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Proceedings Seminar 1981-1982 Mathematical structures in field theories

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PREFACE

During the last decades a revival of mutual interest and scientific co-operation between mathematicians and physicists has become more manifest.
As examples of this we mention the activities of the International Association of Mathematical Physics and of the European Society of Mathematical Physics, the "Rencontres entre Mathématiciens et Physiciens" in France and the program in Bielefeld, Germany.
One of the reasons of this more intensive contact between the two disciplines may lie in the circumstance, that on the one hand very sophisticated theories from many branches of mathematics are needed nowadays in theoretical physics, and on the other hand the overwhelming results in mathematical physics, substantiated by experiments, are a source of inspiration and motivation for many mathematicians.
Whatever may be the cause, also in the Netherlands there is growing a mutual interest between mathematiciens and physicists; as a result of this a national seminar "Mathematical Structures in Field Theories" has been started in 1981 at the University of Amsterdam.
The reader finds in this book the contents of the lectures given during the seminar of the academic year 1981-1982. The program of this first year seminar was mainly directed to differential geometry and gauge field theory; in order to give several participants a better understanding of theories in which they are not fully conversant, some of the lectures are of an introductory character.
This text will be followed at least by the proceedings of the seminars 1982-1983 and 1983-1984.
The organizers appreciate the contributions of the participants of the seminar, and in particular those of the lecturers, who made this text possible.
Also the fine job made by the secretary, Y.Voorn, is greatly acknowledged she had to type the complicated formulae, so beloved by mathematicians and physicists.
Finally, we thank the Centre for Mathematics and Computer Science for the publication of these proceedings.

October, 1984.

E.M. de Jager
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ELEMENTS OF QUANTUM THEORY AND FIELD THEORY

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INTRODUCTION

The theoretical description of processes in high-energy particle physics is based on quantum theory and special relativity. The amalgamation of these two is quantum field theory.

In a process like e^+ (positron) and e^- (electron) annihilation new particles may appear e.g.

$$e^+ + e^- \to \text{proton} + \text{anti-proton} + \pi\text{-mesons}$$
$$e^+ + e^- \to \text{fotons}.$$

To describe such a process one needs a formalism in which particles may be created and annihilated. The appropriate frame-work is the Fock-space formalism. Operator field $\underline{\underline{\Omega}}(\vec{x},t)$ appear quite naturally in this formalism.

Although the emphasis of the seminar will be on the classical solutions of Yang-Mills theories the ultimate goal might be the quantum theory. With this in mind we are going to give a brief survey of quantum theory and field theory.

§1. QUANTUM THEORY (*Schrödinger Picture*)

The two basic notions needed to describe the behaviour of a system are *state* and *observable*. We will give a summary of quantum theory by means of a number of postulates. Later on we will discuss as an example the harmonic oscillator.

P.1 States.

On a fixed (but arbitrary) time the quantum states of a system are represented by unit vectors in a separable Hilbert space $\mathcal{H}$. The vectors $\phi \in H$ and $e^{i\alpha}\phi$, $\alpha \in \mathbb{R}$ represent the same state.

COMMENT.

This Hilbert space description accounts for the superposition principle. If ϕ and χ are states then

$$\psi = \frac{\alpha\phi + \beta\chi}{\|\alpha\phi + \beta\chi\|}, \quad \alpha, \beta \in \mathbb{C} \quad \text{is again a state.}$$

The inner product of two vectors ϕ and χ is denoted by $<\phi,\chi>$. It satisfies $<\phi,\chi> = <\chi,\phi>^*$; $<\phi,\alpha\chi> = \alpha<\phi,\chi>$; $\alpha \in \mathbb{C}$ (complex conjugation is denoted by $*$).

P.2 Observables.

Observable quantities like the energy, the momentum etc. of a system are represented by hermitian operators on $\mathcal{H}$.

COMMENT.

In classical mechanics observable quantities like the position x and the momentum p of a particle depend on time $x = x(t)$, $p = p(t)$. In the Schrödinger picture observables are *time-independent* hermitian operators.

P.3 Measurements.

Let $\underline{\underline{\Omega}}$ be a hermitian operator corresponding to some observable quantity.

A measurement of the observable yields an eigenvalue of $\underline{\underline{\Omega}}$. We elaborate a bit on this point.

Let the eigenvalue equation of $\underline{\underline{\Omega}}$ be given by

(1.1) $$\underline{\underline{\Omega}}\phi_\omega = \omega\phi_\omega , \quad <\phi_\omega , \phi_{\omega'}> = \delta_{\omega\omega'} .$$

Let the system be in a state ϕ. Quantummechanics does not tell you which eigenvalue will be found in the measurement, it *does*, however, tell you the *probability* to find some specific eigenvalue ω'.

This probability is given by

(1.2) $$|<\phi_{\omega'} , \phi>|^2 .$$

The average of many measurements on the system is given by the expectation value

(1.3) $$E_\phi(\underline{\underline{\Omega}}) = \sum_\omega \omega |<\phi_\omega , \phi>|^2 = <\phi , \underline{\underline{\Omega}}\phi> .$$

Here we have assumed that the eigenvectors $\{\phi_\omega\}$ are a basis in H.

The complex number $<\phi_\omega , \phi>$ is called the *probability amplitude*.

Let $\underline{\underline{A}}$ be an operator on H. The complex numbers $(\underline{\underline{A}})_{\omega'\omega} = <\phi_{\omega'} , \underline{\underline{A}}\phi_\omega>$ are called the *matrixelements* of $\underline{\underline{A}}$ with respect to the basis $\{\phi_\omega\}$.

P.4 Evolution.

The operator $\underline{\underline{H}}$ representing the observable "energy" is the generator of the evolution in time. This is expressed by the Schrödinger equation.

(1.4) $$i\hbar \frac{d}{dt}\phi = \underline{\underline{H}}\phi ; \quad \phi_{t+\varepsilon} = [\mathbb{1} - \frac{i\underline{\underline{H}}}{\hbar}\varepsilon + O(\varepsilon^2)]\phi_t$$

h is Planck's constant and $\hbar = h/2\pi$

(1.5) $$\phi(t) = \{\exp - \frac{i\underline{\underline{H}}}{\hbar} \cdot (t-t_0) = \underline{\underline{U}}(t,t_0)\phi(t_0)$$

COROLLARY.

The expectation value is in general a function of t. The time-dependence is governed by the equation

(1.6) $$i\hbar \frac{d}{dt} < \phi, \underline{\underline{\Omega}}\phi > \; = \; < \phi, [\underline{\underline{\Omega}}, \underline{\underline{H}}]\, \phi >$$

$[\underline{\underline{\Omega}}, \underline{\underline{H}}] = \underline{\underline{\Omega}}\,\underline{\underline{H}} - \underline{\underline{H}}\,\underline{\underline{\Omega}}$ is the commutator of $\underline{\underline{\Omega}}$ and $\underline{\underline{H}}$.

The observable corresponding to $\underline{\underline{\Omega}}$ is a constant of the motion if $[\underline{\underline{\Omega}}, \underline{\underline{H}}] = 0$. This stresses the importance of the *Hamiltonian* $\underline{\underline{H}}$.

Taking $\underline{\underline{\Omega}} = \underline{\underline{\ }}$ it follows that $< \phi, \chi >$ and $\| \phi \|$ are time-independent quantities.

P.5 Symmetries.

Symmetries of a quantumsystem are implemented by either unitary or anti-unitary operators on $\mathcal{H}$.

EXAMPLE. Rotational invariance. Suppose a system is invariant under rotations in space $(\vec{x}' = R\vec{x})$. On $\mathcal{H}$ the rotation is represented by a unitary operator $\underline{\underline{U}}(R)$ which commutes with the hamiltonian operator $\underline{\underline{H}}$. Considering infinitesimal rotations $\underline{\underline{U}}(R_{inf}) = \mathbb{1} - i\varepsilon\, \vec{n}\cdot\vec{\underline{\underline{J}}} + O(\varepsilon^2)$ one obtains $[\underline{\underline{H}}, \vec{\underline{\underline{J}}}_i] = 0$ (i=1,2,3). The hermitian operators $\vec{\underline{\underline{J}}}$ are the angular momentum operators. For a rotational invariant system the angular momentum operators are conserved quantities (constants of the motion). This should show the importance of symmetry groups. They lead to conserved quantities.

Time reversal invariance is represented by an anti-unitary operator $\underline{\underline{T}}$. $\underline{\underline{T}}$ has the following properties

$$\underline{\underline{T}}(\alpha\phi + \beta\chi) = \alpha^* \underline{\underline{T}}\,\phi + \beta^* \underline{\underline{T}}\,\chi$$
$$(T\underline{\underline{\phi}}, T\chi) = (\phi, \chi)^*$$

α^* is the complex conjugate of α.

Stationary states; Perturbation theory.

Eigenvectors ϕ_E of the hamiltonian $\underline{\underline{H}}$ are called stationary states.

(1.7) $$\underline{\underline{H}}\,\phi_E = E\,\phi_E\,, \qquad <\phi_{E'}, \phi_E> = \delta_{E'E}\,.$$

The observed quantization of energy (e.g. energy levels in the hydrogen atom) is accounted for by the discrete eigenvalues of $\underline{\underline{H}}$. The time-dependence of a stationary state ϕ_E is given by the phase-factor $\exp\frac{-iE}{\hbar}t$.
In practice it is in general not possible to solve the eigenvalue equation. There are, however, many cases in which the Hamiltonian can be split into two parts

$$\underline{\underline{H}} = \underline{\underline{H}}^{(0)} + \lambda\underline{\underline{H}}^{(1)} \qquad (\lambda \in \mathbb{R})$$

in such a way that the eigenvalue equation for $\underline{\underline{H}}^{(0)}$ can be solved, while $\lambda\underline{\underline{H}}'$ can be considered as being "small" compared to $\underline{\underline{H}}^{(0)}$. In such cases one uses *perturbation theory* to obtain approximate eigenvalues of $\underline{\underline{H}}$. We will not go into the details of this so-called stationary state perturbation theory. We give only one result, the first order correction to an eigenvalue of $\underline{\underline{H}}^{(0)}$.
Let the eigenvalue equation for $\underline{\underline{H}}^{(0)}$ be written as

(1.8) $$\underline{\underline{H}}^{(0)}\phi_{E_n^{(0)}} = E_n^{(0)}\phi_{E_n^{(0)}}\,.$$

The first order correction to $E_n^{(0)}$ is then given by

(1.9) $$E_n^{(1)} = <\phi_{E_n^{(0)}}, \lambda\underline{\underline{H}}^{(1)}\phi_{E_n^{(0)}}>$$

i.e. by the expectation value of the perturbation $\lambda\underline{\underline{H}}^{(1)}$ in the state $\phi_{E_n^{(0)}}$.

§2. EQUIVALENT DESCRIPTIONS, HEISENBERG PICTURE, INTERACTION PICTURE.

In §1 we have discussed the Schrödinger picture of quantum mechanics. The essential point is the time-dependence of the statevectors. Observables are represented by time-independent hermitian operators. In the Heisenberg picture, which we are going to discuss now, the state vectors are independent of time, the operators do depend on time. The Heisenberg picture is closer to classical mechanics.

DEFINITION.

Let $\phi(t)$ be a state vector obeying equation (1.4). The state vector in the Heisenberg picture is defined as

$$(2.1) \qquad \phi_H = \left(\exp + \frac{i\underline{\underline{H}}}{\hbar}(t-t_0)\right) \phi(t) = \underline{\underline{U}}^\dagger(t,t_0)\phi(t).$$

The subscript H refers to Heisenberg.

Comparing (2.1) and (1.5) one sees immediately that ϕ_H is independent of t, $\phi_H = \phi(t_0)$.

DEFINITION.

Let $\underline{\underline{\Omega}}$ be an operator in the Schrödinger picture. The operator $\underline{\underline{\Omega}}_H$ is defined as

$$(2.2) \qquad \underline{\underline{\Omega}}_H(t) = \underline{\underline{U}}^\dagger(t,t_0)\underline{\underline{\Omega}}\,\underline{\underline{U}}(t,t_0) .$$

The evolution of the system is now governed by the equation of motion for $\underline{\underline{\Omega}}_H$

$$(2.3) \qquad i\hbar\frac{d}{dt}\underline{\underline{\Omega}}_H(t) = \left[\underline{\underline{\Omega}}_H(t), \underline{\underline{H}}\right]$$

If $[\underline{\underline{\Omega}}, \underline{\underline{H}}] = 0$ then $\underline{\underline{\Omega}}_H$ is independent of time.

Equation (2.3) resembles very much the classical equation of motion. For a simple system with hamiltonian function H(p,x) (p momentum, x position) one has

(2.4)
$$\begin{cases} \dfrac{dx}{dt} = \{x, H\}_{x,p} = \dfrac{\partial x}{\partial x}\dfrac{\partial H}{\partial p} - \dfrac{\partial x}{\partial p}\dfrac{\partial H}{\partial x} = \dfrac{\partial H}{\partial p} \\[2ex] \dfrac{dp}{dt} = \{p, H\}_{x,p} = \dfrac{\partial p}{\partial x}\dfrac{\partial H}{\partial p} - \dfrac{\partial p}{\partial p}\dfrac{\partial H}{\partial x} = -\dfrac{\partial H}{\partial x} \end{cases}$$

$\{f, g\}_{x,p} := \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial p} - \dfrac{\partial f}{\partial p}\dfrac{\partial g}{\partial x}$ is the Poisson bracket.

In quantum mechanics $\underline{\underline{H}} = H(\underline{\underline{p}},\underline{\underline{x}})$ is an operator, and

$$\frac{d}{dt}\underline{\underline{x}}_H = \frac{1}{i\hbar}\left[\underline{\underline{x}}_H, \underline{\underline{H}}\right].$$

It is very important to take note of the fact that quantities like expectation values and probability amplitudes are independent of the picture one uses, i.e.

(2.5) $$<\phi_H, \chi_H> = <\phi, \chi>$$

(2.6) $$E_\phi(\underline{\underline{\Omega}}) = <\phi, \underline{\underline{\Omega}}\phi> = <\phi_H, \underline{\underline{\Omega}}_H\,\phi_H>.$$

We now discuss a picture which is intermediate between the Schrödinger picture and the Heisenberg picture. It is called the interaction picture or the Dirac picture. The starting point here is a Hamiltonian $\underline{\underline{H}}$ which is splitted into two parts

(2.7) $$\underline{\underline{H}} = \underline{\underline{H}}^{(0)} + \underline{\underline{H}}^{(1)}.$$

<u>DEFINITION</u>.

Let $\phi(t)$ be a state vector in the Schrödinger picture, the corresponding state vector $\phi_I(t)$ in the interaction picture is given by

(2.8) $$\phi_I(t) = (\exp \frac{i}{\hbar}\underline{\underline{H}}^{(0)}\cdot(t-t_0))\,\phi(t) = \underline{\underline{U}}^{(0)\dagger}(t,t_0)\phi(t).$$

Operators $\underline{\underline{\Omega}}_I(t)$ are defined by

(2.9) $$\underline{\underline{\Omega}}_I(t) = \left[\exp\frac{i\underline{\underline{H}}^{(0)}}{\hbar}(t-t_0)\right]\underline{\underline{\Omega}}\left[\exp\frac{-i}{\hbar}\underline{\underline{H}}^{(0)}(t-t_0)\right].$$

In the interaction picture both the state vectors and the operators depend on time. Investigating the time dependence of ϕ_I one finds that it is due to the interaction $\underline{\underline{H}}_I^{(1)}$.

$$(2.10) \qquad i\hbar \frac{d}{dt}\phi_I(t) = \underline{\underline{H}}_I^{(1)}(t)\ \phi_I(t);\ \underline{\underline{H}}_I^{(1)}(t) = \underline{\underline{U}}^{(0)^\dagger}(t,t_0)\cdot\underline{\underline{H}}^{(1)}\underline{\underline{U}}^{(0)}(t,t_0)$$

One might say that the evolution due to the "free" Hamiltonian $\underline{\underline{H}}^{(0)}$ is eliminated from $\phi(t)$.

For the operators one obtains

$$(2.11) \qquad i\hbar \frac{d}{dt}\underline{\underline{\Omega}}_I(t) = \left[\underline{\underline{\Omega}}_I(t),\ \underline{\underline{H}}^{(0)}\right];\quad H^{(0)} = \underline{\underline{H}}_I^{(0)}\ .$$

The time dependence of $\underline{\underline{\Omega}}_I$ is completely determined by the "free" Hamiltonian $\underline{\underline{H}}^{(0)}$.

I have used the words free Hamiltonian and interaction Hamiltonian to smooth the way to field theory. In field theory $\underline{\underline{H}}^{(0)}$ is the sum of the Hamiltonians for free fields, say the electronfield $\underline{\psi}(\vec{x},t)$ and the electronmagnetic field $\underline{\underline{A}}_\mu(\vec{x},t)$. $\underline{\underline{H}}^{(1)}$ contains the interaction, i.e. the coupling of the fields.

The interaction picture is the basis for perturbative calculations in scattering processes. To explain this we introduce the so-called evolution operator $\underline{\underline{U}}_I(t,t_1)$. The evolution of ϕ_I (2.10) can be considered as a succession of unitary transformations.

$$(2.12) \qquad \begin{aligned} \phi_I(t) &= \underline{\underline{U}}_I(t,t_1)\phi_I(t_1) \\ \underline{\underline{U}}_I(t,t) &= \mathbb{1}\ . \end{aligned}$$

From (2.10) and (2.12) one obtains

$$(2.13) \qquad i\hbar \frac{d}{dt}\underline{\underline{U}}_I(t,t_1) = \underline{\underline{H}}_I^{(1)}(t)\underline{\underline{U}}(t,t_1);\quad \underline{\underline{U}}_I(t,t) = \mathbb{1}\ .$$

The differential equation (2.13) and the equal-time condition can be put in an integral equation for $\underline{\underline{U}}_I$.

(2.14) $$\underline{\underline{U}}_I(t,t_1) = \mathbb{1} + \frac{1}{i\hbar} \int_{t_1}^{t} d\tau\, \underline{\underline{H}}_I^{(1)}(\tau)\; \underline{\underline{U}}_I(\tau,t_1).$$

This equation is the starting point for a perturbation expansion of $\underline{\underline{U}}_I$. Solving (2.14) by iteration one obtains

$$\underline{\underline{U}}_I(t,t_1) = + \sum_{n=1}^{\infty} \frac{1}{(i\hbar)^n} \int_{t_1}^{t} d\tau_1 \int_{t_1}^{\tau_1} d\tau_2 \ldots \int_{t_1}^{\tau_{n-1}} d\tau_n\; \underline{\underline{H}}_I^{(1)}(\tau_1)\underline{\underline{H}}_I^{(1)}(\tau_2) \ldots \underline{\underline{H}}_I^{(1)}(\tau_n) =$$

(2.15) $$= + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{(i\hbar)^n} \int_{t_1}^{t} d\tau_1 \int_{t_1}^{t} d\tau_2 \ldots \int_{t_1}^{t} d\tau_n \cdot T(\underline{\underline{H}}_I^{(1)}(\tau_1) \ldots \underline{\underline{H}}_I^{(1)}(\tau_n))$$

$$= T\exp \frac{1}{i\hbar} \int_{t_1}^{t} \underline{\underline{H}}_I^{(1)}(\tau)\,d\tau .$$

Capital T is Dyson's time ordering prescription.

(2.16) $$T[\underline{\underline{\Omega}}(\tau_1)\underline{\underline{\Omega}}(\tau_2) \ldots \cdot \underline{\underline{\Omega}}(\tau_n)] = \underline{\underline{\Omega}}(\tau_{\pi(1)})\cdot \Omega(\tau_{\pi(2)})\ldots\Omega(\tau_{\pi(n)})$$

for $\tau_{\pi(1)} > \tau_{\pi(2)} \ldots > \tau_{\pi(n)}$

with $\pi(1),\ \pi(2)\ldots,\ \pi(n)$ a permutation of $1,2,\ldots n$.

Taking $t \to +\infty$ and $t_1 \to -\infty$ one obtains the scattering operator $\underline{\underline{S}}$

(2.17) $$\underline{\underline{S}} = T\exp \frac{1}{i\hbar} \int_{-\infty}^{\infty} \underline{\underline{H}}_I^{(1)}(t)\,dt \qquad \text{(Dyson)}$$

Almost all results in quantumfield theory are obtained from this formula. It must be said, however, that a lot of problems, such as divergent Feyman integrals originate from that fact that one uses Dyson's formula for singular objects like operator field.

§3. THE HARMONIC OSCILLATOR

We consider a one-dimensional harmonic oscillator to illustrate some of the foregoing ideas. We have chosen the oscillator not only because the eigenvalue problem for the Hamiltonian can be solved exactly but also because of its importance in quantumfield theory. The Hamiltonian of a free quantumfield theory is in fact an infinite system of free harmonic oscillators.

The classical Hamilton function for a harmonic oscillator reads

$$H(p,x) = \frac{p^2}{2m} + \tfrac{1}{2}m\omega^2 x^2 = \text{Kin.energy} + \text{potential energy}. \tag{3.1}$$

The system is quantized by considering p and x as time-independent hermitian operators satisfying

$$[\underline{\underline{p}},\underline{\underline{x}}] = \frac{\hbar}{i}\mathbb{1}. \tag{3.2}$$

The Hamiltonian is then given by

$$\underline{\underline{H}} = \frac{\underline{\underline{p}}^2}{2m} + \tfrac{1}{2}m\omega^2 \underline{\underline{x}}^2 . \tag{3.3}$$

The realization of (3.2) in $\mathcal{H} = L_2(x)$ is usually called the coordinate-representation. The momentum operator is then given by $\underline{\underline{p}} = \frac{\hbar}{i}\frac{d}{dx}$; $\underline{\underline{x}}$ is a multiplication operator $\underline{\underline{x}}\phi(x) = x\phi(x)$. Another possibility is $\mathcal{H} = L_2(p)$, this is the momentum representation. The position operator is then given by $\underline{\underline{x}} = -\frac{\hbar}{i}\frac{d}{dp}$ and the momentum operator $\underline{\underline{p}} = p\mathbb{1}$, i.e. a multiplication operator. The eigenvalue problem for the Hamiltonian reads (in the coordinate-representation)

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \tfrac{1}{2}m\omega^2 x^2\right)\phi_E(x) = E\phi_E(x) . \tag{3.4}$$

The solutions are

$$E_n = (n+\tfrac{1}{2})h\omega \qquad (n=0,1,2,\ldots) \tag{3.5}$$

$$(3.6) \qquad \phi_n(x) = N_n \exp - \frac{m\omega x^2}{2\hbar} \cdot H_n\left(\sqrt{\frac{m\omega}{\hbar}}\, x\right).$$

The functions $\{H_n\}_{n=0}^{\infty}$ are Hermite polynomials, N_n's are normalisation constants.

The functions $\{\phi_n(x)\}_{n=0}^{\infty}$ form a complete orthonormal basis in $L_2(x)$.

An arbitrary state $\phi(x)$ of the oscillator can be expanded in these basis states. The "wave function" $\phi(x)$ is the probability amplitude density for the position of the particle. The probability to find the particle in a region $[a,b]$ is given by

$$(3.7) \qquad P_\phi(x \in [a,b]) = \int_a^b |\phi(x)|^2 \, dx.$$

Measuring the energy of the system in the state $\phi(x)$ yields an eigenvalue E_n of $\underline{\underline{H}}$. The probability for obtaining E_n is given by

$$(3.8) \qquad P_\phi(E_n) = |<\phi_n, \phi>|^2 = \left| \int_{-\infty}^{\infty} dx\, \phi_n^*(x)\phi(x) \right|^2$$

and the expectation value is

$$(3.9) \qquad E_\phi(\underline{\underline{H}}) = \sum_n E_n |<\phi_n, \phi>|^2 = <\phi, \underline{\underline{H}}\phi>.$$

The expectation value for the position operator $\underline{\underline{x}}$ is given by

$$(3.10) \qquad E_\phi(\underline{\underline{x}}) = \int_{-\infty}^{\infty} \phi^*(x) x \phi(x) = <\phi, \underline{\underline{x}}\phi>$$

The foregoing considerations can be given also in the momentum representation.

The wave function $\tilde{\phi}(p)$ is related to $\phi(x)$ by a Fourier transformation

$$(3.11) \qquad \phi(x) = \frac{1}{(2\pi\hbar)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \tilde{\phi}(p) e^{\frac{ipx}{\hbar}} \, dp.$$

Quantities like (3.8), (3.9) and (3.10) are independent of the representation. A well-known shortcoming of the Hilbert space formalism is the fact that the eigenvalue problem for the position operator and the momentum operator cannot be solved in Hilbert space. For the position operator one would have

(3.12) $\underline{\underline{x}}\phi_{x'}(x) = x\phi_{x'}(x) = x'\phi_{x'}(x)$ (x' an eigenvalue)

while the momentum operator would lead to

(3.13) $\frac{\hbar}{i}\frac{d}{dx}\phi_p(x) = p\phi_p(x).$

A solution of (3.12) is given by $\phi_{x'}(x) = c\delta(x-x')$, while (3.13) is volved by $\phi_p(x) = c \exp ipx/\hbar$. If one considers these matters in a light-hearted way one is on the verge of some nice discoveries. The interpretation of $\phi(x)$ as a probability amplitude for the position, if the system is in the abstract state ϕ, can then be considered as a generalisation of a quantity like $<\phi_n, \phi>$ which is merely the component of ϕ along the basisvector ϕ_n. $\phi(x)$ is the component of ϕ along the basisvector ϕ_x. In the same spirit $\tilde{\phi}(p)$ is the component of ϕ along the basisvector ϕ_p. All this can be put on a solid base by considering instead of the Hilbert space the so-called Rigged-Hilbert space, which is also called Gelfand-Triple. An example of a Rigged-Hilbert space is given by the triple of spaces

(3.14) $S(\mathbb{R}) \subset L_2(\mathbb{R}) \subset S^*(\mathbb{R})$

where $S(\mathbb{R})$ is the Schwartz-space of C^∞ functions of fast decrease. Eigenvectors of $\underline{\underline{x}}$ and $\underline{\underline{p}}$ are called generalized eigenvectors.

§4. THE HARMONIC OSCILLATOR AGAIN

The eigenvalue problem for the harmonic oscillator can be solved by purely algebraic methods. We discuss briefly how this is done.

The Hamiltonian (3.3) is written in the form

(4.1) $$\underline{\underline{H}} = \tfrac{1}{2}(\underline{\underline{P}}^2+\underline{\underline{Q}}^2)\hbar\omega$$

with

(4.2) $$\underline{\underline{P}} = \frac{p}{\sqrt{m\hbar\omega}} \quad \text{and} \quad \underline{\underline{Q}} = \sqrt{\frac{m\omega}{\hbar}}\,\underline{\underline{x}}$$

(4.3) $$[\underline{\underline{P}}, \underline{\underline{Q}}] = \frac{1}{i}\mathbb{1}.$$

The Hamiltonian is essentially $\underline{\underline{P}}^2 + \underline{\underline{Q}}^2$ and can be written as

(4.4) $$\underline{\underline{H}} = \left\{ \frac{(\underline{\underline{Q}} - i\underline{\underline{P}})}{\sqrt{2}} \cdot \frac{(\underline{\underline{Q}} + i\underline{\underline{P}})}{\sqrt{2}} + \tfrac{1}{2}\mathbb{1} \right\} \hbar\omega$$

(4.5) $$= (\underline{\underline{a}}^{\dagger}\underline{\underline{a}}+\tfrac{1}{2})\hbar\omega = (\underline{\underline{N}}+\tfrac{1}{2})\hbar\omega.$$

The operators $\underline{\underline{a}}$ and $\underline{\underline{a}}^{\dagger}$ satisfy the commutation relations

(4.6) $$[\underline{\underline{a}}, \underline{\underline{a}}^{\dagger}] = \quad ; \quad \underline{\underline{a}} = \frac{\underline{\underline{Q}} + i\underline{\underline{P}}}{\sqrt{2}},\ \underline{\underline{a}}^{\dagger} = \frac{Q - i\underline{\underline{P}}}{\sqrt{2}}$$

(4.7) $$[\underline{\underline{H}}, \underline{\underline{a}}] = -\hbar\omega\underline{\underline{a}}$$

(4.8) $$[\underline{\underline{H}}, a^{\dagger}] = +\hbar\omega\underline{\underline{a}}^{\dagger}.$$

The operator $\underline{\underline{a}}^{\dagger}$ is called a creation operator, $\underline{\underline{a}}$ is an annihilation operator. To explain these names we consider an eigenvector ϕ_E of $\underline{\underline{H}}$.

$$\underline{\underline{H}}\phi_E = E\phi_E.$$

Acting on ϕ_E with $\underline{\underline{a}}^{\dagger}$ and using (4.8) one obtains

(4.9) $$\underline{\underline{H}}(\underline{\underline{a}}^{\dagger}\phi_E) = (E+\hbar\omega)(\underline{\underline{a}}^{\dagger}\phi_E).$$

So $\underline{\underline{a}}^{\dagger}\phi_E$ is again an eigenvector of $\underline{\underline{H}}$, the eigenvalue is raised by the quantum $\hbar\omega$.

For $\underline{\underline{a}}$ one finds

(4.10) $$\underline{\underline{H}}(\underline{\underline{a}}\phi_E) = (E-\hbar\omega)\underline{\underline{a}}\phi_E$$

The operator $\underline{\underline{a}}$ annihilates a quantum $\hbar\omega$ from the state ϕ_E and the state is mapped on $\underline{\underline{a}}\phi_E$.

Starting from an eigenvector ϕ_E one can construct an infinite chain of eigenvectors. This chain has a lower-bound. This is due to the fact that $\underline{\underline{H}}$ is a non-negative operator.

$$<\phi, \underline{\underline{H}}\phi> = <\underline{\underline{a}}\phi, a\phi> + \tfrac{1}{2}<\phi, \phi> = \| \underline{\underline{a}}\,\phi\|^2 + \tfrac{1}{2}\| \phi \|^2 > 0 .$$

From (4.5) one sees that a state ϕ_0 with

(4.11) $$\underline{\underline{a}}\phi_0 = 0$$

is an eigenvector of $\underline{\underline{H}}$

(4.12) $$\underline{\underline{H}}\phi_0 = \tfrac{1}{2}\hbar\omega\phi_0$$

ϕ_0 is the ground state of the oscillator. Excited states are obtained from ϕ_0 by the action of $\underline{\underline{a}}^{\dagger}$.

Defining

(4.13) $$\phi_n := \frac{1}{\sqrt{n!}}(\underline{\underline{a}}^{\dagger})^n\phi_0 \qquad (n=0,1,2,\ldots)$$

one has $\underline{\underline{H}}\phi_n = (n+\tfrac{1}{2})\hbar\omega\phi_n$

(4.14) $$<\phi_n, \phi_{n'}> = \delta_{nn'} .$$

The states ϕ_n have important properties

(4.15) $$\underline{\underline{a}}^{\dagger}\phi_n = \sqrt{n+1}\;\phi_{n+1}$$

(4.16) $$\underline{\underline{a}}\phi_n = \sqrt{n}\ \phi_{n-1}.$$

The relationship between these results and the previous section is most easily obtained from (4.11). Using (4.2) one obtains

(4.17) $$\left(\sqrt{\frac{m\omega}{\hbar}}\,\underline{\underline{x}} + i\frac{1}{\sqrt{m\hbar\omega}}\underline{\underline{p}}\right)\phi_0 = 0.$$

In the coordinate representation $\underline{\underline{p}} = \frac{\hbar}{i}\frac{d}{dx}$ and $\underline{\underline{x}} = x\mathbb{1}$, so

(4.18) $$\frac{d}{dx}\,\phi_0(x) = -\frac{m\omega}{\hbar}x\phi_0(x)$$

(4.19) $$\Rightarrow \phi_0(x) = C.\exp\left(-\frac{m\omega}{2\hbar}x^2\right).$$

This is the wave function for the ground state (see (3.6)).

The excited states are obtained from $\phi_0(x)$ by the application of $\underline{\underline{a}}^{\dagger}$.

<u>REMARK</u>. In the classical hamiltonfunction (3.1) one can introduce new classical variables $P(t) = \frac{p(t)}{\sqrt{m\hbar\omega}}$, $Q(t) = \sqrt{\frac{m\omega}{\hbar}}\,x(t)$.

This gives $H(p,x) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \left(\frac{P^2+Q^2}{2}\right)\hbar\omega$. Despite the appearance of h .this is still the classical Hamiltonian.

Introducing variables $a(t) = \frac{Q+iP}{\sqrt{2}}$, $a^*(t) = \frac{Q-iP}{\sqrt{2}}$ one obtains

$$H(p,x) = H'(a,a^*) = a^*(t)a(t)\hbar\omega = \frac{a^*a+aa}{2}\,h\omega\ .$$

Expression (4.5) may now be obtained by imposing the quantisation condition $[\underline{\underline{a}},\underline{\underline{a}}^{\dagger}] = \mathbb{1}$.

§5. DIRAC'S <bra|ket> NOTATION

We discuss briefly a notation which was introduced by Dirac in his book, *The Principles of Quantum Mechanics*. Consider a Hilbert $\mathcal{H}$ and its dual $\mathcal{H}^*$; $\mathcal{H}^*$ is the linear space of continuous linear functionals on $\mathcal{H}$.

With each vector $\phi \in \mathcal{H}$ we can associate an element $F_\phi \in \mathcal{H}^*$, by using the scalar product on $\mathcal{H}$. The functional F_ϕ is defined by

$$(5.1) \qquad F_\phi(\chi) := \langle \phi, \chi \rangle ; \qquad \phi \;\&\; \chi \in \mathcal{H}.$$

The functional associated with $\alpha\phi (\alpha \in \mathbb{C})$ is $\alpha^* F_\phi$.

$$(5.2) \qquad F_{\alpha\phi}(\chi) = \langle \alpha\phi, \chi \rangle = \alpha^* \langle \phi, \chi \rangle = \alpha^* F_\phi(\chi).$$

The reverse is also true. If F is an element of $\mathcal{H}^*$, then there exists a unique vector $\phi \in \mathcal{H}$, such that

$$(5.3) \qquad F(\chi) = \langle \phi ; \chi \rangle \qquad \forall \chi \in \mathcal{H}.$$

This is the content of Riesz's Representation Theorem. Let F_1 and F_2 be in $\mathcal{H}^*$ and let ϕ_1 and ϕ_2 be the vectors in $\mathcal{H}$ associated with these functionals. One defines an inner product on $\mathcal{H}^*$,

$$(5.4) \qquad \begin{aligned} &\langle F_1, F_2 \rangle := \langle \phi_2, \phi_1 \rangle ; \quad \| F \| = \| \phi \| \\ &\langle F_1, \alpha F_2 \rangle = \langle \alpha^* \phi_2, \phi_1 \rangle = \alpha \langle \phi_2, \phi_1 \rangle = \alpha \langle F_1, F_2 \rangle . \end{aligned}$$

$\mathcal{H}^*$ is again a Hilbert space. There is a $1-1$ anti-linear relatationship between $\mathcal{H}$ and $\mathcal{H}^*$

$$\begin{aligned} F_1 &\leftrightarrow \phi_1 \\ F_2 &\leftrightarrow \phi_2 \\ \alpha_1 F_1 + \alpha_2 F_2 &\leftrightarrow \alpha_1^* \phi_1 + \alpha_2^* \phi_2 \end{aligned}$$

Dirac denotes vectors in H by the symbol $|\ \rangle$ (ket-vector). So instead of ϕ, χ etc. we have $|\phi\rangle$, $|\chi\rangle$ etc., $\alpha\ \phi$ becomes $\alpha|\phi\rangle$.
The linear functionals F_ϕ, F_χ corresponding to ϕ and χ are denoted by $\langle\phi|$, $\langle\chi|$ and are called bra-vectors. The inner product $\langle\phi,\chi\rangle$ of two vectors ϕ and χ in H is then written (5.1) as the action of $\langle\phi|$ on $|\chi\rangle$,

$$\langle\phi|(|\chi\rangle) = \langle\phi,\chi\rangle .$$

Dirac writes the left hand side as $\langle\phi|\chi\rangle$ and calles this a *bra* c *ket* (bracket).
The inner product of $|\phi\rangle$ and $|\chi\rangle$ is now

(5.5) $$\langle\phi|\chi\rangle = \langle\chi|\phi\rangle^*.$$

It satisfies the following rules

(5.6) $$\langle\phi|\alpha\chi\rangle = \alpha\langle\phi|\chi\rangle$$
$$\langle\alpha\phi|\chi\rangle = \alpha^*\langle\phi|\chi\rangle .$$

The notation is further developed by introducing quantities like $\underline{\underline{P}}_\phi = |\phi\rangle\langle\phi|$. $\underline{\underline{P}}_\phi$ is an operator on H, defined by

(5.7) $$\underline{\underline{P}}_\phi|\chi\rangle = |\phi\rangle\langle\phi|\chi\rangle .$$

With $\| |\phi\rangle \| = \langle\phi|\phi\rangle^{\frac{1}{2}}$ we have for $\underline{\underline{P}}_\phi$

(5.8) $$\underline{\underline{P}}_\phi^2 = \underline{\underline{P}}_\phi .$$

Operators on H are denoted by $\underline{\underline{A}}$, $\underline{\underline{B}}$ etc. The hermitian conjugate $\underline{\underline{A}}^+$ of $\underline{\underline{A}}$ is defined by

(5.9) $$\langle\phi|\underline{\underline{A}}|\chi\rangle = \langle\chi|\underline{\underline{A}}^\dagger|\phi\rangle^* \qquad \forall\phi\ \&\ \chi$$

An operator $\underline{\underline{A}}$ is called Hermitian if $\underline{\underline{A}} = \underline{\underline{A}}^{+}$. Expectation values of such operators are real numbers.

(5.10) $$<\phi|\underline{\underline{A}}|\phi> = <\phi|\underline{\underline{A}}|\phi>^{*} .$$

Consider the eigenvalue equation for a hermitian operator $\underline{\underline{A}}$

(5.11) $$\underline{\underline{A}}|a> = a|a>$$
$$<a \mid a'> = \delta_{aa'}$$

The expansion of a vector $|\phi>$ is given by

(5.12) $$|\phi> = \sum_a |a><a \mid \phi> = \left(\sum_a |a><a|\right) |\phi>$$

(5.13) $$\sum_a |a><a| = \quad .$$

The operators $\underline{\underline{P}}_a := |a><a|$ are projection operators.

The above formalism is also used for operators having generalized eigenvectors. As an example we consider the momentum operator $\underline{\underline{p}}$.

(5.14) $$\underline{\underline{p}}|p> = p|p> ; \quad <p|\underline{\underline{p}} = p<p|$$
$$<p \mid p'> = \delta(p-p') \qquad \text{(Dirac's deltafunction)}$$

A state $|\phi>$ is expanded in the momentum representation

(5.15) $$|\phi> = \int_{-\infty}^{\infty} |p>dp<p \mid \phi>$$

(5.16) $$<\phi \mid \phi> = \int <\phi \mid p>dp<p \mid \phi> = \int |<p \mid \phi>|^2 dp = \int |\phi(p)|^2 dp$$

(5.17) $$\phi(p) := <p \mid \phi> .$$

In analogous way one has in the coordinate representation

(5.18) $$\underline{\underline{x}}|x> = x|x>$$
$$<x \mid x'> = \delta(x-x') .$$

$$|\phi\rangle = \int |x\rangle\, dx\, \langle x|\phi\rangle = \int \phi(x)\, |x\rangle\, dx \tag{5.19}$$

$$\langle \phi | \phi \rangle = \int |\phi(x)|^2 dx\,. \tag{5.20}$$

The relation between $\phi(x)$ and $\phi(p)$ is given by the "transformation matrix" $U_{xp} = \langle x | p \rangle$

$$\langle x | \phi \rangle = \int \langle x | p \rangle dp \langle p | \phi \rangle\,. \tag{5.21}$$

The "matrix" $U_{xp} = \frac{1}{\sqrt{2\pi\hbar}}\, e^{\frac{ipx}{\hbar}}$ (see 3.11).

§6. FIELD QUANTISATION. RELATIVISTIC QUANTUM MECHANICS

In this section we discuss a primitive method to define, starting from a classical real valued function $\phi(x^0,\vec{x})$ $(x^0=ct)$, an operator field in space-time. Consider the relation between energy and momentum for a free particle with mass m.

(6.1) $$E^2 = \vec{p}^{\,2}c^2 + m^2c^4$$

Replacing $\vec{p}$ by $\frac{\hbar}{i}\nabla$ and E/c by $i\hbar\frac{\partial}{\partial ct}$ one obtains from (6.1) a differential operator. Acting on a function $\phi(x^0,\vec{x})$ this gives

(6.2) $$\left(\Delta-\frac{\partial^2}{\partial x^{0^2}}-\mu^2\right)\phi(x^0,\vec{x}) = 0$$
$$\mu = \frac{mc}{\hbar}.$$

This is the Klein-Gordon equation or relativistic Schrödinger equation. The Klein-Gordon equation can*not* be considered as the evolution equation for a quantum mechanical wave function. This has to do with the second order time-derivative. Nevertheless the equation plays an important rôle in elementary particle physics. The classical field $\phi(x^0,\vec{x})$ must be replaced by an operator field $\underline{\phi}(x^0,\vec{x})$ which satisfies (6.2). We will explain how this is done.

For the time being we consider ϕ as a real valued function. Equation (6.2) may be obtained from a Lagrangian

(6.3) $$L = \tfrac{1}{2}(\partial_\nu\phi\partial^\nu\phi - \mu^2\phi^2); \quad \partial_\nu = \frac{\partial}{\partial x^\nu}; \quad \partial^\mu = g^{\mu\nu}\partial_\nu$$

(6.4) $$L = \tfrac{1}{2}\left(\frac{\dot{\phi}^2}{c^2} - \nabla\phi\cdot\nabla\phi - \mu^2\phi^2\right) = T - V, \quad T = 2\frac{1}{c^2}\dot{\phi}^2 .$$

The energy density $H = T + V$

(6.5) $$H = \tfrac{1}{2}\left(\frac{\dot{\phi}^2}{c^2} + \nabla\phi\cdot\nabla\phi + \mu^2\phi^2\right) \geqslant 0$$

and the energy is given by

$$H = \int_{-\infty}^{\infty} d^3x \, \tfrac{1}{2}\left(\frac{\dot{\phi}^2}{c^2} + \nabla\phi\cdot\nabla\phi + \mu^2\phi^2\right). \tag{6.6}$$

The energy is independent of time for solutions of the equation of motion (6.2). Equation (6.2) has plane-wave solutions

$$\begin{aligned} U_{\vec{k}}(x) &= \text{Const.exp.}-i(k^0x^0 - \vec{k}\cdot\vec{x}) \\ &= \text{Const.exp.}-ikx \\ V_{\vec{k}}(x) &\ \text{Const.exp.}-i(-k^0x^0 - \vec{k}\cdot\vec{x}) \end{aligned} \tag{6.7}$$

with

$$k^0 := \sqrt{\vec{k}^2 + \mu^2} \ .$$

The general real valued solution of (6.2) is given by

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d^3k}{2k^0} \left(a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx}\right) = \phi^*(x) \\ kx &= k^0x^0 - \vec{k}\cdot\vec{x} \ . \end{aligned} \tag{6.8}$$

Substituting (6.8) in (6.6), one finds

$$H = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left(\frac{a(\vec{k})\,a^*(\vec{k}) + a^*(\vec{k})\,a(\vec{k})}{2}\right) k^0 \ . \tag{6.9}$$

Comparing this expression with the remark on page 10 we conclude that the energy H can be considered as the energy of an infinite system of harmonic oscillators.

The system is quantized by replacing the classical variable $a(\vec{k})$ and $a^*(\vec{k})$ by operators $\underline{\underline{a}}(\vec{k})$ and $\underline{\underline{a}}^\dagger(\vec{k})$ satisfying the following commutation relations

$$[\underline{\underline{a}}(\vec{k}), \underline{\underline{a}}^\dagger(\vec{k}')] = \hbar c(2\pi)^3\, 2k\delta^3(\vec{k}-\vec{k}') \tag{6.10}$$

$$[\underline{\underline{a}}(\vec{k}), \underline{\underline{a}}(\vec{k}')] = [\underline{\underline{a}}^\dagger(\vec{k}), \underline{\underline{a}}^\dagger(\vec{k}')] = 0 \ . \tag{6.11}$$

The classical Hamiltonian becomes an operator $\underline{\underline{H}}$

(6.12) $$\underline{\underline{H}} = \frac{1}{(2\pi)^3} \cdot \int \frac{d^3k}{2k^0} \underline{\underline{a}}^\dagger(\vec{k})\underline{\underline{a}}(\vec{k})k^0$$

We have omitted the infinite zero-point energy.

The field $\phi(x^0,\vec{x})$ becomes an operator field

(6.13) $$\underline{\underline{\phi}}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} (\underline{\underline{a}}(\vec{k})e^{-ikx} + \underline{\underline{a}}^\dagger(k)e^{ikx}).$$

This operatorfield satisfies the equation

(6.14) $$(\Box-\mu^2)\underline{\underline{}}(x) = 0$$
$$\Box = \Delta - \frac{\partial^2}{\partial x^{02}} .$$

$\underline{\underline{\phi}}(x)$ is a time-dependent operator, it is the field operator in the Heisenberg Picture.

The commutator of $\underline{\underline{\phi}}(x)$ and $\underline{\underline{\phi}}(y)$ is found to be

(6.15) $$[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}(y)] = i\hbar c\, \Delta(x-y) \qquad \text{(Pauli-Jordan)}$$

with

(6.16) $$\Delta(x-y) = \frac{-i}{(2\pi)^3} \int \frac{d^3k}{2k^0} (e^{-ik(x-y)} - e^{ik(x-y)}).$$

The distribution $\Delta(x-y)$ is a Lorentz-invariant solution of the Klein-Gordon equation which has the important property

(6.17) $$\Delta(x-y) = 0 \quad \text{for} \quad (x-y)^2 < 0$$

where $$(x-y)^2 = (x^0-y^0)^2 - (\vec{x}-\vec{y})^2.$$

Space-time points x and y which have $(x-y)^2 < 0$ are said to be space-like separated. No physical signal can be exchanged between such points because one would need a velocity exceeding the speed of light.

§7. INTERPRETATION OF THE QUANTIZED FIELD

The results of §6 can be interpreted as the description of a many-particle system of identical particles with mass m and spin zero. To point out how one arrives at such a picture we first consider the commutation relations of $\underline{\underline{a}}^{\dagger}(\vec{k})$ and $\underline{\underline{a}}(\vec{k})$ with the Hamiltonian (6.12)

$$(7.1) \qquad [\underline{\underline{H}}, \underline{\underline{a}}^{\dagger}(\vec{k})] = \hbar c k^0 \underline{\underline{a}}^{\dagger}(\vec{k}) = \hbar\omega(\vec{k})\underline{\underline{a}}^{\dagger}(\vec{k})$$

$$(7.2) \qquad \begin{cases} [\underline{\underline{H}}, \underline{\underline{a}}(\vec{k})] = -\hbar c k^0 \underline{\underline{a}}(\vec{k}) = -\hbar\omega(\vec{k})\underline{\underline{a}}(\vec{k}) \\ \omega(\vec{k}) = k^0 c = \sqrt{\vec{k}^2 c^2 + \mu^2 c^2} = \sqrt{\vec{k}^2 c^2 + \frac{m^2 c^4}{\hbar^2}} \\ \hbar\omega(\vec{k}) = \sqrt{\vec{p}^2 c^2 + m^2 c^4} \; ; \; \vec{p} = \hbar\vec{k}. \end{cases}$$

Comparing (7.1), (7.2) with (4.7) and (4.8) we conclude that $\underline{\underline{a}}^{\dagger}(\vec{k})$ and $\underline{\underline{a}}(\vec{k})$ are *construction operators*.

From the expression (6.12) for the Hamiltonian we see that a state $|0>$ with the property

$$(7.3) \qquad \underline{\underline{a}}(\vec{k})|0> = 0 \qquad \forall\vec{k}$$

is an eigenstate of $\underline{\underline{H}}$ with eigenvalue zero.

$$(7.4) \qquad \underline{\underline{H}}|0> = 0|0> .$$

The state $|0>$ is called the *vacuum state*.

From the vacuum state we obtain *one-particle states* $\{|\vec{k}>\}$

$$(7.5) \qquad |\vec{k}> := \underline{\underline{a}}^{\dagger}(\vec{k})|0>$$

A one-particle state $|\vec{k}>$ is an eigenstate of the Hamiltonian $\underline{\underline{H}}$ of the momentum operators $\vec{\underline{\underline{P}}}$.*)

(7.6) $$\underline{\underline{H}}|\vec{k}> = \underline{\underline{H}}\,\underline{\underline{a}}^{\dagger}(\vec{k})|0> = \hbar\omega(\vec{k})|\vec{k}>$$

(7.7) $$\vec{\underline{\underline{P}}}|\vec{k}> = \hbar\,\vec{k}|\vec{k}> = \vec{p}\,|\vec{k}> .$$

Acting on $|0>$ with $\underline{\underline{a}}^{\dagger}(\vec{k}_1)\ \underline{\underline{a}}^{\dagger}(\vec{k}_2)$ we obtain a *two-particle* state.

(7.8) $$|\vec{k}_1,\vec{k}_2> = \underline{\underline{a}}^{\dagger}(\vec{k}_1)\ \underline{\underline{a}}^{\dagger}(\vec{k}_2)|0>$$

(7.9) $$\underline{\underline{H}}\ \vec{k}_1,\vec{k}_2> = (\hbar\omega(\vec{k}_1)+\hbar\omega(\vec{k}_2))|\vec{k}_1,\vec{k}_2>$$

(7.10) $$\vec{\underline{\underline{P}}}|\vec{k}_1,\vec{k}_2> = (\hbar\,\vec{k}_1+\hbar\,\vec{k}_2)|\vec{k}_1,\vec{k}_2> .$$

Notice that $|\vec{k}_1,\vec{k}_2> = |\vec{k}_2,\vec{k}_1>$, the two-particle states are symmetric. This is due to the fact that the operators $\underline{\underline{a}}$ and $\underline{\underline{a}}^{\dagger}$ satisfy *commutation relations*.

States with N-particles are given by

(7.11) $$|\vec{k}_1,\vec{k}_2,\ldots\vec{k}_N> = \underline{\underline{a}}^{\dagger}(\vec{k}_1)\ \ldots\ \underline{\underline{a}}^{\dagger}(\vec{k}_N)|0>$$

N-particle states are completely symmetric.

....

Elementary particles are divided into two classes, *bosons* and *fermions*. Bosons are particles with integral spin, $s = 0$, $s = 1$, $s = 2$, ... Fermions have half-integral spin, $s = \frac{1}{2}$, $s = \frac{3}{2}$, ...
For identical bosons the states are symmetric, for identical fermions the states are completely anti-symmetric. To describe fermions one needs *anti-commutation* relations.

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*) The momentum operators: $\vec{\underline{\underline{P}}} = \int \frac{d^3k}{(2\pi)^3 2k^0}\,\underline{\underline{a}}^{\dagger}(\vec{k})\underline{\underline{a}}(\vec{k})\,\vec{k}$ follow from the classical expressions in the same way as the Hamiltonian $\underline{\underline{H}}$.

The Hilbert space for the many particle system is the Fock space

(7.12) $\quad H = \overset{\infty}{\underset{n=0}{\oplus}} H^{(n)}$.

The vacuum state $|0>$ is in $H^{(0)}$. One-particle states are in $H^{(1)}$ etc.

The operator field $\underline{\underline{\phi}}(x)$ is an operator on H. Acting on the vacuum state it creates a one-particle state

(7.13) $$\underline{\underline{\phi}}(x)|0> = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} e^{ikx} |\vec{k}> .$$

The "wave function" of this state in the momentum representation reads

(7.11) $$<k| \underline{\underline{\phi}}(x)|0> = \hbar c\, e^{ikx} .$$

An important quantity in the applications of field theory in scattering processes is the *Feynman propagator* function, $<0| T\underline{\underline{\phi}}(x)\underline{\underline{\phi}}(y)|0>$.

The symbol T is Dyson's time-ordening prescription

(7.15) $$T\underline{\underline{\phi}}(x)\underline{\underline{\phi}}(y) = \theta(x^0-y^0)\underline{\underline{\phi}}(x)\underline{\underline{\phi}}(y) + \theta(y^0-x^0)\underline{\underline{\phi}}(y)\underline{\underline{\phi}}(x) .$$

The propagator is the vacuum-expectationvalue of this time-ordered product*)

(7.16) $$<0| T\underline{\underline{\phi}}(x)\underline{\underline{\phi}}(y)|0> = -i\Delta_F^{(\varepsilon)}(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik(x-y)}}{k^2-\mu^2+i\varepsilon} .$$

The Feynman propagator is a Lorentz invariant solution of the equation.

(7.17) $$(\Box-\mu^2)G(x,y) = +i\delta^{(4)}(x-y) .$$

---

*) We have omitted $\hbar$ and c in (7.16). This can be justified by introducing a system of units in which $\hbar = c = 1$.

The propagator is sometimes interpreted as the amplitude for the creations of a particle at $(y^0,\vec{y})$ and the subsequent annihilation at $(x^0,\vec{x})$.

<u>SUMMARY</u>

We have, starting from a classical i.e. non-quantized field, constructed an operator field $\underline{\underline{\phi}}(x)$. This field describes particles with mass m and spin zero. The many particle space is the Fock-space.

Operator fields can be obtained in a different way. One can take as a starting point the unitary irreducible representation $[m, s]$ of the Poincaré group $P_+^\uparrow$.

From the Hilbert space of one particle states (the representation space) one constructs the Fock space and one defines creation and annihilation operators (construction operators). The operator field is then introduced as the Fourier-transform of the construction operators.

§8. THE DIRAC FIELD AND THE ELECTROMAGNETIC FIELD

We start from the relativistically invariant wave equations for particles with mass m and spin $s = \frac{1}{2}$.

(8.1) $$(i\hbar\gamma^{\mu}\partial_{\mu} - mc)\psi(x) = 0 \qquad \text{(Dirac equation)}$$

$\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are 4×4 matrices satisfying

(8.2) $$\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$$

$$\psi(x) = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$$ is a 4-component spinor; $\psi_{1,2,3,4}$ are complex valued functions.

The adjoint equation reads:

(8.3) $$i\hbar\partial_{\mu}\bar{\psi}\gamma^{\mu} + mc\bar{\psi} = 0$$

where $$\bar{\psi}(x) = \psi^{\dagger}(x)\gamma^0 = (\psi_1^*\psi_2^*\psi_3^*\psi_4^*)\begin{pmatrix}\gamma^0\end{pmatrix}.$$

Equations (8.1) and (8.2) are the Euler-Lagrange equations for the Lagrangian

(8.4) $$L = \bar{\psi}(\gamma^{\mu} i\hbar\partial_{\mu} - mc)\psi = \sum_{\alpha,\beta=1}^{4} \bar{\psi}_{\alpha}(\gamma^{\mu} i\hbar\partial_{\mu} - mc)_{\alpha\beta}\psi_{\beta}.$$

The 4-vector field $J^{\mu}(x)$ given by

(8.5) $$J^{\mu}(x) = \bar{\psi}\gamma^{\mu}\psi$$

is conserved for solutions of the Dirac equations.

(8.6) $$\partial_{\mu}J^{\mu}(x) = 0; \quad Q := \int J^0(x)d^3x; \quad \frac{dQ}{dt} = 0.$$

Second quantization of the Dirac field yields an operator field $\underline{\underline{\psi}}(x)$ (4 components) which satisfy *anti*-commutation relations

(8.7) $$\left\{\underline{\underline{\psi}}_{\alpha}(x), \underline{\underline{\bar{\psi}}}_{\beta}(y)\right\} = (i\hbar\gamma^{\mu}\partial_{\mu} + mc)_{\alpha\beta}\cdot -i\Delta(x-y).$$

The expansion of $\underline{\underline{\psi}}$ in creation- and annihilation operators reads

$$(8.8) \qquad \underline{\underline{\psi}}(x) = \frac{1}{(2\pi)^3} \int \frac{d^3k}{2k^0} \left[\sum_{\sigma=-\frac{1}{2}}^{+\frac{1}{2}} (u(\vec{k};\sigma)\underline{\underline{a}}(\vec{k};\sigma)e^{-ikx} + v(\vec{k};\sigma)\underline{\underline{b}}^{\dagger}(\vec{k};\sigma)e^{ikx}) \right]$$

$u(\vec{k},\sigma)$ and $v(\vec{k},\sigma)$ are 4-component spinors satisfying

$$(8.9) \qquad (\gamma^{\mu}\hbar k_{\mu}-mc)u(\vec{k},\sigma) = 0$$

$$(8.10) \qquad (\gamma^{\mu}\hbar k_{\mu}+mc)v(\vec{k},\sigma) = 0 .$$

The label σ has to do with the spin.

For $\vec{k} = 0$ equation (8.9) reduces to

$$(8.11) \qquad (\gamma^0-\mathbb{1})u(\vec{0},\sigma) = 0 .$$

Taking the representation of the γ-matrices

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}; \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

one finds that (8.11) has 2-independent solutions.

The operators $\underline{\underline{a}}(\vec{k},\sigma)$ and $\underline{\underline{a}}^{\dagger}(\vec{k},\sigma)$ are the construction operators for electron-states. The operators $\underline{\underline{b}}(\vec{k},\sigma)$ and $\underline{\underline{b}}^{\dagger}(\vec{k},\sigma)$ are interpreted as the construction operators for positron states. (The positron is the anti-particle of the electron). The anti-commutation relations read

$$(8.12) \qquad \left\{\underline{\underline{a}}(\vec{k},\sigma),\ \underline{\underline{a}}^{\dagger}(\vec{k}',\sigma')\right\} = (2\pi)^3\, 2k^0\, \delta(\vec{k}-\vec{k}')\,\delta_{\sigma\sigma'},$$

$$(8.13) \qquad \left\{\underline{\underline{b}}(k,\sigma),\ \underline{\underline{b}}^{\dagger}(k',\sigma')\right\} = (2\pi)^3\, 2k^0\, \delta(\vec{k}-\vec{k}')\,\delta_{\sigma\sigma'} .$$

All other anti-commutation relations of these operators are zero.

The anti-commutation relations lead to anti-symmetric many-particle states.

This is in accordance with the Pauli-principle. Identical fermions can never be in the same state. To give an example we consider a two-electron state

$$(8.14)\qquad \begin{aligned} |(\vec{k}_1,\sigma_1);\ (\vec{k}_2,\sigma_2)\rangle &= \underline{\underline{a}}^{\dagger}(\vec{k}_1,\sigma_1)\,\underline{\underline{a}}^{\dagger}(\vec{k}_2,\sigma_2)|0\rangle \\ |(\vec{k}_2,\sigma_2);\ (\vec{k}_1,\sigma_1)\rangle &= \underline{\underline{a}}^{\dagger}(\vec{k}_2,\sigma_2)\,\underline{\underline{a}}^{\dagger}(\vec{k}_1,\sigma_1)|0\rangle = \\ &= -|(\vec{k}_1,\sigma_1);\ (\vec{k}_2,\sigma_2)\rangle\ . \end{aligned}$$

The state vector is zero for $\vec{k}_1 = \vec{k}_2$, $\sigma_1 = \sigma_2$ i.e. in the case that the two electrons have the same quantum numbers. The propagator for the Dirac-field is given by

$$(8.15)\qquad \begin{aligned} \langle 0|\, T\,\underline{\underline{\psi}}_{\alpha}(x)\,\bar{\underline{\underline{\psi}}}_{\beta}(y)\,|0\rangle &= -i\,S^{F}_{\alpha\beta}(x-y) = (i\gamma^{\mu}\partial_{\mu}+m_{\alpha\beta})\cdot -i\Delta_{F}(x-y) \\ &= \int \frac{d^4k}{(2\pi)^4}\,\frac{i(\gamma^{\mu}k_{\mu}+m)}{k^2-m^2+i\varepsilon}\,\alpha\beta\cdot e^{-ik(x-y)}\ . \end{aligned}$$

We now consider the electromagnetic field. The space-time behaviour is described by Maxwell's equations

$$(8.16)\qquad \begin{aligned} &\text{i)} && \operatorname{rot}\vec{E}(\vec{x},t) = -\frac{\partial\vec{B}}{\partial t}(\vec{x},t) \\ &\text{ii)} && \operatorname{rot}\vec{B}(\vec{x},t) = \mu_0\vec{j}(\vec{x},t) + \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E}(\vec{x},t) \\ &\text{iii)} && \operatorname{div}E = \frac{1}{\varepsilon_0}\rho(\vec{x},t) \qquad ;\ c^2 = \frac{1}{\varepsilon_0\mu_0} \\ &\text{iv)} && \operatorname{div}\vec{B} = 0\ . \end{aligned}$$

$\rho(\vec{x},t)$ is the electric charge density, $\vec{j}(\vec{x},t)$ is the current density. These quantities satisfy the continuity equation,

$$(8.17)\qquad \frac{\partial\rho}{\partial t} + \operatorname{div}\vec{j} = 0$$

$\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are the field strengths.

In the Lagrangian the charge and current density are coupled to potentials ϕ and $\vec{A}$.

From the above equations one has

$$\text{(8.18)} \qquad \text{iv)} \Rightarrow \quad \vec{B} = \text{rot}\,\vec{A}$$

$$\text{i) \& iv)} \qquad \vec{E} = -\nabla\phi - \frac{\partial\vec{A}}{\partial t}\,.$$

Defining

$$\text{(8.19)} \qquad A^{\mu} = (\tfrac{\phi}{c},\vec{A})\,; \; J^{\mu} = (c\rho,\vec{j})$$

$$\text{(8.20)} \qquad F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}; \quad \partial^{\mu} = \frac{\partial}{\partial x_{\mu}} = g^{\mu\nu}\frac{\partial}{\partial x}\nu$$

we have

$$\text{(8.21)} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E^1}{c} & -\frac{E^2}{c} & -\frac{E^3}{c} \\ \frac{E^1}{c} & 0 & -B^3 & B^2 \\ \frac{E^2}{c} & B^3 & 0 & -B^1 \\ \frac{E^3}{c} & -B^2 & B^1 & 0 \end{pmatrix}$$

Equations (ii) and (iii) (8.16) are obtained from

$$\text{(8.22)} \qquad \partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu}.$$

Defining the dual $^{*}F^{\mu\nu}$ of $F^{\mu\nu}$ by

$$\text{(8.23)} \qquad {}^{*}F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu k\lambda}F_{\lambda} \qquad\qquad (\varepsilon^{0123}=1)$$

equations (i) and (iv) (8.16) are contained in

$$\text{(8.24)} \qquad \partial\mu\,{}^{*}F^{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu k\lambda}\partial_{\mu}F_{k\lambda} = 0\,.$$

Expressed in the potentials A^{μ} equation (8.22) reads

$$\text{(8.25)} \qquad \partial_{\mu}\partial^{\mu}A^{\nu} - \partial^{\nu}(\partial_{\mu}A^{\mu}) = \mu_0 J^{\nu} \qquad (\nu=0,1,2,3).$$

These equations may be obtained from the Lagrangian

(8.26)
$$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \mu_0 j^\mu A_\mu$$
$$= -\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \mu_0 j^\mu A_\mu \ .$$

The Euler-Lagrange equations for A_μ are given by

(8.27)
$$\frac{\partial L}{\partial A_\mu} - \partial_\nu \frac{\partial L}{\partial \partial_\nu A_\mu} = 0 \ .$$

From (8.26) one has

(8.28)
$$\frac{\partial L}{\partial A_\mu} = -\mu_0 j^\mu$$
$$\frac{\partial L}{\partial \partial_\nu A_\mu} = \frac{\partial L}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial \partial_\nu \mu} = -F^{\nu\mu} = -(\partial^\nu A^\mu - \partial^\mu A^\nu) \ .$$

Maxwell's theory is our first example of a so-called *gauge* theory. From (8.18) and (8.20) one sees that the potentials A^μ are not uniquely determined by the psysical fields $\vec{E}$ and $\vec{B}$. Adding a gradient term $\nabla\chi$ to $\vec{A}$ does not change the $\vec{B}$-field. The $\vec{E}$-field remains the same if $\vec{A}$ is replaced by $\vec{A} + \nabla\chi$ and ϕ by $\phi - \frac{\partial}{\partial t}\chi$.

The freedom in the choice of the potentials is called the gauge freedom.

In the relativistic notation a gauge transformation, i.e. a transformation

(8.29)
$$\begin{cases} \vec{A} \longmapsto \vec{A} + \nabla\chi \\ \phi \longmapsto \phi - \frac{\partial}{\partial t}\chi \end{cases}$$

is given by

(8.30)
$$A^\mu(x) \longmapsto A^\mu(x) + \partial^\mu\chi := \hat{A}^\mu \ .$$

The electromagnetic field tensor $F^{\mu\nu}$ is invariant under this transformation

(8.31)
$$\hat{F}^{\mu\nu} = \partial^\mu \hat{A}^\nu - \partial^\nu \hat{A}^\mu = F^{\mu\nu}$$

$F^{\mu\nu}$ is a gauge invariant quantity. The collection of gauge transformations is a group. Performing two gauge transformations one after another we have

(8.32) $$A^{\mu} \longmapsto A^{\mu} + \partial^{\mu}\chi_1 \longmapsto (A^{\mu}+\partial^{\mu}\chi_1) + \partial^{\mu}\chi_2 = A^{\mu} + \partial^{\mu}\chi_1 + \partial^{\mu}\chi_2 .$$

Taking $\chi = 0$ we have the identity mapping. The inverse of the transformation generated by χ is the transformation given by $-\chi$.
It is clear that we have an abelian group. Electromagnetism is an abelian gauge theory.
The equations of motion (8.25) for the gauge potentials are gauge invariant. (The physical sources $\rho(\vec{x},t)$ and $\vec{j}(\vec{x},t)$ are invariant).
Replacing A^{μ} by $\hat{A}^{\mu} = A^{\mu} + \partial^{\mu}\chi$ we obtain

(8.33) $$\begin{aligned}\partial_{\mu}\partial^{\mu}\hat{A}^{\nu} - \partial^{\nu}(\partial_{\mu}\hat{A}^{\mu}) &= \partial_{\mu}\partial^{\mu}A^{\nu} + \partial_{\mu}\partial^{\mu}\partial^{\nu}\chi - \partial^{\nu}(\partial_{\mu}A^{\mu}) - \partial^{\nu}(\partial_{\mu}\partial^{\nu}\chi)\\ &= \mu_0 j^{\nu}\end{aligned}$$

where we have used

$$\partial_{\mu}\partial^{\mu}\partial^{\nu}\chi = \partial^{\nu}(\partial_{\mu}\partial^{\mu}\chi) .$$

All gauge-equivalent configurations, i.e. the set $\{A^{\mu}\}$ of potentials which are mutually related by gauge transformations describe the same physical situation. One can use the freedom in the potentials to simplify equation (8.25). One often imposes the so-called Lorentz condition

(8.34) $$\partial_{\mu}A^{\mu} = 0.$$

Using the gauge freedom this can always be achieved.

Suppose $\partial_{\mu}A^{\mu} \neq 0$, taking $\hat{A}^{\mu} = A^{\mu}\Lambda$ and requiring

(8.35) $$\partial_{\mu}\hat{A}^{\mu} = 0$$

one obtains an equation for Λ

(8.36) $$\partial_{\mu}\partial^{\mu}\Lambda = -\partial_{\mu}A^{\mu} .$$

A solution of (8.36) is given by

(8.37) $$\Lambda(x) = \int d^4y\, \Delta_F(x-y)\,\partial_\mu A^\mu(y)$$

with

(8.38) $$\Delta_F(x-y) = \lim_{\varepsilon\downarrow 0} \int d^4k \frac{1}{k^2+i\varepsilon}\, e^{-ik(x-y)}$$

(8.39) $$\partial_\mu\partial^\mu \Delta_F(x-y) = -\delta^4(x-y)\,; \qquad \frac{1}{k^2+i0} = P\frac{1}{k^2} - i\pi\delta(k^2).$$

A condition like $\partial_\mu A^\mu = 0$ is called a *gauge-fixing* condition. Other conditions are possible e.g. $A^0 = 0$, or $\operatorname{div}\vec{A} = 0$.

From now on we consider the free Maxwell field, i.e. the equations (8.16), (8.22), (8.25), (8.26) with $\rho = 0$ and $\vec{j} = 0$.

The equations for the potentials A^μ are then given by

(8.40) $$\partial_\mu\partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = 0$$

and the Lagrangian reads

(8.41) $$L = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

Imposing the Lorentz condition $\partial_\mu A^\mu = 0$, equation (8.40) simplifies and one obtains

(8.42) $$\begin{cases} \partial_\mu\partial^\mu A^\nu = 0 \\ \partial_\mu A^\mu = 0\,. \end{cases}$$

The general real-valued solution of these equations is given by

(8.43) $$A^\mu(x) = \frac{1}{(2\pi)^3}\int \frac{d^3k}{2k^0} \sum_{\lambda=0}^{3} \left(\varepsilon^\mu_\lambda(\vec{k})\,a_\lambda(\vec{k})\,e^{-ikx} + \varepsilon^\mu_\lambda(\vec{k})\,a^*_\lambda(\vec{k})\,e^{ikx}\right)$$

with the subsidiary condition

$$(8.44)\qquad k_\mu \varepsilon^\mu_\lambda(\vec{k}) a_\lambda = k_\mu \varepsilon^\mu_\lambda(\vec{k}) a^*_\lambda = 0, \quad k^2 = k^{0^2} - \vec{k}^2 = 0 .$$

The real vectors $\varepsilon^\mu_\lambda(\vec{k})$ $(\lambda=0,1,2,3)$ are called polarization vectors. A convenient choice of these vectors is given by

$$(8.45)\qquad \varepsilon_0 = (1,\vec{0});\ \varepsilon_1 = (0,\vec{n}_1);\ \varepsilon_2 = (0,\vec{n}_2);\ \varepsilon_3 = (0,\frac{\vec{k}}{|\vec{k}|})$$

with

$$\vec{n}_1 \wedge \vec{n}_2 = \frac{\vec{k}}{|\vec{k}|} \qquad \text{(cyclic)}$$

and

$$\vec{n}_1 \cdot \vec{n}_2 = \vec{n}_1 \cdot \vec{k} = \vec{n}_2 \cdot \vec{k} = 0; \quad \vec{n}_1^2 = \vec{n}_2^2 = 1.$$

With this choice we have

$$(8.46)\qquad \varepsilon_\lambda \cdot \varepsilon_{\lambda'} = g_{\lambda\lambda'}$$

and

$$(8.47)\qquad \varepsilon^\mu_\lambda(\vec{k}) \varepsilon^\nu_{\lambda'}(\vec{k}) g_{\lambda\lambda'} = g^{\mu\nu} \qquad \text{(summation over } \lambda \text{ and } \lambda'\text{)}.$$

The quantization of the electromagnetic field is plagued by all kinds of problems. If one wants to keep the theory manifestly covariant one must consider the potentials $A^\mu(x)$ as operatorfields and the classical amplitudes $a_\lambda(\vec{k})$ and $a^*_\lambda(\vec{k})$ are to be replaced by annihilation and creation operators. Due to the redundancy in the 4-vector potential only two of the four operators $\underline{\underline{a}}^\dagger_\lambda(\vec{k})$ $(\lambda=0,1,2,3)$ can be considered as the creation-operators for photons. This can already be seen at the classical level. A plane wave solution of (8.12) is given by

$$(8.48)\qquad A^\mu(x) = \sum_{\lambda=0}^{3} \varepsilon^\mu_\lambda(\vec{k}) a_\lambda(\vec{k}) e^{-ikx} + \varepsilon^\mu_\lambda(\vec{k}) a^*_\lambda(\vec{k}) e^{ikx}$$

with

$$\sum_{\lambda=0}^{3} k_\mu \varepsilon^\mu_\lambda(\vec{k}) a_\lambda(\vec{k}) = 0 \quad \text{and} \quad \sum_{\lambda=0}^{3} k_\mu \varepsilon^\mu_\lambda(\vec{k}) a^*_\lambda(\vec{k}) = 0$$

and

$$k^0 = |\vec{k}| .$$

Using the polarization vectors (8.45) we obtain

(8.49) $$k_\mu \varepsilon^\mu_1(\vec{k}) = k_\mu \varepsilon^\mu_2(\vec{k}) = -\vec{k}\cdot\varepsilon_1 = -\vec{k}\cdot\varepsilon_2 = 0$$

and

(8.50) $$k_0 a_0 - |\vec{k}| a_3 = 0 \Rightarrow a_0 = a_3 .$$

Calculating the field $\vec{E}$ and $\vec{B}$ belonging to (8.48) we find

$$\vec{E} = -\nabla\phi - \frac{\partial A}{\partial t} = -(A^0\cdot c) - \frac{\partial\vec{A}}{\partial t}$$

$$= -\sum_{\lambda=0}^{3} (\varepsilon^0_\lambda a_\lambda\ i\vec{k}ce^{-ikx} - \varepsilon^0_\lambda a^*_\lambda\ i\vec{k}ce^{+ikx}) - \sum_{\lambda=0}^{3} (\vec{\varepsilon}_\lambda a_\lambda \cdot -ik^0ce^{-ikx} + $$

$$+ \vec{\varepsilon}_\lambda a^*_\lambda\ ik^0ce^{ikx})$$

$$= -(a_0e^{-ikx} - a^*_o e^{ikx})\cdot i\vec{k}c + \vec{\varepsilon}_3(a_3e^{-ikx} - a^*_3e^{ikx})\cdot ik^0c +$$

$$+ \vec{\varepsilon}_1(a_1e^{-ikx} - a^*_1e^{ikx})\cdot ik^0c + \vec{\varepsilon}_2(a_2e^{-ikx} - a^*_2\cdot e^{ikx}) ik^0c .$$

Using (8.45) and (8.50) we have

$$\vec{\varepsilon}_3 k^0 = \frac{\vec{k}}{|\vec{k}|} k^0 = \vec{k} ; \qquad a_3\vec{\varepsilon}_3k^0 = a_3\vec{k} = a_0\vec{k}$$

and only the *transverse* components of $A^\mu(x)$ contribute to the field strength

(8.51) $$\vec{E}(x) = \vec{\varepsilon}_1(\vec{k})(a_1e^{-ikx} - a^*_1e^{ikx})\cdot ik^0c + \vec{\varepsilon}_2(a_2e^{-ikx} - a^*_2e^{ikx})\cdot ik^0c .$$

The magnetic field $\vec{B} = \nabla \wedge \vec{A}$ is found to be

$$\vec{B} = - \sum_{\lambda=0}^{3} \left[a_\lambda (\vec{\varepsilon}_\lambda \wedge i\vec{k}) e^{-ikx} - a^*(\vec{\varepsilon}_\lambda \wedge i\vec{k}) e^{ikx} \right]$$

(8.52)
$$= -i(a_1 e^{-ikx} - a_1^* e^{ikx}) \vec{\varepsilon}_1 \wedge \vec{k} - i(a_2 e^{-ikx} - a_2^* e^{ikx}) \vec{\varepsilon}_2 \wedge \vec{k}$$

$$= +i(a_1 e^{-ikx} - a_1^* e^{ikx}) \vec{\varepsilon}_2 |\vec{k}| - i(a_2 e^{-ikx} - a_2^* e^{ikx}) \vec{\varepsilon}_1 |\vec{k}| \ .$$

Again only the transverse polarizations contribute to the field.
We remark that the same holds for the fields $\vec{E}$ and $\vec{B}$ obtained from the general solution (8.43) and (8.44). The plane-wave solution (8.51), (8.52) satisfies $\vec{E}\cdot\vec{k} = \vec{B}\cdot\vec{k} = 0$, this is the transversality of the electromagnetic waves.

The quantization of the field potentials $A^\mu(x)$ may be achieved by imposing the following commutation relations on $\underline{\underline{a}}_\lambda(\vec{k})$ and $\underline{\underline{a}}_\lambda^\dagger(\vec{k})$

(8.53)
$$\left[\underline{\underline{a}}_\lambda(\vec{k}),\ \underline{\underline{a}}_{\lambda}^\dagger(\vec{k}') \right] = -g_{\lambda\lambda'} (2\pi)^3 2k^0 \delta^{(3)}(\vec{k}-\vec{k}')$$

From (8.43) we obtain using (8.47) and (8.53)

(8.54)
$$\left[\underline{\underline{A}}^\mu(x),\ \underline{\underline{A}}^\nu(y) \right] = -g^{\mu\nu} \int \frac{d^3k}{(2\pi)^3 2k^0} \left(e^{-ik(x-y)} - e^{ik(x-y)}\right).$$

It is clear that the Lorentz condition $\partial_\mu A^\mu = 0$ cannot be maintained for the operators $\underline{\underline{A}}^\mu(x)$. This would violate the commutation relations (8.54). Considering (8.53) we find that the "Fock Space" constructed by applying the operators $\underline{\underline{a}}_\lambda^\dagger(\vec{k})$ $(\lambda=0,1,2,3)$ contains non-physical states e.g. the states created by $\underline{\underline{a}}_0^\dagger(k)$.
The Lorentz condition is used to determine the subspace of physical states by imposing

(8.55)
$$\langle \psi_{\text{physical}} | \partial_\mu \underline{\underline{A}}^\mu(x) | \phi_{\text{physical}} \rangle = 0 \ .$$

As an example we consider

(8.56)
$$\langle 0 | \partial_\mu \underline{\underline{A}}^\mu(x) | \vec{k},\lambda \rangle_{\text{phys.}} = 0.$$

with $|\vec{k},\lambda\rangle = \underline{\underline{a}}_\lambda^\dagger(\vec{k})|0\rangle$.

This gives

$$\langle 0| \int \frac{d^3k'}{(2\pi)^3 2k^{0'}} \sum_{\lambda'=0}^{3} -ik'_\mu \cdot \varepsilon^\mu_{\lambda'}(\vec{k}')\, a_{\lambda'}(\vec{k}')\, e^{-ik'x}\, \underline{\underline{a}}_\lambda^\dagger(\vec{k})|0\rangle = 0$$

$$= \sum_{\lambda'=0}^{3} + ik_\mu \varepsilon^\mu_{\lambda'}(\vec{k})\, g_{\lambda\lambda'}\, e^{-ikx} = 0.$$

This gives the condition $k_\mu \varepsilon^\mu_\lambda(k) = 0$ for physical photon states. Using (8.45) we see that only $\lambda = 1$ and $\lambda = 2$ satisfy this condition, so $\lambda = 0$ and $\lambda = 3$ give non-physical states.

The propagator for the e.m. field is given by

$$\langle 0| T A^\mu(x) A^\nu(y)|0\rangle = +g^{\mu\nu}\, i\, \Delta_F(x-y; m=0)$$

$$= \lim_{\varepsilon \downarrow 0} \int \frac{d^4k}{(2\pi)^4} \frac{-ig^{\mu\nu}}{k^2+i\varepsilon}\, e^{-ik(x-y)}.$$

§9. GAUGE TRANSFORMATIONS AND THE ELECTROMAGNETIC INTERACTION

In quantummechanics the states of a system are described by complex valued wave functions. If $\psi(\vec{r},t)$ is such a wave function then $\psi(\vec{r},t)$ and $e^{i\alpha}\psi(\vec{r},t)$ $(\alpha\in\mathbb{R})$ describe the same state. This means that an overall change of the phase of $\psi(\vec{r},t)$ has no observable effects.

H. Weyl (Z.Physic 56 (1929)) and P.A.M. Dirac (Proceedings of the Royal Society of London, A, 133 (1931)) have investigated the consequences of making α space-time dependent.

The result is that a theory which describes originally a free particle is turned into a theory in which the particle has interactions with an external field determined by the function $\alpha = \alpha(\vec{r},t)$.

Dirac's paper is centered around the problem of the quantization of electric charge. Electric charge comes in integer multiples of a unit charge e.g. the charge of the electron. Dirac introduced the notion of non-integrable phase factors, i.e. functions $\alpha(\vec{r},t)$ with $\partial_\mu\partial_\nu\alpha \neq \partial_\nu\partial_\mu\alpha$. By studying the change in the phase of the wave function around a closed curve, he found that due to the fact that the phase is determined up to multiples of 2Π there might be singularities which could be interpreted as magnetic monopoles. Electric charge e and the monopole strength μ are related by

$$e\mu = \tfrac{1}{2}hc$$

where h is Planck's constant and c is the speed of light. This is Dirac's quantization condition for electric charge. We will not pursue these historical developments. To discuss gauge transformations we start from the free Dirac equation

$$(9.1) \qquad (i\gamma^\mu\partial_\mu - m)\psi(x) = 0$$

and the Lagrangian

(9.2) $$L_D = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi.$$

Both the Lagrangian and the equations of motion are invariant under the transformation

(9.3) $$\psi \to \psi'(x) = U(\alpha)\psi(x) = e^{i\alpha}\psi(x) \quad (\alpha\in\mathbb{R})$$

These unitary transformations are called (global) gauge transformations. $\psi'(x)$ describes the same state as does $\psi(x)$ but in a different gauge. In this respect there is some analogy with electromagnetism. The potentials $A_\mu(x)$ and $A_\mu(x) + \partial_\mu\chi(x)$ differ only in the gauge, they lead to the same field strength.

From the invariance of the Lagrangian under the transformation (9.3) one obtains, using Noether's theorem the conservation law

(9.4) $$\partial_\mu J^\mu(x) = 0 ; \qquad J^\mu = \bar{\psi}\gamma^\mu\psi$$

for solutions of the equation of motion.

The gauge transformations (9.3) constitute the abelian group U(1).

Local gauge transformations *)

Let us now consider the case that the parameter α in $e^{i\alpha}$ is a function of x, $x = (x^0,x^1,x^2,x^3)$.

The transformation law is then

(9.5) $$\psi(x) \to \psi'(x) = U(x)\psi = e^{i\alpha(x)}\psi(x).$$

Roughly speaking we have a U(1) group at each point x. At first sight such an extension seems quite harmless, a quantity like $J^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$

*) Mathematicians will call these transformations *global* gauge transformations.

is invariant under these transformations. This is, however, misleading, because $J^\mu(x)$ has a physical meaning only for solutions of the equation of motion. Suppose $\psi(x)$ satisfies (9.1) then $\psi'(x)$ is found to obey the equation.

$$(i\gamma^\mu\partial_\mu - m)\psi'(x) = -\gamma^\mu\psi'(x)\partial_\mu\alpha(x). \tag{9.6}$$

This is not the free Dirac equation. The field $\psi'(x)$ is coupled to the vectorfield $\partial_\mu\alpha(x)$.

At the Lagrangian level we find

$$L(\psi') = \bar{\psi}'(i\gamma^\mu\partial_\mu - m)\psi' + \bar{\psi}'\gamma^\mu\psi'\cdot\partial_\mu\alpha . \tag{9.7}$$

From these considerations it is clear that a theory for electrons alone i.e. with no other forms of matter or radiation present is not invariant under local changes of the phase of the electronfield. Local phase transformations have far reaching consequences, they turn a free-field theory into a theory with interactions.

It is clear that, if we perform a second transformation $U(\beta) = e^{i\beta(x)}$ (9.6) and (9.7) keep the same form but they are not invariant. To construct a theory which allows for local gauge transformations, we must introduce an auxiliarly field $A_\mu(x)$ coupled to $J^\mu = \bar{\psi}\gamma^\mu\psi$.

Consider the Lagrangian

$$L(\psi,A) = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + q\bar{\psi}\gamma^\mu\psi A_\mu \tag{9.8}$$

(q is called the coupling parameter).

If we now perform the transformation on ψ

$$\psi \to \psi' = U(\alpha(x))\psi = e^{iq\alpha(x)}\psi \tag{9.9}$$

the Lagrangian $L(\psi,A)$ will be invariant if we also allow for a change of A_μ such that the term $q\bar{\psi}'\gamma^\mu\psi'\partial_\mu\alpha$ is compensated by the term arising from the change in A_μ. We must then take

(9.10) $$A_\mu \longmapsto A'_\mu = A_\mu + \partial_\mu \alpha.$$

The equation of motion corresponding to (9.8) is given by

(9.11) $$(i\gamma^\mu \partial_\mu - m)\psi = -q\bar{\psi}\gamma^\mu \psi A_\mu .$$

From now on we will call gauge transformations the combined transformation

(9.12) $$\begin{cases} \psi(x) \longmapsto \psi'(x) = U(\alpha(x))\psi(x) = e^{iq\alpha(x)}\psi(x) \\ A_\mu(x) \longmapsto A'_\mu(x) = A_\mu(x) + \partial_\mu \alpha \quad . \end{cases}$$

The Lagrangian (9.8) is invariant under gauge transformations. In the Lagrangian (9.8) the field A_μ does not play a dynamical rôle i.e. there is no equation of motion for A_μ. Adding to (9.8) the gauge invariant expression with

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

we obtain

(9.13) $$\begin{aligned} L &= \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \quad -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad + q\bar{\psi}\gamma^\mu \psi A_\mu \\ &= L_D \qquad\qquad\qquad + L_{e.m} \qquad + L_{int} . \end{aligned}$$

This is the Lagrangian for quantumelectrodynamics. In the applications L_{int} is treated as a perturbation term and one can make a perturbation expansion for scattering amplitudes.

The physical content of gauge invariance can be stated as follows:

A local change of phase of the electronfield $\psi(x)$, accompanied by the appropriate change in the potential $A_\mu(x)$ has no observable effects. $\{\psi(x),\ A_\mu(x)\}$ and $\{e^{iq\alpha(x)}\psi(x),\ A_\mu(x) + \partial_\mu \chi\}$ describe the same state.

REMARKS

1) The basic object in our considerations was the matter field $\psi(x)$. Requiring local gauge invariance led to the introduction of the radiation-field $A_\mu(x)$.

2) Gauge invariance forbids a term of the form $m^2 A_\mu A^\mu$. This means that the field A_μ is a massless field, mediating forces of infinite range.

3) The Lagrangian (9.13) can be rewritten in the form

$$L = \bar{\psi} i \gamma^\mu D_\mu \psi - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{9.14}$$

$$D_\mu \equiv \partial_\mu - iq A_\mu \, .$$

$D_\mu \psi$ is called the gauge covariant derivative of ψ. Performing a gauge transformation (9.12) we have

$$D'_\mu \psi' = (\partial_\mu - iqA'_\mu)\psi' = U(\alpha(x)) D_\mu \psi \tag{9.16}$$

i,e, $D_\mu \psi$ transforms in the same way as the field ψ. The field strength $F_{\mu\nu}$ may be defined as the "curvature"

$$\left[D_\mu , D_\nu\right] \psi = -ig F_{\mu\nu} \psi \, . \tag{9.17}$$

§10. NON-ABELIAN GAUGE TRANSFORMATIONS, YANG-MILLS FIELD

The idea of extending a global gauge symmetry to a local symmetry, thereby introducing interactions of the original field with gauge potentials, has been applied by Yang and Mills to the non-abelian group SU(2).
In their nowadays famous paper: "Conservation of Isotopic Spin and Isotopic gauge invariance" (Phys.Rev. 96, 191, (1954)) they argued that the global SU(2) invariance is not consistent with local field theory and should be extended to local SU(2) invariance.
The notion of isotopic (isobaric) spin is based on the experimentally well established charge independence of the nunclear forces. This means that the strong forces between two neutrons, a neutron and a proton, or between two protons are the same. This leads to the picture that proton and neutron can be considered as two independent states of one particle called the nucleon.
Let us denote these independent states by

$$(10.1) \qquad e_1 = \begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix}1\\0\end{pmatrix} .$$

The quantum states of a nucleon are then described by wave functions of the form

$$(10.2) \qquad \psi(\vec{r},t) = \begin{pmatrix}\psi_1(\vec{r},t)\\ \psi_2(\vec{r},t)\end{pmatrix} = \psi_1(\vec{r},t)e_1 + \psi_2(\vec{r},t)e_2$$

where ψ_1 and ψ_2 are four-component Dirac spinors. (Proton and neutron are particles with spin $s = \frac{1}{2}$). Conventionally one calles e_1 the proton state and e_2 the neutron state.
The probability density of finding the nucleon in the proton state is given by $|\langle e_1,\psi\rangle|^2 = |\psi_1(\vec{r},t)|^2$.
Instead of the basis (10.1) one can of course take another basis to describe the internal degrees of freedom of the nucleon.
Taking into account the normalisation condition

(10.3) $$\int \psi^\dagger(\vec{r},t)\psi(\vec{r},t)d^3x = \int \psi_1^\dagger\psi_1 + \psi_2^\dagger\psi_2)d^3x = 1$$

one easily finds that a change of basis must be given by a unitary transformation

(10.4) $$e'_a = Ue_a \qquad (a=1,2).$$

In matrixform U is given by

(10.5) $$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

with

(10.6) $$U^\dagger U = \mathbb{1}$$

From (10.6) follows that the determinant of U satisfies

$$|\det U| = 1, \quad \text{i.e.} \quad \det U = e^{i\alpha} \qquad (\alpha \in \mathbb{R})$$

Defining U' by

(10.7) $$U = e^{i\alpha}U'$$

we obtain a unitary matrix U' with $\det U' = 1$, i.e. $U' \in SU(2)$. As the factor $e^{i\alpha}$ is merely an overall phase factor we conclude that the isotopic spinstates of the nucleon are completely described by the group SU(2).

Lagrangian formalism, local SU(2).

The Lagrangian for the free nucleon field which is invariant under SU(2) is easily written down

(10.8) $$L_N(\psi) = \sum_{a=1}^{2} \bar{\psi}_a(i\gamma^\mu\partial_\mu - m)\psi_a = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi .$$

Both ψ_1 and ψ_2 satisfy the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi_a = 0 \qquad (a=1,2)\ . \tag{10.9}$$

The transformation

$$\psi \to \psi'(x) = U\psi(x), \quad U \in SU(2)$$

is a symmetry of (10.8), i.e. L_N is invariant

$$L_N(\psi) = L_N(\psi'). \tag{10.10}$$

Let us now see what happens to this symmetry if we make the SU(2) transformations space-time dependent. Suppose we have a mapping from Minkowski space to the group SU(2)

$$x \longmapsto U(x). \tag{10.11}$$

Applying $U(x)$ to a nucleon state $\psi(x) = \sum_1^2 \psi_a(x) e_a$ we obtain

$$\begin{aligned} \psi'(x) &= U(x)\psi(x) = \sum_1^2 \psi_a(x) U(x) e_a \\ &= \sum_1^2 \psi_a(x) e'_a(x). \end{aligned} \tag{10.12}$$

Instead of the original basis $\{e_a\}$ which is the same for all space-time point, we have now a local basis to describe the internal degrees of freedom. It will be clear that $\psi'(x)$ does not satisfy the Dirac equation. At the Lagrangian level it is easily seen that L_N is not invariant under local SU(2) transformations. From (10.8) and (10.12) one obtains

$$\begin{aligned} L_N &= \bar{\psi}' U (i\gamma^\mu \partial_\mu - m) U^{-1} \psi' \\ &= \bar{\psi}'(i\gamma^\mu \partial_\mu - m)\psi' + \bar{\psi}' i\gamma^\mu U \partial_\mu U^{-1} \psi' . \end{aligned} \tag{10.13}$$

To construct a theory for nucleons which is invariant under local transformations we must, just as in the abelian case, introduce new fields coupled to the nucleon field. In this case we need three vectorfields

$$\{A_\mu^a(x)\}_{a=1}^3$$

because we have three independent isotopic spin rotations in SU(2). To obtain the form of the interaction we consider the expression $U\,\partial_\mu U^{-1}$ occurring in (10.13). Writing $U(x)$ in the exponential form

$$U(x) = \exp(i\vec{\alpha}(x)\cdot\vec{\tau}/2) \quad (10.14)$$

with τ_1, τ_2, τ_3 the Pauli matrices we obtain for an infinitesimal transformation

$$U\,\partial_\mu U^{-1} = -i(\partial_\mu\vec{\alpha}(x))\cdot\vec{\tau}/2 + O(\alpha^2). \quad (10.15)$$

In general we have

$$U\,\partial_\mu U^{-1} = -i\vec{\beta}(x)\cdot\vec{\tau}/2. \quad (10.16)$$

This is a Lie algebra valued function.

To obtain an invariant Lagrangian we must introduce a Lie algebra valued vectorfield

$$\underline{A}_\mu = \sum_{a=1} A_\mu^a(x)\frac{\tau_a}{2} \quad (10.17)$$

and couple this field to the nucleon current.

Consider

$$L(\psi,A) = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi + g\bar{\psi}\gamma^\mu\underline{A}_\mu\psi \quad (10.18)$$

where g is a coupling constant.

$L(\psi,A)$ is invariant under local SU(2) transformations of the field ψ if we also transform the field $\underline{A}_\mu$, the transformation of $\underline{A}_\mu$ must be such that the term containing $U\,\partial_\mu U^{-1}$ is cancelled.

This local gauge transformation is easily found.

$$\psi(x) \rightarrow \psi'(x) = U(x)\psi(x)$$

(10.19)

$$\underline{A}_\mu(x) \rightarrow \underline{A}'_\mu(x) = U\underline{A}_\mu U^{-1} + \frac{i}{g} U \partial_\mu U^{-1} .$$

For the Lagrangian (10.18) we have

$$\begin{aligned} L(\psi',A') &= \bar{\psi}'(i\gamma^\mu \partial_\mu - m)\psi' + g\bar{\psi}'\gamma^\mu \underline{A}'_\mu \psi' \\ &= L(\psi,A). \end{aligned}$$

Just as in the case of the electromagnetic interaction we can introduce a gauge covariant derivative of ψ.

(10.20) $$\underline{D}_\mu(A)\psi \equiv (1\partial_\mu - ig\underline{A}_\mu)\psi(x)$$

From (10.19) and (10.20) one obtains

(10.21) $$\underline{D}_\mu(A')\psi'(x) = U(x)\underline{D}_\mu(A)\psi$$

or equivalently

(10.22) $$\underline{D}_\mu(A')U(x) = U(x)\underline{D}_\mu(A).$$

The Lagrangian (10.18) can be written in the manifestly invariant form

(10.23) $$L(\psi,A) = \bar{\psi} i\gamma^\mu \underline{D}_\mu(A)\psi - m\bar{\psi}\psi .$$

To obtain a model in which the fields $A^a_\mu(x)$ are not merely auxiliary objects we must construct an expression which is invariant under the local transformations and which determines the dynamics of the A-fields. Taking the electromagnetic case as an example we define a tensorfield $\underline{F}_{\mu\nu}$ by

(10.24) $$\left[\underline{D}_\mu, \underline{D}_\nu\right]\psi = -ig\underline{F}_{\mu\nu}\psi .$$

This gives

$$\underline{F}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{A}_\mu - ig\left[\underline{A}_\mu, \underline{A}_\nu\right] = \sum_{a=1}^{3} F^a_{\mu\nu} \frac{\tau_a}{2} . \tag{10.25}$$

The fields $\{A^a_\mu(x)\}_{a=1}^3$ are called the gauge potentials or the Yang-Mills fields, $\{F^a_{\mu\nu}\}_{a=1}^3$ are the Yang-Mills field strengths.

Using the commutation relations

$$\left[\tau_a, \tau_b\right] = 2i\varepsilon_{abc}\tau_c \tag{10.26}$$

one finds for the components of $\underline{F}_{\mu\nu}$

$$F^a_{\mu\nu} = \partial_\mu A^a_\mu - \partial^a_\nu A_\mu + g\varepsilon^{abc} A^b_\mu A^c_\nu \quad (\varepsilon_{abc} = \varepsilon^{abc}) . \tag{10.27}$$

Under the gauge transformation $U(x)$ the field tensor transforms as follows

$$\underline{F}_{\mu\nu} \to \underline{F}'_{\mu\nu} = \underline{F}_{\mu\nu}(A') = U(x)\underline{F}_{\mu\nu}(A)U^{-1}(x) . \tag{10.28}$$

One defines the Yang-Mills Lagrangian as the gauge invariant expression

$$L_{Y.M} = -\tfrac{1}{2}\mathrm{Tr}(\underline{F}_{\mu\nu}F^{\mu\nu}) = -\tfrac{1}{4}\sum_{a=1}^{3} F^a_{\mu\nu} F^{\mu\nu}_a . \tag{10.29}$$

Notice that, due to the quadratic term in (10.27), the Yang-Mills Lagrangian contains so-called self-interactions of the potentials of the form (AAA) and (AAAA). These give rise to the non-linearity of the equations of motion for the gauge potentials. Adding the expression (10.29) to (10.23) we obtain the total Lagrangian

$$L_{\text{invariant}} = \bar{\psi}(i\gamma^\mu \underline{D}_\mu - m)\psi - \tfrac{1}{2}\mathrm{Tr}\underline{F}_{\mu\nu}\underline{F}^{\mu\nu} . \tag{10.30}$$

This Lagrangian describes interactions of the nucleon field with the Yang-Mills potentials.

The procedure leading to the invariant Lagrangian can easily be generalized to models containing other matter fields and to other gauge groups.

In fact all gauge models studied nowadays are obtained along the lines described above. We briefly describe a model in which we have apart from the nucleon field three scalar fields $\{\phi_i\}_{i=1}^{3}$

(10.31) $$L = \bar{\psi}(i\gamma^{\mu}\partial_{\mu}-m)\psi + \sum_{i=1}^{3} \tfrac{1}{2}(\partial_{\mu}\phi_i\partial^{\mu}\partial_i - m^2\phi_i\phi_i).$$

Suppose the fields ϕ_1, ϕ_2, ϕ_3 belong to the real 3-dimensional orthogonal representation of SU(2).

A global SU(2) transformation of the fields is then given by

(10.32) $$\begin{cases} \psi \to \psi' = U(\vec{\alpha})\psi = e^{i\vec{\alpha}\cdot\vec{\tau}/2} \\ \phi \to \phi' = R(\vec{\alpha})\phi = (e^{\vec{\alpha}\cdot\vec{I}})\phi = (e^{-i\alpha\cdot(i\vec{I})})\phi \ . \end{cases}$$

The matrices I_1, I_2 and I_3 are given by

$$I_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} ; \quad I_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} ; \quad I_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The Lagrangian (10.31) is invariant under the transformation (10.32). This global invariance can be extended to local gauge invariance by replacing the derivatives $\partial_{\mu}\psi$ and $\partial_{\mu}\phi$ by gauge covariant derivatives. For the field ψ we have

$$\underline{D}_{\mu}(A)\psi = (\partial_{\mu}-ig\underline{A}_{\mu})\psi = (\partial_{\mu}-igA_{\mu}^{a}\frac{\tau_a}{2})\psi.$$

The gauge covariant derivative of ϕ is given by

(10.33) $$\underline{\nabla}\phi = (\partial_{\mu}+gA^{a}(x)I_a)\phi \ .$$

$A^a(x)I_a$ is a function on the 3-dimensional representation of the Lie algebra.

The total Lagrangian, invariant under local SU(2), is obtained from (10.31) by the replacements $\partial_\mu \psi \to \underline{D}_\mu \psi$, $\partial_\mu \phi \to \underline{\nabla}_\mu \phi$ and the addition of $L_{Y.M}$ (10.29).

$$(10.34) \qquad L = \bar{\psi}(i\gamma^\mu \underline{D}_\mu - m)\psi + \tfrac{1}{2}(\underline{\nabla}_\mu \vec{\phi} \cdot \underline{\nabla}_\mu \vec{\phi} - m^2 \vec{\phi} \cdot \vec{\phi}) - \tfrac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a \quad (\vec{\phi} = (\phi_1, \phi_2, \phi_3)).$$

In this model the ψ - particles and ϕ - particles interact via the massless gauge particles. At the time of the discovery of the non-abelian gauge fields there was no immediate application. The interactions of nucleons, being very short-ranged, could not be explained by the exchange of gauge particles. The gauge particles are massless and this leads to interactions of infinite range. Nevertheless, the model introduced by Yang and Mills has caused a major change in particle physics.

The standard model of high energy physics is based on the group $SU(3) \times SU(2) \times U(1)$. The problems of the mass of the gauge fields has been solved by the discovery of the Higgs-Kibble mechanism and spontaneous symmetry breaking. Moreover gauge theories are renormalizable field theories.

REMARK. For a more general approach to gauge field theories one should read the contribution of G. Bauerle to this volume.

§11. REMARKS ON THE FIBRE BUNDLE FORMALISM FOR GAUGE THEORIES

This contribution to the seminar, especially the sections on gauge theories may be rather confusing for mathematicians. It all looks quite artificial and ad hoc, moreover the notion of global and local gauge transformations does not coincide with mathematical usage. We will try to clarify some of these points by sketching briefly the fibre bundle formalism and by pointing out how sections 9 and 10 do fit in this formalism.

a. Principle fibre bundle.

To construct a principle fibre bundle one starts with a Lie group action on a manifold P.

$$P \times G \to P$$
$$(p,g) \to R_g p = pg.$$

For $p \in P$ one defines the orbits $O_p = \{pg | g \in G\}$. The projection Π from P to the orbit space M

$$\Pi\colon p \in P \to O_p \in M$$

is assumed to be a differentiable mapping. For $x \in M$, $\Pi^{-1}(x)$ is called the fibre above x. The fibre is the orbit O_p for $\Pi(p) = x$. Each fibre is diffeomorphic with the Lie group G.

Let $\{U_\alpha\}$ be an open covering of M, then $\Pi^{-1}(U_\alpha)$ is diffeomorphic with $U_\alpha \times G$, i.e. for each U_α there is a diffeomorphism T_α,

$$T_\alpha\colon \Pi^{-1}(U_\alpha) \to U_\alpha \times G$$
$$T_\alpha\colon p \mapsto (\Pi(p),\ \phi_\alpha(p) \in G)$$
$$T_\alpha\colon pg \mapsto (\Pi(pg),\ \phi_\alpha(pg))$$

with $$\phi_\alpha(pg) = \phi_\alpha(p)g$$

T_α is called a local trivialization of P.

b. Local sections of P.

A mapping s_α from U_α into $\Pi^{-1}(U_\alpha)$ with $\Pi \circ S_\alpha = \mathrm{Id}|_{U_\alpha}$ is called a local section of P.

Using a local section $x \in U_\alpha \longmapsto s_\alpha(x) \in \Pi^{-1}(U_\alpha)$ one can define a local trivialization T_α by putting

$$T_\alpha(s_\alpha(x)g) = (x,g).$$

This means that $T_\alpha(s_\alpha(x)) = (x,e)$ where e is the unit element of G.

From a local trivialization T_α one can construct a local section s_α

$$s_\alpha(x) = p[\phi_\alpha(p)]^{-1} = pg[\phi_\alpha(pg)]^{-1}.$$

This definition is such that

$$T_\alpha:\ s_\alpha(x) \to (x,\phi_\alpha(p[\phi_\alpha(p)]^{-1}) = (x,e).$$

A local section $s_\alpha(x)$ and the corresponding local T_α is called a local gauge for P. This nomenclature will become clear in subsection d). Consider now two overlapping regions U_α and U_β. For $x \in U_\alpha \cap U_\beta$ we have

$$s_\alpha(x) = p[\phi_\alpha(p)]^{-1} \quad \text{and} \quad s_\alpha(x) = p[\phi_\beta(p)]^{-1}.$$

This leads to the relation

$$s_\beta(x) = s_\alpha(x)\phi_\alpha(p)[\phi_\beta(p)]^{-1} = s_\alpha(x)g_{\alpha\beta}(x)$$

with

$$g_{\alpha\beta}(x) = \phi_\alpha(p)[\phi_\beta(p)]^{-1}.$$

The mapping $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$

$$x \longmapsto g_{\alpha\beta}(x)$$

is called a *local* gauge transformation. It will be clear that the adjective "local" means with respect to the covering $\{U_\alpha\}$. In this sense the transformations in (10.14) and (10.19) should be called global because it is implicitely assumed that one has a mapping from all of Minkowski space to the group G.

c. Connections on a principle fibre bundle.

A connection on a principle fibre bundle is a prescription to define uniquely vertical and horizontal vectorfields on P. Let $T_p(P)$ be the tangent space at $p \in P$. Given a connection on P we have

$$T_p(P) = V_p(P) \oplus H_p(P) .$$

Vertical vectors $X_p \in V_p(P)$ are tangent vectors to the fibre through p. The prescription is such that $H_{pg}(P)$ is related to H_p by the derivative R_{g*} of the right action $R_g p = pg$, i.e. $H_{pg} = R_{g*}H_p$.
One can equivalently define a connection as a Lie algebra valued one-form ω on P. The one-form ω has the following properties

$$\omega_p(H_p) = 0, \quad \omega_p(V_p) \in L(G)$$
$$R_g^*\omega = Ad(g^{-1})\omega$$

where Ad is the adjoint representation of G on its Lie algebra $L(G)$.

d. Local gauge fields.

Starting from a connection ω, a covering $\{U_\alpha\}$ and local sections $\{s_\alpha\}$ one can define a connection on M in the following way. On U_α we define the one-form ω_α by using the pullback s_α^*

$$\omega_\alpha(x) = s_\alpha^*\omega .$$

For $x \in U_\alpha \cap U_\beta$ we have

$$\omega_\alpha(x) = s_\alpha^*\omega \quad \text{and} \quad \omega_\beta = s_\beta^*\omega = (s_\alpha \circ g_{\alpha\beta})^*\omega .$$

One can prove (see the contribution of Dr. H. Pijls to this volume) that ω_α and ω_β are related by

$$\omega_\beta(x) = \mathrm{Ad}g_{\alpha\beta}^{-1}(x)\omega_\alpha(x) + L_{g_{\alpha\beta*}^{-1}}(x)\circ g_{\alpha\beta*}(x)$$

Choosing a base $\{\tau_a\}_{a=1}^{n}$ in the Lie algebra, $\omega_\alpha(x)$ can be written as

$$\omega_\alpha(x) = \sum_{a=1}^{n} \theta_\alpha^a(x)\tau_a$$

where θ_α^a is a one-form on U_α.
Choosing local coordinates $(x^1,\ldots,x^m)$ for $x \in U_\alpha$ we have

$$\theta_\alpha^a(x) = \sum_{i=1}^{m} (A_i^a(x))_\alpha dx^i \; .$$

The components $A_i^a(x)$ of this one-form are the gauge fields (Yang-Mills potentials). They are defined locally i.e. on the regions U_α. From the relation between ω_α and ω_β one obtains the relation between A_α and A_β on the overlap $U_\alpha \cap U_\beta$. We give this relationship for the case that G is a unitary matrix group. Denoting the matrices of G by R and the basis of the Lie algebra by $\{\tau_a\}$ one obtains

$$(A_j^a(x)\tau_a)_\beta = R_{\alpha\beta}^{-1}(x)(A_j^a\tau_a)_\alpha R_{\alpha\beta}(x) + R_{\alpha\beta}^{-1}(x)\partial_j R_{\alpha\beta}(x).$$

This formula resembles very much formula (10.19) which gives the behaviour of the Yang-Mills potentials under a gauge transformation. The above formula is, however, more general than (10.19).

In (10.19) it is implicitly assumed that the principle fibre bundle has a global trivialization T,

$$T\colon \Pi^{-1}(M) \to M\times G.$$

Using T one defines a global section S of P

$$S(x) = T^{-1}(x,e).$$

Defining now the section $S'(x) = S(x)g^{-1}(x)$ $(g(x) \in G_x)$ we obtain the pull-backs $\theta = S^*\omega$ and $\theta' = S'^*\omega$.

Writing $\Theta(x) = A^a_j(x)\tau_a$ and $\Theta'(x) = A'^a_j(x)\tau_a$ one obtains

$$A'^a_j(x)\tau_a = R(g(x))A^a_j(x)\tau_a R^{-1}(g(x)) + R(g(x))\partial_j R^{-1}(g(x)).$$

This is apart from the factor i/g formula (10.19). It applies to trivial principle fibre bundles only.

e. Gauge transformations on a P.F.B.

A gauge transformation f on a P.F.B. is defined as a diffeomorphism $f: P \to P$ with the following properties:

$$f(pg) = f(p)g$$

$$\Pi(p) = \Pi(f(p))$$

hence p and $f(p)$ are in the same fiber. Now right translations R_g also have the property $\Pi(p) = \Pi(R_g p) = \Pi(pg)$. In general a right translation is not a gauge transformation. This can be seen from $R_{g'}(pg) = pgg'$ and $(R_{g'}(p))g = pg'g$. In general $R_{g'}(pg) \neq (R_{g'}(p))g$. One can prove that given a connection ω on P, the pullback $\omega' = f^*\omega$ is again a connection.

f. Final remark.

In physics things are mostly the other way round i.e. one starts with the Yang-Mills potentials on a covering of some manifold. If the potentials satisfy the relationship discussed in subsection d. they give rise to a uniquely determined connection form on a principle fibre bundle.

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ELEMENTS OF GAUGE THEORY AND THEORY OF GRAVITY

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> ..... *modern theoretical physics is a luxuriant, totally Rabelaisian, vigorous world of ideas, and a mathematician can can find in it everything to satiate himself except the order to which he is accustomed* [1] .

CONTENTS

1. INTRODUCTION

A fundamental physical theory has to comply with the requirements of quantum mechanics and the theory of relativity. Quantum field theory is the most successful theory which satisfies these requirements. Actually, quantum field theory is a general scheme, which amalgamates in a natural way the principles of quantum mechanics and relativity [2]. The theory of elementary particles is almost exclusively written in the language of quantum field theory. Nevertheless the present article is about a certain class of classical i.e. non-quantum mechanical field theories. This is, however, not obsolete at all, because a quantum field theory is obtained by the application of one of the quantization methods to a classical field theory. A most favourable quantization method is, particularly in case of gauge field theories, Feynman's path integral quantization method [3].
Only after quantization of a classical field theory an interpretation in terms of particles becomes available. The latter interpretation is the actual goal in the description of physical reality, because matter appears to consist of particles. In the following we restrict ourselves to the first step: the construction of successful classical field theories. Quantization of a classical field theory is left out of the scope of the present article.
In this article we have no pretensions with regard to full mathematical rigour.

2. CLASSICAL LAGRANGIAN FIELD THEORY

In this section we start with some heuristic considerations and with a definition. Physical phenomena occur somewhere in space and happen in a certain stretch of time. The four-dimensional space-time manifold is the arena where physical (material) systems play their rôle. Important general concepts in the description of a physical system are: measurable quantities (observables) and states.

Let us now specialize to the case of a classical field theory. Suppose for the sake of definiteness that space-time M is the four-dimensional Minkowski space-time of special relativity. Let $(x^\mu) = (x^0, x^1, x^2, x^3, x^3)$ be a Lorentz coordinate system on M, such that $x^0 = ct$ where c is the velocity of light in vacuo.

In a classical field theory the state of the physical system is described by a set $\{\phi_k\}_{k=1}^N$ of complex-valued functions on space-time, called field,

$$\phi_k : (x^\mu) \to \phi_k(x^\mu) \in \mathbb{C} \qquad (k=1,\ldots,N).$$

The index k stands for all tensor and internal indices.

The electrical field strength $\vec{E}(t,\vec{x})$ and the magnetic induction $\vec{B}(t,\vec{x})$ of electrodynamics are prototypes of fields. An important example of a function of these fields is the energy-density $\varepsilon(t,x) := \frac{1}{2}(\vec{E}^2(t,\vec{x})+\vec{B}^2(t,\vec{x}))$ of the electromagnetic field. The energy-density is an example of a measurable quantity (observable).

More generally: an observable $\mathcal{O}$ is a function of the fields $\{\phi_k\}$ and their derivatives $\mathcal{O} = \mathcal{O}(\phi_1,\ldots,\phi_N, \partial_\mu\phi_1,\ldots,\partial_\mu\phi_N)$. The function $\mathcal{O}$ is mostly a polynomial.

The evolution of the system, i.e. the time-dependence of the fields, is determined by the field equations

$$(2.1) \qquad \partial_\mu \frac{\partial L}{\partial(\partial_\mu \phi_k)} - \frac{\partial L}{\partial \phi_k} = 0 \qquad (k=1,\ldots,N)$$

(summation convention and $\partial_\mu := \frac{\partial}{\partial x^\mu}$), where $L = L(\phi_k, \partial_\mu \phi_k)$ is a given function, the so-called Lagrangian density or Lagrangian for short. The field equations (2.1) are called Lagrangian equations. Not every function is acceptable as a Lagrangian, for instance $L = \phi$ leads to a contradiction. Hence the field equations are partial differential equations for the fields. It is to be noted that the only input for a quantum field theory is the specification of the classical fields $\{\phi_k\}$ and a Lagrangian L.

In the preceding lectures the Lagrangian of the Dirac field and the Maxwell field were introduced. After quantization the first describes a system of free electrons and positrons and the second a system of photons. A system of electrons, positrons and photons in interaction is described at the level of a classical field theory by the

$$(2.2) \qquad L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma^\mu \psi A_\mu$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ and the tensor field $F_{\mu\nu}$ is called the *field-strength tensor*. The first term on the right-hand side of (2.2) is the Dirac Lagrangian and $\{\gamma^\mu\}$ are the 4×4 - Dirac matrices, characterized by the anti-commutation relations $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ $((\eta^{\mu\nu}) := \mathrm{diag}(+1,-1,-1,-1))$. The second term in (2.2) is the Lagrangian of the electromagnetic field. Both are discussed in the preceding article. The last term of (2.2) is responsible for the (electromagnetic) interaction of the spin-$\frac{1}{2}$ particles (electrons, positrons). The original introduction of the Lagrangian (2.2) in physics was based on a mixture of phenomelogy and theoretical ideas. After quantization it gives a theory called quantum electrodynamics (Q.E.D), which predicts experimental results with an unprecedented accuracy. Putting $\phi_k = \bar{\psi}$ resp. A_μ the field equations follow from (2.1) and (2.2):

(2.3) $(i\gamma^{\mu}\partial_{\mu}-m)\psi = e\gamma^{\mu}A_{\mu}\psi$ (Dirac)

(2.4) $\partial_{\nu}F^{\nu\mu} = e\bar{\psi}\gamma^{\mu}\psi$ (Maxwell)

The Lagrangian equations of notion are equivalently to the action principle (variational principle of Hamilton). The action S is defined as the functional

(2.5) $$S: (\phi_k) \longmapsto S[\phi_k] := \int_{\Omega} L(\phi,\partial_{\mu}\phi)d^4x$$

where Ω is a region in space-time M. The action principle reads

(2.6) $$\frac{\delta S}{\delta\phi_k(x^{\mu})} = 0 \qquad (k=1,\ldots,N;\ (x^{\mu}) \in \Omega)$$

where on the left the variational (functional) derivative occurs. Equation (2.6) is equivalent with (2.1) and (2.6) requires the action to be stationary at solutions of the field equations (2.1).

DEFINITION. A Lagrangian field theory is a triple $(M, \{\phi_k\}_{k=1}^{N}, L)$ where M is the space-time manifold, $\{\phi_k\}$ is a set of fields on M and $L = L(\phi_k,\partial_{\mu}\phi_k)$ is a function, called the Lagrangian, such that the field equation (2.1) are a consistent set of equations.

The field equations (2.1) are Lorentz-covariant equations if the Lagrangian is a Lorentz-scalar. In the following we suppose that this is the case.

3. SYMMETRY AND CONSERVATION LAWS IN LAGRANGIAN FIELD THEORY

In this section the connection between symmetries of the Lagrangian and conservation laws is discussed.

An *internal symmetry transformation* is a map $(\phi_k) \longmapsto (\hat{\phi}_k)$ where

(3.1) $$\hat{\phi}_k(x^\mu) = F_k(\phi_1(x^\mu),\ldots,\phi_N(x^\mu))$$

such that the Lagrangian is invariant under (3.1) i.e.

(3.2) $$L(\phi_k,\partial_\mu\phi_k) = L(\hat{\phi}_k,\partial_\mu\hat{\phi}_k).$$

Let $\hat{\phi}_1 = F_k(\phi_1,\ldots,\phi_N,\alpha)$ be a differentiable one-parameter family of symmetry transformations, then

(3.3) $$\hat{\phi}_k(x^\mu) = \phi_k(x^\mu) + \left.\frac{dF_k}{d\alpha}\right|_{\alpha=0} \alpha + o(\alpha^2) \quad \text{for} \quad \alpha \to 0.$$

Equations (2.1), (3.2) and (3.3) imply with $G_k := \left.\frac{dF}{d\alpha}\right|_{\alpha=0}$

(3.4) $$0 = \left.\frac{dL(\hat{\phi}_k,\partial_\mu\hat{\phi}_k)}{d\alpha}\right|_{\alpha=0} = \frac{\partial L}{\partial\phi_k}G_k + \frac{\partial L}{\partial(\partial_\mu\phi_k)}\partial_\mu G_k = \partial_\mu\left(\frac{\partial L}{\partial(\partial_\mu\phi_k)}G_k\right)$$

(summation convention for μ and k). The vector field

(3.5) $$J^\mu := \frac{\partial L}{\partial(\partial_\mu\phi_k)}G_k$$

is called a current. From (3.4) and (3.5) follows

(3.6) $$\partial_\mu J^\mu = 0$$

and this is a so-called differential conservation law. A current satisfying (3.6) is called a conserved current. Hence the existence of a differentiable family of internal symmetry transformations implies the existence of a conserved current. This statement is a special case of Noether's theorem. A special case of Klein's theorem reads: Let J^μ be a vector field on Minkowski space-time satisfying the differential conservation law (3.6) and

(3.7) $$Q := \int J^0 d^3x \qquad (d^3x = dx^1dx^2dx^3)$$

then Q is a scalar and time-independent (conserved) i.e.

(3.8) $$\frac{dQ}{dt} = 0.$$

Q is called a *conserved charge*. The above family of symmetry transformations leads to the existence of a conserved charge. In physics one observes the conservation laws and concludes therefrom to symmetries in Lagrangian field theories. More precisely, one obtains a Lie group of symmetry transformations

$$\hat{\phi}_k = F_k(\phi_1,\ldots,\phi_N,\alpha^1,\alpha^2,\ldots,\alpha^r) \qquad (\alpha^i \in \mathbb{R})$$

where (α^i) are the coordinates of a Lie group element. Such a Lie group is called a *symmetry group*.

EXAMPLE.

The Lagrangian of the Dirac field ψ is given by

(3.9) $$L = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$$

where $\bar{\psi} := \psi^\dagger\gamma^0$ and c and $\hbar$ are taken to be equal to one. Note that the transformation $\psi \to \hat{\psi}$ where

(3.10) $$\hat{\psi}(x^\mu) = e^{-i\alpha}\psi(x^\mu) \qquad (\alpha \in \mathbb{R})$$

is a symmetry transformation for all $\alpha \in \mathbb{R}$. The symmetry group is $U(1)$, the group of all unitary 1×1 - matrices. From (3.5), (3.9) and (3.10) one finds the conserved current

(3.11) $$J^\mu = \frac{\partial L}{\partial(\partial_\mu\psi)}(-i\psi) = \bar{\psi}\gamma^\mu\psi.$$

This is up to a multiplicative constant the *electromagnetic current*

(3.12) $$J^{\mu}_{e.m.} := e\bar{\psi}\gamma^{\mu}\psi$$ (e = charge of the electron)

which occurs in the right-hand side of the inhomogeneous Maxwell equation (2.4). The conserved quantity

(3.13) $$Q := \int j^{0}_{e.m.} d^{3}x$$

is the total electric charge.

4. QUANTUM ELECTRODYNAMICS REGAINED

In section 3 it was noted that the Dirac Lagrangian (3.9) has a global U(1) - symmetry, given by (3.10). In physics the adjective "global" means that α in (3.10) is independent of (x^μ). Now we consider the so-called local transformation $\psi \to \hat{\psi}$ given by

$$\hat{\psi}(x^\mu) = e^{-i\alpha(x^\mu)}\psi(x^\mu) \qquad (\alpha(x^\mu) \in \mathbb{R}) \tag{4.1}$$

where α depends on (x^μ). If α is not constant, then L is not invariant under (4.1), because of the second term in the right-hand side of

$$\partial_\mu\psi \longmapsto \partial_\mu\hat{\psi} = e^{-i\alpha(x^\mu)}\partial_\mu\psi + (\partial_\mu e^{-i\alpha(x^\mu)})\psi. \tag{4.2}$$

The Dirac Lagrangian becomes invariant under (4.1) if we replace the derivative $\partial_\mu\psi$ by the *gauge-covariant derivative* $D_\mu\psi$, defined by

$$D_\mu\psi := \partial_\mu\psi + ieA_\mu\psi \qquad (e \in \mathbb{R}) \tag{4.3}$$

and satisfying the additional requirement (4.4) (see below). The constant e is called a coupling-constant and the substitution $\partial_\mu\psi \longmapsto D_\mu\psi$ is called *minimal coupling*. This substitution already shows up in elementary quantummechanics e.g. the Schrödinger equation of a charged particle in an electromagnetic field. The field $A_\mu = A_\mu(x^k)$ is called a *gauge potential (connection)* and it will become the electromagnetic four-potential. The additional requirement in the definition of the gauge-covariant derivative $D_\mu\psi$ is that it transforms like ψ (i.e. (4.1)), hence

$$D_\mu(\hat{A})\hat{\psi}(x^\lambda) = e^{-i\alpha(x^\lambda)}D_\mu(A)\psi(x^\lambda). \tag{4.4}$$

The formulae (4.3) and (4.4) imply the following transformation law of the gauge potential

$$A_\mu \to \hat{A}_\mu = A_\mu + \frac{i}{e}\, e^{i\alpha}\partial_\mu e^{-i\alpha} = A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{4.5}$$

The free Dirac Lagrangian $L = L(\psi,\partial_\mu\psi,\bar{\psi})$ becomes after minimal coupling

(4.6)
$$\begin{aligned}\tilde{L} = \tilde{L}(\psi,\partial_\mu\psi,\bar{\psi},A_\mu) &:= L(\psi,D_\mu\psi,\bar{\psi}) \\ &= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu\psi A_\mu .\end{aligned}$$

The Lagrangian $\tilde{L}$ is invariant under the *gauge transformation (local symmetry transformation)*

(4.7)
$$\begin{cases}\psi(x^\mu) \to \hat{\psi}(x^\mu) = e^{-i\alpha(x^\mu)}\psi(x^\mu) \\ A_\mu(x^\kappa) \to \hat{A}_\mu(x\ ) = A_\mu(x^\kappa) + e^{-1}\partial_\mu\alpha(x^\kappa)\end{cases}$$

i.e.

(4.8)
$$\tilde{L}(\psi,\partial_\mu\psi,\bar{\psi},A_\mu) = \tilde{L}(\hat{\psi},\partial_\mu\hat{\psi},\bar{\hat{\psi}},\hat{A}_\mu).$$

The Lagrangian equations of $\tilde{L}$ with respect to A_μ do not give an equation of motion for A_μ, because they read $\bar{\psi}\gamma^\mu\psi = 0$. To cure this a "kinetic energy term" $L_\gamma = L_\gamma(A_\mu,\partial_\nu A_\mu)$ is added to $\tilde{L}$. In a theory with the Lagrangian (4.6) A_μ is considered as an *external field*, which has to be given, and in $\tilde{L} + L$ it is a *dynamical variable*. We require that L_γ is Lorentz-invariant, gauge invariant and quadratic (because of the superposition principle). The gauge-invariance $L_\gamma(A_\mu,\partial_\nu A_\mu) = L_\gamma(\hat{A}_\mu,\partial_\nu\hat{A}_\mu)$ implies that L_γ only depends on

(4.9)
$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu .$$

The tensor $F_{\mu\nu}$ is called the *field strength tensor*. There are only two independent quadratic scalars of $F_{\mu\nu}$, i.e.

$$F_{\mu\nu}F^{\mu\nu}, \qquad \varepsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}.$$

The latter is a four-divergence and its Lagrange derivative is zero. Hence it does not contribute to the Lagrangian field equations. Hence we put

(4.10) $$L_\gamma = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

with a suitable normalization. It is to be noted that under a gauge transformation $\hat{F}_{\mu\nu} = F_{\mu\nu}$ i.e. $F_{\mu\nu}$ is gauge-invariant. Now there is a reason to be surprised. We started with the free Dirac Lagrangian, which has a global U(1) - invariance. We modify this into a local U(1) - invariance (minimal coupling) and finally turn the connection A_μ into a dynamical variable by adding L_γ to $\tilde{L}$, and we get the Lagrangian (2.2)

(4.11) $$L = \tilde{L} + L_\gamma = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} .$$

It is to be noted that for the field strength tensor

(4.12) $$F_{\mu\nu} = (ie)^{-1}[D_\mu, D_\nu]$$

holds, where [,] denotes the commutator.

Starting with the free Dirac Lagrangian the above recipe gives the Lagrangian (4.11) which is at the basis of quantumelectrodynamics. The free Dirac Lagrangian describes after quantization a system of non-interacting electrons and positrons. Hence by the above procedure this theory is turned into quantumelectrodynamics i.e. a theory of electrons, positrons and photons interacting electromagnetically.

However, at least the following four fundamental interactions manifest themselves in nature

gravitational interaction	(e.g. earth-moon)
weak interaction	(e.g. $n \to p+e+\bar{\nu}_e$)
electromagnetic interaction	(e.g. $e^+ + e^- \to \gamma+\gamma$)
strong interaction	(e.g. $p+n \to p+n$) .

Hence the above successful scheme calls for a generalization. In the following section this generalization is presented.

5. GAUGING A LAGRANGIAN FIELD THEORY

Physics (phenomenology and theoretical ideas) gives us the motivation to write down in concrete cases a specific Lagrangian L_0 with fields $(\phi_k) = (\phi_1,\ldots,\phi_N) := \phi$, i.e.

(5.1) $$L_0 = L_0(\phi_k, \partial_\mu \phi_k)$$

which has a global r-dimensional symmetry Lie group

(5.2) $$G = G_1 \times G_2 \times \ldots \times G_n \times \underbrace{U(1) \times \ldots \times U(1)}_{m} \qquad (n,m=0,1,\ldots)$$

where G_i $(i=1,\ldots,n)$ is a simple and compact Lie group. This means that the fields ϕ transform according to a unitary representation U of the group G, i.e.

(5.3) $$\phi \to \hat{\phi} = U(g)\phi$$

and

(5.4) $$L_0(\phi, \partial_\mu \phi) = L_0(\hat{\phi}, \partial_\nu \hat{\phi}) \ .$$

The Lagrangian L_0 and the symmetry group G are the generalization of the free Dirac Lagrangian and the symmetry group $U(1)$ of section 4. Now we discuss minimal coupling for the symmetry group G [4] [5]. Let the (representation of the) group G, be parametrized in a neighbourhood of $e \in G$ by canonical coordinates (α^i) i.e.

(5.5) $$U(g(\alpha^1,\alpha^2,\ldots,\alpha^r)) = \exp(\alpha^a T_a) \ ,$$

where $\{T_a\}_{a=1}^{r}$ is a basis of the Lie algebra $L(U(G))$ of $U(G)$ satisfying

(5.6) $$[T_a, T_b] = C_{ab}{}^{c}\, T_c$$

and normalized such that

$$(5.7)\qquad \langle T_a, T_b \rangle = \delta_{ab}$$

where $\langle\ ,\ \rangle$ denotes the Cartan-Killing inner product. If the transformations (5.3) are global transformations on M, i.e. the parameters (α^a) are independent of (x^μ), then (5.4) holds for all (α^a) and G is a symmetry group. Under the local symmetry transformation

$$(5.8)\qquad \hat{\phi}(x^\mu) = U(g(\alpha^1(x^\mu),\ldots,\alpha^r(x^\mu))\ \phi(x^\mu)$$

the derivative transforms inhomogeneously

$$(5.9)\qquad \begin{aligned} \partial_\mu \hat{\phi} &= U(g)\partial_\mu \phi + (\partial_\mu U(g))\phi \\ &= U(g)[\partial_\mu \phi + U^{-1}(g)\partial_\mu U(g)]\phi\ . \end{aligned}$$

The analogy with (4.2) is obvious. Although L_0 is invariant under the global symmetry transformation of G, it is in general not invariant under the local transformations (5.8) due to the second term in the right-hand side of equation (5.9). Again the Lagrangian can be made invariant under the local symmetry transformation (5.8) if the derivative $\partial_\mu \phi$ is replaced by the gauge-covariant derivative

$$(5.10)\qquad D_\mu \phi := \partial_\mu \phi + A_\mu \phi\ .$$

This substitution is again called minimal coupling. The rôle of the term $A_\mu \phi$ in (5.10) is to compensate the term $(\partial_\mu U(g))\phi$ in (5.9), which is responsible for the non-invariance of the Lagrangian L_0 under (5.8). Note that $U^{-1}(g)\partial_\mu U(g)$ lies in the Lie algebra $L(G)$ of G, hence we choose $A_\mu \in L(G)$. Hence

$$(5.11)\qquad A_\mu = gA_\mu^a T_a \in L(U(G)) \qquad\qquad (g\in\mathbb{R}).$$

The vector fields A^a_μ $(a=1,\ldots,r)$ are called *gauge potentials*. The real number g in (5.11) is called a coupling constant and for simplicity we have restricted in (5.11) to the case of a simple compact symmetry group G. In the general case where G has the form (5.2) each factor gets its own coupling constant. Let the basis of the Lie algebra of G_i $(i=1,\ldots,n)$ be $\{T^{(i)}_{a_i}\}^n_{i=1}$ satisfying (cf. (5.6) and (5.7))

$$\left[T^{(i)}_{a_i}, T^{(j)}_{b_j}\right] = \delta^{ij}\, C_{a_i b_j}^{\;\;c_i} T_{c_i} \tag{5.12}$$

and

$$< T_{a_i}, T_{b_i} > = \delta_{a_i b_i} \tag{5.13}$$

then (5.11) reads in the general case

$$A_\mu = \sum_{i=1}^{n} g_i A_\mu^{(i)a_i} T_{a_i} + \sum_{j=n+1}^{n+m} g_j A_\mu^{(j)} . \tag{5.14}$$

The gauge covariant derivative has the transformation law

$$D_\mu(A)\phi \to D_\mu(\hat{A})\hat{\phi} = U(g)(D_\mu(A)\phi) . \tag{5.15}$$

From (5.9), (5.10) and (5.15) the transformation rule

$$\hat{A}_\mu = U(g)A_\mu U(g)^{-1} - \partial_\mu U(g))U^{-1}(g) \tag{5.16}$$

follows.

After minimal coupling (5.10) in (5.1) the Lagrangian becomes

$$\tilde{L} := L_0(\phi_k, D_\mu\phi_k) . \tag{5.17}$$

Just as (4.6) $\tilde{L}$ does not give a field equation for the gauge potentials A^a_μ. To cure this a kinetic energy term $L_g = L_g(A^a_\mu, \partial_\nu A^a_\mu)$ is added to $\tilde{L}$. The system is then described by the Lagrangian

(5.18) $$L = \widetilde{L} + L_g$$

and the gauge field is then a dynamical variable. Again L_g is required to be gauge and Lorentz invariant. The Lagrangian L_g of the gauge fields is constructed from the field strength tensor (curvature) which is defined by (cf. (4.12))

(5.19) $$F_{\mu\nu} := [D_\mu(A), D_\nu(A)] = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \in L(U(G)).$$

The components $F^a_{\mu\nu}$ of $F_{\mu\nu}$ are defined by

(5.20) $$F_{\mu\nu} = gF^a_{\mu\nu}T_a$$

or the obvious generalization in case of (5.14). Hence

(5.21) $$F^a_{\mu\nu} = \partial_\mu A^a_\nu + gA^b_\mu A^c_\nu C^a_{bc}.$$

If G is non-Abelian, then $F^a_{\mu\nu}$ contains a term quadratic in the gauge-potentials. Formule (5.19) implies

(5.22) $$F_{\mu\nu} \to \hat{F}_{\mu\nu} = U(g)F_{\mu\nu}U^{-1}(g).$$

For a non-Abelian group $F_{\mu\nu}$ is not gauge-invariant, but gauge-covariant. The gauge fields are made into a dynamical variable by adding their kinetic energy

(5.23) $$L_g = -\frac{1}{4g^2}<F_{\mu\nu}, F^{\mu\nu}> = \frac{-1}{4}\sum_a F^a_{\mu\nu}F^{a\mu\nu}$$

to $\widetilde{L} = L_0(\phi, D_\mu\phi)$. This results in the following Lagrangian of the gauge field theory

(5.24) $$L = L_0(\phi, \partial_\mu\phi + A_\mu\phi) - \frac{1}{4}\sum_{a=1}^{r} F^a_{\mu\nu}F^{a\mu\nu}.$$

Substitution of (5.21) into (5.23) shows that the gauge fields of non-Abelian groups have cubic and quartic terms in the Lagrangian. The field equations of non-Abelian gauge fields are not linear and consequently the superposition principle does not hold.

Let us summarize the recipe of this section. Input is a Lagrangian field theory with Lagrangian $L_0 = L_0(\phi, \partial_\mu \phi)$ and a global symmetry group (5.2). This Lagrangian is made invariant under local symmetry transformations (gauge transformations) (5.8) and (5.16) by minimal coupling (5.10). Finally, the gauge fields are turned into dynamical variables by adding the Lagrangian L_g (5.23) of the gauge field. The end-result is the Lagrangian L of (5.24). The gauge fields herein play an analogous rôle as the photon field A_μ in quantum electrodynamics. The gauge fields A^a_μ mediate the interactions.

The standard model of high-energy physics [3], [6], [7] is a very successful gauge theory with the symmetry group

$$G = SU(3) \times SU(2) \times U(1)$$

which describes the strong, electromagnetic, and weak interaction. The Lagrangian L_0 is build up from scalar and spinor fields. The spinor fields describe the quarks and leptons.

Gauge fields have a tendency to give a long range interaction in a perturbation theoretical approach and this is not acceptable for the weak interaction, which is short-ranged. In order that the weak interactions are short-ranged the corresponding gauge fields are given mass by the so-called Higgs mechanism (spontaneous symmetry breakdown). The Higgs-mechanism is induced by the scalar fields in the Lagrangian L_0. The gauge fields of the factors U(1) and SU(2) of G describe the electroweak interaction. This part of the theory is called Glashow-Weinberg-Salam theory. The gauge fields of SU(3) describe the strong (inter-quark) interactions. This part of the theory is called quantum chromodynamics. The strong interactions are supposed to get their short range by the confinement-mechanism, which is still a topic of current research.

A kind of unification of the three mentioned interactions can be obtained by taking a simple Lie group G which contains $SU(3) \times SU(2) \times U(1)$ as a subgroup. Candidates are $SU(5)$, $SO(10)$, ... Theories of this kind are called grand unified theories (GUT). Interesting consequences of those theories are the instability of the proton, which is under current experimental investigation.

6. EINSTEIN-CARTAN-KIBBLE-SCIAMA THEORY

In this section we show that along similar lines one can establish Einstein's general theory of relativity [8], [9], [10]. Our first objective is the generalization of the Dirac Lagrangian (3.9) to a space-time manifold which is not necessarily Minkowskian. Let space-time M be a four-dimensional Riemannian manifold, with a metric of signature -2. In order to define spinor fields on M we have to introduce tetrad or vierbein fields. A tetrad field is a set of four vector fields $\{e_a\}_{a=0}^{3}$ on M such that they form an orthonormal basis of the tangent space $T_p(M)$ at each point $p \in M$. Denoting the covariant metric tensor field by g this means

$$(6.1) \qquad g(e_a, e_b) = \eta_{ab} \qquad (a,b=0,1,2,3)$$

where $(\eta_{ab}) := \mathrm{diag}(1,-1,-1,-1) =: (\eta^{ab})$ is the Minkowski metric. The Latin indices $a, b, \ldots$ are called bein indices. In a coordinate basis equation (6.1) reads

$$(6.2) \qquad g_{\mu\nu}(x^\sigma) e^\mu_a(x^\sigma) e^\nu_b(x^\sigma) = \eta_{ab}.$$

where $e_a = e^\mu_a \partial_\mu$. Under a coordinate transformation $(x^\mu) \to (x^{\mu'})$ the components of e_a transform as

$$(6.3) \qquad e^{\mu'}_a(x^{\kappa'}) = \frac{\partial x^{\mu'}}{\partial x^\mu} e^\mu_a(x^\kappa),$$

where the Greek indices $\mu, \mu', \ldots$ are called world indices. In $T_p(M)$ there are many orthonormal bases and anyone of them is transformed into another one by a Lorentz transformation. In general these Lorentz transformations Λ depend on the point $p \in M$ and they are called local. Let (x^σ) be the coordinates of $p \in M$ then the transformation rule of the tetrad fields under a local Lorentz transformation $\Lambda^a_{a'}(x^\sigma)$ reads

$$(6.4) \qquad e^\mu_{a'}(x^\sigma) = \Lambda^a_{a'}(x^\sigma) e^\mu_a(x^\sigma),$$

where $(\Lambda^a_{\ a'}(x^\sigma))$ is a Lorentz transformation matrix. The dual basis $\{e^a\}$ of $\{e_a\}$ is defined by

(6.5) $$e^\mu_a e^b_\mu = \delta^b_a .$$

Hence

(6.6) $$e^\mu_a e^a_\nu = \delta^\mu_\nu$$

and this implies together with (6.2)

(6.7) $$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu .$$

From (6.7) it is evident, that the vierbein fields determine the metric. A Dirac spinor field is defined as a four-component complex-valued field with transformation rules under local Lorentz transformations and coordinate transformations which are now formulated. A local Lorentz transformation is given by

(6.8) $$\Lambda(\omega_{ab}(x)) = \exp(-\tfrac{1}{2}\omega_{ab}(x)K^{ab}) \qquad (\omega_{ab}\in\mathbb{R}) ,$$

where $\omega_{ab} = -\omega_{ba}$ and K^{ab} is an infinitesimal generator of the Lorentz group, defined by

(6.9) $$(K^{ab})^c_{\ d} := \eta^{bc}\delta^a_d - \eta^{ac}\delta^b_d = -(K^{ba})^c_{\ d} .$$

A Dirac spinor field transforms under a local Lorentz transformation (6.8) as

(6.10) $$\psi(x) \to \hat{\psi}(x) = S(\omega_{ab}(x))\psi(x),$$

where

(6.11) $$S(\omega_{ab}) := \exp\left(\frac{-i}{4}\omega_{ab}\sigma^{ab}\right), \quad \sigma^{ab} := \frac{i}{2}[\gamma^a, \gamma^b] ,$$

and γ^a (a=0,1,2,3) are the Dirac matrices, characterized by their anti-commutation relations

$$(6.12) \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab} .$$

Actually we restrict ourselves to a sufficiently small neighbourhood of the identity, because otherwise we have to consider the covering group $SL(2,\mathbb{C})$ of the Lorentz group.

Under a coordinate transformation each of the four components of the Dirac spinor field transforms as a scalar field, i.e.

$$(6.13) \qquad \psi'(x^{\mu'}) = \psi(x^\mu).$$

From (6.13) it follows that the derivative $\partial_\mu\psi$ transforms as a vector under coordinate transformations. Under local Lorentz transformations $\partial_\mu\psi$ transforms inhomogeneously.

$$(6.14) \qquad \partial_\mu\psi \to \partial_\mu\hat{\psi} = S(\omega)\partial_\mu\psi + (\partial_\mu S(\omega))\psi,$$

which follows by differentiating (6.10). Again a gauge-covariant derivative is defined

$$(6.15) \qquad \nabla_\mu\psi(x) := \partial_\mu\psi(x) + \frac{i}{4}\sigma^{ab}\omega_{\mu ab}(x)\psi(x) ,$$

such that it transforms homogeneously under local Lorentz transformation i.e.

$$(6.16) \qquad \hat{\nabla}_\mu\hat{\psi} = S(\omega)\nabla_\mu\psi ,$$

and as a vector field under coordinate transformations. The field $\omega_{\mu ab} = -\omega_{\mu ba}$ is called the spin connection. Under a coordinate transformation it transforms as a vector field

$$(6.17) \qquad \omega_{\mu' ab}(x') = \frac{\partial x^\mu}{\partial x^{\mu'}}\omega_{\mu ab}(x)$$

The spin connection transforms under a local Lorentz transformation as

(6.18) $$\hat{\omega}_{\mu ab}(x)\sigma^{ab} = \omega_{ab}(x)S\sigma^{ab}S^{-1} + \frac{4}{i}S(\partial_\mu S^{-1}) .$$

The gauge covariant derivative of $\bar{\psi} := \psi^\dagger\gamma^0$ can be obtained in the following way. Because $\bar{\psi}\psi$ transforms as a scalar field under local Lorentz transformations and coordinate transformations we have

(6.19) $$\nabla_\mu(\bar{\psi}\psi) = \partial_\mu(\bar{\psi}\psi)$$

and by imposing as usual Leibniz's rule on gauge-covariant derivatives

(6.20) $$\nabla_\mu\bar{\psi} = \partial_\mu\bar{\psi} - \frac{i}{4}\bar{\psi}\sigma^{ab}\omega_{\mu ab} =: \bar{\psi}\overleftarrow{\nabla}_\mu$$

follows. A vector field v has bein- and world-components

(6.21) $$v = v^a e_a = v^\mu\partial_\mu .$$

The gauge covariant derivative of a bein vector v^a is given by

(6.22) $$\nabla_\mu v^a = \partial_\mu v^a + \omega_{\mu c}{}^a v^c,$$

as can be seen by considering the prototype

$$\begin{aligned}\nabla_\mu(\bar{\psi}\gamma^a\psi) &= (\nabla_\mu\bar{\psi})\gamma^a\psi + \bar{\psi}\gamma^a(\nabla_\mu\psi)\\ &= \partial_\mu(\bar{\psi}\gamma^a\psi) + \frac{i}{4}\omega_{\mu bc}\,\bar{\psi}[\gamma^a, \sigma^{bc}]\psi\\ &= \partial_\mu(\bar{\psi}\gamma^a\psi) + \omega_{\mu c}^{a}\,\bar{\psi}\gamma^c\psi\end{aligned}$$

because

$$[\gamma^a, \sigma^{bc}] = 2i(\eta^{ab}\gamma^c - \eta^{ac}\gamma^b) .$$

In the covariant derivative of a quantity with world and bein indices the spin connection appears for each bein index and the affine connection $\Gamma^\kappa_{\nu\mu}$ for each world index. An example is given by

$$\nabla_\mu T^a_\nu = \partial_\mu T^a_\nu - \Gamma^\kappa_{\nu\mu} T^a_\kappa + \omega^a_{\mu c} T^c_\nu . \tag{6.23}$$

Equation (6.21) implies

$$v^a e^\mu_a = v^\mu , \tag{6.24}$$

hence

$$v^\mu e^a_\mu = v^a . \tag{6.25}$$

By contraction the tetrad fields e^μ_a and their duals e^a_μ transform world indices into bein indices and the other way around. This generalizes to tensors e.g.

$$e^a_\nu T^\nu_\mu = T^a_\mu . \tag{6.26}$$

Substitution of $T^\nu_\mu = \nabla_\mu v^\nu$ into (6.26) gives, using Leibniz's rule

$$e^a_\nu(\nabla_\mu v^\nu) = \nabla_\mu v^a = \nabla_\mu (e^a_\nu v^\nu) = e^a_\nu(\nabla_\mu v^\nu) + (\nabla_\mu e^a_\nu) v^\nu .$$

for all v^ν. Hence

$$\nabla_\mu e^a_\nu = 0. \tag{6.27}$$

Together with (6.7) this implies the metric postulate

$$\nabla_\mu g_{\kappa\lambda} = 0 \tag{6.28}$$

Equation (6.27) is equivalent with (cf. (6.23))

$$\Gamma^\nu_{\kappa\mu} = e^\nu_a \partial_\mu e^a_\kappa - \omega^b_{\mu a} e^a_\kappa e^\nu_b \tag{6.29}$$

so the affine connection is not an independent field, but determined by the vierbein fields and the spin connection. Now we have all the tools to modify the Dirac Lagrangian (3.9) or the corresponding action S_D into a locally Lorentz invariant one. A Dirac action S_D which gives the Dirac equation in Minkowski space-time reads

$$(6.30) \qquad S_D = \int \left[\tfrac{1}{2}\bar\psi(i\gamma^\mu\partial_\mu - m)\psi - \tfrac{1}{2}\bar\psi(i\overleftarrow{\partial}_\mu\gamma^\mu + m)\psi\right] d^4x.$$

The Lagrangian of (6.30) differs only a four-divergence of the Lagrangian (3.9) and thus both Lagrangians give the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

In the Riemannian space-time M the locally Lorentz invariant action of the Dirac field is given by

$$(6.31) \qquad \tilde S_D = +\int \left[\tfrac{1}{2}\bar\psi(i\gamma^\mu\nabla_\mu - m)\psi - \tfrac{1}{2}\bar\psi(\overleftarrow{\nabla}_\mu i\gamma^\mu + m)\psi\right] e\, d^4x\,,$$

where $e := \det(e^a_\mu)$ and $\gamma^\mu(x) := \gamma^a e^\mu_a(x)$, thus converting the bein-indexed γ-matrices into world-indexed ones. The derivatives in (6.30) are replaced by gauge-covariant derivatives in (6.31), thus the integrand of (6.31) is a Lorentz scalar. The action $\tilde S_D$ is also a scalar under coordinate transformations. The rôle of e in the integrand is to compensate the Jacobian which appears under a coordinate transformation. Because we started with (6.30), instead of the action obtained from (3.9), the resulting action $\tilde S_D$ is real. The vierbein fields and spin-connection become dynamical variables if the total action S is taken to be

$$(6.32) \qquad S = \tilde S_D + S_G,$$

where S_G is the action of the vierbein fields and the spin-connection. The action S_G is constructed from the curvature of the spin-connection, similar to section 5. There is, however, a difference. Here the action is choosen to depend linearly on the curvature instead of quadratically:

$$(6.33) \qquad S_G = \frac{-1}{16\pi G}\int e^{a\mu} e^{b\nu} R_{\mu\nu ab}\, e\, d^4x \qquad (G\in\mathbb{R})\,,$$

where $R_{\mu\nu ab}$ is the curvature of the spin-connection

(6.34) $$R_{\mu\nu ab} = \partial_\mu \omega_{\nu ab} - \partial_\nu \omega_{\mu ab} + \omega_{\mu ca}\omega^c_{\ \nu b} - \omega_{\nu ca}\omega^c_{\ \mu b}$$

and G is a coupling constant.

The variational principle of Hamilton gives by varying e^μ_a, $\omega_{\mu ab}$, $\bar{\psi}$ and ψ, the field equations of e^μ_a, $\omega_{\mu ab}$, ψ and $\bar{\psi}$:

(6.35) $$R^a_\mu - \tfrac{1}{2} e^a_\mu R = 8\pi G T^a_\mu ,$$

(6.36) $$T_{\rho\kappa\nu} = 4\pi G s_{\nu\rho\kappa} ,$$

(6.37) $$(i\gamma^\mu \nabla_\mu - m)\psi - iS^\mu_{\ \rho\mu}\gamma^\rho \psi = 0,$$

(6.38) $$\bar{\psi}(i\nabla_\mu \gamma^\mu + m) - iS^\mu_{\ \rho\mu}\bar{\psi}\gamma^\rho = 0,$$

where

(6.39) $$R^a_\mu := e^\nu_b R^{ab}_{\ \ \mu\nu}$$

is the Ricci tensor,

(6.40) $$R := e^\mu_a R^a_\mu$$

the curvature scalar and

(6.41) $$T^a_\mu := \tfrac{1}{2}\bar{\psi}(i\gamma^a \nabla_\mu - \overleftarrow{\nabla}_\mu i\gamma^a)\psi$$

the energy momentum tensor of the Dirac field.

The torsion and the modified torsion are defined respectively by

(6.42) $$S^\mu_{\ \kappa\lambda} := \tfrac{1}{2}(\Gamma^\mu_{\ \kappa\lambda} - \Gamma^\mu_{\ \lambda\kappa}) = \Gamma^\mu_{\ [\kappa\lambda]}$$

and

(6.43) $$T^\nu_{\ \rho\kappa} := S^\nu_{\ \rho\kappa} + S^\mu_{\ \mu\rho}\delta^\nu_\kappa - S^\mu_{\ \mu\kappa}\delta^\nu_\rho .$$

The spin density is defined by

$$(6.44)\qquad s^{\mu\rho\kappa} := \tfrac{1}{4}\bar{\psi}\{\gamma^{\mu},\sigma^{\rho\kappa}\}\psi = \frac{i}{2}\bar{\psi}\gamma^{[\mu}\gamma^{\rho}\gamma^{\kappa]}\psi = \tfrac{1}{2}\varepsilon^{\mu\rho\kappa\sigma}\bar{\psi}\gamma_{\sigma}\gamma_{5}\psi ,$$

where $\varepsilon_{0123} := 1$ and $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$.
The complete anti-symmetry of the spin density $s^{\mu\rho\kappa}$ implies the complete anti-symmetry of the modified torsion $T^{\mu\rho\kappa}$ and $T^{\mu}_{\ \rho\mu} = 0$ follows. Substitution thereof in

$$(6.45)\qquad S^{\nu}_{\ \rho\kappa} = T^{\nu}_{\ \rho\kappa} - \tfrac{1}{2}\delta^{\nu}_{\kappa}T^{\mu}_{\ \rho\mu} + \tfrac{1}{2}\delta^{\nu}_{\rho}T^{\mu}_{\ \kappa\rho} ,$$

implies that the torsion and modified torsion are equal, i.e.

$$(6.46)\qquad S^{\nu}_{\ \rho\kappa} = T^{\nu}_{\ \rho\kappa} ,$$

and in particular that $S_{\rho\kappa\nu}$ is completely anti-symmetry. Hence, the Dirac-Weyl equation (6.37) becomes

$$(6.47)\qquad (i\gamma^{\mu}\nabla_{\mu}-m)\psi = 0.$$

The complete anti-symmetry of the torsion and the relation

$$(6.48)\qquad \Gamma^{\kappa}_{\mu\lambda} = \left\{ {\kappa \atop \mu\lambda} \right\} + S^{\kappa}_{\ \mu\lambda} + S^{\kappa}_{\ \lambda\mu} + S^{\kappa}_{\ \mu\lambda} ,$$

imply that the symmetric part of the affine connection equals the Christoffel symbol

$$(6.49)\qquad \Gamma^{\kappa}_{(\mu\kappa)} = \left\{ {\kappa \atop \mu\lambda} \right\} .$$

Hence, affine geodesics of $\Gamma^{\kappa}_{\mu\lambda}$ and metrical geodesics coincide in this theory. The conservation laws read

$$(6.50)\qquad \nabla_{\mu}T^{\mu}_{\ \rho} = 2S_{\rho\kappa\mu}T^{[\kappa\mu]} + \tfrac{1}{2}R_{\rho\mu ab}S^{\mu ab} ,$$

and

(6.51) $$\nabla_{\mu} S^{\mu ab} = 2T^{[ab]} .$$

Let us consider the particular case where the spin density $s^{\mu\rho\kappa}$ is zero. Then the torsion is zero and (6.35) becomes the Einstein equation of the theory of general relativity. This is the field equation for the metric tensor field.

We have here an alternative route to the general theory of relativity, which is similar to the approach of section 5. Starting point was the definition of a spinor in curved space-time, leading to vierbein fields and local Lorentz transformations. Then the Dirac equation was made locally Lorentz covariant through the introduction of the gauge-covariant derivative (minimal coupling). Finally, the vierbein fields and the spin connection were made into dynamical variables. The resulting field theory is the Einstein-Cartan-Kibble-Sciama theory of the gravitational field, described by the vierbein field and the spin connection, coupled to a Dirac field. Instead of the Dirac field another matter field could have been taken.

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CONNECTIONS AND COVARIANT DIFFERENTIATION

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The purpose of this chapter is to review the classical concepts of connection and covariant differentiation and to explain their general coordinate free description. Most of the material has been derived from literature [1,2].

1. THE CLASSICAL CONCEPT OF COVARIANT DIFFERENTIATION

Let M be an n-dimensional C^∞ - submanifold in $\mathbb{R}^m$ and $c(t)$ a parameter representation of a C^∞ - curve c on M, with $a < t < b$.

DEFINITION 1. The *covariant* derivative of the vector $Y_{c(t)}$, belonging to the vector field $T(M)$, along the curve c in the point $t = t_0$ is the projection of the vector $\frac{d}{dt} Y_{c(t)}\Big|_{t=t_0}$ on the plane tangent to M through the point $c_0 = c(t_0)$. We use the notation: $\frac{DY}{dt}$.
The vector field Y is called *constant* or *parallel* along the curve c, whenever

$$\frac{DY}{dt} \equiv 0 \quad \text{along} \quad c.$$

Introducing coordinates $x = (x^1,x^2,\ldots,x^n) \in \mathbb{R}^n$ and $y = (y^1,y^2,\ldots,y^m) \in \mathbb{R}^m$ and representing the n-dimensional submanifold M by

$$\{y^1(x^1,x^2,\ldots,x^n), \ \ldots \ y^m(x^1,x^2,\ldots,x^n)\} \qquad (n<m)$$

and the curve c on M by

$$\{y^1(x(t)), \ \ldots \ y^m(x(t)\}, \qquad a < t < b,$$

we have for the covariant derivative of the vector field $Y(t)$ along c:
$Y(t) = b^i(t)X_i(t)$ with $X_i(t) = \frac{\partial y_j}{\partial x^i}\frac{\partial}{\partial y_j}$, the expression

$$\frac{DY}{dt} = \dot{b}^i(t)X_i(t) + b^i(t)\frac{DX_i}{dt}. \qquad (1.1)$$

(the dot denotes differentiation with respect to t).
In order to determine $\frac{DY}{dt}$ one needs to calculate $\frac{DX_i}{dt}$.

According to our definition 1 we have

$$\frac{DX_i}{dt} = \frac{\partial^2 y^j}{\partial x^i \partial x^k}\frac{dx^k}{dt}\Pi\left(\frac{\partial}{\partial y^j}\right)$$

with $\Pi\left(\frac{\partial}{\partial y^j}\right) := a_j^\ell(x)X_\ell(x)$, being the projection of $\frac{\partial}{\partial y^j}$ on the plane tangent to M.

$$\text{Putting}\quad \frac{\partial^2 y^j}{\partial x^i \partial x^k}a_j^\ell(x) = \Gamma_{ik}^\ell(x) = \Gamma_{ki}^\ell(x) \qquad (1.2)$$

$$\text{we get}\quad \frac{DX_i}{dt} = \Gamma_{ik}^\ell(x)\frac{dx^k}{dt}X_\ell(x) \qquad (1.3)$$

$$\text{and}\quad \frac{DY}{dt} = \left(\dot{b}^k + b^i\Gamma_{ij}^k\frac{dx^j}{dt}\right)X_k$$

$$= \left(\frac{\partial b^k}{\partial x^j}\frac{dx^j}{dt} + b^i\Gamma_{ij}^k\frac{dx^j}{dt}\right)X_k \qquad (1.4)$$

The functions $\Gamma_{ij}^k(x)$ are known as the *symbols of Christoffel*. In the particular case that the curve c has the parameter representation $c(x^j) = y(0,0,\ldots,0,x^j,0,\ldots)$ we obtain from (1.3)

$$\frac{DX_i}{\partial x^j} = \Gamma_{ik}^\ell\delta_j^k X_\ell = \Gamma_{ij}^\ell X_\ell \qquad (1.5)$$

Finally we define the vector field

$$X_{ij} := \frac{\partial X_i}{\partial x^j} = \Gamma_{ij}^\ell X_\ell + h_{ij}(x)N,$$

with N normal to the plane tangent to M.

Suppose the first fundamental form on M reads

$$g_{ij}(x)dx^i \otimes dx^j,$$

then we have for the scalar products $<X_i,X_j>$ and $<X_{ij},X_k>$:
$<X_i,X_j> = g_{ij}$ and $<X_{ij},X_k> = \Gamma^{\ell}_{ij}\, g_{\ell k} := \Gamma_{ijk}$. It follows that

$$g_{ij,k} = \frac{\partial}{\partial x^k} g_{ij} = <X_{ik},X_j> + <X_i,X_{jk}> = \Gamma_{ikj} + \Gamma_{jki}$$

and so due to $\Gamma_{ijk} = \Gamma_{jik}$:

$$\Gamma_{ijk} = \tfrac{1}{2}\{g_{jk,i} + g_{ki,j} - g_{ij,k}\} \tag{1.6}$$

From this simple consideration we may draw the conclusion that the Christoffel symbols and hence also the covariant derivative and the parallel displacement of a vector field on M along a curve on M are uniquely defined by the metric on M, induced by the metric in $\mathbb{R}^m$.

<u>REMARK</u>. The Christoffel symbol $\Gamma^k_{ij}(x)$ does not define a tensor fiel; transformation to other coordinates $x' = (x'^1,x'^2,\dots,x'^n)$ yields:

$$\Gamma'^{\gamma}_{\alpha\beta}(x') = \Gamma^k_{ij}(x)\frac{\partial x^i}{\partial x'^{\alpha}}\frac{\partial x^j}{\partial x'^{\beta}}\frac{\partial x'^j}{\partial x^k} + \frac{\partial^2 x^i}{\partial x'^{\alpha}\partial x'^{\beta}}\frac{\partial x'^{\gamma}}{\partial x^i} \tag{1.7}$$

It follows from (1.4) that the covariant derivative of a *given* vector field Y in a point p of a *given* manifold M (with a metric induced by that in $\mathbb{R}^m$) depends only on the vector $\dot{x}(t_0)$ with $x(t_0) = (x^1(t_0),\dots,x^n(t_0))$, being the coordinates of $p \in M$.

Denoting the vector $\dot{x}(t_0)$ by X_p, the covariant derivative of the *given* vector field Y gives the map:

$$T_p(M) \to T_p(M) \quad \text{with} \quad X_p \to \frac{DY}{dt}$$

Henceforth we use the more suggestive notation:

$$\frac{DY}{dt} = \nabla_{X_p} Y \tag{1.8}$$

Fot the particular case of $Y = X_i$ and $X_p = X_j(p)$ we have according to (1.5)

$$\nabla_{X_j} X_i = \Gamma^k_{ij}(x) X_k \tag{1.9}$$

The ∇ - operator, also called the "*del*" operator, satisfies the following theorem.

THEOREM 1. *In any point* $p \in M$ (n-*dimensional* C^∞ - *submanifold*) *and for any* C^∞ - *vector field* $Y \in X(M)$ *there exists a map* $T_p(M) \to T_p(M)$ *with* $X_p \to \nabla_{X_p} Y$ *which satisfies the following properties:*

1. *Wehenever* X *and* Y *are* C^∞ - *vector fields and* $(\nabla_X Y)_p := \nabla_{X_p} Y$ *then* $\nabla_X Y$ *is a* C^∞ - vector field.
2. *The map* $T_p(M) \times X(M) \to T_p(M)$ *with* $(X_p, Y) \to \nabla_{X_p} Y$ *is* $\mathbb{R}$ - *linear as well in* X_p *as in* Y.
3. $\nabla_{(fX)_p} Y = f(p) \nabla_{X_p} Y, \; \forall f: M \to \mathbb{R}$.
4. $\nabla_{X_p}(fY) = (X_p f) Y_p + f(p) \nabla_{X_p} Y, \; \forall f: M \to \mathbb{R}$, (f *differentiable*)
5. $[X, Y] = \nabla_X Y - \nabla_Y X, \; \forall X, Y \in X(M)$
6. $X_p \langle Y_1, Y_2 \rangle = \langle \nabla_{X_p} Y_1, Y_{2_p} \rangle + \langle Y_{1_p}, \nabla_{X_p} Y_2 \rangle, \; \forall X, Y_1, Y_2 \in X(M)$

PROOF. The theorem follows rather easily from the definitions; see lit.1, Boothby, pp. 311-312.

2. THE RIEMANN CONNECTION

Let M be a C^∞-manifold of dimension n.

DEFINITION 2. A Koszul connection on M is a map $\nabla: \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$; notation $(X,Y) \to \nabla_X Y$ which is $\mathbb{R}$-linear in X and Y. Besides this property, the Koszul connection satisfies for all $f,g \in C^\infty(M)$ and for all $X, X', Y, Y' \in \mathcal{X}(M)$ the rules:

$$\text{i)} \quad \nabla_{fX+gX'}Y = f\nabla_X Y + g\nabla_{X'}Y, \tag{2.1}$$

$$\text{ii)} \quad \nabla_X(fY+gY') = f\nabla_X Y + g\nabla_X Y' + (Xf)Y + (Xg)Y', \tag{2.2}$$

In case M is a Riemannian-manifold of dimension n, we have the definition:

DEFINITION 3. A Riemann connection is a Koszul conncetion with the extra properties:

$$\text{iii)} \quad [X,Y] = \nabla_X Y - \nabla_Y X \tag{2.3}$$

$$\text{iv)} \quad X\langle Y,Y'\rangle = \langle \nabla_X Y, Y'\rangle + \langle Y, \nabla_X Y'\rangle \tag{2.4}$$

In Theorem 1 it has been indicated that a Riemann connection exists for a C^∞-submanifold M, embedded in Euclidean space $\mathbb{R}^m$. However the following *fundamental intrinsic* (i.e. without embedding) theorem is valid.

THEOREM 2. *If* M *is a Riemannian* C^∞*-manifold, then there exists on* M *a uniquely determined Riemann connection.*

PROOF. Let $x = (x^1, x^2, \ldots, x^n)$ be the local coordinates in a chart of $U \subset M$ and let

$$X_i = \frac{\partial}{\partial x^i} \quad \text{and} \quad \langle X_i, X_j\rangle_p = g_{ij}(p), \qquad p \in U$$

Let us suppose that a Riemann connection exists on M; we prove first that under this assumption the connection is uniquely determined.
We define the functions $\Gamma^k_{ij}(x)$ by

$$\nabla_{X_i} X_j = \Gamma^k_{ij}(x) X_k,$$

and it follows from (2.3)

$$\Gamma^k_{ij} = \Gamma^k_{ji} \tag{2.5}$$

From (2.4) we get:

$$\frac{\partial}{\partial x^k} g_{ij} := g_{ij,k} = <\nabla_{X_k} X_i, X_j> + <X_i, \nabla_{X_k} X_j> = \Gamma^\ell_{ki} g_{\ell j} + \Gamma^\ell_{kj} g_{\ell i},$$

or

$$g_{ij,k} = \Gamma_{ikj} + \Gamma_{kji} \tag{2.6}$$

It follows now from (2.5)-(2.6) and (1.2)-(1.6) that the function $\Gamma^k_{ij}(x)$ are the Christoffel symbols and hence uniquely determined by the Riemann-metric on M. According to (2.1) and (2.2) we have for arbitrary vector fields $X = a^i(x) X_i$ and $Y = b^i(x) X_i$:

$$\nabla_X Y = a^i \nabla_{X_i}(b^j X_j) = a^i b^j \Gamma^\ell_{ij} X_\ell + a^i \frac{\partial b^j}{\partial x^i} X_j = (Xb^j) X_j + a^i b^j \Gamma^\ell_{ij} X_\ell \tag{2.7}$$

Because the Christoffel symbols are determined uniquely by the metric on M the connection $\nabla_X Y$ is defined unambiguously, however under the assumption that it exists. Moreover, the equations (1.4) and (2.7) yield that $(\nabla_X Y)_p$ is the covariant derivative $\nabla_{X_p} Y$.
Conversely, defining $\Gamma^k_{ij}(x)$ by (2.5) and (2.6) and $\nabla_X Y$ by (2.7), we obtain on M a Riemann connection, which we know to be unique. ∎

For more details see lit[1], pp. 313-318.
Formula (2.7) gives an intrinsic definition of the covariant derivative, without recourse to an embedding in $\mathbb{R}^n$.

3. RIEMANN CONNECTION AND CONNECTION FORMS

Let (U,ϕ) be a chart of the Riemannian-manifold M with coordinates $(x^1,x^2,\ldots,x^n)$, let $p \in U$ and $\{X_1, X_2, \ldots, X_n\}$ a base in $T_p(M)$ with $X_i = \frac{\partial}{\partial x^i}$. The dual base in $T_p^*(M)$ is denoted by $\{\omega^1, \omega^2, \ldots, \omega^n\}$ with $\omega^i X_j = \delta^i_j$. The Riemann connection on M gives for $X = a^i(x)X_i$:

$$\nabla_X X_j = a^i(x)\Gamma^k_{ij}(x)X_k := \omega^k_j(X)X_k \tag{3.1}$$

with $$\omega^k_j(X) = a^i(x)\Gamma^k_{ij}(x) . \tag{3.2}$$

Hence the map ∇ determines n^2 one-forms ω^k_j, with

$$\omega^k_j(X_i) = \Gamma^k_{ij}(x) \tag{3.3}$$

and $$\omega^k_j(X) = \omega^k_j(\omega^i(X)X_i) = \omega^i(X)\Gamma^k_{ij}(x)$$

or $$\omega^k_j = \Gamma^k_{ij}(x)\omega^i . \tag{3.4}$$

Using the relation $d\omega^i(X,Y) = X\omega^i(Y) - Y\omega^i(X) - \omega^i([X,Y])$ and the property (2.3), we obtain after a little calculation:

$$d\omega^i = \omega^j \wedge \omega^i_j . \tag{3.5}$$

On the other hand, the property (2.4) yields

$$dg_{ij}(X) = X(g_{ij}) = X\langle X_i,X_j\rangle = \langle\nabla_X X_i,X_j\rangle + \langle X_i,\nabla_X X_j\rangle$$

$$= \langle\omega^k_i(X)X_k,X_j\rangle + \langle X_i,\omega^k_j(X)X_k\rangle$$

or $$dg_{ij} = g_{kj}\omega^k_i + g_{ik}\omega^k_j . \tag{3.6}$$

Summarizing we have the result that the Riemann connection on M determines n^2 *connection forms* ω^k_j, defined by (3.1) and satisfying the relations (3.5) and (3.6). Also we have conversely: 1-forms ω^k_j can be defined on

M by the relations (3.5) and (3.6) and with the aid of these 1-forms an operator ∇ may be defined as:

$$\nabla_X Y = (Xb^i(x))X_i + b^i(x)\omega_i^k(X)X_k, \tag{3.7}$$

where $Y = b^i(x)X_i$.

It is not difficult to show that (3.2) follows from (3.6) and so the operator ∇, defined by (3.7), is the same as the one defined by (2.7), and hence it satisfies the requirements (2.1)-(2.4) of the uniquely determined Riemann connection. Once again in summary, we have obtained:

THEOREM 3. *The Riemann connection defined on a Riemannian manifold* M *can be defined in two equivakent ways:*

1e. *As an operator* $\nabla: \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$, *satisfying the relations* (2.1), (2.2), (2.3) *and* (2.4).

2e. *As a system of* 1-*forms* ω_j^k, *satisfying the relations* (3.5) *and* (3.6).

4. THE CARTAN CONNECTION

Let $E = (X_1, X_2, \ldots, X_n)$ be a basis in $T_p(M)$ with M a Riemannian manifold and $p \in M$. We assume that E may depend on the point p and so we have for varying p a so-called "*moving frame*" or "*répère mobile*".
The Riemann connection determines n^2 one-forms ω^i_j with

$$\nabla_{X_k} X_j = \omega^i_j(X_k) X_i \tag{4.1}$$

We consider now a transformation to another base

$$E' = (X'_1, X'_2, \ldots, X'_n) = E \cdot a(x), \tag{4.2}$$

where $a(x)$ is the matrix with

$$X'_j = a^\ell_j(x) X_\ell \tag{4.3}$$

(ℓ row index, j column index.)

We have again

$$\nabla_{X'_k} X'_j = \omega'^i_j(X'_k) X'_i \tag{4.1*}$$

and it follows

$$\begin{aligned}\omega'^i_j(\cdot) X'_i = \nabla_{(\cdot)} X'_j &= \nabla_{(\cdot)} a^\ell_j(x) X_\ell \\ &= da^\ell_j(\cdot) X_\ell + a^\ell_j(x) \nabla_{(\cdot)} X_\ell \\ &= da^\ell_j(\cdot) X_\ell + a^\ell_j(x) \omega^i_\ell(\cdot) X_i\end{aligned}$$

or in matrix notation $\omega = (\omega^i_j)$, with i row index and j column index :

$$E'\omega' = E\,da + E\,\omega a.$$

Using (4.2) we get the simple transformation formula

$$E\,a\omega' = E\,da + E\,\omega a, \quad \text{or}$$

$$\omega' = a^{-1}da + a^{-1}\omega a \tag{4.4}$$

This rule leads to the following definition, applicable to any C^{∞} manifold, not necessarily Riemannian:

DEFINITION 4. The Cartan connection is a map from a "moving frame" $E(x)$ to a matrix valued 1-form $\omega(x)$ such that the transformation rule (4.4) holds, i.e.

$$E \to \omega \Rightarrow Ea \to a^{-1}da + a^{-1}\omega a \tag{4.5}$$

5. THE EHRESMANN CONNECTION IN FIBRE BUNDLES.

5.1 Principal fibre bundles.

Let M be a C^∞-manifold of dimension n and G a Lie group with e as unity.

DEFINITION 5. A C^∞-*principal fibre bundle* above M with structure group G is a triple $(P,\Pi,\cdot)$ with the following properties:

1. P is a C^∞ manifold
2. Π is a C^∞ projection map from P onto M with $\Pi(P) = M$; M is called the base space of P.
3. There exists a C^∞ right action of the group G:

$$P \times G \to P \quad \text{with} \quad (p,g) \to p\cdot g := R_g p, \quad p \in P, \quad g \in G$$

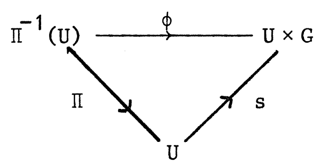
This action satisfies the following conditions

(i) $p\cdot(gh) = (p\cdot g)\cdot h, \quad \forall p \in P, \quad \forall g,h \in G$

(ii) $\Pi(p\cdot g) = \Pi(p) \qquad \forall p \in P, \quad \forall g \in G$

(iii) For any $x \in M$, there exists a neighbourhood U of x and a diffeomorphism $\phi\colon \Pi^{-1}(U) \to U \times G$ such that $\phi(p) = (\Pi(p),\psi(p))$ with $\psi(p\cdot g) = \psi(p)g$. Any fibre $\Pi^{-1}(x)$ is diffeomorph with G, but there does *not* exist a *canonical* diffeomorphism.

The last requirement (iii) means that a principal fibre bundle is *locally* a trivial bundle $U \times G$. A *local section* $s\colon U \to U \times G$ *defines a local trivialisation (local gauge)* $\phi\colon \Pi^{-1}(U) \to U \times G$. We have the following commuting diagram



Any fibre $\Pi^{-1}(x)$ is diffeomorph with the Lie group G and moreover

$$\{p\cdot g \mid g\in G\} = \Pi^{-1}(\Pi(p)).$$

This follows from

$$\Pi(p\cdot g) \overset{3.ii}{=} \Pi(p) \to \{p\cdot g \mid g\in G\} \subset \Pi^{-1}(\Pi(p)) \quad \text{and}$$

from the inclusion $\Pi^{-1}(\Pi(p)) \subset \{p\cdot g \mid g\in G\}$.

The latter inclusion is obtained from 3.iii and 3.ii:

$$q \in \Pi^{-1}(\Pi(p)) \to \Pi(q) = \Pi(p) \xrightarrow{3.iii} \exists h\in G \quad \text{with}$$

$$\underline{\psi(q)} = \psi(p)h = \underline{\psi(p\cdot h)} \quad \text{and} \quad \underline{\Pi(q)} = \Pi(p) \overset{3.ii}{=} \underline{\Pi(p\cdot h)};$$

hence according to 3.iii: $q = p\cdot h$ with $h \in G$.

Furthermore we get from $p\cdot g = p$ for some $g \in G$ that $\psi(p\cdot g) = \psi(p) = \psi(p)g$ and so $g = e$; therefore G *acts without a fix point.*
When i denotes the embedding: $\Pi^{-1}(x) \to P$ then $V_x = i_* T(\Pi^{-1}(x))$ is a *vertical* subspace of the tangent space $T(P)$; whenever a tangent vector $Y \in T(P)$ belongs to V_x for some $x \in M$, then $\Pi_* Y = 0$ and conversely.

We finish this subsection by giving the important example of the so-called *frame bundle*.
Let M be again a C^∞-manifold of dimension n and $T_x(M)$ the tangent space through the point $x \in M$. Further, let
$u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$ be some base for $T_x(M)$. (*coordinate frame*). We denote by $F(M)$ the set of all possible frames $u(x)$ for all tangent spaces $T_x(M)$, where x varies arbitrarily in M.
$F(M)$ is called the *frame bundle* and takes the rôle of the space P.
The projection Π is now the map $F(M) \to M$ with $\Pi(u(x)) = x$, with $u(x)$ *some* base for $T_x(M)$.
Let (x,V) be a chart of M with $x \in V$; denoting the coordinates

of x by $(x_1,x_2,\ldots,x_n)$, any frame u for $T_x(M)$ may be written in the form

$$u_j = x_{ij}(u) \left.\frac{\partial}{\partial x_i}\right|_{\Pi(u)}, \qquad j = 1,2,\ldots,n$$

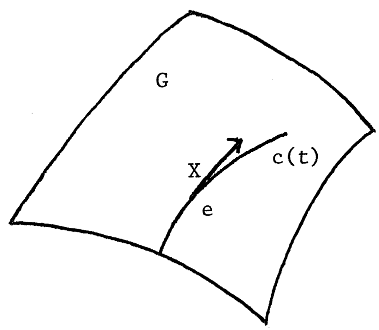
with $x_{ij}(u)$ a non-singular matrix (i row-, j column index); also conversely, any non-singular matrix (x_{ij}) defines a frame. The map $(x,u) \to ((x_1,x_2,\ldots,x_n), (x_{ij})) \in \mathbb{R}^n \times G\ell(n,\mathbb{R})$ is a 1-1 map from $\Pi^{-1}(V) \to V \times G\ell(n,\mathbb{R})$ and so $F(M)$ is a C^∞ manifold.

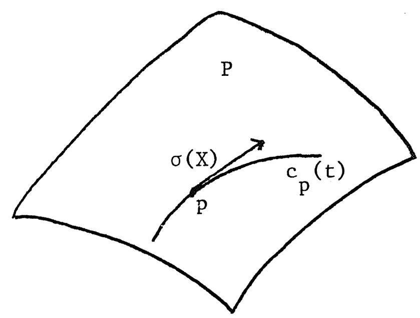
Besides this we have still at our disposal the map $F(M) \times G\ell(n,\mathbb{R}) \to$ $\to F(M)$ defined as

$$(x,u;\ A) \to (x,uA) \quad \text{with} \quad (uA)_j = u_i A_{ij}.$$

$F(M)$ is an example of a principal fibre bundle with structure group $G\ell(n,\mathbb{R})$. For more details on fibre bundles the reader is referred to the chapter by H.G.J. Pijls.

5.2 Fundamental Vectorfields. Let P be a principal fibre bundle with Lie group G and with g the Lie algebra of G. Any $X \in g$ generates a curve $c(t)$, $t \to \exp tX$, in G and for *any* $p \in P$ a curve $c_p(t)$ is defined in P with $c_p(t) = p\cdot\exp(tX) := R_{\exp(tX)}p$.





The vector tangent to $c_p(t)$ at $t = 0$ is denoted by $\sigma(X)(p)$ and so

$$\left(\frac{d}{dt} c_p\right)_{t=0} = \sigma(X)(p)$$

and σ is a map $\mathcal{g} \to \mathcal{X}(P)$ ($\mathcal{X}(P)$ is the set of vector fields on P). When we denote the map $G \to P$ with $g \to p\cdot g$ with p *fixed* by $\sigma_p(g) = p\cdot g$, then we have

$$\sigma(X)(p) = \sigma_{p_*}(X) \tag{5.1}$$

The map $Ad(g)$ from $\mathcal{g} \to \mathcal{g}$ is defined as

$$Ad(g) = (L_g R_{g^{-1}})_* = (R_{g^{-1}} L_g)_* \tag{5.2}$$

(L_g, R_g are respectively left an right multiplication with g)

<u>EXAMPLE</u>. $G = G\ell(n,\mathbb{R})$ and $\mathcal{g} = M(n,\mathbb{R})$ with $M(n,\mathbb{R})$ the set of all $n\times n$ matrices with real entries.

$$Ad(A)M = (L_A R_{A^{-1}})_* M = L_A M R_{A^{-1}} = AMA^{-1}$$

A useful relation is the following:

$$(R_g)_*\sigma(X) = \sigma(Ad(g^{-1})X), \quad \forall X \in \mathcal{g}, \quad \forall g \in G \tag{5.3}$$

<u>PROOF</u>. Applying R_g to the curve $c_p(t) = p\cdot e^{tX}$, we get the curve $p\cdot e^{tX}g = p\cdot g(g^{-1}e^{tX}g) = p\cdot g\exp(t\,Ad\,(g^{-1})X)$.

Hence according to (5.1)

$$(R_g)_*\sigma_{p_*}(X) = \sigma_{p\cdot g_*}(Ad(g^{-1})X) \quad \text{or}$$

$$(R_g)_*\sigma(X) = \sigma(Ad(g^{-1})X). \quad \blacksquare$$

Because G acts without "fix points", is the map $X \to \sigma(X)(p) = \sigma_{p_*}(X)$, with p fixed, an isomorphism, and whenever X traverses the Lie algebra g, then the set $\{\sigma(X)(p)\}_{X \in g}$ is the set of all vertical tangent vectors in p. The vector fields $\sigma(X)$ on P are called the *fundamental vector fields* on P.

5.3 The Cartan and the Ehresmann Connection. We consider again the frame bundle $P = F(M)$ with M a C^∞ manifold of dimension n. A *section* s of this bundle above an open set $V \subset M$ is a C^∞ map $s: V \to F(M)$ with Πs the identity map. A section s is a moving frame above V. This section is in general not global. We reformulate now definition 4 of a Cartan connection for a C^∞ manifold, which needs not to be a Riemannian manifold.

DEFINITION 6. A Cartan connection is a map from local sections s $(V \to F(M))$ to $n \times n$ matrix valued one forms on M: $\omega_s = (\omega^i_{sj})$, such that for any C^∞ function $g: V \to G\ell(n,\mathbb{R})$ the relation holds:

$$\omega_{s \cdot g} = g^{-1} dg + g^{-1} \omega_s g \tag{5.4}$$

with $s \cdot g$ the section $(s \cdot g)(x) = s(x) \cdot g(x)$ and $x \in M$.

It is not difficult to define a $n \times n$ matrix valued 1-form ω on F(M), such that for any local section s

$$\omega_s = s^*(\omega) \tag{5.5}$$

This 1-form ω is defined for any $Y_p \in T_p(F(M))$ as

$$\omega(Y_p) = \omega_s(X_x) \quad \text{with} \quad Y_p = s_* X_x \quad \text{and} \quad \Pi(p) = x.$$

The transformation rule (5.4) reads now

$$(s \cdot g)^*(\omega) = g^{-1} dg + g^{-1} s^*(\omega) g \tag{5.4*}$$

The ∇ operator ("del" operator) connected with the Cartan connection above M, with M a C^∞ n-dimensional manifold (not necessarily Riemannian) is defined as follows

DEFINITION 7. Given a moving frame $X = (X_1, X_2, \ldots, X_n)$, defined by a section s above $V \subset M$, and given the adjoint connection forms $(\omega_{s\,j}^{\ i})$, the operator ∇ is a map from $\mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ with

$$\nabla_X Y = X(b^i(x))X_i + b^i(x)\nabla_X X_i, \tag{5.6}$$

where $Y = b^i(x)X_i$ and $\nabla_X X_i = \omega_{s\,i}^{\ j}(X)X_j$; $x = (x_1, x_2, \ldots, x_n) \in V$. (5.7)

From this definition it follows that

$$\nabla_{X_k} X_i = \omega_{s\,i}^{\ j}(X_k)X_j. \tag{5.8}$$

Moreover ∇ defines a Koszul connection according to definition 2. If we specialize the Cartan connection by taking for M a Riemannian manifold, while $\omega_{s\,j}^{\ i}$ are restricted by the relations (3.5) and (3.6) we obtain the Riemann connection, dealt with in section 3. So the operator ∇, defined by (5.6)-(5.7), is a natural generalization of the covariant derivative on a Riemannian manifold.

The Cartan connection has been defined for the frame bundle $P = F(M)$ by means of formula (5.4*). This formula contains a section $s(V \to F(M))$ and is not completely coordinate free. However, the following theorem gives a useful intrinsic definition of the Cartan connection ω on the frame bundle $F(M)$.

THEOREM 4. *A matrix valued* 1-*form on* $F(M)$ *satisfies the relation* (5.4*), *if and only if*

$$\omega(\sigma(X)(p)) = X, \quad \forall X \in \mathfrak{g} = M(n, \mathbb{R}), \quad \forall p \in F(M), \tag{5.9}$$

and

$$\omega(R_{g*}Y) = Ad(g^{-1})\omega(Y), \quad \forall Y \in \mathcal{X}(P), \quad \forall g \in G \tag{5.10}$$

For the proof we need the following lemma.

<u>LEMMA</u>. Let s be a section of $F(M)$, above some open set $V \subset M$, and let $g: V \to G\ell(n,\mathbb{R})$ be a C^∞ map.
Then for any tangent vector X_x at a point $x \in V$ we have

$$(s\cdot g)_*(X_x) = R_{g(x)*}s_*(X_x) + \sigma(L_{g^{-1}(x)*}X_x(g))(s(x)\cdot g(x)) \tag{5.11}$$

<u>COMMENT</u>. $L_{g^{-1}(x)*}X_x(g)$ is a $n \times n$ matrix and so it may be looked upon as an element of $\mathcal{g}$ and therefore $\sigma(L_{g^{-1}(x)*}X_x(g))$ is for fixed g a vertical vector field on $F(M)$.

<u>PROOF of the lemma</u>. We consider the map $m: F(M) \times G\ell(n,\mathbb{R}) \to F(M)$ with $m(p,g) = p\cdot g \quad (p \in F(M), \quad g \in G\ell(n,\mathbb{R}))$.

The tangent space of $F(M) \times G\ell(n,\mathbb{R})$ at (p,g) is the direct sum $T_p(F(M)) \oplus T_g(G\ell(n,\mathbb{R}))$ and so any element of this tangent space is a pair $(Y_1,Y_2) = Y_1 \oplus Y_2$ with $Y_1 \in T_p(F(M))$ and $Y_2 \in T_g(G\ell(n,\mathbb{R}))$.
Suppose $c(t)$ is the integral curve for X with $c(0) = x$, then we have due to $s\cdot g = m(s,g)$

$$(s\cdot g)_*(X_x) = m_*(s_*(X_x), \ g_*(X_x)) =$$

$$\left[\frac{d}{dt}\, s(c(t))\cdot g(x)\right]\Bigg|_{t=0} + \left[\frac{d}{dt}\, s(x)\cdot g(c(t))\right]\Bigg|_{t=0} \tag{5.12}$$

The first term of the right-hand side is easily seen to be reduced to

$$\frac{d}{dt}[s(c(t))\cdot g(x)]\Bigg|_{t=0} = R_{g(x)*}\ s_*(X_x) \tag{5.13}$$

The second term yields

$$\left[\frac{d}{dt}\, s(x)\cdot g(c(t))\right]\Big|_{t=0} = \frac{d}{dt}[s(x)\cdot g(x)\,\{g^{-1}(x)g(c(t))\}]\Big|_{t=0}$$

$$\overset{(5.1)}{=} \sigma_{s(x)\cdot g(x)*}\left(\frac{d}{dt}\, g^{-1}(x)g(c(t))\right)\Big|_{t=0}$$

$\frac{d}{dt}(g^{-1}(x)g(c(t)))\big|_{t=0}$ is an element of the Lie algebra $\mathfrak{g}$ of $G\ell(n,\mathbb{R})$ and it equals $L_{g^{-1}(x)*}X_x(g)$.

Hence, again according to (5.1)

$$\left[\frac{d}{dt}\, s(x)\cdot g(c(t))\right]\Big|_{t=0} = \sigma_{s(x)\cdot g(x)*}(L_{g^{-1}(x)*}X_x(g)) =$$

$$= \sigma(L_{g^{-1}(x)*}X_x(g))(s(x)\cdot g(x)) \qquad (5.14)$$

Combining (5.12), (5.13) and (5.14) gives the proof of the lemma. ∎

PROOF of THEOREM 4. In order to comply with (5.4^*) we have to look for Lie algebra valued one forms satisfying

$$\omega((s\cdot g)_* X_x) = g^{-1}(x)X_x(g) + g^{-1}(x)\omega(s_* X_x)g(x), \quad \forall X_x \in T_x(M).$$

Using the lemma, this is equivalent to

$$\omega\{R_{g(x)*}s_*(X_x)\} + \omega\{\sigma(L_{g^{-1}(x)*}X_x(g))(s(x)\cdot g(x))\} =$$

$$= g^{-1}(x)X_x(g) + g^{-1}(x)\omega(s_*(X_x))g(x), \quad \forall X_x \in T_x(M) \qquad (5.15)$$

Specializing now to the particular maps $g\colon V \to G\ell(n,\mathbb{R})$ with i) $g(x) = I$ and $g(y) \not\equiv I$ in V (I identity) and ii) $g(y) \equiv \text{constant} = \bar{g}$, $\forall y \in V$ we get respectively:

i) $\omega(s_*(X_x) + \omega\{\sigma(X_x(g))(s(x))\} = X_x(g) + \omega(s_*(X_x))$ or

$$\omega(\sigma(X_x(g)))(s(x)) = X_x(g) \quad \text{for } g(x) = I \qquad (5.16)$$

ii) $$\omega\{R_{\bar{g}*}s_*(X_x)\} = \bar{g}^{-1}\omega(s_*X_x)\bar{g} = Ad(\bar{g}^{-1})\omega(s_*X_x), \qquad (5.17)$$

valid for all sections s of $F(M)$.

Also conversely we have that (5.16) and (5.17) imply (5.15) and so (5.15) is equivalent to (5.16) and (5.17).

In (5.16) we may take for $X_x(g)$ any $n \times n$ matrix, because the only restriction on g is that $g = I$ in the particular point x under consideration, and so we obtain the first condition of Theorem 4:

$$\omega(\sigma(A)(p)) = A, \quad \text{for all } A \in \mathit{g} \quad \text{and all } p \in F(M) \tag{5.18}$$

By choosing $s(x)$ properly, any <u>non vertical</u> vector in $p = s(x)$ may be written as $s_* X_x$ with $X_x \in T_x(M)$. So it follows from (5.17) that we have for non vertical vectors Y:

$$\omega(R_{\bar{g}*} Y) = Ad(\bar{g}^{-1})\omega(Y) \tag{5.19}$$

In case Y is a vertical vector, we remember that we may write $Y = \sigma(B)$ for some $B \in \mathit{g}$; inserting this into the left-hand side of (5.19) and using (5.3) and (5.18) we get:

$$\begin{aligned}\omega(R_{\bar{g}*} Y) = \omega(R_{\bar{g}*}\sigma(B)) &= \omega\{\sigma(Ad(\bar{g}^{-1})B)\} \\ &= Ad(\bar{g}^{-1})\omega(\sigma(B)) = Ad(\bar{g}^{-1})\omega(Y).\end{aligned}$$

Therefore formula (5.19) is valid for all $Y \in T_p(F(M))$ and all $p \in F(M)$ and so also the second condition of the theorem has been proved. ∎

Theorem 4 provides now the following definition of a so-called *Ehresmann connection* on an <u>arbitrary</u> principal fibre bundle.

<u>DEFINITION 7</u>. An *Ehresmann connection* on an arbitrary principal fibre bundle $\Pi: P \to M$ above a C^∞ manifold M with structure group G is a C^∞ Lie algebra valued one form ω, such that

1e $\omega(\sigma(X)) = X, \quad \forall X \in g$ (5.20)

2e $\omega(R_{g*}Y) = Ad(g^{-1})\omega(Y), \quad \forall Y \in X(P)$ and $\forall g \in G$ (5.21)

Consequences: From (5.20) it follows that $\omega: T_p(P) \to g$ is "onto" and so $\ker \omega(p) := H_p$ is a subspace of $T_p(P)$ with the same dimension as M.

This space H_p is called the *horizontal* subspace of the tangent space $T_p(P)$.

Every Ehresmann connection gives rise to a horizontal distribution H on P. We have

i) $T_p(P) = V_p \oplus H_p$, with V_p the subspace of vertical vectors (5.22)

ii) $H_{p \cdot g} = R_{g*}H_p$ (5.23)

As to (5.23) we remark that $\omega(R_{g*}H_p) = Ad(g^{-1})\omega(H_p) = 0$ and so $R_{g*}H_p \subset H_{p \cdot g}$; since left and right-hand side have the same dimension namely the dimension of M, the equation (5.23) is valid.

The projection $\Pi_*: H_p \to T_{\Pi(p)}(M)$ is an isomorphism for all $p \in P$ and hence for any vector field X on M there exists a unique vector field X^* on P such that X^* is everywhere horizontal and

$$\Pi_*(X^*(p)) = X_{\Pi(p)}, \quad \forall p \in P \tag{5.24}$$

The vector field X^* is called the *lift* of the vector field X.

Let s be a section $M \to P$; a useful formula for the lift $X^*_p = X^*_{s(x)}$ of the vector $X_x \in T_x(M)$ is given by the formula

$$X^*_s = s_*X - \sum_{j=1}^{k} \omega^j(s_*X)\sigma(\tilde{X}_j) \tag{5.25}$$

with $(\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_k)$ a basis for the Lie algebra g and with ω^j

defined as

$$\omega(\cdot) = \sum_{j=1}^{k} \omega^j(\cdot)\tilde{X}_j.$$

The field X_s^* is clearly horizontal, because according to (5.20) $\omega(X_s^*) = 0$ and further $\Pi_*(X_s^*) = \Pi_*(s_*X) = X$.

We have the following theorem at our disposal:

THEOREM 5.

1. $(X+Y)^* = X^* + Y^*$, $\forall X,Y \in \mathcal{X}(M)$
2. $(fX)^* = (f\circ\Pi)X^*$, $\forall f\colon M \to \mathbb{R}$, $\forall X \in \mathcal{X}(M)$
3. $[X,Y]^* = h([X^*,Y^*])$, $\forall X,Y \in \mathcal{X}(M)$, *with* $h(\cdot)$ *denoting the horizontal component.*
4. $R_{g_*}(X^*) = X^*$, $\forall X \in \mathcal{X}(M)$, $\forall g \in G$.
5. *Whenever* $Y \in \mathcal{X}(P)$ *with* $R_{g_*}(Y) = Y$, $\forall g \in G$, *then* $Y = X^*$ *with* X *uniquely determined on* M.

PROOF. As to 3 we remark that X^* and X are Π-related and so

$$\Pi_*(h([X^*,Y^*]_p) = \Pi_*([X^*,Y^*]_p) = [X,Y]_{\Pi(p)}.$$

The proof of the other items is left to the reader.

5.4 Parallel transport of vectors in a frame bundle; the covariant derivative.

Let P be an arbitrary principal bundle $\Pi\colon P \to M$ with structure group G.

DEFINITION 8. A C^1 curve $\gamma\colon [0,1] \to P$ is called *horizontal*, whenever all tangent vectors $\gamma'(t)$ are horizontal. A lift of the C^1 curve $c\colon [0,1] \to M$ is a horizontal curve $c^*\colon [0,1] \to P$ with $\Pi(c^*) = c$. It follows that whenever c^* is a lift of c, then also $R_g(c^*)$ is a lift of c (see (5.23)).

We have the following theorem

THEOREM 6. *Let* $c: 0 \leqslant t \leqslant 1 \to M$ *be a (piecewise)* C^1 *curve and* $p_0 \in P$ *with* $\Pi(p_0) = c(0)$. *Then there exists a unique lift* c^* *of* c *with* $c^*(0) = p_0$.

PROOF. On account of the local triviality of the principal bundle there exists a curve $\gamma: 0 \leqslant t \leqslant 1 \to P$ with $\gamma(0) = p_0$ and $\Pi \circ \gamma = c$. Any lift c^* with $c^*(0) = p_0$ must now be of the form

$$c^*(t) = \gamma(t) \cdot g(t), \quad \text{for some } g: 0 \leqslant t \leqslant 1 \to G \text{ and with } g(0) = e.$$

Hence, using (5.11) we get

$$\frac{dc^*(t)}{dt} = R_{g(t)*}\frac{d}{dt}\gamma(t) + \sigma(L_{g(t)_*^{-1}} \frac{d}{dt} g)(c^*(t))$$

and so:

$$\omega(c^{*\prime}(t)) = Ad(g(t)^{-1})\omega(\gamma'(t)) + L_{g(t)_*^{-1}} \frac{d}{dt} g$$

It follows that $c^*(t)$ is horizontal, if and only if the right-hand side is zero. Using the definition (5.2), this results into

$$L_{g(t)_*^{-1}} \frac{d}{dt} g = -L_{g(t)_*^{-1}} R_{g(t)*}\omega(\gamma'(t))$$

or into the differential equation of the first order:

$$\frac{dg}{dt} = - R_{g(t)*}\,\omega(\gamma'(t)),$$

where $\omega(\gamma'(t))$ is now a given curve in $\mathcal{g}$.

This equation has for $g(0) = e$ a unique solution in a right neighbourhood of $t = 0$ and so there exists a unique lift $c^*(t)$ defined in a right neighbourhood of $t = 0$ with $c^*(0) = p_0$.
In the same way one shows that there exists always a unique lift c^*; defined in a neighbourhood of any $\bar{t} \in (0,1)$, while $c^*(\bar{t}) = \bar{p}$ with $\bar{p}$ an arbitrary point of $\Pi^{-1}(c(\bar{t}))$. There is left to prove that the

lift c^* can be defined uniquely on <u>all</u> of $[0,1]$. We know already that a unique lift c^* exists on $[0, t_0)$, with t_0 some positive number between zero and one, and we have to prove that this lift can be extended past t_0. We pick the lift $\tilde{c}^*(t)$ defined in a full neighbourhood of t_0 with an initial condition, which is fixed by the condition $\tilde{c}^*(t_1) = c^*(t_1)$ for some $t_1 < t_0$ and t_1 close enough to t_0 such that $\tilde{c}^*(t_1)$ and $c^*(t_1)$ are defined. Due to the uniqueness on $[0, t_0)$ we have $\tilde{c}^*(t) = c^*(t)$ on $[t_1, t_0)$ and hence $\tilde{c}^*(t)$ extends $c^*(t)$ past t_0. ∎

By means of the last theorem we define now the so-called *parallel transport of fibres* of the principal bundle $\Pi: P \to M$ along any curve $c: [0,1] \to M$.

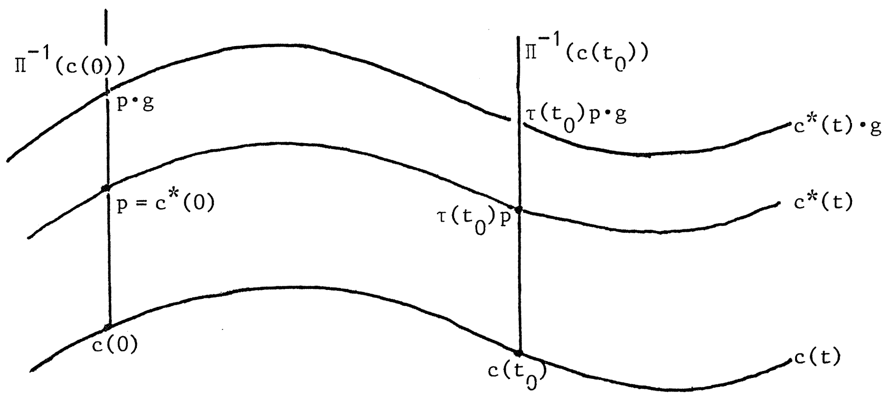
For any $p \in \Pi^{-1}(c(0))$ we require $\tau_t(p) \in \Pi^{-1}(c(t))$, such that $\tau_t(p)$ coincides with the horizontal lift $c^*(t)$ through p.

In this way we get a map

$$\tau_t: \Pi^{-1}(c(0)) \to \Pi^{-1}(c(t)),$$

which is a diffeomorphism between fibres above $c(t)$.

This map is the parallel transport of fibres along the curve c.



We consider now the special case of the frame bundle $F(M)$ and the C^1-curve $c(t) \subset M$ with $c(0) = x$ and X_x is some vector belonging to $T_x(M)$. Let $u = (u_1, u_2, \ldots, u_n)$ be some frame above x, then there exists a unique vector $\vec{\xi} \in \mathbb{R}_n$ such that

$$X_x = u_i \xi^i \overset{\text{not.}}{:=} u_x(\vec{\xi}) \tag{5.26}$$

<u>DEFINITION 9</u>. The *parallel translation* of the vector X_x along $\gamma = c(t)$ is defined as

$$\tau_t(X_x) = c^*(t)(\vec{\xi}) := c_i^*(t)\xi^i \tag{5.27}$$

with $X_x = c_i^*(0)\xi^i$.

This definition means that the components of the vector $X_{c(t)}$ expressed in the base along a lift of $c(t)$, do not change with the transport along $c(t)$. However, the definition has only a well-defined meaning whenever the parallel transport of $X_{c(t)}$ is independent of the choice of the lift of $c(t)$. That this is indeed the case is clear from the following reasoning: Suppose $\bar{c}^* = R_h(c^*)$ is another lift of $\gamma = c(t)$; we have

$$X_x = c^*(0)(\vec{\xi}) = (c^*(0)\cdot h)(h^{-1}\vec{\xi}) = \bar{c}^*(0)(h^{-1}\vec{\xi}) \quad \text{and hence}$$

$$\tau_t(X_x) := \bar{c}^*(t)(h^{-1}\vec{\xi}) = c^*(t)\cdot h)(h^{-1}\vec{\xi}) = c^*(t)(\vec{\xi}) = \tau_t(X_x).$$

τ_t is an isomorphism between the tangent spaces $T_{c(0)}(M)$ and $T_{c(t)}(M)$.

Finally, we are ready for the coordinate free definition of the covariant derivative of the vector Y_x in the point $x \in M$ along the curve $\gamma = c(t)$ with $x = c(0)$.

DEFINITION 10. The *covariant derivative of the vector field* $Y \in \mathcal{X}(M)$ *along the curve* $\gamma = c(t)$ *in the point* $x = c(0)$ *is*

$$\tilde{\nabla}_{X_x} Y := \lim_{h \downarrow 0} \frac{1}{h} (\tau_h^{-1} Y_{c(h)} - Y_x) \tag{5.28}$$

with $X_x = \frac{dc}{dt}(0)$.

A vector field Y is *parallel* along $\gamma = c(t)$, $0 \leqslant t < 1$, when for all t with $0 \leqslant t < 1$

$$\tilde{\nabla}_{X_{c(t)}} Y = 0 \quad \text{with} \quad X_{c(t)} = \frac{dc}{dt}(t) \tag{5.29}$$

Finally, we want to show that the covariant derivative, defined above for arbitrary C^∞ manifolds, is the same as the derivative connected with the Cartan connection and which has been defined in formula (5.6). In order to prove this we still need one lemma.

Suppose we have the vector field $Y \in \mathcal{X}(M)$, $Y_x \in T_x(M)$ with $x = \Pi(v)$ and $v \in F(M)$. According to (5.26)

$$Y_x = v_i \eta^i = v_x(\vec{\eta}) \tag{5.30}$$

We introduce now the map $\Lambda_Y : F(M) \to \mathbb{R}_n$

with $\Lambda_{Y_x}(v_x) = \vec{\eta}$.

So Λ_{Y_x} assigns to the frame v_x the coordinates of Y_x with respect to the frame v_x.

The map Λ_Y may be written as

$$\Lambda_Y(v) = v^{-1}(Y_{\Pi(v)}), \tag{5.31}$$

where v^{-1} is the inverse of the map v, as defined in (5.30). We have the following very useful lemma.

LEMMA. Let X and Y be vector fields on M and $X_x, Y_x \in T_x(M)$. For *any* frame $u \in F(M)$ with $x = \pi(u)$ we have

$$\tilde{\nabla}_{X_x} Y = u(X_u^*(\Lambda_Y)) \tag{5.32}$$

where $X_u^* \in T_u(F(M))$ is the unique horizontal vector with $\pi_*(X_u^*) = X_x$.

COMMENT. According to the notation (5.30) and the definition of Λ_Y the right-hand side of (5.32) equals

$$u(X_u^*(\Lambda_Y)) = \sum_{i=1}^{n} u_i X_u^* \eta^i$$

with $Y = \sum_{i=1}^{n} u_i \eta^i$.

PROOF. Let $\gamma = c(t)$ be a curve in M with $c(0) = x$ and $c'(0) = X_x$; further, let $c^*(t)$ be the lift of $\gamma = c(t)$ with $c^*(0) = u$ and so we have $\left(\frac{d}{dt} c^*\right)(0) = X_u^*$.

We now choose the vector $\vec{\xi} \in \mathbb{R}^n$ with

$$c^*(h)(\vec{\xi}) = Y_{c(h)} \quad \text{or} \quad \vec{\xi} = \{c^*(h)\}^{-1}(Y_{c(h)}).$$

According to parallel transport we have:

$$\tau_h^{-1}(Y_{c(h)}) = c^*(0)(\vec{\xi}) = u(\vec{\xi})$$

and consequently

$$u \circ \{c^*(h)\}^{-1}(Y_{c(h)}) = \tau_h^{-1}(Y_{c(h)}).$$

So we have

$$\tilde{\nabla}_{X_x} Y = \lim_{h \downarrow 0} \frac{1}{h} [\tau_h^{-1}(Y_{c(h)}) - Y_x] =$$

$$= \lim_{h \downarrow 0} \frac{1}{h} [u \circ \{c^*(h)\}^{-1}(Y_{c(h)}) - u \circ u^{-1}(Y_x)] =$$

$$= u(\lim_{h\downarrow 0} \frac{1}{h}[\{c^*(h)\}^{-1}(Y_{c(h)}) - \{c^*(0)\}^{-1}(Y_{c(0)})]) =$$

$$= u(\lim_{h\downarrow 0} \frac{1}{h}[\Lambda_Y(c^*(h)) - \Lambda_Y(c^*(0))]) = u(X_u^*(\Lambda_Y)). \quad \blacksquare$$

Our final theorem reads

<u>THEOREM 7</u>. *The operator* $\tilde{\nabla}_X$, *defined in definition* 10 *by means of parallel transport, is the same as the operator* ∇_X, *defined in definition* 7 *by means of the Cartan connection forms; in formula*

$$\tilde{\nabla}_X Y = \nabla_X Y, \quad \forall X,Y \in \mathcal{X}(M) \tag{5.33}$$

<u>PROOF</u>. In order to prove this theorem we first show that $\tilde{\nabla}_X Y$ defines a Koszul connection (see Definition 2), i.e. for all vector fields $X_i, Y \in \mathcal{X}(M)$ we have:

1. $\tilde{\nabla}_{X_1+X_2} Y = \tilde{\nabla}_{X_1} Y + \tilde{\nabla}_{X_2} Y$
2. $\tilde{\nabla}_{aX} Y = a\tilde{\nabla}_X Y, \quad \forall a \in \mathbb{R}$
3. $\tilde{\nabla}_X (Y_1+Y_2) = \tilde{\nabla}_X Y_1 + \tilde{\nabla}_X Y_2$
4. $\tilde{\nabla}_X (fY) = f\tilde{\nabla}_X Y + X(f)Y, \quad \forall f: M \to \mathbb{R}$, f differentiable.

Before proving 1-4, we show that if X and Y are C^∞ vector fields on M, then also $x \to \tilde{\nabla}_{X_x} Y$ is a vector field on M, $(x \in M)$. This follows readily from the lemma, because we can choose a C^∞-section $x \to u(x)$ in a neighbourhood of any $x \in M$. Also condition 1 follows from the lemma, due to the fact that $(X_1+X_2)^* = X_1^* + X_2^*$ (Theorem 5). The conditions 2, 3 and 4 follow easily from Definition 10 and the proof is left to the reader.

We now compare the Koszul connection $\tilde{\nabla}$ with the Koszul connection ∇, defined in Definition 7 by means of an Ehresmann connection ω on the frame bundle $F(M)$.

We write the matrix valued one form $\omega = (\omega^i_j)$, i row index and j column index, as

$$\omega = \omega^i_j E^j_i,$$

with E^j_i a matrix with all entries zero, except the one in the i^{th} row and the j^{th} column, which equals 1.

Suppose (x,U) is a chart on M and let $s: U \to F(M)$ be the "natural section":

$$s(q) = \left(\frac{\partial}{\partial x^1}\bigg|_q, \ldots, \frac{\partial}{\partial x^n}\bigg|_q \right); \qquad q \in U.$$

The Cartan connection, corresponding with ω, assigns $(s^*\omega^i_j)$ to this frame and according to (5.6), (5.7) and (5.8) the operator ∇ is determined on U by

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = s^*\omega^k_j\left(\frac{\partial}{\partial x^i}\right) \frac{\partial}{\partial x^k} = \omega^k_j\left(s_*\left(\frac{\partial}{\partial x^i}\right)\right) \frac{\partial}{\partial x^k} \tag{5.34}$$

Due to the fact that as well ∇ as $\tilde{\nabla}$ are Koszul connections, the proof of the theorem is completed, when we show

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \tilde{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$$

or according to (5.34) and the last lemma:

$$\omega^k_j\left(s^*\left(\frac{\partial}{\partial x^i}\right)\right) \frac{\partial}{\partial x^k}\bigg|_p = s(p)\left(\left(\frac{\partial}{\partial x^i}\right)^*_{s(p)} \left(\Lambda \frac{\partial}{\partial x^j}\right)\right)$$

Remembering $s(p)(\vec{\xi}) = \xi^k \frac{\partial}{\partial x^k}\Big|_p$ and writing

$$\Lambda_{\frac{\partial}{\partial x^j}}(\cdot) = \{\Lambda^1_{\frac{\partial}{\partial x^j}}(\cdot),\ldots,\Lambda^n_{\frac{\partial}{\partial x^j}}(\cdot)\}$$

we have to show:

$$\omega^k_j\left(s^*\left(\frac{\partial}{\partial x^i}\right)\Big|_p\right) = \left(\frac{\partial}{\partial x^i}\right)^*_{s(p)} \Lambda^k_{\frac{\partial}{\partial x^j}} \tag{5.35}$$

By definition of the function Λ_Y we have for <u>all</u> frames $u = (u_1,u_2,\ldots,u_n) \in F(M)$:

$$\Lambda^k_{\frac{\partial}{\partial x^j}}(u)u_k = \frac{\partial}{\partial x^j} \tag{5.36}$$

and so in particular $\Lambda^k_{\frac{\partial}{\partial x^j}}(s(q)) = \delta^k_j$ (5.37)

Taking for u the frame $s(p)\cdot\exp(tM)$ with M any arbitrary matrix ($M\in\mathcal{g}$) and differentiating (5.36) with respect to t, we get after putting $t = 0$:

$$\frac{d}{dt}\{\Lambda^k_{\frac{\partial}{\partial x^j}}(s(p)\cdot\exp(tM))\}\Big|_{t=0} \frac{\partial}{\partial x^k}\Big|_p +$$

$$+ \Lambda^k_{\frac{\partial}{\partial x^j}}(s(p)) \frac{d}{dt}\{s(p)\cdot\exp(tM)\}_k\Big|_{t=0} = 0$$

and hence

$$\sigma(M)_{s(p)}(\Lambda^k_{\frac{\partial}{\partial x^j}}) \frac{\partial}{\partial x^k}\Big|_p = -\delta^k_j(s(p)\cdot M)_k = -(s(p)\cdot M)_j .$$

Taking for M the matrix E^ν_μ, the latter result becomes

$$\sigma(E^{\nu}_{\mu})_{s(p)}(\Lambda^{k}_{\frac{\partial}{\partial x^j}})\left.\frac{\partial}{\partial x^k}\right|_p = -(s(p)\cdot E^{\nu}_{\mu})_j =$$

$$= -\left.\frac{\partial}{\partial x^{\alpha}}\right|_p (E^{\nu}_{\mu})^{\alpha}_j = -\left.\frac{\partial}{\partial x^{\alpha}}\right|_p \delta^{\alpha}_{\mu}\,\delta^{\nu}_j = -\delta^{\nu}_j \left.\frac{\partial}{\partial x^{\mu}}\right|_p$$

and so

$$\sigma(E^{\nu}_{\mu})_{s(p)}\ \Lambda^{k}_{\frac{\partial}{\partial x^j}} = -\delta^{\nu}_j\ \delta^{k}_{\mu} \tag{5.38}$$

The lift $\left(\frac{\partial}{\partial x^i}\right)^*$ at points $s(p)$ of $\frac{\partial}{\partial x^i}$ can be calculated with the aid of formula (5.25)

$$\left(\frac{\partial}{\partial x^i}\right)^*_{s(p)} = s_*\left.\frac{\partial}{\partial x^i}\right|_p - \omega^{\mu}_{\nu}\left(s_*\left.\frac{\partial}{\partial x^i}\right|_p\right)\sigma(E^{\nu}_{\mu}) .$$

Inserting finally this result into the right-hand side of (5.35) and using (5.37) and (5.38) we obtain:

$$\left(\frac{\partial}{\partial x^i}\right)^*_{s(p)} \Lambda^{k}_{\frac{\partial}{\partial x^j}} = \frac{\partial}{\partial x^i}(\Lambda^{k}_{\frac{\partial}{\partial x^j}}\circ s)(p) - \omega^{\mu}_{\nu}(s_*\left.\frac{\partial}{\partial x^i}\right|_p)\ \sigma(E^{\nu}_{\mu})_{s(p)}(\Lambda^{k}_{\frac{\partial}{\partial x^j}})$$

$$= \frac{\partial}{\partial x^i}(\delta^k_j) + \omega^{\mu}_{\nu}(s_*\left.\frac{\partial}{\partial x^i}\right|_p)\delta^{\nu}_j\ \delta^k_{\mu} =$$

$$= \omega^k_j(s_*\left.\frac{\partial}{\partial x^i}\right|_p),$$

and so the validity of (5.35) is demonstrated, which finishes the proof of our theorem. ∎

In case the Cartan connection is defined on a Riemannian manifold and satisfies the conditions (3.5) and (3.6) then according to Theorems 1, 2, 3 and 7 the covariant derivation $\tilde{\nabla}$ is the same as the covariant derivation ∇ as defined in Definition 1. This proves that the Definition 10 gives a generalization on arbitrary C^{∞} manifolds of the covariant derivation on Riemannian manifolds.

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THE YANG-MILLS EQUATIONS

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CONTENTS

0. INTRODUCTION

Imagine a structured particle, that is a particle which has a location at a point x of $M = \mathbb{R}^4$ and an internal structure, or set of states, labelled by elements g of a Lie group G. We then consider the total space P of all states of such a particle: $P = M \times G$.

In the absence of any external field, however, we consider that all fibres $G_x = \{x\} \times G \subset P$ can be identified to each other.

Now we imagine an external field imposed which has the effect of distorting the relative alignment of the fibres so that no coherent identification is possible between the G_x at different points. However, we assume that G_x and G_y can still be identified if we choose a definite path in $\mathbb{R}^4$ from x to y. In more physical terms we imagine the particle moving from x to y and carrying its internal space with it. This identification of fibres along paths is called parallel transport. If we now imagine two different paths joining x to y then there is no reason for the two different parallel transports to agree and they are assumed to differ by multiplication with a group element, which could be viewed as a generalized "phase shift". This phase shift is interpreted as produced by an external field. In geometric terms it is viewed as the total "curvature" of the fibre bundle over the region enclosed by the two paths.

If we now infinitesimalize this picture we get the infinitesimal parallel transport at a point x in a given direction. This will be an infinitesimal shift A of the fibre G_x into the nearby fibre and is called a *connection*. The infinitesimal curvature F depends on two directions at x and takes values in the Lie algebra of G_x. The curvature F can be thought of as the distortion product by an external field, or it can be identified with the field. The field F is called a *gauge field* and A is called a *gauge potential*.

1. CONNECTIONS, COVARIANT DERIVATIVE, CURVATURE

Let M be a C^∞-manifold of dimension n and let G be a Lie group with unit element e.

(1.1) DEFINITION. A *principal fibre bundle* P with structural group G (principal G-bundle) is defined by a C^∞-map

$$\pi: P \to M$$

and a C^∞ right action

$$P \times G \longrightarrow P$$
$$(u,g) \longmapsto ug := R_g u$$

such that

(i) $\pi^{-1}(x)$ is an orbit for all $x \in M$

(ii) for all $x \in M$ there is a neighbourhood U of x and a diffeomorphism

$$\phi: \pi^{-1}(U) \to U \times G$$

(*local trivialization* or *local gauge map*) such that

(a) the diagram $\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\phi} & U \times G \\ & {\scriptstyle\pi}\searrow \quad \swarrow & \\ & U & \end{array}$ commutes

(b) ϕ commutes with the group action (ϕ is equivariant) i.e.

$\phi(ug) = \phi(u)g \quad (u \in P,\ g \in G).$

From the definition it follows that every fibre $\pi^{-1}(x)$ is diffeomorphic to G but there is no canonical diffeomorphism (the fibre has no natural unit element).

A local section $s: U \to \pi^{-1}(U)$ defines a local trivialization

$$\phi\colon \pi^{-1}(U) \to U \times G$$

by

$$\phi(s(x)) := (x,e).$$

Then

$$\phi(s(x)g) = (x,g).$$

Conversely, if ϕ is a local trivialization, then $s\colon x \longmapsto \phi^{-1}(x,e)$ is the corresponding local section, also called local gauge map.

Note that the bundle P is trivial iff there exists a global gauge map $s\colon M \to P$.

Now consider an open cover $\{U_\alpha\}$ of M together with trivializations

$$\phi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$$

and corresponding sections

$$s_\alpha\colon U_\alpha \to \pi^{-1}(U_\alpha).$$

For $x \in U_\alpha \cap U_\beta$ there is a unique element $g_{\alpha\beta}(x) \in G$ such that

$$s_\beta(x) = s_\alpha(x) g_{\alpha\beta}(x).$$

Then

$$\phi_\beta \circ \phi_\alpha^{-1}\colon (U_\alpha \cap U_\beta) \times G \to (U_\alpha \cap U_\beta) \times G$$

is given by

$$(x,a) \longmapsto (x, g_{\beta\alpha}(x)a).$$

The map $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ is called a *gauge transformation*. The set $\{g_{\alpha\beta}\}$ satisfies the cocycle condition

$$(*) \qquad \begin{cases} g_{\gamma\beta}(x)g_{\beta\alpha}(x) = g_{\gamma\alpha}(x) & \forall x \in U_\alpha \cap U_\beta \cap U_\gamma \\ g_{\alpha\alpha}(x) = e. \end{cases}$$

Conversely, if we start off with an open cover $\{U_\alpha\}$ together with a set $\{g_{\alpha\beta}\}$ where $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ satisfy the cocycle condition, then one can reconstruct the principal G-bundle.

Consider the space $\bigcup_\alpha (U_\alpha \times G)$ (disjoint union) and identify $(x,a) \in U_\alpha \times G$ with $(x,b) \in U_\beta \times G$ if $b = g_{\beta\alpha}(x)a$. The quotient space is then the principal bundle P.

We consider a G-bundle $\pi\colon P \to M$. Let $F = \mathbb{R}^r$ or $\mathbb{C}^r$ and let ρ be a representation of G on F, i.e. there is a left action

$$\begin{aligned} G \times F &\longrightarrow F \\ (g,\xi) &\longmapsto g\xi := \rho(g)\xi . \end{aligned}$$

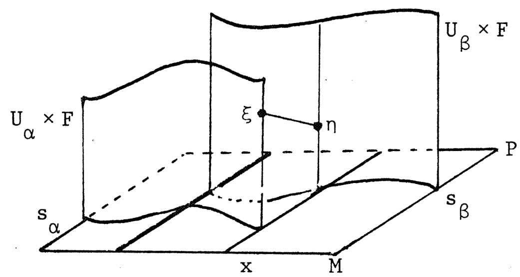
The group G acts from the right on the space $P \times F$ by

$$(u,\xi)g := (ug, g^{-1}\xi).$$

(1.2) DEFINITION. The *associated vector bundle* $P \times_\rho F$ is defined as the orbit space under this action.

We can describe the associated vector bundle in terms of gauge transformations. Consider an open cover $\{U_\alpha\}$ with sections $\{s_\alpha\}$ and local gauge transformations $\{g_{\alpha\beta}\}$. In the disjoint union $\bigcup_\alpha (U_\alpha \times F)$ we introduce the equivalence relation

$$U_\alpha \times F \ni (x,\xi) \sim (x,\eta) \in U_\beta \times F \quad \text{if} \quad \eta = g_{\beta\alpha}(x)\xi.$$



Let $\pi_E : E \to M$ be the natural projection. An element $U \in P$ $(\pi(u) = x \in M)$ determines a basis (frame) in the fibre $\pi_E^{-1}(x)$ as follows. The map

$$u: F \longrightarrow \pi_E^{-1}(x)$$
$$\xi \longmapsto u\xi := \text{orbit of } (u,\xi) = \{(ug, g^{-1}\xi)\}$$

is an isomorphism; the image of the standard basis of F is a frame $f = (e_1,\ldots,e_r)$ in $\pi_E^{-1}(x)$. It is easily checked that the frame $\tilde{f}$ corresponding to the element ug is given by

$$\tilde{f} = (\tilde{e}_1,\ldots,\tilde{e}_r) \quad \text{with}$$
$$\tilde{e}_j = \sum_{j=1}^{r} e_i g_{ij}.$$

We write $\tilde{f} = fg$.

(1.3) EXAMPLES.

1. The frame bundle $F(M)$ is a principal fibre bundle with group $G = G\ell(n,\mathbb{R})$. The associated vector bundle is the tangent bundle $T(M)$.

2. The group $G = \mathbb{Z}_2 = \{+1,-1\}$ acts on S^1 by:

$$\begin{cases} +1 = \text{identity} \\ -1 = \text{antipodal map.} \end{cases}$$

This defines a principal fibre bundle. Let the representation ρ of G on $\mathbb{R}$ be defined by

$$\rho(\pm 1) = \pm \text{ identity}.$$

The associated vector bundle is then the Möbius strip.

3. The group $C^* = \mathbb{C}\setminus\{0\}$ acts on $\mathbb{C}^{n+1}\setminus\{0\}$ by

$$(z_1,\ldots,z_{n+1})\ \alpha := (\alpha z_1,\ldots,\alpha z_{n+1}).$$

This defines a principal fibre bundle over $P_n(\mathbb{C})$ (n-dimensional complex projective space). We define the representation ρ of $\mathbb{C}^*$ on $\mathbb{C}$ by

$$\rho(\alpha)z := \alpha z.$$

The associated vector bundle is called the canonical line bundle over $P_n(\mathbb{C})$. It can be described as follows. Let $[z_1 : \ldots : z_{n+1}]$ be homogeneous coordinates on $P_n(\mathbb{C})$. On the open set $U_i := \{z_i \neq 0\}$ we define the section $s_i\colon U_i \to \mathbb{C}^{n+1}$ by

$$s_i[z_1 : \ldots : z_{n+1}] = \left(\frac{z_1}{z_i},\ldots,1,\ldots,\frac{z_n}{z_i}\right).$$

The local gauge transformations $g_{ij}\colon U_i \cap U_j \to \mathbb{C}^*$ are then given by

$$g_{ij}[z_1 : \ldots : z_{n+1}] = \frac{z_i}{z_j}.$$

<u>REMARK</u>. One can also consider the action of $U(1)$ on the sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. This defines a $U(1)$-bundle on $P_n(\mathbb{C})$. For $n = 1$ one obtains the $U(1)$-bundle S^3 over $P_1(\mathbb{C}) = S^2$; this bundle is nontrivial since S^3 is not $S^2 \times S^1$.

<u>REMARK</u>. The action of $\mathbb{Z}_2$ on $S^n \subset \mathbb{R}^{n+1}$ gives a $\mathbb{Z}_2$-bundle over $P_n(\mathbb{R})$. For $n = 1$ this is again Example 2.

(1.4) Let $\pi\colon P \to M$ be a principal G-bundle and let $\mathfrak{g}$ be the Lie algebra of G. For a fixed element $u \in P$ the map

$$\begin{aligned} G &\longrightarrow P \\ g &\longmapsto ug \end{aligned}$$

(orbit of u under G) is a diffeomorphism of G onto the fibre $\pi^{-1}(x)$ of u. The induced map

$$\sigma_u : \mathfrak{g} \to T_u(P)$$

is an isomorphism of $\mathfrak{g}$ onto the subspace V_u of vertical tangent vectors at u.

DEFINITION. If $A \in \mathfrak{g}$, then the vector field $u \longmapsto \sigma_u(A)$ is called the *fundamental vector field* corresponding to A.
Notation: $\sigma(A)$ or A^*

Note that $\sigma_u(A) = \frac{d}{dt}(u \exp(tA))\Big|_{t=0}$.

Furthermore, if A is seen as a left invariant vector field on G, then the derivative of the map

$$G \longrightarrow P$$
$$g \longmapsto ug$$

maps the vector field A into $\sigma(A)$.

What happens to $\sigma(A)$ if one applies a right translation $R_a : P \to P$ ($a \in G$ fixed)?

The answer is as follows: the map R_a maps the curve $t \longmapsto u\exp(tA)$ into the curve $t \longmapsto u\exp(tA)a = ua\exp(t\,Ad(a^{-1})A)$; so $R_{a*}(\sigma_u(A)) = \sigma_{ua}(Ad(a^{-1})A)$.

(1.5) *Vector-valued differential forms*. The space of all k-forms on M with values in a finite dimensional vector space V is denoted by $A^k(M,V)$. If $\{v_1, \ldots, v_r\}$ is a basis for V, then $\omega \in A^k(M,V)$ can be written as

$$\omega = \sum \omega_i v_i \,.$$

We define

$$d\omega := \sum (d\omega_i) v_i .$$

This definition is independent of the choice of the basis and it defines a map

$$d\colon A^k(M,V) \to A^{k+1}(M,V).$$

Let $\Phi\colon V \times W \to F$ be a bilinear map (V,W and F are finite dimensional vector spaces). Let $\{w_1, \dots, w_p\}$ be a basis for W.
For $\omega = \sum \omega_i v_i \in A^k(M,V)$ and $\eta = \sum \eta_j w_j \in A^\ell(M,W)$ we define the *wedge product with respect to* Φ $\omega \wedge_\Phi \eta \in A^{k+\ell}(M,V)$ by

$$\omega \wedge_\Phi \eta := \sum \omega_i \wedge \eta_j \quad \Phi(v_i, w_j).$$

The intrinsic definition reads

$$(\omega \wedge_\Phi \eta)(X_1, \dots, X_{k+\ell}) = \sum \varepsilon_\sigma \Phi(\omega(X_{\sigma(1)}, \dots, X_{\sigma(k)}), \eta(X_{\sigma(k+1)}, \dots, X_{\sigma(k+\ell)})),$$

where the sum is taken over all permutations σ with $\sigma(1) < \dots < \sigma(k)$ and $\sigma(k+1) < \dots < \sigma(k+\ell)$.

Take e.g. $V = W = M(r,\mathbb{C})$ ($(r\times r)$-matrices). For matrix-valued forms $\omega = (\omega^i_j)$ and $\eta = (\eta^i_j)$ one may take the wedge product $\omega \wedge \eta$ with respect to matrix multiplication and the wedge product $[\omega \wedge \eta]$ with respect to the Lie product. We have

a) if $\theta = \omega \wedge \eta$, then $\theta^i_j = \sum_k \omega^i_k \wedge \eta^k_j$

b) if ω is a k-form and η is an ℓ-form, then
$[\omega \wedge \eta] = \omega \wedge \eta - (-1)^{k\ell} \eta \wedge \omega$.

PROOF of b) for the case $k = \ell = 1$:

$$\begin{aligned}[\omega \wedge \eta](X,Y) &= [\omega(X), \eta(Y)] - [\omega(Y), \eta(X)] \\ &= \omega(X)\eta(Y) - \eta(Y)\omega(X) - \omega(Y)\eta(X) + \eta(X)\omega(Y) \\ &= (\omega\wedge\eta + \eta\wedge\omega)(X,Y). \quad \blacksquare\end{aligned}$$

In particular, for a matrix-valued 1-form ω we have $[\omega\wedge\omega] = 2\omega\wedge\omega$.

(1.6) After these preparations we come to our subject: the theory of connections. Let $\pi\colon P \to M$ be a principal G-bundle, we now give two equivalent definitions of a connection.

DEFINITION 1. A *connection* on P is a distribution $\{H_u\}$ $(u \in P$ and $H_u \subset T_u(P))$ satisfying:

(i) $H_u \oplus V_u = T_u(P)$,

(ii) $R_{a*}(H_u) = H_{ua}$ $(u \in P,\ a \in G)$.

DEFINITION 2. A *connection* on P is a $\mathfrak{g}$-valued 1-form $\omega \in A^1(P,\mathfrak{g})$ satisfying:

(a) $\omega_u \circ \sigma_u = \mathrm{id}$,

(b) $R_a^*\omega = \mathrm{Ad}(a^{-1})\omega$ $(a\in G)$.

The relation between the two definitions is given by $H_u = \ker\omega_u$.

Def.1 ⇒ Def.2. Define ω_u by $\begin{cases}\omega_u(H_u) = 0,\\ \omega_u = \sigma_u^{-1} \text{ on } V_u.\end{cases}$

Then ω satisfies (a) (trivial) and (b):

$$\begin{aligned}(R_a^*)(\sigma_u(A)) &= \omega((R_{a*})\sigma_u(A))\\ &= \mathrm{Ad}(a^{-1})\omega(\sigma_u(A)).\end{aligned}$$

Def.2 ⇒ Def.1. Define H_u by $H_u = \ker\omega_u$.

Then $\{H_u\}$ satisfies (i) (trivial) and (b):

$$\omega(H_u) = 0 \Rightarrow \omega(R_{a*}(H_u)) = \mathrm{Ad}(a^{-1})\omega(H_u) = 0,$$

$$\text{so } R_{a*}(H_u) \subset \ker\omega_{ua} = H_{ua}.$$

(1.7) Consider an open cover $\{U_\alpha\}$ of M with local gauge maps $s_\alpha\colon U_\alpha \to \pi^{-1}(U_\alpha)$ and local gauge transformations $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$. As we have seen, the principal G-bundle P and the associated bundle E can be reconstructed from the $\{g_{\alpha\beta}\}$.

Furthermore, consider a connection ω on P. For each α we define the $\mathfrak{g}$-valued 1-form ω_α on U_α by

$$\omega_\alpha := s_\alpha^*\omega .$$

Then

$$\omega_\alpha = \sum_\mu A_\mu^\alpha dx^u \quad (A_\mu^\alpha \in \mathfrak{g}),$$

where (x^μ) are coordinates on U_α.

<u>THEOREM</u>. $\omega_\beta(x) = Ad(g_{\alpha\beta}(x)^{-1})\omega_\alpha(x) + g_{\alpha\beta}(x)^{-1}dg_{\alpha\beta}(x) \quad (x \in U_\alpha \cap U_\beta)$.

<u>REMARK</u>. If $g: U \to G$ is a C^∞-map, then the composition

$$T_x(M) \xrightarrow[g_*]{} T_{g(x)}(G) \xrightarrow[L_{g(x)^{-1}*}]{} \mathfrak{g} = T_e(G)$$

is briefly denoted by $g(x)^{-1}dg(x)$.

If G is a linear group and $g(x) = (g_{ij}(x))$, then this map is indeed equal to the matrix $g(x)^{-1}dg(x)$.

The formula can also be written as

$$A_\mu^\beta = g_{\alpha\beta}^{-1} A_\mu^\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1}(\partial_\mu g_{\alpha\beta}).$$

<u>PROOF</u>. Consider the map

$$\begin{array}{ccccc} U_\alpha \cap U_\beta & \longrightarrow & P \times G & \xrightarrow{m} & P \\ x & \longmapsto & (s_\alpha(x), g_{\alpha\beta}(x)) & \longmapsto & s_\alpha(x)g_{\alpha\beta}(x) = s_\beta(x). \end{array}$$

Instead of s_α and $g_{\alpha\beta}$ we write s and g respectively. For $X \in T_x(M)$ we have $s_{\beta_*}(X) = m_*\ (s_*(X),\ g_*(X))$. Now m_* at the point $(s(x),\ g(x)) \in P \times G$ is given by

$$m_*(Z_1,Z_2) = m_{1_*}Z_1 + m_{2_*}Z_2,$$

where

$$m_1 = R_{g(x)} : P \longrightarrow P$$

and
$$m_2 = L_{s(x)} : G \longrightarrow P$$
$$a \longmapsto s(x)a.$$

Now m_2 maps the left invariant vector field $A \in g$ into $\sigma(A)$. Hence

$$s_{\beta *}(X) = R_{g(x)*}\, s_*(X) + Y,$$

where Y is the fundamental vector field corresponding to the left invariant vector field that at $g(x)$ coincides with $g_*(X)$. Applying ω we get the described result. ∎

THEOREM. *Conversely, if for a given cover* $\{U_\alpha\}$ *and a given set of sections* $\{s_\alpha\}$ *and gauge transformations* $\{g_{\alpha\beta}\}$ *there is given a set of* 1-*forms* $\omega_\alpha \in A^1(U_\alpha, g)$ *satisfying the above mentioned transformation rules, then there exists a unique connection* $\omega \in A^1(P,g)$ *such that* $\omega_\alpha = s_\alpha^*\omega$.

PROOF. At a point $u = s_\alpha(x)$ we define:

$$\begin{cases} \omega_u = \sigma_u^{-1} \quad \text{on } V_u \\ \omega_u(s_{\alpha *}(X)) = \omega_\alpha(X). \end{cases}$$ ∎

(1.8) Now consider a connection $\{H_u\}$ or ω on P. In general, the distribution $\{H_u\}$ is not involutive and hence (Frobenius) not integrable. If H_u is integrable, then there exist local horizontal sections and conversely.

A section $s: U \to \pi^{-1}(U) \subset P$ is called horizontal if $\omega = 0$ on $s(U)$, or equivalently if $s_*\omega = 0$. An integrable connection is called *flat*.

REMARK. If $s_*\omega = 0$, then $(sg)^*\omega = g^{-1}dg$ ("pure gauge").

The distribution $\{H_u\}$ is integrable if and only if $[X,Y]$ is horizontal for horizontal vector fields X and Y, or equivalently: $\omega([X,Y]) = 0$ if $\omega(X) = \omega(Y) = 0$.

Since $d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$, we have:

$$\begin{aligned} &\{H_u\} \text{ completely integrable} \\ \iff\ & d\omega(X,Y) = 0 \text{ if } \omega(X) = \omega(Y) = 0 \\ \iff\ & d\omega(hX,hY) = 0, \end{aligned}$$

where $h_u : T_u(P) = V_u \oplus H_u \to H_u$ is the projection onto H_u with kernel V_u.

DEFINITION. The $\mathfrak{g}$-valued 2-form $\Omega \in A^2(P,\mathfrak{g})$ defined by

$$\begin{aligned} \Omega(X,Y) &:= d\omega(hX,hY) \\ &= \omega([hX,hY]) \end{aligned}$$

is called the *curvature form* of ω.

So a connection is completely integrable iff its curvature form vanishes.

(1.9) THEOREM. $\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ (*Cartan's structural equation*),

$d\Omega = [\Omega \wedge \omega]$ (*Bianchi's identity*).

REMARK. If G is a linear group, then ω and Ω are matrix-valued forms and these equations can be written as

$$\begin{aligned} \Omega &= d\omega + \omega \wedge \omega \\ d\Omega &= \Omega \wedge \omega - \omega \wedge \Omega. \end{aligned}$$

PROOF of Cartan's structural equation.

1. If X and Y are horizontal vector fields (so $X = hX$, $\omega(X) = 0$ etc.), then $\Omega(X,Y) = d\omega(X,Y)$ as follows from $(d\omega+\frac{1}{2}[\omega \wedge \omega])(X,Y) = d\omega(X,Y) + [\omega(X), \omega(Y)] = d\omega(X,Y)$.
2. If X and Y are both vertical vector fields, then $\Omega(X,Y) = 0$. We

prove that also $(d\omega+\frac{1}{2}[\omega\wedge\omega])(X_u,Y_u) = 0$ for any $u \in P$. We may assume that $X = \sigma(A)$ and $Y = \sigma(B)$ with $A, B \in g$ (otherwise we consider fundamental vector field $\tilde{X}$ and $\tilde{Y}$ with $\tilde{X}_u = X_u$ and $\tilde{Y}_u = Y_u$). Then $\omega_u(Y) = B$ for all $u \in P$ and hence $X\omega(Y) = 0$.
Similarly $Y\omega(X) = 0$. Then we have $(d\omega+\frac{1}{2}[\omega\wedge\omega])(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + [\omega(X), \omega(Y)] = 0$ since $[\sigma(A), \sigma(B)] = \sigma([A,B])$.

3. Finally the case that X is a horizontal and Y is vertical vector field. Then again $\Omega(X,Y) = 0$. We may assume that $Y = \sigma(B)$ $(B \in g)$. Then $(d\omega+\frac{1}{2}[\omega\wedge\omega])(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]) + [\omega(X), \omega(Y)] = -\omega([X,Y])$. This vanishes since we have the following lemma:

LEMMA. *If* X *is a horizontal vector field, then* $[X, \sigma(B)]$ *is also horizontal.*

PROOF. Let $b_t = \exp(tB)$. Then $[\sigma(B), X] = \lim_{t\to 0} \frac{1}{t}(X - (R_{b_t})_* X)$.
Now $(R_{b_t})_* X$ is horizontal since X is. So $[\sigma(B), X]$ is horizontal. ∎

PROOF of Bianchi's identity.

$$\begin{aligned} d\Omega &= d(d\omega+\tfrac{1}{2}[\omega\wedge\omega]) \\ &= \tfrac{1}{2}[d\omega\wedge\omega] - \tfrac{1}{2}[\omega\wedge d\omega] \\ &= [d\omega\wedge\omega]. \end{aligned}$$

On the other hand

$$[\Omega\wedge\omega] = [\omega\wedge\omega] + \tfrac{1}{2}[[\omega\wedge\omega]\wedge\omega].$$

Now $[[\omega\wedge\omega]\wedge\omega] = 0$, for if we put $\theta = [\omega\wedge\omega]$ then

$$[\theta\wedge\omega](X,Y,Z) = [\theta(X,Y), \omega(Z)] - [\theta(X,Z), \omega(Y)] + [\theta(Y,Z), \omega(X)] = 0$$

because of Jacobi's identity. ∎

DEFINITION. The *exterior covariant differential* (with respect to a connection ω) of a p-form θ on P is the form $D\theta$ defined by

$$D\theta(X_1,\dots,X_{p+1}) := d\theta(hX_1,\dots,hX_{p+1}).$$

So we have $\Omega = D\omega$. Also $D\Omega = D^2\omega = 0$ since

$$D\Omega(X,Y,Z) = d\Omega(hX,hY,hZ) = [\Omega \wedge \omega]\,(hX,hY,hZ) = 0.$$

(1.10) Again consider an open cover $\{U_\alpha\}$ with sections $\{s_\alpha\}$ and gauge transformations $\{g_{\alpha\beta}\}$. We define $\mathfrak{g}$-valued 2-forms $\Omega_\alpha \in A^2(U_\alpha,\mathfrak{g})$ by

$$\Omega_\alpha := s_\alpha^*(\Omega) = \sum_{\mu<\nu} F^\alpha_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

where (x^μ) are coordinates on U_α.

THEOREM. $\Omega_\beta(x) = \mathrm{Ad}(g_{\alpha\beta}(x)^{-1})\Omega_\alpha(x) \quad (x \in U_\alpha \cap U_\beta)$.

REMARK. For a linear group G we have

$$F^\beta_{\mu\nu} = g^{-1}_{\alpha\beta} F^\alpha_{\mu\nu}\, g_{\alpha\beta}\,.$$

PROOF. It is easily verified that $R_a^*\Omega = \mathrm{Ad}(a^{-1})\Omega$. Instead of s_α and $g_{\alpha\beta}$ we write s and g respectively. If $X \in T_x(M)$ then

$$s_{\beta *}X = R_{g(x)*}\, s_*(X) + \text{a vertical vector.}$$

So

$$\Omega(s_{\beta *}X, s_{\beta *}Y) = \Omega(R_{g(x)*}s_*(X),\ R_{g(x)*}s_*(Y))$$
$$= \mathrm{Ad}(g(x)^{-1})\Omega(s_*(X), s_*(Y)).\ \blacksquare$$

__THEOREM.__ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu + [A_\mu, A_\nu]$ *(here the index* α *is omitted).*

__PROOF.__ $\overset{*}{s}_\alpha(\omega) = \sum_\mu A^\alpha_\mu dx^\mu$ and $\overset{*}{s}_\alpha(\Omega) = \sum_{\mu<\nu} F^\alpha_{\mu\nu} dx^\mu \wedge dx^\nu$.

So

$$\begin{aligned}
\overset{*}{s}_\alpha \Omega &= d(\overset{*}{s}_\alpha \omega) + \tfrac{1}{2}[\overset{*}{s}_\alpha \omega \wedge \overset{*}{s}_\alpha \omega] \\
&= \sum_{\mu,\nu} \partial_\mu A_\nu dx^\mu \wedge dx^\nu + \tfrac{1}{2} \sum_{\mu,\nu} [A_\mu, A_\nu] dx^\mu \wedge dx^\nu \\
&= \sum_{\mu<\nu} \{\partial_\mu A_\nu - \partial_\nu A_\mu) + [A_\mu, A_\nu]\} dx^\mu \wedge dx^\nu . \quad \blacksquare
\end{aligned}$$

The horizontal lift of the vector field $\frac{\partial}{\partial x^\mu}$ is denoted by D_μ. Since

$$\omega(s_{\alpha *} \frac{\partial}{\partial x^\mu}) = (\overset{*}{s}_\alpha \omega)(\frac{\partial}{\partial x^\mu}) = A^\alpha_\mu ,$$

it follows that at the point $s_\alpha(x)$ we have

$$D_\mu = s_{\alpha *} \frac{\partial}{\partial x^\mu} - \sigma(A^\alpha_\mu).$$

__THEOREM.__ $F_{\mu\nu} = -[D_\mu, D_\nu]$. *Precisely: the value of* $-[D_\mu, D_\nu]$ *at the point* $u = s_\alpha(x)$ *equals* $\sigma_u(F^\alpha_{\mu\nu}(x))$.

__PROOF.__
$$\begin{aligned}
F^\alpha_{\mu\nu} &= \Omega(s_{\alpha *} \frac{\partial}{\partial x^\mu}, s_{\alpha *} \frac{\partial}{\partial x^\nu}) \\
&= \Omega(D_\mu, D_\nu) \\
&= -\omega_u([D_\mu, D_\nu]) . \quad \blacksquare
\end{aligned}$$

Note that the vector field $[D_\mu, D_\nu]$ is vertical.

(1.11) Let $\pi: P \to M$ be a principal G-bundle and let ρ be a representation of G on the vector space F ($F = \mathbb{R}^r$ or $\mathbb{C}^r$). Let $\pi_E: E \to M$ be the associated vector bundle. For a given connection ω on P we will define the parallel transport of fibres of E along a path in M.

Let $t \longmapsto x(t)$ be a curve in M with $x(0) = x_0$ and let $t \longmapsto u(t)$ be the horizontal lift of this curve through $u(0) = u_0$ ($\pi(u_0) = x_0$). The map

$$\tau_h : \pi_E^{-1}(x_0) \longrightarrow \pi_E^{-1}(x(h))$$
$$u(0)\xi \longmapsto u(h)\xi$$

is called *parallel transport* along the curve $t \longmapsto x(t)$.

Let $\phi: M \to E$ be a C^∞-section. The *covariant derivative* $\nabla_X\phi$ of ϕ in x_0 in the direction $X = \dot{x}(0)$ is by definition

$$(\nabla_X\phi)(x_0) := \lim_{h\to 0} \frac{1}{h}\{\tau_h^{-1}\phi(x(h)) - \phi(x_0)\} \in \pi_E^{-1}(x_0).$$

This result only depends on $X = \dot{x}(0)$ and not on the curve. If now X is a vector field on M, then $\nabla_X\phi$ is again a C^∞-section.

(1.12) Let $\phi: U \to E$ be a section. Corresponding to ϕ we define a function $\tilde{\phi}: \pi^{-1}(U) \to F$ as follows. If $u \in P$ with $\pi(u) = x$ and $\phi(x) = u\xi$, then we define

$$\tilde{\phi}(u) := \xi.$$

From this definition it follows that $\tilde{\phi}(ug) = \tilde{\rho}(g^{-1})\xi$.

<u>THEOREM</u>. *Let* $\tilde{X} \in T_u(P)$ *be the horizontal lift of* $X \in T_x(M)$, *then*

$$(\nabla_X\phi)(x) = u(\tilde{X}\,\tilde{\phi}) .$$

<u>PROOF</u>. Let $x(t)$ be a curve with $\dot{x}(0) = X$ and let $u(t)$ be the horizontal lift of this curve through $u(0) = u$.

Then

$$\tilde{X}\,\tilde{\phi} = \lim_{u\to 0} \frac{1}{h}\{\tilde{\phi}(u(h)) - \tilde{\phi}(u(0))\}.$$

We put $\phi(x(h)) = u(h)\xi(h)$. So $\tilde{\phi}(u(h)) = \xi(h)$ and

$$\tilde{X}\,\tilde{\phi} = \lim_{h\to 0} \frac{1}{h}\{\xi(h) - \xi(0)\}.$$

Now $u(0)\xi(h) = \tau_h^{-1}u(h)\xi(h) = \tau_h^{-1}\,(x(h))$, so the theorem follows. ∎

Let (x^μ) be coordinates on U and let us choose a section (gauge map) $s: U \to P$. We put

$$s^*\omega = \sum_\mu A_\mu dx^\mu .$$

This section defines a trivialization

$$\begin{array}{ccc} \pi^{-1}(U) & \longrightarrow & U \times G \\ s(x)g & \longmapsto & (x,g) \end{array} .$$

As we have seen the derivative of this map sends D_μ at the point $s(x)$ into $(\frac{\partial}{\partial x^\mu}, -A_\mu)$ at the point (x,e). The section s also defines a trivialization

$$\begin{array}{ccc} \pi_E^{-1}(U) & \longrightarrow & U \times F \\ s(x)\xi & \longmapsto & (x,\xi) \end{array} .$$

Hence the section s determines in each fibre $\pi_E^{-1}(x)$ $(x \in U)$ a frame

$$f = (e_1,\ldots,e_r).$$

If $\phi: U \to E$ is a section, then we can write

$$\phi(x) = s(x)\xi(x).$$

The coordinates of $\xi(x)$ with respect to the standard basis of F are the same as the coordinates of $\phi(x)$ with respect to the frame f. We write

$$\xi = \phi(x) \quad \text{or} \quad \xi = \phi(f).$$

Note that $\phi(s): U \to F$. It is easy to see that

$$\phi(sg) = \rho(g^{-1})\phi(s).$$

For the corresponding map $\tilde{\phi}: \pi^{-1}(U) \to F$ we have

$$\tilde{\phi}(s(x)g) = \rho(g^{-1})\xi(x).$$

Now we will use the notation

$$\nabla_\mu := \nabla_{\partial/\partial x^\mu} .$$

We want to compute $(\nabla_\mu \phi)(s)$. Hence we have to compute $D_\mu \tilde{\phi}$ at the point $s(x)$:

$$(D_\mu \tilde{\phi}) \quad \text{at} \quad s(x) = (\frac{\partial}{\partial x^\mu}, -A_\mu)(\rho(g^{-1})\xi(x)) \quad \text{at} \quad (x,e)$$

$$= \frac{d}{dt}\bigg|_{t=0} \rho(g(t)^{-1})\xi(x(t)),$$

where $g(t) = \exp(-t\,A_\mu)$ and $x(0) = x$ and $\dot{x}(0) = \frac{\partial}{\partial x^\mu}$.

Now using the fact that $\frac{d}{dt}\Big|_{t=0} \rho(g(t)^{-1}) = \rho_* \frac{d}{dt}\Big|_{t=0} (\exp(t\,A_\mu)) = \rho_*(A_\mu)$, we conclude that

$$(\nabla_\mu \phi)(s)(x) = \frac{\partial \xi}{\partial x^\mu}(x) + \rho_*(A^\mu)\xi(x).$$

So the following theorem has been proved.

<u>THEOREM</u>. $(\nabla_X \phi)(s) = X\phi(s) + \theta(s)(X)\phi(s)$, *where* $\theta = \theta(s) = \sum \rho_*(A_\mu)dx^\mu$ *is a matrix-valued* 1*-form called the connection matrix with respect to the gauge map* s *and the representation* ρ.

The formula can also be written as

$$(\nabla\phi)(s) = d(\phi(s)) + \theta(s)\phi(s).$$

If $\theta(s) = (\theta^i_j(s))$, then

$$\nabla e_j = \sum_i \theta^i_j(s) e_j$$

where $(e_1,\ldots,e_r)$ is the frame corresponding to s.

<u>REMARK</u>. If we take $P = F(M)$ (frame bundle) and $E = TM$ (tangent bundle)

with frame

$$f = \left(\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^n}\right),$$

then

$$\theta^i_j(s) = \sum_\mu \Gamma^i_{j\mu} dx^\mu .$$

So $(\rho_* A_\mu)^i_j = \Gamma^i_{j\mu}$.

(1.13) Let $A^p(M,E)$ denote the space of p-forms on M with coefficients in $\Gamma(M,E)$ (the space of section on M with values in E). The covariant derivative can be considered as a map

$$\nabla : A^0(M,E) \to A^1(M,E).$$

It has the following property

$$\nabla(f\phi) = df\cdot\phi + f\nabla\phi,$$

where $\phi \in A^0(M,E)$ and f is a C^∞-function on M.

It is possible to extend ∇ to a map

$$\nabla : A^p(M,E) \to A^{p+1}(M,E)$$

called *exterior covariant differentiation.* With respect to a local section $s: U \to P$ the definition reads

$$(\nabla\phi)(s) := d(\phi(s)) + \theta(s) \wedge \phi(s),\ \phi \in A^p(U,E).$$

One has to verify that this defines indeed an element of $A^{p+1}(U,E)$; this means one has to verify that

$$(\nabla\phi)(sg) = \rho(g^{-1})(\nabla\phi)(s).$$

This verification depends on the following property:

$$\theta(sg) = \rho(g^{-1})\theta(s)\rho(g) + \rho(g^{-1})d(\rho(g)).$$

The exterior covariant differentiation enjoys the following property

$$\nabla(\omega\phi) = d\omega\cdot\phi + (-1)^k\omega \wedge \nabla\phi,$$

where $\phi \in A^0(M,E)$ and ω is a k-form on M.

(1.14) The *curvature matrix* of the connection with respect to the local section $s\colon U \to P$ and the representation ρ is defined by

$$\Theta(s) := \rho_*(s^*\Omega).$$

It is a matrix-valued 2-form. From Cartan's structural equation it follows that

$$\Theta(s) = d\theta(s) + \theta(s) \wedge \theta(s).$$

It is also easily verified that

$$\Theta(sg) = \rho(g^{-1})\Theta(s)\rho(g)$$

which implies that Θ can be viewed as an element of $A^2(M, \mathrm{Hom}(E,E))$. There is a relation between Θ and the map $\nabla^2\colon A^0(M,E) \to A^2(M,E)$.

THEOREM. $(\nabla^2\phi)(s) = \Theta(s)\phi(s)$.

PROOF. Omitting the letter s we have the following:

$$\begin{aligned}(d+\theta)(d+\theta)\phi &= d^2\phi + \theta\wedge d\phi + d(\theta\phi) + (\theta\wedge\theta)\phi \\ &= \theta\wedge d\phi + (d\theta)\phi - \theta\wedge d\phi + (\theta\wedge\theta)\phi \\ &= (d\theta+\theta\wedge\theta)\phi \\ &= \Theta\phi. \ \blacksquare\end{aligned}$$

REMARK. Another way to compute $\nabla^2\phi$ is the following. From

$$(\nabla\phi)(s) = \sum_\mu (\nabla_\mu\phi)(s)dx^\mu$$

it follows that

$$\nabla^2\phi = \sum_{\mu,\nu} \nabla_\mu \nabla_\nu \phi \, dx^\mu \wedge dx^\nu$$

$$= \sum_{\mu<\nu} [\nabla_\mu, \nabla_\nu]\phi \, dx^\mu \wedge dx^\nu .$$

From this relation we conclude that

$$\rho_*(F_{\mu\nu}) = [\nabla_\mu, \nabla_\nu] .$$

<u>THEOREM</u>. $\Theta(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \in A^0(M, \mathrm{Hom}(E,E))$.

<u>PROOF</u>. The proof is a straightforward computation:

$$\Theta(X,Y)\phi = (d\theta+\theta\wedge\theta)(X,Y)\phi$$

$$= (X(\theta(Y)))\phi - (Y(\theta(X)))\phi$$

$$-\theta([X,Y])\phi + [\theta(X), \theta(Y)]\phi,$$

$$\nabla_X\nabla_Y\phi = \nabla_X(Y+\theta(Y))\phi$$

$$= XY\phi + X(\theta(Y))\phi + \theta(X)Y\phi + \theta(X)\theta(Y)\phi.$$

In the same way the other terms are computed. ∎

<u>REMARK</u>. From this theorem it follows again that

$$\rho_*(F_{\mu\nu}) = \Theta\left(\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right) = [\nabla_\mu, \nabla_\nu] .$$

Finally, the Bianchi identity in this context reads as follows:

<u>THEOREM</u>. $d\Theta(s) = \Theta(s) \wedge \theta(s) - \theta(s) \wedge \Theta(s)$.

(1.15) We return to Bianchi's identity. For a principal fibre bundle this identity can be written as $d\Omega = [\Omega \wedge \omega]$.

If we take a section $s: U \to P$ and if we put

$$\omega_s := s^*\omega$$

$$= \sum_\mu A_\mu dx^\mu$$

and

$$\Omega_s := s^*\Omega$$
$$= \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu ,$$

then

$$d\Omega_s = [\Omega_s \wedge \omega_s]$$

or $$\sum_{\mu<\nu}\sum_{\lambda} \partial_\lambda F_{\mu\nu}\, dx^\lambda \wedge dx^\mu \wedge dx^\nu = \sum_{\mu<\nu}\sum_{\lambda} [A_\lambda , F_{\mu\nu}] dx^\lambda \wedge dx^\mu \wedge dx^\nu .$$

So Bianchi's identity can also be formulated as

$$D_\lambda F_{\mu\nu} + D_\mu F_{\nu\lambda} + D_\nu F_{\lambda\mu} = 0,$$

where $$D_\lambda F_{\mu\nu} := \partial_\lambda F_{\mu\nu} + [A_\lambda , F_{\mu\nu}]$$

and $$F_{\mu\nu} = -F_{\nu\mu} .$$

Now we examine the relation with the covariant derivative. We proved $[\nabla_\mu , \nabla_\nu] = \rho_*(F_{\mu\nu})$.

LEMMA. $[\nabla_\lambda , [\nabla_\mu , \nabla_\nu]]\phi(s) = \rho_*(D_\lambda F_{\mu\nu})\phi(s)$.

PROOF. We know

$$(\nabla_\lambda \phi)(s) = \partial_\lambda(\phi(s)) + \rho_*(A_\lambda)\phi(s).$$

So

$$\begin{aligned}[\nabla_\lambda , [\nabla_\mu , \nabla_\nu]]\phi(s) &= \nabla_\lambda \rho_*(F_{\mu\nu})\phi(s) - \rho_*(F_{\mu\nu})\nabla_\lambda \phi(s)\\ &= \partial_\lambda(\rho_*(F_{\mu\nu})\phi(s)) + \rho_*(A_\lambda)\rho_*(F_{\mu\nu})\phi(s)\\ &\quad -\rho_*(F_{\mu\nu})\partial_\lambda \phi(s) - \rho_*(F_{\mu\nu})\rho(A_\lambda)\phi(s)\\ &= \rho_*(\partial_\lambda F_{\mu\nu} + [A_\lambda , F_{\mu\nu}])\phi(s). \quad \blacksquare\end{aligned}$$

Hence still another formulation of Bianchi's identity is

$$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + [\nabla_\mu, [\nabla_\nu, \nabla_\lambda]] + [\nabla_\nu, [\nabla_\lambda, \nabla_\mu]] = 0.$$

(1.16) Next we discuss the covariant derivative in the principal bundle. Let $\pi: P \to M$ be a principal G-bundle and let ω be a connection on P. The definition of the covariant exterior derivative $D\alpha$ of a k-form α has already been given.

<u>DEFINITION</u>. A k-form α on P is called *horizontal* if $\alpha(X_1,\dots,X_k) = 0$ whenever one of the vectors X_i is vertical.

Let ρ be a representation of G on F.

<u>DEFINITION</u>. A k-form $\alpha \in A^k(P,F)$ is of *type* ρ if

$$R_g^*\alpha = \rho(g^{-1})\alpha \qquad (g\in G).$$

Note that the curvature form $\Omega \in A^2(P,\mathfrak{g})$ is horizontal and of type Ad. Furthermore, if α is horizontal and of type ρ, then $D\alpha$ is also horizontal and of type ρ.

Let $\pi_E: E \to M$ be the vector bundle associated with the representation ρ. Then there is a one-one correspondence between the set of k-forms $\phi \in A^k(M,E)$ and the set of k-forms $\alpha \in A^k(P,F)$ that are horizontal and of type ρ.

For $k = 0$ we have already seen this. For general $k \in \mathbb{N}$ the definition is as follows.

<u>DEFINITION</u>. The form $\alpha \in A^k(P,F)$ corresponding to $\phi \in A^k(M,E)$ is defined as follows: Let $u \in P$ and $\pi(u) = x$. Suppose $\phi_x(X_1,\dots,X_k) = u\xi$, then define

1) $\alpha_u(\tilde{X}_1,\ldots,\tilde{X}_k) := \xi$

(here $\tilde{X}_i \in T_u(P)$ is the horizontal lift of $X_i \in T_x(M)$),

2) $\alpha_u(Y_1,\ldots,Y_k) := 0$ when one of the vectors Y_i is vertical.

It is easily seen that this k-form α is horizontal and of type ρ.

Conversely, if $\alpha \in A^k(P,F)$ is horizontal and of type ρ and $\alpha_u(Y_1,\ldots,Y_k) = \xi$ then the form $\phi \in A^k(M,E)$ can be recovered by putting

$$\phi_x(\pi_* Y_1,\ldots,\pi_* Y_k) := u\xi.$$

Now we have the following formula for the exterior covariant derivative.

<u>THEOREM</u>. *If* $\alpha \in A^k(P,F)$ *is horizontal and of type* ρ, *then*

$$D\alpha = d\alpha + \rho_*\omega \wedge \alpha,$$

where

$$(\rho_*\omega \wedge \alpha)(X_1,\ldots,X_{k+1}) = \sum_{i=1}^{k+1} \rho_*(\omega(X_i))\alpha(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}).$$

<u>PROOF</u>. We have to prove that

$$d\alpha(hX_1,\ldots,hX_{k+1}) = d\alpha(X_1,\ldots,X_{k+1}) + \sum_{i=1}^{k+1} \rho_*(\omega(X_i))\alpha(X_1,\ldots,\hat{X}_i,\ldots,X_{k+1}).$$

We consider three cases.

1) All X_i's are horizontal. Then $X_i = hX_i$ and $\omega(X_i) = 0$. So the formula holds.

2) At least two vectors are vertical. Suppose X_1 and X_2 are vertical. Then $D\alpha(X_1,\ldots,X_{k+1}) = 0$,

$$d\alpha(X_1,\ldots,X_{k+1}) = \sum_j (-1)^{j+1} X_j \alpha(X_1,\ldots,\hat{X}_j,\ldots,X_{k+1}) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j],\ldots) = 0.$$

Also $(\rho_*\omega\wedge\alpha)(X_1,\dots,X_{k+1}) = 0$ (since α is horizontal).

3) At least one vector is vertical. Suppose X_1 is vertical without loss of generality we may assume now that $X_2, \dots, X_k$ are horizontal. Furthermore, we may assume that $X_1 = \sigma(A)$ $(A\in\mathfrak{g})$ and that $X_2,\dots,X_{k+1}$ are G-invariant, i.e.

$$R_{g*}X_j = X_j \quad (j=2,\dots,k).$$

Then we have:

$$D\alpha(X_1,\dots,X_{k+1}) = 0,$$

$$d\alpha(X_1,\dots,X_{k+1}) = X_1\alpha(X_2,\dots,X_{k+1})$$

(note that $[X_1, X_j] = 0$ $(j\geq 2)$ since X_j is invariant under the flow $\{R_{\exp(tA)}\}$),

$$(\rho_*\omega\wedge\alpha)(X_1,\dots,X_{k+1}) = \rho_*(A)\alpha(X_2,\dots,X_{k+1}).$$

Now consider tye function $f \in A^0(P,F)$ defined by $f(u) := \alpha_u(X_2,\dots,X_{k+1})$. Since the X_j's are G-invariant, the function f is of type ρ. The proof will be completed when we show

$$X_1 f + \rho_*(A)f = 0.$$

We put $a_t := \exp(tA)$. Then

$$X_1 f(u) = (\sigma(A)f)(u) = \frac{d}{dt}\Big|_{t=0} f(ua_t) =$$

$$= \frac{d}{dt}\Big|_{t=0} \rho(a_t^{-1})f(u) = -\rho_*(A)f(u). \quad \blacksquare$$

The relation with the covariant derivative in the associated vector bundle is given in the next theorem.

THEOREM. *If* $\alpha \in A^k(P,F)$ *is horizontal and of type* ρ *and corresponds to* $\phi \in A^k(M,E)$, *then* $D\alpha$ *corresponds to* $\nabla\phi$.

PROOF. Let $s: U \to P$ be a local gauge map. We have to prove that

$$(d\phi(s)+\theta(s)\wedge\phi(s))\ (X_1,\ldots,X_{k+1}) =$$

$$(d\alpha+\rho_*\omega\wedge\alpha)_{s(x)}\ (\tilde{X}_1,\ldots,\tilde{X}_{k+1}).$$

One has to write out both members remembering that

1) $\qquad \phi(s)\ (X_1,\ldots,X_k) = \alpha_{s(x)}\ (\tilde{X}_1,\ldots,\tilde{X}_k)$

2) $\qquad \theta(s)\ (X_i) = \rho_*\omega(s_*X_i)$

3) $\qquad d\alpha + \rho_*\omega\wedge\alpha$ is horizontal. ∎

(1.17) We consider the case that $F = g$ and $\rho = \mathrm{Ad}$. The curvature form $\Omega \in A^2(P,g)$ is horizontal and of type Ad. So the covariant derivative $D\Omega$ is given by

$$D\Omega = d\Omega + \rho_*\omega\wedge\Omega .$$

Now

$$(\rho_*\omega\wedge\Omega)(X_1,X_2,X_3) = \rho_*\omega(X_1)\Omega(X_2,X_3) + \text{cycl. permut.}$$
$$= [\omega(X_1),\ \Omega(X_2,X_3)] + \text{cycl. permut.}$$

(remember that the derivative $\rho_*: g \to L(g)$ of $\rho = \mathrm{Ad}$ is given by $\rho_*(A)(B) = [A, B]$).

So $\qquad \rho_*\omega\wedge\Omega = [\omega\wedge\Omega]$.

Hence $\qquad D\Omega = d\Omega + [\omega\wedge\Omega]$.

Let E be the vector bundle associated with the representation Ad. The form corresponding to Ω is denoted by $R \in A^2(M,E)$.

Since $D\Omega = 0$ (Bianchi) we have

$$\nabla R = 0.$$

For ∇R we have the formula

$$(\nabla R)(s) = dR(s) + \theta(s) \wedge R(s)$$

with $$\theta(s) = \rho_*(s^*\omega).$$

Now $$\theta(s) \wedge R(s) = \rho_*(s^*\omega) \wedge R(s) = [s^*\omega \wedge R(s)].$$

So $$(\nabla R)(s) = dR(s) + [s^*\omega \wedge R(s)] = 0.$$

CONNECTION, CURVATURE

principal fibre bundle		associated vector bundle
$\omega \in A^1(P,\mathfrak{g})$	$\omega_\alpha := s^*\omega$ connection forms $\omega_\alpha(x) =$ $\mathrm{Ad}g(x)^{-1}\omega_\alpha(x) + g(x)^{-1}dg(x)$	$\theta(s) := \rho_*(s^*\omega)$ connection matrix $\theta(sg) =$ $\rho(g^{-1})\theta(s)\rho(g) + \rho(g^{-1})d\rho(g)$
	$\omega_\alpha = \sum_\mu A_\mu dx^\mu$ gauge potential $A^\beta_\mu = g^{-1}A^\alpha_\mu g + g^{-1}\partial_\mu g$	$\theta(s) = \sum_\mu \rho_*(A_\mu)dx^\mu = (\theta^i_j)$ $\theta^i_j = \sum_\mu \Gamma^i_{j\mu} dx^\mu$
$\Omega := D\omega \in A^2(P,\mathfrak{g})$	$\Omega_\alpha := s^*\Omega$ curvature form $\Omega_\beta(x) = \mathrm{Ad}g(x)^{-1}\Omega_\alpha(x)$	$\Theta(s) := \rho_*(s^*\Omega)$ curvature matrix $\Theta(sg) = \rho(g^{-1})\Theta(s)\rho(g)$
	$\Omega_\alpha = \sum_{\mu<\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu$ gauge field $F^\beta_{\mu\nu} = g^{-1}F^\alpha_{\mu\nu} g$	$\Theta(s) = \sum_{\mu<\nu} \rho_*(F_{\mu\nu})dx^\mu \wedge dx^\nu = (\Theta^i_j)$ $\Theta^i_j = \sum_{\mu<\nu} R^i_{j\mu\nu} dx^\mu \wedge dx^\nu$
$\Omega = d\omega + \frac{1}{2}[\omega \wedge \omega]$ (Cartan)	$\Omega_\alpha = d\omega_\alpha + \frac{1}{2}[\omega_\alpha \wedge \omega_\alpha]$	$\Theta(s) = d\theta(s) + \theta(s) \wedge \theta(s)$
	$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu]$	relation between Γ's and R's
$D\Omega = d\omega + [\omega \wedge \Omega] = 0$ (Bianchi)	$d\Omega_\alpha = [\Omega_\alpha \wedge \omega_\alpha]$	$d\Theta(s) = \Theta(s) \wedge \theta(s) - \theta(s) \wedge \Theta(s)$
	$\sum(\partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]) = 0$ (cyclic sum, $F_{\mu\nu} = -F_{\nu\mu}$), $\binom{n}{3}$ equations	relation between Γ's and R's

COVARIANT DERIVATIVE

principal fibre bundle P		associated vector bundle E
$D\alpha(X_1,\ldots,X_{k+1}) := d\alpha(hX_1,\ldots,hX_{k+1})$		∇ defined by parallel transport
$\alpha \in A^k(P,F)$ horizont. type ρ	$\longleftrightarrow$	$\phi \in A^k(M,E)$
$D\alpha = d\alpha + \rho_* \omega \wedge \alpha$	$\longleftrightarrow$	$(\nabla\phi)(s) = d\phi(s) + \theta(s) \wedge \phi(s)$
$D^2\alpha = \rho_* \Omega \wedge \alpha$		$(\nabla^2\phi)(s) = \Theta(s)\phi(s)$
if $\alpha \in A^0(P,F)$ of type ρ, then at $s(x)$		$\phi \in A^0(M,E)$
$[D_\mu, D_\nu]\alpha = \rho_*(F_{\mu\nu})\alpha$		$[\nabla_\mu, \nabla_\nu]\phi(s) = \rho_*(F_{\mu\nu})\phi(s)$
		$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]]\phi(s) = \rho_*(\partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}])\phi(s)$
		$[\nabla_\lambda, [\nabla_\mu, \nabla_\nu]] + \text{cycl. permut.} = 0$ (Bianchi)

$F = \mathfrak{g}$ and $\rho = \mathrm{Ad}$

$\Omega \in A^2(P,\mathfrak{g})$	$\longleftrightarrow$	$R \in A^2(M,E)$
$D\Omega = d\Omega + [\omega\wedge\Omega]$		$(\nabla R)(s) = dR(s) + [s^*\omega\wedge R(s)]$
$D\Omega = 0$ (Bianchi)		$\nabla R = 0$ (Bianchi)

2. THE YANG-MILLS EQUATIONS

(2.1) Let P be a principal fibre bundle over M with structural group $G = U(n)$ or $G = SU(n)$.

<u>DEFINITION</u>. A *gauge transformation* $f\colon P \to P$ is an equivariant bundle isomorphism that induces the identity on the base space M, i.e.

$$f(ua) = f(u)a \quad (a \in G,\ u \in P).$$

If f is a gauge transformation, then $f(u)$ can be written as $f(u) = u\gamma(u)$ where the map $\gamma\colon P \to G$ satisfies

$$\gamma(ua) = a^{-1}\gamma(u)a.$$

Let ω be a connection on P and let f be a gauge transformation, then the connection $f(\omega)$ is defined by

$$f(\omega) := f^*\omega.$$

If $s\colon U \to P$ is a local section, then the relation between $s^*\omega$ and $s^*(f(\omega))$ is given by the following lemma.

<u>LEMMA</u>. *Let* $g(x) := \gamma(s(x))$. *Then* $s^*(f\omega) = (sg)^*\omega$.

<u>PROOF</u>.
$$f(s(x)) = s(x)\gamma(s(x)) = s(x)g(x).$$

Hence $$s^*(f(\omega)) = s^*f^*\omega = (fs)^*\omega = (sg)^*\omega. \blacksquare$$

(2.2) Let M be a 4-dimensional (pseudo)-Riemannian manifold with an orientation.

For a differential form $\alpha \in A^p(M)$ the form $*\alpha \in A^{4-p}(M)$ is defined by

$$\beta \wedge *\alpha = (\beta,\alpha)\sigma, \qquad \beta \in A^p(M),$$

where σ = volume form corresponding to (pseudo)-metric and orientation and (β,α) = (pseudo)-inner product of β and α defined by the (pseudo)-metric.

In the case that $M = \mathbb{R}^4$ with Euclidean metric the $*$-operator maps $A^2(\mathbb{R}^4)$ into $A^2(\mathbb{R}^4)$ and

$$*(dx_1 \wedge dx_2) = dx_3 \wedge dx_4$$
$$*(dx_1 \wedge dx_3) = -dx_2 \wedge dx_4$$
$$*(dx_2 \wedge dx_3) = dx_1 \wedge dx_4 .$$

Furthermore $*^2$ = identity and $*$ has eigenvalues ± 1 with eigenspace $A^2_\pm$. A^2_+ has as a base

$$\begin{cases} dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \\ dx_1 \wedge dx_3 - dx_2 \wedge dx_4 \\ dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \end{cases}$$

Furthermore, if $\alpha \in A^2_+(M)$ and $\beta \in A^2_-(M)$ then $\alpha \wedge \beta = 0$.

On 1-forms and 3-form is $*^2 = -$identity.

The $*$-operator depends only on the conformal structure of M (and the orientation). We say that two pseudo-metrics g and g' determine the same conformal structure on M when $g' = hg$ for some strictly positive C^∞-function h on M.

The stereographic projection $\mathbb{R}^4 \to S^4$ is a conformal mapping and it commutes with the $*$-operator. This means that if we project the coordinates $x_1, \ldots, x_4$ of $\mathbb{R}^4$ onto S^4, then the formulas for the $*$-operator on S^4 are the same.

The $*$-operator can be extended to differential forms that take values in a vector bundle.

Let $\pi: P \to M$ be a principal G-bundle and let E be the vector bundle associated with the Ad-representation on its Lie algebra $\mathfrak{g}$. When ω is a connection on P, then the curvature Ω can be viewed as an element of $A^2(M,E)$ (we called it R in (1.17)). Now we consider connections ω for which the *Yang-Mills functional* (Lagrangean)

$$L(\omega) := -\int_M \mathrm{tr}(\Omega \wedge *\Omega)$$

is finite. If there exists a global section $s: M \to P$ (so P is trivial), then

$$L(\omega) = -\int_M \mathrm{tr}((s^*\Omega) \wedge *(s^*\Omega)).$$

Since $(A,B) \longmapsto -\mathrm{tr}(AB)$ is an Ad-invariant inner product on the Lie algebra of $U(n)$, the result does not depend on the section s. Indeed,

$$\mathrm{tr}((sg)^*\Omega \wedge *(sg)^*\Omega) = \mathrm{tr}(\mathrm{Ad}(g^{-1})\Omega \wedge *\mathrm{Ad}(g^{-1})\Omega) = \mathrm{tr}(\Omega \wedge *\Omega).$$

As a consequence we see that L is gauge invariant, i.e. $L(\omega) = L(f\omega)$ when f is a gauge transformation.

Let $C(P)$ be the space of all connection forms ω with $L(\omega) < \infty$ and let $G(P)$ denote the group of all gauge transformations, then the Yang-Mills functional is well-defined on the orbit space $M(P) := C(P)/_{G(P)}$. We are now interested in the critical points of $L: M(P) \to \mathbb{R}$.

<u>REMARK</u>. In the Euclidean case we define for $\eta_1, \eta_2 \in A^p(M,E)$ the inner product

$$\langle \eta_1,\eta_2 \rangle := -\int \mathrm{tr}(\eta_1 \wedge *\eta_2).$$

and the corresponding norm

$$\|\eta_1\|^2 := \langle \eta_1,\eta_1 \rangle$$

Then $L(\omega) = \| \Omega \|^2 \geqslant 0$.

Writing $s^*\Omega = \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$

we obtain

$$\| \Omega \|^2 = -\int_M \operatorname{tr}(F_{12}^2 + F_{13}^2 + \ldots + F_{34}^2)\, dx^1 \wedge \ldots \wedge dx^4 .$$

(2.3) <u>THEOREM</u>. *If* $\omega \in C(P)$, *then*:

ω *is a critical point of* $L \Longleftrightarrow D(*\Omega) = 0$ (*Yang-Mills equation*).

<u>REMARK</u>. If $s: U \to P$ is a local section and $s^*\omega = \sum_\mu A_\mu dx^\mu$, $s^*\omega = \sum_{\mu<\nu} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$, then the Yang-Mills equation is

$$d(*(s^*\Omega)) + [(s^*\omega) \wedge *(s^*\Omega)] = 0.$$

If we write out this equation for the Euclidean case we obtain a set of 4 equations:

$$\sum_\mu \partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 \qquad (\text{with } F_{\mu\nu} = -F_{\nu\mu}).$$

This is a system of second order nonlinear partial differential equations for the potentials (A_μ).

<u>PROOF</u> of the theorem for the Euclidean case. For the base space M we take S^4 in order to insure the convergence of the integrals.

The space $C(P)$ is an affine space, i.e. if ω_0 and ω_1 are in $C(P)$, then $\omega_t = (1-t)\omega_0 + t\omega_1 \in C(P)$.

We compute

$$\left.\frac{d}{dt}\right|_{t=0} L(\omega_t).$$

Writing $\omega_t = \omega_0 + t_\eta$, we obtain

$$\begin{aligned}\Omega_t &= d(\omega_0+t\eta) + \tfrac{1}{2}[(\omega_0+t\eta) \wedge (\omega_0+t\eta)] \\ &= \Omega_0 + t(d\eta + [\omega_0 \wedge \eta]) + \tfrac{1}{2}t^2[\eta \wedge \eta] \\ &= \Omega_0 + t\,D\eta + \mathcal{O}(t^2)\end{aligned}$$

(note that η is horizontal and of type Ad).

We consider these forms as elements of $A^2(M,E)$ (in fact we have to use the notation $\nabla\eta$ instead of $D\eta$).

We obtain $L(\omega_t) = L(\omega_0) + 2t < D\eta, \Omega_0 > + \mathcal{O}(t^3)$.

Now $< D\eta, \Omega_0 > = -< \eta, *D*\Omega_0 >$, since

1) $$< d\eta, \Omega_0 > = -\int tr(d\eta \wedge *\Omega_0) = -\int tr(\eta \wedge d*\Omega_0) = -< \eta, *d*\Omega_0 >$$

2) $$tr([\omega_0 \wedge \eta] \wedge *\Omega_0) = tr(\eta \wedge [\omega_0 \wedge *\Omega_0]).$$

Hence $$\left.\frac{d}{dt}\right|_{t=0} L(\omega_t) = -2 < \eta, *D*\Omega_0 >.$$

This means that L is stationary at ω_0 iff $D*\Omega_0 = 0$. ∎

Since $D\Omega = 0$ (Bianchi's identity), the Yang-Mills equation is satisfied when $*\Omega = \pm\Omega$.

The equation

$$\Omega = *\Omega$$

is called the *self-dual Yang-Mills equation* and

$$\Omega = -*\Omega$$

the *anti-self-dual Yang-Mills equation*. In the Euclidean case the self-dual Yang-Mills equations are

$$F_{12} = F_{34},\ F_{13} = -F_{24},\ F_{23} = F_{14}.$$

These form a system of first order nonlinear partial differential equations for the potential (A_μ).

(2.4) In physics one is interested in the Euclidean Yang-Mills equation on $\mathbb{R}^4$. For P one takes the trivial bundle with group $G = SU(2)$. Let ω be a connection on P and let $s: M \to P$ be a global section. Again we write

$$s^*\omega = \sum_\mu A_\mu \, dx^\mu$$

and

$$s^*\Omega = \sum_{\mu<\nu} F_{\mu\nu} \, dx^\mu \wedge dx^\nu .$$

We consider potentials for which the action L is finite, so that the integral over $\mathbb{R}^4$ converges. To achieve this we assume that the field $F_{\mu\nu}$ decays sufficiently fast as $|x| \to \infty$. For the potential (A_μ) this means that, for large $|x|$, we can find a gauge transformation $g(x)$ so that

$$A_\mu(x) \sim g(x)^{-1} \partial_\mu g(x) \quad \text{as} \quad |x| \to \infty ,$$

where $\sim$ implies asymptotic behaviour including first derivatives. The important point is that the gauge transformation $g(x)$ need only be defined for large $|x|$. In fact it may be impossible to extend the definition of $g(x)$ continuously to the whole space $\mathbb{R}^4$. To see this consider the restriction of $g(x)$ to a sphere $|x| = R$ of large radius. Then g gives a continuous map

$$g: S^3_R \to SU(2).$$

Now the function g can be extended continuously to $|x| \leqslant R$ if and only if the degree k of the map $g: S^3_R \to SU(2)$ is zero ($SU(2)$ is diffeomorphic to S^3).

In order to describe the asymptotic behaviour of the potential we consider the compactification S^4 of $\mathbb{R}^4$.

This is a conformal compactification. Since the Yang-Mills equation is conformally invariant, the transformed Yang-Mills equation on S^4 has the same form.

We only consider potentials that can be extended to S^4. If the degree k is non-zero, then we cannot describe our potential using a single gauge. We need one gauge in a neighbourhood U_1 of $\{x \mid |x| \leq R\}$ and another gauge in a neighbourhood U_2 of $\{x \mid |x| \geq R, x = \infty\}$, the two gauges being related on $U_1 \cap U_2$ by the gauge transformation $g(x)$. Now we have the ingredients to construct a principal G-bundle over S^4. The connection ω can be extended to this bundle.

Now it turns out, that the fibre bundles that appear in this way, are precisely classified by the same integer k (the bundle being trivial iff $k = 0$).

The connection ω in our fibre bundle now has a well-defined curvature Ω on the whole of S^4. Note that $F_{\mu\nu}(x) \to 0$ as $|x| \to \infty$, but the curvature Ω of ω need not be zero on S^4 at the point ∞ (the coordinates $x_1, \ldots, x_4$ are singular at $\infty \in S^4$).

Since S^4 is compact, $L(\omega) < \infty$ (because of the conformal invariance the Yang-Mills functional takes the same value whether computed on S^4 or $\mathbb{R}^4$).

(2.5) Again we consider $M = S^4$ and $G = SU(2)$. Let ω be a connection on the principal fibre bundle P with curvature form $\Omega \in A^2(S^4, E)$. One can prove that

$$k = \frac{-1}{8\pi^2} \int_M \operatorname{tr}(\Omega \wedge \Omega).$$

We decompose $\Omega = \Omega_+ + \Omega_-$ with $\Omega_\pm = \pm * \Omega_\pm \in A^2_\pm(S^4, E)$.

Since $\Omega_+ \wedge \Omega_- = 0$, it follows that

$$\Omega \wedge \Omega = \Omega_+ \wedge \Omega_+ + \Omega_- \wedge \Omega_- \quad \text{and} \quad \Omega \wedge *\Omega = \Omega_+ \wedge \Omega_+ - \Omega_- \wedge \Omega_- .$$

Hence $L(\omega) = -\int_M \operatorname{tr}(\Omega \wedge *\Omega) = \|\Omega_+\|^2 + \|\Omega_-\|^2.$

Also $8\pi^2 k = -\int_M \operatorname{tr}(\Omega\wedge\Omega) = \|\Omega_+\|^2 - \|\Omega_-\|^2$.

So
$$L(\omega) \geqslant 8\pi^2|k|$$
$$L(\omega) = 8\pi^2 k \iff \Omega = \Omega^+ \quad (\Omega \text{ self-dual})$$
$$L(\omega) = -8\pi^2 k \iff \Omega = \Omega^- \quad (\Omega \text{ anti-self-dual}).$$

This proves the following theorem.

THEOREM. *For fixed* k *(fixed bundle* P*) the (anti)-self-dual solutions of the Yang-Mills equation correspond to the absolute minimum of the Yang-Mills functional.*

DEFINITION. An (anti)-self-dual solution of the Yang-Mills equation is called an *instanton*.

The degree k of the map $g\colon S^3_R \to SU(2)$ is called the *instanton number*.

REMARK. The number k determines the bundle P and hence the vector bundle E associated with the standard representation of $SU(2)$ on $\mathbb{C}^2$.

It turns out that k is equal to the second Chern number $c_2(E)$ of the vector bundle E. This interpretation of k can be used when G is another group than $SU(2)$.

For the proof of the next theorem we need the following.

Consider the algebra $\mathbb{H} = \{x = x_1 + x_2 i + x_3 j + x_4 k \mid x_i \in \mathbb{R}\}$ of the quaternions ($i^2=j^2=k^2=-1$ and $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$).

If $x = x_1 + x_2 i + x_3 j + x_4 k$, then x_1 is called the real part and $x - x_1$ the imaginary part; $\bar{x} := x_1 - x_2 i - x_3 j - x_4 k$ is called the conjugate of x.

One has
$$|x|^2 := x\bar{x} = \bar{x}x = \sum_\mu x_\mu^2 \quad \text{and}$$
$$x^{-1} = \frac{\bar{x}}{|x|^2}\,.$$

The algebra $\mathbb{H}$ is isomorphic with a subalgebra of the 2×2 complex matrices in which i, j, k are the matrices

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In particular SU(2) is isomorphic to the unit sphere $\{x \in \mathbb{H} \mid |x| = 1\}$. The Lie algebra of SU(2) is isomorphic to the subspace of the pure imaginary quaternions with natural basis i, j, k.

<u>THEOREM</u>. $k = -\frac{1}{8\pi^2} \int_{S^4} \mathrm{tr}(\Omega \wedge \Omega)$.

<u>PROOF</u>. We take a local section $s: \mathbb{R}^4 \to P$ and use the notation: $\omega_s = s^*\omega$ and $\Omega_s = s^*\Omega$.

Write ω_s and $\omega_s = \omega_1 i + \omega_2 j + \omega_3 k$. Then

$$\begin{aligned} \Omega_s \wedge \Omega_s &= (d\omega_s + \omega_s \wedge \omega_s) \wedge (d\omega_s + \omega_s \wedge \omega_s) \\ &= d\omega_s \wedge d\omega_s + 2d\omega_s \wedge \omega_s \wedge \omega_s + \omega_s \wedge \omega_s \wedge \omega_s \wedge \omega_s . \end{aligned}$$

Now
$$\omega_s \wedge \omega_s = 2i\omega_2 \wedge \omega_3 + 2j\omega_3 \wedge \omega_1 + 2k\omega_1 \wedge \omega_2,$$
$$\omega_s \wedge \omega_s \wedge \omega_s = -6\omega_1 \wedge \omega_2 \wedge \omega_3 \quad \text{and}$$
$$\omega_s \wedge \omega_s \wedge \omega_s \wedge \omega_s = 0.$$

Since $d(\omega_s \wedge d\omega_s) = d\omega_s \wedge d\omega_s$ and $d(2/3\omega_s \wedge \omega_s \wedge \omega_s) = 2d\omega_s \wedge \omega_s \wedge \omega_s$,

it follows that

$$\Omega_s \wedge \Omega_s = d(\omega_s \wedge d\omega_s + 2/3\omega_s \wedge \omega_s \wedge \omega_s) .$$

Let $B_R = \{x \mid |x| \leqslant R\}$ and $S_R = \{x \mid |x| = R\}$.

Then
$$-\int_{B_R} \mathrm{tr}(\Omega_s \wedge \Omega_s) = -\int_{S_R} \mathrm{tr}(\omega_s \wedge d\omega_s + \tfrac{2}{3}\omega_s \wedge \omega_s \wedge \omega_s) .$$

Now assume that ω_s is flat outside B_R (R large), i.e. assume that there exists a neighbourhood U of ∞ and a gauge transformation $g: U \to SU(2)$ such that

$$\omega_s = g^{-1}dg \quad \text{on } U.$$

Then $$d\omega_s = d(g^{-1}) \wedge dg = -g^{-1}dg\, g^{-1} \wedge dg$$
$$= -\omega_s \wedge \omega_s \ ;$$

hence $$-\int_{B_R} tr(\Omega_s \wedge \Omega_s) = \frac{1}{3}\int_{S_R} tr(\omega_s \wedge \omega_s \wedge \omega_s).$$

Now $\omega_s = g^*(\eta)$ on U, where η is the Maurer-Cartan form on $SU(2)$. Since $tr(\eta\wedge\eta\wedge\eta) = -12$(volume form), it follows that

$$\int_{S_R} tr(\omega_s \wedge \omega_s \wedge \omega_s) = -12\int_{S_R} g^*(vol)$$
$$= -12 \deg(g) \int_{SU(2)} vol$$
$$= -24\pi^2 k. \quad \blacksquare$$

(2.6) It is our aim to describe the instanton on $\mathbb{R}^4$ with $k = 1$ (cf. [2]).

We identify $\mathbb{R}^4$ with $\mathbb{H}$ and an $SU(2)$-potential will be given by

$$\omega = \sum_\mu A_\mu dx^\mu$$

with $A_\mu(x) \in Im(\mathbb{H})$. Furthermore, we shall consider the forms

$$dx := dx^1 + dx^2 i + dx^3 j + dx^4 k \quad \text{and}$$
$$d\bar{x} := dx^1 - dx^2 i - dx^3 j - dx^4 k.$$

If $f\colon \mathbb{H} \to \mathbb{H}$, then the expression

$$\omega = Im(f(x)dx) = \tfrac{1}{2}\{f(x)dx - d\bar{x}\,\overline{f(x)}\}$$

will represent an $SU(2)$-potential ($f(x)dx$ is computed formally). Then

$$\Omega = d\omega + \omega \wedge \omega$$
$$= Im(df\wedge dx + f dx \wedge f dx).$$

We compute $dx \wedge d\bar{x}$:

$$dx \wedge d\bar{x} = -2\{(dx^1 \wedge dx^2 + dx^2 \wedge dx^4)i \\ (dx^1 \wedge dx^3 + dx^4 \wedge dx^2)j \\ (dx^1 \wedge dx^4 + dx^2 \wedge dx^3)k\},$$

and see that $dx \wedge d\bar{x}$ is self-dual. Similarly $d\bar{x} \wedge dx$ is anti-self-dual.

We shall now exhibit the basic instanton with $k = +1$ by choosing $f(x) = \frac{\bar{x}}{1+|x|^2}$.

Then $$\omega = \mathrm{Im}\{\frac{\bar{x}dx}{1+|x|^2}\} \quad \text{and} \quad A_1(x) = \frac{-x_2 i - x_3 j - x_4 k}{1+|x|^2} \quad \text{etc.}$$

$$\Omega = \mathrm{Im}\{\frac{d\bar{x}\wedge dx}{1+|x|^2} + \bar{x}d(1+x\bar{x})^{-1} \wedge dx + \frac{\bar{x}dx + \bar{x}dx}{(1+|x|^2)^2}\}$$

$$= \frac{d\bar{x}\wedge dx}{(1+|x|^2)^2}.$$

Hence $\Omega = -*\Omega$ (anti-self-dual) and $\omega \sim \mathrm{Im}(x^{-1}dx) = g(x)^{-1}dg(x)$ as $|x| \to \infty$, where $g(x) = \frac{x}{|x|}$. So ω is asymptotically the gauge transform of 0 by the gauge transformation $g(x)$, or equivalently if we apply the inverse gauge transformation $g(x)^{-1}$ to ω we get 0 asymptotically.

On the unit sphere $|x| = 1$ in $\mathbb{H}$ we have $g(x)^{-1} = \bar{x}$ and the map $x \longmapsto \bar{x}$ of S^3 to itself has degree -1.

In order to see that the instanton can be extended to S^4 one applies the gauge transformation $g(x)^{-1} = |x|x^{-1}$ and then substitutes $y = x^{-1}$.

<u>REMARK</u>. If we replace x by $\bar{x}$ we will obtain an instanton with $k = 1$.

3. COMPLEX VECTOR BUNDLES, CHERN CLASSES

(3.1) Let M be an n-dimensional complex manifold. This means that M can be covered by charts (U_α, ϕ_α), $\phi_\alpha: U_\alpha \to \mathbb{C}^n$, such that the maps

$$\phi_\alpha \circ \phi_\beta^{-1}: \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$$

are biholomorphic.

A function $f: M \to \mathbb{C}$ is called C^∞ if all functions $f \circ \phi_\alpha^{-1}: \phi_\alpha(U_\alpha) \to \mathbb{C}$ are C^∞. Holomorphic functions are defined in an analogous way.

Examples of complex manifolds are: $\mathbb{C}^n$, $P_n(\mathbb{C})$, Riemann surfaces.

Let $x \in M$ and let $(z_1, \ldots, z_n)$ be holomorphic coordinates on a neighbourhood U of x.

We put $z_k = x_k + iy_k$ $(k=1,\ldots,n)$.

<u>Definition</u>. The *real tangent space* $T_x(M)$ is the 2n-dimensional vector space over $\mathbb{R}$ with basis

$$\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}.$$

The complexification of this space

$$T_x^c(M) := T_x(M) \oplus iT_x(M)$$

is called the *complex tangent space* at x (it has dimension $2n$ over $\mathbb{C}$).

<u>REMARK</u>. Let $F^r(x)$ denote the ring of germs of all real C^∞-functions at x; similarly $F^c(x)$ is the ring of germs of all complex C^∞-functions at x. Then

> $T_x(M)$ coincides with the set of all $\mathbb{R}$-linear derivations $X: F^r(x) \to \mathbb{R}$

and

$T_x^c(M)$ is the same as the set of all $\mathbb{C}$-linear derivations $Y: F^c(x) \to \mathbb{C}$.

An element $X \in T_x(M)$ can be extended to a $\mathbb{C}$-linear derivation

$$\tilde{X}: F^c(x) \to \mathbb{C}$$

by setting

$$\tilde{X}(f+ig) := X(f) + iX(g).$$

So $$T_x(M) \subset T_x^c(M) = T_x(M) \oplus iT_x(M)$$

and $$\tilde{X} = (X,0).$$

REMARK. Let $\mathcal{O}(x)$ be the ring of germs of all holomorphic functions at x. If

$$X = \sum a_k \frac{\partial}{\partial x_k} + b_k \frac{\partial}{\partial y_k} \in T_x(M)$$

$(a_k, b_k \in \mathbb{R})$ and $f \in \mathcal{O}(x)$, then

$$X(f) = \sum a_k \frac{\partial f}{\partial x_k} + b_k \frac{\partial}{\partial y_k} = \sum (a_k + ib_k) \frac{\partial f}{\partial x_k}$$

since $\frac{\partial f}{\partial y_k} = i \frac{\partial}{\partial x_k}$ (Cauchy-Riemann equations).
This implies that $T_x(M)$ can be identified with the space of all $\mathbb{C}$-linear derivations on $\mathcal{O}(x)$. This makes $T_x(M)$ into a n-dimensional vector space over $\mathbb{C}$ with basis $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$, the complex multiplication being defined by $i \frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$.

REMARK. Let V be a 2n-dimensional vector space over $\mathbb{R}$. A *complex structure* on V is a map $J: V \to V$ with $J^2 = -\text{identity}$. If one has a complex structure J, then V can be made into an n-dimensional vector space over $\mathbb{C}$ by setting

$$(\alpha + i\beta)v := \alpha v + J\beta v \quad (\alpha, \beta \in \mathbb{R}, v \in V).$$

In the case $V = T_x(M)$ we have

$$J(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}.$$

Let N be a real 2n-dimensional C^∞-manifold. Suppose that for each $x \in N$ there is given complex structure $J_x: T_x(N) \to T_x(N)$ which depends on x in a C^∞ way, then N is called an *almost complex manifold*. So a complex manifold has an *almost complex structure*.

We use the following notation:

$$\frac{\partial}{\partial z_k} := \tfrac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}) \in T_x^c(M)$$

$$\frac{\partial}{\partial \bar{z}_k} := \tfrac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}) \in T_x^c(M) .$$

The n-dimensional vector space over $\mathbb{C}$ spanned by $\left\{\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}\right\}$ is called the *holomorphic tangent space*, which is denoted by $T_x^{(1,0)}(M)$. Similarly, the *anti-holomorphic tangent space* $T_x^{(0,1)}(M)$ has basis $\left\{\frac{\partial}{\partial \bar{z}_1}, \ldots, \frac{\partial}{\partial \bar{z}_n}\right\}$.

So

$$T_x^c(M) = T_x^{(1,0)}(M) \oplus T_x^{(0,1)}(M).$$

The following facts are easily verified:

$$X \in T_x^{(1,0)}(M) \iff X(\bar{f}) = 0 \quad \forall f \in \mathcal{O}(x)$$

$$X \in T_x^{(0,1)}(M) \iff X(f) = 0 \quad \forall f \in \mathcal{O}(x).$$

This immediately implies that the definition of holomorphic tangent space is independent of the choice of the holomorphic coordinates $(z_1, \ldots, z_n)$.

Now we will consider complex-valued differential forms on M. Let ω be a complex-valued 1-form i.e. for each $x \in M$ is $\omega_x: T_x(M) \to \mathbb{C}$ an $\mathbb{R}$-linear map. One can extend ω_x to a $\mathbb{C}$-linear map $\omega_x: T_x^c(M) \to \mathbb{C}$. If we define

$$dz_k := dx_k + idy_k$$

$$d\bar{z}_k := dx_k - idy_k,$$

then ω can be written as

$$\omega = \sum a_k\, dz_k + b_k\, d\bar{z}_k,$$

where a_k and b_k are complex-valued C^∞ - functions.

If $\omega = \sum a_k\, dz_k$, then ω is called of *type* (1,0) and if $\omega = \sum b_k\, d\bar{z}_k$, then is called of type (0,1). It follows that

ω is of *type* (1,0) iff ω vanishes on holomorphic vector fields and

ω is of *type* (0,1) iff ω vanishes on anti-holomorphic vector fields.

This implies that the definitions of type (1,0)- and (0,1)-forms are coordinate independent.

<u>REMARK</u>. Another characterization is the following:

ω is of type $(1,0) \Longleftrightarrow \omega_x\colon T_x(M) \to \mathbb{C}$ is $\mathbb{C}$ - linear,

where $T_x(M)$ is considered as an n-dimensional vector space over $\mathbb{C}$ with basis $\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\}$ with $i\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$.

Similarly:

ω is of type $(0,1) \Longleftrightarrow \omega_x\colon T_x(M) \to \mathbb{C}$ is $\mathbb{C}$ - antilinear.

<u>DEFINITION</u>. A complex valued differential form ω on M is called of *type* (p,q) if with respect to local holomorphic coordinates ω can be written as

$$\omega = \sum a_{IJ}\, dz_I \wedge d\bar{z}_J$$

with

$$dz_I := dz_{i_1} \wedge \ldots \wedge dz_{i_p}$$

$$d\bar{z}_J := d\bar{z}_{j_1} \wedge \ldots \wedge d\bar{z}_{j_q}$$

and a_{IJ} complex C^∞ - functions.

This definition is again independent of the coordinates.

REMARK. At each point $x \in M$ the map

$$\omega_x : T_x(M) \times \ldots \times T_x(M) \to \mathbb{C}$$

is an $\mathbb{R}$ - multilinear alternating map. Now ω of type (p,q) iff

$$\omega_x(cX_1,\ldots,cX_\ell) = c^p\bar{c}^q\omega_x(X_1,\ldots,X_\ell),$$

where $c \in \mathbb{C}$, $\ell = p+q$ and $X_1, \ldots, X_\ell \in T_x(M)$. ($T_x(M)$ considered as $\mathbb{C}$ - vector space with $i\frac{\partial}{\partial x_k} = \frac{\partial}{\partial y_k}$).

The space of complex-valued differential forms of type (p,q) is denoted by $A^{(p,q)}(M)$.

The space of complex-valued k-forms can be decomposed as

$$A^k(M) = \bigoplus_{p+q=k} A^{(p,q)}(M).$$

If $f \in A^0(M)$ (f is a complex-valued C^∞ - function on M), then

$$\begin{aligned} df &= \sum \frac{\partial f}{\partial x_k} dx_k + \frac{\partial f}{\partial y_k} dy_k \\ &= \sum \frac{\partial f}{\partial z_k} dz_k + \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k. \end{aligned}$$

If we define

$$\partial f := \sum \frac{\partial f}{\partial z_k} dz_k \quad \text{en} \quad \bar{\partial} f := \sum \frac{\partial f}{\partial \bar{z}_k} d\bar{z}_k,$$

then $\qquad df = \partial f + \bar{\partial} f.$

Note that f is holomorphic if $\bar{\partial} f = 0$.

If $\omega = \sum a_{IJ} dz_I \wedge dz_J$ is a form of type (p,q), then $d\omega$ can be written as

$$d\omega = \partial\omega + \bar{\partial}\omega$$

with $\partial\omega := \sum \partial a_{IJ} \wedge dz_I \wedge d\bar{z}_J$ and $\bar{\partial}\omega := \sum \bar{\partial} a_{IJ} \wedge dz_I \wedge d\bar{z}_J$

So $\partial\omega$ is of type $(p+1, q)$ and $\bar{\partial}\omega$ of type $(p, q+1)$,
$\partial: A^{(p,q)} \to A^{(p+1,q)}$ and $\bar{\partial}: A^{(p,q)} \to A^{(p,q+1)}$.

Since $0 = d^2\omega = \partial^2\omega + (\partial\bar{\partial}+\bar{\partial}\partial)\omega + \bar{\partial}^2\omega$ and all these forms are of different type, we have

$$\partial^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0, \quad \bar{\partial}^2 = 0.$$

(3.2) DEFINITION. A *holomorphic vector bundle* E of rank r is given by $\pi: E \to M$, where E and M are complex manifolds and π is a holomorphic surjective map such that

(i) the fibre $E_x := \pi^{-1}(x)$ is a $\mathbb{C}$-vector space of dimension r $(x \in M)$,

(ii) for each $x \in M$ there is open neighbourhood U of x and a biholomorphic map

$\phi: \pi^{-1}(U) \to U \times \mathbb{C}^r$ that maps the fibre E_y onto $\{y\} \times \mathbb{C}^r$

(ϕ is called a holomorphic gauge map).

If $\{U_\alpha\}$ is an open cover of M and if $\phi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ are holomorphic gauge maps, then the maps

$$\phi_\beta \circ \phi_\alpha^{-1}: (U_\alpha \cap U_\beta) \times \mathbb{C}^r \to (U_\alpha \cap U_\beta) \times \mathbb{C}^r$$
$$(x,\xi) \mapsto (x,\eta)$$

induce maps $g_{\alpha\beta}: (U_\alpha \cap U_\beta) \to G\ell(r,\mathbb{C})$ (holomorphic gauge transformations)

defined by $\xi = g_{\alpha\beta}(x)\eta$.

The bundle $\pi: E \to M$ is completely determined by the set $\{g_{\alpha\beta}\}$.

An example of a holomorphic vector bundle is the holomorphic tangent bundle $T^{(1,0)}(M)$.

(3.3) Let us consider a holomorphic line bundle $\pi: E \to M$ (rank $r = 1$) that is given by a set $\{g_{\alpha\beta}\}$ of holomorphic gauge transformations. Then $g_{\alpha\beta}: U_\alpha \cap U_\beta \to G\ell(1,\mathbb{C}) = \mathbb{C}\setminus\{0\}$. Now suppose that there exist holomorphic functions $h_\alpha: U_\alpha \to \mathbb{C}\setminus\{0\}$ with $g_{\alpha\beta}(x) = h_\alpha(x)h_\beta(x)^{-1}$. Then the bundle is trivial. Indeed, the gauge maps

$$\begin{aligned}\phi_\alpha: \pi^{-1}(U\ ) &\longrightarrow U_\alpha \times \mathbb{C}\\ p &\longmapsto (x,\psi_\alpha(p))\end{aligned}$$

can be replaced by new gauge maps

$$\begin{aligned}\pi^{-1}(U_\alpha) &\longrightarrow U_\alpha \times \mathbb{C}\\ p &\longmapsto (x, h_\alpha(x)^{-1}\,\psi_\alpha(p)).\end{aligned}$$

These new gauge maps define in fact one global map, since

$$\psi_\alpha(p) = g_{\alpha\beta}(x)\psi_\beta(p) = h_\alpha(x)h_\beta(x)^{-1}\psi_\beta(p)\ ,$$

so

$$h_\alpha(x)^{-1}\psi_\alpha(p) = h_\beta(x)^{-1}\psi_\beta(p).$$

(3.4) Another example of a holomorphic line bundle is the so-called canonical line bundle $E \to P_1(\mathbb{C})$ which we define as follows. Let

$$U_i = \{[z_0 : z_1] \mid z_i \neq 0\}\quad (i=0,1).$$

On U_0 we take the coordinate $z = \dfrac{z_1}{z_0}$ and on U_1 we take $w = \dfrac{z_0}{z_1}$ (so $w = \frac{1}{z}$).

The gauge transformation

$$g_{10}: U_1 \cap U_0 \to \mathbb{C}\setminus\{0\}$$

is given by

$$g_{10}([z_0 : z_1]) = \frac{z_1}{z_0} = z = \frac{1}{w}.$$

The corresponding bundle is determined by $(U_0 \times \mathbb{C}) \cup (U_1 \times \mathbb{C})$ where $(p,\xi_0) \in U_0 \times \mathbb{C}$ is identified with $(p,\xi_1) \in U_1 \times \mathbb{C}$ when $\xi_1 = g_{01}(p)^{-1}\xi_0 = z\xi_0$.

In the same way the bundle $E(k) \to P_1(\mathbb{C})$ is defined by the gauge transformation

$$g_{10}([z_0 : z_1]) = \left(\frac{z_1}{z_0}\right)^k = z^k \quad (k \in \mathbb{Z}) .$$

More general holomorphic line bundles on $P_1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ can be constructed by considering the cover $\{U_0, U_1\}$,

$$U_0 = \{z \mid |z| < R\}, \quad U_1 = \{z \mid |z| > r\} \cup \{\infty\} \quad (r<1<R),$$

and a gauge transformation $g_{10}(z) = g(z)$ which is holomorphic and $\neq 0$ on $U_0 \cap U_1$. If $g(z) = \frac{h_0(z)}{h_1(z)}$ with h_0 holomorphic on U_0 and h_1 holomorphic on U_1, then the corresponding vector bundle is trivial. This occurs if

$$\frac{1}{2\pi i} \int_{|z|=1} \frac{g'(z)}{g(z)} dz = 0 .$$

Indeed, in that case there exists a holomorphic branch of $\log g(z)$ on $U_0 \cap U_1$ with Laurent series

$$\log g(z) = \sum_0^{\infty} a_n z^n - \sum_{-\infty}^{-1} a_n z^n$$
$$= g_0(z) - g_1(z),$$

so that

$$g = \frac{e^{g_0}}{e^{g_1}} = \frac{h_0}{h_1} .$$

In general $g(z)$ can be written as

$$g(z) = z^k \frac{h_0(z)}{h_1(z)} \quad \text{with } h_i \text{ holomorphic on } U_i \text{ and}$$

$$k = \frac{1}{2\pi i} \int_{|z|=1} \frac{g'(z)}{g(z)} dz .$$

The corresponding bundle is then holomorphically equivalent with $E(k)$. This argument can be generalized in order to prove that every holomorphic line bundle on $P_1(\mathbb{C})$ is holomorphically equivalent to some $E(k)$.

We determine the space of holomorphic sections $f: P_1(\mathbb{C}) \to E(k)$. On U_0 the section f can be represented by a function f_0 analytic in z and on U_1, f is represented by f_1 analytic in w:

$$f_0(z) = \sum_0^\infty a_n z^n \quad \text{and} \quad f_1(w) = \sum_0^\infty b_n w^n$$

with $\quad f_1(w) = z^k f_0(w).$

So $\quad \sum_0^\infty b_n \frac{1}{z^n} = z^k \sum_0^\infty a_n z^n .$

This implies that

$$\mathcal{O}(P_1(\mathbb{C}), E(k)) = \begin{cases} 0 & k > 0 \\ \mathbb{C} & k = 0 \\ \text{homogeneous polynomials on } \mathbb{C}^2 \text{ of degree } -k & k < 0 \end{cases}$$

(3.5) Connections on complex vector bundles with hermitian metric.

Let $\pi: E \to M$ be a complex C^∞-vector bundle.

DEFINITION. $\pi: E \to M$ is called a vector bundle with *hermitian metric* if on each fibre E_x is given an inner product $< .,. >_x$ so that for all C^∞-sections $\phi_1, \phi_2 \in A^0(U,E)$ (U open in M) the functions

$$x \longmapsto <\phi_1,\phi_2>(x) := <\phi_1(x),\phi_2(x)>_x$$

are C^∞.

If $f = (e_1,\dots,e_r)$ is a C^∞-frame over U, then define the matrix-valued C^∞-map $h(f)$ by

$$h(f)_{ji} := \langle e_i, e_j \rangle .$$

The frame f is called *unitary* (*orthonormal*) if $h(f)$ is the identity at each point $x \in M$.

If $g: U \to G\ell(r,\mathbb{C})$, then the frame $fg = (\tilde{e}_1,\dots,\tilde{e}_r)$ is defined by

$$\tilde{e}_i = \sum_j g_{ji} e_j .$$

One easily verifies that $h(fg) = {}^t\bar{g}h(f)g$.

The metric can be extended to differential forms that take values in the vector bundle E as follows. If

$$\phi_1,\ \phi_2 \in A^0(M,E) \quad \text{and} \quad \omega_1 \in A^k(M), \quad \omega_2 \in A^\ell(M),$$

then $$\omega_1\phi_1 \in A^k(M,E) \quad \text{and} \quad \omega_2\phi_2 \in A^\ell(M,E),$$

and we define

$$\langle \omega_1\phi_1, \omega_2\phi_2 \rangle := \omega_1 \wedge \omega_2 \langle \phi_1, \phi_2 \rangle .$$

This defines the wedge product.

$$A^k(M,E) \times A^\ell(M,E) \to A^{k+\ell}(M,E).$$

Let ∇ be a connection on the complex vector bundle E, i.e. ∇ is a $\mathbb{C}$-linear map

$$\nabla: A^0(M,E) \to A^1(M,E)$$

which satisfies

$$\nabla(g\phi) = d\phi\cdot\phi + g\nabla\phi$$

for $\phi \in A^0(M,E)$ and $g \in A^0(M)$.

The *connection matrix* $\theta(f)$ with respect to a frame $f = (e_1,\dots,e_r)$ is defined by

$$\nabla e_j = \sum_i \theta(f)_{ij} e_i .$$

Then the following formula holds

$$(\nabla\phi)(f) = d(\phi(f)) + \theta(f)\phi(f).$$

DEFINITION. The connection ∇ is called *compatible with the metric* when

$$d<\xi,\eta> = <\nabla\xi,\eta> + <\xi,\nabla\eta> \quad (\xi,\eta \in A^0(M,E)).$$

This means that the inner product is preserved under parallel transport. Indeed, if $\nabla_X\xi = \nabla_X\eta = 0$, then

$$X(<\xi,\eta>) = <\nabla_X\xi,\eta> + <\xi,\nabla_X\eta> = 0.$$

If ∇ is compatible with the metric, then (writing $h(f) = h$ and $\theta(f) = \theta$)

$$\begin{aligned} dh_{ji} &= d<e_i,e_j> = <\nabla e_i,e_j> + <e_i,\nabla e_j> \\ &= <\sum \theta_{ki}e_k,e_j> + <e_i,\sum \theta_{mj}e_m> \\ &= \sum \theta_{ki}h_{jk} + \sum \bar{\theta}_{mj}h_{mi} \\ &= (h\theta)_{ji} + ({}^t\bar{\theta}h)_{ji}. \end{aligned}$$

So $$dh = h\theta + ({}^t\bar{\theta}h).$$

This condition is also dufficient for ∇ to be compatible with the metric. If the frame f is unitary the condition reduces to

$$\theta(f) + {}^t\bar{\theta}(f) = 0.$$

(3.6) Connections on holomorphic vector bundles with hermitian metric.

Let $\pi: E \to M$ be a holomorphic vector bundle with hermitian metric. Let $A^{(p,q)}(M,E)$ be the space of C^∞-differential forms of type (p,q) with values in E.

Since $A^1(M,E) = A^{(1,0)}(M,E) \oplus A^{(0,1)}(M,E)$, ∇ can be decomposed as

$$\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$$

with

$$\nabla^{(1,0)}: A^0(M,E) \to A^{(1,0)}(M,E)$$
$$\nabla^{(0,1)}: A^0(M,E) \to A^{(0,1)}(M,E).$$

DEFINITION. The connection ∇ is called *compatible with the holomorphic structure* when

$$\nabla^{(0,1)}\phi = 0 \quad \text{for each holomorphic section} \quad \phi \in \mathcal{O}(U,E)$$
$$(U \text{ open in } M).$$

It means that the connection matrix $\theta(f)$ with respect to a holomorphic frame f is a matrix-valued 1-form of type $(1,0)$. Indeed,

$$(\nabla\phi)(f) = (d+\theta(f))\phi(f)$$
$$= (\partial+\theta^{(1,0)}(f))\phi(f) + (\bar\partial+\theta^{(0,1)}(f))\phi(f).$$

So, if ϕ is a holomorphic section and f is a holomorphic frame, then

$$(\nabla^{(1,0)}\phi)(f) = (\partial+\theta^{(1,0)}(f))\phi(f)$$
$$(\nabla^{(0,1)}\phi)(f) = (\bar\partial+\theta^{(0,1)}(f))\phi(f) = \theta^{(0,1)}(f)\phi(f).$$

THEOREM. *Let* $\pi: E \to M$ *be a holomorphic vector bundle with hermitian metric. Then there exists a unique connection* ∇ *which is compatible with the* metric and the holomorphic structure.

If f is a holomorphic frame and $\theta(f)$ and $\Theta(f)$ are the connection and curvature matrix respectively of ∇ with respect to f, then

$$\theta(f) \text{ is of type } (1,0),$$
$$\Theta(f) = \bar{\partial}\theta(f) \text{ is of type } (1,1).$$

PROOF. Suppose ∇ is a connection compatible with metric and holomorphic structure. Let f be a holomorphic frame and let θ be the connection matrix with respect to f, then

$$\begin{cases} \theta \text{ is of type } (1,0) \\ dh = h\theta + {}^t\bar{\theta}h \end{cases}.$$

Since $h\theta$ is of type $(1,0)$ and ${}^t\bar{\theta}h$ is of type $(0,1)$, it follows that

$$\partial h = h\theta \quad \text{and} \quad \bar{\partial}h = {}^t\bar{\theta}h.$$

So
$$\theta = h^{-1}\partial h.$$

This formula serves now as a definition for θ. It has to be verified that

$$\theta(fg) = g^{-1}\theta(f)g + g^{-1}dg$$

so that θ defines a connection.

Since $\partial h^{-1} = -h^{-1}\partial h\, h^{-1}$, it follows that

$$\partial\theta = \partial(h^{-1}\partial h) = -h^{-1}\partial h\, h^{-1} \wedge \partial h = -(h^{-1}\partial h) \wedge (h^{-1}\partial h)$$
$$= -\theta \wedge \theta.$$

Hence
$$\Theta = d\theta + \theta \wedge \theta = \bar{\partial}\theta. \quad \blacksquare$$

(3.7) Invariant polynomials.

Definition. An *invariant polynomial* on $M(r,\mathbb{C})$ (complex $r\times r$ matrices) is a map

$$\hat{\Phi}: M(r,\mathbb{C}) \to \mathbb{C}$$

that can be written as a complex polynomial in the matrix entries satisfying

$$\hat{\Phi}(TAT^{-1}) = \hat{\Phi}(A)$$

for every nonsingular matrix T.

EXAMPLE. If $A \in M(r,\mathbb{C})$, then

$$\det(I+tA) = 1 + t\hat{\Phi}_1(A) + \dots + t^r\hat{\Phi}_r(A),$$

where $\hat{\Phi}_k(A)$ is a polynomial in the matrix entries of A homogeneous of degree k, that is invariant (in fact $\hat{\Phi}_k(A)$ is the k-th elementary symmetric function in the eigenvalues of A).

Thus

$$\hat{\phi}_1(A) = \mathrm{tr}(A) = \sum_i a_{ii}$$

$$\hat{\phi}_2(A) = \sum_{j<k} \begin{vmatrix} a_{jj} & a_{jk} \\ a_{kj} & a_{kk} \end{vmatrix}$$

etc.

and $\hat{\phi}_r(A) = \det(A)$.

Let G be a matrix Lie group. If $\hat{\phi}$ is an invariant polynomial on the Lie algebra $\mathfrak{g}$ of G homogeneous of degree k. One can associate with it a map

$$\Phi: \underbrace{\mathfrak{g} \times \dots \times \mathfrak{g}}_{k \text{ times}} \longrightarrow \mathbb{C}$$

such that $\hat{\Phi}(A) = \hat{\Phi}(A,\ldots,A)$ and

(i) Φ is k-linear

(ii) Φ is symmetric

(iii) Φ is invariant in the sense that

$$\Phi(\mathrm{Ad}(g)A_1, \ldots, \mathrm{Ad}(g)A_k) = \Phi(A_1,\ldots,A_k) \quad (g \in G).$$

This process is called *polarization*. For $k = 2$ one has the formula

$$\Phi(A_1,A_2) = \tfrac{1}{2}\{\hat{\Phi}(A_1+A_2) - \Phi_1(A) - \Phi_2(A)\}.$$

EXAMPLE. Take $G = SU(2)$ so that g is the set of anti-hermitian 2×2 matrices of trace 0.

If

$$A = \begin{pmatrix} ia & b+ic \\ -b+ic & -ia \end{pmatrix} \in g$$

then $\hat{\phi}_2(A) = \det A = a^2 + b^2 + c^2$.

We know that $(A,B) \longmapsto -\mathrm{tr}(AB)$ is an invariant inner product on g. For the corresponding norm holds $-\mathrm{tr}(A^2) = 2\det(A)$.

Thus

$$\phi_2(A,B) = -\tfrac{1}{2}\,\mathrm{tr}(AB).$$

(3.8) The Chern-Weil map.

Let $\pi: P \to M$ be a principal fibre bundle with group $G \subset G\ell(r,\mathbb{C})$. Let ω be a connection on P with curvature form $\Omega \in A^2(P,g)$. Let $\hat{\Phi}$ be an invariant polynomial on g homogeneous of degree k with corresponding k-linear map Φ.

Now we consider the k-fold wedge product of Ω taken with respect to the k-linear map Φ. This form is denoted by $\Phi(\Omega^k)$ and belongs to $A^{2k}(P,\mathbb{C})$.

Now we have

(i) $\Phi(\Omega^k)$ is horizontal (since Ω is)

(ii) $\Phi(\Omega^k)$ is invariant under G, i.e.

$$R_g^*\Phi(\Omega^k) = \Phi(\Omega^k) \qquad (g\in G)$$

(since Ω is of type Ad and Φ is Ad-invariant).

As a consequence $\Phi(\Omega^k)$ is the lift of a 2k-form on M which is also denoted by $\Phi(\Omega^k) \in A^{2k}(M,\mathbb{C})$.

THEOREM.

a) $\Phi(\Omega^k) \in A^{2k}(M,\mathbb{C})$ *is closed,*

b) *its cohomology class does not depend on the connection (it only depends on the bundle* P *and on the invariant polynomial* $\hat{\Phi}$).

The map that associates to $\hat{\Phi}$ the cohomology class of $\Phi(\Omega^k)$ is called the *Chern-Weil map.*

Sketch of the proof. a) Since Φ is symmetric and Ω is a 2-form, we have

$$d\Phi(\Omega^k) = k\Phi(d\Omega\wedge\Omega^{k-1}) = k\Phi([\Omega\wedge\omega]\wedge\Omega^{k-1}).$$

Since Φ is invariant, it follows that

$$\Phi(\mathrm{Ad}(g_t)Y_1,\ldots,\mathrm{Ad}(g_t)Y_k) = \Phi(Y_1,\ldots,Y_k),$$

where $\quad g_t := \exp(tY_0)$ and $Y_0, Y_1, \ldots, Y_k \in \mathfrak{g}$.

Differentiation of this identity at $t = 0$ gives

$$\sum_{i=1}^{k} \Phi(Y_1,\ldots,[Y_0, Y_i], \ldots Y_k) = \sum_{i=1}^{k} \Phi([Y_0, Y_i], Y_1,\ldots,\hat{Y}_i,\ldots,Y_k) = 0.$$

From this identity it follows that

$$\Phi([\Omega \wedge \omega] \wedge \Omega^{k-1}) = 0.$$

b) For the proof part b) we need the following lemma the proof of which is omitted.

LEMMA. *Let* $h: A^k(M \times [0, 1]) \to A^{k-1}(M)$ *be the map that maps*

$$\eta = ds \wedge \alpha + \beta$$

(α and β *do not contain* ds) *to*

$$h(\eta) := \int_{s=0}^{s=1} \alpha$$

(*integration of the coefficients of* α).

Then

$$dh(\eta) + h(d\eta) = i_1^*\eta - i_0^*\eta$$

where

$$i_0: M \to M \times [0,1], \quad i_0(x) = (x,0)$$
$$i_1: M \to M \times [0,1], \quad i_1(x) = (x,1).$$

Suppose now that ω_0 and ω_1 are connections with curvature Ω_0 and Ω_1 respectively. Consider on the bundle $P \times [0,1] \to M \times [0,1]$ the connection

$$\omega = (1-s)\omega_0 + s\omega_1 \in A^1(P \times [0,1], g)$$

with curvature Ω.

Now $i_0^*\Omega = \Omega_0$ and $i_1^*\Omega = \Omega_1$. In view of the lemma we have

$$dh\Phi(\Omega^k) + hd\Phi(\Omega^k) = i_1^*\Phi(\Omega^k) - i_0\Phi(\Omega^k)$$

and hence

$$dh\Phi(\Omega^k) = \Phi(\Omega_1^k) - \Phi(\Omega_0^k).$$

So $\Phi(\Omega_1^k)$ an $\Phi(\Omega_0^k)$ are in the same cohomology class. ∎

REMARK. The form $\Phi(\Omega^k) \in A^{2k}(M,\mathbb{C})$ can also be defined in terms of the vector bundle E associated with the standard representation of G on $\mathbb{C}^r$. Let $\Theta \in A^2(M, \mathrm{Hom}(E,E))$ be the curvature form. If we take a frame f over an open set $U \subset M$, then $\Theta(f)$ is a matrix-valued 2-form on U. The form

$$\Phi(\Theta(f)^k) \in A^{2k}(U,\mathbb{C})$$

is independent of the choice of the frame f since

$$\Theta(fg) = g^{-1}\Theta(f)g$$

and Φ is Ad-invariant. In this manner one defines a 2k-form on M which is of course the same as $\Phi(\Omega^k)$.

(3.9) The Chern class of a vector bundle.

Let $\hat{\Phi}_k$ $(k=0,1,\ldots,r)$ be the invariant polynomials on $M(r,\mathbb{C})$ defined by

$$\det(I+tA) = \sum_{k=0}^{r} \hat{\Phi}_k(A)t^k.$$

Let $E \to M$ be a complex vector bundle of rank r.

Definition. The k-th *Chern class* of E is by definition the cohomology class $c_k(E)$ of the 2k-form

$$\Phi_k((-\frac{1}{2\pi i}\Theta)^k)$$

where Θ is the curvature of some connection on E.

So $c_k(E) \in H^{2k}(M,\mathbb{C})$. Note that $c_0(E) = 1$.

Furthermore, $c(E) := \sum_{k=0}^{r} c_k(E) \in H^*(M,\mathbb{C}) := \bigoplus_k H^k(M,\mathbb{C})$

is called the *total Chern class* of E.

REMARK. $H^*(M,\mathbb{C}) = \bigoplus_k H^k(M,\mathbb{C})$ has the structure of a ring (de Rham cohomology ring):

if $c, c' \in H^*(M,\mathbb{C})$ and $c = [\eta]$, $c' = [\eta']$,

then $c \cdot c' := [\eta \wedge \eta']$.

THEOREM. a) *Let* $E \to M$ *be a complex vector bundle and let* $\psi: N \to M$ *be* C^∞*-map. Let* $\psi^*E \to N$ *be the pull-back of* E *under* ψ *(this means: if* $f = (e_1,\dots,e_r)$ *is a frame for* E *over* $U \subset M$, *then* $f^* = (e_1^*,\dots,e_r^*)$ *with* $e_i^* = e_i \circ \psi$ *is by definition a frame for* ψ^*E *over* $\psi^{-1}(U)$). *Then we have the relation*

$$c(\psi^*E) = \psi^*c(E).$$

b) *If* $E' \to M$ *is another complex vector bundle then*

$$c(E \oplus E') = c(E) \cdot c(E'),$$

where the product is taken in the de Rham cohomology ring.

PROOF. See [4], Ch.III. ∎

THEOREM. *Let* $E \to M$ *be a complex vector bundle of rank* r.

a) *If* E *is trivial* $(E \simeq M \times \mathbb{C}^r)$, *then*

$$c_k(E) = 0 \quad \textit{for} \quad k = 1,2,\dots,r.$$

So $c(E) = 1$.

b) *If* $E \simeq E' \oplus E''$, *where* E'' *is a trivial bundle of rank* s, *then*

$$c_k(E) = 0 \quad \textit{for} \quad k = r-s+1,\dots,r.$$

PROOF. a) Take the trivial connection (curvature zero).

b) $$\begin{aligned} c(E) &= c(E' \oplus E'') \\ &= c(E') \cdot c(E'') \\ &= c(E') \cdot 1 \\ &= 1 + c_1(E') + \ldots + c_{r-s}(E'). \quad \blacksquare \end{aligned}$$

(3.10) EXAMPLES. In order to compute the Chern class of a complex vector bundle one starts with a connection on this vector bundle. For holomorphic vector bundles one tries a hermitian metric and one takes the unique connection which is compatible with the metric and the holomorphic structure.

1. The canonical line bundle on $P_1(\mathbb{C})$. On $U_0 := \{z_0 \neq 0\}$ we take the coordinate $z = \frac{z_1}{z_0}$ and on $U_1 := \{z_1 \neq 0\}$ we take the coordinate $w = \frac{z_0}{z_1}$. Then the canonical line bundle can be represented as $E = (U_0 \times \mathbb{C}) \cup (U_1 \times \mathbb{C})/_\sim$ where $(z,\xi) \sim (w,\eta)$ iff $w = \frac{1}{z}$ and $\eta = z\xi$.
For the metric we take

$$\begin{aligned} h(z,\xi) &= (1 + |z|^2)|\xi|^2, \\ h(w,\eta) &= (1 + |w|^2)|\eta|^2. \end{aligned}$$

The metric is well-defined since $h(z,\xi) = h(w,\eta)$ if $(z,\xi) \sim (w,\eta)$. This metric can be viewed as the metric of $\mathbb{C}^2$ when the canonical line bundle is identified with the one-dimensional subspaces of $\mathbb{C}^2$. As a holomorphic frame over U_0 we take the frame $\xi = 1$. With respect to this frame $h(z) = 1 + |z|^2$. The connection compatible with metric and holomorphic structure has matrix $\theta = h^{-1}\partial h$ and has curvature

$$\Theta = \bar{\partial}(h^{-1}\partial h) = \bar{\partial}\partial \log(1+z\bar{z}) .$$

The Chern form is then given by

$$c_1(E) = -\frac{1}{2\pi i}\bar{\partial}\partial \log(1+z\bar{z})$$

$$= -\frac{1}{2\pi i}\frac{1}{(1+|z|^2)^2} d\bar{z}\wedge dz = -\frac{1}{\pi}\frac{dxdy}{(1+r^2)^2} .$$

It is not exact since

$$\int_{P_1(\mathbb{C})} c_1(E) = \int_0^{2\pi}\int_0^{\infty} -\frac{1}{\pi}\frac{rdrd\phi}{(1+r^2)^2} = -1 .$$

Thus the canonical line bundle is nontrivial.

2. The holomorphic tangent bundle of $P_1(\mathbb{C})$. The tangent bundle is given by $E = (U_0\times\mathbb{C}) \cup (U_1\times\mathbb{C})/_\sim$ where $(z,\xi) \sim (w,\eta)$ iff $w = \frac{1}{z}$ and $\eta = -\frac{1}{z^2}\xi$ (if $w = \frac{1}{z}$, then $\frac{\partial}{\partial z} = \frac{1}{z^2}\frac{\partial}{\partial w}$; so $\xi\frac{\partial}{\partial z} = \eta\frac{\partial}{\partial w}$ if $\eta = -\frac{1}{z^2}\xi$).

As holomorphic frame we take $\frac{\partial}{\partial z}$. The metric is defined by

$$h\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{(1+|z|^2)^2} .$$

It is easily verified that $h\left(\frac{\partial}{\partial w}, \frac{\partial}{\partial w}\right) = \frac{1}{(1+|w|^2)^2}$. Then $\theta = h^{-1}\partial h$ and

$$\Theta = \bar{\partial}(h^{-1}\partial h) = -2\bar{\partial}\partial \log(1+z\bar{z}) .$$

Hence $\int_{P_1(\mathbb{C})} c_1(E) = 2.$

(3.11) Let P be a principal fibre bundle on $M = S^4$ with group $G \subset G\ell(r,\mathbb{C})$ and let ω be a connection on P. Furthermore, let E be the complex vector bundle of rank r associated with the standard representation of G on $\mathbb{C}^r$. Let f be a frame on $U = S^4\setminus\{\infty\} = \mathbb{R}^4$ and let $\Theta = (\Theta^i_j)$ be the curvature matrix with respect to f.

The form that represents the first Chern class is

$$c_1(E) = -\frac{1}{2\pi i}\Phi_1(\Theta) = -\frac{1}{2\pi i}\mathrm{tr}(\Theta) = -\frac{1}{2\pi i}\sum_{j=1}\Theta^j_j .$$

The second Chern class is represented by

$$c_2(E) = \left(-\frac{1}{2\pi i}\right)^2 \Phi_2(\Theta^2) = -\frac{1}{4\pi^2}\,\Theta\wedge_{\Phi_2}\Theta .$$

Now $\Theta = \sum_{i,j} \Theta^i_j E^i_j$ where E^i_j is the matrix whose entries are zero except the one on the place (i,j) which is 1. So

$$c_2(E) = -\frac{1}{4\pi^2} \sum \Phi_2(E^{i_1}_{j_2}, E^{i_2}_{j_2})\, \Theta^{i_1}_{j_1} \wedge \Theta^{i_2}_{j_2}$$

$$= -\frac{1}{4\pi^2} \Big(\sum_{i<j} \Theta^i_i \wedge \Theta^j_j - \Theta^i_j \wedge \Theta^j_i \Big) .$$

We have used that

$$\Phi_2(E^{i_1}_{j_1}, E^{i_2}_{j_2}) = \tfrac{1}{2}\Big\{\hat{\Phi}_2(E^{i_1}_{j_1} + E^{i_2}_{j_2}) - \hat{\Phi}_2(E^{i_1}_{j_1}) - \hat{\Phi}_2(E^{i_2}_{j_2})\Big\}$$

$$= \begin{cases} \tfrac{1}{2} & \text{if } i_1 = j_1 \text{ and } i_2 = j_2 \\ -\tfrac{1}{2} & \text{if } i_1 = j_2 \text{ and } i_2 = j_1 \\ 0 & \text{in the other cases.} \end{cases}$$

<u>REMARK</u>. If $G = SU(2)$, then $c_2(E) = -\frac{1}{4\pi^2}\, \Theta \wedge_{\Phi_2} \Theta = \frac{1}{8\pi^2}\, \mathrm{tr}(\Theta \wedge \Theta)$. The integral $\int_{S^4} c_2(E)$ is called the *Chern number* of the bundle.

<u>REMARK</u>. For the Chern classes of complex line bundles see [4] (Ch.III, Sec.4).

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TWISTOR THEORY AND YANG-MILLS FIELDS

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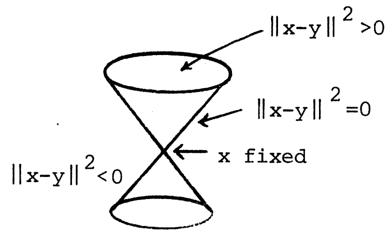
§1. MINKOWSKI SPACE, LORENTZ GROUP, SL(2,$\mathbb{C}$).

The real Minkowski space M is the real affine 4-dimensional space, provided with a metric $\| \ \|$ with signature (+,-,-,-). This metric divides the space according to causality:



If $\|x-y\|^2$ is positive, zero or negative then x and y are said to be timelike, null and spacelike separated respectively.
If x and y are null separated then they can be joined by a light ray.
If we choose an origin for M, then we can find coordinates such that the metric takes the form

(1.1) $$\|x\|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

The *Lorentz group* L (with respect to the chosen origin) is the group of isometries of M preserving the origin.
If $(\Lambda x)^\mu = \Lambda^\mu_\nu x^\nu$ then $\Lambda \in L$ if and only if

(1.2) $$g_{\mu\nu}\Lambda^\mu_\rho \Lambda^\nu_\sigma = g_{\rho\sigma} \quad \text{with} \quad (g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Taking determinants at both sides of (1.2) we get

(1.3) $\det \Lambda = \pm 1, \quad \forall \Lambda \in L.$

Taking $\rho = \sigma$ in (1.2) we obtain

(1.4) $|\Lambda^0_0| \geqslant 1, \quad \forall \Lambda \in L.$

Elements $\Lambda \in L$ with $\det \Lambda = 1$ and $\Lambda^0_0 \geqslant 1$ form a subgroup of L, which is called the restricted Lorentz group $L^{\uparrow}_{+}$, the subgroup which preserves space-time orientation (the connected component containing the identity of $L \cong O(1,3)$.

The group $SL(2,\mathbb{C})$ is the group consisting of all complex (2×2)-matrices with determinant $+1$. This group is closely related to $L^{\uparrow}_{+}$ as we shall see. Let $H(2)$ be the set of all complex Hermitian (2×2)-matrices and let $\sigma_\mu (\mu=0,1,2,3)$ be the elements of $H(2)$ defined by

(1.5)
$$\sigma_0 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \qquad \sigma_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$
$$\sigma_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} \qquad \sigma_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

For any $x \in M$ we define $A_x \in H(2)$ by

(1.6)
$$A_x = x^\mu \sigma_\mu = \frac{1}{\sqrt{2}}\begin{pmatrix}x^0 + x^3 & x^1 - ix^2\\ x^1 + ix^2 & x^0 - x^3\end{pmatrix}$$

This defines a linear bijective mapping from M onto $H(2)$. It is easy to see that

(1.7) $\|x\|^2 = 2 \det A_x .$

Let $B \in SL(2,\mathbb{C})$ and $A_x \in H(2)$. Then $BA_xB^* \in H(2)$, where $B^* = {}^t\bar{B}$ is the Hermitian conjugate of B. So there exists a $x' \in M$ with $BA_xB^* = A'_x$. So B defines a linear mapping $\Lambda(B): M \to M$ by $x \longmapsto x' = \Lambda(B)x$. This mapping is a Lorentz transformation; this follows immediately from (1.7). Furthermore it is easy to see that

$$\Lambda(B)\cdot\Lambda(B') = \Lambda(BB') \tag{1.8}$$

and

$$\Lambda(\mathbb{1}) = \tilde{\mathbb{1}}\,, \tag{1.9}$$

where $\mathbb{1}$ and $\tilde{\mathbb{1}}$ are the identities of $SL(2,\mathbb{C})$ and L respectively. Moreover it is easy to check that

$$\Lambda(B) = \Lambda(B') \iff B = \pm B'. \tag{1.10}$$

It can be proved that the image of $SL(2,\mathbb{C})$ under Λ equals the restricted Lorentz group $L_+^\uparrow$ (see Halpern [6]). So the mapping $B \longmapsto \Lambda(B)$ is a homomorphism from $SL(2,\mathbb{C})$ onto $L_+^\uparrow$ with kernel $\{\pm\mathbb{1}\}$ and describes a 2 - 1 covering of L by $SL(2,\mathbb{C})$.

Suppose as before that we have chosen an origin and coordinates (x^0,x^1,x^2,x^3) such that the metric takes the form

$$\|x\|^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 .$$

We embed $\mathbb{R}^4$ in $\mathbb{C}^4$ in the usual way and extend the Minkowski metric to a holomorphic metric

$$\|z\|^2 = (z^0)^2 - (z^1)^2 - (z^2)^2 - (z^3)^2 . \tag{1.11}$$

Observe that the metric does not possess a well-defined signature any longer. The resulting space is called the *complexified Minkowski space* $M_{\mathbb{C}}$.

The bijective linear mapping from M onto H(2) defined by $x \longmapsto A_x$ may be extended to a bijective mapping from $M_{\mathbb{C}}$ onto the space of all complex (2×2)-matrices defined by

$$A_z = z^\mu \sigma_\mu = \frac{1}{\sqrt{2}} \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^1 + iz^2 & z^0 - z^3 \end{pmatrix} \tag{1.12}$$

We see that $\|z\|^2 = 2 \det A_z$ and that z is real if and only if A_z is Hermitian.

§2. COMPLEX STRUCTURES, SPINORS

Let V be a real vector space and suppose that $J: V \to V$ is a $\mathbb{R}$-linear mapping with $J^2 = -I$ (I identity in V). J is called a *complex structure* on V. Then we can equip V with the structure of a complex vector space in the following manner: define

$$(\alpha+i\beta)v = \alpha v + \beta Jv, \qquad \alpha, \beta \in \mathbb{R}.$$

It is easy to check that V indeed becomes a complex vector space. Conversely, if V is a complex vector space, then V can be considered as a real vector space and then multiplication by i is a $\mathbb{R}$-linear mapping J from V onto V with $J^2 = -I$.

Furthermore, if $\{v_1,\ldots,v_n\}$ is a basis for V over $\mathbb{C}$ then $\{v_1,Jv_1,\ldots,v_n,Jv_n\}$ is a basis for V over $\mathbb{R}$.

<u>EXAMPLE</u>. $V = \mathbb{R}^{2n}$. We can consider $\mathbb{R}^{2n}$ as $\mathbb{C}^n$ if we put

$$z_1 = x_1 + ix_2; \quad z_2 = x_2 + ix_3, \ldots, \; z_n = x_{2n-1} + ix_{2n}.$$

Multiplication by i corresponds to the mapping $J: \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with $J^2 = -I$ defined by

(2.1) $$J(x_1,x_2,\ldots,x_{n-1},x_n) = (-x_2,x_1,\ldots,-x_{2n},x_{2n-1}).$$

This is the *standard complex structure* of $\mathbb{R}^{2n}$. It is clear that $A^{-1}JA$ is again a complex structure on $\mathbb{R}^{2n}$. Moreover, each complex structure on $\mathbb{R}^{2n}$ is of this form. Indeed, let J' be a second complex structure, then we can find a basis $(e_1,J'e_1,\ldots,e_n,J'e_n)$ such that J' is described on this basis by J; i.e.

$J = AJ'A^{-1}$ with transformation A of the basis. Thus
$J' = A^{-1}JA$.

Finally, $A^{-1}JA$ holds precisely when A commutes with J, the standard complex structure. So the corresponding mapping in $\mathbb{C}^n$ is $\mathbb{C}$ - linear. The complex structures on $\mathbb{R}^{2n}$ are therefore described by the coset space $GL(2n,\mathbb{R})/_{GL(n,\mathbb{C})}$. If in addition we fix an orientation and a metric on $\mathbb{R}^{2n}$, then the complex structures on $\mathbb{R}^{2n}$ preserving both orientation and metric are described by $SO(2n,\mathbb{R})/_{U(n)}$. Then in $\mathbb{R}^{2n}$ the complex structure is completely determined.

Now let V again be a real vector space with complex structure J and let $V_{\mathbb{C}} = V \oplus iV$ the complexification of V.
The $\mathbb{R}$ - linear mapping J is extended $\mathbb{C}$ - linearly by defining $J(v_1+iv_2) = Jv_1 + iJv_2$.
Moreover $J^2 = -I$ remains in force, and the eigenvalues of J are $\pm i$. Let $V^{1,0}$ be the eigenspace corresponding to the eigenvalue $+i$ and $V^{0,1}$ the eigenspace corresponding to $-i$. Then we have

$$I = \tfrac{1}{2}(I-iJ) + \tfrac{1}{2}(I+iJ) = P^+ + P^-$$

$$(P^{\pm})^2 = \tfrac{1}{4}(I\mp J)^2 = \tfrac{1}{2}(I\mp J) = P^{\pm}$$

$$P^+P^- = P^-P^+ = 0$$

$$J(w\mp iJw) = \pm i(w\mp iJw)$$

$$P^+V_{\mathbb{C}} = V^{1,0}, \quad P^-V_{\mathbb{C}} = V^{0,1}$$

$$V_{\mathbb{C}} = V^{1,0} \oplus V^{1,0}.$$

Furthermore, it is easy to see that $V^{1,0}$ and $V^{0,1}$ are respectively $\mathbb{C}$-linear and conjugate $\mathbb{C}$ - linear isomorphic with the complex vector space V_J, obtained by considering V as a complex vector space with respect to the complex structure J.

We apply the foregoing to the vector space $S = \mathbb{R}^4$ equipped with the standard complex structure J. It follows that

(2.2) $$S_{\mathbb{C}} = S^{1,0} \oplus S^{0,1}$$

where $S^{1,0}$ and $S^{0,1}$ are respectively $\mathbb{C}$-linear and conjugate $\mathbb{C}$-linear isomorphic with S, considered as a complex vector space with respect to the standard complex structure J.

Consider the basis $f_1 = (1,0,0,0)$, $f_2 = (0,0,1,0)$ in S as a complex vector space. The corresponding basis in $S^{1,0}$ equals: $e_1 = \frac{1}{2}(1,-i,0,0)$, $e_2 = \frac{1}{2}(0,0,1,-i)$ and in $S^{0,1}$: $\bar{e}_1 = \frac{1}{2}(1,i,0,0)$, $\bar{e}_2 = \frac{1}{2}(0,0,0,i)$.

Now

$$e^1 = (1,i,0,0), \quad e^2 = (0,0,1,i)$$
$$\bar{e}^1 = (1,-i,0,0), \quad \bar{e}^2 = (0,0,1,-i)$$

is the dual basis in $S^*_{\mathbb{C}}$. The space spanned by e^1 and e^2 is denoted by $S^*_{1,0}$ and by $\bar{e}^1$ and $\bar{e}^2$ $S^*_{0,1}$.

It is clear that $S^*_{\mathbb{C}} = S^*_{1,0} \oplus S^*_{0,1}$.

Next we consider the tensor product $S^{1,0} \otimes S^{0,1}$. An element ξ of this tensor product can be written in the form

(2.3) $$\xi = \xi^{A\bar{A}} e_A \otimes \bar{e}_{\bar{A}},$$

here we have applied summation convention with respect to the indices A and $\bar{A}$. Now the mapping (1.12) may be interpreted as a $\mathbb{C}$-linear isomorphism from the complexified Minkowski space to $S^{1,0} \otimes S^{0,1}$, given by

(2.4) $$z \longmapsto z^{A\bar{A}} e_A \otimes \bar{e}_{\bar{A}}, \quad (z^{A\bar{A}}) = \frac{1}{\sqrt{2}} \begin{pmatrix} z^0 + z^3 & z^1 - iz^2 \\ z^0 + iz^2 & z^0 - z^3 \end{pmatrix}$$

$SL(2,\mathbb{C})$ acts on S as a complex vector space in a natural way. Now we consider the action induced on $S^{1,0} \otimes S^{0,1}$. The matrix on the basis f_1, f_2 in S of an element $T \in SL(2,\mathbb{C})$ is the same as the matrix of T on the basis e_1, e_2 in $S^{1,0}$ and the conjugate matrix on the corresponding basis $\bar{e}^1, \bar{e}^2$ in $S^{0,1}$. This is a consequence of S and $S^{1,0}$ being isomorphic. The transformation induced in $S^{1,0}$ and $S^{0,1}$ becomes

$$\begin{aligned} T\xi &= T\xi^{A\bar{A}} e_A \otimes \bar{e}_{\bar{A}} = \xi^{A\bar{A}} Te_A \otimes T\bar{e}_{\bar{A}} = \\ &= \xi^{A\bar{A}} T_{BA} e_B \otimes \bar{T}_{\bar{B}\bar{A}} \bar{e}_{\bar{B}} = T_{BA} \xi^{A\bar{A}} \bar{T}_{\bar{B}\bar{A}} e_B \otimes \bar{e}_{\bar{B}} = \\ &= [T(\xi^{A\bar{A}})T^*]_{B\bar{B}} e_B \otimes \bar{e}_{\bar{B}}, \end{aligned}$$

which corresponds with the $SL(2,\mathbb{C})$ - action of (2×2) - matrices, induced by Lorentz transformations.

For the tangent space $T_x(\mathbb{R}^4) \cong \mathbb{R}^4$, identified with $\mathbb{R}^4$ and with basis: $\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$ the corresponding basis in $S_x^{1,0}$ becomes:

in $S_x^{1,0}$: $\quad \frac{1}{2}\left(\frac{\partial}{\partial x^0} - i\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial w^0}, \quad \frac{1}{2}\left(\frac{\partial}{\partial x^2} - i\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial w^1}$

and in $S_x^{0,1}$: $\quad \frac{1}{2}\left(\frac{\partial}{\partial x^0} + i\frac{\partial}{\partial x^1}\right) = \frac{\partial}{\partial \bar{w}^0}, \quad \frac{1}{2}\left(\frac{\partial}{\partial x^2} + i\frac{\partial}{\partial x^3}\right) = \frac{\partial}{\partial \bar{w}^1}$,

the usual notation in complex analysis. The dual basis in the complexified cotangent space becomes

$S^*_{1,0_x}$: $\quad dx^0 + idx^1 = dw^0, \quad dx^2 + idx^3 = dw^1$

$S^*_{0,1_x}$: $\quad dx^0 - idx^1 = d\bar{w}^0, \quad dx^2 - idx^3 = d\bar{w}^1$.

A *spinor* is a member of a tensor product

$$S^{1,0} \otimes S^{1,0} \otimes \ldots \otimes S^{0,1} \otimes \ldots \otimes S^*_{1,0} \otimes \ldots \otimes S^*_{0,1} \otimes \ldots$$

just as a tensor is a member of

$$T_x(M) \otimes T_x(M) \otimes \ldots \otimes T^*_x(M) \otimes T^*_x(M) \otimes \ldots$$

REMARK ON NOTATION. Penrose and his disciples write $z^{AA'}$ instead of $z^{A\bar{A}}$, since this is easier on the typesetter. Moreover, they denote the spaces $S^{1,0}$, $S^{0,1}$, $S^*_{1,0}$, $S^*_{0,1}$ by $\mathbb{C}^A$, $\mathbb{C}^{A'}$, $\mathbb{C}_A$, $\mathbb{C}_{A'}$ respectively.

In most physical literature $z^{A\dot{A}}$ is used. We shall use the notation of Penrose in the following. Finally, it is easy to check that,

$$(2.5) \qquad z^{AA'} = \zeta^A \tau^{A'} \Longleftrightarrow \| z \| = 0 \, ,$$

for $\| z \|^2 = 2 \det(z^{AA'})$.

§3. TWISTOR GEOMETRY

In the preceding sections we have seen that the Minkowski space, after the choice of an origin, may be considered as a subspace of complex (2×2) - matrices $\mathbb{C}^{AA'}$. The complexified Minkowski space may be considered as the space of all complex (2×2) - matrices. We can interprete such a matrix as a linear mapping

$$\mathbb{C}_{A'} \longrightarrow \mathbb{C}^{A}$$

$$\pi_{A'} \longmapsto iz^{AA'}\pi_{A'}$$

(the factor i is convention) and a linear transformation is completely determined by its graph in $\mathbb{C}_{A'} \otimes \mathbb{C}^{A}$.

This 4-dimensional complex vector space is called the *twistor space* $\mathbb{T}$. A point $z^{AA'}$ in the complexified Minkowski space $M_{\mathbb{C}}$ generates a plane (a 2-dimensional complex subspace), defined by

$$\{(\omega^{A},\pi_{A'}) \in \mathbb{T} \mid \omega^{A} = iz^{AA'}\pi_{A'}\}.$$

Almost all planes in $\mathbb{T}$ arise in this manner. More precise, if $\bar{M}_{\mathbb{C}}$ is the Grassmannian of all 2-planes in $\mathbb{T}$, then $M_{\mathbb{C}}$ is an open dense subset of $\bar{M}_{\mathbb{C}}$. Thus $\bar{M}_{\mathbb{C}}$ is a compactification of $M_{\mathbb{C}}$, for the Grassmannian of all 2-planes in $\mathbb{T}$ is compact. Usually one constructs a projective space $P\mathbb{T}$ out of $\mathbb{T}$, which for convenience we will denote by $\mathbb{P}$. It is easy to see that two points $z^{AA'}$ and $w^{AA'}$ in $M_{\mathbb{C}}$ are null separated if and only if the planes $\{\omega^{A} = iz^{AA'}\pi_{A'}\}$ and $\{\omega^{A} = iw^{AA'}\pi_{A'}\}$ intersect according to a line in $\mathbb{T}$. Indeed two planes intersect nontrivially if and only if the equation

$$i(z^{AA'}-w^{AA'})\pi_{A'} = 0$$

admits a nontrivial solution. Hence if and only if $\det(z^{AA'} - w^{AA'}) = 0$ or equivalently $\|z-w\| = 0$. In terms of $\mathbb{P}$, two points z and w are null separated if and only if the corresponding lines L_z and L_w intersect. This is known as *twistor correspondence*

$$\{\text{points in } M_{\mathbb{C}}\} \longleftrightarrow \{\text{lines in } \mathbb{P}\}$$

$$z \longleftrightarrow L_z$$

$$z \text{ and } w \text{ are null separated} \longleftrightarrow L_z \cap L_w \neq \emptyset .$$

If we take a point Z in $\mathbb{P}$ and if we consider the set of all lines through Z then we get a $P_2(\mathbb{C})$ embedded in $M_{\mathbb{C}}$ and such that each pair of points are null separated. Such a subspace, considered in $M_{\mathbb{C}}$, is called an α-*plane*. So we have the correspondence

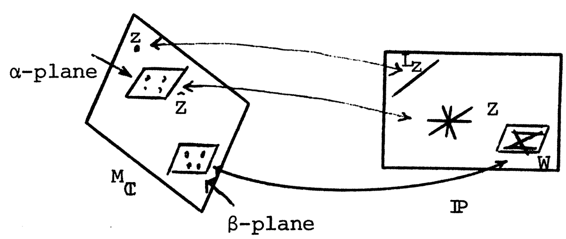
$$\{\alpha\text{-planes in } M_{\mathbb{C}}\} \longleftrightarrow \{\text{points in } \mathbb{P}\}$$

$$\hat{Z} \longleftrightarrow Z$$

Note that an α-plane is a maximal isotropic subspace of $M_{\mathbb{C}}$, since the metric in $M_{\mathbb{C}}$ is nondegenerated. If we fix a plane W in $\mathbb{P}$ (a linearly embedded $P_2(\mathbb{C})$) then there corresponds a β-*plane* in $M_{\mathbb{C}}$ with the set of all lines in that plane

$$\{\beta\text{-planes in } M_{\mathbb{C}}\} \longleftrightarrow \{\text{planes in } \mathbb{P}\}$$

$$\hat{W} \longleftrightarrow W$$



Twistor correspondence

Now we will determine the lines in $\mathbb{P}$ corresponding to points is the real Minkowski space. To this end we introduce a complex conjugation:

$$(3.1)\qquad \begin{array}{ll} \mathbb{C}^A \longleftrightarrow \mathbb{C}^{A'} & \mathbb{C}_A \longleftrightarrow \mathbb{C}_{A'} \\ \xi^A \longmapsto \bar{\xi}^{A'} & \xi_A \longmapsto \bar{\xi}_{A'} \\ \bar{\xi}^A \longleftarrow\!\shortmid\ \xi^{A'} & \bar{\xi}_A \longleftarrow\!\shortmid\ \xi_A \end{array}$$

Note that this is in agreement with the preceding section. Conjugation is extended to the whole spinor algebra in a natural way.

Moreover, we introduce a Hermitian form Φ on $\mathbb{T}$. Let $Z = (\omega^A, \pi_{A'})$, then we set

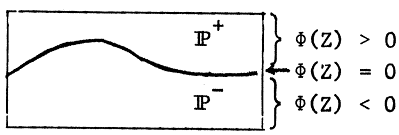
$$(3.2)\qquad \Phi(Z) = \omega^A \bar{\pi}_A + \bar{\omega}^{A'} \pi_{A'} = (\bar{\omega}^{A'}, \bar{\pi}_{A'}) \left(\begin{array}{c|c} 0 & I_2 \\ \hline I_2 & 0 \end{array}\right) \begin{pmatrix} \omega^A \\ \pi_{A'} \end{pmatrix} .$$

The form Φ divides the twistor space into three parts

$$(3.3)\qquad \mathbb{T}^+ = \{Z \mid \Phi(Z) > 0\},\quad \mathbb{T}^0 = \{Z \mid \Phi(Z) = 0\},\quad \mathbb{T}^- = \{Z \mid \Phi(Z) < 0\}$$

and the projective twistor space in corresponding three parts

(3.4) $\mathbb{P}^0 \longrightarrow$



Now let $Z = [\omega^A, \pi_{A'}]$ be a point on L_z, i.e. $\omega^A = iz^{AA'}\pi_{A'}$, then we have

$$\Phi(Z) = iz^{AA'}\pi_{A'}\bar{\pi}_{A'} + \overline{iz^{AA'}\pi_{A'}}\pi_{A'} = iz^{AA'}\pi_{A'}\bar{\pi}_A - i\bar{z}^{A'A}\bar{\pi}_A\pi_{A'} .$$

Thus $\Phi(Z) = 0$ if $(z^{AA'})$ is a Hermitian matrix, i.e. if L_z corresponds with a point of the real Minkowski space M. Conversely, $\Phi(Z) = 0$ for all

Z on L_z implies that $(z^{AA'})$ is Hermitian. So we have shown that:

(3.5) $$z \in M \iff L_z \subset \mathbb{P}^0 .$$

Now it is easy to see that most lines in $\mathbb{P}^0$ arise in this way, i.e. are of the form: L_z with $z \in M$. The collection of lines which are missing are formed by the special line L_∞ with equation: $\pi_{A'} = 0$, together with all lines intersecting this special line. This can be easily seen as follows. Let L be the line through the points $Z = [\omega^A, \pi_{A'}]$ and $\tilde{Z} = [\tilde{\omega}^A, \tilde{\pi}_{A'}]$, that is to say L is described in homogeneous coordinates by

$$\lambda(\omega^A, \pi_{A'}) + \mu(\tilde{\omega}^A, \tilde{\pi}_{A'}) = (\lambda\omega^A + \mu\tilde{\omega}^A,\ \lambda\pi_{A'} + \mu\tilde{\pi}_{A'}).$$

L intersects L_∞ precisely when the equation: $\lambda\pi_{A'} + \mu\tilde{\pi}_{A'} = 0$ admits a nontrivial solution, i.e. $\det(\pi_{A'}, \tilde{\pi}_{A'}) = 0$. It is easy to see that this is precisely the condition on L (in homogeneous coordinates) not to correspond to the graph of a linear mapping $\mathbb{C}_{A'} \to \mathbb{C}^{A'}$, for the linear mapping with matrix $(z^{AA'})$ is represented by

$$\lambda(iz^{00'}, iz^{10'}, 1, 0) + \mu(iz^{01'}, iz^{11'}, 0, 1).$$

Now let $\bar{M}$ be the space of all lines in $\mathbb{P}^0$; i.e. $\Phi(Z) = 0$ for all $Z \in L$, then the lines $\{L_z \mid z \in M\}$ which correspond with points of the real Minkowski space form an open dense subset of the compact space $\bar{M}$. So $\bar{M}$ is a compactification of the real Minkowski space, obtained by adding to M a point = L_∞ together with its light-cone = set of lines intersecting L_∞.

It can be proved that $\bar{M}$ is diffeomorphic with $S^3 \times S^1$ and that $\mathbb{P}^0$ is diffeomorphic with $S^3 \times S^2$. $\mathbb{P}^+$ and $\mathbb{P}^-$ turn out not to be Stein manifolds.

From the twistor theoretical point of view now it would be an appropriate moment to discuss the zero-rest-mass fields (e.g. Maxwell equations) and the related Penrose transformation. To this the equations have to be translated, by using spinors, into a twistor theoretically more suitable **form.**

The Penrose transformation is described using cohomology theory with some sheaves $\mathcal{O}(n)$, related to the canonical line bundle over $P_3(\mathbb{C})$. See Wells [9] over Eastwood [5].

§4. THE FIBRATION $P_3(\mathbb{C}) \to S^4$.

As a complex vector space we can identify the quaternions $\mathbb{H}$ with $\mathbb{C}^2$ as follows

$$\mathbb{C}^2 \to \mathbb{H}$$

$$(4.1) \qquad (z_1,z_2) = (x_1+iy_1,\ x_2+iy_2) \longmapsto z_1 + z_2 j = x_1 + iy_1 + jx_2 + y_2 k,$$

which induces a corresponding isomorphism $\mathbb{C}^4 \xrightarrow{\cong} \mathbb{H}$

$$(4.2) \qquad (z_1,z_2,z_3,z_4) \longmapsto (z_1+z_2 j,\ z_3+z_4 j).$$

As $\mathbb{C} \subset \mathbb{H}$ this last isomorphism induces a mapping

$$(4.3) \qquad \pi\colon P_3(\mathbb{C}) = \mathbb{C}^4/_{\mathbb{C}^*} \longrightarrow P_1(\mathbb{H}) = \mathbb{H}^2/_{\mathbb{H}^*}$$

between projective spaces and in fact yields a vector bundle with $P_3(\mathbb{C})$ as total space and with $P_1(\mathbb{C}) \cong S^2$ (see 4.6) as typical fibre. In homogeneous coordinates the fibre through $[z_1,z_2,z_3,z_4]$ consists of all points

$$(4.4) \qquad w_1(z_1,z_2,z_3,z_4) + w_2(-\bar{z}_2,\bar{z}_1,-\bar{z}_4,\bar{z}_3), \quad w_{1,2} \in \mathbb{C}$$

as left multiplication by j in $\mathbb{H}^2$ induces an anti-linear mapping σ in $P_3(\mathbb{C})$ with $\sigma^2 = \text{id.}$, in homogeneous coordinates given by

$$(4.5) \qquad \sigma(z_1,z_2,z_3,z_4) = (-\bar{z}_2,\bar{z}_1,-\bar{z}_4,\bar{z}_3).$$

The mapping σ defines a *real structure* on $P_3(\mathbb{C})$, which is different from the usual structure, given by conjugating all coordinates.

Note that σ does not possess real points $(\sigma[z] = [z])$ as for any $z \in \mathbb{C}^4$ with $z \neq 0$ we know that z and jz are independent and so $[z]$ and $\sigma[z]$ are distinct points in $P_3(\mathbb{C})$. σ does possess real lines, for given $[z]$ the line through $[z]$ and $\sigma[z]$ is a real line. So the real lines are precisely the fibres of the bundle $\pi\colon P_3(\mathbb{C}) \longrightarrow P_1(\mathbb{H})$. Hence σ preserves the fibration. It will turn out that σ acts as an anti-podal mapping in each fibre.

Using stereographic projection from the north pole $(0,1)$ we obtain the mapping

$$(4.6) \qquad P_1(\mathbb{C})\setminus[0,1] \longrightarrow \mathbb{C} \cong \mathbb{R}^2 \longrightarrow S^2\setminus\{(0,1)\}$$

$$[z_1,z_2] \longmapsto \frac{z_2}{z_1} = \frac{\bar{z}_1 z_2}{|z_1|^2} \longmapsto \left(\frac{2\bar{z}_1 z_2}{|z_1|^2+|z_2|^2}, \frac{-|z_1|^2+|z_2|^2}{|z_1|^2+|z_2|^2}\right)$$

which can be extended to a diffeomorphism $P_1(\mathbb{C}) \xrightarrow{\cong} S^2$ and so that the mapping

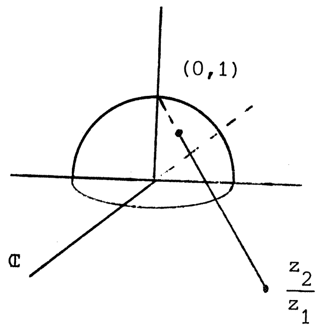
$$P_1(\mathbb{C}) \longrightarrow P_1(\mathbb{C})$$

$$[z_1,z_2] \longmapsto [-\bar{z}_2,\bar{z}_1]$$

corresponds to the mapping

$$S^2 \rightarrow S^2$$

$$a \longmapsto -a$$



Moreover we have $P_1(\mathbb{H}) \cong S^4$. Again using stereographic projection from the north pole $(0,1)$ we obtain an explicit diffeomorphism

$$(4.7) \qquad P_1(\mathbb{H})\setminus[0,1] \longrightarrow \mathbb{H} \cong S^2 \longrightarrow S^4\setminus\{(0,1)\}$$

given by

$$[z_1+z_2j,\ z_3+z_4j] \longmapsto (z_1+z_2j)^{-1}(z_3+z_4j) =$$

$$= \frac{\overline{(z_1+z_2j)}(z_3+z_4j)}{|z_1|^2+|z_2|^2} = \frac{\bar{z}_1z_3+z_2\bar{z}_4+(\bar{z}_1z_4+z_2\bar{z}_3)j}{|z_1|^2+|z_2|^2} \longmapsto$$

$$\longmapsto \left(\frac{2\{\bar{z}_1z_3+z_2\bar{z}_4+(\bar{z}_1z_4-z_2\bar{z}_3)j\}}{|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2}, \frac{-|z_1|^2-|z_2|^2+|z_3|^2+|z_4|^2}{|z_1|^2+|z_2|^2+|z_3|^2+|z_4|^2}\right)$$

which can be extended to a diffeomorphism $P_1(\mathbb{H}) \xrightarrow{\cong} S^4$. So the fibration (4.3) may be considered as a fibration

$$\pi\colon P_3(\mathbb{C}) \to S^4 .$$

In the literature it is well-known that the lines in $P_3(\mathbb{C})$ can be parametrized by so-called Plücker coordinates, also known as the *Klein representation* of lines in $P_3(\mathbb{C})$. Let $[z_1,z_2,z_3,z_4]$ and $[w_1,w_2,w_3,w_4]$ be distinct point in $P_3(\mathbb{C})$. The point $[p] \in P_5(\mathbb{C})$ with homogeneous coordinates

(4.8) $$p_{ij} = z_i w_j - w_i z_j, \qquad i < j$$

is uniquely defined by the line $\overline{zw}$ and satisfies the quadratic equation

(4.9) $$p_{12}p_{34} - p_{13}p_{24} + p_{14}p_{23} = 0.$$

Conversely any point $[p]$ satisfying this equation determines a line in $P_3(\mathbb{C})$.

By $\overline{zw} \longmapsto [p]$ we obtain the Plücker embedding of the Grassmannian $Gr_1(P_3(\mathbb{C}))$ consisting of all lines in $P_3(\mathbb{C})$ into $P_5(\mathbb{C})$, and so can be identified in this way with the non-singular quadric $Q \subset P_5(\mathbb{C})$ given by equation (4.9).

A real structure on $P_3(\mathbb{C})$ induces a real structure on Q. For the standard real structure on $P_3(\mathbb{C})$ the real structure on Q is given by conjugation of the coordinates p_{ij} and so equation (4.9) turns out to be a real equation for Q with respect to this standard real structure. This is a quadratic form of type (3,3) that means 3 plus and 3 minus signs in diagonal form.

For the real structure on $P_3(\mathbb{C})$ given by σ we obtain another real form of Q, a quadratic form of type (5,1). This we can see as follows. Note that σ has the following action on Plücker coordinates.

(4.10) $$p_{12} \mapsto \bar{p}_{12},\ p_{13} \mapsto \bar{p}_{24},\ p_{14} \mapsto -\bar{p}_{23},\ p_{23} \mapsto -\bar{p}_{14},$$
$$p_{24} \mapsto \bar{p}_{13},\ p_{34} \mapsto -\bar{p}_{34}.$$

So the real lines are the lines for which the following quantities are real in the usual sense

(4.11) $$X_1 = p_{12},\ X_2 = ip_{34},\ X_3 = p_{13} + p_{24},\ X_4 = i(p_{13}-p_{24}),$$
$$X_5 = i(p_{14}+p_{23}),\ X_6 = p_{14} - p_{23}.$$

With respect to this coordinates the equation for Q becomes

(4.12) $$4X_1X_2 + X_3^2 + X_4^2 + X_5^2 + X_6^2 = 0.$$

Defining

(4.13) $$Y_1 = X_1 - X_2,\ Y_2 = X_1 + X_2,\ Y_3 = X_3,\ Y_4 = X_4,\ Y_5 = X_5,\ Y_6 = X_6$$

yields the equation for Q

(4.14) $$Y_1^2 = Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 .$$

So the real points of Q are given in affine coordinates by the equation

(4.15) $$Y_2^2 + Y_3^2 + Y_4^2 + Y_5^2 + Y_6^2 = 1,$$

and represents S^4 for $Y_1 \neq 0$ when $Y \in Q$.

Finally it is worth to compare the Klein representation with the splitting (3.4) of the projective twistor space $\mathbb{P}$ into the three parts $\mathbb{P}^+, \mathbb{P}^0, \mathbb{P}^-$. Putting $Z = (\omega^A, \pi_{A'}) = (z_1, z_2, z_3, z_4) = z$ we can write the Hermitian form from (3.2) into the form

(4.16) $$\Phi(z) = z_3\bar{z}_1 + z_4\bar{z}_2 + \bar{z}_3 z_1 + \bar{z}_4 z_2 = p_{14}(z) - p_{23}(z) = Y_6,$$

here $p_{ij}(z)$ are the Plücker coordinates of the line $\overline{z, \sigma z}$. So choosing

$Y_6 = 1$ as north pole we have

$$\mathbb{P}^+ = \pi^{-1} \quad \text{(northern hemisphere)}$$

$$\mathbb{P}^0 = \pi^{-1} \quad \text{(equator)}$$

$$\mathbb{P}^- = \pi^{-1} \quad \text{(southern hemisphere)}.$$

§5. TWISTOR INTERPRETATION OF INSTANTONS, THEOREM OF ATIYAH-WARD

In this section we shall translate the selfdual Yang-Mills equations into complex analytic data on the space $P_3(\mathbb{C})$.

To this end we have to describe the equation $*\alpha = \pm\alpha$ for 2-forms α on $\mathbb{R}^4$ with respect to complex coordinates.

If we identify $\mathbb{R}^4$ with $\mathbb{C}^2$ by some complex structure J then we can write for 2-forms on $\mathbb{R}^4$

$$A^2 = A_+^2 \oplus A_-^2 ,$$

here A_+ and A_- are selfdual and anti-selfdual forms on $\mathbb{R}^4$ respectively. On the other hand

$$A^2 = A^{(2,0)} \oplus A^{(1,1)} \oplus A^{(0,2)},$$

where $A^{(2,0)}$, $A^{(1,1)}$, $A^{(0,2)}$ are forms of type $(2,0)$, $(1,1)$ and $(0,2)$. If $x = (x_1,x_2,x_3,x_4)$ are the coordinates with respect to the basis (e_1,Je_1,e_2,Je_2), oriented and orthonormally chosen, then A_+^2 and A_-^2 are spanned respectively by

$$A_+^2 \begin{cases} \alpha_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 \\ \alpha_2 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4 \\ \alpha_3 = dx_2 \wedge dx_3 + dx_1 \wedge dx_4 \end{cases} \qquad A_-^2 \begin{cases} \alpha_4 = dx_1 \wedge dx_2 - dx_3 \wedge dx_4 \\ \alpha_5 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4 \\ \alpha_6 = dx_2 \wedge dx_3 - dx_1 \wedge dx_4 \end{cases},$$

A_+^2 and A_-^2 are both 3-dimensional. On the other hand $A^{(2,0)}$ is spanned by $dz_1 \wedge dz_2$, $A^{(0,2)}$ by $d\bar{z}_1 \wedge d\bar{z}_2$ and $A^{(1,1)}$ by $dz_1 \wedge d\bar{z}_1$, $d\bar{z}_1 \wedge d\bar{z}_2$, $dz_2 \wedge d\bar{z}_1$, $dz_2 \wedge d\bar{z}_2$. So both $A^{(2,0)}$ and $A^{(0,2)}$ are 1-dimensional and $A^{(1,1)}$ is 4-dimensional.

It is easy to see that

$$\begin{cases} \beta_1 = dz_1 \wedge dz_2 \in A^2_+ \\ \beta_2 = d\bar{z}_1 \wedge d\bar{z}_2 \in A^2_+ \\ \beta_3 = dz_1 \wedge d\bar{z}_2 + dz_2 \wedge d\bar{z}_2 = -2i\alpha_1 \in A^2_+ \end{cases}$$

$$\begin{cases} dz_1 \wedge d\bar{z}_2 \in A^2_- \\ dz_2 \wedge d\bar{z}_1 \in A^2_- \\ dz_1 \wedge d\bar{z}_1 - dz_2 \wedge d\bar{z}_2 \in A^2_- \end{cases}$$

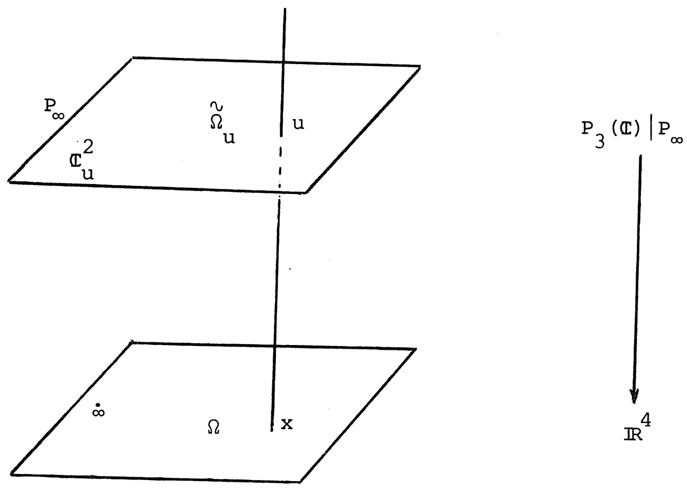
From this we may conclude that: $\alpha \in A^2_- \Rightarrow \alpha \in A^{(1,1)}$. So A^2_- is contained in the intersection $V = \bigcap_J A_J^{(1,1)}$, consisting of 2-forms of type (1,1) with respect to any complex structure (compatible with metric and orientation). Now V is invariant under the action of $SO(4)$, contains A^2_-, but is smaller than any fixed $A_J^{(1,1)}$ because e.g. $\beta_3 = -2i\alpha_1$ can be transformed into $-2i\alpha_3 \notin A^{(1,1)}$ by a $SO(4)$ - transformation. We conclude

$$V = \bigcap_J A_J^{(1,1)}.$$

Thus we have the following result.

<u>LEMMA 1</u>. *A 2-form* α *on* $\mathbb{R}^4$ *is anti-dual if and only if* α *is of type* (1,1) *with respect to each complex structure.*

We will study once more the fibration $\pi: P_3(\mathbb{C}) \to S^4$ from the Penrose's point of view. For simplicity we shall work on $\mathbb{R}^4 \subset S^4$ and so we have removed a point (the point at infinity) from S^4 and at the same time the projective line in $P_3(\mathbb{C})$ above this point at infinity. In this way we obtain a fibration $P_3(\mathbb{C})\backslash P_\infty \to \mathbb{R}^4$, where P_∞ denotes the fibre above ∞. Intersecting projective planes in $P_3(\mathbb{C})$ which intersect according to the line P_∞ now become parallel in $P_3(\mathbb{C})\backslash P_\infty$. We obtain the following picture of our fibration



Above the given point $x \in \mathbb{R}^4$ we can imagine the fibre $P_x \cong S^2$ to be parametrized by u. The plane through P_∞ and a point $u \in P_x$ can be considered to be a copy $\mathbb{C}^2_u$ of $\mathbb{C}^2$ and is identified by the projection with $\mathbb{R}^4$. In this way $\mathbb{R}^4$ is provided with a complex structure depending on u. As the identification of $\mathbb{R}^4$ varies with u, the complex structure which is induced on $\mathbb{R}^4$ by the identification with the corresponding $\mathbb{C}^2_u$ is changed.

We use this point of view of the fibration $\pi\colon P_3(\mathbb{C}) \to S^4$ to describe a 2-form which we have obtained by lifting of an anti-selfdual form on S^4.

Let Ω be an anti-selfdual 2-form on S^4. The lifted form $\tilde{\Omega} = \pi^*\Omega$ of the form Ω then is a 2-form on $P_3(\mathbb{C})$. Now we consider $\tilde{\Omega}$ in a fixed point u above $x \neq \infty$. $\tilde{\Omega}_u$ is pure horizontal, that means, $\tilde{\Omega}_u = 0$ when $\tilde{\Omega}_u$ is applied to two tangent vectors in u of which at least one is tangent to the fibre P_x.

Next we have a look at $\tilde{\Omega}$ transversal to the fibre in u, for instance by studying $\tilde{\Omega}$ on $\mathbb{C}^2_u$ the form $\tilde{\Omega}$ is the lift of a form $\Omega|_{\mathbb{R}^4}$ which is of type $(1,1)$ with respect to any complex structure on $\mathbb{R}^4$. Therefore also with respect to the complex structures which are induced on $\mathbb{R}^4$ by identifying $\mathbb{R}^4$ with $\mathbb{C}^2_u$. Hence $\tilde{\Omega}$ is on $\mathbb{C}^2_u$ of type $(1,1)$ too. As $\tilde{\Omega}$ vanishes along the fibre it follows that $\tilde{\Omega}$ is completely of type $(1,1)$ in u.

Conversely, it can be proved by some topological arguments that all complex structures are passed through and are parametrized by
$u \in P_x \cong S^2 \cong SO(4)/U(2)$, see Atiyah [1], ch.III, §3. Now we have the following result

LEMMA 2. *A form* Ω *on* S^4 *is anti-selfdual if and only if its lift* $\tilde{\Omega}$ *to* $P_3(\mathbb{C})$ *is of type* $(1,1)$.

Now we consider a complex vector bundle $p\colon E \to S^4$ provided with a Hermitian structure $\langle\, ,\, \rangle$ and a compatible connection ∇. If we lift the bundle E over S^4 by $\pi\colon P_3(\mathbb{C}) \to S^4$ then we obtain a bundle $\tilde{E} = \pi^* E$ over $P_3(\mathbb{C})$. The Hermitian structure, the connection and the corresponding curvature θ can be lifted also. We will describe this in somewhat more details. We have $\tilde{E}_z = E_{\pi z}$.

$$\begin{array}{ccc} E & & \tilde{E} \\ \downarrow p & & \downarrow \tilde{p} \\ S^4 & \xleftarrow{\ \pi\ } & P_3(\mathbb{C}) \end{array}$$

If $f = (e_1,\dots,e_r)$ is a frame for E over U, then $\tilde{f} = f \circ \pi = (e_1 \circ \pi,\dots,e_r \circ \pi)$ is a frame for $\tilde{E}$ over $\pi^{-1}(U)$ and the frames $\tilde{f}$ cover $P_3(\mathbb{C})$.

If g is a gauge transformation in $\tilde{E}$ over U then $\tilde{g} = g \circ \pi$ is a gauge transformation in $\tilde{E}$ over $\pi^{-1}(U)$.

The Hermitian structure in E is defined by

$$\langle\, ,\, \rangle_{\pi z} = \langle\, ,\, \rangle_z , \quad \text{e.g.} \quad \tilde{h}(\tilde{f}) = h(f) \circ \pi .$$

If $\theta(f)$ is the connection matrix of ∇ with respect to the frame

$f = (e_1, \dots, e_r)$, e.g. $\nabla e_i = \theta_{ji}(f) e_j$, then we set

$$\tilde{\theta}(\tilde{f}) = \pi^* \theta(f), \quad \text{e.g.} \quad \tilde{\nabla} \tilde{e}_i = \tilde{\theta}_{ji}(\tilde{f}) \tilde{e}_j .$$

Moreover it is easy to see that

$$\tilde{\theta}(\tilde{f}\, \tilde{g}) = \tilde{g}^{-1} \tilde{\theta}(\tilde{f}) \tilde{g} + \tilde{g}^{-1} d\tilde{g} .$$

So $\tilde{\theta}$ transforms by gauge transformations such that $\tilde{\theta}$ defines a global connection $\tilde{\nabla}$ on $\tilde{E}$. For the curvature matrix $\Theta(\tilde{f})$ we have

$$\tilde{\Theta}(\tilde{f}) = d\,\tilde{\theta}(\tilde{f}) + \tilde{\theta}(\tilde{f}) \wedge \tilde{\theta}(\tilde{f}) = d\pi^* \theta(f) + \pi^* \theta(f) \wedge \pi^* \theta(f) =$$

$$= \pi^* (d\theta(f) + \theta(f) \wedge \theta(f)) = \pi^* \Theta(f).$$

Furthermore, the connection $\tilde{\nabla}$ is compatible with the Hermitian structure on $\tilde{E}$. Indeed,

$$d\tilde{h}(\tilde{f}) = dh(f) \circ \pi = \pi^* dh(f) = \pi^* (h(f) \theta(f) + {}^t\bar{\theta}(f) h(f)) =$$

$$= \tilde{h}(\tilde{f}) \tilde{\theta}(\tilde{f}) + {}^t\bar{\tilde{\theta}}(\tilde{f}) \tilde{h}(\tilde{f}).$$

To study the bundle $\tilde{p}: \tilde{E} \to P_3(\mathbb{C})$ in more detail we need the following theorem.

<u>THEOREM 1</u>. *Let* E *be a* C^∞ *complex vector bundle with Hermitian structure over a complex manifold* X. *Suppose that on* E *is given a connection, which is compatible with the Hermitian structure and with the curvature* Θ *of type* $(1,1)$. *Then there is a natural unique holomorphic structure on* E *such that* ∇ *is compatible with this holomorphic structure.*

The proof is based on the Newlander-Nirenberg theorem concerning the integrability of almost complex structures and is described in Atiyah-Hitchin-Singer [2], theorem 5.1.

Now suppose that the curvature of the connection ∇ in the bundle $p: E \to S^4$

is anti-selfdual, then it follows from lemma 2 that the lift $\tilde{\theta} = \pi^*\theta$ of this curvature is of type (1,1). $\tilde{\theta}$ is the curvature of the connection $\tilde{\nabla}$ in the bundle $\tilde{E}$ which is compatible with the Hermitian structure. In combination with theorem 1 this implies the following important result.

THEOREM 2. *If the connection* ∇ *in the bundle* $p: E \to S^4$ *is anti-selfdual, then the bundle* $\tilde{p}: \tilde{E} \to P_3(\mathbb{C})$ *is holomorphic.*

From the construction of $\tilde{E}$ it follows that the restriction $\tilde{E}|P_x$ of $\tilde{E}$ to each fibre P_x, a real line, of the bundle $P_3(\mathbb{C}) \to S^4$ is trivial.

Moreover, it is clear that $\tilde{E}|P_x$ has Hermitian trivial structure too. Liouville's theorem from complex analysis implies that $\tilde{E}|P_x$ is holomorphic trivial also. Note that $P_x \cong S^2$, the Riemann-sphere.

Conversely, it can be shown that if $\tilde{E} \to P_3(\mathbb{C})$ is a holomorphic vector bundle which is trivial when restricted to the fibres of $\pi: P_3(\mathbb{C}) \to S^4$ then the bundle can be reconstructed by taking for E_x the space of all holomorphic sections $\Gamma(P_x \to \tilde{E}|P_x) \cong \mathbb{C}^r$. Given a Hermitian structure on $\tilde{E}$ which is trivial at the fibres we can reconstruct the Hermitian structure on E as well.

Now a holomorphic vector bundle provided with a Hermitian structure possess a unique connection $\tilde{\nabla}$ which is compatible with both the metric as well as the holomorphic structure. The curvature of the connection $\tilde{\nabla}$ is of type (1,1).

It can be shown by techniques from complex analysis that this connection comes from a connection ∇ in E with anti-selfdual curvature.

Now we suppose that the bundle $E \to S^4$ is of rank 2 and is provided with a $SU(2)$-*structure*, e.g. we suppose that gauge transformations are $SU(2)$-valued. Furthermore, we suppose that frames are unitary (orthonormal) and the connection a $SU(2)$-connection. These additional structure of the bundle $E \to S^4$ we have to translate too. The $SU(2)$ - structure on the bundle $E \to S^4$ enables us to define invariantly an anti-linear mapping $\sigma \colon E \to E$ by taking for σ the mapping $\sigma^0 \colon \mathbb{C}^2 \to \mathbb{C}^2$, $\sigma^0(w_1,w_2) = (-\bar{w}_2,\bar{w}_1)$ in some gauge. Note that a $g \in GL(2,\mathbb{C})$ commutes with σ^0 precisely when $g \in SU(2)$. There holds: $< \sigma s, \sigma t > = \overline{< s,t >}$, $\sigma^2 = -1$. Conversely, given the mapping $\sigma \colon E \to E$ we can recover the $SU(2)$ - structure on E as follows.

A gauge ϕ corresponds to the $SU(2)$ - structure precisely when σ equals σ^0 in the gauge ϕ, e,g, $\phi\sigma^{-1} = \sigma^0$. So given the mapping $\sigma \colon E \to E$ we recover the $SU(2)$ - structure in the form of all gauges ϕ with the property that σ equals σ^0 in the gauge ϕ.

Now we can lift the mapping σ to a mapping $\tilde{\sigma} \colon \tilde{E} \to \tilde{E}$ such that the following diagram commutes

$$\begin{array}{ccc} \tilde{E} & \xrightarrow{\tilde{\sigma}} & \tilde{E} \\ \downarrow & & \downarrow \\ P_3(\mathbb{C}) & \xrightarrow{\sigma} & P_3(\mathbb{C}) \end{array} \qquad \begin{array}{ccc} E_x & \xrightarrow{\sigma_x} & E_x \\ \downarrow & & \downarrow \\ \tilde{E}_z & \xrightarrow{\tilde{\sigma}_x} & \tilde{E}_z \\ \downarrow & & \downarrow \\ z & \longrightarrow & \sigma z \end{array} \qquad x = \pi z$$

Note that $\tilde{\sigma}$ is anti-linear but depends in a holomorphic way on $z \in P_3(\mathbb{C})$. By putting $(u,v) = < u, -\tilde{\sigma} v >$ we define a (holomorphic) non-degenerated skew-symmetric Hermitian form on E with $(\tilde{\sigma} u, \tilde{\sigma} v) = \overline{(u,v)}$. Furthermore the Hermitian form $< u,v > = < u,\tilde{\sigma} v >$ is positive definite. Conversely, suppose that $\tilde{E}$, such a form $(\, , \,)_{\tilde{E}}$ and an anti-linear mapping $\tilde{\sigma}$ is given. Then these induce a form $(\, , \,)_E$ on $E_x = \Gamma(P_x \to E|P_x) \cong \mathbb{C}^2$ and an anti-linear mapping σ with the same properties.

We take on E the Hermitian structure defined by $\langle s,t \rangle = (s,\sigma t)_E$.

A gauge ϕ corresponds to the $SU(2)$-structure we are looking for precisely when σ equals σ^0 in this gauge, e.g. $\phi\sigma\phi^{-1} = \sigma^0$.

A gauge transformation with respect to two of these gauges is $SU(2)$-valued.

It is easy to see that the domains of these cover S^4. Indeed, without restriction we may assume that E is a $U(2)$-bundle. For gauges can be changed in such a way that frames become unitary and consequently gauge transformations become $U(2)$-valued.

Next change the gauges by unitary transformations such that σ obtains the canonical form σ^0.

For instance this can be achieved by the transformation $e_1 \longmapsto e_1$, $e_2 \longmapsto \sigma e_1$, $f = (e_1,e_2)$. As

$$\langle \sigma e_1, \sigma e_2 \rangle = \langle \overline{e_1, e_2} \rangle = 1$$

and

$$\langle \sigma e_1, e_1 \rangle = (\sigma e_1, \sigma e_1) = 0$$

this indeed yields a unitary transformation.

<u>SUMMARIZING</u>: An anti-selfdual $SU(2)$-connection on S^4 corresponds with a holomorphic vector bundle $\tilde{E} \to P_3(\mathbb{C})$ of rank 2 with

i) $\tilde{E}|P_x$ is trivial for each $x \in S^4$

ii) $\tilde{E}$ possess a holomorphic anti-linear mapping $\sigma\colon \tilde{E} \to \tilde{E}$ lifting $\sigma\colon P_3(\mathbb{C}) \to P_3(\mathbb{C})$ and with $\tilde{\sigma}^2 = -1$.

iii) A holomorphic non-degenerate skew-symmetric form $(\ ,\ )$ with $(\tilde{\sigma}u,\tilde{\sigma}v) = (\overline{u,v})$ and such that the Hermitian form $\langle u,v \rangle = (u,\tilde{\sigma}v)$ is positive definite.

It can be shown that the Chern-class $c_1(\tilde{E}) = 0$. From this it can be derived that the non-degenerated skew form exists and is uniquely determined up to a constant factor. Moreover, it can be shown that the bundle does not possess any other holomorphic automorphisms than scalars. This implies that the anti-linear mapping $\tilde{\sigma}: \tilde{E} \to \tilde{E}$ is unique. Indeed, if $\tilde{\sigma}_1$, and $\tilde{\sigma}_2$ are two such mappings then $\tilde{\sigma}_1^{-1}\tilde{\sigma}_2$ is a holomorphic automorphism of $\tilde{E}$. Thus $\tilde{\sigma}_2 = c\tilde{\sigma}_1$, for some c.

The condition $\tilde{\sigma}^2 = -1$ now yields $c\bar{c} = 1$ and finally the condition that the Hermitian form $\langle u,v \rangle = (u,\tilde{\sigma}v)$ is positive determines $\tilde{\sigma}$ uniquely (with chosen $(\ , \ )$).

So we have given a sketch of the proof of the following theorem.

THEOREM 3. (Atiyah-Ward). *There exists a bijective correspondence between*

I) *anti-selfdual* SU(2) - *Yang-Mills fields on* S^4

II) *holomorphic vector bundles* $\tilde{E}$ *of rank* 2 *over* $P_3(\mathbb{C})$ *with*

i) $\tilde{E}|P_x$ *is trivial for each real line* P_x, $x \in S^4$

ii) $\tilde{E}$ *possess a holomorphic anti-linear mapping* $\tilde{\sigma}: E \to E$ *lifting* $\sigma: P_3(\mathbb{C}) \to P_3(\mathbb{C})$ *and with* $\tilde{\sigma}^2 = -1$.

REMARKS. 1) There are similar theorems for other types of Yang-Mills fields. For instance SU(n) or Sp(n). Part IIii of the formulation, reflecting the translation of the field type is different. The statement in IIi that $\tilde{E}$ is holomorphic is not changed.

2) In the literature the mapping $\tilde{\sigma}$ in theorem 3, IIii is sometimes called a *symplectic* or a *real structure* on E.

§6. HOLOMORPHIC VECTOR BUNDLES OVER $P_3(\mathbb{C})$

The problem to identify the rank 2 vector bundles has attracted many people. We mention Horrocks, Barth, Maruyama, Hartshorne.

There are two invariants, the Chern-classes c_1 and c_2 which can take any integer under the restriction that the product c_1c_2 is even. For fixed c_1, c_2 there are infinitely many families, depending on parameter spaces (moduli) with increasing complexity. In particular, they exhibit a complicated topological behaviour. For instance they can be not separated. In order to get improvement Mumford introduced the notion "*stable vector bundle*". Given the numbers c_1, c_2 the stable bundles are parametrized by varieties which are called *moduli spaces*.

It turns out that the bundles emerging in instanton problems are all stable and so have nice moduli spaces. Anyhow, the structure of these moduli spaces remains rather complicated.

An important method to study bundles over $P_3(\mathbb{C})$ is the method using *monads*. The word monad was introduced by Horrocks(1964) to denote a sequence

(1) $$0 \longrightarrow F' \xrightarrow{\alpha} F \xrightarrow{\beta} F'' \longrightarrow 0$$

of vector bundles and bundle morphisms with $\beta\alpha = 0$, α is injective and β is surjective. The bundle $\operatorname{Ker}\beta/\operatorname{Im}\alpha$ is called the *cohomology* of the monad.

Horrocks proved: *Each vector bundle over* $P_3(\mathbb{C})$ *can be obtained as the cohomology of a monad, where each bundle* F', F, F'' *is a direct sum of line bundles.*

So a monad is some kind of a "two-sided development" of the bundle E. Although a monad is more complicated than a simple bundle E, it can be explicitly described more easier because the individual bundles F', F, F'' are rather simple and because the mappings α, β can be represented by matrices. So in principle the use of monads reduce the study of vector bundles to linear algebra.

Barth proved: *Stable bundles* E *of rank* 2 *over* $P_3(\mathbb{C})$ *with* $c_1 = 0$ *and* $c_2 = k$ *under the extra assumption* $H^1(P_3(\mathbb{C}), E(-2)) = 0$ *can be described by monads of the following special type*

(2) $$0 \longrightarrow \mathcal{O}(-1)^k \xrightarrow{\alpha} \mathcal{O}^{2k+2} \xrightarrow{\beta} \mathcal{O}(1)^k \longrightarrow 0.$$

Here $\mathcal{O}(n)$ is the line bundle with cocycle $g_{ij}[z] = \left(\frac{z_j}{z_i}\right)^n$, and $E(n) = E \otimes \mathcal{O}(n)$.

A bundle morphism $\mathcal{O}(n)^k \to \mathcal{O}(m)^1$ can be identified with a $k\times 1$ - matrix consisting of homogeneous polynomials of degree $m-n$.

So to describe such a monad we only have to specify the mappings α and β, $(2k+2) \times k$ and $k \times (2k+2)$ matrices of linear forms in the homogeneous coordinates (z_1,z_2,z_3,z_4) of $P_3(\mathbb{C})$. The matrices α and β have to satisfy the conditions: $\operatorname{rank}\alpha = \operatorname{rank}\beta$ in each point $z \neq 0$ and $\beta\alpha = 0$.

Barth proved furthermore that this monad is selfdual in the sense that β equals the transpose of α with respect to the alternating bilinear from J on $\mathcal{O}^{2k+2}$.

Finally the condition $H^1(P_3(\mathbb{C}), E(-2)) = 0$ is explained by Atiyah-Hitchin-Drinfeld-Manin [3]. They proved,

> $H^1(P_3(\mathbb{C}), \tilde{E}(m)) = 0$ $(m \leq 2)$ *for all bundles* $\tilde{E}$ *which appear in instanton problems* (described in the preceding section).

The proof is based on the fact that the linear differential equation $(\Delta+\frac{R}{6})u = 0$ can have no global nontrivial solution on S^4. $R > 0$ is the scalar curvature of S^4 and $\Delta \geq 0$.

Conclusion: *Every (anti)-instanton bundle can be described by a monad of type* (2).

Drinfeld and Manin [4] showed that for vector bundles of rank 2 the condition $H^1(P_3(\mathbb{C}), E(-2))$ is equivalent with the possibility to describe E by special monads of the form

$$0 \longrightarrow A \otimes \mathcal{O}(-1) \xrightarrow{\alpha} B \otimes \mathcal{O} \xrightarrow{\beta} C \otimes \mathcal{O}(1) \longrightarrow 0 ,$$

where A, B and C are complex vector spaces.

As we have seen a SU(2) - bundle $\tilde{E} \to P_3(\mathbb{C})$ is provided with the extra real structure $\tilde{\sigma}$, which has to be translated in terms of the monad.

RESULT: Every SU(2)-(anti)-instanton bundle with instanton number k corresponds to a stable vector bundle over $P_3(\mathbb{C})$ of rank 2, which can be represented by a monad of type (2). This monad is selfdual and the structure $\tilde{\sigma}$ can be carried over to the monad.

To describe the monad we only have to specify the mapping α and the conditions the monad has to satisfy.

The mapping α is given by a $(2k+2) \times k$ - matrix $A(z)$ of linear forms in the homogeneous coordinates $(z_1,z_2,z_3,z_4) = z$ and has to satisfy the conditions

(4.i) $\quad {}^tA(z)\,J\,A(z) = 0, \qquad \forall z$

(4.ii) $\quad \operatorname{rank} A(z) = k, \qquad \forall z \neq 0$

(4.iii) $\quad J\,\overline{A(z)} = A(\sigma z),$

here J is the $(2k+2) \times (2k+2)$ skew-symmetric matrix $\left(\begin{array}{c|c} 0 & I \\ \hline -I & 0 \end{array}\right)$ and $\sigma(z_1,z_2,z_3,z_4) = (-\bar{z}_2,\bar{z}_1,-\bar{z}_4,\bar{z}_3)$.

Condition (4.i) reflects the relation $\beta\alpha = 0$, where β equals the transpose of α with respect to the selfduality of the monad.

(4.ii) is the condition about the rank which implies that α is injective and β surjective

(4.iii) is the translation of the real structure $\tilde{\sigma}$.

The bundle $\mathrm{Ker}\ {}^tAJ/_{\mathrm{Im}\,A} \to P_3(\mathbb{C})$ satisfies the hypothesis of the theorem of Atiyah-Ward if (4.i-4.iii) are satisfied.

There is some arbitrariness in choosing a basis for the bundles left, right and in the middle of the monad. So two matrices $A(z)$ and $A'(z)$ determine the same (anti)-instanton if and only if

$$A'(z) = S \cdot A(z) \cdot T$$

with $\quad S \in Sp(k+1)$ and $T \in GL(k,\mathbb{R})$.

So the moduli spaces of (anti)-instantonen can be obtained as the space of all possible matrices $A(z)$ up to the action of the group $Sp(k+1) \times GL(k,\mathbb{R})$.

Explicit calculations of moduli spaces have been carried out in the literature. Despite the fact that the matrix formulation is said to provide "the complete solution" there remain stil many questions to be answered.

Finally to elucidate the last we give the Horrocks' construction as described by Atiyah-Hitchin-Drinfeld-Manin [3]. The construction starts with a complex linear mapping $A(z): W \to V$ with $\dim W = k$, $\dim V = 2k+2$, $z = (z_1,z_2,z_3,z_4)$ and $A(z)$ is linear in z, e.g.

$$A(z) = \sum_{i=1}^{4} A_i z_i, \quad A_i: W \to V \text{ constant linear mappings.}$$

We suppose that V is provided with a non-degenerated skew-symmetric form $(\, , \,)$.

For any subspace $U \subset V$ the annihilator of U is denoted by U^0, e.g. U^0 consists of all elements $v \in V$ which satisfy: $(u,v) = 0$ for all $u \in U$. We suppose that $A(z)$ satisfies the following condition.

(5) For all $z \neq 0$ the space $U_z := A(z)W$ has dimension k and is isotropic, e.g. $U_z \subset U_z^0$.

If condition (5) is satisfied we define $E_z = U_z^0/U_z$.

As $\dim U_z = k$, $\dim U_z + \dim U_z^0 = 2k+2$ we know that $\dim U_z^0 = k+2$ and $\dim E_z = 2$.

Furthermore E inherits a non-degenerated skew-symmetric form and there holds $E_z = E_{\lambda z}$ for all $\lambda \neq 0$.

So the family E_z defines a complex vector bundle over $P_3(\mathbb{C})$ with structure group $SL(2,\mathbb{C})$.

Note that when $k = 1$, U_z is 1-dimensional and so automatically isotropic For $k > 1$ the condition on U to be isotropic is described in matrix from by

(6) $$^tA(z) J A(z) = 0 ,$$

where $^tA(z)$ is the transpose of $A(z)$ and J the matrix of the skew-symmetric form (,). (6) represents a quadratic equation in the coefficients of the four matrices A_1, A_2, A_3, A_4.

We have

$$0 \longrightarrow V \xrightarrow{A} W \xrightarrow{B} N \longrightarrow 0$$

with $B(= {}^tAJ)$ equals the transpose of A with respect to the form (,) and $N = \operatorname{Im} B$. Moreover $\operatorname{Ker} B = U_z^0$ and so $\operatorname{Ker} B(z)/_{\operatorname{Im} A(z)} = U_z^0/U_z = E_z$.

To obtain a bundle corresponding with a SU(2)-(anti)-instanton we have to introduce a real structure.

We fix an anti-linear mapping σ on W with $\sigma^2 = 1$, on V with $\sigma^2 = -1$ and the well-known σ on $\mathbb{C}^4$ defined by

$$\sigma(z_1,z_2,z_3,z_4) = (-\bar{z}_2,\bar{z}_1,\bar{z}_4,\bar{z}_3).$$

On V we assume that σ is compatible with the skew-symmetric form, e.g. $(\sigma v_1,\sigma v_2) = \overline{(v_1,v_2)}$. In this way on V a Hermitian structure is defined by

$< v_1, v_2 > = (v_1, -\sigma v_2)$, which we assume to be positive definite.

The reality condition on $A(z)$ is the compatibility with σ, e.g.

(7) $$\sigma[A(z)w] = A(\sigma z)\sigma w.$$

The definition of the Hermitian form implies that for the orthogonal complement $U^{\perp}$ of $U \subset V$ we have:

(8) $$U^{\perp} = (\sigma U)^0$$

and from (7) it follows that

(9) $$\sigma U_z = U_{\sigma z}.$$

So $U_z^0 = U_{\sigma z}^{\perp}$ and hence by the positivity of the Hermitian form we know $U_z^0 \cap U_{\sigma z} = 0$. So we have the decomposition

(10) $$V = U_z \oplus R_x \oplus U_{\sigma z},$$

where $R_z = U_z^0 \cap U_{\sigma z}$ depends only on the point $x \in S^4$, parametrizing the line $\overline{z, \sigma z}$ in $P_3(\mathbb{C})$. From this it follows that E inherits a real structure with the properties described in the theorem of Atiyah-Ward.

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