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CWI
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands
Telephone 31-20 592 9333, telex 12571 (mactr nl),
telefax 31-20 592 4199

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CWI Syllabus 32

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F. den Hollander
H. Maassen (eds.)

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PREFACE

This syllabus contains a selection of reports of lectures delivered at the 'Mark Kac seminar on probability and physics' during the academic years 1987-1992. This seminar is a monthly meeting between probabilists and statistical physicists, held in Amsterdam since 1985. (Cf. CWI Syllabus 17, 1985/1987.)
Each year in the Spring a foreign speaker is invited to give a sequence of seminars on a subject chosen for its current interest. The following researchers have honoured us by accepting this invitation:

1988: *B. Souillard* (Paris)
The mathematics and physics of electron and wave propagation in disordered media

1989: *B. Kümmerer* (Tübingen)
Non-commutative probability theory

1990: *H. Spohn* (München)
Large scale dynamics of interacting particle systems

1991: *C. Maes* (Leuven)
Stochastic cellular automata

1992: *E. Bolthausen* (Zürich)
Large deviatons with applications

We thank all speakers for their contribution.

F. den Hollander
H. Maassen

PROGRAM 1987-1992

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A. van Enter (Haifa)
Absence of phase transitions in 1-dimensional long range spin glasses

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A 0-1 law for one-dimensional random walk in random environment

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Self-similar processes

M. Keane (Delft)
One-dependent processes and renormalisation

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H. Maassen (Nijmegen)
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March 4

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The mathematics and physics of electron and wave propagation in disordered media II

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(*National Sympsium Mathematical Physics, Lunteren*)

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J. van den Berg (Amsterdam)
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Mark Kac Seminar, Amsterdam December 4, 1987

PROPERTIES OF GENERAL SELF-SIMILAR PROCESSES

WIM VERVAAT
Katholieke Universiteit Nijmegen

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Mathematisch Instituut
Katholieke Universiteit
Toernooiveld 1
6525 ED Nijmegen
The Netherlands
vervaa@sci.kun.nl

0. Introduction

In the present paper we review general properties of broad classes of self-similar processes and their sample paths. We do not consider convergence to specific processes nor statistical analysis. Furthermore, we restrict our attention to one-dimensional time and state space.

Consequently, this survey covers only part of the vast literature on self-similar processes, and even for the aspects treated by this survey the coverage is incomplete almost surely. For other and supplementary information the reader is referred to the survey Taqqu (1981) for the state of the art in 1979, the bibliography Taqqu (1986) and the exposition Maejima (1989). Special topic surveys are Taqqu & Czado (1985) and Vervaat (1986).

Self-similar processes are one, random, aspect of the fractional geometry of nature, for which Mandelbrot's (1982) famous monograph is the best introduction and a source of inspiration.

1. Lamperti's Theorem

We consider real-valued stochastic processes $X = (X(t))_{t\in T}$ with $T = \mathsf{R}$, $(0,\infty)$ or $[0,\infty)$. Equality in distribution (denoted by $=_d$) and convergence in distribution (denoted by $\to_d$) always refer to finite-dimensional distributions. We say that X is *self-similar* with exponent $H \in \mathsf{R}$ (H-*ss*) if

$$X(u\,\cdot) =_d u^H X(\,\cdot) \qquad \text{for all } u > 0. \tag{1.1}$$

A well-known example is Brownian motion, which is $\frac{1}{2}$-ss. According to Donsker's theorem, Brownian motion is limit in distribution of finite-variance zero-mean random walks under joint rescaling of space and time. Therefore it has to be self-similar as shows the following theorem, originally due to Lamperti (1962) (cf. also Weissman (1975), Vervaat (1981) and Laha & Rohatgi (1982)).

Theorem 1.1 (Lamperti). *Let $T = (0,\infty)$ and suppose that $(Y(t))_{t\in T}$ is measurable in distribution and that*

$$a_n Y(n\,\cdot) \to_d X(\,\cdot) \qquad \text{as } n \to \infty \text{ through the reals,} \tag{1.2}$$

where $a_n > 0$, $|\log a_n| \to \infty$ and $X(1) \neq 0$ with positive probability. Then there is an $H \in \mathsf{R}$ such that

$$a_n a_{nu}^{-1} \to u^H \qquad \text{for } u > 0$$

and X is H-ss.

Some comments and explanations. We say that Y is measurable in distribution if the mapping $T \ni t \mapsto \operatorname{law} Y(t)$ into the probability measures on R is measurable, what amounts to $t \mapsto \mathsf{P}[Y(t) \le x]$ being measurable for all $x \in \mathsf{R}$. The original condition in Lamperti (1962) was continuity in probability. This, together with his context $T = [0,\infty)$ rather than $T = (0,\infty)$, excluded self-similarity with negative exponents.

It is obvious that each H-ss process X occurs as limit in (1.2) for some Y and a_n. Take $Y = X$ and $a_n = n^{-H}$ to arrive even at equality in distribution. *So self-similar processes can be characterized as limits in distribution of processes Y under joint rescaling of space and time.*

The central idea in the proof is simple. Comparing the limits in distribution of $a_n Y(nu\,\cdot)$ and $a_{nu} Y(nu\,\cdot)$ we find $X(ut) =_d c(u)X(t)$ for some $c(u) = \lim a_n a_{nu}^{-1} > 0$. Iterating this result for $X(uvt)$ we find $c(uv) = c(u)c(v)$, and all measurable positive functions c on $(0,\infty)$ satisfying this functional equation are given by $c(u) = u^H$ for some real H.

Already Lamperti allowed for translations in the limit relation:

$$a_n Y(n\,\cdot) + b_n \to_d X(\,\cdot\,). \tag{1.3}$$

To formulate the resulting more general theorem, it is convenient to introduce the set Aff of affine transformations

$$\mathsf{R} \ni x \mapsto ax + b \qquad a, b \in \mathsf{R},\ a > 0$$

with generic element $\gamma = a\cdot + b$. With composition of functions as product, Aff becomes a noncommutative group with inverse $\gamma^{-1} = a^{-1}\cdot - a^{-1}b$ and unit element id_{R}. We say that $\gamma_n = a_n\cdot + b_n$ converges to $\gamma = a\cdot + b$ in Aff if $a_n \to a$ in $(0,\infty)$ and $b_n \to b$ in R. Real powers of $\gamma \in$ Aff can be defined by

$$\gamma^t(x) = \begin{cases} a^t(x-c) + c & \text{if } a \neq 1, \text{ where } c := b/(1-a), \\ x + tb & \text{if } a = 1. \end{cases} \tag{1.4}$$

Theorem 1.2 (generalization of Theorem 1.1). *Let $T = (0,\infty)$ and suppose that $(Y(t))_{t\in T}$ is measurable in distribution and that*

$$\gamma_n(Y(n\,\cdot)) \to_d X(\,\cdot\,) \qquad \text{as } n \to \infty \text{ through the reals,}$$

where Aff $\ni \gamma_n = a_n\cdot + b_n$, $|\log a_n| + |b_n| \to \infty$ and $X(1)$ has a nondegenerate distribution. Then there is a $\beta \in$ Aff such that $\gamma_n \gamma_{nu}^{-1} \to \beta^{\log u}$ in Aff and

$$X(u\,\cdot) =_d \beta^{\log u}(X(\,\cdot\,)) \qquad \text{for } u > 0. \tag{1.5}$$

We say that X is *β-self-similar* if (1.5) holds. If X is γ-ss with γ as in (1.4), then $X - c$ is H-ss with $H = \log a$ in case $a \neq 1$, and e^X is H-ss with $H = b$ in case $a = 1$. So all cases of γ-self-similarity with $\gamma \in$ Aff can be reduced to self-similarity as in (1.1) by simple transformations.

The term 'self-similarity' is coined by B. Mandelbrot in the 1960s (cf. Mandelbrot & Van Ness (1968)). Lamperti (1962) originally used the term 'semi-stability'. From the beginning, self-similar processes were considered also with more-dimensional time and state space (cf. Dobrushin & Major (1979), Major (1981, 1982), Surgailis (1981)). In this context it is natural to replace the multiplication parts $x \mapsto ax$ in the affine transformations of the state space of X by more general linear transformations (cf. Laha & Rohatgi (1982), Hudson & Mason (1982)).

Especially in physics this generality is important, in the study of critical phenomena. Let T now be Euclidean space serving to indicate the location of particles, and let $Y_\tau(t)$ be some physical parameter of the particle at location t (e.g. spin). Here τ is some external parameter, often temperature. If Y_τ is invariant in distribution for motions of T, then the correlation of $Y_\tau(t)$ and $Y_\tau(s)$ depends only on the distance between t and s. The distance l_τ indicating decrease of correlation to e^{-1} is known as the 'correlation distance'. It is known that for fixed locations t and s the correlation of $Y_\tau(t)$ and $Y_\tau(s)$ tends to 1 as τ approximates a critical value τ_c of phase transition. Thus the correlation distance diverges to infinity as $\tau \to \tau_c$. Making the correlation distance to unit of length and rescaling the (cumulative) states of the particles we may hope for interesting limits of $a_\tau Y_\tau(l_\tau \cdot)$ as $\tau \to \tau_c$ (from below or from above). If Y_τ depends sufficiently smoothly on τ on either side of τ_c, then we may expect limits as in Lamperti's theorem, where Y_τ does not depend on τ. This part of physics is referred to as 'renormalization theory'.

2. Self-similarity and other notions of invariance

Before discussing more specific cases, we want to compare self-similarity as defined in the previous section with related notions. First of all, the same term is also used for invariance properties of (nonrandom) sets as e.g. fractals (cf. Mandelbrot (1982)). Here self-similarity of a subset of R^d means its invariance under a (mostly) discrete group G of transformations of R^d, say $G = (\vartheta^n)_{n\in\mathsf{Z}}$. For instance, ϑ^n may be scalar multiplication by 2^n. (Actually, this description applies only to space-filling fractals; for compact fractals things are a bit more complicated (cf. Falconer (1985))).

In the previous section, self-similarity meant *invariance in distribution* of random functions X under a continuous group of transformations of the sample path space: $\vartheta^s X(\cdot) = e^{-sH} X(e^s \cdot)$. Note that the sample paths themselves are not invariant at all.

The following is something in the direction of self-similarity of sample paths of self-similar processes X. If the sample path space is provided with a topology that is sufficiently well related to the measurable structure of the sample path space together with $G = (\vartheta^s)_{s\in\mathsf{R}}$, then there is with probability 1 (wp1) for each neighborhood of a realization of X an unbounded set of s such that $\vartheta^s X$ belongs to this neighborhood. This is Poincaré's recurrence theorem.

On the other hand, let the fractal C be a subset of R^d invariant under the discrete group $(\vartheta^n)_{n\in\mathsf{Z}}$, a subgroup of a continuous group $G = (\vartheta^s)_{s\in\mathsf{R}}$ of transformations of R^d. Then $\vartheta^U C$, with U uniformly distributed in [0,1], is a G-invariant random set: its distribution is invariant under G (but its realizations are not). For more sophisticated interplay between fractals and randomness, see Falconer (1986), Graf (1987), Graf, Mauldin & Williams (1988) and Zähle (1989b). Deterministic anologues of processes as in §5 are treated in Kamae (1986), Kôno (1986b, 1988) and Kamae & Keane (1990).

Having interpreted self-similarity of processes as invariance of their distributions under a group of transformations applied to their sample paths,

we recognize many analogous situations in probability, and more generally in mathematics.

If $\vartheta^s X(\cdot) = X(\cdot + s)$ and X is invariant under (ϑ^s), then X is said to be *(strictly) stationary* (AMS 1980 subject classification: 60G10). Actually, X is H-ss iff Y defined by $Y(t) := e^{-Ht}X(e^t)$ is stationary, as was already observed by Lamperti (1962).

If G is a group of measure-preserving transformations of compact subsets of the time domain of X, then invariance in distribution under G is known as *exchangeability* (AMS 1980 subject classification 60G09).

In this context Ressel (1985) has obtained interesting representation results, which may be applicable to self-similar processes.

Most importantly, invariance in distribution under a group of transformations is a central topic in the theory of dynamical systems (ergodic theory) (AMS 1980 subject classifications: 28Dxx, 34C35, 54H20, 58Fxx). So far, the similarity in subject has not led to links between the various fields, but they are bound to develop in the near future.

For self-similar processes there is not yet a clear AMS classification. I propose 60G11: 'Self-similar processes and other processes invariant under groups of transformations'.

3. Self-similarity without additional assumptions

We now present some basic facts about self-similarity in full generality. For H-ss processes with time domain $T = (0, \infty)$ we have $X(t) =_d t^H X(1)$, and the right-hand side converges wpl in the extended real line $\bar{\mathsf{R}} := [-\infty, \infty]$ as $t \downarrow 0$. From this observation we conclude:

Lemma 3.1. *If X is H-ss, then $X(t)$ converges as $t \downarrow 0$ in distribution to*

$$\begin{cases} 0 & \text{if } H > 0, \\ X(1) & \text{if } H = 0, \\ \infty \cdot X(1) & \text{if } H < 0. \end{cases}$$

We see that for $H > 0$ it is natural to extend the time domain T to $[0, \infty)$ and set $X(0) := 0$.

If X is H-ss, then X^α is αH-ss for $\alpha \geq 0$. If $X(1) \neq 0$ wpl, then by self-similarity $X(t) \neq 0$ wpl for each $t \neq 0$ separately (which does not exclude the presence of zeros with positive probability at random times). Then $1/X(t)$ is defined wpl for each $t \neq 0$ separately. Allowing processes to be undefined or infinite-valued at random times we may say that X^α is also αH-ss for negative α.

The next lemma lists some other operations that preserve self-similarity. In points (a) and (b) we allow processes to be infinite-valued. For point (c) it is important that the time domain of many self-similar processes can be extended in a natural way to all of R.

Lemma 3.2. *Let X be H-ss.*
(a) *If X is a separable process, then $X^{\uparrow}$ defined by $X^{\uparrow}(t) := \sup_{0<s\leq t} X(s)$ is H-ss and nondecreasing.*
(b) *If X is a nondecreasing process, then $X^{\sim}$ defined by $X^{\sim}(s) := \inf\{t : X(t) \geq s\}$ is $1/H$-ss, nondecreasing and right-continuous.*
(c) (Vervaat (1985)). *If X is measurable and Y is an H'-ss process independent of X and with range in the time domain of X, then $X \circ Y = (X(Y(t)))_t$ is HH'-ss.*

We now consider the law of $X(1)$. If it has an atom at b, say $p := \mathsf{P}[X(1)=b] > 0$, then by $X(t) =_d t^H X(1)$ we also have $p = \mathsf{P}[X(t)=t^H b]$ for $t \neq 0$. If X is a measurable process, i.e., $(t,\omega) \mapsto X(t,\omega)$ is jointly measurable, then it follows by Fubini's theorem that

$$\mathsf{E}\mathrm{Leb}\{s \in (0,t]\colon X(s) = s^H b\} = pt.$$

Here is an example of such a process with $p = \frac{1}{2}$. All randomness comes from a random variable U with uniform distribution in [0,1].

$$X(t) := \begin{cases} t^H & \text{if } t > 0 \text{ and } U + 2n - 1 \leq \log t < U + 2n \text{ for some } n \in \mathsf{Z}, \\ 0 & \text{if } t > 0 \text{ otherwise.} \end{cases}$$

However, other self-similar processes will be more interesting for us, and in most cases the law of $X(1)$ will be continuous.

One deeper result about self-similar processes in full generality can be found in Kôno (1983), where almost sure bounds are obtained for $X(t)/t^H\varphi(t)$ as $t \to \infty$. Results on the Hausdorff dimension of sample paths and the existence of square integrable local time are in Kôno (1986), but the basic assumptions point in the direction of self-similar processes with stationary increments, the most important more special case of self-similar processes.

Indeed, for richer theories we must combine self-similarity with other general properties of processes. Three combinations occur in the literature. In the remainder we will review them in historical order: Markov processes (one section), processes with stationary increments (almost all sections) and stationary extremal processes (the last section).

4. Self-similar Markov processes

The first results on self-similar Markov processes already occur in Lamperti (1962). The case $H > 0$ with state space $[0,\infty)$ is handled in detail in Lamperti (1972). Some extensions to state space R^d are made in Kiu (1980) and Graversen & Vuolle-Apiala (1986a,b).

If X is an H-ss Markov process, then its transition kernel K satisfies

$$K^{at}(x,B) = K^t(a^{-H}x, a^{-H}B) \tag{4.1}$$

for $t > 0$, $a > 0$, $x \in \mathsf{R}$ and $B \in \mathrm{Bor}\,\mathsf{R}$. If $H > 0$, then X is an H-ss Markov process iff (4.1) holds and $X(0) = 0$.

The zero of the state space plays a central role in the theory. The way X approaches 0 from other states is a major topic in Lamperti (1972) and the main subject of Kiu (1980) and Graversen & Vuolle-Apiala (1986a,b) (for state space R^d). The way 0 is left is dealt with in Lamperti (1972) and Vuolle-Apiala (1989). The set of zeros in the sample path is studied by Stone (1963) and Wendel (1964).

Here is Lamperti's (1972) characterization of the approach to 0.

Theorem 4.1. *Let X be a strong Markov process with state space $[0,\infty)$ and H-ss ($H > 0$) transition kernel (i.e., (4.1) holds), with sample paths in $D[0,\infty)$. Let $X(0) \neq 0$ wpl and set $\xi := \inf\{t : X(t) = 0\}$, $\tau(t) := \int_0^t X^{-1/H}(s)\mathrm{d}s$ for $0 \leq t < \xi$. Then $\log X \circ \tau^{-1}$ is a process with stationary independent increments, killed at an independent exponentially distributed random time in case X jumps to 0, but with infinite lifetime if X reaches 0 by continuous approach or not at all.*

The case $H < 0$ with state space $(-\infty,0)$ or $(0,\infty)$ and the case of β-self-similarity with $\beta = \cdot + b$ and state space R can be transformed into the case dealt with in the theorem, by the transformations $x \mapsto -1/x$, $x \mapsto 1/x$, $x \mapsto e^{x\operatorname{sgn} b}$, which preserve the Markov property. In particular, Theorem 4.1 applies to all classical limiting extremal processes, as considered in e.g. Resnick & Rubinovitch (1973).

5. Self-similar processes with stationary increments

In most papers self-similar processes are assumed (or turn out) to have stationary increments as well. So common is this combination, that it is often understood that the latter property is part of the former. A process $X = (X(t))_{t\in T}$ with $T \supset (0,\infty)$ has *stationary increments* (*si*) if the joint distributions of the vectors of increments $\big(X(b+t_k) - X(b+t_{k-1})\big)_{k=1}^n$ do not depend on $b \in T$ for $n \in \mathsf{N}$, $0 =: t_0 < t_1 < \cdots < t_n$, i.e.,

$$X(b_1 + \cdot) - X(b_1) =_d X(b_2 + \cdot) - X(b_2) \qquad \text{for } b_1, b_2 \in T. \tag{5.1}$$

Let us consider H-ss si X. Is it possible that $H \leq 0$ for nontrivial processes? (The process $X \equiv 0$ is H-ss si for all H.) The answer is yes if we do not exclude processes with very ill-behaved sample paths. For instance, the collection $(X(t))_{t\in T}$ of independent identically distributed random variables is 0-ss si. We exclude such possibilities by requiring X to be a *measurable process* (cf. lines after Lemma 3.2). Measurable processes have some continuity in probability almost everywhere (see Vervaat (1985, Th.1.1) for details). Unless $X(b+1)-X(b) = 0$ wpl, this comes into conflict with the following observation:

$$\begin{aligned} X(b+t) - X(b) &=_d X(t+t) - X(t) \\ &=_d t^H(X(1+1) - X(1)) =_d t^H(X(b+1) - X(b)) \\ &\begin{cases} =_d X(b+1) - X(b) & \text{if } H = 0 \\ \to_d \infty \cdot \big(X(b+1) - X(b)\big) & \text{if } H < 0 \end{cases} \end{aligned} \tag{5.2}$$

as $t \downarrow 0$. We conclude that H-ss si measurable X with $H < 0$ satisfy $X(t) = X(1)$ wpl for each $t \in T$, where $X(1) = 0$ wpl in case $H < 0$, by Lemma 3.1. Another regularity assumption, that X be a *separable process* in the sense of Doob (1953), simplifies the above conclusion to $X \equiv X(1)$ wpl. (Recall that each process has a separable version with the same distribution.)

In the next sections we will consider H-ss si measurable separable processes X with $H > 0$. If $T = (0, \infty)$, then X can be extended continuously in probability to $T = [0, \infty)$ by setting $X(0) := 0$ (Lemma 3.1), and (5.1) can be phrased more conveniently as

$$X(b + \cdot) - X(b) =_d X(\cdot) \qquad \text{for } b \in T. \tag{5.3}$$

From

$$X(b + t) - X(b) =_d X(t) =_d t^H X(1) \to 0 \text{ wpl} \quad \text{as } t \downarrow 0$$

we see that X is uniformly continuous in probability. Nevertheless, the sample paths of X can behave badly, as shows Maejima (1983b), where sample paths are dense in the plane wpl.

Although many trivial processes have been excluded by our regularity assumptions, there are some others left that fall within our requirements: linear functions with random slope $X(t) \equiv tX(1)$ are 1-ss si.

So far we have considered only H-ss si processes. Do we get more if we replace H-self-similarity by β-self-similarity as in (1.5)? Consulting the lines after (1.5) we see that the reduction to H-self-similarity preserves stationarity of increments iff β is not a translation. In case β is a translation, $\beta = \cdot + d$ with $d \neq 0$, we see as in (5.2) that for all $b \in T$

$$X(b + t) - X(t) =_d X(b + 1) - X(b) + d \log t \to_d -\infty \cdot d \qquad \text{as } t \downarrow 0,$$

which contradicts measurability of X. So H-self-similarity with $H > 0$ is sufficiently general.

Theorem 1.1 now can be specialized to the following result.

Theorem 5.1. *Let $(\xi_k)_{k=1}^{\infty}$ be a stationary sequence of* R*-valued random variables with partial sum process $Y(t) := \sum_{k=1}^{\lfloor t \rfloor} \xi_k$ for $t \geq 0$. If for some real c*

$$a_n\big(Y(n\,\cdot) - cn\,\cdot\big) \to_d X(\,\cdot\,) \qquad \text{as } n \to \infty \text{ through the reals,} \tag{5.4}$$

where $a_n > 0$, $|\log a_n| \to \infty$ and $X(1) \neq 0$ with positive probability, then there is an $H > 0$ such that

$$a_n a_{nu}^{-1} \to u^H \qquad \text{for } u > 0$$

and X is H-ss si. Conversely, all H-ss si X with $H > 0$ can be obtained this way.

Some comments and explanations. The left-hand side of (5.4) has stationary increments on $n^{-1}\mathsf{N}$, so the right-hand side has stationary increments on the closure of $\bigcup_{n=1}^{\infty} n^{-1}\mathsf{N}$ in $[0, \infty)$, which is all of $[0, \infty)$. For the last statement of the theorem, take $c = 0$ and $\xi_k := X(k) - X(k-1)$ for $k \in \mathsf{N}$.

In the most recent literature there is a trend to use the term 'self-affine' for the combination of ss and si.

6. The first examples: independent increments and stable processes

If we strengthen the assumption of stationary increments to stationary *independent* increments, we enter a classical domain of probability. Note that Theorem 5.1 remains true with the same substitution and (ξ_k) a sequence of independent identically distributed random variables. In this way Theorem 5.1 specializes to one of Lévy's classical results.

Almost by definition, self-similar processes X with stationary independent increments are strictly stable motions. The marginal distributions of $X(1)$, or rather their characteristic functions, are known (cf. Hall (1981)). It turns out that $H \geq \frac{1}{2}$, with $H = \frac{1}{2}$ corresponding to Brownian motion. In the more special case of $X(1)$ having a symmetric distribution, the characteristic function is

$$\mathsf{E}e^{i\lambda X(1)} = e^{-c|\lambda|^{1/H}} \tag{6.1}$$

for some $c > 0$. There is an unfortunate divergence in terminology around self-similar and stable processes. The characteristic function in (6.1) is called $1/H$-stable, and $\alpha := 1/H$ its *stability exponent.*

The terminology around stable processes is shifting towards new standards, so we give some definitions here. See Weron (1984), Hardin (1984) and the forthcoming monograph Samorodnitsky & Taqqu (1991) for further information.

Let ξ be an R^d-valued random variable and $(\xi_k)_{k=1}^{\infty}$ a sequence of independent copies of ξ. We say that ξ is *stable* if for each n there are reals $a_n > 0$ and b_n such that $\xi_1 + \xi_2 + \cdots + \xi_n =_d a_n\xi_n + b_n$. We say that ξ is *strictly stable* if $b_n = 0$ for all n. It turns out that the only possibilities for a_n are $a_n = n^{1/\alpha}$ with $0 < \alpha \leq 2$ (no surprise in view of Theorem 5.1, except for the restriction $\alpha \leq 2$). We then say that ξ is *α-stable.* A process X is called α-stable if all its finite-dimensional distributions are α-stable. We say that X is an *α-stable motion* if X is α-stable and has stationary independent increments. A 2-stable process is Gaussian; 2-stable motion is Brownian motion. In the older terminology stable processes were what we now call stable motions.

An R^d-valued random variable ξ is strictly α-stable iff all linear combinations of its components are strictly α-stable (well-known for $\alpha = 2$). Without 'strictly' the equivalence is no longer true (Marcus (1983)).

Having identified ss processes with stationary independent increments as strictly stable motions, we may ask whether there are other stable self-similar processes with stationary (dependent) increments. The following preliminary result looks obvious, but has a surprisingly complicated proof (Kasahara, Maejima & Vervaat (1988)).

Theorem 6.1. *If X is an H-ss si stable process, then X is strictly stable in case $H \neq 1$ and $(X(t) - ct)_{t\geq 0}$ is strictly stable for some real c in case $H = 1$.*

With 2-stable (= Gaussian) processes we are in the comfortable situation that distributions are determined by mean and covariance. Translating the si and

ss requirements of X into invariance properties of its mean and covariance, one can deduce that $0 < H \leq 1$, $\mathsf{E}X(t) = 0$ unless $H = 1$ and

$$\mathsf{E}X(s)X(t) = \mathsf{E}X^2(1) \cdot \frac{1}{2}(|s|^{2H} + |t|^{2H} - |s-t|^{2H}). \tag{6.2}$$

So apart from scaling there is a unique H-ss 2-stable process for each $H \in (0,1]$. For $H = 1$ it is the linear process $X(t) \equiv tX(1)$ with random Gaussian slope $X(1)$. For $H = \frac{1}{2}$ it is Brownian motion. For other H the process X turns out to be *fractional Brownian motion*, which will be described further in the next section. For $\alpha < 2$ these examples can be generalized to *fractional stable processes* (cf. next section), but uniqueness is lost. In this context Kasahara, Maejima & Vervaat (1988) investigate whether strictly α-stable $1/\alpha$-ss si processes are necessarily strictly α-stable motions. The answer is yes for $\alpha = 2$ (as we knew already), no for $1 \leq \alpha < 2$, and again yes (see Samorodnitsky & Taqqu (1990)) for $0 < \alpha < 1$. For further results on ss stable processes, see Kôno & Maejima (1991a,b).

7. Subordination, fractional processes and random scenery

In the literature, many self-similar processes with stationary increments are obtained from basic processes Z by the technique of subordination:

$$X(t) = \int_{\mathsf{R}} L(t,s)Z(\mathrm{d}s) \qquad \text{for } t \geq 0. \tag{7.1}$$

Here L is a deterministic or random function on $[0,\infty) \times \mathsf{R}$ with values in $\bar{\mathsf{R}}$ and such that the integral converges in some theory of stochastic integration. We always suppose that these integrals exist whenever we write them down. If L is random, it is usually assumed to be independent of Z.

Let us say that L is (H_1, H_2)-self-similar if, regarded as random functions of s and t,

$$L(ut, u^{H_2}s) =_d u^{H_1}L(t,s) \qquad \text{for } u > 0. \tag{7.2}$$

Theorem 7.1. *If L and Z are independent, L is (H_1, H_2)-ss and Z is H_3-ss, then X in (7.1) is $(H_1 + H_2H_3)$-ss.*

Proof. With the following regarded as random functions of t we have

$$\begin{aligned} X(ut) &= \int_{\mathsf{R}} L(ut,s)Z(\mathrm{d}s) = \int_{\mathsf{R}} L(ut, u^{H_2}s')Z(u^{H_2}\mathrm{d}s') \\ &=_d \int_{\mathsf{R}} u^{H_1}L(t,s')u^{H_2H_3}Z(\mathrm{d}s') = u^{H_1+H_2H_3}X(t). \end{aligned}$$ □

Let us say that L has stationary increments if there are random variables $W(b)$ such that, with both sides regarded as random functions of s and t,

$$L(b+t,s) - L(b,s) =_d L(t, s+W(b)) \qquad \text{for } b,t \geq 0,\ s \in \mathsf{R}. \tag{7.3}$$

Theorem 7.2. *If L and Z are independent and have stationary increments, then X in (7.1) has stationary increments.*

Proof. With the following regarded as random functions of t we have

$$X(b+t)-X(b)=\int_{\mathsf{R}}\big(L(b+t,s)-L(b,s)\big)Z(\mathrm{d}s)=_d\int_{\mathsf{R}}L\big(t,s+W(b)\big)Z(\mathrm{d}s)$$
$$=\int_{\mathsf{R}}L(t,s')Z\big(-W(b)+\mathrm{d}s'\big)=_d\int_{\mathsf{R}}L(t,s')Z(\mathrm{d}s')=X(t).\,\square$$

Let us first consider the case that L (and W) are nonrandom. Setting $\varphi(t):=L(t,0)$ we obtain from (7.3) with $s=0$

$$L\big(t,W(b)\big)=\varphi(b+t)-\varphi(b)\qquad\text{for } b,t\geq 0. \tag{7.4}$$

Substituting this in (7.3) for fixed φ we obtain a complicated functional equation for W. Solutions are $W(b)=cb$, in particular $W\equiv 0$ and $L(t,s)=f(s)t$ with arbitrary measurable f, and these exhaust the possibilities in case φ vanishes on the negative half-line (Cambanis & Maejima (1990)).

Let us continue with $W(b)=cb$ with $c\neq 0$, or rather $W(b)=b$, to which this case can be reduced by a simple transformation. Now (7.4) specializes to

$$L(t,b)=\varphi(t+b)-\varphi(b)\qquad\text{for } t,b\geq 0, \tag{7.5}$$

and L and φ turn out to be extendable to negative domains such that (7.3) and (7.5) remain valid. It is not hard to find all (H_1,H_2)-ss L that can be obtained by (7.5). Obviously, $H_2=1$ and $\varphi(0)=0$ in all cases. For $H_1\neq 0$ and $t\neq 0$ we have

$$\varphi(t)=c_+(t^+)^{H_1}+c_-(t^-)^{H_1}\qquad(c_+,c_-\in\mathsf{R}), \tag{7.6a}$$

where $t^+:=t\vee 0$, $t^-:=(-t)\vee 0$. For $H_1=0$ and $t\neq 0$ we have

$$\varphi(t)=c_{\operatorname{sgn} t}\qquad(c_+,c_-\in\mathsf{R}), \tag{7.6b}$$

or (surprise!)

$$\varphi(t)=c\log|t|\qquad(c\in\mathsf{R}). \tag{7.6c}$$

When Z is strictly α-stable motion and L and φ are as in (7.5) and (7.6a), then X in (7.1) is called a *fractional stable process* (or *motion*) (Maejima (1983a,b)), and for $\alpha=2$ *fractional Brownian motion* (Rosenblatt (1961), Mandelbrot & Van Ness (1968), the last reference also for traces in the older literature). The integral in (7.1) converges iff $H:=\alpha^{-1}+H_1\in(0,1)$, and then X is a strictly α-stable H-ss si process. When φ is as in (7.6c), X is called a *log-fractional stable process* (Kasahara, Maejima & Vervaat (1988)). The integral in (7.1) converges iff $1<\alpha\leq 2$, and then X is a strictly α-stable α^{-1}-ss si process. For $\alpha=2$ it is Brownian motion, but for $1<\alpha<2$ it is *not* strictly α-stable motion (Kasahara, Maejima & Vervaat (1988)).

For an example of (7.1) with random L, let L be local time of a process Y, i.e., L is a random function, nondecreasing and right-continuous in t, such that for all measurable $f\colon[0,\infty)\times\mathsf{R}\to[0,\infty)$ we have

$$\int_{[0,\infty)}\int_{\mathsf{R}}L(\mathrm{d}t,x)f(t,x)\mathrm{d}x=\int_{[0,\infty)}f(t,Y(t))\mathrm{d}t \tag{7.7}$$

(cf. Geman & Horowitz (1980)). If Y has stationary increments and $Y(0) = 0$ wpl, then (5.3) implies that both sides of (7.7), considered jointly for finitely many f, are equal in distribution to

$$\begin{aligned}\int_{[0,\infty)} f\big(t, Y(b+t) - Y(b)\big)\mathrm{d}t &= \int_{[b,\infty)} f\big(t'-b, Y(t') - Y(b)\big)\mathrm{d}t' \\ &= \int_{[b,\infty)} \int_{\mathsf{R}} L(dt', x) f\big(t'-b, x - Y(b)\big)\mathrm{d}x \\ &= \int_{[0,\infty)} \int_{\mathsf{R}} L\big(b+dt, x' + Y(b)\big) f(t, x')\mathrm{d}x'.\end{aligned}$$

So L has stationary increments, as (7.3) holds with $W(b) = -Y(b)$. If Y is H-ss, then (1.1) implies that both sides of (7.7), considered jointly for finitely many f, are equal in distribution to

$$\begin{aligned}\int_{[0,\infty)} f\big(t, u^H Y(t/u)\big)\mathrm{d}t &= u \int_{[0,\infty)} f\big(ut', u^H Y(t')\big)\mathrm{d}t' \\ &= u \int_{[0,\infty)} \int_{\mathsf{R}} L(dt', x) f(ut', u^H x)\mathrm{d}x \\ &= u \int_{[0,\infty)} \int_{\mathsf{R}} L(dt/u, u^{-H} x') f(t, x') u^{-H}\mathrm{d}x'.\end{aligned}$$

So L is $(1-H, H)$-ss in the sense of (7.2). We conclude from Theorems 7.1 and 7.2

Theorem 7.3 (Major (1981, p.115–117)). *If L is local time of an H-ss si process Y, and Z is an independent H'-ss si process such that the integral in (7.1) is well-defined, then X is $(1 - H + HH')$-ss si.*

Processes as in the theorem occur as limiting processes of 'random walks in random scenery' in Kesten & Spitzer (1979). There Z is strictly β-stable motion ($0 < \beta \leq 2$) and L is the local time of a strictly α-stable motion (so $1 < \alpha \leq 2$, otherwise local time does not exist), resulting in $(1 - \alpha^{-1} + (\alpha\beta)^{-1})$-ss X.

The integration in (7.1) can be generalized to more dimensions:

$$X(t) = \int_{\mathsf{R}^d} L(t, \mathbf{s}) Z(\mathrm{d}\mathbf{s}) \qquad \text{for } t \in [0, \infty)$$

(so $\mathbf{s} \in \mathsf{R}^d$). The case with deterministic L and Brownian motion Z in R^d produces the *Rosenblatt processes* of order d (Rosenblatt (1961), Taqqu (1975, 1979), Dobrushin & Major (1979)). The case with L being local time of a strictly stable motion in R^d is treated by Lang & Nguyen Xuan Xanh (1983).

When Z is a pure jump process, variations on (7.1) are possible with much more flexibility, even for nonrandom L. They will be discussed in Section 12. The combined approach to subordination and random scenery in the present section seems to be new.

8. Other recipes for new examples

Here are two other ways of producing new ss si processes from one or more old ones.

Lemma 8.1 (Vervaat (1985)). *If X and Y are independent processes with stationary increments, X is measurable, $Y(0) = 0$ wpl and Y has range in the time domain of X, then $X \circ Y$ has stationary increments.*

Note that the lemma does not hold true with 'stationary *independent* increments' in hypothesis and conclusion, unless Y is nondecreasing. Lemmas 3.1, 3.2(c) and 8.1 combine into

Theorem 8.2 (Lou (1985), Vervaat (1985)). *If X and Y are independent processes such that X is measurable H-ss si, and Y is H'-ss ($H' > 0$) si with range in the time domain of X, then $X \circ Y$ is HH'-ss si.*

For the sequel, recall that we take separable measurable versions of ss si processes X. Suppose that the sample paths of such a version turn out to have left and right limits everywhere. Then we can choose a version of X whose sample paths are in addition right-continuous, so are elements of the function space $D[0,\infty)$, since X is wpl continuous at each fixed $t \in T$. To see the latter, note that the set of t's where X is not continuous is countable (= finite or countably infinite) (Billingsley (1969, p.124)), and invariant under $t \mapsto ut$ for $u > 0$ by self-similarity, thus empty. We always take versions of X with sample paths in $D[0,\infty)$, whenever possible (it may be not possible, cf. lines below (5.3) and Maejima (1983b)).

If $X = (X(t))_{t\in[0,\infty)}$ has sample paths of locally bounded variation, then X can be written as

$$X = X(0) + X_+ - X_-, \tag{8.1a}$$

where X_+ and X_- are nondecreasing processes defined by

$$X_\pm(t) := \sup\left\{\sum_{k=1}^{n} (X(t_k) - X(t_k - 1))^\pm : n \in \mathsf{N},\ 0 =: t_0 < t_1 \cdots < t_n = t\right\}. \tag{8.1b}$$

We call X_+ (X_-) the *positive (negative) variation process* of X. From the definition of X_+ and X_- we see immediately

Lemma 8.3 (Vervaat (1985)). *If X is H-ss si and of locally bounded variation, then its positive and negative variation processes are H-ss si as well.*

9. Moments and marginal distributions

A complication for the results of the present section is the possibility that $X \equiv 0$. The following result of O'Brien & Vervaat (1983, L.3) shows how to recognize it in the marginal distribution of $X(1)$.

Lemma 9.1. *If X is H-ss si separable with $H > 0$, then $[X(1)=0] = [X \equiv 0]$ modulo null events.*

The following result about moments is due to Maejima (1986).

Theorem 9.2. *Let X be H-ss si with $H > 0$ and such that $X(1) \neq 0$ with positive probability. If $0 < p < 1$ and $\mathsf{E}|X(1)|^p < \infty$, then $Hp < 1$.*

Corollary. If $H > 1$, then $\mathsf{E}|X(1)|^{1/H} = \infty$.

Proof. By Lemma 9.1 and stationary increments we have $X(1)\big(X(2) - X(1)\big) \neq 0$ with positive probability. If $0 < p < 1$ and $\mathsf{E}|X(1)|^p < \infty$, then

$$\mathsf{E}|X(1)|^p = \mathsf{E}|2^{-H}X(2)|^p = 2^{-H}p\mathsf{E}\big|X(1) + \big(X(2) - X(1)\big)\big|^p$$
$$< 2^{-Hp}\mathsf{E}\big(|X(1)|^p + |X(2) - X(1)|^p\big) = 2^{-Hp} \cdot 2\mathsf{E}|X(1)|^p = 2^{1-Hp}\mathsf{E}|X(1)|^p.$$

So $1 - Hp > 0$. □

More detailed results can be obtained about the first moment of $X(1)$, by considering

$$\frac{1}{n}X(n) =_d n^{H-1}X(1) \to_{\text{wpl}} \begin{cases} 0 & \text{if } H < 1, \\ X(1) & \text{if } H = 1, \\ \infty \cdot X(1) & \text{if } H > 1. \end{cases} \tag{9.1a}$$

If $\mathsf{E}X(1)$ exists, i.e., if at least one of $\mathsf{E}X^+(1)$ and $\mathsf{E}X^-(1)$ is finite, then we may apply the Birkhoff ergodic theorem to the stationary sequence $\big(X(k) - X(k-1)\big)_{k=1}^{\infty}$, to obtain

$$\frac{1}{n}X(n) = \frac{1}{n}\sum_{k=1}^{n}\big(X(k) - X(k-1)\big) \to \mathsf{E}^{\mathcal{I}}X(1) \quad \text{wpl.} \tag{9.1b}$$

Here $\mathsf{E}^{\mathcal{I}}$ denotes conditional expectation with respect to the σ-field $\mathcal{I}$ of events that are invariant for the shift of $\big((X(k) - X(k-1)\big)_{k=1}^{\infty}$. It follows that the limits in (9.1a) and (9.1b) are equal in distribution. From this one can derive (Vervaat (1985, Th.3.1))

Theorem 9.3. *Let X be H-ss si separable with $H > 0$ and such that $X(1) \neq 0$ with positive probability. If $\mathsf{E}X(1)$ exists, then*
(a) $\mathsf{E}X(1) = 0$ *if* $H < 1$,
(b) $X(t) \equiv tX(1)$ *if* $H = 1$,
(c) *either* $\mathsf{E}X(1) = \infty$ *and X is strictly increasing unless $X \equiv 0$, or the same holds for $-X$.*

The marginal distributions of $X(t)$ are investigated in O'Brien & Vervaat (1983). In view of Lemma 9.1 the relevant case is $\mathsf{P}[X \equiv 0] = 0$, otherwise the distribution of $X(t)$ has an atom at zero. It is proved that for each fixed $H \neq 1$ there is a function $Q_H : (0, \infty) \to (0, 1]$ such that $Q_H(y) \to 0$ as $y \downarrow 0$ and

$$\sup_{b \in \mathsf{R}} \mathsf{P}[e^b < X(t) < e^{b+y}] \leq Q_H(y)$$

for *all* H-ss si processes X and all $t > 0$. It follows that the distributions of $X(t)$ are continuous for $t > 0$. The authors guess that they are even absolutely continuous, but cannot prove this. It is obvious that all such results cannot hold for $H = 1$, unless the case $X(t) \equiv tX(1)$ is excluded, in which $X(1)$ may have any distribution.

From the known examples there is some evidence that the marginal distributions cannot have thin tails in case $H > 1$ and $\mathsf{P}[X(1)>0] > 0$:

$$\liminf_{x\to\infty} x^{1/H}\mathsf{P}[X(1)>x] > 0.$$

O'Brien & Vervaat (1983) obtain only partial and weaker results in this direction. In all known examples of H-ss si processes X the support of the distribution of $X(1)$ has the form $[a,\infty)$, $(-\infty,b]$ or $(-\infty,b]\cup[a,\infty)$ $(b \le 0 \le a)$ in case $H > 1$ and R in case $H < 1$. O'Brien & Vervaat (1983) indicate some properties of H-ss si processes with other supports, but do not know whether such processes actually exist.

10. Properties of sample paths

The most obvious examples of self-similar processes with stationary increments are strictly stable motions (cf.§6). Much is known about their sample paths, as is the case for all processes with stationary independent increments (Taylor (1973), Fristedt (1974)). It is known that α-stable motions have sample paths of locally bounded variation wpl iff $\alpha < 1$ ($\Leftrightarrow H > 1$), and have sample paths of nowhere bounded variation wpl iff $1 \le \alpha \le 2$ ($\Leftrightarrow \frac{1}{2} \le H \le 1$). So naturally the question arises whether these results are representative for all ss si processes.

Before we answer this question, let us first remark that for ss si processes the dichotomy between bounded or unbounded variation on one fixed compact interval extends to all compact intervals jointly: wpl their sample paths are of locally bounded variation either everywhere or nowhere (Vervaat (1985, Th.2.2)). For $0 < H \le 1$ the general result reflects what we know for stable motions.

Theorem 10.1 (Vervaat (1985, §3)). *If X is an H-ss si process with $H \le 1$ and $\mathsf{P}[X(t)\equiv tX(1)] = 0$, then the sample paths of X have nowhere bounded variation wpl.*

Sketch of the proof. If X has sample paths of locally bounded variation, then write $X = X_+ - X_-$ as in (8.1), to obtain increasing H-ss si processes X_+ and X_-. For both processes $\mathsf{E}X_\pm(1)$ exists in $[0,\infty]$. For $0 < H < 1$ we obtain $\mathsf{E}X_\pm(1) = 0$ by Theorem 9.3.(a), so $X_\pm \equiv 0$, so $X \equiv 0$. For $H = 1$ we obtain $X_\pm(t) \equiv tX_\pm(1)$ by Theorem 9.3.(b), so $X(t) \equiv tX(1)$. □

The situation is more complicated for $H > 1$. The most obvious examples of H-ss si processes do have sample paths of locally bounded variation, but Vervaat (1985) provides several examples with nowhere bounded variation. The simplest (Vervaat (1985, Ex.6.5)) is an application of Theorem 8.2. Let X be

$\frac{1}{4}$-stable increasing motion and Y be an independent Brownian motion, then $X \circ Y$ is 2-ss si with nowhere bounded sample paths.

If X has sample paths of locally bounded variation, then the derivative X' exists almost everywhere, by Lebesgue's differentiation theorem. It is not hard to see that X' is $(H-1)$-ss and *stationary,* so $X'(t) = 0$ wpl for each fixed t in case $H > 1$. It follows that wpl $X' = 0$ almost everywhere (Vervaat (1985, Th.3.3)). So sample paths of locally bounded variation of H-ss si processes are singular unless $H = 1$ and $X(t) \equiv tX(1)$.

Results on the Hausdorff dimension of the sample paths can be found in Kôno (1986a) and Zähle (1988, 1989a), on Hölder continuity in Kôno & Maejima (1991b), and on further growth properties in Takashima (1989).

There is much literature on functional and marginal laws of the iterated logarithm, for the case $H < 1$. For an excellent review, see Taqqu & Czado (1985). There are no results at all in this direction for the case $H \geq 1$, except for the subcase of stationary *independent* increments, i.e., stable motions: Wichura (1974b) and Pakshirajan & Vasudeva (1981). Of related interest are Wichura (1974a), Mori & Oodaira (1976)) and O'Brien & Vervaat (1991a,b).

11. Jump processes

Among all self-similar processes with stationary increments the jump processes allow a special and more detailed analysis. We say that $X = \big(X(t)\big)_{t\geq 0}$ is a *jump process* if X has sample paths in $D[0,\infty)$ (consequently with countably many jump discontinuities), and

$$X(t) = X(0) + \sum_{0<u\leq t} \big(X(u) - X(u-)\big) \qquad \text{for } t > 0.$$

Here the series is understood to converge in some mode of convergence, let us say, absolute convergence to start with. Let Π be the point process in

$$E := [0,\infty) \times \bar{\mathsf{R}}\backslash\{0\}$$

with support

$$\{(t,\, X(t)-X(t-))\colon X(t) \neq X(t-)\}.$$

The point process Π is locally finite in E, in particular towards the levels $x = \pm\infty$, but its atoms may and often will accumulate towards the level $x = 0$. We now have

$$X(t) = X(0) + \sum_{\substack{0<u\leq t \\ (u,x)\in \operatorname{supp}\Pi}} x = X(0) + \int_{\mathsf{R}\backslash\{0\}} x\Pi\big((0,t],\mathrm{d}x\big).$$

It is not hard to see that X has stationary increments iff Π is invariant in distribution under $(t,x) \mapsto (t+b,x)$ for $b > 0$, and that X is H-ss iff Π is invariant in distribution under $(t,x) \mapsto (at, a^H x)$ for $a > 0$.

For $\beta > 0$, let us define Π^β to be the point process in E with support $\{(t, x^{\uparrow\beta}): (t,x) \in \Pi\}$, where

$$x^{\uparrow\beta} := |x|^\beta \operatorname{sgn} x \qquad \text{for } x \in \mathsf{R}.$$

Then Π is invariant in distribution under $(t,x) \mapsto (at, a^H x)$ iff Π^β is under $(t,x) \mapsto (at, a^{\beta H} x)$.

Combining all these observations, we see that each H-ss si jump process X_H can be represented by

$$X_H(t) = \sum_{\substack{0<u\le t \\ (u,x)\in \operatorname{supp}\Pi}} x^{\uparrow H} = \int_{\mathsf{R}\setminus\{0\}} x^{\uparrow H} \Pi\big((0,t], \mathrm{d}x\big), \tag{11.1}$$

where Π is a point process in E that is invariant in distribution under

$$(t,x) \mapsto (at + b, ax) \qquad \text{for } a, b > 0. \tag{11.2}$$

Such point processes are called *Poincaré* in O'Brien & Vervaat (1985) and *1 self-affine* in O'Brien, Torfs & Vervaat (1990). As in the previous sections we exclude $X \equiv 0$, so $\operatorname{supp}\Pi \neq \phi$ wp1. In fact, one can prove that $\{t: (t,x) \in \operatorname{supp}\Pi\}$ is dense in R wp1.

The intensity measure $\mathsf{E}\Pi$ of Π is deterministic and also invariant under (11.2). Consequently, it has the form

$$\mathsf{E}\Pi(\mathrm{d}t, \mathrm{d}x) = c_{\operatorname{sgn} x} \mathrm{d}t\, x^{-2} \mathrm{d}x, \tag{11.3}$$

where $c_+, c_- \in [0, \infty]$. We say that Π has *finite intensity* if both c_+ and c_- are finite.

In (11.1) we wrote X_H instead of X, in order to express that for fixed Poincaré Π we have a whole collection of H-ss si processes for varying H, provided that the integral in (11.1) converges wp1. One can prove that for fixed H the integral in (11.1) converges wp1 either for all $t > 0$ or for none of them. Consequently, the set of all H's for which the integral converges for some (or equivalently, all) $t > 0$, is a random interval $\mathcal{H}_a$ of the form $[H_a, \infty)$, (H_a, ∞) or ϕ.

Theorem 11.1 (O'Brien & Vervaat (1985, Th.2.1)).
(a) $\mathcal{H}_a \subset (1, \infty)$ *wp1.*
(b) *If* Π *has finite intensity, then* $\mathcal{H}_a = (1, \infty)$ *wp1.*

There is much delicate analysis about other modes of convergence and related examples in O'Brien & Vervaat (1985), of which we mention one simple result. Let $\mathcal{H}_c$ be the random set of all H such that the integral in (11.1) converges wp1 for each $t > 0$ separately in the sense that

$$\lim_{\varepsilon \downarrow 0} \int_{\mathsf{R}\setminus(-\varepsilon,\varepsilon)} x^{\uparrow H} \Pi((0,t], \mathrm{d}x) \tag{11.4}$$

exists. It turns out that $\mathcal{H}_c$ is again a random interval with ∞ as right end. Let Π be symmetric in the sense that Π is invariant in distribution for $(t,x) \mapsto$

$(t, -x)$. Then $\mathcal{H}_c = \frac{1}{2}\mathcal{H}_a$ wpl, in particular $\mathcal{H}_c = (\frac{1}{2}, \infty)$ wpl if Π has finite intensity.

Ss si jump processes are a tractable class for the production of examples. The most obvious choice for a Poincaré process Π is a Poisson process with intensity as in (11.3). The resulting X_H has independent increments, so is a strictly stable motion, produced by absolute convergence in (11.1) for $H > 1$, and by convergence as in (11.4) in case $c_+ = c_-$ for $\frac{1}{2} < H \leq 1$.

Another example of a Poincaré process Π is the g-adic lattice process (see O'Brien & Vervaat (1985, §3.2) for details), which produces what we may consider to be the most deterministic H-ss si process X_H for $H > 1$: given the location and size of one jump, all jumps of equal and smaller size are determined together with their locations.

12. Subordination to jump processes; self-similar Palm measures

Self-similar jump proceses with stationary increments allow many other types of subordination than are considered in §7. Several are studied in O'Brien & Vervaat (1985, §3.5). Here we discuss one class treated in Vervaat (1985, §4).

For convenience, let us restrict our attention to processes X which are, moreover, of locally bounded variation. Such processes can be represented by random Radon measures N on R, via

$$N(t, u] = X(u) - X(t),$$
$$X(t) = N(0, t].$$

We call N H-ss and stationary if X is H-ss si. If Π is a Poincaré point process in E and μ is a fixed (deterministic) Radon measure in R, then

$$N_H := \sum_{(t,x)\in \operatorname{supp}\Pi}\sum x^{\uparrow H}\mu\left(\frac{\cdot - t}{|x|}\right) = \int_E\int x^{\uparrow H}\mu\left(\frac{\cdot - t}{|x|}\right)\Pi(\mathrm{d}t, \mathrm{d}x) \qquad (12.1)$$

is an H-ss stationary random Radon measure, provided that the integral in (12.1) converges wpl. The following theorem provides curious sufficient conditions for this, which seem to be close to necessary.

Theorem 12.1 (Vervaat (1985, Th.4.5)). *Suppose that $\mu(\mathsf{R}) < \infty$ and $\operatorname{supp}\mu$ is a compact Lebesgue null set in* R. *Let $(l_n)_{n=1}^{\infty}$ be an enumeration of the lengths of the open holes in $\operatorname{supp}\mu$. If $\sum_n l_n|\log l_n| < \infty$, then the integral in (12.1) converges wpl for all $H > 1$.*

There is an obvious generalization of (12.1) to independent identically distributed random measures μ in the integrand of (12.1). In this way Vervaat (1985) produces various examples of sample path behavior, in particular examples of H-ss si processes with $H > 1$ and continuous sample paths, whether or not with locally bounded variation. Subordination to Poisson Π has been studied by Surgailis (1981) and Taqqu & Wolpert (1983).

Self-similar stationary random measures generate self-similar Palm measures (σ-finite, not necessarily finite). They are studied (in higher dimensions) in Zähle (1988, 1989a).

13. Domains of attraction

Self-similar processes with stationary increments have practical interest, because they are the limits in distribution of partial sum processes of stationary sequences under joint scaling of space and time, as shows Theorem 5.1. Let us say that a stationary sequence $(\xi_k)_{k=1}^{\infty}$ is in the *domain of attraction* of a given ss si process X if (5.4) holds. By the last clause of Theorem 5.1 this domain is not empty. Much work has been done in specific cases to explore its extent.

Domains of attraction of fractional Brownian motion and, more generally, Rosenblatt processes (analogous subordination by multiple integrals to more-dimensional Brownian motion) have been studied most extensively. For references, see the bottom of p.138 in Taqqu (1986). Actually, Rosenblatt processes arise quite naturally as limits in distribution of partial-sum processes of stationary sequences.

Similarly, Kesten & Spitzer (1979) obtained their ss si process (local time of one process subordinated to another, cf.§7) as limits in distribution of rescaled discrete-time random walks in random scenery.

Domains of attraction of (non-Gaussian) stable fractional processes have been studied by Astrauskas (1983), Avram & Taqqu (1986, 1991), Kasahara & Maejima (1988) and Kasahara, Maejima & Vervaat (1988) (log-fractional stable processes).

14. Self-similar stationary extremal processes

On this topic there is a survey in Vervaat (1986). Therefore we restrict ourselves to a few headlines.

The problem posed and solved in O'Brien, Torfs & Vervaat (1990) is the following. Consider the situation of Theorem 5.1, but with maxima instead of sums: replace $Y(t) := \sum_{k=1}^{\lfloor t \rfloor} \xi_k$ by

$$Y(t) := \max_{k=1}^{\lfloor t \rfloor} \xi_k =: \bigvee_{k=1}^{\lfloor t \rfloor} \xi_k.$$

In this situation, it is more convenient to make Y a random set function of time intervals $A \subset [0,\infty) =: T$, so

$$Y(A) := \bigvee_{k \in A} \xi_k,$$

and to look for corresponding random set functions X. This calls for the development of an appropriate analysis for the set functions involved. This is done in Vervaat (1981, 1988) and Norberg (1986) (see also O'Brien & Vervaat (1991a)). The results are the following.

A *sup measure* is an $\bar{\mathbb{R}}$-valued function m on the open subsets of T such that

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigvee_{n=1}^{\infty} m(A_n)$$

for all sequences of open sets (A_n) in T. Each sup measure corresponds one-to-one to an *upper semicontinuous* (*usc*) $\bar{\mathbb{R}}$-valued function f on T by

$$m(A) = \bigvee_{t \in A} f(t) \quad \text{for open } A \subset T, \tag{14.1a}$$

$$f(t) = \bigwedge_{\text{open } G \ni t} m(G) \quad \text{for } t \in T. \tag{14.1b}$$

The collection of sup measures (and thus also the collection of usc functions) becomes a compact metrizable space by the notion of *sup vague convergence*:

$$m_n \to m :\Longleftrightarrow \begin{cases} \limsup m_n(K) \le m(K) & \text{for compact } K \subset T, \\ \liminf m_n(G) \ge m(G) & \text{for open } G \subset T. \end{cases}$$

With the derived Borel field, the sup measures become a measurable space. An *extremal process* is a random sup measure, and can be identified with a random usc function on T via (14.1).

All limits in Theorem 5.1 as modified in the present section are identified in O'Brien, Torfs & Vervaat (1990) as simple transformations in the state space of random usc functions X on T such that

$$X(a \cdot + b) =_d aX(\cdot) \qquad \text{for } a, b > 0. \tag{14.2}$$

We call the corresponding extremal processes *stationary* (the b-part of (14.2)) and 1-*ss* (the a-part of (14.2)). Mostly, X arises from a Poincaré point process Π in $[0, \infty) \times (0, \infty]$ (cf.§11) by

$$X(t) = \sup\{x : (t, x) \in \operatorname{supp} \Pi\},$$

where $\sup \phi := 0$. The classical extremal processes are generated by Poisson Π. There are cases in which X as in (14.2) cannot be generated by a point process. However, all finite-valued random usc functions X that satisfy (14.2) have supports of Lebesgue measure 0 wpl.

We have found that Poincaré point processes Π generate both ss si jump processes and stationary ss extremal processes. The classical case of Poisson Π is treated this way in Resnick (1986, 1987).

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Conformal Field Theories on Riemann Surfaces

Omar Foda
Institute for Theoretical Physics
University of Nijmegen, Toernooiveld
6525 ED Nijmegen, The Netherlands

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Abstract

We introduce the Coulomb gas representation of the simplest class of conformal field theories: the A_n-series of central charges $c = 1 - 6/\Big(m(m+1)\Big)$, $m \geq 3$, on compact Riemann surfaces, of genus g. Our starting point is the critical limit of the corresponding statistical mechanical models defined on a discrete square lattice: the restricted solid-on-solid models of Andrews, Baxter and Forrester.

We obtain the Coulomb gas represenation of these models formulated on a lattice that has the geometry of a torus, then on tori with one puncture, and with two punctures. We form compact Riemann surfaces of arbitrary topology by joining these punctured tori at their boundaries, and obtain expressions for the partitions functions

1 Introduction:

The purpose of these notes is to introduce a class of exactly-solvable statistical mechanical models, and the corresponding conformal field theories that describe their critical behaviour.

There are different ways to introduce conformal field theory. I choose the Coulomb gas representation, since I find it elementary and intuitive. It is also familiar to physicists and mathematicians, from other backgrounds, though under different names: bosonization, the Dotsenko-Fateev representation, or the Feigen-Fuchs representation.

Next, I would like to turn to Riemann surfaces. Boundary conditions play an important rôle in the study of two-dimensional critical behaviour: critical models exhibit different parts of their content under different boundary conditions. For example, the partition functions are obviously boundary condition dependent. We can further our understanding of new models by formulating them under as many different boundary conditions as possible. By putting our models on Riemann surfaces we are doing exactly that.

There are excellent technical reviews both of conformal field theory, and of the theory of Riemann surfaces. We will refer to them for a review of the elements that of these subjects. There is also [10], where the basics of the statistical mechanical models, that we are interested in here, and their relation with the corresponding conformal field theories, have been carefully reviewed. We wish to regard the material that can be found in the first few sections of [10] as a pre-requisite for these notes.

Exact solutions

There has been considerable progress in our understanding of 2-dimensional critical phenomena, and exactly-solvable statistical mechanical models in the past decade. By an exactly-solvable statistical mechanical model, we mean an off-critical model, defined on a discrete lattice, with Boltzmann weights that satisfy the Yang-Baxter equation. One can also consider exact solutions that exist only at criticality, but we like to think of these as being incomplete.

Off criticality, the Boltzmann weights can be written most explicitly in terms of elliptic functions, which can be regarded as infinite series in a complex parameter called the modulus, or the nome. The importance of the nome is that parameterizes the departure from criticality. In the limit that nome goes to zero, elliptic functions reduce to trigonometric functions.

Critical behaviour is obtained in the trigonometric limit. For further details, we refer to [1].

Brief history

To appreciate the rate at which the subject has progressed in the past few years, let us consider a brief survey of the known exact solutions. Up to 1984, there were no more than a few: the Ising [2] and dimer [3, 4] models, that are actually related; the 6-vertex model [5], and the 8-vertex model [1] which can be considered as a generalization of 6-vertex model to off-criticality. Then there are the 3-spin, and hard-hexagon models, both of which are now understood as special cases of the 8-vertex model [1].

Furthermore, there are two exact solutions: the 6-vertex model in an external field, and the spherical model, but these two are quite different from the others: they do not fit into the same classification scheme as the rest [1]. It is good, however, to keep in mind that they exist, as one more reminder of what remains to be explained.

Since 1984, the situation has drastically changed. Since the work of Andrews, Baxter and Forrester (ABF) [6], exact solutions come no longer as isolated examples, but as infinite series. The reason behind this progress is the realization of the rôle that certain mathematical structures play in exact solutions. In the work of ABF these were the Rogers-Ramanujan (RR) identities. The point is that the sum sides of the RR identities appear in the computations of the 1-point functions o these models, using Baxter's corner transfer matrix method. But the RR identities are directly related to the theory of affine Lie algebras. From the latter connection further Lie-algebraic structures began to emerge. This was made very clear in the work of the Kyoto group [7].

These mathematical structures are particularly manifest in the class of models called the restricted solid-on-solid (RSOS) models first introduced in [6]. These models play a unifying rôle in the subject, since a number of already known models such as the Ising, Potts, and hard hexagon models are included in them. In these notes, we will not consider these models in all generality. We will consider only their critical behaviour.

The Coulomb gas representation

One convenient way to study the critical behaviour of a new model is to map it, if possible, on another model that is already well understood. One

model that is quite well-understood is the gaussian model. Because the partition and correlation functions of the gaussian model can be interpreted in terms of configurations of interacting electric and magnetic charges, this is called the Coulomb gas (CG) representation. In these notes, we will consider a simple class of RSOS models, at criticality. We will map them on a Coulomb gas, then we will study their partition functions on compact Riemann surfaces.

The Riemann surfaces that we will consider have the topology of a sphere with attached handles. The number of handles gives the genus of the Riemann surface. The simplest example is the sphere with a single handle attached: the torus. One can always cut a Riemann surface open, and lay it on a plane. This way, it looks like a piece of the plane, with specific boundary conditions. This is basically the reason why we are interested in investigating models on Riemann surfaces: they allow us to probe these models in the presence of some rather peculiar boundary conditions. That allows us to study aspects of these models that would not be so obvious in the presence of the usual, e.g. free or fixed, boundary conditions.

Conformal field theories

At criticality, these models can be described in terms of conformal field theories: 2-dimensional local quantum field theories, that are scale invariant. Scale invariance is a direct consequence of criticality. (Recall the definition of criticality from the renormalization group point of view: the absence of a characteristic length scale. Probing the system at an arbitrary length scale, one always sees the same physics. In that sense, the system is scale invariant.)

In a local quantum field theory, scale invariance plus locality imply local scale invariance, or equivalently: conformal invariance, invariance under angle-preserving transformations. Actually, we have to be a little careful with the word 'invariance' here. What shall actually observe that the correlation functions, for example, will transform in a 'simple' way under conformal transformations. For example, they will scale by a factor that will depend on the dimensions of the operators involved.

In two dimensions, and in contrast to the situation in all higher dimensions, the conformal group is infinte dimensional, since all analytic transformations are conformal transformations. Therefore, the conformal group is infinite dimensional, and conformal invariance provides an infinite number of conditions that allow one to compute the partition and correlation

functions exactly.

Another consequence of conformal symmetry is that conformal field theories are characterized, to a great extent, by a single real variable called the central charge. It is basically the Casimir energy, per unit area, that shows up in the presence of finite geometry.

If one considers the operator content of conformal field theories, one can show that it contains operators that transform as tensors under conformal transformations. These are called primary fields. Conformal field theories that contain a finite number of primary fields are called minimal. The class of minimal models that we are interested in here are characterized by the central charges

$$c = 1 - \frac{6}{m\,(m+1)}, \quad m \geq 3 \tag{1}$$

2 Riemann surfaces

Let us focus on the A_n-seriesof conformal field theories that have $c < 1$. These are characterized by the fact all primary (or scaling) fields are scalars. In other words, they transform as scalars under conformal transformations [8]. Its CG representation on the Riemann sphere (the plane compactified by adding a point at infinity) is well-understood: one has a consistent set of rules to compute all correlation functions. These rules can be obtained starting from the underlying restricted-solid-on-solid (RSOS) models [9, 10]. They have been explained in detail in [10]. The partition function on the sphere is normalized to 1.

The torus

On the torus, one demands that all expressions be invariant under 'large' reparametrizations of the coordinates: reparametrization that cannot be deformed smoothly to the identity. These are called 'modular transformations'. Invariance under modular transformations is a very stringent requirement that can be used to compute partition functions on the torus, and to classify models accordingly [10]. We do have a complete classification of all modular invariant partition functions [11].

On the other hand, our knowledge of the correlation functions is not as complete. We do have prescriptions to compute *certain* correlations (those that require all screening charges to be of the same type; only then do

we obtain expressions that are manifestly doubly-periodic) [10, 12]. Consequently, we do not have, as yet, an explicit construction of the general n-point functions on the torus, that is manifestly doubly-periodic.

One way to address this issue is to re-derive the torus correlation functions from a different approach. Since operator insertions on a compact Riemann surface Σ of genus g can be considered as punctures on the surface, with appropriate boundary conditions for the quantum fields that live on it, any correlation function can be obtained from a partition function on a higher genus surface through degeneration, and suitable projections. This is *one* reason to consider formulating the minimal models on $g > 1$ surfaces. Another reason is that modular invariance on multi-loop surfaces should give us more information about the structure of the models, in addition to what we learnt on the torus.

A number of attempts have been made in this direction. In [13, 14] differential equations for the conformal blocks were derived. However, we do not know of explicit solutions to these equations, apart from simple cases. In [15], the fermionic formulation of the Ising model was extended to genus two. But this formulation is peculiar to the Ising model: there are no similar formulations for the other minimal models. In fact, this is precisely the reason we are interested in the CG representation of these models: we have no other way to perform explicit computations.

It is possible to shift the central charge from its gaussian value, by coupling the gaussian field to the background curvature: a procedure suggested by experience with the b c systems in string theory, and justified for the minimal models in [10]. However, this is not sufficient to project out the null states, as can be seen by specializing the expressions obtained to the torus: one obtains the partition functions of $c = 1$ models, rather than what we know to be the correct expressions:

$$Z_{min}\left(m\right) = Z_{gauss}\left(m/(m+1), m+1\right) - Z_{gauss}\left(m/(m+1), 1\right) \quad (2)$$

where $Z_{min}(m)$ is the partition function of a minimal model, with $c = 1 - 6/\left(m(m+1)\right)$, $Z_{gauss}(\mathrm{g}, r)$ is the partition function of a gaussian model, with $c = 1$, coupling constant g, and a compactification radius r.

For g $r^2 = p/q$:

$$\begin{aligned} Z_{gauss} &= 1/\eta(q)\overline{\eta}(q) \times \\ &\sum_{\alpha,\beta,\gamma} \Theta \begin{bmatrix} \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma \\ 0 \end{bmatrix} (0|2pq\tau)\overline{\Theta} \begin{bmatrix} \frac{1}{2}\alpha - \frac{1}{2}\beta + \gamma \\ 0 \end{bmatrix} (0|2pq\tau) \end{aligned} \quad (3)$$

where $\alpha \in N_p/p$, $\beta \in N_q/q$, $\gamma \in N_2/2$, N_n stands for the set $\{N \bmod n\}$ [19, 20].

$$\eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{4}$$

$$q = \exp(2\pi i\tau) \tag{5}$$

and the theta function

$$\Theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(z|\tau) = \sum_{n\in Z^g} \exp\left[i\pi(n+\alpha)\cdot\tau\cdot(n+\alpha) + 2\pi i(n+\alpha)\cdot(z+\beta)\right] \tag{6}$$

The characteristics α and β are g-dimensional vectors, and τ is the $g \times g$ period matrix of the genus g Riemann surface. For our purposes z is simply an argument of the theta function as defined in equ.(6). For a clear introduction to the theory of Riemann surfaces, see [20]. Z_{gauss} satisfies the identity

$$Z_{gauss}(g, r) = Z_{gauss}(g r^2, 1) \tag{7}$$

Therefore, Z_{gauss} (g,r) is a function of the combination $g r^2$ only. The subtraction on the r.h.s. of equ.(2) is crucial: it projects out the null states: states that have zero norm. It is important, for our purposes, to realize that this projection *is not* obtained from a local action, even in the presence of a coupling to the background curvature: we do not know how to modify the action to achieve that. We wish to emphasize this point, since decoupling the null states is the main issue to be resolved on multi-loop surfaces. What is the origin of the projection in equ.(2) in the CG picture?

In [18], DiFrancesco, Saleur and Zuber obtained a derivation of the torus partition functions starting from the underlying critical statistical mechanical models. They found phase factors that weigh the different soliton sectors differently, *in addition* to the real weights obtained from the exponential of the action of the model. Rearranging terms into gaussian combinations—where all soliton sectors have a common phase that factors out—one ends up with equ.(2). Combining these results with those obtained in [5, 6], one has a coherent picture of how the CG representation, on the sphere and the torus, is rooted in the statistical models. We will refer to this approach to the minimal models that starts from the statistical models as the "lattice" approach. It has the advantage of starting from a faithful representation, which ensures that one ends up with physically meaningful results, that include the decoupling of the null states.

To extend this approach to higher genus compact Riemann surfaces, our strategy will be to decompose the problem into a series of smaller ones. We

will find that each of these sub-problems has essentially already been solved on the sphere or the torus. Our effort will be to set these pieces together. More specifically, we proceed as follows:

- Start with a RSOS model, based on the Dynkin diagram of a A_n group, formulated on a lattice with the topology of a genus-g compact Riemann surface.
- In order to proceed with the mapping on a CG, we need to define the action of the transfer matrix on the lattice. For that we cut the lattice along surface-dividing contour lines: cycles, into g separate parts. Each part has the topology of a torus with either one or two punctures.
- Map the configurations on each part, called "patch", on a linear combinations of the configurations of "cyclic" RSOS models, based on the Dynkin diagrams of the affine groups $\hat{A}_n$.
- Re-attach the punctured tori to recover the original Riemann surface.
- Map the cyclic RSOS configurations, on the multi-loop surface, on a CG.

Our result will be in the form of the difference of two expressions. Each of these is a gaussian functional integral of a scalar field compactified on a circle, and coupled to the background curvature, in the presence of a charge due to the coupling to the curvature, and screening charges, integrated over the surface, that ensure charge neutrality. This functional integral has been evaluated in connection with the bosonized ghost systems, in string theory. We will make use of this to write our expressions in terms of integrals over known functions.

In §3, we recall an intermediate step in the mapping of the RSOS models on a Coulomb gas. It involves representing the original model in terms of random closed self-avoiding walks, that are also non-overlapping and dense on the lattice. We will refer to them as "polygons". They will turn into equipotential lines at the CG level. These will be classified into three types, that can be treated separately, and perform the mapping of the first type: the contractible polygons that do not enclose surface curvature. In §4 we consider the contractible polygons that do enclose surface curvature, and pay attention to the way the transfer matrix can act consistently on a curved surface. In §5, we consider the non-contractible polygons, and compute the

Figure 1: A Dynkin diagram of the A_n-series.

phases associated with the different soliton sectors. In §6 we put what we learnt in the previous sections together, write down modular-invariant expressions for the partition functions, and check their consistency. §7 contains remarks on what remains to be done.

3 The Coulomb gas:

Mapping the A_n-series of minimal models, represented critical RSOS models, on a Coulomb gas has been reviewed in detail in [10]. We start by recalling some basic facts.

The RSOS models can be defined as follows: One starts with a Dynkin diagram of a Lie group in the A_n-series, Fig.1, and assigns each node an integer, serially from 1 to m (the number of nodes = the rank of the group). (In [9], models based on the Dynkin diagrams of the other simply-laced Lie groups were also constructed.)

The configurations of an RSOS model of a given m, are obtained by assigning to the sites of a square diagonal lattice "height" variables that take values in the integers defined above, with the constraint that the heights on nearest-neighbouring sites differ only by ± 1. In particular a zero height difference is not allowed. A typical configuration is shown in Fig.2, where we took $m = 4$.

In the following, we outline the basic mappings that will take us from the RSOS models to the CG:

The random cluster representation

The RSOS configurations can be mapped on random clusters : Fig.3.

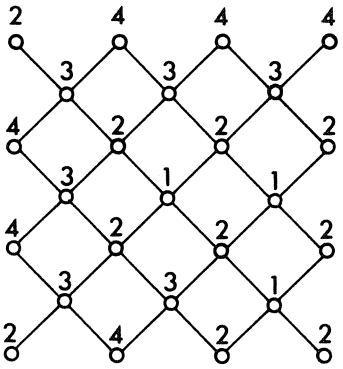

Figure 2: A RSOS configuration.

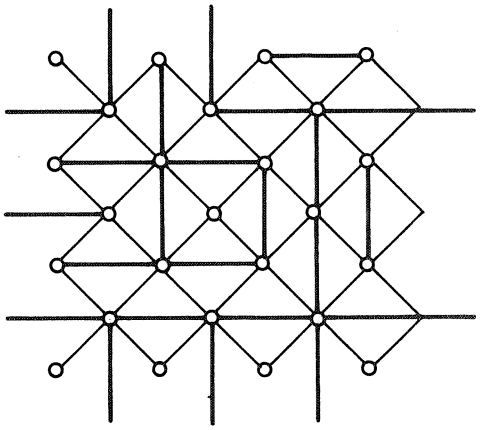

Figure 3: A random cluster configuration.

The random clusters inherit the Boltzmann weights of the initial RSOS configurations, in such a way that one could read the weights directly from the clusters. They are assigned to them by the action of the transfer matrix, that starts from a given initial one-dimensional configuration, and generates all allowed two-dimensional configurations. In Fig.3, consider the transfer matrix to act from left to right. Let us call the bonds parallel to the direction of action *horizontal*, and those orthogonal to it *vertical*. With a suitable choice of the spectral parameter that appears in the transfer matrix, a horizontal bond acquires a Boltzmann weight one. A vertical bond acquires a weight, that can be written in the symmetric form

$$\left(S_1^{h_i} S_1^{h_{i'}} / S_1^{h_{i-1}} S_1^{h_{i+1}}\right)^{1/2}, \quad h_{i-1} = h_{i+1} \tag{8}$$

where $S_1^{h_i} = \sin\left(h_i \pi/(m+1)\right)$ is a component of the eigenvector corresponding to the largest eigenvalue, $2\cos\left(\pi/(m+1)\right)$, of the adjacency matrix of the Dynkin diagram we started with. h_i is a height: $1 \leq h_i \leq m$. For further details see [9, 10]. Although the random clusters are convenient as a definition of the height configurations, they do not look in any way like a Coulomb gas. For that we have to perform the following mapping.

The polygon representation:

The basic step towards a CG representation is to map the clusters on random polygons, as shown in Fig.4.

They will end up being the equipotential lines of the scalar potential of a problem in electrostatics on the Riemann surface. Hence the name, a Coulomb gas.

On arbitrary compact Riemann surfaces, the polygons can be classified—for our purposes—into *three* classes:

- Contractible polygons that enclose no surface curvature.
- Contractible polygons that *do* enclose surface curvature.
- Non-contractible polygons.

On multi-loop surfaces, we will distinguish between homotopically non-trivial cycles that are homologically trivial, i.e. surface-dividing, and homologically non-trivial, i.e. not surface-dividing. Finally, we mention one more mapping:

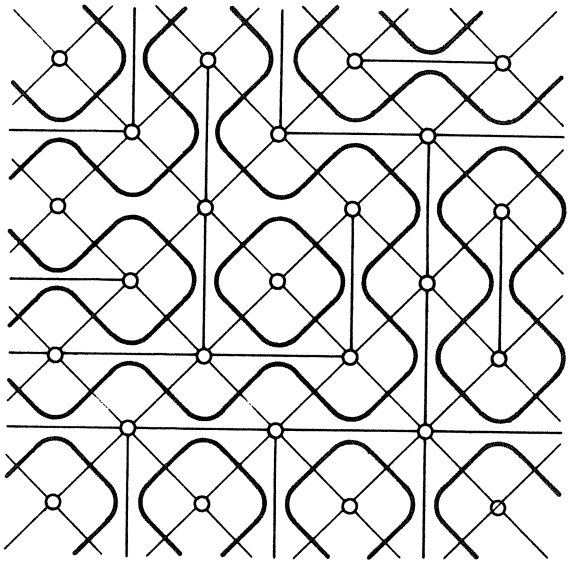

Figure 4: A random polygon configuration.

The oriented polygon representation:

One can assign the polygons orientations, and sum over all possible combinations of oriented polygons. Regarding a polygon as a domain wall that separates regions of different heights, the orientation of a polygon can be set—by convention—to indicate which region on the side of that polygon is higher: crossing a clockwise-oriented (counter-clockwise oriented) polygons indicates taking a step upwards (downwards) height-wise.

What we learnt from the lattice approach is that in mapping the minimal models on a CG, on the sphere and the torus, the basic data that characterize the model show up in three different aspects of the final CG representation. These are in one-to-one corespondence with the polygon classes listed above:

- In the specific value of the gaussian coupling constant, or equivalently the compactification radius. This determines the magnitudes of the CG charges.
- In the appearence of extra charges at the conical singularities, on the Riemann sphere. This causes the shift in the central charge.
- In the appearence of additional phase factors that weigh the different soliton sectors, of the torus partition function, differently. This is the origin of the projection of the null states.

To extend the CG picture to higher genera, we start from an RSOS model defined on a lattice with the topology of a multi-loop Riemann surface, and

map that on a CG. We proceed—as should by now be anticipated—in three steps: in each step we consider one of the polygon classes listed above.

In fact, we can readily say something about the first step. Polygons that are both contractible, and do not enclose curvature are "local": they do not know anything about the curvature of the surface, or its topology: the number of handles. They can be treated exactly as they were on the sphere and the torus: their weights are completely distributed on their corners, and one ends up with the local vertices of a six-vertex model with a definite coupling constant. In the continuum limit, it renormalizes on a gaussian model with an action

$$A[\varphi] = \frac{\mathrm{g}}{4\pi} \int d^2z \, \sqrt{|g|} \, g^{zz} \, \partial_z\varphi \, \partial_z\varphi \ , \quad \mathrm{g} = \frac{m}{m+1} \tag{9}$$

where $|g|$ is the determinant of the metric tensor g^{zz}. Notice that in our normalization the coupling constant g at the Kosterlitz-Thouless point is 1. In equ.(6) the radius of compactification r is not known yet. This will be determined in the following step.

4 The curvature

Let us recall the way the bonds, that formed the random polygons, were assigned weights. The weights depended not only on the height variables h_i, but also on whether the bond is horizontal or vertical. (This does not imply that the model is anisotropic, as can be seen in the polygon representation.) But this way of assigning weights is meaningful only on the plane, where one has a well-defined direction for the action of the transfer matrix: a Euclidean "time-like" direction. Only then can one consistently tell which bonds are horizontal, and which are vertical. There is no such obvious assignments on non-flat surfaces, such as the surface of a cube, or on surfaces with non-trivial topology. How can we determine the weights of the random clusters in these cases? This section is devoted to this question.

Consider a model defined on a planar region with free boundary conditions. To obtain out of that the corresponding model defined on a cylinder (or a torus) one restricts oneself to those configurations that are periodic in one direction (or both). This amounts to discarding all configurations that do not satisfy our restriction. On the surface of a cube, one has to do something similar: find a set of boundary conditions, on the plane with boundaries, that coincide with the configurations allowed on the surface of the cube.

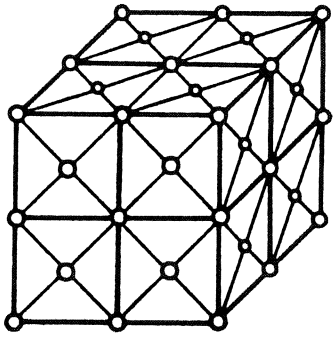

Figure 5: An RSOS model on the surface of a cube.

Take a lattice with RSOS variables on the surface of a cube: Fig. 5.

Choose two opposite faces, and open the surface along their diagonals, to obtain a surface with the topology of a cylinder. This can be opened flat on the plane: Fig.6.

To do that, certain sites are *split* as shown in Fig.6. All data about the shape of the original surface are now in the fact that all parts of a certain "split-site"share the same height variable. The lattice we obtain this way is not convenient for the transfer matrix to act on, because of the missing triangular regions. However, one can fill these gaps with *inert* sites: sites connected by *horizontal bonds only*—they do not independently fluctuate: they are bound to carry the same height as the split-sites connected to them by the horizontal bonds, and therefore do not contribute to the dynamics. Since the horizontal bonds have Boltzmann weight one, the weight of the entire configuration remains unchanged. In a sense, we could have started with free boundary conditions, generated all possible configurations, and retained only those which satisfy the boundary conditions we require. We will refer to any compact surface that we start with as Σ, and to the corresponding open surfaces as Σ_{open}.

Consider the polygons in Fig.7.

Computing their weights, one finds that—in the absence of external operator insertions—each one has exactly the same weight: $2\cos\left(\pi/(m+1)\right)$ [9, 10]. This is an expression of conformal invariance. This includes polygons that, on Σ, would enclose a corner, where curvature is concentrated. But

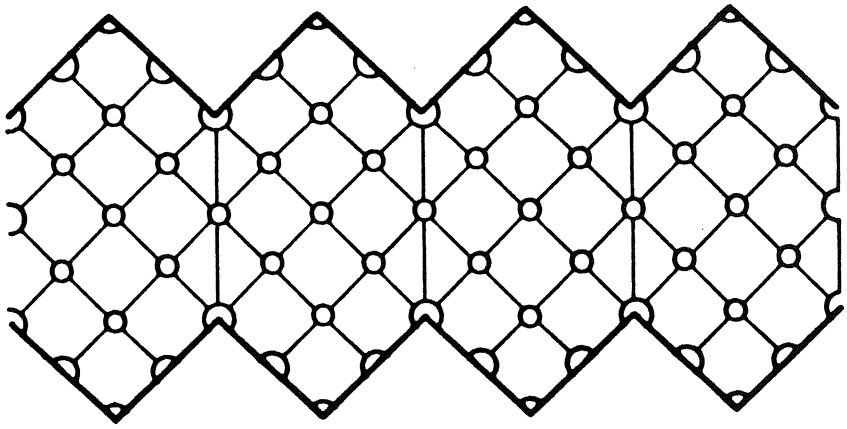

Figure 6: The RSOS lattice on the surface of the cube opened on the plane, with no inert sites.

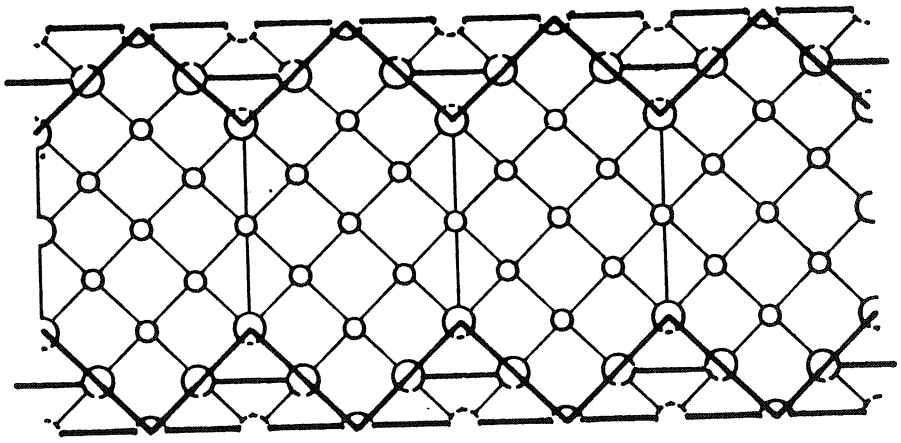

Figure 7: The same surface, as above, with inert sites and bonds.

these are the weights of *entire* polygons. To obtain local expressions—in order to get closer to a local field theory—we distribute them on the segments that constitute the polygons. To distribute the weights, one goes to the oriented-polygon representation. A clockwise-oriented polygon gets a weight $\exp\left(i\pi/4(m+1)\right)$, counter-clockwise-oriented polygons get $\exp\left(-i\pi/4(m+1)\right)$. These phases can be split into a product of smaller phases and distributed on the polygon segments in proportion to their line curvature [10]. This must be done on those segments that live on Σ, and not on the fill-up regions $\Sigma_{open} - \Sigma$. This is because the fill-up regions must remain with weight one, to retain the equivalence with the original surface configurations. Going back to Σ, and distributing the weights on the *available* segments, one is left with extra phase factors that translate, in the CG picture, to charge insertions at the conical singularities. In [10], it was *proposed* that all contractible polygons should have the same weights, including those encircling corners, as part of the definition of the model.

Notice that on Σ_{open} one will also have polygons that *appear* open, given the way we go from Σ to Σ_{open}. To obtain the correct weights of these, we notice that the original surface is isotropic: all faces are equivalent. The polygon weights should not depend on the way we open the surface. But one can always find a way to open the surface such that any closed polygon *remains* closed. Therefore, polygons that are cut open, should be given the same weight as the rest. If one wishes to do so, one can enforce this explicitly by supplementing a given slicing of the surface with a *seam line* of the form introduced in [21]. This would correct the anisotropy introduced by the way we choose to open the surface. We will not go into this here.

The fact that all polygons—including those that enclose corners—have the same weight, is the origin of the background charge introduced in [16]. Notice that until now we have not encountered any polygons that are non-contractible in the sense of winding around the handles of a Riemann surface. These are first met on the cylinder and the torus, and require a different treatment. We delegate this to 4, and go directly to contractible polygons on multi-loop surfaces.

Consider a square lattice on a genus-two surface Σ as in Fig.8.

Our discussion extends readily to $g > 1$ surfaces. To allow the transfer matrix to act on Σ, we wish to open it in a suitable way. We propose to do this as follows:

- Cut the lattice open in two parts, along a surface-dividing cycle. We refer to each separate part as a "patch".

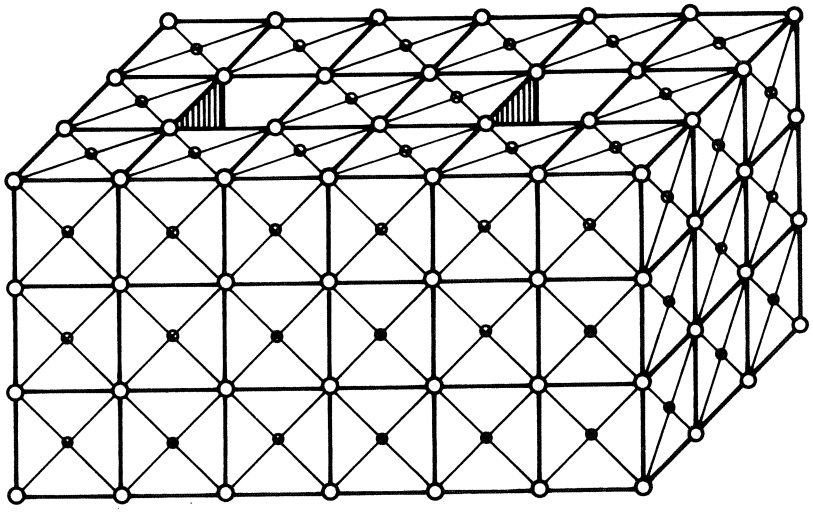

Figure 8: A diagonal square lattice with the topology of a genus-2 surface.

- Consider either patch: it is basically a lattice with the topology of a torus, but with one face missing. (Going through this excercise on $g > 2$, some of the patches will have *two* faces missing.) Each patch can now be opened the way one would open a torus:

  Open the torus into a cylinder with missing regions.

  Open the cylinder onto a planar surface. Notice that the latter has *two* types of missing regions:

  regions with boundaries that are due to the way we cut the original surface along its surface-dividing cycles.

  regions due to opening the cylinder-like surfaces on the plane.

- Since regions of the first type are rectangular, one can still define a direction for the action of the transfer matrix in their presence. Initially, we will assign them free boundary conditions. To recover the configurations allowed on Σ_γ we have to restrict the configurations to match along the boundaries where two patches should be "glued" together in the sense that domain walls should run continuously across the boundaries. Let us refer to these boundaries as "glue-lines". Regions of the second type have triangular parts, and we will fill them with inert sites connected by bonds that are horizontal with respect to the action of the transfer matrix, on that patch: Fig.9.

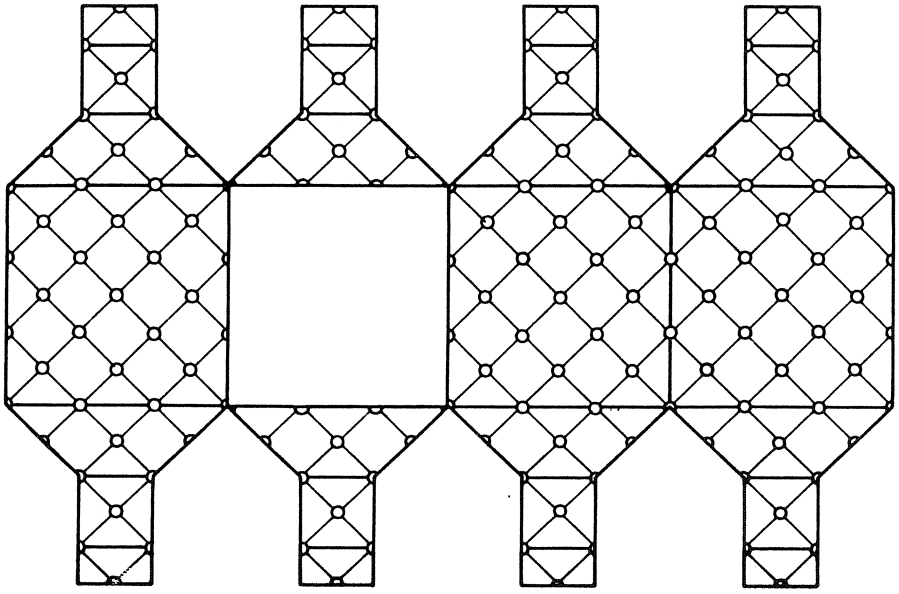

Figure 9: An RSOS model on an open patch of a $g > 1$ Riemann surface.

- Allow the transfer matrix to generate all possible configurations, and retain only those consistent with the Σ boundary conditions.

It is straightforward now to show—following the procedure detailed in [10]—that all contractible loops do indeed get the same weight, and that, on Σ, one is left with extra phase factors, that translate to charges proportional to the background curvature. The only complication —as far as the contractible polygons are concerned—is due to polygons that divide into separate open parts, each on a separate patch, because of they way we open Σ. On a given patch, they terminate on a boundary with "free" boundary conditions: the glue-lines, so it is not straightforward to evaluate their weights. However, the polygon-weights should be independent of the way we open the surface, and—once again—there is always a way to do so such that any contractible polygon remains connected on a single patch. One can then evaluate its weight on that patch, and find it equal to the others.

The background charges can be included *via* a modification of the action [10]. The modified action is

$$A[\varphi] = \frac{\mathrm{g}}{4\pi} \int d^2z \, \sqrt{|g|} \, \left[g^{zz} \, \partial_z\varphi \, \partial_z\varphi \; - \; \frac{i}{4\pi(m+1)} \, R\varphi \, \right] \qquad (10)$$

Where $\mathrm{g} = m/(m+1)$. The functional integral based on (7) will van-

ish unless we include screening charges that neutralize the background charge. Their positions $\{z_i\}$ will be integrated over the surface. We wish to discuss this action:

The structure of equ.(10) is identical to that of the action used to describe the bosonized ghost system: in both cases one has a scalar field compactified on a circle of a definite radius, coupled to the background curvature in a way that changes the energy momentum tensor, and shifts the central charge. To evaluate the functional integrals based on equ.(10), one can follow the treatment of the bosonized ghosts [17].

For simplicity, we have not included in equ.(10) topological terms required to make it invariant under deformations in the choice of homology basis on the Riemann surface. They can be found in [17].

The most intriguing aspect of equ.(10) is that it does *not* fully describe the physics of the minimal models. In particular, it does not lead to a decoupling of the null states. It is unclear whether this can be achieved by a suitable addition to equ.(10). In the following sections we will be mainly interested in this problem.

Remarks

Changing the way we open Σ so that a certain contractible polygon remains on the same patch amounts to shifting the location of the glue-lines along the surface. Can this be done consistently, in the sense that the weights of the configurations remain invariant? The action of the transfer matrix on each patch is well-defined: it is the same as on the corresponding punctured tori. But once we glue the different patches together to recover Σ, the "time" directions on either side of the glue-lines can either be parallel to each other—which is acceptable—or opposite. The latter case does not lead to inconsistencies, since the opposite directions still agree on whether a certain bond is horizontal or vertical, which is all we need to deform the glue-lines without changing the weights.

Once we distribute the total weight of a polygon on its segments, and use the fact that there are two segments *per* lattice-plaquette, we obtain vertex configurations that have the same form and weights as the vertices of the six-vertex model, which renormalizes on a gaussian model [10]. But the six-vertex model has $c = 1$. How can we for-

mulate it on the surface of a cube, and end up with no background charges? The answer is that the six-vertex model is defined *locally* by its vertices. If we wish to formulate it on the surface of a cube, and subsequently open the surface, then the missing regions will have to be filled by *vertices* of weight one.

This means that the polygons of the six-vertex model do *not* all have same weight. Their weights are proportional to their line curvature: the total angle they enclose. This is related to the fact that in the fill-up regions, the vertices of weight one do not necessarily correspond to horizontal bonds. Therefore, in the corresponding polygon representation there will be open polygons, that terminate on the boundaries. Their weights are *not* $2\cos\left(\pi/(m+1)\right)$. Going back to Σ, the polygon weights can be consistently distributed on the available segments, with no left-over phases. In the presence of curvature there will be no background charges and the central charge is 1.

This is different from the situation in the RSOS models, which are defined by their height variables. In their random cluster formulation on has to consider entire clusters that have a definite *global* weight. In passing from that global definition of the weights, to the local definition of the weights of the six-vertex model, one obtains background charges at the conical singularities.

5 The soliton sectors:

Finally, we come to the non-contractible polygons. As shown in [9], each configuration of non-contractible polygons with a fixed assignment of heights has weight one. To obtain the full weight of a configuration, one has to sum over all height assignments that are consistent with the RSOS rules. We can think of a height as a colour, and refer to the number of ways one can assign consistent heights as the number of colourings. Let us outline how the mapping of the non-contractible polygons on the torus goes:

- Since the non-contractible polygons do not intersect, one can always perform a set of modular transformations to rearrange them in a convenient way. We will start from configurations that wind only around the a-cycle of the torus: Fig.10.

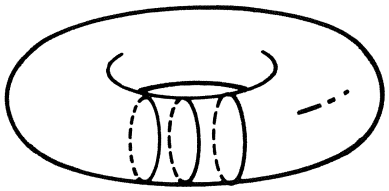

Figure 10: Non-contractible polygons winding on the a-cycle of the torus.

This set is complete—in the sense that we include *all* non-contractible polygons that wind around the a-cycle—but not modular invariant, since we do not include polygons that wind around the b-cycle too.

- Re-express the number of allowed RSOS height-assignments in terms of linear combinations of gaussian height-assignments.
- Perform modular transformations to obtain a modular invariant result.

A typical configuration on the torus is given in Fig.10. The total number of height assignments is given by the number of closed walks of length N on the corresponding Dynkin diagram. This can be computed using the adjacency matrix of the Dynkin diagram. The answer is the trace of the adjacency matrix raised to power N:

$$\sum_{m_j=1}^{m} \left(\cos m_j \pi/(m+1)\right)^N \tag{11}$$

Next, we follow [9] and re-express equ.(11) in terms of the number of walks, of length N, on the Dynkin diagrams of gaussian models. These are the diagrams of the affine $\hat{A}_{2n+1}$ groups, Fig.11: they have the topology of a circle with $2n+2$ nodes. (We choose to write things this way to simplify the discussion below.)

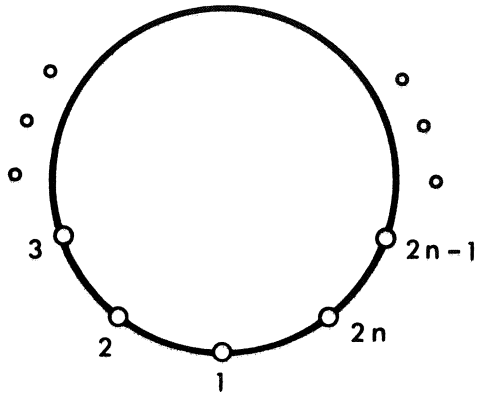

Figure 11: The Dynkin diagram of $\hat{A}_{2n-1}$.

The corresponding models are special cases of the six-vertex model that renormalizes on a gaussian model, compactified on a circle. The compactification radius is a function of the number of nodes we start with.

In the following paragraphs, we digress to motivate the proof given in [9]. The reader who is not interested in these combinatoric details may skip this part. Consider the number of closed walks length N (N-walks), on the A_3 Dynkin diagram. How can we re-express this in terms of walks on affine diagrams? Consider two disconnected copies of the A_3-diagram: Fig.12, and connect them as in Fig.13, where we have introduced two extra nodes.

Let us refer to these extra nodes as "poles". This is the Dynkin diagram of $\hat{A}_7$. The N-walks on A_3 now correspond to N-walks on $\hat{A}_7$, that remain on one half of the diagram, and do *not* touch the poles. Equivalently, they are equal to one-half of { the *total* number of N-walks on the $\hat{A}_7$ diagram − the number of N-walks that touch the poles }. The main point of the proof given in [9], is that the latter number is exactly equal to the number N-walks on the $\hat{A}_1$ diagram of Fig.14.

The reason is that there is a one-to-one corespondence between them: each walk on a $\hat{A}_{2n+1}$ diagram that touches one of the poles can be mapped *uniquely* on a walk on the $\hat{A}_1$ diagram, and *vice versa*. This goes as follows:

- A walk of length N on an affine Dynkin diagram can be described by:

  A starting point,

  A sequence of instructions to proceed serially: either clockwise ,

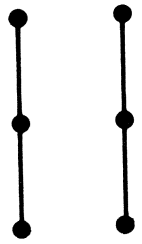

Figure 12: Two copies of the A_3 diagram.

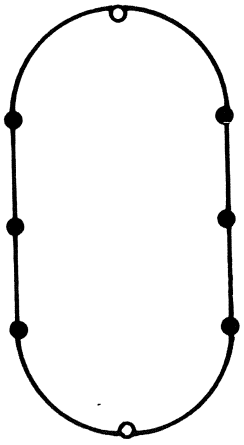

Figure 13: The $\hat{A}_7$ diagram

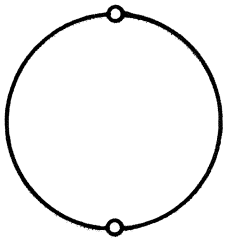

Figure 14: The $\hat{A}_1$ diagram.

to that we assign a numerical value $+1$, or counterclockwise: , to that we assign a numerical value -1.

Example: $+1, +1, -1, -1, -1, +1, \ldots$

Furthermore, the segments that constitute a walk can be classified by their topology, as follows:

- Segments that describe walks that are contractible to a point. They describe "fluctuations". The sum over their numerical values is 0.
- Segments that describe full turns around the affine Dynkin diagram, and as such are not contractible to a point. The sum over their numerical values is 0 mod (2n + 2).

Let us mention one more feature of the walks that touch a pole:

- Any closed walk on $\hat{A}_{2n+1}$ that touches a pole either:
- starts at that pole, or
- starts at another point, and touches the pole later on during the walk.

Remark: A walk that touches a pole more than once, or more than one pole, will be referred, in the following mappings, to the "first pole"it touches.

Let us map each of these walks separately.

- A walk that starts at a pole can be mapped on walk that starts on the corresponding pole on $\hat{A}_1$ (north corresponds to north, and south to south), and proceed, from there on, following the same sequence of instructions, that describe it on the $\hat{A}_{2n+1}$ diagram.

Example: $+1,+1,-1,-1,-1,+1,\ldots \rightarrow +1,+1,-1,-1,-1,+1,\ldots$

Remark: This type of walk is straightforward to map and, more importantly, is trivial to reconstruct given its image on $\hat{A}_1$, since one knows the starting point, and the sequence of instructions.

We turn now to the second type. The point is that all walks on the $\hat{A}_1$ diagram start at one of the poles. How can we encode the starting point on the original diagram? We proceed as folows:

- A walk on $\hat{A}_{2n+1}$ that touches a pole, but does not start at it, is divided into two segments, neither is separately a closed walk:

- A segment that starts at the starting point, and ends at the first pole the walk touches. The sum over the numerical values of the corresponding sequence is $x \neq 0$. Notice that this value is not "mod (2n + 2)", since we are referring to the *first pole* : there can be no intervening full turns. The distance d from the pole is $d = -x$.

- A second segment that starts from that first pole, and continues all the way to the end of the walk. The sum over the numerical values of the corresponding sequence is $-x$ mod (2n + 2).

As mentioned above, in order for the map to be invertible, we wish to encode the information about the starting point in the sequence of instructions that describe the walk on the $\hat{A}_1$ diagram. We will know the starting point, firstly: if we know which pole is the first pole, and secondly: if we know the sequence of steps from the starting point to that pole. We map as follows:

- The first segment is mapped on a walk on $\hat{A}_1$ that follows the original sequence of instructions *up to a revesal of the orientations given by the instructions.* That is, in the sequence of instructions that describe the first segment, we replace each instruction to move clockwise (+1) by one to move counterclockwise (−1), and *vice versa*:

Example: $+1,+1,-1,-1,-1,+1,\ldots \rightarrow -1,-1,+1,+1,+1,-1,\ldots$

The sum over the numerical values of the sequence is also inverted: $x \rightarrow -x$.

- On the other hand, the sequence that describes the second segment is mapped on a walk that starts from the point where the first segment ends, and proceeds faithfully according to the original instructions: *without orientation reversal.* The sum over the numerical values of the sequence does not change. The sum over the values of the entire sequence is $-2x$ mod (2n +2).

The above mapping from a walk on $\hat{A}_{2n+1}$ to a walk on $\hat{A}_1$ is unique. But is it invertible? Given a walk on $\hat{A}_1$, can one reconstruct the original walk on $\hat{A}_{2n+1}$. The answer is yes. To reconstruct the original walk, one proceeds as follows:

- A walk on $\hat{A}_1$ always starts at a pole: it has no where else to start from. The corresponding walk on $\hat{A}_{2n+1}$ either starts from the corresponding pole, or this is the first pole it touches. How can we find out the starting point on the $\hat{A}_{2n+1}$ diagram?

- Add up the numerical values of the sequence of instructions that describe the walk on the $\hat{A}_1$ diagram. There are two possible outcomes:

- If they add up to 0 mod (2n +2), then the walk on the $\hat{A}_{2n+1}$ starts indeed at the corresponding pole.

- If they do not add up to 0 mod (2n +2), then the sum is $-2x$ mod (2n +2). From that we compute x, and determine the position of the starting point with respect to the first pole. But this is not the whole story. We still have to re-invert the instructions that correspond to the first segment of the original walk that precedes the first pole.

- Consider the sequence of instructions that describe the walk on $\hat{A}_1$. Add up the corresponding numerical values *serially* from the beginning, up to the first time that sum reaches $-x$. Invert this partial sequence, then add it to the rest, to recover the original sequence that describes the walk on $\hat{A}_{2n+1}$. This completes the reconstruction.

In general:

$$2\,[\ \#\ \text{N-walks on } A_n\ ] = [\ \#\ \text{N-walks on } \hat{A}_{2n+1}\ ] - [\ \#\ \text{N-walks on } \hat{A}_1\ ]$$

Following [9], one can show that the $\hat{A}_{2n+1}$ Dynkin diagram corresponds to $Z_{gauss}\left(m/(m+1), m+1\right)$, while the $\hat{A}_1$ diagram corresponds to Z_{gauss}

$\left(m/(m+1),1\right)$. Re-expressed in terms of the partition functions, this is equ.(2).

The RSOS configurations are such that a closed walk on a homology cycle corresponds to a closed walk on the A_n diagram. Therefore, one never meets height discontinuities other than ± 1. But once we have translated everything in terms of walks on the affine diagrams, it is possible that a closed walk on a homology cycle traces a walk on the affine diagram that is topologically non-trivial: it winds around the circle that describes the Dynkin diagram. In the continuum limit, these will correspond to discontinuous field configurations. These are the "solitons".

Now we can consider the non-contractible polygons on multi-loop surfaces. For simplicity, we begin with genus-two:

We choose to start with a set of configurations that resembles the one-loop case most. Let us draw our Riemann surface in a way that emphasizes the fact that it is basically two *punctured* tori, with a connecting cylinder. We start from configurations that are restricted in two ways:

- No polygons wind around the b-cycles of the Riemann surface: This is not a drastic restriction.
- All polygons that wind around the connecting-cylinder are homologically trivial. This is the restriction that we will be most interested in.

We are left with configurations of the type shown in Fig.15.

These describe closed walks on the Dynkin diagrams that have the topologies shown in Fig.16.

In particular, we have not yet included the configurations in Fig.17.

They correspond to walks of the topology shown in Fig.18, on the Dynkin diagrams.

We will show that each of these restrictions can be removed by modular transformations.

Consider the configurations in Fig.15. We wish to express these in terms of walks on affine diagrams. The polygons that originally wind around the connecting cylinder cannot contribute to the discontinuities that form the solitons. The reason is that they are homologically trivial: there are no non-contractible paths that intersects them *once*. The fluctuations associated with them will contribute only to the quantum part of the partition function, but not to the classical part that we are now interested in. Thus one can

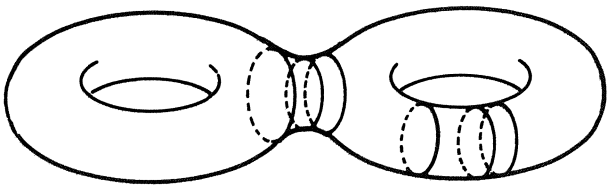

Figure 15: Non-contractible polygons on a $g = 2$ Riemann surface.

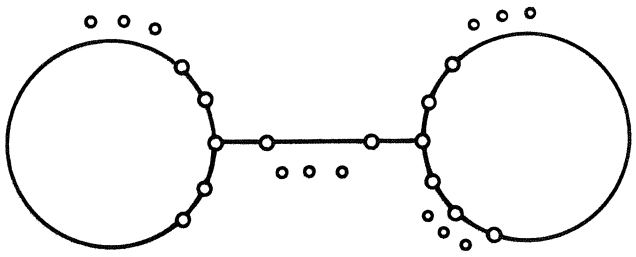

Figure 16: The topology of the walks of Fig.15 on the Dynkin diagram.

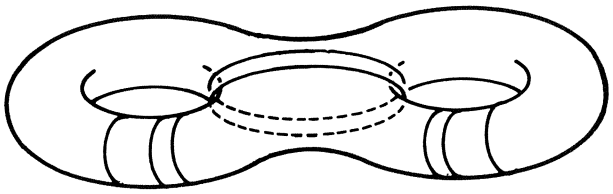

Figure 17: Non-contractible polygons on a $g = 2$ Riemann surface.

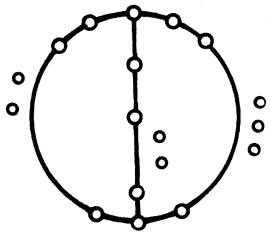

Figure 18: The topology of the walks of Fig.17 on the Dynkin diagram.

factor them out, and concentrate on the polygons that wind around the a-cycles.

But these are exactly the configurations that appear on the torus. Therefore, we can go ahead and re-express their number in terms of the soliton configurations of gaussian models: the set of all soliton configurations on a given a-cycle can be expressed in terms of the difference between the soliton configurations of two gaussian theories, just as in equ.(11). But here we do not have a single a-cycle, but two. Once we glue the patches, we take the product, and end up, up to a multiplicative factor of $\frac{1}{4}$, with *four* different types of soliton configurations:

+ [both handles carry solitons of $r = m(m+1)$]

− [the first handle carries solitons of $r = m(m+1)$, the second has solitons of $r = m/(m+1)$]

− [the reverse of the previous situation]

+ [both handles carry solitons $r = m/(m+1)$]

Is this division of configurations modular invariant? On a multi-loop Riemann surface, we can classify all modular transformations into two types:

- a. those which mix the a_i and b_i cycles that belong to the *same* handle.
- b. those which mix the cycles that belong to *different* handles.

Since the modular group is generated by the Dehn twists, on genus-two, we have five generators that can be represented in matrix form as

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad (12)$$

(In fact, only four of these are linearly independent, but that does not matter here.) The first four generators are of the type-a, mentioned above. On degenerating the two-loop surface along the surface-dividing cycle they reduce to the generators of the modular groups of the resultant tori. Acting with these on the solitons generated by the polygon configurations in Fig.15, we obtain the set of all possible solitons that live only on one handle or the

other, while the polygons on the connecting cylinders remain unchanged. This removes our first restriction on the polygons. We obtain the classical contribution

$$
\begin{aligned}
Z_{cl} = \; & \frac{1}{4}\left(\frac{m}{m+1}\right)^2 \times \\
& \Bigg\{ \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in(m+1)Z \\ \{m_2,n_2\}\in(m+1)Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right) \\
& - \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in(m+1)Z \\ \{m_2,n_2\}\in Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right) \\
& - \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z \\ \{m_2,n_2\}\in(m+1)Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right) \\
& + \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z \\ \{m_2,n_2\}\in Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right)\Bigg\}
\end{aligned}
\tag{13}
$$

However, the second and third terms are *not* modular invariant. The reason is that we have not yet included those configurations generated by the cycle-mixing fifth generator of the modular group: we have not yet removed the second restriction on the configurations we started with. This will not generate any new soliton configurations as far as the first and fourth terms in equ.(13) are concerned, but it will violate the separation of solitons that belong to two different compactification radii, in the second and third terms, to separate handles. This can easily be seen by picking a general configuration that belongs to the second or third terms in equ.(13), and performing a modular transformation that adds a cycle with a small-radius soliton to a cycle with a large-radius one. The result is always a configuration with small-radius solitons only. To be more explicit, take a general configuration from the second term in equ.(13). One can perform a series of modular transformations that makes all solitons on the first handle wind around the a-cycle a_1 , and those on the second handle wind around b_2. We can represent this configuration by the vector $\left(n_{a_1}, n_{a_2}, n_{b_1}, n_{b_2}\right) = \left(n_{a_1}, 0, 0, n_{b_2}\right)$, where $n_{a_1} = 0$ mod (m+1), $n_{b_2} \in Z$. Performing a type-b modular transformation, we obtain

$$\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} n_{a_1} \\ 0 \\ 0 \\ n_{b_2} \end{pmatrix} = \begin{pmatrix} n_{a_1} + n_{b_2} \\ -n_{b_2} \\ 0 \\ n_{b_2} \end{pmatrix} \tag{14}$$

Since $\{(n_{a_1} + n_{b_2}), (n_{b_2})\} \in Z$ it is clear that we have solitons belonging to the small compactification radius on both handles. Therefore, the second term in equ.(13) is not modular invariant. Not even in combination with the third term.

To obtain the fully modular invariant result we should include the orbits generated by the fifth modular transformation. The result is to turn the second and third terms into classical solutions with solitons compactified on the small radius. Each of these is identical to the last term but then with the opposite sign. Thus we obtain

$$\begin{aligned} Z_{cl} &= \frac{1}{4}\left(\frac{m}{m+1}\right)^2 \times \\ &\Bigg\{ \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in(m+1)Z \\ \{m_2,n_2\}\in(m+1)Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right) \\ &- \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z \\ \{m_2,n_2\}\in Z}} \exp\left(-\pi\left(\frac{m}{m+1}\right)(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right)\Bigg\} \end{aligned} \tag{15}$$

Performing a Poisson resummation, and generalizing to multi-loop surfaces, one can rewrite this as

$$\left(\frac{1}{2}\right)^g (\det \mathrm{Im}\tau)^{1/2} \times$$

$$\Bigg\{\sum_{\alpha\beta\gamma} \Theta\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\gamma 0\right)(0|2m(m+1)\tau)\overline{\Theta}\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\gamma 0\right)(0|2m(m+1)\tau)$$

$$-\sum_{\alpha\beta\gamma} \Theta\left(\tfrac{1}{2}\alpha+\tfrac{1}{2}\beta+\gamma 0\right)(0|2m(m+1)\tau)\overline{\Theta}\left(\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta+\gamma 0\right)(0|2m(m+1)\tau)\Bigg\} \tag{16}$$

where α, β, and γ, are g-dimensional vectors. Their components range over the same values indicated following equ.(3). The first term is the multi-loop classical part of a gaussian theory that has g $r^2 = m(m+1)$, the second has g $r^2 = m/(m+1)$: notice the relative minus sign in the characteristics of the second term.

Remark

The fact that the extension to multi-loop surfaces has the form given in equ.(15), can be easily deduced as follows: We are sewing punctured tori. Each contributes the difference of two terms: $(a-b)$. ewing n-tori one obtaines $n+1$ terms, with binomial coefficients, with alternating signs. The first has only large-radius solitons. The last has small-radius solitons. Both are modular invariant. The "cross"terms are mixed, and not modular invariant. Performing modular transformations turns all cross terms to terms that carry only small-radius solitons, as can be seen by considering any two handels, one with large solitons, and acting on the surface with modular transformations that mix them. Adding the contributions of all cross terms to the last, gives a small-radius term with coefficient -1: the sum over all binomial coefficients, with alternating signs, modulo the first, is -1. This is how we end up with the difference of two terms, for any genus.

Eq.(15) is our main result. In the full partition function, the classical solution will be evaluated in the presence of the background and screening charges, that will change the first argument of the theta functions, however the structure of equ.(15) remains the same.

6 The partition functions:

We are finally in a position to put everything together, and write down the partition functions. A minimal RSOS model is mapped, for any m and g, on a linear combination of cyclic RSOS models. Each of these describes a scalar field compactified on a circle, of a prescribed radius, and coupled to the background curvature, in the presence of screening charges that neutralize the background charge. The partition function of the minimal model is given by a linear combination of functional integrals that describe the scalar theories:

$$Z_{min}(m) = \int [d\varphi] \int \cdots \int \prod V_{\alpha_+} \prod V_{\alpha_-} \qquad (17)$$

$$\left\{ \exp - \left[(\frac{1}{4\pi}) m(m+1) \right] \left[\varphi'\varphi' + i \frac{R}{4\pi(m+1)} \varphi \right] \right\}$$

$$- \int [d\varphi] \int \cdots \int \prod V_{\alpha_+} \prod V_{\alpha_-}$$

$$\left\{ \exp - \left[(\frac{1}{4\pi}) m/(m+1) \right] \left[\varphi'\varphi' + i \frac{R}{4\pi(m+1)} \varphi \right] \right\}$$

where, following the functional integral over the quantum field ϕ, one has indicated the surface integrals over the positions of the screening charges. These functional integrals are the same as what one obtains in studies of the bosonized ghost system. They are evaluated in [17, 19]. The result is

$$\begin{aligned} Z^g(m) &= \left(\frac{1}{2}\right)^g \delta \left(\sum \alpha_i + 2\alpha_0(g-1) \right) |\det \bar{\partial}_0|^{-1} \\ &\times \int \cdots \int d^2 z_1 \ldots d^2 z_N \\ &\times \prod_{i<j} |E(z_i, z_j)|^{2\alpha_i \alpha_j} \prod_{n=1}^{N} |\sigma(z_i)|^{-4\alpha_0 \alpha_i} |F_*|^{-8\alpha_0^2} \sum_{\alpha\beta\gamma} A^g_{\alpha\beta\gamma} \overline{A}^g_{\alpha\beta\gamma} \\ &- \sum_{\alpha\beta\gamma} A^g_{\alpha\beta\gamma} \overline{A}^g_{\alpha(-\beta)\gamma} \end{aligned} \tag{18}$$

where

$$A^g_{\alpha\beta\gamma} = \Theta \begin{bmatrix} \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma \\ 0 \end{bmatrix} \left(\sum_i \alpha_i z_i + 2\alpha_0 \Delta | 2m(m+1)\tau \right) \tag{19}$$

and Δ is a divisor of degree $(g-1)$: the location of the zeros and the poles of the metric, and the position of the background charges $2\alpha_0$.

This is our final result. It consists of three parts:

- A quantum part due to the contractible polygons, and the topologically-trivial deformations of the nōn-contractible ones. This is proportional to $|\det \bar{\partial}_0|^{-1}$, where $\bar{\partial}_0 = \partial_{\bar{z}}$ [17, 20]. This includes the contributions of the non-contractible homologically-trivial polygons of Fig.15. They can be consistently absorped in the quantum part since it is r-independent.

- A classical part due to the non-contractible loops. It has a structure that decouples of null state

- Interactions terms due to the background charge left over by the contractible polygons that enclose curvature, and the screening charges required for charge neutrality.

The interactions of the holomorphic components of screening charges are given by the prime form $E(z_i, z_j)$: the exponential of the scalar propagator on a multi-loop Riemann surface:

$$E(z,w) = \frac{\Theta\left[\alpha_*\right](z-w|\tau)}{h(z)h(w)} \tag{20}$$

where α_* is an arbitrary odd characteristic:

$$\Theta\left[\alpha_*\right](-z|\tau) = -\,\Theta\left[\alpha_*\right](z|\tau)$$

and $h(z)$ is a half-differential, with zeros at a set of points $\{R_1, \ldots, R_{p-1}\} \equiv \{R_\alpha\}$. The mapping on a CG is defined up to insertions of these charges [6]. The correct number of insertions is determined by charge neutrality. Since the insertions can be anywhere on the surface, they should be integrated over. The interactions of the screening charges and the background charge are given by σ: a section of a trivial bundle, with no zeros, or poles:

$$\sigma(z) = \frac{h(z)}{\prod_{\alpha=1}^{g-1} E(z, R_\alpha)} \tag{21}$$

For details see [17, 20]. Since our lattice naturally defines a metric that is flat everywhere, up to singularities at isolated points, we included a term $|F_*|^{-4\alpha_0^2}$ that describes the interaction of the isolated background charges:

$$F_* = \frac{\prod_{\alpha<\beta} E(R_\alpha, R_\beta)}{\prod_\alpha h'(R_\alpha)} \tag{22}$$

Remarks

Starting from a *given* lattice, we are working with a *certain* period matrix τ, and a*certain* mertric g, defined by the lattice. Mapping on a CG, gives a continuum theory, with—strictly speaking—these τ and g. The expressions we obtain contain information about the general continuum theory—with arbitrary τ and g: the coupling constant, the compactification radius, the linear combinations one should take in order to decouple the null states, etc. Once we know all this, we identify the general continuum theory, and

proceed to write down our expressions in terms of an arbitrary τ and g. This is what we are implicitly doing all the time.

Eq.(17) has been obtained in a singular metric, however, as we know from [17], the same form holds for any metric, up to the appearence of a Liouville term. We do not wish to include this here, for simplicity.

It is not difficult to convince oneself that starting from different lattice patches, and gluing them with appropriate boundary conditions, one would end up with a minimal model on a Riemann surface with a different metric. For example, on the torus, our way of constructing the lattice would produce a metric with curvature at the corners, that add up to zero. On the other hand, starting from a planar lattice and identifying the boundaries, one obtaines a torus with a flat metric: we are not necessarily bound to singular metrics.

As explained in [6], all heights are measured relative to that of a reference point. In the CG picture, there appears a charge at the reference point equal to the sum over all charges on the surface, but with the opposite sign. We consider configurations where the net charge of all operators on the surface is zero. Thus the charge at the reference point is also zero. Furthermore, since in the oriented-polygon representation the Boltzmann weights are phases defined only up to the contribution of the screening charges, the latter can be moved in and out of a polygon without changing the weights.

Notice that the bosonized bc systems that appear in string theory also admit a polygon representation, but with non-rational m. Consequently, computing their partition functions using the method given in [17], does not give an expression in terms of a linear combination of gaussian expressions: only a single term.

We wish to check the consistency of equ.(18). We immediately see the following:

It is manifestly modular invariant: It is the difference between two gaussian-like correlation functions. Each of these is seperately manifestly modular invariant: i.e. contains a complete set of solitons, all with the same weight.

It does *not* have $c = 1$. This can be seen by looking at the ground state. The lowest energy state: the state with no solitons, cancels out in the subtraction of the two terms in equ.(18). Thus the true ground state is one with a non-trivial soliton, it has a higher energy, which can be absorped in a shift of the central charge, just as on the torus.

But what is the new central charge? Once again we follow what we have learnt from the study of these functional integrals in the context of

the bosonized ghost system [17]: The central charge is determined by the transformation properties of the functional integral in equ.(18) under Weyl transformations. This follows from the form of the action, and *does not* depend on the genus of the Riemann surface. Intuitively, this is due to the fact that the central charge can be understood as an anomaly, and therefore is independent of the surface one formulates the theory on. Instead of actually performing the variation of the action, we can obtain the result as follows: Since the background charge is

(the background charge on the sphere) $\times$ (1 – genus)

we see that the energy momentum tensor is the same as on the sphere, and we have the correct conformal anomaly. Notice that, at the level of equ.(18), the conformal anomaly does not receive contributions only from the quantum part, as in the gaussian model, but also from the interaction terms, which follow from the presence of the background charge, exactly as in the bosonized $b\ c$ systems.

Next we wish to check the decoupling of the null-states. For that we have to look at the spectrum of states that circulate in the handles. The natural way to do that would be to degenerate the surface on tori. There are two ways to degenerate a Riemann surface: by pinching a homologically-trivial cycle, or a non-trivial one [22, 23]. In either case on ends up with one or more punctured Riemann surfaces of lower genus, and a propagator corresponding to the cylinder we pinched. The problem with doing this here, is that we have to find a clear way to separate the leading terms from the subleading ones. This is not straightforward since what we have is an integral representation of the partition functions. Since the integrations are over the whole surface, they depend on the modular parameters. Therefore, one cannot separate the leading terms just by inspecting the integrands. Furthermore, performing the integrations explicitly requires first decomposing them into contour integrals, which is beyond the scope of this work.

We wish to go around this problem. To find the spectrum of states in the partition function, we should simply look at the states that flow through a handle. By modular invariance, they are all equivalent. Instead of actually degenerating the surface, in the sense of decomposing it into smaller parts, we will look at special configurations where states propagate through one handle only and nothing flows in the rest. Let us do this for a $g = 2$ surface.

We restrict the propagation of states to the first handle. This means that we take the second column of characteristics in equ.(18) to be null. Furthermore, we restrict the summation over $\{n_2\}$, the second integer summed over in the definition of the theta function, to $\{0\}$. This leaves us with the one-

loop theta functions. Next we shrink the second handle by a rescaling of the period matrix:

$$\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} \varepsilon\tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22}/\varepsilon \end{pmatrix} \sim \begin{pmatrix} 0 & 0 \\ 0 & \tau_{22} \end{pmatrix} \qquad (23)$$

The quantum part degenerates to that on the torus , and since all curvature is concentrated now in the infinitismal region where the first handle is attached to the second, the leading contributions to the integrals over the screening charges will also come from the integration region where the screening charges are close to the curvature. In the limit, where this region is infinitismally small, all charges are neutralized, the contribution of the infinitismal handle and the screening charges can be factored out and absorbed in an overall normalization N , and we are left with

$$N \frac{1}{\eta(q)\overline{\eta}(q)} \left(\sum_{\alpha\beta\gamma} A^1_{\alpha\beta\gamma} \overline{A}^1_{\alpha\beta\gamma} - \sum_{\alpha\beta\gamma} A^1_{\alpha\beta\gamma} \overline{A}^1_{\alpha(-\beta)\gamma} \right)$$

Up to the normalization factor, these are the torus partition functions of the corresponding minimal models. Since the torus partition functions have decoupled null-states, our two-loop partition functions also do. The same procedure extends to $g > 2$ surfaces, on a loop-by-loop basis, with the same conclusion. This, of course, should be expected. Our construction is basically a sewing of punctured tori, at the level of the lattice. Each torus has only physical states circulating around its handle. After the sewing, performing a set of modular transformations does not change that.

Finally we turn to a discussion of the degeneration on the surface-dividing cycle. What we expect to obtain is the sum over products of all allowed one-point functions on the torus. What we will do is rewrite equ.(15) in the form of a sum of two terms, neither of which is separately invariant under the full genus-two modular group. The first term will degenerate trivially to a product of one-loop partition functions, the second will be argued to be the contribution of one point functions on the torus.

Rewrite equ.(15) as

$$Z_{cl} = \frac{1}{4} \left(\frac{m}{(m+1)} \right)^2$$
$$\times \left\{ \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\} \in (m+1)Z, \\ \{m_2,n_2\} \in (m+1)Z}} \exp\left(-\pi \left(\frac{m}{m+1} \right)^2 (\vec{n} - \vec{m} \cdot \tau) \cdot (\mathrm{Im}\,\tau)^{-1} \cdot (\vec{n} - \overline{\tau} \cdot \vec{m}) \right) \right.$$

$$
\begin{aligned}
&- \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in(m+1)Z, \\ \{m_2,n_2\}\in Z}} \exp\left[-\pi\left(\frac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right] \\
&- \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z, \\ \{m_2,n_2\}\in(m+1)Z}} \exp\left[-\pi\left(\frac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right] \\
&+ \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z, \\ \{m_2,n_2\}\in Z}} \exp\left[-\pi\left(\frac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right] \\
&- \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z\neq(m+1)Z, \\ \{m_2,n_2\}\in Z}} \exp\left[-\pi\left(\frac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right] \\
&- \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z, \\ \{m_2,n_2\}\in Z\neq(m+1)Z}} \exp\left[-\pi\left(\frac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right]\Bigg\}
\end{aligned}
\tag{24}
$$

we degenerate by taking

$$
\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix} \Rightarrow \begin{pmatrix} \tau_{11} & \varepsilon \\ \varepsilon & \tau_{22} \end{pmatrix} \tag{25}
$$

the result is

$$
\begin{aligned}
&\left(\sum_{\alpha\beta\gamma} A^1_{\alpha\beta\gamma}\overline{A}^1_{\alpha\beta\gamma} - \sum_{\alpha\beta\gamma} A^1_{\alpha\beta\gamma}\overline{A}^1_{\alpha(-\beta)\gamma}\right)^2 - \frac{1}{4}\left(\frac{m}{(m+1)}\right)^2 \times \\
&\sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in Z\neq(m+1)Z, \\ \{m_2,n_2\}\in(m+1)Z}} \exp\left[-\pi\left(\tfrac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right] \\
&+ \sum_{\substack{\vec{m},\vec{n} \\ \{m_1,n_1\}\in(m+1)Z, \\ \{m_2,n_2\}\in Z\neq(m+1)Z}} \exp\left[-\pi\left(\tfrac{m}{m+1}\right)^2(\vec{n}-\vec{m}\cdot\tau)\cdot(\mathrm{Im}\,\tau)^{-1}\cdot(\vec{n}-\overline{\tau}\cdot\vec{m})\right]
\end{aligned}
\tag{26}
$$

The first term is the product of two one-loop partition functions—up to the quantum part. What about the rest? There is an arguement that indicates that the rest in eq.(26) are sub-leading one-point functions on the torus: The first term in eq.(26) has its origin in the diagrams shown in

Fig.15, and their transforms under type-a modular transformations, that do not mix the cycles of the different handles.

The second term is the result of all other configurations of the type shown in Fig.17. These are obtained through the action of type-b modular transformations, or sequences of transformations that include it. All configurations in Fig.15 have their domain walls on the connecting-cylinder orthogonal to the axis. In the degeneration limit, the connecting cylinder becomes a propagator, with domain walls positioned as explained above. In the continuum limit the domain walls turn into gradients of the scalar field.

In the case of the propagator of the diagrams in Fig.15, the gradient is in the direction of the axis of the propagator. If one looks at the circles where the final punctured tori attach to the propagator, one finds a *constant* field. This indicates a ground state of one of the sectors: the Verma modules, of the theory, i.e. a primary field. The degeneration of these configurations goes smoothly, and one picks up the leading term corresponding to the insertion of the identity operater. The situation on the configurations shown in Fig.17 is different.

The domain walls must run parallel to the axis of the connecting cylinder as a consequence of the modular transformations that generated them. This results in *non-constant* field configurations around the punctures of the final tori, which indicates that one has an insertion of an excited state. If one could evaluate their contributions explicitly, they would decay asymptotically faster than the identity, in the degeneration limit.

7 Conclusions

We have extended the CG representation of the A_n-series of $c < 1$ models to multi-loop compact Riemann surfaces. The expressions we obtained have a simple, though formal, structure, and present a natural extension of what we had on the sphere, and the torus. However, a lot of work remains to be done. Most importantly, one would like to perform an explicit degeneration of the partition functions obtained, and relate them directly to the torus partition functions. In particular, one would like to see how the degeneration along a surface-dividing cycle works. Degenerating on a primary field other than the identity would give us the correlation functions, which is the main motivation for this work. Probably, this will require re-expressing the surface integrals of the screening charges as sums over products of contour integrals, as a first step.

Furthermore, one wish to see what one learns from the constraints of modular invariance on multi-loop surfaces. We expect it to give us information about the fusion rules. The reason is that states flowing along the canonical b-cycles "fuse" as they go through handles that are common to two b-cycles. The fusion-states are determined by the modular transformation properties of conformal blocks that constitute the multi-loop partition function.

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ENTROPY AND MARKOV PROPERTIES

OF

EQUILIBRIUM STATES OF LATTICE MODELS

by

A.G. Schlijper

Koninklijke/Shell Exploratie en Produktie Laboratorium
Postbus 60, 2280 AB Rijswijk

ENTROPY AND MARKOV PROPERTIES OF EQUILIBRIUM STATES OF LATTICE MODELS

A.G. Schlijper
Koninklijke/Shell
Exploratie en Produktie Laboratorium
Rijswijk, The Netherlands

ABSTRACT

Entropy considerations can be used to establish Markov properties of equilibrium states of classical lattice models; conversely Markov properties can lead to convenient expressions for and bounds on the entropy and free energy densities of equilibrium states. These statements are illustrated.

1. INTRODUCTION

This presentation is concerned with the relation between entropy and Markov properties of equilibrium states of lattice models. The relation is quite an intimate one: it will be shown that entropy considerations can be used to establish Markov properties, and Markov properties can be used to establish convenient expressions for (and estimates of) the entropy density (i.e. entropy per lattice site) of equilibrium states.

The mathematical details below embellish a physical picture that is quite intuitive and easily explained. Consider an Ising model from the binary alloy point of view. One way of producing an equilibrium situation is to start with a small crystal and to let A and B atoms settle, one at a time, onto the surface according to their preferences. These preferences are expressed by conditional probabilities, i.e. the probability of an A (or B) atom settling on the crystal's surface is conditioned by what is there already. In this setup the equilibrium situation is produced by a dynamical process for which the equilibrium state is the stationary state. The natural entropy concept is now a conditional entropy, which measures the randomness associated with adding yet another site to the lattice, given the occupation

of the existing part. In the thermodynamic limit of an infinitely large system, this conditional entropy is equal to the entropy density (entropy per lattice site) of the thermodynamic equilibrium state of the lattice. This thermodynamic equilibrium state turns out to have a certain Markov property: only the boundary layer of the existing lattice contributes to the conditioning; the depth of the boundary layer is determined by the range of the interaction. As a result, the entropy density can be expressed as the conditional entropy of one site with respect to an infinite boundary layer. Convenient estimates may be obtained by restricting the conditioning to a finite (even small, of the order of ten sites) part of the boundary layer.

It is intuitively clear that such an estimate is actually an upper bound: in conditioning with respect to only a part of the boundary layer, fewer restrictions are placed on the freedom with which the new site may be occupied, and this results in more randomness and a larger entropy.

The details of the picture sketched above will be filled in below. We start by establishing terminology and notation.

2. PRELIMINARIES

Consider a classical spin system on Z^d with a translation invariant interaction-round faces potential $\Phi = \{\Phi_X : X \subset Z^d, X \subset$ a unit d-cube$\}$ on the configuration space $\Omega = \Omega_o^{Z^d}$, where Ω_o is a finite set and $\Phi_X : \Omega_o^X \to R$ are real functions. For $\Lambda \subset Z^d$ we denote a configuration on Λ by σ_Λ, i.e. $\sigma_\Lambda \subset \Omega_\Lambda$. We write $\sigma_{Z^d} \equiv \sigma \in \Omega$. Whenever we refer to two configurations σ_Λ on Λ and $\sigma_{\Lambda'}$ on Λ' at the same time it will be understood that these configurations agree on $\Lambda \cap \Lambda'$. The set of continuous functions on Ω that depend only on spins in Λ is denoted by C_Λ.

A <u>state</u> is a probability measure and the set of translation invariant states is denoted by I. When μ is a state on Ω we denote its projection on Ω_Λ by μ_Λ, and we will sometimes refer to μ as an extension of μ_Λ. We denote the conditional measure for a state μ given the configuration $\sigma_\Lambda \in \Omega_\Lambda$ by $\mu(.|\sigma_\Lambda)$ and the projection of this measure on Λ' by $\mu_{\Lambda'}(.|\sigma_\Lambda)$.
When $\Lambda \subset Z^d$, a state ν_Λ on Ω_Λ will be called (locally) translation invariant [12] if $f, g \in C_\Lambda$ and $\mu(f) = \mu(g)$ for all $\mu \in I$ imply that $\nu_\Lambda(f) = \nu_\Lambda(g)$.

Here $\mu(f)$ stands for $\int d\mu f$. Note that if $\mu \in I$ then μ_Λ is locally translation invariant. The set of locally translation invariant states on Ω_Λ will be denoted by I_Λ.

A state μ is a Gibbs state for the potential Φ if it satisfies the DLR equations,

$$\mu_\Lambda(\sigma_\Lambda | \sigma_{\Lambda^c}) = \frac{1}{Z_{\sigma_{\Lambda^c}}} \exp\left[-H_\Lambda(\sigma_\Lambda) - W_\Lambda(\sigma_\Lambda, \sigma_{\Lambda^c})\right] \qquad (1)$$

for each finite $\Lambda \subset Z^d$, where Λ^c is the complement of Λ in Z^d, H_Λ is the Hamiltonian

$$H_\Lambda(\sigma_\Lambda) = \sum_{X \subset \Lambda} \Phi_X(\sigma_X)$$

and W_Λ is the function on $\Omega_\Lambda \times \Omega_{\Lambda^c}$ defined by

$$W_\Lambda(\sigma_\Lambda, \sigma_{\Lambda^c}) = \sum_{\substack{X : X \cap \Lambda \neq \phi \\ X \cap \Lambda^c \neq \phi}} \Phi_X(\sigma_X)$$

$Z_{\sigma_{\Lambda^c}}$ is a constant determined by the normalisation

$$\sum_{\sigma_\Lambda \in \Omega_\Lambda} \mu(\sigma_\Lambda | \sigma_{\Lambda^c}) = 1.$$

For background see e.g. [1, 2].

If one considers the conditional expectation given the spins in $\Lambda' \subset Z^d$, $E(.|\Lambda')$, then the DLR equations immediately imply that a Gibbs state has the following Markov property: for every finite $\Lambda \subset Z^d$ and any function f_Λ that depends only on spins in Λ,

$$E(f_\Lambda | \Lambda^c) = E(f_\Lambda | \partial\Lambda) \qquad (2)$$

where the "border" $\partial\Lambda$ of Λ is

$$\partial\Lambda = \{x \in Z^d \mid \inf_{y \in \Lambda} \sup_{i=1,..,d} |x_i - y_i| = 1\} \qquad (3)$$

Loosely speaking the border of Λ consists of those sites outside Λ that share an interaction with Λ; the Markov property states that fixing the spins in the border of Λ completely determines the state on Λ and that no influence from outside the border region propagates to reach Λ.

Gibbs states thus have the Markov property (2) for all finite Λ. A Markov property for some finite Λ is called a local Markov property. A Markov property that holds for an infinite Λ is called a global Markov property. Since Gibbs states have local Markov properties, one is naturally inclined to try and establish global Markov properties by "continuity" arguments, i.e. by letting a finite Λ grow to infinite size while maintaining property (2).

This, however turns out to be a rather subtle procedure, which cannot always be completed successfully [3, 4]. Indeed, Israel has shown that global Markov properties may fail even for extremal Gibbs states that are translation invariant [4].

Here we employ a technique for proving global Markov properties that is not based directly on local Markov properties. It can be used for translation invariant (or periodic) Gibbs states, since it is based on the variational principle, which states that equilibrium states (i.e. translation invariant Gibbs states) minimise the free energy. A new result, obtained with this approach, is that there is a global Markov property that is satisfied by equilibrium states in general. The line of reasoning extracts information on Markov properties from information on conditional measures, which in turn is obtained from information on conditional entropies, which is obtained from the variational principle [5, 6]. So first we shall proceed to state this variational principle.

For Λ finite the entropy of a state μ, $S_\Lambda(\mu)$, is

$$S_\Lambda(\mu) = - \sum_{\sigma_\Lambda \in \Omega_\Lambda} \mu_\Lambda(\sigma_\Lambda) \log \mu_\Lambda(\sigma_\Lambda) \qquad (4)$$

For $\mu \in I$ the entropy density $s(\mu)$ is

$$s(\mu) = \lim_{\Lambda \to Z^d} \frac{1}{|\Lambda|} S_\Lambda(\mu) \tag{5}$$

where the limit is taken in the van Hove sense and $|\Lambda|$ is the number of lattice sites in Λ. The free energy for the finite volume Λ, $F_\Phi(\Lambda)$, is defined in terms of the Hamiltonian by

$$F_\Phi(\Lambda) = - \log \Big(\sum_{\sigma_\Lambda \in \Omega_\Lambda} \exp(- H_\Lambda(\sigma_\Lambda)) \Big) \tag{6}$$

(Note that we adopt the convention $\beta = (kT)^{-1} = 1$.) The free energy density f_Φ is given by

$$f_\Phi = \lim_{\Lambda \to Z^d} \frac{1}{|\Lambda|} F_\Phi(\Lambda) \tag{7}$$

Define furthermore the function e_Φ by

$$e_\Phi(\sigma) = \sum_X^* \Phi_X(\sigma_X) \tag{8}$$

where Σ^* means that the sum runs over those sets that have $\underline{0} = (0,..,0) \in Z^d$ as their last element in the lexicographic ordering of Z^d. Recall that in the lexicographic order $x < y$ for $x,y \in Z^d$ means: $(x_1 < y_1)$ or $(x_1 = y_1$ and $x_2 < y_2)$ or ... or $(x_1 = y_1$ and ... and $x_{d-1} = y_{d-1}$ and $x_d < y_d)$. The expectation value $\mu(e_\Phi)$ in a translation invariant state μ is then the energy density of this state,

$$\mu(e_\Phi) = \lim_{\Lambda \to Z^d} \frac{1}{|\Lambda|} \mu(H_\Lambda) \tag{9}$$

Since we consider a fixed potential, we often drop the index Φ.

Translation-invariant Gibbs states are characterised by the following variational principle:

Theorem 1 (Variational Principle). For any $\mu \in I$

$$\mu(e_\Phi) - s(\mu) \geq f_\Phi$$

and equality holds if and only if μ is a Gibbs state for the interaction Φ. (For a proof see e.g. [1, 2]).

For Λ finite the conditional entropy given $\Lambda' \subset \mathbb{Z}^d$ of a state μ is defined as

$$S_{\Lambda|\Lambda'}(\mu) = \int \mu_{\Lambda'}(d\sigma_{\Lambda'}) \, S_\Lambda(\mu(.|\sigma_{\Lambda'})) \tag{10}$$

with $S_{\Lambda|\phi}(\mu) = S_\Lambda(\mu)$. This implies that for Λ and Λ' finite

$$S_{\Lambda|\Lambda'}(\mu) = S_{\Lambda \cup \Lambda'}(\mu) - S_{\Lambda'}(\mu) \tag{11}$$

The essential connection between conditional entropy and conditional measures is contained in the following lemma:

Lemma 1 Conditional entropy has the monotonicity property

$$\Lambda'' \subset \Lambda' \implies S_{\Lambda|\Lambda''}(\mu) \geq S_{\Lambda|\Lambda'}(\mu)$$

Equality holds if and only if

$$\mu_\Lambda(.|\sigma_{\Lambda'}) = \mu_\Lambda(.|\sigma_{\Lambda''})$$

(μ - almost surely).

Proof By Jensen's inequality

$$-\mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda''}) \log \mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda''})$$

$$\geq - \int d\mu_{\Lambda'}(d\sigma_{\Lambda'}|\sigma_{\Lambda''}) \, (\mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda'}) \log \mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda'})) \tag{*}$$

The inequality of the lemma is obtained by summing (*) over $\sigma_\Lambda \in \Omega_\Lambda$ and integrating with respect to $\mu_{\Lambda''}(d\sigma_{\Lambda''})$.

If the equality of the lemma is an equality, then (*) has to be an equality μ - a.s. Hence $\mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda'})$ is constant (a.s.) with respect to $\mu_{\Lambda'}(d\sigma_{\Lambda'}|\sigma_{\Lambda''})$ and therefore equal to its expectation value for this measure, which is $\mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda''})$ and therefore equal to its expectation value for this measure, which is $\mu_\Lambda(\sigma_\Lambda|\sigma_{\Lambda''})$. (Q.E.D.)

The statement of this lemma is actually not very surprising: it says that more conditioning removes degrees of freedom and thus reduces the entropy, and that this entropy reduction is zero if and only if the additional conditioning is irrelevant. The possibility of a connection with Markov properties is very obvious.

As the last part of this section we recall the connection between conditional entropy and the entropy density that figures in the variational principle (see e.g. [7, 8]). Order Z^d lexicographically and define

$$Z^d_- = \{x \in Z^d : x < \underline{0} = \{0,..,0\} \notin Z^d\}$$

and $O = \{\underline{0} \in Z^d\}$.

Theorem 2 For $\mu \in I$

$$s(\mu) = S_{O|Z^d_-}(\mu)$$

Proof. Let $V_n = \{|x_i| < n;\ i=1,..,d\}$ for $n \in N$ and let $V_n(y) = V_n \cap \{x<y\}$. Define $\bar{s} = S_{O|Z^d_-}(\mu)$.

Since $\mu_O(.|\sigma_{V_n(\underline{0})}) \to \mu_O(.|\sigma_{Z^d_-})$ for $n \to \infty$ (by monotonicity of the conditional entropy), $\bar{s} = \inf_n S_{O|V_n(\underline{0})}(\mu)$ and since μ is translation invariant $\bar{s} \leq S_{\{y\}|V_n(y)}(\mu)$.

Now

$$S_{V_n}(\mu) = \sum_{y \in V_n} S_{\{y\}|V_n(y)}(\mu)$$

Therefore

$$s(\mu) = \lim_{n\to\infty} \frac{1}{|V_n|} S_{V_n}(\mu) \geq \bar{s}$$

On the other hand take $N \in \mathbb{N}$ and consider

$$W_n = \{y \in V_n \mid \tau V_n(y) \supset V_N(\underline{0}) \text{ for some translation } \tau\}$$

Then $\lim_{n\to\infty} |W_n|/|V_n| = 1$ and $S_{\{y\}|V_n(y)}(\mu) \leq S_{O|V_n(\underline{0})}(\mu)$ for $y \in W_n$. Thus

$$s(\mu) = \lim_{n\to\infty} \frac{S_{V_n}(\mu)}{|V_n|} \leq S_{O|V_N(\underline{0})}(\mu)$$

Hence also $s(\mu) \leq \bar{s}$, so $s(\mu) = \bar{s}$. (Q.E.D.)

All the ingredients for the argument have been prepared now: theorem 1 can be used to get information on a state's entropy density, theorem 2 relates this to a conditional entropy, and lemma 1 is the vehicle to go to conditional measures and thus to Markov properties. The way the ingredients are put together is roughly as follows: from a given equilibrium state μ a state ν is constructed in such a way that

1) ν has the desired Markov property,
2) the entropy density of ν, $s(\nu)$, is not smaller than that of μ, $s(\mu)$ and
3) ν and μ have the same energy density.

Theorem 1 then establishes that ν is an equilibrium state with a global Markov property. Theorem 2 and lemma 1 are used to extract information on μ from the knowledge on ν. Obviously, in the case where there is only one equilibrium state, it follows that $\mu = \nu$.

To illustrate the procedure we shall discuss the one-dimensional case in the next section. There, the crucial element is an extension theorem that was first given (to our knowledge) by Brascamp [9] (see also [10, 11]).

3. THE ONE-DIMENSIONAL CASE

Consider the one-dimensional lattice Z. We denote the set of n consecutive lattice sites $\{-i,-i-1,..,-i-n+1\}$ by L_n^i. Often the specific position of such a set in the lattice will be irrelevant due to translational invariance; in those cases we shall omit the location index i and denote any set of n consecutive sites by the symbol L_n.

Theorem 3. For any $\rho \in I_{L_n}$ there is precisely one $\mu \in I$ such that

(i) μ is an extension of ρ, i.e. $\mu_{L_n} = \rho$

(ii) $s(\mu) = S_{L_n}(\mu) - S_{L_{n-1}}(\mu)$.

Proof. Let $\rho \in I_{L_n}$, i.e. ρ is a (locally) translation invariant state on the configurations of n consecutive lattice sites. Consider the associated density functions $\rho[L_n]$ and $\rho[L_{n-1}]$, i.e. $\rho[L_n](\sigma_{L_n})$ is the probability of the configuration σ_{L_n}, and $\rho[L_{n-1}]$ is similarly the density function associated with the restriction of ρ to $\Omega_{L_{n-1}}$. Note that the property of (local) translation invariance implies the following: if A and B are subsets of L_n and A is a translate of B, then the restrictions of ρ to Ω_A and Ω_B are the same (apart from a trivial isomorphism). For this reason we can speak of the density function $\rho[L_{n-1}]$.

Now, using the density functions $\rho[L_n]$ and $\rho[L_{n-1}]$ we define density functions $\mu[L_{n+m}]$ for sets L_{n+m}, for m=1,2,..., by

$$\mu[L_{n+m}](\sigma) = \frac{\prod_{k=0}^{m} \rho[L_n](\sigma_{L_n^{i+k}})}{\prod_{k=1}^{m} \rho[L_{n-1}](\sigma_{L_{n-1}^{i+k}})} \qquad (12)$$

for all $\sigma \in \Omega_{L_{n+m}^{i}}$, with i arbitrary.

For configurations for which the denominator is zero the numerator is zero as well and we define the quotient to be zero. One easily verifies by explicit calculation that $\mu[L_{n+m}]$, thus defined, is a properly normalised density function and that the set of $\mu[L_{n+m}]$, m=0,1,2,... is compatible in the sense that $\mu[L_{n+m+1}]$ may be reduced to $\mu[L_{n+m}]$ by summation over the spin configurations on $L_{n+m+1} \setminus L_{n+m}$. Hence this set of density functions defines a state μ on Ω, which is a translation-invariant extension of ρ by construction. Note that this is just the construction of a Markov chain from the local measure ρ.

Computing the entropy $S_{L_{n+m}}(\mu)$ from the definition of $\mu[L_{n+m}]$ one finds

$$S_{L_{n+m}}(\mu) = (m+1)\, S_{L_n}(\rho) - m\, S_{L_{n-1}}(\rho)$$

so that

$$s(\mu) = S_{L_n}(\rho) - S_{L_{n-1}}(\rho) = S_{L_n}(\mu) - S_{L_{n-1}}(\mu) \,.$$

Remains to prove the uniqueness of the extension. This we shall do by showing that the construction in Eq. (12) is actually imposed by the requirement

$$s(\mu) = S_{L_n}(\mu) - S_{L_{n-1}}(\mu) \qquad (13)$$

Eq. (11), lemma 1 and theorem 2 combine to give for any $\mu \in I$

$$s(\mu) \leq S_{L_{m+1}}(\mu) - S_{L_m}(\mu) \leq S_{L_m}(\mu) - S_{L_{m-1}}(\mu) \tag{14}$$

with equality holding if and only if for all i

$$\mu[L^i_{m+1}] = \frac{\mu[L^i_m] \cdot \mu[L^{i+1}_m]}{\mu[L^{i+1}_{m-1}]} \tag{15}$$

Imposing Eq. (13) thus implies that Eq. (15) must hold for all $m \geq n$, and this leads to the construction of Eq. (12).

(Q.E.D.)

Theorem 3 can be used to show that for a one-dimensional system with a translation invariant interaction of finite range any equilibrium state is a Markov chain. Indeed, let us consider without loss of generality a translation invariant nearest-neighbour interaction Φ. Let μ be an equilibrium state for Φ. Consider the restriction of μ to Ω_{L_2}, μ_{L_2}. According to theorem 3 there is a unique extension in I, say ν, with $s(\nu) = S_{L_2}(\nu) - S_{L_1}(\nu)$. Since $\nu_{L_2} = \mu_{L_2}$ we have $\nu(e_\Phi) = \mu(e_\Phi)$, i.e. both states have the same energy density, and also, according to Eq. (14),

$$s(\mu) \leq S_{L_2}(\mu) - S_{L_1}(\mu) = S_{L_2}(\nu) - S_{L_1}(\nu) = s(\nu) \tag{16}$$

Since μ is an equilibrium state, the variational principle (theorem 1) yields

$$f_\Phi = \mu(e_\Phi) - s(\mu) \leq \nu(e_\Phi) - s(\nu) \tag{17}$$

hence

$$s(\mu) \geq s(\nu) \tag{18}$$

Combining Eqs. (16), (17) and (18) gives that $s(\mu) = s(\nu)$, that ν is an equilibrium state, and that μ, like ν, satisfies

$$s(\mu) = S_{L_2}(\mu) - S_{L_1}(\mu) \tag{19}$$

But we have seen in the proof of theorem 3 that this property is sufficient to impose the Markovian structure of Eq. (12) on μ. Since $\nu_{L_2} = \mu_{L_2}$, this also now implies that μ and ν must be identical.

Global Markov properties for μ are obvious. In particular we have shown that any equilibrium state has the global Markov property with respect to Z_-. This is implicit in the construction of ν, but also a global Markov property is implicit in Eq. (19). Indeed, by theorem 2 $s(\mu) = S_{0|Z_-}(\mu)$, so using translation invariance to conveniently position L_2 and L_1 we have by Eq. (19)

$$\begin{aligned} S_{0|Z_-}(\mu) &= S_{L_2^0}(\mu) - S_{L_1^{-1}}(\mu) \\ &= S_{0|\{-1\}}(\mu) \\ &= S_{0|\partial((Z_-)^c)}(\mu) \end{aligned}$$

and by lemma 1 this implies the global Markov property

$$\mu_0(.|\sigma_{Z_-}) = \mu_0(.|\sigma_{\partial Z_-^c}) \tag{20}$$

It should be noted that Eq. (20) is a restricted version of a global Markov property, in the sense that it expresses equality of restrictions to Ω_0 of conditional measures.

The next section will present a generalisation of the above discussion to the d-dimensional case. The generalisation is accomplished by utilising the "quasi-one-dimensional" structure that is imposed on Z^d by the lexicographic order relation. Technically this works because it is a total ordering of the lattice that is invariant under lattice translations.

4. MARKOV PROPERTIES AND ENTROPY

The technical tool that puts entropy considerations and the variational principle in tandem can be formulated as follows.

Lemma 2. Let μ be an equilibrium state (i.e. a translation invariant Gibbs state) for Φ and let ν be a translation-invariant state with the same energy density as μ, i.e. $\nu(e_\Phi) = \mu(e_\Phi)$. Let $\Lambda \subset \mathbb{Z}^d$ be such that

(a) $\nu_{\Lambda \cup O} = \mu_{\Lambda \cup O}$ (recall $O = \{\underline{0}\} \in \mathbb{Z}^d$)

(b) $\nu_O(.|\sigma_{\mathbb{Z}^d}) = \nu_O(.|\sigma_\Lambda)$ (ν - a.s.)

Then

(a) ν is an equilibrium state for Φ

(b) $\mu_O(.|\sigma_{\mathbb{Z}^d}) = \mu_O(.|\sigma_\Lambda)$ (μ - a.s.)

Proof: By the variational principle

$$\nu(e_\Phi) - s(\nu) \geq \mu(e_\Phi) - s(\mu) = f_\Phi.$$

Therefore $s(\nu) \leq s(\mu)$. Hence

$$s(\nu) \leq s(\mu) = S_{O|\mathbb{Z}^d}(\mu) \leq S_{O|\Lambda}(\mu)$$

$$= S_{O|\Lambda}(\nu) = S_{O|\mathbb{Z}^d}(\nu) = s(\nu)$$

where we used theorem 2 twice and also the monotonicity property of the conditional entropy, lemma 1. Thus we obtain $s(\nu) = s(\mu)$, which by the

variational principle implies that ν is an equilibrium state, and we obtain $S_{0|Z^d_-}(\mu) = S_{0|\Lambda}(\mu)$, which by lemma 1 implies that $\mu_0(.|\sigma_{Z^d_-}) = \mu_0(.|\sigma_\Lambda)$.

(Q.E.D.)

To apply lemma 2 we must, for an equilibrium state μ, supply a state ν with the required properties. This will typically be done by construction of a Markov chain from a restriction of μ. Actually, to arrive at Theorem 4 below, such a Markovian construction must be iterated. To present the details, we introduce some special subsets of Z^d. Denote the translation over one lattice spacing in the k-th direction by τ_k. Write $\partial^*\Lambda$ for $\partial(\Lambda^c)$. For k = d,..,1, let ρ_k be the "projection" $\rho_k : Z^d \to Z^k$:

$$\rho_k x = \rho_k(x_1,..,x_d) = (x_1,..,x_k) \in Z^k$$

and $\rho_0 x = 0 \in \{0\} \equiv Z^0$. Furthermore we introduce the notation [ ] as $[\rho_k x < \rho_k y] = \{x \in Z^d : \rho_k x < \rho_k y\}$ and $[\rho_k y] = \{x \in Z^d : \rho_k x = \rho_k y\}$. Now, we define for k = d,..,0 and $y \in Z^d$ the set $\Delta^k_y \subset Z^d$ as

$$\Delta^k_y = \partial^*[\rho_k x < \rho_k y] \cup [\rho_k y]$$

Note $\Delta^0_y = Z^d$ and that for $k \geq 1$

$$\Delta^{k-1}_y = \bigcup_{n=-\infty}^{+\infty} \tau_k^n \Delta^k_y .$$

Next we introduce

$$C^{k,+} = \bigcup_{n=0}^{\infty} \tau_k^n \Delta^k_{\underline{0}}$$

$$C^{k,-} = \bigcup_{n=-1}^{\infty} \tau_k^n \Delta^k_{\underline{0}}$$

It is very enlightening to work out these sets for d = 1, 2 and 3. The crucial extension theorem now is the following.

Lemma 3. Let $k \in \{1,..,d\}$. Let ν^k be a translation invariant state on $\Omega_{\Delta_{\underline{0}}^k}$. Then there exists a unique translation invariant state ν^{k-1} on $\Omega_{\Delta_{\underline{0}}^{k-1}}$ such that

(a) $\nu^{k-1}_{\Delta_{\underline{0}}^k} = \nu^k$

(b) $\nu^{k-1}_{[\rho_k \underline{0}]}(.|\sigma_{C^{k,-}}) = \nu^{k-1}_{[\rho_k \underline{0}]}(.|\sigma_{\partial^*[\rho_k x < \rho_k \underline{0}]})$.

Proof: Define $X^n = \rho_k^{-1}\{(0,..,0,n) \in Z^k\}$ and write

$$D^n = \partial^*[\rho_k x < \rho_k \underline{0}] \cup \{X^0 \cup .. \cup X^{n-1}\}$$

$$= \bigcup_{m=0}^{n} \tau_k^m (\partial^*[\rho_k x < \rho_k \underline{0}])$$

The state ν^k may be extended by means of the construction of a Markov chain to a measure ν^+ on $\Omega_{C^{k,+}}$ by defining inductively the projections $\nu^{+,n}$ of ν^+ on Ω_{D^n} for n = 0,1,2,... as follows:

1. $\nu^{+,0} \equiv (\nu^k)_{\partial^*[\rho_k x < \rho_k \underline{0}]}$

2a. $\nu^{+,n}_{D^{n-1}} \equiv \nu^{+,n-1}$

2b. $\nu^{+,n}_{X^{n-1}}(.|\sigma_{D^{n-1}}) \equiv \nu^k_{X^0}(\tau_k^{-n}.|\tau_k^{-n}\sigma_{\tau_k^n \partial^*[\rho_k x < \rho_k \underline{0}]})$

This is the same construction as in Eq. (12): conditional probabilities defined by ν^k are used in combination with translational invariance to

extend the measure ν^k to measures $\nu^{+,n}$. The projections $\nu^{+,n}$ define the translational invariant measure ν^+. The unique measure ν^{k-1} that satisfies the conditions of the lemma is obtained from translation of ν^+.

(Q.E.D.)

The construction of lemma 3 is just a generalisation of the construction in theorem 3. This may be more obvious if one notes that $\Delta_{\underline{0}}^1 = \{x_1 = 0,-1\}$ and thus when k = 1 the state $\nu^{k-1} = \nu^0$ constructed in lemma 3 is precisely the Markov chain arising from $\nu^k = \nu^1$. In other words, for k=1, d=1 the lemma gives the construction of Brascamp's theorem.

We shall iterate lemma 3 to prove our main theorem:

Theorem 4. Any translation-invariant Gibbs state μ for the potential Φ has the Markov property

$$\mu_0(.|\sigma_{z_-^d}) = \mu_0(.|\sigma_{\partial^* z_-^d})$$

and its entropy density is given by

$$s(\mu) = S_{0|\partial^* z_-^d}(\mu) \ .$$

Proof. We shall use lemma 2 to arrive at the first statement of the theorem; the expression for the entropy density then follows from this result and theorem 2. To exploit lemma 2, we shall use lemma 3 to construct a translation invariant state ν on Ω such that

(a) $\nu_{\Delta_{\underline{0}}^d} = \mu_{\Delta_{\underline{0}}^d}$

(b) $\nu_0(.|\sigma_{z_-^d}) = \nu_0(.|\sigma_{\partial^* z_-^d})$ (ν - a.s)

Put $\nu^d = \mu_{\Delta_{\underline{0}}^d}$ and iterate lemma 3 to construct the sequence of states ν^k on $\Omega_{\Delta_{\underline{0}}^k}$ for k = d,d-1,...,0. We contend that $\nu \equiv \nu^0$ is the derived state. It is obvious that ν is translation invariant and satisfies the requirement (a). Remains to prove that ν satisfies (b). Requirement (b) is the following assertion, A(m), for m=d:

$$A(m) : \nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{[\rho_m x < \rho_{m-1}\underline{0}]}) = \nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{\partial^*[\rho_m x < \rho_{m-1}\underline{0}]}) .$$

We prove A(m) for m = 0,..,d by induction on m. A(0) is obviously true. Assume A(m-1) is true for $m \in \{1,..,d\}$.
Then in particular spins in $Y^{m-1} \equiv [\rho_m x < \rho_{m-1}\underline{0}] \setminus [\rho_{m-1} x < \rho_{m-1}\underline{0}]$ are independent of those in $[\rho_{m-1} x < \rho_{m-1}\underline{0}]$ upon conditioning on the spins in $\partial^*[\rho_{m-1} x < \rho_{m-1}\underline{0}]$, and therefore, since

$$[\rho_m x < \rho_{m-1}\underline{0}] = [\rho_{m-1} x < \rho_{m-1}\underline{0}] \cup Y^{m-1} \text{ and } C^{m,-} = \partial^*[\rho_{m-1} x < \rho_{m-1}\underline{0}] \cup Y^{m-1},$$

we have from the induction assumption A(m-1) that

$$\nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{[\rho_m x < \rho_{m-1}\underline{0}]}) = \nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{C^{m,-}}) .$$

But by construction

$$\begin{aligned} \nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{C^{m,-}}) &= \nu^{m-1}_{[\rho_{m-1}\underline{0}]}(.|\sigma_{C^{m,-}}) \\ &= \nu^{m-1}_{[\rho_{m-1}\underline{0}]}(.|\sigma_{\partial^*[\rho_m x < \rho_{m-1}\underline{0}]}) \\ &= \nu_{[\rho_{m-1}\underline{0}]}(.|\sigma_{\partial^*[\rho_m x < \rho_{m-1}\underline{0}]}) \end{aligned}$$

which proves A(m).

Thus A(d) is proved, and the existence of the state ν is established. An application of lemma 2 now proves the theorem.

(Q.E.D.)

As a corollary we have the following varational principle for the free energy density f_Φ; it is stated in terms of states on essentially (d-1) dimensional systems [13].

Corollary 1. Let ν be a translation invariant state on $\Omega_{\Delta_{\underline{0}}^d}$.

Write $s_\partial(\nu) = S_{O|\partial^* Z_{\underline{}}^d}(\nu)$. Then

(a) $\nu(e_\Phi) - s_\partial(\nu) \geq f_\Phi$

(b) $\nu(e_\Phi) - s_\partial(\nu) = f_\Phi$ if and only if there is a translation invariant Gibbs state μ on Ω with $\mu_{\Delta_{\underline{0}}^d} = \nu$.

Proof: Given ν on $\Omega_{\Delta_{\underline{0}}^d}$ the state μ on Ω is constructed as in the proof of theorem 3. The properties of this μ and the variational principle theorem 1 combine to yield this corollary.

(Q.E.D.)

We may thus write

$$f_\Phi = \inf_{\nu \in I_{\Delta_{\underline{0}}^d}} [\nu(e_\Phi) - s_\partial(\nu)]$$

5. BOUNDS ON THE FREE ENERGY

In this final section we shall formulate bounds on the entropy density and free energy of equilibrium states.

Such bounds can be used in analytical calculations or in Monte Carlo simulations of lattice models as approximations for the entropy or free energy [14-17].

A converging sequence of upper bounds on the entropy density $s(\mu)$ of an equilibrium state μ is easily constructed on the basis of the foregoing.

Theorem 5. Let (K_n) be a sequence of subsets of $\partial^* Z^d_-$ such that

$$K_n \subset K_{n+1} \text{ and } K_n \to \partial^* Z^d_-.$$

Let μ be an equilibrium state for the interaction Φ. Then

$$s(\mu) \leq S_{O|K_{n+1}}(\mu) \leq S_{O|K_n}(\mu)$$

and

$$S_{O|K_n}(\mu) \to s(\mu)$$

Consequently, if $f_n \equiv \mu(e_\Phi) - S_{O|K_n}(\mu)$ then $f_\Phi \geq f_{n+1} \geq f_n$ and $f_n \to f_\Phi$.

Proof: Immediate, from theorem 4 and lemma 1.

Thus, the main result of the previous section (theorem 4) leads to lower bounds on the free energy in a straightforward fashion.

It may be surprising that in certain situations the very same arguments ("Markovian extensions" and the variational principle) can be used to arrive at upper bounds on the free energy density f_Φ [13]. We present one example of such a situation.

Take d=2 and let the interaction-round-faces potential Φ be invariant for the reflection θ^i in the line $x_1 = i + \frac{1}{2}$. (Recall Φ is assumed translation-invariant as well.)
We introduce some notation for sets of lattice sites in Z^2:

$$L_n^{i,j} = \{z = (z_1, z_2) \mid z_1 = i,\ z_2 \in \{j-n+1, \ldots, j\}\} \tag{21}$$

$$D_n^{i,j} = \{z = (z_1, z_2) \mid z_1 \in \{i-1, i\},\ z_2 \in \{j-n+1, \ldots, j\}\} \tag{22}$$

(lines and double lines).

The position of a specific set in the lattice will often be irrelevant due to translation invariance; then we shall omit the location indices i,j. For any translation-invariant state ν on Ω we define

$$s_1(\nu) \equiv \lim_{n\to\infty} \frac{S_{L_n}(\nu)}{n} \tag{23}$$

$$s_2(\nu) \equiv \lim_{n\to\infty} \frac{S_{D_n}(\nu)}{2n} \tag{24}$$

$$b(\nu) \equiv 2s_2(\nu) - s_1(\nu) \tag{25}$$

A consequence of the strong subadditivity property of the entropy is that (cf. [1, 2])

$$S_{D_n}(\nu) - S_{L_n}(\nu) \geq n \cdot s(\nu) \tag{26}$$

which leads to the inequality

$$b(\nu) \geq s(\nu)\ , \text{ for any } \nu \in I \tag{27}$$

Similarly

$$S_{D_n}(\nu) - S_{D_{n-1}}(\nu) \geq 2\, s_2(\nu) \tag{28}$$

$$S_{L_n}(\nu) - S_{L_{n-1}}(\nu) \geq s_1(\nu) \tag{29}$$

We now present the following lemma.

Lemma 4. Let ν_n be a state on $\Omega_{D_n^{0,0}}$ that is translation invariant and invariant under the reflection $\Theta^{\vec{0}}$. Then there exists a translation invariant state $\bar{\nu}$ on Ω such that

(a) $\bar{\nu}$ is an extension of ν_n, i.e. $\bar{\nu}_{D_n^{0,0}} = \nu_n$

(b) $s(\bar{\nu}) = 2\, s_2(\bar{\nu}) - s_1(\bar{\nu}) \geq S_{D_n}(\nu_n) - S_{D_{n-1}}(\nu_n) - S_{L_n}(\nu_n) + S_{L_{n-1}}(\nu_n)$

Proof. The proof is a compilation of arguments that have been presented earlier, so we shall be sketchy rather then detailed. The state $\bar{\nu}$ is obtained by construction. In the first stage of the construction ν_n is extended to a state on the infinitely long double line $D_\infty^{0,0}$.
D_∞ may be regarded as a one-dimensional lattice and an appropriate reformulation of theorem 3 yields the existence of an extension ν_∞ to Ω_{D_∞} with

$$2\, s_2(\nu_\infty) = S_{D_n}(\nu_n) - S_{D_{n-1}}(\nu_n) \tag{30}$$

The reflection invariance of ν_n ensures the translation invariance of ν_∞.

The second stage of the construction extends ν_∞ to $\bar{\nu}$ by what is essentially the same device; actually we can use lemma 3 with d=2 and k=1. The entropy of $\bar{\nu}$ is given by

$$s(\bar{\nu}) = 2\, s_2(\nu_\infty) - s_1(\nu_\infty) \tag{31}$$

Since by Eq. (29)

$$s_1(\nu_\infty) \leq S_{L_n}(\nu_\infty) - S_{L_{n-1}}(\nu_\infty) \tag{32}$$

Eqs. (31), (30) and (29) combine to complete the proof. (Q.E.D.)

Theorem 6. Let μ be an equilibrium state for the interaction Φ. Let μ, like Φ, be invariant under the reflections Θ^i. Then

$$s(\mu) \geq S_{D_n}(\mu) - S_{D_{n-1}}(\mu) - S_{L_n}(\mu) + S_{L_{n-1}}(\mu)$$

and

$$f_\Phi \leq \mu(e_\Phi) - [S_{D_n}(\mu) - S_{D_{n-1}}(\mu) - S_{L_n}(\mu) + S_{L_{n-1}}(\mu)] .$$

Proof. Define ν_n as the restriction of μ to $\Omega_{D_n^{0,0}}$.

Since $e_\Phi \in C(\Omega_{D_n^{0,0}})$, $\mu(e_\Phi) = \nu_n(e_\Phi)$. Let $\bar{\nu}$ be the extension of ν_n given by lemma 4. Then, combining lemma 4 with the variational principle, we find:

$$\begin{aligned}
\mu(e_\Phi) - s(\mu) &= f_\Phi \\
&\leq \bar{\nu}(e_\Phi) - s(\bar{\nu}) \\
&\leq \nu_n(e_\Phi) - [S_{D_n}(\nu_n) - S_{D_{n-1}}(\nu_n) - S_{L_n}(\nu_n) + S_{L_{n-1}}(\nu_n)] \\
&= \mu(e_\Phi) - [S_{D_n}(\mu) - S_{D_{n-1}}(\mu) - S_{L_n}(\mu) + S_{L_{n-1}}(\mu)]
\end{aligned}$$

which establishes the theorem.

(Q.E.D.)

The condition that μ be invariant under Θ^i is not essential; a modification of lemma 4 can construct a suitable state $\bar{\nu}$ from a non-reflection invariant state μ.

It is not difficult to prove convergence of the bounds of theorem 6 to $s(\mu)$ and f_Φ, respectively, if $n \to \infty$.

An application of the bounds of theorems 5 and 6 is in the estimation of free energies in Monte Carlo simulations of lattice models. The bounds depend on the probabilities of the configurations of only a finite number of spins; these probabilities may be obtained by Monte Carlo sampling. Examples may be found in [16, 17].

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STOCHASTIC CALCULUS FOR QUANTUM BROWNIAN MOTION OF NON-MINIMAL VARIANCE

— an approach using integral-sum kernel operators

J Martin Lindsay and Hans Maassen

ABSTRACT

The stochastic calculus of non-minimal variance quantum Brownian motion is developed by means of a representation in terms of integral-sum kernels. This representation permits a direct definition of stochastic integrals; a clear view of the structure of martingales; a unified approach to linear quantum stochastic differential equations with explicit expression for their solution; and further insight into the structure of adapted cocycles with direct means of finding their generators. Quantum sde's are means by which dynamical equations for dissipative quantum systems may be solved. The stationary Markov processes resulting from their solution are characterised in terms of quantum detailed balance. Finally an elementary example is treated and the physical interpretation of its constituents is given.

Introduction

Quantum stochastic calculus has flowered from its fundamental papers [HP 1], [BSW 1] into a subject rich, both in structure and applications, with two books on the subject appearing this year ([Par], [Me 3]). The wider subject of quantum, or non-commutative, probability continues to be intensively developed, from the diverse view-points of probability, physics and analysis (see e.g. [QP I-VII]). The present paper is based on an earlier preprint ([LM]). We hope that it is more-or-less self-contained, but our intention is to complement Meyer's lecture notes, where one of the central themes is the interplay between algebraic structures on Hilbert space (especially Fock space) and probabilistic interpretations.

The Barnett–Streater–Wilde theory is based on the Clifford process, which is a precise fermionic analogue of classical Brownian motion ([LM 2]). Barnett's extension ([Bar]) of Segal's non-commutative integration theory ([Seg]) is applied to the Clifford algebra of $L^2(\mathbb{R}_+)$ with its natural trace and filtration of sub-algebras. The Hudson–Parthasarathy calculus is based on a (minimal variance) quantum Brownian motion ([CoH]). Loosely speaking, this consists of a pair of classical Brownian motions (Q,P) satisfying the canonical commutation relations (with probabilists' normalisation):

$$Q_s P_t - P_t Q_s = i\,2\,(s \wedge t). \tag{0.1}$$

Equivalently, quantum Brownian motion may be considered as a non-commutative complex Brownian motion $A = (Q+iP)/2$, whose real and imaginary parts satisfy (0.1).

How can classical processes fail to commute? Each process must be represented as a family of commuting self-adjoint operators on a Hilbert space $\mathcal{H}$, with a unit vector ψ determining the law of the process $Q+iP$. The pair (Q,P) is then a quantum Brownian motion (qBm), of variance σ^2, if the following algebraic, probabilistic and non-degeneracy conditions are satisfied:

$$e^{i(xP_s+yQ_t)} = e^{ixP_s}e^{iyQ_t}e^{ixy(s\wedge t)} \tag{0.2}$$

$$\mathbb{E}^{\psi}\left[e^{i(xP_s+yQ_t)}\right] = e^{-\sigma^2(sx^2+ty^2)/2}$$

$$\left\{e^{i(xP_s+yQ_t)}\psi : x, y \in \mathbb{R},\ s, t \geq 0\right\} \text{ generates } \mathcal{H}. \tag{0.3}$$

As usual, we are working in units in which Planck's constant is 2π. The Weyl relations (0.2), which are a mathematically convenient form for the commutation relations, impose a constraint on the variance of a quantum Brownian motion: $\sigma^2 \geq 1$. This is a manifestation of Heisenberg's uncertainty principle.

There is a qualitative difference between the calculus of minimal, and non-minimal, variance qBm. The *degeneracy* of minimal variance qBm is discussed in [HL 2] from physical, probabilistic and mathematical points of view. The crucial mathematical point is that the state which determines the law of the Brownian motion is not faithful. It is therefore not sufficient to know only how operators act on the single vector ψ – one must work with a convenient dense subspace of $\mathcal{H}$ such as the exponential domain ([HP 1]). The algebra generated by minimal variance qBm is the full algebra of all operators on $\mathcal{H}$. One consequence of this is that the quantum Brownian filtration admits martingales quite different in character to (quantum) Brownian motion, such as the preservation, or number, process and both classical and quantum Poisson processes ([HP 1], [FrM]).

The non-minimal variance (or quasi-free) theory was developed by Barnett, Streater and Wilde ([BSW 2]), Hudson and one of the present authors ([HL 1,2,3], [L 1,2]). Here the state is faithful, so that as well as being cyclic (0.3), the vector ψ is also separating for the algebra $\mathcal{N}^{\sigma}$ generated by the qBm:

$$T_1, T_2 \in \mathcal{N}^{\sigma},\ T_1\psi = T_2\psi \ \Rightarrow T_1 = T_2 .$$

This allows operator questions to be tackled by vector considerations, and leads to a tighter theory. For example, there is a Kunita-Watanabe type representation theorem for square-integrable martingales ([HL 1], [L 2]), which follows from an orthogonal decomposition of the Hilbert space. This fails in the minimal variance case, even when the preservation process is included ([JoM]); and so far there are only partial results ([PS 1,3]).

Symmetric Fock space (over $L^2(\mathbb{R}_+)$) may be identified with *Guichardet space*, which is an L^2-space of functions defined on the finite power set of $\mathbb{R}_+$ ([Gui]). This representation was used by one of us to formulate quantum Itô calculus in terms of integral-sum kernel operators ([M 1,2]). An advantage of this approach is that solutions of linear quantum stochastic differential equations appear in a very explicit form. The key idea is the multiple quantum stochastic integral representation

$$X = \iint x(\sigma,\tau)\, dA_\sigma^*\, dA_\tau \tag{0.4}$$

where $\sigma, \tau \subset \mathbb{R}_+$, $A_\tau = \prod_{t \in \tau} A_t$ and $A_\sigma^* = (A_\sigma)^*$ (adjoint), for operators X on $\mathcal{H}$. This combines with (quantum) Itô relations to reveal x as an integral-sum *kernel* for the operator X, and also to represent the product of operators in terms of a *convolution-like* product of kernels. Meyer extended this idea to incorporate the preservation process ([Me 2]). The integral-sum representation also helps to clarify the relationship between quantum stochastics and the calculus of classical, but anticipating, stochastic processes ([L 3]). The idea of obtaining algebraic structure from multiple (quantum) stochastic integrals, commutation relations and Itô relations is further discussed in [Me 1], [Me 3], [LM 1] and [LiP].

The non-minimal variance theory is *even better* suited to a development in terms of integral-sum kernels. Part of the reason for this is that in this theory *every* L^2-operator is represented by (0.4). In this paper we develop the whole theory from the kernel point of view. New features include a direct definition of the stochastic integrals (Definition 3.2.4); a very simple proof of the stochastic integral representability of martingales (Proposition 7.5.1); a unified approach to linear quantum stochastic differential equations (Section 7.6); explicit expression for the solutions of such equations (Theorems 4.1.1 and 7.6.1); further insight into the structure of adapted cocycles, and a direct means of finding their *generators* (proofs of Propositions 4.3.2 and 8.1.2).

The original physical motivation for quantum stochastic calculus was to integrate dynamical equations describing dissipative quantum systems ([HP 1,2]); in other words, to dilate quantum dynamical semigroups to

(stationary) quantum Markov processes ([Küm], [M 1,2], [Fri]). This is done by solving a quantum stochastic differential equation whose coefficients are related to the generator of the semigroup, and may be done with non-minimal variance qBm if and only if the semigroup satisfies detailed balance ([Ali], [KFGV]). This is shown in Section 9, where physical parameters are re-introduced. In the final section a simple example is described, and the physical interpretation of each of the constituents is explained. For a *field test* of the theory described here see [RoM], where it is used to calculate the dynamical Stark effect. This spectacular phenomenon in quantum optics was predicted on theoretical grounds in the late 60's and observed a few years later. The approach using integral-sum kernels neatly unifies two opposing viewpoints on the phenomenon; one coming from the master equation, and the other from perturbative methods using Feynman diagrams.

The Itô-Clifford theory ([BSW 1]), and the generic variance Fermi theory ([BSW 2], [L 2]), are amenable to a very similar treatment. The formula for the product of Clifford (respectively fermionic) random variables is obtained simply by introducing a ± 1-valued signature function into the Wiener (resp. bose) product (see [Me 3], [LM 1]). The Fock Fermi theory ([ApH]), which has *to some extent* been subsumed by the Bose theory (see [HP 3], [PS 2]), may be described in terms of kernels too.

Before beginning with a brief heuristic discussion of classical Brownian motion from the present point of view, we mention some of the standard notations and conventions used here. $\mathcal{L}^0$ and $\mathcal{L}^p$ will denote the linear spaces of measurable, respectively p-summable functions on a measure space; $\mathcal{B}$ in $(X, \mathcal{B}, \mu)$ the Borel σ-algebra of a topological space X; $C_o(X)$ the space of continuous functions, on a locally compact space, which vanish at infinity; and $C_K^1(I)$ the space of once continuously differentiable functions of compact support on an interval I. The indicator function of a set S will be denoted χ_S; bold symbols always denote n-tuples and inner products follow Dirac's convention of linearity in the *second* argument. A list of special symbols, and where they are introduced, is given at the end.

1. Commutative kernel calculus.

1.1 Wiener-Fock space.

Let $\mathbb{P}$ be the Wiener measure on $\{\gamma : I \mapsto \mathbb{R}$ s.t. γ is continuous and $\gamma(0) = 0\}$ where I is the unit interval $[0,1]$ and B the coordinate (Wiener) process. It is well known that any complex-valued random variable $F \in L^2(\mathbb{P})$ can be expanded as an infinite sum of iterated stochastic integrals:

$$F = \sum_{n=0}^{\infty} \int_{\Omega_n} f_n(t_1,\ldots,t_n) dB_{t_1} \ldots dB_{t_n}, \tag{1.1}$$

where Ω_n is the n-dimensional simplex $\{t \in I : t_1 < t_2 < \ldots < t_n\}$. For $n > 0$, f_n denotes a square integrable function on Ω_n, and f_0 is the constant $\mathbb{E}[F]$. The sequence $f = (f_0, f_1, \ldots)$ will be called the *integral-sum kernel* of F. We have the relation

$$\mathbb{E}[F^2] = \sum_{n=0}^{\infty} \|f_n\|^2$$

indicating that the correspondence between F and f is a unitary equivalence between *Wiener space* $\mathcal{W} := L^2(\mathbb{P})$ and $\mathcal{F} := \oplus_{n=0}^{\infty} L^2(\Omega_n)$, called the *symmetric (or boson) Fock space* of $L^2(I)$ in the physics literature. This isomorphism invites several questions. For instance, what algebraic structure is induced on $\mathcal{F}$ by the multiplication of random variables in $\mathcal{W}$? How is this structure connected to stochastic integration?

In this section we answer these questions on *a formal level*. In the remaining sections we treat in detail the situation which arises when, in the above, the Wiener process is replaced by a *quantum Brownian motion*.

1.2 Set notation.

The n-dimensional ordered simplex Ω_n can be naturally identified with the set $\{\omega \in I : \#\omega = n\}$ and hence the infinite union $\Omega = \bigcup_{n=0}^{\infty} \Omega_n$ may be regarded as the *finite power set* of the interval I:

$$\Omega(I) = \{\omega \subset I : \omega \text{ is finite}\}$$

For this section (only) the measure on $\Omega(I)$ which on $\Omega_n(I)$ is given by the Lebesgue measure $dt_1 \ldots dt_n$, and which has $\varnothing$ as an atom of weight 1 will be

denoted by $d\omega$. We may now write the space of kernels as $L^2(\Omega, d\omega)$ and rewrite (1.1) as

$$F = \int_\Omega f(\omega)\, dB_\omega.$$

1.3 An algebraic structure on Fock space.

Let f and g be the kernels of F and $G \in \mathcal{W}$ respectively, we calculate formally the kernel of the product FG:

$$FG = \int_\Omega f(\omega)\, dB_\omega \cdot \int_\Omega g(\nu)\, dB_\nu$$

Because of the *Itô rule* $(dB)^2 = dt$, the integral over $(\omega, \nu) \in \Omega \times \Omega$ contains non-zero contributions from those regions where points of ω and ν coalesce: $\gamma := \omega \cap \nu \neq \emptyset$. Performing a change of variable $\alpha = \omega \setminus \gamma$ and $\beta = \nu \setminus \gamma$ one obtains

$$FG = \iiint_{\alpha \cap \beta = \emptyset} f(\alpha \cup \gamma) g(\beta \cup \gamma)\, dB_\alpha\, dB_\beta\, d\gamma.$$

Next, the integrals over α and β may be replaced by a single integral over $\sigma = \alpha \cup \beta$, followed by a sum over $\alpha \subset \sigma$:

$$FG = \int_\Omega \left\{ \sum_{\alpha \subset \sigma} \int_\Omega f(\alpha \cup \gamma) g(\bar{\alpha} \cup \gamma)\, d\gamma \right\} dB_\sigma$$

where $\beta = \bar{\alpha}$, the complement of α in σ.

The expression in brackets is thus the integral-sum kernel of FG and (in this section only) will be denoted $f*g$. For details of the proof of this correspondence see [LM 2].

1.4 Kernel calculus.

Taking the product $(f, g) \mapsto f*g$ as a starting point one may build up a stochastic calculus. After specifying a class $\mathcal{K}$ of kernels on which this product is well defined, one introduces $\mathcal{K}$-valued *processes* $\{f_t\}_{t \in I}$ which are *non-anticipating* in the sense that $f_t(\omega) = 0$ as soon as $\max\omega > t$. By the nature of formula (1.1) one easily derives the form of the kernel $(If)_t$ of the stochastic

integral $\int_0^t F_s \, dB_s$ of an L^2-process F :

$$(If)_t(\sigma) = \begin{cases} f_{\max\sigma}(\sigma \setminus \{\max\sigma\}) & \text{if } \sigma \neq \emptyset \text{ and } t \geqslant \max\sigma \\ 0 & \text{otherwise} \end{cases}$$

and similarly, the inverse operation of *stochastic differentiation*, is given by:

$$(\Delta f)_t(\sigma) = f_t(\sigma \cup \{t\})$$

Apart from these operations there are the pathwise integration and differentiation of processes:

$$(Df)_t(\sigma) = \frac{d}{dt} f_t(\sigma).$$

This is the kernel of the forward derivative ([Nel]) of the process F. The following relations hold:

$$\Delta If = f ; \qquad D \int_0^{\cdot} f_s \, ds = f ;$$

and also,

$$f_t - f_0 = (I\Delta f)_t + \int_0^t (Df_s) \, ds$$

which decomposes the semimartingale (f_t) into a martingale part and a bounded variation part. The connection between the algebraic and the differential structure of kernel processes is given by the Leibnitz formula for Δ and Itô formula for D:

$$\Delta(f * g) - \Delta f * g - f * \Delta g = 0$$

$$D(f * g) - Df * g - f * Dg = \Delta f * \Delta g.$$

2. Bose chaos

In the next two sections we describe the algebraic part of non-commutative kernel calculus.

2.1 $\Gamma, \nabla, \rho, \mu_{c_+,c_-}, \dagger$

For each sub-interval I of $\mathbb{R}$ we introduce $\Gamma_I = \Gamma(I)$, the *charged finite power set* of I, consisting of finite subsets of int(I), the interior of I, each point carrying a "charge" — positive or negative.

Definition 2.1.1: $\Gamma_I := \{\sigma : \text{int}(I) \to \{0, +1, -1\} \mid \text{the support of } \sigma \text{ is finite}\}$.

We shall write Γ when the interval I is understood. For $\sigma \in \Gamma$, let $\sigma^{\pm} = \sigma^{-1}(\{\pm 1\})$ and $|\sigma|$, $|\sigma^+|$, $|\sigma^-|$ be the cardinalities of supp σ, σ^+ and σ^- respectively. We continue to think of elements of Γ as sets, using notations like $\sigma \cup \tau$ and $\sigma \setminus \tau$ where there is no danger of confusion. To each $\sigma \in \Gamma$ there corresponds a unique element $(s, \varepsilon) \in int(I)^{|\sigma|} \times \{+1, -1\}^{|\sigma|}$ such that $s_1 < s_2 < \ldots < s_n$, where $\varepsilon_j = \sigma(s_j)$ and $n = |\sigma|$. We shall frequently make these identifications, and write p_1, p_2 for the projections

$$\sigma = (s, \varepsilon) \mapsto s; \qquad \sigma = (s, \varepsilon) \mapsto \varepsilon, \quad (p_i(\varnothing) = \varnothing),$$

respectively. Note the following partitions of Γ:

$$\Gamma = \bigcup_{j,k=0}^{\infty} \Gamma^{j,k} = \bigcup_{n=0}^{\infty} \Gamma^n, \tag{2.1}$$

where $\Gamma^{j,k} = \{\sigma \in \Gamma : |\sigma^+| = j, |\sigma^-| = k\}$ and $\Gamma^n = \{\sigma \in \Gamma : |\sigma| = n\}$. Let $\rho_0 : \Gamma \times \Gamma \to \mathbb{R}_+$ be the map given by

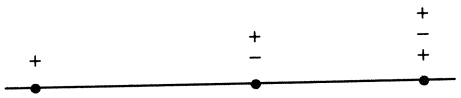
$$(\sigma, \tau) \mapsto \begin{cases} 0 & \text{if } \sigma = \tau = \varnothing \\ 1 \wedge \max\limits_i |s_i - t_i| & \text{if } p_2(\sigma) = p_2(\tau) \\ 1 & \text{otherwise} \end{cases}$$

then (Γ, ρ_0) is a metric space, and we denote its completion by (∇, ρ). $\nabla = \nabla_I$ may be identified with the set

$$\left\{(s, \varepsilon) \in \bigcup_{n=0}^{\infty} I^n \times \{+1, -1\}^n : s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n\right\}$$

and its elements considered as *(charged) generalised subsets* of I, in the sense

that it includes elements of the form



in which sites are occupied by more than one "particle". Under this identification, the union map

$$(\sigma,\tau)\in\nabla\times\nabla\mapsto\sigma\cup\tau\in\nabla$$

is measurable. The partitions (2.1) carry over to ∇ and we shall write $\nabla^{\leq N}$ for $\bigcup_{j+k\leq N}\nabla^{j,k}$. Now *fix* constants $c_+\geq c_->0$. A Borel measure $\mu=\mu_{c_+,c_-}$ on (∇,ρ) is defined as follows. First define a Borel measure λ on (∇,ρ) by

$$\lambda(\varnothing)=1;\qquad \lambda\,|\nabla^{j,k}=\lambda_{j+k}\times d_{j,k}\quad (j+k>0)$$

where λ_n is n-dimensional Lebesgue measure and $d_{j,k}$ is the counting measure on $\{\varepsilon\in\{+,-\}^{j+k}$: exactly j $+$'s and k $-$'s occur$\}$. Then let μ be defined by

$$d\mu=md\lambda\ \text{ where }\ m(\sigma)=c_+^{|\sigma_+|}c_-^{|\sigma_-|},$$

in other words $m(s,\varepsilon)=\prod_i c_{\varepsilon_i}$. Clearly $\nabla_I\backslash\Gamma_I$ is μ-null and, if I is bounded,

$$\mu(\nabla_I)=\exp\{(c_++c_-)\lambda_1(I)\}. \tag{2.2}$$

The measure therefore simply counts the positive charges and the negative charges, and weights accordingly, whereas the metric is sensitive to the way in which the charges are distributed.

Now let $\mathcal{A}$ be an involutive Banach algebra with unit I and involution *.

Definition 2.1.2: Let $\dagger:\nabla_I\to\nabla_I$ be the *charge changing map*

$$(s,\varepsilon)\to(s,-\varepsilon),$$

$\mu^\dagger$ the *induced measure*

$$\mu^\dagger(U)=\mu(\{\sigma:\sigma^\dagger\in U\}),$$

and, for $x:\nabla\to\mathcal{A}$, let $x^\dagger$ be its *involute*:

$$\sigma \to x(\sigma^\dagger)^* .$$

2.2 Smooth kernels.

We now introduce the class of $\mathcal{A}$-valued functions on Γ which will form the basis of the present treatment of the kernel calculus. The choice of class is motivated by our requirement that it support both an *algebraic* and a *stochastic differential structure*.

Definition 2.2.1: For $x : \Gamma_I \to \mathcal{A}$, consider the properties

$\mathcal{K}$i: $\exists\, J_x$ a compact sub-interval of I such that x vanishes outside Γ_{J_x};

$\mathcal{K}$ii $\exists\, K_x \geqslant 1$ such that $\|x(\sigma)\| \leqslant K_x^{|\sigma|+1}\ \forall \sigma \in \Gamma_I$;

$\mathcal{K}$iii $\exists\, K_x \geqslant 1$ such that $\|x(\sigma)-x(\tau)\| \leqslant \rho_0(\sigma,\tau)K_x^{|\sigma|+1}$ as soon as $\rho_0(\sigma,\tau) < 1$.

Denote the class of functions satisfying $\mathcal{K}$i, $\mathcal{K}$ii and $\mathcal{K}$iii by $\mathcal{K}_0^{\mathcal{A}}(I)$, or $\mathcal{K}_0$, and call the elements of $\mathcal{K}_0$ *smooth kernels*. Also denote the class of strongly $\mathcal{A}$-measurable functions ([Yos]) satisfying $\mathcal{K}$i and $\mathcal{K}$ii by $\mathcal{K}_b^{\mathcal{A}}(I)$. Each of the properties is clearly preserved under the involution $\dagger$. For subintervals J of I, $\mathcal{K}_0^{\mathcal{A}}(J)$ is naturally included in $\mathcal{K}_b^{\mathcal{A}}(I)$ but not in $\mathcal{K}_0^{\mathcal{A}}(I)$. By $\mathcal{K}$iii, any smooth kernel has a unique extension to ∇_I, now satisfying $\mathcal{K}$i, $\mathcal{K}$ii and $\mathcal{K}$iii with Γ replaced by ∇ and ρ_0 by ρ, and with the same J_x and K_x. Smooth kernels will therefore frequently be defined only on Γ_I but will thereafter be *considered as functions on the whole of* ∇_I with no notational change.

Example 2.2.2: For $f \in C_\kappa^1(I)$, the following are smooth $\mathbb{C}$-valued kernels

$$\begin{aligned} &\pi_f : \sigma \mapsto \textstyle\prod_{s\in\sigma^+} f(s)\cdot\prod_{t\in\sigma^-}\overline{(-f(t))}\,; \\ &w_f = \hat{\mu}(f)\pi_f\,; \\ &\pi_f^{j,k} = \chi_{\Gamma^{j,k}}\pi_f\,; \qquad \pi_f^{(n)} = \chi_{\Gamma^n}\pi_f\,; \end{aligned} \tag{2.3}$$

$$a_f : \sigma \mapsto \begin{cases} \overline{f(t)} & \text{if } \sigma = \{t^-\} \in \Gamma^{0,1} \\ 0 & \text{otherwise} \end{cases}\,; \qquad r_f = -i\pi_f^{(1)} = i^{-1}\{a_f^\dagger - a_f\}$$

where $\hat{\mu} : C_\kappa^1(I) \to \mathbb{R}_+$ is the map $f \mapsto \exp\{-\frac{1}{2}(c_+ + c_-)\,\|f\|^2_{L^2(I)}\}$. The w_f's, $a_f^\dagger$'s and a_f's will be called respectively the (smooth) *Weyl*, *creation* and

annihilation kernels. For $w_0 = \pi_0$ we shall sometimes write $\delta_\varnothing$.

2.3 The Bose product.

We next introduce the product on $\mathcal{K}_b$ whose form is dictated by the non-commutative duality transform (see section 6). For $x \in \mathcal{K}_b$, the maps $\omega \mapsto x(\omega \cup \alpha)$ and $\omega \mapsto x(\omega^\dagger \cup \alpha)$ are strongly measurable for each subset α of σ, so the next definition is a good one.

Proposition 2.3.1: For $x, y \in \mathcal{K}_b^{\otimes}(I)$ the following map also belongs to $\mathcal{K}_b$:

$$z : \sigma \mapsto \int \sum_{\alpha \subset \sigma} x(\omega^\dagger \cup \alpha)\, y(\omega \cup \bar{\alpha})\, d\omega$$

where the sum is over disjoint partitions $\alpha \cup \bar{\alpha}$ of σ. Moreover if $x, y \in \mathcal{K}_0$ then $z \in \mathcal{K}_0$ also.

Note that the sum is a finite one, and we have abbreviated $d\mu(\omega)$ to $d\omega$.

Proof: Let J be a compact sub-interval of I containing J_x and J_y. Notice that for each $\alpha \in \Gamma_I$,

$$\begin{aligned}\int_\Gamma \| x(\omega^\dagger \cup \alpha) y(\omega \cup \bar{\alpha}) \|\, d\omega &\leq \int_{\Gamma_J} K_x^{|\omega|+|\alpha|+1} K_y^{|\omega|+|\bar{\alpha}|+1}\, d\omega \\ &\leqslant K_x^{|\alpha|+1} K_y^{|\bar{\alpha}|+1} \int_{\Gamma_J} (K_x K_y)^{|\omega|}\, d\omega \\ &= K_x^{|\alpha|+1} K_y^{|\bar{\alpha}|+1} \exp\{(c_+ + c_-) K_x K_y \lambda_1(J)\},\end{aligned}$$

so that z is well-defined as a Bochner integral. Next, since

$$\sum_{\alpha \subset \sigma} K_x^{|\alpha|} K_y^{|\bar{\alpha}|} = (K_x + K_y)^{|\sigma|},$$

z satisfies $\mathcal{K}$ii so that $z \in \mathcal{K}_b$. Now suppose that x and y are smooth. Then if $\rho_0(\sigma, \tau) < 1$— $\sigma = (s, \varepsilon)$, $\tau = (t, \varepsilon)$, say — let $\pi : 2^\sigma \to 2^\tau$ be the bijective map between power sets induced by the pointwise map $s_j \mapsto t_j$ ($j = 1, 2, \ldots |\sigma|$), then

$$\| z(\sigma) - z(\tau) \| \leqslant \int_{\Gamma_J} \sum_{\alpha \subset \sigma} \| x(\omega^\dagger \cup \alpha) y(\omega \cup \bar{\alpha}) - x(\omega^\dagger \cup \pi(\alpha)) y(\omega \cup \pi(\bar{\alpha})) \|\, d\omega$$

which is bounded by $2\rho_0(\sigma,\tau)(K_x+K_y)^{|\sigma|}\exp\{(c_+ + c_-)K_xK_y\lambda_1(J)\}$ so that z also satisfies $\mathcal{K}$iii. Hence $z\in\mathcal{K}_0^{\mathcal{A}}(I)$.

■

Definition 2.3.2: For $x,y\in\mathcal{K}_b^{\mathcal{A}}(I)$ we denote the kernel z defined above by $x*y$.

Some immediate properties are listed next.

1. $\mathcal{K}_0$ is also closed under the *point-wise* product ($x,y\in\mathcal{K}_0 \Rightarrow \sigma\mapsto x(\sigma)y(\sigma)\in\mathcal{K}_0$).

2. When $\mathcal{K}=\mathbb{C}$, $\langle x,y\rangle_{L^2(\mathcal{V},\mu)} = (x^\dagger * y)(\varnothing)$.

3. For $f,g\in C_\kappa^1(I)$,

$$a_f * a_g^\dagger(\varnothing) - a_g^\dagger * a_f(\varnothing) = \langle a_f^\dagger, a_g^\dagger\rangle - \langle a_g, a_f\rangle, = c_+\langle f,g\rangle_{L^2(I)} - c_-\langle \bar{g},\bar{f}\rangle_{L^2(I)}$$
$$= (c_+ - c_-)\langle f,g\rangle_{L^2(I)},$$

whereas,

$$(a_f * a_g^\dagger)(\{t^+,s^-\}) = a_f(\{s^-\})a_g^\dagger(\{t^+\}).$$

Since $a_f * a_g^\dagger$ and $a_g^\dagger * a_f$ are supported by $\mathcal{V}^0\cup\mathcal{V}^{1,1}$ these kernels satisfy the *canonical commutation relations*

$$a_f * a_g^\dagger - a_g^\dagger * a_f = (c_+ - c_-)\langle f,g\rangle\,\delta_\varnothing. \tag{2.4}$$

We next state two combinatorial facts, the first of which will be repeatedly used in the sequel.

Lemma 2.3.3: (a) For an integrable Banach space valued function g on $\Gamma\times\Gamma$

$$\iint g(\alpha,\beta)\,d\alpha\,d\beta = \int \Big\{\sum_{\alpha\subset\omega} g(\alpha,\bar{\alpha})\Big\}\,d\omega.$$

(b) For a vector space valued function f on $\Gamma\times\Gamma$

$$\sum_{\alpha\subset\sigma}\sum_{\beta\subset\tau} f(\alpha\cup\beta,\bar{\alpha}\cup\bar{\beta}) = \sum_{\gamma\subset\sigma\cup\tau} f(\gamma,\bar{\gamma})$$

whenever σ and τ are disjoint.

Proof: (a) For $n \in \mathbb{N}$, let 2^n denote the power set of $n := \{1,2,\ldots,n\}$ considered as a measure space with the counting measure, and for $\omega = (t,\varepsilon) \in \Gamma^{(n)}$, $\alpha \subset \omega$, let $S(\alpha,\omega) = \{j \in n : t_j \in \mathrm{supp}(\alpha)\}$. The mapping $\iota : (\alpha,\beta) \to (\alpha \cup \beta, S(\alpha,\alpha\cup\beta))$ defines a bijection between the set $\{(\alpha,\beta) \in \Gamma\times\Gamma : \alpha\cap\beta = \varnothing\}$, which has full measure in $\Gamma\times\Gamma$, and the set $\bigcup_{n=0}^{\infty} \Gamma^n \times 2^n$. Since ι is measure preserving, the result follows.

(b) Immediate. ■

Proposition 2.3.4: $(\mathcal{K}_b^{\mathcal{A}}(I), *)$ and $(\mathcal{K}_0^{\mathcal{A}}(I), *)$ are associative involutive algebras with unit $1_{\mathcal{A}}\delta_{\varnothing}$ and involution $\dagger$.

Proof: For $x,y,z \in \mathcal{K}_b$ define a function k on Γ by

$$\sigma \mapsto \iiint \sum x(\alpha \cup \omega_2^\dagger \cup \omega_3^\dagger)\, y(\omega_1^\dagger \cup \beta \cup \omega_3)\, z(\omega_1 \cup \omega_2 \cup \gamma)\, d\omega_1\, d\omega_2\, d\omega_3,$$

where the sum is over partitions of σ into a disjoint union of α,β and γ. Applying Lemma 2.3.3 (a) to (ω_2,ω_3) and then Lemma 2.3.3 (b) to $(\omega,\bar{\alpha})$, where ω is the new variable $\omega_2 \cup \omega_3$, gives the following expression for $k(\sigma)$:

$$\int \sum_{\alpha\subset\sigma} d\omega_1 \int d\omega \sum_{\omega_3\subset\omega} \sum_{\beta\subset\bar{\alpha}} x(\alpha\cup\omega^\dagger)\, y(\omega_1^\dagger \cup \beta \cup \omega_3) z(\omega_1 \cup \bar{\omega}_3 \cup \bar{\beta})$$

$$= \int d\omega \sum_{\alpha\subset\sigma} x(\alpha\cup\omega^\dagger) \int d\omega_1 \sum_{\delta\subset\omega\cup\bar{\alpha}} y(\omega_1^\dagger \cup \delta)\, z(\omega_1 \cup \bar{\delta})$$

$$= x*(y*z)(\sigma).$$

On the other hand, applying Lemma 2.3.3 (a) to (ω_1,ω_2) and then Lemma 2.3.3 (b) to $(\omega^\dagger,\bar{\gamma})$, where ω is the new variable $\omega_1 \cup \omega_3$, yields $k(\sigma) = (x*y)*z(\sigma)$, establishing the associativity of $*$. Since the involution on $\mathcal{A}$ is conjugate linear and isometric,

$$(x*y)^\dagger(\sigma) = \left\{\int \sum_{\alpha\subset\sigma^\dagger} x(\alpha\cup\omega^\dagger) y(\bar{\alpha}\cup\omega)\, d\omega\right\}^*$$

$$= \int \sum_{\alpha\subset\sigma} y(\bar{\alpha}^\dagger \cup \omega)^* x(\alpha^\dagger \cup \omega^\dagger)^*\, d\omega$$

$$= \int \sum_{\alpha \subset \sigma} y^\dagger(\bar{\alpha} \cup \omega^\dagger) x^\dagger(\alpha \cup \omega)\, d\omega$$

$$= (y^\dagger * x^\dagger)(\sigma).$$

So † is an involution. Since $1_{\mathcal{A}} \delta_\varnothing$ is obviously a unit the result follows. ■

Remark: 1. If $\mathcal{A}_1$ is a sub-algebra of $\mathcal{A}_2$ then in a natural way $\mathcal{K}_0^{\mathcal{A}_1}(I)$ is a subalgebra of $\mathcal{K}_0^{\mathcal{A}_2}(I)$.

2. $\mathcal{K}_0$ is non-commutative unless $c_+ = c_-$.

2.4 The Weyl Relations.

For $a, b, c \neq 0$ let $\gamma_{a,b} : \mathcal{V}_I \to \mathbb{C}$ be the map $\sigma \to a^{|\sigma^+|} b^{|\sigma^-|}$, and let $\gamma_c = \gamma_{c,c^{-1}}$.

Lemma 2.4.1: For $\alpha, \beta \in \mathcal{V}_I$, $x, y \in \mathcal{K}_0^{\mathcal{A}}(I)$

(i) $\gamma_{a,b}(\alpha \cup \beta) = \gamma_{a,b}(\alpha)\gamma_{a,b}(\beta)$

(ii) $\gamma_{a,b} x \in \mathcal{K}_0^{\mathcal{A}}(I)$

(iii) $\gamma_{a,b}^\dagger = \gamma_{b,a}$

(iv) $\gamma_c(x * y) = \gamma_c x * \gamma_c y$

Proof: The estimate $|\gamma_{a,b}(\sigma)| \leq (|a| + |b|)^{|\sigma|}$ and the identity

$$\gamma_c(\alpha \cup \omega^\dagger)\gamma_c(\bar{\alpha} \cup \omega) = \gamma_c(\sigma)$$

for $a \subset \sigma, \omega \in \mathcal{V}$, suffice to establish the lemma. ■

We shall write γ for γ_c when $c = c_-/c_+$. Thus $\gamma = m^\dagger/m$ and $m = \gamma_{c_+, c_-}$.

Definition 2.4.2: Define the following maps on $\mathcal{K}_0^{\mathbb{C}}(I)$:

$$\Gamma_0 : x \to \sqrt{\gamma} x \qquad G_y : x \mapsto y * x$$

$$S_0 : x \to x^\dagger$$

$$J_0 : x \to \sqrt{\gamma} x^\dagger \qquad D_y : x \mapsto x * y \quad (y \in \mathcal{K}_0^{\mathbb{C}}(I)).$$

Proposition 2.4.3: The following relations obtain

(i) $J_o = \Gamma_o S_o = S_o \Gamma_o^{-1}$

(ii) $J_o^2 (= S_o^2) = \mathrm{id}_{\mathcal{K}_o}$

(iii) J_o is L^2-isometric

(iv) $G_y G_z = G_{y*z}$; $D_z D_y = D_{y*z}$

(v) $G_y D_z = D_z G_y$

(vi) $J_o G_y J_o = D_{J_o y}$, $(y, z \in \mathcal{K}_o^{\mathbf{C}}(I))$

Proof: (i) follows from Lemma 2.4.1, and (ii) follows from (i). Since

$$\int |J_o x|^2 \, d\mu = \int \gamma |x^\dagger|^2 m \, d\lambda = \int m^\dagger |x^\dagger|^2 \, d\lambda = \int m |x|^2 \, d\lambda = \int |x|^2 \, d\mu,$$

(iii) follows. (iv) and (v) are a consequence of the associativity of $*$, and (vi) of the identities

$$\begin{aligned} J_o G_y J_o x &= J_o (y * \sqrt{\gamma} x^\dagger) \\ &= \sqrt{\gamma} \left(\frac{1}{\sqrt{\gamma}} x * y^\dagger \right) = x * J_o y = D_{J_o y} x. \end{aligned}$$

■

Definition 2.4.4: Let $\zeta = \zeta_{c_+, c_-} : C_\kappa^1(I) \times C_\kappa^1(I) \to \mathbb{R}$ be the *symplectic form*

$$(f, g) \mapsto -(c_+ - c_-) \operatorname{Im} \int_I \bar{f} g \, d\lambda_1 .$$

For $f \in C_\kappa^1$ let

$$v_f = J_o w_{\bar{f}} = \sqrt{\gamma} w_{-\bar{f}}.$$

Proposition 2.4.5: For $f, g \in C_\kappa^1(I)$, $x \in \mathcal{K}_o^{\mathbf{C}}(I)$:

(i) $w_f * w_g = e^{i\zeta(f,g)} w_{f+g}$; $w_f^\dagger = w_{-f}$

(ii) $v_g * v_f = e^{i\zeta(f,g)} v_{f+g}$; $v_f^\dagger = \gamma^{-1} v_{-f}$

(iii) $\int |w_f * x|^2 \, d\mu = \int |x * v_g|^2 \, d\mu = \int |x|^2 \, d\mu$

(iv) $\int \bar{w}_0 w_f \, d\mu = \int \bar{v}_0 v_f \, d\mu = \hat{\mu}(f)$.

Proof:

(i) $\pi_f * \pi_g(\sigma) = \sum_{\alpha \subset \sigma} \pi_f(\alpha)\pi_g(\bar{\alpha}) \int \pi_{-\bar{f}} \pi_g \, d\mu$

$= \pi_{f+g}(\sigma) \exp\left\{-c_+ \int \bar{f}g \, d\lambda_1 - c_- \int f\bar{g} \, d\lambda_1\right\}$

$= \pi_{f+g}(\sigma) \exp\left\{-(c_+ + c_-)\mathrm{Re} \int \bar{f}g \, d\lambda_1 - i(c_+ - c_-)\mathrm{Im} \int \bar{f}g \, d\lambda_1\right\}$

$= e^{i\zeta(f,g)} \left[\frac{\hat{\mu}(f+g)}{\hat{\mu}(f)\hat{\mu}(g)}\right] \pi_{f+g}(\sigma)$.

(ii) $v_g * v_g = \sqrt{\gamma} w_{-\bar{g}} * \sqrt{\gamma} w_{-\bar{f}}$

$= \sqrt{\gamma} e^{i\zeta(f,g)} w_{-\bar{f}-\bar{g}} = e^{i\zeta(f,g)} v_{f+g}$

$v_f^\dagger = \sqrt{\gamma} w_{-\bar{f}}^\dagger = \frac{1}{\sqrt{\gamma}} w_{\bar{f}} = \gamma^{-1} v_{-f}$.

(iii) $\int |w_f * x|^2 \, d\mu = (x^\dagger * w_{-f} * w_f * x)(\varnothing)$

$= (x^\dagger * x)(\varnothing) = \int |x|^2 \, d\mu$,

and since $D_{v_g} = J_0 G_{w_{\bar{g}}} J_0$, the L^2-isometry of D_{v_g} follows from that of J_0 (Proposition 2.4.3) and of $G_{w_{\bar{g}}}$.

(iv) is immediate.

■

2.5 L^2-density of kernels

In this subsection we establish some density results for $\mathcal{K}_0^C$ (I) and the smooth Weyl kernels. Let $\mathcal{W}$ and $\mathcal{V}$ be respectively the linear spans of w_f and v_f with f running through $C_K^1(I)$, let $\mathcal{W}_J = \{x \in \mathcal{W}: \mathrm{supp}\, x \subset \mathcal{D}_J\}$ and let $\mathcal{W}_J^{j,k} = \chi_{\mathcal{D}^{j,k}} \mathcal{W}_J$ where J is a compact interval.

Lemma 2.5.1: For $f \in C_\kappa^1(I)$

(i) $$\pi_f^{(n)} = (n!)^{-1} \frac{d^n}{ds^n}\Big|_{s=0} \pi_{sf}$$

(ii) $$\pi_f^{j,k} = (2\pi)^{-1} \int_0^{2\pi} e^{-i(j-k)\theta}\, \pi_{e^{i\theta}f}^{(j+k)}\, d\theta$$

Proof:

(i) For $s \in \mathbb{R}$,

$$\pi_{sf} = \sum_{n=0}^{\infty} s^n \pi_f^{(n)}$$

and (i) follows.

(ii) For $c \in \mathbb{C}$,

$$\pi_{cf}^{(n)} = \pi_f^{(n)} \cdot \sum_{j+k=n} c^j \bar{c}^k \chi_{\mathcal{V}^{j,k}},$$

thus

$$\pi_{e^{i\theta}f}^{(n)} = \sum_{j+k=n} e^{i(j-k)\theta}\, \pi_f^{j,k},$$

from which (ii) follows. ∎

Proposition 2.5.2: $\bar{\mathcal{W}}^{u.} \subset C_0(\mathcal{V}_I)$ (uniform closure).

Proof: Since $\mathcal{W}_J^{j,k}$ is an algebra under pointwise multiplication the Stone–Weierstrass theorem implies that $C(\mathcal{V}_J^{j,k})$ is the uniform closure of $\mathcal{W}_J^{j,k}$. Moreover Lemma 2.5.1 yields $\mathcal{W}_J^{j,k} \subset \bar{\mathcal{W}}_J^{u.}$, thus $\bar{\mathcal{W}}^{u.} \supset C(\mathcal{V}_J^{\leq N})$ for each N and compact J. But any compact set in $\mathcal{V}$ is a subset of some $\mathcal{V}_J^{\leq N}$, therefore $\bar{\mathcal{W}}^{u.} \supset \overline{C_\kappa(\mathcal{V})}^{u.} = C_0(\mathcal{V})$. ∎

Corollary 2.5.3: $\mathcal{W}$ and $\mathcal{V}$ are dense in $L^2(\mathcal{V}_I, \nu)$, where $\nu = \mu + \mu^\dagger$.

For $f \in L^2(I)$ the Weyl kernel w_f given in (2.3) is still a well-defined element of $L^2(\mathcal{V}_I, \mu)$, but not necessarily an element of $\mathcal{K}_0^{\mathbb{C}}(I)$, i.e. not necessarily a *smooth* kernel.

Proposition 2.5.4: The map $w: L^2(I,\lambda_1) \to L^2(\mathcal{V}_I, v)$ is continuous.

Proof: Since

$$\langle \pi_f, \pi_g \rangle_{L^2(v)} = 2e^{(c_+ + c_-)\mathrm{Re}\langle f,g\rangle}\cos\{(c_+ - c_-)\,\mathrm{Im}\langle f,g\rangle\},$$

$$\|\pi_f - \pi_g\|^2_{L^2(v)} = 2\big[e^{(c_+ + c_-)\|f\|^2} + e^{(c_+ + c_-)\|g\|^2}$$

$$- 2e^{(c_+ + c_-)\mathrm{Re}\langle f,g\rangle}\cos\{(c_+ - c_-)\,\mathrm{Im}\langle f,g\rangle\}\big]$$

which tends to 0 as f approaches g. Since $\hat{\mu}$ is clearly continuous, the result follows.

■

Corollary 2.5.5: If D is dense in $L^2(I)$, then the linear span of $\{w_f : f \in D\}$ is dense in $L^2(\mathcal{V}_I, v)$.

3. Adapted processes

In this section subintervals I of the real line will be *assumed to have a left end point* 0. In order to discuss processes we introduce the *adapted power set.*

$$\Gamma_{\mathrm{ad.}}(I) := \{(\sigma,t) \in \Gamma_I \times I : \max\sigma < t \text{ or } \sigma = \varnothing\}.$$

Thus $(\sigma,t) \in \Gamma_{\mathrm{ad.}}(I)$ when $\mathrm{supp}(\sigma) \subset I_{t)} := I \cap (-\infty, t)$. Notice that the maps $\iota^{\pm}: (\sigma,t) \mapsto \sigma \cup \{t^{\pm}\}$ are injective $\Gamma_{\mathrm{ad.}} \to \Gamma$ with images

$$\Gamma^{\pm} := \{\sigma \in \Gamma : \sigma \neq \varnothing, \max\sigma \text{ has charge } \pm\}$$

and, that if $\Gamma^0 = \{\varnothing\}$ then

$$\Gamma^0 \cup \Gamma^+ \cup \Gamma^- \tag{3.1}$$

is a disjoint partition of Γ.

3.1 Smooth adapted processes.

Any map $x: \Gamma_{\mathrm{ad.}} \to \mathcal{A}$ determines a map $k_x: \Gamma \to \mathcal{A}$ by

$$k_x(\sigma) = \begin{cases} x(\sigma\backslash\{\max\sigma\}, \max\sigma) & \text{if } \sigma \neq \varnothing, \\ 0 & \text{if } \sigma = \varnothing. \end{cases} \tag{3.2}$$

Definition 3.1.1: $x : \Gamma_{\text{ad.}}(I) \to \mathcal{A}$ is a *smooth adapted (kernel) process* if for each compact sub-interval J of I, $k_x^J := \chi_{\Gamma_J} k_x$ belongs to $\mathcal{K}_0^{\mathcal{A}}(J)$. We denote the class of smooth adapted processes by $\mathcal{P}_0^{\mathcal{A}}(I)$.

Each smooth adapted process x will be considered as a function on the whole of $\nabla_I \times I$ as follows:

$$x(\sigma,t) = \begin{cases} \lim\limits_{n\to\infty} x(\sigma_n,t) & \text{if } \exists\ \sigma_n \in \Gamma(I_t) \text{ s.t. } \rho(\sigma_n,\sigma) \to 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$, $\sigma \in \nabla_I$, let x_t, x^σ denote the functions $x(\cdot,t)$, $x(\sigma,\cdot)$ on ∇_I and I respectively. Thus, for each $\sigma \in \nabla_I$, x^σ is a locally Lipschitz function on $I_{[\max\sigma}$ and, for each $t \in I$, $x_t \in \mathcal{K}_0^{\mathcal{A}}(I_t)$. In particular, since $\Gamma(\{0\})$ (= $\nabla(\{0\})$) = $\{\varnothing\}$, $x_0 = a\delta_\varnothing$ for some $a \in \mathcal{A}$. If $x, y \in \mathcal{P}_0^{\mathcal{A}}(I)$, then for each $t \in I$, $x_t * y_t \in \mathcal{K}_0^{\mathcal{A}}(I_t)$—in fact, $*$ extends to $\mathcal{P}_0$:

$$(\sigma,t) \mapsto x_t * y_t(\sigma), \qquad (\sigma,t) \in \Gamma_{\text{ad.}}(I)$$

determines an element of $\mathcal{P}_0^{\mathcal{A}}(I)$, denoted $x * y$.

3.2 Kernel differential and integral operators

We are now in a position to introduce the differential operators of the kernel calculus.

Definition 3.2.1: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$ let $\Delta^+x, \Delta^-x : \Gamma_{\text{ad.}}(I) \to \mathcal{A}$ be given by

$$\Delta^+x(\sigma,t) = x(\sigma \cup \{t^+\}, t)$$

$$\Delta^-x(\sigma,t) = x(\sigma \cup \{t^-\}, t)$$

where for each t the continuous extension of x_t to ∇_{I_t} is invoked (see (3.3)).

Proposition 3.2.2: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $\Delta^\pm x \in \mathcal{P}_0^{\mathcal{A}}(I)$.

Proof:: Let $\sigma \in \Gamma_I$, $\sigma \neq \varnothing$, then

$$\begin{aligned} k_{\Delta^\pm x}(\sigma) &= \Delta^\pm x(\sigma\backslash\{\max\sigma\}, \max(\sigma)) \\ &= x(\sigma', \max\sigma) \text{ where } \sigma' = \sigma\backslash\{\max\sigma\} \cup \{\max\sigma\}^+ \end{aligned}$$

$$= k_x(\sigma' \cup \{\max\sigma\}^{\pm}),$$

so for $\sigma \in \Gamma_J$, J a compact subinterval of I,

$$\|k_{\Delta^{\pm}x}(\sigma)\| < (K_x^J)^{|\sigma|+2} \leq \{(K_x^J)^2\}^{|\sigma|+1}.$$

If $\sigma, \tau \in \Gamma_J$ and $\rho_0(\sigma, \tau) < 1$ then

$$\|k_{\Delta^{\pm}x}(\sigma) - k_{\Delta^{\pm}x}(\tau)\| = \|k_x(\sigma' \cup \{\max\sigma\}^{\pm}) - k_x(\tau' \cup \{\max\tau^{\pm}\| < \rho_0(\sigma, \tau)(K_x^J)^{|\sigma|+2}$$

$$\leq \rho_0(\sigma, \tau)([K_x^J]^2)^{|\sigma|+1}$$

and the proof is complete. ■

Definition 3.2.3: Let $\mathcal{P}_d = \{x \in \mathcal{P}_0 : x^{\sigma}$ is differentiable on $I_{(\max\sigma}\ \forall \sigma \in \Gamma_I\}$ and, for $x \in \mathcal{P}_d$ let $\Delta^{\circ}x : \Gamma_{\text{ad.}} \to \mathcal{A}$ be given by

$$\Delta^{\circ}x(\sigma, t) = (x^{\sigma})'(t).$$

Let $\mathcal{P}_1 = \{x \in P_d : \Delta^{\circ}x \in P_0\}$— the *domain* of the *pathwise derivative operator* Δ°.

We next introduce the integral operators.

Definition 3.2.4: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $(\sigma, t) \in \Gamma_{\text{ad.}}(I)$ let

$$I^{\pm}x(\sigma, t) = \begin{cases} x(\sigma \setminus \{\max\sigma\}, \max\sigma) & \text{if } \sigma \in \Gamma^{\pm}, \\ 0 & \text{otherwise;} \end{cases}$$

$$I^{\circ}x(\sigma, t) = \int_0^t x^{\sigma}(s)\, ds.$$

These define smooth adapted processes $I^+x, I^-x, I^{\circ}x$ and the relations

$$\Delta^+I^+ = \Delta^-I^- = \Delta^{\circ}I^{\circ} = \text{id}_{\mathcal{P}_0^{\mathcal{A}}(I)} \tag{3.4}$$

are immediate. Moreover, the following *fundamental theorem* holds:

Proposition 3.2.5: For $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$

$$x_t - x_0 = I^+\Delta^+x_t + I^-\Delta^-x_t + I^{\circ}\Delta^{\circ}x_t. \tag{3.5}$$

Proof: On Γ^0 (3.5) is an immediate consequence of the fundamental theorem of calculus since the first two terms on the right hand side vanish. On Γ^+, $I^-\Delta^- x(\sigma,t) = 0$, $I^+\Delta^+ x(\sigma,t) = x(\sigma,\max\sigma)$ and

$$I^\circ\Delta^\circ x(\sigma,t) = \int_0^t (x^\sigma)'(s)\,ds = \int_{\max\sigma}^t (x^\sigma)'(s)\,ds = x(\sigma,t) - x(\sigma,\max\sigma)$$

again by the fundamental theorem of calculus. Since x_0 vanishes on Γ^+, (3.5) holds there. Similarly the identity is valid on Γ^-. ■

3.3 Itô relation.

Lemma 3.3.1: Let $x \in \mathcal{P}_0^{\mathcal{A}}(I)$, $t \in I$, then

$$\int_0^t \int_{\Gamma_I} x_s\,d\mu\,ds = c_+^{-1}\int_{\Gamma_I} (I^+x)_t\,d\mu = c_-^{-1}\int_{\Gamma_I} (I^-x)_t\,d\mu.$$

Proof:

$$\int_{\Gamma_I} (I^+x)_t\,d\mu = \int_{\Gamma^+_{I_{[t]}}} (I^+x)(\sigma,t)\,d\sigma$$

$$= \int_{\Gamma^+_{I_{[t]}}} x(\sigma\setminus\{\max\sigma\}, \max(\sigma))\,d\sigma$$

$$= c_+\int_0^t \int_{\Gamma_I} x(\tau,s)\,d\tau\,ds,$$

which gives the first equality. For the second replace + by −. ■

Proposition 3.3.2: For $x \in \mathcal{P}_1^{\mathcal{A}}(I)$, $t \mapsto \int_{\Gamma_I} x_t\,d\mu$ is differentiable on I with derivative

$$\int_{\Gamma_I} \sum_\kappa c_\kappa \Delta^\kappa x_t\,d\mu, \quad (c_0 = 1).$$

Proof: $\int_\Gamma x_t\,d\mu - x_t(\varnothing) = \int_{\Gamma^+\cup\Gamma^-} \sum_\kappa I^\kappa\Delta^\kappa x_t\,d\mu$

$$= \int_{\Gamma^+} I^+\Delta^+ x_t \, d\mu + \int_{\Gamma^-} I^-\Delta^- x_t \, d\mu + \int_{\Gamma\setminus\Gamma^0} I^\circ\Delta^\circ x_t \, d\mu$$

$$= \int_0^t \int_\Gamma \{c_+\Delta^+ x_s + c_-\Delta^- x_s\} \, d\mu \, ds + \int_{\Gamma\setminus\Gamma^0} \left\{\int_0^t (x^\sigma)'(s) \, ds\right\} d\sigma,$$

but

$$x_t(\varnothing) = \int_{\Gamma^0} x^\sigma(t) \, d\sigma = x(\varnothing, 0) + \int_{\Gamma^0} \left\{\int_0^t (x^\sigma)'(s) \, ds\right\} d\sigma,$$

so that the result follows by an application of Fubini's theorem.

■

Proposition 3.3.3: The operators $\Delta^\pm$ are derivations on $(\mathcal{P}_0^{\Delta}(I), *)$:

$$\Delta^\pm(x * y) = \Delta^\pm x * y + x * \Delta^\pm y. \tag{3.6}$$

Moreover, if $x, y \in \mathcal{P}_1$ then $x * y \in \mathcal{P}_d$ and Δ° satisfies the Itô rule

$$\Delta^\circ(x * y) = \Delta^\circ x * y + x * \Delta^\circ y + c_+\Delta^- x * \Delta^+ y + c_-\Delta^+ x * \Delta^- y. \tag{3.7}$$

In particular $(\mathcal{P}_1, *)$ is a subalgebra of $(\mathcal{P}_0, *)$.

Proof: For $x, y \in \mathcal{P}_0$ and $(\sigma, t) \in \Gamma_{\text{ad.}}$.

$$\Delta^+(x * y)_t(\sigma) = x * y(\sigma \cup \{t^+\}, t)$$

$$= \int \sum\nolimits_{\alpha \subset \sigma \cup \{t^+\}} x_t(\alpha \cup \omega^\dagger) y_t(\bar{\alpha} \cup \omega) \, d\omega$$

$$= \int \sum\nolimits_{\beta \subset \sigma} x_t(\beta \cup \{t^+\} \cup \omega^\dagger) y_t(\bar{\beta} \cup \omega) \, d\omega$$

$$+ \int \sum\nolimits_{\gamma \subset \sigma} x_t(\gamma \cup \omega^\dagger) y_t(\bar{\gamma} \cup \{t^+\} \cup \omega) \, d\omega$$

$$= \Delta^+ x_t * y_t(\sigma) + x_t * \Delta^+ y_t(\sigma)$$

and since the same holds for Δ^-, (3.6) follows. To prove the Itô relation, let $x, y \in \mathcal{P}_1$, and $(\sigma, t_0) \in \Gamma_{\text{ad.}}$. Since $t_0 > \max\sigma$ there is an interval $I_1 := [t_0 - \varepsilon, t_0 + \varepsilon)$ not containing any point of σ. Let $I_2 := [0, t_0 - \varepsilon)$ denote the remaining part of $I_{t_0]}$ and, for $\alpha, \beta \in \Gamma(I_2)$ let the process $z^{\alpha,\beta} \in \mathcal{P}_1(I_1)$ be

given by

$$z^{\alpha,\beta}\colon (\omega_1, t) \to x_t(\alpha \cup \omega_1^\dagger) y_t(\beta \cup \omega_1)$$

so that $\Delta^{\pm} z_t^{\alpha,\beta}(\omega_1) = \Delta^{\mp} x_t(\alpha \cup \omega_1^\dagger) \Delta^{\pm} y_t(\beta \cup \omega_1)$. Now for $\alpha, \beta, \omega_2 \in \Gamma(I_2)$ and $t \in I_1$, put

$$f_t^{\alpha,\beta}(\omega_2) = \int_{\Gamma_{I_1}} z_t^{\alpha \cup \omega_2^\dagger, \beta \cup \omega_2}(\omega_1)\, d\omega_1.$$

Then by the previous proposition, $t \to f_t^{\alpha,\beta}(\omega_2)$ is differentiable at t_0 with derivative

$$\begin{aligned}
&\int_{\Gamma_{I_1}} \sum_\kappa c_\kappa \left(\Delta^\kappa z^{\alpha \cup \omega_2^\dagger, \beta \cup \omega_2}\right)(\omega_1, t_0)\, d\mu(\omega_1) \\
&\quad = \int_{\Gamma_{I_1}} \Big\{ c_+ \Delta^- x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^+ y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \\
&\qquad + c_- \Delta^+ x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^- y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \\
&\qquad + \Delta^\circ x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \\
&\qquad + x_{t_0}(\alpha \cup \omega_2^\dagger \cup \omega_1^\dagger) \Delta^\circ y_{t_0}(\beta \cup \omega_2 \cup \omega_1) \Big\} d\omega_1.
\end{aligned}$$

Since this derivative (considered as a function of ω_2) is dominated by an (integrable) function of the form $\omega_2 \to K^{|\omega_2|+1}$, we may conclude that $(x * y)^\sigma$ is differentiable at t_0 with derivative

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=t_0} \int_{\Gamma_{I_2}} \sum_{\alpha \subset \sigma} f_t^{\alpha, \bar{\alpha}}(\omega_2)\, d\omega_2 \\
&\quad = \int_{\Gamma(I_2)} \sum_{\alpha \subset \sigma} \frac{d}{dt}\Big|_{t=t_0} f_t^{\alpha, \bar{\alpha}}(\omega_2)\, d\omega_2 \\
&\quad = c_+ \Delta^- x_{t_0}(\sigma) * \Delta^+ y_{t_0}(\sigma) + c_- \Delta^+ x_{t_0} * \Delta^- y_{t_0}(\sigma) + \Delta x_{t_0} * y_{t_0}(\sigma) + x_{t_0} * \Delta y_{t_0}(\sigma),
\end{aligned}$$

in other words (3.7). Each term on the right hand side of (3.7) being $\mathcal{P}_0$, the process $x * y$ must be $\mathcal{P}_1$. ∎

4. Kernel differential equations.

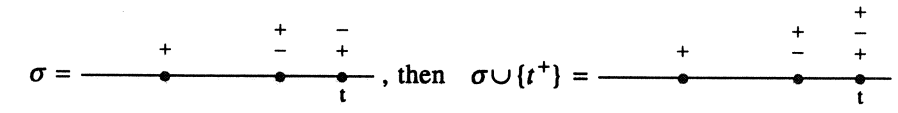
We now demonstrate the ease with which linear stochastic differential equations may be treated in this kernel calculus—moreover we obtain an explicit form for the solutions of such equations (4.3). Again let I have left end point 0. We first extend the definition of Δ^+ and Δ^- as follows. Let

$$\nabla_{\text{ad.}}(I) = \{(\sigma,t) \in \nabla_I \times I : t \geq \max\sigma \text{ or } \sigma = \varnothing\}$$

and, for a function x on $\nabla_{\text{ad.}}(I)$, let

$$\Delta^{\pm}x(\sigma,t) = x(\sigma \cup \{t^{\pm}\}, t)$$

with the convention that if for example



By an adapted (kernel) process we simply mean an $\mathcal{A}$-valued function on $\nabla_{\text{ad.}}(I)$.

4.1 Existence and uniqueness.

A *linear kernel differential equation* is a system

$$\Delta x_t = L(t)x_t, \tag{4.1}$$

where $L : t \to (L^+(t), L^-(t), L^\circ(t)) \in \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A}) \times \mathcal{L}(\mathcal{A})$, and a solution of (4.1) is an adapted process x for which the left-hand side is defined (i.e. each path x^σ is differentiable on $I_{[\max\sigma}$ and $\Delta^\kappa x(\sigma,t) = L^\kappa(t)[x(\sigma,t)]$ for all $(\sigma,t) \in \nabla_{\text{ad.}}(I)$). Thus an adapted process x satisfies (4.1) if and only if

$$\text{(i)}\quad x^\sigma(\max\sigma) = L^{\pm}(\max\sigma)\big[x(\sigma\backslash\{\max\sigma\}, \max\sigma)\big], \quad \sigma \in \nabla^{\pm}; \tag{4.2}$$

$$\text{(ii)}\quad \frac{\mathrm{d}}{\mathrm{d}t}x^\sigma(t) = L^\circ(t)\big[x^\sigma(t)\big] \text{ for } t \geq \max\sigma, \quad \sigma \in \nabla.$$

Note that x satisfies (4.1) if and only if $x^\dagger$ satisfies the *adjoint* k.d.e.:

$$\Delta y_t = L^\dagger(t)y_t,$$

where $(L^\dagger)^\pm(t)[b] = (L^\mp(t)[b^*])^*$ and $(L^\dagger)^\circ(t)[b] = \big(L^\circ(t)[b^*]\big)^*$ $(b \in \mathcal{A})$.

Theorem 4.1.1: Let $L^\circ : I \to \mathcal{L}(\mathcal{A})$ be strongly continuous and locally uniformly bounded. Then for each $b \in \mathcal{A}$ there is a unique solution to (4.1) for which $x(\varnothing, 0) = b$.

Proof: Let $y : \Gamma_{\text{ad.}}(I) \to \mathcal{A}$ be given by

$$y(\sigma, t) = V(t, s_n)L^{\varepsilon_n}(s_n)V(s_n, s_{n-1})\ldots L^{\varepsilon_1}V(s, 0)[b] \tag{4.3}$$

if $\sigma = (s, \varepsilon) \in \Gamma^n$, where $V : I \times I \to \mathcal{L}(\mathcal{A})$ is the solution of the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}V(t, s) = L(t)V(t, s) \qquad V(s, s) = \mathrm{id}_{\mathcal{A}}, \quad (s, t \in I).$$

y defines a pathwise differentiable process satisfying (4.2) and $y(\varnothing, 0) = b$. Moreover it is clearly the unique such process. ∎

One could define and solve non-linear kernel differential equations, however, these would appear to be uninteresting from the point of view of corresponding operator stochastic differential equations. We next isolate sufficient conditions for the solution to be a smooth adapted process.

Theorem 4.1.2: Let $L : I \to \mathcal{L}(\mathcal{A})^3$ be locally Lipschitz. Then the unique solution to (4.1) belongs to $\mathcal{P}_1^{\mathcal{A}}(I)$.

Proof: For a compact subinterval J of I let

$$L_J = \max_\kappa \sup_{t \in J} \|L^\kappa(t)\|_{\mathcal{L}(\mathcal{A})}; \qquad V_J = \exp\{\lambda_1(J)L_J\}$$

and let M_J be a Lipschitz constant for L on J, so that

$$\|V(t, s)\| \leqslant V_J; \qquad \|V(t, s) - V(t', s')\| \leqslant 2\eta L_J V_J^2, \quad (s, t, s', t' \in J),$$

as soon as $|t - t'|, |s - s'| < \eta$. The estimate

$$\|k_v(\tau)\| \leqslant V_J^{|\tau|} L_J^{|\tau|-1} \|b\|, \quad (\tau \in \Gamma_J \setminus \{\varnothing\}),$$

ensures that k_v^J satisfies $\mathcal{K}$ii. Now suppose that $\rho(\tau, \tau') < \eta \leqslant 1$ and let

$$z_{2j} = L^{\varepsilon_j}(t_j);\quad z'_{2j} = L^{\varepsilon_j}(t'_j);\quad z_{2j-1} = V(t_j, t_{j-1});\quad z'_{2j-1} = V(t'_j, t'_{j-1}),$$

where $j = 1,\ldots,n$ and $\tau = (t,\varepsilon)$, $\tau' = (t',\varepsilon') \in \mathcal{V}^n(J)$. Then

$$\begin{aligned}\|k_v(\tau) - k_v(\tau')\| &= \|(z_{2n-1}\ldots z_1 - z'_{2n-1}\ldots z'_1)b\| \\ &= \left|\sum_{j=1}^{2n-1} z_{2n-1}\ldots z_{j+1}(z_j - z'_j)z'_{j-1}\ldots z'_1 b\right| \\ &\leqslant \left\{2\eta n L_J^n V_J^{n+1} + \eta(n-1)L_J^{n-1}V_J^n M_J\right\}\|b\|,\end{aligned}$$

so that k_v^J also satisfies $\mathcal{K}$iii. Thus $v \in \mathcal{P}_0^{\mathcal{A}}(I)$, but since L° is locally Lipschitz it maps $\mathcal{P}_0$ into itself and $\Delta^\circ x = L^\circ x \in \mathcal{P}_0$, that is $x \in \mathcal{P}_1$.

■

4.2 Unitarity.

Now consider the following important special case. Let $q^\kappa : I \to \mathcal{A}$ be locally Lipschitz and let $x \in \mathcal{P}_1$ satisfy the k.d.e.

$$\Delta x_t = q(t)x_t. \tag{4.4}$$

Then one calculates, using the Itô rule (3.7) that for all $b \in \mathcal{A}$,

$$\Delta(x^\dagger * bx)_t = x_t^\dagger * L_q(t)[b]x_t \tag{4.5}$$

and

$$\Delta(x * bx^\dagger)_t = M_q(t)(x * bx^\dagger)_t \tag{4.6}$$

where L_q and M_q are given by

$$\begin{aligned}L_q^\pm(t)[b] &= q^\mp(t)^* b + bq^\pm(t), \\ L_q^\circ(t)[b] &= q^\circ(t)^* b + bq^\circ(t) + c_+ q^+(t)^* bq^+(t) + c_- q^-(t)^* bq^-(t); \qquad (4.7) \\ M_q^\pm(t)[b] &= q^\pm(t)b + bq^\mp(t)^*, \\ M_q^\circ(t)[b] &= q^\circ(t)b + bq^\circ(t)^* + c_+ q^-(t)bq^-(t)^* + c_- q^+(t)bq^+(t)^*. \qquad (4.8)\end{aligned}$$

Proposition 4.2.1: Let $x \in \mathcal{P}_1$ satisfy the k.d.e. (4.4) with initial conditions $x_0 = u_0\delta_\varnothing$ for some unitary $u_0 \in \mathcal{A}$. Then the following are equivalent:

(i) $x*x^\dagger = \mathbb{1}_{\mathcal{A}}\delta_\varnothing$;

(ii) $L_q(\cdot)[\mathbb{1}_{\mathcal{A}}] = 0$;

(iii) $x*x^\dagger = x^\dagger*x = \mathbb{1}_{\mathcal{A}}\delta_\varnothing$.

Moreover, if $t \to q(t)$ is constant, then these are equivalent to

(iv) $x^\dagger*x = \mathbb{1}_{\mathcal{A}}\delta_\varnothing$.

Proof: First note that for all t: $L(t)[\mathbb{1}] = 0 \Leftrightarrow M(t)[\mathbb{1}] = 0$.
(i) $\Rightarrow$ (ii): If $x*x^\dagger = \mathbb{1}_{\mathcal{A}}\delta_\varnothing$ then $M(t)[\mathbb{1}] = 0\ \forall t$ by (4.6).
(ii) $\Rightarrow$ (iii): If $L(\cdot)[\mathbb{1}] = 0$ then, by (4.5), $x^\dagger*x - \mathbb{1}\delta_\varnothing$ satisfies $\Delta y = 0$; $y_0 = 0$ and, by (4.6), $x*x^\dagger - \mathbb{1}\delta_\varnothing$ satisfies $\Delta y_t = M(t)y_t$; $y_0 = 0$, but the unique solution of these is 0.
(iv) $\Rightarrow$ (ii): If $x^\dagger*x = \mathbb{1}\delta_\varnothing$ then, by (4.5), $x^\dagger*L(\cdot)[\mathbb{1}]x = 0$. If q is constant then, since x_0 is *unitary*, $L[\mathbb{1}] = x_0*(x_0^\dagger*L[\mathbb{1}]x_0)*x_0^\dagger = 0$.
Since (iii) obviously implies (i) and (iv) the proof is complete. ■

Remark: Under the equivalent conditions of the above proposition

$$q(t) = \begin{pmatrix} v(t) \\ -v(t)^* \\ -\frac{1}{2}[c_+v(t)^*v(t) + c_-v(t)v(t)^*] + ih(t) \end{pmatrix}$$

for certain Lipschitz functions $h, v : I \to \mathcal{A}$ such that $h(t) = h(t)^*$. In particular, if q is constant, L_q° is given by

$$L_q^\circ(b) = L_v(b) - i[h, b], \tag{4.9}$$

where

$$L_v(b) = c_+\left\{v^*bv - \tfrac{1}{2}(v^*vb + bv^*v)\right\} + c_-\left\{vbv^* - \tfrac{1}{2}(vv^*b + bvv^*)\right\} \tag{4.10}$$

4.3 Adapted kernel cocycles

Let III_t ($t \in \mathbb{R}$) denote the right shift on functions defined on $\mathcal{V}(\mathbb{R})$:

$$(\mathit{III}_t f)(\omega) = f(\omega - t),$$

where $(s,\varepsilon)-t := (s-t,\varepsilon)$. The following class of kernel processes will play a role in the construction of quantum Markov processes.

Definition 4.3.1: $x = \{x_t : t \geqslant 0\}$ is an *adapted kernel cocycle* if

(akc i) $x \in \mathcal{P}_0^{\mathcal{A}}(\mathbb{R}_+)$;

(akc ii) $x_{s+t} = \mathit{\Pi}_s(x_t) * x_s; \qquad x_0 = \mathbb{1}_{\mathcal{A}}\delta_\varnothing, \quad (s,t \geqslant 0).$

Remark: If x is an adapted kernel cocycle then the two parameter family $\{x_{s,t} := \mathit{\Pi}_s(x_{t-s}),\ s \leqslant t\}$ satisfies

(i)′ $\quad \operatorname{supp} x_{r,t} \in \nabla_{[r,t]}$,

(ii)′ $\quad x_{r,t} = x_{s,t} * x_{r,s}\ ;\ x_{0,0} = \mathbb{1}_{\mathcal{A}}\delta_\varnothing;$

(iii)′ $\quad \mathit{\Pi}_u(x_{s,t}) = x_{s+u,t+u};$

(iv)′ $\quad t \to x_{0,t} \in \mathcal{P}_0^{\mathcal{A}}(\mathbb{R}_+)$, for all $(r \leqslant s \leqslant t,\ u \in \mathbb{R})$.

Conversely a two parameter family of kernels $\{x_{s,t} : s \leqslant t\}$ satisfying (i)′-(iv)′ determines an adapted cocycle: $\{x_{0,t} : t \geqslant 0\}$.

Proposition 4.3.2: For a family of kernels $x := \{x_t : t \geqslant 0\}$ the following are equivalent:

(a) x is an adapted kernel cocycle;

(b) x satisfies a kernel differential equation of the form

$$\Delta x = qx, \quad (q \in \mathcal{A}^3);$$

$$x_0 = \mathbb{1}_{\mathcal{A}}\delta_\varnothing; \tag{4.11}$$

(c)

$$x_t(\sigma) = \begin{cases} e^{(t-s_n)q}\, q^{\varepsilon_n}\, e^{(s_n-s_{n-1})q} \ \ldots\ q^{\varepsilon_1}\, e^{s_1 q} & \text{if } \sigma = (s,\varepsilon) \in \nabla_{[0,t]} \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

Proof: The equivalence of (b) and (c) is contained in the proof of Theorem 4.1.1 while if x satisfies (4.11) then, by Theorem 4.1.2, $x \in \mathcal{P}_1$, and (akc ii) follows from the explicit expression for x (4.12); it therefore remains to

establish the implication (a) $\Rightarrow$ (c).

Let x be an adapted kernel cocycle, and for $s \leqslant t$ let $x_{s,t} = \mathit{III}_s(x_{t-s})$. Then $t \to x_t(\varnothing)$ is continuous, by (akc i), and a semigroup since

$$\begin{aligned} x_{s+t}(\varnothing) &= x_{t,s+t} * x_{0,t}(\varnothing) \quad \text{by } (ii)' \\ &= x_{t,s+t}(\varnothing)x_{0,t}(\varnothing) \quad \text{by } (i)' \\ &= x_s(\varnothing)x_t(\varnothing) \quad (\text{since } \varnothing - t = \varnothing). \end{aligned}$$

Let q be its generator. For $0 < \eta < t$,

$$x_{t-\eta,t+\eta}(\{t^{\pm}\}) = (\mathit{III}_{t-\eta}x_{2\eta})(\{t^{\pm}\}).$$

So, by (iv)', $q^{\pm} = \lim_{\eta \downarrow 0} x_{t-\eta,t+\eta}(\{t^{\pm}\})$ exists and is independent of t. By repeated application of (ii)', if $(\sigma, t) \in \Gamma_{\text{ad.}}(\mathbb{R}_+)$ then

$$x_t(\sigma) = p_{(t-s_n-\eta)}x_{s_n-\eta,s_n+\eta}(\{(s_n,\varepsilon_n)\})p_{(s_n-s_{n-1}-2\eta)} \cdots x_{s_1-\eta,s_1+\eta}(\{(s_1,\varepsilon_1)\})p_{s_1-\eta}$$

for $0 < \eta < \min_{i,j}|s_i - s_j|$, where $\sigma = (s,\varepsilon)$. Finally, letting $\eta \downarrow 0$ we obtain (4.12) and the proof is complete. ■

4.4 Generator:

The *generator* of an adapted kernel cocycle is the $q \in \mathcal{A}^3$ which determines its explicit expression (4.12).

Proposition 4.4.1: Let x be an adapted kernel cocycle with generator q, then $T_t : b \to x_t^{\dagger} * bx_t(\varnothing)$ $(t \in \mathbb{R}_+)$ is a one parameter semigroup on $\mathcal{A}$ with generator L_q°.

Proof: The third component of (4.5), evaluated at $\varnothing$, reads

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t(b) = T_t(L_q^{\circ}[b]).$$

Since $T_0(b) = b$ the result follows. ■

5. Quantum probability.

We now describe some theory from operator algebras, associated unbounded operators and quantum dynamical semigroups. Takesaki's books [Ta 1,3] are standard references for the operator algebra theory. We also present Kümmerer's formulation of quantum dynamical semigroups and their dilations (in which invariance of the state is incorporated in the definitions) [Küm]. The material of this section will be used to construct a stochastic calculus for operator valued processes from the kernel calculus described above, thereby streamlining the existing constructions [BSW 2], [HL 1,2], [L 1].

5.1 Some generalities

A *von Neumann algebra* acting on a Hilbert space $\mathfrak{h}$ is a unital $*$-subalgebra of $\mathcal{L}(\mathfrak{h})$, the algebra of bounded linear operators on $\mathfrak{h}$, which is closed in the strong operator topology. For a subset $\mathcal{X}$ of $\mathcal{L}(\mathfrak{h})$, $\mathcal{X}'$ denotes its *commutant*: $\{T \in \mathcal{L}(\mathfrak{h}) : TX = XT \ \forall X \in \mathcal{X}\}$. If $\mathcal{X}$ is self-adjoint then $\mathcal{X}'$ is a von Neumann algebra, and the von Neumann algebra *generated* by $\mathcal{X}$ is $(\mathcal{X}')'$. The tensor product $\mathcal{B}_1 \otimes \mathcal{B}_2$ of two von Neumann algebras $\mathcal{B}_1, \mathcal{B}_2$ acting on $\mathfrak{h}_1, \mathfrak{h}_2$ respectively, is the von Neumann subalgebra $\{T_1 \otimes T_2 : T_i \in \mathcal{B}_i\}''$ of $\mathcal{L}(\mathfrak{h}_1 \otimes \mathfrak{h}_2)$. The non-trivial relation $(\mathcal{B}_1 \otimes \mathcal{B}_2)' = \mathcal{B}_1' \otimes \mathcal{B}_2'$ holds.

An unbounded operator cannot *belong* to a von Neumann algebra $\mathcal{B}$, however we say that an operator T is *affiliated* to $\mathcal{B}$ (written $T \eta \mathcal{B}$) if $\mathcal{B}'\mathrm{Dom}(T) \subset \mathrm{Dom}(T)$ and $TB'\varphi = B'T\varphi \ \forall \varphi \in \mathrm{Dom}(T)$, $B' \in \mathcal{B}'$ where $\mathrm{Dom}(T)$ denotes the *domain* of T. Equivalently if $\mathrm{Gr}(T)$, the graph of T, is considered as a subspace of $\mathfrak{h} \otimes \mathbb{C}^2 = \mathfrak{h} \oplus \mathfrak{h}$, T is affiliated to $\mathcal{B}$ if and only if $(\mathcal{B}' \otimes I)\mathrm{Gr}(T) \subset \mathrm{Gr}(T)$. When T is closed this is equivalent to $P_{\mathrm{Gr}(T)} \in \mathcal{B} \otimes M_2(\mathbb{C}) = M_2(\mathcal{B})$, where $P_{\mathrm{Gr}(T)}$ is the orthogonal projection onto the *graph* of T. If $T \eta \mathcal{B}$ then $T^* \eta \mathcal{B}$, in particular, if $T \eta \mathcal{B}$ is closable then its closure $\bar{T}$ is also affiliated. For operators X and Y, Y is an *extension of* X (or X is a restriction of Y), written $X \subset Y$, means that $\mathrm{Gr}(X)$ is a subspace of $\mathrm{Gr}(Y)$.

The following result will be useful later.

Proposition 5.1.1: Let $\mathcal{B}$ be a von Neumann algebra and $X \eta \mathcal{B}$ be closed. If $T \subset X$ and $\mathcal{C}_0 \mathrm{Dom}(T) \subset \mathrm{Dom}(T)$ for some strongly dense $*$-subalgebra $\mathcal{C}_0$ of

$\mathcal{B}'$, then $\bar{T}\eta\mathcal{B}$.

Proof: Since Dom(T) is invariant under $\mathcal{C}_0$ and T has an affiliated extension, Gr(T) is invariant under $\mathcal{C}_0 \otimes I$. Hence $P_{\mathrm{Gr}(\bar{T})} \in (\mathcal{C}_0 \otimes I)' = (\mathcal{B}' \otimes \mathbb{C}I)' = \mathcal{B} \otimes M_2(\mathbb{C})$. ∎

5.2 Quantum probability spaces.

A *quantum probability space* Q is a triple $(\mathfrak{h}, \mathcal{B}, \xi)$ where $\mathfrak{h}$ is a Hilbert space, $\mathcal{B} \subset \mathcal{L}(\mathfrak{h})$ is a von Neumann algebra and $\xi \in \mathfrak{h}$ is a vector which is both

$$\textit{cyclic} : \mathcal{B}\,\xi \text{ is dense in } \mathfrak{h}$$

$$\text{and } \textit{separating} : T \in \mathcal{B}, T\xi = 0 \Rightarrow T = 0 \tag{5.1}$$

for $\mathcal{B}$. Associated to a quantum probability space Q are three operators: S_Q, J_Q and Δ_Q. S is the closure of the conjugate linear operator with domain $\mathcal{B}\xi$ which maps $T\xi$ to $T^*\xi$ and $S = J\Delta^{1/2}$ is its polar decomposition—J being a conjugate linear isometric involution and Δ a positive self-adjoint operator. Tomita's fundamental lemma states that

$$J\mathcal{B}J = \mathcal{B}', \tag{5.2}$$

$$\Delta^{it}\mathcal{B}\Delta^{-it} = \mathcal{B}, \quad (t \in \mathbb{R}). \tag{5.3}$$

The map $\sigma^Q : t \to \Delta^{it} \cdot \Delta^{-it}$ on $\mathcal{B}$ is called the *modular automorphism group*.

We shall denote Dom(S), considered as a Hilbert space with the graph norm $x \to \{\|x\|^2 + \|Sx\|^2\}^{1/2}$, by Σ_Q.

5.3 *-affiliation.

Since ξ is cyclic for $\mathcal{B}'$ if (and only if) it is separating for $\mathcal{B}$, the prescription

$$\mathrm{Dom}(\dot{X}) = \mathcal{B}'\xi,$$

$$\dot{X}T'\xi = T'x, \quad (T' \in B'), \tag{5.4}$$

associates to each vector $x \in \mathfrak{h}$, a densely defined operator $\dot{X}$ affiliated to $\mathcal{B}$. In general $\dot{X}$ will fail to be closable, however *if* $\xi \in \mathrm{Dom}(\dot{X}^*)$ then, since $\dot{X}^*\eta\mathcal{B}, \dot{X}^*$ will be densely defined and so $\dot{X}$ will be closable. Now

$\xi \in \mathrm{Dom}(\dot{X}^*)$ if and only if $x \in \Sigma_Q$ in which case $\dot{X}^*\xi = Sx$. This is the class of operators with which we shall be dealing.

Definition 5.3.1: Let $Q = (\mathfrak{h}, \mathcal{B}, \xi)$ be a quantum probability space. A closed operator T is **-affiliated* to Q if

(i) $T\eta\mathcal{B}$

(ii) $\xi \in \mathrm{Dom}(T) \cap \mathrm{Dom}(T^*)$

(iii) $B'\xi$ is a core for T.

The set of *-affiliated operators will be denoted $\eta^*(Q)$.

Let $Q = (\mathfrak{h}, \mathcal{B}, \xi)$ be a quantum probability space. There is a bijective correspondence between vectors x in Σ_Q and operators X *-affiliated to Q which is determined by the relation

$$X\xi = x.$$

$\eta^*(Q)$ is a linear space under the strong sum $X+Y := \overline{(X+Y)|_{\mathcal{B}'\xi}}$ with a conjugation given by

$$X^+ = \overline{X^*|_{\mathcal{B}'\xi}}\ .$$

The strong product $X \cdot Y = \overline{XY|_{\mathcal{B}'\xi}}$ is also defined for pairs of $*$-affiliated operators X, Y for which $\xi \in \mathrm{Dom}(S_Q XY)$. We shall drop the dots with the understanding that sums, differences and products (when defined) are in the strong sense. By Proposition 5.1.1 $*$-affiliated operators have common core $\mathcal{B}_0'\xi$ whenever B_0' is a strongly dense $*$-subalgebra $\mathcal{B}_0'$ of $\mathcal{B}'$.

5.4 Quantum dynamical semigroups.

If $\mathcal{B}_1, \mathcal{B}_2$ are von Neumann algebras then $T \in \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is *completely positive* if

$$T \otimes \mathrm{id}_{M_n} : M_n(\mathcal{B}_1) \to M_n(\mathcal{B}_2), \quad [x_{ij}] \to [T(x_{ij})]$$

is positivity preserving for each n, equivalently if

$$\sum_{i,j=1}^{n} y_i^* T(x_i^* x_j) y_j \geq 0$$

for each $n \in \mathbb{N}$, $x_1,\ldots,x_n \in \mathcal{B}_1$, and $y_1,\ldots,y_n \in \mathcal{B}_2$. A *morphism* between quantum probability spaces $Q_i = (\mathfrak{h}_i, \mathcal{B}_i, \xi_i)$ $(i = 1,2)$ will be an element T of $\mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ satisfying

(i) T is completely positive;

(ii) $T(\mathbb{1}_{\mathcal{B}_1}) = \mathbb{1}_{\mathcal{B}_2}$;

(iii) $\langle \xi_2, T(x)\xi_2 \rangle = \langle \xi_1, x\xi_1 \rangle$, $(x \in \mathcal{B}_1)$.

If $u \in \mathcal{B}$ is unitary then Ad $u : b \to u^*bu$ is a morphism of Q if and only if

$$\sigma_t^Q(u) = u, \quad \forall\, t \in \mathbb{R}. \tag{5.5}$$

A morphism T on Q satisfying

$$T(bc) = T(b)c \qquad (b \in \mathcal{B}, c \in \mathcal{C}),$$

where $\mathcal{C}$ = Range (T), is called a *conditional expectation onto* $\mathcal{C}$. Clearly the range of a conditional expectation is an algebra. Given a subalgebra $\mathcal{C}$ of $\mathcal{B}$, a conditional expectation onto $\mathcal{C}$ exists if and only if $\sigma_t^Q(\mathcal{C}) = \mathcal{C}$ $\forall\, t \in \mathbb{R}$ ([Tak 2]). In particular, contrary to classical probability, conditional expectation onto a subalgebra does not always exist. This is a fundamental distinction between classical and quantum probability.

A *quantum dynamical semigroup* is a one parameter semigroup of morphisms of a quantum probability space which is continuous in the pointwise weak-* topology, that is $t \mapsto \langle \psi, T_t(b)\psi \rangle$ is continuous on $\mathbb{R}_+$ for each $b \in \mathcal{B}$, $\psi \in \mathfrak{h}$. If T is a quantum dynamical semigroup on $Q = (\mathfrak{h}, \mathcal{B}, \xi)$, and each T_t is an automorphism of Q (equivalently, if each T_t is an automorphism of $\mathcal{B}$ preserving the state $\langle \xi, \cdot\, \xi \rangle$) then T extends to a *quantum dynamical group*: $T_t = (T_{-t})^{-1}$, $(t < 0)$.

A *dilation* of a dynamical semigroup $\{T_t^0 : t \geqslant 0\}$ on Q_0 consists of a dynamical group $\{T_t : t \in \mathbb{R}\}$ on a quantum probability space Q together with morphisms $j : Q_0 \to Q$ and $\mathbb{P} : Q \to Q_0$ such that

$$\mathbb{P} \circ T_t \circ j = T_t^0, \quad (t \geqslant 0).$$

j will then be an injective $*$-homomorphism $\mathcal{B}_0 \to \mathcal{B}$ and $j \circ \mathbb{P}$ a conditional expectation. Conditional expectations $\mathbb{E}_I : \mathcal{B} \to \mathcal{B}_I = \{T_t \circ j(\mathcal{B}_0) : t \in I\}$ exist for each subinterval I of $\mathbb{R}$ and a dilation is called *Markov* if

$$\mathbb{E}_{(-\infty,0]} \circ T_t \circ j = \mathbb{E}_{\{0\}} \circ T_t \circ j, \quad (t \geqslant 0). \tag{5.6}$$

6. From kernels to operators.

In this section we construct operators from kernels via the $*$ product, using the results from Section 5, thereby defining a (non-commutative) duality transform between vectors and $*$-affiliated operators of the quantum probability space of interest to us here.

6.1 Canonical commutation relations

In view of Propositions 2.4.5 and 2.4.3 we may define operators $W(f)$, $W'(f)$ and J_1 $(f \in C_K^1(I))$ to be the unique isometric extensions to $L^2(\mathcal{V}_I, \mu)$ of G_{w_f}, D_{v_f} and J_0 respectively (see Definition 2.4.2). The following relations are immediate.

Proposition 6.1.1: For $f, g \in C_K^1(I)$,

(i) $W(f)W(g) = e^{i\zeta(f,g)}W(f+g)$; $W(0) = \mathbb{1}$;

(ii) $W'(f)W'(g) = e^{i\zeta(f,g)}W'(f+g)$; $W'(0) = \mathbb{1}$;

(iii) $W'(f)W(g) = W(g)W'(f)$;

(iv) $W'(f) = J_1 W(\bar{f}) J_1$;

(v) $\langle w_0, W(f)w_0 \rangle = \langle w_0, W'(f)w_0 \rangle$

Let $\mathcal{N}_0$ and $\mathcal{M}_0$ be the linear spans of the sets $\{W(f) : f \in C_K^1(I)\}$ and $\{W'(f) : f \in C_K^1(I)\}$, and let $\mathcal{N} = \mathcal{N}_0''$ and $\mathcal{M} = \mathcal{M}_0''$ be the respective von Neumann algebras they generate. In view of Corollary 2.5.3, w_0 is a cyclic vector for both $\mathcal{N}$ and $\mathcal{M}$, and since $\mathcal{N} \subset \mathcal{M}'$ by 6.1.1 (iii) w_0 is also a separating vector for both $\mathcal{N}$ and $\mathcal{M}$. Let S, J and Δ be the Tomita operators for $(\mathcal{N}, w_0)$ (see Section 5). We now relate these to the operators S_0, J_0 and Γ_0

introduced in Definition 2.4.2.

Proposition 6.1.2:

$$S = \overline{S_0}; \qquad J = J_1 = \overline{J_0}; \qquad \Delta^{1/2} = \overline{\Gamma_0}.$$

Proof: $\mathcal{W} = \mathcal{N}_0 w_0$ is a core for $\Gamma := M_{\sqrt{\gamma}}$ by Corollary 2.5.3 and also a core for S and $\Delta^{1/2}$ by Kaplansky's density theorem [Tak 3]. But $S|_{\mathcal{W}} = S_0$, so $J\Delta^{1/2} = S = \bar{S}_0 = \overline{J_1\Gamma_0} = J_1\Gamma$ and the result follows by the uniqueness of polar decompositions. ■

Corollary 6.1.3:

$$\mathcal{N} = \mathcal{M}'$$

Proof: $\mathcal{M}$ = strong closure of $J\mathcal{N}_0 J$ (by Propositions 6.1.1, 6.1.2)

$= J\mathcal{N}J$ (by the isometry of J)

$= \mathcal{N}'$ (by Tomita's relation (5.1)). ■

W and W' are therefore a pair of commuting (cyclic) representations of the canonical commutation relations over the symplectic space $\left(C_K^1(I), \zeta_{c_+,c_-}\right)$ with generating functional $\hat{\mu}_{c_+,c_-}$ [BrR] which may justifiably be called *commutant representations*.

Notice that the algebras $\mathcal{N}$ and $\mathcal{M}$ are equally the von Neumann algebras generated by *bounded* left and right multiplication operators (in the $*$-sense) respectively:

$$\left\{\overline{G_x} : x \in \mathcal{K}_0^{\mathbb{C}}(I), \sup_{y \neq 0} \frac{\|x * y\|}{\|y\|} < \infty\right\},$$

$$\left\{\overline{D_z} : z \in \mathcal{K}_0^{\mathbb{C}}(I), \sup_{y \neq 0} \frac{\|y * z\|}{\|y\|} < \infty\right\}.$$

We shall denote the quantum probability space $\left(L^2(\mathcal{V}_I, \mu), \mathcal{N}, w_0\right)$ by Q_1^I. It follows from Proposition 6.1.2 that $\Sigma_{Q_1^I} = L^2(\mathcal{V}_I, \nu)$.

6.2 Initial space.

Now let $Q_0 = (\mathfrak{h}_0, \mathfrak{A}_0, \xi_0)$ be a quantum probability space and $Q = Q^I = Q_0 \otimes Q_1^I$, in other words $Q^I = (\mathfrak{h}, \mathfrak{A}, \xi)$ where $\mathfrak{h} = \mathfrak{h}_0 \otimes L^2(\mathcal{V}_I, \mu) = L^2(\mathcal{V}_I, \mu; \mathfrak{h}_0)$, $\mathfrak{A} = \mathfrak{A}_0 \otimes \mathcal{N}$ and $\xi = \xi_0 \otimes w_0$. Then Q^I is a quantum probability space, $\Sigma_{Q^I} = L^2(\mathcal{V}_I, \nu; \Sigma_{Q_0})$ and, (for ν-almost all σ),

$$(\Delta_{Q^I}^{1/2} f)(\sigma) = \sqrt{\gamma(\sigma)} \Delta_{Q_0}^{1/2} [f(\sigma)]$$

$$(S_{Q^I} f)(\sigma) = S_{Q_0}[f(\sigma^\dagger)].$$

6.3 Non-commutative duality transform

We next introduce left multiplication operators for $\mathfrak{A}_0$-valued smooth kernels and show that they are *-affiliated to Q^I.

Lemma 6.3.1: For $x, y \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$, $u, v \in \mathfrak{h}_0$,

$$\langle x(\cdot)u, y(\cdot)v \rangle_{\mathfrak{h}} = \langle u, x^\dagger * y(\varnothing)v \rangle_{\mathfrak{h}_0}.$$

Proof:

$$\int \langle x(\sigma)u, y(\sigma)v \rangle_{\mathfrak{h}_0} \, d\sigma = \langle u, \int_{\mathcal{V}_I} x^\dagger(\sigma^\dagger) y(\sigma) \, d\sigma \, v \rangle_{\mathfrak{h}_0}.$$

■

Definition 6.3.2: For $x \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$ let G_x be the map

$$y(\cdot)v \to (x * y)(\cdot)v, \qquad v \in \mathfrak{h}_0, y \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I).$$

Lemma 6.3.3: (i) G_x is a densely defined closable operator on $\mathfrak{h}$

(ii) $G_x^* \supset G_{x^\dagger}$

(iii) $G_{x_1} G_{x_2} = G_{x_1 * x_2}$

Proof: Since, by an application of Fubini's theorem, $y(\cdot)v = 0$ almost everywhere implies that $x * y(\cdot)v = 0$ almost everywhere, the map is well-defined. Let $x, y, z \in \mathcal{K}_b^{\mathcal{L}(\mathfrak{h}_0)}(I)$, then by Lemma 6.3.1 and the associativity of $*$,

$$\langle y(\cdot)u, (x * z)(\cdot)v \rangle_{\mathfrak{h}} = \langle u, (y^\dagger * x * z)(\varnothing)v \rangle_{\mathfrak{h}_0}$$

$$= \langle x^\dagger * y(\cdot)u,\ z(\cdot)v\rangle_{\mathfrak{h}},\quad (u,v\in\mathfrak{h}_0).$$

So $G_x^* \supset G_{x^\dagger}$ which is densely defined—this proves (i) and (ii). (iii) follows from the associativity of $*$. ∎

Definition 6.3.4: Now let $x\in\mathcal{K}_b^{\mathfrak{A}_0}(I)$ and G_x^0 be the restriction of G_x to $(\mathfrak{A}_0'\otimes_{\text{alg.}}\mathcal{M}_0)\xi$.

Proposition 6.3.5: If $X\eta^*Q^I$ is the $*$-affiliated operator corresponding to the vector $x(\cdot)\xi_0$, where $x\in\mathcal{K}_b^{\mathfrak{A}_0}(I)$, then $\overline{G_x^0} = X \subset \overline{G_x}$.

Proof: $\overline{G_x^0}\subset X$ and, by Proposition 5.1.1, $(\mathfrak{A}_0'\otimes_{\text{alg.}}\mathcal{M})\xi$ is a core for X. ∎

Notation 6.3.6: For $x\in\Sigma_{Q^I}$ let $\hat{x}$ be the corresponding $*$-affiliated operator and for $X\eta^*Q^I$, $X^\vee$ will denote the corresponding vector. Thus

$\hat{x}$ is the closure of the operator $T'\xi\mapsto T'x\quad (T'\in\mathfrak{A}')$

and

$X^\vee$ is the vector $X\xi$.

The next result justifies the name *non-commutative duality transform* for the map $\hat{}$.

Proposition 6.3.7: Let $x,y\in\mathcal{K}_b^{\mathfrak{A}_0}(I)$ be such that $\xi\in\text{Dom}(\hat{x}\hat{y})$, then

$$(x*y)\hat{}\ \subset \hat{x}\hat{y}, \tag{6.1}$$

(where, as in future, we abbreviate $z(\cdot)\xi_0$ to $\hat{z}$ when $z\in\mathcal{K}_b^{\mathfrak{A}_0}$).

Proof: Suppose that $\xi\in\text{Dom}(\hat{x}\hat{y})$, in other words $y\in\text{Dom}(\hat{x})$, then $\hat{x}\hat{y}\xi = \hat{x}y = G_x y = x*y = (x*y)\hat{}\,\xi$ and (6.1) follows. ∎

Corollary 6.3.8: Let $x\in\mathcal{K}_b^{\mathfrak{A}_0}(I)$

(i) $\hat{x} = \mathbb{1}_{\mathfrak{X}} \Leftrightarrow x = \mathbb{1}_{\mathfrak{X}_0}\delta_{\varnothing}$;

(ii) $\hat{x}$ is isometric $\Leftrightarrow x^{\dagger} * x = \mathbb{1}_{\mathfrak{X}_0}\delta_{\varnothing}$.

Proof: (i) is immediate and so (ii) follows from the proposition above. ∎

Notation 6.3.9: For a function f on $\mathcal{V}$ and interval I, let f^I denote the function on $\mathcal{V}$:

$$\sigma \to f(\sigma \cap I).$$

Proposition 6.3.10: Let $x \in \mathcal{K}_b^{\mathfrak{X}_0}(\mathbb{R})$ have support in $\mathcal{V}_I$, then $\mathrm{Dom}(\hat{x}) \supset L^2(\mathcal{V}_{\mathbf{R}\setminus I}; \mathfrak{h}_0)$ and for $\varphi \in L^2(\mathcal{V}_{\mathbf{R}\setminus I}; \mathfrak{h}_0)$

$$\hat{x}\varphi = x^I(\cdot)\varphi^{\mathbf{R}\setminus I}(\cdot). \tag{6.2}$$

Proof: Let $\varphi \in L^2(\mathcal{V}_{\mathbf{R}\setminus I}; \mathfrak{h}_0)$ and choose a sequence $\{T_n' : n = 1, 2, \ldots\}$ in $\mathfrak{X}_0' \otimes_{\mathrm{alg.}} \mathcal{M}_0^{\mathbf{R}\setminus I}$ such that $\varphi_n := T_n'\xi \to \varphi$. Then $\varphi_n \in \mathrm{Dom}(\hat{x})$ and

$$\hat{x}\varphi_n(\sigma) = \sum\nolimits_{\alpha \subset \sigma} x(\alpha)\varphi_n(\bar{\alpha}) = x(\sigma \cap I)\varphi_n(\sigma \cap (\mathbb{R}\setminus I)),$$

so that, by the combinatorial Lemma 2.3.3(a):

$$\begin{aligned}
\|\hat{x}(\varphi_n - \varphi_m)\|_h^2 &= \int_{\mathcal{V}} \|x(\sigma \cap I)(\varphi_n - \varphi_m)(\sigma \cap (\mathbb{R}\setminus I))\|_{\mathfrak{h}_0}^2 \, d\sigma \\
&= \int_{\mathcal{V}} \sum\nolimits_{\alpha \subset \sigma} \|x(\alpha)(\varphi_n - \varphi_m)(\bar{\alpha})\|_{\mathfrak{h}_0}^2 \, d\sigma \\
&= \int_{\mathcal{V}} \int_{\mathcal{V}} \|x(\alpha)(\varphi_n - \varphi_m)(\beta)\|_{\mathfrak{h}_0}^2 \, d\alpha \, d\beta \\
&\leqslant \int_{\mathcal{V}} \|x(\alpha)\|_{\mathfrak{X}_0}^2 \, d\alpha \cdot \|\varphi_n - \varphi_m\|_{\mathfrak{h}}^2 \\
&\to 0 \quad \text{as } n, m \to \infty.
\end{aligned}$$

In other words, since $\hat{x}$ is closed, $\varphi = \lim \varphi_n \in \mathrm{Dom}(\hat{x})$ and (6.2) holds. ∎

Corollary 6.3.11: Let $x \in \mathcal{K}_b^{\mathfrak{A}_0}(\mathbb{R})$. Then $\mathrm{Dom}(\hat{x}) \supset \mathfrak{h}_0\delta_\varnothing$

$$\hat{x}v\delta_\varnothing = x(\cdot)v, \qquad (v \in \mathfrak{h}_0).$$

The modular automorphism group σ^Q is expressed most conveniently with the use of the non-commutative duality transform:

$$\sigma_t^Q(\hat{x}) = \left(\gamma^{it}(\cdot)\sigma_t^{Q_0}(x(\cdot))\right)^\wedge \quad (x \in \Sigma_Q). \tag{6.3}$$

7. Operator stochastic calculus.

In this section we again take I to have left end point 0, and we abbreviate Σ_{Q_0} and Σ_Q to Σ_0 and Σ respectively, where Q_0 is a fixed initial quantum probability space and $Q = Q_0 \otimes Q_1^I$ (Section 6.2).

7.1 Extension of the differential and integral operators

First embed $\mathcal{P}_0$ and $\mathcal{P}_1$ into the set of *adapted kernel processes*

$$\mathfrak{p} := \left\{f \in \mathcal{L}^0(I, \mathcal{B}; \Sigma) : f(t) \in \Sigma_{[t]} \quad \forall\, t\right\}$$

by the prescription

$$x \to x(\cdot)\xi_0,$$

and denote the resulting subspaces of $\mathfrak{p}$ by $\mathfrak{p}_0$ and $\mathfrak{p}_1$ respectively. In this way smooth kernel processes may be thought of as Hilbert space-valued rather than algebra valued. Now consider the *locally square integrable kernel processes* and the *martingale kernels*:

$$\mathfrak{l}^2 := \left\{f \in L^2_{\mathrm{loc.}}(I, \mathcal{B}, \lambda; \Sigma) : f_t \in \Sigma_{[t]} \text{ for almost all } t\right\}$$

$$\mathfrak{m} := \left\{f \in \mathfrak{p} : \chi_{\Gamma_{[0,s]}} f_t = f_s \quad \forall\, s \leqslant t\right\}$$

For $T \in I$, $x \in \mathfrak{l}^2$ let

$$\|x\|_T = \left\{\int_0^T \int_{\Gamma_t} \|x(\sigma, t)\|_{\mathfrak{h}_0}^2 \, \mathrm{d}\sigma \, \mathrm{d}t\right\}^{1/2} = \left\{\int_0^T \|x_t\|_{\mathfrak{h}}^2 \, \mathrm{d}t\right\}^{1/2}$$

—the seminorms $\{\|\cdot\|_T : T \in I\}$ clearly separate $\mathfrak{l}^2$.

The operators I^κ, $\Delta^\pm$ and Δ° extend to $\mathfrak{l}^2$, $\mathfrak{p}_0+\mathfrak{m}$ and $\mathfrak{p}_1+\mathfrak{m}$ respectively with the same definitions as in Section 3. In particular, if $x \in \mathfrak{m}$ then $\Delta^\circ x = 0$ and, for $(\sigma,t) \in \Gamma_{\text{ad.}}(I)$:

$$\Delta^\pm x(\sigma,t) = x(\sigma \cup \{t^\pm\},T), \quad (T > t).$$

Moreover, if $x \in \mathfrak{p}$ and $x_t = 0$ for almost all t, then $I^\pm x_t = 0$ for all t, so these integrals do not distinguish versions. Kernel stochastic integrals of locally square integrable processes are *martingales*, kernel stochastic derivatives of martingale kernels are $\mathfrak{l}^2$ (as is seen by an application of the combinational Lemma 2.3.3(a)), and the fundamental theorem (3.2.5) continues to hold for $\mathfrak{p}_1+\mathfrak{m}$ (Proposition 7.5.1 establishes this). The new element here is the isometry/orthogonality relation.

Proposition 7.1.1: Let $x, y \in \mathfrak{l}^2$, then for $T \in I$

$$\|I^+x_T + I^-y_T\|_{\mathfrak{h}}^2 = c_+\|x\|_T^2 + c_-\|y\|_T^2. \tag{7.1}$$

Proof: For $z \in \mathfrak{l}^2$,

$$\begin{aligned}\|I^\pm z_T\|^2 &= \int_{\nabla^\pm_{[0,T]}} \|z(\sigma\backslash\{\max\sigma\},\max\sigma)\|^2 \,d\sigma \\ &= c_\pm \int_0^T \int_{\nabla} \|z(\tau,t)\|^2 \,d\tau\,dt,\end{aligned}$$

and, since I^+x_T and I^-y_T are supported by the disjoint sets ∇^+ and ∇^- respectively, (7.1) follows. ■

The following extension of the previous result is useful.

Proposition 7.1.2: Let $f^\kappa \in \mathfrak{l}^2$ and $x = I^+f^+ + I^-f^- + I^\circ f^\circ$, then

$$\|x_t\|_{\mathfrak{h}}^2 = \int_0^t \{c_+\|f_s^+\|_{\mathfrak{h}}^2 + c_-\|f_s^-\|_{\mathfrak{h}}^2 + 2\text{Re}\langle f_s^\circ, x_s\rangle_{\mathfrak{h}}\}\,ds. \tag{7.2}$$

Proof: By (7.1)

$$\|x_t\|^2 = c_+\|f^+\|_t^2 + c_-\|f^-\|_t^2 + \int_{\Gamma} \|\int_0^t f_s^\circ(\sigma)\, ds\|^2 d\sigma$$

$$+2\mathrm{Re}\int_{\Gamma} \langle \int_0^t f_s^\circ(\sigma)\, ds,\, I^+f_t^+(\sigma) + I^-f_t^-(\sigma)\rangle d\sigma,$$

but (for almost all σ)

$$\langle f_s^\circ(\sigma), I^\pm f_t^\pm(\sigma)\rangle = \langle f_s^\circ(\sigma), I^\pm f_s^\pm(\sigma)\rangle \quad (s \leq t),$$

since $I^\pm f^\pm$ are martingales, and

$$\|\int_0^t f_s^\circ(\sigma)\, ds\|^2 = 2\mathrm{Re}\int_0^t \langle f_s^\circ(\sigma), \int_0^s f_r^\circ(\sigma)\, dr\rangle\, ds$$

so (7.2) follows. ■

7.2 Simple approximation.

The *simple* and *continuous kernel processes* are defined by:

$$\mathfrak{s} = \{f \in \mathfrak{p} : f \text{ is a step function and } f(t) \in \mathcal{K}_0^{\mathfrak{A}_0}(I)\xi_0 \cap \mathfrak{A}\xi \ \ \forall\, t\}.$$

$$\mathfrak{c} = \mathfrak{p} \cap \mathcal{C}(I;\Sigma).$$

Thus martingale kernels and kernel stochastic integrals are continuous, and continuous kernel processes are locally square integrable. The next result establishes the density of $\mathfrak{s}$ in $\mathfrak{l}^2$.

Proposition 7.2.1: Let $f \in \mathfrak{l}^2$, then there is a sequence $f^{(n)} \in \mathfrak{s}$ such that for all $t \in I$,

$$\int_0^t \|f_s - f_s^{(n)}\|_\Sigma^2\, ds \to 0 \tag{7.3}$$

as $n \to \infty$.

Proof: For each $N > 0$,

$$f\chi_{[0,N]} \in L^2(\Gamma_{\mathrm{ad.}}(I), \mathcal{B}, \mu\times\lambda_1;\, \Sigma_0) = L^2(\Gamma_{\mathrm{ad.}}(I), \mathcal{B}, \mu\times\lambda_1) \otimes \Sigma_0.$$

Now

$$\{\chi_{B\times[b,c]} : B \subset \Gamma_{[0,a]} \text{ compact}, \quad a < b < c\}$$

is total in $L^2(\Gamma_{\text{ad.}}(I), \mathcal{B}, \mu\times\lambda_1)$ moreover, by Kaplansky's density theorem, each $\chi_B \otimes v$ $(v \in \Sigma_0)$ may be approximated, in Σ, by elements from $\mathfrak{A}\xi$ (or $\mathcal{R}\xi$ for any strongly dense $*$-subalgebra $\mathcal{R}$ of $\mathfrak{A}$). Combining these facts, the result follows.

■

We write $f^{(n)} \to f$ when $f^{(n)}, f \in \mathfrak{t}^2$ satisfy (7.3) for each t. In view of (7.1) we therefore have

Corollary 7.2.2: Let $f \in \mathfrak{t}^2$. Then there is a sequence $f^{(n)} \in \mathfrak{s}$ such that, for each $t \in I$

$$(I^{\pm}f)_t = \Sigma\text{-}\lim_{n\to\infty} (I^{\pm}f^{(n)})_t. \tag{7.4}$$

7.3 Conditional expectations.

For each subinterval J of I let P_J be the orthogonal projection of multiplication by χ_{Γ_J}, $\Sigma_J = P_J\Sigma$ and $\mathfrak{A}_J = \mathfrak{A} \cap (\Sigma_J)^{\wedge}$.

Proposition 7.3.1: Let $x \in \Sigma$, then

$$x \in \Sigma_J \Leftrightarrow \hat{x}\,\eta\,\mathfrak{A}_J.$$

Proof: If $x \in \Sigma_J$ then it follows from Theorem 2.2 of [LiW] that $\hat{x} = \bar{Y}$ where $\text{Dom}(Y) = \mathfrak{A}_J'\xi$, $YR\xi = Rx$ $(R \in \mathfrak{A}_J')$. In particular, $\hat{x}\,\eta\,\mathfrak{A}_J$. Conversely if $\hat{x}\,\eta\,\mathfrak{A}_J$ then, since $P_J \in \mathfrak{A}_J'$, $\chi_{\Gamma_J}x = x$, in other words $x \in \Sigma_J$.

■

Remark 7.3.2: The duality transform establishes a bijection between operator valued maps $F : I \to \eta^*(Q)$ for which $F_t\,\eta\,\mathfrak{A}_{[t]}$—called *adapted operator processes*—and maps $f : I \to \Sigma$ for which $f(t) \in \Sigma_{[t]}$. In particular, *to each of the classes* $\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{t}^2, \mathfrak{m}, \mathfrak{s}$ and $\mathfrak{c}$ *corresponds a class of adapted operator processes* $\hat{\mathfrak{p}}_0, \ldots, \hat{\mathfrak{c}}$.

Definition 7.3.3: For each subinterval J of I, the conditional expectation $\mathbb{E}_J$ on $\eta^*(Q)$ is given by

$$\mathbb{E}_J[\hat{x}] = (P_J x)^{\wedge}.$$

By Proposition 7.3.1, $\mathbb{E}_J[X]\,\eta\,\mathfrak{A}_J$ for each $*$-affiliated operator X. $\mathbb{E}_J$ extends a conditional expectation in the sense of Section 5 to $*$-affiliated operators. We shall list some of the properties enjoyed by these maps. Let $A, B \in \mathfrak{A}$ and $X \in \eta^*(Q)$ be such that XA and $\overline{BX} \in \eta^*(Q)$, then

(i) $\mathbb{E}_J[X^+] = \mathbb{E}_J[X]^+$ — adjoint preserving

(ii) if $J_1 \subset J_2$ then $\mathbb{E}_{J_1} \circ \mathbb{E}_{J_2} = \mathbb{E}_{J_1}$ — projectivity

(iii) if $X \in \hat{\Sigma}_J$ then $\mathbb{E}_J[XA] \subset X\mathbb{E}_{J[A]}$

if $A \in \mathfrak{A}_J$ then $\mathbb{E}_J[XA] \subset \mathbb{E}_J[X]A$

(iv) if $B \in \mathfrak{A}_J$ then $\mathbb{E}_j[\overline{BX}] = \overline{B\mathbb{E}_J[X]}$

if $X \in \hat{\Sigma}_J$ then $\mathbb{E}_J[\overline{BX}] = \overline{\mathbb{E}_J[B]X}$

(v) if $J_1 \cap J_2 = \varnothing$ then $\mathbb{E}_{J_1}$ agrees with $\mathbb{E}_{\{0\}}$ on $\hat{\Sigma}_{J_2}$ (7.5)

(vi) $\mathbb{E}_{\{0\}}[\hat{x}] = \left(\delta_{\varnothing}(\cdot)x(\varnothing)\xi_0\right)^{\wedge}$.

These properties are straightforward to verify, for instance (i) follows from the fact that P_J commutes with S_Q and (iv) from the fact that P_J commutes with $\mathfrak{A}_J$.

Remarks: 1. If $T \in \eta^*(Q)$ then $t \to \mathbb{E}_t[T]$ is a *martingale* (i.e. belongs to $\hat{\mathfrak{m}}$).

2. An adapted operator process X is a martingale if and only if $\mathbb{E}_s[X_t] = X_s \forall\, s \leqslant t$.

7.4 Quantum stochastic integration ([BSW 2], [L 1,2], [HL 1]]).

We now introduce *operator* creation and annihilation processes which together constitute a *quantum Brownian motion* ([CoH]) and define the stochastic integral with respect to these processes. We shall then argue that our present definition agrees with previous definitions.

Let $a_t := \mathbb{1}_{\mathfrak{A}_0} a_{\chi_{[0,t]}}$ $(t \in I)$ where $a_f, a_f^\dagger$ are the kernels of (2.3), then $a, a^\dagger \in p_1$. The operator process $A^- := \hat{a}$, $A^+ := (a^\dagger)\hat{}$ are called the *annihilation* and *creation processes*; they are mulually adjoint martingales $((A_t^\pm)^\dagger = A_t^\mp \quad \forall\, t)$.

Definition 7.4.1: We define (operator) quantum stochastic integrals as follows. For $F \in (\mathfrak{l}^2)\hat{}$,

$$\int_0^{\cdot} F \mathrm{d}A^\pm := (I^\pm f)\hat{}$$

where $f = F^\vee$.

Lemma 7.4.2: Let $f = x\chi_{(u,v]}$ where $x \in \left(\mathcal{K}_b^{\mathfrak{A}_0}\xi_0\right) \cap \left(\mathfrak{A}_{[u]}\xi\right)$ and $u < v$. Then for $t \geqslant v$,

$$\int_0^t \hat{f}\, \mathrm{d}A^\pm = (A_v^\pm - A_u^\pm)\hat{x}. \tag{7.6}$$

Proof: First note that

$$(A_v^+ - A_u^+)\hat{x} = a^\dagger_{\chi_{(u,v]}} * x$$

But, since the supports of x and $a^\dagger_{\chi_{(u,v]}}$ only have $\varnothing$ in common,

$$\begin{aligned} a^\dagger_{\chi_{(u,v]}} * x(\sigma) &= \sum\nolimits_{\alpha \subset \sigma} a^\dagger_{\chi_{(u,v]}}(\alpha) x(\bar{\alpha}) \\ &= \begin{cases} x(\sigma \backslash \{\max\sigma\}) & \text{if } \sigma \in \Gamma^+, \max\sigma \in (u,v] \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} f(\sigma \backslash \{\max\sigma\}, \max\sigma) & \text{if } \sigma \in \Gamma^+ \\ 0 & \text{otherwise} \end{cases} \\ &= I^+ f(\sigma, t) \qquad \text{for } t \geqslant v. \end{aligned}$$

By a similar argument for A^- the result now follows.

■

When these stochastic integrals have been defined in the past, the procedure has been the familiar one of first defining them for elementary processes by (7.6), then extending by linearity to simple processes, then invoking the density result (Lemma 7.2.1) and the isometry property (Proposition 7.1.1)—

proved for simple processes—to extend the integral to (locally) square integrable processes by the *prescription* (7.4) preserving isometry. Our approach has been to both define the stochastic integrals and establish their isometry/orthogonality properties *directly*—always invoking the duality transform (6.9) to pass back and forth between vectors and *-affiliated operators. The previous lemma, together with the density result establish the equivalence of our definition with the previous ones, modulo variations in domain which, in view of Proposition 5.1.1 and [LiW], have no significance.

7.5 Martingale representation theorem.

Since $I^{\pm}$ map $\mathfrak{t}^2$ into $\mathfrak{m}$, stochastic integrals of adapted operator processes are a source of martingales. In this section it is shown that all martingales arise in this way ([HL 1,2], [L 2]).

Proposition 7.5.1: Let $X \in \hat{\mathfrak{m}}$ then there are $F^{\pm} \in \hat{\mathfrak{t}}^2$ such that

$$X_t = X_0 + \int_0^t F^+ \, dA^+ + \int_0^t F^- \, dA^- \quad (t \in I) \tag{7.7}$$

the processes $F^{\pm}$ being unique up to a Lebesgue null set.

Proof: Let $F^{\pm} = (\Delta^{\pm} x)^{\wedge}$ where $x = X^{\vee}$. Then

$$\int_0^t F^+ \, dA^+ + \int_0^t F^- \, dA^- = (I^+ \Delta^+ x_t)^{\wedge} + (I^- \Delta^- x_t)^{\wedge} = (\chi_{\mathcal{P}^+} x_t)^{\wedge} + (\chi_{\mathcal{P}^-} x_t)^{\wedge}$$

so $X_t - \int_0^t F^+ \, dA^+ - \int_0^t F^- \, dt^- = (\chi_{\mathcal{P}^0} x_t)^{\wedge} = \hat{x}_0$, and (7.7) follows. Uniqueness follows from the isometry/orthogonality result (Proposition 7.1.1). ∎

7.6 Stochastic differential equations.

The kernel formalism allows a unified approach to the existence problem for linear (operator) stochastic differential equations, and, since the solution of corresponding kernel differential equations may be given explicitly, this approach gives more information than a purely operator approach [L 1], [HL 2].

Theorem 7.6.1: Let F_i^κ, G_i^κ be locally Lipschitz $\mathfrak{A}_{\{0\}}$-valued functions on I. Then the operator sde

$$\begin{cases} dX = \sum_{i=1}^{l} F_i^+ X G_i^+ \, dA^+ + \sum_{i=1}^{m} F_i^- X G_i^- \, dA + \sum_{i=1}^{n} F_i^\circ X G_i^\circ \, dt; \\ X(0) = X_0, \quad (X_0 \in \mathfrak{A}_{\{0\}}) \end{cases} \tag{7.8}$$

has a solution. In other words, there is an adapted process X such that $G_i^\kappa(s)\xi \in \mathrm{Dom}(X(s))\ \forall\, s, \kappa, i$, and

$$X(t) - X(0) = \int_0^t Y^+(s)\, dA^+ + \int_0^t Y^-(s)\, dA^- + \int_0^t Y^\circ(s)\, ds, \tag{7.9}$$

where Y^κ is the process $s \mapsto$ closure of $\sum_i F_i^\kappa(s) X(s) G_i^\kappa(s)$. Moreover X^+ satisfies the *conjugate equation* to (7.8).

Proof: Let x be the $\mathfrak{p}_1$ solution of the corresponding kde, that is the solution of (4.1) in which $L^\kappa(t)[b] = \sum_i f_i^\kappa(t) b g_i^\kappa(t)$ where $f_i^\kappa(s) \otimes \mathbb{1}_\eta = F_i^\kappa(s)$ and $g_i^\kappa(s) \otimes \mathbb{1}_\eta = G_i^\kappa(s)$. Now $G_i^\kappa(s)\xi = g_i^\kappa(s)\xi_0 \delta_\varnothing \in D(\hat{x}_s)$ by Corollary 6.3.11. By the fundamental theorem (3.2.5)

$$x_t - x_0 = \sum_\kappa I^\kappa \Big[\sum_i g_i^\kappa(\cdot) x(\circ, \cdot) f_i^\kappa(\cdot) \Big]_t$$

which implies (7.9) for $X = \hat{x}$. By symmetry $X^+ = (x^\dagger)\hat{}$ satisfies the conjugate equation.

■

The proof of uniqueness does not use kernels in any essential way.

Theorem 7.6.2: ([L 1], [HL 2]) Let F^κ, G^κ be locally bounded maps $I \to \mathfrak{A}_{\{0\}}$ which are strongly measurable. There is at most one solution to each of the operator sde's

$$dX = F^+ X\, dA^+ + F^- X\, dA^- + F^\circ X\, dt\ ; \quad X(0) = X_0 \quad (X_0 \in \mathfrak{A}_{\{0\}}) \tag{7.10}$$

$$dY = Y G^+\, dA^+ + Y G^-\, dA^- + Y G^\circ dt\ ; \quad Y(0) = Y_0 \quad (Y_0 \in \mathfrak{A}_{\{0\}}) \tag{7.11}$$

Proof: (i) Let X_1 and X_2 be processes satisfying (7.10). Putting $Z = X_1 - X_2$ and applying Proposition 7.1.2 we have

$$\|z(t)\|^2 = \int_0^t \{c_+\|f^+z\|^2 + c_-\|f^-z\|^2 + 2\mathrm{Re}\langle z, f^\circ z\rangle\}\, ds$$

$$\leqslant c_T \int_0^t \|z(s)\|^2\, ds \quad \text{for } t \in [0,T],$$

where $c_T = 2\sum_\kappa \sup_{[0,T]} c_\kappa \|F^\kappa(\cdot)\|^2$. Iterating this inequality yields $z = 0$, i.e. $X_1 = X_2$.

(ii) Let $Z = Y_1 - Y_2$ where, Y_1 and Y_2 satisfy (7.11), then Z^+ satisfies (7.10) with $X_0 = 0$, $F^+ = G^{-*}$, $F^- = G^{+*}$, $F = G^*$, but so does $X \equiv 0$, so $Z^+ = 0$ by (i), and therefore $Y_1 = Y_2$. ∎

Combining these results we have

Theorem 7.6.3: Let F^κ be locally Lipschitz $\mathfrak{A}_{\{0\}}$-valued maps on I. Then there is a unique solution to the operator sde

$$dX = F^+X\, dA^+ + F^-X\, dA^- + F^\circ X\, dt\ ; \qquad X(0) = X_0 \quad (X_0 \in \mathfrak{A}_{\{0\}})$$

Its adjoint is the unique solution to the operator sde

$$dY = YF^{-*}\, dA^+ + YF^{+*}\, dA^- + YF^{\circ *}\, dt\ ; \qquad Y(0) = (X_0)^*$$

Moreover the operator processes possess smooth kernels which satisfy corresponding kde's and may be written explicitly (4.3).

Remark: A class of operator stochastic differential equations, not covered by the results here, were considered by Barnett, Streater and Wilde ([BSW 2]). They established existence and uniqueness for equations of the form

$$dY = f(Y,t)\, dB + g(Y,t)\, dt$$

where B is a linear combination of the creation and annihilation process and f and g are adapted (in the obvious sense) and satisfy Lipschitz and continuity conditions.

8. Adapted cocycles

In this section we characterize adapted operator cocycles (cf. Section 4) as groundwork for the construction of quantum Markov processes in the next section. Q_o will denote a fixed initial quantum probability space and θ_t ($t \in \mathbb{R}$) will be the right shift on $\eta^*(Q)$, (where $Q = Q_o \otimes Q_1$, as in Section 6.2 and $I = \mathbb{R}$):

$$\theta_t(\hat{x}) := (\Theta_t x)\hat{}, \quad t \in \mathbb{R}$$

$$= \Theta_t \hat{x} \Theta_{-t}$$

(see Section 4.3). For an early paper on Markovian cocycles, see [AcF].

Definition 8.0.1: An $\mathfrak{A}$-valued process $\{X_t : t \geq 0\}$ is an *adapted (operator) cocycle* if

(aci) $X \in \hat{\mathfrak{e}}$.

(acii) $\theta_t(X_s)X_t = X_{s+t}$; $\quad X_0 = \mathbb{1}$ $\quad (s, t \geq 0)$.

Remark: If X is an adapted cocycle then the two parameter family $\{X_{s,t} : \theta_s(X_{t-s}),\ s \leq t\}$ satisfies:

(i)′ $X_{s,t} \in (\Sigma_{[s,t]})\hat{}$

(ii)′ $X_{s,t}X_{r,s} = X_{r,t}$; $\quad X_{0,0} = \mathbb{1}_{\mathfrak{A}}$

(iii)′ $\theta_u(X_{s,t}) = X_{s+u,t+u}$

(iv)′ $t \mapsto (X_{u,t})^{\vee}$ is continuous on $[u, \infty)$, $(u \in \mathbb{R},\ r \leq s \leq t)$.

Conversely a two parameter family $\{X_{s,t} : s \leq t\}$ from $\mathfrak{A}$ satisfying (i)′ - (iv)′ determines an adapted cocycle: $\{X_{0,t} : t \geq 0\}$.

It is now a simple matter (easy part of Proposition 8.1.2) to show that bounded operator valued solutions of stochastic differential equations of the form

$$dX = Q^{+}X\,dA^{+} + Q^{-}X\,dA^{-} + Q^{\circ}X\,dt\,; \quad X(0) = \mathbb{1}_{\mathfrak{A}}, \quad (Q^{\kappa} \in \mathfrak{A}_{\{0\}}), \quad (8.1)$$

provide adapted cocycles. In the converse direction, it was established in [HL 2] that *unitary valued* adapted cocycles (there called covariant adapted evolutions) are necessarily solutions of equations of the form (8.1). Here we apply the kernel formalism to establish this for adapted cocycles over a *finite dimensional* quantum probability space. The idea of the proof is again to apply the duality transform to Proposition 4.3.2, but since the continuity condition (iv)′ is considerably weaker than the corresponding one for kernel cocycles we have to work a little harder.

8.1 Characterization.

For a function f on ∇ and an interval I, let f^I denote the map $\sigma \to f(\sigma \cap I)$.

Lemma 8.1.1: Let $x \in \Sigma_I$ be (almost everywhere) $\mathfrak{A}_0\xi$-valued and such that $\hat{x} \in \mathfrak{A}$. Then for $y \in \Sigma_{\mathbf{R}\setminus I}$

$$\hat{x}y = x^I(\cdot)y^{\mathbf{R}\setminus I}(\cdot) \qquad \text{(a.e.)}$$

(using notation 6.3.9).

Proof: First choose a countable dense set D' from $(\mathfrak{A}_0' \otimes_{\text{alg.}} \mathcal{M}_0)\xi$. Let $\{X_m \in \mathfrak{A}_0 \otimes_{\text{alg.}} \mathcal{N}_0^I : m = 1,2,\ldots\}$ be a sequence which strongly approximates $\hat{x}$. By taking subsequences we may assume that $X_m d'$ is pointwise convergent (outside a null set Ξ_1) to $\hat{x}d'$ for each $d' \in D'$. Now let $\{y_n = Y_n\xi : n = 1,2,\ldots\}$ be a sequence in D' converging to y but also, again by taking subsequences, such that $\hat{x}y_n$ converges pointwise (outside a null set Ξ_2) to $\hat{x}y$. Then, since x is $\mathfrak{A}_0$-valued and the y_n are $\mathfrak{A}_0'$-valued, Proposition 6.3.10 gives, for $\sigma \notin \Xi_1 \cup \Xi_2$,

$$\begin{aligned}
\hat{x}y(\sigma) &= \lim_{n\to\infty} (\hat{x}y_n)(\sigma) \\
&= \lim_{n\to\infty} \lim_{m\to\infty} x_m^I(\sigma) y_n^{\mathbf{R}\setminus I}(\sigma) \\
&= \lim_{n\to\infty} y_n^{\mathbf{R}\setminus I}(\sigma) x^I(\sigma) \\
&= \lim_{n\to\infty} x^I(\sigma) y_n^{\mathbf{R}\setminus I}(\sigma) \\
&= x^I(\sigma) y^{\mathbf{R}\setminus I}(\sigma),
\end{aligned}$$

and the result is proved.

■

Proposition 8.1.2: Let $\{Y_t : t \geq 0\}$ be a family of operators in $\mathfrak{A}$ with $\mathfrak{A}_0$ finite dimensional, then the following are equivalent:

(a) Y is an adapted cocycle;

(b) Y satisfies a stochastic differential equation of the type (8.1);

(c) $Y^{\vee}$ has a version which satisfies the equivalent conditions of Proposition 4.3.2.

Proof: (c) $\Leftrightarrow$ (b): By the fundamental Theorem (3.2.5) the duality transform maps the unique solution of $\Delta z = qz$, $z_0(\varnothing) = \mathbb{1}_{\mathfrak{A}_0}$ to the unique solution of

$$dZ = Q^+Z\,dA^+ + Q^-Z\,dA^- + Q^\circ Z\,dt, \quad Z_0 = \mathbb{1}.$$

(c) $\Rightarrow$ (a): By Proposition 6.3.7 the duality transform maps adapted kernel cocycles to adapted operator cocycles.

(a) $\Rightarrow$ (c): Let Y be an adapted cocycle. Define $y_{s,t} \in \Sigma$ $(s \leq t)$ by $y_{s,t} = \mathbb{W}_s(Y_{t-s})$, then we immediately have, for $r \leq s \leq t$, $a \in \mathbb{R}$

$$y_{s,t}(\omega) = y_{s-a,t-a}(\omega - a) \text{ for a.a. } \omega \tag{8.3}$$

in particular,

$$y_{s,t}(\varnothing) = y_{t-s}(\varnothing) \tag{8.4}$$

and, by Lemma 8.1.1,

$$y_{r,t}^{[r,t]} = y_{s,t}^{[s,t]} y_{r,s}^{[r,s]} \quad \text{a.e.} \tag{8.5}$$

which may be extended to

$$y_{r_0,r_n}^{[r_0,r_n]} = y_{r_{n-1},r_n}^{[r_{n-1},r_n]} y_{r_{n-2},r_{n-1}}^{[r_{n-2},r_{n-1}]} \cdots y_{r_0,r_1}^{[r_0,r_1]} \quad \text{a.e.,} \tag{8.6}$$

where $r \in \mathbb{R}_{\leq}^{n+1} := \{r \in \mathbb{R}^{n+1} : r_0 \leq r_1 \leq \ldots \leq r_n\}$. By (8.4) and (8.5) the map

$$p : \mathbb{R}_+ \to \mathfrak{A}_0,\ t \to y_t(\varnothing)$$

is a semigroup which, by (aci) is continuous. Let q be its generator. Let

$(s_1, s_5) \in R^2_{\leqslant}$, then by (8.6), for almost all $\{s_3^+\}$, $s_1 < s_3 < s_5$

$$y_{s_2,s_4}(\{s_3^+\}) = p^{-1}_{s_5-s_4} y_{s_1,s_5}(\{s_3^+\}) p^{-1}_{s_2-s_1}$$

for (s_2, s_4) outside a null set $N_{(s_1,s_3,s_5)}$, $s \in \mathbb{R}^5_{\leqslant}$. So

$$\lim_{\substack{s_2 \uparrow s_3,\, s_4 \downarrow s_3 \\ (s_2,s_4) \notin N}} y_{s_2,s_4}(\{s_3^+\})$$

exists, by the continuity of p, equals

$$p^{-1}_{s_5-s_3} y_{s_1,s_5}(\{s_3^+\}) p^{-1}_{s_3-s_1}$$

and moreover is clearly independent of (s_1, s_5). Thus there is a map $q^+ : \mathbb{R} \to \mathfrak{A}_0$ and a subset V of $\mathbb{R}^3_{\leqslant}$ of full Lebesgue measure such that

$$F(s) := p^{-1}_{s_3-s_2} y_{s_1,s_3}(\{s_2^+\}) p^{-1}_{s_2-s_1} = q^+(s_2) \quad (s \in V)$$

By covariance (8.3), $\{(s,a) \in \mathbb{R}^3_{\leqslant} \times \mathbb{R} : F(s) = F(s-a)\}$ has full Lebesgue measure. Moreover,

$$\{(s,a) \in \mathbb{R}^3_{\geqslant} \times \mathbb{R} : q^+(s_2-a) = F(s-a)\} = T(V \times \mathbb{R})$$

where T is the measure preserving map $(x,a) \to (x-a,a)$. Hence for almost all $(s,a) \in \mathbb{R}^3_{\leqslant} \times \mathbb{R}$,

$$q^+(s_2-a) = F(s-a) = F(s) = q^+(s_2)$$

In particular, q^+ is almost everywhere constant. Applying the same argument to $\mathcal{V}^{0,1}$ we obtain $q^{\pm} \in \mathfrak{A}_0$ satisfying

$$y^{[s_1,s_3]}_{s_1,s_3}(\{s_2^+\}) = p_{s_3-s_2} q^{\pm} p_{s_2-s_1} \quad \text{for a.a. } s \in \mathbb{R}^3_{\geqslant}.$$

Combining this with (8.6) gives

$$\begin{aligned} y_{r_0,r_n}((s,\varepsilon)) &= (p_{r_n-s_n} q^{\varepsilon_n} p_{s_n-r_{n-1}}) \ldots (p_{r_1-s_1} q^{\varepsilon_1} p_{s_1-r_0}) \\ &= p_{r_n-s_n} (q^{\varepsilon_n} p_{s_n-s_{n-1}} \ldots p_{s_2-s_1} q^{\varepsilon_1}) p_{s_1-r_0} \\ &\qquad \text{for a.a.} (r_0, s_1, s_1, \ldots, s_n, r_n) \quad \mathbb{R}^{2n+1}_{\geqslant}. \end{aligned}$$

and, using (8.6) again,

$$y_t(\sigma) = p_{t-v}\{y_{u,v}(\sigma)\}p_u \quad \text{for a.a } \sigma \in \Gamma_{[u,v]}, \quad 0 \leqslant u < v \leqslant t$$

$$= p_{t-s_n} q^{\varepsilon_n} p_{s_n - s_{n-1}} \cdots p_{s_2 - s_1} q^{\varepsilon_1} p_{s_1} \quad \text{for a.a. } (u,s,v) \in \mathbb{R}^{n+2}_{\geqslant}$$

Letting $u_n \downarrow 0$ and $v_n \downarrow 0$ with care, we see that y has the form (4.12) and the proof is complete.

■

8.2 The reduced semigroup.

Proposition 8.2.1: If X is an adapted cocycle, then the family of operators $\{R_t := \mathit{III}_{-t} X_t, t \geqslant 0\}$ satisfies

$$R_{s+t} = R_s R_t \tag{8.7}$$

Proof:

$$R_{s+t} = \mathit{III}_{-s-t} \mathit{III}_t(X_s) X_t$$

$$= \mathit{III}_{-s-t} \mathit{III}_{s+t} \mathit{III}_{-s} X_s \mathit{III}_{-t} X_t$$

$$= R_s R_t$$

■

In view of Propositions 4.2.1 and 4.4.1 and Corollary 6.3.8 the following is immediate.

Proposition 8.2.2: An adapted cocycle X is unitary valued if and only if its generator has the form $(V, -V^*, -\frac{1}{2}(c_+ V^* V + c_- V V^*) + iH)$ where $H = H^*$. In this case the one parameter semigroup T^0 on $\mathfrak{A}_0$ given by

$$j \circ T^0_t = E_{\{0\}} \circ \mathrm{Ad}\, X_t \circ j$$

(consisting of completely positive maps) has generator $L_v - i[H, \cdot]$.

T^0 is called the *reduced semigroup* of the adapted cocycle X.

9. Quantum Markov Processes

In this section we shall be concerned with the construction of dilations using Bose noise, and the characterization of the class of quantum dynamical

semigroups which admit such a dilation (cf. [ApF]).

9.1 Multidimensional Bose Noise

Let $\Gamma(I;l)$ denote the *charged and coloured finite power set of I with l colours:*

$$\Gamma(I;l) = \bigcup_{N=0}^{\infty} \Gamma^N(I)\times\{1,\ldots,l\}^N.$$

let ρ_0 be the metric on $\Gamma(I;l)$ given by

$$\rho_0((s,\varepsilon,k),(s',\varepsilon',k')) = \begin{cases} 0 & \text{if both equal } \varnothing \\ 1 \wedge \max_i |s_i - s_i'| & \text{if } (\varepsilon,k) = (\varepsilon',k') \\ 1 & \text{otherwise} \end{cases}$$

and let $(\bar{\Gamma}(I;l),\rho)$ be the completion of $(\Gamma(I;l),\rho_0)$. The constants $c_\pm$ will now be *colour dependent*:

$$c_{\pm,j} = c_\pm(\beta_j,h_j), \quad j = 1,\ldots,l,$$

where $(\boldsymbol{\beta},\boldsymbol{h}) \in (0,\infty)^l\times[0,\infty)^l$ is fixed and the functions $c_\pm$ are determined by continuity and the relations

$$c_+(\beta,h) - c_-(\beta,h) = h; \qquad c_+(\beta,h)/c_-(\beta,h) = e^{\beta h}.$$

Explicitly:

$$c_-(\beta,h) = \begin{cases} h(e^{\beta h}-1)^{-1} \\ \beta^{-1} \end{cases}; \qquad c_+(\beta,h) = \begin{cases} h(1-e^{-\beta h})^{-1} & (h>0) \\ \beta^{-1} & (h=0). \end{cases}$$

The origin of this parametrisation of the pair (c_+,c_-) by (β,h) lies in physics: When a small quantum-mechanical system, such as an atom, is coupled to the electromagnetic field (a Bose field) it is usually sensitive only to certain small spectral regions of this field around frequencies ν_j $(j = 1,\ldots,l)$, say. When the field is at a temperature T, these regions are described to a good approximation by copies of Bose noise with parameters

$$\beta_j = 1/kT \quad \text{and} \quad h_j = \hbar\nu_j,$$

where $\hbar$ and k are the constants of Planck (divided by 2π) and Boltzmann

respectively. The j-dependence of β_j leaves open the possibility of coupling to fields of different temperatures. A classical field corresponds to Bose noise with $h = 0$.

Let $\mu = \mu^{\beta,h}$ be the Borel measure on $\Gamma(I;l)$ given by $d\mu = m d\lambda$ where

$$m((s,\varepsilon,k)) = \prod_j c_{\varepsilon_j,k_j}. \qquad \text{(cf. 2.1)}$$

A kernel calculus may be constructed on $(\Gamma(I;l)\,\mu^{\beta,h}))$ as before, involving operators $I^{\varepsilon,k}$, I°, $\Delta^{\varepsilon,k}$ and Δ° ($\varepsilon = \pm, k = 1,\ldots,l$), and a quantum probability space $Q_1^{\beta,h} = (L^2(d\mu^{\beta,h}), \mathcal{N}^{\beta,h}, \delta_{\emptyset}^{\beta,h})$. Corresponding to Propositions 8.1.2 and 8.2.2 we have:

Proposition 9.1.1: For a family $\{Y_t : t \geq 0\}$ of operators in $\mathfrak{A}^{\beta,h}$, the following are equivalent:

(a) Y is an $\mathcal{N}^{\beta,h}$-adapted cocycle

(b) Y satisfies an s.d.e. of the type

$$dY = \sum_{i=1}^{l} (Q^{+,i}\, dA_i^+ + Q^{-,i}\, dA_i^-) + Q^\circ dt\ ;\ Y_0 = I \quad (Q^\lambda \in \mathfrak{A}_{\{0\}})$$

(c) $Y = \hat{y}$ where y satisfies a k.d.e. of the type

$$\Delta y = qy; \qquad y_0(\emptyset) = \mathbb{1}_{\mathfrak{A}_0} \quad (q \in \mathfrak{A}_o^{2l+1})$$

(d) $Y = \hat{y}$ where y is an $\mathcal{N}^{\beta,h}$-adapted kernel cocycle

(e) $Y = \hat{y}$ where

$$y_t(s,\varepsilon,k) = e^{(t-s_N)q}\, q^{\varepsilon_N,k_N}\, e^{(s_N-s_{N-1})q} \ldots\, q^{\varepsilon_1,k_1}\, e^{s_1 q} \quad \text{for } t \geq s_N$$

as long as *either* Q_0 is finite dimensional, *or* Y is unitary valued.

Proposition 9.1.2: The $\mathcal{N}^{\beta,h}$-adapted *unitary* cocycles are those with generator Q of the form

$$Q^{+,k} = V_k, \quad Q^{-,k} = -V_k^*, \quad k = 1,\ldots,l,$$

$$Q = iE - \tfrac{1}{2}\sum_{k=1}^{l} \{c_+(\beta_k,h_k)V_k^*V_k + c_-(\beta_k,h_k)V_kV_k^*\}, \quad (E = E^*).$$

The generator of the corresponding reduced semigroup on $\mathfrak{A}_0$ is then

$$i[\,\cdot\,,e] + \sum_{k=1}^{l} L_{v_k}^{\beta_k,h_k} \quad (\text{where } j(e) = E,\ j(v_k) = V_k).$$

We write $L_v^{\beta,h}(b)$ for $c_+(\beta,h)\big[v^*v - \tfrac{1}{2}\{v^*v,b\}\big] + c_-(\beta,h)\big[vbv^* - \tfrac{1}{2}\{vv^*,b\}\big]$ where $\{.,.\}$ is the anti-commutator $a,b \mapsto ab+ba$).

9.2 Bose dilations

Lemma 9.2.1: Let Y be an $\mathcal{N}^{\beta,h}$-adapted cocycle with generator $q \otimes \mathbb{1}_{\mathcal{N}}$. If either Y is unitary valued, or Q_0 is finite dimensional, then for each $t \in \mathbb{R}$,

$$\sigma_t^{\beta,h}(Y_s) = Y_s \ \ \forall s \geqslant 0 \quad \Leftrightarrow \quad \begin{cases} \sigma_t^0(q) = q \\ \sigma_t^0(q^{\varepsilon,j}) = e^{\varepsilon it\beta_j h_j} q^{\varepsilon,j} \end{cases} \quad (\varepsilon = \pm,\ j = 1,\ldots,l).$$

Proof: The modular automorphism group $\sigma^{\beta,h}$ is given by (6.3) where $\gamma := m^\dagger/m$ is the map

$$(r,\varepsilon,k) \to e^{-\sum_j \varepsilon_j \beta_j h_j}$$

so that

$$\sigma_t(Y_s) = Y_s \ \ \forall s \geqslant 0$$

$$\Leftrightarrow e^{-it\sum_j \varepsilon_j\beta_j h_j} \sigma_t^0\big(e^{(s-r_N)q}\, q^{\varepsilon_N,k_N} \ldots\, q^{\varepsilon_1,k_1}\, e^{r_1 q}\big)$$

$$= e^{(s-r_N)q}\, q^{\varepsilon_N,k_N} \ldots\, q^{\varepsilon_1,k_1}\, e^{r_1 q} \quad \forall\, \varepsilon,k \text{ and } s \geqslant r_N \geqslant \ldots \geqslant r_1$$

$$\Leftrightarrow \sigma_t^0(q) = q, \quad \sigma_t^0(q^{\varepsilon,j}) \quad (\varepsilon = \pm,\ j = 1,\ldots,l).$$

■

Proposition 9.2.2: Let U be an $\mathcal{N}^{\beta,h}$-adapted unitary cocycle with generator $Q = q \otimes \mathbb{1}_{\mathcal{N}}$. Then

$$t \mapsto (\mathrm{Ad}\, U_t) \circ \mathbb{m}_t \quad (t \geqslant 0)$$

determines a quantum dynamical group T on $Q^{\beta,h}$ if and only if

$$\sigma_t^0(q^\circ) = q^\circ, \quad \sigma_t^0(q^{\varepsilon,j}) = e^{\varepsilon it\beta_j h_j} q^{\varepsilon,j}, \quad (\varepsilon = \pm, j = 1,\ldots,l).$$

In this case the reduced semigroup T^o of U is a quantum dynamical semigroup on Q_o admitting the Markov dilation $(Q, j, \mathbb{P}, T)$ where $j: b \to b \otimes \mathbb{1}_{\mathcal{N}}$ and $\mathbb{P}$ is characterized by $\mathbb{P}(X) \otimes \mathbb{1} = \mathbb{E}^{\beta,h}_{\{0\}}[X]$.

Proof: Since ξ is invariant under the shift on $L^2(\Gamma(I; l), \mathfrak{h}_0)$, $\{\mathit{III}_t : t \in \mathbb{R}\}$ is a quantum dynamical group on Q. By Proposition 8.2.1, T satisfies the semigroup property, so the equivalence follows from (5.5) and the lemma. Each T^o_t is then a composition of morphisms:

$$T^o_t = \mathbb{P} \circ \mathrm{Ad}\, U_t \circ j = \mathbb{P} \circ T_t \circ j \quad (\text{since } \mathit{III}_t \circ j = j),$$

so T^o is a quantum dynamical semigroup and T a dilation of T^o. The Markov property follows from (7.5).

■

Definition 9.2.3: A (Markov) dilation $(Q, j, \mathbb{P}, T)$ of a quantum dynamical semigroup T^o on Q^o is a *Bose dilation* if for some (β, h)

$$Q = Q^o \otimes Q_1^{\beta,h}; \quad j(b) = b \otimes \mathbb{1}; \quad T_t = \mathrm{Ad}\, U_t \circ \mathit{III}_t$$

where U is an $\mathcal{N}^{\beta,h}$-adapted unitary cocycle.

Proposition 9.2.4: Let T^o be a quantum dynamical semigroup on Q^o with generator L. The following are equivalent:

(a) T^o admits a Bose dilation

(b) $$L = i[e, \cdot] + \sum_{k=1}^{l} L^{\beta_k, h_k}_{v_k} \tag{9.1}$$

for some $(l, \beta, h, e = e^*, v)$ satisfying $\sigma^o_t(e) = e$,

$$\sigma^o_t(v_k) = e^{it\beta_k h_k} v_k \quad (k = 1, \ldots, l). \tag{9.2}$$

Proof: (a) $\Rightarrow$ (b): This is immediate from Propositions 9.1.2 and 9.2.2.

(b) $\Rightarrow$ (a): Let L be determined by $(l, \beta, h, d = d^*, v)$ as in (9.1) then the $\mathcal{N}^{\beta,h}$-adapted unitary cocycle with generator q where

$$q^{\circ} = \mathrm{id.} - \tfrac{1}{2}\sum_{k=1}^{l}\{c_{+}(\beta_k,h_k)v_k^{*}v_k + c_{-}(\beta_k,h_k)v_kv_k^{*}\}$$

$$q^{+,j} = v_j, \quad q^{-,j} = -v_j^{*} \quad (j = 1,\ldots,m).$$

leads to a Bose dilation, since $v_k^{*}v_k$ and $v_kv_k^{*}$ are fixed under σ_t°, by Proposition 9.2.2. ■

9.3 Detailed Balance

In the physics literature one finds the condition of *detailed balance* for the transition probabilities between the energy levels of a quantum-mechanical system. This condition says that the transition probabilities between any pair of levels balance each other in the equilibrium state of the whole system. For a long time it was believed that detailed balance was a necessary condition for the dynamical semigroup determined by these transition probabilities to be physically realisable, i.e. to possess a dilation. Although this belief is now known to be erroneous in general ([KüM], [FrM]), we shall prove it correct for the case of dilations using Bose noise.

Definition 9.3.1: A norm continuous quantum dynamical semigroup T° on Q° with generator L satisfies *detailed balance* if there is a quantum dynamical semigroup S° on Q° satisfying

(db i) $\quad \langle S_t^{\circ}(a)\xi_{\circ}, b\xi_{\circ}\rangle = \langle a\xi_{\circ}, T_t^{\circ}(b)\xi_{\circ}\rangle \qquad \forall\, a,b \in \mathfrak{A}_{\circ},\ t \geq 0$

(T° has a Q°-adjoint) and whose generator M satisfies

(db ii) $\quad L - M = i[\,e\,,\cdot\,]$ for some $e = e^{*}$ in $\mathfrak{A}_{\circ}$.

Remarks:

1. T° has a Q°-adjoint if and only if it commutes with the modular automorphism group $\sigma^{Q^{\circ}}$.
2. If T° has a dilation $(Q,j,\mathbb{P},T)$ then it has a Q°-adjoint, namely $\mathbb{P}\circ T_{-t}\circ j$ $(t \geq 0)$.

Proposition 9.3.2: Let T^o be a quantum dynamical semigroup on Q^o. Suppose that $\mathfrak{A}_o$ is finite dimensional, and a factor (i.e. has trivial centre). Then T^o admits a Bose dilation if and only if T^o satisfies detailed balance.

Proof: Let T be a Bose dilation of T_t^o given by the cocycle U with generator $q \otimes \mathbb{1}$. The generator of $\mathbb{P} \circ T_{-t} \circ j = \mathbb{P} \circ Ad U_t^* \circ j$ is M_q (see 4.7) and since $L_q - M_q = [q - q^*, \cdot]$, T^o satisfies detailed balance.

Conversely ([Ali]) suppose T^o satisfies detailed balance. $\mathfrak{A}_o$, being a finite dimensional matrix algebra, which is also a factor, is isomorphic to $M_n(\mathbb{C}) \times I_m$ for some n, m, where I_m is the $m \times m$ identity matrix. The generator L of the completely positive semigroup T_o is therefore expressible in the form ([Lin], [GKS])

$$i[u, \cdot] + \sum_{k=1}^{p} \left(v_k^* \cdot v_k - \tfrac{1}{2}\{v_k^* v_k, \cdot\} \right)$$

for some $p \in \mathbb{N}, u = u^*, v_k \in \mathfrak{A}$, $k = 1, \ldots, p$. Let $\sum_{\chi \in sp(\ln \Delta_o)} e^{\chi} P_\chi$ be the spectral decomposition of Δ_{Q^o} and define x_χ and $v_{\chi,k}$ $(k = 1, \ldots, p)$ in $\mathfrak{A}_o$ by

$$u_\chi \xi_o = P_\chi u \xi_o$$

$$v_{\chi,k} \xi_o = P_\chi v_k \xi_o$$

then,

$$\sigma_t^o(u_\chi) = e^{i\chi t} u_\chi$$

$$\sigma_t^o(v_{\chi,k}) = e^{i\chi t} v_{\chi,k} \tag{9.3}$$

for $\chi \in \mathrm{sp}(\ln \Delta_o)$, and, since L commutes with σ^o, we have

$$\begin{aligned} L &= \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} \sigma_t^o \circ L \circ \sigma_{-t}^o \, dt \\ &= \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} \Big[i \sum_\chi e^{i\chi t} [u_\chi, \cdot] \\ &\qquad + \sum_{k=1}^{p} \sum_{\chi, \chi'} e^{i(\chi - \chi')t} \left(v_{\chi',k}^* \cdot v_{\chi,k} - \tfrac{1}{2}\{v_{\chi',k}^* v_{\chi,k}, \cdot\} \right) \Big] \, dt \\ &= i[u_o, \cdot] + \sum_{k,\chi} \left(v_{\chi,k}^* \cdot v_{\chi,k} - \tfrac{1}{2}\{v_{\chi,k}^* v_{\chi,k}, \cdot\} \right) \end{aligned} \tag{9.4}$$

Elementary manipulations with the Tomita operators, such as

$$v_{\chi,k}\xi_0 = e^{-\chi}\Delta_0 v_{\chi,k}\xi_0, \qquad v^*_{\chi,k}v_{\chi,k}\xi_0 = \Delta v^*_{\chi,k}v_{\chi,k}\xi_0$$

yield the identities

$$\langle a\xi_0\,,\, i[u_0,b]\xi_0\rangle = \langle -i[u_0,a]\xi_0\,,\, b\xi_0\rangle$$

$$\langle a\xi_0\,,\, v^*_{\chi,k}bv_{\chi,k}\xi_0\rangle = \langle e^{-\chi}v_{\chi,k}av^*_{\chi,k}\xi_0\,,\, b\xi_0\rangle$$

$$\langle a\xi_0\,,\, \{v^*_{\chi,k}v_{\chi,k}\,,\, b\}\xi_0\rangle = \langle \{v^*_{\chi,k}v_{\chi,k}\,,\, a\}\xi_0\,,\, b\xi_0\rangle$$

for $a, b \in \mathfrak{A}_0$. Thus the generator L^* of the Q^0-adjoint semigroup equals

$$-i[\,u_0\,,\,\cdot\,] + \sum_{k,\chi}\,\left[e^{-\chi}v_{\chi,k}\cdot v^*_{\chi,k} - \tfrac{1}{2}\{v^*_{\chi,k}v_{\chi,k},\,\cdot\,\}\right].$$

Since $L^*(\mathbb{1})$ must vanish, we have

$$L^* = -i[u_0,\cdot\,] + \sum_{k,\chi}\, e^{-\chi}\left[v_{\chi,k}\cdot v^*_{\chi,k} - \tfrac{1}{2}\{v_{\chi,k}v^*_{\chi,k},\,\cdot\,\}\right] \tag{9.5}$$

By (db ii)

$$L - L^* = i[\,d\,,\,\cdot\,] \text{ for some } d = d^* \in \mathfrak{A}_0 \tag{9.6}$$

and, since L and L^* commute with σ^0, $\sigma^0_t(d) - d$ lies in the centre of $\mathfrak{A}_0$, hence

$$\sigma^0_t(d) = d \quad (t \in \mathbb{R}) \tag{9.7}$$

Since $L = \frac{1}{2}(L+L^*) + \frac{1}{2}(L-L^*)$ and $c_-(\beta,h)/c_+(\beta,h) = e^{-\beta h}$, we see, by combining (9.3–9.7) with Proposition 9.6, that T^0 admits a Bose dilation. ∎

10. The Wigner-Weisskopf Atom

10.1 Description

Let $M_2(\mathbb{C})$ be the algebra of observables of an atom possessing two energy levels: a higher level 1 and a lower level 2. The equilibrium state on $M_2(\mathbb{C})$ at inverse temperature β is given by

$$\varphi_\beta : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \frac{e^{-\beta h}a+d}{e^{-\beta h}+1}$$

where $\hbar = h/2\pi$ and h is Plank's constant. Working in units for which $\hbar = 1$ the appropriate quantum probability space Q_0 is $(\mathfrak{h}_0, \mathfrak{A}_0, \xi_0)$ where

$\mathfrak{h}_0 = M_2(\mathbb{C})$ with inner product $(x,y) \to \varphi_\beta(x^*y)$

$\xi_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\mathfrak{A}_0 = \pi(M_2(\mathbb{C})$ where $\pi : M_2(\mathbb{C}) \to \mathcal{L}(\mathfrak{h}_0)$ is the representation given by

$$a_\pi = \pi(a) : x \to ax.$$

Note that $\mathfrak{h}_0$ is isomorphic to $\mathbb{C}^2 \oplus \mathbb{C}^2$ with the inner product

$$((\psi_1, \psi_2), (\chi_1, \chi_2)) \to \frac{e^{-\beta}}{1+e^{-\beta}}\langle \psi_1, \chi_1 \rangle + \frac{1}{1+e^{-\beta}}\langle \psi_2, \chi_2 \rangle,$$

ξ_0 corresponding to $\left(\binom{1}{0}, \binom{0}{1}\right)$ under this isomorphism. The modular group $\sigma_t^0 := \sigma_t^{Q_0}$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\pi \mapsto \begin{pmatrix} a & e^{-i\beta t}b \\ e^{i\beta t}c & d \end{pmatrix}_\pi$$

The *lowering operator* of the atom $v := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\pi$ clearly satisfies (9.2) (with $l = 1$) and e^{tL_v} is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_\pi \mapsto \frac{e^{-\beta}a+d}{e^{-\beta}+1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_\pi + e^{-t\coth\beta/2}\frac{a-d}{e^{-\beta}+1}\begin{pmatrix} 1 & 0 \\ 0 & e^{-\beta} \end{pmatrix}_\pi$$

$$+ e^{(-t/2)\coth(\beta/2)}\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}_\pi$$

This quantum dynamical semigroup is known as the approach to thermal equilibrium of the two-level atom coupled to a quantum field at inverse temperature β, a system first described by Wigner and Weisskopf ([WeW]). The construction of Proposition 9.2.2 immerses this atom in the flow of a Bose noise $Q_1(\mathbb{R})$ governed by the shift III_t. The development in time of the

coupled system

$$T_t : X \mapsto U_t^* \amalg_t(X) U_t = U_t^* \amalg_t X \amalg_{-t} U_t$$

can be described, in the *Schrodinger picture*, by the unitary group $R_t = \amalg_{-t} U_t$ for $t \geq 0$ and $U_{-t}^* \amalg_{-t}$ for $t < 0$.

10.2 Interpretation of $\mathfrak{h}$

The natural configuration space for the two-level atom is $\{1,2\}$ and a pure state of the atom is described by a *wave function* $\psi \in L^2(\{1,2\}) = \mathbb{C}^2$, associating the expectation $\langle \psi, b\psi \rangle$ to the observable $b \in M_2(\mathbb{C})$. However, the atom may be found in a mixed state, i.e. a convex combination of pure states, such as $\phi_\beta = (1+e^{-\beta})^{-1}(e^{-\beta}\langle e_1, \cdot e_1 \rangle + \langle e_2, \cdot e_2 \rangle)$. One may then think of it as residing in one of these pure states, each with a probability given by its coefficient. This may be substantiated by representing the observable algebra of the atom on $\mathbb{C}^2 \oplus \mathbb{C}^2 \simeq \mathfrak{h}_0$ as was done above. Since in quantum mechanics one is free to represent the Hilbert space of a system as $L^2(\Omega, \mu)$ for different choices of (Ω, μ), thereby obtaining equally valid *configuration spaces* (Ω, μ), we may now call $\{1,2\}\times\{1,2\}$ a configuration space of the atom. The state ϕ_β is then given by the wave function ξ_0 on $\{1,2\}^2$. When in this state the atom may be *excited* by the operator v^* to $v^*\xi_0 = \pi(e_{12})$, or *de-excited* by v to $v\xi_0 = \pi(e_{21})$. Further excitation of $v^*\xi_0$ or de-excitation of $v\xi_0$ is not possible, but $vv^*\xi_0 = \pi(e_{22})$ and $v^*v\xi_0 = \pi(e_{11})$ are permissible wave functions (when normalised). We recover ξ_0 as the superposition $v^*v\xi_0 + vv^*\xi_0$.

On the other hand Γ is a configuration space for the noise in thermal equilibrium at inverse temperature β. Its interpretation resembles that for the atom: excitations added to the thermal background are described as points with positive charge, the negative charges standing for de-excitations or *particles* removed from the background. In this way we view $\mathfrak{h} = L^2(\{1,2\}^2 \times \Gamma)$ as the space of wave functions of atom and noise.

10.3 The Schrodinger Evolution

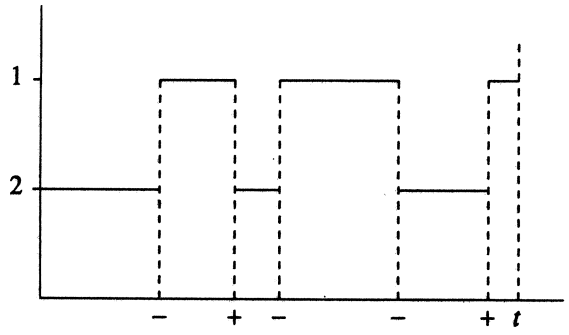
If at time 0 the system of atom and noise has the wave function ψ, then at a later time t its wave function will be $R_t\psi$, given by

$$(i,j,\sigma) \mapsto \sum_{\alpha\subset\sigma+t} \sum_{k\in\{1,2\}} \int_\Gamma u_t(\alpha\cup\omega^\dagger)_{ik}\,\psi(\bar{\alpha}\cup\omega)_{kj}\,d\omega \qquad (10.1)$$

where $u \in \mathcal{P}_1([0,\infty))$ is the adapted kernel cocycle with generator q given by

$$q^+ = v, \quad q^- = -v^*, \quad q^\circ = -\tfrac{1}{2}(1+e^{-\beta})^{-1}\begin{pmatrix} e^{-\beta} & 0 \\ 0 & 1 \end{pmatrix}_\pi$$

The cocycle may be represented diagrammatically: for $\sigma = (s,\varepsilon)$, $u_t(\sigma)_{ij} = 0$ unless the charges $\varepsilon_1,\ldots,\varepsilon_n$ alternate and have sum $(i-j)$, in which case $u_t(\sigma)_{ij} = \exp\{-\frac{1}{2}(1+e^{-\beta})^{-1}[l_1+e^{-\beta}l_2]\}$, where l_1(*respectively* l_2) is the total length of the higher (lower) plateaux in the following diagram.



The case $(i,j) = (1,2)$

The Schrödinger evolution R_t has the following interpretation: (10.1) expresses $R_t\psi$ in terms of ψ by summation over all configurations $(k,j,\bar{\alpha}\cup\omega)$ which may lead to (i,j,σ) by the combined effect of a left shift by t and an interaction with the atom. The atom, located at the origin, can emit bosons (here written as α) and absorb others (written ω), leaving a part $\bar{\alpha} = \sigma+t\backslash\alpha$ of the initial configuration intact. The shift then takes the result $\sigma+t$ to σ. The adaptedness of u ensures that only those particles which pass the origin during the time interval $[0,t]$ may be absorbed, and only such are emitted which end up in the *space* interval $[-t,0]$.

The value $u_t(\alpha\cup\beta^\dagger)_{ik}$ of the coefficient in the summation (10.1) is understood by considering (10.2) as a picture of a possible history of excitations and de-excitations during the time interval $[0,t]$.

Symbol	Description	Subsection
$\mathscr{L}^0$		0
$\mathscr{B}$	Borel σ-algebra	0
Γ	charged finite power set	2.1.1
∇	completion of above	2.1
μ	measure on ∇	2.1
$c_+ \geq c_- > 0$	fixed constants	2.1
$(\mathscr{A}, *)$	fixed involutive Banach algebra	2.1
$\dagger$	involution on ∇, μ and $\mathscr{F}(\nabla;\mathscr{A})$	2.1.2
$\mathscr{K}_b^{\mathscr{A}}, \mathscr{K}_o^{\mathscr{A}}$	bounded, smooth kernels	2.2
π_f, w_f, a_f	various kernels	2.2.2
$\hat{\mu}$	quasi-free characteristic function	2.2.2
$*$	bose convolution product	2.3.2
$\gamma_{a,b}, \gamma_c$		2.4.1
Γ_o, J_o, S_o	modular operators	2.4.1
G_y, D_x	left, right multiplication operators	2.4.2
ζ	symplectic form	2.4.4
v_f	commutant Weyl kernel	2.4.4
$\mathscr{W}, \mathscr{V}$	Weyl, commutant Weyl algebras	2.5.0
ν	$\mu + \mu^\dagger$	2.5.3
$I_{t)}$	$I \cap (-\infty, t)$	
$\Gamma_{\text{ad.}}$	adapted simplex	3.0
Γ^+	{maxσ has + charge}	3.0
k_x		3.1
$\mathscr{P}_o^{\mathscr{A}}$	smooth adapted kernel process	3.1.1
$\Delta^\kappa, \kappa = 0, +, -$	differential operators	3.2.1
I^κ	integral operators	3.2.4
$\boldsymbol{\Delta}$	$(\Delta^+, \Delta^-, \Delta^\circ)$	4.1
III_t	Shift	4.3.0
$\mathcal{Q}$	quantum probability space	5.2
$\sigma^{\mathcal{Q}}$	modular automorphism group	5.2
$S_{\mathcal{Q}}, J_{\mathcal{Q}}, \Delta_{\mathcal{Q}}$	modular operators	5.2
$\eta^*(\mathcal{Q})$	*-affiliated operators	5.3.1
X^+	conjugation on $\eta^*(\mathcal{Q})$	5.3

$\hat{x}$, $X^{\vee}$	non-commutative duality	6.3.6
$\mathfrak{p}_0, \mathfrak{p}_1, \mathfrak{l}^2, \mathfrak{m}$	kernel processes	6.7.1
$\mathfrak{s}, \mathfrak{c}$	more kernel processes	7.2
$\mathbb{E}$	conditional expectation	7.3.3
$\mathfrak{m}_t$	shift	8.0
$\hbar, K$	Planck, Botzmann constants	9.1.0

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Mathematics Department
University Park
GB-Nottingham NG7 2RD

Mathematisch Instituut
Katholieke Universiteit
NL-6525 ED Nijmegen

An Invitation to Probabilistic Cellular Automata

CHRISTIAN MAES[1]

Instituut voor Theoretische Fysica, K.U. Leuven

Abstract. Probabilistic Cellular Automata are, in their simplest form, discrete time Markov processes on the spin configurations of a regular lattice. While their construction is rather simple, they possibly give rise to a rich and complex behavior. Their study is part of probability theory but certain features make them relevant to a variety of problems in statistical physics, fluid mechanics,... as well as in computer science, biology, sociology, ... and many others.

We investigate the behavior of probabilistic cellular automata from the point of view of statistical mechanics. We summarize some recent results and introduce some of the key concepts.

Contents :

- General context, motivation and definitions
- Relation with Gibbs states (the method of equilibrium statistical mechanics)
- High noise regime (general results, Dobrushin regime, return to equilibrium)
- Low noise regime (non–ergodicity, phase transitions, erosion)
- Important examples (Stavskaya, Toom, Voter models)
- Self–Organization, hydrodynamics (role of conserved quantities, study of fluctuations)

[1] Aangesteld Navorser N.F.W.O., Belgium

1.INTRODUCTION

Cellular Automata (CA) consist of a large number of individual components or cells which may take on a finite number of positions. They are updated via a local and parallel mechanism in which, at every step, each cell's position takes on a new value depending on the previous time configuration of cells in a finite neighborhood.

They appear under various forms as (simple) complex systems, in neural networks, in models of self–organized criticality,... in a variety of contexts including biology, the social sciences and many others. Very often one is interested in their stochastic version, the so called Probabilistic Cellular Automata (PCA), obtained by adding noise : that is, independently at each cell and at each time, the automaton rule is followed only with some probability $1-\epsilon$ with noise parameter $0 \leq \epsilon \leq 1$.

We start by formalizing this idea. Consider the regular lattice $\mathbf{Z}^{d+1}$ in $d+1$ dimensions, $d = 0, 1, ...$ We look at it as the collection of translations of $\mathbf{Z}^d$, i.e. every site $x = (n, i) \in \mathbf{Z}^d$ has a *time*-coordinate $n \in \mathbf{Z}$ and a *space*-coordinate $i \in \mathbf{Z}^d$. To each site $x \in \mathbf{Z}^{d+1}$ we assign a random variable e_x taking values in the finite set $I, |I| = N$, with independent and identical distribution

$$Q(e_x = \alpha) = q_\alpha, \qquad \sum_\alpha q_\alpha = 1, \alpha \in I \tag{1.1}$$

The realization $e = \{e_x, x \in \mathbf{Z}^{d+1}\}$ determines which automaton rule must be followed. Choose a finite region $U \subset \mathbf{Z}^d$ which contains the origin and let there also be specified N functions

$$M^\alpha : \{+1, -1\}^U \to \{-1, +1\} \tag{1.2}$$

Their translations $M_i^\alpha, i \in \mathbf{Z}^d$ are functions of the infinite Ising spin configurations $\eta = \{\eta_i = \pm 1, i \in \mathbf{Z}^d\}, M_i^\alpha(\eta) = \pm 1$, which only depend on the configuration in the region $U + i$. For a given realization e and a given initial configuration $\sigma_{0,.} = \xi$ of spins on the time zero layer, we iteratively put

$$\sigma_{n,i} = M_i^\alpha(\sigma_{n-1,.}) \text{ if } e_{n,i} = \alpha \tag{1.3}$$

to obtain a spin configuration $\sigma = \sigma(e, \xi)$ on the (positive time) halfspace. It is now easy to see that the induced measure on the $\{\sigma_{n,.}, n = 0, 1, ...\}$ is a *Markov process* P^ξ with transition operator P acting on all continuous functions $f \in C(\Omega)$ of the configurations $\zeta \in \Omega = \{-1, +1\}^{\mathbf{Z}^d}$, as

$$Pf(\xi) = \int f(\zeta)\, p(d\zeta|\xi) = \int \hat{f}(\sigma(e, \xi))\, dQ(e) \tag{1.4}$$

where $\hat{f}(\sigma) = f(\sigma_{1,.})$ and $p(d\zeta|\xi)$ is the product measure on Ω with mean

$$h_i(\xi) = \int \zeta_i\, p(d\zeta|\xi) = \sum_\alpha q_\alpha M_i^\alpha(\xi), i \in \mathbf{Z}^d \tag{1.5}$$

Equivalently, this Markov process with initial data

$$P^\xi(\sigma : \sigma_{0,.} = \xi) = 1 \tag{1.6}$$

has *transition probabilities*

$$P^\xi(\sigma_{n+1,.} = \zeta | \sigma_{n,.} = \zeta') = \prod_{i \in \mathbf{Z}^d} p_i(\zeta_i|\zeta') \tag{1.7}$$

with $p_i(\zeta_i|\zeta') = \frac{1}{2}(1 + \zeta_i h_i(\zeta'))$.

This is a simple example of what we call a probabilistic cellular automaton.

It includes the deterministic version, the cellular automata, as the special case where $h_i(\zeta) = \pm 1$. From the above it is clear that many different choices for the functions $\{M^\alpha\}$ and the distribution Q lead to the same P^ξ or, in other words, every PCA can be viewed as a convex combination of CA but this decomposition is certainly non–unique. The representation with the e-field is however particularly useful when the PCA is a stochastic perturbation of one CA–rule. In that case

$$h_0(\zeta) = (1 - 2\epsilon)M(\zeta) \tag{1.8}$$

for one given function $M(\zeta) = \pm 1$, and $\epsilon \in [0, 1/2]$ is the *noise* level. The PCA can now be described via the stochastic evolution equation :

$$\sigma_{n,i} = e_{n,i} + (1 - |e_{n,i}|)M_i(\sigma_{n-1,\cdot}) \tag{1.9}$$

where the e_x take values -1, 1 both with probability ϵ and 0 with probability $1 - 2\epsilon$.

The addition of noise is the usual way to incorporate unknown factors present in the system. For example, in computer science, the reliability of large parallel computations must be understood for noisy environments.

The assumptions we make (such as considering a local, translation invariant and time homogeneous updating of Ising spins on a regular lattice) are certainly serious restrictions but we hope to convince the reader that even these simplest of PCA are worth our attention and still have many secrets to be discovered.

The basic questions concern the characterization of invariant measures, the asymptotic behavior and classification accordingly. A probability measure ν on Ω is said to be *invariant* or *stationary* if $\nu P = \nu$ and the PCA is called *ergodic* if $\mu P^n \to \nu$ (weakly as $n \to \infty$) for all $\mu_0 = \mu$. Here $\mu P^n = \mu P^{n-1} P$ and $\mu P(f) = \mu(Pf)$. The set of invariant measures is non–empty and convex. It is very well possible to have a unique invariant measure and still the PCA not being ergodic. A typical situation is when the system enters a cycle. Finally, to avoid confusion we must add that — as a dynamical system — the stationary process obtained by the Markov extension P^ν of some extremal invariant measure ν with time translation as measure preserving transformation, is always ergodic, see e.g. [R]. The time averages converge

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_\eta P^k \to \nu \tag{1.10}$$

weakly as $n \to \infty$ for ν-a.e. η. This point of view (the ergodic theory in the proper sense) is analyzed in [S] but it will however not be taken up here anymore.

2. The relation with Gibbs states

We consider here PCA with strictly positive transition probabilities and write

$$p_0(\zeta_0|\xi) = \exp[-H_0(\zeta_0, \xi)] \tag{2.1}$$

for some local function H_0 of the configuration ξ in the region U and of the spin ζ_0 which we think off as at time 1. We write $H_x(\sigma)$ for the usual translation over $x \in \mathbf{Z}^{d+1}$ of $H_0(\sigma)$ where

σ is now a space–time configuration. The Markov process in a finite space–time volume V with fixed spin configurations outside this volume, is a Gibbs measure with Hamiltonian

$$H_V(\sigma_V, \sigma_{V^c}) = \sum_{x \in \bar{V}} H_x(\sigma_V, \sigma_{V^c}) \tag{2.2}$$

where $\bar{V} = \{x \in \mathbf{Z}^{d+1} : x \in V \text{ or } U + x \cap V \neq \emptyset\}$. It is therefore possible to find a local *interaction potential* $\{J_A, A \subset \mathbf{Z}^{d+1}\}$ ($J_A = 0$ if A is too big) and formal Hamiltonian

$$H(\sigma) = \sum H_x(\sigma) = -\sum J_A \prod_{x \in A} \sigma_x \tag{2.3}$$

such that, on the halfspace, the Markov process P^ξ becomes a Gibbs measure for Hamiltonian H. Moreover, if we let the process evolve from an arbitrary time invariant measure ν at negative time $-N$ and next let $N \to \infty$, then a Gibbs measure P^ν is obtained on the full space–time lattice $\mathbf{Z}^{d+1}$. Its conditional distribution for the spin at site $x = (n, i)$ is

$$\begin{aligned} P^\nu(\sigma_x = \eta_x | \sigma_y = \eta_y, y \neq x) &= \frac{\prod' p_j(\eta_{m,j} | \eta_{m-1,\cdot})}{Z(\eta_y, y \neq x)} \\ &= \frac{1}{1 + \exp[-2 \sum_{A \ni x} J_A \prod_{y \in A} \eta_y]} \end{aligned} \tag{2.4}$$

where the product $\prod'$ is over all (m, j) such that $m = n + 1, i \in U + j$, or, $(m, j) = x$; Z is a normalization.

Note that this interaction potential obviously does not depend on the invariant measure ν and we can thus ask whether **all** Gibbs measures with respect to H, i.e. solutions $\mathcal{P}$ of (2.4), are Markov processes with transition operator P. In particular, the question arises because the P^ν can be viewed as Gibbs measures obtained from some very special PCA–like boundary conditions ; what then is the influence of also specifying the spins in the present and in the future ? The answer was given (at least partially) in [GKLM] : all space–time translation invariant Gibbs measures $\mathcal{P}$ **must** equal P^ν for some stationary measure ν. This implies a one-to-one relation between invariant Gibbs measures for Hamiltonian (2.3) and invariant measures of the PCA. It remains an open problem whether this spatial translation invariance is essential here.

Some consequences of this relation will be discussed in the following sections. A systematic study was carried out in [LMS]. We just add here that because of the normalization condition

$$\sum_{\xi_0 = \pm} p_0(\xi_0 | \xi') = 1 \tag{2.5}$$

applied to (2.1), the interaction potential is severely constrained. One of the more important consequences is that it implies that the free energy density of these Gibbs measures is always zero.

3. The high noise regime

The purpose of this section is to show that we can easily find a regime in which the PCA behaves essentially as was it an uninteracting system. That is, we give a condition

(which is easy to verify) under which there is an exponential decay to the unique invariant measure. This measure itself then has exponentially decaying correlation functions. In computer science language, such a PCA corresponds to an unreliable machine. The analog in equilibrium statistical mechanics is the high temperature regime for uniqueness of Gibbs measures. From the previous section it is in fact clear that any condition on the interaction potential ensuring uniqueness of the corresponding Gibbs measures, implies uniqueness of the PCA invariant measures. There is however a more natural way to end up at the same result.

The technique we use to find this criterion is interesting in itself and has many more applications. We wish to *couple* two processes starting from different initial configurations and the condition will be that there is a coupling which is almost concentrated on the diagonal : that is, the processes become identical. To explain this, we start by considering the simplest situation of two probability measures ρ^1 and ρ^2 on a single spin variable. We are thus given two numbers $p = \rho^1(+)$ and $q = \rho^2(+)$ and a coupling of ρ^1 and ρ^2 is a probability measure ρ on $\{+,-\} \times \{+,-\}$ characterized by non-negative numbers $a = \rho(+,+), b = \rho(+,-), c = \rho(-,+)$ and $d = \rho(-,-)$ such that $a+b = p, a+c = q, a+b+c+d = 1$. By trivial algebra it is clear that $b+c = p+q-2a \geq |p-q|$. We can thus choose $a = \min(p,q)$ to get that $\rho(\sigma^1 \neq \sigma^2) = |p-q|$.

Let us now take $\rho^1 = p_i(.|\xi)$ and $\rho^2 = p_i(.|\xi^j)$ where ξ^j is obtained from the configuration $\xi \in \Omega$ by flipping the spin at site $j \in \mathbf{Z}^d$. It is clear that the above constructed coupling ρ can be translated over all $i \in \mathbf{Z}^d$ to obtain a coupling $T^{\xi,\xi^j}(d\zeta^1, d\zeta^2)$ of $p(d\zeta|\xi)$ and $p(d\zeta|\xi^j)$ which is the product over all those translates. If we let for some local function f

$$\delta_i f = \sup_{\zeta} |f(\zeta) - f(\zeta^i)| \tag{3.1}$$

then, from

$$Pf(\xi) - Pf(\xi^j) = \int [f(\zeta^1) - f(\zeta^2)] T^{\xi,\xi^j}(d\zeta^1, d\zeta^2) \tag{3.2}$$

we get that

$$\delta_j Pf \leq \sum_i \delta_i f \, \alpha_{ij} \tag{3.3}$$

for $\alpha_{ij} = \sup_\xi T^{\xi,\xi^j}(\zeta_i^1 \neq \zeta_i^2)$. From the above we optimally choose

$$\alpha_{ij} = \sup_{\xi} |p_i(1|\xi^j) - p_i(1|\xi)| \tag{3.4}$$

If $\gamma = \sum_j \alpha_{ij}$, then

$$|||Pf||| = \sum_j \delta_j Pf \leq \gamma \sum_i \delta_i f = \gamma |||f||| \tag{3.5}$$

Hence, P is a contraction in the semi-norm $|||.|||$ whenever $\gamma < 1$.

It is then easy to see that $P^n f$ has to converge exponentially fast to some constant which is the expectation of f in the unique (by construction) invariant measure ν, i.e. in the sup-norm $||.||$

$$||P^n f - \int f d\nu|| \leq c\gamma^n |||f||| \tag{3.6}$$

with c some constant. To obtain exponential decay in space for this invariant measure is standard.

It is interesting to see that this high noise condition $\gamma < 1$ immediately implies a proof of the ergodicity condition for continuous time Markov processes (interacting particle systems, see [L], the $M < \epsilon$ condition). The reason is that there exists a $\delta \to 0$ approximation of a continuous time process via PCA by multiplying each spin flip probability with δ and at the same time taking time steps of length δ. We briefly sketch the argument here (the idea was found in [S]). Consider a sequence of PCA with transition operators P_δ defined as in (1.4)-(1.7) from the transition probabilities

$$p_{\delta,i}(\zeta_i|\xi) = (1-\delta)\frac{1}{2}(1+\zeta_i\xi_i) + \delta\, p_i(\zeta_i|\xi) \tag{3.7}$$

It is well known that $|||P_\delta^{t/\delta} f - S(t)f||| \to 0$ as $\delta \to 0$, where $S(t), t \geq 0$, is the Markov semigroup corresponding to the formal generator

$$Lf(\xi) = \sum_i c(i,\xi)[f(\xi^i) - f(\xi)] \tag{3.8}$$

where $c(i,\xi) = p_i(-\xi_i|\xi)$ is the spin flip rate. This is just a consequence of the suitable convergence $\frac{1}{\delta}(P_\delta - 1) \to L$ on the level of generators. In terms of these functions, we define

$$M = \sum_{j\neq 0} \sup_\eta |c(0,\eta^j) - c(0,\eta)|$$

and

$$\epsilon = \inf_\eta [c(0,\eta^0) + c(0,\eta)] \tag{3.9}$$

The PCA determined by P_δ is ergodic whenever the corresponding $\gamma_\delta < 1$. With (3.4) applied to (3.7), it is easy to show that this $\gamma_\delta = 1 + \delta(M - \epsilon)$ for $\delta \to 0$ so that

$$|||S(t)f||| \leq \gamma_\delta^{t/\delta}\, |||f||| \to e^{t(M-\epsilon)}\, |||f||| \tag{3.10}$$

and exponential ergodicity follows whenever $M < \epsilon$.

We end this section be referring to the recent work of [MS] for an optimized version of "natural" high noise criteria. There, a sequence of constructive conditions is given which are — in some sense — also **necessary** to have exponential ergodicity.

4. Low noise regime

The low noise regime corresponds to PCA which are a small perturbation of some deterministic CA-rule. For example, if in (1.7) ϵ is close to zero. There is no need to argue that to understand the behavior for small noise, we must know something about the CA trajectories themselves. This is the same idea as in equilibrium statistical mechanics where one tries to construct the low temperature phases as perturbations of certain groundstates. One could in principle even refer to section 2 and give the problem to the equilibrium statistical mechanician. The equilibrium theory however does hardly help us here because of the specific nature of the generated interactions (see (2.1) and (2.5)) and the, in general, rather complex structure of the groundstates or CA-trajectories. Yet, it turns out that similar

ideas can be successfully applied in the analysis of low noise PCA. Although there are still many open problems, recently there have emerged reasonably powerful methods to show phase transitions in PCA. We limit ourselves to some basic concepts.

For simplicity, let us consider a PCA of the form (1.7) with CA function $M(\zeta)$ having the $\pm$ symmetry (so that we can switch all $+$ into $-$ and vice versa). Suppose moreover that $M(+) = +$ (and then of course, by symmetry, also $M(-) = -$), so that both the all plus and the all minus configurations are left invariant for the CA evolution. A natural question is to see whether we find back two different invariant measures for the PCA, one which looks like a perturbation of the all plus state, the other of the all minus state. If this happens we say that there is a phase transition. The PCA has more than one invariant measure. For this to happen, it helps that the all plus configuration is *stable* under the CA evolution against inserting minus islands. In other words, we want to see an *erosion* mechanism which washes out finite islands in a sea of all plus. After a finite time, the CA evolution automatically restores the all plus configuration.

It is interesting to see that this is in fact never possible in one dimension ($d = 1$) for CA rules $M(\zeta)$ which are increasing in the configuration ζ. That means that if one spin in ζ is turned $+$, then M cannot decrease. A proof can be found in [LMS]. This lends support to the so called Positive Rates Conjecture, which says that for strictly positive transition probabilities there can be no phase transition in one dimension. It is however possible to find one-dimensional eroders and it is therefore hoped to be able to construct examples of non-ergodic PCA also in $d = 1$, see [G] and [GKL].

Examples of such eroders can be easily found in higher dimensions and in the next section we will encounter the best known example — the Toom model. It is of course still a long way to prove from an erosion property that the PCA has different invariant measures. There are at least two methods which one can try.

The first is reminiscent of the Peierls argument in statistical mechanics and can be found in [T], [BG]. One constructs contours (which may be quite complicated) in configurations where, started from the all plus state, after a large time the spin at the origin is minus. Clearly, errors (deviating from the CA rule) have been made. Every such error costs probability ϵ and there have to be quite sufficient given the eroder property. If even more, the geometry is such that we can restrict the number of possible contours in terms of the numbers of errors that have been made, we are ready for an energy–entropy argument to conclude that the probability of having a minus at the origin is very small, uniform in time.

The second method uses ideas from renormalization group and is applied in [D], [BD]. The idea is to use the eroder mechanism and look at the dynamics at a larger scale (in space and time) to study the flow of the noise. If the noise level goes down, then the CA invariant configurations, all plus and all minus one, are stable fixed points and there is a phase transition at low noise.

The next section discusses some low noise results in three prototype examples.

5. Important examples

a) Stavskaya's model. : It is a one-dimensional PCA defined by the transition probabilities

$$\begin{aligned} p_0(1|\xi) &= 1 \text{ if } \xi_0 = \xi_{-1} = 1 \\ &= \epsilon \text{ otherwise} \end{aligned} \tag{5.1}$$

Equivalently, we let $e_x \in \{0,1\}, x \in \mathbf{Z}^2$, be the i.i.d. random field with distribution $q_0 = 1-\epsilon, q_1 = \epsilon$, cf. (1.1), and functions (1.2) given by

$$\begin{aligned} M^0(\zeta) = 1 \text{ if } \zeta_0 = \zeta_{-1} = 1 \\ = -1 \text{ otherwise} \end{aligned} \tag{5.2}$$

and $M^1(\zeta) = 1$ always. We say that there is an *error* at $x \in \mathbf{Z}^2$ if $e_x = 1$. Observe that the all one state δ_+ ($\xi_i = 1$, for all $i \in \mathbf{Z}$), is invariant, no matter what value ϵ has. From the $\gamma < 1$ condition of Section 3, we find that for $\gamma = 2(1-\epsilon) < 1$ or $\epsilon > .5$, Stavskaya's model is exponentially ergodic and δ_+ is the **only** invariant measure. For $\epsilon = 0$ the all minus configuration is invariant as well. One of the first results concerning phase transitions for PCA was the proof (see [T2]) that for $\epsilon > 0$ small enough, there is another invariant measure (perturbation of the all minus state) besides δ_+.

Suppose we start the PCA from a configuration where all spins are minus. If at time $n > 0$ the origin $\sigma_{n,0} = -1$, then certainly $e_{n,0} = 0$ and either $\sigma_{n-1,-1} = -1$ or $\sigma_{n-1,0} = -1$. At any rate, repeating this observation, also at time $n-2$ at least one of the spins $\sigma_{n-2,-2}, \sigma_{n-2,-1}, \sigma_{n-2,0}$ must be minus. Iterating up to the time zero layer we obtain a path of minus spins and corresponding values $e_x = 0$ (no error !). On the other hand, whenever such a path exists we must have $\sigma_{n,0} = -1$. Hence, showing that with overwhelming probability the origin carries a minus spin, uniformly in time, can be mapped on a problem in oriented percolation. It is then easy to prove that there is a critical ϵ_c such that for $\epsilon > \epsilon_c$ there is ergodicity and for $\epsilon < \epsilon_c$ there are different invariant measures. It is estimated that $\epsilon_c \approx 0.31$. The importance of Stavskaya's model lies, besides being the first of such models, in being the simplest prototype of models for which oriented percolation plays an important role. There are in fact many systems which, when viewed on suitable length and time scales, are dominated by 1-dependent oriented percolation. For this recent insight, we refer to [D] for a summary.

b) Majority Vote model. : The one-dimensional version of this model has transition probabilities

$$\begin{aligned} p_i(\zeta_i|\xi) = 1-\epsilon \text{ if } \zeta_i = \text{sgn}(\xi_{i-1} + \xi_i + \xi_{i+1}) \\ = \epsilon \text{ otherwise} \end{aligned} \tag{5.3}$$

We thus take $M(\zeta) = \text{sgn}(\zeta_{-1} + \zeta_0 + \zeta_1)$ in (1.8) and apply (1.9). In words, the spin takes on the majority of itself and its two nearest neighbors with probability $1-2\epsilon$; with probability 2ϵ the spin chooses randomly between ± 1. From the $\gamma < 1-$criterion, there is exponential ergodicity for $\frac{1}{3} < \epsilon < \frac{2}{3}$.

Gray has proven that for **all** $\epsilon > 0$ this PCA has a unique invariant state, [Gr]. The idea is to look at the process ***simultaneously*** for the all 1 and the all -1 initial conditions. We define new variables :

$$\begin{aligned} s_x = 1 \text{ if } \sigma_x^+ = \sigma_x^- = 1 \\ = 0 \text{ if } \sigma_x^+ = 1, \sigma_x^- = -1 \\ = -1 \text{ if } \sigma_x^+ = \sigma_x^- = -1 \end{aligned} \tag{5.4}$$

where $\sigma^{\pm} = \sigma(e, \pm)$ are defined as in (1.3) and (1.9). Note that always $\sigma_x^+ \geq \sigma_x^-$. In terms of the $s-$variables, the process starts from $s_{0,i} = 0$ for all $i \in \mathbf{Z}$ and we must show that

all these zeroes disappear in the course of time. The reason that happens is the occurence (with probability one whenever $\epsilon > 0$) of blocks of 1 or -1 in the middle of which no new zeroes can be created and thus, roughly speaking are only changing through the walks of their boundaries. One has the combination of two effects : 1. in blocks of $s = \pm 1$, there is stability against zeroes 2. the boundaries of the $s = \pm 1$ regions perform a random walk–like motion which (we are in one dimension !) tends to create even bigger regions when they meet. Gray's proof can be applied to many other (but similar) one-dimensional PCA.

c) Toom's model. : Toom's North East Center model provides an important example of a PCA for which a contour argument can be set up showing a phase transition. It is a stochastic perturbation of a two-dimensional majority vote CA possessing the so called eroder property. On the square lattice consider the triples of spins formed by the spin at site i, its northern and its eastern neighbor. If the majority of these spins is 1, we put the spin at site i at the next time equal to 1 with probability $1 - p$; if this majority was -1, we let it be -1 with probability $1 - q$. Toom proves (among many other beautiful results) that whenever p and q are small, there is more than one invariant measure.

The prime feature of the model that makes the argument work is, as was mentioned earlier, the eroder property. That is, the all 1 (or all -1) configuration is invariant under the CA ($p = q = 0$) and any configuration which is 1 on all but a finite number of sites, evolves after a finite time into the all 1 configuration. Moreover, the intersection of any two different North-East-Center triples contains at most one site. If we find a minus spin at the origin at time $n >> 1$, then, if initially all spins were 1, a lot of "errors" (not following the majority rule) must have been made. More details are to be found in the original [T] paper as well as in [LMS]. A more recent analysis which seems to come close to the heart of the matter is in [BG2].

6. Further Remarks

We end these notes by mentioning some of the interesting activity that goes on for PCA-related models. They do not fall however in the context of the previous sections. We wish to warn the reader that these remarks are highly incomplete and only serve the purpose of referring to other work.

a) Conservation laws. : The PCA we considered in these notes are defined via simultaneaous (or parallel) and independent updating of all spins. Therefore a strict conservation law, e.g. of the total spin, cannot be present. It would imply a relation between the spins. PCA which are used to model hydrodynamics evolve by particle exchange and their special behavior strongly depends on the existence of corresponding conservation laws. Due to these constraints their study is in general more complicated. On the other hand, the presence of conservation laws makes the PCA physically different and interesting. An important example is in the study of fluctuations. The question is how the induced time evolution looks like for the coarse grained quantities. Central limit phenomena are quite different when a conservation law is present. The fluctuations of nonconserved quantities change in time on a much faster scale than those of the conserved quantity. The result is that we may expect that at a macroscopic scale the BBGKY-hierarchy of microscopic evolution equations for the correlation functions closes up. There are many discussions on this phenomenon and we prefer to leave it to the reader to consult [HS], [Sp], [GVV] and the references therein.

b) Self-organized criticality (SOC). : By now, there are many (P)CA models of SOC. The recent revival of SOC was started by the paper [BTW]. We limit us here to mentioning

just one type of models in continuous time and with a conservation law. They are again very different from the PCA we have considered in the previous section. They are exclusion processes in which nearest neighbor spins are exchanged with rates which depend not only on the neighboring configuration but also on the orientation of this bond. This way the model becomes anisotropic. It was realized in [LVWZ] and [GLMS] that these models generically have long range spatial stationary correlations. E.g. the two points function decays only like a power. This behavior is drastically different from the one in section 3 where close to independence one always has exponential decay. These models are therefore *critical* and this without tuning some special parameter. A rigorous proof for microscopic systems is lacking here and we invite the reader to look for some simple PCA for which this effect can be rigorously demonstrated.

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