pi)$ for any self-financing strategy π^N satisfy for $n = 1, \dots, N$ the relations (3.3.14), see Part I, Proposition 3.3.4. For another point of view on these equations, rewrite them by using $p^N_{2k-1\,n} = 1 - p^N_{2k\,n}$ in the following form: for $k = 1, \dots, 2^{n-1}$

$$\grave{v}^N_{2k-1\,n}(\pi) - \grave{v}^N_{k\,n-1}(\pi) = -p^N_{2k\,n}(\grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{2k-1\,n}(\pi)) \tag{4.3.46}$$

or, equivalently,

$$\grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{k\,n-1}(\pi) = p^N_{2k-1\,n}(\grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{2k-1\,n}(\pi)) \tag{4.3.47}$$

with the increments in time on the left hand side and the increments in the state space on the right, cf (3.3.17). Hence, in (4.3.46) and (4.3.47) the risk neutral probabilities take on the rôle of the proportionality coefficients. In virtue of Lemma 4.3.8 we can give to the relations (4.3.46) an approximate form asserted in the next proposition. Later, in Section 4.4, we will trace similarity to the equations (4.4.29) for the limiting Poisson model.

Proposition 4.3.9. *Let π^N be a self-financing strategy and $\grave{V}^N(\pi)$ its discounted value process. Then for each $n = 1, \dots, N$ the set of states $\{\grave{v}^N_{kn}(\pi)\}_{k=1,.}$ given by (4.3.32), satisfies the following identities:*

$$\frac{\grave{v}^N_{2k-1\,n}(\pi) - \grave{v}^N_{k\,n-1}(\pi)}{t_n - t_{n-1}} = \tag{4.3.48}$$

$$-(\lambda + \lambda^N_{2k\,n}) \quad (\grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{2k-1\,n}(\pi)) \tag{4.3.49}$$

with λ and $\{\lambda^N_{2k\,n}\}_{k=1,\dots,2^{n-1}}$ as in (4.3.19) and (4.3.42), respectively.

If, moreover, the conditions 4.3.1 and 4.3.2 hold, then

$$\frac{\grave{v}^N_{2k-1\,n}(\pi) - \grave{v}^N_{k\,n-1}(\pi)}{t_n - t_{n-1}} \sim -\lambda(\grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{2k-1\,n}(\pi)). \tag{4.3.50}$$

Proof. It is easily verified that the equations (4.3.49) and (4.3.50) take the following undiscounted form:

$$\begin{aligned} \frac{v^N_{2k-1\,n}(\pi) - v^N_{k\,n-1}(\pi)}{t_n - t_{n-1}} &- \quad (r + \varrho^N_n)v^N_{k\,n-1}(\pi) \\ = -(\lambda + \lambda^N_{2k\,n}) &\quad (v^N_{2k\,n}(\pi) - v^N_{2k-1\,n}(\pi)) \end{aligned} \tag{4.3.51}$$

and

$$\begin{aligned} \frac{v^N_{2k-1\,n}(\pi) - v^N_{k\,n-1}(\pi)}{t_n - t_{n-1}} &- \quad r v^N_{k\,n-1}(\pi) \\ \sim &- \quad \lambda(v^N_{2k\,n}(\pi) - v^N_{2k-1\,n}(\pi)). \end{aligned} \tag{4.3.52}$$

We will prove these undiscounted equations (4.3.51) and (4.3.52). By the undiscounted version of equation (4.3.46)

$$(1+\rho_n^N)v_{k\,n-1}^N(\pi) - v_{2k-1\,n}^N(\pi) = p_{2k\,n}^N(v_{2k\,n}^N(\pi) - v_{2k-1\,n}^N(\pi))$$

which by (4.3.41) yields

$$\begin{aligned}&(1+\rho_n^N)v_{k\,n-1}^N(\pi) - v_{2k-1\,n}^N(\pi)\\ &= (v_{2k\,n}^N(\pi) - v_{2k-1\,n}^N(\pi))(\lambda + \lambda_{2k\,n}^N)\Delta t_n.\end{aligned}$$

Due to (4.3.9), this is equivalent to

$$\begin{aligned}&v_{k\,n-1}^N(\pi) - v_{2k-1\,n}^N(\pi) + (r+\varrho_n^N)v_{k\,n-1}^N(\pi)\Delta t_n\\ &\quad = (v_{2k\,n}^N(\pi) - v_{2k-1\,n}^N(\pi))(\lambda + \lambda_{2k\,n}^N)\Delta t_n\end{aligned}$$

which yields (4.3.51). Moreover, (4.3.51) implies (4.3.52), since ϱ_n^N and $\{\lambda_{2k\,n}^N\}_{k=}$ are negligible remainder terms. □

4.3.3 Completeness, hedging strategy and option valuation

Completeness and hedging strategies. Let $W^N(T)$ be a certain wealth to be attained at the terminal date $t = T$, that is a variable with 2^N possible states $\{\grave{w}_k^N(T)\}_{k=1,\dots,2^N}$. As we know from Part I, Section 3.4, in the binary market any such wealth is attainable by a certain self-financing strategy, called the *hedging strategy* against $W^N(T)$, if only the investment amounts to

$$w_{10} = \sum_{k=1}^{2^N} P_{kN}\grave{w}_k^N(T), \tag{4.3.53}$$

cf Part I, Formula (3.4.3). This property of the binary market is called the *completeness*. Investor's hedging strategy is described as follows. By using (4.3.44), the states of variables $\grave{W}_{N-n}^N$ are defined by

$$\grave{w}_{j\,N-n}^N = \sum_{2^n(j-1)<k\le 2^n j} P_{kN}^n\grave{w}_k^N(T),\ \ j=1,\dots,2^{N-n}, \tag{4.3.54}$$

with $n = 1,\dots,N$. Then the process $\grave{W}^N = \{\grave{W}_t^N\}_{t\in[0,T]}$ is formed by

$$\grave{W}^N(t) = \sum_{n=0}^{N} \grave{W}_n^N I_{[t_n,t_{n+1})}(t). \tag{4.3.55}$$

Obviously, the process starts from $\grave{W}^N(0) = \grave{w}_{10}^N$, i.e. from (4.3.53). At the terminal date $t = T$ it arrives at the desired wealth $\grave{W}^N(T)$. Following the procedure of Part I, Section 3.4.2, we can find a unique strategy π^N so that the corresponding discounted value process (4.3.31) is identified to the process (4.3.55). This is the hedging strategy against $\grave{W}^N(T)$ we are looking for.

Option valuation. In Part I, Section 3.5, formula (4.3.53) is applied to the following problem of option pricing. Suppose that today, at time $t = 0$, we are going to sign a contract that gives us the right to buy one share of a stock at a specified price K, called the *exercise price*, and at a specified time T, called the *maturity* or *expiration time*. If the stock price $S^N(T)$ is below the exercise price, i.e. $S^N(T) \leq K$, then the contract is worthless to us. On the other hand, if $S^N(T) > K$, we can exercise our option: we can buy one share of the stock at the fixed price K and then sell it immediately in the market for the price $S^N(T)$. Thus this option, called the *European call option*, yields a profit at maturity T equal to

$$\max\{0, S^N(T) - K\} = (S^N(T) - K)^+. \tag{4.3.56}$$

The function (4.3.56) of the stock price $S^N(T)$ is called the *payoff function* for the European call option.

A contract with some fixed payoff function (not necessarily of form (4.3.56)) is called a *contingent claim.* The European call option is thus a special contingent claim with payoff (4.3.56). In general, a payoff $\grave{H}^N(T)$ is a function that maps each of 2^N possible trajectories of the discounted stock price to some nonnegative numerical value. In our usual terms, $\grave{H}^N(T)$ is assumed to occupy 2^N possible states, say $\{\grave{h}_k^N(T)\}_{k=1,\dots,2^N}$.

In order to decide how much would we be willing to pay at time $t = 0$ for a ticket which gives the right to buy at maturity $t = T$ one share of stock with exercise price K, we need to construct the hedging strategy that duplicates the payoff (4.3.56). Recall that the hedging strategy π^N against a contingent claim with a payoff function $\grave{H}^N(T)$ is a unique strategy whose value process $\grave{V}^N(\pi)$ is identical to the process $\{\grave{H}_t^N\}_{t\in[0,T]}$ that is defined by the same considerations as in the previous subsection:

$$\grave{H}^N(t) = \sum_{n=0}^{N} \grave{H}_n^N I_{[t_n,t_{n+1})}(t).$$

where $\grave{H}_{N-n}^N$ is a variable with the states

$$\sum_{2^n(j-1)<k\leq 2^n j} P_{kN}^n \grave{w}_k^N(T), \quad j = 1, \dots, 2^{N-n},$$

cf (4.3.54) and (4.3.55). This strategy indeed duplicates the payoff. It requires the initial wealth that yields the fair price $C^N = C(H^N)$ of the contingent claim with the payoff function $H^N(T)$, that amounts to

$$C(\grave{H}^N) = \sum_{k=1}^{2^N} P_{kN} \grave{h}_j^N(T).$$

The fair price of the European call option (4.3.56), in particular, amounts to

$$C^N = \sum_{k=1}^{2^N} P_{kN}(\grave{s}_k^N(T) - \grave{K})^+. \tag{4.3.57}$$

4.4 Poisson market

4.4.1 Asset pricing

Bond and stock. In this section we consider the limiting model for a securities market that stems from the approximate relations of Section 4.3.1. In view of (4.3.11) and (4.3.12), the model for the bond is defined by the linear return process $\mathcal{R}^\circ = \{\mathcal{R}_t^\circ\}_{t\in[0,T]}$ with

$$\mathcal{R}_t^\circ = rt \tag{4.4.1}$$

and the exponential price process $B^\circ = \{B_t^\circ\}_{t\in[0,T]}$ with

$$B_t^\circ = e^{rt}, \tag{4.4.2}$$

where $r > 0$ is a riskless interest rate on the bond.

The stock is again a risky asset: its return process $R^\circ = \{R_t^\circ\}_{t\in[0,T]}$ may jump unexpectedly at certain instants. Let m be a number of jumps, a nonnegative integer that equals to 0 if only no jump occurs. Otherwise, if $m > 0$ jumps occur, we denote by $T_1, \ldots, T_m$ the consecutive instants of their occurrence. All the jumps are assumed to be of an equal size, say $a > 0$. For convenience, we also introduce $T_0 = 0$ and $T_{m+1} \geq T$. Clearly, it is assumed that $T_k < T_{k+1}$ for all $k = 0, 1, \ldots, m$. Thus the net returns on the stock are non-zero only at instants $T_1, \ldots, T_m$ and

$$\Delta R^\circ(T_k) = a > 0, \quad k = 1, \ldots, m.$$

The cumulative return process R° will be defined with the help of the so-called *counting process* $\mathcal{N} = \{\mathcal{N}_t\}_{t\in[0,T]}$ with $\mathcal{N}(t)$ that counts the number of jumps observed up to time $t \in [0, T]$. By assumption $\mathcal{N}(0) = 0$. Further $\mathcal{N}(t) = 0$ if no jump occurs up to time t and $\mathcal{N}(t) = k$ if $T_k \leq t$ for $k \in \{1, \ldots, m\}$. In particular $\mathcal{N}(T) = m$. We define next the limiting return process R° on the stock in accordance with the right-hand side of (4.3.17): the cumulative effect on the upward displacement yields $a\mathcal{N}(t)$ and that of the downward displacement yields $-bt$. This leads to the model

$$R^\circ(t) = a\mathcal{N}(t) - bt. \tag{4.4.3}$$

Consequently, the price process on the stock $S^\circ = \{S_t^\circ\}_{t\in[0,T]}$ is now defined by

$$\begin{aligned} S^\circ(t) &= s\mathcal{E}(R^\circ)_t \\ &= s(1+a)^{\mathcal{N}(t)}e^{-bt} \end{aligned} \tag{4.4.4}$$

(cf Section 4.2.1) where $s > 0$ is a fixed current price on the stock $S^\circ(0) = s$.

Due to (4.4.2) and (4.4.4), the discounted stock price process is defined by

$$\begin{aligned} \grave{S}^\circ(t) &\doteq \frac{S^\circ(t)}{B^\circ(t)} \\ &= s(1+a)^{\mathcal{N}(t)}e^{-a\lambda t} \end{aligned} \tag{4.4.5}$$

and the corresponding return process $\grave{R}^{\circ} \doteq \{\grave{R}^{\circ}_t\}_{t\in[0,T]}$ by

$$\grave{R}^{\circ}(t) \doteq a(\mathcal{N}(t) - \lambda t), \tag{4.4.6}$$

cf (4.3.18). The relation $\grave{S}^{\circ} = s\mathcal{E}(\grave{R}^{\circ})$ is obtained in Section 4.2.1.

Differential equations for states of the stock price process. The price process on the stock may be presented alternatively in terms of its state $s^{\circ}_k(t)$ within each interval $[T_k, T_{k+1}), k = 0, 1, \dots, m$, that is

$$s^{\circ}_k(t) = s(1+a)^k e^{-bt} \tag{4.4.7}$$

by (4.4.4) (note that in the present case no distinction is needed between states of the stock price and their numerical values, due to one to one correspondence). We then have at each $t \in [0, T]$ that

$$S^{\circ}(t) = \sum_{k=0}^{m} s^{\circ}_k(t) I_{[T_k, T_{k+1})}(t).$$

Likewise,

$$\grave{S}^{\circ}(t) = \sum_{k=0}^{m} \grave{s}^{\circ}_k(t) I_{[T_k, T_{k+1})}(t) \tag{4.4.8}$$

with the states

$$\grave{s}^{\circ}_k(t) = s(1+a)^k e^{-a\lambda t} \tag{4.4.9}$$

within the interval $[T_k, T_{k+1}), k = 0, 1, \dots, m$. Similarly to (4.3.5), we have

$$S^{\circ}(t-) = s^{\circ}_0(t) I_{[0,T_1]}(t) + \sum_{k=1}^{m} s^{\circ}_k(t) I_{(T_k, T_{k+1}]}(t). \tag{4.4.10}$$

Suppose that $S^{\circ}(t-)$ is in state $s^{\circ}_k(t)$. Then at time t the stock price either stays in state $s^{\circ}_k(t)$ or jump to state $s^{\circ}_{k+1}(t)$. This observation led in the last subsection of Section 4.2.1 to the following definition of the difference operator $DS^{\circ}(t)$ in the state space of the present market: if $S^{\circ}(t-)$ is in state $s^{\circ}_k(t)$, then $DS^{\circ}(t)$ is in the state

$$D_{k+1}(S^{\circ}_t) = s^{\circ}_{k+1}(t) - s^{\circ}_k(t). \tag{4.4.11}$$

Hence we have

Proposition 4.4.1. *The states (4.4.7) and (4.4.9) of the stock price process and its discounted version satisfy the following differential equations:*

$$\frac{ds^{\circ}_k(t)}{dt} - rs^{\circ}_k(t) = -\lambda D_{k+1}(S^{\circ}_t), \quad t \in (T_k, T_{k+1}], \tag{4.4.12}$$

and

$$\frac{d\grave{s}^{\circ}_k(t)}{dt} = -\lambda D_{k+1}(\grave{S}^{\circ}_t), \quad t \in (T_k, T_{k+1}]. \tag{4.4.13}$$

Proof. By (4.4.7) it follows from (4.4.11) that

$$D_k(S_t^\circ) = as(1+a)^{k-1}e^{-bt} = as_{k-1}^\circ(t).$$

Therefore, by differentiating both sides of (4.4.7) we get

$$\frac{ds_k^\circ(t)}{dt} - rs_k^\circ(t) = -(r+b)s_k^\circ(t) = -\lambda a s_k^\circ(t) = -\lambda D_{k+1}(S_t^\circ),$$

which yields (4.4.12). To verify that equation (4.4.13) follows from (4.4.12), take into consideration that by (4.2.13) and (4.4.2)

$$DS^\circ(t) = e^{-rt}DS(t) \tag{4.4.14}$$

and

$$\frac{d\grave{s}_k^\circ(t)}{dt} = e^{-rt}\left(\frac{ds_k^\circ(t)}{dt} - rs_k^\circ(t)\right). \tag{4.4.15}$$

The proof is complete. □

Compare (4.4.13) with (4.2.16) to see from the assertion of Proposition 4.2.3 that the states (4.4.9) of the discounted stock price process $\grave{S}^\circ$ satisfy the relations

$$\grave{s}_k^\circ(t) = \sum_{j=0}^\infty p_j(\lambda(T-t))\grave{s}_{k+j}^\circ(T). \tag{4.4.16}$$

In particular

$$s = \sum_{j=0}^\infty p_j(\lambda T)\grave{s}_j^\circ(T). \tag{4.4.17}$$

This can also be verified directly, since

$$\sum_{j=0}^\infty \frac{(\lambda(T-t))^j}{j!}e^{-\lambda(T-t)}s(1+a)^{k+j}e^{-a\lambda T} = s(1+a)^k e^{-a\lambda t}.$$

Cf (4.4.16) and (4.4.17) with (3.3.26) and (3.3.27) in Part I.

4.4.2 Self-financing strategies

Portfolio. Consider an investor who invests an amount $v \geq 0$ in the present market and then follows a trading strategy $\pi = (\Psi, \Phi)$ with the portfolio components $\Psi = \{\Psi_t\}_{t\in[0,T]}$ and $\Phi = \{\Phi_t\}_{t\in[0,T]}$ which yield the value process $V^\circ(\pi) = \Psi B^\circ + \Phi S^\circ$ defined at $t \in [0,T]$ by

$$V^\circ(t;\pi) = \Psi(t)B^\circ(t) + \Phi(t)S^\circ(t). \tag{4.4.18}$$

Clearly, the initial endowment amounts to

$$v \doteq V^\circ(0;\pi) = \Psi(0)B^\circ(0) + \Phi(0)S^\circ(0).$$

Discounting (4.4.18), we get

$$\grave{V}^{\circ}(t;\pi) = \Psi(t) + \Phi(t)\grave{S}^{\circ}(t). \tag{4.4.19}$$

Since between two consecutive jumps the stock price process does evolve smoothly the investor selects both components as piecewise continuous functions of type

$$\Psi(t) = \psi_1(t)I_{[0,T_1]}(t) + \sum_{k=1}^{m} \psi_{k+1}(t)I_{(T_k,T_{k+1}]}(t) \tag{4.4.20}$$

and

$$\Phi(t) = \phi_1(t)I_{[0,T_1]}(t) + \sum_{k=1}^{m} \phi_{k+1}(t)I_{(T_k,T_{k+1}]}(t), \tag{4.4.21}$$

where ψ_k and ϕ_k are continuously differentiable functions with $\Psi(0) = \psi_1(0)$ and $\Phi(0) = \phi_1(0)$. Note that these processes are predictable, i.e. of the same type as S_-, cf (4.4.10). It is important to notice that the process $\grave{V}^{\circ}(\pi)$ for a self-financing strategy π is of the same type as the stock price process, since at each $t \in [0,T]$ it may be represented similarly to (4.4.8) as follows:

$$\grave{V}^{\circ}(t;\pi) = \sum_{k=0}^{m} \grave{v}_k^{\circ}(t;\pi)I_{[T_k,T_{k+1})}(t).$$

In view of definitions in Section 4.3.1, the states $\{\grave{v}_k^{\circ}(t;\pi)\}_{k=0,1,\dots,m}$ at $t \in [T_k, T_{k+1})$ satisfy

$$\grave{v}_k^{\circ}(t,\pi) - \grave{v}_k^{\circ}(T_k;\pi) = \int_{T_k}^{t} \phi_{k+1}(u)d\grave{s}_k^{\circ}(u) \tag{4.4.22}$$

with $\grave{v}_0^{\circ}(T_0;\pi) = v$ and

$$\grave{v}_k^{\circ}(T_k;\pi) = v + \sum_{j=1}^{k} \left\{ \int_{T_{j-1}}^{T_j} \phi_j(u)d\grave{s}_{j-1}^{\circ}(u) + \phi_j(T_j)D_j(\grave{S}_{T_j}^{\circ}) \right\}.$$

Recall that $D_j(\grave{S}_{T_j}^{\circ}) = \grave{s}_j^{\circ}(T_j) - \grave{s}_{j-1}^{\circ}(T_j)$, cf (4.4.11). Arguing as before, we introduce the predictable process $DV^{\circ}(\pi)/DS^{\circ}$ defined at $t \in [0,T]$ by

$$\frac{DV^{\circ}(t;\pi)}{DS^{\circ}(t)} = \frac{D_1(V_t^{\circ}(\pi))}{D_1(S_t^{\circ})}I_{[0,T_1]}(t) + \sum_{k=1}^{m} \frac{D_{k+1}(V_t^{\circ}(\pi))}{D_{k+1}(S_t^{\circ})}I_{(T_k,T_{k+1}]}(t),$$

cf (4.2.11) and (4.2.12). Due to the predictability of the portfolio components, we can apply formula (4.2.13) to both (4.4.18) and (4.4.19) to get

$$DV^{\circ}(\pi) = \Phi DS^{\circ} \quad \text{and} \quad D\grave{V}^{\circ}(\pi) = \Phi D\grave{S}^{\circ}. \tag{4.4.23}$$

This means, in particular, that

$$D_k(V_t^{\circ}(\pi)) = \phi_k(t)D_k(S_t^{\circ}). \tag{4.4.24}$$

Self-financing. According to the definition in Section 4.3.2, a trading strategy $\pi = (\Psi, \Phi)$ is self-financing if for each $t \in (0, T]$

$$B^\circ_- \cdot \Psi_t + S^\circ_- \cdot \Phi_t = 0 \tag{4.4.25}$$

In the present case Propositions 4.3.5 and 4.3.7 may be reformulated as follows:

Proposition 4.4.2. *A trading strategy π is self-financing if and only if its discounted value process $\grave{V}^\circ(\pi) = \{\grave{V}^\circ_t(\pi)\}_{t\in[0,T]}$ admits the following integral representation: at each $t \in [0, T]$*

$$\grave{V}^\circ(t;\pi) = v + \Phi \cdot \grave{S}^\circ_t. \tag{4.4.26}$$

In view of (4.4.23), this can be given the form of Clark's formula

$$\grave{V}^\circ(t;\pi) = v + \frac{DV^\circ(\pi)}{DS^\circ} \cdot \grave{S}^\circ_t. \tag{4.4.27}$$

It will be proved in the next proposition that the value process for a self-financing strategy satisfies differential equations (4.4.28), similar to (4.4.12). Note that the discounted versions of these equations (4.4.13) and (4.4.29) play in the present market the same rôle as the equations (4.3.38) and (4.3.46) in the binary market.

Proposition 4.4.3. *In a Poisson market a trading strategy π is self-financing if and only if the states of the value process and its discounted version satisfy the following differential equations: at $t \in (T_k, T_{k+1}]$*

$$\frac{dv^\circ_k(t;\pi)}{dt} - rv^\circ_k(t) = -\lambda D_{k+1}(V^\circ_t(\pi)) \tag{4.4.28}$$

and

$$\frac{d\grave{v}^\circ_k(t;\pi)}{dt} = -\lambda D_{k+1}(\grave{V}^\circ_t(\pi)). \tag{4.4.29}$$

Proof. (i) Under the self-financing condition (4.4.25) we have (4.4.22) that implies

$$\frac{d\grave{v}^\circ_k(t;\pi)}{dt} = \phi_{k+1}(t)\frac{d\grave{s}^\circ_k(t)}{dt}.$$

So, (4.4.29) follows from (4.4.13) and (4.4.24). The equivalence of (4.4.28) and (4.4.29) is straightforward, since the relations (4.4.14) and (4.4.15) easily extend to the states of the value processes.

(ii) Conversely, assume (4.4.29) with $t \in [T_k, T_{k+1})$. This and (4.4.23) yield

$$\begin{aligned}
\frac{d\grave{v}^\circ_k(t;\pi)}{dt} &= -\lambda\frac{D_{k+1}(\grave{V}^\circ_t(\pi))}{D_{k+1}(\grave{S}^\circ_t)}D_{k+1}(\grave{S}^\circ_t) \\
&= -\lambda\phi_{k+1}(t)D_{k+1}(\grave{S}^\circ_t) = \phi_{k+1}(t)\frac{d\grave{s}^\circ_k(t)}{dt}.
\end{aligned}$$

For the latter identity, see (4.4.13). Integrate the result from T_k up to $t \in [T_k, T_{k+1})$ to get (4.4.22) which means the self-financing. The proof is complete. □

Like in the case of the stock price process, compare (4.4.29) and (4.2.16) to see by the assertion of Proposition 4.4.3 that the states of the discounted value process $\grave{V}^\circ(\pi)$ satisfy the relations

$$\grave{v}_k^\circ(t;\pi) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{v}_{k+j}^\circ(T;\pi). \tag{4.4.30}$$

In particular

$$v = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{v}_j^\circ(T;\pi). \tag{4.4.31}$$

4.4.3 Completeness, hedging strategy and option valuation

Completeness and hedging strategies. The differential-difference equations (4.4.29) and their solutions (4.4.30) and (4.4.31) play here the same rôle as the equations (3.3.14) and their solutions (3.3.30) and (3.3.31) in a binary market of Part I. They yield, in particular, the completeness of a Poisson market to be shown next. The basic condition for this is that in the market in question only the number of jumps of the net returns on the stock is counted up to the terminal date T, and no distinction is made between different trajectories with equal number of jumps.

Let $\grave{W}(T)$ be a discounted wealth desired by an investor at the terminal date T. In accordance to the conditions in the Poisson market, $\grave{W}(T)$ depends only on the number of jumps occurred up to the terminal date T. Moreover, it is chosen so that if $\grave{S}^\circ(T)$ occurs in state $\grave{s}_k^\circ(T)$ with some nonnegative integer k, then the corresponding state of $\grave{W}(T)$ is $\grave{w}_k(T)$. The definitions in Section 4.3.3 of the completeness and the hedging strategy extend straightforwardly to the present situation. Let us, however, present anew the exact formulation.

Definition 4.4.4. *A Poisson market is complete if any desired wealth $\grave{W}(T)$ with the set of possible states $\{\grave{w}_k(T)\}_{k=0,1,\dots}$ is attainable with a certain initial endowment: there is a self-financing strategy π whose value process at the terminal date T attains the identity $\grave{V}(T;\pi) = \grave{W}(T)$. The necessary initial endowment is then $v = V(0;\pi)$. This particular strategy is called the hedging strategy against $\grave{W}(T)$.*

Similarly to Proposition 3.4.3 in Part I, we have

Proposition 4.4.5. *A Poisson market is complete. The hedging strategy π against a desired wealth $W(T)$ of the above type is uniquely defined by the portfolio components (4.4.20) and (4.4.21) with*

$$\psi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\frac{(1+a)\grave{w}_{j+k}(T) - \grave{w}_{j+k+1}(T)}{a} \tag{4.4.32}$$

and

$$\phi_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t)) \frac{\grave{w}_{j+k+1}(T) - \grave{w}_{j+k}(T)}{a\grave{s}_k^\circ(t)}, \tag{4.4.33}$$

where $\{\grave{w}_k(T)\}_{k=0,1,\ldots}$ *are the states of the discounted wealth* $\grave{W}(T)$. *The initial endowment needed amounts to*

$$v = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{w}_j(T). \tag{4.4.34}$$

Proof. Let $t \in [T_k, T_{k+1})$. By definition (4.4.19) the discounted value process for the present strategy π is in state

$$\grave{v}_k^\circ(t;\pi) = \psi_k(t) + \phi_k(t)\grave{s}_k^\circ(t)$$

which by (4.4.32) and (4.4.33) coincides with

$$\grave{w}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{w}_{k+j}(T). \tag{4.4.35}$$

By Proposition 4.2.3, (4.4.35) solves the differential equation (4.2.16) subject to the boundary condition (4.2.17) with $\grave{w}_k(T)$ instead of $\grave{h}_k(T)$. Then by Proposition 4.4.3 the present strategy is self-financing. Moreover, at the terminal date T its value process attains the desired wealth $W(T)$. Thus π is the hedging strategy against $W(T)$. As $\{\grave{w}_k(T)\}_{k=0,1,\ldots}$ in (4.4.35) are arbitrary, the Poisson market is complete. Finally, (4.4.34) is an easy consequence of (4.4.35). □

Option valuation. Let us continue to assume that in the Poisson market under consideration only the total number of jumps of the net returns on the stock is taken into account, and all the financial derivatives traded at the terminal date T depend only on the terminal stock price. A particular discounted contingent claim $\grave{H}(T)$ is then supposed to be a variable with the set of possible states $\{\grave{h}_j(T)\}_{j=0,1,\ldots}$, fixed by means of a certain contract function of a single argument H as follows:

$$\grave{H}(T) = H(\grave{S}^\circ(T))$$

and

$$\grave{h}_j(T) = H(\grave{s}_j^\circ(T)), \quad j = 0, 1, \ldots$$

To price this financial derivative, we can apply Proposition 4.4.5. According to (4.4.34), this yields the fair price of the contingent claim $\grave{H}(T)$:

$$C(\grave{H}) = \sum_{j=0}^{\infty} p_j(\lambda T)\grave{h}_j(T) \tag{4.4.36}$$

In the special case of the European call option

$$\grave{h}_j(T) = (\grave{s}_j^\circ(T) - \grave{K})^+ \tag{4.4.37}$$

(with a certain exercise price K, cf (4.3.56)), we have

Proposition 4.4.6. *For a nonnegative integer* j_0 *denote*

$$F(j_0; \lambda) = \sum_{j > j_0} p_j(\lambda), \tag{4.4.38}$$

cf (4.2.14). Then the fair price C *of the European call option with the payoff function (4.4.37) may be presented as follows:*

$$\begin{aligned} C &= sF\left(\lfloor \log(\frac{K}{s} + bT)/\log(1+a) \rfloor ; (1+a)\lambda T \right) \\ &+ e^{-rT} KF\left(\lfloor \log(\frac{K}{s} + bT)/\log(1+a) \rfloor ; \lambda T \right) \end{aligned} \tag{4.4.39}$$

Proof. By (4.4.36) and (4.4.37)

$$C = \sum_{j=0}^{\infty} p_j(\lambda T)(\grave{s}_j^\circ(T) - \grave{K})^+ \tag{4.4.40}$$

where $\grave{K} = e^{-rT}K$, as usual. Therefore, by (4.2.14) and (4.4.9)

$$C = e^{-\lambda T} \sum_{j=0}^{\infty} \frac{(\lambda T)^j}{j!} (s(1+a)^j e^{-a\lambda T} - \grave{K})^+ . \tag{4.4.41}$$

Obviously, the terms with

$$j \leq j_0 = \lfloor \log(\frac{K}{s} + bT)/\log(1+a) \rfloor$$

equal zero so that (4.4.41) reduces to

$$C = e^{-\lambda T} \sum_{j > j_0} \frac{(\lambda T)^j}{j!} (s(1+a)^j e^{-a\lambda T} - \grave{K})$$

which yields (4.4.39) by definition (4.4.38). □

Formula (4.4.40) is often called the Cox–Ross option pricing formula (see e.g. [12] or [39], Section 6.2; cf also [13] and [14]). Its probabilistic interpretation is as follows: the right hand side is the expectation with respect to the Poisson distribution $\mathcal{P}_{\lambda T}$ of a random variable taking on the value $(s_j^\circ(T) - K)^+$ with probability $p_j(\lambda T), j = 0, 1, \ldots$. In its specific form (4.4.39), this formula is comparable with the well-known Black-Scholes formula for the geometric Brownian motion model, see [6], [39], Formula (1.5), or [48], Section 5.8.

4.5 On the Poisson approximation

4.5.1 Approximation of the assets

In the present section the link is sought between the binary model of Section 4.3 and the Poisson model of Section 4.4. By using certain heuristic arguments we show that under the conditions of Section 4.3.1 the Poisson model can serve as an approximation to the binary model. This is already visible by the simple comparison of equations (4.3.50) and (4.4.29) (or (4.3.52) and (4.4.28)), for the left hand side of (4.3.50) may be viewed as a prelimiting version (in the form of finite differences) of the time derivative in (4.4.29).

We let the number of trading periods N to increase unboundedly and the length of each trading period $\Delta t_n = t_n - t_{n-1}$ to tend to zero for $n = 1, \dots, N$. For simplicity, we focus our attention to the special case when the new prices are announced regularly so that the trading times are equidistant, given by $t_n = nT/N$ with $n = 0, 1, \dots, N$, and the stretch of each trading period is $\Delta t_n = T/N$ with $n = 1, \dots, N$. For, asymptotically, this makes no difference (in fact one can proceed without this specification, however at the expense of some details which we want to avoid here). At the same time, the formulations are somewhat simplified. Instead of (4.3.1) and (4.3.2), for instance, we may write

$$B^N(t) = B^N_{\lfloor tN/T \rfloor} \quad \text{and} \quad S^N(t) = S^N_{\lfloor tN/T \rfloor}.$$

As usual, $\lfloor x \rfloor$ is the integer part of x, i.e. the largest integer not exceeding x.

Concerning the bond, the situation is simple, since Condition 4.3.1 means that at each fixed $t \in [0, T]$

$$B^N(t) \sim B^\circ(t),$$

as we have already seen, cf (4.3.12) and (4.4.2).

As for a risky asset, the stock, the desired statement that at each fixed $t \in [0, T]$

$$S^N(t) \sim S^\circ(t) \tag{4.5.1}$$

concerns the trajectories of the processes on the both sides. The idea behind this approximation is quite simple, as we will see below. Its exact formulation, however, would require probabilistic considerations that lay beyond the scope of the present paper.

At fixed $t \in [0, T]$

$$S^N(t) = s \prod_{n=0}^{\lfloor tN/T \rfloor} (1 + \Delta R^N_n) \tag{4.5.2}$$

(cf (4.3.13)) may occupy one of the $2^{\lfloor tN/T \rfloor}$ states, i.e. up to time t the stock price may evolve along one of $2^{\lfloor tN/T \rfloor}$ trajectories. The states of the net returns ΔR^N_n in (4.5.2), given by (4.3.14), are under condition 3.1.2 approximated by

(4.3.17), with the right hand side independent of the index k. Therefore those trajectories of the stock price development up to time $t \in [0,T]$ that jump upwards exactly j times and downwards remaining $\lfloor tN/T \rfloor - j$ times, are all approximated by the same expression

$$s(1+a)^j \left(1 - \frac{bT}{N}\right)^{\lfloor tN/T \rfloor - j}. \tag{4.5.3}$$

This is obtained by substituting in the product on the right hand side of (4.5.2) the net returns ΔR_n^N by their approximate states according to (4.3.17), i.e. j times by a and $\lfloor tN/T \rfloor - j$ times by $-bT/N$. The latter expression tends to $s_j^\circ(t)$ of form (4.4.7), since for fixed $j \geq 0$ and $t \in [0,T]$

$$\lim_{N\to\infty} \left(1 - \frac{bT}{N}\right)^{\lfloor tN/T \rfloor - j} = \lim_{N\to\infty} \left(1 - \frac{bT}{N}\right)^{\frac{tN}{T}} = e^{-bt}.$$

As $N \to \infty$ the index j in (4.5.3) tends to take on any nonnegative integer value, so that the approximate states are indeed $\{s_j^\circ(t)\}_{j=0,1,\dots}$. These are heuristic arguments behind the statement (4.5.1).

4.5.2 Approximate option pricing

According to lemma 4.3.8, for each $n = 1, \dots, N$ the risk neutral probabilities $\{p_{kn}^N\}_{k=1,\dots,2^n}$ are approximated independently of the indices k and n by

$$\begin{aligned} p_{2k\,n}^N &\sim \frac{\lambda T}{N} \\ p_{2k-1\,n}^N &\sim 1 - \frac{\lambda T}{N}. \end{aligned} \tag{4.5.4}$$

Indeed, we have (4.3.40) with Δt_n substituted in (4.3.40) by T/N. This allows us to the approximate the probability distribution $\{P_{kN}\}_{k=1,\dots,2^N}$ defined by (4.3.45). Recall for this purpose the partition introduced in Chapter 2, Example 2.3.1, of the set of all possible terminal states $\{s_{kN}\}_{k=1,\dots,2^N}$ into $N+1$ disjoint subsets $\Pi_0, \Pi_1, \dots, \Pi_N$, so that if a state belongs to Π_j, then the corresponding trajectory evolves with exactly j upward and $N-j$ downward displacements. The number of all such trajectories equals to $\binom{N}{j}$. In virtue of (4.3.45) and (4.5.4), in all these cases, i.e. in all cases when $s_{kN} \in \Pi_j$, we have the same approximation

$$P_{kN}^N \sim \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-j}. \tag{4.5.5}$$

We want to apply (4.5.5) to the option pricing formula (4.3.57). By taking into consideration the approximation of Section 4.4.1 to the states of the stock price, we obtain

$$C^N \sim \sum_{j=0}^{N} \binom{N}{j} \left(\frac{\lambda T}{N}\right)^j \left(1 - \frac{\lambda T}{N}\right)^{N-j} (\grave{s}_j^\circ(T) - \grave{K})^+. \tag{4.5.6}$$

The expression on the right hand side may be simplified, since for each fixed integer $j \geq 0$

$$\begin{aligned} & \binom{N}{j}\left(\frac{\lambda T}{N}\right)^j\left(1-\frac{\lambda T}{N}\right)^{N-j} \\ = & \frac{(\lambda T)^j}{j!}\left(1-\frac{\lambda T}{N}\right)^{N-j}\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{j-1}{N}\right) \end{aligned}$$

tends to $\frac{(\lambda T)^j}{j!}e^{-\lambda T}$ as $N \to \infty$. The limit is $p_j(\lambda T)$, cf (4.2.14). Thus (4.5.6) yields

$$C^N \sim \sum_{j=0}^{\infty} p_j(\lambda T)(\grave{s}_j^\circ(T) - \grave{K})^+,$$

cf (4.4.40).

4.5.3 Approximate hedging strategy

In this section the heuristic arguments of the previous Sections 4.4.1 and 4.4.2 are applied to the hedging strategy against the European call option. The construction of this strategy is based on the formula (4.3.54) with the discounted states

$$\grave{w}_k^N(T) = (\grave{s}_k^N(T) - \grave{K})^+, \;\; 2^n(j-1) < k \leq 2^n j, \tag{4.5.7}$$

and the transition probabilities

$$P_{kN}^n, \;\; 2^n(j-1) < k \leq 2^n j. \tag{4.5.8}$$

We already know from Section 4.4.1 how to approximate the states (4.5.7). The set of the transition probabilities (4.5.8) has to be approximated as well, for all $j \in \{1, \dots, 2^{N-n}\}$. By arguments similar to that of the previous section, the approximation is free of the index k. Indeed, all entries in the subset of (4.5.8) corresponding to the subset of the trajectories with exactly j upward and $n-j$ downward displacements, $j \in \{0, 1, \dots, n\}$, are approximated by the same number

$$\left(\frac{\lambda T}{N}\right)^j\left(1-\frac{\lambda T}{N}\right)^{n-j},$$

due to (4.3.44) and (4.5.4). For each $j \in \{0, 1, \dots, n\}$ this subset consists of $\binom{n}{j}$ entries. Consequently, the process $\grave{W}^N$ of Section 4.3.3 occupies at time $t \in [0, T]$ one of the states $\grave{w}_{k\tau}^N$ with $\tau \doteq \lfloor tN/T \rfloor$ and with some $k = 1, \dots, 2^\tau$. Each of these states is approximated as follows:

$$\grave{w}_{k\tau}^N \sim \sum_{j=0}^{N-\tau}\binom{N-\tau}{j}\left(\frac{\lambda T}{N}\right)^j\left(1-\frac{\lambda T}{N}\right)^{N-\tau-j}\grave{w}_{k+j}^N(T) \tag{4.5.9}$$

where $\grave{w}_j^N(T)$ is given by (4.5.7). The expression on the right hand side may be simplified by the following considerations. Firstly, by the results of section 5.1 and by (4.5.7)

$$\grave{w}_j^N(T) \sim \grave{w}_j(T) \tag{4.5.10}$$

with

$$\grave{w}_j(T) = (\grave{s}_j^\circ(T) - \grave{K})^+.$$

Secondly, for each integer $j \geq 0$ and $t \in [0,T]$

$$\begin{aligned} \binom{N-\tau}{j}\left(\frac{\lambda T}{N}\right)^j &= \frac{(\lambda T)^j}{j!}\left(1-\frac{\tau}{N}\right)\cdots\left(1-\frac{\tau-j}{N}\right) \\ &\to \frac{\{\lambda(T-t)\}^j}{j!} \end{aligned} \tag{4.5.11}$$

as $N \to \infty$, with the same integer $\tau \doteq \lfloor tN/T \rfloor$. Finally,

$$\begin{aligned} \lim_{N\to\infty}\left(1-\frac{\lambda T}{N}\right)^{N-\tau-j} &= \lim_{N\to\infty}\left(1-\frac{\lambda T}{N}\right)^{N\left(1-\frac{t}{T}\right)} \\ &= e^{-\lambda(T-t)}. \end{aligned} \tag{4.5.12}$$

In view of (4.2.14), the equations (4.5.9)–(4.5.12) result in

$$\grave{w}_{k\tau}^N \sim \grave{w}_k(t) \tag{4.5.13}$$

where

$$\grave{w}_k(t) = \sum_{j=0}^{\infty} p_j(\lambda(T-t))\grave{w}_{k+j}(T)$$

(cf (4.4.35)). Thus we have derived the relation (4.5.13) between the states of the processes $\grave{W}^N$ and $\grave{W}$ which yield the value processes for the hedging strategies against the European call option in the prelimiting binary market of Section 4.3 and the Poisson market of Section 4.4, respectively.

Chapter 5

Gaussian Approximation

5.1 Introduction

5.1.1 Outline of this chapter

In this chapter we consider the situation in which a binary securities market allows for the Gaussian approximation to be carried out in Section 5.4.3. For the necessary material on binary markets we refer to Part I and the previous Chapter 4, Section 3.

Throughout the present chapter the basic Conditions 4.3.1 and 5.1.1 are assumed, see the next section. They restrict the limiting behaviour of the asset prices. Condition 4.3.1 on the bond is inherited from Chapter 4, while Condition 5.1.1 on the stock differs completely from Condition 4.3.2 in Chapter 4. It is in fact the adaptation of the conditions well-known in the probability theory, under which random walk admits diffusion approximation (see e.g. [11], [34] and various papers in [70]). But since the reader is not supposed to be familiar with advanced methods of the probability theory, the presentation is kept at the same low technical level as in the previous chapters, with the help of certain unsophisticated algebraic considerations. The emphasis then will be on phenomenological understanding the cash flow mechanism in the market under consideration, that is shown to reveal strong similarity to physical Brownian motion (more generally, to diffusion with drift) or to the molecular mechanism of heat flow, cf Sections 5.2 and 5.3. For more details on the economical laws governing the market, we refer to the seminal paper [6] by Black and Scholes; cf also Wilmott. The results obtained in this manner in the Sections 5.4.3 and 5.5, are of heuristic nature (comparable to e.g. [13], [40] and [75]), for the full rigour would require higher technical level of the general theory of stochastic processes, cf [15], [17], [23], [28], [29], and [75] – [77]. For further reading we would recommend a selective list of papers [46], [50] and [51], which are related to various extend to the present subject but leading the reader far afield. The point of view taken in the present chapter is somewhat different: an attempt is made to facilitate reading without an advanced probabilistic prerequisite.

For instance, Section 5.2 introduces the reader to Brownian motion and its original mathematical model by Wiener. The relation to diffusion and thermal conductance is discussed in Section 5.3. Usage of heat equations for the description of the cash flow dynamics in Section 5.5 is preceded by Proposition 5.4.3 that asserts an approximate heat equation for value processes in binary markets (somewhat in the spirit of Kac's paper on random walk in [70]). Moving towards the continuous-time model, in Section 5.4.3 the approximation is discussed to the option pricing formula for the European call option. The final result is the celebrated Black-Scholes formula (5.5.21), cf [6]. The seminal paper [39] inspired some authors to integrate this formula in the mathematical textbooks on martingales and stochastic calculus, cf e.g. [48], Section 5.8, [60], Chapter 12, or [73], Section 15.2. Of course, it constitutes the backbone of all recent textbooks on mathematical finance. In Section 5.5 the Black-Scholes model is described in which the stock price process is assumed to be a geometrical Brownian motion. The theory is developed along the same lines as in the previous parts: the properties of the possible states of the discounted stock prices asserted in Proposition 5.5.1 are shared by every discounted value process for a self-financing strategy (see Proposition 5.5.3) and all these stem from the fact that the Gaussian transition probability density u in (5.5.6) and (5.5.12) satisfies the heat equation (5.3.2). In Chapter 4 we have had the similar situation: the Poisson distribution has been shown to possess the property asserted in Lemma 4.2.2 and therefore we have had (4.4.13) and (4.4.29). The counterpart in Part I of these are (3.3.4) and (3.3.14), respectively. These are basically the arguments for the completeness of our markets; cf Chapter 3, Proposition 3.4.3, Chapter 4, Proposition 4.4.5, and the present chapter, Proposition 5.5.6. In these propositions we construct the hedging strategies against any wealth desired at the terminal date T. Finally, pricing of contingent claims is accomplished by the procedure described at the end of Section 5.4.

5.1.2 Basic conditions

In this section the assumptions will be made to guarantee desired approximations. The basic setup is same as in Chapter 4, Section 3. Condition 4.3.1 on the bond prices is retained unaltered, but the condition on the stock prices essentially differ from previous Condition 4.3.2. New prices on both assets are again announced at certain fixed trading times, say $t_0 < t_1 < \cdots < t_N$ where $t_0 = 0$ is the current date and $t_N = T$ the terminal date. Moreover, the simplification introduced in Section 4.5 is also retained and new prices are supposed to be announced regularly so that trading times are equidistant, given by $t_n = nT/N$ for $n = 0, 1, \ldots, N$, and the stretch of the trading periods $\Delta t_n = t_n - t_{n-1}$ for $n \in \{1, \ldots, N\}$ are all given by $\Delta t_n = T/N$ (in fact one can proceed without this specification, however at the expense of some details which we want to avoid). The formulations are indeed somewhat simplified. For instance, the corresponding price processes $B^N = \{B_t^N\}_{t\in[0,T]}$ and $S^N = \{S_t^N\}_{t\in[0,T]}$ are defined in the entire time interval $[0,T]$ by $B^N(t) = B^N_{\lfloor tN/T \rfloor}$ and $S^N(t) = S^N_{\lfloor tN/T \rfloor}$, where $\lfloor x \rfloor$ denotes the largest integer not exceeding x.

Put $B^N(0) = 1$ and $S^N(0) = s$ for simplicity, where s is a certain positive number. The discounted stock price process is denoted as in the previous chapters by $\grave{S}^N = \{\grave{S}_t^N\}_{t\in[0,T]}$ with $\grave{S}^N(t) = S^N(t)/B^N(t)$.

Like in the previous chapter, it is supposed throughout that the number N of the trading times is very large, and the approximation is sought to the option pricing formulas of Chapter 3. Letting $N \to \infty$ we will impose upon the grid $\{t_0, t_1, \dots, t_N\}$ to become finer and finer, since the mesh width is T/N. Moreover, the asset prices ought to be made dependent on the index N in a certain special manner. Regarding the bond prices, see Conditions 4.3.1 in Section 4.3.1. The conditions on the behaviour of the returns on stock $\Delta R^N(t_n) = R^N(t_n) - R^N(t_{n-1})$ at trading times $\{t_n\}_{n=1,\dots,N}$ are formulated as in Chapter 4 in terms of their states

$$r_{kn}^N \doteq \frac{s_{kn}^N}{s_{k_1\, n-1}^N} - 1, \quad k = 1, \dots, 2^n, \tag{5.1.1}$$

where $k_1 = \lceil \frac{k}{2} \rceil$ as usual. But the present conditions are completely different.

Condition 5.1.1. *At the trading time t_n with some $n \in \{1, \dots, N\}$ the return on the stock $\Delta R^N(t_n)$ is in one of the 2^n states*

$$r_{kn}^N = \begin{cases} \sigma\sqrt{\Delta t_n} + (a + \eta_{kn}^N)\Delta t_n & \textit{if } k \textit{ is even} \\ -\sigma\sqrt{\Delta t_n} - (b + \eta_{kn}^N)\Delta t_n & \textit{if } k \textit{ is odd} \end{cases} \tag{5.1.2}$$

where $\sigma > 0$, a and b are some constants, while $\{\eta_{kn}^N\}_{k=1,\dots,2^n}$ are negligible remainder terms as $N \to \infty$.

The negligibility of these remainder terms is understood as in Chapter 4, Remark 4.3.4. Using the same sign $\sim$ as in Section 4.3.1 we may express (5.1.2) in the following form

$$r_{kn}^N \sim \begin{cases} \sigma\sqrt{\Delta t_n} + a\Delta t_n & \text{if } k \text{ is even} \\ -\sigma\sqrt{\Delta t_n} - b\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{5.1.3}$$

If Condition 4.3.1 holds as well, then the states $\{\grave{r}_{kn}^N\}_{k=1,\dots,2^n}$ of the discounted net return $\Delta\grave{R}^N(t_n)$ with $n \in \{1, \dots, N\}$ are approximated as follows. Due to (4.3.10) it follows from the relation (2.3.16) in Part I, Chapter 2, that the net returns on the stock $\Delta\grave{R}^N(t_n)$ are approximated by the excess returns $\Delta(R^N - \mathcal{R}^N)_n$. Apply (4.3.10) and (5.1.3) to the latter returns. We get

$$\grave{r}_{kn}^N \sim \begin{cases} \sigma\sqrt{\Delta t_n} + (a - r)\Delta t_n & \text{if } k \text{ is even} \\ -\sigma\sqrt{\Delta t_n} - (b + r)\Delta t_n & \text{if } k \text{ is odd.} \end{cases} \tag{5.1.4}$$

By obvious reasons, the parameter $\sigma > 0$ determining the amplitude of the leading terms in these formulas is often called *volatility* (as well as *diffusion coefficient* or *thermal diffusivity*, depending on the context).

To anticipate the idea behind the forthcoming approximations in Section 5.4.3, note the following. Like in the previous chapters, let the even state indices correspond to the upward displacements, and the odd indices to the downward displacements. If now the same weight $\frac{1}{2}$ are assigned to both of these displacements, then in virtue of (5.1.4) the average return is approximated by

$$\frac{1}{2}(\grave{r}^N_{2k\,n} + \grave{r}^N_{2k-1\,n}) \sim \mu \Delta t_n \tag{5.1.5}$$

with a constant

$$\mu = \frac{1}{2}(a-b) - r \tag{5.1.6}$$

called the *drift coefficient*, since the drift μt in (5.5.3) and thereafter is in fact the accumulation of instant drifts on the right hand side of (5.1.5). Let us now average differently to get rid of the drift. Namely, let us correct the weights for upward and downward displacements to be

$$\frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta t_n}) \quad \text{and} \quad \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t_n}), \tag{5.1.7}$$

respectively. This results in the zero mean on the right hand side: the average return is approximated as follows.

Lemma 5.1.2. *Under the conditions 4.3.1 and 5.1.1 the weights (5.1.7) neutralize the upward displacements of the discounted net return on the stock against downward displacements in the sense that*

$$\frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}^N_{2k\,n} + \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}^N_{2k-1\,n} \sim 0. \tag{5.1.8}$$

Proof. Under the required conditions we have (5.1.4) and hence (5.1.5) and (5.1.6). These give

$$\begin{aligned} &\frac{1}{2}(1 - \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}^N_{2k\,n} + \frac{1}{2}(1 + \frac{\mu}{\sigma}\sqrt{\Delta t_n})\grave{r}^N_{2k-1\,n} \\ \sim\ & \mu \Delta t_n - \frac{\mu}{2\sigma}\Delta t_n(2\sigma + (a+b)\sqrt{\Delta t_n}) = -\frac{a+b}{2}\frac{\mu}{\sigma}(\Delta t_n)^{3/2}. \end{aligned}$$

The right hand side of this relation is of a lower magnitude then that of (5.1.5). Hence, this relation is equivalent to (5.1.8). □

In Section 5.4.1 the weights just used occur again in Formula (5.4.6), to serve as the approximation to the so-called *risk neutral probabilities*. The usage of the last term is in clear connection with the fact expressed by (5.1.8) that the drift is eliminated by suitable averaging. Compare Lemma 5.1.2 with its counterpart, Lemma 4.3.3 in Chapter 4. The difference is apparent. In contrast to (4.3.20), in (5.1.8) the weights are almost equally shared between the upward and downward displacements. Next, let us look at the terms proportional to $\sqrt{\Delta t_n}$ that are all the same in (5.1.2), (5.1.3) and (5.1.4). The sequence of the corresponding upward and downward displacements (as the index n runs trough $\{0, 1, \ldots, N\}$) are symmetrical and form the so-called *symmetrical random walk*, cf the special subsection in Part I, Section 2.2.2. This is a mathematical model of the hypothetical situation in which a minute particle, immersed in a liquid, suffers many collisions with the molecules of the medium. These molecules, being in the thermal motion, impart energy and momentum to the particle, so that it undergoes very irregular and erratic motion. If we imagine the collisions regularly spread in time with intervals equal $\Delta t_n = T/N$ that results in either upward or downward displacements of equal chance with small steps of size $\sigma\sqrt{\Delta t_n}$, then we end up in the situation under consideration. Since the number of molecules N is very large, it is convenient to let $N \to \infty$ and to look for suitable approximations. One of the central results of the probability theory tells us that since the displacements are of considerably larger order of magnitude $\sim \sqrt{\Delta t_n}$ then the width Δt_n of the time interval, the degeneracies are excluded and the limiting process turns out to be mathematical Brownian motion. This process, denoted in the present paper by $\mathcal{W}$, will occur in the definitions (5.5.2) and (5.5.3), presenting the risky component in securities market models with continuous-time trading. Section 5.2 devoted to Wiener's original construction of Brownian motion (or Wiener process, as it is often called) and to various properties of its trajectories, although it may be not so easy to imagine their appearance. But if we try to imagine a very long realization of our symmetrical random walk plotted on a graph with regular small time intervals and with displacements per time interval proportional to the square root of its length, then we are led to expect that the trajectory of the limiting process, although continuous, has an infinite number of small spikes in any finite interval and is therefore non-differentiable. This is indeed the case, see Section 5.2.4 for further comments.

5.1.3 Gaussian distribution

In the present chapter an important rôle is played by the so-called *probability integral*

$$G(x) = \int_{-\infty}^{x} g(y)dy \tag{5.1.9}$$

with the density

$$\frac{dG(x)}{dx} \doteq g(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}. \tag{5.1.10}$$

Due to the property

$$\int_{-\infty}^{\infty} g(y)dy = 1 \tag{5.1.11}$$

this is the probability distribution, called the standard *Gaussian* or *normal* distribution. Generally, the Gaussian distribution $G(\cdot;\mu,\sigma^2)$ with two parameters $\mu \in (-\infty,\infty)$ and $\sigma^2 \in (0,\infty)$ called the *expectation* and *variance* respectively, is defined by $G(x;\mu,\sigma^2) = G(\frac{x-\mu}{\sigma})$, with the density

$$\frac{dG(x;\mu,\sigma^2)}{dx} \doteq g(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

This density will frequently occur in our considerations trough the function $u(\cdot,\cdot)$ of space coordinate $x \in (-\infty, \infty)$ and time $t \in (0, \infty)$, defined by

$$u(x,t) \doteq g(x; x_0 + \mu t, \sigma^2 t), \tag{5.1.12}$$

where μ and σ^2 are certain parameters, while x_0 is a certain initial site so that

$$\lim_{t \downarrow 0} u(x,t) = u(x,0) = \delta(x - x_0). \tag{5.1.13}$$

Here δ is Dirac's delta function with the following reproducing property: for any bounded continuous function f

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0). \tag{5.1.14}$$

It is indeed not hard to see that for a such f

$$\lim_{t \downarrow 0} \int_{-\infty}^{\infty} u(x,t) f(x) dx = f(x_0).$$

To this end let $t \downarrow 0$ in the integral

$$\begin{aligned} &\frac{1}{\sqrt{2\pi t}\sigma} \int_{-\infty}^{\infty} e^{-\frac{(x-x_0-\mu t)^2}{2t\sigma^2}} f(x) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} f(x_0 + \mu t + y\sigma\sqrt{t}) dy, \end{aligned}$$

where the substitution $\frac{x-x_0-\mu t}{\sigma\sqrt{t}} = y$ is made. Since the limit can be taken under the integration sign, from (5.1.10) and (5.1.11) we get the convergence to the desired $f(x_0)$.

For brevity, we will use the following notations for partial derivatives:

$$u_t = \frac{\partial u}{\partial t}, \quad u_x = \frac{\partial u}{\partial x} \quad \text{and} \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}. \tag{5.1.15}$$

5.2 Brownian motion

5.2.1 Wiener's construction

In this section we will discuss Wiener's measure-theoretical analysis of the physical Brownian motion that had been discovered at about century earlier by Brown in 1820's, who made microscopic observations on the minute particles contained in the pollen of plants suspended in a liquid. He observed the highly irregular motion of these particles and made first attempts to interpret this strange phenomenon. However, the true cause of the motion became known much later. It was understood that highly irregular and erratic displacements arise from thermal motion of the molecules of the liquid in which the particles are immersed, as the result of extremely large number of collusions with these molecules. One of the first attempts to establish the mathematical framework for Brownian motion was undertaken at around the year 1900 by Bachelier, whose goal in his thesis on "theory of speculation" was to develop methods for option valuation (see [39], p 217, for a short description of these ideas). Few years later Einstein proposed the mathematical theory which we will touch upon in Section 5.3.1. Meanwhile, in this Section 5.2 we focus on Wiener's ideas as presented in his later books [71] and [72]. First, we will describe a set of trajectories along which the mathematical Brownian motion is allowed to evolve (called below Brownian paths, for brevity). Then we indicate various properties of this set of functions with which one is able to establish a sophisticated integration theory. We will start with setting up a mapping between certain sets of functions called quasi-intervals, and certain subintervals of the unit interval $0 \leq \alpha \leq 1$. This will be done in such a manner that the obtained functions $x(\cdot,\cdot)$, measurable in both arguments (t,α), will be continuous in time t for almost every α. These will be trajectories of our Brownian motion.

5.2.2 Quasi-intervals

Let $\mathcal{C}_0$ be a class of all real valued functions of time which start from the origin, i.e. if $f \in \mathcal{C}_0$, then $f(0) = 0$. We are now going to define particular subsets of $\mathcal{C}_0$, called *quasi-intervals* by the reason to become clear soon. Let n be any positive integer, and let $\mathbf{t}_n = \{t_j\}_{j=1,\dots,n}$ be a set of n instants so that $0 < t_1 < \dots < t_n \leq T$. These are n points on the t-axis. Through each of these points we pass the line perpendicular to the t-axis. On each such line, say j^{th} one, $j = 1,\dots,n$, we choose an interval ι_j of the real axis. A quasi-interval $I_n(\mathbf{t}_n;\iota_1,\dots,\iota_n)$ consists of all real-valued functions $f \in \mathcal{C}_0$ whose values at t_j are confined to ι_j, i.e. $f(t_j) \in \iota_j$, $j = 1,\dots,n$. For example, take $n = 2$ and $\mathbf{t}_2 = \{\frac{1}{2}T, T\}$ and consider the following four quasi-intervals:

$$I_2(\mathbf{t}_2;\iota_{\ell_1},\iota_{\ell_2}),\ \ \ell_1,\ell_2 = 1,2, \tag{5.2.1}$$

with

$$\iota_1 = [-\infty,0],\ \ \iota_2 = (0,\infty]. \tag{5.2.2}$$

Obviously, these four quasi-intervals partitioning the entire class $\mathcal{C}_0$, since each particular pair is disjoint (no function can belong to two different quasi-intervals) and at the same time their union coincides with $\mathcal{C}_0$ (each function necessarily belongs to one of these quasi-intervals). For instance, $I_2(\mathbf{t}_2; \iota_1, \iota_1)$ consists of functions from $\mathcal{C}_0$ whose values at $t = \frac{1}{2}T$ and $t = T$ are non-positive, i.e. if $f \in I_2(\mathbf{t}_2; \iota_1, \iota_1)$, then $f(\frac{1}{2}T) \leq 0$ and $f(T) \leq 0$. In this section we will be only interested in such sets of quasi-intervals, partitioning $\mathcal{C}_0$. Moreover, starting from the above partition, we will construct a sequence of finer and finer partitions of a special type. At the first stage just described the subindex n was equal 2. At the next stage it will be equal 2^2, then 2^3 and so forth. The number of quasi-intervals involved at each stage will increase rapidly: starting from 4, at the n^{th} stage it will become $(2^n)^{2^n}$. But let us first describe the second stage. Take $n = 4$ and $\mathbf{t}_4 = \{\frac{1}{4}T, \frac{1}{2}T, \frac{3}{4}T, T\}$. This $\mathbf{t}_4$ contains the previous $\mathbf{t}_2$, of course. Put

$$\iota_1 = \left[-\infty, \tan\left(-\frac{\pi}{4}\right)\right], \qquad \iota_2 = \left(\tan\left(-\frac{\pi}{4}\right), 0\right],$$
$$\iota_3 = \left(0, \tan\frac{\pi}{4}\right], \qquad \iota_4 = \left(\tan\frac{\pi}{4}, \infty\right].$$

This refines the previous segmentation (5.2.2) of a real line (drawn perpendicularly to the t-axis). Recall that $\tan x$ is a monotonically increasing function of $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, with $\tan 0 = 0$ and $\tan\left(\pm\frac{\pi}{2}\right) = \pm\infty$. Note that this concrete choice is immaterial, since any other monotonically increasing real-valued function will do as well. Define now a partition of $\mathcal{C}_0$ by the 4^4 quasi-intervals

$$I_4(\mathbf{t}_4; \iota_{\ell_1}, \dots, \iota_{\ell_4}), \;\; \ell_1, \dots, \ell_4 = 1, \dots, 4. \tag{5.2.3}$$

Clearly, this is a partition finer then the previous one since, for instance,

$$\bigcup_{\ell_1, \ell_3 = 1, \dots, 4} \; \bigcup_{\ell_2, \ell_4 = 1, 2} I_4\left(\mathbf{t}_4; \iota_{\ell_1}, \dots, \iota_{\ell_4}\right) = I_2(\mathbf{t}_2; \iota_1, \iota_1),$$

and each quasi-interval of the previous stage is obtained by the union of a certain number of quasi-intervals of the next stage.

Carrying on in this manner, at the n^{th} stage we fix the set of 2^n dyadic instants $\mathbf{t}_{2^n} = \left\{\frac{j}{2^n}T\right\}_{j=1,\dots,2^n}$ and partition $\mathcal{C}_0$ by the following quasi-intervals:

$$I_{2^n}\left(\mathbf{t}_{2^n}; \iota_{\ell_1}, \dots, \iota_{\ell_{2^n}}\right), \;\; \ell_1, \dots, \ell_{2^n} = 1, \dots, 2^n,$$

where

$$\iota_1 = \left[-\infty, \tan\frac{(1 - 2^{n-1})\pi}{2^n}\right],$$

and

$$\iota_\ell = \left(\tan\frac{(\ell - 1 - 2^{n-1})\pi}{2^n}, \tan\frac{(\ell - 2^{n-1})\pi}{2^n}\right], \;\; \ell = 2, 3, \dots, 2^n.$$

For example, $I_{2^n}\left(\mathbf{t}_{2^n}; \iota_1, \dots, \iota_1\right)$ consists of all real-valued functions $f \in \mathcal{C}_0$ whose values at the 2^n dyadic instants $t = \frac{1}{2^n}T, \frac{2}{2^n}T, \dots, T$ are restricted by the following inequalities:

$$f\left(\frac{j}{2}T\right) \leq \tan \frac{\left(1-2^{n-1}\right)\pi}{2^n}, \quad j=1,\ldots,2^n.$$

5.2.3 Equicontinuity

Of course, we have in the previous section purposely chosen for the exponential increase of dyadic instants of successive partitioning, in order to guarantee the convergence of certain procedures which will be set up below.

For each $n=1,2,\ldots$ we have constructed $(2^n)^{2^n}$ quasi-intervals partitioning the entire class $\mathcal{C}_0$, since they do not intersect and their union coincide with $\mathcal{C}_0$. Following Wiener (cf [72]), we associate with each such quasi-interval, say $I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)$, a positive number $p\left\{I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)\right\}$ equal to

$$\int_{\iota_{\ell_1}}\cdots\int_{\iota_{\ell_{2^n}}}\prod_{j=1}^{2^n} u\left(y_j-y_{j-1},t_j-t_{j-1}\right)dy_1\cdots dy_{2^n}, \tag{5.2.4}$$

where u is related to the Gaussian density g through (5.1.12) with $x_0=\mu=0$. For convenience we put $t_0=y_0=0$. Using property (5.1.11) we obtain for each positive integer n that

$$\sum_{\ell_1\cdots\ell_{2^n}=1}^{2^n} p\left\{I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)\right\}=1. \tag{5.2.5}$$

Therefore the number defined by (5.2.4) is called the *probability* associated with the quasi-interval $I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)$.

As is mentioned in Section 5.2.2, each quasi-interval of the n^{th} stage may be represented by the union of certain number of quasi-intervals of the next $n+1^{\text{th}}$ stage. Suppose

$$I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)=\bigcup I_{2^{n+1}}\left(\mathbf{t}_{2^{n+1}};\iota_{\ell_1},\ldots,\iota_{\ell_{2^{n+1}}}\right).$$

Then it is easily verified that

$$p\left\{I_{2^n}\left(\mathbf{t}_{2^n};\iota_{\ell_1},\ldots,\iota_{\ell_{2^n}}\right)\right\}=\sum p\left\{I_{2^{n+1}}\left(\mathbf{t}_{2^{n+1}};\iota_{\ell_1},\ldots,\iota_{\ell_{2^{n+1}}}\right)\right\}, \tag{5.2.6}$$

where the summation extends over the same set of indices as in the preceding union.

This procedure of ascribing probabilities to quasi-intervals may be nicely characterized by the following mapping. At the first stage map the quasi-intervals (5.2.1) to the unit interval by starting at the origin and laying out on the α-axis 4 adjoining segments whose lengths are $p\left\{I_2\left(\mathbf{t}_2;\ell_1,\ell_1\right)\right\}$, $p\left\{I_2\left(\mathbf{t}_2;\ell_1,\ell\right.\right.$ $p\left\{I_2\left(\mathbf{t}_2;\ell_2,\ell_1\right)\right\}$ and $p\left\{I_2\left(\mathbf{t}_2;\ell_2,\ell_2\right)\right\}$, respectively. At the second stage map the quasi-intervals (5.2.3) to the unit interval by continuing to translate the probabilities into lengths and by arranging the necessary segments in such a way that if a given quasi-interval $I_4(\mathbf{t}_4; \iota_{\ell_1}, \ldots, \iota_{\ell_4})$ is a portion of a certain quasi-interval of the first stage, then the corresponding segments stand in the same relation. It should be clear now that if this procedure is kept up indefinitely, the lengths of the image intervals on $0 \leq \alpha \leq 1$ will tend to zero.

In fact, no specification (5.2.4) is needed to satisfy the properties (5.2.5) and (5.2.6), but the following property, called *equicontinuity*, is based upon this special form of the associated probabilities (Gaussian form as specified in Section 5.1.3). Let us formulate this in the separate theorem. For $h \in (0, \infty)$ and $\lambda \in \left[0, \frac{1}{2}\right)$ denote by $\mathcal{C}_0(h, \lambda)$ the subset of $\mathcal{C}_0$ consisting of functions $f \in \mathcal{C}_0$ such that for all dyadic instants t_1 and t_2

$$|f(t_1) - f(t_2)| \leq h|t_1 - t_2|^\lambda. \tag{5.2.7}$$

Note that if $h < h'$, then $\mathcal{C}_0(h, \lambda) \subset \mathcal{C}_0(h', \lambda)$. Denote by $\mathcal{C}_0(h, \lambda)^c$ the complement set to $\mathcal{C}_0$.

Proposition 5.2.1. *For $h \in (0, \infty)$ and $\lambda \in \left[0, \frac{1}{2}\right)$ the subset $\mathcal{C}_0(h, \lambda)^c$ of $\mathcal{C}_0$ can be enclosed in the union of quasi-intervals whose probability (the sum of the probabilities of the involved quasi-intervals) is less then certain number $p(h)$ which tends to vanish, i.e. $p(h) \to 0$ as $h \to \infty$.*

For the present we assume this result, since the proof is too lengthy to be reproduced here, though not difficult technically, see e.g. WIENER et. al. [72], Section 2.3. For a general context on equicontinuity sets of functions and related results, including Ascoli's theorem which we are going to apply below, see DIEUDONNÉ [20], Section 7.5. In order to use these general arguments we note first that Proposition 5.2.1 allows us to associate with any $\varepsilon > 0$ a positive number $h_0 = h_0(\varepsilon)$ and to confine our considerations to h's such that $p(h) < \varepsilon$ for $h > h_0$. We then delete a denumerable set of quasi-intervals of total probability $< \varepsilon$ so as to obtain the remainder in $\mathcal{C}_0$ that consists of functions satisfying (5.2.7) for all dyadic couples of instants from $[0, T]$. But we seek more - we want a set of functions satisfying (5.2.7) at all couples of instants from $[0, T]$, not necessarily dyadic. Therefore we have to modify the set obtained above (of functions f satisfying (5.2.7) at all dyadic couples of instants), associating with each element f a unique continuous function F by identity $F(t) = f(t)$ at a dyadic t and $F(t) = \lim_{t_n \to t} f(t_n)$ where $\{t_n\}_{n=1,2,\ldots}$ is a sequence of dyadic instants converging to t. This modification yields the set of continuous functions F with property (5.2.7) at each $t_1, t_2 \in [0, T]$. This is an equicontinuous class of uniformly bounded (by the above h) functions and in virtue of Ascoli's theorem mentioned above every sequence within this class has a uniformly convergent subsequence. The limit is itself continuous, with the equicontinuity property (5.2.7) at each $t_1, t_2 \in [0, T]$.

In view of the above construction and of Proposition 5.2.1, we are led to shrink the interval $0 \leq \alpha \leq 1$ by removal of a set of points of a negligible total length, so that every point α that remains has threefold characterization:
a) by a sequence of intervals closing down on it,
b) by the sequence of corresponding quasi-intervals and

c) by a uniquely determined function F common to this sequence of quasi-intervals, equicontinuous in the above sense.
We denote this function be $w(\cdot, \alpha)$. For variable α these are the trajectories of our Brownian motion.

5.2.4 Quadratic variation

The Brownian paths, although almost all continuous, turn out to be very irregular in nature. It can be shown, for instance, that for almost all α and each fixed instant t

$$\limsup_{\Delta t \to 0} \frac{w(t+\Delta t, \alpha) - w(t, \alpha)}{\Delta t} = \infty.$$

In other words, at each instant t the Brownian paths have infinite upper derivatives. Some explanation of this phenomenon is provided by the fact that in the prelimiting situation our symmetric random walk was allowed to make steps of considerably larger order of magnitude then the length of time intervals (see the discussion at the end of Section 5.1.2).

The next uncommon fact about the irregular character of the Brownian paths is that they almost all have infinite length. This follows from the following general remark concerning any continuous function f of time $t \in [0, T]$: for any grid $\{t_0, t_1, \dots, t_N\}$

$$\sum_{j=1}^{N} [\Delta f(t_j)]^2 \leq \max_{j \in \{1, \dots, N\}} |\Delta f(t_j)| \sum_{j=1}^{N} |\Delta f(t_j)|$$

with $\Delta f(t_j) = f(t_j) - f(t_{j-1})$. Hence, if the continuous function is of bounded variation (in the sense that the sum on the right is bounded independently of the choice of the grid or, geometrically, that the graph has infinite length) the sum on the left vanishes as the mesh width tends to zero. The following proposition asserts that this sum tends to zero for almost no Brownian path, hence almost no path has bounded variation.

Proposition 5.2.2. *For any time interval $[0, t]$, let $\{t_0, t_1, \dots, t_N\}$ be the dyadic grid of equidistant instants $t_j = \frac{j}{N}t$ with $N = 2^n$. Then for almost all α the sum of squares of the increments converges as $n \to \infty$:*

$$\sum_{j=1}^{N} [\Delta w(t_j, \alpha)]^2 \to \sigma^2 t. \tag{5.2.8}$$

The proof of this important result lays beyond the scope of the present paper (though probabilistic proofs are not very complicated, see e.g. [21], Section 8.2, or [42], Section 2.2). Instead, let us provide some intuitive explanation by turning back to the prelimiting situation of the symmetric random walk: within each time interval the square of the step size equals to the length of this interval multiplied by σ^2. Hence, the sum of squares yields the length t of the whole interval $[0, t]$ multiplied by the same σ^2.

Note that the right hand side in (5.2.8) is the same for almost all α. The limiting function $\sigma^2 t$ is special: within a wide class of stochastic processes (martingales with continuous sample paths, see [67], proposition 7.3) with convergent sums of type (5.2.8), only Brownian motion possesses this limit. Generally, such a limit, obtained from the increments of a certain underlying process X, depends on α and therefore constitutes a stochastic process with non-decreasing sample paths of bounded variation. It is usually denoted by $\langle X \rangle$ and called the *quadratic variation process* for X. If, for instance, X is a stochastic process of bounded variation, then $\langle X \rangle_t \equiv 0$. As we have already seen $\langle \mathcal{W} \rangle_t = \sigma^2 t$. Regarding quadratic variation processes, this is all we need for the present, see [67] for further references. In addition we only observe, anticipating Section 5.5, that also $\langle \grave{R}^\circ \rangle_t = \sigma^2 t$ for the discounted return process $\grave{R}^\circ = \{\grave{R}^\circ_t\}_{t \in [0,T]}$, since the presence of the drift μt in (5.5.3) is immaterial.

5.2.5 Itô's integral

Since almost no Brownian path is of bounded variation, the integral with respect to $\mathcal{W}$ cannot be defined in any conventional way, unless an integrand itself is of bounded variation, as in the latter case it can be defined in the usual Riemann-Stieltjes sense, see e.g. [37], p 55, or [56], pp 14–15. Indeed, if the integrand Φ has sample paths $\phi(\cdot, \alpha)$ of bounded variation for almost all α, then one can overcome difficulties on defining $\int_0^t \Phi_\theta d\mathcal{W}_\theta$ by means of the integrating by parts formula which yields

$$\int_0^t \phi(\theta, \alpha) dw(\theta, \alpha) = \phi(t, \alpha) w(t, \alpha) - \int_0^t w(\theta, \alpha) d\phi(\theta, \alpha). \tag{5.2.9}$$

This device is efficient for our purposes in Section 5.5, since the integrands are interpreted there as investor's portfolios during a finite trading period that by nature cannot have sample paths of infinite length.

However in theory we need more, for instance, the integral of form $\int_0^t \mathcal{W}_\theta d\mathcal{W}$ which cannot be defined in the above sense. An attempt to evaluate this integral explicitly would not lead to usual answer $\frac{1}{2}\mathcal{W}_t^2$ (as would be the case, if $\mathcal{W}$ were of bounded variation). Instead, we get

$$2 \int_0^t \mathcal{W}_\theta d\mathcal{W}_\theta = \mathcal{W}_t^2 - \langle \mathcal{W} \rangle_t \tag{5.2.10}$$

with $\langle \mathcal{W} \rangle_t = \sigma^2 t$ the quadratic variation of $\mathcal{W}$, see the previous section. In order to understand how the additional term (referred sometimes to as Itô's correction term) enters into consideration, look at the following Riemann-Stieltjes sum for this integral: with the notations of Proposition 5.2.2

$$2 \sum_{j=1}^{N} w(t_{j-1}, \alpha) \Delta w(t_j, \alpha) = w^2(t, \alpha) - \sum_{j=1}^{N} [\Delta w(t_j, \alpha)]^2.$$

The identity is obtained by elementary algebra, cf [37], p 60, [42], p 157. Let now $N \to \infty$. According to Itô's integration theory, the left hand side tends to $2\int_0^t w(\theta,\alpha)dw(\theta,\alpha)$, while the second term on the right tends to $\sigma^2 t$ by Proposition 5.2.2.

Surely, it is just impossible to enter here in details of stochastic calculus. For a good introduction we have already referred to [67] where further references can be found. For the same purposes one can also use [56], Section 1.1.3, [37], Chapter 4, [42], Section 4.5 (or [63] for more advanced theory). In the sequel we only intend to give some insight in formulas needed in Section 5.5. Firstly, we observe that the work on integration begins with a class of elementary integrands so that there is no confusion about how to integrate them. These are so-called simple processes whose sample paths are piecewise constant functions with discontinuities at certain fixed instants. They depend at each $t \in [0,T]$ only on the past of Brownian motion. See [67], formula (5.2). Then the integral with respect to $\mathcal{W}$ over any interval $[0,t]$ is defined in an obvious manner as the corresponding Riemann-Stieltjes sum: if Φ is an integrand whose sample path is observed to jump at instant t_j with $\phi_j(\alpha), j = 0,1,\dots,n$, then $\int_0^t \Phi_\theta d\mathcal{W}_\theta$ is defined by

$$\int_0^t \phi(\theta,\alpha)dw(\theta,\alpha) = \sum_{j=0}^{n} \phi_j(\alpha)[w(t_{j+1}\wedge t,\alpha) - w(t_j \wedge t,\alpha)],$$

cf [67], formula (5.3); for properties of this integral, see [67], proposition 5.2. We have, for instance, that the quadratic variation of the continuous process $X_t = \int_0^t \Phi_\theta d\mathcal{W}_\theta$ is

$$\langle X\rangle_t = \int_0^t \Phi_\theta^2 d\langle \mathcal{W}\rangle_\theta = \sigma^2 \int_0^t \Phi_\theta^2 d\theta. \tag{5.2.11}$$

Next, the definition is extended to a class of integrands approachable in a certain sense by sequences of simple processes. The integral of an integrand from this class is defined as the limit of the corresponding integrals of simple processes (the limit has to be independent of a particular choice of the approximating sequence). This is quite a delicate task. What makes theory operational is that one can take care about integration rules known from ordinary calculus. For instance, the chain rule says that if X is as above and Ψ is a suitable integrand, then

$$\int_0^t \Psi_\theta dX_\theta = \int_0^t \Psi_\theta \Phi_\theta d\mathcal{W}_\theta. \tag{5.2.12}$$

Next another important rule – Itô's formula (called sometimes the change of variables rule). Let u be a certain sufficiently smooth function of its arguments x and t, a space coordinate and time respectively, at least continuously differentiable in time and twice differentiable in space. Then with X possessing property (5.2.11) and with the notations (5.1.15) we have

$$\begin{aligned} u(X_t,t) &= u(X_0,0) + \int_0^t u_x(X_\theta,\theta)dX_\theta + \int_0^t u_t(X_\theta,\theta)d\theta \\ &+ \frac{1}{2}\sigma^2 \int_0^t u_{xx}(X_\theta,\theta)d\langle X\rangle_\theta. \end{aligned} \tag{5.2.13}$$

The proof is based on the fact that the quadratic variation $\langle X \rangle$ is non-zero, given by (5.2.11). Hence, when the function $u(\cdot,\cdot)$ is developed in Taylor's expansion, there is a (unconventional) contribution from the second term that yields Itô's correction term. This is a fundamental formula of stochastic calculus and its proof can be found in the textbooks we refer to, cf e.g. [48], Section 3.3. This section is closed by two applications.
(i) Put $u(x,t) = x^2$. From (5.2.13) we obtain an extension of (5.2.10) to X.
(ii) Put $u(x,t) = e^{x+\mu t}$ with some constant μ and apply (5.2.13) to $X = \mathcal{W} - \frac{1}{2}\langle \mathcal{W} \rangle$, as is done in [67], p 371. We obtain that $Z_t = e^{\mathcal{W}_t + \mu t - \frac{1}{2}\sigma^2 t}$ satisfies the following linear integral equation $Z_t = 1 + \int_0^t Z_\theta d\mathcal{W}_\theta$. The solution Z is known as Doléans-Dade (or stochastic) exponential and is denoted by $Z = \mathcal{E}(\mathcal{W})$. Clearly, if $\mathcal{W}$ in this integral equation were an ordinary function of bounded variation, say F, then we would simply have the ordinary exponential $Z = \mathcal{E}(\mathcal{W}) = e^F$.

5.3 Heat equations

5.3.1 Fokker-Planck equation

In this section we follow HIDA [42] in his short account of Einstein's ideas concerning Brownian motion. We need to consider only the projection of the motion onto a line. The density of pollen grains per unit length at an instant t will be denoted by $u(\cdot, t)$. This is a function of a space coordinate $x \in (-\infty, \infty)$. Suppose that the movement occurs uniformly in both time and space, so that the proportion of the pollen grains moved from x to $x + y$ in a time interval $(t, t+\Delta t)$ of length Δt may be written $p(y, \Delta t)$, as this is independent of x and t. For this time interval we thus obtain the superposition

$$u(x, t + \Delta t) = \int_{-\infty}^{\infty} u(x - y, t)p(y, \Delta t)dy \tag{5.3.1}$$

(valid under suitable smoothness conditions on the functions u and p, none of our present concern). Further, the function p is supposed to be a probability density possessing property (5.1.11), with the first two moments proportional to Δt:

$$\int_{-\infty}^{\infty} y^i p(y, \Delta t)dy = \begin{cases} \mu\Delta t & \text{if } i = 1 \\ \sigma^2\Delta t & \text{if } i = 2 \end{cases}$$

where the proportionality parameters $\mu \in (-\infty, \infty)$ and $\sigma^2 \in (0, \infty)$ are called the *drift* and *diffusion* coefficients, respectively, by the reason to become clear soon. Then the Taylor expansion of (5.3.1) for small increments Δt

$$\begin{aligned} & u(x, t) + \Delta t\ u_t(x, t) + \cdots \\ = & \int_{-\infty}^{\infty} \{u(x, t) - yu_x(x, t) + \frac{1}{2}y^2 u_{xx}(x, t) - \cdots\}p(y, \Delta t)dy \end{aligned}$$

(recall notations (5.1.15) for corresponding partial derivatives) reduces to the following second order partial differential equation

$$u_t = \frac{1}{2}\sigma^2 u_{xx} - \mu u_x. \tag{5.3.2}$$

This is the well-known *Fokker-Planck equation* for diffusion with drift (see e.g. [34], Section 14.6 or [11], Section 5.6). Suppose that initially the grain is at certain site x_0 say, which yields the initial condition (5.1.13). Then integrating (5.3.2), we obtain

Proposition 5.3.1. *The solution of the Fokker-Planck equation (5.3.2) subject to the initial condition (5.1.13) is given by (5.1.12).*

Proof. To verify that u given by (5.1.12) satisfies (5.3.2), make use of $g'(x) = -xg(x)$ and calculate directly from (5.1.12) the corresponding partial derivatives (5.1.15). As was already seen in Section 5.1.3 the initial condition (5.1.13) is satisfied as well. The proof is complete. $\square$

According to the theory of stochastic processes in the particular case of $x_0 = 0$, $\mu = 0$ and $\sigma = 1$ the u thus obtained turns out to be the transition probability function for standard Brownian motion (viewed as a Markov process, see e.g. [42], Section 2.4). In general this is the transition probability function for *diffusion* that consists of two terms, a deterministic term plus a stochastic term. The latter term is Brownian motion (which starts from an arbitrary site x_0, not necessarily the origin) scaled by σ, a constant associated with the medium. The deterministic term occurs only in presence of external field of source (e.g. gravity) which causes the drift μt.

In the next section we will discuss the strong similarity between diffusion and the molecular mechanism of thermal conductance. Of course, in contrast to diffusion there are no actual migration particles bearing heat, so that in this case relevant partial differential equations of type (5.3.2) have to be derived by phenomenological considerations.

5.3.2 Thermal conductance

In physics equation (5.3.2) emerge again in the following problem of thermal conductance. Let u be a certain sufficiently smooth function of its arguments x and t, a space coordinate and time respectively, at least continuously differentiable in time and twice differentiable in space. Consider the partial differential equation (5.3.2) with $\mu = 0$. In the present context, this is called the *heat equation*, because it describes the temperature distribution of a certain homogeneous isotropic body, in the absence of any heat sources within the body (it is enough for our purposes to restrict the consideration to special case of 'scalar body', a rod; see e.g. [9], [36], [48] or [69]). Otherwise, if a certain heat source causes temperature change proportional to time, with a proportionality parameter μ, then the heat equation is (5.3.2). In the present context the parameter σ^2 is called *thermal diffusivity* (as was pointed out by Einstein, $\sigma^2 = \frac{RT}{Nf}$ where N is universal constant depending on suspending material, T the absolute temperature, N the Avogadro number and f the coefficient of friction, see e.g. [72], Section 2.1).

Furthermore, in cases where the heat flow consists of both conduction and radiation, the heat equation has to be altered to include the effects of radiation. For instance, consider the temperature distribution in a rod which has a longitudinal heat flow due to conduction, as well as heat radiation from the surface. By Newton's law of surface heat transfer the rate of cooling per unit length of bar can be estimated by $r\bar{u}(\cdot,\cdot)$ where r is a positive constant and $\bar{u}$ a function of x and t expressing the excess of temperature of the bar over its surroundings. The heat equation then becomes

$$\bar{u}_t + r\bar{u} = \frac{1}{2}\sigma^2\bar{u}_{xx} - \mu\bar{u}_x. \tag{5.3.3}$$

But the latter equation is easily reduced to the previous (5.3.2) by a change of variable

$$\bar{u}(x,t) = e^{-rt}u(x,t) \tag{5.3.4}$$

with u satisfying (5.3.2). Thus, by integrating (5.3.2) and substituting the result into (5.3.4) we get the solution to (5.3.3). If, for instance, we look for a particular solution of (5.3.3) which yields the temperature distribution within the body due to a unit source of heat that at the initial date $t = 0$ is totally concentrated at a site x_0, then according to Proposition 5.3.1 the solution to (5.3.3) is given by

$$\bar{u}(x,t) = e^{-rt} g(x; x_0 + \mu t, \sigma^2 t). \tag{5.3.5}$$

Consider now the situation in which the distribution of temperature throughout the body at the initial date $t = 0$ is not concentrated at a certain site x_0 as before, but presented by a sufficiently smooth function f. Then the problem of thermal conductance consists of integrating the heat equation (5.3.3) subject to the initial condition

$$\bar{u}(x,0) = f(x). \tag{5.3.6}$$

Proposition 5.3.2. *The solution of the heat equation (5.3.3) subject to the initial Condition (5.3.6) is given by*

$$\hat{v}(x,t) = \int_{-\infty}^{\infty} \bar{u}(y-x,t) f(y) dy \tag{5.3.7}$$

with $\bar{u}$ of form (5.3.5) where $x_0 = 0$.

Proof. As was already noted, one can set $r = 0$ without loss of generality. Then it is easily seen that (5.3.7) satisfies (5.3.2) since the required partial differentiation can be carried out under the integral sign. To complete the proof, evaluate (5.3.7) at $t = 0$ by taking into consideration (5.1.12) and the property (5.1.14) of the limit (5.1.13). □

As was mentioned in Section 5.1.1, the option valuation problem of Section 5.5.3 reveals strong similarity to the problem of heat conduction where at a certain instant T (call it *maturity*, as usual) the distribution of temperature throughout the body is given and it is required to restore the previous process.

Suppose first that at maturity $t = T$ the temperature distribution is totally concentrated at a certain site x_0. Then the problem consists in solving equation

$$-\bar{u}_t + r\bar{u} = \frac{1}{2}\sigma^2 \bar{u}_{xx} - \mu \bar{u}_x \tag{5.3.8}$$

subject to the boundary condition $\bar{u}(x,T) = \delta(x - x_0)$. By the same arguments as above, we get the solution (5.3.5) but now with time to maturity $\tau = T - t$ on the right hand side instead of t.

Finally, let the temperature distribution at maturity $t = T$ be presented by a certain sufficiently smooth function f, so that the boundary condition becomes

$$\bar{u}(x,T) = f(x). \tag{5.3.9}$$

Then Proposition 5.3.2 yields

Corollary 5.3.3. *The solution of the heat equation (5.3.8) subject to the boundary condition (5.3.9) is given by*

$$\hat{u}(x,t) = \int_{-\infty}^{\infty} \bar{u}(x-y;\tau) f(y) dy \tag{5.3.10}$$

with time to maturity $\tau = T - t$ and with $\bar{u}$ of form (5.3.5) where $x_0 = 0$.

5.3.3 Modified heat equation

The problem considered at the end of the previous section is a special case of the general *Cauchy problem*, the solution of which is known as the *Feynman–Kac formula*, see e.g. [48], Section 5.7. In this section another special case will be treated, the solution of which yields in Section 5.5 the celebrated *Black–Scholes* formula (5.5.21) for option valuation, cf [6].

Let h be a function of a positive space coordinate $x \in (0,\infty)$ and time $t \in (0,\infty)$. Consider the following partial differential equation: for these x and t

$$-h_t + rh = \frac{1}{2}\sigma^2 x^2 h_{xx} - \mu x h_x. \tag{5.3.11}$$

First look for the particular solution of this equation subject to the boundary condition that for sufficiently smooth functions f of a positive argument we have at any positive site x_0

$$\lim_{t \uparrow T} \int_0^{\infty} h(\frac{x}{x_0}, t) f(x) \frac{dx}{x} \to f(x_0). \tag{5.3.12}$$

It is easily verified by the same arguments as above that the solution is given by $h(x,t) = \bar{u}(\log x, \tau)$ where $\bar{u}$ is again given by (5.3.5) but this time with $x_0 = 0$ and with $\mu + \frac{1}{2}\sigma^2$ instead of μ, i.e.

$$h(x,t) = e^{-r\tau} g(\log x; \mu\tau + \frac{1}{2}\sigma^2\tau, \sigma^2\tau). \tag{5.3.13}$$

Indeed, (5.3.11) is obtained by differentiating $h(x,t) = \bar{u}(\log x, \tau)$ and taking (5.3.8) into consideration, and (5.3.12) by taking the limit as $\tau \to 0$ on the right hand side of

$$\int_0^{\infty} h(\frac{x}{x_0}, t) f(x) \frac{dx}{x} = e^{-r\tau} \int_{-\infty}^{\infty} g(y; \mu\tau + \frac{1}{2}\sigma^2\tau, \sigma^2\tau) f(x_0 e^y) dy.$$

We apply this to the following boundary problem. Let H be a sufficiently smooth function of a positive argument (e.g. $(x-K)^+$ as in Section 5.4.3). Integrate (5.3.11) subject to the boundary condition

$$\bar{h}(x,T) = H(x). \tag{5.3.14}$$

Proposition 5.3.4. *The solution of the modified heat equation (5.3.11) subject to the boundary Condition (5.3.14) is given by*

$$\bar{h}(x,t) = \int_0^\infty h\left(\frac{x}{y}, t\right) H(y)\frac{dy}{y} \tag{5.3.15}$$

with h defined by (5.3.13).

Proof. Write (5.3.15) in the form

$$\bar{h}(x,t) = \int_{-\infty}^\infty \bar{u}(\log x - y, \tau) H(e^y) dy \tag{5.3.16}$$

and apply the same arguments as in the course of proving Proposition 5.3.2. □

By obvious change of variables we get from (5.3.13) and (5.3.16) that

$$\bar{h}(x,t) = e^{-r\tau} \int_{-\infty}^\infty g(y, -\mu\tau, \sigma^2\tau) H(xe^{y-\frac{1}{2}\sigma^2\tau}) dy, \tag{5.3.17}$$

the formula to which we will return in the Sections 5.4.3 and 5.5.

5.4 Towards the Black–Scholes model

5.4.1 Risk neutral probabilities

The considerations of the present section follow at certain extend the pattern of Section 4.3, though the results are completely different, of course. Via Lemma 4.3.3 we have proceeded to the representation (4.3.41) and the corresponding approximation (4.3.40). Alike, we will depart from Lemma 5.1.2 to get the representation (5.4.7) and the approximation (5.4.6) for the risk neutral probabilities. Next, we will use the equations (4.3.46), i.e.

$$\grave{v}^N_{2k-1\,n}(\pi) - \grave{v}^N_{k\,n-1}(\pi) = -p^N_{2k\,n} D_k(\grave{V}^N_n(\pi)), \;\; k = 1, \dots, 2^{n-1}, \qquad (5.4.1)$$

with

$$D_k(\grave{V}^N_n(\pi)) = \grave{v}^N_{2k\,n}(\pi) - \grave{v}^N_{2k-1\,n}(\pi) \qquad (5.4.2)$$

for $n = 1, \dots, N$. From these equations and from Lemma 4.3.8 we have obtained Proposition 4.3.9. The latter was intended as a prelimiting version of Proposition 4.4.3 that provides an important characterization of the Poisson market. In the next subsection the same equations (5.4.1) will be combined with Lemma 5.4.2 to prove Proposition 5.4.3 that will provide a prelimiting version of the heat equation. Like the assertion of Proposition 4.3.9, the new result will be expressed in terms of the increments in time and space, as well. Since the heat equations involve the first and second order derivatives in the space coordinate, however, not only (5.4.2) but also the second differences will be involved

$$D^2_k(V_n(\pi)) = v_{4k\,n}(\pi) - v_{4k-1\,n}(\pi) - v_{4k-2\,n}(\pi) + v_{4k-3\,n}(\pi) \qquad (5.4.3)$$

for $n = 2, \dots, N$ and $k = 1, \dots, 2^{n-2}$.

In the remainder of the present subsection we are going to provide two lemmas. The second one is Lemma 5.4.2 mentioned above. It is preceded by Lemma 5.4.1 concerning the gross returns on the stock. From Section 2.3.2 we know that these returns satisfy $DZ^N = DS^N/S^N_-$. In virtue of (5.1.2) we have the following approximation.

Lemma 5.4.1. *Under Condition 5.1.1 we have for $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$ that*

$$D_k(Z^N_n) = \frac{D_k(S^N_n)}{s^N_{k\,n-1}} = 2\sigma\sqrt{\Delta t_n}(1 + \xi^N_{k\,n-1}) \qquad (5.4.4)$$

with remainder terms $\{\xi^N_{kn}\}_{k=1,\dots,2^n}$ negligible in the sense explained in Remark 4.3.4.

Proof. By (5.1.2)

$$D_k(Z^N_n) = r^N_{2k\,n} - r^N_{2k-1\,n} = 2\sigma\sqrt{\Delta t_n} + (a+b+\eta^N_{2k\,n} + \eta^N_{2k-1\,n})\Delta t_n.$$

This satisfies (5.4.4) with

$$\xi_{k\,n-1}^N = \frac{a+b+\eta_{2k\,n}^N+\eta_{2k-1\,n}^N}{2\sigma}\sqrt{\Delta t_n}, \tag{5.4.5}$$

which under Condition 5.1.1 are indeed negligible remainder terms. □

It will be seen below that the risk neutral probabilities of the upward and downward displacements are asymptotically equivalent to the weights (5.1.7) used in Lemma 5.1.2, i.e. for $n = 1, \dots, N$

$$p_{kn}^N \sim \frac{1}{2}\left(1+(-1)^{k-1}\frac{\mu}{\sigma}\sqrt{\Delta t_n}\right), \quad k=1,\dots,2^n, \tag{5.4.6}$$

with the parameters $\sigma > 0$ same as in (5.1.2) and μ as in (5.1.6). This is indeed a consequence of the assertion of the following

Lemma 5.4.2. *Under the conditions 4.3.1 and 5.1.1 the risk neutral probabilities $\{p_{kn}^N\}_{k=1,\dots,2^n}$ for $n = 1, \dots, N$ may be represented as follows:*

$$p_{kn}^N = \frac{1}{2}\left(1+(-1)^{k-1}\left(\frac{\mu-\gamma_{k\,n-1}^N}{\sigma}\right)\sqrt{\Delta t_n}\right) \tag{5.4.7}$$

with remainder terms $\{\gamma_{kn}^N\}_{k=1,\dots,2^n}$ negligible (cf Remark 4.3.4).

Proof. For $n = 1, \dots, N$ let $\{\gamma_{kn}^N\}_{k=1,\dots,2^n}$ be defined by the equality

$$(1+\xi_{k\,n-1}^N)\gamma_{k\,n-1}^N = \varrho_n^N + \mu\xi_{k\,n-1}^N - \frac{\eta_{2k\,n}^N - \eta_{2k-1\,n}^N}{2} \tag{5.4.8}$$

with r and ϱ_n^N as in Condition 4.3.1, σ and $\{\eta_{kn}^N\}_{k=1,\dots,2^n}$ as in Condition 5.1.1 and $\{\xi_{kn}^N\}_{k=1,\dots,2^n}$ as in (5.4.5). Hence under the Conditions 4.3.1 and 5.1.1 all of $\{\gamma_{kn}^N\}_{k=1,\dots,2^n}$ are indeed negligible. Since by (4.3.10) and (5.1.2)

$$\rho_n^N - r_{2k-1\,n}^N = \sigma\sqrt{\Delta t_n} + \left(r+b+\varrho_n^N+\eta_{2k-1\,n}^N\right)\Delta t_n$$

and by (5.1.2) and (5.4.5)

$$r_{2k\,n}^N - r_{2k-1\,n}^N = 2\sigma(1+\xi_{k\,n-1}^N)\sqrt{\Delta t_n},$$

it follows from (4.3.43) that

$$\begin{aligned}\sigma(1+\xi_{k\,n-1}^N)\left(1-2p_{2k\,n}^N\right) &= \sigma\xi_{k\,n-1}^N - \left(r+b+\varrho_n^N+\eta_{2k-1\,n}^N\right)\sqrt{\Delta t_n} \\ &= \left(\mu-\varrho_n^N+\frac{\eta_{2k\,n}^N-\eta_{2k-1\,n}^N}{2}\right)\sqrt{\Delta t_n}.\end{aligned}$$

Thus

$$p_{2k\,n}^N = \frac{1}{2}\left(1-\frac{\mu-\varrho_n^N+\frac{\eta_{2k\,n}^N-\eta_{2k-1\,n}^N}{2}}{\sigma\left(1+\xi_{k\,n-1}^N\right)}\sqrt{\Delta t_n}\right)$$

which is equivalent to (5.4.7), since

$$\mu - \varrho_n^N + \frac{\eta_{2k\,n}^N - \eta_{2k-1\,n}^N}{2} = \left(1 + \xi_{k\,n-1}^N\right)\left(\mu - \gamma_{k\,n-1}^N\right)$$

by (5.4.8). □

5.4.2 Heat equation in finite differences

It will be shown in this subsection that under Conditions 4.3.1 and 5.1.1 an identity in finite differences holds that resembles the modified heat equation (5.3.11) upon neglecting remainder terms. This identity concerns the states $v_{kn} = v_{kn}^N(\pi)$ of the value process $V = V^N(\pi)$ for a self-financing strategy π. For simplicity, the argument π is suppressed, as well as all the upper indices N. As was already mentioned in Section 5.4.1, we are going to use both the first differences $D_k(V_n) = v_{2k\,n} - v_{2k-1\,n}$ for $n = 1, \dots, N$ and $k = 1, \dots, 2^{n-1}$, as well as the second differences $D_k^2(V_n) = v_{4k\,n} - v_{4k-1\,n} - v_{4k-2\,n} + v_{4k-3\,n}$ for $n = 2, \dots, N$ and $k = 1, \dots, 2^{n-2}$. The explicit expressions of the remainder terms and the consequent arguments, which we will need in the course of the proof, are rather clumsy and presented here solely for the sake of completeness (Kac's paper in [70] is a closest reference we know). Otherwise, the proof is in fact immaterial for further reading and may be taken granted.

Proposition 5.4.3. *Let the Conditions 4.3.1 and 5.1.1 hold. Let π be a self-financing strategy and $V = \{V_n\}_{n=0,1,\dots,N}$ its value process. Then for each $n = 2, \dots, N$ and $k = 1, \dots, 2^{n-2}$ the following identity hold:*

$$\frac{1}{2}\sigma^2 \frac{D_k^2(V_n)}{D_k(Z_{n-1})^2}\left(1 + \xi_{k\,n-2}\right)^2\left(1 + \varsigma_{k\,n-2}\right) - \left(\mu + \mu_{k\,n-2}\right)\frac{D_k(V_{n-1})}{D_k(Z_{n-1})}$$
$$= -\frac{v_{4k-1\,n} - v_{k\,n-2}}{t_n - t_{n-2}} + \left(r + \delta_{n-2}\right) v_{k\,n-2} + \delta_{k\,n-2}$$

where δ_n, $\{\delta_{kn}\}_{k=1,\dots,2^n}$, $\{\varsigma_{kn}\}_{k=1,\dots,2^n}$ and $\{\mu_{kn}\}_{k=1,\dots,2^n}$ are negligible remainder terms, alike $\{\xi_{kn}\}_{k=1,\dots,2^n}$ given by (5.4.5).

Proof. By the first identity in (5.4.4) the desired identity is equivalent to

$$\begin{aligned}
& \frac{1}{2}\left(\sigma s_{k\,n-2}\right)^2 \frac{D_k^2(V_n)}{D_k(S_{n-1})^2}\left(1 + \xi_{k\,n-2}\right)^2\left(1 + \varsigma_{k\,n-2}\right) \\
- \quad & \left(\mu + \mu_{k\,n-2}\right) s_{k\,n-2}\frac{D_k(V_{n-1})}{D_k(S_{n-1})} \\
= \quad & -\frac{v_{4k-1\,n} - v_{k\,n-2}}{t_n - t_{n-2}} + \left(r + \delta_{n-2}\right) v_{k\,n-2} + \delta_{k\,n-2}
\end{aligned} \tag{5.4.9}$$

We will prove that the latter equation is satisfied with the remainder terms

$$\varsigma_{k\,n-2} = 4p_{4k\,n}p_{4k-3\,n} - 1, \tag{5.4.10}$$

$$\mu_{k\,n-2} = \left(1 + \rho_n\right)\left(1 + \xi_{k\,n-2}\right)\left(\mu - \frac{\gamma_{k\,n-2} + \gamma_{2k\,n-1}}{2}\right) - \mu, \tag{5.4.11}$$

$$2\delta_{n-2} = \varrho_{n-1} + \varrho_n + (r + \varrho_{n-1})(r + \varrho_n)\Delta t_n \tag{5.4.12}$$

and

$$\begin{aligned}\delta_{k\,n-2} &= p_{4k\,n}((1+\xi_{k\,n-1})\, s_{2k\,n-1}\frac{D_{2k}(V_n)}{D_{2k}(S_n)}\frac{\gamma_{2k\,n-1}-\gamma_{2k-1\,n-1}}{2} \\ &+ \frac{v_{4k-1\,n}-v_{k\,n-2}}{t_n-t_{n-2}} - \frac{v_{4k-2\,n}-v_{k\,n-2}}{t_n-t_{n-2}}).\end{aligned} \tag{5.4.13}$$

This will imply the desired assertion, since the remainder terms, given by (5.4.10)–(5.4.13) are negligible under the Conditions 4.3.1 and 5.1.1. Indeed, the terms $\{\varsigma_{kn}\}_{k=1,\ldots,2^n}$ are negligible due to (5.4.6) and the terms $\{\mu_{kn}\}_{k=1,\ldots,2^r}$ due to (4.3.10), (5.4.5) and (5.4.8). Next, the terms $\{\delta_{kn}\}_{k=1,\ldots,2^n}$ are negligible due to (5.4.8) and to the fact that the difference of the second and third terms on the right-hand side is negligible. Finally, the negligibility of δ_n is obvious.

To prove (5.4.9), we proceed as follows. It will be shown first that (5.4.9) is equivalent to (5.4.14) below. Then the latter identity will be proved. Let us examine (5.4.9) term by term. Due to (5.4.4) and (5.4.10), the first term on the left hand side may be written in the form

$$p_{4k\,n}p_{4k-3\,n}\frac{D_k^2(V_n)}{2\Delta t_n}.$$

The second term may be rewritten as

$$(1+\rho_n)\,\frac{1-p_{2k\,n-1}-p_{4k\,n}}{2\Delta t_n}D_k(V_{n-1}),$$

since by (5.4.7) and (5.4.11)

$$\begin{aligned}(1+\rho_n)\,\sigma\frac{1-p_{2k\,n-1}-p_{4k\,n}}{\sqrt{\Delta t_n}} &= (1+\rho_n)\left(\mu - \frac{\gamma_{k\,n-2}+\gamma_{2k\,n-1}}{2}\right) \\ &= \frac{\mu+\mu_{k\,n-2}}{1+\xi_{k\,n-2}}.\end{aligned}$$

Next, let us handle the right-hand side of (5.4.9). The sum of the first two terms on this side equals

$$\frac{(1+\rho_n)\,(1+\rho_{n-1})\,v_{k\,n-2}-v_{4k-1\,n}}{2\Delta t_n}$$

by (4.3.10) and (5.4.12). On verifying this take into consideration that the trading times are equidistant and $\Delta t_n = T/N$ independently of n. Finally, by the same assumptions and by (5.4.4) the third term on the right, given by (5.4.13), may be written in the form

$$\frac{p_{4k\,n}}{2\Delta t_n}\left(v_{4k-1\,n} - v_{4k-2\,n} + D_{2k}(V_n)\sqrt{\Delta t_n}\frac{\gamma_{2k\,n-1}-\gamma_{2k-1\,n-1}}{2\sigma}\right).$$

Thus we obtain the following identity, equivalent to (5.4.9):

$$
\begin{aligned}
& p_{4k\,n}p_{4k-3\,n}D_k^2(V_n) + (1+\rho_n)\,(p_{2k\,n-1} + p_{4k\,n} - 1)\,D_k(V_{n-1}) \\
= \quad & (1+\rho_n)\,(1+\rho_{n-1})\,v_{k\,n-2} - v_{4k-1\,n} \\
+ \quad & p_{4k\,n}\left(v_{4k-1\,n} - v_{4k-2\,n} + D_{2k}(V_n)\sqrt{\Delta t_n}\frac{\gamma_{2k\,n-1}-\gamma_{2k-1\,n-1}}{2\sigma}\right).
\end{aligned}
$$

The latter identity, in turn, is equivalent to

$$
\begin{aligned}
& (1+\rho_n)\{v_{2k\,n-1}p_{4k-3\,n} + v_{2k-1\,n-1}p_{4k\,n} \\
+ \quad & (p_{2k\,n-1}+p_{4k\,n}-1)D_k(V_{n-1})\} - (v_{4k-1\,n}p_{4k-3\,n} + v_{4k-2\,n}p_{4k\,n}) \\
= \quad & (1+\rho_n)(1+\rho_{n-1})v_{k\,n-2} - v_{4k-1\,n} \\
+ \quad & p_{4k\,n}\{v_{4k-1\,n} - v_{4k-2\,n} + (p_{4k\,n} - p_{4k-2\,n})D_{2k}(V_n)\} \qquad (5.4.14)
\end{aligned}
$$

in virtue of the following two identities. Firstly

$$
\sqrt{\Delta t_n}\frac{\gamma_{2k\,n-1} - \gamma_{2k-1\,n-1}}{2\sigma} = p_{4k\,n} - p_{4k-2\,n}
$$

which is a consequence of (5.4.7). Secondly, due to

$$
\begin{aligned}
p_{4k\,n}p_{4k-3\,n}D_k^2(V_n) \quad = \quad & (1+\rho_n)\,(v_{2k\,n-1}p_{4k-3\,n} + v_{2k-1\,n-1}p_{4k\,n}) \\
- \quad & (v_{4k-1\,n}p_{4k-3\,n} + v_{4k-2\,n}p_{4k\,n})
\end{aligned}
$$

which is a consequence of (5.4.3) and (5.4.1), since

$$
D_k^2(V_n) = \frac{(1+\rho_n)\,v_{2k\,n-1} - v_{4k-1\,n}}{p_{4k\,n}} - \frac{v_{4k-2\,n} - (1+\rho_n)\,v_{2k-1\,n-1}}{p_{4k-3\,n}}.
$$

Thus the equivalence of (5.4.9) and (5.4.14) is proved. It remains to prove (5.4.14). By (5.4.1) in its non-discounted form

$$
\begin{aligned}
& v_{2k\,n-1}p_{4k-3\,n} + v_{2k-1\,n-1}p_{4k\,n} - (1+\rho_{n-1})\,v_{k\,n-2} \\
& = v_{2k\,n-1}\,(p_{4k-3\,n} - p_{2k\,n-1}) + v_{2k-1\,n-1}\,(p_{4k\,n} - p_{2k-1\,n-1}) \\
& = (v_{2k\,n-1} - v_{2k-1\,n-1})\,(1 - p_{2k\,n-1} - p_{4k\,n}) + v_{2k\,n-1}\,(p_{4k\,n} - p_{4k-2\,n})
\end{aligned}
$$

and

$$
\begin{aligned}
& v_{4k-1\,n}p_{4k-3\,n} + v_{4k-2\,n}p_{4k\,n} - v_{4k-1\,n} = v_{4k-2\,n}p_{4k\,n} - v_{4k-1\,n}p_{4k-2\,n} \\
& = v_{4k-1\,n}\,(p_{4k\,n} - p_{4k-2\,n}) - (v_{4k-1\,n} - v_{4k-2\,n})\,p_{4k\,n}.
\end{aligned}
$$

These two identities imply (5.4.14). The proof is complete. $\square$

5.4.3 Approximate option pricing

In the present section the link is sought between the binary market of the present Section 5.4 and the Black-Scholes market of the next Section 5.5. By using certain heuristic arguments we show that under the Conditions 4.3.1 and 5.1.1 the Black-Scholes model can serve as an approximation to the binary model, when the number of trading periods N increases unboundedly and so the length of each trading period T/N tends to zero. For clear distinction from the prelimiting situation, we will denote the bond and stock prices in the Black-Scholes market by $B^\circ(t)$ and $S^\circ(t)$, respectively (the coincidence with the notations of the previous case of the Poisson market can't cause ambiguities, as two limiting cases are clearly separated). Trading in the latter market is going on continuously so that $t \in [0,T]$. Regarding the bond, the situation is simple, since Condition 4.3.1 means that at each fixed $t \in [0,T]$ the approximation $B^N(t) \sim B^\circ(t)$ holds. Actually, the approximate bond price is led to be $B^\circ(t) = e^{rt}$, cf (4.3.12) and (5.5.1) below. As for a risky asset, the stock, in the present setup we will be only able to demonstrate certain aspects of the approximation of the process S^N by S°, for more detailed treatment would lead us too far afield. It will be shown, in particular, how to approximate the fair price of the European call option, see Proposition 5.4.5.

We will use expression (5.4.7) for the approximate risk neutral probabilities, but suppress the negligible remainder terms $\{\gamma_{kn}^N\}_{k=1,\dots,2^n}$, as it is not hard to verify that they play no part in the asymptotic considerations below. Thus the approximation (5.4.6) may be used. Since the right hand side in (5.1.3) is independent of indices k and n, the situation here is asymptotically similar to that of the homogeneous binomial model. Thus we may use the Cox-Ross-Rubinstein option pricing formula (3.5.6) of Chapter 3. Upon the substitutions (4.3.10), (5.1.3) and (5.4.6), this yields the first approximation to the option pricing formula:

$$\begin{aligned} C^N &\sim (1+r\frac{T}{N})^{-N}\sum_{j=0}^{N}\binom{N}{j}\frac{1}{2^N}(1-\frac{\mu}{\sigma}\sqrt{\frac{T}{N}})^j(1+\frac{\mu}{\sigma}\sqrt{\frac{T}{N}})^{N-j} \\ &\times\ H(s(1+\sigma\sqrt{\frac{T}{N}}+a\frac{T}{N})^j(1-\sigma\sqrt{\frac{T}{N}}-b\frac{T}{N})^{N-j}) \qquad (5.4.15)\end{aligned}$$

with $H(x) = (x-K)^+$, the payoff function for the European call option. It will be shown below that this approximation can be considerably simplified, see Proposition 5.4.5. This is preceded by the following

Lemma 5.4.4. *Fix positive integers n and $j \le n$. Denote $t_j = j\sqrt{2/n}$ so that $\Delta t_j \equiv \sqrt{2/n}$. Then (i) for a nonnegative constant c*

$$\left(\frac{1+\sqrt{\frac{c}{2n}}}{1-\sqrt{\frac{c}{2n}}}\right)^j \sim e^{\sqrt{c}t_j}$$

and (ii) with g the standard normal density (cf (5.1.10))

$$\frac{1}{2^{2n}}\binom{2n}{n+j} \sim g(t_j)\Delta t_j. \qquad (5.4.16)$$

Proof. (i) It follows from $\log(1+x) \sim x$ that

$$j\left\{\log\left(1+\sqrt{\frac{c}{2n}}\right) - \log\left(1-\sqrt{\frac{c}{2n}}\right)\right\} \sim j\sqrt{\frac{2c}{n}} = \sqrt{c}t_j,$$

which yields the desired result.
(ii) Presenting the left hand side as the product of

$$a_n = \frac{1}{2^{2n}}\binom{2n}{n}$$

and

$$b_n = \frac{n(n-1)\cdots(n-j+1)}{(n+j)\cdots(n+1)} = \frac{1}{(1+\frac{j}{n})(1+\frac{j}{n-1})\cdots(1+\frac{j}{n-j+1})}$$

we get to show

$$a_n \sim \frac{1}{\sqrt{\pi n}} = \frac{\Delta t_j}{\sqrt{2\pi}}$$

and

$$\log b_n \sim -\frac{j^2}{n} = -\frac{t_j^2}{2}.$$

The former relation follows from Stirling's formula $n! \sim e^{-n}n^n\sqrt{2\pi n}$, truly from its consequence

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}},$$

see [7], Section 1.2. The latter one follows from $\log(1+x) \sim x$, since

$$\log b_n = -\sum_{k=0}^{j-1}\log\left(1+\frac{j}{n-k}\right) \sim -\sum_{k=0}^{j-1}\frac{j}{n-k} \sim -\frac{j^2}{n} = -\frac{t_j^2}{2}.$$

The proof of (i) is complete. □

There are various methods for proving the next proposition presented in textbooks on the probability theory. In the sequel we will continue to follow BREIMAN [7].

Proposition 5.4.5. *Under Conditions 4.3.1 and 5.1.1 the fair price of the European call option with the payoff function $H(x) = (x-K)^+$, is approximated as follows:*

$$C^N \sim \int_{-\infty}^{\infty} g(y;0,\sigma^2 T)\left(se^{y-\frac{1}{2}\sigma^2 T} - \grave{K}\right)^+ dy \tag{5.4.17}$$

where $\grave{K} = \frac{K}{B^\circ(T)} = e^{-rT}K$ is the discounted exercise price, cf (5.5.1).

Proof. Without loss of generality we assume that N is even. Put $N = 2n$. Then the summation in (5.4.15) may be changed to $\{-n, \dots, n\}$ that yields

$$C^N \sim C_n^n \sum_{j=-n}^{n} g_{jn} c_n^j H\left(sK_n^n k_n^j\right) \tag{5.4.18}$$

where g_{jn} denotes the left hand side of (5.4.16) and

$$C_n = \frac{1 - \frac{\mu^2}{\sigma^2}\frac{T}{2n}}{\left(1 + r\frac{T}{2n}\right)^2}, \qquad K_n = (1+\sigma\sqrt{\frac{T}{2n}}+a\frac{T}{2n})(1-\sigma\sqrt{\frac{T}{2n}}-b\frac{T}{2n}),$$

$$c_n = \frac{1 - \frac{\mu}{\sigma}\sqrt{\frac{T}{2n}}}{1 + \frac{\mu}{\sigma}\sqrt{\frac{T}{2n}}}, \qquad k_n = \frac{1 + \sigma\sqrt{\frac{T}{2n}} + a\frac{T}{2n}}{1 - \sigma\sqrt{\frac{T}{2n}} - b\frac{T}{2n}}.$$

By the well known property of exponentials

$$C_n^n \sim e^{-rT - \frac{1}{2}\frac{\mu^2 T}{\sigma^2}}$$

and

$$K_n^n \sim (1 + (a - b - \sigma^2)\frac{T}{2n})^n \sim e^{\frac{a-b}{2}T - \frac{1}{2}\sigma^2 T} = e^{(r+\mu)T - \frac{1}{2}\sigma^2 T},$$

cf (5.1.6). Apply now Lemma 5.4.4. Assertion (ii) gives an approximation to g_{jn} and assertion (i) gives

$$c_n^j \sim e^{-\frac{\mu\sqrt{T}t_j}{\sigma}} \quad \text{and} \quad k_n^j \sim e^{\sigma\sqrt{T}t_j}.$$

These approximations reduce (5.4.18) to

$$C^N \sim \frac{e^{-rT}}{\sqrt{2\pi}} \sum_{j=-n}^{n} e^{-\frac{1}{2}\frac{(\sigma\sqrt{T}t_j + \mu T)^2}{\sigma^2 T}} H(se^{\sigma\sqrt{T}t_j + (r+\mu)T - \frac{1}{2}\sigma^2 T})\Delta t_j.$$

Put $\tau_j = \sigma\sqrt{T}t_j$. Then

$$C^N \sim e^{-rT} \sum_{\tau_j \in \mathcal{T}_n} g(\tau_j; -\mu T, \sigma^2 T) H(se^{\tau_j + (r+\mu)T - \frac{1}{2}\sigma^2 T})\Delta\tau_j$$

with $\mathcal{T}_n$ the set $\{j\sigma\sqrt{\frac{2T}{n}}\}_{j=0,\pm1,\dots,\pm n}$ whose lowest entry $-\sigma\sqrt{2nT}$ tends to $-\infty$ and the largest entry $\sigma\sqrt{2nT}$ to ∞ as $n \to \infty$. So the sum in the latter expression is actually the Riemann sum for the integral

$$\int_{-\infty}^{\infty} g(y + \mu T, 0, \sigma^2 T)\left(se^{y + \mu T - \frac{1}{2}\sigma^2 T} - \grave{K}\right)^+ dy$$

which is independent of μ and equals to the integral on the right hand side of (5.4.17). The proof is complete. □

5.5 Black-Scholes model

5.5.1 Assets

In this section we consider the limiting model for a securities market. According to (4.3.11) and (4.3.12), the model for the bond is defined by the linear return process $\mathcal{R}^\circ = \{\mathcal{R}_t^\circ\}_{t\in[0,T]}$ and the exponential price process $B^\circ = \{B_t^\circ\}_{t\in[0,T]}$ with

$$\mathcal{R}_t^\circ = rt \quad \text{and} \quad B_t^\circ = e^{rt} \tag{5.5.1}$$

where $r > 0$ is a riskless interest rate on the bond. (Note that $B^\circ = \mathcal{E}(\mathcal{R}^\circ)$ in the sense given at the very end of Section 5.2.5.)

The stock is again a risky asset: its return process $R^\circ = \{R_t^\circ\}_{t\in[0,T]}$ is defined in accordance with the right-hand side of (5.1.3): the cumulative impact up to time t of the terms proportional to $\sqrt{\Delta t_n}$ yields $\mathcal{W}_t$ and that of the terms proportional to Δt_n yields the drift $\frac{1}{2}(a-b)t$, see discussion at the end of Section 5.1.2. This leads to the following diffusion model

$$R_t^\circ = \mathcal{W}_t + \frac{1}{2}(a-b)t.$$

Consequently, the price process on the stock $S^\circ = \{S_t^\circ\}_{t\in[0,T]}$ is now defined by

$$S_t^\circ = s\mathcal{E}(R^\circ)_t = se^{\mathcal{W}_t + \frac{1}{2}(a-b)t - \frac{1}{2}\sigma^2 t} = se^{R_t^\circ - \frac{1}{2}\langle R^\circ\rangle_t} \tag{5.5.2}$$

(see application (ii) at the end of Section 5.2.5) where $s > 0$ is a fixed current price on the stock $S_0^\circ = s$ and $\langle R^\circ\rangle_t = \sigma^2 t$ as in Section 5.2.4. The discounted stock price process is defined by

$$\grave{S}_t^\circ \doteq \frac{S_t^\circ}{B_t^\circ} = se^{\mathcal{W}_t + \mu t - \frac{1}{2}\sigma^2 t} = se^{\grave{R}_t^\circ - \frac{1}{2}\langle \grave{R}^\circ\rangle_t} \tag{5.5.3}$$

where $\grave{R}_t^\circ = \mathcal{W}_t + \mu t$ is the corresponding return at instant t, cf (5.1.6). The relation $\grave{S}^\circ = s\mathcal{E}(\grave{R}^\circ)$ is obtained in Section 5.2.5.

As we know, Brownian motion takes its rise at the origin but afterwards at any consecutive instant $t > 0$ it may visit any site $-\infty < x < \infty$. Accordingly, the non-discounted and discounted stock prices, starting from a fixed state $s > 0$, may occupy at any instant $t \in [0,T]$ and site $-\infty < x < \infty$ one of the states

$$\begin{aligned} s^\circ(x,t) &= se^{xI_{\{t\neq 0\}} + \frac{1}{2}(a-b)t - \frac{1}{2}\sigma^2 t} \\ \grave{s}^\circ(x,t) &= se^{xI_{\{t\neq 0\}} + \mu t - \frac{1}{2}\sigma^2 t} \end{aligned} \tag{5.5.4}$$

where $I_{\{t\neq 0\}}$ is the indicator function equal 1 everywhere except at the origin $t = 0$ where it equals to 0. Concerning these states, the following simple proposition holds true.

Proposition 5.5.1. *(i) At each instant $t > 0$, the discounted stock price is in one of the states (5.5.4) that satisfies the second order partial differential equation (5.3.8) with $r = 0$, i.e.*

$$-\grave{s}_t^\circ = \frac{1}{2}\sigma^2 \grave{s}_{xx}^\circ - \mu \grave{s}_x^\circ. \tag{5.5.5}$$

(ii) With u given by (5.1.12) where $x_0 = 0$, we have

$$\grave{s}^\circ(x,t) = \int_{-\infty}^{\infty} u(x-y, \Delta t)\grave{s}^\circ(y, t+\Delta t)dy. \tag{5.5.6}$$

Proof. (i) The required partial derivatives $\grave{s}_t^\circ$, $\grave{s}_x^\circ$ and $\grave{s}_{xx}^\circ$ are simply calculated. Thus (5.3.2) is easily verified.
(ii) By the obvious property $\grave{s}^\circ(x+\Delta x, t+\Delta t) = \grave{s}^\circ(\Delta x, \Delta t)\grave{s}^\circ(x,t)$ of the states (5.5.4), it suffices to show

$$\int_{-\infty}^{\infty} u(x,t)\grave{s}^\circ(-x,t)dx = 1.$$

But this is easily verified by (5.1.11) and (5.1.12). □

Remark 5.5.2. By analogy to the prelimiting situation, we want to define the difference operator D in the state space. We depart from (5.4.4) and let $\Delta t_n = \frac{T}{N} \to 0$. The terms $\{\xi_{kn}^N\}_{k=1,\dots,2^n}$ are negligible. According to (5.1.2) the denominator $2\sigma\sqrt{\Delta t_n}$ is the first approximation to the difference between two alternative states of the return. Then by the same arguments as above Brownian motion enters into consideration: we are led to define the limit on the left as $\frac{DS_t^\circ}{DW_t}$. This yields $\frac{DS^\circ}{DW} = S^\circ$, one of the first formulas of the Malliavin calculus, see [56], exercise 2.2.1 on p 107.

5.5.2 Self-financing strategies

Let us consider an investor who invests an amount $v \geq 0$ in the present market and then follows a trading strategy $\pi = (\Psi, \Phi)$ with portfolio components $\Psi = \{\Psi_t\}_{t\in[0,T]}$ and $\Phi = \{\Phi_t\}_{t\in[0,T]}$. The corresponding value process $V^\circ(\pi) = \Psi B^\circ + \Phi S^\circ$ is defined at $t \in [0,T]$ by

$$V^\circ(t;\pi) = \Psi(t)B^\circ(t) + \Phi(t)S^\circ(t) \tag{5.5.7}$$

with $v = V^\circ(0;\pi)$. If π is a self-financing strategy in the sense of the definition in Section 5.4.1, then one can apply the integrating by parts formula of Section 5.2.5 (with respect to the geometric Brownian motion instead of the ordinary Brownian motion in (5.2.9); this substitution is allowed by the chain rule (5.2.12)). This yields the same integral representation for discounted value process $\grave{V}^\circ(\pi) = \{\grave{V}^\circ(t;\pi)\}_{t\in[0,T]}$ as before: at each $t \in [0,T]$

$$\grave{V}^\circ(t;\pi) = v + \Phi \cdot \grave{S}_t^\circ, \tag{5.5.8}$$

cf Section 4.3, Proposition 4.3.5. Analogously to the trading in binary markets, the self-financing of a strategy $\pi = (\Psi, \Phi)$ means that the portfolio components

$\Psi(t)$ and $\Phi(t)$ yield not only the market value of the holding at instant $t \in (0,T]$ (given by (5.5.7)) but also at an immediate future instant $t+\Delta t$:

$$V^\circ(t+\Delta t;\pi) = \Psi(t)B^\circ(t+\Delta t) + \Phi(t)S^\circ(t+\Delta t) \tag{5.5.9}$$

cf (4.3.28) and (4.3.29). Let us denote the possible states of the portfolio components $\Psi(t)$ and $\Phi(t)$ by

$$\{\psi(x,t), -\infty < x < \infty\}$$

$$\{\phi(x,t), -\infty < x < \infty\}$$

and the states of the discounted market value of this holding $\grave{V}^\circ(t;\pi)$ by

$$\{\grave{v}^\circ(x,t;\pi), -\infty < x < \infty\}$$

. Then (5.5.7) and (5.5.9) mean both

$$\grave{v}^\circ(x,t;\pi) = \psi(x,t) + \phi(x,t)\grave{s}^\circ(x,t) \tag{5.5.10}$$

and

$$\grave{v}^\circ(x+\Delta x,t+\Delta t;\pi) = \psi(x,t) + \phi(x,t)\grave{s}^\circ(x+\Delta x,t+\Delta t). \tag{5.5.11}$$

This fact has the following implication:

Proposition 5.5.3. *(i) At each instant $t > 0$, the discounted market value $\grave{V}^\circ(t;\pi)$ of a self-financing strategy π is in one of the states $\{\grave{v}^\circ(x,t;\pi), -\infty < x < \infty\}$ that satisfies the second order partial differential equation (5.3.8) with $r = 0$.*
(ii) With u given by (5.1.12) where $x_0 = 0$, we have

$$\grave{v}^\circ(x,t;\pi) = \int_{-\infty}^{\infty} u(x-y,\Delta t)\grave{v}^\circ(y,t+\Delta t;\pi)dy. \tag{5.5.12}$$

Proof. (i) can be obtained from (ii). Indeed, take (5.5.12) with $t+\Delta t = T$ and apply to both sides the operator $\frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2} - \mu\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$. On the right the operator is allowed to act under the integral sign. But this yields 0, since $u(x-y,\tau)$ satisfies (5.3.8) with $r = 0$. As usual $\tau = T - t$ is time to maturity.
(ii) In (5.5.11) substitute $x + \Delta x$ by a variable y, multiply both sides by $u(x-y,\Delta t)$ and integrate with respect to dy. By (5.5.6) and (5.5.10) we get (5.5.12). □

Remark 5.5.4. Like in the previous remark at the end of Section 5.5.1, we recall the prelimiting situation in which the stock component was expressed as $\Phi^N(t) = \frac{DV^N(t;\pi)}{DS^N(t)}$, see Clark's formula (4.3.34) of Proposition 4.3.7. This structure is retained as $N \to \infty$, thus giving the possibility to rewrite (5.5.8) in Clark's form

$$\grave{V}^\circ(t;\pi) = v + \frac{DV^\circ(\pi)}{DS^\circ} \cdot \grave{S}^\circ_t. \tag{5.5.13}$$

This is again an elementary formula in the Malliavin calculus, see [56], definition 1.2.1 on p 24. In the remaining part of this section more light will be shed on this formula, cf (5.5.16) below.

There is yet another relationship between the value process for a self-financing strategy and the stock price process. Namely, let $\grave{h}$ be a solution of the partial differential equation (5.3.11) with $r = \mu = 0$. Then for each $t \in [0, T]$ the following useful representation can be proved:

$$\grave{V}^\circ(t;\pi) = \grave{h}(\grave{S}^\circ(t), t). \tag{5.5.14}$$

In the next proposition the proof is provided in the form of the relationship between the states of the variables on the left and right hand side.

Proposition 5.5.5. *(i) At each instant $t > 0$ and site x the states of the discounted stock price $\grave{S}^\circ(t)$ and the discounted value $\grave{V}^\circ(t;\pi)$ for a self-financing strategy π are related by*

$$\grave{v}^\circ(x, t; \pi) = \grave{h}(\grave{s}^\circ(x, t), t) \tag{5.5.15}$$

where $\grave{h}$ is the same as in (5.5.14).

Proof. By taking the relevant partial derivatives on the both sides of (5.5.15) we get

$$\frac{1}{2}\sigma^2 \grave{v}^\circ_{xx} - \mu \grave{v}^\circ_x + \grave{v}^\circ_t = (\frac{1}{2}\sigma^2 \grave{s}^\circ_{xx} - \mu \grave{s}^\circ_x + \grave{s}^\circ_t)\grave{h}_x + \frac{1}{2}\sigma^2 \grave{s}^{\circ 2}_x \grave{h}_{xx} + \grave{h}_t$$

It remains now to recall the assertion (i) in Proposition 5.5.1 and to apply (5.3.11) with $r = \mu = 0$ to the sum of the last two terms on the right. Indeed, by taking into consideration $\grave{s}^\circ_x = \grave{s}^\circ$ we get on the right hand side $\frac{1}{2}\sigma^2 \grave{s}^\circ(x,t)^2 \grave{h}_{xx}(\grave{s}^\circ(x,t), t) + \grave{h}_t(\grave{s}^\circ(x,t), t)$ which equals 0 in view of the assumption that $\grave{h}$ satisfies (5.3.11) with $r = \mu = 0$. This means that (5.5.15) satisfies (5.3.8) with $r = 0$, cf Proposition 5.5.3, assertion (i). □

Let us apply Itô's formula (5.2.13) to (5.5.14). Taking into consideration also the expression (5.2.11) for the quadratic variation and the chain rule (5.2.12), we get

$$\begin{aligned}
\grave{V}^\circ(t;\pi) &= \grave{V}^\circ(0;\pi) + \int_0^t \grave{h}_x(\grave{S}^\circ(\theta), \theta) d\grave{S}^\circ(\theta) + \int_0^t \grave{h}_t(\grave{S}^\circ(\theta), \theta) d\theta \\
&+ \frac{1}{2}\sigma^2 \int_0^t \grave{h}_{xx}(\grave{S}^\circ(\theta), \theta) \grave{S}^\circ(\theta)^2 d\theta,
\end{aligned}$$

cf [39], formula (1.12). But the last two terms vanish, since $\grave{h}$ satisfies (5.3.11) with $r = \mu = 0$. We thus have the integral representation (5.5.8) (or (5.5.13)) with

$$\Phi(t) = \frac{DV^\circ(t, \pi)}{DS^\circ(t)} = \grave{h}_x(\grave{S}^\circ(t), t). \tag{5.5.16}$$

Formula (5.5.14) is often presented in its nondiscounted form (see e.g. [39], formula (1.8), or [48], formula (5.8.36)):

$$V^\circ(t, \pi) = \bar{h}(S^\circ(t), t) \tag{5.5.17}$$

where $\bar{h}$ satisfies (5.3.11) with $r = -\mu$. Formula (5.5.16) for the stock component of the portfolio becomes $\Phi(t) = \bar{h}_x(\grave{S}^\circ(t), t)$. One can easily verify these claims by taking into consideration that

$$\bar{h}(x,t) = e^{rt}\grave{h}(e^{-rt}x, t)$$

. Of course, the factor e^{rt} means discounting, so (cf (5.5.1) and (5.5.3))

$$\grave{h}(\grave{S}^\circ(t), t) = \frac{\bar{h}(S^\circ(t), t)}{B^\circ(t)}. \tag{5.5.18}$$

5.5.3 Hedging strategies and option pricing

In the present section the same question as in Chapter 3, Section 3.4.1 arises whether the Black-Scholes market is complete or not. In order to formulate this question explicitly, consider again an investor whose goal is to attain at the terminal date T a certain wealth $W(T) = W(S^\circ(T))$ which is a certain function W of the stock price $W(S^\circ(T))$. According to (5.5.4) this wealth may be in one of the states

$$w^\circ(x,T) = W(s^\circ(x,T)), \quad -\infty < x < \infty. \tag{5.5.19}$$

If we now define the completeness of the present market similarly to the definition in Chapter 3, Section 3.4.1 or Chapter 5, Section 4.3.3, then we get

Proposition 5.5.6. *The Black–Scholes market is complete: any desired wealth $W(T)$ of the above type is attainable with a certain initial endowment, since there is a uniquely defined self-financing strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$, called the hedging strategy against $W(T)$, whose value process $V^\circ(\pi^\circ) = \{V^\circ(t;\pi^\circ)\}_{t\in[0,T]}$ attains at the terminal date T the identity $V^\circ(T;\pi^\circ) = W(T)$. The necessary initial endowment is then $v = V^\circ(0;\pi^\circ)$.*

Proof. This follows from the explicit construction of the hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against $W(T)$ which is provided below. □

The hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against $W(T)$ is constructed as follows. Let $\bar{h}$ be the solution of (5.3.11) with $r = -\mu$, subject to the boundary condition (5.3.14) where $H(x)$ is identified with $w^\circ(x,T)$ given by (5.5.19). According to Proposition 5.3.4 this function has the representation (5.3.17) with $r = -\mu$ and with H substituted by W. Alternatively, we may work as in the previous section with $\grave{h}$ which solves (5.3.11) with $r = \mu = 0$ and is related to $\bar{h}$ via (5.5.18). Recall that $\grave{h}_x = \bar{h}_x$. Use the notations

$$\psi^\circ(x,t) = \grave{h}(x,t) - x\grave{h}_x(x,t) = e^{-rt}\bar{h}(e^{rt}x, t) - x\bar{h}_x(x,t)$$

and

$$\phi^\circ(x,t) = \grave{h}_x(x,t) = \bar{h}_x(x,t)$$

to define the strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ with $\Psi^\circ(t) = \psi^\circ(\grave{S}^\circ(t), t)$ and $\Phi^\circ(t) = \phi^\circ(\grave{S}^\circ(t), t)$. By definition (5.5.7) the market value of the latter holding is simply determined:

$$V^\circ(t, \pi^\circ) = \psi^\circ(\grave{S}^\circ(t), t)B^\circ(t) + \phi^\circ(\grave{S}^\circ(t), t)S^\circ(t) = \bar{h}(S^\circ(t), t),$$

cf (5.5.17). This is indeed the hedging strategy, since at maturity the market value $V^\circ(T, \pi^\circ) = \bar{h}(S^\circ(T), T)$ amounts to the desired wealth $W(S^\circ(T))$ in virtue of the boundary condition fixed above. Thus the desired wealth is attained. In view of (5.3.17) with $r = -\mu$, the necessary initial endowment amounts to

$$v = V^\circ(0; \pi^\circ) = e^{-rT} \int_{-\infty}^{\infty} g(y, 0, \sigma^2 T) H(xe^{y+rT-\frac{1}{2}\sigma^2 T}) dy. \tag{5.5.20}$$

with H substituted by W.

Observe that in the special case of $H(x) = (x - K)^+$ the right hand side reduces to the integral in (5.4.17). This is in a full agreement with the method of option pricing by means of a hedging trading strategy that duplicates the payoff, see Chapter 3, Section 3.5, or Chapter 4 Section 4.3.3. According to this method, formula (5.5.20) serves for pricing contingent claims. Let the payoff function of a contingent claim be determined by a certain nonnegative function H of the stock price at maturity $S^\circ(T)$, i.e. $H(T) = H(S^\circ(T))$. Then we have

Proposition 5.5.7. *In the Black–Scholes market the fair price $C(H)$ of a contingent claim with the payoff function $H(T) = H(S^\circ(T))$ is identified with the right hand side of (5.5.20).*

As was already mentioned, formula (5.5.20) applied to the special payoff $(x - K)^+$ determines the fair price C of the European call option, which coincides with the integral in (5.4.17). It is now easy to present C in the form (5.5.21) below, suitable for calculations by using tables of the standard normal distribution. This is called the *Black–Scholes option valuation formula.*

Corollary 5.5.8. *In the Black–Scholes market the fair price C of the European call option is presented as follows*

$$\begin{aligned} C &= sG\left(\frac{\log\frac{s}{K} + rT + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \\ &- e^{-rT} KG\left(\frac{\log\frac{s}{K} + rT - \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}\right) \end{aligned} \tag{5.5.21}$$

with G the standard normal distribution, cf (5.1.9).

Proof. The right hand side in (5.4.17), equal to

$$\int_{-\infty}^{\infty} g(y; \frac{1}{2}\sigma^2 T, \sigma^2 T) \left(se^{-y} - \grave{K}\right)^+ dy,$$

reduces to

$$\begin{aligned} & \int_{-\infty}^{\log \frac{s}{K}} g(y; -rT + \frac{1}{2}\sigma^2 T, \sigma^2 T) \left(se^{-(y+rT)} - \grave{K} \right) dy \\ = & \int_{-\infty}^{\log \frac{s}{K}} sg(y; -rT - \frac{1}{2}\sigma^2 T, \sigma^2 T) dy \\ - & \grave{K} \int_{-\infty}^{\log \frac{s}{K}} g(y; -rT + \frac{1}{2}\sigma^2 T, \sigma^2 T) dy. \end{aligned}$$

By (5.1.9) the right hand side coincides with that of (5.5.21). The proof is complete. □

Using similar considerations one can specify the hedging strategy $\pi^\circ = (\Psi^\circ, \Phi^\circ)$ against the European call option by the portfolio components at $t \in [0, T]$ and $\tau = T - t$

$$\Psi^\circ(t) = -e^{-rt} K G \left(\frac{\log \frac{S^\circ(t)}{K} + r\tau - \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}} \right)$$

and

$$\Phi^\circ(t) = G \left(\frac{\log \frac{S^\circ(t)}{K} + r\tau + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}} \right).$$

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