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Preface

This monograph covers a series of lectures presented by Maarten Bergvelt and Fons ten Kroode in the seminar "Mathematical Structures in Field Theories" during the academic year 1986–1987. At that time Bergvelt was a research fellow at the Max Planck Institute in Bonn, Germany, while ten Kroode was preparing his thesis on Kac–Moody algebras at the Mathematical Institute of the University of Amsterdam. Notwithstanding the pressure they both were under they managed to give a very comprehesive series of seminars for an audience of mathematicians and theoretical physicists As is shown by the table of contents they covered, starting with finite dimensional semisimple Lie algebras and ending with infinite dimensional matrix algebras, all fundamental concepts needed to fully understand the structure of Kac–Moody algebras and the integrable highest weight representations. All this is richly illustrated by the homogeneous and principal realization of the basic module $L(\Lambda_0)$ of the Kac–Moody algebra $A_1^{(1)}$.

The editors want to apologize for the delay in publication of the volume. This delay is mainly due to the problems in getting the manuscript typed. In the end ten Kroode did it himself after having finished his thesis.

The organizers of the seminar want to express their acknowledgement to Bergvelt and ten Kroode for their most inspiring lectures and to the people of the Centre for Mathematics and Computer Science for the printing job.

The editors
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Chapter 1 simple Lie algebras

1.0 introduction

In this chapter we give a survey of the theory of finite dimensional simple Lie algebras in such a way that the generalization to Kac-Moody algebras is not too surprising. We will not attempt to prove everything, but we will illustrate the theory with many examples. As a standard reference to the theory of finite dimensional simple Lie algebras we use [Hum].

1.1 simple Lie algebras, definitions

A Lie algebra over $\mathbb{C}$ is a vector space over $\mathbb{C}$ equipped with an operation on itself, called bracket or commutator and denoted by $(x,y) \rightarrow [x,y]$, which is bilinear, antisymmetric (i.e., $[x,y] = -[y,x]$) and which satisfies the Jacobi identity:

(1.1.1) $$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$$

The standard example of a Lie algebra is the space EndV of endomorphisms of a vector space V, equipped with the bracket

(1.1.2) $$[x,y] := xy - yx \qquad \forall x,y \in \text{End}V$$

EndV together with this bracket is also denoted by $gl(V)$. More concretely, if V has (complex) dimension n we can, by choosing a basis, identify $gl(V)$ with $gl(n,\mathbb{C})$, the space of $n \times n$ complex matrices.

Let from now on $\underline{g}$ be a fixed Lie algebra. Let $\underline{a}$, $\underline{b}$ be two subspaces of $\underline{g}$. Then we denote by $[\underline{a}\ , \underline{b}]$ the vector space spanned by elements $[x,y]$ with $x \in \underline{a}$ and $y \in \underline{b}$.

A subalgebra $\underline{a}$ of $\underline{g}$ is a subspace of $\underline{g}$ such that $[\underline{a}\ , \underline{a}] \subset \underline{a}$. An ideal $\underline{i}$ of $\underline{g}$ is a subalgebra such that $[\underline{g}\ , \underline{i}] \subset \underline{i}$. For instance $\{0\}$ and $\underline{g}$ itself are ideals of $\underline{g}$. We will call these the trivial ideals. A nontrivial ideal in $gl(n,\mathbb{C})$ is formed by the multiples of the identity matrix. Possibly nontrivial ideals are the center $\underline{c}$, defined by the requirement $[\underline{c}\ , \underline{g}] = 0$, and the so-called derived algebra $\underline{g}' := [\underline{g}\ , \underline{g}]$; this depends on $\underline{g}$. A Lie algebra is called abelian if [g , g] = 0. For example every one dimensional Lie algebra is abelian, just as the set of diagonal matrices in $gl(n,\mathbb{C})$.

A Lie algebra g is called simple if it is nonabelian and if it has no nontrivial ideals. The Lie algebra $gl(n,\mathbb{C})$ is not simple, since it contains the ideal $\{\lambda I_n, \lambda\in\mathbb{C}\}$. In this chapter we will study the simple Lie algebras not so much because the property of simplicity is of great interest to us, but because these Lie algebras happen to be special cases of a class of in general infinite dimensional Lie algebras, the so-called Kac-Moody algebras, which are (except when they happen to be finite dimensional) not simple.

1.2 simple Lie algebras, examples

To have some examples of simple Lie algebras, we will briefly describe in this section the four classical series of Lie algebras, denoted by A_l, B_l, C_l and D_l. We will skip the proof that these algebras are indeed simple.

The algebra A_l is also known as $sl(l+1,\mathbb{C})$ and consists of $(l+1)\times(l+1)$ traceless matrices. The other three series B_l, C_l and D_l consist of subalgebras of A_l (for suitable l). To describe them, we need a little digression on symmetries of bilinear forms. Let V be a vector space and f a bilinear form on V. Then we will call $x\in gl(V)$ an (infinitesimal) symmetry of f if

$$f(xv,w) + f(v,xw) = 0 \qquad \forall v,w\in V \tag{1.2.1}$$

One easily checks that the symmetries of f form a Lie algebra, i.e., that if $x,y\in gl(V)$ are symmetries of f, their commutator [x,y] is also a symmetry of f.

Using this concept, B_l is defined as the subalgebra of $A_{2l} = sl(2l+1,\mathbb{C})$, consisting of infinitesimal symmetries of a nondegenerate symmetric form on $\mathbb{C}^{2l+1}$. One also denotes B_l by $so(2l+1,\mathbb{C})$ and checks that it consists of skew symmetric matrices.

In the same way C_l is defined as the Lie algebra of infinitesimal symmetries of a nondegenerate antisymmetric bilinear form (= symplectic form) on $\mathbb{C}^{2l}$. This algebra is also denoted by $sp(2l,\mathbb{C})$ and it consists of matrices of the form

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^t \end{pmatrix} \tag{1.2.2}$$

with Z_i $l{\times}l$ matrices and Z_2, Z_3 symmetric.

The Lie algebra D_l is defined in a similar way as B_l, but now f is a symmetric bilinear form on an even dimensional space $\mathbb{C}^{2l}$. D_l is also known as so($2l$,$\mathbb{C}$) and, just as B_l, consists of skew symmetric matrices.

Beside the examples above there exist five more simple Lie algebras, the so-called exceptional ones, denoted by F_4, G_2, E_6, E_7 and E_8. We will not discuss these here, but we remark that especially E_8 is very interesting from the point of view of string theory.

1.3 the Killing form

In the theory of Lie algebras an important rôle is played by a certain bilinear form on a Lie algebra g. In this section we study some of the properties of this so-called Killing form.

We define for every $x \in g$ a linear transformation $adx \in gl(g)$ by $adx(y) := [x,y]$. This is called the adjoint action of g on itself. If g has dimension n, adx can be represented (after choosing a basis for g) by an $n{\times}n$ matrix. Using this, we introduce the Killing form (on finite dimensional Lie algebras) by

(1.3.1) $$(x|y) := \mathrm{trace}(\, adx\ ady\,)$$

By a fundamental property of the trace this definition is independent of the choice of basis made to calculate the right hand side of (1.3.1). One also sees that the Killing form is bilinear and symmetric. Furthermore, we have

(1.3.2) $$([x,y]|z) = (x|[y,z])$$

Indeed: $$([x,y]|z) = \mathrm{trace}(\, ad[x,y]\ adz\,)$$

$$= \mathrm{trace}(\, [adx\ ,\ ady]\ adz)$$

$$= \mathrm{trace}(\, adx\ [ady\ ,\ adz]\,)$$

$= (x|[y,z])$

Property (1.3.2) is referred to as the invariance of the Killing form. (In fact it shows that $\mathrm{ad}\,\underline{g} \subset gl(\underline{g})$ consists of infinitesimal symmetries of the Killing form, see section 1.2.)

Very important is the following theorem, which we shall not prove.

theorem (Cartan)

The Killing form on a simple (finite dimensional) Lie algebra is nondegenerate.♦

1.4 Cartan subalgebras

Let form now on $\underline{g}$ be a fixed finite dimensional Lie algebra over $\mathbb{C}$. An element $x \in \underline{g}$ is called semisimple if $\mathrm{ad}x$ can be diagonalized. For example $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in A_1 = sl(2,\mathbb{C})$ is semisimple while $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not. A Cartan subalgebra $\underline{h}$ of $\underline{g}$ is a maximal abelian subalgebra consisting of semisimple elements. So $\mathbb{C}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are Cartan subalgebras while $\mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is not (although it is maximal abelian).

theorem 1.4.1

Any simple Lie algebra has a Cartan subalgebra.♦

definition 1.4.2

An automorphism of $\underline{g}$ is an invertible linear mapping $\phi : \underline{g} \to \underline{g}$ such that $\phi([x,y]) = [\phi(x),\phi(y)]\ \forall x,y \in \underline{g}$.

theorem 1.4.3

Let $\underline{h}_1$ and $\underline{h}_2$ be two Cartan subalgebras of $\underline{g}$, then there exists an automorphism ϕ of $\underline{g}$ such that $\phi(\underline{h}_1) = \underline{h}_2$.♦

We remark that this theorem does not hold for real simple Lie algebras. Therefore, their structure theory is more complicated.

As an example of theorem 1.4.3 we mention the Cartan subalgebras $\mathbb{C}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which are related through conjugacy by the element $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Theorem 1.4.3 tells us that all Cartan subalgebras are equivalent. In particular, their dimension is an invariant of $\underline{g}$ and is called the rank of $\underline{g}$. The subscripts in the Lie algebras $A_l, \ldots, E_8$ denote their rank. From now on we will fix some Cartan subalgebra $\underline{h}$ of $\underline{g}$. For the classical Lie algebras A_l and C_l one always chooses $\underline{h}$ to consist of diagonal matrices.

1.5 roots and root spaces

The importance of Cartan subalgebras lies in the fact that the endomorphisms adh, $h \in \underline{h}$ commute. (Indeed we have: $adh_1\, adh_2(x) = [h_1,[h_2,x]] = [x,[h_2,h_1]] + [h_2,[h_1,x]] = adh_2\, adh_1(x)$, where we have used the Jacobi identity and the fact that $\underline{h}$ is abelian.) As is well known, a set of commuting, diagonalizable operators on a vector space has a collection of common eigen vectors which form a basis for this vector space. We will study the common eigenvectors of $\{adh, h \in \underline{h}\}$.

Let $\underline{h}^*$ be the dual of $\underline{h}$ (i.e., the space of linear functions $\alpha : \underline{h} \rightarrow \mathbb{C}$) and denote the dual contraction by $<\alpha,h>$, $\alpha \in \underline{h}^*$, $h \in \underline{h}$. Define for $\alpha \neq 0 \in \underline{h}^*$ the (possibly trivial) space

(1.5.1) $\qquad \underline{g}_\alpha := \{x \in \underline{g} \mid [h,x] = <\alpha,h>x, \forall h \in \underline{h}\}$

If $\underline{g}_\alpha \neq 0$ it is called a root space, α a root and $x \in \underline{g}_\alpha$ a root vector. The set of all roots is called the root system Δ:

(1.5.2) $\qquad \Delta := \{\alpha \in \underline{h}^* - \{0\} \mid \underline{g}_\alpha \neq 0\}$

The Cartan subalgebra $\underline{h}$ itself consists of eigenvectors of $ad\underline{h}$ with eigenvalues zero, so we can write $\underline{h} = \underline{g}_0$. However, one prefers to exclude 0 from the set of roots Δ.

Since the eigenvectors of $ad\underline{h}$ form a basis of $\underline{g}$, we have a direct sum decomposition

(1.5.3) $\qquad \underline{g} = \underline{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \underline{g}_\alpha$

This is called the root space decomposition of $\underline{g}$.

1.6 roots and root spaces for A_l

Recall that $A_l = sl(l+1,\mathbb{C})$. We choose as Cartan subalgebra $\underline{h}$ the set of diagonal matrices (of trace zero). Let $\mathrm{diag}(a_1, a_2,...,a_{l+1})$ denote the diagonal matrix with $a_i \in \mathbb{C}$ as its entries. Then

(1.6.1) $$\underline{h} = \{\mathrm{diag}(a_1, a_2,...,a_{l+1}) \mid a_i \in \mathbb{C}, \sum a_i = 0\}$$

Define the elements λ_i of $\underline{h}^*$ by

(1.6.2) $$<\lambda_i , \mathrm{diag}(a_1, a_2,...a_{l+1})> = a_i \qquad i = 1, 2,...l,+1$$

We have of course the following relation between the λ_i's

(1.6.3) $$\lambda_1 + \lambda_2 + ... + \lambda_{l+1} = 0$$

and any choice of l of the λ_i's yields a basis for $\underline{h}^*$.

Let E_{ij} be the matrix with a 1 on the $(i,j)^{th}$ entry and 0 elsewhere. Then one calculates

(1.6.4) $$[\mathrm{diag}(a_1, a_2,...,a_{l+1}) , E_{ij}] = (a_i - a_j) E_{ij}$$

$$= <\lambda_i - \lambda_j , \mathrm{diag}(a_1, a_2,...,a_{l+1})> E_{ij}$$

Hence we find that $\lambda_i - \lambda_j \in \underline{h}^*$ is a root for all $i \neq j$ and that the associated root space is given by

(1.6.5) $$g_{\lambda_i - \lambda_j} = \mathbb{C}\, E_{ij} \qquad i \neq j$$

Since the E_{ij} together with $\underline{h}$ form a basis for $sl(l+1,\mathbb{C})$, we find that all roots are of the form $\lambda_i - \lambda_j$. In other words

(1.6.6) $$\Delta = \{ \lambda_i - \lambda_j \mid i \neq j, \ 1 \leq i,j \leq l+1 \}$$

1.7 further properties of roots and root spaces

We want to know which root spaces $\underline{g}_\alpha$ are orthogonal to each other with respect to the Killing form (1.3.1); let $x_\alpha \in \underline{g}_\alpha$, $y_\beta \in \underline{g}_\beta$, then by the invariance of the Killing form we have for all $h \in \underline{h}$:

(1.7.1) $$0 = ([h,x_\alpha] \mid y_\beta) + (x_\alpha \mid [h,y_\beta])$$

$$= \langle\alpha+\beta , h\rangle (x_\alpha | y_\beta)$$

Hence, if $\alpha + \beta \neq 0$ the root spaces $\underline{g}_\alpha$ and $\underline{g}_\beta$ are orthogonal. From this we see that, if α is a root, $-\alpha$ must also be a root (for else by (1.5.3) $(x_\alpha | y) = 0$ for all $y \in \underline{g}$, contradicting the nondegeneracy of the Killing form). Since $\underline{h}$ is orthogonal to all $\underline{g}_\alpha$, $\alpha \in \Delta$, the restriction of the Killing form to $\underline{h}\times\underline{h}$ must be nondegenerate. This leads to the introduction of an isomorphism between $\underline{h}$ and $\underline{h}^*$:

(1.7.2) $$\nu : \underline{h} \rightarrow \underline{h}^*$$

$$\langle\nu(h) , h'\rangle := (h \mid h') \qquad \forall h,h' \in \underline{h}$$

Using this isomorphism, we can transport the Killing form to $\underline{h}^*$. This form will also be denoted by $(\mid)$.

In the example of $sl(l+1,\mathbb{C})$ we found that the root spaces $\underline{g}_\alpha$ are one dimensional. This holds for all simple Lie algebras. A simple calculation, using the Jacobi identity, shows that

(1.7.3) $$[\underline{g}_\alpha , \underline{g}_\beta] \subset \underline{g}_{\alpha+\beta} \qquad \forall \alpha,\beta \in \Delta$$

In fact even more is true; if α,β and $\alpha+\beta$ are roots, then

(1.7.4) $$[\underline{g}_\alpha , \underline{g}_\beta] = \underline{g}_{\alpha+\beta}$$

From (1.7.3) we see that $[\underline{g}_\alpha , \underline{g}_{-\alpha}]$ belongs to $\underline{h}$. Let $x_{\pm\alpha} \in \underline{g}_{\pm\alpha}$, then we can write

(1.7.5) $$(h \mid [x_\alpha, x_{-\alpha}]) = ([h, x_\alpha] \mid x_{-\alpha})$$

$$= <\alpha, h> (x_\alpha \mid x_{-\alpha})$$

$$= (h \mid v^{-1}(\alpha)) (x_\alpha \mid x_{-\alpha})$$

$$= (h \mid (x_\alpha \mid x_{-\alpha}) v^{-1}(\alpha)) \qquad \forall h \in \underline{h}$$

Since (|) is nondegenerate on $\underline{h}$, we obtain

(1.7.6) $$[x_\alpha, x_{-\alpha}] = (x_\alpha \mid x_{-\alpha})\, v^{-1}(\alpha)$$

So we find the important result that the triple x_α, $x_{-\alpha}$, $v^{-1}(\alpha)$ forms a subalgebra of $\underline{g}$, isomorphic to $sl(2,\mathbb{C})$. To see this, we choose a standard basis for $sl(2,\mathbb{C})$:

(1.7.7) $$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and choose $(x_\alpha \mid x_{-\alpha})$ to be $\frac{2}{|\alpha|^2}$. Then we have an isomorphism

(1.7.8) $$e \to x_\alpha \qquad f \to x_{-\alpha} \qquad h \to \frac{2}{|\alpha|^2} v^{-1}(\alpha) =: \alpha^\vee$$

The $\alpha^\vee$'s are called coroots. We learn from this that a simple Lie algebra is built up from $sl(2,\mathbb{C})$ subalgebras (one for each pair $\alpha,-\alpha$ in Δ), spanned by x_α, $x_{-\alpha}$ and $\alpha^\vee$.

Next we discuss the connection between the different $sl(2,\mathbb{C})$ subalgebras. Let $\alpha,\beta \in \Delta$, $\alpha \neq \beta$. Then we want to know for which $i \in \mathbb{Z}$ the element $\beta + i\alpha$ is again a root. It turns out that these integers i form a closed interval: $-r \leq i \leq q$ and that

(1.7.9) $$r - q = \frac{2(\beta \mid \alpha)}{(\alpha \mid \alpha)} = <\beta, \alpha^\vee>$$

The set $\{\beta + i\alpha \mid -r \leq i \leq q\}$ is called the α-root string through β. For formula (1.7.9) to make sense, the right hand side must be an integer and, miraculously, $<\beta,\alpha^\vee>$ turns out to be an integer for all α,β in the root system of a simple Lie algebra.

1.8 simple roots

First note that Δ spans $\underline{h}^*$, for if it did not, there would exist an element $h \in \underline{h}$, such that $\langle\alpha,h\rangle = 0\ \forall \alpha \in \Delta$, implying that $[h,x_\alpha] = 0\ \forall \alpha \in \Delta$, or in other words that h would belong to the center of $\underline{g}$, which, being an ideal, is zero. Next consider the formula (1.7.4). It suggests that we might try to generate all root spaces by taking commutators of a subset of the root spaces, or, more or less equivalently, that one might try to write the roots as integral linear combinations of a subset of the roots.

These considerations lead to the concept of a root basis Π of the root system Δ. This is a subset $\{\alpha_1, \alpha_2, \ldots, \alpha_l\}$ of Δ that forms a basis of $\underline{h}^*$ and such that every $\alpha \in \Delta$ can be written as

$$\alpha = \sum_{i=1}^{l} n_i \alpha_i \qquad n_i \in \mathbb{Z} \tag{1.8.1}$$

with either all $n_i \geq 0$ or all $n_i \leq 0$ (and not all $n_i = 0$).

theorem
Every root system Δ associated to a simple Lie algebra $\underline{g}$ has a root basis Π.♦

The elements α_i of Π are called the simple roots, the corresponding root vectors, denoted by e_i are called simple root vectors. The root vectors associated to the roots $-\alpha_i$ are denoted by f_i. We always normalize things in such a way that

$$[e_i\,, f_i] = \alpha_i^\vee \tag{1.8.2}$$

with

$$\alpha_i^\vee := \frac{2}{|\alpha_i|^2} v^{-1}(\alpha_i) \tag{1.8.3}$$

1.9 generators and relations

The simple root vectors e_i, f_i and their associated simple coroots $\alpha_i^\vee$ satisfy the following relations

(1.9.1) $[e_i , f_j] = \delta_{ij}\, \alpha_i^\vee$

(1.9.2) $[\alpha_i^\vee , e_j] = <\alpha_j , \alpha_i^\vee> e_j$

(1.9.3) $[\alpha_i^\vee , f_j] = -<\alpha_j , \alpha_i^\vee> f_j$

(1.9.4) $[\alpha_i^\vee , \alpha_j^\vee] = 0$

(1.9.5) $(\mathrm{ad}\, e_i)^{1-<\alpha_j,\alpha_i^\vee>} e_j = 0$

(1.9.6) $(\mathrm{ad}\, f_i)^{1-<\alpha_j,\alpha_i^\vee>} f_j = 0$

Relation (1.9.1) for $i = j$ is the definition of $\alpha_i^\vee$, while for $i \neq j$ (1.9.1) follows from the relation $[e_i , f_j] \subset \underline{g}_{\alpha_i - \alpha_j}$ and the fact that $\alpha_i - \alpha_j$ is not a root (because α_i and α_j are simple roots). Relations (1.9.2) and (1.9.3) just state that e_j and f_j are root vectors associated to the roots α_j and $-\alpha_j$ respectively. Relation (1.9.4) is the commutativity of $\underline{h}$ and finally, relations (1.9.5) and (1.9.6) tell us that the α_i-root string through α_j is given by α_j, $\alpha_j + \alpha_i$, $\alpha_j + 2\alpha_i$,..., $\alpha_j + (-<\alpha_j , \alpha_i^\vee>)\alpha_i$.

Note in these relations the appearance of the matrix of integers

(1.9.7) $a_{ij} := <\alpha_j , \alpha_i^\vee>$ $\qquad 1 \leq i,j \leq l$

This is called the Cartan matrix of $\underline{g}$. It is uniquely determined by the root system Δ of $\underline{g}$ and satisfies the following conditions

(1.9.8) (1) $a_{ii} = 2$

(2) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0, \quad a_{ij} \in \mathbb{Z}, \ a_{ij} \leq 0$ if $i \neq j$

(3) det $A \neq 0$ and all principal minors of A are positive

The relations (1.9.1)-(1.9.6) are very important because they are defining relations. This means the following: one can start with symbols e_i, f_i and $\alpha_i^\vee$ and consider the Lie algebra generated by them, subject to the relations (1.9.1)-(1.9.6) (with a_{ij} a matrix satisfying the conditions (1.9.8)). Then one can prove that the resulting Lie algebra is a direct sum of finite dimensional simple Lie algebras. Imposing one more condition on the Cartan matrix ("indecomposibility"), one obtains in this way precisely the finite dimensional simple Lie algebras.

This construction of the simple Lie algebras by choosing a Cartan matrix satisfying the conditions (1.9.8) and imposing the relations (1.9.1)-(1.9.6) can be generalized. If one drops the condition det $A \neq 0$ and requires that all proper principal minors are positive, one obtains by the same construction the so-called affine Kac-Moody algebras. These will be the subject of the rest of these lectures.

Chapter 2 affine Lie algebras [Kac]

2.0 introduction

In this chapter we introduce a second class of Kac-Moody algebras, namely the affine ones. In contrast with the finite dimensional simple Lie algebras, which were surveyed in the preceding chapter, these algebras are neither finite dimensional nor simple. They may be defined in two equivalent ways. The first one is purely algebraic and was already hinted at in section 1.9; it defines the affine algebras as algebras generated by symbols e_i, f_i and $\alpha_i^\vee$ subject to relations (1.9.1)-(1.9.6). This method has the advantage that it may also be used to define arbitrary Kac-Moody algebras.

In this chapter however, we have chosen for an explicit construction of affine algebras as the universal central extensions of Lie algebras associated to certain infinite dimensional groups. In the next chapter we will study the structure of the affine algebras and discover that it is quite similar to the structure of finite dimensional simple Lie algebras. As a result we will eventually find the defining relations (1.9.1)-(1.9.6) for the affine case.

2.1 loop groups

In this section we will define the loop group $\tilde{G}$ associated to a finite dimensional Lie group G. For simplicity we will restrict ourselves to the case where the finite dimensional Lie group is $G_0 = SU(n,\mathbb{C})$. The group $\tilde{G}_0$ is the collection of all mappings from the unit circle to the group G_0 with pointwise multiplication, i.e.,

(2.1.1) $$\tilde{G}_0 := \{g : S^1 \to G_0\}$$

$$\forall g_1, g_2 \in \tilde{G}_0 : \qquad (g_1 \cdot g_2)(\theta) := g_1(\theta)\, g_2(\theta)$$

(A mapping from the unit circle to G_0 is represented by a periodic mapping in θ.)
To give $\tilde{G}_0$ the structure of a Lie group we have to put some differentiable structure on $\tilde{G}_0$. In order to do this, one must specify the class of mappings $g : S^1 \to G_0$ under consideration, e.g., continuous, smooth, real analytic, etc. We will not go into such

details here and assume that the mappings are "as nice as we want". The interested reader is referred to [Pre&Seg].

Let $\tilde{V}$ be the space of all "nice" vector valued functions $f : S^1 \to \mathbb{R}^n = V$. The group $\tilde{G}_0$ acts naturally on $\tilde{V}$;

$$(2.1.2) \qquad g = \begin{pmatrix} g_{11} & \cdot & \cdot & g_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ g_{n1} & \cdot & \cdot & g_{nn} \end{pmatrix} \in \tilde{G}_0; \quad g_{ij} : S^1 \to G_0$$

$$v = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \in \tilde{V}; \qquad f_i : S^1 \to \mathbb{R}$$

$$g.v := \sum_{ij=1}^{n} (g_{ij} f_j)\, e_i$$

where $\{e_1, \ldots, e_n\}$ is of course the standard basis for $\mathbb{R}^n$.

2.2 loop algebras

Since we have a (faithful) representation of $\tilde{G}_0$ as a subgroup of $GL(\tilde{V})$, it is not difficult to determine its Lie algebra as a subalgebra of $gl(\tilde{V})$ = End $\tilde{V}$. Let $g(t) = (g_{ij}(t))_{ij=1}^{n}$ be a smooth curve in $\tilde{G}_0$ passing through the identity for $t = 0$;

$$(2.2.1) \qquad g_{ij}(0) : S^1 \to \mathbb{R}$$

$$g_{ij}(0)(\theta) := \delta_{ij} \qquad \forall \theta$$

Its tangent vector at $t = 0$ is the endomorphism h of $\tilde{V}$ defined by

$$(2.2.2) \qquad h = (h_{ij})_{ij=1}^{n}; \qquad h_{ij}(z) := \frac{d}{dt}\{g_{ij}(t)(\theta)\}|_{t=0}$$

The Lie algebra $\tilde{g}_0$ of $\tilde{G}_0$ is therefore the collection of mappings $h : S^1 \to g_0 = su(n,\mathbb{C})$ with pointwise commutator;

(2.2.3) $\forall h_1, h_2 \in \tilde{g}_0 : \; [h_1 , h_2](\theta) := [h_1(\theta) , h_2(\theta)]$

In the sequel we will restrict ourselves to the so-called polynomial loop algebra, consisting of all su(n,$\mathbb{C}$)-valued functions on the unit circle, whose entries have finite Fourier expansions;

(2.2.4) $$\tilde{g}_0^{pol} := \{x : S^1 \to g_0 \mid x(z) = \sum_{k=N}^{M} a_k \cos k\theta + \sum_{k=N}^{M} b_k \sin k\theta,\; a_k,\, b_k \in g_0,\; N,M \in \mathbb{Z}\}$$

Complexifying this algebra, we find the algebra $\tilde{g}^{pol}$ given by

(2.2.5) $$\tilde{g}^{pol} := \{\, x : S^1 \to g = sl(n,\mathbb{C}) \mid x(z) = \sum_{k=N}^{M} a_k\, z^k,\; a_k \in sl(n,\mathbb{C}),\; N,M \in \mathbb{Z}\}$$

where $z = e^{i\theta}$. From (2.2.5) it is clear that $\tilde{g}^{pol}$ is isomorphic to the tensor product of Laurent polynomials in z and z^{-1} with the finite dimensional algebra $g = sl(n,\mathbb{C})$;

(2.2.6) $$\tilde{g}^{pol} \cong \mathbb{C}[z,z^{-1}] \otimes g$$

$$[z^k \otimes x , z^j \otimes y] = z^{k+j} \otimes [x,y] \qquad \forall k,j \in \mathbb{Z},\; x,y \in g$$

Remark: Since we will only work with the polynomial loop algebra, we will write $\tilde{g}$ instead of $\tilde{g}^{pol}$ in the sequel.

2.3 Diff S^1 and $\tilde{G}_0 \ltimes$ Diff S^1

Here we study another interesting infinite dimensional group, namely the group of diffeomorphisms of the circle;

(2.3.1) $$\text{Diff } S^1 := \{\phi : S^1 \to S^1 \mid \phi \text{ diffeom.}\}$$

$$\forall \phi_1, \phi_2 \in \text{Diff } S^1: \quad (\phi_1.\phi_2)(\theta) := \phi_1(\phi_2(\theta)) \qquad \forall \theta$$

This group is closely related to the loop group $\tilde{G}_0$, introduced in the preceding section, since it acts as a group of automorphisms on $\tilde{G}_0$; this action is defined by:

$$(2.3.2) \qquad \phi.g := g\circ\phi^{-1} \in \tilde{G}_0 \qquad \forall\phi\in \mathrm{Diff}\, S^1,\ g\in\tilde{G}_0$$

One easily checks that

$$(2.3.3) \qquad \forall\phi_1,\phi_2\in \mathrm{Diff}\, S^1,\ g_1,g_2\in\tilde{G}_0:$$

$$\phi_1.\phi_2.g_1 = (\phi_1\phi_2).g_1 \qquad \text{(action)}$$

$$\phi_1.(g_1g_2) = (\phi_1.g_1)\,(\phi_1.g_2) \quad \text{(automorphism)}$$

This enables us to define the semi direct product group $\tilde{G}_0 \ltimes \mathrm{Diff}\, S^1$ as the collection of pairs (g,ϕ) with multiplication law

$$(2.3.4) \qquad (g_1,\phi_1)\,(g_2,\phi_2) = (g_1(\phi_1.g_2),\ \phi_1\phi_2)$$

It is easy to construct faithful representations of Diff S^1 and $\tilde{G}_0\ltimes\mathrm{Diff}\, S^1$ on the space $\tilde{V}$ of all functions $f: S^1\to\mathbb{R}^n$;

$$(2.3.5) \qquad v = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \in \tilde{V}; \qquad f_i : S^1\to\mathbb{R}^n$$

$$\forall\phi\in\mathrm{Diff}\, S^1: \qquad \phi.v := \begin{pmatrix} f_1.\phi^{-1} \\ \cdot \\ \cdot \\ f_n.\phi^{-1} \end{pmatrix}$$

$$\forall(g,\phi)\in\tilde{G}_0\ltimes\mathrm{Diff}\, S^1: \qquad (g,\phi).v := g.\phi.v$$

We check the multiplication law (2.3.4);

$$(2.3.6) \qquad (g_1,\phi_1)\,(g_2,\phi_2).v = (g_1,\phi_1)\,.\,g_2.\begin{pmatrix} f_1.\phi_2^{-1} \\ \cdot \\ \cdot \\ f_n.\phi_2^{-1} \end{pmatrix}$$

$$= g_1\,.\,(g_2.\phi_1^{-1}).\begin{pmatrix} f_1.\phi_2^{-1}\phi_1^{-1} \\ \cdot \\ \cdot \\ f_n.\phi_2^{-1}\phi_1^{-1} \end{pmatrix}$$

$$= g_1 \cdot (\phi_1 . g_2) \cdot (\phi_1 \phi_2) \cdot v$$

$$= (g_1 (\phi_1 . g_2) , (\phi_1 \phi_2)) \cdot v$$

2.4 the Lie algebras δ and $\tilde{\underline{g}} \oplus \delta$

The Lie algebras of Diff S^1 and $\tilde{G}_0 \ltimes$Diff S^1 may again be constructed as sub-algebras of End $\tilde{V}$. Let $\phi(t)$ be a smooth curve in Diff S^1 passing through the identity for t = 0. Its tangent vector at t = 0 is the endomorphism of $\tilde{V}$ given by

$$(2.4.1) \qquad \frac{d}{dt}\{ \phi(t) \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \}|_{t=0} = \frac{d}{dt}\{ \begin{pmatrix} f_1(\phi(t)^{-1}(\theta)) \\ \cdot \\ \cdot \\ f_n(\phi(t)^{-1}(\theta)) \end{pmatrix} \}|_{t=0}$$

$$= - g(\theta) \frac{d}{d\theta} \begin{pmatrix} f_1(\theta) \\ \cdot \\ \cdot \\ f_n(\theta) \end{pmatrix}$$

where

$$(2.4.2) \qquad g(\theta) := \frac{d}{dt}\{\phi(t)(\theta)\}|_{t=0}$$

The Lie algebra Vect S^1 of Diff S^1 therefore consists of operators of the form $g(\theta)\frac{d}{d\theta}$, $g: S^1 \to \mathbb{R}$. Note that this is just the collection of vectorfields on the circle, which was to be expected. We will again restrict ourselves to the case that $g(\theta)$ has a finite Fourier expansion. After complexification we find the algebra

$$(2.4.3) \qquad \delta^{pol} := \{ \sum_{k=N}^{M} a_k z^{k+1} \frac{d}{dz} \mid a_k \in \mathbb{C},\ N, M \in \mathbb{Z} \}$$

As before we will omit the subscript pol and write δ for this algebra.

It is obvious that the operators $d_k := z^{k+1}\frac{d}{dz}$ form a basis for δ. They satisfy the following commutation relations

(2.4.4) $[d_k, d_l] = -(k-l)d_{k+l}$

It is now easy to construct the complexification of the (polynomial) Lie algebra associated to the semi direct product group $\tilde{G}_0 \ltimes \mathrm{Diff}\ S^1$; it is the vector space $\tilde{g} \oplus \delta$ with Lie bracket

(2.4.5) $$[z^k \otimes x + \alpha d_l , z^m \otimes y + \beta d_n] =$$

$$z^{k+m} \otimes [x,y] + \alpha m\, z^{l+m} \otimes y - \beta k\, z^{k+n} \otimes x - \alpha\beta(l-n)\, d_{l+n}$$

Note that the algebra δ acts as an algebra of derivations on $\tilde{g}$, i.e.,

(2.4.6) $$\forall d \in \delta: \ \mathrm{ad}d\{[z^k \otimes x , z^l \otimes y]\} =$$

$$[\mathrm{ad}d\{z^k \otimes x\} , z^l \otimes y] + [z^k \otimes x , \mathrm{ad}d\{z^l \otimes y\}]$$

(This was to be expected, since the group Diff S^1 acts as a group of automorphisms on $\tilde{G}_0$.)

2.5 central extensions of groups

Let G be a finite dimensional complex Lie group. A representation of G on a vector space V is a homomorphism $\pi : G \rightarrow GL(V)$. We thus have:

(2.5.1) $$\pi(g_1)\, \pi(g_2) = \pi(g_1 g_2);$$

$$\pi(e) = I_V$$

In physical applications one also encounters a slightly more general situation; (2.5.1) is then replaced by

(2.5.2) $$\pi(g_1)\, \pi(g_2) = \varepsilon_\pi(g_1, g_2)\, \pi(g_1 g_2)$$

$$\pi(e) = I_V$$

where $\varepsilon_\pi(g_1, g_2)$ is an element of $\mathbb{C}^\times$, the set of non vanishing complex numbers. (N.B.: in the case of unitary operators on a Hilbert space we of course want ε_π to have values in U(1) rather than in $\mathbb{C}^\times$.) In this case we speak of a projective representation of G on V.

In the sequel we will assume that the mapping $\varepsilon_\pi : G\times G \to \mathbb{C}^\times$ is smooth. Because of the associativity of the multiplication (2.5.2) and the fact that $\pi(e) = I_V$, ε_π has to satisfy

(2.5.3a) $$\varepsilon_\pi(g_1, g_2)\, \varepsilon_\pi(g_1 g_2, g_3) = \varepsilon_\pi(g_1, g_2 g_3)\, \varepsilon_\pi(g_2, g_3)$$

(2.5.3b) $$\varepsilon_\pi(g,e) = \varepsilon_\pi(e,g) = 1$$

In general any smooth mapping $\varepsilon : G\times G \to \mathbb{C}^\times$, satisfying (2.5.3a,b) will be called a smooth 2-cocycle with values in $\mathbb{C}^\times$.

We now define the central extension of G associated to the 2-cocycle as the set $\mathbb{C}^\times \times G$ with multiplication law

(2.5.4) $$(\lambda_1, g_1)\,(\lambda_2, g_2) = (\lambda_1 \lambda_2 \varepsilon_\pi(g_1, g_2), g_1 g_2) \qquad \forall \lambda_1, \lambda_2 \in \mathbb{C}^\times, g_1, g_2 \in G$$

The properties (2.5.3a,b) guarantee that this defines indeed a group and that the set $\mathbb{C}^\times \times \{e\}$ is a central subgroup. Moreover, the mapping $\tilde{\pi} : \mathbb{C}^\times \times G \to GL(V)$ defined by

(2.5.5) $$\tilde{\pi}(\lambda, g) = \lambda\, \pi(g)$$

is a homomorphism;

(2.5.6) $$\begin{aligned}\tilde{\pi}(\lambda_1, g_1)\, \tilde{\pi}(\lambda_2, g_2) &= \lambda_1 \lambda_2\, \pi(g_1)\pi(g_2) \\ &= \lambda_1 \lambda_2 \varepsilon_\pi(g_1, g_2)\, \pi(g_1 g_2) \\ &= \tilde{\pi}(\lambda_1 \lambda_2 \varepsilon_\pi(g_1, g_2), g_1 g_2)\end{aligned}$$

$$= \tilde{\pi}((\lambda_1,g_1)\,(\lambda_2,g_2))$$

We conclude that a projective representation of G is in fact a representation of a central extension of G.

Sometimes the mapping $\pi : G \to GL(V)$ can be redefined in such a way that the factors $\varepsilon_\pi(g_1,g_2)$ are removed from (2.5.2); Let $\sigma : G \to \mathbb{C}^\times$ be a smooth mapping, $\sigma(e) = 1$. Define $\pi' : G \to GL(V)$ by

$$\pi'(g) := \sigma(g)\,\pi(g) \tag{2.5.7}$$

We then have:

$$\pi'(g_1)\,\pi'(g_2) = \frac{\sigma(g_1)\sigma(g_2)}{\sigma(g_1g_2)}\,\varepsilon_\pi(g_1,g_2)\,\pi'(g_1g_2) \tag{2.5.8}$$

so that the factors disappear if

$$\varepsilon_\pi(g_1,g_2) = \frac{\sigma(g_1g_2)}{\sigma(g_1)\sigma(g_2)} \tag{2.5.9}$$

A 2-cocycle of the form (2.5.9) is called a 2-coboundary and 2-cocycles are called equivalent if they differ by a 2-coboundary. Since two projective representations associated to equivalent 2-cocycles can be transformed into each other by (2.5.7), we are only interested in the collection of 2-cocycles modulo 2-coboundaries. This is called the second cohomology group of G with values in $\mathbb{C}^\times$; it is denoted by $H^2(G,\mathbb{C}^\times)$.

2.6 central extensions of Lie algebras

We will now construct the Lie algebra of the central extension $\mathbb{C}^\times \times G$ of G defined in the preceding section. The tangent space in the identity (1,e) is of course

(2.6.1) $$T_{(1,e)}(\mathbb{C}^{\times}\times G) = T_1\mathbb{C}^{\times} \oplus T_eG \cong \mathbb{C} \oplus \underline{g}$$

Let X be a left invariant vectorfield on $\mathbb{C}^{\times}\times G$. At (1,e) it is given by

(2.6.2) $$X_{(1,e)} = a\frac{\partial}{\partial z}\Big|_{(1,e)} + \xi = \frac{d}{dt}(e^{ta}, \exp t\xi)\Big|_{t=0}$$

where $a \in \mathbb{C}$, $\xi \in \underline{g}$ and $\frac{\partial}{\partial z}$ is a tangent vector to $\mathbb{C}^{\times}$. Since X is left invariant, we have

(2.6.3) $$\begin{aligned} X_{(z,g)} &= L_{(z,g)*}X_{(1,e)} \\ &= \frac{d}{dt}(z,g)(e^{ta}, \exp t\xi)\Big|_{t=0} \\ &= \frac{d}{dt}(z\,e^{ta}\,\varepsilon(g,\exp t\xi), g\exp t\xi)\Big|_{t=0} \\ &= \{az + z\frac{d}{dt}\varepsilon(g,\exp t\xi)\Big|_{t=0}\}\frac{\partial}{\partial z}\Big|_{(z,g)} + L_{g*}\xi \\ &= a(z,g)\frac{\partial}{\partial z}\Big|_{(z,g)} + L_{g*}\xi \end{aligned}$$

where $a(z,g) := az + z\frac{d}{dt}\varepsilon(g,\exp t\xi)$. If Y is a second left invariant vectorfield, which is given in the identity by $Y_{(1,e)} = b\frac{\partial}{\partial z} + \eta$, we can compute the commutator;

(2.6.4) $$\begin{aligned} [X, Y]_{(1,e)} &= [a(z,g)\frac{\partial}{\partial z}\Big|_{(z,g)} + L_{g*}\xi,\ b(z,g)\frac{\partial}{\partial z}\Big|_{(z,g)} + L_{g*}\eta]_{(1,e)} \\ &= \{a\frac{\partial}{\partial z}b(z,g)_{(1,e)} - b\frac{\partial}{\partial z}a(z,g)_{(1,e)}\}\frac{\partial}{\partial z}(1,e) \\ &\quad + \{\frac{d}{dt}b(1,\exp t\xi)\Big|_{t=0} - \frac{d}{dt}a(1,\exp t\eta)\Big|_{t=0}\}\frac{\partial}{\partial z}\Big|_{(1,e)} + [\xi,\eta] \\ &= [\xi,\eta] + \omega(\xi,\eta)\frac{\partial}{\partial z}\Big|_{(1,e)} \end{aligned}$$

where

(2.6.5) $$\omega(\xi,\eta) := \frac{d}{dt}\frac{d}{ds}\{\varepsilon(\exp t\xi, \exp s\eta) - \varepsilon(\exp t\eta, \exp s\xi)\}\Big|_{t=s=0}$$

(We have used that $\varepsilon(e,g) = 1\ \forall g \in G$.)

If we denote the tangent vector $\frac{\partial}{\partial z}|_{(1,e)}$ by c, we may rewrite the bracket (2.6.4) as

$$(2.6.6) \qquad [\xi + ac\, ,\, \eta + bc] = [\xi,\eta] + \omega(\xi,\eta)\, c$$

NB: the bracket on the left hand side is the bracket on the Lie algebra $\underline{g} \oplus \mathbb{C}c$, the bracket on the right hand side is the bracket on the "old" Lie algebra $\underline{g}$. Formula (2.6.6) tells us that $\mathbb{C}c$ is a one dimensional central subalgebra of $\underline{g} \oplus \mathbb{C}c$. Therefore, $\underline{g} \oplus \mathbb{C}c$ is called a central extension of $\underline{g}$.

The mapping $\omega : \underline{g}\times\underline{g}\to\mathbb{C}$ defined by (2.6.5) is closely related to the 2-cocycle ε on G; it is called a 2-cocycle on $\underline{g}$ with values in $\mathbb{C}$. Because of the antisymmetry of the bracket (2.6.6) and the Jacobi identity, ω has to satisfy

$$(2.6.7a) \qquad \omega(\xi,\eta) = -\omega(\eta,\xi)$$

$$(2.6.7b) \qquad \omega(\xi,[\eta,\zeta]) + \omega(\zeta,[\xi,\eta]) + \omega(\eta,[\zeta,\xi]) = 0$$

We now compute what happens, if we take for ε a 2-coboundary in (2.6.5); if $\sigma:G\to\mathbb{C}^\times$ is a smooth function, we have

$$(2.6.8) \qquad \frac{d}{dt}\frac{d}{ds}\,\sigma(\exp t\xi\, \exp s\eta)\,|_{t=s=0} = \frac{d}{dt}\sigma_*\,|_{\exp t\xi}\,(L_{\exp t\xi *}\,\eta)|_{t=0}$$

$$= \frac{d}{dt}(R^*_{\exp t\xi}\,\sigma_*)|_e\,(R_{\exp -t\xi *}L_{\exp t\xi *}\eta)|_{t=0}$$

$$= \frac{d}{dt}(R^*_{\exp t\xi}\,\sigma_*)|_e\,(\eta)|_{t=0} + \sigma_*\,|_e([\xi,\eta])$$

$$= \frac{d}{dt}\frac{d}{ds}\,\sigma(\exp s\eta\, \exp t\xi)\,|_{t=s=0} + \sigma_*\,|_e([\xi,\eta]$$

Using this formula and the fact that $\sigma(e) = 1$, we find for (2.6.5) in the case $\varepsilon_\pi(g_1,g_2) = \sigma(g_1g_2)/\sigma(g_1)\sigma(g_2)$:

$$(2.6.9) \qquad \omega(\xi,\eta) = \theta([\xi,\eta])$$

where $\theta := \sigma_*\,|_e : \underline{g}\to\mathbb{C}$ is an element of the dual $\underline{g}^*$ of $\underline{g}$. Such 2-cocycles are of course called 2-coboundaries on $\underline{g}$ and two 2-cocycles are again called equivalent if they differ by a 2-coboundary. Dividing out the space of 2-cocycles on $\underline{g}$ with respect to this equivalence relation, we obtain the second cohomology group of $\underline{g}$ with values in $\mathbb{C}$, denoted by $H^2(\underline{g},\mathbb{C})$. With these definitions it is clear that the mapping (2.6.5) induces a homomorphism $H^2(G,\mathbb{C}^\times)\rightarrow H^2(\underline{g},\mathbb{C})$.

At this point one may ask if this mapping is invertible, i.e., if we can, given a 2-cocycle ω on the Lie algebra $\underline{g}$, find a globally defined smooth 2-cocycle ε on the Lie group G, such that ω and ε are related by (2.6.5). If G is finite dimensional, connected and simply connected, the answer to this question is affirmative. For infinite dimensional groups such as the loop group $\tilde{G}$ associated to G however, the situation is different.

In section 2.8 we will construct a 2-cocycle on the loop algebra $\tilde{\underline{g}}$. Since the corresponding central extension $\hat{\underline{g}} := \tilde{\underline{g}} \oplus \mathbb{C}c$ is infinite dimensional, it is in the first place not clear if there exists a central extension $\hat{G}$ of the loop group $\tilde{G}$, which has Lie algebra $\hat{\underline{g}}$. In the sequel we will use the representation theory of the algebra $\hat{\underline{g}}$ to construct $\hat{G}$ explicitly, so that there are no problems concerning existence. However, $\hat{G}$ turns out to be topologically non trivial, i.e., it cannot globally be written as $\mathbb{C}^\times\times\tilde{G}$. Instead $\hat{G}$ is a fiber bundle over $\tilde{G}$ with fiber $\mathbb{C}^\times$. From this we conclude that there cannot exist a globally defined, smooth 2-cocycle on $\tilde{G}$, describing the multiplication in $\hat{G}$.

2.7 $H^2(\underline{g},\mathbb{C})$ for $\underline{g}$ simple

Here we compute $H^2(\underline{g},\mathbb{C})$ for a simple finite dimensional Lie algebra $\underline{g}$. The calculation is interesting, because it suggests how to calculate $H^2(\tilde{\underline{g}},\mathbb{C})$, which is what we are really interested in.

Recall that a derivation of $\underline{g}$ is a linear mapping d, satisfying

(2.7.1) $$d([x,y]) = [d(x),y] + [x,d(y)] \qquad \forall x,y\in \underline{g}$$

The Jacobi identity for $\underline{g}$ states that for any fixed $u\in\underline{g}$ the adjoint action $\mathrm{ad}u : \underline{g}\rightarrow\underline{g}$ is a derivation on $\underline{g}$. It is well known that for $\underline{g}$ simple all derivations are of this type (see, e.g., [Hum])

Let $\omega : \underline{g} \times \underline{g} \to \mathbb{C}$ be an arbitrary 2-cocycle on $\underline{g}$. For any $x \in \underline{g}$ the mapping $i_x\omega : \underline{g} \to \mathbb{C}$ is linear and therefore, $i_x\omega \in \underline{g}^*$. Using the Killing form on $\underline{g}$, we can identify $\underline{g}$ and $\underline{g}^*$;

$$(2.7.2) \qquad \nu : \underline{g} \to \underline{g}^*$$

$$\langle \nu(x), y \rangle := (x|y)$$

(Compare with (1.7.2) where this was done for the subalgebra $\underline{h} \subset \underline{g}$.)
Therefore we may write

$$(2.7.3) \qquad (i_x\omega)(y) = \omega(x,y) = (d(x)|y) \qquad \forall x,y \in \underline{g}$$

where $d : \underline{g} \to \underline{g}$ is a linear mapping. Using the invariance property (1.3.2) of the Killing form and the properties (2.6.7a,b) of a 2-cocycle, we deduce that d is a derivation;

$$(2.7.4) \qquad (d([x,y]) \mid z) = \omega([x,y],z)$$

$$= -\omega(y,[x,z]) + \omega(x,[y,z])$$

$$= -(d(y)|[x,z]) + (d(x)|[y,z])$$

$$= ([x,d(y)] + [d(x),y] \mid z) \qquad \forall z \in \underline{g}$$

As remarked above, d must then be of the form $\mathrm{ad}\,u$ for some $u \in \underline{g}$ and we obtain

$$(2.7.5) \qquad \omega(x,y) = i_x\omega(y)$$

$$= (\mathrm{ad}\,u(x)|y)$$

$$= (u|[x,y]) \qquad \forall x,y \in \underline{g}$$

This shows that any 2-cocycle on a simple Lie algebra $\underline{g}$ is a 2-coboundary whence $H^2(\underline{g},\mathbb{C}) = 0$.

2.8 $H^2(\tilde{\underline{g}},\mathbb{C})$ for $\underline{g}$ simple

The calculation of $H^2(\underline{g},\mathbb{C})$ in the preceding section suggests the following construction for nontrivial elements of $H^2(\tilde{\underline{g}},\mathbb{C})$; suppose that we have an invariant bilinear form $(\ |\ ) : \tilde{\underline{g}} \times \tilde{\underline{g}} \to \mathbb{C}$ and a derivation $d : \tilde{\underline{g}} \to \tilde{\underline{g}}$, satisfying

(2.8.1) $$(d(\tilde{x}) \mid \tilde{y}) = -(\tilde{x} \mid d(\tilde{y})) \qquad \forall \tilde{x}, \tilde{y} \in \tilde{\underline{g}}$$

The reader may easily check that $\omega(x,y) := (d(x)|y)$ is then a 2-cocycle on $\tilde{\underline{g}}$.

For a simple algebra $\underline{g}$ this procedure yields only 2-coboundaries, since in that case all derivations are of the type $\mathrm{ad}\,u$, $u \in \underline{g}$. For the loop algebra $\tilde{\underline{g}}$ we have seen that there are also other derivations, namely the elements of the algebra δ, introduced in 2.4; this opens the possibility to construct nontrivial 2-cocycles on $\tilde{\underline{g}}$.

It remains to show, that there exists an invariant form on $\tilde{\underline{g}}$; let us assume that $(\ |\ ) : \tilde{\underline{g}} \times \tilde{\underline{g}} \to \mathbb{C}$ is such a form. Define mappings $c_{kl} : \underline{g} \times \underline{g} \to \mathbb{C}$ by

(2.8.2) $$c_{kl}(x,y) := (z^k \otimes x \mid z^l \otimes y) \qquad \forall k,l \in \mathbb{Z},\ x,y \in \underline{g}$$

For fixed k and l c_{kl} is an invariant bilinear form on $\underline{g}$. On a simple Lie algebra such a form has to be a scalar multiple of the Killing form on $\underline{g}$;

(2.8.3) $$c_{kl}(x,y) = \lambda_{kl} \cdot \mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y) \qquad \forall k,l \in \mathbb{Z},\ x,y \in \underline{g}$$

Now we use the invariance of the Killing form and the form $(\ |\ ) : \tilde{\underline{g}} \times \tilde{\underline{g}} \to \mathbb{C}$

(2.8.4) $$([z^k \otimes x\ ,\ z^l \otimes y] \mid z^m \otimes u) = (z^k \otimes x \mid [z^l \otimes y\ ,\ z^m \otimes u]) \qquad \forall k,l,m \in \mathbb{Z},\ x,y,u \in \underline{g}$$

to derive

(2.8.5) $$\lambda_{k+l,m} = \lambda_{k,l+m} \qquad \forall k,l,m \in \mathbb{Z}$$

Substituting m = 0, we conclude that the complex constants λ_{kl} depend only on k+l:

(2.8.6) $$\lambda_{kl} = \lambda_{k+l,0} = \lambda_{lk}$$

A possible choice for the λ's is

(2.8.7) $$\lambda_{kl} := \delta_{k+l,i} \qquad \text{for some fixed } i\in\mathbb{Z}$$

The reader easily checks that the form $(\ |\ )_i : \underline{g} \times \underline{g} \to \mathbb{C}$ defined by

(2.8.8) $$(z^k\otimes x \mid z^l\otimes y)_i := \delta_{k+l,i}\, \mathrm{tr}(\mathrm{ad}x\ \mathrm{ad}y)$$

is indeed an invariant form on $\tilde{\underline{g}}$. Therefore, the space of invariant bilinear forms is infinite dimensional and has basis $\{\ (\ |\ )_i\ \}_{i\in\mathbb{Z}}$.

Now fix an $i\in\mathbb{Z}$ and let $d\in\delta$ be a derivation. A short calculation shows that the condition $(d(\tilde{x})|\tilde{y})_i = -\ (d(\tilde{y})|\tilde{x})_i$ is only satisfied if we take $d = \alpha d_i$, $\alpha\in\mathbb{C}$. We then have:

(2.8.9) $$(d_i(\tilde{x}) \mid \tilde{y})_i = (z^{i+1}\frac{d}{dz}(\tilde{x}) \mid \tilde{y})_i$$

$$= (z\frac{d}{dz}(\tilde{x}) \mid \tilde{y})_0$$

$$= (d_0(\tilde{x}) \mid \tilde{y})_0$$

We conclude that, although there are infinitely many derivations and infinitely many invariant forms, the construction yields only one 2-cocycle, (2.8.9) on $\tilde{\underline{g}}$. One can actually prove that this is essentially (i.e., modulo 2-coboundaries) the only 2-cocycle on $\tilde{\underline{g}}$; dim $H^2(\tilde{\underline{g}},\mathbb{C}) = 1$. The central extension $\hat{\underline{g}} = \tilde{\underline{g}} \oplus \mathbb{C}c$ associated to this 2-cocycle is therefore universal, i.e., any other nontrivial central extension of $\tilde{\underline{g}}$ is isomorphic to $\hat{\underline{g}}$.

The commutation relations in $\hat{\underline{g}}$ become

(2.8.10) $$[z^k\otimes x\ ,\ z^l\otimes y] = z^{k+l}\otimes[x,y] + k\delta_{k+l,0}\,(x|y)\ c \qquad \forall k,l\in\mathbb{Z},\ x,y\in \underline{g}$$

2.9 A central extension of $\tilde{g} \oplus \delta$

In section 2.8 we have constructed the universal central extension of the loop algebra $\tilde{g}$ associated to a simple finite dimensional algebra g. This was done by means of the 2-cocycle ω defined by

$$\omega(z^k \otimes x, z^l \otimes y) = k\delta_{k+l,0}\,(x|y) \qquad \forall k,l \in \mathbb{Z},\ x,y \in g \tag{2.9.1}$$

Leaving out the tensor product symbol and thinking again of $\tilde{g}$ as the algebra of (polynomial) mappings from S^1 to g, we may also write

$$\omega(\tilde{x},\tilde{y}) = \frac{1}{2\pi i} \int_{(S^1)^+} (\frac{d}{dz}\tilde{x}(z) \mid \tilde{y}(z))\, dz \qquad \forall \tilde{x},\tilde{y} \in \tilde{g} \tag{2.9.2}$$

This formula is easily checked by substituting $\tilde{x}(z) = z^k x$, $\tilde{y}(z) = z^l y$; its relevance is that it shows clearly the invariance of ω under the group $(\mathrm{Diff}S^1)^+$ of orientation preserving diffeomorphisms of the circle;

$$\omega(\psi.\tilde{x},\psi.\tilde{y}) = \frac{1}{2\pi i} \int_{(S^1)^+} (\frac{d}{dz}\tilde{x}(\psi^{-1}(z)) \mid \tilde{y}(\psi^{-1}(z)))\, dz \tag{2.9.3}$$

$$= \frac{1}{2\pi i} \int_{(S^1)^+} (\frac{d}{dz'}\tilde{x}(z') \mid \tilde{y}(z'))\, dz'$$

where $\psi \in (\mathrm{Diff}S^1)^+$ and $z' := \psi^{-1}(z)$.

Infinitesimally this becomes

$$\omega(d_k(\tilde{x}),\tilde{y}) + \omega(\tilde{x},d_k(\tilde{y})) = 0 \qquad \forall k \in \mathbb{Z},\ \tilde{x},\tilde{y} \in \tilde{g} \tag{2.9.4}$$

which can of course also be verified with (2.9.1). Formula (2.9.4) enables us to extend the 2-cocycle ω on $\tilde{g}$ to a 2-cocycle $\bar{\omega}$ on $\tilde{g} \oplus \delta$ defined by

$$\bar{\omega}(\tilde{x}+d\,,\,\tilde{y}+d') := \omega(\tilde{x}\,,\,\tilde{y}) \qquad \forall \tilde{x},\ \tilde{y} \in \tilde{g},\ d,d' \in \delta \tag{2.9.5}$$

We check the cocycle identity (2.6.6.b);

(2.9.6) $\bar{\omega}([\tilde{x}+d_i\,,\,\tilde{y}+d_j]\,,\,\tilde{z}+d_k)$ + cyclic permutations =

$$= \omega([\tilde{x}\,,\,\tilde{y}] + d_i(\tilde{y}) - d_j(\tilde{x})\,,\,\tilde{z}) + \text{c.p.}$$

$$= \omega(d_i(\tilde{y})\,,\,\tilde{z}) - \omega(d_j(\tilde{x})\,,\,\tilde{z}) + \text{c.p.}$$

$$= \omega(d_i(\tilde{y})\,,\,\tilde{z}) - \omega(d_i(\tilde{z})\,,\,\tilde{y}) + \text{c.p.}$$

$$= 0$$

This means that we have constructed a nontrivial element of $H^2(\tilde{g}\oplus\delta,\mathbb{C})$. The associated central extension $\tilde{g} \oplus \delta \oplus \mathbb{C}c$ has commutation relations

(2.9.7) $[z^k\otimes x + \alpha_1 d_l + \beta_1 c\,,\, z^m\otimes y + \alpha_2 d_n + \beta_2 c] =$

$$= z^{k+m}\otimes[x,y] + m\alpha_1 z^{l+m}\otimes y - k\alpha_2 z^{n+k}\otimes x + \alpha_1\alpha_2(n-l)\, d_{l+n}$$

$$+ k\delta_{k+m,0}\,(x|y)\, c$$

2.10 invariant bilinear forms

In section 2.8 we have seen that the Lie algebra $\tilde{g}$ admits infinitely many invariant bilinear forms. One easily checks that these forms are nondegenerate. For the construction of the 2-cocycle (2.8.9) it is sufficient to consider only

(2.10.1) $(\,|\,)_0 : \tilde{g}\times\tilde{g}\to\mathbb{C}$

$$(z^k\otimes x \,|\, z^l\otimes y)_0 = \delta_{k+l,0}\,(x|y) \qquad \forall k,l\in\mathbb{Z},\ x,y\in g$$

Here we will try to construct an invariant bilinear form on $\tilde{g}\oplus\delta\oplus\mathbb{C}c$, whose restriction to the subspace $\tilde{g}$ is given by $(\,|\,)_0$.

We will start to show that the restriction of such a form to the subalgebra δ is trivial. Let $(\mid) : \delta \times \delta \rightarrow \mathbb{C}$ be an arbitrary bilinear form and define $a_{ij} := (d_i \mid d_j)$ $\forall i,j \in \mathbb{Z}$. The invariance of the form $(\mid)$ gives

(2.10.2) $\quad ([d_i , d_j] \mid d_k) = (d_i \mid [d_j , d_k]) \qquad \forall i,j,k \in \mathbb{Z}$

or, in terms of the a_{ij}'s

(2.10.3) $\quad (j-i)\, a_{i+j,k} = (k-j)\, a_{i,j+k}$

For j = 0 this becomes

(2.10.4) $\quad (i+k)\, a_{ik} = 0$

and we conclude

(2.10.5) $\quad a_{ik} = 0 \qquad \text{if } i+k \neq 0$

Substituting $i+j = -k$ in (2.10.3), we find

(2.10.6) $\quad (2j+k)\, a_{-kk} = (k-j)\, a_{-(k+j),(k+j)}$

and for k = 0 this becomes

(2.10.7) $\quad 2j\, a_{00} = -j\, a_{-jj}$

From this we obtain

(2.10.8) $\quad a_{j,-j} = -2\, a_{00} \quad \text{if } j \neq 0$

Finally, substituting i = j, we find

(2.10.9) $\quad (k-j)\, a_{j,j+k} = 0$

For k = -2j this reads

$$(2.10.10) \qquad -3j\, a_{j,-j} = 0$$

Combining (2.10.5), (2.10.8) and (2.10.10), we indeed find

$$(2.10.11) \qquad a_{ij} = 0 \qquad\qquad \forall i,j \in \mathbb{Z}$$

Having shown that $(d_i \mid d_j) = 0\ \forall i,j \in \mathbb{Z}$ for any invariant bilinear form $(\ \mid\ )$ on $\tilde{\underline{g}} \oplus \delta$, we proceed to calculate $(\tilde{\underline{g}} \mid c)$, $(c \mid d_k)$ and $(c \mid c)$. For this we need two simple lemmas;

lemma 2.10.1
Let $\underline{g}$ be a finite dimensional simple Lie algebra, then $[\underline{g}, \underline{g}] = \underline{g}$

proof:
$[\underline{g}, \underline{g}]$ is an ideal in $\underline{g}$ and must therefore be either zero or $\underline{g}$. In the first case $\underline{g}$ would be abelian, contradicting simplicity.♦

As a corollary to this lemma we mention that every element $x \in \underline{g}$ can be written as

$$(2.10.12) \qquad x = \sum_{i=1}^{N} [u_i, v_i] \qquad\qquad u_i, v_i \in \underline{g}$$

lemma 2.10.2
There exist elements $h_1, h_2 \in \underline{h}$ such that

$$(2.10.13) \qquad c = [z \otimes h_1, z^{-1} \otimes h_2]$$

proof:
The Killing form on $\underline{g}$ remains nondegenerate when restricted to $\underline{h}$, so for any $h_1 \in \underline{h}$ we can find an $h_2 \in \underline{h}$ such that $(h_1 \mid h_2) = 1$.♦

Using (2.10.12), we calculate

$$(2.10.14) \qquad (z^k \otimes x \mid c) = \sum_{i=1}^{N} ([z^k \otimes u_i, 1 \otimes v_i] \mid c)$$
$$= \sum_{i=1}^{N} (z^k \otimes u_i \mid [1 \otimes v_i, c]) = 0$$

and

$$(2.10.15) \qquad (z^k \otimes x \mid d_j) = \sum_{i=1}^{N} (z^k \otimes u_i \mid [1 \otimes v_i , d_j]) = 0$$

The expression (2.10.13) for the center is used to compute

$$(2.10.16) \qquad (c \mid d_k) = (z \otimes h_1 \mid [z^{-1} \otimes h_2 , d_k])$$

$$= (z \otimes h_1 \mid z^{-1+k} \otimes h_2)$$

$$= \delta_{k0} (h_1 | h_2) = \delta_{k0}$$

and

$$(2.10.17) \qquad (c|c) = (z \otimes h_1 \mid [z^{-1} \otimes h_2 , c]) = 0$$

Summarizing, we have seen that any invariant bilinear form on $\tilde{g} \oplus \delta \oplus \mathbb{C}c$, whose restriction to $\tilde{g}$ is given by $(\ |\ )_0$, is completely determined by

$$(2.10.18) \qquad (\tilde{g} \mid c) = 0 \qquad (\tilde{g} \mid \delta) = 0$$

$$(c|c) = 0 \qquad (d_i \mid d_j) = 0 \quad \forall i,j$$

$$(c \mid d_k) = \delta_{k0}$$

Therefore, its total isotropic subspace is given by $\bigoplus_{k \neq 0} \mathbb{C}d_k$. Throwing away this subspace, we obtain the subalgebra $\hat{g}^e$ of $\tilde{g} \oplus \delta \oplus \mathbb{C}c$ given by

$$(2.10.19) \qquad \hat{g}^e := \tilde{g} \oplus \mathbb{C}c \oplus \mathbb{C}d_0$$

on which the bilinear form (2.10.18) is nondegenerate. Note that $\hat{g}^e$ is an extension of $\hat{g}$ by $\mathbb{C}d_0$. It is called the full affine algebra associated to g and has commutation relations

(2.10.20) $[z^k \otimes x + \alpha_1 c + \beta_1 d_0 \, , z^l \otimes y + \alpha_2 c + \beta_2 d_0] =$

$$z^{k+l} \otimes [x,y] + l\beta_1 z^l \otimes y - k\beta_2 z^k \otimes x + k\delta_{k+l,0} \, (x|y) \, c$$

2.11 the Virasoro algebra

In this section we study central extensions of the algebra δ; let $\omega : \delta \times \delta \rightarrow \mathbb{C}$ be a 2-cocycle on δ and define $\omega_{ij} := \omega(d_i, d_j)$. We then have

(2.11.1a) $\omega_{ij} = -\,\omega_{ji}$

(2.11.1b) $(j-i)\,\omega_{i+j,k} + (i-k)\,\omega_{i+k,j} + (k-j)\,\omega_{j+k,i} = 0$

Substituting k = 0 in (2.11.1b), we find

(2.11.2) $\omega_{ij} = \frac{i-j}{i+j}\,\omega_{i+j,0}$ if $i+j \neq 0$

We will show that (2.11.2) enables us to modify ω by a 2-coboundary in such a way that the result satisfies

(2.11.3) $\omega_{ij} = 0$ if $i+j \neq 0$

Indeed, let $\theta : \delta \rightarrow \mathbb{C}$ be defined by

(2.11.4) $\theta(d_i) = \frac{1}{i}\,\omega_{i,0}$ for $i \neq 0$

$= 0$ for $i = 0$

We then have

(2.11.5) $\theta([d_i \, , d_j]) = (j-i)\,\theta(d_{i+j}) = \frac{j-i}{j+i}\,\omega_{i+j,0}$ if $i+j \neq 0$

$= 0$ if $i+j = 0$

Assuming now that ω satisfies (2.11.3), we only have to compute $\omega_{i,-i}$. For this we substitute i+j = -k in (2.11.1b) and obtain

$$(2.11.6) \qquad (j-i)\,\omega_{i+j,-(i+j)} + (2i+j)\,\omega_{-j,j} - (i+2j)\,\omega_{-i,i} = 0 \qquad \forall i,j$$

To solve (2.11.6), one may temporarily replace i and j by continuous variables x and y and $\omega_{i,-i}$ by a smooth odd function f(x); relation (2.11.6) then becomes

$$(2.11.7) \qquad (y-x)\,f(x+y) - (2x+y)\,f(y) + (x+2y)\,f(x) = 0$$

Differentiating (2.11.7) with respect to the variable $\xi := x+y$ in $\xi = 0$ we obtain

$$(2.11.8) \qquad x\,f'(x) - 3f(x) = -2x\,f'(0)$$

The solution of this o.d.e. is

$$(2.11.9) \qquad f(x) = ax^3 + f'(0)\,x$$

The general solution of (2.11.6) is therefore

$$(2.11.10) \qquad \omega_{i,-i} = ai^3 + bi \qquad a,b \in \mathbb{C}$$

We conclude that any 2-cocycle on δ is equivalent to

$$(2.11.11) \qquad \omega_{ij} = \delta_{i+j,0}\,(ai^3 + bi) \qquad a,b \in \mathbb{C}$$

Remark that the linear term in (2.11.11) is in fact a 2-coboundary; it is associated to the one form $\theta : \delta \rightarrow \mathbb{C}$ defined by

$$(2.11.12) \qquad \theta(d_i) := \frac{1}{2}\delta_{i0}$$

so: $$\theta([d_i\,,d_j]) = \frac{1}{2}(j-i)\,\delta_{i+j,0} = -\,i\,\delta_{i+j,0}$$

Therefore, $H^2(\delta,\mathbb{C})$ is one dimensional and the algebra δ has a universal central extension $\delta \oplus \mathbb{C}\kappa$ defined by the 2-cocycle $\omega_{ij} := \delta_{i+j,0}i^3$. In physics one traditionally uses the 2-cocycle (2.11.11) with $a = -b = \frac{1}{12}$. The commutation relations then become

$$[d_i , d_j] = (j-i)\, d_{i+j} + \frac{1}{12}\delta_{i+j,0}\,(i^3-i)\,\kappa \qquad (2.11.13)$$

The algebra $\delta \oplus \mathbb{C}\kappa$ is called the Virasoro algebra. It plays an important role in string theory.

2.12 the Virasoro extension of affine algebras

In section 2.8 we have seen that the algebra $\tilde{g} \oplus \delta$ admits a central extension by the 2-cocycle (2.9.5). Let us denote this 2-cocycle by $\bar{\omega}_1$. Section 2.11 shows that there is another central extension of $\tilde{g} \oplus \delta$ by a 2-cocycle $\bar{\omega}_2$:

$$\bar{\omega}_2(\tilde{x} + d\, , \tilde{y} + d') = \omega_2(d,d') \qquad (2.12.1)$$

where ω_2 is the 2-cocycle (2.11.11) on δ. We will now show that $H^2(\tilde{g}\oplus\delta,\mathbb{C})$ is two dimensional and that it is spanned by $\bar{\omega}_1$ and $\bar{\omega}_2$.

Let ω be an arbitrary 2-cocycle on $\tilde{g} \oplus \delta$. Remark that its restrictions to the subalgebras $\tilde{g}$ and δ are 2-cocycles on these algebras and must therefore be scalar multiples of $\bar{\omega}_1$ and $\bar{\omega}_2$ respectively. Therefore, we only have to compute cross terms; using (2.10.12), we find:

$$\begin{aligned} \omega(z^k\otimes x\, , d_l) &= \sum_{i=1}^{N} \omega([z^k\otimes u_i\, , 1\otimes v_i]\, , d_l) \\ &= -\sum_{i=1}^{N} \{\omega([d_l\, , z^k\otimes u_i]\, , 1\otimes v_i)+\omega([1\otimes v_i\, , d_l\,]\, , z^k\otimes u_i)\} \\ &= -k\sum_{i=1}^{N} \omega(z^{k+l}\otimes u_i\, , 1\otimes v_i) \\ &= k\lambda\sum_{i=1}^{N} \bar{\omega}_1(z^{k+l}\otimes u_i\, , 1\otimes v_i) \qquad \text{for some } \lambda\in\mathbb{C} \\ &= 0 \end{aligned} \qquad (2.12.2)$$

Therefore we have

$$(2.12.13) \qquad \omega(\tilde{x} + d , \tilde{y} + d') = \omega(\tilde{x} , \tilde{y}) + \omega(d , d')$$

$$= (\lambda \,\bar{\omega}_1 + \mu \,\bar{\omega}_2) (\tilde{x} + d , \tilde{y} + d')$$

We conclude that the universal central extension of $\tilde{g} \oplus \delta$ is two dimensional and is given by the commutation relations

$$(2.12.14) \qquad [z^k \otimes x + \alpha_1 d_l + \beta_1 c + \gamma_1 \kappa , z^m \otimes y + \alpha_2 d_n + \beta_2 c + \gamma_2 \kappa] =$$

$$z^{k+m} \otimes [x,y] + m\alpha_1 z^{l+m} \otimes y - k\alpha_2 z^{n+k} \otimes x + \alpha_1 \alpha_2 (n-l) d_{l+n}$$

$$+ k\delta_{k+m,0} (x|y) c + \alpha_1 \alpha_2 \frac{1}{12} \delta_{l+n,0} (l^3 - l) \kappa$$

This algebra is called the Virasoro extension of the affine algebra $\hat{g}$. It turns up naturally in the representation theory of $\hat{g}$.

Chapter 3 structure of affine algebras

3.1 root space decomposition of $\hat{\underline{g}}^e$

We recall from chapter 1 that the finite dimensional simple Lie algebra $\underline{g}$ has, with respect to a given Cartan subalgebra $\underline{h}$, a root space decomposition

(3.1.1) $$\underline{g} = \bigoplus_{\alpha\in\Delta} \underline{g}_\alpha \oplus \underline{h}$$

$$\underline{g}_\alpha := \{x\in \underline{g} \mid [h,x] = <\alpha,h>x,\ \forall h\in \underline{h}\}$$

This decomposition induces a decomposition of the full affine algebra $\hat{\underline{g}}^e = \bigoplus_{j\in\mathbb{Z}} z^j\otimes\underline{g} \oplus \mathbb{C}c \oplus \mathbb{C}d_0$;

(3.1.2) $$\hat{\underline{g}}^e = \bigoplus_{\substack{j\in\mathbb{Z},\alpha\in\Delta\cup\{0\}\\ (j,\alpha)\neq(0,0)}} z^j \otimes \underline{g}_\alpha \oplus \{1 \otimes \underline{h} \oplus \mathbb{C}c \oplus \mathbb{C}d_0\}$$

We have the commutation relations (see (2.10.20))

(3.1.3) $$[1 \otimes h\,, z^j \otimes x_\alpha] = <\alpha,h>\, z^j \otimes x_\alpha$$

$$[c\,, z^j \otimes x_\alpha] = 0$$

$$[d_0\,, z^j \otimes x_\alpha] = j\, z^j \otimes x_\alpha \qquad \forall\alpha\in\Delta\cup\{0\},\ x_\alpha\in \underline{g}_\alpha,\ h\in \underline{h}$$

From these relations it is clear that $z^j \otimes x_\alpha$ is for all $j\in\mathbb{Z}$ and for all $\alpha\in\Delta\cup\{0\}$ a common eigenvector of the abelian subalgebra

(3.1.4) $$\hat{\underline{h}}^e := 1 \otimes \underline{h} \oplus \mathbb{C}c \oplus \mathbb{C}d_0$$

This subalgebra is called the Cartan subalgebra of $\hat{\underline{g}}^e$. Sometimes we will also work with $\hat{\underline{h}} := 1 \otimes \underline{h} \oplus \mathbb{C}c$.

Every element $\alpha\in \underline{h}^*$ can be considered as an element of $(\hat{\underline{h}}^e)^*$ -which will be denoted by the same symbol- by defining

(3.1.5) $\quad <\alpha, c> = 0 = <\alpha, d_0>$

If we also introduce a special element $\delta \in (\hat{\underline{h}}^e)^*$ by

(3.1.6) $\quad <\delta, 1 \otimes \underline{h}> = \{0\}, \quad <\delta, d_0> = 1, \quad <\delta, c> = 0$

we can rewrite the commutation relations (3.1.3) as

(3.1.7) $\quad [\hat{h}, z^j \otimes x_\alpha] = <j\delta+\alpha, \hat{h}> z^j \otimes x_\alpha \qquad \forall \hat{h} \in \hat{\underline{h}}^e$

This means that the decomposition (3.1.2) is a root space decomposition for the algebra $\hat{\underline{g}}^e$;

(3.1.8) $\quad \hat{\underline{g}}^e = \oplus_{\gamma \in \hat{\Delta}} \hat{\underline{g}}^e_\gamma \oplus \hat{\underline{h}}^e$

where the root system $\hat{\Delta} \subset (\hat{\underline{h}}^e)^*$ is given by

(3.1.9) $\quad \hat{\Delta} = \{\gamma = j\delta + \alpha \mid j \in \mathbb{Z}, \alpha \in \Delta \cup \{0\}, (j,\alpha) \neq (0,0)\}$

and the associated root spaces are

(3.1.10) $\quad \hat{\underline{g}}^e_{j\delta+\alpha} = z^j \otimes \underline{g}_\alpha$

Note that we have excluded the pair $(j,\alpha) = (0,0)$ corresponding to the common eigenvalue 0 from the root system $\hat{\Delta}$ just as in the case of the finite root system Δ.

From the definition (3.1.9) of $\hat{\Delta}$ it is clear that $\hat{\Delta}$ can be written as a disjoint union

(3.1.11) $\quad \hat{\Delta} = \hat{\Delta}^{re} \cup \hat{\Delta}^{im}$

where

(3.1.12) $\quad \hat{\Delta}^{re} := \{j\delta + \alpha \mid j \in \mathbb{Z}, \alpha \in \Delta\}$

and

(3.1.13) $\hat{\Delta}^{im} := \{j\delta \mid j \in \mathbb{Z}-\{0\}\}$

The roots in $\hat{\Delta}^{re}$ are called real roots; they share the following two properties with roots of the finite root system Δ:

a) if $\gamma \in \hat{\Delta}^{re}$, then the only multiples of γ in $\hat{\Delta}$ are $\pm\gamma$
b) $\forall\ \gamma \in \hat{\Delta}^{re}$: $\dim \hat{g}^e_\gamma = \dim z^j \otimes g_\alpha = 1$

The roots in $\hat{\Delta}^{im}$ are called imaginary roots. For these roots we have

a) if $\gamma \in \hat{\Delta}^{im}$, then $n\gamma \in \hat{\Delta}^{im}$ $\forall\ n \in \mathbb{Z}-\{0\}$
b) $\forall\ \gamma \in \hat{\Delta}^{im}$: $\dim \hat{g}^e_\gamma = \dim z^j \otimes \underline{h} = \text{rank } \underline{g}$

3.2 generators for $\hat{\underline{g}}$

In this section we will construct a set of generators for the algebra $\hat{g}$. Let us start with the underlying finite dimensional algebra g; it is generated by the simple root vectors e_i and f_i, $i = 1, 2, \ldots l$. There is however, a smaller set of generators, which we will discuss below.

lemma 3.2.1
Let $\underline{g}$ be finite dimensional and simple. Then there exists a unique root $\theta = \sum_{i=1}^{l} a_i\alpha_i$ such that $\theta+\alpha_i \notin \Delta$ $\forall i \in \{1, 2, \ldots l\}$. If α is any other root, $\theta-\alpha$ is an integral linear combination of the α_i's with non negative coefficients. Moreover, $<\theta,\alpha_i^\vee> \geq 0$ $\forall$i.

proof:
Since the root system Δ is finite, there must be a $\theta \in \Delta$ such that $\theta+\alpha_i \notin \Delta$ for all i. For a proof of the uniqueness of this θ we refer to [Hum]. Let $\alpha = \sum_{i=1} k_i\alpha_i$ be a root and suppose that $\alpha \neq \theta$. Then there must be an index j such that $\alpha' := \alpha + \alpha_j \in \Delta$ (otherwise α would be θ). Going on in this way, and using the finiteness of Δ, we find a root $\alpha'' = \alpha + \alpha_j + \alpha_k + \ldots + \alpha_l$ such that $\alpha'' + \alpha_m \notin \Delta$ for all m. Since θ is the unique root with this property, α'' must be θ and hence $\theta-\alpha = \alpha_j + \alpha_k + \ldots + \alpha_l$. To prove the final

assertion, remark that the α_i-string through θ is given by $\theta - r\alpha_i, \ldots\ldots \theta - \alpha_i, \theta$ where $r = \langle\theta, \alpha_i^\vee\rangle \geq 0$ (see (1.7.9)).♦

The unique root θ from this lemma is called the highest root of g; similarly, $-\theta$ is the lowest root. By multiplying the Killing form on g with a suitable factor, we can always achieve

(3.2.1) $$|\theta|^2 = 2$$

We proceed to choose root vectors $e_{-\theta} \in g_{-\theta}$, $e_\theta \in g_\theta$ normalized such that

(3.2.2) $$(e_\theta \mid e_{-\theta}) = \frac{2}{|\theta|^2} = 1$$

Forming commutators of $e_{-\theta}$ with the simple root vectors e_i, $i = 1, 2, \ldots l$, we can construct root vectors associated to higher roots then $-\theta$;

(3.2.3) $$[e_{i_1}, [e_{i_2}, [e_{i_3}, \ldots [e_{i_k}, e_{-\theta}] \ldots]]] \in g_{-\theta + \alpha_{i_1} + \alpha_{i_2} + \alpha_{i_3} \ldots + \alpha_{i_k}}$$

It is possible to show that any element of g can be written as a linear combination of elements of the form (3.2.3). Therefore, $e_{-\theta}, e_1, \ldots, e_l$ generate g.

With this knowledge it is easy to construct a set of generators for $\hat{g}$; since e_i, f_i, $i = 1, 2, \ldots, l$ generate g, $1 \otimes e_i$, $1 \otimes f_i$, $i = 1, 2, \ldots, l$ generate the subalgebra $1 \otimes g \subset \hat{g}^e$. For simplicity we will write e_i, f_i for the root vectors $1 \otimes e_i$, $1 \otimes f_i$ in $\hat{g}^e$. Moreover, we define

(3.2.4) $$e_0 := z \otimes e_{-\theta}, \qquad f_0 := z^{-1} \otimes e_\theta$$

for the root vectors associated to the roots

(3.2.5) $$\alpha_0 := \delta - \theta, \quad -\alpha_0 = -\delta + \theta$$

Using the fact that any element of g can be written as a linear combination of elements of the form (3.2.3), we derive that any element from the subspace $z \otimes g \subset \hat{g}^e$ can be constructed by commuting the elements $e_0, e_1, \ldots, e_l$. Similarly, the elements e_0 and

$z \otimes e_i$, i = 1, 2,...., l generate the subspace $z^2 \otimes \underline{g}$ and so on for $z^k \otimes \underline{g}$, k>0. In the same manner we show that $z^k \otimes \underline{g}$ for k<0 is generated by $f_0, f_1, ..., f_l$.

The center c can be written as $c = [z \otimes h_1 , z^{-1} \otimes h_2]$ as we saw in the preceding chapter. Finally, the derivation d_0 cannot be obtained by commuting generators e_i, f_i, i = 0, 1,...., l. Summarizing, we have:

proposition 3.2.2

The root vectors e_i, f_i, i = 0, 1,...., l generate $\hat{\underline{g}}$. ♦

Analogous to (1.8.2) we define elements

(3.2.6) $$\alpha_i^\vee := [e_i , f_i] \qquad i = 0, 1, 2, ..., l$$

For i = 1, 2, ..., l we obtain the simple coroots from formula (1.8.2), which form a basis for $1 \otimes \underline{h}$. For i = 0 we find, using (1.7.6) and (3.2.2):

(3.2.7) $$\begin{aligned} \alpha_0^\vee &= [e_0 , f_0] \\ &= [z \otimes e_{-\theta} , z^{-1} \otimes e_\theta] \\ &= -1 \otimes v^{-1}(\theta) + c \end{aligned}$$

Using (1.8.3), we derive

(3.2.8) $$v^{-1}(\theta) = \sum_{i=1}^{l} a_i v^{-1}(\alpha_i) = \sum_{i=1}^{l} \tfrac{1}{2} a_i |\alpha_i|^2 \alpha_i^\vee$$

Combining this with (3.2.7), we get

(3.2.9) $$c = \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee ; \qquad a_0^\vee = 1,\ a_i^\vee = \tfrac{1}{2} a_i |\alpha_i|^2$$

It is possible to show that the numbers $a_i^\vee$ are positive integers.

With (3.2.9) and (3.1.4) we easily see that the set $\{d_0, \alpha_0^\vee, \alpha_1^\vee, ..., \alpha_l^\vee\}$ is a basis for $\hat{\underline{h}}^e$.

3.3 triangular decomposition of $\underline{g}$ and $\hat{\underline{g}}^e$

From chapter 1 we know that the set $\{\alpha_1, \alpha_2, ..., \alpha_l\}$ is a root basis for the finite root system Δ of $\underline{g}$, i.e., any root α can be written as an integral linear combination $\sum_{i=1}^{l} n_i \alpha_i$ with either all $n_i \geq 0$ or all $n_i \leq 0$. Therefore, the root system Δ can be decomposed in a set of positive roots (all $n_i \geq 0$) Δ_+ and a set of negative roots (all $n_i \leq 0$) Δ_-;

(3.3.1) $$\Delta = \Delta_+ \cup \Delta_-$$

Let $\underline{n}_+$ and $\underline{n}_-$ be the subalgebras defined by

(3.3.2) $$\underline{n}_+ := \bigoplus_{\alpha\in\Delta_+} \underline{g}_\alpha \qquad\qquad \underline{n}_- := \bigoplus_{\alpha\in\Delta_-} \underline{g}_\alpha$$

One can show that $\underline{n}_+$ is generated by $e_1, e_2, ..., e_l$ and $\underline{n}_-$ by $f_1, f_2, ..., f_l$ and one obviously has the following "triangular" decomposition of $\underline{g}$

(3.3.3) $$\underline{g} = \underline{n}_- \oplus \underline{h} \oplus \underline{n}_+$$

For the case $\underline{g} = sl(n,\mathbb{C})$ this simply corresponds to the decomposition of a traceless matrix in a lower triangular part, a diagonal part and an upper triangular part (see 1.6).

For the algebra $\hat{\underline{g}}^e$ the situation is quite similar;

proposition 3.3.1
$\{\alpha_0, \alpha_1, ..., \alpha_l\}$ is a root basis for $\hat{\Delta}$.

proof:
All roots are of the form $\gamma = j\delta+\alpha$, $j\in\mathbb{Z}$, $\alpha\in\Delta\cup\{0\}$, $(j,\alpha) \neq (0,0)$. Substituting $\delta=\alpha_0+\theta$, we find $\gamma=j\alpha_0+j\theta+\alpha$. From lemma 3.2.1 it is clear that for $j>0$ $j\theta+\alpha$ is for any $\alpha\in\Delta$ an integral linear combination of $\alpha_1, ..., \alpha_l$ with non negative coefficients. ♦

From this proposition it is clear that we can again write

(3.3.4) $$\hat{\Delta} = \hat{\Delta}_- \cup \hat{\Delta}_+$$

where

(3.3.5) $\hat{\Delta}_+ = \{\gamma = j\delta+\alpha \mid (j>0, \alpha\in \Delta) \vee (j = 0, \alpha\in \Delta_+\}$

$\hat{\Delta}_- = \{\gamma = j\delta+\alpha \mid (j<0, \alpha\in \Delta) \vee (j = 0, \alpha\in \Delta_-\}$

Analogous to (3.3.2) we define

(3.3.6) $\hat{\underline{n}}_+ := \bigoplus_{\gamma\in\hat{\Delta}_+} \hat{\underline{g}}^e{}_\gamma , \qquad \hat{\underline{n}}_- := \bigoplus_{\gamma\in\hat{\Delta}_-} \hat{\underline{g}}^e{}_\gamma$

and we have the following triangular decomposition of $\hat{\underline{g}}^e$:

(3.3.7) $\hat{\underline{g}}^e = \hat{\underline{n}}_- \oplus \hat{\underline{h}}^e \oplus \hat{\underline{n}}_+$

3.4 Cartan matrix and defining relations for $\hat{\underline{g}}^e$

We finally arrive at the defining relations for affine algebras. Let us start to define the Cartan matrix associated to $\hat{\underline{g}}^e$ as the $(l+1)\times(l+1)$-matrix $A = (a_{ij})_{ij=0}^{l}$ given by

(3.4.1) $a_{ij} = <\alpha_j , \alpha_i^\vee>$

Remark that for $1\leq i,j\leq l$ we reobtain the Cartan matrix associated to the finite dimensional Lie algebra $\underline{g}$ (see (1.9.7)). The other cases are

(3.4.2) $a_{00} = <\alpha_0 , \alpha_0^\vee> = <\delta - \theta , -1 \otimes v^{-1}(\theta) + c>$

$= <\theta , v^{-1}(\theta)> = |\theta|^2 = 2$

(3.4.3) $a_{0i} = <\alpha_i , \alpha_0^\vee> = <\alpha_i , -1 \otimes v^{-1}(\theta) + c>$

$= -<\alpha_i , v^{-1}(\theta)> = - <\theta , v^{-1}(\alpha_i)>$

$= = - \dfrac{2}{(\alpha_i|\alpha_i)} <\theta,\alpha_i^\vee> \leq 0 \qquad (i \neq 0)$

(3.4.4) $a_{i0} = \langle\alpha_0 , \alpha_i^\vee\rangle = \langle\delta - \theta , \alpha_i^\vee\rangle = - \langle\theta , \alpha_i^\vee\rangle \leq 0 \qquad (i \neq 0)$

From (3.4.2-3.4.4) we see that A shares the properties (1.9.8) (1) and (2) with the Cartan matrix of a finite dimensional simple Lie algebra;

(3.4.5) (1) $a_{ii} = 2$

(2) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0, \qquad a_{ij} \in \mathbf{Z}_{\leq 0}$ for $i \neq j$

We will now show that the determinant of the Cartan matrix of an affine Lie algebra is zero. Using (3.2.5) and the expression $\theta = \sum_{i=1}^{l} a_i \alpha_i$ we may write

(3.4.6) $\delta = \alpha_0 + \theta = \sum_{i=0}^{l} a_i \alpha_i, \qquad a_0 = 1$

With the definition (3.1.6) of δ we compute

(3.4.7) $\langle\delta , \alpha_i^\vee\rangle = 0 \qquad$ if $i = 1, 2, ..., l$

and

(3.4.8) $\langle\delta , \alpha_0^\vee\rangle = \langle\delta , -1 \otimes v^{-1}(\theta) + c\rangle = 0$

Hence:

(3.4.9) $\langle\delta , \alpha_j^\vee\rangle = \sum_{i=0}^{l} a_i \langle\alpha_i , \alpha_j^\vee\rangle = \sum_{i=0}^{l} a_{ji} a_i = 0 \qquad \forall j$

showing that the columns of A are dependent whence $\det A = 0$. Since the submatrix $(a_{ij})_{1\leq ij \leq l}$ is the Cartan matrix of a finite dimensional Lie algebra, the $(l+1)\times(l+1)$-matrix A has rank l. One can show that it satisfies in stead of (1.9.8)(3):

(3.4.10) $\det A = 0$ and all proper principal minors are positive

Remark: Apart from the Cartan matrices of loop algebras there are also other matrices satisfying (3.4.5)(1), (2) and (3.4.10). These matrices are related to the so-called twisted loop algebras, which we will not discuss here.

We can now write down the defining relations for $\hat{\underline{g}}^e$ analogous to (1.9.1-1.9.6). We have seen that $\hat{\underline{g}}^e$ is generated by e_i, f_i, $i = 0, 1, \ldots, l$ and the Cartan subalgebra $\hat{\underline{h}}^e = \bigoplus_{i=0}^{l} \mathbb{C}\alpha_i^\vee \oplus \mathbb{C}d_0$ and these generators are subject to the following relations

(3.4.11) $\quad [e_i , f_j] = \delta_{ij}\, \alpha_i^\vee \qquad 0 \leq ij \leq l$

(3.4.12) $\quad [h , e_i] = \langle\alpha_i,h\rangle\, e_i \qquad \forall h \in \hat{\underline{h}}^e,\ 0 \leq i \leq l$

(3.4.13) $\quad [h , f_i] = -\langle\alpha_i,h\rangle\, f_i \qquad \forall h \in \hat{\underline{h}}^e,\ 0 \leq i \leq l$

(3.4.14) $\quad [h , h'] = 0 \qquad \forall h, h' \in \hat{\underline{h}}^e$

(3.4.15) $\quad (\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0 \qquad 0 \leq ij \leq l$

(3.4.16) $\quad (\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0 \qquad 0 \leq ij \leq l$

Relations (3.4.11)-(3.4.14) are clear, while (3.4.15) and (3.4.16) coincide with (1.9.5) and (1.9.6) for $1 \leq ij \leq l$. For $i = 0$, $j \neq 0$ (3.4.15) becomes:

(3.4.17) $$(\mathrm{ad}\, e_0)^{1-a_{0j}}\,(e_j) = z^{1-a_{0j}} \otimes (\mathrm{ad}\, e_{-\theta})^{1+(\alpha_j|\theta)}\,(e_j) = 0$$

where we have used (1.7.9) for the θ-root string through α_j and the fact that the multiple commutator (3.4.17) cannot yield a central term. The other cases are treated similarly.

Remark: It is possible to study algebras defined by relations (3.4.11)-(3.4.16), where the integers a_{ij} satisfy only (3.4.5)(1) and (2). In this way one obtains more general Kac-Moody algebras than the finite dimensional and affine ones. We refer the interested reader to [Kac].

3.5 involutions on $\hat{\underline{g}}^e$

Here we exploit the symmetry between positive and negative roots to define an involution ω on $\hat{\underline{g}}^e$ (i.e., an automorphism $\omega : \hat{\underline{g}}^e \to \hat{\underline{g}}^e$ such that $\omega^2 = 1$), which will be important in the sequel. Since e_i, f_i, $i = 0, 1, \ldots, l$ and $\hat{\underline{h}}^e$ generate $\hat{\underline{g}}^e$, it is sufficient to define ω on these generators. This is done as follows:

$$\begin{aligned} &\omega(e_i) = -f_i \\ &\omega(f_i) = -e_i \\ &\omega(h) = -h \qquad \forall h \in \hat{\underline{h}}^e \end{aligned} \tag{3.5.1}$$

One easily checks that (3.5.1) is compatible with the defining relations (3.4.11)-(3.4.16), i.e., that it indeed defines an automorphism of $\hat{\underline{g}}^e$.

The image of an arbitrary element of $\hat{\underline{g}}^e$ can now be computed by means of

$$\begin{aligned} &\omega(\alpha x + \beta y) = \alpha\omega(x) + \beta\omega(y) \\ &\omega([x, y]) = [\omega(x), \omega(y)] \qquad \forall x, y \in \hat{\underline{g}}^e, \alpha, \beta \in \mathbb{C} \end{aligned} \tag{3.5.2}$$

It is clear that ω interchanges positive and negative root vectors and that $\omega(\hat{\underline{n}}_+) = \hat{\underline{n}}_-$.

Closely related to the involution ω is the antilinear involution ω_0, which is defined on generators by

$$\begin{aligned} &\omega_0(e_i) = -f_i \\ &\omega_0(f_i) = -e_i \\ &\omega_0(\alpha_i^\vee) = -\alpha_i^\vee \\ &\omega_0(d_0) = -d_0 \end{aligned} \tag{3.5.3}$$

and is extended to $\hat{\underline{g}}^e$ by

(3.5.4)
$$\omega_0(\alpha x + \beta y) = \alpha^* \omega_0(x) + \beta^* \omega_0(y)$$

$$\omega_0([x , y]) = [\omega_0(x) , \omega_0(y)] \qquad \forall x, y \in \hat{\underline{g}}^e, \alpha, \beta \in \mathbb{C}$$

Chapter 4 representation theory

4.0 introduction

In this chapter we study representations of affine Kac-Moody algebras. Representation theory is a subject with many aspects. Here we will mainly concentrate on the so-called integrable highest weight representations. Integrable means that the action of the algebra can be "exponentiated" (or "integrated") to the action of an associated group. Highest weight means that the representation contains a "vacuum vector".

We will see that these representations have a pre-Hilbert space structure, which makes them interesting for applications in quantum mechanics. They also play an important rôle in the theory of soliton equations.

4.1 representations and modules

Let $\underline{g}$ be a Lie algebra and V some (possibly infinite dimensional) vector space. A representation of $\underline{g}$ on V is a Lie algebra homomorphism

(4.1.1) $\rho : \underline{g} \to gl(V)$

In other words: ρ is a linear map, which satisfies

(4.1.2) $\rho([x,y]) = [\rho(x) , \rho(y)] \qquad \forall x, y \in \underline{g}$

The bracket on the right hand side is the commutator in $gl(V)$, as defined in (1.1.2).

A concept equivalent to that of a representation is that of a $\underline{g}$-module. This is a vector space with an action of $\underline{g}$ defined by

(4.1.3) $\underline{g} \times V \to V$

$(x,v) \to x.v \qquad x \in \underline{g},\ v \in V$

such that $\forall\ x,y \in \underline{g},\ v,\ v_1,\ v_2 \in V,\ \lambda,\mu \in \mathbb{C}$:

(4.1.4) (1) $x.(\lambda v_1 + \mu v_2) = \lambda\, x.v_1 + \mu\, x.v_2$

(2) $(\lambda x + \mu y).v = \lambda\, x.v + \mu\, y.v$

(3) $[x,y].v = x.(y.v) - y.(x.v)$

It is easy to see that a pair (V,ρ), with ρ a representation of $\underline{g}$ on V, defines a $\underline{g}$-module structure on V and vice versa. We will use the terms representation and $\underline{g}$-module as well as the notations $\rho(x)(v)$ and x.v more or less interchangeably.

A submodule of a $\underline{g}$-module is a subspace W of V such that

$$x.w \in W \qquad \forall x \in \underline{g},\ \forall w \in W \tag{4.1.5}$$

The zero subspace and V itself are always submodules. These will be called the trivial submodules.

4.2 reducible and irreducible modules

Recall the loop algebra $\tilde{\underline{g}}$ consisting of maps $x : S^1 \rightarrow gl(n,\mathbb{C})$ as discussed in chapter 2. Geometrically, the most natural $\tilde{\underline{g}}$-module seems to be the vector space $\tilde{V} = \tilde{\mathbb{C}}^n$ consisting of maps $f : S^1 \rightarrow \mathbb{C}^n$, with as action the pointwise matrix multiplication:

$$(x.f)(z) := x(z)\, f(z) \qquad \forall z \in S^1 \tag{4.2.1}$$

However, from the point of view of representation theory this is a very complicated representation. To understand this, consider an interval I on the circle and the collection $\tilde{V}_I \subset \tilde{V}$ of all vector valued functions on the circle that vanish on I. It is clear then that if $f \in \tilde{V}_I$, also $x.f \in \tilde{V}_I$ for all $x \in \tilde{\underline{g}}$. In other words: $\tilde{V}_I$ is a nontrivial submodule of $\tilde{V}$.

A $\tilde{\underline{g}}$-module is called irreducible if it has only trivial submodules and reducible otherwise. So $\tilde{V}$ is highly reducible, since it contains many nontrivial submodules.

In representation theory one tries to use the irreducible modules as building blocks for the construction of arbitrary modules. In the sequel we will be interested in a special class of irreducible modules for $\tilde{g}$ (or rather for its central extension $\hat{g}$, discussed in the previous chapter).

4.3 integrable representations

We are interested in representations for which the action of the Lie algebra can be integrated to the action of some associated group, using the exponential map. Since we will be working with infinite dimensional spaces, we want to avoid questions about convergence.

To illustrate how this may be done, we first consider the Lie algebra $sl(2,\mathbb{C})$. It has a natural (irreducible) representation on $\mathbb{C}^2$, and the Chevalley generators e, f and h are represented by the matrices

$$e=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad f=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix} \tag{4.3.1}$$

Since h is diagonal with eigenvalues ± 1, $\exp th$ can be defined as multiplication by $\exp\pm t$ on the ± 1 eigenspaces.

More generally, an operator A acting on a vector space V such that

$$V=\bigoplus_{\lambda} V_\lambda; \qquad Av_\lambda=\lambda v_\lambda \qquad \forall v_\lambda\in V_\lambda \tag{4.3.2}$$

can be exponentiated: $\exp tA$ acts as multiplication by $\exp t\lambda$ on V_λ.

Now consider the elements e and f. They can be exponentiated

$$\begin{aligned}\exp te&=\begin{pmatrix}1&0\\0&1\end{pmatrix}+t\begin{pmatrix}0&1\\0&0\end{pmatrix}\\ \exp tf&=\begin{pmatrix}1&0\\0&1\end{pmatrix}+t\begin{pmatrix}0&0\\1&0\end{pmatrix}\end{aligned} \tag{4.3.3}$$

because $e^2=0=f^2$ and consequently the power series expansion for exp is a finite sum.

More generally, an operator B on V is called locally nilpotent if for every $v\in V$ there exists an integer $n_v>0$ such that for $n>n_v$ we have

(4.3.4) $\quad B^{n}v = 0$

It is clear that the operator exptB is well defined in this case, since it acts on every $v \in V$ as a polynomial (of maximal degree n_v) in B.

lemma 4.3.1
The group $SL(2,\mathbb{C}) = \{g \in gl(2,\mathbb{C}) \mid \det g = 1\}$ is generated by exp te and exp t'f, $t, t' \in \mathbb{C}$.

proof:
See [Car]. ♦

This means that every element of $SL(2,\mathbb{C})$ can be written as a product of a finite number of exp te's and exp t'f 's. The proof is a simple exercise in matrix multiplication. What is interesting about this result is that if we did not know of the existence of $SL(2,\mathbb{C})$ before, we could have defined it as the group associated to the algebra $sl(2,\mathbb{C})$ and the representation $V = \mathbb{C}^2$, i.e., as the group generated by the operators exp te, exp t'f. In this construction the concept of convergence is never used.

Next we return to the Kac-Moody algebra $\hat{\underline{g}}$. The analogue of $\mathbb{C}h$ in $sl(2,\mathbb{C})$ is the Cartan subalgebra $\hat{\underline{h}}$ and the analogues of e and f are the Chevalley generators e_i and f_i, $i = 0, 1,..., l$. We want the Cartan subalgebra to act by multiplication operators on eigenspaces, so we introduce a class of representations for which this is true.

definition 4.3.2
A $\hat{\underline{g}}$-module is called $\hat{\underline{h}}$-diagonalizable if

(4.3.5) $\quad V = \bigoplus_{\lambda \in (\hat{\underline{h}})^*} V_\lambda$

with

(4.3.6) $\quad V_\lambda := \{v \in V \mid h.v = \langle\lambda, h\rangle v, \forall h \in \hat{\underline{h}}\}$

If $V_\lambda \neq 0$, then λ is called a weight, V_λ a weight space and elements of V_λ weight vectors. These concepts are generalizations of the concepts of roots, root spaces and root vectors for the adjoint representation (see section 1.5). In the sequel we will consider only representations for which the weight spaces are finite dimensional, although one can easily construct modules for which this is not true.

We now define a class of representations in which the e_i's and the f_i's can be exponentiated.

definition 4.3.3

A $\hat{\underline{g}}$-module is called integrable if it is $\hat{\underline{h}}$-diagonalizable and if the e_i's and f_i's are locally nilpotent operators.

For every integrable $\hat{\underline{g}}$-module V one can define the associated group $\hat{G}_V$ to be the group of automorphisms of V generated by the operators $\exp te_i$, $\exp t'f_i$, $i = 0, 1, \ldots, l$, $t, t' \in \mathbb{C}$.

4.4 highest and lowest weight modules

The generalization of formula (1.7.3) for the adjoint representation to an arbitrary $\hat{\underline{h}}$-diagonalizable $\hat{\underline{g}}$-module is given by the following lemma.

lemma 4.4.1

Let $V = \oplus V_\lambda$ be an $\hat{\underline{h}}$-diagonalizable $\hat{\underline{g}}$-module, $e_\alpha \in \hat{\underline{g}}_\alpha$ a root vector, then

(4.4.1) $$e_\alpha . V_\lambda \subset V_{\lambda+\alpha}$$

proof:

Let $v \in V_\lambda$, then $\forall h \in \hat{\underline{h}}$: $$h.e_\alpha.v = [h , e_\alpha].v + e_\alpha.h.v$$

$$= \langle\alpha,h\rangle\, e_\alpha.v + \langle\lambda,h\rangle e_\alpha.v$$

$$=\langle\alpha+\lambda,h\rangle\, e_\alpha.v \;\blacklozenge$$

Define for an arbitrary $\hat{\underline{h}}$-diagonalizable $\hat{\underline{g}}$-module the weight system to be

(4.4.2) $\quad P(V) := \{\lambda \in (\hat{\underline{h}})^* \mid V_\lambda \neq 0\}$

Let λ, λ' be two elements of the weight system. We will call λ' a higher weight than λ if there exists an element $\alpha = \sum_{i=1} n_i \alpha_i$, $n_i \in \mathbb{Z}_{\geq 0}$, $\alpha \neq 0$, such that

(4.4.3) $\quad \lambda' = \lambda + \alpha$

This introduces a partial ordering on P(V). Because of (4.4.1) we can say that the operator e_α increases the weight of V_λ, if $\alpha \in \Delta_+$. In the language of quantum mechanics e_α ($\alpha \in \Delta_+$) is an annihilation operator and similarly $e_{-\alpha}$ ($\alpha \in \Delta_+$) is a creation operator.

In quantum mechanics one considers systems where all states can be obtained from the vacuum by the application of creation operators. In representation theory there is an analogous concept.

definition 4.4.2

A $\hat{\underline{g}}$-module V is called a highest weight module with highest weight Λ if there exists a $v \in V$ such that

(1) $\quad h.v = \langle \Lambda, h \rangle v \qquad \forall h \in \hat{\underline{h}}$

(2) $\quad \hat{\underline{n}}_+.v = \{0\}$

(3) $\quad U(\hat{\underline{n}}_-).v = V$, i.e., V is generated by the action of $\hat{\underline{n}}_-$ on the highest weight vector

Similarly, one defines a lowest weight module V^* with lowest weight Λ to be a representation space containing a vector v^* such that

(1) $\quad h.v^* = \langle \Lambda, h \rangle v^* \qquad \forall h \in \hat{\underline{h}}$

(2) $\quad \hat{\underline{n}}_-.v^* = \{0\}$

(3) $U(\hat{\underline{n}}_+).v^* = V$, i.e., V is generated by the action of $\hat{\underline{n}}_+$ on the lowest weight vector

Next we want to know which of the integrable modules are highest or lowest weight modules. To answer this question, we have to study the action of the Cartan subalgebra and in particular the action of the central element $c \in \hat{\underline{h}}$ on integrable modules.

theorem 4.4.3

Let V be an integrable $\hat{\underline{g}}$-module and $\lambda \in P(V)$, then

(4.4.4) $$<\lambda , \alpha_i^\vee> \in \mathbb{Z} \qquad i = 0, 1,..., l \;\blacklozenge$$

This means that the fundamental coroots $\alpha_i^\vee = [e_i,f_i]$ act as integers on the weight spaces V_λ. The proof of this theorem uses some facts from the representation theory of $sl(2,\mathbb{C})$, which are however somewhat too technical to discuss here.

In the previous chapter we have seen that the canonical central element c can be expressed as an integral linear combination of the coroots

(4.4.5) $$c = \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee \qquad a_i^\vee \in \mathbb{N}$$

Since c is central, it acts on an irreducible module as a multiple of the identity operator. Using the theorem, we find that c acts by multiplication by an integer n, called the level of V. Note that the representations of level zero are representations of the loop algebra, since c is zero in such a representation.

theorem 4.4.4 [Cha]

Let V be an irreducible integrable $\hat{\underline{g}}$-module with finite dimensional weight spaces; let n be the level of V, then

(1) if $n>0$, V is a highest weight module

(2) if $n<0$, V is a lowest weight module

(3) if $n = 0$, V is neither $\blacklozenge$

The irreducible $\hat{g}$-modules of level 0 (which are loop algebra representations) have only recently been discovered and classified. They seem to have found as yet no applications in physics.

Next we describe the highest and lowest weights for integrable representations. Define the weight lattice P by:

(4.4.6) $$P := \{\lambda \in (\underline{\hat{h}})^* \mid \langle \lambda , \alpha_i^\vee \rangle \in \mathbb{Z},\ i = 0, 1, \ldots, l\}$$

(By theorem 4.4.3 P consists of all weights that can possibly appear in P(V) for V integrable.) Define the so-called dominant integral weights by:

(4.4.7) $$P_+ := \{\lambda \in P \mid \langle \lambda , \alpha_i^\vee \rangle \geq 0,\ i = 0, 1, \ldots, l\}$$

theorem 4.4.5

For every $\Lambda \in P_+$ there exists a unique irreducible integrable highest weight module with highest weight Λ, denoted by $L(\Lambda)$, and a unique irreducible integrable lowest weight module with lowest weight $-\Lambda$, denoted by $L^*(\Lambda)$. Conversely, every irreducible integrable highest (lowest) weight module has highest (lowest) weight Λ ($-\Lambda$) with $\Lambda \in P_+$.♦

Define the elements Λ_i of P_+ by

(4.4.8) $$\langle \Lambda_i , \alpha_j^\vee \rangle = \delta_{ij} \qquad ij = 0, 1, \ldots, l$$

These elements Λ_i are called fundamental weights. It is clear that every element $\Lambda \in P_+$ can be written as

(4.4.9) $$\Lambda = \sum_{i=0}^{l} k_i \Lambda_i \qquad k_i \in \mathbb{Z}_{\geq 0}$$

The highest weight modules $L(\Lambda_i)$ are called the fundamental highest weight modules. One can obtain every $L(\Lambda)$ with Λ given by (4.4.8), from the fundamental representation by taking tensor products.

Especially important is the module $L(\Lambda_0)$. It has always level one and is called the basic representation. In the next chapter we will give an explicit construction for $L(\Lambda_0)$ for some of the affine Kac-Moody algebras $\hat{\underline{g}}$.

4.5 the contravariant Hermitian form

In this section we will discuss the pre-Hilbert space structure of the integrable highest weight modules $L(\Lambda)$. Consider the weight space decomposition

(4.5.1) $$L(\Lambda) = \bigoplus_{\lambda\in(\hat{\underline{h}})^*} L(\Lambda)_\lambda$$

Define the restricted dual $L^*(\Lambda)$ of $L(\Lambda)$ by

(4.5.2) $$L^*(\Lambda) = \bigoplus_{\lambda\in(\hat{\underline{h}})^*} (L(\Lambda)_\lambda)^*$$

where $(L(\Lambda)_\lambda)^*$ is the dual of the finite dimensional weight space $L(\Lambda)_\lambda$. An element α of $(L(\Lambda)_\lambda)^*$ can be seen as an element of $L(\Lambda)^*$, which is denoted by the same symbol, by setting $<\alpha\, , L(\Lambda)_\mu> = 0\ \forall\mu \neq \lambda$. It is then clear that the restricted dual $L^*(\Lambda)$ is the subspace of the full dual $L(\Lambda)^*$, consisting of all linear functions on $L(\Lambda)$ which are zero on almost all weight spaces $L(\Lambda)_\lambda$.

Now define on $L^*(\Lambda)$ a $\hat{\underline{g}}$-action by

(4.5.3) $$<x.\alpha\, , v> := - <\alpha\, , x.v> \qquad \forall\alpha\in L^*(\Lambda),\ x\in\hat{\underline{g}},\ v\in L(\Lambda)$$

It is easy to verify that this action defines a $\hat{\underline{g}}$-module structure on $L^*(\Lambda)$;

(4.5.4) $$\begin{aligned} <[x,y].\alpha\, ,v> &= - <\alpha\, , [x,y].v> \\ &= <\alpha\, , y.x.v - x.y.v> \\ &= <x.y.\alpha - y.x.\alpha\, , v> \end{aligned}$$

With this action $L^*(\Lambda)$ is $\hat{\underline{h}}$-diagonalizable; let $\alpha\in (L(\Lambda)_\lambda)^*$, then for $v\in L(\Lambda)_\mu$, $h\in\hat{\underline{h}}$

(4.5.5) $\quad <h.\alpha\, ,\, v> \, = - <\alpha\, ,\, h.v> \, = -<\mu\, ,\, h> <\alpha\, ,\, v>$

This is zero if $\mu \neq \lambda$, hence $h.\alpha \in (L(\Lambda)_\lambda)^*$ and

(4.5.6) $\quad h.\alpha = - <\lambda\, ,\, h> \alpha$

So (4.5.2) is the weight space decomposition of $L^*(\Lambda)$.

Furthermore $L^*(\Lambda)$ is a lowest weight module: let $\alpha \in (L(\Lambda)_\Lambda)^*$, then for all $v \in L(\Lambda)$

(4.5.7) $\quad <n_-.\alpha\, ,\, v> \, = - <\alpha\, ,\, n_-.v> \, = 0 \qquad \forall n_- \in \hat{\underline{n}}_-$

since $n_-.v$ has always lower weight than Λ. Furthermore, by (4.5.7), $\alpha \in (L(\Lambda)_\Lambda)^*$ has weight $-\Lambda$. So the notation $L^*(\Lambda)$ is consistent with the one introduced in theorem 4.4.5.

Next we introduce an alternative $\hat{\underline{g}}$-module structure on $L(\Lambda)$. Recall the linear involution ω, defined in section 3.5. Using this, we define on $L(\Lambda)$

(4.5.8) $\quad \pi^*_\Lambda(x)(v) := \pi_\Lambda(\omega(x))(v) \qquad \forall x \in \hat{\underline{g}},\ v \in L(\Lambda)$

One easily checks that $(L(\Lambda)\, ,\, \pi^*_\Lambda)$ is a lowest weight module of lowest weight $-\Lambda$. Hence, by the uniqueness of lowest weight modules, we have an isomorphism $\Psi : L(\Lambda) \to L^*(\Lambda)$ such that the following diagram commutes

(4.5.9)
$$\begin{array}{ccc} L(\Lambda) & \overset{\Psi}{\rightarrow} & L^*(\Lambda) \\ \pi^*_\Lambda(x) \downarrow & & \downarrow \pi_\Lambda(x) \\ L(\Lambda) & \underset{\Psi}{\rightarrow} & L^*(\Lambda) \end{array}$$

We use this isomorphism to define a bilinear form on $L(\Lambda)$:

(4.5.10) $\quad B_\Lambda(v,w) := <\Psi(v)\, ,\, w> \qquad \forall v, w \in L(\Lambda)$

Since $L(\Lambda)$ and $L^*(\Lambda)$ are non degenerately paired, the form B_Λ is also non degenerate.

Next we want to modify B_Λ to obtain a Hermitian form, i.e., a form H_Λ that is antilinear in the first argument. To this end we consider the antilinear involution ω_0. It has eigenvalues ± 1 on $\hat{\underline{g}}$. If $\hat{\underline{g}}_1$ is the eigenspace corresponding to the eigenvalue +1,

$i\hat{g}_1$ is the eigenspace corresponding to the eigenvalue -1 and we have $\hat{g} = \hat{g}_1 \oplus i\hat{g}_1$. ($\hat{g}_1$ and $i\hat{g}_1$ are vector spaces over $\mathbb{R}$!). Since every element of $L(\Lambda)$ can be obtained from the vacuum vector $v_\Lambda \in L(\Lambda)_\Lambda$, we also have a decomposition of the module $L(\Lambda)$:

$$(4.5.11) \qquad L(\Lambda) = L(\Lambda)_1 + i\, L(\Lambda)_1$$

given by

$$(4.5.12) \qquad x.v_\Lambda = (x_1 + i\, x_2).v_\Lambda = x_1.v_\Lambda + i\, x_2.v_\Lambda = v_1 + i\, v_2$$

Then we define for $v = v_1 + i\, v_2,\ \ w = w_1 + i\, w_2$

$$(4.5.13) \qquad H_\Lambda(v,w) = B_\Lambda(v_1, w_1) + B_\Lambda(v_2, w_2) - iB_\Lambda(v_2, w_1) + iB_\Lambda(v_1, w_2)$$

This form satisfies:

$$(4.5.14) \qquad (a) \quad H_\Lambda(v,w) = H_\Lambda(w,v)^*$$

$$(b) \quad H_\Lambda(\lambda v,w) = \lambda^* H_\Lambda(v,w) = H_\Lambda(v,\lambda^* w)$$

$$(c) \quad H_\Lambda(x.v,w) = -\, H_\Lambda(v,\omega_0(x).w)$$

A form with property (4.5.14)(c) is called contravariant. This property expresses the fact that all elements of $\hat{g}_1$ (for which $\omega_0(x) = x$) act as anti Hermitian operators on $L(\Lambda)$.

It is clear that the construction for the Hermitian form H_Λ above holds for any $\Lambda \in (\hat{h})^*$. The following theorem states that only in the integrable case, i.e., $\Lambda \in P_+$, we obtain a pre-Hilbert space structure. In physics such a condition is called a quantization condition.

theorem 4.5.1

The Hermitian form $H_\Lambda : L(\Lambda) \times L(\Lambda) \to \mathbb{C}$ is positive definite if and only if $\Lambda \in P_+$.

proof: See [Kac], [Kac&Rai]. ♦

Chapter 5 the basic representation $L(\Lambda_0)$

5.0 introduction

In this chapter we present two explicit constructions of the basic representation $L(\Lambda_0)$ in the case of the simplest affine Kac-Moody algebra, namely the universal central extension of the loop algebra $\tilde{\underline{g}}$, where $\underline{g} = sl(2,\mathbb{C})$ (see chapter 2). Both constructions will turn out to be closely related to the construction of the Fock space by means of creation and annihilation operators in quantum field theory.

In sections 5.1-5.4 we will discuss the so-called homogeneous realization of the module $L(\Lambda_0)$. This method was discovered by Frenkel and Kac in 1980 [Fre&Kac], and independently by Segal [Seg], and is therefore often referred to in literature as the Frenkel-Kac, Segal construction of the basic representation. It is interesting to remark that the formulas for the so-called vertex operators (section 5.4) were already known in physics for almost 20 years by that time from the theory of dual models (strings).

Historically the first explicit construction of $L(\Lambda_0)$ was given by Lepowsky and Wilson [Lep&Wil] in 1978. This so-called principal realization of $L(\Lambda_0)$ will be given in section 5.5.

At this point we recall from chapter 4 that the module $L(\Lambda_0)$ is unique up to isomorphism. This means that the realizations of $L(\Lambda_0)$ mentioned above must be isomorphic. In practice one uses both realizations depending on the specific application one has in mind.

5.1 the homogeneous Heisenberg subalgebra

Let $\hat{\underline{g}} := \bigoplus_{k\in\mathbf{Z}} z^k \underline{g} \oplus \mathbb{C}c \oplus \mathbb{C}d_0$, where $\underline{g} = sl(2,\mathbb{C})$. Throughout this chapter we will use the standard basis

(5.1.1) $$e=\begin{pmatrix}0&1\\0&0\end{pmatrix}, \qquad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \qquad f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

for $sl(2,\mathbb{C})$. The commutation relations are:

(5.1.2) $$[h\,,e] = 2e, \quad [h\,,f] = -2f, \quad [e\,,f] = h$$

Note that in the terminology of chapter 3 we have: $e \equiv e_1$, $f \equiv f_1$, $h \equiv \alpha_1^\vee$.

For k>0 we define

(5.1.3) $$p_k := \frac{1}{2} z^k h$$

$$q_k := \frac{1}{k} z^{-k} h$$

These elements satisfy the so-called Heisenberg commutation relations, familiar from quantum mechanics;

(5.1.4) $$[p_k, q_j] = \delta_{kj}\, c$$

Therefore, the subalgebra $\hat{\underline{s}} := \bigoplus_{k>0} \mathbb{C}q_k \oplus \mathbb{C}\, c \bigoplus_{k>0} \mathbb{C}p_k$ is called an infinite Heisenberg subalgebra (HSA). In this particular example one speaks of the homogeneous HSA.

Remember from chapter 4 that the center c acts as multiplication by 1 in the basic representation $L(\Lambda_0)$. Moreover, the highest weight vector v_{Λ_0} is killed by the action of $\hat{\underline{n}}_+ := \bigoplus_{k>0} z^k \underline{g} \oplus \mathbb{C}e$, so we have in particular:

(5.1.5) $$p_k . v_{\Lambda_0} = 0$$

It is very easy to construct an irreducible representation over $\hat{\underline{s}}$ such that c acts as unity and such that there is a vector which is killed by all p_k's; one simply takes the space $\mathbb{C}[x_i; i \geq 1]$ of polynomials in all variables $x_1, x_2, \ldots$, and defines the action of $\hat{\underline{s}}$ by

(5.1.6) $$p_k . P := \frac{\partial P}{\partial x_k}$$

$$q_k . P := x_k P$$

$$c.P = P \qquad \qquad \forall P \in \mathbb{C}[x_i]$$

In fact, one has the following theorem, which can be regarded as an algebraic version of the well known Stone-von Neumann theorem.

theorem 5.1.1
Let V be an irreducible $\underline{\hat{s}}$-module, such that c acts as the identity and such that there exists a so-called vacuum vector, which is annihilated by the p_k's, then V is isomorphic to $\mathbb{C}[x_i]$ with the action of $\underline{\hat{s}}$ given by (5.1.6).♦

The problem is of course that the module $L(\Lambda_0)$ is -by definition- irreducible under the action of $\underline{\hat{g}}$ but not necessarily under the action of the subalgebra $\underline{\hat{s}}$. Kac has proved that $L(\Lambda_0)$ is completely reducible under the action of $\underline{\hat{s}}$, i.e., it is a direct sum of irreducible $\underline{\hat{s}}$-modules of type (5.1.6). If we take the vacuum vectors out of each term of this direct sum, we obtain the so-called vacuum space $\Omega(\Lambda_0)$ for the action of $\underline{\hat{s}}$;

(5.1.7) $$\Omega(\Lambda_0) := \{v \in L(\Lambda_0) \mid p_k.v = 0 \ \forall k\}$$

The structure of $L(\Lambda_0)$ is therefore given by

(5.1.8) $$L(\Lambda_0) \cong \mathbb{C}[x_i] \otimes \Omega(\Lambda_0)$$

5.2 the centralizer $\hat{S}$ of $\underline{\hat{s}}$

In order to determine the structure of $\Omega(\Lambda_0)$ we make the following simple observation; if $A \in \mathrm{End}L(\Lambda_0)$ commutes with the action of $\underline{\hat{s}}$ on $L(\Lambda_0)$, it maps the vacuum space into itself;

(5.2.1) $$p_k.(Av) = A(p_k.v) = 0 \qquad \forall v \in \Omega(\Lambda_0)$$

Therefore, we can, once we have such operators A, construct new elements of $\Omega(\Lambda_0)$ starting from the vacuum vector v_{Λ_0}, which is obviously in $\Omega(\Lambda_0)$ (see (5.1.5)).

In order to find such operators, recall from chapter 4 that the representation $L(\Lambda_0)$ is integrable, i.e., the operators e_i, f_i, $i = 0,1$ are locally nilpotent. This enabled us to define the group $\hat{G}_{L(\Lambda_0)}$ as the group generated by $\exp te_i$, $\exp tf_i$, $i = 0,1$, $t \in \mathbb{C}$. A possible supply of operators A commuting with the action of $\underline{\hat{s}}$ is given by the centralizer $\hat{S}$ of $\underline{\hat{s}}$ in $\hat{G}_{L(\Lambda_0)}$;

(5.2.2) $$\hat{S} := \{\hat{g}\in \hat{G}_{L(\Lambda_0)} \mid \hat{g}x\hat{g}^{-1} = x \ \forall x\in \hat{\underline{s}}\}$$

One can argue that $\hat{G}_{L(\Lambda_0)}$ is an extension by a one dimensional center of the polynomial loop group $\tilde{G} = SL_2(\mathbb{C}[z,z^{-1}])$ introduced in chapter 2 and that there exists a surjective homomorphism $\pi : \hat{G}_{L(\Lambda_0)} \to \tilde{G}$. On the level of the algebras $\hat{g}$ and $\tilde{g}$ this corresponds of course to the projection

(5.2.3) $$\pi_* : \hat{g} \to \tilde{g}$$

$$\pi_*(\tilde{x} + \alpha c) = \tilde{x} \qquad \forall \tilde{x}\in \tilde{g},\ \alpha\in \mathbb{C}$$

Using these projections, it is more or less natural to study instead of $\hat{S}$ the group $\tilde{S}\subset\tilde{G}$ defined by

(5.2.4) $$\tilde{S} := \{\tilde{g}\in \tilde{G} \mid \tilde{g}x\tilde{g}^{-1} = x \ \forall x\in \tilde{\underline{s}}\}$$

where $\tilde{\underline{s}} := \pi_*(\hat{\underline{s}})$ is a commutative subalgebra of $\tilde{g}$. The group $\tilde{S}$ is easily determined; its elements are matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}[z,z^{-1}])$, satisfying

(5.2.5) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} z^k & 0 \\ 0 & -z^k \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} z^k & 0 \\ 0 & -z^k \end{pmatrix} \qquad \forall k\in \mathbb{Z}$$

which is equivalent to

(5.2.6) $$ad + bc = 1$$
$$ab = cd = 0$$

Combining this with $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc = 1$, we find:

(5.2.7) $$a = xz^{-i}, \qquad d = x^{-1}z^{i} \qquad x\in \mathbb{C}^{\times},\ i\in \mathbb{Z}$$
$$b = c = 0$$

Therefore, the group $\hat{S}$ is generated by the matrices

(5.2.8) $$\tilde{T} := \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}, \qquad \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \qquad x\in \mathbb{C}^{\times}$$

The group $H := \{ \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \mid x \in \mathbb{C}^{\times} \}$ is the Cartan subgroup of $G = SL_2(\mathbb{C})$; its elements are of the form exp(lnx.h). The group $\tilde{T}r := \{ \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}^k \mid k \in \mathbb{Z} \}$ is called the translation group. One easily establishes the following relations

$$(5.2.9) \qquad \tilde{T}^{-1} z^k e \, \tilde{T} = z^{k+2} e$$

$$\tilde{T}^{-1} z^k f \, \tilde{T} = z^{k-2} f$$

$$\tilde{T}^{-1} z^k h \, \tilde{T} = z^k h$$

We proceed to construct lifts of the groups $\tilde{T}r$ and H to subgroups of $\hat{G}_{L(\Lambda_0)}$. For the Cartan subgroup H this is done as follows: consider the matrix $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} \in H$. Since h acts as a diagonal operator on $L(\Lambda_0)$, we can define the isomorphism exp(lnx.h) of $L(\Lambda_0)$ by

$$(5.2.10) \qquad \exp(\ln x.h).v = \sum_{k \geq 0} \frac{1}{k!} (\ln x.h)^k.v \qquad \forall v \in L(\Lambda_0)$$

In fact one can prove that exp(lnx.h) is an element of $\hat{G}_{L(\Lambda_0)}$, i.e., that it can be written as a product of generators $\exp(te_i)$, $\exp(tf_i)$, $t \in \mathbb{C}$, $i = 0, 1$. Therefore, exp(lnx.h) is a lift of the matrix $\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ to $\hat{G}_{L(\Lambda_0)}$. Such a lift is of course determined up to an element of the central subgroup $\exp(\alpha c) \subset \hat{G}_{L(\Lambda_0)}$, $\alpha \in \mathbb{C}$. In this way we obtain the Cartan subgroup $\hat{H}$ of $\hat{G}_{L(\Lambda_0)}$;

$$(5.2.11) \qquad \hat{H} := \{\exp(\alpha c + \beta h) \mid \alpha, \beta \in \mathbb{C}\}$$

One easily verifies that $\hat{H} \subset \hat{S}$;

$$(5.2.12) \qquad \exp(\alpha c + \beta h)\, z^k h \exp(-\alpha c - \beta h).v = z^k h\,.v \qquad \forall v \in L(\Lambda_0)$$

Next we construct a lift of the group $\tilde{T}r$ to $\hat{G}_{L(\Lambda_0)}$. For this we remark that:

$$(5.2.13) \qquad \tilde{T} = \begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \begin{pmatrix} 1 & -z^{-1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= \exp(zf)\exp(-z^{-1}e)\exp(zf)\exp(e)\exp(-f)\exp(e)$$

Recall from chapter 3 that the generators of $\tilde{g}$ are given by

$$(5.2.14) \qquad e_0 = z\, e_{-\theta} = z\, f \qquad\qquad f_0 = z^{-1}\, e_\theta = z^{-1}\, e$$

$$e_1 = e \qquad\qquad f_1 = f$$

so that we can also write

$$(5.2.15) \qquad \tilde{T} = \; = \exp(e_0)\exp(-f_0)\exp(e_0)\exp(e_1)\exp(-f_1)\exp(e_1)$$

Viewing e_0, f_0, e_1 and f_1 as locally nilpotent operators on $L(\Lambda_0)$, (5.2.15) defines a lift $\hat{T}$ of $\tilde{T}$. The group $\hat{T}r$ is then defined as the group generated by the operator $\hat{T}$. We now have the following

proposition 5.2.1

The following operator identities hold on $L(\Lambda_0)$: (compare with (5.2.9)):

$$(5.2.16) \qquad \text{a)}\ \hat{T}^{-1}\, z^k e\, \hat{T} = z^{k+2} e$$

$$\text{b)}\ \hat{T}^{-1}\, z^k f\, \hat{T} = z^{k-2} f$$

$$\text{c)}\ \hat{T}^{-1}\, z^k h\, \hat{T} = z^k h + 2\delta_{k,0} c$$

$$\text{d)}\ \hat{T}^{-1}\, d_0\, \hat{T} = d_0 - h - c$$

sketch of the proof:

If A is a nilpotent operator, B an arbitrary operator, one has the well-known formula

$$(5.2.17) \qquad e^A\, B\, e^{-A} = e^{adA}\,(B)$$

Using this, we can write

$$\hat{T}^{-1}\, x\, \hat{T} = e^{-ade_1} e^{adf_1} e^{-ade_1} e^{-ade_0} e^{adf_0} e^{-ade_0} (x) \qquad \forall x \in \hat{\underline{g}} \tag{5.2.18}$$

Expanding the exponentials and using the commutation relations, one obtains (5.2.16).♦

corollary 5.2.2

$\hat{Tr} \subset \hat{S}$, i.e, the operator $\hat{T}$ centralizes the action of $\hat{\underline{s}}$.

proof:

Immediate from (5.2.16c) (recall that $h \notin \hat{\underline{s}}$).♦

This completes the construction of $\hat{S}$; it is the group generated by $\hat{Tr}$ and $\hat{H}$. Remark that, while $\tilde{S}$ is abelian, $\hat{S}$ is not; from (5.2.16c) we read off:

$$\hat{T}^{-1} \exp(\alpha c + \beta h)\, \hat{T} = \exp(\, (\alpha+2\beta)c + \beta h) \tag{5.2.19}$$

5.3 the structure of $\Omega(\Lambda_0)$

Now we are in a position to determine the structure of $\Omega(\Lambda_0)$. First of all we remark that

$$\exp(\alpha c + \beta h).v_{\Lambda_0} = e^{<\Lambda_0, \alpha c+\beta h>}.v_{\Lambda_0} = e^{\alpha}\, v_{\Lambda_0} \tag{5.3.1}$$

since $<\Lambda_0 , h> = <\Lambda_0 , \alpha_1^\vee> = 0$ (see (4.4.8)). Hence the Cartan subgroup $\hat{H}$ maps the highest weight space $L(\Lambda_0)_{\Lambda_0}$ into itself and we obtain no new elements of $\Omega(\Lambda_0)$ in this way.

Fortunately, this is not true for the action of the translation group $\hat{Tr}$;

lemma 5.3.1

$$\hat{T}^k . L(\Lambda_0)_{\Lambda_0} = L(\Lambda_0)_{\Lambda_0+k\alpha_1-k^2\delta} \tag{5.3.2}$$

proof:

Using (5.2.16c-d) and an easy induction with respect to k, we prove:

$$(5.3.3)\qquad \hat{T}^{-k}(\lambda h + \mu c + \nu d_0)\,\hat{T}^{k} = (\lambda-\nu k)\,h + (\mu+2k\lambda-\nu k^2)\,c + \nu d_0 \qquad \forall \lambda, \mu, \nu \in \mathbb{C},\ k \in \mathbb{Z}$$

Hence:

$$(5.3.4)\qquad (\lambda h + \mu c + \nu d_0).\hat{T}^{k}.v_{\Lambda_0} = \hat{T}^{k}(\,(\lambda-\nu k)\,h + (\mu+2k\lambda-\nu k^2)\,c + \nu\, d_0\,)\,v_{\Lambda_0}$$

$$= (\mu + 2k\lambda - \nu k^2)\,\hat{T}^{k}\,.\,v_{\Lambda_0}$$

$$= <\Lambda_0 + k\alpha_1 - k^2\delta\,,\,\lambda h + \mu c + \nu d_0>\hat{T}^{k}\,.\,v_{\Lambda_0}$$

where we have used: $<\Lambda_0\,, d_0> = 0$, $<\delta\,, d_0> = 1$.♦

The following theorem is due to Frenkel and Kac. It states that the vacuum space $\Omega(\Lambda_0)$ is irreducible under the action of $\hat{S}$;

theorem 5.3.2

The vectors $v_k := \hat{T}^{k}.v_{\Lambda_0}$, $k\in\mathbb{Z}$ form a basis for $\Omega(\Lambda_0)$.♦

For the sake of completeness we mention

corollary 5.3.3

The weight system $P(\Lambda_0)$ of $L(\Lambda_0)$ is given by

$$(5.3.5)\qquad P(\Lambda_0) = \{\Lambda_0 + k\alpha_1 - k^2\delta - l\delta \mid k\in\mathbb{Z},\ l\in\mathbb{Z}_{\geq 0}\}$$

proof:

According to lemma 5.3.1, the vectors v_k have the weight $\Lambda_0 + k\alpha_1 - k^2\delta$. Each v_k is a vacuum vector for $\hat{\underline{s}}$. Let $v_{k,l} := x_l \otimes v_k$ be the vector created from this vacuum by the operator $q_l := \frac{1}{l}\lambda^{-l}h$; it has weight $\Lambda_0 + k\alpha_1 - k^2\delta - l\delta$.♦

5.4 vertex operators

In the preceding sections we have given an explicit realization of the module $L(\Lambda_0)$ as the vector space $\mathbb{C}[x_i] \otimes \Omega(\Lambda_0)$, where $\Omega(\Lambda_0) = \oplus \mathbb{C} v_k$, $v_k = \hat{T}^k.v_{\Lambda_0}$. In this section we want to establish the action of the algebra $\hat{g} = \underset{j\in\mathbf{Z}}{\oplus} z^j g \oplus \mathbb{C}c \oplus \mathbb{C}d_0$ on this space. From the construction of $L(\Lambda_0)$ the action of $z^k h$, c and d_0 is already clear; for a typical element $P \otimes v_k$, $P\in\mathbb{C}[x_i]$ we have

$$\text{(5.4.1)} \qquad z^i h.P \otimes v_k = 2p_i.P \otimes v_k = 2\frac{\partial P}{\partial x_i} \otimes v_k \qquad (i>0)$$

$$z^{-i} h.P \otimes v_k = iq_i.P \otimes v_k = (ix_i P) \otimes v_k \qquad (i>0)$$

$$c.P \otimes v_k = P \otimes v_k$$

while the action of h and d_0 is given by

$$\text{(5.4.2)} \qquad h.x_l \otimes v_k = <\Lambda_0 + k\alpha_1 - k^2\delta - l\delta\, , h> x_l \otimes v_k = 2k\, x_l \otimes v_k$$

$$d_0.x_l \otimes v_k = <\Lambda_0 + k\alpha_1 - k^2\delta - l\delta\, , d_0> x_l \otimes v_k = -(k^2+l)\, x_l \otimes v_k$$

It remains to find formulas for the action of $z^i e$, $z^i f$, $i\in\mathbf{Z}$. For this purpose we introduce the so-called vertex operators:

$$\text{(5.4.3)} \qquad X(\alpha,\zeta) := \sum_{i\in\mathbf{Z}} \zeta^{-i} z^i e$$

$$X(-\alpha,\zeta) := \sum_{i\in\mathbf{Z}} \zeta^{-i} z^i f$$

Here ζ is just a formal parameter and the expressions (5.4.3) ought to be considered as formal operator valued power series. It will turn out to be relatively easy to give explicit formulas for the action of $X(\alpha,\zeta)$ and $X(-\alpha,\zeta)$. Extracting the coefficient of ζ^{-i} of these formulas, one obtains the action of the operators $z^i e$ and $z^i f$ respectively.

We concentrate on $X(\alpha,\zeta)$, the procedure for $X(-\alpha,\zeta)$ being analogous. First of all we compute the commutation relations of $X(\alpha,z)$ with the elements of the homogeneous HSA;

(5.4.4) $$[p_k , X(\alpha,\zeta)] = \sum_{i\in\mathbb{Z}} \zeta^{-i} [z^k \tfrac{h}{2}, z^i e]$$
$$= \zeta^k \sum_{i\in\mathbb{Z}} \zeta^{-(i+k)} z^{i+k} e$$
$$= \zeta^k X(\alpha,\zeta)$$

$$[q_k , X(\alpha,\zeta)] = \sum_{i\in\mathbb{Z}} \zeta^{-i} [\tfrac{1}{k} z^{-k} h , z^i e]$$
$$= \tfrac{2}{k} \zeta^{-k} \sum_{i\in\mathbb{Z}} \zeta^{-(i-k)} z^{i-k} e$$
$$= \tfrac{2}{k} \zeta^{-k} X(\alpha,\zeta)$$

Remember that p_k is represented by $\frac{\partial}{\partial x_k}$, q_k by x_k. This and (5.4.4) motivates the definition of the following operators:

(5.4.5) $$E_-(\alpha,\zeta) := \exp(\sum_{i>0} \zeta^i x_i)$$
$$E_+(\alpha,\zeta) := \exp(-2\sum_{i>0} \frac{\zeta^{-i}}{i} \frac{\partial}{\partial x_i})$$

These operators satisfy

(5.4.6) $$[p_k , E_-(\alpha,\zeta)] = \zeta^k E_-(\alpha,\zeta)$$

$$[q_k , E_-(\alpha,\zeta)] = 0$$

$$[p_k , E_+(\alpha,\zeta)] = 0$$

$$[q_k , E_+(\alpha,\zeta)] = \tfrac{2}{k} \zeta^{-k} E_+(\alpha,\zeta)$$

To proceed we define the operator $Z(\alpha,\zeta)$ by

(5.4.7) $$Z(\alpha,\zeta) = E_-(\alpha,\zeta)^{-1} X(\alpha,\zeta) E_+(\alpha,\zeta)^{-1}$$

Using (5.4.4) and (5.4.6) one easily derives:

(5.4.8) $$[p_k\, , Z(\alpha,\zeta)] = 0$$

$$[q_k\, , Z(\alpha,\zeta)] = 0$$

We see that $Z(\alpha,\zeta)$ commutes with the action of $\underline{\hat{s}}$ and hence it maps the vacuum space $\Omega(\Lambda_0)$ into itself. We will need the following

lemma 5.4.1

(5.4.9) $$\hat{T}^{-k}\, Z(\alpha,\zeta)\, \hat{T}^{k} = \zeta^{2k}\, Z(\alpha,\zeta)$$

(5.4.10) $$[h\, , Z(\alpha,\zeta)] = \langle\alpha,h\rangle\, Z(\alpha,\zeta) = 2Z(\alpha,\zeta)$$

(5.4.11) $$[d_0\, , Z(\alpha,\zeta)] = -\zeta \frac{d}{d\zeta} Z(\alpha,\zeta)$$

proof:

We have:

(5.4.12) $$\begin{aligned} \hat{T}^{-k}\, X(\alpha,\zeta)\, \hat{T}^{k} &= \sum_{i\in\mathbf{Z}} \zeta^{-i}\, \hat{T}^{-k}\, z^{i}e\, \hat{T}^{k} \\ &= \zeta^{2k} \sum_{i\in\mathbf{Z}} \zeta^{-(i+2k)}\, z^{i+2k}e \\ &= \zeta^{2k}\, X(\alpha,\zeta) \end{aligned}$$

and since $\hat{T}$ commutes with the action of $\underline{\hat{s}}$, it commutes with the operators $E_-(\alpha,\zeta)$ and $E_+(\alpha,\zeta)$. This proves (5.4.9); (5.4.10) is proved analogously. For (5.4.11) we remark that ad d_0 is the derivation $z\frac{d}{dz}$. Since z^k is always accompanied by ζ^{-k}, this is equivalent to $-\zeta\frac{d}{d\zeta}$. ♦

Using (5.4.9), we calculate

(5.4.13) $$\begin{aligned} Z(\alpha,\zeta).v_k &= Z(\alpha,\zeta)\, .\hat{T}^{k}.v_{\Lambda_0} \\ &= \zeta^{2k}\, \hat{T}^{k}\, Z(\alpha,\zeta).v_{\Lambda_0} \end{aligned}$$

So we are ready if we know the action of $Z(\alpha,\zeta)$ on the highest weight vector. For this we use (5.4.10) and the fact that $h.v_{\Lambda_0} = 0$; this yields:

$$h.Z(\alpha,\zeta).v_{\Lambda_0} = [h\,,\,Z(\alpha,\zeta)].v_{\Lambda_0} \tag{5.4.14}$$

$$= \langle\alpha,h\rangle\, Z(\alpha,\zeta).v_{\Lambda_0}$$

Since $Z(\alpha,\zeta).v_{\Lambda_0}$ must be in $\Omega(\Lambda_0)$ and the weights occurring in this vacuum space are of the form $\Lambda_0 + k\alpha_1 - k^2\delta$ we conclude that $Z(\alpha,\zeta).v_{\Lambda_0} \in L(\Lambda_0)_{\Lambda_0+\alpha_1-\delta}$;

$$Z(\alpha,\zeta).v_{\Lambda_0} = f(\zeta).v_1 = f(\zeta).\hat{T}.v_{\Lambda_0} \tag{5.4.15}$$

The function $f(\zeta)$ is determined as follows:

$$d_0.Z(\alpha,\zeta).v_{\Lambda_0} = [d_0\,,\,Z(\alpha,\zeta)].v_{\Lambda_0} = -\zeta\frac{df}{d\zeta}.v_{\Lambda_0} \tag{5.4.16}$$

$$= \langle\Lambda_0 + \alpha_1 - \delta\,,\,d_0\rangle\, Z(\alpha,\zeta).v_{\Lambda_0}$$

$$= -\,f(\zeta).v_{\Lambda_0}$$

Hence:

$$\zeta\frac{df}{d\zeta} = f(\zeta) \tag{5.4.17}$$

and we conclude that $f(\zeta) = a\zeta$ for some constant $a \in \mathbb{C}$. A small calculation which we shall not display here, shows that $a = 1$.
Eventually we find, using (5.4.13,15,17):

$$Z(\alpha,\zeta).v_k = \zeta^{2k+1}\, v_{k+1} \tag{5.4.18}$$

We briefly summarize this section with the explicit formula for the action of the vertex operator $X(\alpha,\zeta)$ on $L(\Lambda_0) \cong \mathbb{C}[x_i] \otimes \Omega(\Lambda_0)$; we have:

$$X(\alpha,\zeta) = E_-(\alpha,\zeta)\, Z(\alpha,\zeta)\, E_+(\alpha,\zeta) \tag{5.4.19}$$

$$= E_-(\alpha,\zeta)E_+(\alpha,\zeta)\ Z(\alpha,\zeta)$$

The differential operator

$$(5.4.20) \qquad E_-(\alpha,\zeta)E_+(\alpha,\zeta) = \exp\Big(\sum_{i>0} \zeta^i x_i\Big)\ \exp\Big(-2\sum_{i>0} \frac{\zeta^{-i}}{i}\frac{\partial}{\partial x_i}\Big)$$

only acts on the polynomial part of the tensor product, while the operator $Z(\alpha,z)$ only acts on $\Omega(\Lambda_0)$ according to (5.4.18).

5.5 the principal realization of $L(\Lambda_0)$

In this section we will give a different construction of the module $L(\Lambda_0)$. The essential ingredients for the homogeneous realization of $L(\Lambda_0)$ are the homogeneous HSA $\underline{\hat{s}}$ and its centralizer $\hat{S}$ in $\hat{G}_{L(\Lambda_0)}$. The question arises if there are other HSA's $\underline{\hat{s}}'$ and associated centralizers $\hat{S}'$ and, if there are, whether the homogeneous construction can be generalized to the pair $(\underline{\hat{s}}', \hat{S}')$. Kac and Peterson [Kac&Pet 1] have classified all inequivalent HSA's and have formulated the following

theorem 5.1
Let $\underline{\hat{g}} = \bigoplus_{j\in\mathbb{Z}} z^j\underline{g} \oplus \mathbb{C}c \oplus \mathbb{C}d_0$ be the (untwisted) affine algebra, associated to the finite dimensional simple algebra $\underline{g}$, let $\underline{\hat{s}}$ be a (maximal) HSA of $\underline{\hat{g}}$ and $\hat{S}$ its centralizer in $\hat{G}_{L(\Lambda_0)}$. Then the module $L(\Lambda_0)$ is isomorphic to the tensor product $V \otimes \Omega(\Lambda_0)$, where V is an irreducible $\underline{\hat{s}}$-module as in theorem 5.1.1 and $\Omega(\Lambda_0)$ the vacuum space of $\underline{\hat{s}}$, defined in (5.1.6). Moreover, if $\underline{g}$ has a symmetric Cartan matrix, $\Omega(\Lambda_0)$ is irreducible under the action of the group $\hat{S}$.♦

In our simple case $\underline{g} = sl(2,\mathbb{C})$ the classification yields only one other HSA $\underline{\hat{s}}'$. It is called the principal HSA and is defined by

$$(5.5.1) \qquad \underline{\hat{s}}' = \bigoplus_{k\geq 0} \mathbb{C}\, q_k' \oplus \mathbb{C}c \bigoplus_{k\geq 0} \mathbb{C}\, p_k'$$

$$p_k' := z^k e + z^{k+1} f$$

$$q_k' := \frac{1}{2k+1}(z^{-k-1}e + z^{-k}f)$$

The reader easily verifies the Heisenberg commutation relations:

(5.5.2) $$[p_k', q_k'] = \delta_{kj}\, c$$

The centralizer $\tilde{S}'$ of $\underline{\tilde{s}}' := \pi_*(\hat{\underline{s}}')$ in $\tilde{G}$ is again easily determined; it consists of all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}[z,z^{-1}])$ such that

(5.5.3) $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & z^k \\ z^{k+1} & 0 \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 0 & z^k \\ z^{k+1} & 0 \end{pmatrix} \qquad \forall k \in \mathbb{Z}$$

The polynomials a, b, c, $d \in \mathbb{C}[z,z^{-1}]$ have to satisfy

(5.5.4)
$$\begin{aligned} &bdz - ac = 0 \\ &-b^2z + a^2 = 1 \\ &d^2z - c^2 = z \\ &ad - bc = 1 \end{aligned}$$

The only solutions to these equations are $b = c = 0$, $a = d = 1 \vee a = d = -1$ and hence $\tilde{S}' = \{\pm I\} = \text{center } SL_2(\mathbb{C})$. For $\hat{S}'$ we find $\hat{S}' = \{\exp(\alpha c) \mid \alpha \in \mathbb{C}\}$.

It is obvious that $\hat{S}'$ maps the highest weight space $L(\Lambda_0)_{\Lambda_0}$ into itself. Therefore, we have, using theorem 5.1.1:

(5.5.5) $$L(\Lambda_0) = \mathbb{C}\, v_{\Lambda_0}$$

This can also be formulated by saying that the module $L(\Lambda_0)$ is irreducible under the action of the principal HSA $\hat{\underline{s}}'$. From this we conclude: $L(\Lambda_0) \cong \mathbb{C}[x_i, i \geq 0]$.

The action of $p_k' := z^k e + z^{k+1} f$ is given by $\frac{\partial}{\partial x_k}$ and the action of $q_k' := \frac{1}{2k+1}(z^{-k-1}e + z^{-k}f)$ by multiplication with x_k. Remark that the set $\{z^k e + z^{k+1} f, z^k e - z^{k+1} f, z^k h, c, d_0\}$ is a basis of $\hat{\underline{g}}$. Therefore, we are ready if we know the action of the following vertex operator

(5.5.6) $$X(\zeta) = \sum_{i \in \mathbb{Z}} \{ (z^i e - z^{i+1} f)\, \zeta^{(-2i+1)} + z^i h\, \zeta^{-2i} \}$$

The commutation relations of $X(\zeta)$ with p_k' and q_k' are easily worked out;

(5.5.7) $$[p_k', X(\zeta)] = -2\,\zeta^{2k+1}\{X(\zeta) - \tfrac{1}{2}c\}$$

$$[q_k', X(\zeta)] = -2\,\frac{\zeta^{-(2k+1)}}{2k+1}\,\{X(\zeta) - \tfrac{1}{2}c\}$$

Defining $\hat{X}(\zeta) := X(\zeta) - \frac{1}{2}c$, this becomes

(5.5.8) $$[p_k', \hat{X}(\zeta)] = -2\,z^{2k+1}\,\hat{X}(\zeta)$$

$$[q_k', \hat{X}(\zeta)] = -2\,\frac{\zeta^{-(2k+1)}}{2k+1}\,\hat{X}(\zeta)$$

Hence $\hat{X}(\zeta)$ is given by the differential operator

(5.5.9) $$\hat{X}(\zeta) = a\exp(-\sum_{k\geq 0} 2\,\zeta^{2k+1}x_k)\,\exp(\sum_{k\geq 0} 2\,\frac{\zeta^{-(2k+1)}}{2k+1}\frac{\partial}{\partial x_k})$$

for some complex constant a. One easily shows that $a = -\frac{1}{2}$. This completes the principal realization of $L(\Lambda_0)$.

chapter 6 Lie algebras of infinite matrices

6.0 introduction

In the previous chapters we have seen that one can construct interesting Lie algebras by choosing a (generalized) Cartan matrix A, introducing generators e_i, f_i, $\alpha_i^\vee$ and imposing relations between them involving A (see sections 1.9 and 3.4). Until now we have taken A to be a finite matrix.

In this chapter we will study an example where A is a certain infinite matrix. The Lie algebra one obtains by the same construction as before can be realized as a Lie algebra of infinite traceless matrices ("$sl(\infty)$"). We also consider a completion of this algebra, denoted by A_∞ and we will see that all affine Kac-Moody algebras of type $A_n^{(1)}$ are contained in A_∞.

The representation theory for these algebras is the same as before. We will give two explicit realizations of the fundamental representations $L(\Lambda_k)$, the so-called (fermionic) wedge realization and the (bosonic) Fock-realization. The fact that these are realizations of the same abstract modules $L(\Lambda_k)$ is known in quantum field theory as Bose-Fermi correspondence.

6.1 the Lie algebras $gl(\infty)$, $sl(\infty)$ and $g'(A_\infty)$

Consider matrices $(g_{ij})_{i,j\in\mathbb{Z}}$, such that all but a finite number of g_{ij}'s are zero. Matrix multiplication between such matrices is a well defined operation and we can define a commutator in the usual way. The resulting Lie algebra is denoted by $gl(\infty)$.

For elements of $gl(\infty)$ the trace is also well defined and we may consider the subalgebra of traceless elements of $gl(\infty)$:

(6.1.1) $$sl(\infty) := \{g\in gl(\infty) \mid tr(g) = 0\}$$

Let $E_{i,j}$ be the matrix in $gl(\infty)$ which is zero everywhere except for its (i,j)th entry, which is one. These matrices satisfy

(6.1.2) $$[E_{i,j}, E_{k,l}] = \delta_{jk} E_{i,l} - \delta_{il} E_{k,j}$$

and they form a basis for $gl(\infty)$; every element of $gl(\infty)$ can be written as a finite linear combination of the $E_{i,j}$'s and conversely every finite linear combination of the $E_{i,j}$'s belongs to $gl(\infty)$.

To give a basis for $sl(\infty)$ we introduce the traceless diagonal matrices

$$\alpha_i^\vee := E_{i,i} - E_{i+1,i+1} \tag{6.1.3}$$

Then the collection

$$\{\alpha_i^\vee,\ i\in\mathbb{Z}\ ;\ E_{i,j},\ i,j\in\mathbb{Z}\ ,\ i\neq j\} \tag{6.1.4}$$

forms a basis for $sl(\infty)$.

Define

$$\underline{h} := \bigoplus_{i\in\mathbb{Z}} \mathbb{C}\alpha_i^\vee \tag{6.1.5}$$

This is a maximal abelian subalgebra of $sl(\infty)$ and indeed a Cartan subalgebra. As in chapter 1 we will construct the roots and root vectors with respect to $\underline{h}$. First we define matrices

$$e_i := E_{i,i+1}, \qquad f_i := E_{i+1,i} \qquad\qquad i\in\mathbb{Z} \tag{6.1.6}$$

These are eigenvectors for $\underline{h}$; we have

$$\begin{aligned}[\alpha_i^\vee , e_j] &= [E_{i,i} - E_{i+1,i+1} , E_{j,j+1}] \\ &= \delta_{ij} E_{i,j+1} - \delta_{i,j+1} E_{j,i} - \delta_{i+1,j} E_{i+1,j+1} + \delta_{j+1,i+1} E_{j,i+1} \\ &= (2\,\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1})\, e_j \qquad\qquad \forall i,j\in\mathbb{Z}\end{aligned} \tag{6.1.7}$$

and similarly

$$[\alpha_i^\vee , f_j] = -\,(2\,\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1})\, f_j \qquad\qquad \forall i,j\in\mathbb{Z} \tag{6.1.8}$$

Introduce elements $\alpha_i \in \underline{h}^*$ by

(6.1.9) $$<\alpha_i , \alpha_j^\vee> = 2\,\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1} =: a_{ij}$$

This defines an infinite Cartan matrix

(6.1.10) $$A_\infty := (a_{ij})_{ij\in\mathbb{Z}}$$

Using the α_i's we see that

(6.1.11) $$[h , e_j] = <\alpha_j , h> e_j$$

$$[h , f_j] = - <\alpha_j , h> f_j \qquad \forall h\in \underline{h},\ j\in\mathbb{Z}$$

and e_j, f_j are root vectors with roots α_j and $-\alpha_j$ respectively (see section 1.5).

Now an arbitrary matrix $E_{i,j}$, $i \neq j$ can be obtained by commuting e_i's or f_i's: let i<j, then

(6.1.12) $$E_{i,j} = [e_i , [e_{i+1}, [e_{i+2},...[e_{j-2} , e_{j-1}]...]]]$$

and if i>j then

(6.1.13) $$E_{i,j} = [f_{i-1} , [f_{i-2}, [f_{i-3},...[f_{j+1} , f_j]...]]]$$

If $\underline{g}_\alpha$ and $\underline{g}_\beta$ are root spaces, we have

(6.1.14) $$[\underline{g}_\alpha , \underline{g}_\beta] \subset \underline{g}_{\alpha+\beta}$$

So we find from (6.1.12-14) that $E_{i,j}$ is a root vector with root $\alpha_i + \alpha_{i+1} + ...+ \alpha_{j-1}$ (if i<j) or $-(\alpha_j + \alpha_{j+1} + ...+ \alpha_{i-1})$ (if i>j). Since the $E_{i,j}$, $i \neq j$ form a basis for the elements of $sl(\infty)$ not in $\underline{h}$, the root system is given by:

(6.1.15) $$\Delta = \{\pm (\alpha_i + \alpha_{i+1} + ...+ \alpha_j) \mid i\leq j,\ i,j\in\mathbb{Z}\}$$

The roots α_i, $i\in\mathbb{Z}$ are simple roots (every root can be expressed as an integral linear combination of the α_i's with a single sign). The root spaces $sl(\infty)_\alpha$, $\alpha\in\Delta$ are all one dimensional, just as in the case of finite dimensional simple Lie algebras.

Using the Cartan matrix (6.1.10), we can write down relations between the generators e_i, f_i and $\alpha_i^\vee$ (cf. sections 1.9 and 3.4); we have $\forall i,j\in\mathbb{Z}$:

(6.1.16) $$[e_i , f_j] = \delta_{ij}\, \alpha_i^\vee$$

$$[\alpha_i^\vee , e_j] = <\alpha_j , \alpha_i^\vee> e_j$$

$$[\alpha_i^\vee , f_j] = -<\alpha_j , \alpha_i^\vee> f_j$$

$$[\alpha_i^\vee , \alpha_j^\vee] = 0$$

$$(\mathrm{ad}\, e_i)^{1-<\alpha_j,\alpha_i^\vee>} e_j = 0$$

$$(\mathrm{ad}\, f_i)^{1-<\alpha_j,\alpha_i^\vee>} f_j = 0$$

One can prove that all other Lie algebraic relations between the generators of $sl(\infty)$ can be derived from (6.1.16). This means that $sl(\infty)$ is isomorphic to the Kac-Moody algebra associated to the Cartan matrix A_∞. In this context one refers to the abstract Lie algebra with generators e_i, f_i and $\alpha_i^\vee$ subject to the relations (6.1.16) as $g'(A_\infty)$ and distinguishes it from its concrete realization $sl(\infty)$.

We can define elements of $g'(A_\infty)$ by:

(6.1.17) $$e_\alpha := [e_{\alpha_i} , [e_{\alpha_{i+1}}, [e_{\alpha_{i+2}},\ldots[e_{\alpha_{j-2}} , e_{\alpha_{j-1}}]\ldots]]]$$

$$f_\alpha := [f_{\alpha_{j-1}} , [f_{\alpha_{j-2}}, [f_{\alpha_{j-3}},\ldots[f_{\alpha_{i+1}} , f_{\alpha_i}]\ldots]]]$$

$$\alpha = \alpha_i + \alpha_{i+1} + \ldots\alpha_{j-1} \in \Delta$$

Of course we have a Lie algebra isomorphism

(6.1.18) $\quad \phi : g'(A_\infty) \to sl(\infty)$

$$\phi(e_\alpha) = E_{i,j}, \qquad \phi(f_\alpha) = E_{j,i}, \qquad \alpha = \alpha_i + \alpha_{i+1} + ... \alpha_{j-1} \in \Delta$$

$$\phi(\alpha_i^\vee) = E_{i,i} - E_{i+1,i+1}$$

The homomorphism property is expressed by

(6.1.19) $\quad [x , y] = \phi([\phi^{-1}(x) , \phi^{-1}(y)]) \qquad \forall x, y \in sl(\infty)$

6.2 the center of a Kac-Moody algebra and completions of $sl(\infty)$

The Cartan subalgebra $\underline{h}$ of a Kac-Moody algebra is maximal abelian, hence it contains in particular the center $\underline{c}$ of the Kac-Moody algebra. For every simple root vector we have

(6.2.1) $\quad [c , e_i] = <\alpha_i , c> e_i = 0 \qquad \forall c \in \underline{c}$

or $<\alpha_i , c> = 0 \ \forall i \in \mathbb{Z}$. In the situation where the e_i and f_i generate the whole algebra (which is the case we are interested in) one easily finds:

(6.2.2) $\quad \underline{c} = \{c \in \underline{h} \mid <\alpha_i , c> = 0 \ \forall i\}$

Now consider the center of $g'(A_\infty)$ (we think of it as an abstract Kac-Moody algebra, not as $sl(\infty)$). We write for $c \in \underline{c} \subset \underline{h}$:

(6.2.3) $\quad c = \sum_{k \in \mathbb{Z}} \lambda_k \alpha_k^\vee$

Then by (6.1.9) and the condition (6.2.2) on the center we must have:

(6.2.4)
$$0 = <\alpha_i , \sum_{k \in \mathbb{Z}} \lambda_k \alpha_k^\vee> = \sum_{k \in \mathbb{Z}} \lambda_k <\alpha_i , \alpha_k^\vee>$$

$= 2\lambda_i - \lambda_{i-1} - \lambda_{i+1}$

Now the sum in (6.2.3) is finite (by definition of $g'(A_\infty)$), so there will exist an $i>>0$, such that $\lambda_{i+k} = 0 \; \forall k \geq 0$. But then we find from (6.2.4) that also $\lambda_{i-1} = 0$. Repeating the argument, shows that $\lambda_i = 0 \; \forall i$ and that $g'(A_\infty)$ has no center. (This can also be deduced from the realization of $g'(A_\infty)$ as $sl(\infty)$.)

Note that this argument is entirely based on the fact that we have defined $g'(A_\infty)$ to consist of finite linear combinations of root vectors e_α, f_α and elements $\alpha_i^\vee$. Let us provisionally allow infinite sums and investigate the center in that case. Of course we must be careful in specifying which infinite sums we allow in order to ensure that the resulting space is a Lie algebra (we will discuss this later).

By induction one easily sees that the solution of the recursion relation (6.2.4) for λ_k is given by

(6.2.5) $$\lambda_k = k\,\lambda_1 - (k-1)\,\lambda_0$$

with λ_0 and λ_1 arbitrary constants. Hence

(6.2.6) $$c = (\lambda_1 - \lambda_0) \sum_{k\in\mathbf{Z}} k\,\alpha_k^\vee + \lambda_0 \sum_{k\in\mathbf{Z}} \alpha_k^\vee$$

and we find

(6.2.7) $$\underline{c} = \mathbb{C}c_1 \oplus \mathbb{C}c_2; \qquad c_1 = \sum_{k\in\mathbf{Z}} k\,\alpha_k^\vee\,, \qquad c_2 = \sum_{k\in\mathbf{Z}} \alpha_k^\vee$$

So, allowing infinite sums (in a way yet to specify), we obtain a two dimensional center for the resulting Lie algebra, which is therefore no longer $g'(A_\infty)$.

Naively extending the isomorphism $\phi : g'(A_\infty) \rightarrow sl(\infty)$ (see (6.1.18)) to a linear mapping $\bar{\phi}$, we find that the elements c_1 and c_2 are mapped on the matrices:

(6.2.8) $$\bar{\phi}(c_1) = \sum_{k\in\mathbf{Z}} k\,(E_{k,k} - E_{k+1,k+1}) = \sum_{k\in\mathbf{Z}} E_{k,k} = I$$

$$\bar{\phi}(c_2) = \sum_{k\in\mathbf{Z}} (E_{k,k} - E_{k+1,k+1}) = 0$$

The first relation says that we are no longer dealing with traceless matrices; the algebra $sl(\infty)$ should be extended to the algebra $gl(\infty)$. The second relation tells us that, even if we replace $sl(\infty)$ by $gl(\infty)$, the mapping $\bar{\phi}$ is not an isomorphism anymore. Therefore, the algebra $gl(\infty)$ should be extended further by a one dimensional center. We will come back to this, but first we will be more precise about which infinite sums we will allow.

The algebra $g'(A_\infty)$ has a root space decomposition:

(6.2.9) $$g'(A_\infty) = \underline{h} \oplus \bigoplus_{\alpha \in \Delta} g_\alpha$$

where $\underline{h}$ is defined by (6.1.5) and the root system Δ is given by (6.1.15). We stress that the symbol $\oplus$ means that there are only finite sums. Now consider the vector space

(6.2.10) $$\prod_{i \in \mathbb{Z}} \mathbb{C} \alpha_i^\vee \oplus \prod_{\alpha \in \Delta} g_\alpha$$

Here the symbol $\prod$ means that we allow all infinite sums. We write for an element of (6.2.10):

(6.2.11) $$x = \sum_{i \in \mathbb{Z}} c_i \alpha_i^\vee + \sum_{\alpha \in \Delta} c_\alpha e_\alpha$$

We would like to define the commutator of two elements of the form (6.2.11) by linearity on the summands, but this is not always possible. For instance let

(6.2.12) $$x_k := [e_0 , [e_1 ,...[e_{k-2} , e_{k-1}]...]] \qquad k>0$$

$$x_{-k} := [f_{k-1} , [f_{k-2} ,...[f_1 , f_0]...]] \qquad k>0$$

Then one checks (for instance by calculation in $sl(\infty)$) that

(6.2.13) $$[x_k , x_{-l}] = \delta_{kl} (\alpha_0^\vee + \alpha_1^\vee + ... \alpha_k^\vee) + y_{kl}$$

where $y_{kl} \notin \underline{h}$. The commutator of the infinite sums

(6.2.14) $$x_+ := \sum_{k>0} x_k, \qquad x_- := \sum_{k>0} x_{-k}$$

would therefore contain all $\alpha_i^\vee$, i>0 with infinite coefficients.

So we impose on the sums (6.2.11) the following constraint. Let $\alpha = \pm \sum_{k=0}^{j-1} \alpha_{i+k}$ be a root, then we define the height of α to be the number of simple roots in α;

$$(6.2.15) \qquad \mathrm{ht}(\alpha) = \mathrm{ht}(\pm \sum_{k=0}^{j-1} \alpha_{i+k}) = j$$

By abuse of language we will also call j the height of the corresponding root vector e_α. Then we demand for an element x of the form (6.2.11) the set

$$(6.2.16) \qquad \{k \mid k = \mathrm{ht}(\alpha), c_\alpha \neq 0\}$$

to be finite. So we allow in x an arbitrary sum of elements of the Cartan subalgebra plus a finite number of heights. We denote the set of elements satisfying this condition by $\overline{g'(A_\infty)}$. Note that $\mathrm{ht}(x_k) = k$ and hence x_+ and x_- do not belong to $\overline{g'(A_\infty)}$.

Let us check that $\overline{g'(A_\infty)}$ is indeed a Lie algebra. We have a triangular decomposition

$$(6.2.17) \qquad \overline{g'(A_\infty)} = \underline{\bar{n}}_- \oplus \underline{\bar{h}} \oplus \underline{\bar{n}}_+$$

where $\underline{\bar{n}}_-$ ($\underline{\bar{n}}_+$) consists of sums of negative (positive) root vectors with a finite number of heights and $\underline{\bar{h}}$ consists of arbitrary sums of elements of the Cartan subalgebra. First let x, y$\in \underline{\bar{n}}_+$, then

$$(6.2.18) \qquad [x,y] = [\sum_{\alpha\in\Delta_+} \lambda_\alpha e_\alpha , \sum_{\beta\in\Delta_+} \mu_\beta e_\beta] = \sum_{\substack{\alpha,\beta \\ \alpha+\beta\in\Delta_+}} \lambda_\alpha \mu_\beta \, c(\alpha,\beta)\, e_{\alpha+\beta}$$

(The elements e_α were defined in (6.1.17).) This is well-defined since if $\alpha+\beta\in\Delta_+$, there are only a finite number of pairs (α',β'), $\alpha',\beta'\in\Delta_+$, such that $\alpha'+\beta' = \alpha+\beta$. Hence the coefficient of $e_{\alpha+\beta}$ is finite and the commutator of two elements of $\underline{\bar{n}}_+$ is well defined. Due to the symmetry between $\underline{\bar{n}}_+$ and $\underline{\bar{n}}_-$ the same holds for the commutator in $\underline{\bar{n}}_-$.

Now take h$\in\underline{\bar{h}}$, x$\in\underline{\bar{n}}_+$, then

$$(6.2.19) \qquad [h,x] = [\sum_{i\in\mathbf{Z}} \lambda_i \alpha_i^\vee \,,\, \sum_{\alpha\in\Delta_+} \mu_\alpha e_\alpha]$$

$$= \sum_{\alpha\in\Delta_+} \mu_\alpha (\sum_{i\in\mathbf{Z}} \lambda_i <\alpha \,,\, \alpha_i^\vee>)\, e_\alpha$$

The root α is of the form $\alpha = \alpha_j + \alpha_{j+1} + \ldots \alpha_{j+k}$ and by (6.1.9) $<\alpha \,,\, \alpha_i^\vee> \neq 0$ only for a finite number of i's. Hence the coefficient of e_α in the right hand side of (6.2.19) is again finite. Of course the same holds for a commutator between elements $h\in\overline{\underline{h}}$ and $y\in\overline{\underline{n}}_-$.

Finally we take $x\in\overline{\underline{n}}_+$ and $y\in\overline{\underline{n}}_-$. Then:

$$(6.2.20) \qquad [x,y] = [\ \sum_{\alpha\in\Delta_+} \lambda_\alpha e_{-\alpha} \,,\, \sum_{\beta\in\Delta_+} \mu_\beta e_\beta]$$

$$= \sum_{\substack{\alpha,\beta \\ \beta-\alpha\in\Delta_+}} \lambda_\alpha \mu_\beta \ c(-\alpha,\beta)\ e_{\beta-\alpha}$$

where $c(-\alpha,\beta)$ is some structure constant. Now for a given $\gamma\in\Delta$ there is only a finite number of pairs (α,β) of positive roots such that $\gamma = \beta - \alpha$: suppose $\gamma>0$, then we have $\gamma = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+k}$ and we can take $\beta = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{i+k+l}$ and $\alpha = \alpha_{i+k+1} + \alpha_{i+k+2} + \ldots + \alpha_{i+k+l}$. This gives an infinite number of possibilities. However, if we restrict α and β by demanding that their heights can only take a finite number of values, there are only a finite number of possibilities. This gives a finite coefficient for $e_{\beta-\alpha}$ in (6.2.20) in the case $\beta-\alpha>0$. of The situation for $\beta-\alpha\leq 0$ is similar.

We conclude that we have a well defined commutator on $\overline{g'(A_\infty)}$ and hence $\overline{g'(A_\infty)}$ is indeed a Lie algebra.

Next we consider the completion on the level of infinite matrices. Recall that the matrix $E_{i,j}$ corresponds to the root vector $e_{\alpha_i+\ldots\alpha_{j-1}}$ if i<j and to the root vector $e_{-(\alpha_{i-1}+\ldots\alpha_j)}$ if i>j. In both cases the height is given by $|i - j|$. If we demand that a matrix $x = \sum_{ij} a_{ij} E_{i,j}$ has only a finite number of heights, it is clear that x must be of finite width around the diagonal. We denote the algebra of such matrices by $\overline{gl(\infty)}$. Note that $\overline{gl(\infty)}$ is not isomorphic to $\overline{g'(A_\infty)}$ since the latter contains the central element $c_2 = \sum_{k\in\mathbf{Z}} \alpha_k^\vee$, which is mapped on the zero matrix in $\overline{gl(\infty)}$ under the linear extension $\overline{\phi}$ of the isomorphism ϕ (see (6.2.8)).

To repair this unfortunate situation, we introduce an extension $A_\infty := \overline{gl(\infty)} \oplus \mathbb{C}c$. (We hope that the reader will not be too much confused by the fact that we use the symbol A_∞ both for the infinite Cartan matrix (6.1.10) and for this extension of $\overline{gl(\infty)}$.) We also introduce an isomorphism (of vector spaces)

$$\bar{\phi} : \overline{g'(A_\infty)} \to A_\infty \tag{6.2.21}$$

by demanding that $\bar{\phi}$ coincides with ϕ outside $\underline{h}$:

$$\bar{\phi}(e_\alpha) = E_{i,j} \tag{6.2.22}$$

$$\bar{\phi}(f_\alpha) = E_{j,i} \qquad \alpha = \alpha_i + \alpha_{i+1} + \ldots \alpha_{j-1}$$

and that the restriction of $\bar{\phi}$ to the center of $\overline{g'(A_\infty)}$ is given by:

$$\bar{\phi}(c_1) = I \qquad \bar{\phi}(c_2) = c \tag{6.2.23}$$

This can be implemented by

$$\bar{\phi}(\alpha_i^\vee) = \phi(\alpha_i^\vee) + \delta_{i0}\, c \tag{6.2.24}$$

$$= E_{i,i} - E_{i+1,i+1} + \delta_{i0}\, c$$

Consequently, we have:

$$\bar{\phi}^{-1}(E_{i,i}) = \sum_{k \geq i} \alpha_k^\vee \qquad \text{if } i>0 \tag{6.2.25}$$

$$\bar{\phi}^{-1}(E_{i,i}) = \sum_{k \geq i} \alpha_k^\vee - c_2 \qquad \text{if } i \leq 0$$

$$\bar{\phi}^{-1}(E_{i,j}) = \phi^{-1}(E_{i,j}) \qquad \text{if } i \neq j$$

Note that the extension A_∞ is at the moment only a vector space, not a Lie algebra. We can change this situation by demanding that $\bar{\phi}$ is a Lie algebra isomorphism, i.e., we introduce the following Lie algebra structure on A_∞ (cf. formula (6.1.19)):

$$(6.2.26) \qquad [x,y]_{A_\infty} := \bar{\phi}\{ [\bar{\phi}^{-1}(x) , \bar{\phi}^{-1}(y)]_{\overline{g'(A_\infty)}} \}$$

(One easily checks that this indeed defines a Lie algebra structure on A_∞.) It is clear that c is central in A_∞ (as it should be). Moreover, for $x,y \in sl(\infty) \subset A_\infty$ we can write:

$$(6.2.27) \qquad [x,y]_{A_\infty} = \bar{\phi}\{ [\bar{\phi}^{-1}(x) , \bar{\phi}^{-1}(y)]_{\overline{g'(A_\infty)}} \}$$

$$= \bar{\phi}\{ [\phi^{-1}(x) , \phi^{-1}(y)]_{g'(A_\infty)} \}$$

$$= \bar{\phi}\{ \phi^{-1}([x,y]_{sl(\infty)}) \}$$

$$= [x,y]_{sl(\infty)} + \Psi(x,y)\, c$$

Here $\Psi : sl(\infty) \times sl(\infty) \to \mathbb{C}$ is a two cocycle. Its value on the matrices $E_{i,j}$ and $E_{k,l}$ is given by the coefficient of $\alpha_0^\vee$ in $[\phi^{-1}(E_{i,j}) , \phi^{-1}(E_{k,l})]$ or, what is the same, the coefficient of $E_{0,0} - E_{1,1}$ in $[E_{i,j} , E_{k,l}]$. Hence

$$(6.2.28) \qquad \Psi(E_{i,j} , E_{k,l}) = \begin{cases} 1 & \text{if } i = l,\ j = k,\ i \le 0,\ j \ge 1 \\ -1 & \text{if } i = l,\ j = k,\ i \ge 1,\ j \le 0 \\ 0 & \text{in all other cases} \end{cases}$$

It is easy to verify that Ψ can be extended to a cocycle $\Psi: \overline{gl(\infty)} \times \overline{gl(\infty)} \to \mathbb{C}$ (denoted by the same symbol) by linearity; let $x = \sum_{ij} a_{ij} E_{i,j}$, $y = \sum_{kl} b_{kl} E_{k,l}$ be two elements of $\overline{gl(\infty)}$, then

$$(6.2.29) \qquad \Psi(x,y) = \sum_{ij} a_{ij} b_{ji} \Psi(E_{i,j} , E_{j,i})$$

$$= \sum_{\substack{i \le 0 \\ j \ge 1}} a_{ij} b_{ji} - \sum_{\substack{i \ge 1 \\ j \le 0}} a_{ij} b_{ji}$$

which is finite because x and y are of finite width around the main diagonal.

We conclude that the algebraic structure of A_∞ is given by:

$$(6.2.30) \qquad [x + \alpha c , y + \beta c]_{A_\infty} = [x,y]_{\overline{gl(\infty)}} + \Psi(x,y)\, c$$

where Ψ is the two cocycle on $\overline{gl(\infty)}$ defined by (6.2.28).

Remark: The calculation (6.2.27) shows that the restriction of the two cocycle Ψ to $sl(\infty)$ is a two coboundary; define the linear mapping $\theta : sl(\infty) \to \mathbb{C}$ by:

$$\begin{aligned} &\theta(E_{i,i} - E_{i+1,i+1}) := \delta_{i0} \\ &\theta(E_{i,j}) := 0 \text{ if } i \neq j \end{aligned} \tag{6.2.31}$$

then:

$$\begin{aligned} \theta([E_{i,j}, E_{k,l}]) &= \theta(\delta_{jk} E_{i,l} - \delta_{il} E_{k,j}) \\ &= \delta_{jk}\, \delta_{il}\, \theta(E_{i,i} - E_{j,j}) \\ &= \Psi(E_{i,j}, E_{k,l}) \end{aligned} \tag{6.2.32}$$

However, if we would try to extend θ to a linear mapping $\overline{gl(\infty)} \to \mathbb{C}$, we would find

$$\theta(E_{i,i}) = \theta(\sum_{j \geq i} E_{j,j} - E_{j+1,j+1}) = \begin{cases} 1 & \text{if } i \leq 0 \\ 0 & \text{if } i > 0 \end{cases} \tag{6.2.33}$$

and therefore this θ is not defined on the identity matrix. This shows that Ψ is not a two coboundary on the full algebra $\overline{gl(\infty)}$. This fact will also be clear from the next section.

6.3 connection with loop algebras and their central extensions

Consider the vector space $\mathbb{C}^\infty$ of all vectors $(v_i)_{i \in \mathbb{Z}}$ with almost all v_i zero. Introducing a basis $\{e_i\}_{i \in \mathbb{Z}}$ of $\mathbb{C}^\infty$ we can write

$$\mathbb{C}^\infty = \bigoplus_{i \in \mathbb{Z}} \mathbb{C} e_i \tag{6.3.1}$$

The Lie algebra $gl(\infty)$ acts naturally on $\mathbb{C}^\infty$; we define

(6.3.2) $\qquad E_{i,j}\, e_k = \delta_{jk}\, e_i$

and extend by linearity. The completion $\overline{gl(\infty)}$ also acts on $\mathbb{C}^\infty$.

Recall from chapter 2 the Lie algebra $\tilde{gl}_n = \mathbb{C}[z,z^{-1}] \otimes gl_n$ (see (2.2.6)). A natural representation space for $\tilde{gl}_n$ is $\tilde{\mathbb{C}}^n = \mathbb{C}[z,z^{-1}] \otimes \mathbb{C}^n$ with action

(6.3.3) $\qquad z^k \otimes A \,.\, z^l \otimes x = z^{k+l} \otimes A.x \qquad\qquad A \in gl_n,\ x \in \mathbb{C}^n$

Let us now identify $\tilde{\mathbb{C}}^n$ and $\mathbb{C}^\infty$ as follows:

(6.3.4) $\qquad z^{-k} \otimes e_i \rightarrow e_{i+nk}$

where $\{e_i\}_{1\le i\le n}$ is the standard basis for $\mathbb{C}^n$. Let $E_{i,j}$ be the standard basis for gl_n. Then the element $z^k \otimes E_{i,j}$ acts on $\mathbb{C}^\infty$ by the identification (6.3.4): we have

(6.3.5) $\qquad z^k \otimes E_{i,j} \,.\, z^{-l} \otimes e_m = \delta_{jm}\, z^{-(l-k)} \otimes e_i$

Because $z^{-l} \otimes e_m$ corresponds to e_{m+nl} and $z^{-(l-k)} \otimes e_i$ to $e_{i+n(l-k)}$, the element $z^k \otimes E_{i,j}$ corresponds to the matrix

(6.3.6) $\qquad \sum_{p\in\mathbb{Z}} E_{i+(p-k)n,j+pn}$

Note that this matrix belongs to the completion $\overline{gl(\infty)}$ and that the coefficients of this matrix satisfy the periodicity condition

(6.3.7) $\qquad c_{ij} = c_{i+n,j+n}$

So what we have done is defining an injection

(6.3.8) $\qquad \iota : \tilde{gl}_n \rightarrow \overline{gl(\infty)}$

$$z^k \otimes E_{i,j} \rightarrow \sum_{p\in\mathbb{Z}} E_{i+(p-k)n,j+pn}$$

This is of course a Lie algebra homomorphism and the image is precisely the subalgebra of $\overline{gl(\infty)}$, consisting of matrices satisfying the condition (6.3.7).

Next we consider the central extension $A_\infty = \overline{gl(\infty)} \oplus \mathbb{C}c$. Using the injection ι, we can pull back the cocycle Ψ on $\overline{gl(\infty)}$ to $\tilde{gl}_n$ to define a central extension $\tilde{gl}_n \oplus \mathbb{C}c$ with commutator

(6.3.9) $$[z^k \otimes A , x^l \otimes B] = z^{k+l} \otimes [A,B] + \Psi(\iota (z^k \otimes A) , \iota (z^l \otimes B))\, c$$

We calculate the central term explicitly; let $A = \sum_{i,j=1}^{n} a_{ij} E_{i,j}$, $B = \sum_{i,j=1}^{n} b_{ij} E_{i,j}$ be two elements of gl_n, then

(6.3.10) $$\Psi(\iota(z^k \otimes A) , \iota(z^l \otimes B)) = \sum_{i,j,p,q=1}^{n} a_{ij}\, b_{pq}\, \Psi(\iota(z^k \otimes E_{i,j}) , \iota(z^l \otimes E_{p,q}))$$
$$= \sum_{i,j,p,q=1}^{n} a_{ij}\, b_{pq} \sum_{r,s\in\mathbf{Z}} \Psi(E_{i+n(r-k),j+nr} , E_{p+n(s-l),q+ns})$$

We only get contributions from the terms for which

(6.3.11) $$j+nr = p+n(s-l), \qquad i+n(r-k) = q+ns$$

which is equivalent to

(6.3.12) $$j = p,\ \ r = s-l, \qquad i = q,\ \ r-k = s$$

from which we find $k+l = 0$. Hence, performing the p, q, r summations, we find

(6.3.13) $$\Psi(\iota(z^k \otimes A) , \iota(z^l \otimes B)) =$$
$$\delta_{k+l,0} \sum_{i,j=1}^{n} \alpha_{ij}\, \beta_{ji} \sum_{s\in\mathbf{Z}} \Psi(E_{i+ns,j+n(s-l)} , E_{j+n(s-l),i+ns})$$

The cocycles on the right hand side of (6.3.13)give a contribution + 1 if $i+ns \le 0$ and $j+n(s-l) \ge 1$ and a contribution - 1 if $i+ns \ge 1$ and $j+n(s-l) \le 0$. The first case can only occur if $l-1<s<0$ and the second case only if $0\le s<l$.

So if $l\le 0$, we are in the first case and we get:

(6.3.14) $$\Psi(\iota(z^k \otimes A), \iota(z^l \otimes B)) = \delta_{k+l,0} \sum_{i,j=1}^{n} \alpha_{ij}\, \beta_{ji} \sum_{s=l}^{-1} (+\ 1)$$

$$= k\, \delta_{k+l,0}\, \mathrm{tr}(AB)$$

and if $l>0$:

(6.3.15) $$\Psi(\iota(z^k \otimes A), \iota(z^l \otimes B)) = \delta_{k+l,0} \sum_{i,j=1}^{n} \alpha_{ij}\, \beta_{ji} \sum_{s=0}^{l-1} (-\ 1)$$

$$= -l\, \delta_{k+l,0}\, \mathrm{tr}(AB) = k\, \delta_{k+l,0}\, \mathrm{tr}(AB)$$

So we find that the pull-back of the two cocycle on $\overline{gl(\infty)}$ gives precisely the usual two cocycle on the loop algebras $\tilde{gl}_n$ (see (2.8.10)). In other words: all affine Kac-Moody algebras $A_n^{(1)} \cong \hat{sl}_n$ are included in the completed Kac-Moody algebra A_∞. (Here we use that the trace form coincides with the Killing form for $sl(n,\mathbb{C})$.)

6.4 representation theory of $sl(\infty)$ and A_∞

The representation theory as described in chapter 4 is also valid for the Lie algebra $g'(A_\infty) \cong sl(\infty)$. In particular, let

(6.4.1) $$P = \{\lambda \in \underline{h}^* \mid \langle\lambda, \alpha_i^\vee\rangle \in \mathbb{Z} \quad \forall i \in \mathbb{Z}\}$$

and

(6.4.2) $$P_+ = \{\lambda \in P \mid \langle\lambda, \alpha_i^\vee\rangle \geq 0\}$$

Then for every $\Lambda \in P_+$ there exists a unique integrable, irreducible highest weight module $L(\Lambda)$ with highest weight vector v_Λ.

By applying the lowering operators, we can obtain any vector in $L(\Lambda)$ and therefore we have a weight space decomposition

(6.4.3) $$L(\Lambda) = \bigoplus_{\vec{k}} L(\Lambda)_{\Lambda - \sum k_i \alpha_i}$$

where the sum is over vectors $\vec{k} = (k_i)_{i\in\mathbb{Z}}$ with all but a finite number of k_i's zero.

The fundamental representations $L(\Lambda_i)$ are the ones associated to the fundamental weights Λ_i defined by

$$(6.4.4) \qquad <\Lambda_i , \alpha_j^\vee> = \delta_{ij}$$

Next we want to extend the action of $g'(A_\infty)$ on $L(\Lambda)$ to an action of $\overline{g'(A_\infty)} \cong A_\infty$. Again some care must be taken, since we are dealing with infinite sums. Consider an element of the completed Cartan subalgebra $\underline{\bar{h}} = \prod \mathbb{C}\alpha_i^\vee$:

$$(6.4.5) \qquad h = \sum_{i\in\mathbb{Z}} \lambda_i \alpha_i^\vee$$

Acting with h on the highest weight vector, we find

$$(6.4.6) \qquad h.v_\Lambda = <\Lambda, \sum_{i\in\mathbb{Z}} \lambda_i \alpha_i^\vee> v_\Lambda$$

and this will in general diverge. So we impose an extra condition on the highest weight Λ:

$$(6.4.7) \qquad <\Lambda , \alpha_i^\vee> = 0 \text{ for all but a finite number of } i\in\mathbb{Z}$$

Note that this condition is satisfied for the fundamental weights. With this restriction we find that h acts as multiplication by a finite constant on any weight space; let $v\in L(\Lambda)_{\Lambda-\sum k_i\alpha_i}$, then

$$(6.4.8) \qquad h.v = <\Lambda - \sum k_i\alpha_i , \sum_{j\in\mathbb{Z}} \lambda_j \alpha_j^\vee> v$$
$$= \{\sum_{j\in\mathbb{Z}} \lambda_j <\Lambda , \alpha_j^\vee> - \sum k_i <\alpha_i , \alpha_j^\vee>\} v$$

which is finite by (6.4.7), the fact that all but a finite number of k_i's are zero and (6.1.9).

A slightly more technical argument shows that the other elements of the completion $\overline{g'(A_\infty)}$ also act in a well defined manner on $L(\Lambda)$ (if Λ satisfies (6.4.7)), so that we may think of $L(\Lambda)$ as a module both over $g'(A_\infty)$ and over $\overline{g'(A_\infty)} \cong A_\infty$.

6.5 the wedge realization of $L(\Lambda_k)$ [Kac&Pet 2]

Consider the Lie algebra $sl(n,\mathbb{C})$. As is well known, the n-1 fundamental highest weight representations $L(\Lambda_k)$, k = 1, 2, ...n-1, can be realized as a space of exterior forms

$$(6.5.1) \qquad L(\Lambda_k) \cong \wedge^k \mathbb{C}^n, \qquad k = 1, 2, ..., n-1$$

(See, e.g., [Var], chapter 4, section 4.7.) In this section we will construct realizations of $L(\Lambda_k)$ over $g'(A_\infty)$ (and hence also over its completion $\overline{g'(A_\infty)}$) analogous to these "wedge" representations.

Recall the vector space $\mathbb{C}^\infty = \underset{i\in\mathbb{Z}}{\oplus} \mathbb{C}e_i$. This is a $g'(A_\infty) \cong sl(\infty)$ module but not a highest weight module (there is no vector $v\in\mathbb{C}^\infty$, which is annihilated by all positive root vectors $e_i = E_{i,i+1}$), and neither is any of its finite exterior powers $\wedge^k \mathbb{C}^\infty$.

Therefore, we introduce the space of semi infinite exterior products $\wedge^\infty \mathbb{C}^\infty$. This is a vector space with a basis consisting of infinite exterior products of basis elements e_i of $\mathbb{C}^\infty$:

$$(6.5.2) \qquad e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....$$

such that $i_0 > i_1 > i_2 > ....$ and such that for $l >> 0$ $i_{l+1} = i_l - 1$. On this space $sl(\infty)$ acts as usual: for $x\in sl(\infty)$ we have

$$(6.5.3) \qquad \sigma(x)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....) := (x.e_{i_0}) \wedge e_{i_1} \wedge e_{i_2} \wedge ....$$

$$+ e_{i_0} \wedge (x.e_{i_1}) \wedge e_{i_2} \wedge ....$$

$$+ e_{i_0} \wedge e_{i_1} \wedge (x.e_{i_2}) \wedge .... + ...$$

Note that $\sigma : sl(\infty) \rightarrow \text{End} \wedge^\infty \mathbb{C}^\infty$ can be extended to a representation $\tau : gl(\infty) \rightarrow \text{End} \wedge^\infty \mathbb{C}^\infty$.

We can distinguish the basis elements (6.5.2) by their behaviour at large l; we will say that an element (6.5.2) has charge k if for all $l >> 0$ $i_l = k - l$. For instance

$$(6.5.4) \qquad v_k := e_k \wedge e_{k-1} \wedge e_{k-2} \wedge ....$$

has charge k. We refer to v_k as the k^{th} vacuum. The vector space of all vectors of charge k is denoted by $\wedge_k^\infty(\mathbb{C}^\infty)$ and we clearly have a decomposition of the total semi infinite wedge space in sectors of fixed charge;

(6.5.5) $$\wedge^\infty(\mathbb{C}^\infty) = \bigoplus_{k\in\mathbb{Z}} \wedge_k^\infty(\mathbb{C}^\infty)$$

The element v_k is a highest weight vector for the space of charge k vectors $\wedge_k^\infty(\mathbb{C}^\infty)$: denoting the restriction of the natural $sl(\infty)$ action σ to $\wedge_k^\infty(\mathbb{C}^\infty)$ by σ_k, we can write

(6.5.6) $$\sigma_k(\alpha_i^\vee)(v_k) = \{(E_{i,i} - E_{i+1,i+1}).e_k\} \wedge e_{k-1} \wedge e_{k-2} \wedge ....$$

$$+ e_k \wedge \{(E_{i,i} - E_{i+1,i+1}).e_{k-1}\} \wedge e_{k-2} \wedge ....+ ...$$

$$= \{\delta_{ik} - \delta_{i+1,k} + \delta_{i,k-1} - \delta_{i+1,k-1} + ...\}\, v_k$$

$$= \delta_{ik}\, v_k = <\Lambda_k , \alpha_i^\vee>\, v_k$$

and

(6.5.7) $$\sigma_k(e_i)(v_k) = (E_{i,i+1}.e_k) \wedge e_{k-1} \wedge e_{k-2} \wedge ....$$

$$+ e_k \wedge (E_{i,i+1}.e_{k-1}) \wedge e_{k-2} \wedge ....+ ...$$

$$= 0$$

Even more is true: any element of $\wedge_k^\infty(\mathbb{C}^\infty)$ can be obtained from v_k by application of elements of $sl(\infty)$, showing that $\wedge_k^\infty(\mathbb{C}^\infty)$ is a highest weight module with highest weight Λ_k. To prove this assertion we take an element

(6.5.8) $$v = e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....$$

of $L(\Lambda_k)$, i.e., for $l \geq N_0$ we have $i_l = k-l$. Then

(6.5.9) $$v = (E_{i_0,k}\cdot e_k) \wedge E_{i_1,k-1}\cdot e_{k-1} \wedge \,....(E_{i_{N_0-1},k-N_0+1}\cdot e_{k-N_0+1}) \wedge e_{k-N_0} \wedge ...$$

$$= \sigma_k(E_{i_0,k})\, \sigma_k(E_{i_1,k-1}) \ldots \sigma_k(E_{i_{N_0-1},k-N_0+1})\, (v_k)$$

Finally one can show that $\wedge_k^\infty(\mathbb{C}^\infty)$ is irreducible under the action of $sl(\infty)$, showing that

(6.5.10) $$\wedge_k^\infty(\mathbb{C}^\infty) \cong L(\Lambda_k)$$

and that the decomposition (6.5.5) of the complete semi infinite wedge space is a decomposition in irreducible submodules, namely the fundamental $sl(\infty)$ modules.

Next we introduce elementary creation and annihilation operators on $\wedge^\infty(\mathbb{C}^\infty)$. For every basis vector e_i we define linear operators

(6.5.11) $$\varepsilon(e_i) : \wedge^\infty(\mathbb{C}^\infty) \to \wedge^\infty(\mathbb{C}^\infty)$$

$$i(e_i) : \wedge^\infty(\mathbb{C}^\infty) \to \wedge^\infty(\mathbb{C}^\infty)$$

by their action on basis vectors:

(6.5.12) $$\varepsilon(e_i)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....) := e_i \wedge e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....$$

$$i(e_i)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....) = \sum_{k=0}^{\infty} (-)^k \delta_{i,i_k}\, e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge ....\wedge \hat{e}_{i_k} \wedge ...$$

where the notation $\hat{e}_{i_k}$ means that the vector e_{i_k} is deleted. The restrictions of these operators to a fixed charge sector raise and lower the charge:

(6.5.13) $$\varepsilon(e_i) : \wedge_k^\infty(\mathbb{C}^\infty) \to \wedge_{k+1}^\infty(\mathbb{C}^\infty)$$

$$i(e_i) : \wedge_k^\infty(\mathbb{C}^\infty) \to \wedge_{k-1}^\infty(\mathbb{C}^\infty)$$

Furthermore, these operators satisfy anticommutation relations

(6.5.14) a) $\{\varepsilon(e_i), \varepsilon(e_j)\} = 0$

b) $\{i(e_i), i(e_j)\} = 0$

c) $\{i(e_i), \varepsilon(e_j)\} = \delta_{ij}$

We only check the last one:

$$(6.5.15) \qquad i(e_i)\,\varepsilon(e_j)\,(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots) = i(e_i)\, e_j \wedge e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots$$

$$= \delta_{ij}\, e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots - e_j \wedge \{i(e_i)(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots)\}$$

$$= \delta_{ij}\, e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots - \varepsilon(e_j)i(e_i)\,(e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge \ldots)$$

which indeed proves (6.5.14c).

Mathematically the importance of the operators $i(e_i)$ and $\varepsilon(e_j)$ is that the action of the basis elements of $sl(\infty)$ can be expressed as a product of these operators; we clearly have

$$(6.5.16) \qquad \sigma_k(E_{i,j}) = \varepsilon(e_i)\, i(e_j) \qquad\qquad \forall i, j, k \in \mathbb{Z}$$

In physical terminology the $i(e_i)$'s and $\varepsilon(e_j)$'s are creation and annihilation operators for fermions and the vacua v_k are called "Dirac seas".

Next we turn to the action of the completion A_∞ on $\wedge_k^\infty(\mathbb{C}^\infty)$. First we will introduce some notation; recall the isomorphisms

$$(6.5.17) \qquad \phi : g'(A_\infty) \to sl(\infty)$$

$$\bar{\phi} : \overline{g'(A_\infty)} \to \overline{gl(\infty)} \oplus \mathbb{C}c$$

and let ψ be the isomorphism (6.5.10) from $\wedge_k^\infty(\mathbb{C}^\infty)$ to $L(\Lambda_k)$. Denote by π_k the action of $g'(A_\infty)$ on $L(\Lambda_k)$, by $\bar{\pi}_k$ the action of its completion $\overline{g'(A_\infty)}$ on $L(\Lambda_k)$ and by $\bar{\sigma}_k$ the action of A_∞ on $\wedge_k^\infty(\mathbb{C}^\infty)$. We then have

$$(6.5.18)\qquad \sigma_k(x) = \psi^{-1} \circ \pi_k(\phi^{-1}(x)) \circ \psi \qquad \forall x \in sl(\infty)$$

$$\bar{\sigma}_k(x) = \psi^{-1} \circ \bar{\pi}_k(\bar{\phi}^{-1}(x)) \circ \psi \qquad \forall x \in A_\infty = \overline{gl(\infty)} \oplus \mathbb{C}c$$

With these formulas, we can calculate the action of a basis $\{E_{i,j}, c\}$ for A_∞. For c we have by (6.2.23):

$$\begin{aligned}(6.5.19)\qquad \bar{\sigma}_k(c) &= \psi^{-1} \circ \bar{\pi}_k(\bar{\phi}^{-1}(c)) \circ \psi \\ &= \psi^{-1} \circ \bar{\pi}_k(\sum_{j \in \mathbb{Z}} \alpha_j^\vee) \circ \psi \\ &= \psi^{-1} \circ I_{L(\Lambda_k)} \circ \psi \\ &= I_{\wedge_k^\infty(\mathbb{C}^\infty)}\end{aligned}$$

since $\langle \Lambda_k, \alpha_j^\vee \rangle = 1$ for all k.

If $i \neq j$ we have $\bar{\phi}^{-1}(E_{i,j}) = \phi^{-1}(E_{i,j})$ and hence

$$\begin{aligned}(6.5.20)\qquad \bar{\sigma}_k(E_{i,j}) &= \psi^{-1} \circ \bar{\pi}_k(\bar{\phi}^{-1}(E_{i,j})) \circ \psi \\ &= \psi^{-1} \circ \pi_k(\phi^{-1}(E_{i,j})) \circ \psi \\ &= \sigma_k(E_{i,j})\end{aligned}$$

So the representation $\bar{\sigma}_k$ coincides with the natural representation σ_k for off diagonal matrices. Finally, we calculate $\bar{\sigma}_k$ for diagonal matrices. By (6.2.25) we have for $i>0$

$$\begin{aligned}(6.5.21)\qquad \bar{\sigma}_k(E_{i,i}) &= \psi^{-1} \circ \bar{\pi}_k(\bar{\phi}^{-1}(E_{i,i})) \circ \psi \\ &= \sum_{j \geq i} \psi^{-1} \circ \pi_k(\bar{\phi}^{-1}(E_{j,j} - E_{j+1,j+1})) \circ \psi \\ &= \sum_{j \geq i} \psi^{-1} \circ \pi_k(\phi^{-1}(E_{j,j} - E_{j+1,j+1})) \circ \psi \\ &= \sum_{j \geq i} \sigma_k(E_{j,j} - E_{j+1,j+1}))\end{aligned}$$

$$= \tau_k(E_{i,i})$$

while for $i \le 0$

(6.5.22) $$\bar{\sigma}_k(E_{i,i}) = \psi^{-1} \circ \bar{\pi}_k(\bar{\phi}^{-1}(E_{i,i})) \circ \psi$$

$$= \sum_{j \ge i} \psi^{-1} \circ \pi_k(\bar{\phi}^{-1}(E_{j,j} - E_{j+1,j+1} + \delta_{i0}c) - c_2) \circ \psi$$

$$= \sum_{j \ge i} \psi^{-1} \circ \pi_k(\phi^{-1}(E_{j,j} - E_{j+1,j+1})) \circ \psi - I_{\wedge_k^{\infty}(\mathbb{C}^{\infty})}$$

$$= \sum_{j \ge i} \sigma_k(E_{j,j} - E_{j+1,j+1})) - I_{\wedge_k^{\infty}(\mathbb{C}^{\infty})}$$

$$= \tau_k(E_{i,i}) - I_{\wedge_k^{\infty}(\mathbb{C}^{\infty})}$$

We can express the representation $\bar{\sigma}_k$ in terms of fermionic creation and annihilation operators; define normal ordering by

(6.5.23) $$: \varepsilon(e_i)\, i(e_j) : := \begin{cases} \varepsilon(e_i)\, i(e_j) & \text{if } i>0 \\ -\, i(e_j)\, \varepsilon(e_i) & \text{if } i \le 0 \end{cases}$$

Note that, if $i \neq j$, the anti commutation relations (6.5.14) tell us that the normal ordering operation has no effect. For $i = j$ we get:

(6.5.24) $$: \varepsilon(e_i)\, i(e_i) : = \begin{cases} \varepsilon(e_i)\, i(e_i) & \text{if } i>0 \\ \varepsilon(e_i)\, i(e_i) - I & \text{if } i \le 0 \end{cases}$$

This means that

(6.5.25) $$\bar{\sigma}_k(E_{i,j}) = : \varepsilon(e_i)\, i(e_j) :$$

So the effect of completing $g'(A_\infty)$ to $\overline{g'(A_\infty)}$ is incorporated by performing the normal ordering operation on the creation and annihilation operators.

6.6 the bosonic realization of $L(\Lambda_k)$ over A_∞

In this section we will construct a bosonic realization of $L(\Lambda_k)$ over A_∞ analogous to the construction in section 5.5 of $L(\Lambda_0)$ over $A_1^{(1)}$. We first introduce a Heisenberg subalgebra in A_∞. Define a matrix E in $\overline{g'(A_\infty)}$ by

$$(6.6.1) \qquad E = \bar{\phi}(\sum_{i\in\mathbb{Z}} e_i) = \sum_{i\in\mathbb{Z}} E_{i,i+1}$$

This matrix is invertible; its inverse is

$$(6.6.2) \qquad E^{-1} = \sum_{i\in\mathbb{Z}} E_{i+1,i} = \bar{\phi}(\sum_{i\in\mathbb{Z}} f_i)$$

In fact one easily checks that

$$(6.6.3) \qquad E^k = \sum_{i\in\mathbb{Z}} E_{i,i+k} \qquad \forall k\in\mathbb{Z}$$

Now define

$$(6.6.4) \qquad p_k := E^k, \qquad q_k := \frac{1}{k}E^{-k} \qquad k>0$$

If we consider these elements as elements of the central extension $A_\infty = \overline{gl(\infty)} \oplus \mathbb{C}c$, we have

$$(6.6.5) \qquad [p_k, q_j] = \frac{1}{j}\sum_{m,n\in\mathbb{Z}} [E_{m,m+k}, E_{n+j,n}]$$

$$= \frac{1}{j}\sum_{m,n\in\mathbb{Z}} \{\delta_{m+k,n+j} E_{m,n} - \delta_{m,n} E_{n+j,m+k}$$

$$+ \Psi(E_{m,m+k}, E_{n+j,n})\, c\}$$

$$= \frac{1}{j}\sum_{m,n\in\mathbb{Z}} \Psi(E_{m,m+k}, E_{n+j,n})\, c$$

$$= \frac{1}{j}\delta_{kj}\sum_{m\in\mathbb{Z}} \Psi(E_{m,m+k}, E_{m+k,m})\, c$$

$$= \frac{1}{j}\delta_{kj}\sum_{m=-k+1}^{0} \Psi(E_{m,m+k}, E_{m+k,m})\, c$$

$$= \delta_{kj}\, c$$

This means that the subalgebra

(6.6.6) $$\underline{s} := \bigoplus_{k\in\mathbb{Z}} \mathbb{C}p_k \bigoplus_{k\in\mathbb{Z}} \mathbb{C}q_k \oplus \mathbb{C}c$$

is a Heisenberg subalgebra of A_∞. It is easy to verify that the intersection $\underline{s}\cap A_1^{(1)}$ coincides with the principal Heisenberg subalgebra $\hat{\underline{s}}'$ introduced in (5.5.1). Therefore, $\underline{s}$ is called the principal Heisenberg subalgebra of A_∞.

One can prove that the fundamental modules $L(\Lambda_k)$ remain irreducible under the action of $\underline{s}$ and therefore we can identify

(6.6.7) $$L(\Lambda_k) \cong \mathbb{C}[x_i, i>0]$$

The elements of the principal Heisenberg subalgebra act by the assignments

(6.6.8) $$p_k \to \frac{\partial}{\partial x_k}, \quad q_k \to x_k, \quad c \to I_{\mathbb{C}[x_i, i>0]}$$

In order to find the action of the rest of the algebra A_∞, we introduce the generating matrix

(6.6.9) $$\hat{A}(u,v) := \sum_{i,j\in\mathbb{Z}} u^i v^{-j} E_{i,j}$$

(u and v are formal parameters comparable to the formal parameter ζ in (5.4.3)) and calculate the commutators of $\hat{A}(u,v)$ with the elements of the principal Heisenberg subalgebra;

(6.6.10) $$[p_k, \hat{A}(u,v)] = [\sum_{m\in\mathbb{Z}} E_{m,m+k}, \sum_{i,j\in\mathbb{Z}} u^i v^{-j} E_{i,j}]$$

$$= \sum_{i,j,m\in\mathbb{Z}} u^i v^{-j} \{\delta_{i,m+k} E_{m,j} - \delta_{jm} E_{i,m+k} + \delta_{mj}\, \delta_{m+k,i}\, \Psi(E_{j,i}, E_{i,j})\, c\}$$

$$= \sum_{i,j\in\mathbb{Z}} u^i v^{-j} \{E_{i-k,j} - E_{i,j+k} + \delta_{i,j+k}\, \Psi(E_{j,i}, E_{i,j})\, c\}$$

$$= u^k \sum_{i,j\in\mathbb{Z}} u^i v^{-j} E_{i,j} - v^k \sum_{i,j\in\mathbb{Z}} u^i v^{-j} E_{i,j}$$

$$+ \sum_{i,j\in\mathbb{Z}} u^{j+k} v^{-j} \Psi(E_{j,j+k}, E_{j+k,j})\, c$$

$$= (u^k - v^k)\, \hat{A}(u,v) + u^k \sum_{j=-k+1}^{0} \left(\frac{u}{v}\right)^j c$$

$$= (u^k - v^k)\, [\hat{A}(u,v) + \frac{u}{u-v} c]$$

where the notation $\frac{u}{u-v}$ stands for the formal power series $\sum_{k\geq 0} \left(\frac{v}{u}\right)^k$. With this result it is natural to define:

$$\text{(6.6.11)} \qquad A(u,v) := \hat{A}(u,v) + \frac{u}{u-v} c$$

Then A(u,v) satisfies

$$\text{(6.6.12)} \qquad [p_k \,,\, A(u,v)] = (u^k - v^k)\, A(u,v)$$

Similarly one calculates

$$\text{(6.6.13)} \qquad [q_k \,,\, A(u,v)] = \frac{1}{k}(u^{-k} - v^{-k})\, A(u,v)$$

As in chapter 5 we see that the action of A(u,v) on $L(\Lambda_k) \cong \mathbb{C}[x_i, i>0]$ can be represented by the vertex operator

$$\text{(6.6.14)} \qquad \Gamma_k(u,v) = \sum_{i,j\in\mathbb{Z}} u^i v^{-j} \bar{\sigma}_k(E_{i,j}) + \frac{u}{u-v} \bar{\sigma}_k(c)$$

$$= c_k(u,v) \exp\{ \sum_{i>0} (u^i - v^i) x_i \} \exp\{ -\sum_{i>0} \frac{1}{i} (u^{-i} - v^{-i}) \frac{\partial}{\partial x_i} \}$$

where $c_k(u,v)$ is a formal power series, which can be calculated by acting with $\Gamma(u,v)$ on the highest weight vector v_k. After a small calculation one finds:

$$\text{(6.6.15)} \qquad c_k(u,v) = \left(\frac{u}{v}\right)^k \frac{u}{u-v}$$

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