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PREFACE

These proceedings cover part of the lectures given in the seminar 'Mathematical Structures in Field Theories', held at the University of Amsterdam during the academic year 1984-1985 and 1985-1986.

The first contribution by P.J.M. Bongaarts treats the 'axiomatics' free quantum fields, including Wightman functions and the Fock space formalism. The emphasis is on the underlying mathematical aspects, this has the advantage of leading to a unified picture of free fields.

The second part by E.A. de Kerf is more or less based on Bongaarts' lectures. In contrast to the latter the emphasis lies on the physical aspects. The starting point is the 'particle picture', based on the unitary representations of the inhomogeneous Poincaré group.

The third contribution by P.H.M. Kersten studies infinitesimal symmetries using the local jet bundle formulation from differential geometry. As an interesting application the SU(2) instanton and magnetic monopole are obtained.

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The editors
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The Mathematical Structure of Free Quantum Fields
Gaussian Systems

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0. SUMMARY

Quantum field theory is the theoretical framework of present day elementary particle physics. It is a general theory of great power, but riddled with deep mathematical problems. The theory of free fields is an elementary part of it, mathematically well understood and of rather trivial physical content. Free fields are nevertheless important as a starting point and as a constant frame of reference for all further nontrivial developments.

Our exposition in this paper, aimed in the first place at readers with an interest in mathematical aspects, will try to give a unified picture of free fields, stressing more than is usual the essentially simple mathematical structure underlying the variety of different cases. In doing so it will leave on one hand the explicit treatment of the various particular cases to the detailed but rather heuristic discussions in standard text books on quantum field theory, and on the other hand the technical details necessary for a completely rigorous formulation to the few specialized books devoted to this subject.

Sections 1 and 2 give a general but brief discussion of the main features of quantum field theory, with emphasis on the characterization of quantum fields by means of systems of n-point functions, the non-commutative analogous of correlation functions for ordinary stochastic systems. Using this language section 3 describes free fields as "Gaussian" systems. The general construction of free fields is given. As an illustration of this, the example of the free scalar field is worked out in detail in section 4. The connection with the heuristic formulations of standard text books is explained. In the construction of free fields, as operator theories, from given systems of n-point functions, a key rôle is played by the Fock many-particle formalism. This is of some mathematical interest in its own right, and has also applications outside quantum field theory. It is therefore discussed separately in section 5, an appendix.

1. INTRODUCTION

It is generally accepted at present, that physical phenomena on the level below that of atoms and nuclei are governed by *quantum field theories*.

In the formulation discussed here the basic objects in quantum field theory are *field operators*, i.e. operator valued functions of space-time. There are scalar fields $\phi(x)$, with $x = (\vec{x},t)$, vector fields $A_\mu(x)$, with $\mu = 0,1,2,3$, spinor fields $\psi_\alpha(x)$, with $\alpha = 1,2,3,4$, tensor fields $F_{\mu\nu}(x)$, etc.

Remember the mathematical framework of quantum theory in general:

1. A Hilbert space H as space of states vectors.
2. Operators in H as physical observables.
3. A unitary group $U(t) = e^{-itH}$ for the time evolution. $U(t)$ acts either on vectors as $\psi(t) = e^{-itH}\psi$ (Schroedinger picture), or on operators as $A(t) = e^{itH} A e^{-itH}$ (Heisenberg picture). The choice between these two possibilities is a matter of mathematical convenience, not of physical substance, because physical results are given in terms of inner product expressions $(\psi,A\psi)$.

In quantum field theory one prefers the Heisenberg picture. Field operators such as $\phi(x)$ are Heisenberg picture operators.

$$\phi(x) = \phi(\vec{x},t) = e^{itH}\,\phi(\vec{x},0)e^{-itH} \tag{1}$$

Different quantum field theory models are used for different classes of phenomena. The ultimate goal is a single "unified" theory for all phenomena. A particular quantum field theory is characterized by a choice of one or more fields, from such types as scalar, vector, etc., together with appropriate field equations, leading to a time evolution in the sense of (1).

The simplest example is the *free scalar field*, a field $\phi(x)$, satisfying the Klein-Gordon equation

$$(\Box+m^2)\phi(x) = 0 \tag{2}$$

with $\Box = \frac{\partial^2}{\partial t^2} - \sum_{j=1}^{3} \frac{\partial^2}{\partial x_j^2} = \partial^\mu \partial_\mu$, m a nonnegative parameter. Less simple is the so-called ϕ^4-model, a scalar field with nonlinear field equation

$$(\Box+m^2)\phi(x) = -\lambda\phi(x)^3 \tag{3}$$

in which the parameter λ, the coupling constant, measures the deviation from the free case (2). Note that in general the field equation are, as equations for "classical" i.e. number valued equations, the Euler-Lagrange equations of a variational problem, characterized by a so-called Lagrangian function. For the ϕ^4-model this function contains a $\phi(x)^4$ term.

The direct physical significance of scalar fields is small. Due to their relative simplicity they are, however, of theoretical importance. Models such as the ϕ^4-model are mathematically still far from trivial and have been the object of extensive investigation in the last few years.

Of more immediate physical importance and with its predictions confirmed by experiments of great precision is *quantum electrodynamics*. It has vector and scalar fields A_μ and ψ_α, and a set of coupled equations which are usually written in a shorthand notation, that need not be explained here, as

$$\begin{aligned} \partial^\mu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu &= e\tilde{\psi}\gamma_\nu\psi \\ (i\gamma^\mu\partial_\mu - m)\psi &= e\gamma^\mu A_\mu \psi \end{aligned} \tag{4}$$

Generalization of ideas inherent in quantum electrodynamics has led to *quantum gauge field theories*. Such theories have come to dominate sub-nuclear physics. As mathematical models, however, they are very incompletely understood sofar.

Quantum mechanics can be obtained from classical mechanics by quantization, a procedure in which number valued quantities are replaced by Hilbert

space operators. Such quantization procedures are of general usefulness in setting up explicit examples of quantum theories, although the classical theories involved may not longer be valid physical theories in their own right, but only convenient mathematical devices thought up specially for the purpose of quantization. This is the case in field theory, where there exist only meaningful physical classical theories for the electromagnetic and gravitational fields.

The physical interpretation of quantum field theory is not immediately obvious or straightforward. The basic variables on one hand are the *fields*, in experiments one measures on the other hand the behaviour of particles, annihilation, creation and scattering. The presence of particles with certain definite energies and linear momenta corresponds, roughly speaking, to the intensities of specific frequencies in the quantum field, considered as a wave phenomenon.

The distinction between wave functions and field operators is important. In ordinary quantum mechanics the primary objects are *wave functions* $\psi(x)$, i.e. vectors in a L^2 Hilbert space of states. In quantum electrodynamics $\psi(x)$ stands for an *operator field*, a function defined on space-time with values in the operators of a Hilbert space of states. It is rather confusing that one sometimes considers approximations in which this distinction is less clear or even completely lost.

The concept of *symmetry* plays an important rôle in quantum field theory. Symmetries appear as unitary representations of *symmetry groups*, acting in the Hilbert space of states. A basic symmetry is that of translations in space-time. It is realized by unitary operators $U(a)$, $\forall a \in \mathbb{R}^4$, whose action connects the field operators at different space-time points according to

$$U(a)\phi(x)U(a)^{-1} = \phi(x+a) \tag{5}$$

Note that for a pure time translation over $a = (0,t)$, this is in fact formula (1). The next important symmetry is the symmetry due to the requirements of special relativity. Combined with translation symmetry this gives symmetry with respect to the inhomogeneous Lorentz group, or rather its covering group, which is called the Poincaré group. All the field theories of sub-nuclear physics, except gravitation, have Poincaré symmetry in the sense indicated here. The irreducible representations that occur have a characteristic parameter s, the spin, a generalized angular momentum, which takes integer or half integer values. This leads to a division in two fundamentally different types of quantum field theories, *boson* fields (integer s), and *fermion* fields (half integer s). The scalar field, for example, is a boson field $(s = 0)$, the vector field of quantum electrodynamics is also a boson field $(s = 1)$, and the spinor field is a fermion field $(s = \frac{1}{2})$.

After these introductory remarks and some further considerations on the general case in the next section, we shall discuss as our main topic the particular case of *free fields*, theories with each a single type of field, satisfying *linear* equations such as

$$\begin{aligned} &(\Box+m^2)\phi(x) = 0 \\ &(i\gamma^\mu\partial_\mu - m)\psi(x) = 0 \\ &\partial^\mu\partial_\mu A_\nu - \partial_\nu\partial^\mu A_\mu = 0 \end{aligned} \tag{6}$$

Free quantum fields are mathematically well understood. They all have the same structure, which it is worthwhile to understand in its general form before working out explicitly separate particular examples. The physical content of free quantum fields is not very sophisticated. A free quantum field describes a system of an arbitrary but constant number of non-interacting, freely moving particles of one particular type. As Hilbert space theory the

free fields fit naturally in the so-called Fock space or many particle formalism. Free quantum fields are important, not so much as physical theories on their own, but as a necessary background for the much more difficult non-linear theories needed for a realistic description of sub-nuclear particle behaviour.

2. GENERAL QUANTUM FIELD THEORY. N-POINT FUNCTIONS

The discussion of a particular field theory usually starts on the classical level with the specification of a *Lagrangian*, a function of the fields and its partial derivatives. The field equations then follow as Euler-Lagrange equations from a variation principle for the action, the integral of the Lagrangian function over a space-time region. The Lagrangian is also the point of departure for the re-writing of a classical field theory in *Hamiltonian* form.

In the example of the ϕ^4-model the Lagrangian is

$$L(\phi,\partial_\mu\phi) = \tfrac{1}{2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2) - \tfrac{1}{4}\lambda\phi^4 \tag{7}$$

with $\partial_\mu\phi = \frac{\partial\phi}{\partial x^\mu}$, $\mu = 0,1,2,3$, $x^0 = t$, $g^{\mu\nu}$ the Minkowski space metrical tensor with off-diagonal elements zero and $g^{00} = -g^{kk} = 1$, $k = 1,2,3$, and the usual summation convention for repeated indices.

The field equations (3) then follow from the general form of the Euler-Lagrange equations

$$\frac{\partial L}{\partial\phi(x)} - \frac{\partial}{\partial x^\mu}\frac{\partial L}{\partial(\partial_\mu\phi(x))} = 0 \tag{8}$$

For a Hamiltonian formulation the field $\phi(x) = \phi(\vec{x},t)$ may be regarded as an infinite set of canonical "coordinates". Conjugate "momenta" are then introduced as

$$\pi(\vec{x},t) = \frac{\partial L}{\partial(\partial_0\phi(x))} \tag{9}$$

Further development of a Hamiltonian formulation is for this scalar field example quite straightforward. In general this is not the case and various complications arise. Apart from technical aspects connected with the infinite dimension of the symplectic manifolds involved, there is the presence of *constraints*, in quantum electrodynamics, and in a general manner in gauge

fields such as the Yang-Mills field. A more fundamental problem is the fact that the Hamiltonian formulation as we know it so far makes sense only for *boson* systems.

Quantization is the construction of a Hilbert space theory in which the classical variables have become operators, under the general rule that Poisson brackets $\{a,b\}$ of pair of classical variables correspond to cummutator expressions $\frac{2\pi}{i\hbar}[A, B]$ for the associated operators, or more simply $-i[A, B]$ if one employs, as is usual in field theory, units such that $\hbar = \frac{\hbar}{2\pi} = 1$.

The basic algebraic relations between the operators of a quantized theory are the relations between the operators associated with coordinates and conjugate momenta. In classical mechanics one has

$$\begin{aligned} \{q_j,q_k\} &= \{p_j,p_k\} = 0 \\ \{q_j,p_k\} &= \delta_{jk} \end{aligned} \tag{10}$$

which in quantum mechanics becomes

$$\begin{aligned} [Q_j,Q_k] &= [P_j,P_k] = 0 \\ [Q_j,P_k] &= i\hbar\delta_{jk} \end{aligned} \tag{11}$$

In field theory there is a similar situation. In the example of the scalar field one obtains from the classical description the following operator relations, which should hold for all times, because we are in the Heisenberg picture

$$\begin{aligned} [\phi(\vec{x},t),\ \phi(\vec{x}',t)] &= [\Pi(\vec{x},t),\ \Pi(\vec{x}',t)] = 0 \\ [\phi(\vec{x},t),\ \Pi(\vec{x}',t)] &= i\delta(\vec{x}-\vec{x}') \end{aligned} \tag{12}$$

The dynamics, i.e. the time evolution, in the sense of formula (1), is determined by the Hamiltonian operator H, given as a space integral of an ex-

pression in the basic operators $\phi(\vec{x},t)$ and $\Pi(\vec{x},t)$, suggested again by the classical description.

The presence of a Dirac delta-function in (12) is significant. It implies that the concept of field operator at a sharp space-time point x is, strictly speaking, meaningless. The operator fields are operator valued generalized functions, or *operator valued distributions*. For a scalar field for example the proper formulation should be in terms of an operator valued functional $\phi(f)$, depending linearly on test functions $f(x)$ belonging to a linear function space V, and connected with the heuristic field operator $\phi(x)$ by the symbolic relation

$$\phi(f) \text{ "=" } \int \phi(x)f(x)d^4x \tag{13}$$

It will not come as a surprise that this distribution character of the field operators leads to serious mathematical difficulties, at all places where nonlinear expressions in the fields appear. A nonlinear equation such as (3) is a priori meaningless as operator equation. To find a Hilbert space operator which is well-defined and corresponds in a reasonable manner with a heuristic expression like $\phi(x)^4$ or $\int \phi(x)^4 d^3\vec{x}$ is a major mathematical problem which is still far removed from a complete solution. It is only for free quantum fields that we possess an adequate mathematical formulation.

Summarizing one can say that quantum field theory is a formal scheme of great beauty and effectiveness. However, the mathematical meaning of an important part of its concepts and methods have remained unclear and basic questions of existence and consistency of its various models have not been answered.

Of theoretical importance for the clarification of the mathematical structures that seem to be involved is the concept of vacuum expectation values of products of field operators. Every field theory is supposed to

possess a *vacuum vector* Ω, describing a state in which there are no particles of any kind present. It has various other properties, is invariant under all symmetry transformations and is comparable to the ground state of a system of harmonic oscillators. Determining this vector, or even proving its existence is almost as difficult as the complete construction of the field theory in question. Its expectation values are therefore also not very tractable. Nevertheless their general properties are worth studying.

The vacuum expectation values of products of fields are usually called n-*point functions*, or *Wightman functions*. For a scalar field model one has as n-point functions a sequence of generalized functions W_n

$$\begin{aligned} W_o &= (\Omega,\Omega) = 1 \\ W_n(x_1,x_2,\ldots,x_n) &= (\Omega,\phi(x_1)\phi(x_2)\ldots\phi(x_n)\Omega) \qquad n = 1,2,3,\ldots \end{aligned} \tag{14}$$

More rigorously formulated one has a sequence of n-linear forms W_n on the test function space V

$$W_n : V\times V\times\ldots\times V \to \mathbb{C} \tag{15}$$

defined as

$$\begin{aligned} W_o &= 1 \\ W_n(f_1,\ldots,f_n) &= (\Omega,\phi(f_1)\ldots\phi(f_n)\Omega) \\ &\qquad \text{for} \quad f \in V, \quad j = 1,2,\ldots,n\,, \quad n = 1,2,\ldots \end{aligned} \tag{16}$$

In the case of a vector field the n-point functions depend also on space-time indices μ_j

$$W_{n,\mu_1\ldots\mu_n}(x_1,\ldots,x_n) = (\Omega,A_{\mu_1}(x_1)\ldots A_{\mu_n}(x_n)\Omega) \tag{17}$$

For the rigorous version the operator field may be taken to depend on 4-component test functions $f = f^{\mu}(x)$, according to the heuristic formula

$$A(f) \text{ "=" } \int A_\mu(x) f^\mu(x) d^4x \tag{18}$$

The n-point functions are then again n-linear forms on a test function space V

$$W_n(f_1,\ldots,f_n) = (\Omega, A(f_1)\ldots A(f_n)\Omega) \tag{19}$$

which is essentially the same formula as (16).

This shows incidentally, that one of the advantages of the rigorous, more intrinsic formulation is, that it brings out the structural similarity of different particular cases. The usual, more heuristic formulation in generalized function language is nevertheless very often quite practical and effective, apart from the fact that it is used almost exclusively in most standard texts on quantum field theory. Ideally one should have both formulations at one's disposal, with the rigorous version to fall back on in the case of ambiguities.

The n-point functions of field theory are very similar to the *correlation functions* of probability theory, or more particularly, stochastic processes. They become in every respect a straightforward generalization if one takes the point of view that a system of quantum variables, like the field operators at different space-time points x_j, is a noncommutative version of an ordinary system of random variables, like the values of a stochastic process at different times.

The properties of a stochastic process can be read off from its correlation functions. This is even more so for the n-point functions. They characterize a field theory completely, in such a manner, that the full field theory as an operator theory in Hilbert space can be recovered, in a unique and constructive way, from the knowledge of the n-point functions as a sequence of number valued distributions. This is the content of Wightman's

reconstruction theorem, which we shall not prove or even state in a precise manner here. We observe only that an n-point function as an n-linear form on V defines a linear form on the n-fold tensor product $\otimes^n V$. A sequence of n-point functions determines therefore a single linear functional on the direct sum $\sum_{n=0}^{\infty} \oplus(\otimes^n V)$, which is in a natural way an algebra under tensor multiplication. The reconstruction theorem is not much more than a particular application of a general representation theory for algebras, connected with the names of Gelfand, Naimark and Segal, in which linear functionals on an algebra induce operator representations of that algebra.

N-point functions provide us with an alternative way of characterizing quantum field theories, which is of great theoretical importance. The n-point functions of the ϕ^4-model are as impossible to obtain explicitly as the corresponding operator system, but they are more open to methods that study the model in question as a limit of simpler, more accessible models. In the case of free fields the n-point functions have a simple and very characteristic form and can moreover be found explicitly. On this our discussion of free fields in the next section will be based.

3. FREE QUANTUM FIELDS. A GENERAL APPROACH.

Free fields have quadratic Lagrangians leading to field equations that are linear, and - at least, in the case of boson systems - of second order. A canonical procedure can be followed, which starts with the fields as classical systems, and gives the corresponding quantum systems in the manner indicated in the preceding section. Due to the complications that arise in the particular cases, the method, certainly as presented in most of the text books, is less convincing than the final result, which is essentially the same for all the different cases. Explaining this general result is our main purpose. We shall therefore avoid the canonical quantization altogether and follow a different line of approach, which is made possible by a theorem, stating that the n-point functions of a free quantum field, i.e. a field satisfying a certain type of linear equation and having certain reasonable physical properties, have a very characteristic form, which we shall call "*Gaussian*", and are in fact uniquely determined. This enables us to discuss the free quantum field from a general point of view, starting from the particular form of the n-point functions, and then recovering the corresponding operator theory, in the spirit of the reconstruction theorem.

In this section we shall consider the general formalism, in the next section the free scalar field will be discussed in explicit detail, as an illustrative example. The two sections should be read alongside each other.

We restrict the discussion first to theories based on a single field, We suppose moreover that the field operators are "hermitian" i.e. symmetric or possibly self-adjoint. The physical meaning of this is that one has particles that have no electric charge.

The choice of a basic field implies a choice of test functions. For a scalar field one has single-component test functions, as in (13), for a vector field vector-valued test functions, as in (18), and so on. The

components of the test functions are usually taken to be Schwartz's- S-type functions of fast decrease. For the field operators $\phi(f)$, $A(f)$, etc. to be "hermitian", the test functions must be real. It turns out to be convenient however to extend to a space of complex test functions.

Summarizing we suppose now to be given, as the first element of the mathematical formalism describing a particular quantum field theory, a *complex vector space* V, which is the complexification of a given real space, or equivalently a complex vector space in which a *conjugation* $f \longmapsto f^*$ is specified.

The elements of V, in general multi-component test functions of some sort, shall be denoted as f, g, etc. . The conjugation is often but not always complex conjugation of separate components. As a test function space from distribution theory V has a locally convex topology of a certain kind. This belongs to the technical aspects that we shall not discuss.

According to the reconstruction theorem one may think of a quantum field theory as something characterized by a system of n-point functions, which means here a sequence $\{W_n\}_{n=o}^{\infty}$ of (continuous) n-linear forms on V

$$W_n : V\times V\times\ldots\times V \to \mathbb{C} \tag{20}$$

Because of the eventual identification with inner product expressions according to

$$W_n(f_1,\ldots,f_n) = (\Omega,\phi(f_1)\ldots\phi(f_n)\Omega) \tag{21}$$

additional properties are necessary for the W:

Hermiticity of the field operators, which because the use of complex test functions, must be expressed as

$$(\Omega_1,\phi(f)\Omega_2) = (\phi(f^*)\Omega_1,\Omega_2) = (\Omega_2,\phi(f^*)\Omega_1)^* \tag{22}$$

for all pairs of vectors Ω_1, Ω_2 in the Hilbert space H, is guaranteed by the n-point functions being *hermitian symmetric* in the sense of

$$W_n(f_1,\ldots,f_n) = W_n(f_n^*,\ldots,f_1^*)^* \tag{23}$$

(Note that $\phi(f)$ denotes the field operator in general now, as already in (21)).

Positivity of the inner product in the Hilbert space H is connected with a *positivity condition* that has to be imposed on the set of n-point functions. We shall not give its general form, a quadratic relation involving all the W_n's simultaneously. One of the simpler special cases that follow from it is for instance the condition

$$W_2(f^*,f) \geqslant 0 \qquad \forall f \in V \tag{24}$$

This is obviously related to the Hilbert space property

$$(\phi(f)\Omega,\phi(f)\Omega) \geqslant 0 \qquad \forall f \in V$$

We are now in a position to give the particular simple form of systems of n-point functions for free fields:

We define a *Gaussian system* of n-point functions as a system determined by a given bilinear form $b(\cdot,\cdot)$ on V in the following way:

$$W_o = 1$$

$$W_{2n+1}(f_1,\ldots,f_{2n+1}) = 0 \qquad n = 0,1,2,\ldots$$

$$W_{2n}(f_1,\ldots,f_{2n}) = \sum_{\sigma}{}' (\operatorname{sign}\sigma)^{\varepsilon}\ b(f_{\sigma(1)},f_{\sigma(2)})\ldots b(f_{\sigma(2n-1)},f_{\sigma(2n)}) \qquad n = 1,2,\ldots \tag{26}$$

in which the summation is over all permutations σ of the indices $1,2,\ldots,2n$, satisfying $\sigma(1) < \sigma(3) < \ldots < \sigma(2n-1)$; $\sigma(1) < \sigma(2),\ldots, \sigma(2n-1) < \sigma(2n)$,

and with $\varepsilon = 0$ (boson case) or $\varepsilon = 1$ (fermion case).
Written out explicitly we have the sequence

$$
\begin{aligned}
&W_0 = 1 \\
&W_1(f) = 0 \\
&W_2(f_1,f_2) = b(f_1,f_2) \\
&W_3(f_1,f_2,f_3) = 0 \\
&W_4(f_1,f_2,f_3,f_4) = b(f_1,f_2)b(f_3,f_4) + (-1)^{\varepsilon} b(f_1,f_3)b(f_2,f_4) + \\
&\qquad\qquad + b(f_1,f_4)b(f_2,f_3) \qquad\qquad (27) \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

All properties of the n-point functions of a Gaussian system follow from properties of the two-point function, the bilinear form $b(\cdot,\cdot)$. In particular one can show that a necessary and sufficient condition for hermitian symmetry of the n-point functions, in the sense of (23), is

$$b(f,g) = b(g^*,f^*)^* \qquad \forall f,g \in V \qquad (28)$$

and for the general positivity property

$$b(f^*,f) \geqslant 0 \qquad \forall f \in V \qquad (29)$$

In order to be able to recover from a Gaussian system of n-point functions a"hermitian" quantum field, in terms of operators in a proper Hilbert space, the fundamental bilinear form $b(\cdot,\cdot)$ must have (28) and (29) as additional properties.

A Gaussian stochastic process, or any other Gaussian system of randum variables from ordinary probability theory, is characterized completely by its covariance or two-point correlation function. The higher correlation functions are simple expressions in the two-point function, in fact just the expressions given by (26), with $\varepsilon = 0$. In our point of view a quantum field system is a noncummutative generalization of an ordinary system of

random variables. The n-point functions correspond to correlation functions. This justifies our use of the term Gaussian in quantum field theory.

Free quantum fields are Gaussian systems with additional properties:

a. The field operator satisfies a linear *field equation*, such as one of the equations in (6). A necessary and sufficient condition for this is that every n-point function satisfies the equation in each of its n variables separately. In the Gaussian case it is then enough that this holds for the two-point function.

b. There is *symmetry* with respect to the Poincaré group, and possibly with respect to other groups. In a proper formulation symmetry in quantum field theory means usually that there is a group G acting in the first place by linear transformations $T(a)$ in the basic test function space V and at the same time is represented by unitary operators $U(a)$ in the Hilbert space H, such that

$$\begin{aligned} &U(a)\phi(f)U(a)^{-1} = \phi(T(a)f) \\ &U(a)\Omega_o = \Omega_o \qquad\qquad \forall a \in G \end{aligned} \tag{30}$$

with Ω_o the vacuum state in H. This implies, as one easily shows, that the n-point functions are invariant as n-linear forms on V

$$\begin{aligned} &W_n(T(a)f_1,\ldots,T(a)f_n) = W_n(f_1,\ldots,f_n) \\ &\qquad \forall a \in G;\quad f_j \in V;\quad j = 1,2,\ldots,n,\quad n = 1,2,\ldots \end{aligned} \tag{31}$$

c. There are further properties that we shall not discuss here, such as locality and positivity of the energy, and in the Gaussian case again expressed as properties of the two point function.

The properties of a, b and c together are in most cases enough to deter-

mine a free quantum field in a unique way, by fixing a two-point function. Roughly speaking such a two-point function is a function of a single space-time variable, because of translation invariance. As such it will be a Lorentz invariant solution of the field equation, satisfying a few extra requirements.

Before proceeding to the construction of the free fields as operator theories we want to indicate how the restrictions imposed in the beginning of this section can be removed.

For a theory with several different fields one must use a basic test function space V obtained by taking the direct sum of the test function spaces associated with these different fields. With this the formulation of general field theory in terms of n-point functions remains the same. The form of the n-point functions for a free theory containing bosons and fermions is more complicated than that given by (26). The complications of such a mixed free field theory are however not of an essential nature, and will therefore not be discussed here.

Electrically charged particles are described by fields that are complex valued on the classical level and consist of non-hermitian operators on the quantum level. For a hermitian field one starts with a real space of test functions which is, for the sake of mathematical convenience, extended by complexification to a complex space V. For a non-hermitian field one starts already with a complex space which also has to be extended to a larger space. The way to do this is slightly complicated but is a familiar device for situations where two complex structures appear simultaneously in the same vector space: One, in this case the original complex structure from V, is represented by a linear operator J with $J^2 = 1$. We shall not discuss this further here but give an explicit example in the next section. The result is in any case that one retains in this manner for complex fields

the general formulation based on a complex vector space V with a conjugation $*$.

A quantum field in terms of operators in Hilbert space can be obtained from its n-point functions by a general construction. For Gaussian systems there is a special construction which does not follow immediately from the general procedure, but is very simple and leads in a straightforward manner to the Fock space realization of the free fields. This we shall now discuss.

The data for this construction are the test function space, a complex vector space V with conjugation $*$, together with a bilinear form $b(\cdot,\cdot)$ on V, the two-point function, with $b(f,g)^* = b(g^*,f^*) \geqslant 0$. Note that $h(f,g) := b(f^*,g)$ is a positive definitive hermitian form on V.

Let V_L be the left degeneration space of $b(\cdot,\cdot)$, defined as

$$V_L := \{f \in V \mid b(g,f) = 0, \quad \forall g \in V\} \tag{32}$$

Using the Schwartz inequality one verifies that V_L is also the null space of $h(\cdot,\cdot)$

$$V_L := \{f \in V \mid h(f,f) = 0\} \tag{33}$$

The quotient space V/V_L is a (complex) pre-Hilbert space with respect to the inner product induced by $h(\cdot,\cdot)$, according to

$$(\chi f, \chi g) = h(f,g) = b(f^*,g) \tag{34}$$

for all f, g in V and with χ the canonical surjection from V onto V/V_L. Its completion is denoted as $H^{(1)}$, and is called the *one-particle Hilbert space*

$$H^{(1)} := \overline{V/V_L} \tag{35}$$

A symmetry appears in the theory as invariance of the bilinear form $b(\cdot,\cdot)$ under a group of linear transformations $T(a)$ in V

$$b(T(a)f, T(a)g) = b(f,g) \qquad \forall f,g \in V; \quad \forall a \in G \tag{36}$$

The linear transformations $T(a)$ in V induce operators $U^{(1)}(a)$ in $H^{(1)}$ according to

$$U^{(1)}(a)\chi f := \chi T(a)f \tag{37}$$

The $U^{(1)}(a)$ are in fact *unitary operators* in the one-particle space $H^{(1)}$

$$(U^{(1)}(a)\chi f, U^{(1)}(a)\chi g) = (\chi T(a)f, \chi T(a)g = b((T(a)f)^*, T(a)g) =$$
$$= b(T(a)f^*, T(a)g) = b(f^*,g) = (\chi f,\chi g) \qquad \forall f,g \in V; \quad \forall a \in G \tag{38}$$

In this derivation we have used the fact that the $T(a)$ are always complexifications of operators in the real part of V and commute therefore with the conjugation operator $*$.

We are now in a position to make contact with the Fock space formalism, a general description of many-particle systems in quantum theory which is discussed separately in an appendix. Our procedure so far, starting from the n-point functions of a Gaussian system, constructing $H^{(1)}$ and using this now in a Fock space formulation looks rather arbitrary. Justification will be found in the final result of all this.

Let H be the *many-particle space* over $H^{(1)}$, either in the boson or the fermion version, as explained in the appendix. H will be the Hilbert space of states of our quantum field, the vacuum state for the field being the vacuum vector in the many-particle formalism.

The next and, most important step in the construction is the definition of *field operators* $\phi(f)$. For this we have at our disposal the annihilation

and creation operators $a(u)$ and $a^*(u)$, depending anti-linearly, respectively linearly on vectors u from $H^{(1)}$, and satisfying simple algebraic relations, commutation relations in the case of bosons, anti-commutation relations for fermions. We define, for every f in V, a field operator $\phi(f)$ as

$$\phi(f) := a^*(\chi f) + a(\chi f^*) \tag{39}$$

The first thing to note is that the field operator defined in this manner depends indeed linearly on f. Various other properties of $\phi(f)$ can be established using properties of the creation and annihilation operators. The main thing to verify is however the fact that the vacuum expectation values of these field operators are indeed the n-point functions as originally given. This then contains everything else and justifies the whole procedure. The verification of the form (26) for the vacuum expectation values is straightforward, it uses the commutation or anti-commutation ralations and the property $a(u)\Omega_o = 0$, for all u in $H^{(1)}$. For the two-point function one has for instance

$$\begin{aligned} &(\Omega_o, \phi(f_1)\phi(f_2)\Omega_o) = (\phi(f_1^*)\Omega_o,\ \phi(f_2)\Omega_o) = (a^*(\chi f_1^*)\Omega_o,\ a^*(\chi f_2)\Omega_o) = \\ &= (\Omega_o, a(\chi f_1^*)a^*(\chi f_2)\Omega_o) = (\Omega_o,\ [a(\chi f_1^*),\ a^*(\chi f_2)]\Omega_o) = \\ &= (\chi f_1^*, \chi f_2) = b(f_1, f_2) \end{aligned} \tag{40}$$

The general case for arbitrary n is left to the reader.

In the case of a *symmetry* one has unitary operators $U^{(1)}(a)$ in $H^{(1)}$ as indicated. These can be extended to product operators, unitary operators $U(a)$, in the many-particle space H. Using the relations between product operators and the creation and annihilation operators, given in the appendix, one easily deduces the properties (30) for the $U(a)$, which confirms

their status as proper symmetry operators for the quantum field.

With this we have completed the general procedure for the construction of free quantum fields. Application to a particular case means the search for a suitable bilinear form on some given vector space of test functions. Once this has been found, everything else follows. Written out explicitly the resulting representation may look quite complicated, particularly, because $H^{(1)}$ is defined as a space of equivalence classes of test functions, for which in practice some realization in terms of single functions will be used.

4. THE FREE SCALAR FIELD.

We consider a field $\phi(x), x = (\vec{x},t)$, real as a classical field, a hermitian operator field on the quantum level. It should satisfy the field equation

$$(\partial_\mu \partial^\mu + m^2)\phi = 0 \tag{41}$$

This is the Euler-Lagrange equation of a variational problem given by the Lagrange function

$$L(\phi, \partial_\mu \phi) = \tfrac{1}{2}(g^{\mu\nu}\partial_\mu \phi \partial_\nu \phi - m^2 \phi^2) \tag{42}$$

As a quantum field $\phi(x)$ describes a system of an arbitrary but fixed number of free particles such as π^o-mesons, particles with zero electric charge and a positive mass m. Scalar particles have spin zero; $\phi(x)$ is therefore a *boson* field. The only symmetry is that with respect to the inhomogeneous Lorentz group, which must be represented by unitary operators $U(a,\Lambda)$ in the Hilbert space of states H such that

$$\begin{aligned} U(a,\Lambda)\phi(x)U(a,\Lambda)^{-1} &= \phi(\Lambda x + a) \\ U(a,\Lambda)\Omega_o &= \Omega_o \end{aligned} \tag{43}$$

for all translations a and Lorentz transformations Λ and with Ω_o the "no-particle" vacuum state. (Lorentz transformations are linear transformations $x' = \Lambda x$ that leave the bilinear form $xy = g_{\mu\nu}x^\mu y^\nu = x^o y^o - \sum_{j=1}^{3} x^j y^j$ invariant.)

The canonical quantization procedure for this scalar field should be based on the equal time commutation relations

$$\begin{aligned} &[\phi(\vec{x},t),\ \phi(\vec{x}',t)] = [\partial_o \phi(\vec{x},t),\ \partial_o \phi(\vec{x}',t)] = 0 \\ &[\phi(\vec{x},t),\ \partial_o \phi(x',t)] = i\delta(\vec{x}-\vec{x}') \end{aligned} \tag{44}$$

The Dirac δ-function indicates that the use of operators defined at sharp space-time points x is heuristic. In our formulation here the field operators are "smeared" with test functions $f(x)$ on space-time, according to the heuristic formula

$$\phi(f) = \int \phi(x) f(x) d^4x \tag{45}$$

The *basic test function space* V consists therefore in this case of all complex "scalar" test functions $f(x)$, infinitely differentiable and of fast decrease in the sense of Schwartz. Conjugation is ordinary complex conjugation.

The free scalar field is characterized by a *Gaussian system of* n*-point functions*, as defined rigorously in (26) and (27), with $\varepsilon = 0$, and written in heuristic manner

$$\begin{aligned} &W_0 = 1 \\ &W_{2n+1}(x_1,\ldots,x_{2n+1}) = 0 \\ &W_{2n}(x_1,\ldots,x_{2n}) \sum_\sigma W_2(x_{\sigma(1)},x_{\sigma(2)}) \cdots W_2(x_{\sigma(2n-1)},x_{\sigma(2n)}) \end{aligned} \tag{46}$$

with a summation over permutations σ as in (26).

With this general form, the only remaining task is to find the two-point function $W_2(x_1,x_2)$. For this the conditions of hermitian symmetry in the sense of (28), positivity as expressed by (29), Lorentz invariance, (31) and (43), together with the field equation (41) and finally the requirement of positivity of the energy are sufficient, and determine the two-point function completely, up to a positive scale factor.

To discuss this we use heuristic generalized function language. Everything can be rephrased in rigorous test function language, but this is not very enlightening. In any case the result will be stated in precise form.

We introduce Fourier transforms. For test functions we use the convention

$$\hat{f}(k) = \frac{1}{(2\pi)^2} \int e^{ikx} f(x) d^4x \tag{47}$$

with $k \in \mathbb{R}^4$, $k = (k^0, k^1, k^2, k^3)$ and the Lorentz product $kx = g_{\mu\nu} k^\mu x^\nu = k^0 x^0 - \sum_{j=1}^{3} k^j x^j$.

Translation invariance of $W_2(x_1, x_2)$ allows us to write the two-point function as

$$\begin{aligned} W_2(f_1, f_2) &= \iint W_2(x_1, x_2) f_1(x_1) f_2(x_2) d^3x_1 d^3x_2 = \\ &= \int \hat{W}(k) \hat{f}_1(-k) \hat{f}_2(k) d^4k \end{aligned} \tag{48}$$

with $\hat{W}(k)$ a distribution in a single 4-vector variable k. Lorentz invariance, hermitian symmetry and the field equation imply then together that $\hat{W}(k)$ is a real, Lorentz invariant solution of

$$(k^2 - m^2) \hat{W}(k) = 0 \tag{49}$$

which is of course the Fourier transformed version of the Klein-Gordon equation (46). It is well-known that (49) has two linearly independent invariant solutions. The general invariant solution can be written as a linear combination

$$c_+ \delta_+(k^2 - m^2) + c_- \delta_-(k^2 - m^2) \tag{50}$$

The invariant solutions $\delta_+(k^2 - m^2)$ and $\delta_-(k^2 - m^2)$ are distributions, in fact measures, which have support on the so-called forward, respectively backward mass-shells V_+^m and V_-^m defined as

$$V_\pm^m = \{k \in \mathbb{R}^4 \mid k^2 = m^2; k^o \gtrless 0\} \tag{51}$$

and can be written in rigorous form as functionals depending on test functions

$$\int h(k)\delta_{\pm}(k^2-m^2)d^4k \text{ "=" } \int h(k^o = \pm\omega(\vec{k}),\vec{k})\,\frac{d^3\vec{k}}{2\omega(\vec{k})} \tag{52}$$

with $\omega(\vec{k}) = \sqrt{\vec{k}^2+m^2}$, $\vec{k}^2 = \sum_{j=1}^{3}(k^j)^2$. Note that $\frac{d^3\vec{k}}{2\omega(\vec{k})}$ is a Lorentz invariant measure on the three dimensional mass-shells $V_{\pm}^m$, and that the $\delta_{\pm}(k^2-m^2)$ as functionals are nothing else than integrals of the restrictions of functions to the mass-shells $V_{\pm}^m$, with these measures.

For general reasons that we can not discuss, the variable k in Fourier transforms has here the physical interpretation of an energy-momentum variable. In order to have only positive energies in the theory, the constant c_- in the solution (50) for $\hat{W}(k)$ must be zero. Positivity in the sense of (29) then requires the remaining constant c_+ to be positive. We fix c_+ by convention as $c_+ = 2\Pi$, this leads eventually to agreement with the canonical equal-time commutation relations (44).

This brings us to the final result, the two-point function for the real scalar field of mass m, formulated rigorously as a bilinear form on the space of test functions

$$W_2(f_1,f_2) = 2\Pi \int_{k^o=\omega(\vec{k})} \hat{f}_1(-k)\hat{f}_2(k)\,\frac{d^3\vec{k}}{2k^o} \tag{53}$$

Note that the conjugation in the test function space becomes in terms of Fourier transformed test functions $\hat{f}(k) \longmapsto \hat{f}(-k)^*$. The associated hermitian form $h(\cdot,\cdot)$ is therefore

$$h(f_1,f_2) = 2\Pi \int_{k^o=\omega(\vec{k})} \hat{f}_1(k)^*\hat{f}_2(k)\,\frac{d^3\vec{k}}{2k^o} \tag{54}$$

In standard texts on quantum field theory one finds only the heuristic two-point function $W_2(x_1,x_2)$, which is then written usually as

$$W_2(x_1,x_2) = i\Delta^{(+)}(x_1-x_2;m) \tag{55}$$

with the "singular function" $\Delta^{(+)}$ defined as

$$\Delta^{(+)}(y;m) = -\frac{i}{(2\pi)^3}\int_{k^o=\omega(\vec{k})} e^{-ikx}\frac{d^3\vec{k}}{2k^o} \tag{56}$$

The verification that (52) is indeed the rigorous version of (55) is left to the reader.

Having a description of the free scalar field in terms of explicit n-point functions based on (52), we can proceed to the construction of the corresponding operator theory, along the line of the preceding section.

Inspection of (52) shows that the left degeneration space V_L of $W_2(\cdot,\cdot)$ consists of test functions $f(x)$ with Fourier transforms that vanish on the forward mass-shell V_+^m. The quotient space V/V_L is a space of equivalence classes of test functions, two test functions being equivalent when the restriction of their Fourier transforms to the forward mass-shell are identical. This suggests a natural realization of V/V_L as a space of functions on the forward mass-shell, or because of the relation $k^2-m^2=0$ as functions of the 3-vector variable $\vec{k} = (k^1,k^2,k^3)$

$$\phi(\vec{k}) = (\chi f)(\vec{k}) = \sqrt{2\pi}\,\hat{f}(k^o=\omega(\vec{k}),\vec{k}) \tag{57}$$

with a factor $\sqrt{2\pi}$ added for a convenient normalization. As a result the inner product (34) on V/V_L becomes

$$(\phi_1,\phi_2) = (\chi f_1,\chi f_2) = W_2(f_1^*,f_2) = \int \phi_1(k)^*\phi_2(\vec{k})\frac{d^3\vec{k}}{2\omega(\vec{k})} \tag{58}$$

The completed one-particle space $H^{(1)} = \overline{V/V_L}$ is in this realization therefore

$$H^{(1)} = L^2\left(\mathbb{R}^3, \frac{d^3\vec{k}}{2\omega(\vec{k})}\right) \tag{59}$$

In the physical interpretation the functions $\phi(\vec{k})$, as vectors in $H^{(1)}$ represent states of the system in which there is exactly one particle present. In accordance with the general principles of quantum theory $|\phi(\vec{k})|^2$, or rather $\frac{|\phi(\vec{k})|^2}{2\omega(\vec{k})}$ is the probability density for the linear momentum $\vec{k}$ of this particle.

In most quantum field theory text books one employs heuristic one-particle state vectors $|\vec{k}>$ which are supposed to have sharp momentum $\vec{k}$. A one-particle state as here described by a function $\phi(\vec{k})$ may be regarded as a continuous superposition of such basic states

$$|\phi> = \int \phi(\vec{k})\, |\vec{k}> \frac{d^3\vec{k}}{2\omega(\vec{k})} \tag{60}$$

The $|\vec{k}>$ have of course no rigorous meaning as vectors in the Hilbert space $H^{(1)}$. They are nevertheless quite convenient in heuristic and explicit manipulations and are used extensively, alongside similar heuristic objects we shall meet further on in this discussion.

In the test function formulation of quantum field theory the action of the inhomogeneous Lorentz group is given in the first place by linear transformations of the test function space V, in this scalar field case

$$(T(a,\Lambda)f)(x) = f(\Lambda^{-1}(x-a)) \tag{61}$$

The hermitian form $h(f,g) = W_2(f^*,g)$ is invariant under these transformations. One has therefore induced unitary operators $U^{(1)}(a,\Lambda)$ in $H^{(1)}$, according to (40), acting on the functions $\phi(\vec{k})$ as

$$(U^{(1)}(a,\Lambda)\phi)(\vec{k}) = e^{ika}\phi(\Lambda^{-1}\vec{k}) \tag{62}$$

with $(\Lambda^{-1}\vec{k})^j = (\Lambda^{-1})^j{}_\mu k^\mu$, $k^o = \omega(\vec{k})$. The $U^{(1)}(a,\Lambda)$ are indeed unitary operators because the measure $\frac{d^3\vec{k}}{2\omega(\vec{k})}$ is Lorentz invariant.

From the one-particle space $H^{(1)}$ we construct the full state space of the system as the many-particle space H of the Fock space formalism defined over $H^{(1)}$, in the manner discussed in the appendix.

The unitary operators $U^{(1)}(a,\Lambda)$ form an irreducible representation of the inhomogeneous Lorentz group. It is of such a type that we use the version of the Fock space construction with symmetric tensors: The scalar field is a boson quantum field.

In H we have the vacuum or no-particle state Ω_o, in this context often denoted by $|0>$. Because we have $H^{(1)}$ as space of momentum functions $\phi(\vec{k})$, vectors from an $H^{(n)}$ for $n > 1$ will be represented by symmetric functions of n momentum variables $\vec{k}_1, \ldots, \vec{k}_n$, with an obvious integral expression for the inner product. An arbitrary vector from H will be an infinite sequence of such functions, for $n = 1,2,\ldots$. The $U^{(1)}(a,\Lambda)$ generate as product operators unitary symmetry operators $U(a,\Lambda)$ in the full state space H. They leave by definition the vacuum vector $|0>$ invariant.

The creation and annihilation operators depend linearly, respectively anti-linearly on vectors from $H^{(1)}$, in this case functions $\phi(\vec{k})$. This suggest regarding the $a^*(\phi)$ and $a(\phi)$ as operator-valued distributions with heuristic operators $a^*(\vec{k})$ and $a(\vec{k})$, defined for sharp values of $\vec{k}$, according to the relations

$$\begin{aligned} a^*(\phi) &= \int a^*(\vec{k})\phi(\vec{k})\frac{d^3\vec{k}}{2\omega(\vec{k})} \\ a(\phi) &= \int a(\vec{k})\phi(\vec{k})^*\frac{d^3\vec{k}}{2\omega(\vec{k})} \end{aligned} \tag{63}$$

The commutation relations

$$\begin{aligned} &[a(\phi_1), a(\phi_2)] = [a^*(\phi_1), a^*(\phi_2)] = 0 \\ &[a(\phi_1), a^*(\phi_2)] = (\phi_1,\phi_2) = \int \phi_1(\vec{k})^*\phi_2(\vec{k})\frac{d^3\vec{k}}{2\omega(\vec{k})} \end{aligned} \tag{64}$$

imply for the heuristic operators

$$[a(\vec{k}_1),\ a(\vec{k}_2)] = [a^*(\vec{k}_1),\ a^*(\vec{k}_2)] = 0$$
$$[a(\vec{k}_1),\ a^*(\vec{k}_2)] = 2\omega(\vec{k})\delta(\vec{k}_1-\vec{k}_2) \tag{65}$$

The use of the operators $a(\vec{k})$ and $a^*(\vec{k})$ instead of the rigorous $a(\phi)$ and $a^*(\phi)$ is a universal feature of standard field theory text books. It fits in quite well with the use of the sharp momentum one-particle states $|\vec{k}>$, because (60) and (63) gives

$$|\vec{k}> = a^*(\vec{k})|0> \tag{66}$$

One considers in fact heuristic general n-particle states $|\vec{k}_1, \ldots, \vec{k}_n>$, introduced in a way similar to the one-particle states $|\vec{k}>$ and satisfying the relation

$$|\vec{k}_1, \ldots, \vec{k}_n> = \frac{1}{\sqrt{n!}}\, a^*(\vec{k}_1) \ldots a^*(\vec{k}_n)|0> \tag{67}$$

Finally there are field operators $\phi(f)$ defined by (39). This definition is simple but in this rigorous form not very obvious. The non-rigorous version which appears again in all standard texts is much more enlightening. We "derive" it by writting in (39) the symbolic integrals over test functions and operators at sharp x and $\vec{k}$ values. One one hand one has

$$\phi(f) = \int \phi(x)f(x)d^4x \tag{68}$$

On the other hand one may write, using (63), (57) and (47),

$$a^*(\chi f) + a(\chi f^*) = \int a^*(\vec{k})\sqrt{2\Pi}\,\hat{f}(\omega(\vec{k}),\vec{k})\frac{d^3\vec{k}}{2\omega(\vec{k})} + \int a(\vec{k})\sqrt{2\Pi}\,\hat{f}(-\omega(\vec{k}),-\vec{k})\frac{d^3\vec{k}}{2\omega(\vec{k})} =$$

$$= \frac{1}{(2\Pi)^{\frac{3}{2}}}\int\Big(\int_{k^o=\omega(\vec{k})} (a^*(\vec{k})e^{ikx} + a(\vec{k})e^{-ikx})\frac{d^3\vec{k}}{2\omega(\vec{k})}\Big)f(x)d^4x \tag{69}$$

Omitting the test functions $f(x)$ this results in

$$\phi(x) = \frac{1}{(2\Pi)^{\frac{3}{2}}}\int_{\omega(\vec{k})=\omega(\vec{k})} (a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx})\frac{d^3\vec{k}}{2\omega(\vec{k})} \tag{70}$$

This formula although heuristic has a clear physical meaning: The $a(\vec{k})$ and $a^*(\vec{k})$ are Fourier coefficients of the field operator $\phi(x)$ as wave phenomenon; $\vec{k}$ is a wave vector, $k^o = \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$ the corresponding frequency.

The relation between the unitary operators $U(a,\Lambda)$ as product operators and the creation and annihilation operators, as given in the appendix, leads easily to the covariance relation

$$U(a,\Lambda)\phi(f)U(a,\Lambda)^{-1} = \phi(T(a,\Lambda)f) \tag{71}$$

or in non-rigorous form

$$U(a,\Lambda)\phi(x)U(a,\Lambda)^{-1} = \phi(\Lambda x+a) \tag{72}$$

for every x in $\mathbb{R}^4$, translation a and Lorentz transformation Λ. Together with this unitary representation of the symmetry group there is a representation of the corresponding Lie algebra in terms of sum operators. Useful heuristic expressions for these generators follow easily from the general formalism sketched in the appendix. One obtains for instance for the translations $U(a,1)$, written as

$$U(a) = e^{ik_\mu P^\mu} \tag{73}$$

the bilinear expressions

$$P_\mu = \int k_\mu a^*(\vec{k}) a(\vec{k}) \frac{d^3\vec{k}}{2\omega(\vec{k})} \tag{74}$$

By using the inverse of the Fourier formula (70) the P_μ can be written as equally heuristic expressions in the field operator $\phi(x)$ and its partial derivatives. P_o is of course the Hamiltonian operator from (1). In this way one establishes in an explicit manner the symmetry properties with respect to the inhomogeneous Lorentz group, the symmetry group of special relativity, which completes our presentation of the free scalar quantum field as operator theory.

5. APPENDIX: The Fock space formalism.

The principal element in the operator representattion of free quantum fields is the use of the so-called Fock space or "many particle" construction. This is a Hilbert space construction of general significance in quantum theory, applied in various other areas, outside quantum field theory, such as quantum statistical mechanics and solid state physics. It is essentially a general procedure for assembling new quantum systems from given ones. It is in this general spirit that we shall in this appendix discuss it, stressing the basic mathematical structures involved in terms of tensor products and direct sums of Hilbert spaces, without worrying too much about the technical details connected with infinite dimensional linear algebra, details that can all be supplied by standard methods of functional analysis.

We have two main prescriptions for joining Hilbert spaces describing separate physical systems, to form the state space of a new "total" system:

1. Tensor products. Let $H_1, H_2, \ldots, H_n$ describe separate systems. When the systems are supposed to be different, e.g. different single particle systems, one uses the tensor product Hilbert space $H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$ to describe the situation in which all the given system are supposed to be present at the same time, e.g. a system of n different particles.

If the systems are identical, e.g. n single electrons, fundamental principles of quantum theory force us to use a reduced state space. There are two cases, corresponding with two different sort of fundamental systems occurring in nature, *bosons* (or bosonic systems), and *fermions* (or fermionic systems). An electron is for example a fermion, a photon is a boson. For bosonic systems we have to use the *symmetrized* tensor product of the n identical given Hilbert spaces, for fermionic systems the *anti-symmetrized* tensor product.

2. Direct sums. Consider again systems described by the Hilbert spaces $H_1, H_2, \ldots$ (this time the sequence is allowed to be infinite). The direct sum space $H = H_1 \oplus H_2 \oplus \ldots = \sum_j \oplus H_j$ describes a situation in which, with a certain probability, one may find one self either in a state ψ_1 from H_1, or in a state ψ_2 from H_2, or in a state ψ_3 from H_3, etc.

The two operations, the taking of tensor products and of direct sums, are the ingredients of a general construction by which the so-called Fock space is obtained from an arbitrarily given Hilbert space in two different ways, depending on whether one has a boson or a fermion system. In order to have a single set of formulas for both cases we employ an index ε taking two values, 0 for the boson case, 1 for that of fermions. Manipulations with ε should be modulo 2. There is a $\mathbb{Z}_2$ grading underlying the structure of boson and fermion Fock space, especially when the two are combined. In this respect the terms even and odd will sometimes be used. The $\mathbb{Z}_2$ gradedness is of great interest in further developments. The advantages of using it explicitly have so far not been fully realized. This will not be discussed further.

The construction of the Fock space over a given Hilbert space.
Suppose to be given a (complex) Hilbert space $H^{(1)}$, the so-called "1-*particle space*", with inner product $(\cdot,\cdot)^{(1)}$. (Note the physicists convention for the inner product; $(u,\lambda v) = \lambda(u,v)$, $\forall \lambda \in \mathbb{C}$).

Define, for a fixed choice of ε, and for $n = 1,2,3,\ldots$, as the symmetrized ($\varepsilon=0$), respectively antisymmetrized ($\varepsilon=1$) n-fold tensor product space

$$H_\varepsilon^n = (\otimes^n H^{(1)})_\varepsilon \tag{A1}$$

with in addition

$$H^{o} = \mathbb{C} \tag{A2}$$

The spaces H^{n}_{ε} are called "n-*particle spaces*"

REMARK. $(\otimes^{n}H^{(1)})$ is the image of the projection operator $P^{(n)}_{\varepsilon}$ in the ordinary tensor product space $\otimes^{n}H^{(1)}$; it is defined by linear extension of

$$P^{(n)}_{\varepsilon}(u_1\otimes\ldots\otimes u_n) = (u_1\otimes\ldots\otimes u_n)_{\varepsilon} = \frac{1}{n!}\sum_{\sigma}(\operatorname{sign}\sigma)^{\varepsilon}\, u_{\sigma(1)}\otimes\ldots\otimes u_{\sigma(n)} \tag{A3}$$

The sum is over all permutations σ of the indices $1, \ldots, n$. Every $H^{(n)}_{\varepsilon}$ is in an obvious way again a Hilbert space, with inner product defined by extension of

$$((u_1\otimes\ldots\otimes u_n)_{\varepsilon},\ (v_1\otimes\ldots\otimes v_n))^{(n)} = \frac{1}{n!}\sum(\operatorname{sign}\sigma)^{\varepsilon}\,(u_1,v_{\sigma(1)})^{(1)}(u_2,v_{\sigma(2)})^{(1)}\ldots(u_n,v_{\sigma(n)})^{(1)} \tag{A4}$$

In $H^{(0)}$ one has of course

$$(z,z') = \bar{z}z' \qquad z,z'\in\mathbb{C} = H^{(o)} \tag{A5}$$

REMARK. The inner product is nothing else but the restriction to symmetric, respectively anti-symmetric elements of the usual inner product in the ordinary tensor product space $\otimes^{n}H^{(1)}$, which is given by

$$(u_1\otimes\ldots\otimes u_n,\ v_1\otimes\ldots\otimes v_n)^{(n)} = (u_1,v_1)^{(1)}\ldots(u_n,v_n) \tag{A6}$$

We finally define the two versions of the "*many-particle space*" or *Fock space* as the direct sum of the $H^{(n)}_{\varepsilon}$

$$H_{\varepsilon} = \sum_{n=o}^{\infty}\oplus H^{(n)}_{\varepsilon} = \sum_{n=o}^{\infty}\oplus(\otimes^{n}H^{(1)})_{\varepsilon} \tag{A7}$$

H_ε is of course again a Hilbert space . An element ψ in H_ε is a possibly infinite sequence $\{\psi^{(n)}\}_{n=o}^{\infty}$, $\psi^{(n)} \in H_\varepsilon^{(n)}$, with $\sum_{n=o}^{\infty} (\psi^{(n)},\psi^{(n)})^{(n)} < \infty$. The inner product in H_ε is

$$(\psi,\psi') = \sum_{n=o}^{\infty} (\psi^{(n)},\psi'^{(n)})^{(n)} \tag{A8}$$

A special rôle is played by a unit vector Ω_o in $H^o(=\mathbb{C})$, the "*vacuum vector*"

$$\Omega_o = 1 \ (\in \mathbb{C} = H^{(0)}) \tag{A9}$$

Creation and annihilation operators in Fock space.

We define in the Fock space H for every vector u in the 1-particle space $H^{(1)}$ two operators: an operator $c(u)$ (*creation operator*), and an operator $a(u)$ (*annihilation operator*), by means of linear extension of

$$\begin{cases} c(u)\Omega_o = u \\ c(u)(u_1\otimes\ldots\otimes u_n)_\varepsilon = \sqrt{n+1}(u\otimes u_1\otimes\ldots\otimes u_n)_\varepsilon \end{cases} \tag{A10}$$

$$\begin{cases} a(u)_o = 0 \\ a(u)(u_1\otimes\ldots\otimes u_n)_\varepsilon = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^{\varepsilon(j+1)} (u,u_j)^{(1)} (u_1\otimes\ldots\otimes u_{j-1}\otimes\ldots\otimes u_n)_\varepsilon \end{cases} \tag{A11}$$

for all 1-particle vectors $u, u_1, \ldots, u_n$ and for $n = 1,2,3,\ldots$. Note that we dropped the index ε at places where its rôle is not important. In our discussion where we consider either bosons or fermions and not a combination of the two types of systems at the same time, this makes sense in the interest of a transparant notation.

For these creation and annihilation operators a few simple but important properties can be easily established:

$$\begin{cases} c(u) & \text{depends (complex) linearly on } u \text{ in } H^{(1)}. \\ a(u) & \text{depends (complex) anti-linearly on } u \text{ in } H^{(1)}. \end{cases} \tag{A12}$$

$$(\psi, a(u)\psi') = (c(u)\psi, \psi') \qquad \forall u \in H^{(1)}, \quad \psi,\psi' \in H \tag{A13}$$

This means in the loose non-technical manner of our discussion that $a(u)$ and $c(u)$ are hermitian adjoints to each other. One writes therefore usually $c(u)$ as $a^*(u)$.

Characteristic are the following algebraic properties, *commutation relations*, respectively *anti-commutation relations*

$$\begin{aligned} [a(u), a(v)]_\varepsilon &= [c(u), c(v)]_\varepsilon = 0 \\ [a(u), c(v)] &= (u,v)^{(1)} \mathbb{1} \end{aligned} \tag{A14}$$

for all $u,v \in H^{(1)}$ and with the bracket $[\cdot,\cdot]_\varepsilon$ defined as

$$[A,B]_\varepsilon = AB - (-1)^\varepsilon BA \tag{A15}$$

With $c(u) = a^*(u)$ one has the more conventional form

$$\begin{aligned} [a(u), a(v)]_\varepsilon &= [a^*(u), a^*(v)]_\varepsilon = 0 \\ [a(u), a^*(v)]_\varepsilon &= (u,v)^{(1)} \mathbb{1} \end{aligned} \tag{A16}$$

REMARK. The Hilbert space H is generated by the operators $a^*(u)$ from the vacuum vector, i.e. H is spanned linearly by Ω_o and vectors of the form

$$a^*(u_1)a^*(u_2) \ldots a^*(u_n)\Omega_o = \sqrt{n!}(u_1 \otimes \ldots \otimes u_n)_\varepsilon \tag{A17}$$

Operators in H induced by operators in $H^{(1)}$.

There are two important ways in which operators in $H^{(1)}$ ("1-particle operators") generate operators in the full Fock space:

1. Product operators. (of special importance for *unitary* operators). Suppose an operator $U^{(1)}$ given in $H^{(1)}$. Define the (tensor) product operator $p(U^{(1)})$ in H by linear extension of

$$\begin{cases} p(U^{(1)})\Omega_o = \Omega_o \\ p(U^{(1)})(v_1 \otimes \ldots \otimes v_n)_\varepsilon = (U^{(1)}v_1 \otimes \ldots \otimes U^{(1)}v_n)_\varepsilon \end{cases} \tag{A18}$$

One has

$$\begin{cases} p(u^{(1)})\big|_{H^{(1)}} = U^{(1)} \quad (p(U^{(1)}) \text{ indeed extends } U^{(1)}) \\ U^{(1)} \text{ unitary} \Rightarrow P(U^{(1)}) \text{ unitary.} \\ p(1_{H^{(1)}}) = 1_H \\ p(U^{(1)}V^{(1)}) = p(U^{(1)})p(V^{(1)}) \quad \text{(multiplicativity)} \end{cases} \tag{A19}$$

In this manner one may extend a unitary representation of a *group* of operators in $H^{(1)}$ to a unitary representary in H.

Useful relations are also

$$\begin{cases} a(U^{(1)}v) = p(U^{(1)})a(v)p(U^{(1)})^{-1} \\ a^*(U^{(1)}v = p(U^{(1)})a^*(v)p(U^{(1)})^{-1} \end{cases} \tag{A20}$$

2. Sum operators. (of special importance for "hermitian", i.e. self-adjoint operators). Suppose an operator $A^{(1)}$ given in $H^{(1)}$. Define the sum operator $s(A^{(1)})$ in H by linear extension of

$$\begin{cases} s(A^{(1)})\Omega_o = 0 \\ s(A^{(1)})(u_1\otimes\ldots\otimes u_n) = (A^{(1)}u_1\otimes u_2\otimes\ldots\otimes u_n)_\varepsilon + \\ \qquad + (u_1\otimes A^{(1)}u_2\otimes\ldots\otimes u_n)_\varepsilon + \ldots + (u_1\otimes u_2\otimes\ldots\otimes A^{(1)}u_n)_\varepsilon \end{cases} \tag{A21}$$

One has

$$s(A^{(1)})\Big|_{H^{(1)}} = A^{(1)} \quad (s(A^{(1)}) \text{ indeed extends } A^{(1)})$$

$$\begin{cases} A^{(1)} \text{ hermitian (self-adjoint)} \Rightarrow s(A^{(1)}) \text{ hermitian (self-adjoint)} \\ s(1_{H^{(1)}}) = N, \text{ the "number operator" defined by } N\psi^{(n)} = n\psi^{(n)}, \\ \qquad \psi^{(n)} \in H^{(n)}, \quad n = 1,2,\ldots \\ s(\lambda A^{(1)}+\mu B^{(1)}) = \lambda s(A^{(1)}) + \mu s(B^{(1)}). \quad \text{(linearity)} \end{cases} \tag{A22}$$

In general the property of multiplicativity does not hold for $s(\cdot)$ as it does for $p(\cdot)$

$$s(A^{(1)})s(B^{(1)}) \neq s(A^{(1)})s(B^{(1)}) \tag{A23}$$

Instead one has the important *Lie algebra property*

$$s([A^{(1)}, B^{(1)}]) = [s(A^{(1)}), s(B^{(1)})] \tag{A24}$$

for each pair of 1-particle operators $A^{(1)}$, $B^{(1)}$ and with the ordinary $\varepsilon = 0$ commutator bracket for bosons as well as for fermions. This means that the sum operator procedure extends a representation of a Lie algebra in $H^{(1)}$ to a representation in H.

$$\begin{cases} a(A^{(1)}u) = -[s(A^{(1)}), a^*(u)] \\ a^*(A^{(1)}u) = [s(A^{(1)}), a(u)] \end{cases} \tag{A25}$$

with again the ordinary commutator bracket for both cases.

It is obvious that the relation between a Lie group representation in $H^{(1)}$ and its corresponding Lie algebra representation is preserved when the group operators are extended to H by the product procedure and the Lie algebra operators by the sum procedure. In particular one has the relation

$$p(e^{iA^{(1)}}) = e^{is(A^{(1)})} \tag{A26}$$

FURTHER REMARKS.

1. We have given here only the essential "algebraical" properties of the Fock space formalism. In general infinite dimensional Hilbert spaces are involved. (In the boson case the many-particle space H is always infinite dimensional, in the fermion case H is infinite dimensional if and only if the one-particle space $H^{(1)}$ is infinite dimensional.) Infinite dimensional Hilbert space entails all sorts of mathematical complications and subtleties: Unbounded operators with their domain problems. (Creation and annihilation operators are always bounded in the case of fermions and unbounded in the boson case.) The distinction between symmetric and self-adjoint operators. The technicalities of Hilbert space tensor products, etc. All these mathematical details can be treated in a completely satisfactory manner by standard methods from functional analysis, without changing the essential features of the picture given here.

2. For practical applications the following is of great importance. Let $e_1, e_2, \ldots$ be an orthonormal base in the one-particle space $H^{(1)}$. Instead of the full set of creation and annihilation operators $a^*(u)$, $a(u)$, depending on general vectors u from $H^{(1)}$ on uses

$$\begin{aligned} a_j &= a(e_j) \\ a_j^* &= a^*(e_j) \qquad j = 1,2,3,\ldots \end{aligned} \tag{A27}$$

In this way one obtains a system of creation and annihilation operators depending on a discrete index, with the fundamental algebraic relations

$$[a_j, a_k]_\varepsilon = [a_j^*, a_k^*]_\varepsilon = 0$$
$$[a_j, a_k^*]_\varepsilon = \delta_{jk} \mathbb{1} \qquad j,k = 1,2,3,\ldots \tag{A28}$$

This leads among other things to a practical formula for sum operators. Suppose the one-particle space operator $A^{(1)}$ is given by its matrix elements

$$A^{(1)}_{jk} = (e_j, A^{(1)} e_k) \tag{A29}$$

It is then easy to show that the sum operator $s(A^{(1)})$ in H can be written as

$$s(A^{(1)}) = \sum_{j,k=1\ldots} A^{(1)}_{jk}\, a_j^*\, a_k \tag{A30}$$

If in particular the matrix $A^{(1)}_{jk}$ is diagonal, i.e. $A^{(1)}_{jk} = \alpha_j \delta_{jk}$, this becomes

$$s(A^{(1)}) = \sum_{j=1,\ldots} \alpha_j\, a_j^*\, a_j \tag{A31}$$

The "particle number" operator N is, for every orthonormal base $e_1, e_2, \ldots$ in $H^{(1)}$

$$N = s(\mathbb{1}_{H^{(1)}}) = \sum_{j=1,2\ldots} a_j^*\, a_j \tag{A32}$$

3. The next step in the same direction is of even greater practical value. It consists of the introduction of *heuristic creation and annihilation operators*, depending on a continuous index. Let, as a typical example, $H^{(1)}$ be given as an L^2 function space, e.g. $H^{(1)} = L^2(\mathbb{R}^3, d\vec{k})$, $\vec{k} = (k^1, k^2, k^3)$. Vectors u in $H^{(1)}$ are functions of $\vec{k}$, with inner product

$(u,v) = \int u^*(\vec{k})v(\vec{k})d\vec{k}$. The n-particle spaces $H^{(n)}$ are again function spaces n.l. $L^2_\varepsilon(\mathbb{R}^{3n}, d\vec{k}_1 \ldots d\vec{k}_n)$, consisting of symmetric (ε=0) or anti-symmetric (ε=1) functions of $\vec{k}_1, \vec{k}_2, \ldots, \vec{k}_n$. The creation and annihilation operators depend linearly, respectively anti-linearly on functions u from $H^{(1)}$. This suggests the symbolic expressions

$$\begin{aligned} a^*(u) &= \int a^*(\vec{k})u(\vec{k})d\vec{k} \\ a(u) &= \int a(\vec{k})u^*(\vec{k})d\vec{k} \end{aligned} \tag{A33}$$

In this way a system of heuristic creation and annihilation operators depending on a continuum of values $\vec{k}$ in $\mathbb{R}^3$ is obtained. The characteristic commutation/anti-commutation relations are

$$\begin{aligned} [a(\vec{k}_1), a(\vec{k}_2)]_\varepsilon &= [a^*(\vec{k}_1), a^*(\vec{k}_2)]_\varepsilon = 0 \\ [a(\vec{k}_j), a^*(\vec{k}_2)]_\varepsilon &= \delta(\vec{k}_1 - \vec{k}_2)\,\mathbb{1} \end{aligned} \tag{A34}$$

The operators $a^*(\vec{k})$ and $a(\vec{k})$ lead to equally heuristic expressions for sum operators: Let the one-particle operator $A^{(1)}$ in $H^{(1)}$ be characterized rigorously or, as is usually the case non-rigorously, by an integral kernel $A^{(1)}(\vec{k},\vec{k}')$, i.e. $(A^{(1)}u)(\vec{k}) = \int A^{(1)}(\vec{k},\vec{k}')u(\vec{k}')d\vec{k}'$. The sum operator $s(A^{(1)}$ is then, in analogy with formula (A30) written as

$$s(A^{(1)}) = \iint A^{(1)}(\vec{k}_1,\vec{k}_2)a^*(\vec{k}_1)a(\vec{k}_2)d\vec{k}_1 d\vec{k}_2 \tag{A35}$$

When $A^{(1)}$ is diagonal, i.e. acts by multiplication as $(A^{(1)}u)(\vec{k}) = \alpha(\vec{k})u(\vec{k})$, this becomes

$$s(A^{(1)}) = \int \alpha(\vec{k})a^*(\vec{k})a(\vec{k})d\vec{k} \tag{A36}$$

The particle number operator N has the symbolic form

$$N = \int a^*(\vec{k})a(\vec{k})d\vec{k} \tag{A37}$$

The heuristic form of the Fock space formalism based on operators such as $a^*(\vec{k})$, $a(\vec{k})$ and expressions as (A35) is used almost universally in standard quantum field theory textbooks. We introduced it very briefly already in chapter 4, with a slightly different normalization convention, more natural in connection with Lorentz covariance properties.

4. The Fock space formalism as presented here may be easily generalized to the case where the 1-particle space $H^{(1)}$ is not a Hilbert space but a vector space with an inner product in a more general sense, i.e. not necessarily positive definite, but only nondegenerate. The mathematical details are technically different, but for a suitably chosen locally convex $H^{(1)}$ the results are quite satisfactory and slightly more elegant than for the Hilbert space case. Such a generalization seems to be required in a proper mathematical formulation of quantum electrodynamics, in particular for the quantum field describing photons, where a 1-particle space with indefinite inner product arises naturally. It is probably also relevant for quantum gauge fields in general.

5. Starting from separate boson and fermion Fock spaces one can construct mixed Fock spaces. Let $H_B^{(1)}$ and $H_F^{(1)}$ be a boson, respectively a fermion 1-particle space, with boson, respectively fermion Fock space H_B and H_F

$$\begin{aligned} H_B &= \sum_{n=0}^{\infty} \oplus H_B^{(n)} = \sum_{n=0}^{\infty} \oplus (\otimes^n H^{(1)})_{\varepsilon=0} \\ H_F &= \sum_{n=0}^{\infty} \oplus H_F^{(n)} = \sum_{n=0}^{\infty} \oplus (\otimes^n H_F^{(1)})_{\varepsilon=1} \end{aligned} \tag{A38}$$

The mixed "total" Fock space is then defined as

$$H = H_B \otimes H_F \tag{A39}$$

In H the vacuum vector is

$$\Omega_o = \Omega_o^B \otimes \Omega_o^F \tag{A40}$$

Boson and fermion creation and annihilation operators $a_B^*(u)$, $a_B(u)$, $\forall u \in H_B^{(1)}$ and $a_F^*(u)$, $a_F(u)$, $\forall u \in H_F^{(1)}$ are defined in $H_B^{(1)}$ and $H_F^{(1)}$ and re-appear in an obvious manner in $H = H_B \otimes H_F$ ($a_B(u) \longmapsto a_B(u) \otimes \mathbb{1}_{H_F}$, $a_F(u) \to \mathbb{1}_{H_B} \otimes a_F(u)$, etc.) Together with the separate commutation, respectively anti-commutation relations one has that all boson operators commute with all fermion operators.

The mixed Fock space H can be looked upon as built on a "total" 1-particle space, that is a direct sum of the separate 1-particle spaces $H_B^{(1)}$ and $H_F^{(1)}$

$$H^{(1)} = H_B^{(1)} \oplus H_F^{(1)} \tag{A41}$$

Representations of groups and associated representations of Lie algebras can be extended by the sum and product operator procedure if and only if the the representations in $H^{(1)}$ leave the subspaces $H_B^{(1)}$ and $H_F^{(1)}$ invariant. The physical meaning of this is that no proper symmetries exist between bosons and fermions in this many-particle formalism. $H^{(1)}$ can however be regarded as a $\mathbb{Z}_2$-graded vector space. A proper Lie algebra which leaves the subspaces invariant may be extended to a so-called "*graded Lie algebra*" which is not a Lie algebra but something more general. It does mix up the boson and fermion 1-particle spaces and has a natural extension to the full Fock space H. This leads to the concept of *supersymmetry*, which has recently become of interest.

6. REFERENCES.

The theory of free quantum fields can be found in the first chapters of any one of the many standard textbooks on quantum field theory. The treatment is always heuristic. Typical modern examples are: C. Itzykson, J-B. Zuber, "Quantum field theory", New York, etc. 1980, a very extensive book, and N.N. Bogoliubov, D.V. Shirkov, "Quantum fields", Reading Mass., 1983, a book of more managable proportions. Mathematically rigorous formulations with all technical details are given in the few books devoted to mathematical aspects of quantum field theory. An example is N.N. Bogolubov, A.A. Logunov, I.T. Todorov, "introduction to axiomatic quantum field theory". See also M. Reed, B. Simon, "Methods of modern mathematical physics. II". These presentations stress the mathematical details, but are less explicit on the underlying structure.

The first rigorous treatment of the Fock space formalism was given by J.M. Cook, Trans.Amer.Math.Soc. 74, 222(1953). A now standard presentation of the mathematical aspects of the formalism, although somewhat uneven in its attention to detail, is F.A. Berezin, "The method of second quantization", New York 1966. For the mathematical theory of free fields and Fock space formalism in terms of indefinite metric spaces from distribution theory, see P.J.M. Bongaarts, J.Math.Phys. 23, 1881(1982).

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Particles, Fields and Scattering in Relativistic Quantum Theory

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PARTICLES, FIELDS AND SCATTERING IN RELATIVISTIC QUANTUM THEORY

1. INTRODUCTION

In discussing the program of this year's seminar it was suggested that a contribution on scattering theory in relativistic field theory including the perturbation expansion, Feynman diagrams, divergences and renormalization theory would be a fine topic to round off the introduction to field quantization presented in the first year of the seminar [1] and Dr. Bongaart's lectures on path integral quantization and the scattering matrix [2]. Soon after the discussion it was realized that this subject would be a rather technical contribution which would not really fit in the framework of Mathematical structures in field theories. For this reason we changed the subject. Instead of a lot of technicalities on Feynman diagrams we are going to discuss the concept of operator fields and the scattering operator in relativistic field theory from a more fundamental point of view, called the particle point of view.

Our starting point will be the experimentally well established fact that the restricted Poincaré group is a symmetry group for the interactions between elementary particles. From the symmetries of this group follow the conservation laws of energy-momentum and angular momentum. These conservation laws allow a rather detailed analysis of the kinematics of a process without knowing the precise form of the interaction. We illustrate this by a well known simple example. For the process of scattering of a photon by a free electron (Compton effect) one easily finds, using the conservation of energy and momentum, that the change $\Delta\lambda$ in the wavelength of the photon is related to the scattering angle Θ, the angle between the momentum of the photon before and after scattering, by the simple formula

$$\Delta\lambda = \frac{h}{mc}\,(1 - \cos\Theta)$$

where h is Planck's constant, m the mass of the electron and c the speed of light.

Taking into account conservation laws following from other symmetries on obtains a global picture of physical processes. Symmetry considerations are a powerful tool in particle physics. New particles have been predicted on symmetry grounds and have been found in experiments. The recent (1983) discovery of the intermediate vector bosons Z and $W^{\pm}$ which together with the photon are the carriers of weak- and electromagnetic interactions is of major importance for the gauge symmetry foundation of particle physics.

For a detailed analysis i.e. the calculation of the probability amplitude to obtain a definite final state from a given initial state one needs the precise form of the interaction between the particles taking part in a process. In relativistic quantum theory the formalism used to describe particles and interactions is local field theory. Quantum field theory started in 1927 with the quantization of the electromagnetic field by Dirac. Classically the free electromagnetic field can be considered as an infinite number of harmonic oscillators which can be quantized in the usual way. In doing this one is led to the operator fields $\underline{\underline{A}}^{\mu}(x^{o}, \vec{x})$ for the vector potential. Soon after Dirac's work Jordan and Wigner showed that a system consisting of an arbitrary number of identical free fermions e.g. electrons can alternatively be described by a fermion operator field $\underline{\underline{\psi}}(x^{o}, \vec{x})$ i.e. a field which satisfies "canonical" *anti*-commutation relations. After these achievements it was only a minor step, be it with unforeseen consequences, to write down the interaction lagrangian for electromagnetic interactions.

$$L_{e.m} = q\, \underline{\underline{\bar{\psi}}}(x)\gamma^{\mu}\underline{\underline{\psi}}(x)\, \underline{\underline{A}}_{\mu}(x)$$

Inspired by this Fermi (1934) succeeded in constructing a lagrangian for weak interactions, the famous four-fermion interaction

$$L_{weak} = G_F \bar{\underline{\psi}}_p \underline{\psi}_n \underline{\psi}_e \underline{\psi}_{\bar{\nu}}$$

This lagrangian describes the decay of the neutron. (The labels on $\underline{\psi}$ refer to proton, neutron, electron and anti-neutrino)..

This in turn led Yukawa to a model lagrangian for the strong interactions of nucleons(N) and mesons(Π)

$$L_{strong} = f \underline{\underline{N}} \vec{\tau} \underline{\underline{N}} \cdot \vec{\underline{\underline{\Pi}}}$$

We will not go into all the modifications based on experimental data and symmetry principles which have led to refinements of the models for weak and strong interactions.

Nowadays one needs not learn anymore about Fermi's and Yukawa's lagrangian. Modern particle physics is based on gauge theory. Electro-weak and strong interaction are non-abelian gauge theories based on the gauge group

$$SU(3) \times SU(2) \times U(1).$$

After these divertimenti we make two important observations concerning scattering processes.

In a scattering process one is dealing with a physical system which is prepared in a well defined initial state and which ends up in some final state. We need a theory which enables us to describe the evolution of the system and to calculate the probality amplitude to obtain a specific final state from a given initial state. The framework for this is quantum theory. The evolution is described by an operator S, called the evolution (or scattering) operator.

The second observation concerns the construction of the evolution operator. In almost all cases all one can do is compare the evolution of a system of interacting particles with the evolution of a free system, in other words, in almost all cases the interaction must be treated as a perturbation

on the free motion. If one considers the enormous amount of literature on the evolution operator, varying from the highly mathematical approach of axiomatic field theory via the analytic S-matrix theory to the pure perturbation theory approach one realizes that it is almost impossible to obtain any results at all in a few lectures.

We propose the following program.
We start with a description of the unitary irreducible representations of the restricted Poincaré group. This gives us the appropriate description of one-particle states, characterized by mass and spin.

From the one-particle states we obtain the formalism describing a system of non-interacting identical particles, the creation- and annihilation operators, i.e. the Fock-space formalism. An equivalent description of a system of identical particles is given by covariant operator fields on space-time. These fields are defined from the creation- and annihilation operators by suitable Fourier transforms. The relation between spin and statistics, i.e. commutation relations for bosons (integer spin) and anti-commutation relations for fermions (half integral spin) for the operator fields, is derived from the condition of microcausality. This condition ensures that observables $O(x)$ constructed from the operator fields have a vanishing commutator for spacelike separated points:

$$[O(x), O(y)] = o \quad \text{for} \quad (x-y)^2 < o$$

i.e. measurements at such points have no correlation whatsoever.

After these preliminaries (which will take some time) we sketch the description of the evolution operator as it was given by Lehmann, Symanzik and Zimmermann. This yields an expression for the scattering matrix S which contains apart from wave functions for the particles participating in a process the Green's function $\tau(x_1,\ldots,x_n)$ as the vacuum expectation value of

a time-ordered product of operator fields

$$\tau(x_1,\ldots,x_n) = \langle o|T\,\underline{\underline{\phi}}(x_1)\ \ldots\ \underline{\underline{\phi}}(x_n)|\,o\rangle$$

At this point we make contact with Dr. Bongaarts contribution on path integrals. Green's functions can be obtained by functional derivation from a generating functional $Z[J]$,

$$Z[J] = N \int D[\phi]\, e^{i\int (L(\phi,\ldots) + \phi(x)J(x))d^4x}$$

$$\tau(x_1,\ldots x_n) = \frac{1}{i^n}\frac{\delta}{\delta J(x_1)}\ \cdots\ \frac{\delta}{\delta J(x_n)}\ Z[J]\Big|_{J=o}$$

REMARK.

We assume that the reader is familiar with the contributions of Bauerle [3], Bongaarts [2] and De Kerf [1] to the seminar.

2. UNITARY IRREDUCIBLE REPRESENTATIONS OF THE RESTRICTED POINCARÉ GROUP.

2.1. The Poincaré group and the Lorentz group.

The Poincaré group P is the group of inhomogeneous linear transformations on the four-dimensional space-time manifold which preserve a certain quadratic form. If we denote space-time coordinates with respect to some Lorentz frame (initial frame) by $x^\mu = (x^o, x^1, x^2, x^3)$ with $x^o = ct$ then the invariant form reads

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \tag{2.1}$$

with $\eta = \mathrm{diag}(1,-1,-1,-1)$. (Throughout this chapter summation convention is used.) The transformations are the mappings $x \longmapsto \hat{x}$ with

$$\hat{x}^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{2.2}$$

where Λ is a real 4×4 matrix and $a^\mu = (a^o, a^1, a^2, a^3)$ an arbitrary set of four real numbers. Invariance of (2.1) under the mappings (2.2) means

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = \eta_{\mu\nu} d\hat{x}^\mu d\hat{x}^\nu \tag{2.3}$$

This leads to restrictions on the matrices Λ:

$$\eta_{\mu\nu} \Lambda^\mu_\lambda \Lambda^\nu_\rho = \eta_{\lambda\rho} \tag{2.4}$$

From (2.4) one obtains by taking $\lambda = \rho = o$

$$(\Lambda^o_o)^2 \geqslant 1 \tag{2.5}$$

and by taking the determinant

$$(\det \Lambda)^2 = 1 \tag{2.6}$$

So one has either $\det \Lambda = 1$ or $\det \Lambda = -1$ and $\Lambda^o_o \geqslant 1$ or $\Lambda^o_o \leqslant -1$.

We will denote the transformations of $\underline{P}$ by (a,Λ). Transformations of type $(a,\mathbb{1})$ with $\mathbb{1}$ the unit matrix are called *translations*, transformations $(0,\Lambda)$ are called *Lorentz transformations*. The composition law of the group reads

$$(a_2,\Lambda_2)(a_1,\Lambda_1) = (a_2+\Lambda_2 a_1, \Lambda_2\Lambda_1) . \tag{2.7}$$

The identity element is $(o,\mathbb{1})$. The inverse of (a,Λ) is the group element $(a,\Lambda)^{-1} = (-\Lambda^{-1}a,\Lambda^{-1})$. Both the translations $T = \{(a,\mathbb{1})\}$ and the Lorentz transformations $L = \{(o,\Lambda)\}$ are subgroups of $\underline{P}$. The abelian translation subgroup T is an invariant subgroup:

$$(b,\Lambda)(a,\mathbb{1})(b,\Lambda)^{-1} = (\Lambda a,\mathbb{1}), \qquad (b,\Lambda) \in \underline{P}. \tag{2.8}$$

From now on we restrict ourselves to the subgroup $P_+^\uparrow$, this is the group of transformations (a,Λ) with $\det\Lambda = 1$ and $\Lambda_o^o \geqslant 1$.
The group $L_+^\uparrow$ is defined as

$$L_+^\uparrow = \{\Lambda,\ \Lambda\in L \mid \det\Lambda = 1,\ \Lambda_o^o \geqslant 1\} \tag{2.9}$$

it is the *proper orthochronous Lorentz group*.

EXAMPLES OF LORENTZ TRANSFORMATIONS

Rotations.

Taking $\Lambda_o^o = 1$ we obtain from (2.4) that Λ must have the form

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}$$

with the 3×3 matrix R satisfying $R^TR = \mathbb{1}$ and $\det R = 1$. So $\hat{x}^o = x^o$ and $\hat{x}^i = R^i_j x^j$. These transformations are isomorphic with 3-dimensional rotations. Specifying a rotation by the rotational axis $\vec{n}$ (with $\vec{n}^2 = 1$) and the angle θ we have

$$\begin{cases} \hat{x}^0 = x^0 \\ \vec{\hat{x}} = \vec{x} + (\vec{n}\wedge\vec{x})\sin\theta + \vec{n}\wedge(\vec{n}\wedge\vec{x})\ (1-\cos\theta) \end{cases} \tag{2.10}$$

We will sometimes use the notation

$$\hat{x}^\mu = R(\vec{n},\theta)^\mu{}_\nu x^\nu \tag{2.11}$$

where it is understood that $R^0_0 = 1$ and $R^0_i = R^i_0 = 0$. The rotations form a compact subgroup of $L^\uparrow_+$.

SPECIAL LORENTZ TRANSFORMATIONS

From the passive point of view special Lorentz transformations give the relation between the coordinates x^μ of a space-time event with respect to an inertial frame K and the coordinates $\hat{x}^\mu$ of the same event with respect to an inertial frame $\hat{K}$ which moves with respect to K with constant velocity $\vec{v}$. The spatial axis of $\hat{K}$ are parallel to those of K (no rotation). If we take $\hat{K}$ and K such that their origins coincide at $x^0 = o$, a special Lorentz transformation reads

$$\begin{aligned} \hat{x}^0 &= \gamma(x^0 - \vec{\beta}\cdot\vec{x}) \\ \vec{\hat{x}} &= -\gamma\vec{\beta}x^0 + \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}) + \vec{x} \end{aligned} \tag{2.12}$$

with $\vec{\beta} = \vec{v}/c$, $\gamma = (1-\vec{\beta}^2)^{-\frac{1}{2}}$, $|\vec{\beta}| < 1$ and c the speed of light in vacuum.

We will adopt the active point of view. The special Lorentz transformation is then the mapping $x \longmapsto \hat{x}$ with

$$\begin{cases} \hat{x}^0 = \gamma(x^0 + \vec{\beta}\cdot\vec{x}) \\ \vec{\hat{x}} = \vec{x} + \gamma\vec{\beta}x^0 + \frac{\gamma^2}{\gamma+1}\vec{\beta}(\vec{\beta}\cdot\vec{x}) \end{cases} \tag{2.13}$$

A special Lorentz transformation is completely determined by the three components of $\vec{\beta}$. Instead of these parameters one can use a unit vector $\vec{m} = \vec{\beta}/|\vec{\beta}|$ and a parameter $\chi \geqslant 0$ defined by $\gamma = \cosh\chi$ and $\gamma|\vec{\beta}| = \sinh\chi$. In this parametrization (2.13) reads

$$\begin{cases} \hat{x}^0 = x^0 \cosh\chi + \vec{m}\cdot\vec{x} \sinh\chi \\ \\ \hat{\vec{x}} = \vec{x} + x^0\vec{m} \sinh\chi + \vec{m}(\vec{m}\cdot\vec{x})(\cosh\chi - 1) \end{cases} \tag{2.14}$$

Special Lorentz transformations do not form a subgroup. Special Lorentz transformations may be represented by points $\vec{m}\chi$ with $\chi \geqslant 0$ but otherwise arbitrary. This shows that the Lorentz group is a non-compact Lie group.

It can be shown that any Lorentz transformation $\Lambda \in L_+^\uparrow$ can be written as the product of a rotation and a special Lorentz transformation. A Lorentz transformation can therefore be labelled by six parameters $(\vec{n},\theta;\ \vec{m},\chi)$.

The tangent space of $L_+^\uparrow$ in $\Lambda = \mathbb{1}$ (i.e. the Lie algebra) is easily obtained from (2.10) and (2.14). Taking $\vec{n}_1 = (1,0,0)$, $\vec{n}_2 = (0,1,0)$, $\vec{n}_3 = (0,0,1)$ we get by differentiation with respect to θ and putting $\theta = 0$ the tangent vectors

$$I^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \equiv I^{23}, \quad I^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \equiv I^{31}$$

$$I^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = I^{12} \tag{2.15}$$

From (2.14) one obtains by taking $\vec{m}_1 = (1,0,0)$, $\vec{m}_2 = (0,1,0)$ and $\vec{m}_3 = (0,0,1)$ and by differentiation with respect to χ

$$(2.16)\qquad I^{o1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};\quad I^{o2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};\quad I^{o3} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

The matrices (2.15) and (2.16) form a basis for the Lie algebra of $L_+^\uparrow$. An arbitrary element F of the algebra can be written in the form

$$F = \tfrac{1}{2}\omega_{\mu\nu} I^{\mu\nu} \tag{2.17}$$

where we have defined $I^{\mu\nu} = -I^{\nu\mu}$ so that $\omega_{\mu\nu}$ can be taken anti-symmetric: $\omega_{\mu\nu} = -\omega_{\nu\mu}$. The matrices $I^{\mu\nu}$ can be written in an elegant form

$$(I^{\mu\nu})^{\rho}_{\lambda} = \eta^{\mu\rho}\eta^{\nu}_{\lambda} - \eta^{\nu\rho}\eta^{\mu}_{\lambda} \tag{2.18}$$

where $\eta^{\mu\rho} = \eta_{\mu\rho}$ and $\eta^{\mu}_{\rho} = \delta^{\mu}_{\rho}$.

The exponential mapping gives for the group elements

$$\Lambda = e^{\frac{1}{2}\omega_{\mu\nu} I^{\mu\nu}} \tag{2.19}$$

In physics one uses mostly a slightly different convention. With the definition

$$K^{\mu\nu} = iI^{\mu\nu} \tag{2.20}$$

a group element takes the form

$$\Lambda = e^{-\frac{1}{2}i\omega_{\mu\nu} K^{\mu\nu}} \tag{2.21}$$

The matrices $K^{\mu\nu}$ have the following commutation relations

$$[K^{\mu\nu}, K^{\rho\lambda}] = i(\eta^{\mu\lambda}K^{\nu\rho} + \eta^{\nu\rho}K^{\mu\lambda} - \eta^{\mu\rho}K^{\nu\lambda} - \eta^{\nu\lambda}K^{\mu\rho}) \tag{2.22}$$

The hermitian matrices $J^1 = K^{23}$, $J^2 = K^{31}$, $J^3 = K^{12}$ are called the generators of rotations. The anti-hermitian matrices $K^1 = K^{o1}$, $K^2 = K^{o2}$,

$K^3 = K^{o3}$ are the generators of special Lorentz transformations.

2.2. The group $SL(2,\mathbb{C})$ and the homomorphism onto $L_+^\uparrow$.

The group $SL(2,\mathbb{C})$ is the group of 2×2 complex matrices A with $\det A = 1$. We show that there is a two-to-one homomorphism from $SL(2,C)$ onto $L_+^\uparrow$. The elements A and $-A$ have the same image $\Lambda(A) = \Lambda(-A)$. To obtain an explicit form of $\Lambda(A)$ we associate with a 4-vector $x^\mu = (x^o,x^1,x^2,x^3)$ the hermitian matrix

$$X = \begin{pmatrix} x^o+x^3 & -x^1+ix^2 \\ -x^1-ix^2 & x^o-x^3 \end{pmatrix} = x^o\sigma^o - \vec{x}\cdot\sigma = x^\mu\sigma_\mu \tag{2.23}$$

with $\sigma^o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

The matrix X satisfies

$$X^\dagger = X, \quad \det X = (x^o)^2 - \vec{x}^2 = x^2, \quad \mathrm{Tr}\, X = 2x^o \tag{2.24}$$

There is a one-to-one correspondence between 2×2 hermitian matrices and elements of $\mathbb{R}^4$.

Consider now the transformation $X \longmapsto \hat{X}$ given by

$$\hat{X} = AXA^\dagger, \qquad A \in SL(2,C) \tag{2.25}$$

The matrix $\hat{X}$ satisfies

$$\hat{X}^\dagger = \hat{X} \quad \text{and} \quad \det\hat{X} = \det X \tag{2.26}$$

Because of the first property $\hat{X}$ can be written in the form (2.23)

$$\hat{X} = \hat{x}^\mu\sigma_\mu = A(x^\nu\sigma_\nu)A^\dagger \tag{2.27}$$

As $\det\hat{X} = \det X$ we have

$$\hat{x}^2 = x^2 \tag{2.28}$$

The transformation (2.25) gives a linear mapping $x \longmapsto \hat{x}$ which preserves the Lorentz length. We conclude that this mapping is a Lorentz transformation

$$\hat{x}^\mu = \Lambda(A)^\mu_\nu x^\nu \tag{2.29}$$

It is readily verified that the mapping $A \longmapsto \Lambda(A)$ is a homomorphism. To obtain the explicit form of $\Lambda(A)$ we need some properties of the matrices σ. The set $\sigma^\mu = (\sigma^0,\sigma^1,\sigma^2,\sigma^3)$ contains the unit matrix and the Pauli matrices. To streamline the notation we define $\sigma_\mu = \eta_{\mu\nu}\sigma^\nu = (\sigma_0,-\sigma^1,-\sigma^2,-\sigma^3)$. We will use the notation $\tilde{\sigma}_\mu = \sigma^\mu$ and $\tilde{\sigma}^\mu = \sigma_\mu$. It is now easy to see that

$$\mathrm{Tr}(\tilde{\sigma}^\mu\sigma_\rho) = 2\delta^\mu_\rho \tag{2.30}$$

Considering again (2.27)

$$\hat{x}^\mu\sigma_\mu = A(x^\nu\sigma_\nu)A^\dagger = (A\sigma_\nu A^\dagger)x^\nu$$

we obtain using (2.30)

$$\hat{x}^\mu = \tfrac{1}{2}(\mathrm{Tr}\,\tilde{\sigma}^\mu A\sigma_\nu A^\dagger)x^\nu$$

which yields

$$\Lambda(A)^\mu_\nu = \tfrac{1}{2}\mathrm{Tr}\,\tilde{\sigma}^\mu A\sigma_\nu A^\dagger \tag{2.31}$$

The kernel of the homomorphism consists of the matrices I and $-I$ in $SL(2,C)$. We have therefore a two-to-one mapping from $SL(2,\mathbb{C})$ to L. To show that the image is $L^\uparrow_+$ we use the polar decomposition of $SL(2,\mathbb{C})$. *Any* element $A \in SL(2,\mathbb{C})$ can *uniquely* be written as the product of a unitary matrix U with $\det U = 1$ and a positive definite hermitian matrix H with $\det H = 1$

$$A = UH \tag{2.32}$$

The unitary unimodular matrices in SL(2,C) constitute the subgroup $SU(2,\mathbb{C})$. These matrices can be parametrized by a unit vector $\vec{n}$ and an angle $\theta(0 \leqslant \theta < 2\pi)$

$$U(\vec{n},\theta) = I\cos\frac{\theta}{2} - i\vec{n}\cdot\vec{\sigma}\sin\frac{\theta}{2} = \exp - i\theta\vec{n}\cdot\frac{\vec{\sigma}}{2} \tag{2.33}$$

Substituting (2.33) in (2.31) we obtain the Lorentz transformations corresponding to the unitary subgroup. A direct calculation reveals that $\Lambda(U) = \Lambda(\vec{n},\theta)$ is just the rotation given by (2.11). So the unitary subgroup is mapped onto the subgroup of rotations of $L_+^{\uparrow}$.

The subset of hermitian matrices is not a subgroup. A matrix H with the properties mentioned above can be written in the form

$$H = h_o\sigma^o + h_1\sigma^1 + h_2\sigma^2 + h_3\sigma^3$$

with h_o, h_1, h_2, h_3 real and

$$\det H = h_o^2 - \vec{h}^2 = 1\,, \quad \mathrm{Tr}\,H = 2h_o > 0$$

Instead of the parameters h we can use the unit vector $\vec{m} = \vec{h}/|\vec{h}|$ and a parameter $\chi \geqslant 0$ defined by

$$h_o = \cosh\chi/2\,, \qquad |\vec{h}| = \sinh\chi/2 \tag{2.35}$$

With this parametrization (2.35) becomes

$$\begin{aligned} H(\vec{m},\chi) &= \sigma^o\cosh\chi/2 + \vec{m}\cdot\vec{\sigma}\sinh\chi/2 \\ &= \exp\chi\vec{m}\cdot\frac{\vec{\sigma}}{2} = \exp -i\chi\vec{m}\cdot i\frac{\vec{\sigma}}{2} \end{aligned} \tag{2.36}$$

Substitution of (2.36) in (2.31) yields after some calculation the Lorentz transformation $\Lambda(H) = \Lambda(\vec{m},\chi)$ corresponding to $H(\vec{m},\chi)$. The result is precisely the matrix for a special Lorentz transformation given in (2.13).

From the way of parametrization it is clear that one gets all of $L_+^\uparrow$.

EXAMPLES

Special Lorentz transformations.

Consider the energy-momentum four-vector $\underset{o}{p} = (mc, \vec{o})$ for a particle at rest $(\vec{p}=o)$. We ask for the special Lorentz transformation which transforms $\underset{o}{p}$ in $p = (p^o, \vec{p})$. In SL(2,C) we look for the matrix H satisfying

$$H \underset{o}{P} H = P \tag{2.37}$$

with $\underset{o}{P} = p^\mu \underset{o}{\sigma}_\mu = \begin{pmatrix} mc & o \\ o & mc \end{pmatrix}$, $P = p^\mu \sigma_\mu = \begin{pmatrix} p^o p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^o + p^3 \end{pmatrix}$

Solving (2.37) for H gives

$$H(p) = \frac{mc\sigma^o + p^\mu \sigma_\mu}{(2mc(p^o+mc))^{\frac{1}{2}}} \tag{2.38}$$

The inverse matrix $H^{-1}(p)$ transforms the 4-momentum p^μ satisfying $(p)^2 = m^2c^2$, $p^o > 0$ into $\underset{o}{p} = (mc,o)$. This shows that any 4-momentum on the hyperboloid $(p)^2 = (p^o)^2 - \vec{p}^2 = m^2c^2$ $(p^o > 0)$ can be mapped into the point $\underset{o}{p}$. In fact for any two points p and q on this hyperboloid there exists a matrix A in SL(2,C) which connects these points. An example of such a matrix is given by

$$A_{q \leftarrow p} = H(q)H^{-1}(p) \tag{2.39}$$

We call the hyperboloid an *orbit*. The group SL(2,C) (and therefore $L_+^\uparrow$) acts transitively on this orbit.

Little groups.

The problem is to determine the subgroup in $L_+^\uparrow$ which leaves invariant an arbitrary but fixed p^μ on the hyperboloid $(p)^2 = m^2c^2$ $(p^o > 0)$.

In $SL(2,C)$ we have to look for the subgroup of matrices A_p which satisfy

$$A_p(p^\mu\sigma_\mu)A_p^\dagger = p^\mu\sigma_\mu \tag{2.40}$$

The subgroup of matrices A_p is called the *little group* (or stabilizer) belonging to p. As the little groups belonging to different points q and p on the same orbit are isomorphic we can choose for p the most simple vector on the orbit i.e. $\underset{o}{p} = (mc, \vec{o})$. The matrices $A_{\underset{o}{p}}$ satisfy

$$A_{\underset{o}{p}} \begin{pmatrix} mc & o \\ o & mc \end{pmatrix} A_{\underset{o}{p}}^\dagger = \begin{pmatrix} mc & o \\ o & mc \end{pmatrix} \tag{2.41}$$

Form (2.41) we see that $A_{\underset{o}{p}} A_{\underset{o}{p}}^\dagger = I$. The little group for a time-like vector is therefore the unitary subgroup $SU(2,C)$.

For light-like vectors k ($k^2 = 0$, $k^o > 0$) things a more complicated. Any light-like vector k with $k^o > 0$ can be obtained from $\underset{o}{k} = (1,0,0,1)$ by a special Lorentz transformation in the z-directions which transforms $\underset{o}{k}$ into $\hat{k} = (k^o,0,0,k^o)$ and a rotation which transforms $\hat{k}$ into k. To determine the structure of the little group of a light-like vector we can therefore study the matrices $A_{\underset{o}{k}}$ in $SL(2,\mathbb{C})$ which satisfy

$$A_{\underset{o}{k}} \begin{pmatrix} o & o \\ o & 2 \end{pmatrix} A_{\underset{o}{k}}^\dagger = \begin{pmatrix} o & o \\ o & 2 \end{pmatrix} \tag{2.42}$$

One finds that the little group of a light-like vector is isomorphic to the group $E(2)$, the euclidean group of the 2-dimensional euclidean plane.

3. CONSTRUCTION OF THE UNITARY IRREDUCIBLE REPRESENTATIONS OF $P_+^\uparrow$

3.1. One-particle states, mass and spin.

The statement that the restricted Poincaré group $P_+^\uparrow$ is a symmetry group for a quantum system has far-reaching consequences. For a thorough discussion on symmetry transformations in quantum theory we refer to Bauerle [4]. Here we give the results for the group $P_+^\uparrow$.

We consider a quantum system with Hilbert space H, for which $P_+^\uparrow$ is a symmetry group. One can prove that if one takes instead of $P_+^\uparrow$ the inhomogeneous $SL(2,C)$ group $\{(a,A) \mid A \in SL(2,C)\}$ as the symmetry group one obtains a *unitary* representation on H.

The inhomogeneous $SL(2,C)$ group acts on $\mathbb{R}^4$ in the following way

$$x \longmapsto \hat{x} = \Lambda(A)x + a$$

where $\Lambda(A)$ is given by (2.29).

The multiplication law reads

$$(a_2,A_2)(a_1,A_1) = (a_2+\Lambda(A_2)a_1,\ A_2A_1) \tag{3.1}$$

To each transformation (a,A) corresponds a unitary operator $\underline{\underline{U}}(a,\Lambda)$ on H and one has

$$\underline{\underline{U}}(a_2,A_2)\underline{\underline{U}}(a_1,A_1) = \underline{\underline{U}}(a_2,\Lambda(A_2)a_1,A_2A_1) \tag{3.2}$$

The problem at hand is to determine all irreducible sets of unitary operators satisfying (3.2).

To solve this problem we consider first the operators corresponding to the abelian subgroup $\{(a,I)\}$ of translations:

$$\underline{\underline{U}}(a,I)\underline{\underline{U}}(b,I) = \underline{\underline{U}}(a+b,I) \tag{3.3}$$

$$(a^\mu = a^o,a^1,a^2,a^3)$$

The unitary operators $U(a,I)$ can be represented by

$$\underline{\underline{U}}(a,I) = e^{ia_\mu P^\mu/\hbar} \tag{3.4}$$

with

$$P^\mu = \frac{\hbar}{i}\frac{\partial}{\delta a^\mu}\underline{\underline{U}}(a,I)\Big|_{a=o} \tag{3.5}$$

The hermitian operators $P^\mu = (P^o,\vec{P})$ being the generators of translations are interpreted as the operators of energy (P^o) and momentum $(\vec{P})$. As the translation group is an abelian group we have

$$[P^\mu,P^\nu] = 0 \tag{3.6}$$

We now use the fact that the translation group is an invariant subgroup. From (2.8) we obtain

$$\underline{\underline{U}}^{-1}(b,A)U(a,I)\underline{\underline{U}}(b,A) = \underline{\underline{U}}(\Lambda^{-1}(A)a,I) \tag{3.7}$$

so

$$\underline{\underline{U}}^{-1}(b,A)e^{ia_\mu P^\mu}\underline{\underline{U}}(b,A) = e^{i(\Lambda^{-1}(A)a)_\mu P^\mu} \quad *) \tag{3.8}$$

Differentiating with respect to a_μ and taking $a_\mu = 0$ afterwards gives

$$\underline{\underline{U}}^{-1}(b,A)P^\mu\underline{\underline{U}}(b,A) = \Lambda(A)^\mu_\nu P^\nu \tag{3.9}$$

Clearly the operators P^o, P^1, P^2, P^3 transform as 4-vector under the transformations of the group.

The operator $(P)^2$ defined as

$$(P)^2 = \eta_{\mu\nu}P^\mu P^\nu = (P^o)^2 - \vec{P}^2 \tag{3.10}$$

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*) We omit Planck's constant $\hbar$.

is an invariant operator i.e.

$$\underline{\underline{U}}^{-1}(b,A)(P)^2\underline{\underline{U}}(b,A) = (P)^2 \tag{3.11}$$

In an irreducible representation $(P)^2$ is a multiple of the identity operator

$$(P)^2 = m^2 I \tag{3.12}$$

with m^2 real.

We restrict ourselves to the cases $m \geqslant 0$.

Using the properties of the operators P^μ we start the construction of a basis for an irreducible representation. To keep the treatment as short as possible we feel free to use Dirac's concept of generalized (improper) states. We take as a basis the common eigenvectors $|p,\alpha;m>$ of the commuting operators P^μ

$$P^\mu|p,\alpha;m> = p^\mu|p,\alpha;m> \tag{3.13}$$

The eigenvalues $p^\mu = (p^o,p^1,p^2,p^3)$ are real. The label α account for a possible degeneracy of energy-momentum states and will be investigated in a moment. The label m characterizes (partly) an irreducible representation, it will be omitted unless there is danger for confusion.

From (3.12) and (3.13) we obtain

$$p_\mu p^\mu = m^2, \quad p^o = \pm\sqrt{\vec{p}^2 + m^2} \tag{3.14}$$

Again we make a restriction. We consider only the case $p^o = \sqrt{\vec{p}^2 + m^2}$. Notice that (3.14) is precisely the relation between energy and momentum for a relativistic particle with mass m. With the plus sign for p^o the energy and momentum of a basis state $|p,\alpha>$ are completely specified by the momentum $\vec{p}$. We will, however, use the notation $|p,\alpha>$. As we have restricted

ourselves to the group $P_+^\uparrow$ the sign of p^o cannot change under Lorentz transformations. The continuum states $|p,\alpha>$ will be normalized invariantly according to

$$<p',\alpha'|p,\alpha> = (2\pi)^3 2p\delta(\vec{p}-\vec{p}')\delta_{\alpha\alpha'} \tag{3.15}$$

where we already assumed that α is a discrete label.

We now investigate the transformation properties of these states. Under translation (a,I) we have

$$\underline{\underline{U}}(a,I)|p,\alpha> = e^{ia_\mu P^\mu}|p,\alpha> = e^{ia_\mu p^\mu}|p,\alpha> \tag{3.16}$$

Hence the "physical state" remains the same.

To obtain the effect of a Lorentz transformation we use (3.8) with $b = o$.:

$$\begin{aligned} P^\mu \underline{\underline{U}}(o,A)|p,\alpha> &= \underline{\underline{U}}(o,A)\Lambda(A)^\mu_\nu P^\nu|p,\alpha> \\ &= \Lambda(A)^\mu_\nu p^\nu \underline{\underline{U}}(o,A)|p,\alpha> \end{aligned} \tag{3.17}$$

This tells us that the transformed state is again an eigenstate of P^μ, the eigenvalues $\Lambda(A)^\mu_\nu p^\nu$ are just the Lorentz transforms of p^μ.

From the transitivity of $SL(2,\mathbb{C})$ on the orbit $p^2 = m^2$ $(p^o > 0)$ it follows that all energy-momentum eigenstates $|p,\alpha>$ can be obtained from the set of states $|\underset{o}{p},\alpha>$ with $\underset{o}{p} = (m,\vec{o})$.

Equation (3.17) gives

$$\underline{\underline{U}}(o,A)|p,\alpha> = C_{\beta\alpha}(\Lambda(A)p,A)\ |\ \Lambda(A)p,\beta) \tag{3.18}$$

(Summation on β)

Performing two transformations

$$\underline{\underline{U}}(o,A_2)\underline{\underline{U}}(o,A_1) = \underline{\underline{U}}(o,A_2A_1)$$

we obtain for the matrices $(C_{\alpha\beta})$ the equation

$$C(p,A_2)C(\Lambda(A_2^{-1})p,A_1) = C(p,A_2A_1) \tag{3.19}$$

We are interested in solutions of (3.19) for which $C(p,A)$ is a finite dimensional *unitary* matrix for A in $SL(2,\mathbb{C})$. Without the p-dependence there would be no solution except the trivial one. This is because of the fact that $SL(2,\mathbb{C})$ being a non-compact group has no finite dimensional unitary representations.

Let us consider the subspace $H(p) = \{|p,\alpha>, p \text{ fixed}\}$ and the little group $L(p)$ belonging to p.

Denoting the elements of $L(p)$ by A_p (see 2.40) we obtaining from (3.19)

$$C(p,(A_2)_p)C(p,(A_1)_p) = C(p,(A_2)_p(A_1)_p) \tag{3.20}$$

Clearly the matrices $C(p,A)$ must be in a representation of $L(p)$ and we must investigate the irreducible unitary representations of the little group.

Once we have found these representations we can construct representations of the full group by the method of induced representations.

We will consider in detail the case $m > o$ and we give the result for $m = o$.

As we have seen in section 2 the little group for a time-like vector p^μ is the unitary group $SU(2,\mathbb{C})$. The unitary irreducible representation are well-known, they are denoted by $D^{(s)}(U)$, $U \in SU(2)$. The label s can take values $o, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The matrices $D^{(s)}(U)$ give a representation in a $(2s+1)$ dimensional vector space.

Taking the standard 4-vector $\underset{o}{p} = (m,\vec{o})$ we have in $H(\underset{o}{p})$

$$\underline{\underline{U}}(o,U)|\underset{o}{p};\sigma>^{(s)} = D^{(s)}_{\sigma'\sigma}(U)|\underset{o}{p};\sigma'>^{(s)} \qquad (U \in SU(2))$$
$$(\sigma,\sigma' = s, s-1, \ldots, -s) \tag{3.21}$$

The label s has a physical interpretation. With the standard momentum $\underset{o}{p}$

we have in $H(p_o)$ precisely $(2s+1)$ states $|p_o;\sigma>$. Under a "rotation" these states transform according to the unitary irreducible representation $D^{(s)}$. The states $|p_o;\sigma>$ are the $(2s+1)$ spin states of a massive particle with spin s. In the conventional basis the states are eigenstates of the commuting operators $\vec{S}^2$ and S^3 where

$$S^1 = \hbar\,\Sigma^1, \quad S^2 = \hbar\,\Sigma^2, \quad S^3 = \hbar\,\Sigma^3 \tag{3.22}$$

and $\Sigma^1, \Sigma^2, \Sigma^3$ are the generators in the $(2s+1)$-dimensional representation of the Lie algebra of $SU(2)$.

We have

$$\begin{aligned} S^3|p_o,\sigma> &= \hbar\,\sigma\ |p_o,\sigma'> \qquad (\sigma = s, s-1, \ldots, -s) \\ (\vec{S}^2)|p_o,\sigma> &= s(s+1)\hbar^2|p_o,\sigma'> \end{aligned} \tag{3.23}$$

From the results obtained sofar one can construct the representations of $P_+^\uparrow$ for $m > o$. We start with

$$\underline{\underline{U}}(o,U)|p_o,\sigma> = D^{(s)}_{\sigma'\sigma}(U)|p_o,\sigma'>$$

for a representation which mass $m > o$ and spin s. The $(2s+1)$ vectors $|p_o,\sigma>$ constitute a basis in $H(p_o)$. As we have shown in (2.37) and (2.38) any vector p^μ with $p^2 = m^2$ and $p^o > o$ can be obtained from $p_o = (m,\vec{o})$ by a special Lorentz transformation $H(p)$ (in $SL(2,\mathbb{C})$). We use these transformations to define a $(2s+1)$-dimensional basis in $H(p)$:

$$|p,\sigma> := \underline{\underline{U}}(o, H(p))|p_o,\sigma> \tag{3.24}$$

We must now show that this leads to a representation of the symmetry group, i.e. we must investigate the behaviour of $|p,\sigma>$ under $\underline{\underline{U}}(o,A)$. To do this we use the fact that any $A \in SL(2,\mathbb{C})$ can be obtained from an element A_{p_o} of the little group.

Consider the matrices A, $H(p)$ and $H(\Lambda(A)p)$. The matrix $H(p)$ transforms $\overset{p}{\underset{o}{}}$ into p, the matrix $H(\Lambda(A)p)$ transforms $\overset{p}{\underset{o}{}}$ into $\Lambda(A)p$ and $H^{-1}(\Lambda(A)p)$ transforms $\Lambda(A)p$ into $\overset{p}{\underset{o}{}}$. If we define

$$\overset{A_p}{\underset{o}{}} := H^{-1}(\Lambda(A)p)AH(p) \tag{3.25}$$

we have a matrix $\overset{A_p}{\underset{o}{}}$ which maps $\overset{p}{\underset{o}{}}$ in $\overset{p}{\underset{o}{}}$. Clearly $\overset{A_p}{\underset{o}{}}$ belongs to the little group of the standard vector $\overset{p}{\underset{o}{}} = (m,\vec{o})$.

The unitary matrix $\overset{A_p}{\underset{o}{}}$ defined in (3.25) and the rotation. $\Lambda(\overset{A_p}{\underset{o}{}}) \in L_+^\dagger$. are called Wigner rotations. It is clear that $\overset{A_p}{\underset{o}{}}$ depends on A *and* on p. We denote a Wigner rotation by $U(p,A)$,

$$U(p,A) = H^{-1}(\Lambda(A)p)\,AH(p) \in SU(2) \tag{3.26}$$

This simple formula is the cruse for the construction of the representations. Using (3.26) we can factorize the unitary operator $\underset{\equiv}{U}(o,A)$ corresponding to A:

$$\underset{\equiv}{U}(o,A) = \underset{\equiv}{U}(o,\, H(\Lambda(A)p)\underset{\equiv}{U}(o,\, U(p,A))\underset{\equiv}{U}(o,\, H^{-1}(p)) \tag{3.27}$$

This enables us to determine the transformation properties of $|p,\sigma>$ under $\underset{\equiv}{U}(o,A)$ by going first from $H(p)$ to $H(\overset{p}{\underset{o}{}})$ (use 3.23)

$$\underset{\equiv}{U}(o,A)\,|p,\sigma> = \underset{\equiv}{U}(o,\, H(\Lambda(A)p))\ \underset{\equiv}{U}(o,\, U(p,A))\,|\overset{p}{\underset{o}{},}\sigma> \tag{3.28}$$

Next we use (3.21) (recall that $U(p,A) \in SU(2)$. The result is

$$\underset{\equiv}{U}(o,A)\,|p,\sigma> = D^{(s)}_{\sigma',\sigma}(U(p,A))\,|\Lambda(A)p,\sigma> \tag{3.29}$$

with $\quad D^{(s)}(U(p,A)) = D^{(s)}(H^{-1}(\Lambda(A)p)\,A\,H(p))$

Formula (3.29) gives the transformation properties of $|p,\sigma>$ for any p^μ with $(p)^2 = m^2$ and $p^o = \sqrt{p^2+m^2}$ and for all $A \in SL(2,\mathbb{C})$.

From (3.16) and (3.29) one obtains

$$\underline{\underline{U}}(a,A)|p,\sigma> = e^{ia_\mu(\Lambda(A)p)^\mu} D^{(s)}_{\sigma',\sigma}(U(p,A))|\Lambda(A)p,\sigma'> \tag{3.30}$$

It is not difficult to show that the operators $\underline{\underline{U}}(o,A)$ defined by (3.30) satisfy the group property (3.2). The representation defined by (3.30) is characterized by two parameters namely $m > o$ and s.

We have derived the quantum states of a free particle with rest mass $m > o$ and spin s and the transformation law of these states under the inhomogeneous $SL(2,\mathbb{C})$ group.

The parameter m is related to the Casimir operator $(P)^2 = m^2 \mathbb{1}$. In section 3.3 where we discuss the Lie algebra of the Poincaré group we show that s is also related to a Casimir operator.

We will briefly discuss the results for $m = o$ $p \neq o$. The little group is then the Euclidean group $E(2)$ of the plane. To obtain a discrete set of states with the standard momentum $k^\mu_o = (1,0,0,1)$ the two commuting generators of $E(2)$ must be mapped in null operators in the representation. The remaining generator generates rotations and is denoted by J^3.

The irreducible representations are characterized by $m = o$ and a parameter λ with values $0, \pm\frac{1}{2}, \pm 1, \ldots$. This parameter is called the helicity.

In an irreducible representation $(m=o,\lambda)$ there is precisely one state for each four-momentum k. Denoting the states by $|k,[\lambda]>$ one derives:

$$\underline{\underline{U}}(o,A)|k,[\lambda]> = e^{-i\lambda\phi(k,A)}|\Lambda(A)k,[\lambda]> \tag{3.31}$$

The angle ϕ depends on k and A.

We conclude this section with the transformation properties of normalizable states ϕ.

$$\hat{\phi}_\sigma(p) := (\underline{\underline{U}}(a,A)\phi)_\sigma(p) = e^{ia_\mu p^\mu} D^{(s)}_{\sigma\sigma'}(H^{-1}(p)\, A\, H(\Lambda(A^{-1})p))\phi_{\sigma'}(\Lambda^{-1}(A)p)$$

$$\int(\sum_\sigma \phi^*_\sigma(p)\phi_\sigma(p)\, \frac{d^3p}{(2\pi)^3 2p^o} = 1 \tag{3.32}$$

3.2. The Lie algebra of the Poincaré group.

In this section we discuss briefly the Lie algebra of the Poincaré group.

For the generators of translations we have

$$[\underline{\underline{P}}^{\mu}, \underline{\underline{P}}^{\nu}] = o \tag{3.33}$$

From (3.8) with $b = o$ we obtain the commutation relations of $\underline{\underline{P}}^{\mu}$ with the generators of Lorentz transformation.

An infinitesimal transformation A in $SL(2,\mathbb{C})$ is of the form

$$A = \mathbb{1} + \tfrac{1}{2}\omega_{\mu\nu}\Sigma^{\mu\nu} + O(\omega^2) \tag{3.34}$$

The corresponding Lorentz transformation reads

$$\Lambda = \mathbb{1} + \tfrac{1}{2}\omega_{\mu\nu} I^{\mu\nu} + O(\omega^2) \tag{3.35}$$

with the matrices $I^{\mu\nu}$ given by (2.18).

We write the unitary operator $\underline{\underline{U}}(o,A)$ with A given by (3.34) as

$$\underline{\underline{U}}(o,A) = \underline{\underline{\mathbb{1}}} - \frac{i}{2}\omega_{\mu\nu}\underline{\underline{M}}^{\mu\nu} + O(\omega^2) \tag{3.36}$$

The operators $\underline{\underline{M}}^{\mu\nu}$ introduced here are hermitian operators.

Substituting (3.35) and (3.36) in (3.8) one obtains

$$\frac{i}{2}\omega_{\kappa\lambda}[\underline{\underline{M}}^{\kappa\lambda}, \underline{\underline{P}}^{\mu}] = \tfrac{1}{2}\omega_{\kappa\lambda}(I^{\kappa\lambda})^{\mu}_{\ \nu}\underline{\underline{P}}^{\nu}$$

Using (2.18) leads to

$$[\underline{\underline{M}}^{\kappa\lambda}, \underline{\underline{P}}^{\mu}] = -i\,(\eta^{\kappa\mu}\underline{\underline{P}}^{\lambda} - \eta^{\lambda\mu}P^{\kappa}) \tag{3.37}$$

The operators $\underline{\underline{M}}^{\kappa\lambda}$ are interpreted as the operators of the total relativistic angular momentum.

The commutation relations between the $\underline{\underline{M}}$'s can be read off from (2.22)

$$[\underline{\underline{M}}^{\mu\nu}, \underline{\underline{M}}^{\rho\lambda}] = i(\eta^{\mu\lambda}\underline{\underline{M}}^{\nu\rho} + \eta^{\nu\rho}\underline{\underline{M}}^{\mu\lambda} - \eta^{\mu\rho}\underline{\underline{M}}^{\nu\lambda} - \eta^{\nu\lambda}\underline{\underline{M}}^{\mu\rho}) \tag{3.38}$$

In section (3.1) we have seen that the unitary representations are constructed from the momentum eigenstates and representations of the little group. We now construct the generators of the little group L_p belonging to p by considering infinitesimal transformations. For $\Lambda_p \in L_p$ we have

$$\Lambda_p p = p$$

Taking $\Lambda_p = 1 + \frac{1}{2}\omega_{\kappa\lambda} I^{\kappa\lambda} + O(\omega^2)$ we obtain a condition on $\omega_{\kappa\lambda}$:

$$\omega_{\kappa\lambda} (I^{\kappa\lambda})^{\mu}_{\ \nu} p^{\nu} = o$$

Using (2.18) leads to

$$\omega_{\kappa\lambda} p^{\lambda} = o \qquad (\omega_{\kappa\lambda} = -\omega_{\lambda\kappa}) \tag{3.39}$$

Taking for p the standard vector $\underset{o}{p} = (m,\vec{o})$ gives $\omega_{10} = \omega_{20} = \omega_{30} = o$ and ω_{23}, ω_{31}, ω_{12} arbitrary. This confirms what we know. The little group is just the rotation group, or better $SU(2)$ in $SL(2,C)$.
In (3.39) we have four equations for six parameters. The equations are dependent as can be seen by contracting we p^{κ} :

$$\omega_{\kappa\lambda} p^{\kappa} p^{\lambda} = o$$

There are in fact just three independent equations. The solution of (3.39) can be given in a very elegant form

$$\omega_{\kappa\lambda} = \Sigma_{\kappa\lambda\mu\nu} \omega^{\mu} p^{\nu} \tag{3.40}$$

where ω^{μ} is an arbitrary infinitesimal four-vector.
Using (3.40) we obtain for Λ_p

$$\begin{aligned} \Lambda_p(\omega) &= \mathbb{1} + \tfrac{1}{2}\Sigma_{\kappa\lambda\mu\nu} I^{\kappa\lambda} \omega^{\mu} p^{\nu} \\ &= \mathbb{1} + \omega^{\mu} W_{\mu} \end{aligned}$$

where the matrices W_μ are defined by

$$W_\mu = \tfrac{1}{2}\Sigma_{\mu\nu\kappa\lambda} I^{\kappa\lambda} p^\nu$$

The four matrices W_μ are not independent, they satisfy

$$W_\mu p^\mu = o$$

The unitary operator $U(o,\Lambda_p(\omega))$ representing the infinitesimal transformations $\Lambda_p(\omega)$ are given by

$$\underline{\underline{U}}(o,\Lambda_p(\omega)) = \underline{\underline{1}} - i\omega^\mu \underline{\underline{W}}_\mu + O(\omega^2)$$

where the hermitian operators $\underline{\underline{W}}_\mu$ are defined as

$$\begin{aligned} \underline{\underline{W}}_\mu &= \tfrac{1}{2}\Sigma_{\mu\nu\kappa\lambda} \underline{\underline{P}}^\nu \underline{\underline{M}}^{\kappa\lambda} \\ &= \tfrac{1}{2}\Sigma_{\mu\nu\kappa\lambda} \underline{\underline{M}}^{\kappa\lambda} \underline{\underline{P}}^\nu \end{aligned} \tag{3.41}$$

Clearly the operators W_μ satisfy the equation

$$\underline{\underline{W}}_\mu \underline{\underline{P}}^\mu = o \tag{3.42}$$

The commutation relations are

$$\begin{aligned} [\underline{\underline{W}}_\mu, \underline{\underline{P}}_\nu] &= o \\ [\underline{\underline{W}}_\mu, \underline{\underline{W}}_\nu] &= -i\Sigma_{\mu\nu\kappa\lambda} \underline{\underline{W}}^\kappa \underline{\underline{P}}^\lambda \end{aligned} \tag{3.43}$$

The operator $(\underline{\underline{W}})^2 = \underline{\underline{W}}^\mu \underline{\underline{W}}_\mu$ commutes with the generators $\underline{\underline{P}}^\lambda$ and $\underline{\underline{M}}^{\mu\nu}$, hence $(\underline{\underline{W}})^2$ is a multiple of the identity operator in an irreducible representation of the group.

On the states $|_o^p, \sigma>$ the operators $\underline{\underline{W}}_\mu$ act in the following way

$$\underline{W}_\mu |\underset{o}{p},\sigma> = \tfrac{1}{2}\Sigma_{\mu\nu\kappa\lambda}\underline{M}^{\kappa\lambda}\underline{P}^{\nu}|\underset{o}{p},\sigma>$$

$$= \frac{m}{2}\Sigma_{\mu o\kappa\lambda}\underline{M}^{\kappa\lambda}|\underset{o}{p},\sigma>$$

This gives

$$\underline{W}_o|\underset{o}{p},\sigma> = o$$
$$\underline{W}_1|\underset{o}{p},\sigma> = -m\,\underline{J}^1|\underset{o}{p},\sigma>$$
$$\underline{W}_2|\underset{o}{p},\sigma> = -m\,\underline{J}^2|\underset{o}{p},\sigma>$$
$$\underline{W}_3|\underset{o}{p},\sigma> = -m\,\underline{J}^3|\underset{o}{p},\sigma>$$

As we have seen the angular momentum for the particle at rest is just the spin angular momentum, hence we can replace $\vec{\underline{J}}$ by $\vec{\Sigma}$. The result is

$$(\underline{W})^2|\underset{o}{p},\sigma> = m^2(\vec{\Sigma})^2|\underset{o}{p},\sigma> = m^2(s(s+1)|\underset{o}{p},\sigma>$$

As $[(\underline{W}^2), \underline{U}(o,A)] = o$ we have

$$(\underline{W})^2|p,\sigma> = m^2\, s(s+1)\, |p,\sigma>$$

We conclude that for a particle with $m > o$ the Casimir operator $(\underline{W})^2$ determines the spin s.

4. IDENTICAL PARTICLES

4.1. Many-particle states, construction operators.

We consider massive particles $(m > o)$ with spin s. The space of one-particle states $\{|p,\sigma>\}$ is denoted by $H^{(1)}$.

For a two-particle state we define

$$|p_1\sigma_1; p_2\sigma_2> = |p_1\sigma_1>_1 |p_2,\sigma_2>_2$$

In this state we have particle 1 with quantum labels $p_1\sigma_1$ and particle 2 with labels $p_2\sigma_2$. For identical particles the distinction 1 and 2 makes no sense and one must consider symmetric or antisymmetric states. We define

$$|p_1\ \sigma_1; p_2\sigma_2>^{sym} = \frac{1}{2!}\{|p_1\sigma_1>_1 |p_2\sigma_2>_2 + |p_2\sigma_2>_1 |p_1\sigma_1>_2\}$$

$$|p_1\ \sigma_1; p_2\sigma_2>^{asym} = \frac{1}{2!}\{|p_1\sigma_1>_1 |p_2\sigma_2>_2 - |p_2\sigma_2>_1 |p_1\sigma_1>_2\}$$

and generalize to N-particle states

$$|p_1\sigma_1; p_2\sigma_2 \cdots p_N\sigma_N>^{sym} = \frac{1}{N!}\sum_{perm.} |p_1\sigma_1>_1 |p_2\sigma_2> \cdots |p_N\sigma_N>_N$$

$$|p_1\sigma_1; p_2\sigma_2 \cdots p_N\sigma_N>^{asym} = \frac{1}{N!}\sum_{perm} \varepsilon^P |p_1\sigma_1>_1 |p_2\sigma_2>_2 \cdots |p_N\sigma_N>_N$$

The sum overpermutations on the r.h.s. refers to the quantum labels in the skeleton $|\ >_1 |\ >_2 \cdots |\ >_N$.

The factor ε^P equals 1 for even permutations and -1 for odd permutations.

To avoid a doubling of formulae we use

$$|p_1\sigma_1; \ldots; p_N\sigma_N> = \frac{1}{N!}\sum_{perm} \varepsilon^P |p_1\sigma_1>_1 |p_2\sigma_2>_2 \cdots |p_N\sigma_N>_N \qquad (4.1)$$

for "symmetrized" states. For symmetrical states $\varepsilon^P \equiv 1$ and for antisymmetrical states ε^P is the sign of the permutation.

The normalization of "symmetrized" states is given by

$$\begin{aligned} &\langle p_1'\sigma_1'; p_2'\sigma_2' \ldots; p_N'\sigma_N' \mid p_1\sigma_1; p_2\sigma_2; \ldots p_N\sigma_N\rangle = \\ &= \frac{1}{N!}\sum_P \varepsilon^P \langle p_1'\sigma_1' \mid p_1\sigma_1\rangle\langle p_2'\sigma_2' \mid p_2\sigma_2\rangle \ldots \langle p_N'\sigma_N' \mid p_N\sigma_N\rangle \end{aligned} \tag{4.2}$$

with

$$\langle p'\sigma' \mid p\sigma\rangle = (2\pi)^3 2p^o \delta(\vec{p}-\vec{p}')\delta_{\sigma\sigma'}$$

(see also 5.15))

The sum over permutations in (4.2) refers to one set of quantum labels e.g. $\{p_1\sigma_1, p_2\sigma_2, \ldots p_N\sigma_N\}$.

The states $\{|p_1\sigma_1; p_2\sigma_2, \ldots p_N\sigma_N\rangle\}$ constitute a basis in the N-particle space $H^{(N,S)} := (H^{(1)} \otimes H^{(1)} \ldots \otimes H^{(1)})^S$, N refers to the number of particles and S refers to symmetric or antisymmetric states.

We define the grand state-space $H^{(G)}$

$$H^{(G)} = H^{(0)} \dot{+} H^{(1)} \dot{+} H^{(2,S)} \dot{+} H^{(3,S)} \dot{+} \ldots \tag{4.3}$$

where $H^{(0)} = \mathbb{C}|o\rangle$

The inner-product on $H^{(G)}$ is defined by

$$\begin{aligned} &\langle p_1'\sigma_1'; p_2'\sigma_2' \ldots; p_M'\sigma_M' \mid p_1\sigma_1; \ldots p_N\sigma_N\rangle = \\ &= \delta_{NM}\frac{1}{N!}\sum_P \varepsilon^P \langle p_1'\sigma_1' \mid p_1\sigma_1\rangle\langle p_2'\sigma_2' \mid p_2\sigma_2\rangle \ldots \langle p_N'\sigma_N' \mid p_N\sigma_N\rangle \end{aligned} \tag{4.4}$$

Having defined N-particle states for arbitrary N we can now define the construction operators i.e. the creation- and annihilation operators.

We define creation operators $\underline{\underline{a}}^\dagger(p,\sigma)$ by

$$\underline{\underline{a}}^\dagger(p,\sigma)|p_1\sigma_1; \ldots; p_N\sigma_N\rangle = \sqrt{N+1}\,|p\sigma; p_1\sigma_1 \ldots; p_N\sigma_N\rangle \tag{4.5}$$

$\underline{\underline{a}}^\dagger(p,\sigma)$ maps symmetrized N-particle states onto symmetrized (N+1) particle states.

It is important to put the label $p\sigma$ in a fixed place. In (4.5) we have put $(p\sigma)$ in front of the other labels.

The commutation relations of creation operators are easily obtained, we have

$$\underline{\underline{a}}^{\dagger}(p'\sigma')\underline{\underline{a}}^{\dagger}(p\sigma)|p_1\sigma_1;\ \ldots\ p_N\sigma_N\rangle = \sqrt{(N+2)(N+1)}\,|p'\sigma';\ p\sigma;\ \ldots\ p_N\sigma_N\rangle$$

$$\underline{\underline{a}}^{\dagger}(p\sigma)\underline{\underline{a}}^{\dagger}(p'\sigma')|p_1\sigma_1;\ \ldots\ p_N\sigma_N\rangle = \sqrt{(N+2)(N+1)}\,|p\sigma;\ p'\sigma';\ \ldots\ p_N\sigma_N\rangle$$

$$= \varepsilon\sqrt{(N+2)(N+1)}\,|p'\sigma';\ p\sigma;\ \ldots\ p_N\sigma_N\rangle$$

where $\varepsilon = 1$ for a symmetrical state and $\varepsilon = -1$ for antisymmetrical states. This gives

$$[\underline{\underline{a}}^{\dagger}(p'\sigma'),\ \underline{\underline{a}}^{\dagger}(p,\sigma)]_{-\varepsilon} := \underline{\underline{a}}^{\dagger}(p'\sigma')\underline{\underline{a}}^{\dagger}(p\sigma) - \varepsilon\,\underline{\underline{a}}^{\dagger}(p\sigma)\underline{\underline{a}}^{\dagger}(p'\sigma') = 0 \tag{4.6}$$

Clearly $\varepsilon = 1$ gives commutation relations and $\varepsilon = -1$ yields anticommutation relations.

The hermitian conjugate of $a^{\dagger}(p,\sigma)$ (with respect to 4.4) is an annihilation operator. From (4.5) we have

$$\langle p'_1\sigma'_1\ \ldots\ p'_N\sigma'_N|\ \underline{\underline{a}}(p,\sigma) = \langle p\sigma;\ p'\sigma'\ \ldots\ p'_N\sigma'_N|\sqrt{N+1} \tag{4.7}$$

Taking the scalar product of (4.7) with

$$|p_1\sigma_1\ \ldots\ p_M\sigma_M\rangle$$

one can deduce that

$$\underline{\underline{a}}(p,\sigma)|p_1\sigma_1\ \ldots\ p_M\sigma_M\rangle = \frac{1}{\sqrt{M}}\sum_i \varepsilon^{i-1}\langle p\sigma\,|\,p_i\sigma_i\rangle|p_1\sigma_1;\ \ldots\ \hat{p_i}\sigma_i;\ p_M\sigma_M\rangle \tag{4.8}$$

where $\hat{p_i}\sigma_i$ means that these labels must be deleted.

Clearly $a(p,\sigma)$ maps a state from $H^{(M,S)}$ onto a state in $H^{(M-1,\,S)}$.

From (4.8) one finds

$$[\underline{\underline{a}}(p,\sigma), \underline{\underline{a}}(p',\sigma')] = 0 \tag{4.9}$$

and from (4.5) and (4.8)

$$[\underline{\underline{a}}(p,\sigma), \underline{\underline{a}}^{\dagger}(p',\sigma')]_{-\varepsilon} = (2\pi)^3 2p^o \delta(\vec{p} - \vec{p}')\delta_{\sigma\sigma'} \tag{4.10}$$

4.2. Transformation properties of construction operators.

In section 3.2 we have derived the transformation properties of one-particle states, see (3.30)

$$\underline{\underline{U}}(a,A)|p,\sigma> = e^{ia_\mu(\Lambda(A)p)^\mu} D^{(s)}_{\sigma'\sigma}(U(p,A))|\Lambda(A)p,\sigma'>$$

The operators $\underline{\underline{U}}(a,A)$ defined by (3.30) constitute a unitary irreducible representation of the inhomogeneous $SL(2,\mathbb{C})$ group.

To obtain the transformation properties of the construction operators $a(p,\sigma)$ and $a^{\dagger}(p,\sigma)$ we must define the representation on the grand state space $\mathcal{H}^G$. All this is quite straightforward.

For translations we define $U(a,\mathbb{1})$ by

$$\begin{aligned} &U(a,\mathbb{1})|p_1\sigma_1; \dots p_N\sigma_N> = \\ &= \frac{1}{N!}\sum_P \varepsilon^P \underline{\underline{U}}(a,\ )|p_1\sigma_1>_1 \underline{\underline{U}}(a,\ )\ p_2\sigma_2>_2 \dots \underline{\underline{U}}(a,\ )|p_N\sigma_N>_N \\ &= e^{ia_\mu(p_1+p_2\dots p_N)^\mu} |p_1\sigma_1; \dots; p_N\sigma_N> \end{aligned} \tag{4.11}$$

For $A \in SL(2,\mathbb{C})$ we define $U(o,A)$ by

$$\begin{aligned} &U(o,A)|p_1\sigma_1; \dots p_N\sigma_N> = \\ &= \frac{1}{N!}\sum_P \varepsilon^P \underline{\underline{U}}(o,A)|p_1\sigma_1>_1 \cdot \dots \cdot \underline{\underline{U}}(o,A)|p_N\sigma_N>_N \\ &= \frac{1}{N!}\sum_P \varepsilon^P D^{(s)}_{\sigma_1'\sigma_1}(U(p_1,A))\, D^{(s)}_{\sigma_2'\sigma_2}(U(p_2,A)) \dots D^{(s)}_{\sigma_N'\sigma_N}(U(p_N,A)) \cdot \\ &\qquad\qquad |\Lambda(A)p_1\sigma_1'>_1 \cdot \dots \cdot |\Lambda(A)p_N\sigma_N'>_N \\ &= D^{(s)}_{\sigma_1'\sigma_1}(p_1,A) D^{(s)}_{\sigma_2'\sigma_2}(p_2,A) \dots D^{(s)}_{\sigma_N'\sigma_N}(p_N,A)|\Lambda(A)p_1\sigma_1'; \dots; \Lambda(A)p_N\sigma_N'> \end{aligned} \tag{4.12}$$

From (4.5) and (4.11) one obtains

$$\underline{\underline{a}}^{\dagger}(p,\sigma)U(a,\mathbb{1})|p_1\sigma_1; p_N\sigma_N\rangle = e^{ia_\mu \sum_{j=1}^{} p_j^\mu} \sqrt{N+1}|p\sigma; p_1\sigma_1 \ldots p_N\sigma_N\rangle$$

$$= e^{-ia_\mu p^\mu} U(a,\mathbb{1})\underline{\underline{a}}^{\dagger}(p,\sigma)|p_1\sigma_1; \ldots p_N\sigma_N\rangle$$

and this gives

$$U^{-1}(a,\mathbb{1})\underline{\underline{a}}^{\dagger}(p,\sigma)U(a,\mathbb{1}) = e^{-ia_\mu p^\mu} \underline{\underline{a}}^{\dagger}(p,\sigma) \tag{4.13}$$

Taking the adjoint yields

$$U^{-1}(a,\mathbb{1})\underline{\underline{a}}(p,\sigma)U(a,\mathbb{1}) = e^{ia_\mu p^\mu} \underline{\underline{a}}(p,\sigma) \tag{4.14}$$

To derive the transformation law for $\underline{\underline{a}}^{\dagger}(p,\sigma)$ under homogeneous transformations we use (4.5) and (4.12). The result reads

$$U^{-1}(o,A)\underline{\underline{a}}^{\dagger}(p,\sigma)U(o,A) = D^{(s)}_{\sigma'\sigma}(H^{-1}(\Lambda(A^{-1})p)A^{-1}H(p))\underline{\underline{a}}^{\dagger}(\Lambda(A^{-1})p,\sigma') \tag{4.15}$$

The Wigner "rotation" in (4.15) is given by

$$H^{-1}(\Lambda(A^{-1})p) \cdot A^{-1} \cdot H(p) = U(p,A^{-1}) \in SU(2,\mathbb{C})$$

(see also 3.29)).

The transformation law for the annihilation operator $\underline{\underline{a}}(p,\sigma)$ follows from (4.15) by taking the adjoint; this gives

$$\begin{aligned} U^{-1}(o,A)\underline{\underline{a}}(p,\sigma)U(o,A) &= D^{(s)*}_{\sigma'\sigma}(U(p,A^{-1}))\underline{\underline{a}}(\Lambda(A^{-1})p,\sigma') \\ &= D^{(s)}_{\sigma\sigma'}(H^{-1}(p)A\,H(\Lambda(A^{-1})p))\underline{\underline{a}}(\Lambda(A^{-1})p,\sigma') \end{aligned} \tag{4.16}$$

which is precisely the transformation law for a one-particle wave-function, see (3.32).

In the r.h.s. of (4.16) we have used the unitarity of the matrices $D^{(s)}$ which gives $D^{(s)*} = (D^{(s)^{-1}})^T$ (T is transpose). Using this property we

can rewrite (4.15) in a form in which the summation index σ' is also on the right

$$U^{-1}(o,A)\underline{a}^{\dagger}(p,\sigma)U(o,A) = D^{(s)*}_{\sigma\sigma'}(H^{-1}(p)AH(\Lambda(A^{-1})p)\underline{a}^{\dagger}(\Lambda(A^{-1})p,\sigma') \tag{4.17}$$

The argument of $D^{(s)*}$ in this formula is the same as the argument of $D^{(s)*}$ in (4.16).

We now use an important property of the representations of SU(2): the representations $U \longmapsto D^{(s)}(U)$ and $U \longmapsto D^{(s)*}(U)$ are equivalent. There exists a unitary matrix $C^{(s)}$ such that

$$D^{(s)*}(U) = C^{(s)}D^{(s)}(U)C^{(s)^{-1}} \tag{4.18}$$

The matrices $C^{(s)}$ can be choosen in such a way that

$$(C^{(s)})^2 = (-1)^{2s}\,\mathbb{1} \tag{4.19}$$

For example $C^{(\frac{1}{2})} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$C^{(1)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

In SU(2) one has

$$U^{*} = C^{(\frac{1}{2})}U\,C^{(\frac{1}{2})^{-1}}$$

Using (4.18) and (4.17) one sees that the linear superposition

$$C^{(s)^{-1}}_{\sigma\sigma'}\,\underline{a}^{\dagger}(p,\sigma') \qquad (\sigma = -s,\ldots,s)$$

transforms in the same way as $\underline{a}(p,\sigma)$. One has

$$\begin{aligned} &U^{-1}(o,A)(C^{(s)^{-1}}_{\sigma\sigma'}\,\underline{a}^{\dagger}(p,\sigma'))U(o,A) = \\ &\qquad = D^{(s)}_{\sigma\sigma''}(H^{-1}(p)AH(\Lambda(A^{-1})p))\,C^{(s)^{-1}}_{\sigma''\sigma'}\,\underline{a}^{\dagger}(\Lambda(A^{-1})p,\sigma') \end{aligned} \tag{4.20}$$

We will use this property in the construction of operator fields.

4.3. Particles and antiparticles.

The electron is a particle with mass $m = m_e$ and spin $s = \frac{1}{2}$. One also knows the positron, a particle with mass equal to m_e and spin $s = \frac{1}{2}$. This means that these particles have the same behaviour under the space-time transformation of the inhomogeneous $SL(2,\mathbb{C})$ group. The difference (in this case electric charge) is some property which one used to call internal symmetry. In general particles and their antiparticles differ in quantum numbers such as charge, baryon number, hypercharge etc.

The representation theory discussed in the foregoing sections applies without any modification to antiparticles as well. We will denote antiparticle construction operators by $\underline{\underline{b}}(p,\sigma)$ and $\underline{\underline{b}}^{\dagger}(p,\sigma)$.

The commutation relations are

$$[\underline{\underline{b}}(p,\sigma),\ \underline{\underline{b}}^{\dagger}(p',\sigma')]_{-\varepsilon} = (2\pi)^3 2p^o \delta(\vec{p}-\vec{p}')\delta_{\sigma\sigma'} \tag{4.21}$$

$$[\underline{\underline{b}}(p,\sigma),\ \underline{\underline{b}}(p',\sigma')]_{-\varepsilon} = [\underline{\underline{b}}^{\dagger}(p,\sigma),\ \underline{\underline{b}}^{\dagger}(p',\sigma')]_{-\varepsilon} = 0 \tag{4.22}$$

and the transformation rules read

$$U^{-1}(o,A)\underline{\underline{b}}(p,\sigma)U(o,A) = D^{(s)}_{\sigma\sigma'}(H^{-1}(p)A\,H(\Lambda(A^{-1})p))\underline{\underline{b}}(\Lambda(A^{-1})p,\sigma') \tag{4.23}$$

$$U^{-1}(o,A)C^{(s)\,-1}_{\sigma\sigma'}\underline{\underline{b}}^{\dagger}(p,\sigma')U(o,A) = D^{(s)}_{\sigma\sigma''}(H^{-1}(p)A\,H(\Lambda(A^{-1})p)C^{(s)\,-1}_{\sigma''\sigma'}\underline{\underline{b}}^{\dagger}(\Lambda(A^{-1})p,\sigma') \tag{4.24}$$

Commutation relations of the form $[\underline{\underline{a}},\underline{\underline{b}}]_{\pm}$, $[\underline{\underline{a}},\underline{\underline{b}}^{\dagger}]_{\pm}$ etc. between particle operators and antiparticle operators should also be given. We will use

$$[a(p,\sigma),\ b(p',\sigma')]_{-\varepsilon} = 0 \tag{4.25}$$

for all types of mixed relations.

According to Jordan and Wigner, Zeitschrift für Physik, 47 (1928)631 one can

use commutation as well as anti-commutation relations in the mixed case.

4.4. Operator fields.

A multicomponent operator field $\underline{\phi}_i(x^o,\vec{x})\Big|_{i=1}^n$ defined on $H^{(G)}$ is called a covariant field if

$$U^{-1}(a,A)\underline{\phi}_i(x^o,\vec{x})U(a,A) = D_{ii'}(A)\underline{\phi}_{i'}(\Lambda(A^{-1})(x-a))$$

where $D(A)$ is some finite-dimensional representation of the group $SL(2,\mathbb{C})$.

In field theory operator fields are often obtained by starting from "classical fields" e.g. fields satisfying the Klein-Gordon equation or the Dirac equation and by declaring these fields to be operator fields with certain commutation or anti-commutation relation. The procedure is called canonical quantization.

In our opinion it is far more satisfying to introduce operator fields on $H^{(G)}$ by making use of the construction operators a, $a^\dagger$, b, $b^\dagger$ which are already defined on $H^{(G)}$. We start with a simple example and give some general results afterwards.

Consider the case $m > o$ and $s = o$.

The operator

$$\underline{a}(p) + \xi\underline{b}^\dagger(p) \qquad (\xi\in\mathbb{C}) \tag{4.26}$$

is defined for all $p = (p^o,\vec{p})$ with $p^2 = m^2$ and $p^o = \sqrt{\vec{p}^2+m^2}$.

Under $U(o,A)$ we have according to (4.16) and (4.24) for $s = o$,

$$U^{-1}(o,A)(\underline{a}(p) + \xi\underline{b}^\dagger(p))U(o,A) = \underline{a}(\Lambda(A^{-1})p) + \xi\underline{b}^\dagger(\Lambda(A^{-1})p) \tag{4.27}$$

Under translations (4.26) transform as

$$U^{-1}(a,\mathbb{1})\,(\underline{a}(p) + \xi\underline{b}^\dagger(p))U(a,\mathbb{1}) = e^{ia_\mu p^\mu}\underline{a}(p) + \xi e^{-ia_\mu p^\mu}\underline{b}^\dagger(p) \tag{4.28}$$

(see (4.13) and (4.14)).

It is now easy to construct a covariant operator field $\underline{\underline{\phi}}(x^o,\vec{x})$ by taking the fourier-transform of (4.26) with the allowed $\vec{p}$'s.

$$\begin{aligned}\underline{\underline{\phi}}(x^o,\vec{x}) &:= \int \frac{d^3p}{(2\pi)^3 2p^o}\left(e^{-ipx}\,\underline{\underline{a}}(p) + \xi e^{+ipx}\,\underline{\underline{b}}^\dagger(p)\right) \\ &= \int d^4p\,\frac{\Theta(p^o)\delta(p^2-m^2)}{(2\pi)^3}\left(e^{-ipx}\,\underline{\underline{a}}(p) + \xi e^{ipx}\,\underline{\underline{b}}^\dagger(p)\right)\end{aligned} \tag{4.29}$$

This operator field satisfies the Klein-Gordon equation

$$(\partial_\mu\partial^\mu+m^2)\underline{\underline{\phi}}(x^o,\vec{x}) = o \tag{4.30}$$

Moreover, it has the simple transformation properties of a scalar field

$$U^{-1}(a,A)\underline{\underline{\phi}}(x)U(a,A) = \underline{\underline{\phi}}(\Lambda(A^{-1})(x-a)) \tag{4.31}$$

From $\underline{\underline{\phi}}(x)$ we obtain $\underline{\underline{\phi}}^\dagger(x)$

$$\underline{\underline{\phi}}^\dagger(x) = \int d\mu(p)\left(e^{ipx}\,\underline{\underline{a}}^\dagger(p) + \xi^* e^{-ipx}\,\underline{\underline{b}}(p)\right) \tag{4.32}$$

Using the commutation relations for the construction operators we calculate $[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}^\dagger(y)]_\mp$. The result is:

$$[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}^\dagger(y)]_\mp = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^o}\left[e^{-ip(x-y)} \mp |\xi|^2 e^{ip(x-y)}\right] \tag{4.33}$$

4.5. Micro causality.

Up to now we have no restriction on ξ, particle and antiparticle operators need not occur with the same factor in $\underline{\underline{\phi}}(x)$. We now put a restriction on the fields, we require

$$[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}^\dagger(y)]_\mp = o \quad \text{for} \quad (x-y)^2 < o \tag{4.34}$$

i.e. for space-like separated points x and y the commutator or anti-commutator much vanish. This condition which is called microcausality stems

from the fact that measurements of an observable quantity in two space-like separated points have no correlation whatsoever. Although the fields $\underline{\underline{\phi}}$ and $\underline{\underline{\phi}}^{\dagger}$ are no observables themselves, condition (4.34) guarantees that observables constructed from these fields satisfy the property of causality in the sense described above.

We consider (4.33) for two space-like separated points x and y. By a Lorentz-transformation $(x-y)$ can be transformed into $(\hat{x}-\hat{y}) = \Lambda(x-y)$ with $\hat{x}^o - \hat{y}^o = o$. As both integrals in (4.33) are Lorentz-invariant we can consider (4.33) with $\hat{x}^o - \hat{y}^o = o$. Omitting the hats we finds

$$[\underline{\underline{\phi}}(x), \phi^{\dagger}(y)]_{\mp} = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^o} e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(1 \mp |\xi|^2) \tag{4.35}$$

To satisfy microcausality this must be zero, hence we must take the minus sign and $|\xi|^2 = 1$ in (4.35).

This tells us that for spin zero fields we must take the commutator $[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}^{\dagger}(y)]_- = \underline{\underline{\phi}}(x)\underline{\underline{\phi}}^{\dagger}(y) - \underline{\underline{\phi}}^{\dagger}(y)\underline{\underline{\phi}}(x)$.

Tracing back the origin of the $\mp$ in (4.33) we find that for spinless particles ε must be taken equal to 1 in (4.10) and (4.21) and we conclude that identical particles with spin zero have symmetrical states and commutators.

The parameter ξ in the definition of $\underline{\underline{\phi}}$ (4.29) must have $|\xi|^2 = 1$, and can be incorporated in the operator $b^{\dagger}$. The result is

$$\underline{\underline{\phi}}(x) = \int d\mu(p)\left(\underline{\underline{a}}(p)e^{-ipx} + \underline{\underline{b}}^{\dagger}(p)e^{ipx}\right) \tag{4.36}$$

Particle and antiparticle operators appear with the same "strength".

The commutator reads:

$$\begin{aligned}[\underline{\underline{\phi}}(x), \underline{\underline{\phi}}^{\dagger}(y)]_- &= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^o}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\ &= i\,\Delta(x-y)\end{aligned} \tag{4.37}$$

with $\Delta(x-y)$ the famous Pauli-Jordan commutator function.

From (4.37) one obtains the equal time commutator

$$[\underline{\underline{\phi}}(x^o,\vec{x}),\ \underline{\underline{\dot{\phi}}}^{\dagger}(x^o,\vec{y})] = i\,\delta(\vec{x}-\vec{y}) \tag{4.38}$$

known from canonical quantization.

The generalization of the above to the case $s \neq o$ is rather complicated but it can be done.

One might think that $\left\{\underline{\underline{\phi}}_{\sigma}(x)\right\}_{\sigma=-s}^{\sigma=+s}$ defined as

$$\underline{\underline{\phi}}_{\sigma}(x) = \int d\mu(p)\left(\underline{\underline{a}}(p,\sigma)e^{-ipx} + C^{(s)}_{\sigma\sigma'}\,\underline{\underline{b}}^{\dagger}(p,\sigma')e^{ipx}\right)$$

is a covariant $(2s+1)$ component field but things are not that easy.

Due to the Wigner rotation occurring in (4.16) and (4.20) we arrive at

$$U^{-1}(o,A)\underline{\underline{\phi}}_{\sigma}(x)U(o,A) = \int d\mu(p)\ D^{(s)}_{\sigma\rho}[H^{-1}(p)A\,H(\Lambda(A^{-1})p)]\cdot \left(\underline{\underline{a}}(\Lambda(A^{-1})p,\rho)e^{-ipx} + C^{(s)^{-1}}_{\rho\sigma'}\,\underline{\underline{b}}^{\dagger}(\Lambda(A^{-1})p,\sigma')e^{ipx}\right)$$

The p-dependence of the matrices $D^{(s)}$ prevents the construction. Taking, however, suitable linear superposition of the construction operator makes it possible to obtain covariant operator fields for any spin. One starts with

$$\underline{\underline{\phi}}_{n}(x) = \int d\mu(p)\ \left\{u_n(p,\sigma)\underline{\underline{a}}(p,\sigma)e^{-ipx} + v_n(p,\sigma)C^{(s)}_{\sigma\sigma'}\,\underline{\underline{b}}^{\dagger}(p,\sigma')e^{ipx}\right\} \tag{4.39}$$

(summation on σ and σ') and requires the fields to transform as

$$U^{-1}(o,A)\underline{\underline{\phi}}_{n}(x)U(o,A) = D_{nm}(A)\underline{\underline{\phi}}_{m}(\Lambda(A^{-1})x) \tag{4.40}$$

where $D(A)$ is a representation of $SL(2,\mathbb{C})$.

The requirement (4.40) gives restrictions on the so-called polarization vectors $u_n(p,\sigma)$ and $v_n(p,\sigma)$. One finds

$$D_{nm}(A)u_m(p,\sigma) = u_n(\Lambda(A)p,\sigma')D^{(s)}_{\sigma'\sigma}\left(H^{-1}(\Lambda(A)p)A\,H((p)\right) \tag{4.41}$$

$$D_{nm}(A)v_m(p,\sigma) = v_m(\Lambda(A)p,\sigma')D^{(s)}_{\sigma'\sigma}\left(H^{-1}(\Lambda(A)p)A\,H(p)\right) \tag{4.42}$$

The physical meaning of these equations is easily seen by considering the point $p = p_o = (m,\vec{o})$ and taking $A = U \in SU(2) \subset SL(2,\mathbb{C})$. We then find

$$D_{nm}(U)\ u_m(p_o,\sigma) = u_n(p_o,\sigma')D^{(s)}_{\sigma'\sigma}(U) \qquad (\sigma = -s, \ldots +s) \tag{4.43}$$

This is clearly a condition on u_n and $D(A)$. It says that in order to construct a covariant field for spin $s \neq o$ one must consider those representations D of $SL(2,\mathbb{C})$ which are reducible under $SU(2)$ and which contain a $(2s+1)$ dimensional subspace which is invariant under $D(U)$ and transforms as $D^{(s)}(U)$.

Such representations of $SL(2,\mathbb{C})$ exist. The simplest description is obtained by using a $(2s+1)$ dimensional irreducible representation $D^{(s,o)}(A)$ of $SL(2,\mathbb{C})$. This gives a $(2s+1)$ component field.
An alternative description uses the representation $D^{(o,s)}(A)$ which is also $(2s+1)$ dimensional. Taking the irreducible representation $D^{(s_1,s_2)}$ one can define a covariant field forspin s only if the reducible $D^{(s_1,s_2)}(U)$ $(U \in SU(2))$ contains $D^{(s)}(U)$. Recall that

$$D^{(s_1,s_2)}(U) \cong D^{(s_1+s_2)}(U) \dot{+} D^{(s_1+s_2-1)}(U) \dot{+} \ldots D^{|s_1-s_2|}(U)$$

Using reducible representations of $SL(2,\mathbb{C})$ one often considers fields transforming with

$$D^{(s,o)}(A) \dot{+} D^{(o,s)}(A)$$

An example is the 4-component Dirac spinor field, which transforms according to the reducible representation $D^{(\frac{1}{2},o)} \dot{+} D^{(o,\frac{1}{2})}$ of $SL(2,\mathbb{C})$.

An important result follows from the requirement of microcausality. It turns

out that for *integral* values of s one must take *commutators* while *half-integral* values of s require anti-commutation relations.

We finish this section by giving some results for the case of zero mass. To construct covariant operator fields from the construction operators for $m = o$ one has again group theoretical restrictions. To obtain a field describing particles with helicity λ $(= 0, \pm\frac{1}{2}, \pm 1, \dots)$ using an irreducible representation $D^{(s_1,s_2)}$ one is restricted by the condition

$$s_1 - s_2 = \lambda.$$

It is interesting to consider the case $|\lambda| = 1$ i.e. the photon-field and gauge boson-fields.

The description of the photon-field by the four-vector potential $A^{\mu}(x)$ conflicts the above restriction. The four-vector field transforms according to $D^{(\frac{1}{2},\frac{1}{2})}$ for which $s_1 - s_2 = \frac{1}{2} - \frac{1}{2} = 0$. This is the reason why canonical quantization of $A^{\mu}(x)$ leads to several unpleasant features such as non-physical states which must be excluded by subsidiary conditions.

5. INTERACTIONS, THE SCATTERING OPERATOR

5.1. Introduction.

We summarize what we have done so far. Starting with the unitary irreducible representations of the group $P_+^\uparrow$ which gives the appropriate description of one-particle states we have defined many-particle states of identical particles and the creation- and annihilation operators $\underline{\underline{a}}(p,\sigma)$ and $\underline{\underline{a}}^\dagger(p,\sigma)$. Using these operators we have introduced operator fields in space-time. Imposing the condition of microcausality we have found the relation between spin and statistics: for identical particles with integral spin the statevectors must be symmetrical in the quantum numbers (Bose-Einstein statistics) the operators obey commutation relations, for identical particles with half-integral spin the statevectors are antisymmetric (Fermi-Dirac statistics) and the operators satisfy anti-commutation relations.

We want to stress that the formalism developed up to now applies to non-interacting particles only. The operator fields satisfy free-field equations. In the simplest case, considered in section 4.4 the field $\underline{\underline{\phi}}(x)$ satisfies the Klein-Gordon equation

$$(\partial_\mu \partial^\mu + m^2)\underline{\underline{\phi}}(x) = o$$

In the case of spin $s = \frac{1}{2}$ one can construct a four-component operator field each component of which satisfies the Klein-Gordon equation while the number of independent components is restricted to two by the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\underline{\underline{\psi}}(x) = o$$

For the description of non-interacting particles one does not really need the fields in space-time. With the experimentally well established Pauli Exclusion Principle the symmetry or antisymmetry of identical particle statevectors is sufficiently well-founded. The main reason

for introducing fields in space-time is the description of interactions. In this chapter we will consider the simplest example of a system of interacting particles and introduce the scattering operator according to the scheme of Lehmann, Symanzik and Zimmermann (L.S.Z.), [5].

This scheme is based on the assumption that one can speak of incoming and outgoing particles in a scattering process. This seems an almost trivial assumption. What is meant, however, is not trivial: it is assumed that the incoming particles as well as the outgoing particles are free particles.

In the remainder of this chapter we are going to give a brief description of the L.S.Z. scheme. We will restrict ourselves to the simplest case of *spinless* (s=o) massive (m>o) identical particles.

5.2. In-states, out-states, asymptotic completeness, the scattering operator.

The states of a quantum system are represented by vectors in a Hilbert space H, the observables by operators on H.

To describe the scattering of particles we assume H to have two bases, the in-basis and out-basis.

The in-basis consists of the vacuumstate and many-particle states:

$$|o\,;\,\text{in}\rangle,\ |p;\text{in}\rangle;\ |p_1,p_2;\text{in}\rangle,\ |p_1,p_2,p_3;\text{in}\rangle\ \text{ etc.}$$

The out-basis has:

$$|o;\text{out}\rangle,\ |p;\text{out}\rangle;\ |p_1,p_2;\text{out}\rangle;\ |p_1,p_2,p_3;\text{out}\rangle\ \text{ etc.}$$

We explain the meaning of the labels *in* and *out*.

A state $|\alpha;\text{in}\rangle$ is a statevector in the *Heisenberg picture*, i.e. independent of time, with the property that the system described by $|\alpha;\text{in}\rangle$ consists of free particles with quantum numbers represented by α, at $t = -\infty$.

To be more specific $|p_1,p_2;\text{in}\rangle$ is the statevector for a system which con-

sists at $t = -\infty$ of two free particles with four-momenta p_1 and p_2. With the additional assumption that the interaction is such that bound states of two- or more particles are excluded, the Heisenberg state $|p_1,p_2; in\rangle$ describes the process of scattering.

At an arbitrary choosen time t the number of particles in the state $|p_1,p_2; in\rangle$ can be determined by an appropriately defined counting operator. It should be remembered that operators are in general time-dependent in the Heisenberg picture.

The state $|\alpha; out\rangle$, is that Heisenberg state that consists of free particles, with quantum numbers represented by α, at $t = +\infty$.

It should be clear that $|o; in\rangle$ and $|o; out\rangle$ do not describe any process at all and can be identified

$$|o\rangle = |o; in\rangle = |o; out\rangle$$

Likewise $|p; in\rangle$ and $|p; out\rangle$ being one-particle eigenvectors of the energy-momentum operators P_μ do not describe a scattering process and we take

$$|p; in\rangle = |p; out\rangle$$

To define the scattering operator $\underline{\underline{S}}$ we have to assume that the set of in-states as well as the set of out-states form a complete set in H. This assumption is called *asymptotic completeness*. The scattering operator $\underline{\underline{S}}$ is now defined as the invertible unitary mapping between the in-basis and the out-basis.

Following common usage we write

$$|\alpha; in\rangle = \underline{\underline{S}}\, |\alpha; out\rangle \qquad (5.1)$$

where α denotes the set of quantum numbers specifying the states.

The matrix elements of $\underline{\underline{S}}$ are interpreted as the probability amplitudes for the occurrence of a scattering process e.g.

$$<p_1',p_2',p_3',p_4'; \text{out} \mid p_1,p_2; \text{in}> = <p_1',p_2',p_3',p_4'; \text{out} \mid \underline{\underline{S}} \mid p_1,p_2; \text{out}> =$$

$$<p_1',p_2',p_3',p_4'; \text{in} \mid \underline{\underline{S}} \mid p_1,p_2; \text{in}> \tag{5.2}$$

gives the probability amplitude that the system consisting at $t = -\infty$ of two-particles with momenta p_1 and p_2 ends up as a four-particle state with momenta p_1', p_2', p_3', p_4'.

It should be noted that the interpretation applies in fact to normalizable states only.

To calculate the S-matrix elements for a specific system one needs to know the interaction between the particles. It is at this point that operator fields enter the scene.

5.3. The asymptotic condition, the interpolating field.

The basis of in-states may be described by the repeated application of the creation operator $\underline{\underline{a}}^{\dagger}_{in}(p)$ on the vacuum state.

$$\mid p_1,p_2,\ldots p_N; \text{in}> = \frac{1}{\sqrt{N!}} \; \underline{\underline{a}}^{\dagger}_{in}(p_1)\ldots\underline{\underline{a}}^{\dagger}_{in}(p_N) \mid o> \tag{5.3}$$

Likewise we define $\underline{\underline{a}}^{\dagger}_{out}$

$$\mid p_1,\ldots p_N; \text{out}> = \frac{1}{\sqrt{N!}} \; \underline{\underline{a}}^{\dagger}_{out}(p_1)\ldots\underline{\underline{a}}^{\dagger}_{out}(p_N) \mid o> \tag{5.4}$$

and one easily derives the following relations

$$\underline{\underline{a}}^{\dagger}_{in}(p) = \underline{\underline{S}} \; \underline{\underline{a}}^{\dagger}_{out}(p)\underline{\underline{S}}^{-1} \tag{5.5}$$

$$\underline{\underline{a}}_{in}(p) = \underline{\underline{S}} \; \underline{\underline{a}}_{out}(p)\underline{\underline{S}}^{-1} \tag{5.6}$$

Using the in- and out-construction operators we define the fields $\underline{\underline{\phi}}_{in}(x)$

and $\underline{\underline{\phi}}_{out}(x)$ in space-time

$$\underline{\underline{\phi}}_{\substack{in\\out}}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^o}\left(\underline{\underline{a}}_{\substack{in\\out}}(p)e^{-ipx} + \underline{\underline{a}}^{\dagger}_{\substack{in\\out}}(p)e^{ipx}\right) \tag{5.7}$$

satisfying

$$(\partial_\mu\partial^\mu+m^2)\,\underline{\underline{\phi}}_{\substack{in\\out}}(x) = o$$

and

$$\underline{\underline{\phi}}_{in}(x) = \underline{\underline{S}}\;\underline{\underline{\phi}}_{out}(x)\;\underline{\underline{S}}^{-1} \tag{5.8}$$

It will be clear that the free fields $\underline{\underline{\phi}}_{\substack{in\\out}}$ do not describe the interaction and it seems reasonable to assume the existence of a non-free field $\underline{\underline{\phi}}(x)$ which approaches $\underline{\underline{\phi}}_{in}$ and $\underline{\underline{\phi}}_{out}$ for $t \to -\infty$ and $t \to +\infty$ respectively.

To clarify this point we recall that in the framework of Lagrangian field theory a system of interacting particles is described by a Lagrangian which in our case is of the form

$$L = \tfrac{1}{2}(\partial_\mu\underline{\underline{\phi}}\partial^\mu\underline{\underline{\phi}} - m^2\underline{\underline{\phi}}^2) + P(\underline{\underline{\phi}}) \tag{5.9}$$

when $P(\underline{\underline{\phi}})$ is of polynomial form e.g. $a\underline{\underline{\phi}}^3 + b\underline{\underline{\phi}}^4$.

The equation of motion for ϕ reads

$$(\partial_\mu\partial^\mu+m^2)\underline{\underline{\phi}} = \frac{dP}{d\underline{\underline{\phi}}} \tag{5.10}$$

As $\underline{\underline{\phi}}(x)$ does not satisfy the free Klein-Gordon equation it does not have a plane wave expansion with time-independent coefficients. One can, however, give an expansion with time-dependent coefficients

$$\underline{\underline{\phi}}(x) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^o}\,(\underline{\underline{a}}(p,t)e^{-ipx} + \underline{\underline{a}}^{\dagger}(p,t)e^{ipx} \tag{5.11}$$

We will now state the asymptotic condition which is the central feature of the L.S.Z. scheme.

It is assumed that for a system of interacting particles described by the field $\underline{\underline{\phi}}(x)$ there exists free field $\underline{\underline{\phi}}_{in}(x)$ and $\underline{\underline{\phi}}_{out}(x)$ such that

$$\lim_{t\to-\infty} \langle u| \underline{\underline{a}}(p,t) |v\rangle = \langle u| \underline{\underline{a}}_{in}(p) |v\rangle \tag{5.12}$$

$$\lim_{t\to+\infty} \langle u| \underline{\underline{a}}(p,t) |v\rangle = \langle u| \underline{\underline{a}}_{out}(p) |v\rangle \tag{5.13}$$

for any pair of normalized states $|u\rangle$ and $|v\rangle$. It is in this weak-limit sense that $\underline{\underline{\phi}}(x)$ is required to approach $\underline{\underline{\phi}}_{in}$ and $\underline{\underline{\phi}}_{out}$.

The operator field $\underline{\underline{\phi}}(x)$ is called the interpolating field, it interpolates between $\underline{\underline{\phi}}_{in}(x)$ and $\underline{\underline{\phi}}_{out}(x)$ in the weak-limit sense given by (5.12) and (5.13).

5.4. Reduction formulae, Green functions.

We derive an expression for S-matrix elements which contains the Green functions of the interpolating field and one-particle wave functions.

We start with the one-particle wave function.

Consider the matrix element

$$\langle o| \underline{\underline{\phi}}_{in}(x) |p,in\rangle = \langle o| [\underline{\underline{\phi}}_{in}(x), \underline{\underline{a}}^{\dagger}_{in}(p)] |o\rangle \tag{5.14}$$

$$= e^{-ipx} = f_p(x)$$

We call this matrix element a one-particle wave function.

Next we derive expressions for $\underline{\underline{a}}_{in}(p)$ and $\underline{\underline{a}}(p,t)$ in terms of the operator fields in space-time. We define the so-called Klein-Gordon form $\langle , \rangle_{K.G}$ for complex-valued functions f,g on space-time.

$$\langle f,g\rangle_{K.G} := i \int \left(f^*(x)\dot{g}(x) - \dot{f}^*(x)g(x) \right) d^3x$$

$$= i \int f^*(x) \overleftrightarrow{\frac{\partial}{\partial x^o}} g(x) d^3x .$$

This K.G.-form is a conserved quantity for any pair of solutions of the Klein-Gordon equation which vanish sufficiently fast for $|\vec{x}| \to \infty$.

One easily verifies the following "orthogonality relations" for one-particle wave functions.

$$\langle f_p, f_{p'} \rangle_{K.G} = 2p^o(2\pi)^3\delta(\vec{p}-\vec{p}') \tag{5.16}$$

$$\langle f_p^*, f_{p'} \rangle_{K.G} = \langle f_p, f_{p'}^* \rangle_{K.G} = o \tag{5.17}$$

$$\langle f_p^*, f_{p'}^* \rangle_{K.G} = -2p^o(2\pi)^3\delta(\vec{p}-\vec{p}') \tag{5.18}$$

Using these relations one obtains from (5.7) and (5.11)

$$\underline{\underline{a}}_{in}(p) = i\int d^3x\, f_p^*(x) \overleftrightarrow{\frac{\partial}{\partial x^o}} \underline{\underline{\phi}}_{in}(x) = \langle f_p, \underline{\underline{\phi}}_{in}(x) \rangle_{K.G} \tag{5.19}$$

$$\underline{\underline{a}}(p,t) = i\int d^3x\, f_p^*(x) \overleftrightarrow{\frac{\partial}{\partial x^o}} \underline{\underline{\phi}}(x) = \langle f_p, \underline{\underline{\phi}}(x) \rangle_{K.G} \tag{5.20}$$

Finally, we need the following relation

$$i\int d^4x(f^*K_x g - (K_x f^*)g) = \lim_{t\to\infty}\langle f,g\rangle_{K.G} - \lim_{t\to-\infty}\langle f,g\rangle \tag{5.21}$$

with $K_x = \partial_\mu\partial^\mu + m^2$ the Klein-Gordon wave-operator.
This relation is valid for function which vanish for $|\vec{x}| \to \infty$.
We are now ready for the reduction formulae of S-matrix elements.

Consider the matrix element

$$\begin{aligned} S_{p_1'\ldots p_\ell', p_1\ldots p_m} &= \langle p_1'\ldots p_\ell' \text{ out} \mid p_1\ldots p_m \text{ in}\rangle \\ &= \langle p_1'\ldots p_\ell' \text{ out} \mid \underline{\underline{a}}_{in}^\dagger(p_1) \mid p_2\ldots p_m \text{ in}\rangle \\ &= \lim_{t\to-\infty}\langle p_1'\ldots p_\ell' \text{ out} \mid \underline{\underline{a}}^\dagger(p_1,t) \mid p_2\ldots p_m \text{ in}\rangle \\ &= \lim_{t\to-\infty}\langle p_1'\ldots p_\ell' \text{ out} \mid \langle \underline{\underline{\phi}}^\dagger, f_{p_1}\rangle_{K.G} \mid p_2\ldots p_m \text{ in}\rangle \end{aligned} \tag{5.22}$$

Here we have applied the asymptotic condition (5.12) for the creation operators. Strictly speaking the condition may be applied only with normalizable states. We will stick to our notation and ask the reader to act as if the "plane waves" are replaced by normalized positive frequency wave packet solu-

tions of the Klein-Gordon equation. Let us now apply (5.21) with $g = f_{p_1}$ a normalized solutions of the Klein-Gordon equation and $f = \underline{\underline{\phi}}^{\dagger}$. This leads to

$$S_{p'_1\ldots p'_\ell,\, p_1\ldots p_m} = \langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{a}}^{\dagger}_{\text{out}}(p_1) \mid p_2\ldots p_m \text{ in}\rangle + \qquad (5.23)$$

$$+ i \int d^4x_1 \langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{\phi}}^{\dagger}(x_1) \mid p_2\ldots p_m \text{ in}\rangle \overleftarrow{K}_{x_1} f_{p_1}(x_1)$$

Notice that the first term is zero if all outgoing momenta $p'_1\ldots p'_\ell$ are different from p_1 (see also the conjugate of (4.8)).

To reduce the second term we write

$$\langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{a}}^{\dagger}_{\text{in}}(p_2) \mid p_3\ldots p_m \text{ in}\rangle =$$

$$= \lim_{t_2\to-\infty} \langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{a}}^{\dagger}(p_2,t_2) \mid p_3\ldots p_m \text{ in}\rangle$$

$$= \lim_{t_2\to-\infty} \langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{\phi}}^{\dagger}(x_1) \langle \underline{\underline{\phi}}^{\dagger}(x_2), f_{p_2}\rangle_{K.G} \mid p_3\ldots p_m \text{ in}\rangle$$

At this point we introduce the time-ordered product of operator fields; we write

$$\lim_{t_2\to-\infty} \underline{\underline{\phi}}^{\dagger}(x_1) \langle \underline{\underline{\phi}}^{\dagger}(x_2), f_{p_2}\rangle_{K.G} = \lim_{t_2\to-\infty} \langle T(\underline{\underline{\phi}}^{\dagger}(x_1)\, \underline{\underline{\phi}}^{\dagger}(x_2)), f_{p_2}\rangle_{K.G}$$

Now we apply again relation (5.21) with $g = f_{p_2}$ and $f^* = T\, \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{\phi}}^{\dagger}(x_2)$. This gives

$$\langle p'_1\ldots p'_\ell \text{ out} \mid \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{a}}^{\dagger}_{\text{in}}(p_2) \mid p_3\ldots p_m \text{ in}\rangle$$

$$= \lim_{t_2\to-\infty} \langle p'_1\ldots p'_\ell \text{ out} \mid \langle T\, \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{\phi}}^{\dagger}(x_2), f_{p_2}\rangle_{K.G} \mid p_3\ldots p_m \text{ in}\rangle$$

$$= \lim_{t_2\to+\infty} \langle p'_1\ldots p'_\ell \text{ out} \mid \langle T\, \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{\phi}}^{\dagger}(x_2), f_{p_2}\rangle_{K.G} \mid p_3\ldots p_m \text{ in}\rangle +$$

$$+ i \int dx_2 \langle p'_1\ldots p'_\ell \text{ out} \mid T\, \underline{\underline{\phi}}^{\dagger}(x_1)\underline{\underline{\phi}}(x_2) \mid p_3\ldots p_m \text{ in}\rangle \overleftarrow{K}_{x_2} f_{p_2}(x_2)$$

The term with $\lim_{t_2\to+\infty}$ leads to a matrix element of $\underline{\underline{a}}^{\dagger}_{\text{out}}(p_2)\underline{\underline{\phi}}^{\dagger}(x_1)$. This

term is zero if $p_2 \neq p_i'$ $(i = 1,\ldots\ell)$. We assume this to be the case. The result of the above manipulations is then

$$S_{p_1'\ldots p_\ell',\, p_1\ldots p_m} = \langle p_1'\ldots p_\ell' \text{ out} \mid p_1\ldots p_m \text{ in}\rangle = \tag{5.24}$$

$$= (i)^2 \int dx_1 dx_2 \langle p_1'\ldots p_\ell' \text{ out} \mid T\, \underline{\underline{\phi}}^\dagger(x_1)\underline{\underline{\phi}}^\dagger(x_2) \mid p_3\ldots p_m\rangle \overleftarrow{K}_{x_1} \overleftarrow{K}_{x_2} f_{p_1}(x_1) f_{p_2}(x_2)$$

The procedure leading from (5.22) to (5.24) can be repeated untill all in-particles are reduced and we end up with the vacuum state. The same procedure can be applied to the out-state.

If the out-momenta are different from the in-momenta we find the famous L.S.Z. expression for the S-matrix elements.

$$S_{p_1'\ldots p',\, p_1\ldots p_m} = \langle p_1'\ldots p_\ell' \text{ out} \mid p_1\ldots p_m\rangle =$$

$$= (i)^{\ell+m} \int d^4y_1\ldots d^4y_\ell\, d^4x_1\ldots d^4x_m\, f^*_{p_1'}(y_1)\ldots f^*_{p_\ell'}(y_\ell) K_{y_1}\ldots K_{y_\ell}$$

$$\langle o \mid T\, \underline{\underline{\phi}}(y_1)\ldots (y_\ell)\underline{\underline{\phi}}^\dagger(x_1)\ldots\underline{\underline{\phi}}^\dagger(x_m) \mid o\rangle \overleftarrow{K}_{x_1}\ldots\overleftarrow{K}_{x_m} f_{p_1}(x_1)\ldots f_{p_2}(x_m)$$

One observes the following facts. The kernel of this expression is the Green function

$$G(y_1\ldots y_\ell,\, x_1\ldots x_m) = \langle o \mid T\, \underline{\underline{\phi}}(y_1)\ldots\underline{\underline{\phi}}(y_\ell)\underline{\underline{\phi}}^\dagger(x_1)\ldots\underline{\underline{\phi}}^\dagger(x_m) \mid o\rangle \tag{5.26}$$

the vacuum expectation value of a time-ordered product of operator fields. Each in-particle gives rise to a $\underline{\underline{\phi}}^\dagger$ each out-particle gives a $\underline{\underline{\phi}}$. There is a Klein-Gordon operator $K_x = \partial_\mu\partial^\mu + m^2$ acting on the Green-function for each particle in the matrix element. Finally, we notice that $f_p(x) = e^{-ipx}$ and $f^*_p(y) = e^{ipy}$. This means that the S-matrix element is essentially the Fourier-transform, taken at the physical (on mass-shell) momenta $p_1\ldots p_m$, $p_1'\ldots p_\ell'$, of the expression

$$\overrightarrow{K}_{y_1}\ldots\overrightarrow{K}_{y_\ell} \langle o \mid T\, \underline{\underline{\phi}}(y_1)\ldots\underline{\underline{\phi}}(y_\ell)\underline{\underline{\phi}}^\dagger(x_1)\ldots\underline{\underline{\phi}}^\dagger(x_m) \mid o\rangle \overleftarrow{K}_{x_1}\ldots\overleftarrow{K}_{x_m}. \tag{5.27}$$

This is as far as we want to go with our exposition of the L.S.Z. scheme for the scattering operator.

The central quantity is the Green function (5.26). As has been indicated in the introduction this quantity can be studied (and sometimes calculated) using the path integral formalism.

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Infinitesimal Symmetries of Partial Differential Equations with an Application to the Construction of the Instanton and Monopole Solution of the Self-Dual Yang-Mills Equations

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TABLE OF CONTENTS

0. INTRODUCTION

Infinitesimal symmetries of partial differential equations can be described in an algebraic and elegant way using the local jet bundle formulation.

In section 1 we give a short survey of the notions of local jet bundles, differential equations, contact modules, exterior differential systems, (infinitesimal) symmetries and similarity solutions.

In section 2 we derive the Lie algebra of infinitesimal symmetries of self-dual SU(2) Yang-Mills equations and obtain the Belavin-Polyakov-Schwartz-Tyupkin instanton solution as a similarity solution. The Lie algebra of infinitesimal symmetries of the static self-dual SU(2) gauge field is given and the Prasad-Sommerfield monopole solution is obtained as a similarity solution too.

1. LOCAL JET BUNDLES, DIFFERENTIAL EQUATIONS, EXTERIOR DIFFERENTIAL SYSTEM, SYMMETRIES

Some of the general notions of the local jet bundle formulation are given. A more extensive introduction is given in [1], [2]. We shall adopt the notation from [2].

Partial differential equations are described according to [2] and exterior differential systems on a jet bundle are introduced. The prolongation of a differential equation and of an exterior differential system is defined. Symmetries of exterior differential systems are defined and the criterion for a vector field to generate a local 1-parameter group of symmetries is given. A theorem concerning the structure of the components of infinitesimal symmetries due to Bäcklund [3] is given; the notion of similarity solution [4], [5] is introduced.

Let M,N be C^∞ manifolds and let $C^\infty(M,N)$ denote the collection of local C^∞ maps $f: U \to N$, U open in M.

In applications, M is the space of independent variables and N is the space of dependent variables.

Two maps $f,g \in C^\infty(M,N)$ are said to agree to order k at $x \in M$ if $f(x) = g(x)$ and if there are local charts around $x \in M$, and around $f(x) = g(x) \in N$ in which all derivatives at x, up to and including order k are the same.

The equivalence class of maps which agree with f to order k at x is called the k-*jet of* f *at* x, denoted $j^k_x f$.

If $x^a (a=1,\ldots,\dim M)$ are local coordinates around $x \in M$ and $z(\mu=1,\ldots,\dim N)$ are local coordinates around $f(x) \in N$, then $j^k_x f$ is determined by the quantities

$$x^a, z^\mu = f^\mu(x),\ z^\mu_a = \partial_a f^\mu(x),\ldots,\ z^\mu_{a_1\ldots a_k} = \partial_{a_1\ldots a_k} f(x), \qquad (1.1)$$

where $f(x)$ is the coordinate presentation of f.

In (1.1) and further $\partial_a, \ldots, \partial_{a_1 \ldots a_k}$, denote partial derivatives

$$\partial_a f^\mu(x) = \frac{\partial}{\partial x^a} f^\mu(x), \quad \partial_{a_1 \ldots a_k} f^\mu(x) = \frac{\partial^k}{\partial x^{a_1} \ldots \partial x^{a_k}} f^\mu(x) \tag{1.1a}$$

Latin indices $a, a_1, \ldots$ range and sum over $1, \ldots, \dim M$, while Greek indices range and sum over $1, \ldots, \dim N$.

Conversely, any collection of numbers

$$x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k}$$

where $z^\mu_{a_1 \ldots a_\ell}$ are symmetric in their subscripts, determines a unique equivalence class.

From this we obtain the following definition of the k-*jet bundle*.

The k-jet bundle of M and N, denoted $J^k(M,N)$ is the set of all k-jets $j^k_x f$ with k fixed, $x \in M$, $f \in C^\infty(M,N)$, provided with a natural differentiable structure.

The map

$$\alpha: J^k(M,N) \to M \tag{1.2}$$

defined by

$$j^k_x f \longmapsto x \tag{1.2a}$$

is called the source map, and x is called the source of $j^k_x f$.

The map

$$\beta: J^k(M,N) \to N \tag{1.3}$$

defined by

$$j^k_x f \longmapsto f(x) \tag{1.3a}$$

is called the target map, and $f(x)$ is called the target of $j^k_x f$.

In [2] it is demonstrated that

$$x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k} \tag{1.4}$$

may be chosen as local coordinates around $\xi \in J^k(M,N)$ where x^a are local coordinates around $x = \alpha(\xi) \in M$ and z^μ are local coordinates around $f(x) = \beta(\xi) \in N$.

If $k > \ell$, then ignoring all derivatives above the ℓ-th, yields the natural projection, π^k_ℓ of the k-jet bundle on the ℓ-jet bundle

$$\begin{aligned} \pi^k_\ell &: J^k(M,N) \to J^\ell(M,N) \\ & j^k_x f \mapsto j^\ell_x f. \end{aligned} \tag{1.5}$$

In local coordinates this amounts to

$$\pi^k_\ell(x^a, z^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_k}) = (x^a, x^\mu, z^\mu_a, \ldots, z^\mu_{a_1 \ldots a_\ell}) \tag{1.6}$$

$J^0(M,N)$ may be identified with $M \times N$.

If $f \in C^\infty(M,N)$ then the k-*jet extension of* f is the map

$$j^k f: U \to J^k(M,N) \tag{1.7}$$

defined by

$$x \mapsto j^k_x f \tag{1.7a}$$

In order to decide whether a map

$$\phi: M \to J^k(M,N) \tag{1.8}$$

is the k-jet extension of a function $f \in C^\infty(M,N)$ i.e.,

$$\phi(x) = j^k_x f \tag{1.9}$$

the *contact module* Ω^k over $C^\infty(J^k(M,N))$ is introduced as the module of 1-forms θ on $J^k(M,N)$ such that

$$(j^k f)^* = 0 \tag{1.10}$$

for every function $f \in C^\infty(M,N)$.

In (1.10) $(j^k f)^* \theta$ is the *pullback* of the form θ to $U \subset M$ by the function $j^k f$.

In standard coordinates a basis for Ω^k is given by

$$\begin{aligned}
\theta^\mu &= dz^\mu - z^\mu_c dx^c \\
\theta^\mu_a &= dz^\mu_a - z^\mu_{ac} dx^c \\
&\vdots \\
\theta^\mu_{a_1 \cdots a_{k-1}} &= dz^\mu_{a_1 \cdots a_{k-1}} - z^\mu_{a_1 \cdots a_{k-1} c} dx^c .
\end{aligned} \tag{1.11}$$

The introduction of the module Ω^k is motivated by:

ϕ is the k-jet extension of a function $f \in C^\infty(M,N)$ iff $\phi^*(\Omega^k) = 0$.

For later use we define the total derivative vector fields on $J^k(M,N)$ by

$$D^{(k)}_a = \partial_a + z^\mu_a \partial_{z^\mu} + \ldots + z^\mu_{a a_1 \cdots a_{k-1}} \partial_{z^\mu_{a_1 \cdots a_{k-1}}} \tag{1.12}$$

Total derivative vector fields commute

$$[D^{(k)}_a, D^{(k)}_b] = 0 \tag{1.13}$$

where in (1.13) $[\ ,\ ]$ denotes the Lie bracket of vector fields.

A system of partial differential equations of order k, for short a differential equation, can be described by functions

$$F^h : J^k(M,N) \to \mathbb{R} \qquad (h=1,\ldots,c) \tag{1.14}$$

A *differential equation* is a subset $\mathcal{Y} \subset J^k(M,N)$ being the zero set of

$$F: J^k(M,N) \to \mathbb{R}^c \tag{1.15}$$

$$\xi \in \mathcal{Y} \mapsto 0 \tag{1.15a}$$

A *solution of a differential equation* is a map $f \in C^\infty(M,N)$ such that

$$j^k_x f \in \mathcal{Y} \quad \text{for} \quad x \in U \subset M \tag{1.16}$$

In terms of differential forms a differential equation of order k can be described by a *closed ideal* I of differential forms defined on $J^k(M,N)$. An ideal I of differential forms is *closed* if

$$dI \subset I. \tag{1.17}$$

Without loss of generality we can assume that a closed ideal I is generated by homogeneous forms $\alpha_1, \ldots, \alpha_m$ and their exterior derivatives $d\alpha_1, \ldots, d\alpha_m$

$$I = \langle \alpha_1, \ldots, \alpha_m, d\alpha_1, \ldots, d\alpha_m \rangle \tag{1.18}$$

In the sequel we shall sometimes call a closed ideal I an *exterior differential system*, and we shall always assume an ideal I to be closed.
In section 2 we construct the ideal I of differential forms associated with a differential equation (1.17) in the following way.
Start at the basis of contact 1-forms defined in (1.11) and solve each of

$$F^h(\xi) = 0 \qquad (h=1,\ldots,c)$$

for some $z^\mu_{a_1 \ldots a_k}$ and substitute into (1.11); the ideal I is then generated by these new 1-forms and their exterior derivatives.
In effect we construct the restriction of the contact 1-forms to the differential equation $\mathcal{Y}$ (1.15).

A *solution of an exterior differential system* is a function $f \in C^{\infty}(M,N)$ such that

$$(j^k f)^* I = 0 \tag{1.19}$$

Note that a solution defined by (1.19) of the exterior differential system (1.19) sketched above is a solution of $\mathcal{Y}$ (1.16) and vice versa.

A *symmetry* of an exterior differential system defined on $J^k(M,N)$ is a C^{∞}-map

$$\phi: J^k(M,N) \to J^k(M,N) \tag{1.20}$$

such that

$$\phi^* I \subset I \tag{1.21}$$

If $\phi_t: J^k(M,N) \to J^k(M,N)$ $(|t| < \varepsilon)$ is a local 1-parameter group of symmetries [6], such that ϕ_0 is the identity map, then the generating vector field V of this 1-parameter group is called an *infinitesimal symmetry* of the exterior differential system, and the vector field V has to satisfy the condition

$$L_V I \subset I \tag{1.22}$$

where L_V denotes the Lie derivative with respect to the vector field V. The infinitesimal symmetries of an exterior differential system constitute a Lie algebra under the standard Lie bracket of vector fields.

Given a differential form β, then

$$L_V d\beta = d L_V \beta \tag{1.23}$$

i.e., the Lie derivative with respect to a vector field V and the exterior derivative commute.

Relation (1.23) will be used in the following way:

Let I be generated by $\alpha_1, \ldots, \alpha_m$, $d\alpha_1, \ldots, d\alpha_m$ (1.18), then if the vector field V satisfies

$$L_V \alpha_i \in I, \quad (i=1,\ldots,m) \tag{1.24}$$

the condition

$$L_V d\alpha_i \in I \quad (i=1,\ldots,m) \tag{1.25}$$

is satisfied, due to (1.23).

So, if V satisfies (1.24), it satisfies (1.22) and is by consequence an infinitesimal symmetry of I.

We now state the following theorem about the structure of the components of an infinitesimal symmetry [3].

THEOREM 1.1. (Bäcklund)

Let the vector field V *defined on* $J^k(M,N)$ *be an infinitesimal symmetry of an exterior differential system* I,

$$V = \xi^a \partial_{x^a} + \eta^\mu \partial_{z^\mu} + \eta^\mu_a \partial_{z^\mu_a} + \ldots + \eta_{a_1 \ldots a_k} \partial_{z^\mu_{a_1 \ldots a_k}} \tag{1.26}$$

then there are two cases to be considered

case a: $\dim N > 1$

the components of V (1.26) have the following structure

$$\begin{aligned} \xi^a &= \xi^a(x^a, z^\mu) \\ \eta^\mu &= \eta^\mu(x^a, z^\mu) \\ \eta^\mu_a &= D^{(k)}_a(\eta^\mu) - z^\mu_b D^{(k)}_a(\xi^b) \\ \eta^\mu_{a_1 \ldots a_k} &= D^{(k)}_{a_k}(\eta^\mu_{a_1 \ldots a_{k-1}}) - z^\mu_{a_1 \ldots a_{k-1} b} D^{(k)}_{a_k}(\xi^b) \end{aligned} \tag{1.27}$$

i.e. the vector field V (1.26) generates a local group of *point* transformations $\overline{\phi_t}: J^0(M,N) \to J^0(M,N)$

case b: $\dim N = 1$ $(z^1=z)$
the most general situation is that there is a function

$$W = W(x^a, z, z_a) \tag{1.28}$$

such that the components of V (1.26) are given by

$$\xi^a = -\frac{\partial W}{\partial z_a}, \quad \eta = W - z_b \frac{\partial W}{\partial z_b}, \quad \eta_a = \frac{\partial W}{\partial x^a} + z_a \frac{\partial W}{\partial z} \tag{1.29}$$

while the other components of V are obtained in the same way as in case a (1.27), i.e., the vector field V (1.29) generates a local group of Lie *contact* transformations [3], $\bar{\phi}_t: J^1(M,N) \to J^1(M,N)$

Similarity solutions are introduced according to [5].
Let I be an exterior differential system defined on $J^k(M,N)$ and let the vector field V be an infinitesimal symmetry of I (1.22).
We construct the ideal I' from I (1.18) in the following way

$$I' = \langle \alpha_1, .., d\alpha_m, V \lrcorner \alpha_1, \ldots V \lrcorner d\alpha_m \rangle \tag{1.30}$$

The ideal I' is closed i.e., $dI' \subset I'$.
Moreover the vector field V is a Cauchy characteristic [7] of the ideal I' i.e.,

$$V\ I' \subset I' \tag{1.31}$$

and due to Cartan's theory on exterior differential systems we have a dimension reduction; for instance a partial differential equation in 2 independent variables reduces to an ordinary differential equation.

To demonstrate this technique we discuss

EXAMPLE 1.1. The Heat equation is given by

$$u_t = u_{xx}(x^1=x, x^2=t, z^1=z=u) \tag{1.32}$$

A closed ideal I of differential forms defined on $\mathbb{R}^7 = \{(x,t,u,u_x,u_t,u_{xt},u_{tt})\}$ is generated by

$$\begin{aligned} \alpha_1 &= du - u_x\,dx - u_t\,dt \\ \alpha_2 &= du_x - u_t\,dx - u_{xt}\,dt \\ \alpha_3 &= du_t - u_{xt}\,dx - u_{tt}\,dt \end{aligned} \tag{1.33}$$

and the exterior derivatives $d\alpha_1$, $d\alpha_2$, $d\alpha_3$.

The Lie algebra of infinitesimal symmetries is formed by [5]

$$\begin{aligned} &X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = u\partial_u, \quad X_4 = 2t\partial_x - xu\partial_u \\ &X_5 = x\partial_x + 2t\partial_t, \qquad X_6 = xt\partial_x + t^2\partial_t + \left(-\frac{x^2}{4} - \frac{t}{2}\right)u\partial_u \\ &X_7 = g(x,t)\,\partial_u, \quad \text{where } g(x,t) \text{ is a function satisfying (1.32)} \end{aligned} \tag{1.34}$$

In (1.34) we only write the ∂_x-, ∂_t-, ∂_u-components of the vector fields, the other components are defined by (1.27). A similarity solution associated with X_4 is obtained by the contraction of X_4 and α_1

$$X_4 \;\alpha_1 = -xu - 2tu_x \tag{1.35}$$

It can be shown that the other conditions (1.30) lead to differential consequence of (1.35).

To construct a solution of (1.30) we have to solve

$$u_t = u_{xx} \tag{1.36a}$$

$$-xu - 2tu_x = 0 \tag{1.36b}$$

From (1.36b) we obtain

$$u(x,t) = H(t)e^{-\frac{x^2}{4t}}, \tag{1.37}$$

and substitution of (1.37) into (1.36a) results in an ordinary differential equation for the function $H(t)$

$$\frac{dH(t)}{dt} = -\frac{1}{2t}H(t)$$

from which we obtain the following well-known solution of the heat equation

$$u(x,t) = \frac{C}{\sqrt{t}}e^{-\frac{x^2}{4t}} \quad \blacksquare \tag{1.38}$$

An elegant application of infinitesimal symmetries in the study of nonlinear differential equations is given by the following

THEOREM 1.2. (Kumei & Bluman)[8].

A scalar n-th *order nonlinear differential equation*

$$F(x^a, z, z_a, \ldots, z_{a_1 \ldots a_n}) = 0 \qquad x^a \in \mathbb{R}^m, \qquad z \in \mathbb{R} \tag{1.39}$$

is transformable by a 1-1 *contact transformation to a linear differential equation if and only if the differential equation* (1.39) *admits an infinitesimal symmetry of the form*

$$V = [\sigma(w)Z(\bar{X}^a(w))]\partial_z + \ldots \tag{1.40}$$

where $w = (x^a, z, z_a)$

1°. $Z\colon \mathbb{R}^m \to \mathbb{R}(X^a \longmapsto Z(X^a))$ *is an arbitrary solution of some* n-th *order linear differential equation*

$$A(Z) = A(X^b)Z + A^a(X^b)Z_a + \ldots + A^{a_1 \ldots a_n}(X^b)Z_{a_1 \ldots a_n} = 0 \tag{1.41}$$

2°. $\bar{X}^a\colon R^{2m+1} \to R^m(w \longmapsto \bar{X}^a(w))$ *is a component of a Lie contact transformation*

$$X^a = \bar{X}^a(w),\ Z = \bar{Z}(w),\ Z_a = \bar{Z}_a(w) \tag{1.42}$$

and

$$\begin{aligned} \partial_{z_a} \bar{Z}(w) &= \bar{Z}_b(w)\partial_{z_a} \bar{X}_b(w) \\ \sigma(w) &= (\partial_z \bar{Z}(w) - \bar{Z}_a(w)\partial_z \bar{X}^a(w))^{-1} \end{aligned} \tag{1.43}$$

The transformation (1,42), (1.43) *maps* (1.39) *to a linear differential equation*

$$A(Z) - \Phi(X^a) = 0 \tag{1.44}$$

EXAMPLE 1.2. We consider Thomas' equation [9]

$$z_{xy} + \alpha z_x + \beta z_y + \gamma z_x z_y = 0 \qquad (\alpha,\beta,\gamma \neq 0) \tag{1.45}$$

The Lie algebra of infinitesimal symmetries of (1.1.56) is generated by

$$\begin{aligned} V_1 &= \partial_x,\quad V_2 = \partial_y,\quad V_3 = \partial_z \\ V_4 &= \gamma x\partial_x - \gamma y\partial_y + (\alpha y - \beta x)\partial_z + \ldots \\ V_5 &= e^{-\gamma z}\, v(x,y)\partial_z + \ldots \end{aligned} \tag{1.46}$$

where $v(x,y)$ has to satisfy the linear differential equation

$$v_{xy} + \alpha v_x + \beta v_y = 0 \tag{1.47}$$

First of all V_5 (1.46) is just the vector field V in Theorem 1.2, while the left-hand side of (1.47) is just $A(Z)$ of (1.41).

The Lie contact transformation (1.42) is determined by

$$X^1 = x,\quad X^2 = y \tag{1.48}$$

and Z has to satisfy (1.43), so

$$e^{-z} = (\partial_z \bar{Z})^{-1} \tag{1.49}$$

From equation (1.49) we obtain [8]

$$Z = \frac{e^{\gamma z}}{\gamma} - \phi(x,y) \tag{1.50}$$

In effect the Lie contact transformation (1.42) is in this situation just a Lie point transformation (1.48), (1.50).

From (1.48), (1.50) we derive

$$\begin{aligned} x &= X^1 \\ y &= X^2 \\ z &= \gamma^{-1} \log \gamma(Z+\phi) \end{aligned} \tag{1.51}$$

and

$$\begin{aligned} z_x &= \gamma^{-1}(Z+\phi)^{-1}(Z_{X^1}+\phi_{X^1}) \\ z_y &= \gamma^{-1}(Z+\phi)^{-1}(Z_{X^2}+\phi_{X^2}) \\ z_{xy} &= \gamma^{-1}(Z+\phi)^{-1}(Z_{X^1X^2}+\phi_{X^1X^2}) - \gamma^{-1}(Z+\phi)^{-2}(Z_{X^1}+\phi_{X^1})(Z_{X^2}+\phi_{X^2}) \end{aligned} \tag{1.52}$$

Substitution of (1.51), (1.52) into (1.45) results in

$$\gamma^{-1}(Z+\phi)^{-1}\{Z_{X^1X^2} + \alpha Z_{X^1} + \beta Z_{X^2} - \Phi(X^1,X^2)\} = 0 \tag{1.53}$$

where

$$\Phi(X^1,X^2) = -\phi_{X^1X^2} - \alpha\phi_{X^1} - \beta\phi_{X^2} \tag{1.53a}$$

Equation (1.53) is in agreement with equation (1.44) of Theorem 1.2.

If we take

$$\phi(x,y) = \text{constant} \tag{1.54}$$

then the transformation defined by (1.51), (1.52) transforms the differential equation (1.45) into

$$Z_{X^1X^2} + \alpha Z_{X^1} + \beta Z_{X^2} = 0 \tag{1.55}$$

which is just equation (1.47). ∎

2. INFINITESIMAL SYMMETRIES OF SELF-DUAL SU(2) YANG-MILLS EQUATIONS. THE BELAVIN-POLYAKOV-SCHWARTZ-TYUPKIN INSTANTON AND THE MONOPOLE SOLUTION.

First of all we shall give a very short description of the SU(2)-gauge theory. For a more extensive elementary exposition we refer to the survey paper by M.K. Prasad [10], from which we adopt the notations. Then we shall indicate how to derive the Lie algebra of infinitesimal symmetries of the self-dual SU(2) Yang-Mills equations. Using these results we demonstrate the way in which the BPST-instanton solution can be obtained as a similarity solution [11].

By imposing additional conditions we compute the Lie algebra of infinitesimal symmetries of the static self-dual SU(2) equations. Analogous to the construction of the BSPT-instanton we derive the *ansatz* [12] leading to the monopole solution.

Let M be a 4-dimensional Euclidean space with coordinates $x = (x_1,\ldots,x_4)$ so there will be no distinction between contravariant and covariant indices, $x_\mu = x^\mu$. The basic object in gauge theory is the Yang-Mills gauge potential. The gauge potential is a set of fields $A^a_\mu (a=1,\ldots,3;\ \mu=1,\ldots,4)$.

It is convenient to introduce a matrix valued vector field $A_\mu(x)$, by

$$A_\mu = gT^a A^a_\mu ; \quad T^a = \frac{\sigma^a}{2i} (=1,2,3), \tag{2.1}$$

where σ^a are the Pauli matrices

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}; \tag{2.1a}$$

g being a constant called the gauge coupling constant. Throughout this section we shall use the Einstein summation convention when an index occurs twice. ($a,b,\ldots$ take on the values $1,2,3$; μ,ν take on the values $1,\ldots,4$).

From the matrix valued gauge potential $A_\mu dx_\mu$ one constructs the matrix valued field strength $F_{\mu\nu}(x)$ by

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{2.2}$$

where $\partial_\mu = \partial_{x_\mu}$; $[A_\mu, A_\nu] = A_\mu A_\nu - A_\nu A_\mu$.

If one defines the covariant derivative

$$D_\mu = \partial_\mu + A_\mu \tag{2.3}$$

then (2.2) is rewritten as

$$F_{\mu\nu} = [D_\mu, D_\nu] \tag{2.4}$$

In explicit component form,

$$F_{\mu\nu} = gT^a F^a_{\mu\nu} \ , \tag{2.5a}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\varepsilon_{abc} A^b_\mu A^c_\nu \tag{2.5b}$$

and $$\varepsilon_{abc} = \begin{cases} 1 & \text{if } abc \text{ is an even permutation of } 1\ 2\ 3 \\ -1 & \text{if } abc \text{ is an odd permutation of } 1\ 2\ 3 \\ 0 & \text{otherwise.} \end{cases}$$

We will use the expression "static gauge fields" to refer to gauge potentials that are independent of x_4 (x_4 to be considered as time) i.e.,

$$\partial_4 A_\mu(x) = 0 \qquad (\ =1,\ldots,4) \tag{2.6}$$

For gauge potentials that depend on all four coordinates $x_1, \ldots, x_4$ the action functional is defined by

$$S = \frac{1}{4} \int F^a_{\mu\nu} F_{\mu\nu} (d^4x) \ , \tag{2.7a}$$

the integral taken over $\mathbb{R}^4$, while for static gauge fields we define the energy functional by

$$E = \frac{1}{4} \int F^a_{\mu\nu} F^a_{\mu\nu} (d^3x) \tag{2.7b}$$

whereas in (2.7b) the integral is taken over $\mathbb{R}^3$.

The extremals of the action S (or of the energy E for static gauge fields) are found by standard calculus of variations techniques leading to the Euler-Lagrange equations

$$\partial_\mu F_{\mu\nu} + [A_\mu, F_{\mu\nu}] = 0 = [D_\mu, F_{\mu\nu}] \tag{2.8}$$

or, in components

$$\partial_\mu F^a_{\mu\nu} + g\varepsilon_{abc} A^b_\mu F^c_{\mu\nu} = 0 \tag{2.8a}$$

(2.8a) is a system of *second order* nonlinear partial differential equations for the 12 unknown functions $A^a_\mu(x)$ $(a=1,\ldots,3;\ \mu=1,\ldots,4)$ and seems hard to solve.

Then one introduces the dual gauge field strength $*F_{\mu\nu}$ defined as

$$*F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho} F_{\lambda\rho} \tag{2.9}$$

where $\varepsilon_{\mu\nu\lambda\rho}$ is the completely antisymmetric tensor in M, defined by

$$\varepsilon_{\mu\nu\lambda\rho} = \begin{cases} +1 & \text{if } \mu\nu\lambda\rho \text{ is an even permutation of } 1\ 2\ 3\ 4 \\ -1 & \text{if } \mu\nu\lambda\rho \text{ is an odd permutation of } 1\ 2\ 3\ 4 \\ 0 & \text{otherwise.} \end{cases}$$

Since D_μ (2.3) satisfies the Jacobi identity

$$[D_\lambda, [D_\mu, D_\nu]] + [D_\mu, [D_\nu, D_\lambda]] + [D_\nu, [D_\lambda, D_\mu]] = 0 \tag{2.10}$$

multiplication of (2.10) by $\varepsilon_{\mu\nu\lambda\rho}$ results in

$$[D_\mu, {}^*F_{\mu\nu}] = 0 \tag{2.11}$$

If we compare (2.8), (2.11) we see that any gauge field which is self-dual

$${}^*F_{\mu\nu} = F_{\mu\nu}, \tag{2.12}$$

automatically satisfies (2.6).

(2.9) is a system of *first order* nonlinear partial differential equations. Instanton solutions, and monopole solutions for the static gauge fields satisfy (2.12) [10] and the condition that (2.7a), (2.7b) are finite.

Written out, (2.12) takes the form

$$F_{12} = F_{34}, \quad F_{13} = -F_{24}, \quad F_{14} = F_{23} \tag{2.13}$$

So in components the self-dual Yang-Mills equations is described as a system of 9 nonlinear partial differential equations,

$$\begin{aligned}
&-A^1_{4,1} + A^1_{3,2} - A^1_{2,3} + A^1_{1,4} + g\{-A^2_1A^3_4 + A^2_2A^3_3 - A^2_3A^3_2 + A^2_4A^3_1\} = 0\\
&-A^2_{4,1} + A^2_{3,2} - A^2_{2,3} + A^2_{1,4} + g\{A^1_1A^3_4 - A^1_2A^3_3 + A^1_3A^3_2 - A^1_4A^3_1\} = 0\\
&-A^3_{4,1} + A^3_{3,2} - A^3_{2,3} + A^3_{1,4} + g\{-A^1_1A^2_4 + A^1_2A^2_3 - A^1_3A^2_2 + A^1_4A^2_1\} = 0\\
&A^1_{3,1} + A^1_{4,2} - A^1_{1,3} - A^1_{2,4} + g\{A^2_1A^3_3 + A^2_2A^3_4 - A^2_3A^3_1 - A^2_4A^3_2\} = 0\\
&A^2_{3,1} + A^2_{4,2} - A^2_{1,3} - A^2_{2,4} + g\{-A^1_1A^3_3 + A^1_2A^3_4 + A^1_3A^3_1 + A^1_4A^3_2\} = 0\\
&A^3_{3,1} + A^3_{4,2} - A^3_{1,3} - A^3_{2,4} + g\{A^1_1A^2_3 + A^1_2A^2_4 - A^1_3A^2_1 - A^1_4A^2_2\} = 0\\
&A^1_{2,1} - A^1_{1,2} - A^1_{4,3} + A^1_{3,4} + g\{A^2_1A^3_2 - A^2_2A^3_1 - A^2_3A^3_4 + A^2_4A^3_3\} = 0\\
&A^2_{2,1} - A^2_{1,2} - A^2_{4,3} + A^2_{3,4} + g\{-A^1_1A^3_2 + A^1_2A^3_1 + A^1_3A^3_4 - A^1_4A^3_3\} = 0\\
&A^3_{2,1} - A^3_{1,2} - A^3_{4,3} + A^3_{3,4} + g\{A^1_1A^2_2 - A^1_2A^2_1 - A^1_3A^2_4 + A^1_4A^2_3\} = 0
\end{aligned} \tag{2.14}$$

whereas in (2.14)

$$A^a_{\mu,\nu} = \partial_\nu A^a_\mu \qquad (a = 1,\ldots,3;\ \mu,\nu = 1,\ldots,4)$$

We construct a differential ideal I defined on a 55-dimensional space

$$\mathbb{R}^{55} = \{(x_1,x_2,x_3,x_4, A^1_1,\ldots,A^1_4, A^1_{1,1},\ldots,A^3_{4,4}, \hat{A}^1_{1,4},\ldots,\hat{A}^3_{3,4})\}, \qquad (2.16)$$

generated by the twelve 1-forms

$$\alpha^a_\mu = dA^a_\mu - A^a_{\mu,\nu}dx_\nu, \qquad (a = 1,\ldots,3;\ \mu = 1,\ldots,4) \qquad (2.17)$$

and substitution of $A^1_{1,4}, \ldots, A^3_{3,4}$, obtained by solving (2.14) for these variables. (in (2.16) $\hat{A}^1_{1,4},\ldots,$ denotes deletion)

The symmetry condition

$$L_V I \subset I,$$

where V is a vector field defined on $\mathbb{R}^{55}$ results in an overdetermined system of 48 partial differential equations for the components of this vector field. Since the most general Lie contact symmetries are Lie point symmetries [3], the (16) components of $\partial_1, \ldots, \partial_4, \partial_{A^1_1}, \ldots, \partial_{A^3_4}$ are functions, depending on $x_1, \ldots, x_4, A^1_1, \ldots, A^3_4$.
The general solution of this overdetermined system constitutes a Lie algebra of infinitesimal symmetries [13], generated by the vector fields

```
VF(1) := (D_A11*DF(F(71),X1) + D_A12*DF(F(71),X2) + D_A13*DF(F(71),X3)
        + D_A14*DF(F(71),X4) + D_A21*A31*GE*F(71) + D_A22*GE*A32*F(71)
        + D_A23*GE*A33*F(71) + D_A24*A34*GE*F(71) - D_A31*A21*GE*F(71)
        - D_A32*GE*A22*F(71) - D_A33*GE*A23*F(71) - D_A34*A24*GE*F(71))/GE
```

(2.18)

$$
\begin{aligned}
VF(2) &:= (-D_{A11}*A21*GE*F(81) - D_{A12}*GE*A22*F(81) - D_{A13}*GE*A23*F(81) \\
&\quad - D_{A14}*A24*GE*F(81) + D_{A21}*A11*GE*F(81) + D_{A22}*GE*A12*F(81) \\
&\quad + D_{A23}*GE*A13*F(81) + D_{A24}*GE*A14*F(81) - D_{A31}*DF(F(81),X1) \\
&\quad - D_{A32}*DF(F(81),X2) - D_{A33}*DF(F(81),X3) - D_{A34}*DF(F(81),X4))/GE \\
VF(3) &:= (D_{A11}*A31*GE*F(63) + D_{A12}*GE*A32*F(63) + D_{A13}*GE*A33*F(63) \\
&\quad + D_{A14}*A34*GE*F(63) - D_{A21}*DF(F(63),X1) - D_{A22}*DF(F(63),X2) \\
&\quad - D_{A23}*DF(F(63),X3) - D_{A24}*DF(F(63),X4) - D_{A31}*A11*GE*F(63) \\
&\quad - D_{A32}*GE*A12*F(63) - D_{A33}*GE*A13*F(63) - D_{A34}*GE*A14*F(63))/GE \\
VF(4) &:= D_{X1} \\
VF(5) &:= D_{X2} \\
VF(6) &:= D_{X3} \\
VF(7) &:= D_{X4} \\
VF(8) &:= D_{X1}*X2 - D_{X2}*X1 + D_{A11}*A12 - D_{A12}*A11 + D_{A21}*A22 - D_{A22}*A21 \\
&\quad + D_{A31}*A32 - D_{A32}*A31 \qquad (2.18) \\
VF(9) &:= -D_{X1}*X3 + D_{X3}*X1 - D_{A11}*A13 + D_{A13}*A11 - D_{A21}*A23 + D_{A23}*A21 \\
&\quad - D_{A31}*A33 + D_{A33}*A31 \\
VF(10) &:= -D_{X1}*X4 + D_{X4}*X1 - D_{A11}*A14 + D_{A14}*A11 - D_{A21}*A24 + D_{A24}*A21 \\
&\quad - D_{A31}*A34 + D_{A34}*A31 \\
VF(11) &:= -D_{X2}*X3 + D_{X3}*X2 - D_{A12}*A13 + D_{A13}*A12 - D_{A22}*A23 + D_{A23}*A22 \\
&\quad - D_{A32}*A33 + D_{A33}*A32 \\
VF(12) &:= D_{X2}*X4 - D_{X4}*X2 + D_{A12}*A14 - D_{A14}*A12 + D_{A22}*A24 - D_{A24}*A22 \\
&\quad + D_{A32}*A34 - D_{A34}*A32 \\
VF(13) &:= -D_{X3}*X4 + D_{X4}*X3 - D_{A13}*A14 + D_{A14}*A13 - D_{A23}*A24 + D_{A24}*A23 \\
&\quad - D_{A33}*A34 + D_{A34}*A33 \\
VF(14) &:= D_{X1}*X1 + D_{X2}*X2 + D_{X3}*X3 + D_{X4}*X4 - D_{A11}*A11 - D_{A12}*A12 - D_{A13}*A13 \\
&\quad -D_{A14}*A14 - D_{A21}*A21 - D_{A22}*A22 - D_{A23}*A23 - D_{A24}*A24 - D_{A31}*A31 \\
&\quad - D_{A32}*A32 - D_{A33}*A33 - D_{A34}*A34
\end{aligned}
$$

$$
\begin{aligned}
VF(15) :=\ & D_{X1}*(X4^2+X3^2+X2^2-X1^2) - 2*D_{X2}*X2*X1 - 2*D_{X3}*X3*X1 - 2*D_{X4}*X4*X1 \\
& 2*D_{A11}*(X4*A14+X3*A13+X2*A12+X1*A11) + 2*D_{A12}*(-X2*A11+X1*A12) \\
& + 2*D_{A13}*(-X3*A11+X1*A13) + 2*D_{A14}*(-X4*A11+X1*A14) \\
& + 2*D_{A21}*(A21*X1+X4*A24+X3*A23+X2*A22) + 2*D_{A22}*(-A21*X2+X1*A22) \\
& + 2*D_{A23}(-A21*X3+X1*A23) + 2*D_{A24}*(-A21*X4+X1*A24) \\
& 2*D_{A31}*(X4*A34+X3*A33+X2*A32+X1*A31) + 2*D_{A32}*(-X2*A31+X1*A32) \\
& 2*D_{A33}*(-X3*A31+X1*A33) + 2*D_{A34}*(-X4*A31+X1*A34) \\
VF(16) :=\ & -2*D_{X1}*X2*X1 + D_{X2}*(X4^2+X3^2-X2^2+X1^2) - 2*D_{X3}*X3*X2 - 2*D_{X4}*X4*X2 \\
& + 2*D_{A11}*(X2*A11-X1*A12) + 2*D_{A12}*(X4*A14+X3*A13+X2*A12+X1*A11) \\
& + 2*D_{A13}*(-X3*A12+X2*A13) + 2*D_{A14}*(-X4*A12+X2*A14) \\
& + 2*D_{A21}*(A21*X2-X1*A22) + 2*D_{A22}*(A21*X1+X4*A24+X3*A23+X2*A22) \\
& + 2*D_{A23}*(-X3*A22+X2*A23) + 2*D_{A24}*(-X4*A22+X2*A24) \\
& + 2*D_{A31}*(X2*A31-X1*A32) + 2*D_{A32}*(X4*A34+X3*A33+X2*A32+X1*A31) \\
& + 2*D_{A33}*(-X3*A32+X2*A33) + 2*D_{A34}*(-X4*A32+X2*A34) \\
VF(17) :=\ & -2*D_{X1}*X3*X1 - 2*D_{X2}*X3*X2 + D_{X3}*(X4^2-X3^2+X2^2+X1^2) - 2*D_{X4}*X4*X3 \\
& + 2*D_{A11}*(X3*A11-X1*A13) + 2*D_{A12}*(X3*A12-X2*A13) \\
& + 2*D_{A13}*(X4*A14+X3*A13+X2*A12+X1*A11) + 2*D_{A14}*(-X4*A13+X3*A14) \\
& + 2*D_{A21}*(A21*X3-X1*A23) + 2*D_{A22}*(X3*A22-X2*A23) \\
& + 2*D_{A23}*(A21*X1+X4*A24+X3*A23+X2*A22) + 2*D_{A24}*(-X4*A23+X3*A24) \\
& + 2*D_{A31}*(X3*A31-X1*A33) + 2*D_{A32}*(X3*A32-X2*A33) \\
& + 2*D_{A33}*(X4*A34+X3*A33+X2*A32+X1*A31) + 2*D_{A34}*(-X4*A33+X3*A34) \\
VF(18) :=\ & -2*D_{X1}*X4*X1 - 2*D_{X2}*X4*X2 + 2*D_{X3}*X4*X3 + D_{X4}*(-X4^2+X3^2+X2^2+X1^2) \\
& + 2*D_{A11}*(X4*A11-X1*A14) + 2*D_{A12}*(X4*A12-X2*A14) \\
& + 2*D_{A13}*(X4*A13-X3*A14) + 2*D_{A14}*(X4*A14+X3*A13+X2*A12+X1*A11) \\
& + 2*D_{A21}*(A21*X4-X1*A24) + 2*D_{A22}*(X4*A22-X2*A24) \\
& + 2*D_{A23}*(X4*A23-X3*A24) + 2*D_{A24}*(A21*X1+X4*A24+X3*A23+X2*A22) \\
& + 2*D_{A31}*(X4*A31-X1*A34) + 2*D_{A32}*(X4*A32-X2*A34) \\
& + 2*D_{A33}*(X4*A33-X3*A34) + 2*D_{A34}*(X4*A34+X3*A33+X2*A32+X1*A31)
\end{aligned}
$$

(2.18)

In (2.18) we introduced the following notations

$$Aa\mu = A^{a}_{\mu}\ ;\ D = \partial\ ;\ X1 = X_1\ ;\ X2 = X_2\ ;\ X3 = X_3\ ;\ X4 = X_4, \qquad (2.19)$$

while in VF(1), VF(2), VF(3) the functions F(63), F(71), F(81) are arbitrary, depending on x_1, x_2, x_3, x_4.
VF(1), VF(2), VF(3) are just the generators of the gauge transformations [10]. The vector fields VF(4), VF(5), VF(6), VF(7) are generators of translations while VF(8), ..., VF(13) refer to rotations; VF(4), ..., VF(18) are the generators of the conformal group.

To construct similarity solutions associated to infinitesimal symmetries of self-dual Yang-Mills equations (2.14) we start from the vector fields X_1, X_2, X_3,

$$\begin{aligned} X_1 &= VF(8) + F(1)*VF(1) + F(2)*VF(2) + F(3)*VF(3) \\ X_2 &= VF(9) + F(4)*VF(1) + F(5)*VF(2) + F(6)*VF(3) \\ X_3 &= VF(10) + F(7)*VF(1) + F(8)*VF(2) + F(9)*VF(3) \end{aligned} \qquad (2.20)$$

i.e., we take a combination of a rotation and a gauge transformation. (In (2.29), the meaning of F(1)*VF(1) is: take F(71) = F(1) in VF(1)).
We also construct the commutators of the vector fields X_1, X_2, X_3

$$[X_1,X_2],\ [X_1,X_3],\ [X_2,X_3], \qquad (2.21)$$

and make the following choice for the functions F(i) (i=1,...,9)

$$\begin{aligned} &F(1) = 0,\ F(2) = -1,\ F(3) = 0 \\ &F(4) = 0,\ F(5) = 0,\ F(6) = -1 \\ &F(7) = -1,\ F(8) = 0,\ F(9) = 0 \end{aligned} \qquad (2.22)$$

Now the contraction of the vector fields X_1, X_2, X_3, $[X_1,X_2]$, $[X_1,X_3]$, $[X_2,X_3]$, (2.20), (2.21), (2.22) and the 12 contact 1-forms (2.17) re-

sults in 72 partial differential equations for the functions A^a_μ (a = 1,...,3; μ = 1,...,4). The set of 72 equations is given by

```
 1:      0 =  A12 + A21 - A11,1*X2 + A11,2*X1
 2:      0 =  A22 - A11 - A21,1*X2 + A21,2*X1
 3:      0 =  A32 - A31,1*X2 + A31,2*X1
 4:      0 =  A22 - A11 - A12,1*X2 + A12,2*X1
 5:      0 = -A12 - A21 - A22,1*X2 + A22,2*X1
 6:      0 = -A31 - A32,1*X2 + A32,2*X1
 7:      0 =  A23 - A13,1*X2 + A13,2*X1
 8:      0 = -A13 - A23,1*X2 + A23,2*X1
 9:      0 = -A33,1*X2 + A33,2*X1
10:      0 =  A24 - A14,1*X2 + A14,2*X1
11:      0 = -A14 - A24,1*X2 + A24,2*X1
12:      0 = -A34,1*X2 + A34,2*X1
13:      0 = -A13 - A31 + A11,1*X3 + A11,3*X1
14:      0 = -A23 + A21,1*X3 - A21,3*X1
15:      0 = -A33 + A11 + A31,1*X3 - A31,3*X1
16:      0 = -A32 + A12,1*X3 - A12,3*X1
17:      0 =  A22,1*X3 - A22,3*X1
18:      0 =  A12 + A32,1*X3 - A32,3*X1
19:      0 = -A33 + A11 + A13,1*X3 - A13,3*X1
20:      0 =  A21 + A23,1*X3 - A23,3*X1
21:      0 =  A13 + A31 + A33,1*X3 - A33,3*X1
22:      0 = -A34 + A14,1*X3 - A14,3*X1
23:      0 =  A24,1*X3 - A24,3*X1
24:      0 =  A14 + A34,1*X3 - A34,3*X1
25:      0 = -A14 + A11,1*X4 - A11,4*X1
26:      0 = -A24 - A31 + A21,1*X4 - A21,4*X1
```

(2.23)

```
27:     0 = -A31,4*X1 - A34 + A21 + A31,1*X4
28:     0 = -A12,4*X1 + A12,1*X4
29:     0 = -A22,4*X1 - A32 + A22,1*X4
30:     0 = -A32,4*X1 + A22 + A32,1*X4
31:     0 = -A13,4*X1 + A13,1*X4
32:     0 = -A23,4*X1 - A33 + A23,1*X4
33:     0 = -A33,4*X1 + A23 + A33,1*X4
34:     0 =  A11 - A14,4*X1 + A14,1*X4
35:     0 = -A34 + A21 - A24,4*X1 + A24,1*X4
36:     0 =  A24 + A31 - A34,4*X1 + A34,1*X4
37:     0 =  A11,2*X3 - A11,3*X2
38:     0 = -A31 + A21,2*X3 - A21,3*X2
39:     0 =  A21 + A31,2*X3 - A31,3*X2
40:     0 = -A13 + A12,2*X3 - A12,3*X2
41:     0 = -A23 - A32 + A22,2*X3 - A22,3*X2
42:     0 = -A33 + A22 + A32,2*X2 - A32,3*X2                    (2.23)
43:     0 =  A12 + A13,2*X3 - A13,3*X2
44:     0 = -A33 + A22 + A23,2*X3 - A23,3*X2
45:     0 =  A23 + A32 + A33,2*X3 - A33,3*X2
46:     0 =  A14,2*X3 - A14,3*X2
47:     0 = -A34 + A24,2*X3 - A24,3*X2
48:     0 =  A24 + A34,2*X3 - A34,3*X2
49:     0 =  A31 + A11,2*X4 - A11,4*X2
50:     0 =  A21,2*X4 - A21,4*X2
51:     0 = -A31,4*X2 - A11 + A31,2*X4
52:     0 = -A12,4*X2 - A14 + A32 + A12,2*X4
53:     0 = -A22,4*X2 - A24 + A22,2*X4
54:     0 = -A32,4*X2 - A34 - A12 + A32,2*X4
55:     0 = -A13,4*X2 + A33 + A13,2*X4
```

```
56:     0 = -A23,4*X2 + A23,2*X4
57:     0 = -A33,4*X2 - A13 + A33,2*X4
58:     0 =  A34 + A12 - A14,4*X2 + A14,2*X4
59:     0 =  A22 - A24,4*X2 + A24,2*X4
60:     0 = -A14 + A32 - A34,4*X2 + A34,2*X4
61:     0 =  A21 - A11,3*X4 + A11,4*X3
62:     0 = -A11 - A21,3*X4 + A21,4*X3
63:     0 =  A31,4*X3 - A31,3*X4
64:     0 =  A12,4*X3 + A22 - A12,3*X4                    (2.23)
65:     0 =  A22,4*X3 - A12 - A22,3*X4
66:     0 =  A32,4*X3 - A32,3*X4
67:     0 =  A13,4*X3 + A14 + A23 - A13,3*X4
68:     0 =  A23,4*X3 + A24 - A13 - A23,3*X4
69:     0 =  A34 + A33,4*X3 - A33,3*X4
70:     0 =  A24 - A13 + A14,4*X3 - A14,3*X4
71:     0 = -A14 - A23 + A24,4*X3 - A24,3*X4
72:     0 = -A33 + A34,4*X3 + A34,3*X4
```

where in (2.23) we used the same notation as introduced in (2.19) and

$$Aa\mu,\nu = \partial_\nu A^a_\mu \tag{2.23a}$$

The system can be solved in a straightforward way, leading to the following result

$$\begin{aligned} A^1_1 &= x_4 f(r), \quad A^1_2 = x_3 f(r), \quad A^1_3 = -x_2 f(r), \quad A^1_4 = -x_1 f(r) \\ A^2_1 &= -x_3 f(r), \quad A^2_2 = x_4 f(r), \quad A^2_3 = x_1 f(r), \quad A^2_4 = -x_2 f(r) \\ A^3_1 &= x_2 f(r), \quad A^3_2 = -x_1 f(r), \quad A^3_3 = x_4 f(r), \quad A^3_4 = -x_3 f(r) \end{aligned} \tag{2.24}$$

where

$$r = \left(x_1^2+x_2^2+x_3^2+x_4^2\right)^{\frac{1}{2}} \tag{2.24a}$$

At the derivation of the monopole solution, we shall discuss in some more detail how to solve a system of partial differential equations like (2.24). Substitution of (2.25) into (2.14) yields an *ordinary* differential equation for the function f(r)

$$\frac{df(r)}{dr} + rgf(r)^2 = 0; \tag{2.25}$$

the solution of this equation is given by

$$f(r) = +\left(\frac{1}{2}g\right)^{-1} \frac{1}{r^2+C} \qquad (C \text{ a constant} \tag{2.26}$$

The result (2.24), (2.26) is just the Belavin-Polyakov-Schwartz-Tyupkin instant on solution [11].

More general, if we choose

$$F(2) = \pm 1, \quad F(6) = \pm 1, \quad F(7) = \pm 1 \tag{2.27}$$

and

$$F(2)*F(6)*F(7) = -1, \tag{2.28}$$

or equivalently,

$$F(7) = -F(2)*F(6) \tag{2.28a}$$

we arrive at

$$\begin{aligned}
&A_1^1 = x_4 f(r) \quad , A_2^1 = x_3 f(r) \quad , A_3^1 = -x_2 f(r) \quad , A_4^1 = -x_1 f(r) \\
&A_1^2 = x_3 f(r)F(2), A_2^2 = -x_4 f(r)F(2), A_3^2 = -x_1 f(r)F(2), A_4^2 = x_2 f(r)F(2) \\
&A_1^3 = -x_2 f(r)F(6), A_2^3 = x_1 f(r)F(6), \quad A_3^3 = -x_4 f(r)F(6), A_4^3 = x_3 f(r)F(6)
\end{aligned} \tag{2.29}$$

while $f(r)$ has to satisfy (2.14), which results in

$$\frac{df(r)}{dr} + rg(F(2)*F(6))f(r)^2 = 0 \tag{2.29a}$$

Choosing $F(2)$, $F(6)$, $F(7)$ as in (2.27) and

$$F(2)*F(6)*F(7) = +1 \tag{2.30}$$

the result is

$$\begin{aligned}
&A_1^1 = x_4 f(r) \quad , A_2^1 = -x_3 f(r) \quad , A_3^1 = x_2 f(r) \quad , A_4^1 = -x_1 f(r) \\
&A_1^2 = -x_3 f(r)F(2), A_2^2 = -x_4 f(r)F(2), A_3^2 = x_1 f(r)F(2), A_4^2 = x_2 f(r)F(2) \\
&A_1^3 = x_2 f(r)F(6) \ , A_2^3 = -x_1 f(r)F(6), A_3^3 = -x_4 f(r)F(6), A_4^3 = x_3 f(r)F(6)
\end{aligned} \tag{2,31}$$

while $f(r)$ has to satisfy

$$r\frac{df}{dr}(r) + 4f(r) + gr^2(F(2)*F(6))f(r)^2 = 0 \tag{2.31a}$$

The solution of (2.31a)

$$f(r) = -\frac{2}{g}(F(2)*F(6))^{-1}\frac{a^2}{(r^2+a^2)r^2} \tag{2.32}$$

together with (2.31) is just the 't Hooft instanton solution with instanton number $k = 1$; this solution (2.21), (2.32) can be obtained from (2.24), (2.26) by a gauge transformation.

The equations for the static $SU(2)$ gauge field are described by (2.11) and (2.6).

The infinitesimal symmetries for the static gauge field are obtained from those for the time-dependent case in the following way: Take a general combination of the vector fields $VF(1), \ldots, VF(18)$ (2.18), compute the prolongation components $\partial_{A_{\mu,4}} a$ of this general vector field and impose the condition that these components are zero subject to (2.14), (2.6), [10]

$$L_V A^a_{\mu,4} = 0 \Big|_{\substack{(2.11)\\(2.5)}} \tag{2.33}$$

A straightforward computation then results in the Lie algebra of infinitesimal symmetries for static self dual SU(2) Yang-Mills equations

$$\begin{aligned}
VF(1) &:= (D_{A11}*DF(C(1),X1) + D_{A12}*DF(C(1),X2) + D_{A13}*DF(C(1),X3)\\
&\quad + D_{A21}*GE*A31*C(1) + D_{A22}*GE*A32*C(1) + D_{A23}*GE*A33*C(1)\\
&\quad + D_{A24}*GE*A34*C(1) - D_{A31}*GE*A21*C(1) - D_{A32}*GE*A22*C(1)\\
&\quad - D_{A33}*GE*A23*C(1) - D_{A34}*GE*A24*C(1))/GE\\
VF(2) &:= (-D_{A11}*GE*A21*C(2) - D_{A12}*GE*A22*C(2) - D_{A13}*GE*A23*C(2)\\
&\quad - D_{A14}*GE*A24*C(2) + D_{A21}*GE*A11*C(2) + D_{A22}*GE*A12*C(2)\\
&\quad + D_{A23}*GE*A13*C(2) + D_{A24}*GE*A14*C(2) - D_{A31}*DF(C(2),X1)\\
&\quad - D_{A32}*DF(C(2),X2) - D_{A33}*DF(C(2),X3))/GE\\
VF(3) &:= (D_{A11}*GE*A31*C(3) + D_{A12}*GE*A32*C(3) + D_{A13}*GE*A33*C(3)\\
&\quad + D_{A14}*GE*A34*C(3) - D_{A21}*DF(C(3),X1) - D_{A22}*DF(C(3),X2)\\
&\quad - D_{A23}*DF(C(3),X3) - D_{A31}*GE*A11*C(3) - D_{A32}*GE*A12*C(3)\\
&\quad - D_{A33}*GE*A13*C(3) - D_{A34}*GE*A14*C(3))/GE
\end{aligned} \tag{2.34}$$

$$VF(4) := D_{X1}$$

$$VF(5) := D_{X2}$$

$$VF(6) := D_{X3}$$

$$\begin{aligned}
VF(7) &:= D_{X1}*X2 - D_{X2}*X1 + D_{A11}*A12 - D_{A12}*A11 + D_{A21}*A22 - D_{A22}*A21\\
&\quad + D_{A31}*A32 - D_{A32}*A31\\
VF(8) &:= -D_{X1}*X3 + D_{X3}*X1 - D_{A11}*A13 + D_{A13}*A11 - D_{A21}*A23 + D_{A23}*A21\\
&\quad - D_{A31}*A33 + D_{A33}*A31\\
VF(9) &:= -D_{X2}*X3 + D_{X3}*X2 - D_{A12}*A13 + D_{A13}*A12 - D_{A22}*A23 + D_{A23}*A22\\
&\quad - D_{A32}*A33 + D_{A33}*A32\\
VF(10) &:= D_{X1}*X1 + D_{X2}*X2 + D_{X3}*X3 - D_{A11}*A11 - D_{A12}*A12 - D_{A13}*A13\\
&\quad - D_{A14}*A14 - D_{A21}*A21 - D_{A22}*A22 - D_{A23}*A23 - D_{A24}*A24 - D_{A31}*A31\\
&\quad - D_{A32}*A32 - D_{A33}*A33 - D_{A34}*A34
\end{aligned}$$

In (2.34) C(1), C(2), C(3) are arbitrary functions of $x_1, \ldots, x_3$ VF(1), VF(2), VF(3) are just the generators of the gauge transformations. VF(7), VF(8), VF(9) generate the rotations, while VF(10) is the generator of the scale change.

In order to construct similarity solutions to the static SU(2) gauge field, we proceed in a way analogously to the one for the time-dependent field configuration.

We define the vector field Y_1, Y_2, Y_3 by

$$\begin{aligned} Y_1 &= VF(7) - VF(2) \\ Y_2 &= VF(8) - VF(3) \\ Y_3 &= VF(9) - VF(1) \end{aligned} \tag{2.35}$$

and put C(1), C(2), C(3) equal to 1.

We compute the contractions of these vector fields and the contact 1-forms (2.17), (2.6).

This results in a system of 36 equations for the functions A_μ^a (notations as in (2.19), (2.23a))

```
 1:      0 =  A21 + A12 - A11,1*X2 + A11,2*X1
 2:      0 = -A11 + A22 - A21,1*X2 + A21,2*X1
 3:      0 =  A32 - A31,1*X2 + A31,2*X1
 4:      0 = -A11 + A22 - A12,1*X2 + A12,2*X1
 5:      0 = -A21 - A12 - A22,1*X2 + A22,2*X1
 6:      0 = -A31 - A32,1*X2 + A32,2*X1                    (2.36)
 7:      0 =  A23 - A13,1*X2 + A13,2*X1
 8:      0 = -A13 - A23,1*X2 + A23,2*X1
 9:      0 = -A33,1*X2 + A33,2*X1
10:      0 =  A24 - A14,1*X2 + A14,2*X1
11:      0 = -A14 - A24,1*X2 + A24,2*X1
```

```
12:     0 = -A34,1*X2 + A34,2*X1
13:     0 = -A31 - A13 + A11,1*X3 - A11,3*X1
14:     0 = -A23 + A21,1*X3 - A21,3*X1
15:     0 =  A11 - A33 + A31,1*X3 - A31,3*X1
16:     0 = -A32 + A12,1*X3 - A12,3*X1
17:     0 =  A22,1*X3 - A22,3*X1
18:     0 =  A12 + A32,1*X3 - A32,3*X1
19:     0 =  A11 - A33 + A13,1*X3 - A13,3*X1
20:     0 =  A21 + A23,1*X3 - A23,3*X1
21:     0 =  A31 + A13 + A33,1*X3 - A33,3*X1
22:     0 = -A34 + A14,1*X3 - A14,3*X1
23:     0 =  A24,1*X3 - A24,3*X1
24:     0 =  A14 + A34,1*X3 - A34,3*X1                    (2.36)
25:     0 =  A11,2*X3 - A11,3*X2
26:     0 = -A31 + A21,2*X3 - A21,3*X2
27:     0 =  A21 + A31,2*X3 - A31,3*X2
28:     0 = -A13 + A12,2*X3 - A12,3*X2
29:     0 = -A32 - A23 + A22,2*X3 - A22,3*X2
30:     0 =  A22 - A33 + A32,2*X3 - A32,3*X2
31:     0 =  A12 + A13,2*X3 - A13,3*X2
32:     0 =  A22 - A33 + A23,2*X3 - A23,3*X2
33:     0 =  A32 + A23 + A33,2*X3 - A33,3*X2
34:     0 =  A14,2*X3 - A14,3*X2
35:     0 = -A34 + A24,2*X2 - A24,3*X2
36:     0 =  A24 + A34,2*X3 - A34,3*X2
```

We shall now indicate in some detail how to solve (2.36).

Note that, due to (2.36(12))

$$A_4^3 = f^1(r_{12}, 3) \tag{2.37}$$

where

$$r_{12} = \sqrt{x_1^2 + x_2^2};$$

and due to (3.5.36(24))

$$A_4^1 = x_1\left\{\partial_2 f^1(r_{12},x_3) - \frac{x_3}{r_{12}}\partial_1 f^1(r_{12},x_3)\right\} \tag{2.38}$$

where ∂_i denotes differentiation with respect to the i-th component.

Now let

$$\partial_2 f^1(r_{12},x_3) - \frac{x_3}{r_{12}}\partial_1 f^1(r_{12},x_3) = h(r_{12},x_3) \tag{2.39}$$

Substitution of (2.37), (2.39) into (2.36.(22)) results in

$$f^1(r_{12},x_3) = x_3 h(r_{12},x_3) + \frac{x_1^2 x_3}{r_{12}}\partial_1 h(r_{12},x_3) - x_1^2\,\partial_2 h(r_{12},x_3)$$

or

$$f^1(r_{12},x_3) - x_3 h(r_{12},x_3) = x_1^2\left(\frac{x_3}{r_{12}}\partial_1 h(r_{12},x_3) - \partial_2 h(r_{12},x_3)\right) \tag{2.40}$$

Differentiation of (2.40) with respect to x_1, x_2 yields

$$\frac{x_3}{r_{12}}\partial_1 h(r_{12},x_3) - \partial_2 h(r_{12},x_3) = 0 \tag{2.41a}$$

$$f^1(r_{12},x_3) = x_3 h(r_{12},x_3) \tag{2.41b}$$

From (2.41a) we obtain

$$h(r_{12},x_3) = \ell(r) \tag{2.42}$$

where

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} \tag{2.42a}$$

and finally, due to (2.41b), (2.38) and (2.36(36))

$$A_4^1 = x_1 \ell(r); \quad A_4^2 = x_2 \ell(r); \quad A_4^3 = x_3 \ell(r) \tag{2.43}$$

Handling the remaining system in a similar way, a straightforward but tedious computation leads to the general solution of (2.36) i.e.,

$$\begin{aligned}
A_1^1 &= \tfrac{1}{2}x_1^2 f(r) + k(r), \quad A_2^1 = \tfrac{1}{2}x_1 x_2 f(r) - x_3 u(r), \quad A_3^1 = \tfrac{1}{2}x_1 x_3 f(r) + x_2 u(r) \\
A_1^2 &= \tfrac{1}{2}x_1 x_2 f(r) + x_3 u(r), \quad A_2^2 = \tfrac{1}{2}x_2^2 f(r) + k(r), \quad A_3^2 = \tfrac{1}{2}x_2 x_3 f(r) - x_1 u(r) \\
A_1^3 &= \tfrac{1}{2}x_1 x_3 f(r) - x_2 u(r), \quad A_2^3 = \tfrac{1}{2}x_2 x_3 f(r) + x_1 u(r), \quad A_3^3 = \tfrac{1}{2}x_3^2 f(r) + k(r) \\
A_4^1 &= x_1 \ell(r), \quad A_4^2 = x_2 \ell(r), \quad A_4^3 = x_3 \ell(r)
\end{aligned} \tag{2.44}$$

where u, ℓ, k, f are functions of r.

Substitution of (2.44) into (2.14), 2.6) yields a system of three ordinary differential equations for the functions u, ℓ, k, f

$$\begin{aligned}
&\frac{d\ell(r)}{dr} + \frac{du(r)}{dr} - gru(r)^2 - gru(r)\ell(r) + \tfrac{1}{2}grf(r)k(r) = 0 \\
&r^2\frac{du(r)}{dr} + 2ru(r) - r\ell(r) - gr^3 u(r)\ell(r) + grk(r)^2 + \tfrac{1}{2}gr^3 f(r)k(r) = 0 \\
&\frac{dk(r)}{dr} - \tfrac{1}{2}rf(r) - grk(r)u(r) - gr\ell(r)k(r) - \tfrac{1}{2}gr^3 f(r)u(r) = 0
\end{aligned} \tag{2.45}$$

If we choose

$$f(r) \equiv 0, \quad k(r) \equiv 0, \quad \ell(r) = \frac{h(r)}{r}, \quad u(r) = -\frac{a(r)}{r} \tag{2.46}$$

we are led by (2.46), (2.44) to the monopole solution, obtained by Prasad & Sommerfield [12] by imposing the *ansatz* (2.44), (2.46).

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