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THE STRUCTURE OF REAL SEMISIMPLE LIE GROUPS

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INTRODUCTION

This volume contains the notes of a short course given at the algebra seminar of the Department of Pure Mathematics of the Mathematical Centre. Preceding to this course we studied HUMPHREYS' "Introduction to Lie algebras and representation theory" (see [Hu], in particular Ch.I,II,III and Theorem 25.2); also a few introductory lectures on the theory of Lie groups were given before (see for instance [He,Ch.II], [War] or [Koo]). The principal aim of these notes is to provide the required knowledge about structure theory of real semisimple Lie groups to the beginning research student in analysis on Lie groups who already knows about complex semisimple Lie algebras and general Lie groups. In particular, Chapters I,II and the first section of Chapter III serve this purpose. Certainly, this material is also included in standard texts like HELGASON [He] or WALLACH [Wa], but there it is embedded in a much wider context, and the student looking for a quick introduction may get lost.

A second purpose of this volume is to point out connections between the structure theory of real semisimple Lie groups and other fields of mathematics, notably the geometry of Tits systems (Chapter III) and topological dynamics (Chapter IV). In particular, section 2 of Chapter III and sections 1-4 of Chapter IV can be used as introductions to these other topics, even for people who are not interested in semisimple Lie groups. The final chapter, Ch.V, is of algebraic nature. It discusses the classification of real semisimple Lie algebras, a well-known result of which the proof is often skipped.

All authors have their present address at the Mathematical Centre. I want to thank them for their contribution and also the audience attending the course for stimulating discussions. A.M. Cohen kindly offered hospitality to this course in his algebra seminar and it is due to him that the scope of the course was extended with the topics from Chapters III, IV.

T.H. Koornwinder, editor

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Chapter I

REAL SEMISIMPLE LIE ALGEBRAS

T.H. KOORNWINDER

This chapter presents the basic structure theory of real semisimple Lie algebras. It is assumed that the reader knows the structure theory for complex semisimple Lie algebras, cf. for instance HUMPHREYS [Hu, Ch. I, II, III and Theorem 25.2]. Our development of the theory is parallel to parts of HELGASON's book: see Ch. II (§5,6), Ch. III (§6,7), Ch. V (§2,6), Ch. VI (§3), Ch. VII (Theor. 2.16) and Ch. X (Theor. 3.25) in [He]. However, in contrast with [He] we have formulated everything in terms of the restricted root system Σ of g rather than the root system Φ of $g_{\mathbb{C}}$.

1. GENERALITIES ABOUT SEMISIMPLE LIE ALGEBRAS

Let g be a Lie algebra over a field $\mathbb{F}$ of characteristic 0. Throughout this chapter, when speaking about a Lie algebra, we mean a finite-dimensional Lie algebra.

DEFINITION 1.1. The *Killing form* of g is the bilinear form B defined by

$$B(X,Y) := \mathrm{tr}(\mathrm{ad}\, X \circ \mathrm{ad}\, Y), \quad X,Y \in g.$$

DEFINITION 1.2. The Lie algebra g is called *semisimple* if its Killing form B is a nondegenerate bilinear form.

The easy proofs of (1.1), (1.2) and Prop. 1.3. below are left to the reader:

$$(1.1) \qquad B(\sigma X,\sigma Y) = B(X,Y), \; X,Y \in g, \; \sigma \in \mathrm{Aut}(g),$$

$$(1.2) \qquad B(X,[Y,Z]) = B([X,Y],Z), \; X,Y,Z \in g.$$

PROPOSITION 1.3. Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then B restricted to $\mathfrak{a} \times \mathfrak{a}$ equals the Killing form $B_{\mathfrak{a}}$ of $\mathfrak{a}$. Furthermore,

$$\mathfrak{a}^{\perp} := \{X \in \mathfrak{g} \mid B(X,Y) = 0 \quad \forall Y \in \mathfrak{a}\}$$

is an ideal in $\mathfrak{g}$.

PROPOSITION 1.4. Let $\mathfrak{g}$ be semisimple and let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. Then $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$ (direct sum of Lie algebras) and $\mathfrak{a}$ and $\mathfrak{a}^{\perp}$ are semisimple.

PROOF. $\dim \mathfrak{a} + \dim \mathfrak{a}^{\perp} = \dim \mathfrak{g}$, since B is nondegenerate. If $Z \in \mathfrak{g}$ and $X,Y \in \mathfrak{a} \cap \mathfrak{a}^{\perp}$ then $B(Z,[X,Y]) = B([Z,X],Y) = 0$, so $[X,Y] = 0$ and $\mathfrak{a} \cap \mathfrak{a}^{\perp}$ is an abelian ideal in $\mathfrak{g}$. If $Z \in \mathfrak{g}$ and $X \in \mathfrak{a} \cap \mathfrak{a}^{\perp}$ then $\operatorname{ad} X \operatorname{ad} Z(\mathfrak{g}) \subset \mathfrak{a} \cap \mathfrak{a}^{\perp}$ and $\operatorname{ad} X \operatorname{ad} Z\ (\mathfrak{a} \cap \mathfrak{a}^{\perp}) = \{0\}$, so $B(X,Z) = 0$, hence $X = 0$ and $\mathfrak{a} \cap \mathfrak{a}^{\perp} = \{0\}$. Thus $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^{\perp}$. By Prop. 1.3, $B_{\mathfrak{a}}$ and $B_{\mathfrak{a}^{\perp}}$ are nondegenerate. □

PROPOSITION 1.5. Let $\mathfrak{g}$ be semisimple. Then ad is an isomorphism from $\mathfrak{g}$ onto $\operatorname{Der}(\mathfrak{g})$.

PROOF. See [Hu, Theorem 5.3]. The proof given there does not use that $\mathbb{F}$ is algebraically closed. □

DEFINITION 1.6. Let $\mathfrak{g}$ be a real Lie algebra. The *adjoint group* of $\mathfrak{g}$, denoted by $\operatorname{Int}(\mathfrak{g})$, is the analytic subgroup of $GL(\mathfrak{g})$ whose Lie algebra is $\operatorname{ad}(\mathfrak{g})$.

If $\mathfrak{g}$ is real then clearly $\operatorname{Aut}(\mathfrak{g})$ is a closed subgroup of $GL(\mathfrak{g})$. This observation, together with Prop 1.5 yields:

PROPOSITION 1.7. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $\operatorname{Int}(\mathfrak{g})$ is the identity component of $\operatorname{Aut}(\mathfrak{g})$. In particular, $\operatorname{Int}(\mathfrak{g})$ is a closed subgroup of $\operatorname{Aut}(\mathfrak{g})$ and of $GL(\mathfrak{g})$.

DEFINITION 1.8. A real Lie algebra $\mathfrak{g}$ is said to be *compact* if $\operatorname{Int}(\mathfrak{g})$ (with its own Lie group topology) is compact.

PROPOSITION 1.9. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Then $\mathfrak{g}$ is compact if and only if B is negative definite.

PROOF. Assume that B is negative definite. $\operatorname{Int}(\mathfrak{g})$ is closed in $\operatorname{Aut}(\mathfrak{g})$ and $\operatorname{Aut}(\mathfrak{g})$ is closed in $O(\mathfrak{g},B)$ (the orthogonal group on $\mathfrak{g}$ with respect to the inner product $-B$). Since $O(g,B)$ is compact, the same holds for $\mathrm{Aut}(g)$ and $\mathrm{Int}(g)$.

Conversely, assume that $\mathrm{Int}(g)$ is compact. Then there is an $\mathrm{Int}(g)$-invariant inner product $(.,.)$ on g. Let $O(g)$ be the corresponding orthogonal group. Since $\mathrm{Int}(g) \subset O(g)$ we have $\mathrm{ad}(g) \subset o(g)$, where $o(g)$ is the Lie algebra of skew-symmetric matrices on g. Let $X \in g$ and let ad X have matrix (A_{ij}) with respect to some orthonormal basis. Then

$$B(X,X) = \mathrm{tr}(\mathrm{ad}\, X\ \mathrm{ad}\, X) = \sum_{i,j} A_{ij}A_{ji} = -\sum_{i,j} (A_{ij})^2 \leq 0.$$

Suppose $B(X,X) = 0$ then $(A_{ij}) = 0$, so ad $X = 0$, hence $X = 0$. □

Let g be a real Lie algebra. Let $g_C := g + ig$ be the complexification of the real vector space g and extend the bracket operation for g to a complex bilinear operation on g_C. Then g_C is a complex Lie algebra: the *complexification* of the real Lie algebra g.

If g_C is a complex Lie algebra then $(g_C)_R$ denotes g_C considered as a real Lie algebra.

A *real form* of a complex Lie algebra g_C is a real subalgebra g of $(g_C)_R$ such that $(g_C)_R = g \oplus ig$ (direct sum of real vector spaces). Then g_C is isomorphic to the complexification of g.

LEMMA 1.10. Let g, g_C and $(g_C)_R$ be related to each other as above. Let B_g, B_{g_C} and $B_{(g_C)_R}$ be the corresponding Killing forms. Then

$$(1.3) \qquad B_{g_C}(X,Y) = B_g(X,Y), \qquad X,Y \in g,$$

$$(1.4) \qquad B_{(g_C)_R}(X,Y) = 2\mathrm{Re}\, B_{g_C}(X,Y), \quad X,Y \in g_C.$$

PROOF. (1.3) follows immediately. For the proof of (1.4) choose a basis $X_1,\ldots,X_n$ of g_C and let ad X ad Y have matrix $P + iQ$ with respect to this basis (P,Q real $n\times n$ matrices). Then $X_1,\ldots,X_n, iX_1,\ldots,iX_n$ is a basis of $(g_C)_R$ and ad X ad Y has matrix $\begin{pmatrix} P & -Q \\ Q & P \end{pmatrix}$ with respect to this basis. Thus

$$\mathrm{tr}_{g_C}\ \mathrm{ad}\, X\ \mathrm{ad}\, Y = \mathrm{tr}\, P + i\ \mathrm{tr}\, Q$$

and

$$\mathrm{tr}_{(g_C)_R}\ \mathrm{ad}\, X\ \mathrm{ad}\, Y = 2\ \mathrm{tr}\, P. \quad □$$

PROPOSITION 1.11. *The Lie algebras g, g_C and $(g_C)_R$ are all semisimple if one of them is.*

PROOF. Evident from Lemma 1.10. □

The *radical* of a Lie algebra g is the largest solvable ideal in g.

PROPOSITION 1.12. *Let g be a real Lie algebra. Then g is semisimple if and only if the radical r of g is $\{0\}$.*

PROOF. If g is semisimple then, by Prop. 1.3, each abelian ideal of g equals $\{0\}$. Hence $r = \{0\}$. Conversely assume $r = \{0\}$. Let g_C have radical s_C and let σ be the conjugation of g_C with respect to g, i.e. $\sigma(X+iY) := X-iY$, $X,Y \in g$. Then $\sigma\, s_C$ is again a solvable ideal of g_C, hence $\sigma\, s_C = s_C$. It follows that $s_C = s + is$, with s being a solvable ideal in g. So $s \subset r = \{0\}$. Thus, by [Hu, Theorem 5.1], B_{g_C} is nondegenerate, so g is semisimple by Prop. 1.11. □

DEFINITION 1.13. A subalgebra h of a Lie algebra g is said to be *reductive in g* if $\mathrm{ad}_g(h)$ acts on g in a semisimple way (i.e., for each $\mathrm{ad}_g(h)$ - invariant linear subspace V of g there is a complementary $\mathrm{ad}_g(h)$ - invariant linear subspace W). Furthermore, a Lie algebra g is called *reductive* if it is reductive in itself.

PROPOSITION 1.14. *Let g be a real or complex Lie algebra. The following two statements are equivalent:*

(a) *g is reductive.*

(b) *$g = c \oplus [g,g]$ (direct sum of Lie algebras), where c is the center of g, and $[g,g]$ is semisimple.*

PROOF. If (b) holds then (a) follows by Prop. 1.4. Conversely assume (a). Since g is reductive, each abelian ideal of g is included in c. Again by reductivity, g is the direct sum of c and an ideal h. Then h will not have abelian ideals $\neq \{0\}$. Hence the radical of h is $\{0\}$, so $h = [h,h] = [g,g]$ and h is semisimple. □

2. COMPACT REAL FORMS

Let g_C be a complex Lie algebra. The subalgebra h_C is called a *Cartan subalgebra* if h_C is nilpotent and equal to its normaliser in g_C.

The subalgebra h_C is called *toral* if, for each $H \in h_C$, ad H is a semisimple endomorphism of g_C.

PROPOSITION 2.1. *Let g_C be a complex semisimple Lie algebra with subalgebra h_C. The following three statements are equivalent:*

(a) *h_C is a Cartan subalgebra.*

(b) *h_C is a maximal toral subalgebra.*

(c) *h_C is a maximal abelian subalgebra and h_C is toral.*

PROOF.

(a) $\Longleftrightarrow$ (b): [Hu, Cor. 15.3]

(b) $\Rightarrow$ (c): [Hu, Prop. 8.2]

(c) $\Rightarrow$ (b): [Hu, Lemma 8.1] □

Let g_C be a complex semisimple Lie algebra. Fix a Cartan subalgebra h_C of g_C (CSA's exist, cf. [Hu, §8.1] and they are all conjugate under $\mathrm{Int}(g_C)$, cf. VARADARAJAN [Var, Theorem 4.1.3]). We remember some of the structural properties of the pair (g_C, h_C) (cf. [Hu, §8]). For $\alpha \in h_C^*$ (dual vector space of h_C) let

$$(2.1) \qquad g^\alpha := \{X \in g_C \mid [H,X] = \alpha(H)X \qquad \forall H \in h_C\}.$$

Let the set Φ consists of all $\alpha \in h_C^*\backslash\{0\}$ such that $\dim g^\alpha > 0$. Then dim $\dim g^\alpha = 1$ for $\alpha \in \Phi$ and

$$(2.2) \qquad g_C = h_C + \sum_{\alpha\in\Phi} g^\alpha \quad \text{(direct sum of vector spaces)}.$$

The restriction of the Killing form B of g_C to h_C is a nondegenerate form on h_C. For each $\lambda \in h_C^*$ define $T_\lambda \in h_C$ such that

$$(2.3) \qquad B(T_\lambda, H) = \lambda(H) \qquad \forall H \in h_C.$$

Define a nondegenerate form $\langle.,.\rangle$ on h_C^* by

$$(2.4) \qquad \langle\lambda,\mu\rangle := B(T_\lambda, T_\mu), \quad \lambda,\mu \in h_C^*.$$

Let

$$(2.5) \qquad h_R := \text{real span } \{T_\alpha \mid \alpha \in \Phi\}.$$

Then h_R is a real form of h_C, B is positive definite on h_R and $<.,.>$ is positive definite on h_R^*. The set Φ, as a subset of h_R^* with inner product $<.,.>$ satisfies the axioms of a reduced root system:

DEFINITION 2.2. Let E be a finite-dimensional real vector space with inner product $<.,.>$. A finite subset Σ of $E\backslash\{0\}$ is called a *root system* if:

(a) Σ spans E;

(b) If $\alpha,\beta \in \Sigma$ then $\beta - \frac{2<\beta,\alpha>}{<\alpha,\alpha>}\alpha \in \Sigma$;

(c) If $\alpha,\beta \in \Sigma$ then $\frac{2<\beta,\alpha>}{<\alpha,\alpha>} \in \mathbb{Z}$.

The root system Σ is called *reduced*, if it satisfies the additional condition:

(d) If $\alpha \in \Sigma$ then the only multiples of α in Σ are $\pm\,\alpha$.

Define

$$H_\alpha := \frac{2}{<\alpha,\alpha>}\, T_\alpha, \qquad \alpha \in \Phi. \tag{2.6}$$

For each $\alpha \in \Phi$ choose $X_\alpha \in g^\alpha\backslash\{0\}$. For $\alpha,\beta \in \Phi$ with $\alpha+\beta \in \Phi$ let $c_{\alpha,\beta} \in$ $\in \mathbb{C}\backslash\{0\}$ be defined by

$$[X_\alpha,X_\beta] = c_{\alpha,\beta}X_{\alpha+\beta}. \tag{2.7}$$

THEOREM 2.3. (cf. [Hu, Prop. 25.2]). *With notation as above the* X_α*'s can be chosen such that:*

(a) $\qquad [X_\alpha,X_{-\alpha}] = H_\alpha, \qquad \alpha \in \Phi;$

(b) $\qquad c_{\alpha,\beta} = -c_{-\alpha,-\beta}, \qquad \alpha,\beta,\alpha+\beta \in \Phi.$

Then the $c_{\alpha,\beta}$*'s are real.*

If the X_α's satisfy condition (a) of Theorem 2.3 then, for $\alpha,\beta \in \Phi$:

$$B(X_\alpha,X_\beta) = \begin{cases} 2/<\alpha,\alpha> & \text{if } \beta = -\alpha, \\ 0 & \text{if } \beta \neq -\alpha. \end{cases} \tag{2.8}$$

Provide h_R^* with an ordering $>$ such that each pair of elements of h_R^* is related and such that for $\lambda,\mu,\nu \in h_R^*$ and $t > 0$ we have the implications

$\lambda > \mu \Rightarrow \lambda + \nu > \mu + \nu$ and $\lambda > \mu \Rightarrow t\lambda > t\mu$. Then we call h_R^* an *ordered vector space*. This induces an ordering on Φ. Let

(2.9) $$\Phi^+ := \{\alpha \in \Phi | \alpha > 0\}.$$

Choose the X_α's as in Theorem 2.3. Then we have a decomposition (direct sum of vector spaces)

(2.10) $$g_C = \bigoplus_{\alpha\in\Phi^+} \mathbb{C}(X_\alpha - X_{-\alpha}) \oplus$$
$$\oplus \left(\bigoplus_{\alpha\in\Phi^+} \mathbb{C}(iX_\alpha + iX_{-\alpha})\right) \oplus (\mathbb{C} \otimes (ih_R))$$

which is orthogonal with respect to B, and

(2.11) $$\begin{cases} B(X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}) = -\frac{4}{\langle\alpha,\alpha\rangle}, \\ B(iX_\alpha + iX_{-\alpha}, iX_\alpha + iX_{-\alpha}) = -\frac{4}{\langle\alpha,\alpha\rangle}, \\ B|_{ih_R} \text{ is negative definite.} \end{cases}$$

Furthermore, we find for $\alpha, \beta \in \Phi^+$;

$$[iH_\alpha, X_\beta - X_{-\beta}] = \beta(H_\alpha)(iX_\beta + iX_{-\beta}),$$

$$[iH_\alpha, iX_\beta + iX_{-\beta}] = -\beta(H_\alpha)(X_\beta - X_{-\beta});$$ and, if moreover $\alpha \neq \beta$:

$$[X_\alpha - X_{-\alpha}, X_\beta - X_{-\beta}] = c_{\alpha,\beta}(X_{\alpha+\beta} - X_{-\alpha-\beta}) +$$
$$- c_{\alpha,-\beta}(X_{\alpha-\beta} - X_{-\alpha+\beta}),$$

$$[i(X_\alpha + X_{-\alpha}), i(X_\beta + X_{-\beta})] = - c_{\alpha,\beta}(X_{\alpha+\beta} - X_{-\alpha-\beta}) +$$
$$- c_{\alpha,-\beta}(X_{\alpha-\beta} - X_{-\alpha+\beta}),$$

$$[X_\alpha - X_{-\alpha}, i(X_\beta + X_{-\beta})] = c_{\alpha,\beta}(iX_{\alpha+\beta} + iX_{-\alpha-\beta}) +$$
$$+ c_{\alpha,-\beta}(iX_{\alpha-\beta} + iX_{-\alpha+\beta}),$$

$$[X_\alpha - X_{-\alpha}, i(X_\alpha + X_{-\alpha})] = 2iH_\alpha.$$

COROLLARY 2.4. *Let the* X_α*'s be as in Theorem 2.3. Then* ih_R *together with the vectors* $X_\alpha - X_{-\alpha}$ $(\alpha\in\Phi^+)$ *and* $iX_\alpha + iX_{-\alpha}$ $(\alpha\in\Phi^+)$ *span a real form* u *of* g_C *on which* B *is negative definite.*

Combination with (1.3) and Prop. 1.9 yields:

THEOREM 2.5. *Every complex semisimple Lie algebra has a compact real form.*

REMARK 2.6. If u is a compact real semisimple Lie algebra then $\mathrm{Int}(u)$, being a closed subgroup of the compact group $O(u,B)$, is compact. By [He, Theorem II. 6.9] the universal covering group U^* of $\mathrm{Int}(u)$ is then compact. Hence every connected Lie group U with Lie algebra u is compact.

3. CARTAN DECOMPOSITIONS

Let g_C be a complex semisimple Lie algebra and let u be a compact real form of g_C. Let τ be the conjugation of g_C with respect to u. Then τ is an involutive automorphism of $(g_C)_R$. Since B is negative definite on u (Prop. 1.9), the form

$$B_\tau(X,Y) := -B(X,\tau Y),\ X,Y \in g_C, \tag{3.1}$$

is positive definite Hermitian on g_C.

PROPOSITION 3.1. *Let* g_C, u, τ *be as above. Let* g *be a real form of* g_C *and let* σ *be the conjugation of* g_C *with respect to* g*. Then, for some* $\phi \in \mathrm{Aut}(g_C)$, $\phi.u$ *is invariant under* σ.

PROOF. $\sigma\tau$ is in $\mathrm{Aut}(g_C)$ and hermitian with respect to B_τ. Hence $P := (\sigma\tau)^2$ is in $\mathrm{Aut}(g_C)$ and positive definite hermitian with respect to B_τ. Let $\{X_i\}$ be a basis of eigenvectors for P, i.e. $PX_i = \lambda_i X_i$ with $\lambda_i > 0$. For $t \in \mathbb{R}$ define the linear endomorphism P^t of g_C by $P^t X_i := \lambda_i{}^t X_i$. Let us prove that $P^t \in \mathrm{Aut}(g_C)$. Put

$$[X_i,X_j] = \sum_k c_{ijk}X_k. \text{ Then}$$

$$\sum_k c_{ijk}\lambda_k X_k = P[X_i,X_j] = [PX_i,PX_j] = \sum_k c_{ijk}\lambda_i\lambda_j X_k.$$

Hence $c_{ijk}\lambda_k = c_{ijk}\lambda_i\lambda_j$, which implies $c_{ijk}\lambda_k^t = c_{ijk}\lambda_i^t\lambda_j^t$. Thus $P^t \in \mathrm{Aut}(g_C)$. The proposition will follow from $\sigma\phi\tau\phi^{-1} = \phi\tau\phi^{-1}\sigma$. We will show that $\phi := P^{\frac{1}{4}}$ satisfies this identity. Since $P\sigma\tau = \sigma\tau P$ and $P\tau = \tau P^{-1}$, we also have $P^{\frac{1}{2}}\sigma\tau = \sigma\tau P^{\frac{1}{2}}$, $P^{\frac{1}{4}}\tau = \tau P^{-\frac{1}{4}}$. Hence

$$\sigma P^{\frac{1}{4}}\tau P^{-\frac{1}{4}} = \sigma\tau P^{-\frac{1}{2}} = P^{-\frac{1}{2}}\sigma\tau = P^{\frac{1}{2}}P^{-1}\sigma\tau = P^{\frac{1}{2}}\tau\sigma = P^{\frac{1}{4}}\tau P^{-\frac{1}{4}}\sigma. \qquad \sqcup$$

Let g be a real form of g_C with corresponding conjugation σ and let u be a compact real form of g_C with corresponding conjugation τ. Clearly we have:

$$\sigma(u) = u \Longleftrightarrow \sigma\tau = \tau\sigma \Longleftrightarrow \tau(g) = g. \tag{3.2}$$

It follows from Prop. 3.1 that, for a given g, we can choose u such that the three equivalent statements of (3.2) hold. In that case, put

$$k := g \cap u, \quad p := g \cap iu, \quad \theta := \tau|_g. \tag{3.3}$$

Then

$$g = k + p \tag{3.4}$$

is the decomposition of g into eigenspaces k, p of θ with eigenvalues 1, -1, respectively. Let B_g be the Killing form of g. Then $B_g|_k$ is negative definite and $B_g|_p$ is positive definite. Indeed, by (1.3):

$B_g|_k = B_{g_C}|_k = B_u|_k$ is negative definite, and a similar reasoning applies to $p \subset iu$.

<u>DEFINITION 3.2</u>. Let g be a real semisimple Lie algebra with Killing form B. An involutive automorphism θ of g is called *Cartan involution* and the corresponding eigenspace decomposition $g = k + p$ into 1 and -1 eigenspaces of θ is called *Cartan decomposition* if $B|_k$ is negative definite and $B|_p$ is positive definite.

Note that the conditions of this definition imply that

$$[k,k] \subset k, \; [k,p] \subset p, \; [p,p] \subset k. \tag{3.5}$$

Clearly, for any real form g of g_C we realised a Cartan decomposition by (3.3) and (3.4). Conversely, starting with some Cartan decomposition $g = k + p$ for a real semisimple Lie algebra g, we can consider the complexification g_C of g, which is again semisimple, and we can define

(3.6) $\quad u := k + ip,$

which turns out to be a compact real form of g_C.

THEOREM 3.3. *Let g be a real semisimple Lie algebra. Then g has a Cartan decomposition. If $g = k_1 + p_1 = k_2 + p_2$ are two Cartan decompositions of g then $\psi.k_2 = k_1$, $\psi.p_2 = p_1$ for some $\psi \in \mathrm{Int}(g)$.*

PROOF. We already settled the existence of a Cartan decomposition. Consider the compact real forms $u_j := k_j + ip_j$ of g_C with corresponding conjugations τ_j, $j = 1,2$. Let $P := (\tau_1\tau_2)^2$. By the proof of Proposition 3.1 we have $\tau_1(P^{\frac{1}{4}}u_2) = P^{\frac{1}{4}}u_2$, hence

$$P^{\frac{1}{4}}u_2 = (P^{\frac{1}{4}}u_2) \cap u_1 + (P^{\frac{1}{4}}u_2) \cap iu_1,$$

where B is negative definite on the first summand and positive definite on the second summand. However, B is negative definite on $P^{\frac{1}{4}}u_2$ by (1.1) and Prop 1.9. Hence $P^{\frac{1}{4}}u_2 \subset u_1$, so $P^{\frac{1}{4}}u_2 = u_1$ by equality of dimension. Let σ be the conjugation of g_C with respect to g. Then $\sigma\tau_j = \tau_j\sigma$ $(j=1,2)$. Hence $\sigma P^t = P^t\sigma$ for all real t, so $P^t \in \mathrm{Aut}(g)$. The transformations P^t form a continuous one-parameter group of automorphisms of g. Thus $P^t = \exp tX$ with $X \in \mathrm{Der}(g) = \mathrm{ad}(g)$ (cf. Prop. 1.5). Hence $P^{\frac{1}{4}} \in \mathrm{Int}(g)$. The theorem follows since $P^{\frac{1}{4}}(g \cap u_2) = g \cap u_1$ and $P^{\frac{1}{4}}(g \cap iu_2) = g \cap iu_1$. □

For $i = 1,2$ let u_i be a compact real semisimple Lie algebra with involutive automorphism θ_i. We call (u_1,θ_1) and (u_2,θ_2) *isomorphic* if there is an isomorphism ψ from u_1 onto u_2 such that $\theta_2 = \psi\theta_1\psi^{-1}$.

DEFINITION 3.4. An isomorphy class of real semisimple Lie algebras and an isomorphy class of compact real semisimple Lie algebras with involution are called *dual* to each other if they contain members g and (u,θ) respectively, such that g and u are both real forms of the same complex Lie algebra g_C, g has a Cartan decomposition $g = k + p$, and $u = k + ip$ with $\theta|_k = \mathrm{id}$, $\theta|_{ip} = -\mathrm{id}$.

It follows from Theorem 3.3 that an isomorphy class of the one type has precisely one dual isomorphy class of the other type. If $\mathfrak{g}$ and $(\mathfrak{u},\theta)$ belong to dual isomorphy classes then $\mathfrak{g}$ and $(\mathfrak{u},\theta)$ are called *dual* to each other.

The proof of the following proposition is immediate.

PROPOSITION 3.5. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with involutive automorphism θ. Then*

$$(3.7) \qquad B_\theta(X,Y) := -B(X,\theta Y), \quad X,Y \in \mathfrak{g},$$

defines a nondegenerate bilinear symmetric form on $\mathfrak{g}$. This form is positive definite if and only if θ is a Cartan involution: If θ is a Cartan involution with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ then, for $X \in \mathfrak{g}$, $\mathrm{ad}\, X$ is symmetric (respectively skew-symmetric) with respect to B_θ if and only if $X \in \mathfrak{p}$ (respectively $X \in \mathfrak{k}$).

Observe that, with subalgebras and involutions chosen as in (3.2), (3.3), the form B_θ is the restriction of the form B_τ (cf. 3.1)) to $\mathfrak{g}$.

Let $\mathfrak{g}$ be a real Lie algebra with subalgebra $\mathfrak{h}$. The analytic subgroup of $\mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ will be denoted by $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{h})$. The subalgebra $\mathfrak{h}$ is called a *compactly imbedded* subalgebra of $\mathfrak{g}$ if the group $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{h})$ is compact (with its own Lie group topology).

LEMMA 3.6. *Let $\mathfrak{g}$ be a real Lie algebra. Let $s \in \mathrm{Aut}(\mathfrak{g})$.* Define

$$\tilde{s}(g) := sgs^{-1}, \quad g \in \mathrm{Int}(\mathfrak{g}).$$

Then $\tilde{s} \in \mathrm{Aut}(\mathrm{Int}(\mathfrak{g}))$ and

$$d\tilde{s}(\mathrm{ad}\, X) = \mathrm{ad}(sX), \quad X \in \mathfrak{g}$$

PROOF. Let $X \in \mathfrak{g}$. Then $s(\mathrm{ad}\, X)s^{-1} = \mathrm{ad}\, sX$ and $s(\exp \mathrm{ad}\, X)s^{-1} = \exp(\mathrm{ad}\, sX)$. □

PROPOSITION 3.7. *Let $\mathfrak{g}$ be a real semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Then $\mathfrak{k}$ is a compactly imbedded subalgebra of $\mathfrak{g}$.*

PROOF. Let $X \in \mathfrak{g}$. Then:

$$(d\tilde{\theta})(\mathrm{ad}\ X) = \mathrm{ad}\ X \iff \mathrm{ad}(\theta X) = \mathrm{ad}\ X \iff \theta X = X \iff X \in \mathfrak{k}.$$

Hence $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ is the identity component of the closed subgroup $\{g \in \mathrm{Int}(\mathfrak{g}) \mid \tilde{\theta}(g) = g\}$ of $\mathrm{Int}(\mathfrak{g})$, which in its turn, is a closed subgroup of $GL(\mathfrak{g})$ (cf. Prop. 1.7). It follows that $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ is closed in $GL(\mathfrak{g})$. By Prop. 3.5 we have $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{k}) \subset \mathfrak{o}(\mathfrak{g};B_\theta)$. Hence $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k}) \subset SO(\mathfrak{g};B_\theta)$. We conclude that $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$, being a closed subgroup of a compact Lie group, is compact itself.

4. ROOT SPACE DECOMPOSITIONS

From now on let $\mathfrak{g}$ be a real semisimple Lie algebra with Killing form B, Cartan involution θ and corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Also assume that $\mathfrak{g}$ is a noncompact Lie algebra, i.e. $\dim \mathfrak{p} \neq 0$. Then $\mathfrak{p}$ always contains nontrivial abelian subspaces, namely the 1-dimensional subspaces. Thus the maximal abelian subspaces of $\mathfrak{p}$ have nonzero dimension.

LEMMA 4.1. *Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Choose $X \in \mathfrak{a}$ such that*

$$\mathfrak{p}_X := \ker(\mathrm{ad}_{\mathfrak{g}}X) \cap \mathfrak{p} \tag{4.1}$$

has minimal dimension. Then $\mathfrak{p}_X = \mathfrak{a}$.

PROOF. Clearly $\mathfrak{a} \subset \mathfrak{p}_X \subset \mathfrak{p}$. Suppose there exists $Z \in \mathfrak{p}_X \backslash \mathfrak{a}$. Then for some Y in $\mathfrak{a}$ we have $[Y,Z] \neq 0$ and thus $[X+tY,Z] \neq 0$ if $t \neq 0$. Since ad X and ad Y are symmetric (Prop. 3.5) and mutually commuting endomorphisms, it follows that $\ker(\mathrm{ad}(X+tY)) \subset \ker(\mathrm{ad}X)$ for sufficiently small nonzero t, i.e. $\mathfrak{p}_{X+tY} \subset \mathfrak{p}_X$. But then $\mathfrak{p}_{X+tY} = \mathfrak{p}_X$ by the choice of X. Hence $[X+tY,Z] = 0$, which is a contradiction.

THEOREM 4.2. *The maximal abelian subspaces of $\mathfrak{p}$ are mutually conjugate under $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$. If $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$ then*

$$\mathfrak{p} = \bigcup_{k \in \mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})} k.\mathfrak{a}. \tag{4.2}$$

PROOF. Let $H \in \mathfrak{a}$ such that $\mathfrak{p}_H$(cf. (4.1)) has minimal dimension. Let $X \in \mathfrak{p}$. We will prove that $k.X \in \mathfrak{a}$ for some $k \in K := \mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$. The function $k \to B(H,k.X)$ is continuous on the compact (cf. Prop. 3.7) Lie group K, so it attains a minimum for some $k_0 \in k$. Hence, for all $T \in k$:

$$0 = \frac{d}{dt} B(H, \exp(t \text{ ad } T) k_0 . X)\big|_{t=0} =$$

$$= B(H, [T, k_0 . X]) = B([k_0 . X, H], T).$$

Now $[k_0 . X, H] \in [p, p] \subset k$ and B is negative definite on k, so $[k_0 . X, H] = 0$. Hence $k_0 . X \in p_H = a$ (cf. Lemma 4.1). This settles (4.2).

Finally, let a and H be as above and let a' be another maximal abelian subspace of p. Let $k \in K$ such that $k.H \in a'$. Then $[H, k^{-1}.X] = 0$ for all $X \in a'$, so $k^{-1}.a' \subset p_H = a$. Hence $k^{-1}.a' = a$ because a' is maximal abelian. □

REMARK 4.3. Let a be a maximal abelian subspace of p and let b be a subset of a. Let $z_b(p)$ denote the centraliser of b in p and $Z_b(K)$ the centraliser of b in $\text{Int}_g(k)$. Then, in the same way as for Theorem 4.2, it can be proved that

$$(4.3) \qquad z_b(p) = \bigcup_{k \in Z_b(K)} k.a.$$

DEFINITION 4.4. The *(real) rank* of the real semisimple Lie algebra $g = k + p$ is the dimension of a maximal abelian subspace of p.

In view of Theorem 3.3 and 4.2 the rank of g is well-defined.

Fix a maximal abelian subspace a of p. For $\alpha \in a^*$ *define*

$$(4.4) \qquad g_\alpha := \{X \in g \mid [H, X] = \alpha(H)X \quad \forall H \in a\}.$$

$\alpha \in a^*$ is called a *root* of the pair (g, a) if $\alpha \neq 0$ and $\dim g_\alpha > 0$. Let Σ be the set of all roots. Since the endomorphisms ad H ($H \in a$) of g are symmetric w.r.t. B_θ and mutually commuting, we have the *root space decomposition*

$$(4.5) \qquad g = g_0 + \sum_{\alpha \in \Sigma} g_\alpha.$$

This is a direct sum of vector spaces, orthogonal with respect to B_θ. Clearly, Σ is a finite set. Obviously we have

$$(4.6) \qquad \theta g_\alpha = g_{-\alpha}.$$

In particular, $\theta g_0 = g_0$, so $g_0 = (g_0 \cap k) + (g_0 \cap p)$. Since a is maximal abelian in p, we have $g_0 \cap p = a$. Let m be defined as the centraliser of a in k. Then $m = g_0 \cap k$, so

(4.7) $$g_0 = m + a.$$

It is also clear that

(4.8) $$[g_\alpha, g_\beta] \subset g_{\alpha+\beta},$$

so g_0, and hence m, are subalgebras of g.

It follows from (4.6) that $\alpha \in \Sigma$ iff $-\alpha \in \Sigma$. Make a^* into an ordered vector space. Let Σ^+ denote the set of positive roots. Define

(4.9) $$n := \sum_{\alpha\in\Sigma^+} g_\alpha.$$

Because of (4.8) n is a nilpotent subalgebra of g.

Let $\alpha \in \Sigma^+$, $X \in g_{-\alpha}$. Then $X = (X+\theta X) - \theta X \in (g_\alpha + g_{-\alpha}) \cap k + g_\alpha$. Hence

$$g = g_0 \cap k + a + \sum_{\alpha\in\Sigma^+} (g_\alpha + g_{-\alpha}) \cap k + \sum_{\alpha\in\Sigma^+} g_\alpha.$$

Thus

(4.10) $$g = k + a + n \qquad \text{(direct sum of vector spaces).}$$

This is called the *Iwasawa decomposition*.

5. THE ROOT SYSTEM Σ

The Killing form B restricted to a is positive definite. For $\lambda \in a^*$ define $A_\lambda \in a$ by

(5.1) $$B(H, A_\lambda) = \lambda(H) \qquad \forall H \in a.$$

Define an inner product $\langle .,. \rangle$ on a^* by

(5.2) $$\langle \lambda, \mu \rangle := B(A_\lambda, A_\mu).$$

LEMMA 5.1. *Let* $\alpha \in \Sigma$ *and choose* $X_\alpha \in \mathfrak{g}_\alpha \backslash \{0\}$ *such that, possibly after multiplying* X_α *with a nonzero real scalar,*

$$(5.3) \qquad B_\theta(X_\alpha, X_\alpha) = 2/\langle\alpha,\alpha\rangle$$

Define

$$(5.4) \qquad X_{-\alpha} := -\theta X_\alpha, \qquad H_\alpha := [X_\alpha, X_{-\alpha}].$$

Then $\qquad X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, $H_\alpha \in \mathfrak{a}$,

$$(5.5) \qquad H_\alpha = \frac{2A_\alpha}{\langle\alpha,\alpha\rangle},$$

$$(5.6) \qquad [H_\alpha, X_\alpha] = 2X_\alpha, \qquad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}.$$

Thus $X_\alpha, Y_\alpha, H_\alpha$ *span a subalgebra of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}(2,\mathbb{R})$ *and they form a standard basis for this copy of* $\mathfrak{sl}(2,\mathbb{R})$.

PROOF. $H_\alpha \in \mathfrak{g}_0$ and $\theta H_\alpha = -H_\alpha$, so $H_\alpha \in \mathfrak{a}$. For all $H \in \mathfrak{a}$ we have

$$B(H_\alpha, H) = B([X_\alpha, X_{-\alpha}], H) = B([H, X_\alpha], X_{-\alpha}) =$$

$$= \alpha(H)B(X_\alpha, X_{-\alpha}) = \alpha(H)B_\theta(X_\alpha, X_\alpha) = \frac{2\alpha(H)}{\langle\alpha,\alpha\rangle} = B(\frac{2A_\alpha}{\langle\alpha,\alpha\rangle}, H).$$

This settles (5.5). Now $[H_\alpha, X_\alpha] = \frac{2\alpha(A_\alpha)}{\langle\alpha,\alpha\rangle} X_\alpha = 2X_\alpha$, and similarly for the other formula in (5.6). Finally, the statement about $\mathfrak{sl}(2,\mathbb{R})$ follows from [Hu, Example 2.1]. □

LEMMA 5.2. *Let* X, Y, H *be a standard basis for* $\mathfrak{sl}(2,\mathbb{R})$, *i.e.* $[X,Y] = H$, $[H,X] = 2X$, $[H,Y] = -2Y$. *Let* W *be a finite-dimensional real* $\mathfrak{sl}(2,\mathbb{R})$*-module. Then* W *has a basis of weight vectors* w (i.e. $H.w = mw$) *with integer weights* m *such that for each* $m \in \mathbb{Z}$ *the multiplicities of the weights* m *and* $-m$ *are equal.*

PROOF. Let $W_c := W+iW$. This naturally becomes a $\mathfrak{sl}(2,C)$-module. Let V_c be an irreducible subspace of W_c. Then V_c has a basis $v_0, v_1, \ldots, v_m$ such that $H.v_j = (m-2j)v_j$ and $Y.v_j = (j+1)v_{j+1}$ (cf. [Hu, §7.2]). Write $v_j = v_j' + iv_j''$ with $v_j', v_j'' \in W$. Then, since W is a $\mathfrak{sl}(2,\mathbb{R})$-module, we have $H.v_j' = (m-2j)v_j'$ and $Y.v_j' = (j+1)v_{j+1}'$ and similarly for v_j''. Thus $V_c = V+iV$, where V is an irreducible submodule of W with weights $m, m-2, \ldots, -m+2, -m$ each occurring with multiplicity 1. Since W_c is a direct sum of irreducible submodules (cf. [Hu, Theorem 6.3]), the same holds for W. □

Remember the axioms of a root system, cf. Def. 2.2.

THEOREM 5.3. *Σ, as a subset of $\mathfrak{a}^*$ with inner product $\langle .,. \rangle$, is a root system.*

PROOF. First we prove that Σ spans $\mathfrak{a}^*$. Let $H \in \mathfrak{a}$ such that $\alpha(H) = 0$ for all $\alpha \in \Sigma$. Then $(\mathrm{ad}\, H)(\mathfrak{g}_\alpha) = \{0\}$ for all $\alpha \in \Sigma$, so $(\mathrm{ad}\, H)(\mathfrak{g}) = \{0\}$, so H is in the center of $\mathfrak{g}$. Hence $H = 0$ by semisimplicity of $\mathfrak{g}$, so Σ spans $\mathfrak{a}^*$.

Next we prove (b) and (c) of Def. 2.2. Fix $\alpha \in \Sigma$ and $X_\alpha, X_{-\alpha}, H_\alpha$ as in Lemma 5.1. Then, via the adjoint action of $X_\alpha, X_{-\alpha}$ and H_α on $\mathfrak{g}$, $\mathfrak{g}$ becomes a $\mathfrak{sl}(2,\mathbb{R})$-module. Let $\beta \in \Sigma$. Then $V := \Sigma_{j \in \mathbb{Z}}\, \mathfrak{g}_{\beta+j\alpha}$ is a $\mathfrak{sl}(2,\mathbb{R})$-submodule and

$$[H_\alpha, X] = \left(\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} + 2j\right)X, \qquad X \in \mathfrak{g}_{\beta+j\alpha}.$$

Thus, for $j = 0$, we obtain (c) of Def. 2.2 by applying Lemma 5.2. Again by Lemma 5.2 it follows that

$$\dim \mathfrak{g}_\beta = \dim \mathfrak{g}_{\beta+j\alpha} \text{ if } j = -\frac{2\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle}.$$

This settles (b) of Def. 2.2. □

PROPOSITION 5.4. *Let Σ be a root system. If $\alpha \in \Sigma$ then the only possible multiples of α in Σ are $\pm \frac{1}{2}\alpha$, $\pm \alpha$, $\pm 2\alpha$.*

PROOF. Let $\beta = c\alpha \in \Sigma$. Then $2\langle\beta,\alpha\rangle/\langle\alpha,\alpha\rangle$ and $2\langle\beta,\alpha\rangle/\langle\beta,\beta\rangle$ are integer. □

Note that, if Σ is a root system then

$$\Sigma_1 := \{\alpha \in \Sigma \mid \tfrac{1}{2}\alpha \notin \Sigma\} \tag{5.7}$$

and

$$\Sigma_2 := \{\alpha \in \Sigma \mid 2\alpha \notin \Sigma\} \tag{5.8}$$

are reduced root systems.

DEFINITION 5.5. Let Σ be a root system in E. The *Weyl group* W of Σ is the group of orthogonal transformation of E generated by the reflections

$$\lambda \to \lambda - \frac{2\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}, \quad \alpha \in \Sigma.$$

Let Σ be the root system of $(\mathfrak{g},\mathfrak{a})$. The *multiplicity* μ_α of $\alpha \in \Sigma$ is defined by

$$(5.9) \qquad \mu_\alpha := \dim \mathfrak{g}_\alpha.$$

It follows from the proof of Theorem 5.3 that:

PROPOSITION 5.6. *Let* Σ *be the root system of* $(\mathfrak{g},\mathfrak{a})$ *with Weyl group* W. *If* $\alpha \in \Sigma$, $w \in W$ *then* $\mu_{w.\alpha} = \mu_\alpha$.

We conclude this section with some corollaries of Lemma 5.1 and of the proof of Theorem 5.3.

LEMMA 5.7. *Let* $\alpha \in \Sigma$. *If* $0 \neq X \in \mathfrak{g}_\alpha$ *then* $[\mathfrak{g}_0,X] = \mathfrak{g}_\alpha$.

PROOF (due to G. van Dijk). Without loss of generality we may assume that $X = X_\alpha$ as in Lemma 5.1. The Lie algebra generated by $X_\alpha, X_{-\alpha}$ and H_α acts by the adjoint representation on $\Sigma_{j\in\mathbb{Z}}\ \mathfrak{g}_{j\alpha}$ with weights -2,0,2 and possibly -4,4. By the representation theory of $\mathfrak{sl}(2,\mathbb{R})$ $\mathfrak{g}_\alpha$ has a basis of weight vectors v belonging to irreducible submodules and each such v can be written as $v = X_\alpha.w$ for some $w \in \mathfrak{g}_0$ (cf. [Hu, §7.2] and the proof of Lemma 5.2).

COROLLARY 5.8. *For each* $\alpha \in \Sigma$, $\mathfrak{g}_\alpha$ *is an irreducible* $\mathrm{ad}(\mathfrak{m})$*-module.*

PROOF. Let $0 \neq X \in \mathfrak{g}_\alpha$. Then, by Lemma 7.7:

$$\mathfrak{g}_\alpha = [\mathfrak{g}_0,X] = [\mathfrak{m},X] + [\mathfrak{a},X] =$$

$$= \mathrm{ad}(\mathfrak{m})X + \mathbb{R}X. \quad \square$$

COROLLARY 5.9. *If* $\mathfrak{q}$ *is a subalgebra of* $\mathfrak{g}$ *which includes the subalgebra* $\mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ then, for each $\alpha \in \Sigma^+$, $\mathfrak{q} \cap \mathfrak{g}_{-\alpha} = \mathfrak{g}_{-\alpha}$ *or* $\{0\}$.

COROLLARY 5.10. *If* $\alpha \in \Sigma$, $0 \neq X \in \mathfrak{g}_\alpha$, $0 \neq Y \in \mathfrak{g}_{-\alpha}$ *then* $[X,Y] \neq 0$.

PROOF. Suppose $[X,Y] = 0$. Then, for all $Z \in g_0$: $0 = B(Z,[X,Y]) = B([Z,X],Y)$. Choose Z such that $[Z,X] = -\theta Y$ (cf. Lemma 5.7). Then $B_\theta(Y,Y) = 0$. Hence $Y = 0$. This is a contradiction. □

6. EXAMPLES

EXAMPLE 6.1. Let g_C be a complex semisimple Lie algebra. Use the notation of §2. Let u be the compact real form of g_C obtained in Cor. 2.4 with corresponding conjugation τ. Then τ is an involution of $(g_C)_R$ and $(g_C)_R = u + iu$ is the corresponding decomposition into the 1 and -1 eigenspaces. Let B_R be the Killing form of $(g_C)_R$. Then B_R is negative definite on u and positive definite on iu by (1.4). Thus τ is a Cartan involution of $(g_C)_R$. Now h_R is a maximal abelian subspace of iu. The root system of $((g_C)_R,h_R)$ coincides with the root system Φ of (g_C,h_C). If $\alpha \in \Phi$ then the root spaces g_α and g^α also coincide. However, g^α is considered as a complex vector space of dimension 1 and g_α as a real vector space of dimension 2. Since h_C is maximal abelian in g_C, the centralizer of h_R in u (the space m for $(g_C)_R$) equals ih_R. Thus m is an abelian Lie algebra in this case.

EXAMPLE 6.2. Use again the notation of §2. It follows from Theorem 2.3 that h_R together with the vectors X_α $(\alpha \in \Phi)$ span a real form g of g_C. It also follows from Theorem 2.3 that θ defined by

$$(6.1) \qquad \theta|_{h_R} := -\mathrm{id}, \quad \theta X_\alpha = -X_{-\alpha},$$

is an involution of g. Then 1 eigenspace k of θ is spanned by $X_\alpha - X_{-\alpha}$ $(\alpha\in\Phi)$, the -1 eigenspace p by h_R and $X_\alpha + X_{-\alpha}$ $(\alpha\in\Phi)$. Because of (2.11) and (1.3), θ is a Cartan involution. Note also that the dual compact Lie algebra $k + ip$ equals u as defined by Cor. 2.4. The space h_R is a maximal abelian subspace of p. It is also maximal abelian in g. Real semisimple Lie algebras with this property are called *normal*. It can be proved that the Lie algebra g under consideration is unique up to isomorphism as a normal real form of g_C (cf. [He., Theorem IX. 5.10]). The root system of the pair (g,h_R) coincides with the root system Φ of (g_C,h_C). If $\alpha \in \Phi$ then $g_\alpha = \mathbb{R} X_\alpha$, i.e. one-dimensional. The subspace m of g has dimension zero.

EXAMPLE 6.3. Let $g_C = \mathfrak{sl}(n+1,C)$, the Lie algebra of all complex $(n+1)\times(n+1)$ matrices with trace zero $(n\geq 1)$. Let A^* denote the Hermitian adjoint of of $A \in \mathfrak{sl}(n+1,\mathbb{C})$. Then τ, defined by $\tau A := -A^*$, is a conjugate linear involutive automorphism of $\mathfrak{g}_C$. The fixed point set of τ is $\mathfrak{u} := \mathfrak{su}(n+1)$, the real Lie algebra of all skew-hermitian $(n+1)\times(n+1)$ matrices with trace zero. The compact Lie group $SU(n+1)$ has Lie algebra $\mathfrak{u}$, so $\mathfrak{u}$ is a compact real form of $\mathfrak{g}_C$. Let J be the $(n+1)\times(n+1)$ matrix $\mathrm{diag}(-1,1,\ldots,1)$ and let $\sigma A := -JA^*J$. Then σ is also a conjugate linear involutive automorphism of $\mathfrak{g}_C$ and $\sigma\tau = \tau\sigma$. The fixed point set of σ is $\mathfrak{g} := \mathfrak{su}(n,1)$, the real Lie algebra of all matrices

$$\begin{pmatrix} -\mathrm{tr}B & z_1\ldots z_n \\ \bar{z}_1 & \\ \vdots & B \\ \bar{z}_n & \end{pmatrix}, \quad B \in \mathfrak{u}(n),\ z_1,\ldots,z_n \in \mathbb{C}.$$

Let $\mathfrak{k} := \mathfrak{g} \cap \mathfrak{u}$, $\mathfrak{p} := \mathfrak{g} \cap (i\mathfrak{u})$, $\theta := \sigma\tau$. Then θ is a Cartan involution for $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition. We see that $\mathfrak{k}$ consists of all matrices

$$\begin{pmatrix} -\mathrm{tr}B & 0 \\ 0 & B \end{pmatrix}, \quad B \in \mathfrak{u}(n),$$

so $\mathfrak{k}$ is isomorphic to $\mathfrak{u}(n)$, and that $\mathfrak{p}$ consists of all matrices

$$\begin{pmatrix} 0 & z_1\ldots z_n \\ \bar{z}_1 & \\ \vdots & 0 \\ \bar{z}_n & \end{pmatrix}, \quad z_1,\ldots,z_n \in \mathbb{C}.$$

Let $\mathfrak{a}$ be the abelian subspace of $\mathfrak{p}$ consisting of all matrices

$$H_t := \begin{pmatrix} 0 & \ldots & 0 & t \\ \vdots & & & \\ 0 & & 0 & \\ t & & & \end{pmatrix}, \quad t \in \mathbb{R} \tag{6.2}$$

It follows easily that $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Thus $\mathfrak{g}$ has rank 1.

Let $\alpha \in \mathfrak{a}^*$ such that $\alpha(H_t) = t$. We find that $\Sigma = \{\alpha,2\alpha,-\alpha,-2\alpha\}$. $\mathfrak{g}_\alpha$

consist of all matrices

$$\begin{pmatrix} 0 & z_1 & \cdots & z_{n-1} & 0 \\ \bar{z}_1 & & & & -\bar{z}_1 \\ \vdots & & 0 & & \vdots \\ \bar{z}_{n-1} & & & & -\bar{z}_{n-1} \\ 0 & z_1 & \cdots & z_{n-1} & 0 \end{pmatrix} \quad , \ z_1,\ldots,z_{n-1} \in \mathbb{C},$$

so $\dim \mathfrak{g}_\alpha = 2n-2$.

$\mathfrak{g}_{2\alpha}$ consists of all matrices

$$\begin{pmatrix} it & 0 & \cdots & 0 & -it \\ 0 & & & & 0 \\ \vdots & & 0 & & \vdots \\ 0 & & & & 0 \\ it & 0 & \cdots & 0 & -it \end{pmatrix} \quad , \ t \in \mathbb{R},$$

so $\dim \mathfrak{g}_{2\alpha} = 1$.

$\mathfrak{m}$ consists of all matrices

$$\begin{pmatrix} -\frac{1}{2}\mathrm{tr}C & 0 \ \cdots & 0 \\ 0 & & \vdots \\ \vdots & C & \vdots \\ \vdots & & 0 \\ 0 \ \cdots & 0 & -\frac{1}{2}\mathrm{tr}C \end{pmatrix} \quad , \ C \in \mathfrak{u}(n-1)$$

In these examples, in particular the last one, some important differences can be observed between root space decompositions for complex and real semisimple Lie algebras:

$(\mathfrak{g}_C, \mathfrak{h}_C)$	$(\mathfrak{g}, \mathfrak{a})$
Φ reduced;	Σ may be non-reduced;
$\dim \mathfrak{g}^\alpha = 1 \quad \forall \alpha \in \Phi$;	$\dim \mathfrak{g}_\alpha$ may exceed 1;
$\mathfrak{g}^0 = \mathfrak{h}_C$	$\mathfrak{a}$ may be a proper subspace of $\mathfrak{g}_0$.

7. DYNKIN DIAGRAMS

Let Σ be a root system and let Σ_1 and Σ_2 be the associated reduced root systems, cf. (5.7), (5.8). The root system Σ is called *irreducible* if it cannot be decomposed into two disjoint nonempty mutually orthogonal subsets. Clearly, if one of the root systems Σ, Σ_1, Σ_2 is irreducible, then all of them are irreducible. A subset Δ of Σ is called a *base* of Σ if Δ is a basis of span(Σ) such that each $\beta \in \Sigma$ is a linear combination of elements of Δ with either all coefficients being nonnegative integers or all coefficients being nonpositive integers. Any base Δ for the reduced root system Σ_1 (which exists by [Hu, Theorem 10.1]) is also a base for Σ. We will use the notation

$$(7.1) \qquad a_{\alpha,\beta} := \frac{2\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle}, \quad \alpha,\beta \in \Sigma.$$

<u>THEOREM 7.1</u>. *The irreducible nonreduced root systems are precisely the root systems*

$$(bc)_\ell := \{\pm e_i \pm e_j\,(1\le i<j\le \ell),\ \pm e_i\,(1\le i\le\ell),\ \pm 2e_i\,(1\le i\le \ell)\},\quad \ell = 1,2,\ldots$$

<u>PROOF</u>. It follows by direct verification that $(bc)_\ell$ is a root system. (This can also be seen, because $(bc)_\ell$ is the root system of $su(\ell,\ell)$.). Clearly $(bc)_\ell$ is nonreduced. If $\Sigma = (bc)_\ell$ then $\Sigma_1 = b_\ell$, so $(bc)_\ell$ is irreducible.

Conversely, let Σ be a nonreduced irreducible root system and let Δ be a base of Σ_1. Since each $\alpha \in \Sigma_1$ is Weyl group conjugate to an element in Δ (cf. [Hu., Theorem 10.3(c)]), we can take $\alpha \in \Delta$ such that $2\alpha \in \Sigma$. Let $\beta \in \Delta\backslash\{\alpha\}$. Suppose $\langle\alpha,\beta\rangle \neq 0$. Then $a_{\beta,\alpha}$ and $a_{\beta,2\alpha}$ are negative integers and $a_{\beta,\alpha} = 2a_{\beta,2\alpha}$. Hence $a_{\beta,\alpha} = -2$, $a_{\alpha,\beta} = -1$ (cf. [Hu.,§9.4]), so $\{\alpha,\beta\}$ is a double link in the Dynkin diagram of (Σ_1,Δ) with $\langle\beta,\beta\rangle = 2\langle\alpha,\alpha\rangle$. By the classification theorem (cf. [Hu., Theorem 11.4]) it follows that the Dynkin diagram equals

$$a_1 := \quad \underset{\alpha}{\circ}$$

or

$$b_\ell := \circ - \circ \ \cdots\ \circ - \underset{\beta}{\circ} \Rightarrow \underset{\alpha}{\circ} \qquad (\ell \geq 2).$$

Now we know (cf. [Hu., §12.1]) that there exists an orthonormal basis $\{e_1,\ldots,e_\ell\}$ of span Σ such that

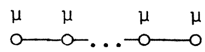
$$\Sigma_1 = \{\pm e_i \pm e_j (1 \le i < j \le \ell),\ \pm e_i (1 \le i \le \ell)\}$$

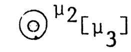
and

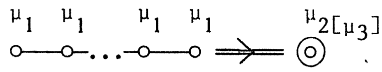
$$\Delta = \{e_i - e_{i+1} (1 \le i \le \ell - 1),\ \alpha = e_\ell\}.$$

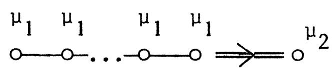
We showed that $2e_\ell \in \Sigma$. The reflection associated with $e_i - e_\ell$ sends $2e_\ell$ to $2e_i$, so $2e_i \in \Sigma$. On the other hand, since $a_{e_i, \pm e_i \pm e_j} = \pm 1$, $2(\pm e_i \pm e_j) \notin \Sigma$. □

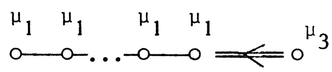
Use the notation of §4 and 5. Let Σ be the root system of $\mathfrak{g}$ and suppose that Σ is irreducible. Then at most three root lengths occur in Σ and all roots of a given length are conjugate under W(cf. Theorem 7.1 and [Hu, §10.4, Lemma C]). Thus, by Prop. 5.6, the multiplicity μ_α only depends on the length of $\alpha \in \Sigma$. Below we list the possible Dynkin diagrams for irreducible Σ, together with the possible multiplicities. A mode $\circ^{\mu}$ denotes a single root with multiplicity μ. A node $\circledcirc^{\mu_1}[\mu_2]$ denotes a double root with multiplicity μ_1 for the smaller root and multiplicity μ_2 for the bigger root. All multiplicities are assumed to be nonzero. We used [Hu, Theorem 11.4] and Theorem 7.1.

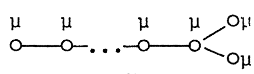
$a_\ell(\ell \ge 1)$: 

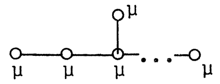
$(bc)_1$: 

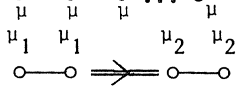
$(bc)_\ell(\ell \ge 2)$: 

$b_\ell(\ell \ge 2)$: 

$c_\ell(\ell \ge 3)$: 

$d_\ell(\ell \ge 4)$: 

$e_\ell(\ell = 6,7,8)$: 

f_4: 

g_2: 

It can be shown that two real semisimple Lie algebras which have the same Dynkin diagram and multiplicity function, are isomorphic. However, the proof is indirect, using the Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$, cf. Ch. V.

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Chapter II

REAL SEMISIMPLE LIE GROUPS

B. HOOGENBOOM

In this chapter the basic structure theory of real semisimple Lie groups is developed. It is assumed that the reader knows the basic theory of Lie groups, cf. for instance ch. II of HELGASON'S book [He].

As in Ch. I, most of the theory is developed parallel to parts of [He]: see ch. IV (§3), ch. V (§6), ch. VI (§1,2,5), ch. VII (§2) and ch. IX (Theorem 1.1). However, in contrast to [He], we do not use any differential geometry, and the Weyl group theory is developed here for symmetric spaces of the noncompact type instead of symmetric spaces of the compact type. Our treatise of the Iwasawa decomposition is mainly based on WALLACH'S book: see [Wal,§7.4].

1. ORTHOGONAL SYMMETRIC LIE ALGEBRAS

DEFINITION 1.1. The pair $(\mathfrak{g},s)$ is called an *orthogonal symmetric Lie algebra* if $\mathfrak{g}$ is a real semisimple Lie algebra, s an involutive automorphism of $\mathfrak{g}$ and the set of fixed points of s, which we shall denote by $\mathfrak{k}$, is a compactly imbedded subalgebra. If, in addition $\mathfrak{k} \cap z = (0)$ (z center of $\mathfrak{g}$), then we shall call $(\mathfrak{g},s)$ *effective*.

Let $(\mathfrak{g},s)$ be an orthogonal symmetric Lie algebra. Then a pair (G,K) is said to be *associated* with $(\mathfrak{g},s)$ if G is a connected Lie group with Lie algebra $\mathfrak{g}$, and K is a Lie subgroup of G with Lie algebra $\mathfrak{k}$.

Clearly, if $\mathfrak{g}$ is a noncompact semisimple real Lie algebra and θ is a Cartan involution of $\mathfrak{g}$, then $(\mathfrak{g},\theta)$ is an effective orthogonal symmetric Lie algebra; such orthogonal symmetric Lie algebras are said to be of the *noncompact type*.

If G is a Lie group, and σ an automorphism of G, then we shall write G_σ for the set of fixed points of σ in G, and $(G_\sigma)_0$ for the identity component of G_σ.

DEFINITION 1.2. The pair (G,K) is called a *symmetric pair* if G is a connected Lie group, K a closed subgroup of G and there exists an analytic involutive automorphism σ of G such that $(G_\sigma)_0 \subset K \subset G_\sigma$. If, in addition, $\mathrm{Ad}_G(K)$ is compact, (G,K) is called a *Riemannian symmetric pair*.

If (G,K) is a symmetric pair, we shall always let π denote the canonical projection of G onto G/K, and we put $o := \pi(e)$. G will be considered as a Lie transformation group of G/K, and for $g \in G$ we define a diffeomorphism $\tau(g)$ of G/K onto G/K by $\tau(g)\cdot g'K := gg'K$ $(g' \in G)$.

In view of what we have said already, the following lemma is obvious:

LEMMA 1.3. *Let* (G,K) *be a Riemannian symmetric pair,* σ *an involutive automorphism of* G *such that* $(G_\sigma)_0 \subset K \subset G_\sigma$. *Let* $\mathfrak{g},\mathfrak{k}$ *be the Lie algebras of* G,K, *respectively. Let* $s := d\sigma$. *Then* $(\mathfrak{g},s)$ *is an orthogonal symmetric Lie algebra.*

In the situation of Lemma 1.3 σ is uniquely determined by s. If, in addition, $(\mathfrak{g},s)$ is of noncompact type, then s is uniquely determined by $\mathfrak{k}$, hence σ is uniquely determined by K.

REMARK 1.4. Let $(\mathfrak{g},s)$ be an orthogonal symmetric Lie algebra, and (G,K) a pair associated with $(\mathfrak{g},s)$. Then (G,K) is not necessarily a Riemannian symmetric pair, because (i) it may be impossible to lift s to G, (ii) K may not be closed. However, if s can be lifted to an involutive automorphism of G, say σ, then we may choose $K := G_\sigma$ (or $K := (G_\sigma)_0$), and (G,K) becomes a Riemannian symmetric pair. For example, let $\mathfrak{g}$ be a noncompact semisimple Lie algebra, θ a Cartan involution of $\mathfrak{g}$. Let $G := \mathrm{Int}(\mathfrak{g})$, or a simply connected Lie group with Lie algebra $\mathfrak{g}$, then θ can be lifted to G (cf. Theorem 3.3).

One of the main facts for Riemannian symmetric pairs is that they give rise to so called Riemannian symmetric spaces (which will be defined in due course), and that all Riemannian symmetric spaces arise in this way. Thus the classification of Riemannian symmetric spaces can, via Lemma 1.3, be reduced to the classification of certain orthogonal symmetric Lie algebras, cf. [He, ch X]. These things, however, lie outside the scope of this book.

DEFINITION 1.5. ([War, p.52]). Let M be a C^∞-manifold. A *Riemannian structure* on M is a choice of a positive definite inner product $Q(.,.)_m$ on each tangent space M_m, such that if X,Y are C^∞-vector fields, then $Q(X,Y)$ is a C^∞-function on M. If Q is not positive definite, but only non-degenerate, we'll speak of a *pseudo-Riemannian structure* on M. A differential manifold together with a (pseudo-) Riemannian structure is called a *(pseudo-) Riemannian manifold*.

DEFINITION 1.6. Let M be a Riemannian manifold. M is called a *Riemannian symmetric space* if for all $m \in M$ there exists an involutive isometry s_m such that m is an isolated fixed point of s_m. s_m is called the *symmetry* around m.

REMARK 1.7. What we have called a Riemannian symmetric space is called in [He] a Riemannian globally symmetric space.

THEOREM 1.8. *Let* (G,K) *be a Riemannian symmetric pair. Then there exists a* G-*invariant Riemannian structure* Q *on* G/K, *and in each such* Q *the manifold* G/K *is a Riemannian symmetric space, with* s_0 *given by* $s_0(gK) = (\sigma g)K$ (σ *the involutive automorphism of* G *such that* $(G_\sigma)_0 \subset K \subset G_\sigma$).

PROOF. Let $d\sigma$ and $d\pi$ denote the differential of σ and π at e. Let $\mathfrak{g},\mathfrak{k}$ denote the Lie algebras of G,K, respectively. Let $\mathfrak{p} := \{X \in \mathfrak{g}: d\sigma(X) = -X\}$, then $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Let $X \in \mathfrak{p}$, $k \in K$. Then $\sigma(\exp \mathrm{Ad}(k)(tX)) = k \exp(-tX)k^{-1} \Rightarrow d\sigma(\mathrm{Ad}(k)X) = -\mathrm{Ad}(k)X$. Hence $\mathfrak{p}$ is $\mathrm{Ad}_G(K)$-invariant.

Let T_0 denote the tangent space to G/K at o, then $d\pi: \mathfrak{g} \to T_0$, $\ker(d\pi) = \mathfrak{k}$, and $d\pi: \mathfrak{p} \to T_0$ is a linear bijection. Now

(1.1) $$d\pi \circ \mathrm{Ad}(k)X = d\tau(k) \circ d\pi(X),\ k \in K,\ X \in \mathfrak{p}.$$

(This follows because $(\exp \mathrm{Ad}(k)(tX)).K = (k \exp(tX)k^{-1}).K = \tau(k)\pi(\exp(tX))$. Now there exists an $\mathrm{Ad}_G(K)$-invariant inner product B on $\mathfrak{p}$ (since $\mathrm{Ad}_G(K)$ is a compact group). Define $Q_0 := B \circ (d\pi)^{-1}$, then Q_0 is a $d\tau(k)$ ($k \in K$) invariant form on T_0. Denote the corresponding symmetric bilinear form on $T_0 \times T_0$ also by Q_0. Now define a bilinear form $Q_{\tau(g).o}$ on $(G/K)_{\tau(g).o} \times (G/K)_{\tau(g).o}$ ($\tau(g).o \in G/K$) by

(1.2) $$Q_{\tau(g).o}\,(d\tau(g)(X), d\tau(g)(Y)) := Q_0(X,Y),\ X,Y \in \mathfrak{p}.$$

$Q_{\tau(g).o}$ is well defined because Q_0 is $\mathrm{Ad}_G(K)$ invariant. Now it follows that $\tau(g).o \to Q_{\tau(g).o}$ is a G-invariant analytic Riemannian structure on G/K (because $\tau(g)$ is analytic for all $g \in G$). But on the other hand, each G-invariant Riemannian structure on G/K arises in this fashion from an Ad(K)-invariant quadratic form on $\mathfrak{p}$.

Let s_0 be defined by $s_0(gK) := (\sigma g)K$, then s_0 is an involutive diffeomorphism of G/K onto itself, and $(ds_0)_o = -\mathrm{Id}$. Now, let $g \in G$, $X,Y \in (G/K)_{\tau(g).o}$. Then $d\tau(g^{-1})(X) \in T_0$, $d\tau(g^{-1})(Y) \in T_0$. It follows from the definition of s_0 that

$$(1.3) \qquad s_0 \circ \tau(g)(xK) = \sigma(gx)K = \sigma(g)\sigma(x)K = (\tau(\sigma(g)) \circ s_0)(xK).$$

Thus $s_0 \circ \tau(g) = \tau(\sigma(g)) \circ s_0$. Let $X_0 := d\tau(g^{-1})(X)$, $Y_0 := d\tau(g^{-1})(Y)$. Then

$$Q(ds_0(X),ds_0(Y)) = Q(ds_0(d\tau(g)(X_0)),ds_0(d\tau(g)(Y_0))) = Q(ds_0(X_0),ds_0(Y_0)) = \\ = Q(X_0,Y_0) = Q(X,Y).$$

Hence s_0 is an isometry. But for an arbitrary point in G/K, say $\tau(g).o$, the symmetry around $\tau(g).o$ is given by

$$(1.4) \qquad s_{\tau(g).o} := \tau(g) \circ s_0 \circ \tau(g^{-1}).$$

This being an isometry the space G/K becomes a Riemannian symmetric space. □

REMARK 1.9. The converse of Theorem 1.8 is also true. This lies, however, again outside the scope of this book, the proof being more difficult than that of Theorem 1.8. But for the sake of completeness, we shall state this converse here, for the proof see [He, Theorem IV.3.3 (i) and (ii)]: *Let* M *be a Riemannian symmetric space, let* m_0 *be a fixed point in* M *and let* s_0 *denote the symmetry around* m_0. *Let* $G := I_0(M)$ *(that is, the connected component of the identity of the group of all isometries of* M*), and let* K *be the subgroup of* G *leaving* m_0 *fixed. Then* (G,K) *is a Riemannian symmetric pair with respect to the involutive automorphism* σ *defined by* $\sigma(g) := s_0 g s_0$ $(g \in G)$, *and* M *is diffeomorphic to* G/K.

2. THE CARTAN DECOMPOSITION OF A SEMISIMPLE LIE GROUP

In this section we shall state and prove the so-called Cartan decomposition of a semisimple Lie group. A refinement of this decomposition, which to some extent gives a uniqueness condition, will be given later on, cf. Theorem 6.2.

Let g be a noncompact semisimple Lie algebra over $\mathbb{R}$, let θ be a Cartan involution of $\mathfrak{g}$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ denote the corresponding Cartan decomposition of $\mathfrak{g}$ (cf. §I.3). Then $(\mathfrak{g},\theta)$ is an orthogonal symmetric Lie algebra of the noncompact type. Let (G,K) be a pair associated to $(\mathfrak{g},\theta)$, and assume K to be connected. Let $\mathfrak{g}_c$ denote the complexification of $\mathfrak{g}$, and let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}_c$, defined by $\mathfrak{u} := \mathfrak{k} + i\,\mathfrak{p}$. Let τ be the conjugation of $\mathfrak{g}_c$ with respect to $\mathfrak{u}$ (thus $\tau|_{\mathfrak{g}} = \theta$), and define, just as in (I.3.1), B_τ by $B_\tau(X,Y) := -B(X,\tau Y)(X,Y \in \mathfrak{g}_c)$, B_θ by $B_\theta(X,Y) := -B(X,\theta Y)$ $(X,Y \in \mathfrak{g})$. Then $B_\theta = B_\tau|_{\mathfrak{g}\times\mathfrak{g}}$.

<u>LEMMA 2.1.</u> (i) *If* $X \in \mathfrak{k}$, *then* ad X *is skew-symmetric with respect to* B_θ.
(ii) *If* $X \in \mathfrak{p}$, *then* ad X *is symmetric with respect to* B_θ.

<u>PROOF</u>. Let $X \in \mathfrak{k}$. Then for all $Y,Z \in \mathfrak{g}$:

$$B_\theta(\mathrm{ad}\,X(Y),Z) = B([Y,X],\theta Z) = B(Y,[X,\theta Z]) = B(Y,\theta[X,Z]) = -\,B_\theta(Y,\mathrm{ad}\,X(Z)).$$

The proof for $X \in \mathfrak{p}$ is similar. □

Now Proposition I.3.7 yields:

<u>LEMMA 2.2.</u> $\mathrm{Ad}_G(K)$ *is a compact subgroup of* $\mathrm{Ad}_G(G)$.

Let $O(n)$ denote the set of all $n \times n$ orthogonal matrices, and $S(n)$ the set of all $n \times n$ symmetric matrices. We shall now prove a *polar decomposition* for G, the prototype of which is given by:

<u>LEMMA 2.3.</u> *The mapping* $(O,S) \to Oe^S : O(n) \times S(n) \to GL(n,\mathbb{R})$ *is a homeomorphism.*

For a proof, cf. [Che, ch.I]. As a second step, we shall prove the polar decomposition for $\mathrm{Ad}(G) = \mathrm{Int}(\mathfrak{g})$.

<u>PROPOSITION 2.4.</u> *The mapping* $(k,X) \to k\exp(\mathrm{ad}\,X)$: $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k}) \times \mathfrak{p} \to \mathrm{Ad}(G)$ *is a homeomorphism.*

<u>PROOF</u>. Let $Z_1,\dots,Z_n$ be an orthonormal basis of $\mathfrak{g}$ with respect to B_θ. Let $Y,U,V \in \mathfrak{g}$. Then a straightforward calculation shows that $B_\theta(\mathrm{ad}\,Y(U),V) = -\,B_\theta(U,(\mathrm{ad}(\theta Y)(V))$, hence ${}^t(\mathrm{ad}\,Y) = -\mathrm{ad}(\theta Y)$. Thus $\psi \in \mathrm{Ad}(G) \Rightarrow {}^t\psi \in \mathrm{Ad}(G)$.

Now according to Lemma 2.3, each $X \in GL(n,\mathbb{R})$ can be written as $X = Oe^S$, where O is an $n \times n$ orthogonal matrix, and S is an $n \times n$ symmetric matrix. Let $\psi \in \mathrm{Ad}(G)$, then $\psi = Oe^S$. Thus $e^{2S} = {}^t\psi.\psi \in \mathrm{Ad}(G)$ by the above remark. Hence $e^{2S} \in \mathrm{Aut}(\mathfrak{g})$. Hence $e^{2tS} \in \mathrm{Aut}(\mathfrak{g})$ $\forall t \in \mathbb{R}$ (cf. proof of Prop. I.3.1), hence $S \in \mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Hence $S = \mathrm{ad}\, Y$ for some $Y \in \mathfrak{g}$. But also $\mathrm{ad}(\theta Y) = -{}^t(\mathrm{ad}\, Y) = -{}^tS = -S = -\mathrm{ad}\, Y$ (S being symmetric). Hence $\theta Y = -Y$, or $Y \in \mathfrak{p}$.

Thus $O \in \mathrm{Ad}(G) \cap O(\mathfrak{g}, B_\theta)$. This group has Lie algebra $\mathrm{ad}(\mathfrak{g}) \cap \mathfrak{o}(\mathfrak{g}, B_\theta) = \mathfrak{k}$. Now we claim that O is in the identity component $\mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$ of $\mathrm{Ad}(G) \cap O(\mathfrak{g}, B_\theta)$. To see this, let $t \to \psi(t)$ be a continuous curve in $\mathrm{Ad}(G)$ such that $\psi(0) = e$, $\psi(1) = \psi$. Hence, by Lemma 2.3 and the preceding remarks, $\psi(t) = O(t)e^{\mathrm{ad}\, Y(t)}$, where $t \to O(t)$ is a continuous curve in $\mathrm{Ad}(G) \cap O(\mathfrak{g}, B_\theta)$ joining e and O. Hence $O \in \mathrm{Int}_{\mathfrak{g}}(\mathfrak{k})$.

Thus the map $\mathrm{Ad}(K) \times \mathfrak{p} \to \mathrm{Ad}(G)$, defined by $(\phi, X) \to \phi e^{\mathrm{ad}\, X}$ is a surjective homeomorphism. □

The (simple) proof of the following lemma is left to the reader.

<u>LEMMA 2.5</u>. *Let* G *be a connected Lie group,* K *a closed Lie subgroup of* G. *If* $^{G}/_{K}$ *is simply connected, then* K *is connected.*

<u>THEOREM 2.6</u>. *Let* G *be a connected semisimple Lie group with Lie algebra* $\mathfrak{g}$. *Let* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ *be a Cartan decomposition of* $\mathfrak{g}$. *Let* K *be the analytic subgroup of* G *corresponding to* $\mathfrak{k}$. *Let* Z *denote the center of* G. *Then*:

(i) $Z \subset K$.

(ii) *The map* $K \times \mathfrak{p} \to G$, *given by* $(k, X) \to k \exp X$ *is a homeomorphism of* $K \times \mathfrak{p}$ *onto* G. *Moreover,* K *is closed.*

<u>PROOF</u>. It follows from the proof of Proposition 2.4 that the map $\mathfrak{p} \to \mathrm{Ad}(G)$ given by $X \to e^{\mathrm{ad}\, X} = \mathrm{Ad}(\exp X)$ is a homeomorphism into, hence the map $\mathfrak{p} \to G$ given by $X \to \exp X$ is a homeomorphism into. Now, let $H := \mathrm{Ad}^{-1}(\mathrm{Ad}(K))$. Then H is closed, $Z \subset H$ and $H = KZ$. From Proposition 2.4 it follows that $\mathrm{Ad}(G)/\mathrm{Ad}(K)$ is homeomorphic with $\mathfrak{p}$. But also G/H is homeomorphic with $\mathrm{Ad}(G)/\mathrm{Ad}(K)$, hence G/H is simply connected. Thus, by Lemma 2.5, H is connected. H has also Lie algebra $\mathfrak{k}$, hence $H = K$. Since Z is discrete ($\mathfrak{g}$ being semisimple, $\ker(\mathrm{ad}) = 0$, and ad is the differential of Ad, hence $\ker(\mathrm{Ad}) = Z$ is discrete), it follows that $Z \subset K$. This proves (i).

To prove (ii), let $g \in G$. Then we have seen that $\mathrm{Ad}(g) = \mathrm{Ad}(k)e^{\mathrm{ad}\, X}$ with $k \in K$, $X \in \mathfrak{p}$. Hence $g = zk \exp(X)$ with $z \in Z$. Since $z \in K$ (by (i)) it follows that the map $K \times \mathfrak{p} \to G$ is injective. To prove that it is also surjective, assume that $k_1 \exp X_1 = k_2 \exp X_2$ ($k_i \in K$, $X_i \in \mathfrak{p}$). Then $\mathrm{Ad}(k_1)e^{\mathrm{ad}\, X_1} = \mathrm{Ad}(k_2)e^{\mathrm{ad}\, X_2}$. But we have proved in Proposition 2.4 that the mapping $\mathrm{Ad}(K) \times \mathfrak{p} \to \mathrm{Ad}(G)$ is a surjective homeomorphism, hence $X_1 = X_2$, and thus also $k_1 = k_2$. □

REMARK 2.7. In the preceeding lemmas, etc., the word "homeomorphism" can be replaced by "diffeomorphism". However, this would make the proofs more complicated, especially 2.3, cf. [Wal, Prop. 3.2.10].

COROLLARY 2.8. (*Cartan decomposition*). $G = KAK$.

PROOF. Let $g \in G$. Then it follows from Theorem 2.6 that g can be written as

$$(2.1) \qquad g = k \exp X, \quad k \in K,\ X \in \mathfrak{p}.$$

But it follows from expression (I.4.2) that X can be written as

$$(2.2) \qquad X = \mathrm{Ad}(k_1).H, \quad k_1 \in K,\ H \in \mathfrak{a}.$$

Combination of (2.1) and (2.2) yields $g = kk_1 \exp Hk_1^{-1}$ which proves the corollary. □

3. MAXIMAL COMPACT SUBGROUPS

Let notations be as in §2. That is, $\mathfrak{g}$ is a noncompact semisimple Lie algebra over $\mathbb{R}$, θ is a Cartan involution of $\mathfrak{g}$. Then, as we have remarked in §1, $(\mathfrak{g},\theta)$ is an orthogonal symmetric Lie algebra of the noncompact type. Let (G,K) be a pair associated to $(\mathfrak{g},\theta)$. In this section we shall prove that K is connected, (G,K) is a Riemannian symmetric pair and all maximal compact subgroups of G are conjugate under inner automorphisms of G.

THEOREM 3.1. (i) *K is connected and closed.*
(ii) *K is compact iff Z is finite.*

PROOF. It was shown in Theorem 2.6 that K_0 is closed (K_0 denoting the identity component of K), and if we define $\phi: K \times \mathfrak{p} \to G$ by $\phi(k,X) := k \exp X$ ($k \in K$, $X \in \mathfrak{p}$), then $\phi(K_0 \times \mathfrak{p}) = G$.

Now suppose there exist $k_1,k_2 \in K$, $X_1,X_2 \in \mathfrak{p}$ such that $k_1\exp X_1 = k_2\exp X_2$. Then, by the fact that $k \in K$ implies that $\mathrm{Ad}(k) \in O(\mathfrak{g},B_\theta)$, and $\mathrm{Ad}(k)$ commutes with θ (because $\mathrm{Ad}(k)\mathfrak{k} \subset \mathfrak{k}$, and thus $\mathrm{Ad}(k)\mathfrak{p} \subset \mathfrak{p}$), we can show that $k_1 = k_2$, $X_1 = X_2$, that is ϕ is 1-1. Since $\phi(K_0\times\mathfrak{p}) = \phi(K\times\mathfrak{p})$ by the above remark, this implies that $K_0 = K$. Hence K is connected and closed.

Let $K^* := K/Z$. Then K^* is compact (since k is a compactly imbedded subalgebra by Definition 1.1, this follows from Remark I.2.6) and Z is discrete. Hence K is compact iff Z is finite. □

<u>COROLLARY 3.2</u>. *Suppose* Z *is finite. Then* K *is maximal compact.*

<u>PROOF</u>. Let K_1 be a compact subgroup of G such that $K \subset K_1 \subset G$. Let k_1 denote the Lie algebra of K_1. Then k_1 is a compactly imbedded subalgebra. But k is a maximal compactly imbedded subalgebra of g. (This follows from the fact that k is compactly imbedded, $B|_{k\times k}$ is negative definite and $B|_{p\times p}$ is positive definite, cf. §I.3.) Hence $k = k_1$. But then by Theorem 3.1 we get $K = K_1$. □

<u>THEOREM 3.3</u>. *There exists an involutive, analytic automorphism* $\tilde{\theta}$ *of* G *such that* $G_{\tilde{\theta}} = K$ and $(d\tilde{\theta})_e = \theta$.

This makes (G,K) *a Riemannian symmetric pair.*

<u>PROOF</u>. θ is an automorphism of g. Let $G^\sim$ be the universal covering group of G. Then θ extends to $G^\sim$. Calling this extension Θ, then $d\Theta = \theta$. By Theorem 2.6 we have $Z^\sim \subset (G^\sim)_\Theta$. Let $\pi^\sim : G^\sim \to G$ denote the covering map, then $\ker(\pi^\sim) \subset Z^\sim$. Θ induces an automorphism of $G = G^\sim/\ker(\pi^\sim)$, say $\tilde{\theta}$, such that $\pi^\sim((G^\sim)_\Theta) = G_{\tilde{\theta}}$. Now it follows from Theorem 2.6 and Theorem 3.1 that $G_{\tilde{\theta}} = K$, and thus is (G,K) a Riemannian symmetric pair. □

Now we have to make use of the following lemma, a proof of which can be found in [Mos]. For about thirty years, the only known proof was Cartan's original proof, which uses quite deep results of Riemannian manifolds of negative curvature (cf. [He]). In Mostow's exposure, some of Cartan's results are proved without using any differential geometry.

<u>LEMMA 3.4</u>. ([Mos,§3]). *Let* (G,K) *be a Riemannian symmetric pair of the noncompact type. Let* K_1 *be a compact subgroup of* G. *Then* K_1 *acting on* G/K *has a fixed point.*

<u>COROLLARY 3.5</u>. *Conditions as in Lemma 3.4. Then* $\exists g \in G$ *such that* $g^{-1}K_1 g \subset K$.

<u>PROOF</u>. $g^{-1}K_1 g \subset K \Leftrightarrow gK$ is a fixed point of G/K under the action of K_1. □

<u>THEOREM 3.6</u>. *Let* (G,K) *be a Riemannian symmetric pair of the noncompact type. Then* K *has a unique maximal subgroup which is also maximal compact in* G.

<u>PROOF</u>. $Ad_G(K)$ is by definition compact, and its Lie algebra is k. Hence k is a compact lie algebra. We can write $k = s + a$, with s semisimple and a

abelian. Let S and A be analytic subgroups of K with Lie algebras $\mathfrak{s}$ and $\mathfrak{a}$, respectively. Then S is compact, by Remark I.2.6 ($\mathfrak{s}$ being a compact Lie algebra) and A is abelian. Hence we can write $A = T \times V$ (direct product), where T is a torus and V is analytically isomorphic to a euclidean space. Put $K' := ST$. Then K' is compact, since S and T are both compact. But K' and V commute, and $K' \cap V$ is a compact subgroup of V. Since V is euclidean this implies that $K' \cap U = \{e\}$. Thus $K' \times V$ is an analytic subgroup of K with Lie algebra $\mathfrak{k}$. Hence $K = K' \times V$ (direct product). Hence K' is a unique maximal subgroup of K. The second assertion follows from Corollary 3.5. □

Combination of Corollary 3.5 and Theorem 3.6 yields:

COROLLARY 3.7. *All maximal compact subgroups of a connected semisimple Lie group* G *are connected and conjugate under an inner automorphism of* G.

4. THE IWASAWA DECOMPOSITION OF A SIMISIMPLE LIE GROUP

In this subsection we shall give a global analogue of the Iwasawa decomposition of a semisimple Lie algebra as given in (I.4.10). Thus, as in (I.4.10), let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of the semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{R}$. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let K,A and N be analytic subgroups of G with Lie algebras $\mathfrak{k}$,$\mathfrak{a}$ and $\mathfrak{n}$, respectively. Then we shall prove that the mapping:

$$(4.1) \qquad \Phi : (k,a,n) \to kan \quad (k\in K, a\in A, n\in N)$$

is an analytic diffeomorphism of the product manifold $K \times A \times N$ onto G.

The decomposition G = KAN given by (4.1) is called an *Iwasawa decomposition of* G.

LEMMA 4.1. *Suppose* $\mathfrak{g} = \mathfrak{h}_1 + \mathfrak{h}_2 + \mathfrak{h}_3$ *(direct sum of vector spaces), with* $\mathfrak{h}_i$ *subalgebra of* $\mathfrak{g}$. *Let* H_i *be an analytic subgroup of* G *with Lie algebra* $\mathfrak{h}_i$, *and suppose that* $[\mathfrak{h}_2,\mathfrak{h}_3] \subset \mathfrak{h}_3$. *Define* $\alpha: H_1 \times H_2 \times H_3 \to G$ *by* $\alpha(h_1,h_2,h_3) := $ $:= h_1h_2h_3$. *Then* α *is everywhere regular.*

PROOF. Identify the tangent space $(H_1 \times H_2 \times H_3)_{(h_1,h_2,h_3)}$ with $H_{h_1} \oplus H_{h_2} \oplus H_{h_3}$. Take $X_i \in \mathfrak{h}_i$.

Then $\alpha(h_1 \exp tX_1, h_2, h_3) = h_1h_2h_3 \exp(t\mathrm{Ad}(h_3^{-1})\mathrm{Ad}(h_2^{-1})X_1)$,

$\alpha(h_1, h_2 \exp tX_2, h_3) = h_1h_2h_3 \exp(t\mathrm{Ad}(h_3^{-1})X_2)$,

$\alpha(h_1, h_2, h_3 \exp tX_3) = h_1h_2h_3 \exp(tX_3)$.

Thus:

$$d\alpha_{(h_1,h_2,h_3)}(dL_{h_1}X_1, dL_{h_2}X_2, dL_{h_3}X_3) = dL_{h_1h_2h_3}(\mathrm{Ad}(h_3^{-1})\mathrm{Ad}(h_2^{-1})X_1 + \mathrm{Ad}(h_3^{-1})X_2 + X_3).$$

(L_ρ denoting left translation by $\rho \in G$), since the tangent vector to the curve $x \exp tX$ at x is $dL_x(X)$. Now suppose $\mathrm{Ad}(h_3^{-1})\mathrm{Ad}(h_2^{-1})X_1 + \mathrm{Ad}(h_3^{-1})X_2 + X_3 = 0$. Then $X_1 + \mathrm{Ad}(h_2)X_2 + \mathrm{Ad}(h_2)\mathrm{Ad}(h_3)X_3 = 0$. But $X_1 \in h_1$, $\mathrm{Ad}(h_2)X_2 \in h_2$ and $\mathrm{Ad}(h_2)\mathrm{Ad}(h_3)X_3 \in h_3$ since $[h_2, h_3] \subset h_3$. Hence $X_1 = 0$, $X_2 = 0$, $X_3 = 0$. □

This lemma implies:

<u>COROLLARY 4.2</u>. *Let* g, h_i, H_i *and* α *be as in Lemma* 4.1. *If, in addition,* $\alpha(H_1 \times H_2 \times H_3)$ *is closed in* G *and* α *is injective, then* α *is a surjective diffeomorphism.*

The next lemma is taken from [Wal, Lemma 7.4.2]; it gives the prototype of the Iwasawa decomposition for $GL(n,\mathbb{R})$:

<u>LEMMA 4.3</u>. *Let* $O(p)$ *be the group of all* $p \times p$ *orthogonal matrices. Let* $D(p)$ *be the group of all* $p \times p$ *real diagonal matrices with positive entries on the diagonal. Let* $N(p)$ *be the group of all* $p \times p$ *upper triangular matrices with* 1's *on the diagonal. Then the map*

$$(k,a,n) \to kan: O(p) \times D(p) \times N(p) \to GL(p,\mathbb{R})$$

is a surjective homeomorphism.

<u>PROOF</u>. Suppose there exist $k,k' \in O(p)$, $a,a' \in D(p)$ and $n,n' \in N(p)$ such that $kan = k'a'n'$. Then $(k')^{-1}k = a'n'n^{-1}a^{-1}$ is on the one hand upper triangular with positive diagonal entries, but on the other hand an orthogonal matrix. Hence it is the identity, hence $k = k'$. Thus $an = a'n'$. This implies that $(a')^{-1}a = n'n^{-1}$. Now $n'n^{-1}$ is upper triangular with 1's on the diagonal, thus $a' = a$. Thus $n' = n$.

Now we shall construct a continuous map of $GL(p,\mathbb{R})$ to $O(p) \times D(p) \times$

$\times N(p)$. This map gives the inverse of the map $(k,a,n) \to kan$, and this proves the lemma. Let $g \in GL(p,\mathbb{R})$. Let $e_1,\ldots,e_p$ be the standard basis of $\mathbb{R}^p$. Let $v_i := g^{-1}e_i$. Apply the Gram-Schmidt orthogonalization process to the basis $v_1,\ldots,v_p$. This gives a basis $u_1,\ldots,u_p$ such that $u_i = \Sigma_{j\le i}\, a_{ji}v_j$ with $a_{ii} > 0$ for all i. Define $a(g) \in D(p)$ and $n(g) \in N(p)$ by

$$a(g) := \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{pp} \end{pmatrix},$$

$$n(g) := a(g)^{-1} \begin{pmatrix} a_{11} & \cdots & a_{1p} \\ & \ddots & \vdots \\ 0 & & a_{pp} \end{pmatrix}.$$

Then $a(g)n(g)v_i = u_i$. Let $k(g)$ be defined by $k(g)u_i = e_i$. Then $k(g) \in O(p)$. Hence $k(g)a(g)n(g)g^{-1}e_i = e_i$ for all i. Hence $g = k(g)a(g)n(g)$, and since $g \to k(g)$, $g \to a(g)$, $g \to n(g)$ are all clearly continuous, the lemma follows. □

Now, identify $ad(\mathfrak{g})$ and $\mathfrak{g}$. Let G_0 denote the adjoint group of G. We shall prove the Iwasawa decomposition for G_0 first. Let K_0,A_0,N_0 be the analytic subgroups of G_0 corresponding to $\mathfrak{k},\mathfrak{a},\mathfrak{n}$, respectively. Since the elements of G_0 are endomorphisms, we can represent them by a matrix once we have chosen a basis for $\mathfrak{g}$. Hence, choose an orthonormal basis relative to B_θ of $\mathfrak{a},\mathfrak{m}$ and $\mathfrak{g}_\alpha$ (for all $\alpha \in \Sigma$). Write this as an ordered orthonormal basis of $\mathfrak{g}$, compatible to the chosen linear ordering on $\mathfrak{a}^*$. Then, with respect to this basis, K_0 consists of orthogonal matrices, A_0 consists of diagonal matrices with positive diagonal entries and N_0 consists of upper triangular matrices with 1's on the diagonal. That is, $K_0 \subset O(p)$, $A_0 \subset D(p)$ and $N_0 \subset N(p)$, where $\dim \mathfrak{g} = p$.

<u>LEMMA 4.4</u>. *The mapping* $\psi\colon K_0 \times A_0 \times N_0 \to G_0$ *given by* $\psi(k,a,n) := kan$ *is an analytic diffeomorphism.*

<u>PROOF</u>. ψ is clearly injective. Since K_0 is compact, it is closed in $O(p)$. A_0 is closed in $D(p)$. Since G_0 and $N(p)$ are both closed in $GL(p,\mathbb{R})$, $G_0 \cap N(p)$ is closed in $N(p)$. $G_0 \cap N(p)$ has Lie algebra $ad(\mathfrak{g}) \cap \mathfrak{n}(p) = ad(\mathfrak{n})$ ($\mathfrak{n}(p)$ being the lie algebra of $N(p)$). Hence N_0 is the identity component of $G_0 \cap N(p)$, hence closed in $N(p)$. Lemma 4.3 implies that $K_0A_0N_0$ is closed in

$GL(p,\mathbb{R})$. The lemma now follows from Corollary 4.2. □

The next theorem gives us the so-called *Iwasawa decomposition*. In fact, it states that the mapping Φ in (4.1) is an analytic diffeomorphism.

THEOREM 4.5. $G = KAN$.

PROOF. First we prove that N_0 is simply connected. By taking inverses we get from Lemma 4.4: $G_0 = N_0A_0K_0$, hence $(n,a) \to naK_0 : N_0 \times A_0 \to G_0/K_0$ is a homeomorphism. But the mapping $X \to \exp(\mathrm{ad}\, X) : \mathfrak{p} \to G_0/K_0$ is also a homeomorphism. Hence $N_0 \times A_0$ is homeomorphic with $\mathfrak{p}$. Since $\mathfrak{p}$ is simply connected, it follows that N_0 is simply connected (NB. It is in general true that a connected nilpotent linear Lie group is simply connected, cf. [He, §VI.4]). Now $\mathrm{Ad}: G \to G_0$ and $\mathrm{Ad}(K) = K_0$, $\mathrm{Ad}(A) = A_0$, $\mathrm{Ad}(N) = N_0$, and $\ker(\mathrm{Ad}) \subset Z(G) \subset K$ (Theorem 2.6). Hence $G_0 = K_0A_0N_0$, thus $G = \ker(\mathrm{Ad})KAN = KAN$. Suppose $k_1a_1n_1 = k_2a_2n_2$. Applying Ad and using Lemma 4.4 this implies that $\mathrm{Ad}(k_1) = \mathrm{Ad}(k_2)$, $\mathrm{Ad}(a_1) = \mathrm{Ad}(a_2)$, $\mathrm{Ad}(n_1) = \mathrm{Ad}(n_2)$. But A_0 and N_0 are both simply connected, hence $a_1 = a_2$, $n_1 = n_2$. Thus $k_1 = k_2$. The theorem now follows from Corollary 4.2. □

5. THE WEYL GROUP OF Σ

Let, just as in §I.5, Σ be the set of restricted roots, that is, the roots of the pair $(\mathfrak{g},\mathfrak{a})$.

DEFINITION 5.1. Let M^* be the normalizer of $\mathfrak{a}$ in K, and let M be the centralizer of $\mathfrak{a}$ in K.
That is: $M^* = \{k \in K : \mathrm{Ad}(k)\mathfrak{a} \subset \mathfrak{a}\}$,
$M = \{k \in K : \mathrm{Ad}(k)H = H \text{ for all } H \in \mathfrak{a}\}$.
Let $\mathfrak{m}$ denote the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$.

LEMMA 5.2. M *and* M^* *have the same Lie algebra, namely* $\mathfrak{m}$. *Moreover, the group* M^*/M *is finite.*

PROOF. The Lie algebra of M is obviously $\mathfrak{m}$. Let $\mathfrak{m}^*$ be the Lie algebra of M^*. Let $X \in \mathfrak{m}^*$. Thus, for $H \in \mathfrak{a}$, $[X,H] \in \mathfrak{a}$. Hence $B(\mathrm{ad}(H)X,\mathrm{ad}(H)X) = -B(\mathrm{ad}(H)^2X,X) = 0$, $\mathfrak{a}$ being abelian. Hence, since B is positive definite on $\mathfrak{p}$, $[X,H] = 0$ for all $H \in \mathfrak{a}$. Hence $X \in \mathfrak{m}$, hence $\mathfrak{m} = \mathfrak{m}^*$.

Let $G_0 := \mathrm{Int}(\mathfrak{g})$, with subgroups K_0, M_0, M_0^* defined just as for G. Then $\mathrm{Ad}^{-1}(M_0) = M$, $\mathrm{Ad}^{-1}(M_0^*) = M^*$ and $\ker(\mathrm{Ad}) \subset M$. Hence $M^*/M = M_0^*/M_0$. But M_0^*/M_0 is compact (since K_0 is compact) and discrete, hence finite. □

REMARK 5.3. Since $m^* \to \mathrm{Ad}(m^*)$ $(m^* \in M^*)$ induces an isomorphism of M^*/M into $GL(\mathfrak{a})$, we can regard M^*/M as a group of linear transformation of $\mathfrak{a}$.

Let W be the Weyl group of Σ as defined in §I.5. We shall prove here that W is isomorphic with M^*/M. Remark that via the correspondences $\lambda \leftrightarrow A_\lambda$, $\mathfrak{a}^* \leftrightarrow \mathfrak{a}$ we can regard W as acting on $\mathfrak{a}$ instead of acting on $\mathfrak{a}^*$. We shall always consider W as acting on $\mathfrak{a}$. We know that W is generated by the reflections s_α $(\alpha \in \Sigma)$, with s_α defined by

$$(5.1) \qquad s_\alpha(H) := H - 2\,\frac{\alpha(H)}{\alpha(A_\alpha)}A_\alpha \qquad (H \in \mathfrak{a}).$$

Thus we have to prove that $s_\alpha \in M^*/M$ and that M^*/M is generated by the s_α.

LEMMA 5.4. *Let* $\alpha \in \Sigma$. *Then* $s_\alpha \in M^*/M$.

PROOF. Let X_α, $X_{-\alpha}$, H_α be as in Lemma I.5.1. Put $Y_\alpha := X_\alpha - \theta X_\alpha$, $Z_\alpha := X_\alpha + \theta X_\alpha$. Then $Y_\alpha \in \mathfrak{p}$, $Z_\alpha \in \mathfrak{k}$, and $[H_\alpha, Y_\alpha] = 2Z_\alpha$, $[H_\alpha, Z_\alpha] = 2Y_\alpha$, $[Z_\alpha, Y_\alpha] = 2H_\alpha$. Thus for all $t \in \mathbb{R}$:

$$\mathrm{Ad}(\exp tZ_\alpha)H_\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{(2t)^n}{(2n)!} A_\alpha + \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(2t)^{2n+1}}{(2n+1)!} Y_\alpha.$$

Hence

$$(5.2) \qquad \mathrm{Ad}(\exp \tfrac{1}{2}\pi Z_\alpha)H_\alpha = -H_\alpha.$$

Let $k_0 := \exp \frac{1}{2}\pi Z_\alpha$, then (5.2) becomes

$$(5.3) \qquad \mathrm{Ad}(k_0)H_\alpha = -H_\alpha.$$

But the hyperplane $\alpha(H) = 0$ is left pointwise fixed by $\mathrm{Ad}(k_0)$, because $\alpha(H) = 0$ implies $[Z_\alpha, H] = 0$. Hence s_α is the restriction to $\mathfrak{a}$ of $\mathrm{Ad}(k_0)$ (cf. [Hu, Lemma 9.1]). Since it is also clear that $\mathrm{Ad}(k_0)\mathfrak{a} \subset \mathfrak{a}$, the lemma follows. □

LEMMA 5.5. *Let* $k \in M^*$. *If* $\lambda \in \mathfrak{a}^*$, *put* $\lambda^k(H) := \lambda(\mathrm{Ad}(k^{-1})H)$ *for* $H \in \mathfrak{a}$. *Then* $\lambda \in \Sigma$ *iff* $\lambda^k \in \Sigma$.

PROOF. Let $X \in \mathfrak{g}_\alpha$. Then for all $H \in \mathfrak{a}$

$$[H,\mathrm{Ad}(k)X] = \mathrm{Ad}(k)[\mathrm{Ad}(k^{-1})H,X]$$
$$= \alpha(\mathrm{Ad}(k^{-1})H)\ \mathrm{Ad}(k)X$$
$$= \alpha^k(H)\ \mathrm{Ad}(k)X.$$

Hence

$$\mathrm{Ad}(k)\mathfrak{g}_\alpha = \mathfrak{g}_{\alpha^k}. \quad \square$$

Let, as usual, the connected components in $\mathfrak{a} - \{H \in \mathfrak{a}:\alpha(H) = 0$ for some $\alpha \in \Sigma\}$ be called *Weyl chambers*.

THEOREM 5.6. *Let* $s \in M^*/M$. *Then* s *permutes the Weyl chambers. Also* M^*/M *is simply transitive on the set of Weyl chambers in* $\mathfrak{a}$.

PROOF. Remark first that it is enough to show the theorem for $\mathrm{Int}(\mathfrak{g})$, since $\ker(\mathrm{Ad}) \subset M$. Let $k \in M^*$ be such that $s = \mathrm{Ad}(k)|_{\mathfrak{a}}$. Lemma 5.5 implies that if some root in Σ vanishes at a point $H \in \mathfrak{a}$, then a root in Σ vanishes at $\mathrm{Ad}(k)H$. Hence s permutes the Weyl chambers.

Now we know from [Hu, Theorem 10.3] that the subgroup of M^*/M generated by all s_α ($\alpha \in \Sigma$) is transitive on the Weyl chambers, hence M^*/M is transitive on the Weyl chambers.

Now suppose $s \in M^*/M$ such that $sC = C$ for a Weyl chamber C. Let N denote the order of s. Choose $H_0 \in C$, and set $H := \frac{1}{N}(H_0+sH_0+\ldots+s^{N-1}H_0)$. Then $sH = H$, and $H \in C$ since Weyl chambers are convex. Let $k_0 \in M^*$ be such that $s = \mathrm{Ad}(k_0)|_{\mathfrak{a}}$. Then $\mathrm{Ad}(k_0)tH = tH\ \forall t \in \mathbb{R}$.

$\mathrm{Ad}(k_0)$ acts as a complex linear transformation on $\mathfrak{g}_c$, and thus we can consider $\mathrm{Ad}(k_0)$ as an element of the analytic subgroup U_0 of $\mathrm{Int}((\mathfrak{g}_c)_R)$ corresponding to the Lie algebra $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ ($M^* \subset K$). U_0 is compact (§I.2). Let Z_H be the centralizer of iH in U_0. Then $\mathrm{Ad}(k_0) \in Z_H$. But Z_H is also the centralizer in U_0 of $\gamma := \{\exp(t\ \mathrm{ad}(iH))\,|\,t \in \mathbb{R}\} \subset U_0$. Let z_H be the Lie algebra of Z_H. Then we claim that $z_H = z_{\mathfrak{u}}(i\mathfrak{a})$:= centralizer of $i\mathfrak{a}$ in $\mathfrak{u}$. To see this, we only have to prove that if $X \in z_H$, then $X \in z_{\mathfrak{u}}(i\mathfrak{a})$. Let $X \in z_H$. Then $X = X_0 + \Sigma_{\alpha\in\Sigma}X_\alpha$, with $X_0 \in \mathfrak{m}_c + \mathfrak{a}_c$, $X_\alpha \in (\mathfrak{g}_\alpha)_c$.
Then:

$$0 = [H,X] = \sum_{\alpha\in\Sigma} \alpha(H)X_\alpha, \quad \alpha(H) \neq 0.$$

Hence $X_\alpha = 0$ for all $\alpha \in \Sigma$. Hence $X \in \mathfrak{m}_c + \mathfrak{a}_c$, hence $X \in z_{\mathfrak{u}}(i\mathfrak{a})$.

Now Z_H is the centralizer in the compact group U_0 of the toral subgroup $\mathcal{Cl}(\gamma)$ ($\mathcal{Cl}$ denoting closure), hence by [He, cor. VII.2.8] Z_H is connected.

Hence Z_H is generated by $\exp(z_H)$. Hence $\mathrm{Ad}(k_0) \in Z_H$ acts as the identity on $i\mathfrak{a}$, hence on $\mathfrak{a}$. Thus s is the identity, hence M^*/M is simply transitive. □

From Theorem 5.6 we obtain:

COROLLARY 5.7. M^*/M *is generated by the reflections* s_α $(\alpha \in \Sigma)$.

Now combination of Lemma 5.4 and Corollary 5.7 yields:

THEOREM 5.8. $W \simeq M^*/M$.

6. THE CARTAN DECOMPOSITION REVISITED

PROPOSITION 6.1. *Let* $\mathfrak{a}_0 \subset \mathfrak{a}$, *and let* $k \in K$ *be such that* $\mathrm{Ad}(k)\mathfrak{a}_0 \subset \mathfrak{a}$. *Then* $\exists s \in W$ *such that*

$$(6.1) \qquad s.A = \mathrm{Ad}(k)A \quad \forall A \in \mathfrak{a}_0.$$

PROOF. Let $H \in \mathrm{Ad}(k^{-1})\mathfrak{a}$ be such that $z_{\mathfrak{p}}(H) = \mathrm{Ad}(k^{-1})\mathfrak{a}$ (such elements exist, cf. Lemma I.4.1). Then $\exists z_0 \in K \cap Z_{\mathfrak{a}_0}$ such that $H \in \mathrm{Ad}(z_0^{-1})\mathfrak{a}$ (by Remark I.4.3). Hence $\mathrm{Ad}(k^{-1})\mathfrak{a} = \mathrm{Ad}(z_0^{-1})\mathfrak{a}$, hence $kz_0^{-1} \in M^*$. But since $z_0 \in K \cap Z_{\mathfrak{a}_0}$, $s := \mathrm{Ad}(kz_0^{-1})|_{\mathfrak{a}}$ satisfies (6.1). □

The Cartan decomposition $G = KAK$ as given in §2 has the following disadvantage: it is not a unique decomposition (note that only the A part can be unique). A refinement of Corollary 2.8 in which the A part indeed is unique is established here in Theorem 6.2.

Recall that the connected components of $\mathfrak{a} - \{H \in \mathfrak{a}\colon \alpha(H) = 0$ for some $\alpha \in \Sigma\}$ are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+$ (called the *positive Weyl chamber*), and call a root $\alpha \in \Sigma$ a *positive root* if $\alpha(H) > 0$ for all $H \in \mathfrak{a}^+$. Let $A^+ := \exp(\mathfrak{a}^+)$. Now it is clear from Corollary 2.8 and Theorem 5.6 that each $g \in G$ can be written as $g = k_1 a k_2$, with $k_i \in K$ and $a \in C\ell(A^+)$. We claim that the $a \in C\ell(A^+)$ in the above decomposition is unique. To see this, suppose that $k_1 \exp H_1 k_1' = k_2 \exp H_2 k_2'$ with $k_i, k_i' \in K$, $H_i \in \mathfrak{a}^+$. Thus for some $k, k' \in K$ we have $k \exp H_1 k' = \exp H_2$. By applying the Cartan involution θ we get $k \exp(-H_1)k' = \exp(-H_2)$. Hence $\exp(\mathrm{Ad}(k)(-2H_1)) = \exp(-2H_2)$. Thus $\mathrm{Ad}(k)H_1 = H_2$, since $\mathrm{Ad}(k)H_1 \in \mathfrak{p}$ and exp is 1-1 on $\mathfrak{p}$. But Proposition 6.1 states that there then exists $s \in W$ such that $sH_1 = H_2$. Now [He, Th.VII.2.22] states that the only element of the Weyl group which sends an element of a Weyl chamber to another element of the same Weyl chamber is the identity. Hence $H_1 = H_2$. Thus we have proved:

THEOREM 6.2. $G = KC\ell(A^+)K$, *that is, each* $g \in G$ *can be written as* $g = k_1 a k_2$ *with* $k_i \in K$ *and* $a \in C\ell(A^+)$. *In this decomposition* a *is unique.*

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Chapter III

SEMISIMPLE LIE GROUPS FROM A GEOMETRIC VIEWPOINT

A.M. COHEN

In Chapter II the Cartan decomposition and the Iwasawa decomposition for a connected real semisimple Lie group G have been discussed. In this chapter we shall deal with yet another decomposition, named after Bruhat, and some of its geometric consequences.

More specifically, we shall discuss the topic of 'B,N-pairs'. The prosaic name of this topic is connected with two particular subgroups of G, usually denoted by N and B. However, in the present text B stands for the Killing form and N is a nilpotent subgroup not contained in M^*A, the (biggest possible) group N of the B,N-pair in G. Therefore, we shall from now on refer to a B,N-pair as a Tits system, thereby displaying the name of its founder (see TITS [T1] and the references therein). The groups B_0, N_0 that appear in the sequel will play the rôles of B,N in Tits' theory.

The geometries of our interest will be based on the cosets with respect to a maximal subgroup of G containing B_0.

1. THE BRUHAT DECOMPOSITION OF A REAL NONCOMPACT CONNECTED SEMI-SIMPLE LIE GROUP.

In this section, G is a real noncompact connected semi-simple Lie group with finite center. $\theta, \mathfrak{g}, \mathfrak{k}, \mathfrak{p}, \mathfrak{a}, \mathfrak{n}, \mathfrak{m}, K, A, \Sigma, N, M, M^*, W$ are as defined in Ch. I and II. The material of this section is taken from HELGASON [He], STEINBERG [S] and WARNER [W].

1.1. Since Ad(m) for $m \in M$ fixes $\mathfrak{a}$ pointwise, it stabilizes each $\mathfrak{g}_\alpha$ ($\alpha \in \Sigma$). Therefore, M centralizes A and normalizes N. Similarly, M^* normalizes A. Thus $B_0 = MAN$ and $N_0 = M^*A$ are subgroups of G. Moreover, B_0 and N_0 are closed in G and have Lie algebras $\mathfrak{b}_0 = \mathfrak{m} + \mathfrak{a} + \mathfrak{n} = \mathfrak{g}_0 + \mathfrak{n}$ and $\mathfrak{m} + \mathfrak{a}$ respectively. Write $H_0 = B_0 \cap N_0$.

LEMMA (i) B_0 *is semidirect product of* M *and (normal)* AN, *but also of* MA *and (normal)* N.

(ii) N *is nilpotent and simply connected, and consists solely of unipotent elements.*

(iii) A *is abelian.*

(iv) $H_0 = MA$ *is normal in* M^*A *and* $W \cong M^*A/MA$.

PROOF. (i), (ii), (iii) Straightforward in view of the previous chapter. (iv) If $x \in H_0 = B_0 \cap N_0$, then there are $m \in M$, $m' \in M^*$, $a,a' \in A$ and $n \in N$ with $x = man = m'a'$ (according to the definition of B_0 and N_0). Thus $(m')^{-1}man = a'$, and $n = 1$ by uniqueness of the Iwasawa decomposition. This yields $x = ma \in MA$, whence $H_0 \subseteq MA$. The converse is trivial.

Finally, $W \cong M^*/M \cong M^*A/MA$ owing to Theorem II.5.8 and a standard isomorphism lemma. □

1.2. From now on, we shall identify the two groups W and M^*/M. For $w \in W$, let m_w be an element of M^* that maps onto w under the natural projection $M^* \to W$ (in other words, such that $w = m_wM$). We are mainly interested in cosets of B_0 in G indexed by elements of M^*. For $w \in W$, the left coset m_wB_0, the right coset B_0m_w and the double coset $B_0m_wB_0$ do not depend on the particular choice of m_w, as $M \subseteq B_0$. Thus there is no harm in writing wB_0, B_0w and B_0wB_0 respectively, and we shall frequently do so. In fact, this remark applies to cosets of any subgroup of G containing M. Also for X a subset of W, we write B_0XB_0 to denote $\bigcup_{x\in X} B_0xB_0$, and so on.

THEOREM (BRUHAT'S LEMMA). *Let* G *be a real noncompact connected semi-simple Lie group with finite center. Retain the notations of above. Then* $G = B_0WB_0$ *and, for* $w,w' \in W$, *the equality* $B_0wB_0 = B_0w'B_0$ *is equivalent to* $w = w'$.

This result is the socalled *Bruhat decomposition.* It states that G consists of double cosets of B_0 indexed by N_0. Thus knowing B_0, N_0 and their interaction under multiplication determines knowledge of G. For complex semi-simple Lie groups, such a decomposition also exists.

Before going into the consequences of this celebrated 'lemma' we shall present Harish-Chandra's proof in five steps, four of which are lemmas. The first of these is a general lemma about nilpotent Lie groups.

1.3. LEMMA. *Let* V *be a connected Lie group with nilpotent Lie algebra* $\mathfrak{v}$.
(i) *The map* $\exp\colon \mathfrak{v} \to V$ *is regular and surjective.*

(ii) *If* V *is simply connected, then* exp: $\mathcal{V} \to V$ *is a diffeomorphism.*

(iii) *If* U *is an analytic subgroup of* V *and* V *is simply connected, then* U *is simply connected and closed in* V.

PROOF. (i) We proceed by induction on dim V. If $\mathcal{V}$ is abelian, then statement (i) is well known.

Suppose V is nonabelian (hence, dim V > 1). Let $\mathcal{C}$ be the center of $\mathcal{V}$, and write $C = \exp \mathcal{C}$. Then C is the connected component of unity of the center of V, so C is a closed connected normal subgroup of V. Therefore, V/C is a connected Lie group with Lie algebra $\mathcal{V}_1 = \mathcal{V}/\mathcal{C}$. As $\dim \mathcal{V}/\mathcal{C} > 0$ and $\dim \mathcal{C} > 0$, the induction hypothesis yields that exp: $\mathcal{V}_1 \to V/C$ and exp: $\mathcal{C} \to C$ are regular and surjective maps. Thus, if $v \in V$, there are $X \in \mathcal{V}$ and $c \in C$ such that $v = (\exp X)c$, and there is $Y \in \mathcal{C}$ with $\exp Y = c$, so that $v = (\exp X)(\exp Y) = \exp(X + Y)$. This shows that $\exp: \mathcal{U} \to U$ is surjective.

In order to establish regularity, suppose that for $X, Y \in \mathcal{V}$ we have $d\exp_X(Y) = 0$. Then $Y \in \mathcal{C}$ as exp: $\mathcal{V}_1 \to \mathcal{V}/\mathcal{C}$ is regular. But $\exp(X + tY) = (\exp X)(\exp tY)$, so $d\exp_X(Y) = 0$ implies $Y = 0$, proving regularity of exp.

(ii) Again, we use induction on dim V, the abelian case being well known.

If V is simply connected, then so is V/C. Therefore, the induction hypothesis yields that exp: $\mathcal{V}_1 \to V/C$ is injective. Let $\mathcal{U}$ be a linear subspace of $\mathcal{V}$ complementary to $\mathcal{C}$. If $\exp X_1 \exp Y_1 = \exp X_2 \exp Y_2$ for $X_1, X_2 \in \mathcal{U}$ and $Y_1, Y_2 \in \mathcal{C}$, then $X_1 - X_2 \in \mathcal{C}$ by the injectivity just established. But also $X_1 - X_2 \in \mathcal{U}$, so that $X_1 = X_2$. This shows that the map $(u_1, u_2) \to u_1 u_2$ from $(\exp \mathcal{U}) \times C$ to V is a diffeomorphism, so that C is simply connected. By the induction hypothesis, exp: $\mathcal{C} \to C$ is injective. Now suppose that $\exp Z_1 = \exp Z_2$ for $Z_1, Z_2 \in \mathcal{V}$. There are $X_1, X_2 \in \mathcal{U}$ and $Y_1, Y_2 \in \mathcal{C}$ such that $Z_i = X_i + Y_i$ $(i=1,2)$. Thus $\exp X_1 \exp Y_1 = \exp X_2 \exp Y_2$, and, reasoning as above we get that $X_1 = X_2$. Injectivity of exp: $\mathcal{C} \to C$ yields that $Y_1 = Y_2$, hence that $Z_1 = Z_2$, so that exp: $\mathcal{V} \to V$ is injective.

Being bijective and regular, this map must be a diffeomorphism.

(iii) This is immediate from (ii). □

1.4. Since N is a nilpotent simply connected Lie group with Lie algebra $\mathcal{n}$, we know that N and $\mathcal{n}$ are diffeomorphic by means of exp from what we have just seen. Here is a collection of diffeomorphisms in the other direction.

LEMMA. *Take* $H \in \mathcal{a}$ *with* $\alpha(H) \neq 0$ *for all* $\alpha \in \Sigma$. *Denote* $\phi: N \to \mathcal{n}$ *the map given by*

$$\phi(n) = (\mathrm{Ad}(n)-I)H \qquad (n \in N).$$

Then ϕ is a diffeomorphism of N onto n.

PROOF. Let $n \in N$. There is $X \in n$ with $n = \exp X$. Clearly, $\phi(n) = (\exp(\mathrm{ad}\, X)-I)H \in n$ as $[a,n] \subseteq n$. If $n^1 \in N$ with $\phi(n) = \phi(n^1)$, then $\mathrm{Ad}(n)\, H = \mathrm{Ad}(n^1)H$, so $\mathrm{Ad}(n^{-1}n^1)$ fixes H. Suppose $Y \in n$ satisfies $n^{-1}n^1 = \exp Y$. Then $\exp(\mathrm{ad}\, Y)\, H = H$. We claim that this implies $Y = 0$. Write $Y = \sum_{\beta\in\Sigma,\beta>0} Y_\beta$, where $Y_\beta \in g_\beta$.

If $Y \neq 0$, then there is $\alpha \in \sum^+$ with $Y = Y_\alpha + \sum_{\beta\in\Sigma,\beta>\alpha} Y_\beta$ and $Y_\alpha \neq 0$, so that $\exp(\mathrm{ad}Y)H \in H - \alpha(H)Y_\alpha + \oplus_{\beta\in\Sigma,\beta>\alpha} g_\beta$.

Thus $\exp(\mathrm{ad}Y)H \neq H$ as $\alpha(H) \neq 0$, contradiction.

Hence $Y = 0$, and $n^{-1}n^1 = 1$, so that $n = n^1$. We have shown that ϕ is injective.

Next we establish surjectivity. Suppose there is $Z \in n - \phi(N)$. Then $Z \neq 0$ as $\phi(1) = 0$. Write $Z = Z_\alpha + \sum_{\beta\in\Sigma,\beta>\alpha} Z_\beta$, where $Z_\alpha \neq 0$. As Σ has finite cardinality, we may choose Z such that $U = \sum_{\beta>\alpha,\beta\in\Sigma} U_\beta$ with $U_\beta \in g_\beta$ implies $U \in \phi(N)$.

Now the choice of H implies that ad H is nonsingular on n. Thus there is $Z_1 \in n\backslash\{0\}$ with $[H,Z_1] = Z$. Put $n_1 = \exp Z_1$ and consider $\mathrm{Ad}(n_1)(H+Z)-H$. This element is contained in $\oplus_{\beta\in\Sigma,\beta>\alpha} g_\beta$ and hence, by the 'minimality' assumption on Z, in $\phi(N)$. Thus, for some $n_2 \in N$, we have

$$(\mathrm{Ad}(n_2)-I)H = \mathrm{Ad}(n_1)(H+Z)-H,$$

which amounts to $\phi(n_1^{-1}n_2) = Z$. This yields surjectivity of ϕ.

It remains to derive that ϕ is regular at any point n of N. Since for free parameter t and $X \in n$, we have $\mu(n \exp tX) = \phi(n) + \mathrm{Ad}(n)(t[X,H]) + O(t^2)$ as $t \to 0$, we get

$$(d\phi)_n(dL_n(X)) = -\mathrm{Ad}(n)(\mathrm{ad}(H)X),$$

where L_n denotes left multiplication by n.

This shows that $(d\phi)_n(dL_nX) = 0$ implies $X = 0$, by nonsingularity of ad H. Hence, ϕ is regular at any $n \in N$. □

1.5. LEMMA

(i) $\quad a + n = \{X \in b_0 \,|\, \mathrm{ad}\, X$ has real eigenvalues only$\}$.

(ii) $\quad n = \{X \in b_0 \mid \text{ad } X \text{ is nilpotent}\}$.

(iii) $\quad a = \{X \in g_0 \mid \text{ad } X \text{ has real eigenvalues only}\}$.

PROOF. (i) Choose a basis for g as in the proof of Lemma II.4.3. Let $Z \in b_0$, $Z = T + H + X$ with $T \in m$, $H \in a$, $X \in n$. Each eigenvalue of ad(T+M) is an eigenvalue of ad Z. Hence, if ad Z has real eigenvalues only, then ad(T + H) has real eigenvalues only. But ad T and ad H are skew-symmetric and symmetric respectively (with respect to the chosen basis) and commute. Therefore ad T = 0, so T = 0. This proves (i).
(ii) is an easy consequence of (i).
(iii) is straightforward. □

1.6. LEMMA. *For each* $x \in G$, $b_0 = b_0 \cap \mathrm{Ad}(x)b_0 + n$.

PROOF. Since b_0 obviously contains the right hand side, we need only show that the dimensions of the spaces at both sides of the equality sign are the same.

Let $x \in G$. Then $x \in kB_0$ for some $k \in K$ by the Iwasawa decomposition. As $\mathrm{Ad}(x)b_0 = \mathrm{Ad}(k)b_0$, we may restrict to the case where $x \in K$.

Now $n \cap \mathrm{Ad}(x)b_0 = n \cap \mathrm{Ad}(x)n$ by Lemma 1.5, and $b_0^{\perp} = \theta n$, so

$$\begin{aligned}
&\dim(b_0 \cap \mathrm{Ad}(x)b_0 + n) = \dim g + \dim(b_0 \cap \mathrm{Ad}(x)b_0 + n) - \dim n \\
&- \dim b_0 = \dim g - \dim(n \cap \mathrm{Ad}(x)b_0) + \dim b_0 \cap \mathrm{Ad}(x)b_0 - \dim b_0 = \\
&= \dim g - \dim(n \cap \mathrm{Ad}(x)n) + \dim b_0 - \dim(b_0 + \mathrm{Ad}(x)b_0) = \\
&= \dim(b_0 + \mathrm{Ad}(x)b_0)^{\perp} - \dim(n \cap \mathrm{Ad}(x)n) + \dim b_0 = \\
&= \dim(\theta n \cap \mathrm{Ad}(x)\theta n) - \dim(n \cap \mathrm{Ad}(x)n) + \dim b_0 = \dim b_0,
\end{aligned}$$

where the last equality holds since $\mathrm{Ad}(x)\theta n = \theta\mathrm{Ad}(x)n$ as $x \in K$. The conclusion is $\dim(b_0 \cap \mathrm{Ad}(x)b_0 + n) = \dim b_0$, as wanted. □

1.7. PROOF OF THEOREM 1.2. Let $x \in G$. We shall first prove that $x \in B_0WB_0$. Pick $H \in a$ with $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. Then by Lemma 1.6, there is $X \in n$ with $H + X \in b_0 \cap \mathrm{Ad}(x)b_0$. In view of Lemma 1.4, $\mathrm{Ad}(n)H = H + X$ for some $n \in N$, so that $\mathrm{Ad}(x^{-1}n)H \in b_0$. But H, and hence $\mathrm{Ad}(x^{-1}n)H$ has real eigenvalues only, so $\mathrm{Ad}(x^{-1}n)H \in a + n$ by Lemma 1.5. Take $H_1 \in a$ and $X_1 \in n$ with $\mathrm{Ad}(x^{-1}n)H = H_1 + X_1$. Now the eigenvalues of ad H_1 and ad H coincide, so $\alpha(H_1) \neq 0$ for all $\alpha \in \Sigma$. Due to Lemma 1.4, there is $n_1 \in N$ with $\mathrm{Ad}(n_1)(H_1+X_1) = H_1$. It follows that $\mathrm{Ad}(n_1x^{-1}n)H = H_1$. Therefore $\mathrm{Ad}(n_1x^{-1}n)$

maps the centralizer in $\mathfrak{g}$ of H into the centralizer in $\mathfrak{g}$ of H_1. But these centralizers both coincide with $\mathfrak{g}_0 = \mathfrak{m} + \mathfrak{a}$ (by an easy argument). Hence $\mathrm{Ad}(n^{-1}xn_1^{-1})\mathfrak{g}_0 = \mathfrak{g}_0$. But then also $\mathrm{Ad}(n^{-1}xn_1^{-1})\mathfrak{a} = \mathfrak{a}$ owing to Lemma 1.5. Let $n^{-1}xn_1^{-1} = n_2ak$ with $k \in K$, $a \in A$, $n \in N$ be the (reverse) Iwasawa decomposition for $n^{-1}xn_1^{-1}$. Then $\mathrm{Ad}(k)\mathfrak{a} \subseteq \mathrm{Ad}(a^{-1}n_2^{-1}\mathfrak{a}) \subseteq \mathfrak{a} + \mathfrak{n}$, hence $\mathrm{Ad}(k)\mathfrak{a} \subseteq (\mathfrak{a}+\mathfrak{n}) \cap \cap\, \mathfrak{p} = \mathfrak{a}$, since $\mathrm{Ad}(k)$ commutes with θ. Thus $k \in M^*$ and n_2 normalizes $\mathfrak{a}$. It follows that $n_2 = 1$ and that $n^{-1}xn_1^{-1} \in AM^*$, so that $x \in nAM^*n_1 \subseteq NAM^*N \subseteq \subseteq B_0WB_0$. This proves $G = B_0WB_0$.

Suppose that for $m, m^1 \in M^*$, we have $B_0mB_0 = B_0m^1B_0$.

Then $m \in B_0m^1B_0 = NAMm^1MAN = Nm^1MAN$, so after replacement of m^1 by an element of m^1M, we may assume the existence of $a^1 \in A$ and $n, n^1 \in N$ with $nm = m^1a^1n^1$. Now for $H \in \mathfrak{a}$, we have $\mathrm{Ad}(nm)H \in \mathrm{Ad}(m)H + \mathfrak{n}$, and $\mathrm{Ad}(m^1a^1n^1)H \in \in \mathrm{Ad}(m^1)H + \mathfrak{g}_0^{\perp}$. Thus $\mathrm{Ad}(m^{-1}m^1)H = H$ for any $H \in \mathfrak{a}$, whence $m^{-1}m^1 \in M$ and $m^1M = mM$, as wanted. This ends the proof of Theorem 1.2. □

1.8. Bruhat's lemma has a refinement which we intend to discuss now. Set $\mathfrak{n}^- = \theta\mathfrak{n}$ $(= \oplus_{\alpha\in\Sigma^-}\mathfrak{g}_\alpha)$ and $N^- = \exp \mathfrak{n}^- = \theta N$.

<u>LEMMA</u>. $B_0 \cap N^- = \{1\}$.

<u>PROOF</u>. Let $x \in B_0 \cap N^-$. Then $x = man$ for certain $m \in M$, $a \in A$, $n \in N$ by Theorem II.4.5. Choose $H \in \mathfrak{a}$ with $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. By (a slight generalization of) Lemma 1.4 there are $X_1, X_2 \in \mathfrak{n}$, $Y_1 \in \mathfrak{n}^-$ such that $H + Y_1 = \mathrm{Ad}(x)H = \mathrm{Ad}(ma)(H+X_1) = H + X_2$. Hence $Y_1 = 0$. Consequently, $\mathrm{Ad}(x)$ fixes H, and by a reasoning as in the proof of Theorem 1.2, we get $x \in MA$. As $MA \cap N^- = \{1\}$, it follows that $x = 1$. □

1.9. For $w \in W$ (pick $m_w \in M^*$ with $w = m_wM$, as before and) set

$$\mathfrak{n}_w = \mathfrak{n} \cap \mathrm{Ad}(m_w)\mathfrak{n},$$
$$\mathfrak{u}_w = \mathfrak{n} \cap \mathrm{Ad}(m_w)\mathfrak{n}^- = \mathfrak{n} \cap \theta\,\mathrm{Ad}(m_w)\mathfrak{u},$$

and set $N_w = \exp(\mathfrak{n}_w)$,

$$U_w = \exp(\mathfrak{u}_w).$$

Note that the notation makes sense as $\mathrm{Ad}(M)$ stabilizes both $\mathfrak{n}$ and $\mathfrak{n}^-$. Moreover, it is easily seen that $\mathfrak{n}_w = \oplus_{\alpha\in\Sigma^+\cap w\Sigma^+}\mathfrak{g}_\alpha$ and $\mathfrak{u}_w = \oplus_{\alpha\in\Sigma^+\cap w\Sigma^-}\mathfrak{g}_\alpha$.

LEMMA. *Let* V *be a nilpotent simply connected Lie group with Lie algebra* ν. *Suppose* $\nu = \nu_0 \supsetneq \nu_1 \supsetneq \cdots \supsetneq \nu_r \supsetneq 0$ *is a chain of ideals of* ν *such that* $[\nu,\nu_i] \subseteq \nu_{i+1}$ *for all* i $(0\le i\le r)$. *If* u_1 *and* u_2 *are two linear subspaces of* ν *such that* $\nu_i = u_1 \cap \nu_i \oplus u_2 \cap \nu_i$ *for all* i, *then the map* $(X,Y) \to \exp X \exp Y$ $(X \in u_1, Y \in u_2)$ *is a bijection from* $u_1 \times u_2$ *onto* V.

REMARK. With a little more effort one can show that this map from $u_1 \times u_2$ to V is in fact a diffeomorphism (cf. [He, Lemma VI.5.2]).

PROOF. Injectivity is a consequence of Lemma 1.3(ii). As for surjectivity, this will be established by induction on dim V. As $V_r = \exp \nu_r$ is a closed subgroup of V contained in the center, we have by induction that $(X,Y) \mapsto (\exp X)(\exp Y)V_r$ $(X\in u_1, Y\in u_2)$ is a surjective map from $u_1 \times u_2$ to V/V_r. Thus, for any $v \in V$, there are $X \in u_1$, $Y \in u_2$, $Z \in \nu_r$ such that $v = (\exp X)(\exp Y)(\exp Z)$.

Writing $Z = Z_1 + Z_2$ with $Z_i \in u_i \cap \nu_r$ $(i=1,2)$, we get $v = \exp(X+Z_1)\exp(Y+Z_2)$ and hence the desired surjectivity. □

1.10. PROPOSITION. N_w *and* U_w *are closed simply connected subgroups of* N. *Furthermore*, $N_w \cap U_w = \{1\}$ *and* $N = N_wU_w = U_wN_w$.

PROOF. Straightforward application of Lemma 1.3 establishes the first statement, and $N_w \cap U_w = \{1\}$. As for $N = N_wU_w = U_wN_w$, this follows from the preceding lemma (where for the $(\nu_i)_{1\le i\le r}$ one may take the collection $(\oplus_{\beta\in\Sigma,\beta\ge\alpha}\mathfrak{g}_\beta)_{\alpha\in\Sigma^+}$). □

1.11. For $w \in W$, and C a subset of G normalized by H, write $C^w = w^{-1}Cw$. Then $N_w^w = N_{w^{-1}}$ and $U_w^w = \theta(U_{w^{-1}})$. The latter subgroup of G will also be denoted $\underline{U}_{w^{-1}}$. These notations enable us to rewrite B_0wB_0 as follows:

$$B_0wB_0 = NAMwB_0 = NwB_0 = U_wN_wwB_0 = U_wwN_w^wB_0 = U_wwB_0 = wU_w^-B_0.$$

We claim that U_w parametrizes the left cosets B_0wB_0/B_0 of B_0 in B_0wB_0 properly:

LEMMA.

(i) $G = \bigcup_{w\in W} U_wwB_0$

(ii) *For* $w \in W$, *the map* $U_w \to B_0wB_0/B_0$ *given by*

$$u \mapsto uwB_0 \qquad (u \in U_w)$$

is regular and bijective.

PROOF. (i) results from Theorem 1.2 and the remark just made.
(ii) Let ϕ be the map $u \mapsto uwB_0$ under consideration. It is surjective by (i). Suppose $uwB_0 = u^1wB_0$ for $u,u^1 \in U_w$. Then $m_w^{-1}u^{-1}u^1m_w \in B_0 \cap U_w^w \subseteq B_0 \cap N^{-1} = \{1\}$ according to Lemma 1.8, so that $u = u^1$. This yields that ϕ is injective. It remains to settle regularity. For $X \in \mathfrak{u}_w$, we have $u(\exp tX)wB_0 = um_w\exp(t\mathrm{Ad}(m_w^{-1})X)B_0$, so that $d\phi_u(X) = dL_{um_w}(\mathrm{Ad}(m_w^{-1})X \bmod \mathfrak{b}_0)$. Therefore, $d\phi_u(X) = 0$ implies $\mathrm{Ad}(m_w^{-1})X \in \mathfrak{b}_0 \cap \mathfrak{n}^- = 0$, whence $X = 0$. □

1.12. Next, topologize the quotient space G/B_0 by its natural quotient topology, i.e., such that the natural map $G \to G/B_0$ is open and continuous. Note that G/B_0 is a Hausdorff space as B_0 is closed in G.

For $\alpha \in \sum$, put $\mu_\alpha = \dim \mathfrak{g}_\alpha$ (as in (5.9) of Chapter I) and for $\Gamma \subseteq \Sigma$ set $\mu(\Gamma) = \sum_{\alpha\in\Gamma} \mu_\alpha$.

PROPOSITION. (i) $\dim B_0wB_0/B_0 = \mu(\sum^+\cap w\sum^-)$ *for any* $w \in W$. (ii) *There is a unique* $w_0 \in W$ *with* $w_0\sum^+ = \sum^-$. *This element is an involution* (i.e., $w_0^2 = 1$). *Moreover*, $B_0w_0B_0$ *is open and dense in* G.

PROOF. (i) $\dim B_0wB_0/B_0 = \dim U_w = \dim \mathfrak{u}_w = \mu(\sum^+\cap w\sum^-)$ in view of 1.11.
(ii) The first statements are well-known properties of Weyl groups, see BOURBAKI [Bou, Ch VI, no° 1.6].

Since $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{b}_0$ and $N^- \cap B_0 = \{1\}$, the map $(n,b) \mapsto nb$ from $N^- \times B_0$ to G is open and injective. It follows that N^-B_0/B_0 is open in G/B_0 and that $n \mapsto m_wnB_0$ defines a diffeomorphism from N^- onto $m_wN^-B_0/B_0$ for any $w \in W$. But $U^-_{w^{-1}}$ is a closed connected subgroup of N^-, so $U_wwB_0/B_0 = m_wU^-_{w^{-1}}B_0/B_0$ is regularly embedded in $m_wN^-B_0/B_0$ (i.e., its topology as a submanifold is the induced topology), which is open in G/B_0. Hence, U_wwB_0/B_0 is regularly embedded in G/B_0. Since, according to (i), $\dim U_wwB_0/B_0 = \mu(\sum^+\cap w\sum^-) < \mu(\sum^+) = \dim U_{w_0}w_0B_0/B_0$ if $w \neq w_0$, we obtain that $B_0w_0B_0/B_0$, being the complement of the strictly lower dimensional subvariety $\bigcup_{w\in W\setminus\{w_0\}}B_0wB_0/B_0$, is open and dense in G/B_0. This yields that B_0wB_0 is open and dense in G. □

1.13. Let R be the set of *fundamental reflections* of W, whose members are the reflections s_α (see (5.1) of Chapter II) for $\alpha \in \Delta$, i.e. the reflections corresponding to fundamental roots.

The set $\sum^+ \cap w\sum^-$ appearing in the foregoing proposition has a very explicit description. For $w \in W$, denote by $\ell(w)$ the length t of a minimal expression $w = r_1r_2\ldots r_t$ of w as a product of fundamental reflections r_i ($1 \le i \le t$), also referred to as the *length of* w.

LEMMA. *Let* $w \in W$ *and* $r \in R$. *Suppose* α *is the fundamental root corresponding to* r. *Then*

(i) $\ell(wr) = \ell(w) \pm 1$.

(ii) $\ell(rw) > \ell(w)$ iff $\alpha \in \sum^+ \cap w\sum^+$

(iii) $\ell(wr) > \ell(w)$ iff $\alpha \in \sum^+ \cap w^{-1}\sum^+$

(iv) $\ell(rw) < \ell(w)$ iff $\alpha \in \sum^+ \cap w\sum^-$

(v) $\ell(wr) < \ell(w)$ iff $\alpha \in \sum^+ \cap w^{-1}\sum^-$

(vi) *If* $w = r_1r_2\ldots r_t$ *is a minimal expression of* w *as a product of fundamental reflections* r_i ($1 \le i \le t$), *then* $\sum^+ \cap w\sum^- = \{r_1r_2\ldots r_{i-1}\alpha_i \mid \alpha_i$ *is a positive root of* r_i; $1 \le i \le t\}$.

(vii) *If* $\sum$ *is reduced, then* $\ell(w) = |\sum^+ \cap w\sum^-|$.

(viii) *The unique element* w_0 *in* W *with* $w_0\sum^+ = \sum^-$ (cf. *Theorem* II. 5.6 *and Proposition* 1.12) *satisfies* $\ell(w) + \ell(ww_0) = \ell(w_0)$.

PROOF. See BOURBAKI [Bou, Ch. IV no$^{\circ}$ 1.7, Ch. VI no$^{\circ}$ 1.6] for (i), (vi) and (viii). The other statements are easy consequences. □

w_0 is called the *opposite involution* or just *longest element* of W.

1.14. We are out to determine the closure $\overline{B_0wB_0}$ of B_0wB_0 in G for arbitrary $w \in W$ and to verify the axioms of a Tits system for B_0, N_0. A collection of special Lie subgroups of G will be used which we shall introduce now.

Recall that Δ is the base (or set of fundamental roots) of $\sum$, determined by an ordering on $\mathfrak{a}$ (cf. Section 7 of Chapter I). Fix a subset J of Δ. Write $\sum(J) = \sum \cap \sum_{\alpha\in J} \mathbb{R}\alpha$. This is again a root system (see [Car, Prop. 2.5.1]), whose Weyl group will be denoted by W_J. Let $\mathfrak{n}(J) = \oplus_{\alpha\in\Sigma(J),\alpha>0}\, \mathfrak{g}_\alpha$ and $\mathfrak{n}^-(J) = \theta\mathfrak{n}(J)$, and let $\mathfrak{g}(\sum)$ denote the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{n}(J)$ and $\mathfrak{n}^-(J)$. Then, clearly, $\mathfrak{g}(J)$ is a subalgebra of $\mathfrak{n}(J) + \mathfrak{g}_0 + \mathfrak{n}^-(J)$. For $\mathfrak{c}$ a linear subspace of $\mathfrak{g}$, put $\mathfrak{c}(J) = \mathfrak{c} \cap \mathfrak{g}(J)$. Note that this is in accordance with $\mathfrak{n}(J)$, $\mathfrak{n}^-(J)$ and $\mathfrak{g}(J)$ as previously defined.

LEMMA. *Let* J *be a subset of* Δ.

(i) $\mathfrak{g}(J)$ *is semisimple.*

(ii) $\mathfrak{g}(J)$ *has Cartan-decomposition* $\mathfrak{g}(J) = \mathfrak{k}(J) + \mathfrak{p}(J)$,

(iii) $\mathfrak{a}(J)$ *is a maximal abelian subalgebra in* $\mathfrak{p}(J)$, *and* $\mathfrak{a}(J) = \sum_{\alpha\in\Sigma(J)} \mathbb{R}A_\alpha$,

where A_α *is defined in* (5.1) *of Chapter* I.
(iv) $(\mathfrak{g}(J),\mathfrak{a}(J))$ *has root system* $\Sigma(J)$.

PROOF. (i) $\mathfrak{g}(J) = \mathfrak{n}(J) + \mathfrak{g}_0(J) + \mathfrak{n}^-(J)$ and $\mathfrak{g}_0(J) \supseteq \oplus_{\alpha\in\Sigma(J)}[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$. Thus

$$\ell = \mathfrak{n}(J) + \bigoplus_{\alpha\in\Sigma(J)} [\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}] + \mathfrak{n}^-(J)$$

is contained in $\mathfrak{g}(J)$. On the other hand, ℓ is a Lie algebra containing $\mathfrak{n}(J),\mathfrak{n}^-(J)$(Use the Jacobi identity to verify that the part of ℓ contained in $\mathfrak{g}_0(J)$ is in fact a Lie subalgebra).

Thus, $\ell = \mathfrak{g}(J)$. Since ℓ is θ-invariant, any ad ℓ-invariant linear subspace of $\mathfrak{g}$ has an ad ℓ-invariant complement (namely, the orthoplement with respect to the inner product B_θ defined in (3.7) of Chapter I). Thus ℓ is reductive in $\mathfrak{g}$, whence reductive (see Definition I.1.13). According to Lemma I.5.1, we also have $\ell = [\ell,\ell]$, so that ℓ is semi-simple by Proposition I.1.14.
(ii) Clearly, $\mathfrak{k}(J)$ and $\mathfrak{p}(J)$ are the eigenspaces of θ in ℓ. In view of Definition I.3.2, it remains to show that $B^1:(X,Y) \mapsto B_\ell(\theta X,Y)$ is a negative definite form on ℓ. For $X \in \mathfrak{g}$, the map ad $X \circ$ ad θX is semi-simple with real non-positive eigenvalues (as it is $-$ ad $X \circ (\mathrm{ad}\, X)^t$, where the transpose $(\mathrm{ad}\, X)^t$ is taken with respect to $-B_\theta$). Therefore $(\mathrm{ad}\, X)|_\ell \circ (\mathrm{ad}\, \theta X)|_\ell$ has real nonpositive eigenvalues only for any $X \in \ell$, so that B^1 is negative semidefinite. But B^1 is nondegenerate, since $(X,Y) \mapsto B_\ell(X,Y)$ is nondegenerate according to (i). This shows that B^1 is negative definite, as wanted.
(iii) Suppose $X \in \mathfrak{p}(J)$ commutes with all of $\mathfrak{a}(J)$. Write $X = X_1 + Y + X_2$ with $X_1 \in \mathfrak{n}$, $X_2 \in \mathfrak{n}^-$ and $Y \in \mathfrak{g}_0$, and take $H \in \mathfrak{a}(J)$ with $\alpha(H) \neq 0$ for all $\alpha \in \Sigma(J)$. Then $0 = [H,X] = [H,X_1] + [H,Y] + [H,X_2]$, so $[H,X_1] = [H,Y] = [H,X_2] = 0$ by decomposition of $\mathfrak{g}$. Since ad H is nonsingular on $\mathfrak{n}^-$ and on $\mathfrak{n}$ it follows that $X_1 = X_2 = 0$, whence $X \in (\mathfrak{m}+\mathfrak{a}) \cap \mathfrak{p}(J) = \mathfrak{a}(J)$, as wanted.
(iv) Is a direct consequence of earlier statements. □

1.15. Let $G(J)$ (respectively $K(J)$, $A(J)$, $N(J)$) denote the connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}(J)$ (respectively $\mathfrak{k}(J)$, $\mathfrak{a}(J)$, $\mathfrak{n}(J)$). Writing $G(J) = K(J)\exp \mathfrak{p}(J)$ yields that in fact $G(J)$ is a closed subgroup of G (note that $K(J)$ is compact). Denote by $M(J)$ (respectively $M^*(J)$) the centralizer (respectively normalizer in $K(J)$) of $\mathfrak{a}(J)$, and set $B_0(J) = M(J)A(J)N(J)$. Then $M(J)$ is in M, since the centralizer of $\mathfrak{a}(J)$ in $\mathfrak{k}(J)$ is contained in $\mathfrak{m}$ (to prove this, use that the centralizer of $H \in \mathfrak{a}$ is

$\mathfrak{g}_0 + \oplus_{\alpha\in\Sigma,\alpha(H)=0}\, \mathfrak{g}_\alpha$). Moreover, the reflection s_α for $\alpha \in \sum(J)$ of Lemma II.5.4 is actually defined by means of k_0 in $M^*(J)$, so that $M^*(J)$ is contained in M^*.

Also, observe that $\sum^+ \cap w\sum^- = \sum^+(J) \cap w\sum^-$ for $w \in W_J$ (as w preserves $\sum^+\backslash\sum^+(J)$), so that $\mathfrak{u}_w = \mathfrak{u}_w(J)$ and U_w is contained in $N(J)$. Later on, we shall use that for the unique involution $w_J \in W_J$ with $w_J\sum^+(J) = \sum^-(J)$, we have $U_{w_J} = N(J)$.

PROPOSITION. *Let* J *be a nonempty subset of* Δ. *Then* $G(J)$ *is a noncompact connected semi-simple Lie subgroup of* G *with finite center. Thus* $G(J) = K(J)A(J)N(J)$ *is an Iwasawa decomposition of* $G(J)$ *and* $G(J) = \cup_{w\in W_J} U_w wB_0(J)$ *is a refined Bruhat decomposition of* $G(J)$, *where* W_J *is the Weyl group of* $\sum(J)$. *Moreover,* $W_J \cong M^*(J)/M(J)$.

PROOF. Since J is nonempty, $\mathfrak{n}(J)$ is nontrivial; hence $\mathfrak{p}(J) \neq 0$. So $G(J)$ is noncompact. Semi-simplicity of $G(J)$ is immediate from the previous lemma. The center of $G(J)$ is finite as $K(J)$ is compact in regard to Theorem II.3.1. Thus, $G(J)$ is a Lie subgroup as specified (cf. Lemma 1.11 and Theorem II.5.8). □

1.16. For the moment, restrict attention to the case where J consists of a single fundamental root α. Then $G(\alpha), A(\alpha), \ldots$ will be used rather than $G(\{\alpha\}), A(\{\alpha\}), \ldots$. Let r be the reflection corresponding to α(thus $r = s_\alpha$). The Bruhat decomposition reads $G(\alpha) = B_0(\alpha) \cup rU_r^- B_0(\alpha)$. Moreover, $\mathfrak{u}_r = \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ (where possibly $\mathfrak{g}_{2\alpha} = 0$).

The following lemma concerns multiplication of double cosets of B_0 in G.

LEMMA. *Let* $w \in W$ *and* $r \in R$. *Then*

(i) $B_0rB_0rB_0 = B_0 \cup B_0rB_0$

(ii) *If* $\ell(wr)>\ell(w)$, *then* $B_0wB_0rB_0 = B_0wrB_0$.

(iii) *If* $\ell(wr)<\ell(w)$, *then* $B_0wB_0rB_0 = B_0wB_0 \cup B_0wrB_0$.

PROOF. Let α be the fundamental root corresponding to r.

(i) As $B_0rB_0 = U_r rB_0$ we have $B_0rB_0rB_0 = B_0rU_r rB_0 \subseteq B_0G(\alpha)B_0 \subseteq B_0B_0(\alpha)B_0 \cup B_0B_0(\alpha)rB_0(\alpha)B_0 = B_0 \cup B_0rB_0$ (where in the one but last equation, the Bruhat decomposition for $G(\alpha)$ is used). On the other hand $B_0 \subseteq B_0rB_0rB_0 \subsetneq B_0$, for otherwise $N(\alpha)^-\backslash\{1\} \subseteq rU_r r \subseteq rB_0r \subseteq B_0$, conflicting Lemma 1.8.

(ii) Suppose $\ell(wr)>\ell(w)$; then $\alpha \in w^{-1}\sum^+$ by Lemma 1.13 (iii). It suffices to establish that $wU_r r$ is contained in B_0wrB_0. Let $u = m_w u r$ for some $u \in U_r$. Take $Y \in \mathfrak{u}_r$ with $u = \exp Y$. Now $x = (m_w u m_w^{-1})m_w r = \exp(\mathrm{Ad}(m_w)Y)m_w r \in$ $\in Nm_w r \subseteq B_0wrB_0$ as $w\alpha \in \sum^+$, so that we are done.

(iii) Suppose $\ell(wr)<\ell(w)$; then $\ell((wr)r)>\ell(wr)$, so $B_0wB_0rB_0 = B_0wrB_0rB_0rB_0 =$ $= B_0wrB_0(B_0 \cup B_0rB_0) = B_0wrB_0 \cup B_0wrB_0rB_0 = B_0wrB_0 \cup B_0wB_0$ by use of (i) and (ii).

This settles the lemma. □

1.17. The results obtained so far on the refined Bruhat decomposition are collected in the theorem below.

THEOREM. *Let* G *be a noncompact connected real semi-simple Lie group with finite center. Retain the usual notations, and put* $B_0 = MAN$ *and* $N_0 = M^*A$. *Then the following holds:*

(i) B_0 *and* N_0 *generate* G.

(ii) *The group* $H_0 = B_0 \cap N_0$ *is a normal subgroup of* N_0 *and* $W = N_0/H_0$.

(iii) W *contains a generating set* R *such that for any* $w \in W$ *and* $r \in R$:

(iii)^1 $B_0wB_0rB_0 \subseteq B_0wB_0 \cup B_0wrB_0$.

(iii)^{11} $rB_0r^{-1} \nsubseteq B_0$

(iv) W *is a Weyl group with root system* $\sum$ *and opposite involution* w_0.

(v) *If* w,r *are as in* (iii) *then* $B_0wB_0rB_0 = B_0wrB_0$ *iff* $\ell(wr)>\ell(w)$, *where* $\ell(w)$ *is the usual length of* w *with respect to* R.

(vi) $\bigcap_{w\in W} B_0^w = B_0 \cap B_0^{w_0} = H_0$.

(vii) $B_0 = H_0N$, *where* N *is a normal subgroup of* B_0 *and* $H_0 \cap N = \{1\}$.

(viii) B_0 *is the normalizer of* N *in* G.

(ix) N_0 *is the normalizer of* H_0 *in* G.

(x) *For any* $w \in W$, *we have* $N = N_wU_w = U_wN_w$, *where* $N_w = N \cap B_0^{w^{-1}}$ *and* $U_w = N_{ww_0}$. *Furthermore* $N_w \cap U_w = \{1\}$.

(xi) $G = \bigcup_{w\in W} B_0wB_0 = \bigcup_{w\in W} U_w wB_0 = \bigcup_{w\in W} m_w^{-1} U_w^- B_0$, *where* $m_w \in N_0$ *is such that* $m_wH_0 = w$, *and* $U_w^- = B_0^{w_0} \cap N^{w^{-1}}$.

(xii) B_0, N_0 *are closed subgroups of* G.

(xiii) A,N,N_w,U_w *(for* $w \in W$*) are closed nilpotent simply-connected subgroups of* G.

(xiv) $H_0 = MA$.

PROOF.

(i) is a direct consequence of Theorem 1.2.

(ii) $H_0 = B_0 \cap N_0 = MAN \cap M^*A = MA$ in M^*A as M is normal in M^*. Furthermore, $N_0/H_0 = M^*A/MA = M^*/M = W$.

(iii),(v) See Lemma 1.16.

(iv) is well known.

(vi) $MA \subseteq \bigcap_{w \in W} B_0^w \subseteq MAN \cap MAN^{w_0} = MA$, since according to Lemma 1.8 $B_0 \cap N^{w_0} = B_0 \cap N^- = \{1\}$. Consequently $H_0 = MA = \bigcap_{w \in W} B_0^w = B_0 \cap B_0^{w_0}$.

(vii) is known from the construction of B_0.

(viii) Suppose $x \in G$ normalizes N. Write $x = b_1 m_w b_2$ for $b_1, b_2 \in B_0$ and $w \in W$ (cf. Theorem 1.2). Then $N^{m_w} = N^{b_1 m_w b_2 b_2^{-1}} \subseteq N^{b_2^{-1}} = N$, so that m_w normalizes N. This implies $w\sum^+ = \sum^+$, whence $w = 1$ and $x \in b_1 M b_2 \subseteq B_0$.

(ix) Suppose $x \in G$ normalizes H_0. Since $\mathfrak{m} + \mathfrak{a}$ is the set of fixed points of $Ad(H_0)$, this subalgebra is stabilized by $Ad(x)$. Thus $Ad(x)\mathfrak{a} \subseteq \mathfrak{m} + \mathfrak{a}$. In view of Lemma 1.5, this yields $Ad(x)\mathfrak{a} \subseteq \mathfrak{a}$, whence $x \in M^*A = N_0$.

(x) Let N_w and U_w be defined as in 1.9. In view of Proposition 1.10 we only have to prove $N_w = N \cap B_0^{w^{-1}}$ and $U_w = N \cap B_0^{w_0 w^{-1}}$. Now, $N_w = \exp(\mathfrak{n} \cap Ad(m_w)\mathfrak{n}) \subseteq \exp \mathfrak{n} \cap \exp(Ad(m_w)\mathfrak{n}) \subseteq N \cap B_0^{w^{-1}}$, and $U_w = \exp(\mathfrak{n} \cap Ad(m_w)\mathfrak{n}^-) \subseteq \exp \mathfrak{n} \cap \exp(Ad(m_w m_{w_0} \mathfrak{n}) \subseteq N \cap B_0^{w_0 w^{-1}}$. Thus $N = N_w U_w = (N \cap B_0^{w^{-1}})(N \cap B_0^{w_0 w^{-1}})$. As $N \cap B_0^{w_0^{-1}} \cap B_0^{w_0 w^{-1}} = N \cap (B_0 \cap B_0^{w_0})^{w^{-1}} = N \cap H_0 = \{1\}$ owing to (vi) and Lemma 1.8, we obtain $N_w = N \cap B_0^{w^{-1}}$ and $U_w = N \cap B_0^{w_0 w^{-1}}$, as wanted.

(xi) $U_w^- = U_{w^{-1}}^{w^{-1}} = (N \cap B_0^{w_0 w})^{w^{-1}} = N^{w^{-1}} \cap B_0^{w_0}$. Now refer to Lemma 1.11 to finish the proof of (xi).

(xii), (xiii), (xiv) are well-known facts by now. □

The setting of the above theorem will later be recognized as that of a Tits system.

1.18. We end this section with a topological consequence of the Bruhat decomposition.

For each fundamental root α in Σ with corresponding reflection r, choose $m_r \in rM(\alpha)$ and define $Y_r = Y_\alpha = U_r m_r$. Then $G(\alpha) = B_0(\alpha) \cup Y_r B_0(\alpha)$ and Y_r is a system of left coset representatives of $B_0(\alpha)$ in $G(\alpha) - B_0(\alpha)$ as well as of B_0 in $B_0 r B_0$ (cf. Theorem 1.17 (xi)). Furthermore, denote by Z_r the intersection $Y_r A(\alpha) N(\alpha) \cap K$. Then Z_r is in bijective correspondence with Y_r. For, in view of Iwasawa's decomposition to each $y \in Y_r$ there belong $z \in Y_r A(\alpha) N(\alpha) \cap K$, $a \in A(\alpha)$ and $n \in N(\alpha)$ with $y = zan$, leading to a bijection $y \mapsto z$ from Y_r to Z_r.

<u>LEMMA</u>. *Let* $w = r_1 r_2 \ldots r_t$ *be a minimal expression of* $w \in W$ *with* $r_i \in R$. *Then*

(i) $B_0wB_0 = Y_{r_1}Y_{r_2}\cdots Y_{r_t}B_0 = Z_{r_1}Z_{r_2}\cdots Z_{r_t}B_0$, *with uniqueness of expression in the middle and the right hand side* (i.e., *if* $y_1y_2\cdots y_tb = y_1'y_2'\cdots y_t'b'$ *for* $b,b' \in B_0$ *and* $y_i,y_i' \in Y_{r_i}$ $(1\le i\le t)$, *then* $b = b'$ *and* $y_i = y_i'$ *for all* i, *and similarly with* Z_{r_i} *instead of* Y_{r_i}).
(ii) $B_0wB_0 = (B_0(\alpha_1)r_1B_0(\alpha_1))(B_0(\alpha_2)r_2B_0(\alpha_2))\cdots(B_0(\alpha_t)r_tB_0(\alpha_t))B_0$, *where* $\alpha_i \in \Delta$ *is the root of* r_i $(1\le i\le t)$.

PROOF. (i) By Lemma 1.6, we have $B_0wB_0 = B_0wr_tB_0r_tB_0$. Using induction, we get $B_0wr_tB_0 = Y_{r_1}Y_{r_2}\cdots Y_{r_{t-1}}B_0$, while $B_0r_tB_0 = Y_{r_t}B_0$ by construction of Y_{r_t}. It follows that $B_0wB_0 = Y_{r_1}Y_{r_2}\cdots Y_{r_{t-1}}B_0r_tB_0 = Y_{r_1}Y_{r_2}\cdots Y_{r_t}B_0$. As for uniqueness, if $y_1\cdots y_tb = y_1'\cdots y_t'b'$ with $b,b' \in B_0$ and $y_i,y_i' \in Y_{r_i}$ $(1\le i\le t)$, then $y_2\cdots y_tb = y_1^{-1}y_1'\cdots y_t'b'$. As $y_1^{-1}y_1' \in m_{r_1}^{-1}U_{r_1}m_{r_1} \subseteq r_1B_0r_1$, we have $y_1^{-1}y_1' \in B_0 \cup B_0r_1B_0$. But $y_1^{-1}y_1' \in B_0r_1B_0$ would imply $y_2\cdots y_tb = y_1^{-1}y_1'\cdots y_t'b' \in B_0r_2\cdots r_tB_0 \cap B_0wB_0$ according to Lemma 1.16, and hence $w = r_2\cdots r_t$ by Theorem 1.2, which is absurd. Thus $y_1^{-1}y_1' \in B_0$, so $y_1 = y_1'$ (cf. Lemma 1.11). Now $y_2\cdots y_tb = y_2'\cdots y_2'\cdots y_t'b'$, and we can finish by induction. As for $B_0wB_0 = Z_{r_1}Z_{r_2}\cdots Z_{r_t}B_0$, the proof is similar.
(ii) $B_0wB_0 = Y_{r_1}\cdots Y_{r_t}B_0 \subseteq (B(\alpha_1)r_1B(\alpha_1))(B(\alpha_2)r_2B(\alpha_2))\cdots(B(\alpha_t)r_tB(\alpha_t))B_0 \subseteq B_0r_1B_0r_2B_0\cdots B_0r_tB_0 = B_0wB_0$. As all inclusions must be equalitites, the proof of (ii) is settled. □

1.19. THEOREM. *Let* G *be a noncompact connected real semi-simple Lie group with finite center. Then*
(i) G/B_0 *is diffeomorphic to* K/M, *and hence compact.*
(ii) *Let* S_w *for* $w = r_1r_2\cdots r_t$ *as in Lemma* 1.18 *be the set of elements in* W *that are products of subsequences of the expression* $(r_1,r_2,\ldots,r_t)$. *Then* S_w *is independent of the chosen expression of* w *and*

$$\overline{B_0wB_0} = \bigcup_{w'\in S_w} B_0w'B_0.$$

PROOF. (i) The map : $K \to G/B_0$ given by $k \mapsto kB_0$ $(k\in K)$ is continuous and constant on xM for any $x \in K$. Therefore, a continuous map of K/M to G/B_0 is induced, which is surjective as $G = KB_0$ (Iwawasa's decomposition) and injective as $M = B_0 \cap K$. Since K/M is compact, the map is open, too. The tangent space $\mathfrak{k}/\mathfrak{m}$ of K/M at kM for $k \in K$, is mapped onto the tangent space $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{k}/\mathfrak{m}$ (as vector spaces). Therefore, the map is regular, whence (i).
(ii) $\overline{B_0wB_0} \supseteq \overline{B(\alpha_1)r_1B(\alpha_1)}\,\overline{B(\alpha_2)r_2B(\alpha_2)}\cdots\cdots\overline{B(\alpha_t)r_tB(\alpha_t)}B_0 = G(\alpha_1)G(\alpha_2)\cdots G(\alpha_t)B_0$ by Proposition 1.12(ii) applied to $G(\alpha_i)$ $(1\le i\le t)$.

Consequently,

$$\overline{B_0wB_0/B_0} \supseteq \overline{B_0wB_0}/B_0 \supseteq G(\alpha_1)G(\alpha_2)\ldots G(\alpha_t)B_0/B_0 \supseteq K(\alpha_1)K(\alpha_2)\ldots K(\alpha_t)B_0/B_0.$$

But the right hand side is the image under a continuous map of a product of compact spaces, hence compact, and therefore closed.

As it contains $Z_{r_1}Z_{r_2}\ldots Z_{r_n}B_0/B_0 = B_0wB_0/B_0$ according to (ii), we get $\overline{B_0wB_0/B_0} = \overline{B_0wB_0}/B_0 = G(\alpha_1)G(\alpha_2)\ldots G(\alpha_t)B_0/B_0$, whence $\overline{B_0wB_0} = G(\alpha_1)G(\alpha_2)\ldots G(\alpha_t)B_0 = \bigcup_{w^1\in S_w} B_0w^1B_0$, as wanted (to settle the latter equality, reason as in the proof of Lemma 1.18(ii)).

Finally, independence of S_w from the minimal expression of w follows since $\overline{B_0wB_0}$ is independent of the chosen expression (use uniqueness of $w^1 \in W$ in $B_0w^1B_0$). □

The above theorem supplies a computational set up for the cohomology of G/B, which in turn is of relevance for certain representations of G. We shall not dwell on these matters any further and refer the interested reader to KLEIMAN [K].

2. TITS SYSTEMS

In this section, G is an arbitrary (abstract) group. The theme of this section is that for groups with a Tits system, in particular those of the previous section, a lot of structure is determined by a relatively small group W. As notation suggests, W is the Weyl group of G if G is a Lie group such as in Section 1. Since many properties of Tits systems are derived in BOURBAKI [Bou], the reader is referred to this excellent exposition for proofs of several statements below.

2.1. DEFINITIONS. A pair (B_0,N_0) of subgroups of G is called a *Tits system* provided there is a group W and a generating subset R of W such that

(i) B_0 and N_0 generate G

(ii) $H_0 = B_0 \cap N_0$ is a normal subgroup of N_0 and $W = N_0/H_0$

(iii) For any $w \in W$ and $r \in R$

(iii)^1 $B_0wB_0rB_0 \subseteq B_0wB_0 \cup B_0wrB_0$

(iii)^{11} $rB_0r^{-1} \not\subseteq B_0$

Note that expressions such as wB_0 for $w \in W$ are well defined as $H_0 \subseteq B_0$. For $w \in W$, denote by m_w an element of N_0 such that $m_wH_0 = w$. The group W will be called the *Coxeter group of the Tits system* (though, in Tits [Ti] it is called the Weyl group of the system; as we shall see below, W is always a Coxeter group but not necessarily a Weyl group; hence the change of terminology). The Tits system is called *saturated* if $\cap_{w\in W} B_0^w = H_0$, and *split* if B_0 contains a normal subgroup N with $B_0 = H_0N$ and $H_0 \cap N = \{1\}$. It is easy to check that if (B_0,N_0) is a Tits system, then (B_0,N_0^1) for $N_0^1 = (\cap_{w\in W} B_0^w)N$ is a saturated Tits system with the same Coxeter group. A tuple (W,R) consisting of a group W and a generating subset R of W is called a *Coxeter system* if there are natural numbers $m_{r,s} > 1$ depending on distinct $r,s \in R$ and satisfying $m_{r,s} = m_{s,r}$ such that the following set of relations for $r,s \in R$ with $r \neq s$ forms a presentation of W:

$$r^2 = 1$$

$$(rs)^{m_{r,s}} = 1.$$

Set $m_{r,r} = 1$.

The matrix $(m_{r,s})_{r,s\in R}$ is called the *Coxeter diagram* and the group W is called the *Coxeter group* of (W,R).

The Coxeter diagram has a pictorial presentation by means of the graph $(R,\sim)$ in which $r \sim s$ for $r,s \in R$ iff $m_{r,s} > 2$ and in which the edge $\{r,s\}$ is labeled $m_{r,s}$ provided $m_{r,s} > 3$. If in a Dynkin diagram, the bonds $\circ\!\Rightarrow\!\circ$ (or $\circ\!\Leftarrow\!\circ$) and $\circ\!\Rrightarrow\!\circ$ (or $\circ\!\Lleftarrow\!\circ$) are replaced by the labeled edges $\circ\overset{4}{—}\circ$ and $\circ\overset{6}{—}\circ$ respectively, a Coxeter diagram results. Thus Dynkin diagrams may (upon disregarding the ">" signs) be viewed as Coxeter diagrams. This embedding also makes sense at the group level: this is stated in Lemma 2.3 below. If $w \in W$, then $\ell(w)$ denotes the *length of* w (*with respect to* R), i.e. the length t of a minimal expression $w = r_1r_2\cdots r_t$ of w as a product of $r_i \in R$ $(1\leq i\leq t)$. Actually, we shall employ this notion of length for any tuple (W,R) of a group W and a generating subset R.

2.2. PROPOSITION. *Suppose* (W,R) *is a pair consisting of a group* W *and a finite generating subset* R *of involutions in* W. *Then the following statements are equivalent*:

(i) (W,R) *is a Coxeter system*

(ii) *There is a collection* $(P_r)_{r\in R}$ *of subsets of* W *such that* (ii)′ $\quad 1 \in P_r$ *for all* $r \in R$

(ii)″ $\quad P_r \cap rP_r = \emptyset$ *for all* $r \in R$

(ii)‴ *If* $r, r' \in R$ *and* $w \in P_r$ *with* $wr' \notin P_r$, *then* $rw = wr'$.

(iii) *For any* $w \in W$, $r \in R$ *with* $\ell(rw) \leq \ell(w)$ *and for each minimal expression* $w = r_1 r_2 \ldots r_t$ *of* w *as a product of* $r_i \in R (1 \leq i \leq t)$, *there is an integer* j $(1 \leq j \leq t)$ *such that* $rr_1 \ldots r_{j-1} = r_1 \ldots r_{j-1} r_j$.

Moreover, if these statements hold, then $P_r = \{w \in W \mid \ell(rw) > \ell(w)\}$.

PROOF. See BOURBAKI [Bou, Chapter IV, §1.6 and §1.7]. □

The property formulated in (iii) above is known as the *Exchange condition*.

2.3. LEMMA. *If* (W,R) *is a pair consisting of a Weyl group* W *and a set* R *of fundamental reflections of* W, *then* (W,R) *is a Coxeter system.*

PROOF. See BOURBAKI [Bou, Chapter V, §3, Theorem 1], or do it yourself by establishing (ii)′ with $P_r = \{w \in W \mid \ell(rw) > \ell(w)\}$ using Lemma 1.13. □

2.4. EXAMPLE. Let G be a noncompact connected real semi-simple Lie group with finite center, and let B_0, N_0, W, R, Σ be as in the previous subsection. Then (B_0, N_0) is a split saturated Tits system whose Coxeter group is a Weyl group of type Σ by (i),(ii),...,(vi),(vii) of Theorem 1.17. In the sequel we shall see that some of the other statements in Theorem 1.17 follow from this observation. Note that B_0 is the connected Lie subgroup of G whose Lie algebra is $\mathfrak{g}_0 + \mathfrak{n}$ and that N_0 is the normalizer in G of $\mathfrak{g}_0$. The parallel between this Tits system and the Tits system (B_0^C, N_0^C) of the complexified Lie group G_C is easy to spot: B_0^C is the connected Lie subgroup of G_C whose Lie algebra is $\mathfrak{g}^0_C$ and N_0^C is the normalizer in G of $\mathfrak{g}^0_C$.

2.5. For $w \in W$, let S_w be the subset of W whose members are products of subsequences of a minimal expression for w (cf. Theorem 1.19).

PROPOSITION. *Suppose* (B_0, N_0) *is a Tits system with Coxeter group* W *and distinguished generating subset* R *of* W. *Then* (i) (W,R) *is a Coxeter system. Moreover for* $w, w' \in W$ *and* $r \in R$ *we have*

(ii) $\quad \ell(rw) > \ell(w) \iff B_0 r B_0 w B_0 = B_0 rw B_0$

(iii) $\quad \ell(wr) > \ell(w) \iff B_0 w B_0 r B_0 = B_0 wr B_0$

(iv) $\quad \ell(rw) < \ell(w) \iff B_0 r B_0 w B_0 = B_0 w B_0 \cup B_0 rw B_0$

(v) $\quad \ell(wr) < \ell(w) \iff B_0 w B_0 r B_0 = B_0 w B_0 \cup B_0 wr B_0$

(vi) $B_0w^1B_0wB_0 \subseteq B_0w^1S_wB_0$.

PROOF. (i) We first show that each element of R is an involution. Thus let $r \in R$. Clearly $r \neq 1$ in view of axiom $(iii)^{11}$ in Definition 2.1.

By Definition 2.1(iii) with $w = r^{-1}$, the union of double cosets $B_0r^{-1}B_0rB_0$ is contained in $B_0rB_0 \cup B_0$ and distinct from B_0. Since it clearly contains B_0, this implies

$$B_0r^{-1}B_0rB_0 = B_0 \cup B_0r^{-1}B_0.$$

Taking inverses left and right yields

$$B_0r^{-1}B_0rB_0 = B_0 \cup B_0rB_0,$$

whence $B_0r^{-1}B_0 = B_0rB_0$.

Applying $(iii)^1$ of Definition 2.1 with $w = r$ yields

$$B_0rB_0rB_0 \subseteq B_0rB_0 \cup B_0r^2B_0.$$

But $B_0rB_0rB_0 = B_0 \cup B_0rB_0$ by what we just saw. Hence $B_0r^2B_0 = B_0$, i.e. $r^2 \in B_0$. As $m_r \in N_0$, we get $m_r^2 \in B_0 \cap N_0 = H_0$, proving $r^2 = 1$.

This shows that (B_0,N_0), or rather (G,B_0,N_0,R) is a Tits system in the sense of BOURBAKI [Bou, Chapter IV, §2.1].

The remainder of the proof can now be found in BOURBAKI [Bou, Chapter IV, §2.4, Theorem 2]. It runs as follows: for $r \in R$ define P_r by $P_r = \{w \in W | BrBwB = BrwB\}$ and verify (ii) of Proposition 2.2.

(ii) follows as $P_r = \{w \in W | \ell(rw) > \ell(w)\}$ by Proposition 2.2.

(iii) follows from (ii) by inversion, since $\ell(w) = \ell(w^{-1})$.

(iv) is equivalent to (ii).

(v) is equivalent to (iii).

(vi) is obtained by repeated application of axiom $(iii)^1$ of Tits systems. □

2.6. Let W be a group with generating subset R of involutions. For J a subset of R, denote by W_J the subgroup of W generated by J. Clearly, this is in accordance with the definition of W_J given in 1.14 for the case where W is a Weyl group with generating set R of fundamental reflections.

Write ${}_JD = \{w \in W | \ell(rw)>\ell(w) \text{ for all } r \in J\} = \bigcap_{r\in J} P_r$ and $D_J = \{w \in W | \ell(wr)>\ell(w) \text{ for all } r \in J\} = \bigcap_{r\in J} P_r^{-1}$. Here P_r is as in the last statement of Proposition 2.2. Furthermore, if I is a(nother) subset of R, put $D_{I,J} = {}_ID \cap D_J$. Thus $D_{\emptyset,J} = {}_JD$ and $D_{J,\emptyset} = D_J$. Clearly, $D_R = \{1\}$, and $D_\emptyset = W$.

LEMMA. *Let* (W,R) *be a Coxeter system. Suppose* I,J,K *are subsets of* R *and* w *is an element of* W. *Then*

(i) (W_J,J) *is a Coxeter system,* $W_J \cap R = J$, *and if* $w \in W_J$, *then* $\ell(w)$ *is the length of* w *with respect to* J.

(ii) $W_I \cap (W_JW_K) = W_{I\cap J}\, W_{I\cap K}$

(iii) *There is a bijection* $D_{I,J} \to W_I\backslash W/W_J$ *such that if* $w_{I,J} \in W_IwW_J \cap D_{I,J}$ *there are* $w_1 \in W_I$, $w_2 \in W_J$ *with* $w = w_1w_{I,J}w_2$ *such that* $\ell(w) = \ell(w_1) + {}$ $+ \ell(w_{I,J}) + \ell(w_2)$.

PROOF. (i), (ii) See BOURBAKI [Bou, Chapter IV, §1.8 and Exercize 1.1]. (iii) See BOURBAKI [Bou, Chapter IV, Exercise 1.3]. □

2.7. From now on, assume that B_0, N_0 *is a Tits system of* G *with Coxeter group* W *and associated generating set* R *of involutions in* W. For J a subset of R, put $G_J = B_0W_JB_0$.

PROPOSITION. *Let* I,J *be subsets of* R. *Then*

(i) G_J *is a subgroup of* G. *In particular* $G = G_R = B_0WB_0$ *and* $G_\emptyset = B_0$.

(ii) *The mapping* $W_IwW_J \mapsto G_IwG_J$ *is a bijection of* $W_I\backslash W/W_J$ *onto* $G_I\backslash G/G_J$.

(iii) $G_I \cap G_J = G_{I\cap J}$

(iv) *For any subgroup* G^1 *of* G *containing* B_0, *there is a subset* K *of* R *such that* $G^1 = G_K$.

(v) *The normalizer in* G *of* G_J *is* G_J.

(vi) *No two distinct subgroups of* G *containing* B_0 *are conjugate.*

PROOF. See BOURBAKI [Bou, Chapter IV, §2]. □

Note that the case $I = J = \emptyset$ of statement (ii) above is the Bruhat decomposition for (arbitrary) Tits systems.

2.8. DEFINITIONS. The subgroups of G containing B_0 are called the *standard parabolic subgroups* of G. A subgroup of G containing a conjugate of B_0 is called a *parabolic subgroup* of G. The *maximal parabolic subgroups* are those parabolic subgroups which are maximal proper subgroups of G (i.e. corresponding to subsets J of R of cardinality $|R|-1$). The *minimal parabolic subgroups* are those parabolic subgroups which are minimal among those that properly contain a conjugate of B_0 (i.e. they correspond to subsets J or R of cardinality 1). In some literature ([W]), minimal parabolics are defined as the conjugates of B_0. Here, the conjugates of B_0 will be called *Borel subgroups*. Note that this definition does not coincide with that of a Borel subgroup of a linear algebraic group, cf. BOREL [Bor].

2.9. EXAMPLE. Let G be a real noncompact connected semi-simple Lie group with finite center. We want to establish, first of all, what the parabolic subgroups of G look like. We shall identify $J \subseteq R$ with the corresponding subset of Δ (consisting of roots in Δ of reflections in J). Recall the definition $n(J) = \sum_{\alpha\in\Sigma(J),\alpha>0} \mathfrak{g}_\alpha$, $\mathfrak{g}(J) = \langle n(J), \theta n(J)\rangle$, etcetera from 1.14. Define $\mathfrak{g}_J = n^-(J) + \mathfrak{g}_0 + n$. Then $\mathfrak{g}_J$ is a Lie subalgebra of $\mathfrak{g}$ containing b. In fact, G_J *is the connected Lie subgroup of* G *whose Lie algebra is* $\mathfrak{g}_J$. For, the latter subgroup contains B_0, so equals G_I for some $I \subseteq R$ according to Proposition 2.7 (iv). If $\alpha \in \Delta$ is a root corresponding to $s_\alpha \in J$, then $G(\alpha)$ has Lie algebra $\mathfrak{g}(\{s_\alpha\}) \subseteq \mathfrak{g}_J$, so $G(\alpha) \subseteq G_I$. In particular, $s_\alpha \in G_I$, and $J \subseteq I$. On the other hand, if $\alpha \in I$, then $\mathrm{Ad}(s_\alpha)\mathfrak{g}_\alpha = \mathfrak{g}_{-\alpha}$, while $\mathfrak{g}_\alpha \subseteq \mathfrak{g}_J$, so $\mathfrak{g}_{-\alpha} \subseteq \mathfrak{g}_J$. Hence $\alpha \in J$. The conclusion is that $I = J$, and $G_I = G_J$.

In the present situation, more is known of the structure of the parabolic subgroups than can be derived from the axioms of Tits systems. Let $J \subseteq R$. By the above paragraph, the pair $(B_0(J), M^*(J)A(J))$ defines a Tits system for $G(J)$ (see 1.14 for notation).

Thus, in particular, for any $w \in W_J$, $r \in J$, we have $m_wB_0(J)m_r \subseteq B_0(J)m_wB_0(J) \cup B_0(J)m_{wr}B_0(J)$. From this inclusion it is readily derived that $B_L = H_0N(J)$ and $N_L = M^*(J)M$ define a Tits system for the subgroup L_J of G generated by B_L and N_L. The Bruhat decomposition of L_J yields

$$L_J = N(J)H_0W_JN(J) = U_{w_J}H_0W_JU_{w_J},$$

where w_J is the longest element in W_J (see 1.13). Thus, $G_J = B_0W_JB_0 = N_{w_J}U_{w_J}H_0W_JU_{w_J} = N_{w_J}L_J$. Moreover,

$$\begin{aligned} N_{w_J} \cap L_J &= N_{w_J} \cap H_0U_{w_J} \qquad \text{(by uniqueness of the Bruhat decomposition)} \\ &= (N \cap B_0^{w_J}) \cap (B_0 \cap B_0^{w_0w_J}) = \\ &= N \cap (B_0 \cap B_0^{w_0})^{w_J} = N \cap H_0 = \{1\}, \end{aligned}$$

while $\mathrm{Ad}(N(J))$, $\mathrm{Ad}\, m_w$ $(w \in W_J)$ stabilize $n_{w_J} = \oplus_{\alpha\in\Sigma^+\setminus\Sigma(J)}\, g_\alpha$, so that L_J normalizes N_{w_J}. We conclude that G_J is the semidirect product of a nilpotent simply connected subgroup N_{w_J} and the socalled *Levi-factor* L_J *of* G *with respect to* J, a group closely related to its subgroup G(J). This decomposition of G_J is known as the *Levi-decomposition* of G_J and plays an important rôle in many inductive arguments in a manner analogous to what we have seen in Theorem 1.19.

2.10. At the end of this section, the quotient topology on G/G_J is discussed for G as in the above example. The remainder of this section is devoted to the proof that statements (x) and (xi) of Theorem 1.17 hold for arbitrary split Tits systems whose Coxeter groups are Weyl groups. As this result is inessential for the next section, it may well be skipped by those who are primarily interested in geometry.

For $w,w^1 \in W$, set $B_0^{\,w} = w^{-1}B_0w$; $B_w = B_0 \cap B_0^{w^{-1}}$, and $B^w_{w^1} = (B_{w^1})^w$. Thus, $B^{w^{-1}}_{w^{-1}} = B_w$.

<u>LEMMA</u>. *Suppose* (B_0,N_0) *is a Tits system with Coxeter group* W *and distinguished generating set* R. *Then for any* $w \in W$ *and* $r \in R$ *with* $\ell(rw)>\ell(w)$ *the following holds:*

(i) $B_0rB_0 \cap B_0^{w^{-1}} = \emptyset$ *and* $B_{rw} = B^r_w \cap B_0$.

(ii) $B_0 = B_rB_w$.

Moreover, if $w,w^1 \in W$ *with* $\ell(ww^1) = \ell(w) + \ell(w^1)$, *then*

(iii) $B_{ww^1} \subseteq B_w$.

<u>PROOF</u>. (i) By Propositions 2.5 and 2.7 we have $B_0rB_0wB_0 \cap B_0wB_0 = \emptyset$. Multiplying from the right by w^{-1} yields $B_0rB_0B_0^{w^{-1}} \cap B_0B_0^{w^{-1}} = \emptyset$, whence $B_0rB_0 \cap B_0^{w^{-1}} = \emptyset$. Now $B_0^r \subseteq B_0 \cup B_0rB_0$, so $B_0^r \cap B_0^{w^{-1}} \subseteq (B_0 \cup B_0rB_0) \cap B_0^{w^{-1}} = B_0 \cap B_0^{w^{-1}} = B_w$. Conjugating by r yields $B_{rw} \subseteq B^r_w$, whence $B_{rw} \subseteq B^r_w \cap B_0$. Since the other inclusion is obvious, this proves (i).

(ii) As $rB_0w \subseteq B_0rwB_0$, we have $B_0 \subseteq rB_0rwB_0w^{-1} = B_0^rB_0^{w^{-1}}$. Let $b \in B_0$ and write $b = xy$ with $x \in B_0^r$ and $y \in B_0^{w^{-1}}$. Then $y = x^{-1}b \in B_0^rB_0 \cap B_0^{w^{-1}} \subseteq (B_0 \cup B_0rB_0) \cap B_0^{w^{-1}} = B_w$ by (i). In particular, $y \in B_0$, whence $x \in B_0$. It follows that $b \in (B_0\cap B_0^{\,r})(B_0 \cap B_0^{w^{-1}}) = B_rB_w$. Thus $B_0 \subseteq B_rB_w$ and we are done as the other inclusion is trivial.

(iii) First of all, assume $w^1 = r$. Then $\ell(wr) > \ell(w)$, so $\ell(rw^{-1}) > \ell(w^{-1})$. Hence by (i),

$$B_{wr} = B^{rw^{-1}}_{rw^{-1}} = (B^r_{w^{-1}} \cap B_0)^{rw^{-1}} \subseteq B^{w^{-1}}_{w^{-1}} = B_w,$$

so that $B_{ww^1} \subseteq B_w$ if $w^1 \in R$. The full statement follows by repeated application of the latter inclusion. □

2.11. From now on in this section, W *is a Weyl group and* R *is a set of fundamental reflections generating* W. Therefore, W is finite and there is a longest element w_0 in W (cf. Lemma 1.13(viii)). For $w \in W$, set $B^-_w = B_{ww_0}$.

LEMMA. *Suppose* (B_0,N_0) *is saturated. Then for* $w \in W$ *and* $r \in R$ *with* $\ell(rw)>\ell(w)$ *we have:*

(i) $B_{w_0} = H_0$

(ii) $B^-_r \subseteq B_w$

(iii) $B^-_{rw} = B^-_r(B^-_w)^r$

PROOF. (i) By Lemmas 2.10 (iii) and 1.13 (viii) we have $B_{w_0} = B_{v(v^{-1}w_0)} \subseteq B_v$ for any $v \in W$. Hence $H_0 = \bigcap_{u\in W} B_0^u = \bigcap_{u\in W} B_u = B_{w_0}$, proving (i).
(ii) According to Lemma 1.13 (viii) and the hypothesis $\ell(w_0) - 1 = \ell(w^{-1}rw_0) + \ell(w^{-1}r) - 1 = \ell(w^{-1}rw_0) + \ell(w)$, or $\ell(w(w^{-1}rw_0)) = \ell(w) + \ell(w^{-1}rw_0)$. Therefore, Lemma 2.10 (iii) yields $B^-_r = B_{w(w^{-1}rw_0)} \subseteq B_w$, as wanted.
(iii) First of all, note that $\ell(r(rww_0)) = \ell(ww_0) = \ell(w_0) - \ell(w) > \ell(w_0) - \ell(rw) = \ell(rww_0)$, so that $B^-_r \subseteq B^-_{rw}$ by (ii).

Now $B_0 = B_rB_{rw_0} = B_rB^-_r = B^-_rB_r$ in view of Lemma 2.10 (ii). Therefore, $B^-_{rw} = B^-_{rw} \cap B_0 = B^-_{rw} \cap B^-_rB_r = B^-_r(B^-_{rw} \cap B_r)$ (recall that $B^-_r \subseteq B^-_{rw}$). Since $B_{r(rww_0)} \subseteq B^r_{rww_0}$ by Lemma 2.10 (i), we obtain $(B^-_w)^r \subseteq B^-_{rw} \subseteq B_0$, so that $B^-_{rw} \cap B_r = B_0 \cap B_0^{w_0w^{-1}r} \cap B_0^r = B_0 \cap (B^-_w)^r = (B^-_w)^r$, and $B^-_{rw} = B^-_r(B^-_{rw} \cap B_r) = B^-_r(B^-_w)^r$, as stated. □

2.12. PROPOSITION. *Suppose* (B_0,N_0) *is a saturated Tits system whose Coxeter group* W *is a Weyl group. Then for* $w \in W$ *the following prevails:*

$$B_0 = B_wB^-_w \text{ with } B_w \cap B^-_w = H_0.$$

PROOF. First of all observe that $B_w \cap B^-_w = B_0 \cap B_0^{w^{-1}} \cap B_0^{w_0w^{-1}} = B_0 \cap B_w^{w^{-1}} = H_0$ in regard to Lemma 2.11 (i). We next show $B_0 = B_wB^-_w$ by induction on $\ell(w)$. Note that the statement is equivalent to $B_0 = B^-_wB_w$ as all terms represent groups.

If $\ell(w) = 0$, then $B_0 = B_0H_0 = B_wB^-_w$ as $w = 1$, and we are done.

If $\ell(w) = 1$, then $w \in R$ and the equality holds as we have seen in the proof of Lemma 2.11 (iii).

Suppose $\ell(w) \geq 2$, and write $w = rw_1$ with $r \in R$ and $\ell(w) > \ell(w_1)$. Then

$$B_wB_w^- = (B_{w_1}^r \cap B_0)(B_{w_1}^-)^r B_r^- \qquad \text{(by Lemmas 2.10 (i) and (2.11 (iii))}$$

$$= ((B_{w_1} \cap B_0^r)B_{w_1}^-)^r B_r^-$$

$$= ((B_{w_1}B_{w_1}^-) \cap B_0^r)^r B_r^- \qquad \text{(as } B_{w_1}^- \subseteq B_0^r \text{ by the proof of Lemma 2.11 (iii))}$$

$$= (B_0 \cap B_0^r)^r B_r^- \qquad \text{(by induction)}$$

$$= B_rB_r^- = B_0$$

This establishes the proposition. □

2.13. From now on, let (B_0,N_0) be a split saturated Tits system (whose Coxeter group W is a Weyl group). Decompose $B_0 = H_0N$ for N a normal subgroup of B_0 with $H_0 \cap N = \{1\}$ and set $N^w = w^{-1}Nw$, $N_w = N \cap B_0^{w^{-1}}$ and $U_w = N_{ww_0}$ as before (cf. Theorem 1.17). The following corollary mimics statement (x) of Theorem 1.17.

COROLLARY. $N = N_wU_w = U_wN_w$ *with* $N_w \cap U_w = \{1\}$.

PROOF. Since $B_0 = B_wB_w^-$, we get $H_0N = H_0N_wU_w$. But $N_wU_w \subseteq N$, so $N_wU_w = N$ in view of $H_0 \cap N = \{1\}$. Finally, $H_0 = B_w \cap B_w^- = H_0(N_w \cap U_w)$, whence $N_w \cap U_w = \{1\}$.

2.14. Next we show that Theorem 1.17 (xi) has a generalization to the present setting. For $I,J \subseteq R$, recall from 2.6 that $D_{I,J} = \{w \in W | \ell(rw) = \ell(ws) > \ell(w)$ for all $r \in I$, $s \in J\}$ and that $D_J = D_{\emptyset,J}$. Write $L_{I,J}(w) = D_J \cap W_IwW_J$.

PROPOSITION. *Let* (B_0,N_0) *be a split saturated Tits system whose Coxeter group* W *is a Weyl group. Then for* $I,J \subseteq R$ *and* $w \in W$, *the following holds:*
(i) *If* $w \in D_J$, *then* $G_J \cap B_0^w \subseteq B_{w^{-1}}$.
(ii) $G_IwG_J = \bigcup_{d \in L_{I,J}(w)} U_ddG_J$,

with uniqueness of expression at the right hand side (up to the choice of a representative m_d *for each* $d \in L_{I,J}(w)$).

PROOF. (i) Let $w \in D_J$. Owing to the exchange condition 2.2, we have $wW_J \cap S_w = \{w\}$, where S_w is defined in 1.19. Hence $S_{w^{-1}}w \cap W_J = \{1\}$, so that

$$G_J \cap B_0^w \subseteq B_0W_JB_0 \cap B_0w^{-1}B_0wB_0 \subseteq$$

$$\subseteq B_0W_JB_0 \cap B_0S_{w^{-1}w}B_0 = B_0(W_J \cap S_{w^{-1}w})B_0$$

$$= B_0$$

This yields $G_J \cap B_0^w \subseteq B_0^w \cap B_0$, ending the proof of (i).
(ii) $G_IwG_J = B_0W_IwW_JB_0 = \cup_{d\in L_{I,J}(w)} B_0dW_JB_0 =$
$= \cup_{d\in L_{I,J}(w)} B_0dG_J = \cup_{d\in L_{I,J}(w)} U_ddG_J$ thanks to Proposition 2.5 (vi). As for the uniqueness, suppose $um_dg = u^1m_{d^1}g^1$ for $u,u^1 \in U_d$; $d,d^1 \in L_{I,J}(w)$; $g,g^1 \in G_J$. Then $d = d^1$ as $B_0dG_J = B_0d^1G_J$ according to Proposition 2.7 (ii). Thus $g^1g^{-1} = m_d^{-1}(u^1)^{-1}um_d \in G_J \cap B_0^d \subseteq B_{d^{-1}}$ by (i), whence $(u^1)^{-1}u \in U_d \cap$ $\cap B_{d^{-1}}^{d^{-1}} = N \cap B_{dw_0} \cap B_d = N \cap (B_{w_0})^{d^{-1}} = N \cap H_0 = \{1\}$. The conclusion is that $u = u^1$ and $g = g^1$. □

2.15. EXAMPLE. Let G be a real noncompact connected semi-simple Lie group with finite center. As a topological space, G/G_J is homeomorphic to $K/(K\cap G_J)$. Hence it is compact. Analogously to the proof of Proposition 1.12 (ii), it can be shown that the submanifolds U_wwG_J/G_J ($w \in D_J$) are regularly embedded in G/G_J. (Note that $U_{w_0}w_0G_J$ is open in G as $U_{w_0}w_0B_0$ is open in G). For $d \in D_J$, the submanifold $B_0dG_J/G_J = U_ddG_J/G_J$ is diffeomorphic to a vector space over $\mathbb{R}$ of dimension dim $\mathfrak{u}_d$. Thus, there is at least one element $d \in D_J$ such that $\mu(\Sigma^+ \cap d\Sigma^-) = \dim G - \dim G_J = \mu(\Sigma^+) - \mu(\Sigma^+(J))$. The nonclosed submanifolds of minimal dimension among the B_0wG_J/G_J ($w \in D_J$) are those of the form B_0rG_J/G_J for $r \in R\backslash J$. The closures of these submanifolds will play the rôle of lines in the geometry on G/G_J to be discussed in the next section.

3. THE GEOMETRY OF TITS SYSTEMS

Throughout this section G is a group with Tits system (B_0,N_0) whose Coxeter group W has a distinguished generating set R of fundamental reflections.

3.1. There are two formally distinct ways to describe geometries. They are illustrated by the following pair of definitions of a projective plane.

(A) A *projective plane* is a set P of *points* together with a collection L of subsets of P, called *lines*, such that

(i) Any two points are in exactly one line

(ii) Any two lines have a point in common

(iii) Lines have at least three points

(iv) There is a point and a line such that the point is not in the line.

(B) A *projective plane* is a tuple (P,L) of sets P (points) and L (lines) together with a relation, called *incidence*, between P and L, such that (i), (ii), (iii), (iv) of (A) hold (here, a point p is said to be in a line ℓ if p and ℓ are incident, and so on).

It is obvious how to obtain one description of a projective plane from the other. The advantage of (B) over (A) is that duality between points and lines is more naturally present. On the other hand in the 'classical approach' (A) a single set P with its structure is studied, rather than such a formal object as in (B).

The Tits system in G gives rise to a geometry in the vein of (B), as we shall now see.

3.2. DEFINITIONS. For $J \subseteq R$, write $G^J = G_{R\setminus J}$.

If r,s are distinct fundamental reflections of W, we say that $xG^{\{r\}}$ and $yG^{\{s\}}$ for $x,y \in G$ are *incident* if $xG^{\{r\}} \cap yG^{\{s\}} \neq \emptyset$.

The tuple $(G/G^{\{r\}})_{r\in R}$ equipped with the incidence relation thus defined, will be called the *Tits geometry associated with* (B_0,N_0). This geometry is closely related to the notion of building, cf. Tits [T1] and [T2].

It is our purpose to derive properties of geometries defined as in (A) and obtained from Tits geometries by letting one of the sets $G/G^{\{r\}}$ for $r \in R$ be the set of points. We shall formulate this in terms of incidence systems.

An *incidence system* (P,L) is defined to be a set P of *points* and a collection L of subsets of P, called *lines*, such that every line has at least two distinct points. If (P,L) is an incidence system, then the *collinearity graph* of (P,L) is the graph whose vertex set is P and whose edges are the pairs of collinear points. The incidence system is called *connected* whenever its collinearity graph is connected. Likewise terms such as *connected components, cliques, paths* will be applied freely to (P,L) when in fact they are meant for the collinearity graph (a *clique* of a graph is a set of vertices in the graph such that each pair of mutually distinct members is an edge). For $x,y \in P$, we let d(x,y) denote the ordinary distance in the collinearity graph and write $x \perp y$ to express that $d(x,y) \leq 1$ (i.e. x and y are collinear). Moreover, $x^\perp = \{y \in P \mid y \perp x\}$. For X a subset of P, we write $X^\perp = \bigcap_{x \in X} x^\perp$.

The subset X is called *nondegenerate* if $X \cap X^\perp = \emptyset$; it is called a *subspace* (of (P,L)) whenever each point of P belonging to a line that has two distinct points in X, is itself in X. A subspace is called *singular* if it is a clique. The length i of a longest chain $X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_i = X$ of nonempty singular subspaces X_j of a singular subspace X is called the *rank* of X and denoted by rk(X). The incidence system is called *linear* if two distinct points are in at most one line.

From now on, fix $r \in R$. If $g \in G$, write $\underline{g} = gG^{\{r\}}$ and $\overline{g} = gG_{\{r\}}G^{\{r\}}/G^{\{r\}}$ $(= gB_0\langle r\rangle G^{\{r\}}/G^{\{r\}})$. For $X \subseteq G$, write $\underline{X} = \{\underline{g} \mid g \in X\}$ and $\overline{X} = \{\overline{g} \mid g \in X\}$.

Then $(\underline{G},\overline{G})$ is the incidence system that we are interested in. Note that for $g \in G$ the line $\overline{g}$ is (indeed) determined by the incidence relation of the Tits geometry via $\overline{g} = \{\underline{x} \in \underline{G} \mid xG^{\{r\}} \cap gG_{\{r\}} \neq \emptyset\}$. Thus $\overline{g}$ has at least two distinct points, namely $\underline{g}$ and $\underline{gr}$.

This incidence system is said to be a *Lie incidence system of type* W *over* r, or just *of type* $X_{n,i}$ where X_n is the type of the Coxeter diagram for which W is the Weyl group, and where i is the node in X_n corresponding to r. To fix notation, we let the nodes in X_n (for connected diagrams) be numbered as depicted in the table.

We shall establish some properties of $(\underline{G},\overline{G})$. Note that G acts transitively on $(\underline{G},\overline{G})$ by left multiplication on $\underline{G}$ as a group of automorphisms. For $I,J \subseteq R$ (recall that $G^J = G_{R\setminus J}$ and) write $W^J = W_{R\setminus J}$.

TABLE

Coxeter diagrams with numbered nodes associated with Dynkin diagrams

A_n $(n\geq 1)$	$1 - 2 - \cdots - (n-1) - n$
$B_n = C_n$ $(n\geq 2)$	$1 - 2 - \cdots - (n-1) = n$
D_n $(n\geq 3)$	$1 - 2 - \cdots - (n-2) - n$, with $(n-1)$ joined to $(n-2)$
E_6	$1 - 2 - 3 - 5 - 6$, with 4 joined to 3
E_7	$1 - 2 - 3 - 4 - 6 - 7$, with 5 joined to 4
E_8	$1 - 2 - 3 - 4 - 5 - 7 - 8$, with 6 joined to 5
F_4	$1 - 2 = 3 - 4$
G_2	$1 \equiv 2$

Finally, W is called *irreducible* if there is no partitioning $R = I \dot{\cup} J$ with $I,J \neq R$ such that $st = ts$ for all $s \in I$, $t \in J$. This is equivalent to connectedness of the Dynkin diagram, and to irreducibility of W in its natural linear representation as a group generated by reflections, cf. Bourbaki [Bou, Ch VI §1.2].

3.3. LEMMA. *Suppose* W *is irreducible. Set* $P_r = \{w \in W \mid \ell(rw) > \ell(w)\}$ *for* $r \in R$ (cf. Proposition 2.2). *Then*

(i) $S_w \subseteq W_J$ *whenever* $w \in W_J$.

(ii) $W_J \cap W_I w W_I = W_{I\cap J} w W_{I\cap J}$ *for any* $I,J \subseteq R$ *and* $w \in W_J$.

(iii) *For any* $J \subseteq R$ *and* $r \in R\backslash J$, *we have* $P_r \cap rW_J r = W_J \cap rW_J r$, *and the elements of this set commute with* r.

PROOF. (i) This is an immediate consequence of Lemma 2.6.
(ii) Clearly, $W_{I\cap J}wW_{I\cap J} \subseteq W_J \cap W_IwW_I$.

In view of (i) and Lemma 2.6, there are $w_1, w_2 \in W_{I\cap J}$ and $w^1 \in W_J \cap D_{I,I}$, such that $w = w_1w^1w_2$. This implies $W_J \cap W_IwW_I = W_J \cap W_Iw^1W_I$ and $W_{I\cap J}wW_{I\cap J} = W_{I\cap J}w^1W_{I\cap J}$, so that upon replacement of w by w^1, we may assume $w \in D_{I,I} \cap W_J$. Due to Lemma 2.6, for $u \in W_IwW_I$, there exist $u_1, u_2 \in W_I$ such that $u = u_1wu_2$ and $\ell(u) = \ell(u_1) + \ell(w) + \ell(u_2)$. So if $u \in W_J \cap W_IwW_I$, statement (i) yields that $u_1, u_2 \in W_J$, whence $u = u_1wu_2 \in W_{I\cap J}wW_{I\cap J}$, and we are done.
(iii) Assume $w \in P_r$ and $w^1 \in W_J$ satisfy $w = rw^1r$. Since $rw = w^1r \notin W_J$, the Exchange condition 2.2 yields $\ell(w) = \ell(rw)-1 = \ell(w^1r)-1 = \ell(w^1)$, i.e., $\ell(w) = \ell(w^1)$. Using the Exchange condition once again, we obtain that either there is $w^{11} \in W_J$ with $w = rw^1r = w^{11}r$ or $rw^1r = w^1$.

But the former case leads to the absurdity $w^1 \in W_J \cap rW_J = \emptyset$, so r and w^1 commute, and $w = w^1$.

This shows that $P_r \cap rW_Jr$ is contained in $W_J \cap rW_Jr$. Since the other inclusion is obvious, this settles the lemma. □

3.4. LEMMA. *Let* $r \in R$ *and* $J \subseteq R\backslash\{r\}$. *Set* $D = W_J\langle r\rangle W_J\langle r\rangle \ldots \langle r\rangle W_J$ (2d+1 terms) *and* $E = W_J \cap rW_Jr$. *Then*
(i) $B_0rB_0W_J\langle r\rangle B_0 \cap B_0W_J\langle r\rangle B_0 \subseteq B_0\langle r\rangle EB_0$.
(ii) $bDB_0 \cap B_0rB_0DB_0 = b(D\cap rD)B_0$ *for any* $b \in B_0$.
(iii) $B_0DB_0 \cap B_0rB_0DB_0 = B_0(D\cap rD)B_0 = rB_0(D\cap rD)B_0$.

PROOF. (i) Note that $rP_r \cap W_J\langle r\rangle = rW_J \cap W_Jr = rE$ due to the previous lemma. Hence,

$$\begin{aligned}
&B_0rB_0W_J\langle r\rangle B_0 \cap B_0W_J\langle r\rangle B_0 \subseteq \\
&\subseteq (B_0rW_J\langle r\rangle B_0 \cup (B_0(rP_r\cap W_J\langle r\rangle)B_0)) \cap B_0W_J\langle r\rangle B_0 \subseteq \\
&\subseteq (B_0(rW_J\langle r\rangle \cap W_J\langle r\rangle)B_0 \cup B_0(rP_r \cap W_J\langle r\rangle)B_0 \subseteq \\
&\subseteq B_0(E \cup rE)B_0 = B_0\langle r\rangle EB_0
\end{aligned}$$

by use of Proposition 2.5. This ends the proof of (i).
(ii) Let $b \in B_0$. Suppose $y \in bwB_0 \cap B_0rB_0w^1B_0$ for $w, w^1 \in D$. If $\ell(rw^1) > \ell(w^1)$, then $B_0rB_0w^1B_0 = B_0rw^1B_0$ by Preposition 2.5, so $w = rw^1 \in D \cap rD$, and $y \in bwB_0 \subseteq b(D\cap rD)B_0$. Assume $\ell(rw^1) < \ell(w^1)$. Then $w^1 \notin W_J$. By the exchange condition there is an expression

$$w^1 = rd \quad \text{with } d \in D$$

(Note that $u \in D$ implies $S_u \subseteq D$).

Thus $w^1, rw^1 \in D \cap rD$ and $y \in B_0rw^1B_0 \cup B_0w^1B_0 \subseteq B_0(D \cap rD)B_0$. Hence $w \in D \cap rD$ and $y \in bwB_0 \subseteq b(D \cap rD)B_0$. We have shown $bDB_0 \cap B_0rB_0DB_0 \subseteq b(D \cap rD)B_0$. Since the other inclusion is obvious, this proves (ii).

(iii) The first equality is obvious from (ii). Since $r(D \cap rD) = D \cap rD$, we have

$$rB_0(D \cap rD)B_0 \subseteq B_0rB_0(D \cap rD)B_0 \subseteq B_0(D \cap rD)B_0 \cup B_0r(D \cap rD)B_0 =$$

$$= B_0(D \cap rD)B_0.$$

Therefore, also $B_0(D \cap rD)B_0 \subseteq rB_0(D \cap rD)B_0$, whence the second equality of (iii). □

3.5. We are now ready to formulate some basic properties of Lie incidence systems.

<u>PROPOSITION</u>. *Let* $(\underline{G},\overline{G})$ *be a Lie incidence system as described above. Then*

(i) $\underline{x},\underline{y}$ *are collinear points of* $(\underline{G},\overline{G})$ *iff* $y^{-1}x \in G^{\{r\}}G_{\{r\}}G^{\{r\}}$.

(ii) G *is transitive on ordered pairs of distinct collinear points.*

(iii) $(\underline{G},\overline{G})$ *is a clique iff* $W = W^{\{r\}}W_{\{r\}}W^{\{r\}}$. *If it is not a clique, it is nondegenerate.*

(iv) *Any two lines of* $(\underline{G},\overline{G})$ *meet iff* $W = W_{\{r\}}W^{\{r\}}W_{\{r\}}$.

(v) $(\underline{G},\overline{G})$ *is a linear incidence system.*

(vi) *For* $d \in \mathbb{N}$, $\underline{x} \in \underline{G}$, *and* $\overline{z} \in \overline{G}$ *such that* $\overline{z}$ *has two points of distance* $\leq d$ *to* $\underline{x}$, *any point of* $\overline{z}$ *has distance* $\leq d$ *to* x.

(vii) *If* $I \subseteq R$ *with* $r \in I$, *then* G_I *is a subspace of* $(\underline{G},\overline{G})$, *which, together with the lines that it contains, is a Lie incidence system of type the Coxeter diagram of* W_I *over* r.

<u>PROOF</u>. (i) $\underline{x},\underline{y}$ are collinear iff there is $\overline{z} \in \overline{G}$ with $\underline{x},\underline{y} \in \overline{z}$. Since $\underline{x},\underline{y} \in \overline{z} \Leftrightarrow x,y \in zG_{\{r\}}G^{\{r\}} \Leftrightarrow z \in xG^{\{r\}}G_{\{r\}} \cap yG^{\{r\}}G_{\{r\}}$, we get that $\underline{x},\underline{y}$ are collinear iff $y^{-1}xG^{\{r\}}G_{\{r\}} \cap G^{\{r\}}G_{\{r\}} \neq \emptyset$, i.e. iff $y^{-1}x \in G^{\{r\}}G_{\{r\}}G^{\{r\}}$. This establishes (i).

(ii) Since G is transitive on $\underline{G}$, we need only check that the stabilizer $G^{\{r\}}$ of $\underline{1}$ in G is transitive on the neighbors of $\underline{1}$. But it is immediate from (i) that the set of neighbors of $\underline{1}$ is $\underline{G^{\{r\}}r}$, so that the check is trivial.

(iii) This is a consequence of (i) and (ii), in view of Proposition 2.5 and

Lemma 2.6.

(iv) $(\forall \bar{x},\bar{y} \in \bar{G})(\bar{x} \cap \bar{y} \neq \emptyset) \Longleftrightarrow (\forall x,y \in G)(xG_{\{r\}}G^{\{r\}} \cap yG_{\{r\}}G^{\{r\}} \neq \emptyset) \Longleftrightarrow$ $\Longleftrightarrow (\forall g \in G)(g \in G_{\{r\}}G^{\{r\}}G_{\{r\}}) \Longleftrightarrow G = G_{\{r\}}G^{\{r\}}G_{\{r\}} \Longleftrightarrow W = W_{\{r\}}W^{\{r\}}W_{\{r\}}$ where in the last equivalence Proposition 2.5 and Lemma 2.6 are used again. Hence (iv).

(v) Suppose $\underline{x},\underline{y} \in \bar{z}$ for $x,y,z \in G$ with $\underline{x} \neq \underline{y}$. Write $y^{-1}x = g_1rg_2$ with $g_1,g_2 \in G^{\{r\}}$. Then $z \in y(y^{-1}xG^{\{r\}}G_{\{r\}} \cap G^{\{r\}}G_{\{r\}})$ so $g_1^{-1}y^{-1}z \in rG^{\{r\}}G_{\{r\}} \cap$ $\cap\, G^{\{r\}}G_{\{r\}} = rB_0W^{\{r\}}\langle r\rangle B_0 \cap B_0W^{\{r\}}\langle r\rangle B_0 \subseteq B_0\langle r\rangle EB_0$, where $E = W^{\{r\}} \cap rWr^{\{r\}}$, by use of Lemma 3.4 (i). Thus $z \in yg_1G_{\{r\}}EB_0$. It follows that if $\underline{x},\underline{y} \in \bar{z}_1$, then

$$z_1G_{\{r\}}G^{\{r\}} = yg_1G_{\{r\}}EB_0\langle r\rangle G^{\{r\}} =$$

$$= yg_1G_{\{r\}}E\langle r\rangle G^{\{r\}} = yg_1G_{\{r\}}\langle r\rangle EG^{\{r\}} =$$

$$= yg_1G_{\{r\}}G^{\{r\}} = zG_{\{r\}}G^{\{r\}},$$

so that $\bar{z}_1 = \bar{z}$.

This settles uniqueness of the line $\bar{z}$ on $\underline{x},\underline{y}$. The conclusion is that $(\underline{G},\bar{G})$ is a linear incidence system, as wanted.

(vi) Since G is transitive on pairs of distinct collinear points, we may assume that $\bar{z}$ is the (unique) line on $\underline{1}$ and $\underline{r}$, and that $\underline{x}$ has distance $\leq d$ to $\underline{1}$ and $\underline{r}$. This means that

$$x \in \underbrace{G^{\{r\}}\langle r\rangle G^{\{r\}}\ldots\langle r\rangle G^{\{r\}}}_{(2d+1)\text{ factors}} \cap \underbrace{rG^{\{r\}}\langle r\rangle\ldots\langle r\rangle G^{\{r\}}}_{(2d+1)\text{ factors}}$$

In other terms, with D as in Lemma 3.4, we have $x \in B_0DB_0 \cap rB_0DB_0$, whence $rbx \in B_0DB_0$.

The previous lemma yields $B_0x \subseteq rB_0(D\cap rD)B_0$, whence $rbx \in B_0DB_0$ for any $b \in B_0$. This shows that $\underline{x}$ has distance $\leq d$ to $\underline{br}$ for any $b \in B_0$, as $\bar{z} = \{\underline{1}\} \cup \underline{B_0r}$, this yields (vi).

(vii) Suppose $x,y \in G_I$ satisfy $\underline{x} \perp \underline{y}$, $\underline{x} \neq \underline{y}$. Write $y^{-1}x = g_1rg_2$ with $g_1,g_2 \in G_{I\setminus\{r\}}$ (note that $G_I \cap G^{\{r\}}G_{\{r\}}G^{\{r\}} = B_0(W^{\{r\}}\langle r\rangle W^{\{r\}} \cap W_I)B_0 =$ $= B_0W_{I\setminus\{r\}}\langle r\rangle W_{I\setminus\{r\}}B_0 = G_{I\setminus\{r\}}G_{\{r\}}G_{I\setminus\{r\}}$ by Lemma 2.6). The line on $\underline{x},\underline{y}$ is $\overline{yg_1}$, so if $\underline{z} \in \overline{yg_1}$, then $\underline{z} = \underline{yg_1h}$ for some $h \in G_{\{r\}}$. In particular $\underline{z} \in \underline{G_I}$. This shows that the line on $\overline{\underline{x},\underline{y}}$ is entirely contained in $\underline{G_I}$. The conclusion is that G_I is a subspace.

Finally, it is easily seen that the map

$$\phi : G_I/G_{I\setminus\{r\}} \to G_I G^{\{r\}}/G_{\{r\}} \text{ given by}$$

$$\phi(xG_{I\setminus\{r\}}) = xG^{\{r\}} = \underline{x} \qquad (x \in G_I)$$

leads to an isomorphism between the incidence systems

$$(G_I/G_{I\setminus\{r\}}, \{zG_{\{r\}}G_{I\setminus\{r\}} \mid z \in G_I\}) \text{ and } (\underline{G}_I, \bar{G}_I).$$

Since the former is a Lie incidence system of type the Coxeter diagram of W_I over r, so is the latter. □

3.6. The next result states some well-known facts of Weyl groups. The Weyl group whose Dynkin diagram is of type X_n will be denoted by $W(X_n)$. The fundamental reflection in $W(X_n)$ corresponding to the node i in the Dykhin diagram (numbered according to the table) is denoted r_i. Recall from 2.6 that $D_{I,J}$ for $I,J \subseteq R$ is the set $\{w \in W \mid \ell(rw) > \ell(w) < \ell(ws)$ for all $r \in I$, $s \in J\}$ and that it is a set of $W_I - W_J$ double coset representatives.

PROPOSITION. (i) *If* $W = W(A_n)$, *then* $D_{R\setminus\{r_1\},R\setminus\{r_1\}} = \{1,r_1\}$.
(ii) *If* $W = W(B_n)$ $(n\geq 2)$ *or* $W = W(D_n)$ $(n\geq 3)$, *then* $D_{R\setminus\{r_1\},R\setminus\{r_1\}} = \{1,r_1,w_1\}$, *where*

$$w_1 = \begin{cases} r_1r_2\cdots r_{n-1}r_nr_{n-1}\cdots r_2r_1 & \textit{if } W = W(B_n) \\ r_1r_2\cdots r_{n-1}r_nr_{n-2}r_{n-3}\cdots r_2r_1 & \textit{if } W = W(D_n) \end{cases},$$

and w_1 *normalizes* $W^{\{r_1\}}$.

PROOF. To facilitate notation, set $U = W^{\{r_1\}}$.
(i) If n=1, there is nothing to prove. Let n>1. Suppose $w \in U$. Then by induction on n, we may assume $w \in U^{\{r_2\}} \cup U^{\{r_2\}}r_2U^{\{r_2\}}$. Because r_1 centralizes $U^{\{r_2\}}$ and $m_{r_1,r_2} = 3$, this yields $r_1wr_1 \in U^{\{r_2\}} \cup U^{\{r_2\}}r_1r_2r_1U^{\{r_2\}} \subseteq U \cup$ $\cup\, Ur_2r_1r_2U = U \cup Ur_1U$. Thus $r_1Ur_1 \subseteq U \cup Ur_1U$, and statement (i) readily follows.

(We remark that another way of proof for this statement would be to identify the permutation representation of W on the cosets of $W^{\{r_1\}}$ as that of the symmetric group on n+1 letters)
(ii) The two cases $W = W(B_n)$ and $W = W(D_n)$ being much alike, we shall only

deal with $W = W(B_n)$. First of all, note that for $1<i<n$ we have

$$\begin{aligned} r_i w_1 &= r_1 r_2 \cdots r_{i-2} r_i r_{i-1} r_i \cdots r_{n-1} r_n \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{i-2} r_{i-1} r_i r_{i-1} r_{i+1} \cdots r_{n-1} r_n \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{n-1} r_n \cdots r_{i+1} r_{i-1} r_i r_{i-1} \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{n-1} r_n \cdots r_{i+1} r_i r_{i-1} r_i r_{i-2} \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{n-1} r_n \cdots r_1 r_i = w_1 r_i, \end{aligned}$$

while

$$\begin{aligned} r_n w_1 &= r_1 r_2 \cdots r_{n-2} r_n r_{n-1} r_n r_{n-1} \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{n-2} r_{n-1} r_n r_{n-1} r_n r_{n-2} \cdots r_2 r_1 = \\ &= r_1 r_2 \cdots r_{n-1} r_n r_{n-1} \cdots r_2 r_1 r_n = w_1 r_n, \end{aligned}$$

so that w_1 centralizes U (if $W = W(D_n)$, then w_1 normalizes U but does centralize U).

Next, we show that $w_1 \notin U \cup Ur_1U$. If there are $u_1, u_2 \in U$ with $w_1 = u_1 r_1 u_2$, then $w_1^{-1} r_1 = u_1^{-1} u_2^{-1} \in U$ as w_1 centralizes U, whence $w_1^{-1} r_1 = r_1 r_2 \cdots r_{n-1} r_n r_{n-1} \cdots r_2 \in r_1 U \cap U = \emptyset$, a contradiction. Therefore, $w_1 \notin Ur_1U$. Suppose $w_1 \in U$. Then $r_1 w_1 r_1 \in U \cap r_1 U r_1$, so that $r_1 w_1 r_1 = w_1$ by Lemma 3.3 (iii). But this yields $r_2 w_1 r_2 = w_1 \in U^{\{r_2\}}$, and so on, leading to $w_1 = r_n$. Since this implies that r_n commutes with r_{n-1} contradicting $W = W(B_n)$, we conclude that $w_1 \notin U$.

By now, it is readily verified that $w_1 \in D_{R\setminus\{r_1\},R\setminus\{r_1\}}$. For, by symmetry of w_1 ($w_1^2 = 1$), it suffices to check that $w_1 \in D_{R\setminus\{r_1\}}$, i.e. that $\ell(w_1 r_i) > \ell(w_1)$ for each i ($1<i\leq n$). In view of the Exchange condition, the converse would lead to an element w_1^1 obtained from w_1 by omission of a single reflection from its minimal expression, such that $w_1^1 \notin U \cup Ur_1U$. Clearly, omission of r_1 is out of the question. Omission of r_i ($1<i<n$) would lead to

$$\begin{aligned} w_1^1 &= r_1 r_2 \cdots r_{i-1} r_{i+1} \cdots r_{n-1} r_n r_{n-1} \cdots r_2 r_1 = \\ &= r_{i+1} \cdots r_{n-1} r_n r_{n-1} \cdots r_{i+1} (r_1 r_2 \cdots r_{i-1}) r_i r_{i-1} \cdots r_1 \in \\ &\in U r_1 r_2 \cdots r_{i-2} r_i r_{i-1} r_i r_{i-2} \cdots r_2 r_1 U = \\ &= U r_1 r_2 \cdots r_{i-2} r_{i-1} r_{i-2} \cdots r_2 r_1 U = \end{aligned}$$

$$= \dots = Ur_1r_2r_1U = Ur_1U,$$

while omission of r_n results in

$$w_1^1 = r_1r_2\dots r_{n-1}r_{n-1}\dots r_2r_1 = 1.$$

The conclusion is that there is no i $(1<i\leq n)$ with $\ell(w_1r_i) \leq \ell(w_1)$, proving $w_1 \in D_{R\backslash\{r_1\},R\backslash\{r_1\}}$.

It remains to establish that W consists of the three double U-cosets (represented by $1,r_1,w_1$) only. To do so, it suffices to derive that $V = U \cup Ur_1U \cup Uw_1U$ is closed under right and left multiplication by r_1.

If $n = 2$, then $V = \langle r_2\rangle \cup \langle r_2\rangle r_1\langle r_2\rangle \cup r_1r_2r_1\langle r_2\rangle = W$, so we are done. Suppose, therefore, that $n > 2$. By induction, $U = U^{\{r2\}} \cup {}^{\{r2\}}r_2U^{\{r2\}} \cup r_1w_1r_1U^{\{r2\}}$. Since r_1 centralises $U^{\{r2\}}$ and $r_1r_2r_1 = r_2r_1r_2$, this yields $r_1Ur_1 = U^{\{r2\}} \cup U^{\{r2\}}r_1r_2r_1U^{\{r2\}} \cup U^{\{r2\}}w_1U^{\{r2\}} \subseteq V$. As $r_1Uw_1 = r_1w_1U \subseteq Ur_1U$, we obtain $r_1V = V_{r_1} = V$, as wanted. □

3.7. Part (i) of the following lemma is a converse to Proposition 3.6 (i).

<u>LEMMA</u>. *Suppose* W *is an irreducible Weyl group and* $r \in R$.
(i) *If* $W = W^{\{r\}} \cup W^{\{r\}}rW^{\{r\}}$, *then there is* $n \in \mathbb{N}$ *such that* $W = W(A_n)$ *and* $r = r_1$, *or* r_n.
(ii) *If* $W = W^{\{r\}} \cup W^{\{r\}}r \cup rW^{\{r\}} \cup rW^{\{r\}}r$, *then* $W = W(A_2)$.

<u>PROOF</u>. (i) In view of Lemma 3.3 (ii) applied to $J \subseteq R$ with $r \in J$, we have $W_J = W_{J\backslash\{r\}} \cup W_{J\backslash\{r\}}rW_{J\backslash\{r\}}$.

Taking $J = \{r,s\}$ for $s \in R$, we get $m_{r,s} \leq 3$. Suppose the Coxeter diagram of W has a subdiagram of type B_n $(n\geq 2)$ or $D_n (n\geq 3)$ containing r as node numbered 1. Then, taking for J the reflections of the nodes in this subdiagram, we get by the above equality that W_J has only two double cosets with respect to $W_{J\backslash\{r\}}$, which conflicts Proposition 3.6 (ii). Therefore, using the table, we obtain that $W = W(A_n)$.
(ii) Set $U = W^{\{r_1\}}$. Since $1 \in U \cap rUr$, we have that there are at most three left U-cosets in W. Clearly, $|W/U| \geq 2$, and if $|W/U| = 2$, then $rU = Ur$, so that r centralizes U, which conflicts the irreducibility of W. Thus W has precisely three left U-cosets. Let $s \in R$ with $rs \neq sr$. Then $W = U \cup rU \cup srU$. Hence the group W has a representation as the full group of permutations on the three cosets of U. It is readily seen (e.g., as $W(A_n)$ is isomorphic to

the symmetric group on n+1 letters) that this representation is faithful, i.e., that $W \cong W(A_2)$ (as abstract groups). But then $W = \langle r,s\rangle = W(A_2)$ and we are done. □

3.8. THEOREM. *Let* (B_0,N_0) *be a Tits system in* G *whose Coxeter group* W *is an irreducible Weyl group. Then*

(i) *The Lie incidence system* $(\underline{G},\bar{G})$ *is of type* $A_{2,1}$ *(or, equivalently, of type* $A_{2,2}$*) iff it strictly contains a line and any two lines meet.*

In this case, $(\underline{G},\bar{G})$ *is a projective plane.*

(ii) *The Lie incidence system* $(\underline{G},\bar{G})$ *is of type* $A_{n,1}$ *(or, equivalently, of type* $A_{n,n}$*) iff* $(\underline{G},\bar{G})$ *is a singular space of rank* n. *If this holds,* $(\underline{G},\bar{G})$ *is a projective space of rank* n.

PROOF. (i) Suppose $(\underline{G},\bar{G})$ is of type $A_{2,1}$. Then $W = W(A_2) = \langle r_1\rangle\langle r_2\rangle\langle r_1\rangle = \langle r_2\rangle\langle r_1\rangle\langle r_2\rangle$, according to Proposition 3.6 (i) (or by direct verification!) so that Proposition 3.5 (iii), (iv) can be applied. Together with statement (v) of the same proposition, this yields that Axioms (i) and (ii) of a projective plane given in 3.1 hold for $(\underline{G},\bar{G})$. As to Axiom (iii), note that $\bar{1}$ has cardinality 2 iff $B_0 rG^{\{r\}} = rG^{\{r\}}$, which is equivalent to $B_0^r \subseteq G^{\{r\}}$. Since $B_0 \subseteq G^{\{r\}}$, this would lead to $G_{\{r\}} = B_0 \cup B_0 rB_0 = B_0 rB_0 rB_0 = B_0 B_0^r B_0 \subseteq G^{\{r\}}$, which is clearly absurd, in view of Lemma 2.6. Hence, $\bar{1}$, and therefore each line, has cardinality > 2. Finally, the point $\underline{r}_2$ is not on the line $\bar{1}$, so Axiom (iv) holds, too. Thus, $(\underline{G},\bar{G})$ is a projective plane. As for the other implication, assume $\underline{G}$ strictly contains a line and any two lines meet. Then by Proposition 3.5, we have $W = W^{\{r\}} \cup W_r^{\{r\}} \cup rW^{\{r\}} \cup rW^{\{r\}}r$, so by Lemma 3.7 (ii), $W = W(A_2)$, and we are done.

(ii) Suppose that $(\underline{G},\bar{G})$ is of type $A_{n,1}$, and put $r = r_1$. Then $W = W^{\{r\}}\langle r\rangle W^{\{r\}}$ by Proposition 3.6 (i), so $(\underline{G},\bar{G})$ is a clique, indeed. If n = 1, then $\underline{G}$ consists of a line. Let n > 1.

We derive that $\underline{G},\bar{G}$ are the points and lines of a projective space by proving the fundamental axiom of Veblen and Young, cf. [VY]. This axiom states that any two lines meeting in a point span a projective plane. In other words, let ℓ,m be two distinct lines intersecting in x, then any two lines intersecting both ℓ,m outside x meet. In order to verify this we may assume $x = \underline{1}$ as G is transitive on $\underline{G}$. Since the stabilizer $G^{\{r\}}$ in G of x is transitive on lines through x, we may assume $\ell = \bar{1}$. The stabilizer of both x and ℓ contains $G^{\{r,r_2\}}$. Since the latter group is transitive on $\underline{1}^\perp\backslash\bar{1}$ (for $\underline{1}^\perp = \underline{G^{\{r\}}\langle r\rangle} = \underline{G^{\{r,r_2\}}\langle r\rangle} \cup \underline{G^{\{r,r_2\}}r_2G^{\{r,r_2\}}\langle r\rangle} =$

$= \underline{B_0\langle r\rangle} \cup \underline{G^{\{r,r_2\}}r_2r}$, so that $\underline{1}^{\perp}\backslash\bar{1} = \underline{G^{\{r,r_2\}}r_2r}$), we may assume that $m = \overline{r_2}$. Thus, x,ℓ,m are contained in $\underline{G_{\{r,r_2\}}}$, so we may restrict attention to the latter subspace (cf. Proposition 3.5 (vii)). By the same proposition and Theorem 3.8, it results that ℓ,m span a projective plane. Thus $(\underline{G},\bar{G})$ is a projective space. An easy induction argument shows that $\underline{1} = G_{\{r\}} \subsetneq \underline{G}_{\{r,r_2\}} \subsetneq \cdots \subsetneq G_{\{r,r_2,\ldots,r_n\}}$ is a maximal chain of nonempty singular subspaces. Consequently, $(\underline{G},\bar{G})$ has rank n.

To settle the reverse, observe that if $(\underline{G},\bar{G})$ is a singular space, then $W = W^{\{r\}}W_{\{r\}}W^{\{r\}}$, so that $W = W(A_m)$ for some $m \in \mathbb{N}$, and $r = r_1$ or r_n. If moreover, $(\underline{G},\bar{G})$ has rank n, so we must have $m = n$ by the previous paragraph.

This ends the proof of the theorem. □

<u>3.9. EXAMPLE</u>. Consider the case where G is a real semi-simple connected non-compact Lie group of type A_n $(n>1)$ with finite centre. The incidence system $(\underline{G},\bar{G})$ of type $A_{n,1}$ is then a projective space of rank n by the above theorem. This incidence system has the topology of a compact Hausdorff space on which G operates as a topological transformation group. Its lines are closed sub-varieties homeomorphic to one point compactifications of real vector spaces of dimension $\mu = \mu_\alpha$ for α a fundamental root of the root system of G (cf. (cf. 1.12 and 1.19).

In fact, the projective space is defined over a real division algebra F of dimension μ. Thus $\mu = 1,2,4$ (or 8 if $n=2$) according as F consists of the real, complex, quaternion (or octonion) numbers. The corresponding Lie groups are $G = SL(n+1,\mathbb{R})$, $SL(n+1,C)$, $SU^*(2n+2) = SL(n+1,\mathbb{H})$ (and $E_{6(-26)}$) in the respective cases.

3.10. Here is another illustration of how to derive geometric properties from the Tits system. We derive the axiom that BUEKENHOUT & SHULT [BS] introduced to characterise Lie incidence systems of type $B_{n,1}$ $(n\geq 2)$ and $D_{n,1}$ $(n\geq 3)$ by means of earlier work of Veldkamp and Tits (see [T1]).

<u>THEOREM</u>. *Let* $(\underline{G},\bar{G})$ *be a Lie incidence system of type* $B_{n,1}$ $(n\geq 2)$ *or* $D_{n,1}$ $(n\geq 3)$. *Then for any point* $p \in \underline{G}$ *and any line* $\ell \in \bar{G}$, *the point* p *is collinear with precisely one or all points of* ℓ.

<u>PROOF</u>. Since $W = W(B_n)$ or $W(D_n)$, we get from Proposition 3.6 (ii) that $W = \langle r\rangle W^{\{r\}}\langle r\rangle W^{\{r\}}$. We shall show that this implies $p^{\perp} \cap \ell \neq \emptyset$.

In view of transitivity of G on lines, we may assume $\ell = \bar{1}$, so that with $p = \underline{x}$, we get

$$p^{\perp} \cap \ell = (G_{\{r\}}G^{\{r\}} \cap xG^{\{r\}}G_{\{r\}}G^{\{r\}})/G^{\{r\}}.$$

Thus, $p^{\perp} \cap \ell \neq \emptyset$ iff $x \in G_{\{r\}}G^{\{r\}}G_{\{r\}}G^{\{r\}}$. As $G_{\{r\}}G^{\{r\}}G_{\{r\}}G^{\{r\}} =$ $= B_0\langle r\rangle W^{\{r\}}\langle r\rangle W^{\{r\}}B_0 = B_0WB_0 = G$, we obtain $p^{\perp} \cap \ell \neq \emptyset$, as wanted. The theorem now follows from Proposition 3.5 (vi). □

3.11. We end this chapter with some references to further work in this area. First of all, [T1] is the basic reference for Tits systems and the geometries called buildings, while [T2] is of use for the connection between Lie incidence systems and buildings. In [Coo1], COOPERSTEIN derives for which Lie incidence systems the lines are of the form $x^{\perp} \cap y^{\perp}$ for distinct collinear points x,y. In [Coo2], he characterizes finite Lie incidence systems of type $A_{n,d}$, $D_{5,5}$ and $E_{6,1}$. The $A_{n,d}$-characterization has been extended to the arbitrary (not necessarily finite) case by COHEN [Coh2]; other extensions of Cooperstein's result are forthcoming. Finally, dual polar spaces, i.e., Lie incidence systems of type $B_{n,n}$ or $D_{n,n}$ have been dealt with by CAMERON [Cam], and metasymplectic spaces, i.e. systems of type $F_{4,1}$, by COHEN [Coh1].

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Chapter IV

THE FURSTENBERG BOUNDARY OF A SEMISIMPLE LIE GROUP

J. de VRIES

In this chapter we discuss the so-called Furstenberg boundary of a connected semisimple Lie group with finite center. Our approach is based on [4]. In particular, our point of view is that of topological dynamics. First we shall introduce some notions and results concerning T-spaces; this will culminate in the proof of the existence of a so-called universal strongly proximal minimal T-space for a given topological group T. Then for the case that T is a connected semisimple Lie group with finite center, this "abstract" entity is identified in terms of an Iwasawa decomposition. In 3 appendices, some results concerning compact convex sets in locally convex topological vector spaces and integration theory are summarized.

1. BASIC DEFINITIONS AND ELEMENTARY RESULTS

1.1. The symbol T shall always denote a topological Hausdorff group. A *T-space* (sometimes called a *flow* with *phase group* T or a *topological transformation group* with acting group T) is a pair $X := \langle X,\pi\rangle$, where X is a topological Hausdorff space and $\pi : T \times X \to X$ is a continuous mapping, satisfying the following conditions

(i) $\pi(e,x) = x$ for all $x \in X$;

(ii) $\pi(s,\pi(t,x)) = \pi(st,x)$ for all $s,t \in T$ and $x \in X$.

In order to simplify the notation, we shall usually write tx instead of $\pi(t,x)$, tB instead of $\pi[\{t\} \times B]$, Ax instead of $\pi[A\times\{x\}]$, and, in general, AB or A.B instead of $\pi[A\times B]$. It is easy to see, that for every $t \in T$ the mapping

$$\pi^t : x \mapsto tx : X \to X$$

is a homeomorphism of X onto itself with inverse $(\pi^t)^{\leftarrow} = \pi^{t^{-1}}$
the conditions (i) and (ii) above can be rewritten as $\pi^e = 1_X$ and $\pi^s \circ \pi^t =$

$= \pi^{st}$ $(s,t \in T)$.

1.2. EXAMPLES.

(i) Let H be a closed subgroup of T and let T/H be the space of all left cosets sH of H in T $(s \in T)$, endowed with the quotient topology (i.e. the finest topology making the quotient mapping $q : s \mapsto sH : T \to T/H$ continuous). Define $\pi : T \times (T/H) \to T/H$ by

$$\pi(t,sH) := tsH \qquad (s,t \in T).$$

Then π is continuous, and it is easily verified that π satisfies the conditions (i) and (ii) of 1.1. So $\langle T/H,\pi\rangle$ is a T-space (that T/H is a Hausdorff space follows from the fact that H is closed).

(ii) Let X be a topological Hausdorff space and let $\xi : X \to X$ be a homeomorphism of X onto itself. Then the mapping

$$\pi : (n,x) \mapsto \xi^n(x) := \underbrace{\xi \circ \ldots \circ \xi \circ \xi}_{n \text{ times}}(x) : \mathbb{Z} \times X \to X$$

defines an action of $\mathbb{Z}$ on X, so that $\langle X,\pi\rangle$ is a $\mathbb{Z}$-space. Note, that $\pi^1 = \xi$; therefore, we shall denote this $\mathbb{Z}$-space by (X,ξ) and call it the *discrete flow, generated by* ξ.

1.3. Let $\mathcal{X} = \langle X,\pi\rangle$ be a T-space. A subset A of X is called *invariant* whenever $TA \subseteq A$. If A is invariant, then also its closure $\bar{A}$ is invariant. The smallest closed invariant subset of X containing a point x is clearly $\overline{Tx}$, the closure of the *orbit* Tx of x ($\overline{Tx}$ is called the orbit-closure of x). It is not always the case that for $x,y \in X$ one has $y \in \overline{Tx}$ iff $x \in \overline{Ty}$ (the orbit-closures may not form a partition of X). Orbit-closures with this property are called *minimal*. Formally, a subset M of X is called *minimal* whenever $M \neq \emptyset$, M is closed and invariant, and if A is a closed, invariant subset of X such that $A \subseteq M$, then either $A = \emptyset$ or $A = M$. If M is a subset of X, then the following statements are equivalent:

(i) M is minimal;

(ii) $M \neq \emptyset$ and $\forall x \in M$: $\overline{Tx} = M$;

In particular, it follows that an orbit-closure $\overline{Tx}$ is minimal iff $\overline{Ty} = \overline{Tx}$ for every $y \in \overline{Tx}$, iff $x \in \overline{Ty}$ for every $y \in \overline{Tx}$.

Since minimal subsets of X are the minimal elements of the partial ordering (with respect to inclusion) of all non-empty, closed, invariant subsets of X, ZORN's lemma implies:

1.4. <u>THEOREM</u>. *Every non-empty compact invariant subset of* X *includes a minimal subset. In particular, if* X *is compact, then there is a minimal subset of* X. □

1.5. <u>CONSTRUCTIONS</u>

(i) Let Y be an invariant subset of X, where $\mathcal{X} = \langle X,\pi\rangle$ is a T-space. Then the restriction of π to $T\times Y$ is an action of T on Y, which will, for convenience, also be denoted by π. Thus, we obtain the T-space $\langle Y,\pi\rangle = \mathcal{Y}$. Notation: $\mathcal{Y} \subseteq \mathcal{X}$.

(ii) Let Λ be an index set and let for every $\lambda \in \Lambda$ be given the T-space $\mathcal{X}_\lambda = \langle X_\lambda,\pi_\lambda\rangle$. Then the product of the T-spaces $\mathcal{X}_\lambda (\lambda\in\Lambda)$ is the T-space $\mathcal{X} = \langle X,\pi\rangle$, where

$$X := \prod_{\lambda\in\Lambda} X_\lambda \quad \text{(cartesian product with the product topology}$$

$$\pi(t,(x_\lambda)_{\lambda\in\Lambda}) := (\pi_\lambda(t,x_\lambda))_{\lambda\in\Lambda} \quad \text{for } t \in T \text{ and } x = (x_\lambda)_{\lambda\in\Lambda} \in X.$$

It is easily verified that $\mathcal{X}$ is, indeed, a T-space. Notation: $\mathcal{X} = \prod_{\lambda\in\Lambda} \mathcal{X}_\lambda$

1.6. A *morphism of* T-*spaces* $\phi\colon \mathcal{X} \to \mathcal{Y}$ is a continuous mapping $\phi\colon X \to Y$ *) such that $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$. If ϕ is a surjection of X onto Y, then $\mathcal{Y}$ is also called a *factor of* $\mathcal{X}$ *by* ϕ, and $\mathcal{X}$ is called *an extension of* $\mathcal{Y}$ *by* ϕ. An isomorphism of T-spaces is, of course, a morphism of T-spaces $\phi\colon \mathcal{X} \to \mathcal{Y}$ such that ϕ is a homeomorphism of X onto Y. In that case, $\phi^{\leftarrow}\colon \mathcal{Y} \to \mathcal{X}$ is also an isomorphism of T-spaces.

If $\phi\colon \mathcal{X} \to \mathcal{Y}$ is a morphism of T-spaces, then ϕ defines an equivalence relation R_ϕ on X, namely,

$$R_\phi := \{(x,x') \in X\times X : \phi(x) = \phi(x')\}.$$

Clearly, R_ϕ is a closed, invariant subset of $X \times X$.

1.7. Let $\mathcal{X}$ and $\mathcal{Y}$ be T-spaces and let $\phi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of T-spaces. The following statements are easily verified:

)* We shall adopt the following convention: unless stated otherwise, $\mathcal{X} := \langle X,\pi\rangle$, $\mathcal{Y} := \langle Y,\sigma\rangle$ and $\mathcal{Z} := \langle Z,\zeta\rangle$.

(i) If A is an invariant subset of X then $\phi[A]$ is invariant in Y, and if B is an invariant subset of Y, then $\overset{\leftarrow}{\phi}[B]$ is invariant in X.
(ii) If A is a minimal subset of X, then $\phi[A]$ is minimal in Y (consider $A \cap \overset{\leftarrow}{\phi}[B]$ for a closed invariant subset B of $\phi[A]$).

In particular, if X is compact (hence $\phi[X]$ closed in Y) and Y is minimal then ϕ must be surjective (so Y is a factor of X), and if X is minimal and ϕ is surjective, then Y is minimal.

1.8. Our aim is, among others, to study factors of compact Hausdorff T-spaces. *So from now on all our* T-*spaces will be compact and Hausdorff* (although for certain definitions and results compactness is not needed).

2. PROXIMAL FLOWS

2.1. Two points x,y in a T space X are called *proximal* whenever $\overline{T(x,y)} \cap \Delta_X = \emptyset$; here $\Delta_X := \{(z,z) : z \in X\}$. If the points x and y are proximal, then (x,y) is called a *proximal pair*. So (x,y) is a proximal pair whenever there exists a point $z \in X$ such that

$$\forall U \in \mathcal{V}_z \quad \exists t \in T\!: \; tx \in U \text{ and } ty \in U$$

(here $\mathcal{V}_z$ denotes the neighbourhood filter of z). Obviously, if (x,y) is a proximal pair, then $\overline{Tx} \cap \overline{Ty} \neq \emptyset$.

A T-space X is called *proximal* whenever each pair of points is a proximal pair.

2.2. <u>LEMMA</u>. *A* T-*Space* X *is proximal iff every minimal subset of* $X \times X$ *is contained in* Δ_X.

<u>PROOF</u>. If X is proximal and M is minimal in $X \times X$, then for $(x,y) \in M$ we have $\overline{T(x,y)} \cap \Delta_X \neq \emptyset$, but also $\overline{T(x,y)} \subseteq M$. Hence $M \cap \Delta_X \neq \emptyset$, and $M \cap \Delta_X$ is a closed, invariant subset of M. By minimality, $M \cap \Delta_X = M$, that is, $M \subseteq \Delta_X$.

Conversely, if every minimal subset of $X \times X$ is contained in Δ_X, then this is in particular the case for the minimal subset, which is included in $\overline{T(x,y)}$ (cf. 1.4). Hence $\overline{T(x,y)} \cap \Delta_X \neq \emptyset$ for arbitrary $(x,y) \in X \times X$. □

2.3. <u>COROLLARY</u>. *If* X_λ *is a proximal T-space for every* $\lambda \in \Lambda$, *then* $\Pi_{\lambda\in\Lambda} X_\lambda$ *is proximal.*

PROOF. Let $X := \Pi_{\lambda\in\Lambda}X_\lambda$, and, for every $\lambda \in \Lambda$, $p_\lambda : X \to X_\lambda$ the canonical projection; obviously, each p_λ is a morphism of T-spaces. If M is a minimal subset of $X \times X$, then $p_\lambda \times p_\lambda[M]$ is minimal in $X_\lambda \times X_\lambda$ (cf. 1.7) hence $p_\lambda \times p_\lambda[M] \subseteq \Delta X_\lambda$, by 2.2. This implies, that $M \subseteq \Delta_X$, hence X is proximal by 2.2. □

2.4. PROPOSITION. *Let X be a proximal T-space. Then:*
(i) *If Z is a closed, invariant subset of X, then Z is proximal.*
(ii) *If $\phi : X \to Y$ is a factor, then Y is proximal.*

PROOF. (i) is obvious, and for (ii), observe that $\phi \times \phi[\overline{T(x,y)}] = \overline{T(\phi x,\phi y)}$. (Indeed, $\phi \times \phi[T(x,y)] = T(\phi x,\phi y)$ because ϕ is a morphism of T-spaces. Hence $\phi \times \phi[\overline{T(x,y)}] \subseteq \overline{T(\phi x,\phi y)}$ because $\phi \times \phi$ is continuous. But $\phi \times \phi[\overline{T(x,y)}]$ is compact, hence closed, so that $\phi \times \phi[\overline{T(x,y)}] = \overline{T(\phi x,\phi y)}$.) Now it is clear that for each pair $z_1,z_2 \in Y$ we have:

$$\overline{T(z_1,z_2)} \cap \Delta_Y = \phi \times \phi[\overline{T(x,y)}] \cap \Delta_Y \supseteq \phi \times \phi[\overline{T(x,y)} \cap \Delta_X] \neq \emptyset;$$

here $x,y \in X$ are such, that $\phi x = z_1$ and $\phi y = z_2$. □

2.5. PROPOSITION. *Let for* $i = 1,2$, $\phi_i : X \to Y$ *be a factor, X and Y minimal and, in addition* Y *proximal. Then* $\phi_1 = \phi_2$.

PROOF. Consider for $i = 1,2$ the graph of ϕ_i:

$$\Gamma_i := \{(x,\phi_i(x)) : x \in X\} \subseteq X \times Y.$$

Then Γ_i is a closed, invariant subset of $X \times Y$. As Γ_i is the image of X under the mapping $x \to (x,\phi_i(x))$, and this mapping is a morphism of T-spaces, Γ_i is minimal (cf. 1.7).

We claim, that two points (x,y) and (x,y') in $X \times Y$ are always proximal. If this is true, then in particular for every $x \in X$ the points $(x,\phi_1(x))$ and $(x,\phi_2(x))$ are proximal. Let $(u,v) \in X \times Y$ be the point such that for every neighbourhood of (u,v), say $U \times V$ ($U \in \mathcal{V}_u$ and $V \in \mathcal{V}_v$), there exists $t \in T$ with

$$t(x,\phi_1(x)) \in U\times V, \quad t(x,\phi_2(x)) \in U \times V,.$$

This implies

$$(u,v) \in \overline{T(x,\phi_1(x))} \text{ and } (u,v) \in \overline{T(x,\phi_2(x))},$$

hence $(u,v) \in \Gamma_1 \cap \Gamma_2$. As Γ_1 and Γ_2 are minimal, this implies that $\Gamma_1 = \Gamma_2$, hence $\phi_1 = \phi_2$.

It remains to prove our claim. So consider (x,y) and (x,y') in $X \times Y$. Since Y is proximal, there exists $z \in Y$ such that

$$(*) \qquad \forall W \in \mathcal{V}_z \; \exists t_W \in T : t_W y \in W \text{ and } t_W y' \in W.$$

Since X is compact, the family of sets

$$F_W := \overline{\{t_{W'}x : W' \in \mathcal{V}_z \;\&\; W' \subseteq W\}}$$

for $W \in \mathcal{V}_z$ has a non-empty intersection in X. Let

$$x' \in \bigcap_{W \in \mathcal{V}_z} F_W.$$

Consider an arbitrary neighbourhood of the point (x',z) in $X \times Y$, say $U \times V$ with $U \in \mathcal{V}_{x'}$ and $V \in \mathcal{V}_z$. Since $x' \in F_V$, we have that

$$U \cap \{t_{W'}x : W' \in \mathcal{V}_z \;\&\; W' \subseteq V\} \neq 0.$$

So there exists $W' \in \mathcal{V}_z$, $W' \subseteq V$, such that

$$t_{W'}x \in U.$$

On the other hand, by $(*)$ we have

$$t_{W'}y \in W' \text{ and } t_{W'}y' \in W'.$$

Hence $t_{W'}(x,y) \in U \times W' \subseteq U \times V$ and $t_{W'}(x,y') \in U \times W' \subseteq U \times V$. This proves our claim. □

2.6. COROLLARY. *Let X be minimal and proximal. Then the only* T-*endomorphism of X is $1_X : X \to X$.* □

2.7. COROLLARY. *If* T *is abelian, then the trivial one-point flow is the only flow which is minimal and proximal.*

PROOF. Suppose X is a minimal and proximal T-space. For every $t \in T$, π^t: $: X \to X$ is an endomorphism, because T is abelian. Hence $\pi^t = 1_X$, i.e. the action is trivial. As X is minimal, this implies that the space consists of one point. □

2.8. REMARK. If T is compact, then every minimal and proximal flow is trivial as well. Indeed, for every $x,y \in X$ we have $T(x,y) = \overline{T(x,y)}$, hence proximality implies that $T(x,y) \cap \Delta_X \neq \emptyset$, i.e. there exists $t \in T$ such that $tx = ty$, hence $x = y$. A group T, which admits no nontrivial minimal proximal flows will be called *strongly amenable*. Thus, abelian groups and compact groups are strongly amenable.

2.9. If $X = \langle X,\pi \rangle$ is a T-space, and S is a subgroup of T, then the action of T on X induces an action of S on X. In general, one cannot expect that if X is minimal, then X is minimal under the action of S. For proximality, however, we have:

2.10. PROPOSITION. *Let* S *be a subgroup of* T *and suppose that there exists a compact subset* K *of* T *such that* T = K.S. *If* X *is proximal, then* X *is proximal under the action of* S.

PROOF. First, observe that for any subset A of $X \times X$ (similarly: for a subset A in any T-space Z) we have $\overline{KA} = K\overline{A}$. (Indeed, since K and $\overline{A}$ are compact, $K\overline{A}$ is compact, hence closed, and $KA \subseteq K\overline{A}$, so $\overline{KA} \subseteq K\overline{A}$. On the other hand, for every $k \in K$ one has $k\overline{A} = \overline{kA} \subseteq \overline{KA}$, so $K\overline{A} \subseteq \overline{KA}$).

Now for every pair $(x,y) \in X \times X$ we apply this to $T(x,y)$; so

$$\overline{T(x,y)} = \overline{KS(x,y)} = K\overline{S(x,y)}.$$

However, $\overline{T(x,y)} \cap \Delta_X \neq \emptyset$, so $K\overline{S(x,y)} \cap \Delta_X \neq \emptyset$. Since Δ_X is invariant under K, it follows that $\overline{S(x,y)} \cap \Delta_X \neq \emptyset$. □

2.11. REMARK. If S is a closed subgroup of T and T = KS with K compact, then the space T/S of left cosets of S (endowed with the usual quotient topology) is a compact Hausdorff space. Conversely, if T/S is compact and, in addition, T is locally compact, then it is easy to show that there exists a compact subset K of T such that T = KS. (Proof: let U be a compact neighbourhood of e in T. Since the quotient mapping $q: t \mapsto tS: T \to T/S$ is open, the image under q of every translate sU of U ($s \in T$) is open in T/S. As T/S is compact, there are finitely many $s_1,\ldots,s_n \in T$ such that $\bigcup_{i=1}^n q[s_iU] = T/S$, that is, $\bigcup_{i=1}^n s_iUS = T$. Now $K := \bigcup_{i=1}^n s_iU$ is the desired compact set.)

3. STRONGLY PROXIMAL FLOWS

For results and notation from functional analysis, we refer to appendix A1 to this chapter. In particular, one can find there the definition of the space $M(X)$ of probability measures on a compact Hausdorff space X.

3.1. Consider a T-space $X = \langle X,\pi\rangle$. Then for every $t \in T$ the mapping $M(\pi^t)$: $M(X) \to M(X)$ is defined according to A1.5. For convenience, we shall write for $t \in T$ and $\mu \in M(X)$:

$$t\mu := M(\pi^t)(\mu) = \mu \circ \tilde{\pi}^t.$$

Then $(t,\mu) \mapsto t\mu : T \times M(X) \to M(X)$ is a mapping, satisfying the following conditions:

(i) $e\mu = \mu$ (for $\tilde{\pi}^e$ is the identity mapping of $C(X)$)

(ii) $\forall s, t \in T$: $s(t\mu) = (st)\mu$ (for $\tilde{\pi}^t \circ \tilde{\pi}^s = \tilde{\pi}^{st}$)

(iii) the mapping $(t,\mu) \mapsto t\mu$: $T \times M(X) \to M(X)$ is continuous.

(The proof is as follows; consider $(t,\mu) \in T \times M(X)$ and observe, that for every $(s,\nu) \in T \times M(X)$ and for every (fixed) $f \in C(X)$:

$$\begin{aligned} |t\mu(f) - s\nu(f)| &= |\mu(f \circ \pi^t) - \nu(f \circ \pi^s)| \leq \\ &\leq |\mu(f \circ \pi^t) - \nu(f \circ \pi^t)| + |\nu(f \circ \pi^t) - \nu(f \circ \pi^s)| \\ &\leq |\mu(f \circ \pi^t) - \nu(f \circ \pi^t)| + \| f \circ \pi^t - f \circ \pi^s \| . \end{aligned}$$

As $f \circ \pi$ is continuous on $T \times X$ and X is compact, there exists for every $\varepsilon > 0$ a neighbourhood V of t in T such that

$$|f \circ \pi(t,x) - f \circ \pi(s,x)| < \varepsilon \quad \text{for all } s \in V,\ x \in X.$$

Hence $\| f \circ \pi^t - f \circ \pi^s \| \leq \varepsilon$ for all $s \in V$. Moreover, if $\nu \in U_\mu(\{f \circ \pi^t\},\varepsilon)$ (see A1.1 for the notation) we have

$$|\mu(f \circ \pi^t) - \nu(f \circ \pi^t)| < \varepsilon.$$

Consequently, for all (s,v) in the neighbourhood $V \times U_\mu(\{f \circ \pi^t\},\varepsilon)$ of (t,f) in $T \times M(X)$ we have $|t\mu(f) - s\nu(f)| < 2\varepsilon$. This proves the desired continuity.)

It follows, that the mapping $(t,\mu) \mapsto t\mu\colon T \times M(X) \to M(X)$ is an action of T on the compact Hausdorff space $M(X)$.

3.2. Let us, again, consider a T-space $X = \langle X,\pi\rangle$, and consider the embedding mapping $\underline{\delta}\colon X \to M(X)$. It follows from the diagram in A1.5, that $\underline{\delta}$ is a morphism of T-spaces; so if we consider X as a closed subset of $M(X)$ (via the embedding $\underline{\delta}$), then X is invariant in $M(X)$ under the action of T on $M(X)$, and the action of T on X is the restriction of the action of T on $M(X)$. Therefore, for convenience (and in accordance with 1.5(i)) the action of T on $M(X)$ will also be denoted by π. So we have the following diagram for each $t \in T$:

$$\begin{tikzcd} M(X) \arrow[r, "\pi^t"] & M(X) \\ X \arrow[u, "\underline{\delta}"] \arrow[r, "\pi^t"] & X \arrow[u, "\underline{\delta}"'] \end{tikzcd}$$

The following observations are easily verified

(i) $\forall t \in T\colon \pi^t\colon M(X) \to M(X)$ is affine (that is, for all $\mu,\nu \in M(X)$ and $a \in [0,1]$ the equality $\pi^t(a\mu + (1-a)\nu) = a\pi^t\mu + (1-a)\pi^t\nu$ holds); this is an immediate consequence of the fact, that π^t is the restriction of a linear operator to a convex set.

(ii) ex $M(X) = \underline{\delta}[X]$ is a closed invariant subset of the T-space $\langle M(X),\pi\rangle$, which is different from $M(X)$. So $\langle M(X),\pi\rangle$ is not minimal.

3.3. The T-space $X = \langle X,\pi\rangle$ is called *strongly proximal* whenever $\langle M(X),\pi\rangle$ is proximal.

Since by 3.2 the T-space X is isomorphic with a closed invariant subset of $M(X)$, and invariant sub-T-spaces of a proximal T-space are also proximal, we have:

3.4. PROPOSITION. *If X is strongly proximal, then X is proximal.* $\square$

The following result characterises strong proximality:

3.5. LEMMA. *The following conditions are equivalent for a T-space X:*

(i) *X is strongly proximal;*
(ii) $\forall\mu \in M(X)$: $\overline{T\mu} \cap \underline{\delta}[X] \neq \emptyset$ (*i.e.* $\exists x \in X$: $\delta_x \in \overline{T\mu}$).

PROOF. (i) $\Rightarrow$ (ii): Consider $\mu \in M(X)$ and $z \in X$. By proximality of $M(X)$ we have (closures are in $M(X)$): $\overline{T\mu} \cap \overline{T\delta_z} \neq \emptyset$ (see 2.1). Since $\underline{\delta}[X]$ is closed and invariant, we have $\overline{T\delta_z} \subseteq \delta[X]$, hence $\overline{T\mu} \cap \underline{\delta}[X] \neq \emptyset$.
(ii) $\Rightarrow$ (i): Consider the mapping

$$\phi:(\mu,\nu) \to \tfrac{1}{2}(\mu+\nu): M(X) \times M(X) \to M(X).$$

Then ϕ is continuous (this is a consequence of the fact, that the w^*-topology in $C(X)'$ is a vector space topology) and ϕ is a morphism of T-spaces (each of the mappings π^t on $M(X)$ is affine). As $M(X)$ is compact, orbit closures are mapped onto orbitclosures by ϕ, so for all $\mu,\nu \in U(X)$:

$$\phi[\overline{T(\mu,\nu)}] = \overline{T(\phi(\mu,\nu))} = \overline{T(\tfrac{\mu+\nu}{2})}.$$

By (ii), $\delta_x \in \overline{T(\frac{\mu+\nu}{2})}$ for some $x \in X$, hence there are $\mu',\nu' \in \overline{T(\mu,\nu)}$ such that $\phi(\mu',\nu') = \delta_x$, that is, $\frac{1}{2}(\mu'+\nu') = \delta_x \in \operatorname{ex} M(X)$. Consequently, $\mu' = \nu'$, hence $\overline{T(\mu,\nu)} \cap \Delta_X \neq \emptyset$. This proves (i). □

3.6. EXAMPLE. Let G be a semisimple connected Lie group with finite center, and let G = KAN be an Iwasawa decomposition of G. Then, if $P := N_G(N)$ (normaliser of N in G), G/P is compact, and, under the natural action of G on G/P (cf. 1.2(i)), G/P is a compact, minimal G-space (minimality: G/P consists of just one G-orbit). We shall show in 4.7 below that G/P *is strongly proximal* (this result is due to C.C. MOORE, 1964). In particular, it follows that G/P *admits no invariant measure* (if μ were an invariant measure on G/P, then $\overline{T\mu} = \{\mu\}$, hence by 3.5, μ is a point measure of the form δ_x for $x \in G/P$. But then x would be an invariant point in G/P (because $\underline{\delta}$ is a morphism of G-spaces), a contradiction).

3.7. If ϕ: $X \to Y$ is a morphism of T-spaces, then ϕ induces a continuous mapping $M(\phi)$: $M(X) \to M(Y)$. Since for all $t \in T$ we have $\phi \circ \pi^t = \sigma^t \circ \phi$, we also have $M(\phi) \circ M(\pi^t) = M(\sigma^t) \circ M(\phi)$. In view of the definition of the actions of T on $M(X)$ and $M(Y)$, respectively, it follows, that $M(\phi)$ is a morphism of T-spaces.

3.8. PROPOSITION. *Suppose that $X = \langle X,\pi\rangle$ is a strongly proximal* T*-space and* *that* Y *is a closed, invariant subset of* X. *Then* $\langle Y,\pi\rangle$ *is strongly proximal.*

PROOF. By appendix A1.5, the inclusion mapping $i:Y \to X$ induces an injection $M(i)\colon M(Y) \to M(X)$. So $M(Y)$ may be considered as a closed invariant subset of the (proximal!) T-space $M(X)$. Hence $M(Y)$ is proximal, that is, $\langle Y,\pi\rangle$ is strongly proximal. □

3.9. PROPOSITION. *Let* J *be an index set and let, for every* $i \in J$, $X_i :=$ $\langle X_i,\pi_i\rangle$ *be a strongly proximal* T-*space. Then the product* $\Pi_{i\in J}X_i$ *is strongly proximal as well.*

PROOF. (1). First we prove the result for finite products. Using induction, it is easy to see that we may restrict ourselves to the case of a product of two strongly proximal T-spaces X_1 and X_2. For $j = 1,2$, let $p_j\colon X_1 \times X_2 \to X_j$ be the projection and let $\psi_j := M(X_1{\times}X_2) \to M(X_j)$ be the induced mapping. Since ψ_1 and ψ_2 are morphisms of T-spaces and all orbit closures under consideration are compact, it follows that

$$(1) \qquad \psi_j[\overline{T\nu}] = \overline{T\psi_j(\nu)} \text{ for all } \nu \in M(X_1{\times}X_2) \ (j=1,2).$$

Let $\mu \in M(X{\times}X)$. Since X_1 is strongly proximal, it follows from 3.5, that $\overline{T\psi_1(\mu)} \cap \delta^{(1)}[X_1] \neq \emptyset$. Hence by (1) there exists $\mu' \in \overline{T\mu}$ such that $\psi_1(\mu') \in$ $\in \delta^{(1)}[X_1]$. This implies that

$$(2) \qquad \psi_1[\overline{T\mu'}] = \overline{T\psi_1(\mu')} \subseteq \delta^1[X_1],$$

because $\delta^1[X_1]$ is closed and invariant in $M(X_1)$. Now 3.5, applied to X_2 implies, that $\overline{T\psi_2(\mu')} \cap \delta^{(2)}[X_2] \neq \emptyset$, so there exists $\mu'' \in \overline{T\mu'}$ with $\psi_2(\mu'') \in \delta^{(2)}[X_2]$. Then (2) implies that

$$\psi_1(\mu'') \in \psi_1[\overline{T\mu'}] \subseteq \delta^{(1)}[X_1].$$

Consequently, we have $x_1 \in X_1$ and $x_2 \in X_2$ such that

$$\psi_1(\mu'') = \delta^{(1)}_{x_1} \text{ and } \psi_2(\mu'') = \delta^{(2)}_{x_2}.$$

Then it is not difficult to show that $\mu'' = \delta(x_1,x_2)$)* (point-measure on $X_1 \times X_2$). So we have

$$\delta(x_1,x_2) \in \overline{T\mu'} \subseteq \overline{T\mu},$$

hence 3.5 implies that $X_1 \times X_2$ is strongly proximal.

(2) Now suppose J is infinite. For every finite subset F of J, let $p_F : X := \Pi_{i \in J} X_i \to \Pi_{j \in F} X_j =: X_F$ be the projection, and let $\psi_F := M(p_F) : M(X) \to M(X_F)$ be the induced mapping. Finally, let $\mathcal{F}$ be the set of all finite subsets of J. Since for each $F \in \mathcal{F}$ we have a morphism of T-spaces $\psi_F : M(X) \to M(X_F)$, we have a morphism of T-spaces

$$\psi : M(X) \to \prod_{F \in \mathcal{F}} M(X_F)$$

such that for every $F \in \mathcal{F}$ and $\mu \in M(X) : \psi(\mu) = (\psi_F(\mu))_{F \in \mathcal{F}}$. By the first part of the proof, each $M(X_F)$ is proximal (for X_F is strongly proximal), hence by 2.3, the full product $\Pi_{F \in \mathcal{F}} M(X_F)$ is proximal. However, by the Stone-Weierstrass theorem, ψ is injective)**, hence a topological embedding (indeed, all spaces under consideration are compact Hausdorff). So $M(X)$, being isomorphic with a closed invariant subset of a proximal T-space is proximal, i.e. X is strongly proximal. □

)* If U_i is open in X_i $(i=1,2)$ and $(x_1,x_2) \notin U_1 \times U_2$, then it is easy to see that $\mu''(U_1 \times U_2) = 0$ (suppose $x_1 \notin U_1$: then we have $\mu''(U_1 \times U_2) \leq \mu''(\overleftarrow{p_1} U_1) = \psi_1(\mu'')(U_1) = \delta_{x_1}(U_1) = 0$). If S is a compact subset of $X_1 \times X_2$, and $(x_1,x_2) \notin S$, then S can be covered by finitely many rectangles, none of which contains (x_1,x_2), i.e. each having μ''-measure zero. Hence $\mu''(S) = 0$. This implies, that the support of μ'' is included in $\{(x_1,x_2)\}$, hence $\mu'' = \delta(x_1,x_2)$.

)** Consider $\mu, \nu \in M(X)$ such that $\psi(\mu) = \psi(\nu)$, that is, $\psi_F(\mu) = \psi_F(\nu)$ for all $F \in \mathcal{F}$. Let

$$A := \{ f \circ p_F : F \in \mathcal{F} \ \& \ f \in C(X_F)\}.$$

It follows immediately from the definitions and assumptions, that for every $g \in A$ we have (say $g = f \circ p_F, F \in \mathcal{F}, f \in C(X_F)$):

$$\mu(g) = \mu(f \circ p_F) = \psi_F(\mu)(f) = \psi_F(\nu)(f) = \nu(f \circ p_F) = \nu(g).$$

Now A is a subalgebra of C(X) containing the constant functions and separating the points of X, so A is dense in C(X) (Stone-Weierstrass). As $\mu|_A = \nu|_A$, this implies that $\mu = \nu$. This shows, that ψ is injective.

3.10. PROPOSITION. *Let $\phi:X \to Y$ be a surjective morphism of* T*-spaces. If X is strongly proximal, then so is Y.*

PROOF. The induced morphism $M(\phi):M(X) \to M(Y)$ is surjective (A1.5). Now apply 2.4(ii)). □

3.11. THEOREM. *There exists a unique (up to isomorphism) universal minimal strongly proximal* T*-space SP_T which is characterized by the following properties*

(i) *SP_T is minimal and strongly proximal;*

(ii) *For every minimal, strongly proximal* Y *there exists a unique morphism of* T*-spaces $\phi:SP_T \to Y$.*

PROOF. Since in every minimal T-space an image of T is dense, there is only a set of isomorphy types of minimal T-spaces)*. Let $\{X_i\}_{i\in J}$ be a complete set of representatives of isomorphism classes of strongly proximal, minimal T-spaces. Let SP_T be the T-space, defined by a *minimal* subset of the product $\Pi_{i\in J} X_i$ (cf. 1.4 and 1.5(i)). Then 3.8 and 3.9 imply, that SP_T is strongly proximal; by choice, SP_T is minimal. So condition (i) is fulfilled.

If Y is an arbitrary minimal, strongly proximal T-space, then Y is isomorphic to X_j for some $j \in J$. Therefore, the restriction to SP_T of the projection $p_j:\Pi_{i\in J} X_i \to X_j$ defines a morphism of T-spaces from SP_T into Y (and, by 1.7, *onto* Y). It follows from 2.5, that such a morphism is unique. So (ii) is satisfied.

Finally, suppose that X' is a T-space with properties (i) and (ii). Then there exists (unique) morphisms of T-spaces $\phi:SP_T \to X'$ and $\psi:X' \to SP_T$. Then by 2.6, $\psi \circ \phi$ is the identity map of SP_T and $\phi \circ \psi$ the identity map of X'. It follows that ϕ (and ψ as well) is an isomorphism. Thus, SP_T is unique up to isomorphism. □

3.12. REMARK. Let G and P be as in Example 3.6. We shall show in section below, that $G/P \cong SP_G$, i.e. G/P is the universal strongly proximal minimal G-space. For the proof we need, however, more knowledge about strongly proximal T-spaces; in particular, we need to know the relationship between strong proximality and (the non-existence of) invariant measures.

)* The cardinality of a minimal T-space X is at most $2^{2^{|T|}} =: \kappa$, because each point of X determines a filter in Tx (for given $x \in X$), namely, the trace of the neighbourhood filter of that point in Tx. On a set of cardinality $\le \kappa$ there are no more than $2^{2^{\kappa}}$ topological structures, and no more than $|T| \times \kappa^2$ possible actions of T.

4. AFFINE T-SPACES; FIXED-POINT THEOREMS

4.1. Let E be a (real) locally convex topological vector space, Q a non-empty compact convex subset of E and σ an action of T on Q. Then the T-space $\mathcal{Q} := \langle Q,\sigma\rangle$ is called *affine* whenever each homeomorphism $\sigma^t: Q \to Q$ $(t \in T)$ is an affine mapping, i.e. $\sigma^t(ax+(1-a)y) = a\sigma^t(x) + (1-a)\sigma^t(y)$ for $x,y \in Q$ and $0 \le a \le 1$.

It is clear, that for every T-space $\mathcal{X}$ (on a compact space X) the corresponding T-space on $M(X)$ is affine (cf.3.2).

Note, that non-trivial affine T-space $\mathcal{Q}$ cannot be minimal: it is obvious that ex Q is invariant, so $\overline{\text{ex}}$ Q is invariant as well. By the Krein-Milman theorem, $\overline{\text{ex}}\ Q \neq \emptyset$, and also $\overline{\text{ex}}\ Q \neq Q$, unless ex Q consists of just one point, in which case Q also consists of one point. In the study of affine T-spaces it is, therefore, not the notion of minimality which is important, but the notion of irreducibility. A non-empty, closed invariant *convex* subset A of an affine T-space $\mathcal{Q}$ is called *irreducible* whenever the only closed, invariant *convex* subsets of A are $\emptyset$ and A. Similar to theorem 1.4 it follows from ZORN's lemma, that

4.2. <u>THEOREM</u>. *Every affine* T-*space* $\mathcal{Q}$ *contains an irreducible subset.* □

For our purposes, the most important result is the following theorem. For a part of its proof, we refer to appendix A2.

4.3. <u>THEOREM</u>. *Let* $\mathcal{Q}$ *be an affine* T-*space. The following conditions are equivalent:*

(i) $\mathcal{Q}$ *is irreducible*

(ii) $\mathcal{Q}$ *is strongly proximal and* $\overline{\text{ex}}$ Q *is minimal.*

(iii) $\overline{\text{ex}}$ Q *is strongly proximal and minimal.*

<u>PROOF</u>.

(i)⇒(ii): Let Y be a closed invariant subset of Q. Then $\overline{\text{co}}$ Y is invariant as well, hence by irreducibility of Q, $\overline{\text{co}}\ Y = Q$. Now A1.3 (second part) implies, that $\overline{\text{ex}}\ Q \subseteq Y$. Since this holds for every closed invariant subset Y of Q, it follows, that $\overline{\text{ex}}$ Q is minimal (in fact, it follows, that $\overline{\text{ex}}$ Q is the *unique* minimal subset of Q).

In order to prove, that $\mathcal{Q}$ is strongly proximal, it is sufficient to show, that $\delta^{(Q)}[\overline{\text{ex}}\ Q]$ is the unique minimal subset of $M(Q)$: indeed, if this is true, then every orbit closure in $M(X)$, since it contains by 1.4 a minimal subset, contains the set $\delta^{(Q)}[\overline{\text{ex}}\ Q]$. As $\delta^{(Q)}[\overline{\text{ex}}\ Q] \subseteq \delta^{(Q)}[0]$, it follows, that every orbit closure in $M(Q)$ meets $\delta^{(Q)}[Q]$. But then the result follows from 3.5.

So consider a closed, invariant subset Y' of $M(Q)$. By A2.7, the *barycenter map* $b^{(Q)}:M(Q) \to Q$ is a morphism of T-spaces, so $b^{(Q)}[Y']$ is a closed, invariant subset of Q. By the first part of our proof, $\overline{\text{ex}}\ Q \subseteq b^{(Q)}[Y']$. Then it follows from A2.4, that $\delta^{(Q)}[\text{ex}\ Q] \subseteq Y'$, and this implies, that $\delta^{(Q)}[\overline{\text{ex}}\ Q] \subseteq \overline{Y'} = Y'$. Since this holds for every closed invariant subset Y', this shows, indeed, that $\delta^{(Q)}[\overline{\text{ex}}\ Q]$ is the unique minimal subset of $M(Q)$.

(ii)⇒(iii): Clear from 3.8.

(iii)⇒(i): Let A be a non-empty compact convex invariant subset of Q. If we put $X := \overline{\text{ex}}\ Q$ and $b^{(X)}: M(X) \to Q$ the corresponding barycenter mapping, then the pre-image $B := b^{(X)\leftarrow}[A]$ of A is a non-empty compact convex invariant (cf.A2.7) subset of $M(X)$. Now X is given to be minimal and strongly proximal, so $M(X)$ is proximal, and $\delta^{(X)}[X]$ is a minimal subset of $M(X)$. If $x \in B$ and $y \in \delta^{(X)}[X]$, then by proximality, their orbit closures intersect each other. Since $\overline{Tx} \subseteq B$ and $\overline{Ty} = \delta^{(X)}[X]$, it follows that $B \cap \delta^{(X)}[X] \neq \emptyset$, and as $\delta^{(X)}[X]$ is minimal, this implies that $B \supseteq \delta^{(X)}[X]$. Hence $M(X) = \overline{\text{co}}\ \delta^{(X)}[X] \subseteq$ $\subseteq B$ (cf.A1.4), so that $A = Q$. This shows, that Q is irreducible. □

4.4. A topological group T is said to have the *fixed point property* whenever every affine T-space Q has a fixed point (= invariant point). A topological group T is said to be *amenable* whenever condition (ii) in the following theorem is satisfied)*.

4.5. <u>THEOREM</u>. *The following properties are equivalent for a topological group* T:

(i) T *has the fixed point property;*

(ii) *Every* T-*space* X (*on a compact Hausdorff space!*) *has an invariant measure*)**, *i.e.* T *is amenable;*

(iii) *Every minimal strongly proximal* T-*space* X *is trivial* (*i.e.* X *consists of one point*);

(iv) Sp_T *is trivial.*

)* Formally, the definition of amenable is different from, but equivalent to, condition (ii). See [5].

)** i.e. $\exists \mu \in M(X) : t\mu = \mu$ for all $t \in T$.

PROOF.
(i)⇒(ii): If X is a T-space, then (i) implies that the induced affine T-space $M(X)$ has an invariant point μ.
(ii)⇒(iii): If μ is an invariant point in $M(X)$ and X is strongly proximal, then by 3.5 there exists $x \in X$ such that $\delta_x \in \overline{T\mu} = \{\mu\}$.
Consequently, $\delta_x = \mu$, hence x is an invariant point in X. If X is also minimal, this implies that $X = \{x\}$.
(iii)⇔(iv): obvious.
(iii)⇒(i): Suppose Q is an affine T-space. Let Q_0 be an irreducible subset of Q (cf.4.2). Then the T-space $\overline{\mathrm{ex}\ Q_0}$ is strongly proximal and minimal by 4.3, hence $\overline{\mathrm{ex}\ Q_0} = \{x\}$ by the assumption. Then x is an invariant point in Q. □

4.6. COROLLARY. *If* T *is strongly amenable, then* T *is amenable.*

PROOF. Cf. 2.8 (definition of strong amenability), 3.4 and 4.5. □

In our next theorem we indicate some amenable groups. It contains the Markov-Kakutani fixed point theorem (cf. [2]). First a lemma:

4.7. LEMMA. *Let* S *be a closed normal subgroup of* T *and suppose that both* S *and* T/S *are amenable. Then* T *is amenable.*

PROOF. We show, that T has the fixed point property. Consider an affine T-space Q, and let

$$Q_1 := \{x \in Q : sx = x \text{ for all } s \in S\}.$$

Since S is amenable, $Q_1 \neq \emptyset$. Moreover, it is easy to see that Q_1 is convex and closed in Q, hence compact. In addition, Q_1 is invariant under T (indeed: if $t \in T$, then for every $s \in S$ there exists $s' \in S$ such that $st = ts'$, so that $s(tx) = t(s'x) = tx$ for all $x \in Q_1$, i.e. $tx \in Q_1$ for all $x \in Q_1$. So Q_1 is a T-space, and because S is included in the stabilizer $\{t \in T : tx = x$ for all $x \in Q_1\}$ of Q_1, there exists an action $\tilde{\pi}$ of T/S on Q_1, defined by

$$\tilde{\pi}^{q(t)} := \pi^t \text{ for all } t \in T$$

(here $q: T \to T/S$ is the quotient map; as q is an open mapping, it is not difficult to show that the action $\tilde{\pi}$ of T/S on Q_1 is continuous as a mapping from $(T/S) \times Q_1$ into Q_1). However, T/S is amenable, so there exists $x_1 \in Q_1$ such that $\tilde{\pi}^u x_1 = x_1$ for all $u \in T/S$, hence $\pi^t x_1 = x_1$ for all $t \in T$ (cf. the definition of $\tilde{\pi}$). So x_1 is a fixed point for T in Q. □

4.8. THEOREM. *The following groups are amenable:*

(i) *all abelian groups;*

(ii) *all compact groups;*

(iii) *all solvable groups;*

(iv) *all compact extensions of solvable groups, i.e. groups* T *which contain a closed normal subgroup* S *such that* T/S *is compact and* S *is solvable.*

PROOF. All abelian and compact groups are strongly amenable (cf. 2.8), hence amenable (cf. 4.6). This proves (i) and (ii).

(iii) If T is solvable, then we have subgroups T_i of T ($i = 1,\dots,n$) such that $\{e\} = T_0 \subset T_1 \subset \dots \subset T_n = T$, and, in addition, T_{i-1} is a normal subgroup of T_i and T_i/T_{i-1} is abelian for $i = 1,\dots,n$. Then also

$$\{e\} = T_0 \subset \overline{T}_1 \subset \dots \subset \overline{T}_n = T,$$

where each $\overline{T}_{i-1}$ is a normal subgroup in $\overline{T}_i$ and $\overline{T}_i/\overline{T}_{i-1}$ is abelian)* (the proof is a straightforward consequence of the principle that the relations which have to be checked hold on a dense subset of the set in question, hence, by continuity, on the whole set). Now lemma 4.7 and case (i) above imply by induction that $\overline{T}_1, \overline{T}_2, \dots, T_n = T$ are amenable.

(iv) Follows from (ii), (iii) and lemma 4.7. □

5. BOUNDARIES OF CONNECTED LIE GROUPS

For the results from integration theory which we need in this section, we refer to appendix A3.

)* We could do without taking closures. Let T_d denote the group T with its *discrete* topology. Then the proof, indicated here, shows that T_d is amenable (using the original sequence $\{e\} = T_0 \subset T_1 \subset \dots \subset T_n$, without taking closures), and this obviously implies that T is amenable. Notice, that the fact that we need a *closed* subgroup in lemma 4.7 comes merely from our agreement to consider only topological Hausdorff groups (T/S is Hausdorff if and only if S is closed in T). However, we could do equally well without this condition.

5.1. By the original definition in [3], a *boundary* of a (Lie) group G is a compact homogeneous space M such that for every probability measure μ on M there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that the sequence $\{t_n\mu\}_{n\in\mathbb{N}}$ converges to a point measure. Here a *homogeneous space* M for G is a quotient space of the form M := G/H for some closed subgroup H of G; we shall always require M to be compact. By 1.2(i), an action of G on M is defined by $t(sH) := ts\,H$ $(s\in G; t\in G)$.

In the case of a Lie group G, G is metrizable, hence every homogeneous space M is matrizable [cf. [6], 8.14). Now by A1.6, if M is a compact metric space, then $M(M)$ is first countable. Hence for $\mu,\nu \in M(M)$, $\nu \in \overline{G\mu}$ if and only if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that $\nu = \lim_{n\to\infty} t_n\mu$. Keeping this in mind, we may restate the definition of a boundary as follows: a *boundary* of a Lie group G is a compact homogeneous space M which is, under the natural action of G, strongly proximal (cf. lemma 3.5).

5.2. Let G be a connected semisimple Liegroup with finite center, and let G = KAN be an Iwasawa decomposition. As G has finite center, K is compact (cf. Theorem II.3.1). Recall from chapter V, that if we put S := AN, N is a normal subgroup of S. In addition, let

$$P := N_G(N), \quad M := Z_K(A).$$

Then P = MAN (cf. Theorem III 1.17(viii)), hence obviously

$$P \subseteq H := N_G(S).$$

In fact, it follows from Lemma III.1.5. that P = H; this result will again be obtained as a by-product of the main result of this section, namely, that the universal minimal strongly proximal G-space SP_G (cf. 3.11) is isomorphic with the homogeneous G-space G/P.

5.3. LEMMA. *The space* SP_G *contains a point* z *which is invariant under the actions of* S *and of* H *on* SP_G.

PROOF. Clearly, the groups S and H, being subgroups of G, act on SP_G. Since S is solvable, there exists an S-invariant measure $\mu \in M(TP)_G$ (cf. 4.8). Notice, that the action of G on $M(SP_G)$ is proximal - just by the definition of strong proximality, - and as G = KS with K compact, proposition 2.10 implies, that $M(SP_G)$ is also proximal under the action of S, i.e. SP_G as an

S-space is strongly proximal. So by 3.5, there exists a point $z \in SP_G$ such that $\delta_z \in \overline{S\mu}$. Since μ is S-invariant, it follows, that $\delta_z = \mu$, whence z is an S-invariant point in SP_G.

In order to prove that z is H-invariant, consider $t \in H$. By the definition of H, for every $s \in S$ there exists $s' \in S$ such that $st = ts'$, hence $s(tz) = t(s'z) = tz$. This shows, that tz is an S-invariant point for every $t \in H$. However, S acts proximally on SP_G (even strongly proximal: see above), so the S-orbit closures $\overline{Sz}$ and $\overline{S(tz)}$ meet each other. As z and tz are S-invariant, it follows, that $z = tz$. This shows, that z is H-invariant. □

5.4. <u>COROLLARY</u>. *Let* $L := \{t \in G : tz = z\}$ *be the stabilizer of the point* $z \in SP_G$, *considered in* 5.3. *Then* $H \subseteq L$, *and the* G-*space* SP_G *is isomorphic with the* G-*space* G/L.

<u>PROOF</u>. The only theory, which is not yet completely obvious from lemma 5.3 is, that SP_G is isomorphic to G/L. To prove this, observe, that $L \supseteq H \supseteq S$. Hence G/L is the continuous image of G/S under the mapping $tS \mapsto tL : G/S \to GL$. But G/S is compact, hence G/L is compact.

Next, consider the mapping

$$\phi \colon tL \mapsto tz \colon G/L \to SP_G .$$

$$\begin{tikzcd} G \arrow[r] \arrow[d] & SP_G \\ G/L \arrow[ur, "\phi"'] & \end{tikzcd}$$

It is easily checked, that this is a morphism of G-spaces. In addition, it is clear from the definition of L, that this mapping is injective. However, the image of G/L in SP_G under this mapping is compact, hence closed, hence it contains the orbit closure of z. As SP_G is minimal, it follows that ϕ is surjective. So ϕ is the desired isomorphism of G-spaces (continuity of $\overset{\leftarrow}{\phi}$ follows from the fact that ϕ is bijective, G/L is compact, and TP_G is Hausdorff). □

5.5. <u>THEOREM</u>. (*FURSTENBERG-MOORE*). *Let* G *be a connected semisimple Liegroup with finite center. Then the* G-*space* SP_G *is isomorphic to the* G-*space* G/P, *which is a boundary. Moreover, for any closed subgroup* G' *of* G, *the homogeneous* G-*space* G/G' *is a boundary iff* G' *contains some conjugate of* P.

<u>PROOF</u>. By the result of C.C. Moore, the G-space G/P is a boundary, that is, G/P is strongly proximal (cf. 4.11 below). By the universality of SP_G, there exists a (unique) morphism of G-spaces from SP_G onto G/P. Using 5.4 it follows that there is a morphism of G-spaces $\phi \colon G/L \to G/P$. For clarity, let

us denote the elements of G/L by $\bar{t}$ (:= tL) and those of G/P by $\tilde{t}$ (:= tP), $t \in G$. If $\phi(\bar{e}) = \tilde{t}_0$, then for all $t \in L$ we have $t\bar{e} = \bar{e}$, hence

$$t\tilde{t}_0 = t\phi(\bar{e}) = \phi(t\bar{e}) = \phi(\bar{e}) = \tilde{t}_0.$$

So L leaves $\tilde{t}_0$ invariant, hence $Lt_0P \subseteq t_0P$, so

$$L \subseteq t_0Pt_0^{-1}.$$

Using the inclusions obtained in 5.2 and 5.5, we have

$$P \subseteq H \subseteq L \subseteq t_0Pt_0^{-1}.$$

This implies, that $P = t_0Pt_0^{-1}$)* hence P = H = L. So the first part of our theorem follows from 5.4.

As to the last assertion of the theorem, the universality of G/P implie that there exists a morphism of G-spaces ψ: G/P → G/G', and similar to the argument, used above, this implies that there exists $t_1 \in G$ such that $Pt_1G' \subseteq t_1G'$, that is, $t_1^{-1}Pt_1 \subseteq G'$. □

5.6. COROLLARY. (See also Cor. II. 3.7, Theor. I.4.2, Lemma II. 5.4). *If* G = KAN *and* G = K'A'N' *are two Iwasawa-decompositions, and* P := $N_G(N)$, P' := $N_G(N')$, *then* P *and* P' *are conjugate.*

PROOF. By theorem 4.4 (which applies also with P replaced by P'), each of the groups P and P' contains a conjugate of the other. □

5.7. COROLLARY. *Every minimal strongly proximal G-space is a boundary.*

PROOF. Let X denote an arbitrary minimal strongly proximal G-space. Then X is a homomorphic image of G/P. This implies, that X consists of *one* orbit, and similar to the proof of 5.4 one shows, that X ≅ G/G' for some closed subgroup of G. So X is a boundary. □

5.8. We close this section with a proof, that G/P is strongly proximal. For

)* This is, because in this situation P and $t_0Pt_0^{-1}$ have the same Lie algebras, P being a closed subgroup with finitely many components.

background knowledge, see the Appendices.

In appendix A3.4 we show, that in order to prove that G/P is strongly proximal, it is sufficient to prove, that there exists *one* quasi-invariant measure μ_0 on G/P and a sequence $\{t_n\}_{n\in\mathbb{N}}$ in G such that the sequence $\{t_n\mu_0\}_{n\in\mathbb{N}}$ converges in $M(G/P)$ to a point measure.

Now we need the following facts concerning the structure of G/P:

- there exists an open and dense subset Q in G/P such that the set $G\backslash \overleftarrow{q}[Q]$ has (left) Haar-measure zero; hence Q: $G \to G/P$ is the canonical projection (in order to see this, recall from Lemma III.1.11 that $G = \bigcup_{w\in W} U_w wP$; then $Q = q[U_{w_0}w_0]$ with w_0 such that $U_{w_0} = N$; for $w \in W$, $w \neq w_0$, $U_w wP$ has lower dimension, hence Haar measure zero).
- the set Q has the structure of a vector space and the action of A on Q (induced by the action of G on G/P) can be diagonalized with respect to a certain basis. With coordinates with respect to this basis, put for $r > 0$:

$$B_r := \{x = (\xi_1,\ldots,\xi_n) \in Q : |\xi_i| < r \text{ for } i = 1,\ldots,r\}.$$

Now $t_0 \in A$ can be chosen such that

$$(*) \qquad t_0^{n+1}B_r \supseteq t_0^n B_r; \quad \bigcup_{n\in\mathbb{N}} t^n B_r = Q$$

(take $t_0 := \exp Z$, Z in the positive Weyl chamber).

Let $\mu_0 \in M(G/P)$ be quasi-invariant. Then for every $r > 0$ and $n \in \mathbb{N}$ we have

$$t_0^{-n}\mu_0(B_r) = \mu_0(t_0^n B_r) \leq \mu_0(Q),$$

hence by (*):

$$(**) \qquad \lim_{n\to\infty} (t_0^{-n}\mu_0)(B_r) = \mu_0(Q).$$

Let λ_G denote left Haar measure on G. Then by $\nu[B] := \lambda_G(\overleftarrow{q}[B])$ for a Borel-subset B of G/P, a measure on G/P is defined, which is easily seen to be quasi-invariant. Hence by one of the results, quoted in A3.1, ν and μ_0 have the same null-sets (both measures are quasi-invariant)$)^*$

$)^*$ One might object that A3.1 concerns only measures in $M(G/P)$, i.e. measures ν with $\|\nu\| \leq 1$. However, since G/P is compact, all (Borel-)measures on G/P are bounded. In particular, for a suitable constant $c > 0$ the measure ν defined above satisfies $c\nu \in M(G/P)$.

Since $\lambda_G(G\backslash \overset{\leftarrow}{q}[Q]) = 0$, it follows that $\mu_0((G/P)\backslash Q) = 0$, hence $\mu_0(Q) = 1$. The sequence $\{t_0^{-n}\mu_0\}_{n\in\mathbb{N}}$ has in the compact, first countable space $M(Q/P)$ a convergent subsequence, say with limit μ_1. Now condition (**) implies, that

$$\mu_1(B_r) = \mu_0(Q) = 1.$$

Since B_r is closed for every $r > 0$, it follows, that the support of μ_1 is included in $\bigcap_{r>0} B_r = 0$ (=zero-vector of Q), hence $\mu_1 = \delta_0$. This shows, that some subsequence of $\{t_0^{-n}\mu_0\}_{n\in\mathbb{N}}$ converges to a point measure.

APPENDIX A1: COMPACT CONVEX SETS: $C(X)$ AND $M(X)$

For proofs of results in this appendix, we refer to [1], [8] and [10].

A1.1. Let X be a compact Hausdorff space, and $C(X)$ the set of all continuous, real-valued functions on X. With the usual pointwise defined operations and the supremum norm, $C(X)$ is a Banach space. Its (topological) dual will be denoted $C(X)'$; it is the space of all continuous linear mappings $\mu: C(X) \to \mathbb{R}$. We shall consider two topologies on $C(X)'$: the topology, induced by the norm $\|\cdot\|$, where

$$\|\mu\| := \sup\{|\mu(f)| \,\vdots\, f \in C(X) \text{ and } \|f\| \leq 1\}$$

for $\mu \in C(X)'$, and, in addition, the w^*-topology (also denoted as the $\sigma(C(X)',C(X))$-topology), which is the weakest topology on $C(X)'$ for which all mappings $\mu \mapsto \mu(f): C(X)' \to \mathbb{R}$ for $f \in C(X)$ are continuous. Thus, the w^*-topology is the relative topology of $C(X)'$ in $\mathbb{R}^{C(X)}$. It is important for our purposes to observe, that $C(X)'$ with the w^*-topology is a locally convex topological vectorspace)*.

Let $S' := \{\mu \in C(X)' \,\vdots\, \|\mu\| \leq 1\}$, the unit ball in $C(X)'$. By Alaoglu's theorem, S' *is a compact Hausdorff space when endowed with the* w^**-topology*. Consequently,

$$M(X) := \{\mu \in C(X)' \,\vdots\, \mu \geq 0 \text{ and } \mu(\underline{1}) = 1\}$$

is compact in the w^*-topology)1. To prove this, it is sufficient to show that $M(X)$ is closed in $C(X)'$, and that $M(X) \subseteq S'$.

(1) $M(X)$ is w^*-closed in $C(X)'$, because $M(X)$ can be written as an intersection of w^*-closed sets, as follows:

$$M(X) = \bigcap_{\substack{f \in C(X) \\ f \geq 0}} \{\mu \in C(X)' \,\vdots\, \mu(f) \geq 0\} \cap \{\mu \in C(X)' \,\vdots\, \mu(\underline{1}) = 1\}.$$

)1 $\underline{1}$ is the constant function on X with value 1. Note, that $\underline{1} \in C(X)$. Moreover, $\mu \geq 0$ means: $f \in C(X)$ and $f \geq 0 \Rightarrow \mu(f) \geq 0$; or equivalently, $f,g \in C(X)$ and $f \leq g \Rightarrow \mu(f) \leq \mu(g)$ (monotonicity).

)* A base for the neighbourhoodsystem of $\mu \in C(X)'$ is formed by the collection of all sets

$$U_\mu(F,\varepsilon) := \{\nu \in C(X)' | \forall f \in F : |\mu(f) - \nu(f)| < \varepsilon\}$$

with $\varepsilon > 0$ and F a finite subset of $C(X)$.

(2) If $\mu \in M(X)$, then by monotonicity we have for every $f \in C(X)$ that $-\mu(|f|) = \mu(-|f|) \leq \mu(f) \leq \mu(|f|)$, that is,

$$|\mu(f)| \leq \mu(|f|) \qquad \text{for } f \in C(X).$$

Clearly, $|f| \leq \|f\| . \underline{1}$ (recall, that $|f|(x) := |f(x)|$) and, consequently,

$$|\mu(f)| \leq \mu(\|f\| . 1) = \|f\| \; \mu(\underline{1}) = \|f\| .$$

Thus, $\|\mu\| \leq 1$ for $\mu \in M(X)$.

(3) It is easy to see, that $M(X)$ is a convex subset of $C(X)'$, that is, if $\mu, \nu \in M(X)$ and $0 \leq a \leq 1$, then $a\mu + (1-a)\,\nu \in M(X)$.

Resuming, we have shown that $M(X)$ *is a compact, convex subset of the locally convex tvs* $C(X)'$ *with its* w^**-topology.*

A1.2. By Riesz's representation theorem there is a 1,1-correspondence between $M(X)$ and the set of all regular probability measures on X, as follows: if $\mu \in M(X)$, then there exists a unique regular Borel measure m_μ on X such that

$$\mu(f) = \int_X f dm_\mu \qquad \text{for all } f \in C(X).$$

As $m_\mu(X) = \mu(\underline{1}) = 1$, it follows that m_μ is a probability measure.

Conversely, it is clear that every regular Borel probability measure m on X induces an element $\mu_m\colon f \mapsto \int_X f dm$ of $M(X)$. As $\mu_{m_\mu} = \mu$ for every $\mu \in M(X)$ and $m_{\mu_m} = m$ for every regular probability measure, we shall just identify $\mu \in M(X)$ with the corresponding measure. So we may write

$$\mu(f) = \int_X f d\mu \qquad \text{for } f \in C(X),\ \mu \in M(X).$$

A1.3. Let E be a locally convex topological vector space. The following result is the famous theorem of Krein and Milman: Let Q be a compact, convex subset of E, $Q \neq \emptyset$. Then the set ex Q of all extreme points)* in Q is non-empty, and $\overline{co}(\text{ex } Q) = Q$.)§

)* and)§ : see footnotes at next page.

Conversely, if B is any *closed* subset of the compact, convex subset Q of E such that $\overline{co}(B) = Q$, then $\overline{ex}(Q) \subseteq B$.

(It should be observed, that usually, ex Q is *not* closed in Q; in this respect the next example is an exception.)

A1.4. Let X be a compact Hausdorff space. For $x \in X$, let $\delta_x : C(X) \to \mathbb{R}$ be defined by $\delta_x(f) := f(x)$ for $f \in C(X)$. Then for all $x \in X$, $\delta_x \in M(X)$. The mapping $\delta : x \mapsto \delta_x : X \to M(X)$ has the following properties:
(1) $\delta : X \to M(X)$ is injective, because C(X) separates the points of X.
(2) $\delta : X \to M(X)$ is continuous ($M(X)$ being endowed with the w^*-topology), because $x \mapsto \delta_x(f) : X \to \mathbb{R}$ is continuous for every $f \in C(X)$.
(3) As X is compact and $M(X)$ is Hausdorff (and compact), then $\delta : X \to M(X)$ is a topological embedding. Its range $\delta[X]$ coincides with ex $M(X)$ - which is therefore a closed subset of $M(X)$. In fact, for $\mu \in C(X)'$ the following equivalent:
(i) $\mu \in \text{ex}\, M(X)$
(ii) $\exists x \in X \,\vdots\, \mu = \delta_x$
(4) By (3) and A1.3(3), we have : $M(X) = \overline{co}(\delta[X])$.

A1.5. If X and Y are compact Hausdorff spaces and $\phi : X \to Y$ is a continuous mapping, then ϕ induces a continuous (norm decreasing) linear mapping

$$\tilde{\phi} : f \mapsto f \circ \phi : C(Y) \to C(X),$$

and this induces a continuous linear mapping

$$(\tilde{\phi})' : \mu \mapsto \mu \circ \tilde{\phi} : C(X)' \to C(Y)'.$$

)* If Q is a convex set, then $x \in Q$ is called an *extreme point* of Q whenever $Q\backslash\{x\}$ is convex. Equivalently, x is an extreme point of the convex set Q iff $x = ay + (1-a)z$ with $y,z \in Q$ and $0 \leq a \leq 1$ implies either $x = y$ or $x = z$. The set of all extreme points of Q will be denoted ex(Q) or ex Q and its closure in Q by $\overline{\text{ex}}$ Q.

)§ If A is a subset of a vector space, then co(A) or co A is the *convex hull* of A : co A := $\cap\{Q \,\vdots\, Q \text{ convex } \& \ A \subseteq Q\} = \{\Sigma_{x \in F}\, a(x)x \,\vdots\, F \subseteq A,\ F \text{ finite},\ a(x) \geq 0 \text{ for all } x \in F,\ \Sigma_{x\in F}\, a(x) = 1\}$. If A is a subset of a tvs, then $\overline{co}(A) := \overline{co(A)}$. Then $\overline{co}(A)$ is also convex: it is the least *closed* convex set, containing A.

It is easy to see, that $(\tilde{\phi})'$ is w^*-continous and that $(\tilde{\phi})'$ maps $M(X)$ into $M(Y)$. Call the restriction of $(\tilde{\phi})'$ to $M(X)$ and $M(Y)$: $M(\phi)$. Thus, we have the continuous mapping

$$M(\phi) : M(X) \to M(Y).$$

If, for simplicity, we denote $M(\phi)(\mu) =: \tilde{\mu} \in M(Y)$ for $\mu \in M(X)$, then, by definition, we have

$$\tilde{\mu}(f) = \mu(f \circ \phi) \qquad \text{for } f \in C(Y)$$

or, in a different notation,

$$\int_Y f d\tilde{\mu} = \int_X f \circ \phi \, d\mu \qquad \text{for } f \in C(Y).$$

A straightforward computation shows, that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d, "\delta^X"'] & Y \arrow[d, "\delta^Y"] \\ M(X) \arrow[r, "M(\phi)"] & M(Y) \end{tikzcd}$$

where δ^X and δ^Y are the canonical embeddings of X into $M(X)$ and Y into $M(Y)$ (according to A1.4). Using this and A1.4(4) it follows easily, that $M(\phi)$ *is a surjection if and only if* ϕ *is a surjection.*

Using the fact that every continuous function on a closed subset of Y can be extended to a continuous function on Y, (Y is a normal space, since Y is compact Hausdorff) it follows that, if ϕ is injective, then $\tilde{\phi} : C(Y) \to C(X)$ is surjective. This implies that $(\tilde{\phi})'$, hence $M(\phi)$, is injective. Since the converse follows easily from the diagram above, we have, that $M(\phi)$ *is injective if and only if* ϕ *is injective.*

A1.6. *If* X *is a compact, metrizable space, then* $M(X)$ *is first countable,* i.e. each point in $M(X)$ has a countable neighbourhood base. The proof is as follow

First, observe, that the space C(X) is separable, i.e. C(X) has a countable subset D which is dense (in the topology of uniform convergence, of course). This is a consequence of the Stone-Weierstrass theorem. We claim, that for $\mu \in M(X)$ the family of all sets

$$U_\mu(F',\frac{1}{k}) := \{\nu \in M(X) : |\nu(f) - \mu(f)| < \frac{1}{k} \text{ for all } f \in F'\},$$

$k \in \mathbb{N}$, F' a finite subset of D, forms a local base at μ. This will prove our statement, since the cardinality of the set of all finite subsets of the countable set D is countable. So consider an arbitrary basical neighbourhood $U_\mu(F,\varepsilon)$ of μ ($\varepsilon > 0$, F a finite subset of C(X)). Let $k \in \mathbb{N}$ be such that $3/k < \varepsilon$. Select for each $f \in F$ an element $f' \in D$ such that $\|f-f'\| < 1/k$, and let $F' := \{f' : f \in F\}$. Then for each $\nu \in M(X)$ and $f \in F$ we have

$$|\nu(f) - \nu(f')| = |\nu(f - f')| \leq \|\nu\| \, \|f - f'\| < \frac{1}{k}.$$

Hence for every $f \in F$ we have

$$\begin{aligned} |\nu(f) - \mu(f)| &\leq |\nu(f) - \nu(f')| + |\nu(f') - \mu(f')| + |\mu(f') - \mu(f)| \\ &< \frac{2}{k} + |\nu(f') - \mu(f')| \end{aligned}$$

This implies, that $U_\mu(F',1/k) \subseteq U_\mu(F,\varepsilon)$, which proves our claim.

APPENDIX A2: THE BARYCENTER MAP

A2.1. If X is a non-empty compact subset of a locally convex topological vector space E, and $\mu \in M(X)$, then we say that a point $x \in E$ is *represented by* μ, or that μ has *barycenter* x, whenever

$$\phi(x) = \int_X \phi \, d\mu$$

for every $\phi \in E'$. As E' separates the points of E (a consequence of the Hahn-Banach theorem), the barycenter of μ, if it exists, is *unique*. In our next theorem it is stated, among others, that such a barycenter always exists; it will be denoted by $b(\mu)$.

A2.2. THEOREM. *Let* X *be a compact non-empty subset of a locally convex topological vector space* E *such that* $Q := \overline{\text{co}}\, X$ *is compact*)*. *Then there exists a unique mapping* $b : M(X) \to Q$ *with the following properties:*

(i) *For every* $\mu \in M(X)$, $b(\mu)$ *is the barycenter of* μ.

(ii) $b : M(X) \to Q$ *is surjective, i.e. every point of* Q *is the barycenter of some probability measure on* X.

(iii) $b : M(X) \to Q$ *is affine, and continuous with respect to the* w^**-topology on* $M(X)$.

PROOF. Cf. [10], Chapter 1. □

The mapping $b : M(X) \to Q$ (sometimes to be denoted by $b^{(X)}$) will be called the *barycenter map*. The two cases which will interest us most are the following ones: (a) $X = \overline{\text{ex}}\, Q$ for a given compact convex set Q and (b) X = = Q for such a Q.

)* Compactness of Q follows from compactness of X if E is complete or if E = F', F a Banach space and E having the w^*-topology; also, if we start with compact Q and put $X := \overline{\text{ex}}\, Q$, the conditions are fulfilled (by A1.3, Q = = $\overline{\text{co}}$ X in this case)

A2.3. EXAMPLE. Let Q be a compact convex subset in a locally convex topological vector space E, $Q \neq \emptyset$, and let $X \subseteq Q$ be closed such that $Q = \overline{\text{co}}\, X$ (particular cases are: $X = \overline{\text{ex}}\, Q$ and $X = Q$). For every $x \in X$ we have $\delta_x^{(X)} \in M(X)$. We claim, that $x = b(\delta_x^{(X)})$. Indeed, for every $\phi \in E'$ we have

$$\phi(x) = \delta_x^{(X)}(\phi) = \int_X \phi \, d\delta_x^{(X)}.$$

Thus, we have $b \circ \delta^{(X)}$ = identity mapping of X.

The following result states, that extreme points of Q can be the barycenter only of the corresponding point measures:

A2.4. THEOREM. *Let* Q *be as above. If* $\mu \in M(X)$ *and if* $b(\mu) \in \text{ex}\, Q$, *then* $\mu = \delta_{b(\mu)}^{(Q)}$.

PROOF. Cf. [10], Prop. 1.4. □

The final result in this appendix will be, that in the case of an affine T-space Q and an invariant compact subset X of Q such that $Q = \overline{\text{co}}\,(X)$, the barycenter map $b^{(X)} : M(X) \to Q$ is a morphism of T-spaces. Again, this result applies to the cases $X = \overline{\text{ex}}\, Q$ (indeed, ex Q is invariant under an affine action of T on Q, hence its closure is invariant as well) and $X = Q$. For this we need the following lemma and its corollary:

A2.5. LEMMA. *Let* Q *be as above. Then the set*

$$\{\phi|_Q + r.\underline{1} \,\vdots\, \phi \in E' \;\&\; r \in \mathbb{R}\}$$

is dense in the space $A(Q)$ *of all affine continuous real-valued functions*)* *on* Q *endowed with the topology of uniform convergence on* Q.

PROOF. Cf. [10], Prop. 4.5. □

A2.6. COROLLARY. *Let* X *and* Q *be as in Theorem* A2.2. *Then the barycenter map* $b : M(X) \to Q$ *has the property*

$$\forall f \in A(Q) : f(b(\mu)) = \int_X f \, d\mu \qquad (\mu \in M(X))$$

)* $f \in C(Q)$ is called affine whenever $f(ax+(1-a)y) = af(x)+(1-a)f(y)$ for all $x,y \in Q$ and $0 \le a \le 1$.

<u>PROOF</u>. By the definition of barycenter, the formula holds if $f = \phi|_Q$ for $\phi \in E'$. It holds also, if $f = r.\underline{1}$ for $r \in \mathbb{R}$, hence it holds for all mappings f of the form $\phi|_Q + r.\underline{1}$ with $\phi \in E'$ and $r \in \mathbb{R}$. Now observe, that the mappings $f \mapsto \int_X f\, d\mu$ and $f \mapsto f(b(\mu))$ from $C(Q)$ to $\mathbb{R}$ are both continuous with respect to the topology of uniform convergence on Q. Since these mappings are equal to each other on a dense subset of $A(Q)$, they are equal on $A(Q)$. □

A2.7. <u>THEOREM</u>. *Let Q be an affine* T*-space and let* X *be a closed invariant subsubset of* Q *such that* $\overline{\text{co}}\, X = Q$. *Then the barycenter map* $b : M(X) \to Q$ *is an affine morphism of* T*-spaces.*

<u>PROOF</u>. We need only to show that b satisfies the condition $b \circ \sigma^t = \sigma^t \circ b$ for all $t \in T$ (cf. A2.2 (iii) for the other properties). Since for all $\phi \in E'$ and $t \in T$ we have $\phi \circ \sigma^t \in A(Q)$, it is clear that

$$\phi(b(\sigma^t\mu)) = \int_X \phi\, d(\sigma^t\mu) = \int_X (\phi \circ \sigma^t) d\mu \underset{\overline{}}{\overset{(A2.6)}{=\!=\!=}}$$
$$= (\phi \circ \sigma^t)(b(\mu)) = \phi(\sigma^t(b(\mu)))$$

$(\mu \in M(X))$. Since E' separates the points of E, it follows that $b(\sigma^t\mu) = \sigma^t b(\mu)$ for all $\mu \in M(X)$, hence $b \circ \sigma^t = \sigma^t \circ b$. □

APPENDIX A3: QUASI INVARIANT MEASURES

A3.1. Let G and P be as in section 4. Although we shall formulate the results here in the context of G/P (which is compact), all results about quasi-invariant measures and convolution of measures are valid in the context of an arbitrary homogeneous space G/H (H a closed subgroup of G) for arbitrary locally compact G, satisfying the second axiom of countability (i.e. separable and metrizable). Of course, in the general case one should consider $K(G/H)$ (= space of continuous real-valued functions with compact support) instead of $C(G/P)$.

First, recall that in an arbitrary set X with σ-field $\mathcal{B}$ for two finite measures μ and ν on $\mathcal{B}$ the following properties are equivalent.)*:

(i) $\forall B \in \mathcal{B} : \mu(B) = 0 \Rightarrow \nu(B) = 0$

(ii) $\forall \varepsilon > 0\ \exists \delta > 0 \vdots \forall B \in \mathcal{B}\ (\mu(B) < \delta \Rightarrow \nu(B) < \varepsilon)$

If these conditions are fulfilled, then we write $\nu << \mu$ and we say that ν is *absolutely continuous* with respect to μ.

Now an arbitrary measure μ on G/H is called *quasi-invariant* whenever for every $t \in G$ we have

$$t\mu << \mu \text{ and also } \mu << t\mu$$

(i.e. μ and $t\mu$ have exactly the same sets of measure zero). The following facts can be shown (cf. [9], Ch.IX):

- if $\mu \in M(G/H)$ then μ is quasi-invariant iff

$$\forall \text{ Borelset } B \subseteq G/P : \mu(B) = 0 \Longleftrightarrow \lambda_G(q^{\leftarrow}[B]) = 0$$

Here $q : G \to G/H$ is the quotient map, and λ_G is left Haar measure on G.

- if $\mu_1, \mu_2 \in M(G/H)$ are both quasi-invariant, then $\mu_1 << \mu_2$ and $\mu_2 << \mu_1$.

)* this is an easy consequence of the Radon-Nikodym theorem, but can also be proved easily in a straightforward way.

- there exists $\mu \in M(G/H)$ which is quasi-invariant.
(Note, that these results also hold for $G = G/\{e\}$; thus there exists $\lambda \in U(G)$, λ quasi-invariant.)

A3.2. If $\nu \in M(G)$ and $\mu \in M(G/P)$, then an element $\nu * \mu$ of $M(G/P)$ is defined by

(1) $$\nu * \mu(f) = \int_{G/P} \int_G f(tx) \quad d\nu(t)d\mu(x) \qquad (f \in C(G/P)).$$

(In order to prove that this makes sense, one has to show that the mapping $x \mapsto \int_G f(tx)d\nu(t) : G/P \to \mathbb{R}$ is continuous. This is not difficult, once one knows that for all $\varepsilon > 0$ there exists a compact subset C of G such that $\nu(G\setminus C) < \varepsilon/2$.)

Some elementary computations (using FUBINI's theorem) show, that

(2) $$\delta_t * \mu = t\mu \qquad \text{for } t \in G,\ \mu \in M(G/P)$$

and

(3) $$t(\nu * \mu) = \delta_t*(\nu * \mu) = t\nu * \mu \qquad \text{for } t \in G,\ \nu \in M(G),\ \mu \in M(G/P).$$

Moreover, for any Borel subset of G/P we have, if $\nu \in M(G)$ and $\mu \in M(G/P)$:

$$\begin{aligned} \nu * \mu(B) &= \int_{G/P} \int_G \chi_B(tx)d\nu(t)d\mu(x) \\ &= \int_G \left(\int_{G/P} \chi_B(tx)d\mu(x) \right) d\nu(t). \end{aligned}$$

So we have

(4) $$\nu * \mu(B) = \int_G (t\mu)(B)d\nu(t).$$

A3.3. <u>LEMMA</u>. *If* $\mu \in M(G/P)$, *and if* $\lambda \in M(G)$ *is quasi-invariant, then* $\lambda * \mu$ *is a quasi-invariant.*

<u>PROOF</u>. For a Borelset B in G/P we have, by (4)

$$\lambda * \mu(B) = \int_G (t\mu)(B)d\lambda(t).$$

This implies, that $\lambda * \mu(B) = 0$ iff $(t\mu)(B) = 0$ for λ - a.e. t in G. For $s \in G$ we have, by (3), $s(\lambda*\mu) = s\lambda * \mu$, so we obtain

$$(s(\lambda*\mu))(B) = 0 \iff (s\lambda*\mu)(B) = 0 \iff (t\mu)(B) = 0 \text{ for } s\lambda \text{ - a.e. } t \text{ in } G.$$

However, λ and $s\lambda$ have the same nullsets in G, so the phrases "λ-a.e. t in G" and "$s\lambda$-a.e. t in G" have exactly the same meaning. Consequently $\lambda * \mu(B) = 0$ iff $s(\lambda*\mu)(B) = 0$. So by definition, $\lambda * \mu$ is quasi-invariant. □

A3.4. <u>LEMMA</u>. *If there exists* $\mu_0 \in M(G/P)$, μ_0 *quasi-invariant, such that* $\{t_n\mu_0\}_{n\in\mathbb{N}}$ *converges for some sequence* $\{t_n\}_{n\in\mathbb{N}}$ *in* G *to a point measure* δ_x *in* $M(G/P)$ $(x \in G/P)$, *then* G/P *is strongly proximal.*

<u>PROOF</u>. Consider $\mu \in M(G/P)$. We have to show that for some sequence $\{s_k\}_{k\in\mathbb{N}}$ in G the sequence $\{s_k\mu\}_{k\in\mathbb{N}}$ converges to δ_x.

There exists a quasi-invariant $\lambda \in M(G)$. By A3.3, $\lambda * \mu$ is quasi-invariant in $M(G/P)$, hence $\lambda * \mu << \mu_0$ (cf. A3.1). So

$$(5) \qquad \forall\varepsilon > 0 \quad \exists\delta > 0 \quad \forall \text{ Borelset } B \text{ in } G/P : \mu_0(B) < \delta \Rightarrow \lambda * \mu(B) < \varepsilon.$$

Let $\{U_k\}_{k\in\mathbb{N}}$ be a nested local base at x, and put $B_k := (G/P)\backslash U_k$. Then B_k is closed, and $x \notin B_k$. Hence

$$0 = \delta_x(B_k) = \lim_{n\to\infty} t_n\mu_0(B_k) = \lim_{n\to\infty} \mu_0(t_n^{-1}B_k).$$

Using (5), this implies, that for every $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that for all $n \geq n_k$:

$$(6) \qquad t_n\lambda * \mu\ (B_k) = t_n(\lambda*\mu)(B_k) = \lambda * \mu(t_n^{-1}B_k) < \frac{1}{k}.$$

However, by formula (4),

$$(7) \qquad t_n\lambda * \mu\ (B_k) = \int_G s\mu(B_k)d(t_n\lambda)(s) = \int_G (t_ns\mu)(B_k)d\lambda(s)$$

Now (6) and (7) clearly imply that there exists $s_k \in G$ such that

$$(8) \qquad s_k\mu(B_k) < \frac{2}{k}.$$

Now the sequence $\{s_k\mu\}_{k\in\mathbb{N}}$ has (in $\mathcal{M}(G/P)$, which is a compact, first countable space) a convergent subsequence, say with limit ν. In passing to this subsequence, we may assume, that $\nu = \lim_{k\to\infty} s_k\mu$. For any closed subset K of G/P, this implies that $\nu(K) = \lim_{k\to\infty} s_k\mu(K)$. However, if $x \notin K$, then $U_k \cap K = \emptyset$ for almost all $k \in \mathbb{N}$, that is $K \subseteq B_k$. Then, by (8),

$$s_k\mu(K) \le s_k\mu(B_k) < \frac{2}{k},$$

hence $\nu(K) = 0$. Since this holds for every closed subset K of G/P with $x \notin K$, this implies, that $\nu = \delta_x$. Therefore, $\lim_{k\to\infty} s_k\mu = \delta_k$. □

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Chapter V

CLASSIFICATION OF REAL SEMISIMPLE LIE ALGEBRAS

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This chapter gives a sketch of the classification of real forms of a complex semi-simple Lie algebra. The approach given here is close to that of ARAKI and SUGIURA (cf. [Satake], Appendix). Different treatments can be found f.i. in CARTAN, GANTMACHER (simplified by MURAKAMI) and HELGASON.

1. NORMALLY RELATED REAL FORMS

Let g_c be a complex Lie algebra and let $(g_c)_{\mathbb{R}}$ denote g_c considered as a real Lie algebra. Put Γ for $\mathrm{Gal}(\mathbb{C},\mathbb{R}) = \{1,\gamma_0\}$. If one has an action of Γ on $(g_c)_{\mathbb{R}}$, then we denote the action of γ_0 by σ. σ is called a *conjugation of* g_c if it satisfies

$$\sigma(\lambda X) = \bar{\lambda}\,\sigma(X) \qquad \text{for } \lambda \in \mathbb{C},\ \ X \in (g_c)_{\mathbb{R}}.$$

We will use the notation g_σ for the *fixed point* set of a conjugation σ. It is a *real form* of g.

From now on assume that g_c is semi-simple. Let g_σ be a non-compact real form of g_c. We know from (Ch.I, Proposition 3.1) that there exists a compact real form g_τ of g_c so that $\sigma \circ \tau = \tau \circ \sigma$. Denote $\sigma \circ \tau$ by θ; it is called the *Cartan involution* of g_σ. Put $k = g_\sigma \cap g_\tau$ and $p = g_\sigma \cap i.g_\tau$, then $g_\sigma =$ $= k + p$ is a *Cartan decomposition* of g_σ in the sense of (ChI., Definition 3.2).

For a subalgebra h of g_σ let $\mathrm{Int}_{g_\sigma}(h)$ denote the analytic subgroup of $\mathrm{Int}(g_\sigma)$ with Lie algebra $\mathrm{ad}_{g_\sigma}(h)$.

DEFINITION 1.1. An *Iwasawa C.S.A.* h of g_σ is a Cartan subalgebra (C.S.A) satisfying $h \supset a$, with a a maximal abelian subalgebra contained in p.

They are obtained in the following way:

PROPOSITION 1.2. Let $\mathfrak{a}$ be a maximal abelian subalgebra contained in $\mathfrak{p}$ and $\mathfrak{h} \supset \mathfrak{a}$ a maximal abelian subalgebra of $\mathfrak{g}_\sigma$. Then $\mathfrak{h}$ is an Iwasawa C.S.A. of $\mathfrak{g}_\sigma$. Furthermore all Iwasawa C.S.A.'s of $\mathfrak{g}_\sigma$ are conjugate under $\mathrm{Int}_{\mathfrak{g}_\sigma}(\mathfrak{k})$.

PROOF. In order to prove that $\mathfrak{h}$ is a C.S.A. of $\mathfrak{g}_\sigma$ it suffices to show that $\mathfrak{h}$ is θ-stable, since all elements of $\mathfrak{k}$ resp. $\mathfrak{p}$ are ad-semisimple.

Denote the centralizer of $\mathfrak{a}$ in $\mathfrak{g}_\sigma$ by $\xi_{\mathfrak{g}_\sigma}(\mathfrak{a})$. Since $\mathfrak{a}$ is θ-stable and maximal abelian in $\mathfrak{p}$ we have:

$$\xi_{\mathfrak{g}_\sigma}(\mathfrak{a}) = \xi_{\mathfrak{g}_\sigma}(\mathfrak{a}) \cap \mathfrak{k} + \mathfrak{a}.$$

Let $H \in \mathfrak{h} \subset \xi_{\mathfrak{g}_\sigma}(\mathfrak{a})$, $H = X + Y$, $X \in \xi_{\mathfrak{g}_\sigma}(\mathfrak{a}) \cap \mathfrak{k}$, $Y \in \mathfrak{a}$. Then $X = H-Y \in \mathfrak{h}$ and the first assertion has been proved.

Since all maximal abelian subalgebras in $\mathfrak{p}$ are conjugate under $\mathrm{Int}\, \mathfrak{g}_\sigma(\mathfrak{k})$ (Ch.I, Theorem 4.2) it suffices to show that two C.S.A.'s $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}_\sigma$, containing $\mathfrak{a}$, are conjugate under $\mathrm{Int}_{\mathfrak{g}_\sigma}(\mathfrak{k} \cap \xi_{\mathfrak{g}_\sigma}(\mathfrak{a})$. This result follows from the fact that $\mathfrak{h}_i = \mathfrak{h}_i \cap \mathfrak{k} + \mathfrak{a}$ $(i=1,2)$, with $\mathfrak{h}_i \cap \mathfrak{k}$ maximal abelian subalgebras of the compact Lie algebra $\xi_{\mathfrak{g}_\sigma}(\mathfrak{a}) \cap \mathfrak{k}$ and that all C.S.A.'s of a compact Lie algebra are conjugate. (cf. [He, Ch.V Theorem 6.4 and Ch.VII Corollary 2.7]). Q.E.D.

Fix now a compact real form $\mathfrak{u} = \mathfrak{g}_\tau$ of $\mathfrak{g}_c$ and a C.S.A. $\mathfrak{t}$ of $\mathfrak{u}$.

DEFINITION 1.3. A real form $\mathfrak{g}_\sigma$ of $\mathfrak{g}_c$ is called *related to* $(\mathfrak{u},\mathfrak{t})$ if $\sigma \circ \tau = \tau \circ \sigma$ and $\sigma(\mathfrak{t}_c) = \mathfrak{t}_c$. $\mathfrak{g}_\sigma$ is called *normally related* to $(\mathfrak{u},\mathfrak{t})$ if $\mathfrak{g}_\sigma$ is related to $(\mathfrak{u},\mathfrak{t})$ and moreover $\mathfrak{a} = i\mathfrak{t} \cap \mathfrak{g}_\sigma$ is maximal abelian in $\mathfrak{p} = i\mathfrak{u} \cap \mathfrak{g}_\sigma$.

REMARK. If $\mathfrak{g}_\sigma$ is related to $(\mathfrak{u},\mathfrak{t})$ and $\mathfrak{k} = \mathfrak{g}_\sigma \cap \mathfrak{u}$, then $\mathfrak{g}_\sigma = \mathfrak{k} + \mathfrak{p}_i$ is a Cartan decomposition of $\mathfrak{g}_\sigma$ with Cartan involution $\theta = \sigma \circ \tau$. If moreover $\mathfrak{g}_\sigma$ is normally related to $(\mathfrak{u},\mathfrak{t})$ then $\mathfrak{a} = i\mathfrak{t} \cap \mathfrak{g}_\sigma$ is maximal abelian in $\mathfrak{p}$ and $\mathfrak{h} = \mathfrak{t}_c \cap \mathfrak{g}_\sigma$ is an Iwasawa C.S.A. of $\mathfrak{g}_\sigma$.

Since all C.S.A's resp. compact real forms of $\mathfrak{g}_c$ are conjugate under the group of inner automorphisms of $\mathfrak{g}_c$, we have:

PROPOSITION 1.4. Every real form of $\mathfrak{g}_c$ is conjugate with one, normally related to $(\mathfrak{u},\mathfrak{t})$.

2. ACTION OF Γ ON THE ROOT SYSTEM

Let $\Phi \subset \mathcal{t}_c^*$ be the *root system* of $\mathcal{g}_c$ w.r.t. the C.S.A. $\mathcal{t}_c$, X_r *the root lattice* of Φ, W the *Weyl group*, $T_\alpha \in \mathcal{t}_c$ the vector corresponding to $\alpha \in \Phi$, $h_{\mathbb{R}} = \Sigma_{\alpha\in\Phi}\ \mathbb{R}.T_\alpha$ (See Ch.I, (2.5)) and $\mathcal{g}_c = \mathcal{t}_c + \Sigma_{\alpha\in\Phi}\ \mathcal{g}^\alpha$ the root space decomposition of $\mathcal{g}_c$.

If $\mathcal{g}_\eta$ is a real form of $\mathcal{g}_c$ s.t. $\eta(\mathcal{t}_c) = \mathcal{t}_c$, then η induces an anti-involution η^* of $\mathcal{t}_c^*$ defined by:

(2.1) $\qquad (\eta^*\lambda)(H) = \overline{\lambda(\eta(H))}$ for all $H \in \mathcal{t}_c$, $\lambda \in \mathcal{t}_c^*$.

For $\alpha \in X_r$ denote $\eta^*(\alpha)$ by α^η.

In the following way one sees that $\eta^*(\Phi) = \Phi$, $\eta(T_\alpha) = T_{\alpha^\eta}$ for all $\alpha \in \Phi$ and $\eta(h_{\mathbb{R}}) = h_{\mathbb{R}}$: if $\alpha \in \Phi$, $H \in \mathcal{t}_c$, $Y \in \mathcal{g}^\alpha$, then:

$$[H,\eta(Y)] = \eta[\eta(H),Y] = \overline{\alpha(\eta(H))\eta(Y)} = \alpha^\eta(H).\eta(Y)$$

Hence $\alpha^\eta \in \Phi$. Furthermore:

$$B(\eta(T_\alpha),H) = \overline{B(T_\alpha,\eta(H))} = \overline{\alpha(\eta(H))} = \alpha^\eta(H) = B(T_{\alpha^\eta},H),$$

so that $\eta(T_\alpha) = T_{\alpha^\eta}$ and $\eta(h_{\mathbb{R}}) = h_{\mathbb{R}}$.

By means of η^* one can define an action of Γ on (X_r,Φ) and we denote the action of $\gamma \in \Gamma$ by : α^γ.

<u>REMARK 2.1</u>. If one takes η equal to τ where $\mathcal{g}_\tau$ is a compact real form then we get for $\alpha \in \Phi$:

(2.2) $\qquad \alpha^\tau = -\alpha$

because if $H \neq 0 \in h_{\mathbb{R}}$ with $\tau(H) = H$, then $B(H,H) = B(H,\tau(H)) < 0$ which contradicts the fact that B is strictly positive definite on $h_{\mathbb{R}} \times h_{\mathbb{R}}$, hence $\tau(H) = -H$ for every $H \in h_{\mathbb{R}}$. Similarly if $\mathcal{g}_\sigma$ is related to $(\mathcal{u},\mathcal{t})$ then $\sigma(h_{\mathbb{R}}) = \tau(h_{\mathbb{R}})$ and the fact that $B|_{h_{\mathbb{R}}\times h_{\mathbb{R}}}$ is positive definite imply that

(2.3) $\qquad h_{\mathbb{R}} \subset i\,k + p.$

The action of Γ on (X_r, Φ) gives rise to an important Γ-stable $\mathbb{Z}$-module of X_r, namely

(2.4) $$X_0 = \{\lambda \in X_r \mid \sum_{\gamma\in\Gamma} \lambda^\gamma = 0\} = \{\lambda \in X_r \mid \lambda + \lambda^{\gamma_0} = 0\}.$$

Let π be the natural projection : $X_r \to X_r/X_0$ and $\Phi_0 = \Phi \cap X_0$; Φ_0 is a closed subsystem of Φ (i.e. if $\alpha,\beta \in \Phi_0$, $\alpha + \beta \in \Phi$ then $-\alpha \in \Phi_0$ and $\alpha + \beta \in \Phi_0$). Denote $\pi(\Phi\setminus\Phi_0)$ by Σ_0.

Let g_σ be normally related to (u,t), $a = it \cap g_\sigma$ and Σ the *set of restricted roots* of the pair (g_σ, a) in the sense of (Ch.I, section 4). Then we have:

LEMMA 2.2. *Let* g_σ, a, Σ *and* Σ_0 *be as above. Then* $\Sigma = \Sigma_0$.

PROOF. Since $\Sigma \cup \{0\}$ is precisely the set of restrictions to a of the elements in Φ, the assertion is clear.

3. Γ-ORDER

In this section let Φ be a not necessarily reduced root system, X_r the root lattice of Φ with an action of Γ denoted by $\alpha \to \alpha^\gamma$ for $\gamma \in \Gamma$ and X_0, Φ_0 defined as in (2.4).

Recall that a *linear order* > on a $\mathbb{Z}$-module X is a partial order satisfying: i) for all $\chi \neq 0$ in X: $\chi > 0$ or $-\chi > 0$.

ii) if $\chi_1 > \chi_2$ then $\chi_1 + \chi > \chi_2 + \chi$ for all $\chi \in X$.

DEFINITION 3.1. A linear order > of X_r which satisfies the condition:

(3.1) $\qquad$ if $\chi > 0$ and $\chi \notin X_0$, then $\chi^\gamma > 0$ for all $\gamma \in \Gamma$

is called a Γ-*order*.

An equivalent condition for (3.1) is:

LEMMA 3.2. A linear order > on X_r is a Γ-order if and only if the following condition is satisfied:

(3.2) if $\chi \notin X_0$, $\chi > 0$ and $\chi' \equiv \chi \pmod{X_0}$ then $\chi' > 0$.

PROOF. Suppose (3.1) is satisfied and $\chi > 0$, $\chi \notin X_0$ and $\chi' \equiv \chi \bmod X_0$. Then $\chi' = \chi + \chi_0$ for some $\chi_0 \in X_0$. As $\Sigma_{\gamma\in\Gamma}\, \chi_0^{\gamma} = 0$ we have by (3.1): $\Sigma_{\gamma\in\Gamma}\, \chi'^{\gamma} = \Sigma_{\gamma\in\Gamma}\, \chi > 0$ hence $\chi' > 0$. Conversely assuming (3.2), then by the definition of X_0, $\chi^{\gamma} \equiv \chi \bmod X_0$ for all $\chi \in X_r$. Q.E.D.

A typical way to obtain a Γ-order is to take the lexicographic order on X_r relative to the following basis of $X_r \otimes_{\mathbb{Z}} \mathbb{Q}$: Choose a basis $\alpha_1,\ldots,\alpha_r$ of $X_0 \otimes_{\mathbb{Z}} \mathbb{Q}$ and extend these with $\alpha_{r+1},\ldots,\alpha_n$ to a basis of $X \otimes_{\mathbb{Z}} \mathbb{Q}$. Define then for $\chi \in X_r$:

$$\chi > 0 \Longleftrightarrow \chi = \sum_{i=1}^{s} \nu_i\, \alpha_i \text{ with } \nu_s > 0.$$

This order satisfies condition (3.2).

Note that a Γ-order on X_r induces a linear order on X_0 and X_r/X_0.

Fix a Γ-linear order on X_r and let Δ be a fundamental system of Φ w.r.t. this order, a so-called Γ-*fundamental system* of Φ. Write Δ_0 for $\Delta \cap \Phi_0$ and $\bar{\Delta}$ for $\pi(\Delta\backslash\Delta_0)$. $\bar{\Delta}$ is called a *restricted fundamental system* of Φ w.r.t. Φ_0. Let W_0 be the subgroup of W generated by the reflections s_α, $\alpha \in \Phi_0$. Write W_σ for $\{w \in W \mid w(X_0) = X_0\}$ then W_0 is a normal subgroup of W_σ. Namely if $w \in W_\sigma$, $\alpha \in \Phi_0$, then $w \,.\, s_\alpha .\, w^{-1} = s_{w(\alpha)} \in W_0$ ($w(\alpha) \in \Phi \cap X_0 = \Phi_0$). Every $w \in W_\sigma$ induces an automorphism of $Y = X_r/X_0$ which we denote by $\pi(w)$. We have now the relation:

$$\pi(w\chi) = \pi(w).\pi(\chi) \qquad \text{for } \chi \in X_r,\ w \in W_\sigma.$$

PROPOSITION 3.3. *Notations being as above, then the following properties hold:*

i) Δ_0 *is a fundamental system of* Φ_0 w.r.t. *the induced order on* X_0.

ii) *If* Δ' *is another* Γ-*fundamental system of* Φ, $\Delta_0' = \Delta' \cap \Phi_0$ *and* $\bar{\Delta}' = \pi(\Delta'-\Delta_0')$, *then* $\Delta = \Delta'$ *iff* $\Delta_0 = \Delta_0'$ *and* $\bar{\Delta} = \bar{\Delta}'$.

iii) *If* $w \in W_\sigma$, *then* $w(\Delta)$ *is also a* Γ-*fundamental system and the subsequent statements are equivalent:*

a) $w \in W_0$

b) $\pi(w) = 1$

c) $\pi(w)(\bar{\Delta}) = \bar{\Delta}$.

PROOF. i) Assume $\alpha \in \Phi_0$, $\alpha > 0$.

Then $\alpha = \Sigma_{i=1}^{n} m_i\alpha_i$, $m_i \geq 0$, $\alpha_i \in \Delta$. Put $\beta = \Sigma_{j|\alpha_j \notin \Delta_0} m_j\alpha_j$. The assumption: $\beta \neq 0$, gives $\beta \in X_0$ and $\beta = m_k\alpha_k + \gamma$ with $m_k > 0$, $\gamma \geq 0$. Since $m_k\alpha_k \notin X_0$ also $\gamma \notin X_0$, consequently

$$0 < \pi(m_k\alpha_k) = \pi(-\gamma) < 0.$$

This contradicts the assumption.

(ii) "$\Leftarrow$" Denote the linear order of Δ' by $>'$. Take any $\alpha \in \Delta$. If $\alpha \in \Delta_0 = \Delta_0'$, then $\alpha >' 0$ (α is positive w.r.t. the order defining Δ'). If $\alpha \notin \Delta_0$, then $\pi(\alpha) \in \bar{\Delta} = \bar{\Delta}'$, hence in both cases $\alpha >' 0$. Since Δ determines $\Phi^+ = \{\alpha \in \Phi | \alpha > 0\}$, one may conclude $\Phi^+ \subset \Phi'^+$, consequently $\Phi^+ = \Phi'^+$ and $\Delta = \Delta'$.

iii) Since $w(X_0) = X_0$, $w(X^+) = \{w(\chi) | \chi > 0\}$ defines a Γ-order on X with $w(\Delta)$ as Γ-fundamental system of Φ (see Lemma 3.2). The inclusions a) $\Rightarrow$ b) and b) $\Rightarrow$ c) being obvious, we are left to prove c) $\Rightarrow$ a). Since $\pi(w)\bar{\Delta} = \pi(w(\Delta))$, it is a consequence of the next lemma:

LEMMA 3.4. *If Δ, Δ' are Γ-fundamental systems of Φ s.t. $\Delta = \bar{\Delta}'$, then there exists an unique* $w_0 \in W_0$ s.t. $\Delta' = w_0\,\Delta$.

PROOF. Δ_0 and Δ_0' are fundamental systems of Φ_0, thus there exists a unique $w_0 \in W_0$ such that $w_0\,\Delta_0 = \Delta_0'$. Now $(w_0\Delta) \cap \Phi_0 = \Delta_0'$ and $\pi(w_0\Delta) = \bar{\Delta} = \bar{\Delta}'$ so that by ii) of Proposition 3.3 : $\Delta' = w_0\Delta$.

Put W_Σ for $\{\pi(w) | w \in W_\sigma\}$, then the foregoing proposition implies

COROLLARY 3.5. $W_\Sigma = W_\sigma / W_0$.

Using similar arguments one can derive the following useful results, which we won't need however for the classification.

LEMMA 3.6. *If $\pi(\alpha_i) = \delta_j$ (where $\alpha_i \in \Delta$, $\delta_j \in \bar{\Delta}$) and $\gamma \in \Gamma$, then*

$$\alpha_i^\gamma = \alpha_r + \sum_{\alpha_k \in \Delta_0} c_{ik}\,\alpha_k \quad \textit{for some } \alpha_r \in \pi^{-1}(\delta_j), c_{ik} \in \mathbb{Z} \geq 0$$

See [Satake, 2.1.6] *or* [Warner, Lemma 1.1.3.2].

PROPOSITION 3.7. *Let $\bar{\Delta} = \{\delta_1, \ldots, \delta_r\}$ (the δ_i mutually distinct). Then the elements $\delta_1, \ldots, \delta_r$ are linearly independent.*

COROLLARY 3.8. *Every $\delta \in \Sigma_0$ can be expressed uniquely in the form:*

$\delta = \pm \Sigma_{i=1}^{r} m_i \delta_i$ *with* $m_i \in \mathbb{Z} \geq 0$.

<u>REMARK</u>. The notion of Γ-order can be introduced in a more general context. Namely one can replace $\mathbb{R}$ by a perfect field k, $\mathbb{C}$ by a finite extension of k and X_r by the character group of a maximal torus of a connected semi-simple algebraic group G, defined over k.

Results analogous to the ones derived here are valid there. They play a role in the classification of K/k forms of G. More details can be found in [Satake, Ch.II]. The same remark holds for the notion of Γ-diagrams, to be introduced in the next paragraph.

4. Γ-DIAGRAMS

We keep to the notations of the previous section.

Γ acts on the Γ-fundamental systems of Φ. Namely if $\gamma \in \Gamma$ and Δ a Γ-fundamental system of Φ, then $\Delta^\gamma = \{\alpha^\gamma | \alpha \in \Delta\}$ is a Γ-fundamental system of Φ corresponding to the set of positive elements: $(X_r^+)^\gamma = \{\chi^\gamma | \chi \in X_r^+\}$ of X_r. (w.r.t. a new Γ-linear order).

Since $\alpha_i \equiv \alpha_i^\gamma \bmod X_0$, for $\alpha_i \in \Delta$, $\gamma \in \Gamma$, it follows that $\bar{\Delta}^\gamma = \bar{\Delta}$, hence there is a unique element $w_\gamma \in W_0$ s.t. $\Delta^\gamma = w_\gamma \Delta$ (lemma 3.4) (Since $\Delta_0^{\gamma_0} = - \Delta_0$, w_γ is the opposition automorphism of W_0.) Using this, one can define a new action of Γ on X_r by:

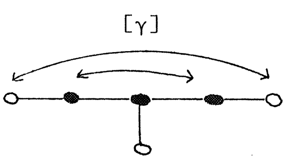
$$\chi^{[\gamma]} = w_\gamma^{-1} \chi^\gamma. \tag{4.1}$$

In particular $\chi \to \chi^{[\gamma]}$ is an involutive automorphism of the triple (X_r, Φ, Δ).

<u>DEFINITION 4.1</u>. A quadruple $(\Phi, \Delta, \Delta_0, [\gamma])$ as defined above is called a Γ-diagram.

A Γ-diagram can be illustrated by giving the vertices of Δ_0 in the Dynkin diagram of (Φ, Δ) a black colour, and indicating the action $[\gamma]$ of Γ on (Φ, Δ) by arrows; for example:

$^2E_6$:



Note that a Γ-diagram is characterized by Δ_0 and a diagram automorphism of order 2 of Δ, leaving Δ_0 invariant.

DEFINITION 4.2. An *isomorphism of* Γ-*diagrams* $S_1 = (\Phi,\Delta,\Delta_0,[\gamma])$ and $S_2 = (\Phi',\Delta',\Delta_0',[\gamma]')$ is a linear bijection ρ between (Φ,Δ,Δ_0) and $(\Phi',\Delta',\Delta_0')$ satisfying

$$[\gamma]' = \rho \circ [\gamma] \circ \rho^{-1}.$$

PROPOSITION 4.3. *Assume* Γ *acts on* Φ. *Let* Δ *and* Δ' *be* Γ-*fundamental bases of* Φ *and let* $S = (\Phi,\Delta,\Delta_0,[\gamma_0])$ *resp.* $S' = (\Phi,\Delta',\Delta_0',[\gamma_0'])$ *be the corresponding* Γ-*diagrams. Then* S *is isomorphic to* S'.

PROOF. Let $W_0 < W$ be the Weyl group of Φ_0.

Since the opposition automorphism of W_0 w.r.t. Δ_0 is conjugate to the opposition automorphism of W_0 w.r.t. Δ_0' (by the same $w \in W_0$ conjugating Δ_0 and Δ_0') we may assume:

$$\Delta_0 = \Delta_0' \text{ and } [\gamma_0]|_{\Delta_0} = [\gamma_0]'|_{\Delta_0}.$$

But $\chi^{\gamma_0} = w_{\gamma_0}\, \chi^{[\gamma_0]} = w_{\gamma_0}\, \chi^{[\gamma_0]'}$ ($w_{\gamma_0} \in W_0$ s.t. $w_{\gamma_0}(\Delta_0) = -\Delta_0$) hence $[\gamma_0]' = [\gamma_0]$. Since the action of $[\gamma_0]$ on Φ is involutive, it leaves an irreducible component of Φ stable, or it permutes two of the irreducible components of Φ. In both cases the diagrams of S and S' are the same, what induces again an isomorphism $\Delta \to \Delta'$ according to the diagram. Q.E.D.

We use notations as in section 1 and 2.

If g_σ is related to (u,t), then the induced Γ-action on (X_r,Φ) determines a Γ-diagram $S = (\Phi,\Delta,\Delta_0,[\gamma_0])$ for every Γ-fundamental basis Δ of Φ.

The above result implies:

PROPOSITION 4.4. *The pair* (g_σ,t_c) *as defined above, determines the* Γ-*diagram* S *up to isomorphism.*

In general real forms g_{σ_1} and g_{σ_2} of g_c, related to (u,t), are not necessarily isomorphic by means of an element of $\mathrm{Aut}(g_c)$, if their Γ-diagrams are so (f.i. if $t \subset g_{\sigma_1} \cap g_\tau$ and $t \subset g_{\sigma_2} \cap g_\tau$). However:

THEOREM 4.5. *Assume* g_{σ_1} *and* g_{σ_2} *to be normally related to* (u,t) *and* S_1 *resp.* S_2 *are the corresponding* Γ-*diagrams. Then* g_{σ_1} *is isomorphic to* g_{σ_2} *if and* *only if* S_1 *is isomorphic to* S_2.

The proof requires a bit more technique than we developed so far and can be found in [ARAKI, Theorem 2.10, 2.11] and [SATAKE, Theorem 2.4.1]. I want to pay more attention to the determination of the Γ-diagrams.

For compact real forms the above correspondence is obvious since they are all isomorphic. In particular take $(\mathfrak{u},\mathfrak{t})$ as in section 1. Since τ^* acts as $-\mathrm{id}$ on X_r we get: $\Delta_0 = \Delta$ and $\Delta^{\gamma_0} = \tau^*(\Delta) = -\Delta = w_{\gamma_0}(\Delta)$ with $w_{\gamma_0} \in W = W_0$ s.t. $w_{\delta_0}(\Delta) = -\Delta$. Consequently w_{γ_0} is the opposition automorphism of Δ (i.e. the longest element of W w.r.t. Δ),

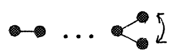
$$\chi^{[\gamma_0]} = w_{\gamma_0}^{-1} \circ \tau^*(\chi) = -w_{\gamma_0}(\chi) \qquad \text{for } \chi \in X_r$$

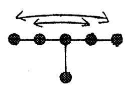
and the Γ-diagram is: $S = (\Phi,\Delta,\Delta,-w_{\gamma_0})$. The different possibilities are listed in:

<u>PROPOSITION 4.6</u>. *The* Γ*-diagram of a compact real simple Lie algebra belongs to one of the following types*:

i) ${}^1X_\ell$ *where* $X_\ell \neq A_\ell$ $(\ell\geq 2)$, D_ℓ (ℓ odd) *and* E_6 *or*

(ii)

(4.2) ${}^2A_\ell(\ell\geq 2)$:  *or* ${}^2D_\ell$ (ℓ odd) 

or ${}^2\bar{t}_6$: 

<u>PROOF</u>. The opposition automorphism of Δ is nontrivial iff Φ belongs to one of the following types: A_ℓ $(\ell\geq 2)$, D_ℓ (ℓ odd) or E_6. Therefore in these cases we get the diagrams (4.2). Q.E.D.

Denote the group of automorphisms of X_r, which leave Δ and Δ_0 invariant by Aut (X_r,Δ,Δ_0).

Whether an arbitrary diagram is a Γ-diagram can be decided with:

<u>PROPOSITION 4.7</u>. *If* Φ *is a root system,* Δ *a fundamental system of* Φ, Δ_0 *a subset of* Δ *and* [.] *a homomorphism:* $\Gamma \to \mathrm{Aut}(X_r,\Delta,\Delta_0)$, *then the quadruple* $S = (\Phi,\Delta,\Delta_0,[\gamma_0])$ *is a* Γ*-diagram of some action of* Γ *on* (X_r,Φ) *iff* $(\Phi_0',\Delta_0,\Delta_0,[\gamma]/_{X_0'})$ *is the* Γ*-diagram of the action* $\chi^{\gamma_0} = -\chi$ *of* Γ *on* (X_0',Φ_0'), *where* $X_0' = \{\Delta_0\}_r$ *and* $\Phi_0' = \Phi \cap X_0'$. *Furthermore the action of* Γ *on* (X_r,Φ) *is in that case completely determined by the corresponding* Γ*-diagram* $(\Phi,\Delta,\Delta_0,[\gamma])$.

PROOF. The assertion "$\Rightarrow$" being obvious, we assume that $(\Phi_0', \Delta_0, \Delta_0, [\gamma]/X_0')$ satisfies the above conditions. Let w_{γ_0} be the unique element in $W_0 = W(\Delta_0) <$ $< W$ satisfying $w_{\gamma_0}(\Delta_0) = -\Delta_0$. Define an action of Γ on (X_r, Φ) by:

$$\chi^{\gamma_0} = w_{\gamma_0}\, \chi^{[\gamma_0]}.$$

Since $(X_0')_{\mathbb{Q}}$ and $(X_0')_{\mathbb{Q}}^{\perp}$ are eigenspaces of both $[\gamma_0]$ and w_{γ_0}, they commute with each other.

Therefore $\chi^{\gamma_0^2} = \chi$ and $\chi \to \chi^{\gamma_0}$ defines an action of Γ on X_r. It is easy to see that $(\Phi, \Delta, \Delta_0, [\gamma])$ is the Γ-diagram of this action. As the definition of χ^{γ_0} given above is the only possible way to define an action of Γ on (X_r, Φ) giving rise to the given Γ-diagram, the last assertion is clear. Q.E.D.

By combining proposition 4.6 and 4.7 we get a result that will be useful at the classification of the Γ-diagrams:

LEMMA 4.8. *The following conditions for a quadruple* $S_0 = (\Phi_0', \Delta_0, \Delta_0, [\gamma]$ *are mutually equivalent:*

i) S_0 *is the* Γ*-diagram of the action* $\chi^{\gamma_0} = -\chi$ *of* Γ *on* (X_0', Φ_0')

ii) $[\gamma_0]$ *is the opposition automorphism of* Δ_0

iii)$-[\gamma_0] \in W_0$

iv) *The quadruple* S_0 *is the* Γ*-diagram of a compact real Lie algebra.*

v) *If* Δ_i *is an irreducible component of* Δ_0*, then* $[\gamma_0]$ *leaves* Δ_i *invariant and* $(\Phi_i, \Delta_i, \Delta_i, [\gamma]/\Delta_i)$ *is the* Γ*-diagram of one of the compact real lie algebras of proposition 4.4.*

Since not every Γ-diagram comes from a real form, normally related to $(\mathfrak{u}, \mathfrak{t})$, we define:

DEFINITION 4.9. A Γ-diagram $S = (\Phi, \Delta, \Delta_0, [\gamma])$, where Φ is a reduced root system, is called *admissible* if there exist a complex semisimple Lie algebra $\mathfrak{g}_c$, a compact real form $\mathfrak{u}$ of $\mathfrak{g}_c$, a C.S.A. $\mathfrak{t}$ of $\mathfrak{u}$ and a real form $\mathfrak{g}_\sigma$ of $\mathfrak{h}_c$ normally related to $(\mathfrak{u}, \mathfrak{t})$ s.t. the induced Γ-diagram is isomorphic to S.

We like to find necessary and sufficient conditions for a Γ-diagram to be admissible. Since the action of Γ on Φ is characterized by an involution of Φ, we are left to determine when an involution of Φ can be lifted to a conjugation of $\mathfrak{g}_c$.

5. LIFTING CONDITIONS

To lift an automorphism ρ of Φ to an automorphism of g_c one uses a Chevalley basis [HU, 14.2]. To lift ρ to an automorphism of a compact real form of g_c, we make use of the Weyl basis:

<u>DEFINITION 5.1.</u> A *Weyl basis of* g_c w.r.t. (u,t) is a basis $\{X_\alpha | \alpha \in \Phi\}$ of g_c mod t_c with the properties:

i) $X_\alpha \in g_\alpha$, $[X_\alpha, X_{-\alpha}] = H_\alpha$ for each $\alpha \in \Phi$.

ii) $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ where $\alpha,\beta,\alpha+\beta \in \Phi$ and the constants $c_{\alpha,\beta}$ satisfy: $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$ $(\in \mathbb{R})$.

iii) $X_\alpha - X_{-\alpha}$ and $i(X_\alpha + X_{-\alpha}) \in u$ for each $\alpha \in \Phi$.

The existence of such a basis is clear from the construction of a compact real form from a given C.S.A. (cf., Ch. I, Corollary 2.4) and the fact that two compact real forms are conjugate under $\mathrm{Int}(g_c)$.

Note that iii) implies:

$$\tau(X_\alpha) = -X_{-\alpha} \qquad (\alpha \in \Phi). \tag{5.1}$$

Fix a Weyl basis of g_c w.r.t. (u,t). If g_σ is a real form of g_c with $\sigma(t_c) = t_c$, then σ acts on Φ as in (2.1). In particular: $\sigma(g^\alpha) = g^{\alpha^\sigma}$. Put $\sigma(X_\alpha) = \kappa_\alpha X_{\sigma^*(\alpha)}$ for $\kappa_\alpha \in \mathbb{C}$, $\alpha \in \Phi$. From 5.1 i) and ii) follows:

$$\begin{aligned} &\kappa_\alpha \cdot \kappa_{-\alpha} = 1 && \text{for every } \alpha,\beta,\alpha+\beta \in \Phi \\ &\kappa_{\alpha+\beta} c_{\alpha,\beta} = \kappa_\alpha \cdot \kappa_\beta \cdot C_{\alpha^\sigma,\beta^\sigma} \end{aligned} \tag{5.2}$$

Hence $\{\kappa_{\alpha_i} | \alpha_i \in \Delta,\ \Delta \text{ a fundamental basis of } \Phi\}$ determines $\{\kappa_\alpha | \alpha \in \Phi\}$.

If σ and τ commute, then:

$$\kappa_\alpha = \kappa_{-\alpha} \qquad (\alpha \in \Phi) \tag{5.3}$$

and that is equivalent to (see (5.2)):

$$|\kappa_\alpha| = 1 \qquad \text{for any } \alpha \in \Phi. \tag{5.4}$$

Moreover the fact that σ is an anti-involution of g_c implies:

$$\bar{\kappa}_\alpha \cdot \kappa_{\alpha\delta} = 1. \tag{5.5}$$

Whether a C.S.A. of a real form $\mathfrak{g}_\sigma$ is an Iwasawa C.S.A. is determined by a property of the coefficients $\{\kappa_\alpha\}_{\alpha\in\Phi}$. From (5.2), (5.3) and (5.5) follows for $\alpha \in \Phi_0$:

$$(5.6) \qquad \kappa_\alpha = \kappa_{-\alpha} = \pm 1.$$

In particular:

$$(5.7) \qquad \kappa_\alpha = 1 \iff \sigma(X_\alpha) = X_{-\alpha} = -\tau(X_\alpha) \iff \theta(X_\alpha) = -X_\alpha \iff X_\alpha \in \mathfrak{p}_c$$

$$(5.8) \qquad \kappa_\alpha = -1 \iff \sigma(X_\alpha) = -X_{-\alpha} = \tau(X_\alpha) \iff \theta(X_\alpha) = X_\alpha \iff X_\alpha \in \mathfrak{k}_c.$$

On the other hand if $\alpha \in \Phi\backslash\Phi_0$ then we have:

$$X_\alpha + \sigma(X_{-\alpha}) \in \mathfrak{p}_c, \qquad X_\alpha - \sigma(X_{-\alpha}) \in \mathfrak{k}_c$$

thus: $\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha^\sigma} = \mathbb{C}(X_\alpha + \sigma(X_{-\alpha})) + \mathbb{C}\,(X_\alpha - \sigma(X_{-\alpha}))$.

Write $\mathfrak{t}_c = \mathfrak{t}_c^+ + \mathfrak{t}_c^-$ with $\mathfrak{t}_c^+ = \mathfrak{t}_c \cap \mathfrak{k}_c$, $\mathfrak{t}_c^- = \mathfrak{t}_c \cap \mathfrak{p}_c$, then we get the following decompositions:

$$(5.9) \qquad \mathfrak{k}_c = \mathfrak{t}_c^+ + \sum_{\alpha\in\Phi_0^-} \mathfrak{g}^\alpha + \sum_{\alpha\in\Phi\backslash\Phi_0} \mathbb{C}(X_\alpha - \sigma(X_{-\alpha}))$$

$$(5.10) \qquad \mathfrak{p}_c = \mathfrak{t}_c^- + \sum_{\alpha\in\Phi_0^+} \mathfrak{g}^\alpha + \sum_{\alpha\in\Phi\backslash\Phi_0} \mathbb{C}(X_\alpha + \sigma(X_{-\alpha})).$$

where $\Phi_0^+ = \{\alpha \in \Phi_0 \mid \kappa_\alpha = 1\}$, $\Phi_0^- = \{\alpha \in \Phi_0 \mid \kappa_\alpha = -1\}$.

Furthermore (5.10) implies:

<u>PROPOSITION 5.2</u>. *Let $\mathfrak{g}_\sigma$ be related to $(\mathfrak{u},\mathfrak{t})$. The following properties are equivalent:*

i) $\mathfrak{g}_\sigma$ *is normally related to* $(\mathfrak{u},\mathfrak{t})$

ii) $\mathfrak{t}_c^-$ *is maximal abelian in* $\mathfrak{p}_c$

iii) $\kappa_\alpha = -1$ *for all* $\alpha \in \Phi_0$.

<u>REMARK 5.3</u>. Let ϕ be an automorphism of $\mathfrak{t}_c^*$ such that $\phi(\Phi) = \Phi$. Assume moreover that one has for every $\alpha_i \in \Delta = \{\alpha_1,\ldots,\alpha_n\}$ an isomorphism π_i: $\mathfrak{g}^{\alpha_i} \to \mathfrak{g}^{\phi(\alpha_i)}$. Then we know from [HU, 14.2] that there exists a $\psi \in \mathrm{Aut}(\mathfrak{g}_c)$ such that: i) $\psi(X_{\alpha_i}) = a_{\alpha_i} X_{\phi(\alpha_i)}$, $a_{\alpha_i} \in \mathbb{C}$.

ii) $\psi^+|_{\mathfrak{t}_c^*} = \phi$

iii) $\psi|_{\mathfrak{g}^{\alpha_i}} = \pi_i$.

One verifies that its proof can be modified such that if all π_i commute with τ, then the same can be organized for ψ.

Note that the condition $\pi_i \circ \tau = \tau \circ \pi_i$ $(i = 1,\dots,n)$ is equivalent to:

$$\bar{a}_{\alpha_i} = a_{-\alpha_i} \qquad (i = 1,\dots,n). \tag{5.11}$$

and because of $a_\alpha . a_{-\alpha} = 1$ this is equivalent to:

$$|a_{\alpha_i}| = 1 \qquad (i = 1,\dots,n). \tag{5.12}$$

By using relation (5.2) the $\{a_{\alpha_i}\}_{\alpha_i \in \Delta}$ can be extended uniquely to a tuple $\{a_\alpha\}_{\alpha\in\Phi}$ with $a_\alpha \in \mathbb{C}$ and $|a_\alpha| = 1$ for all $\alpha \in \Phi$. We will prove now:

PROPOSITION 5.4. *Assume* Γ *acts on* Φ *and let* $S = (\Phi,\Delta,\Delta_0,[\gamma])$ *be the corresponding* Γ*-diagram. Then* S *is admissible iff there exists a tuple* $\{\kappa_{\alpha_i}\}_{\alpha_i\in\Delta}$ s.t. $|\kappa_{\alpha_i}| = 1$; $\kappa_{\alpha_i} = -1$ for $\alpha_i \in \Delta_0$ and if $\{\kappa_\alpha\}_{\alpha\in\Phi}$ is the unique extension satisfying (5.2) (exists by [HU., Theorem 14.2]) then $\kappa_{\alpha_i} . \kappa_{\alpha_i^{\gamma_0}} = 1$ for $\alpha_i \in \Delta \setminus \Delta_0$.

PROOF. That those conditions are necessary is a consequence of (5.4), (5.5) and proposition 5.2.

As for the sufficiency, let $\mathfrak{g}_c$, $\mathfrak{u},\mathfrak{t}_c,\Phi$ be as before. Fix a Weyl basis of $\mathfrak{g}_c$ w.r.t. $(\mathfrak{u},\mathfrak{t}_c)$ and let $(\Phi)_{\mathbb{R}}$ be the real span of Φ in $\mathfrak{t}_c^*$. Extend the action of $\gamma_0 \in \Gamma$ on Φ to $\{\Phi\}_{\mathbb{R}}$ and denote it also by γ_0. Write ϕ for $\gamma_0 \circ \tau^*$ and define for $\alpha_i \in \Delta$; $\pi_i : \mathfrak{g}^{\alpha_i} \to \mathfrak{g}^{\phi(\alpha_i)}$ by: $\pi_i(X_{\alpha_i}) = a_{\alpha_i} X_{\phi(\alpha_i)}$ with $a_{\alpha_i} = -\kappa_{\alpha_i} = -\kappa_{-\alpha_i}$ (See (5.11)). According to remark 5.3 there exists a $\psi \in \mathrm{Aut}(\mathfrak{g}_c)$ s.t. $\psi|\mathfrak{g}_{\alpha_i} = \pi_i$ $(\alpha_i \in \Delta)$ and $\psi^* = \phi$ on $\mathfrak{t}_c^*$. Since $a_{\alpha_i} = \overline{a_{-\alpha_i}}$ we have:

$$\psi \circ \tau = \tau \circ \psi$$

and by ii): $a_{\alpha_i} . a_{-\alpha_i \gamma_0} = 1$, hence $\psi^2 = \mathrm{id}$.

The conjugation $\sigma = \psi \circ \tau$ gives us the desired normally related real form $\mathfrak{g}_\sigma$ of $\mathfrak{h}_c$

6. NORMAL ROOT SYSTEMS

To simplify the notations we denote from now on the action of $\gamma_0 \neq$ $\neq 1 \in \Gamma$ on a root system Φ by a bar and its extension to a conjugation of g_c (if it exists) by σ.

DEFINITION 6.1. A reduced root system Φ equipped with an action of Γ is said to be *normal* iff

$$\bar{\alpha} - \alpha \notin \Phi \text{ for any } \alpha \in \Phi.$$

The corresponding Γ-diagram is also called *normal*.

PROPOSITION 6.2. *If* $S = (\Phi,\Delta,\Delta_0,[\sigma])$ *is admissible then* S *is normal.*

PROOF. Since S is admissible the action of Γ on Φ can be extended to a conjugation σ of g_c as in definition 4.9.

Let $\{\kappa_\alpha\}_{\alpha\in\Phi}$ be the corresponding tuple of scalars, according to proposition 5.4. If $\bar{\alpha} = -\alpha$ then $\bar{\alpha} - \alpha = -2\alpha \notin \Phi$. Assume that $\bar{\alpha} \neq -\alpha$ and $\beta =$ $= \bar{\alpha} - \alpha \in \Phi$. Then $\beta \in \Phi_0$, hence $\sigma(X_\beta) = X_{-\beta}$.

By applying σ on both sides of $[X_\alpha,\sigma(X_{-\alpha})] = \kappa_{-\alpha}c_{\alpha,-\bar{\alpha}}\, X_\beta$ we get since $\beta \in \Phi_0$:

$$[\sigma(X_\alpha),X_{-\alpha}] = -\,\kappa_{-\alpha}\, c_{\alpha,-\bar{\alpha}}\, X_\beta.$$

On the other hand we have:

$$[\sigma(X_\alpha),X_{-\alpha}] = K_\alpha\, c_{\alpha,-\bar{\alpha}}\, X_\beta.$$

Therefore $\bar{\kappa}_{-\alpha} = -\,\kappa_{\alpha}$. Since $\bar{\kappa}_{-\alpha} = \kappa_\alpha$ this would imply $\kappa_\alpha = 0$ and we have arrived at a contradiction. Q.E.D.

For a normal root system Σ_0 appears to be a root system:

PROPOSITION 6.3. *Let* Φ *be a normal root system and* $\Sigma_0 = \pi(\Phi\backslash\Phi_0)$ *as in section 2. Then* Σ_0 *is a root system itself.* [WARNER, page 21].

Araki constructed first the possible normal root systems and determined later the admissible ones. It is however easier to determine first the Γ-diagrams, the normal Γ-diagrams follow then easily from them.

7. REDUCTION TO THE CASE OF REAL RANK ONE

The *real rank* of a real semisimple Lie algebra $\mathfrak{g}_\sigma = \mathfrak{k} + \mathfrak{p}$, related to $(\mathfrak{u},\mathfrak{t})$ is defined as the dimension of a maximal abelian subspace of $\mathfrak{p}$ (cf. Ch. I, Definition 4.3). If $S = (\Phi,\Delta,\Delta_0,[\gamma])$ is the corresponding Γ-diagram, then proposition 3.7 gives that the real-rank is equal to $|\bar{\Delta}|$. Call $|\bar{\Delta}|$ the *real rank of a Γ-diagram*. (So the real rank equals the number of $[\gamma_0]$-equivalent classes of the not-coloured vertices in the Γ-diagram).

We want to reduce the problem of the classification of admissible Γ-diagrams to the case of Γ-diagrams of real rank equal to one.

If $S = (\Phi,\Delta,\Delta_0,[\gamma])$ is a Γ-diagram and Δ' a $[\gamma]$-stable subset of Δ then one can form the subsystem $S_{\Delta'} = (\Phi',\Delta',\Delta_0',[\gamma]')$ of S. Here Φ' is the intersection of the $\mathbb{Z}$- span of Δ' with Φ, $\Delta_0' = \Delta_0 \cap \Delta$ and $[\gamma]' = [\gamma]|_{\Phi'}$.

<u>LEMMA 7.1</u>. *Let the notations be as above. Assume that* S *is admissible and that* Δ' *satisfies*

(7.1) *if* $\alpha \in \Delta'$, $\beta \in \Delta$ *and* $\langle\alpha,\beta\rangle \neq 0$ *then* $\beta \in \Delta'$.

Then $S_{\Delta'}$ *is admissible.*

<u>PROOF</u>. To prove that Γ acts on Φ' it suffices to show that if $\alpha_i \in \Delta'$ then $\alpha_i \in \Phi'$. Now $\bar{\alpha}_i = w_\gamma\, \alpha_i^{[\gamma]}$, where $w_\gamma = w_{\alpha_{i_s}} \circ \cdots\cdots \circ w_{\alpha_{i_1}}$, $\alpha_{ij} \in \Delta_0$. For $1 \le k \le S$ write α_k' for $w_{\alpha_{i_k}} \circ \cdots \circ w_{\alpha_{i_1}} (\alpha_i^{[\gamma]})$ and denote $\alpha_i^{[\gamma]}$ by α_0'. We prove by induction that $\alpha_k' \in \Phi'$. Since Δ' is $[\gamma]$-stable: $\alpha_i^{[\gamma]} \in \Delta'$, so we may assume $\alpha_{k-1}' \in \Phi'$. If $\langle\alpha_{i_k},\Delta'\rangle = 0$ then $\alpha_k' = \alpha_{k-1}'$, because $\alpha_k' = w_{\alpha_{i_k}}(\alpha_{k-1}')$ and $\alpha_{k-1}' \in \Phi'$.

If $\langle\alpha_{i_k},\Delta'\rangle \neq 0$ then by condition (7.1) $\alpha_{i_k} \in \Delta'$ and this implies $\alpha_k' \in \Phi'$. Hence Φ' is stable under the action of Γ on Φ. As S is admissible it suffices now to show that $s_{\Delta'}$ is the Γ-diagram for this action of Γ on Φ'. If W' is the Weyl group of Φ' and $w_\gamma' \in W'$ the unique element of W' s.t. $w_\gamma'(\Delta') = \Delta'^{\gamma}$, then if $w_\gamma = \Pi w_{\alpha_{i_j}}$:

$$w_\gamma|_{\Phi'} = \prod_{\langle\alpha_{ij},\Delta'\rangle \neq 0} w_{\alpha_{ij}}|_{\Phi'} = \prod_{\alpha_{ij} \in \Delta'} w_{\alpha_{ij}}|_{\Phi'} = w_\gamma' \in W'$$

Hence for $\alpha \in \Phi'$. $w_\gamma'(\alpha^\gamma) = w_\gamma|_{\Phi'}(\alpha^\gamma) = \alpha^{[\gamma]|\Phi'}$. Q.E.D.

PROPOSITION 7.2. *Notations as before, assume* $\Delta = \Delta' \cup \Delta''$ *with* Δ' *and* Δ'' $[\gamma]$*-stable subsets of* Δ *that satisfy both* (7.1) *and such that* $\Delta' \cap \Delta'' \subset \Delta_0$. *If* $S_{\Delta'}$ *and* $S_{\Delta''}$ *are admissible, then* S *is admissible.*

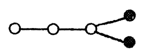
PROOF. Let $\{\kappa'_{\alpha_i}\}_{\alpha_i\in\Delta'}$ resp. $\{\kappa''_{\alpha_i}\}_{\alpha\in\Delta''}$ be the systems of scalars as in proposition 5.4. Since $\Delta' \cap \Delta'' \subset \Delta_0$ they coincide for common α. Define now

$$\kappa_{\alpha_i} = \begin{cases} \kappa'_{\alpha_i} & \text{if } \alpha_i \in \Delta' \\ \kappa_{\alpha_i} & \text{if } \alpha_i \in \Delta'' \end{cases}$$

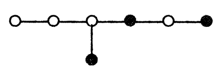
The tuple $\{\kappa_{\alpha_i}\}_{\alpha_i\in\Delta}$ satisfies the conditions of proposition 5.4, hence S is admissible. Q.E.D.

REMARK 7.3. Condition (7.1) on Δ' and Δ'' implies that Δ'_0 and Δ''_0 consist of a number of connected components of Δ_0.

EXAMPLE 7.4. As will turn out later the following Γ-diagrams are admissible:

Δ':  Δ'':

By proposition 7.2 the following Γ-diagram for E_7 is admissible:



REMARK 7.5. In view of the above results, the classification of admissible Γ-diagrams is reduced to the case of real rank equal to one. In fact every admissible Γ-diagram can be "built up" as follows: let $S = \{\Phi,\Delta,\Delta_0,[\gamma]\}$ be an admissible Γ-diagram and put $\bar{\Delta} = \pi(\Delta-\Delta_0) = \{\delta_1,\ldots,\delta_r\}$. For each δ_i, define $\Delta^i = \Lambda_0 \cup (\pi^{-1}(\delta_i) \cap \Delta)$. Then Δ^i satisfies the condition (7.1) and is $[\gamma]$-stable. ($\pi(\alpha) = \pi(\alpha^{[\gamma]}) = \delta_i$ for all $\alpha \in \Delta^i-\Delta_0$). Therefore the canonical subsystem $S_{\Delta'}$ is admissible and has real rank one. The decomposition $\Delta = \Delta^1\cup\ldots\cup\Delta^r$ satisfies the conditions of proposition 7.2.

8. THE RANK ONE CLASSIFICATION

First a definition of irreducibility for a Γ-diagram:

DEFINITION 8.1. The system $S = (\Phi,\Delta,\Delta_0,[\gamma_0])$ is *$\mathbb{R}$-irreducible* if Δ is not the union of two mutually orthogonal $[\gamma_0]$-stable, non-empty, subsystems Δ' and Δ''. The system S is *absolutely irreducible* if Δ is connected.

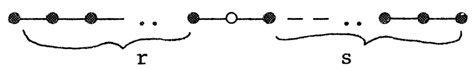
Note that if Δ is $\mathbb{R}$-irreducible but not absolutely irreducible then $\Delta = \Delta' \cup \Delta''$ with Δ' and Δ'' connected and $\Delta'^{[\gamma_0]} = \Delta''$.

Using proposition 4.7 and lemma 4.8 we can state:

PROPOSITION 8.2. *There exist twenty types of $\mathbb{R}$-irreducible Γ-diagrams with the restricted rank one (See table 1).*

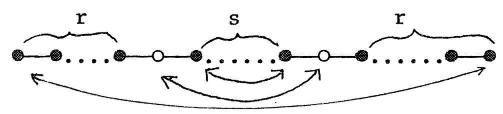
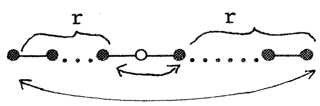
PROOF. Assume S is an absolutely irreducible Γ-diagram of $\mathbb{R}$-rank. We give here a proof for the root system $A_\ell (\ell \geq r)$. For other types of root systems the proof can be carried out similarly.

There are two possibilities: a) $[\gamma_0] = 1$ and b) $[\gamma_0] \neq 1$. In the case a), a Γ-diagram $S = (\Phi,\Delta,\Delta_0,[\gamma_0])$ with the restricted rank one has the following form:

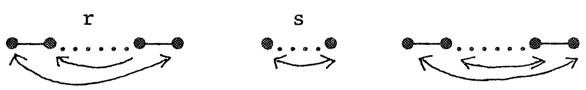


Hence $\Phi_0 = A_\ell \times A_s$. By proposition 4.7 and lemma 4.8, S corresponds to an action of Γ iff $r \leq 1$ and $s \leq 1$. This gives the diagrams 2,3 and 4 in table 1.

In case b), $\ell = \text{rank } \Phi \geq r$ and $[\gamma_0]$ is the unique element in $\mathrm{Aut}(X_r,\Phi,\Delta)$ of order 2. Hence the Γ-diagram S has one of the following forms:

 or 

Now $\Phi_0 = A_r \times A_s \times A_r$, but the Γ-diagram of the compact real form of $A_r \times A_s \times A_r$ looks like (cf. lemma 4.8):



By proposition 4.7, the Γ- diagram S corresponds to an action of Γ if and only if $r = 0$, so the Γ-diagram S must be diagram no. 5 from table 1.

Suppose now that S is $\mathbb{R}$-irreducible but not absolutely irreducible and of restricted rank one. Then $\Delta_0 = \phi$, because every $\mathbb{R}$-irreducible component of $S_0 = (\Phi_0,\Delta_0,\Delta_0,[\gamma_0]|_{X_0'})$ is absolutely irreducible. Hence S can be only diagram no. 1 from table 1. Q.E.D.

PROPOSITION 8.3. *Among the* Γ*-diagrams listed in table 1, the diagrams* 3,7,8, 17,19,20 *are not normal. All other diagrams are normal.*

PROOF. For a non-normal system, we give the action of γ_0 as an element of $\mathrm{Aut}(X_r,\Phi)$, denoted by s_0, and the root $\alpha \in \Phi$ s.t. $\bar{\alpha} - \alpha \in \Phi$. We use the notations of roots in [BOURBAKI, Ch. VI, section 4]

no.5: $s_0 = s_{\varepsilon_2-\varepsilon_3}$, $\alpha = \varepsilon_1-\varepsilon_2$, $\bar{\alpha}-\alpha = \varepsilon_2-\varepsilon_3$

no.7: $s_0 = s_{\varepsilon_1-\varepsilon_2} s_{\varepsilon_3}\ldots s_{\varepsilon_\ell}$, $\alpha = \varepsilon_2$, $\bar{\alpha}-\alpha = \varepsilon_1-\varepsilon_2$

no.8: $s_0 = s_{2\varepsilon_2}\ldots\ldots s_{2\varepsilon_\ell}$, $\alpha = \varepsilon_1+\varepsilon_2$, $\bar{\alpha}-\alpha = -2\varepsilon_2$.

no.17: $s_0 = s_{\varepsilon_1-\varepsilon_2}.s_{\varepsilon_3}s_{\varepsilon_4}$, $\alpha = \varepsilon_1$, $\bar{\alpha}-\alpha = \varepsilon_2-\varepsilon_1$,

no.19: $s_0 = s_{\varepsilon_1-\varepsilon_2}$, $\alpha = \varepsilon_1-\varepsilon_3$, $\bar{\alpha}-\alpha = \varepsilon_2-\varepsilon_1$.

no.20: $s_0 = s_{-2\varepsilon_1+\varepsilon_2+\varepsilon_3}$, $\alpha = \varepsilon_1-\varepsilon_2$, $\bar{\alpha}-\alpha = -2\varepsilon_1+\varepsilon_2+\varepsilon_3$.

To prove that a system is normal, it is sufficient to show:

$$\bar{\alpha}-\alpha \notin \Phi \text{ for all } \alpha \in \Phi-\Phi_0.$$

As an example we prove that the diagrams no.11 and no.13 are normal. The proof for the other types goes analogously.

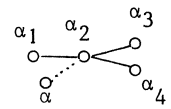
In no.11 and 13, one has: $(X_0)_{\mathbb{Q}} = \mathbb{Q}.(\varepsilon_1-\varepsilon_2) + \Sigma_{i=3}^{\ell}\, \mathbb{Q}\varepsilon_i$. So: $\Phi^+-\Phi_0 = \{\varepsilon_1+\varepsilon_2, \varepsilon_1\pm\varepsilon_i, \varepsilon_2\pm\varepsilon_i\ (i\geq 3)\}$. Since $\overline{\varepsilon_1+\varepsilon_2} - (\varepsilon_1+\varepsilon_2) = 0 \notin \Phi$, $\overline{\varepsilon_1\pm\varepsilon_i} - (\varepsilon_1\pm\varepsilon_i) = \varepsilon_2-\varepsilon_1\pm 2\varepsilon_i \notin \Phi$ and $\overline{\varepsilon_2\pm\varepsilon_i} - \varepsilon_2\pm\varepsilon_i = \varepsilon_1-\varepsilon_2+2\varepsilon_i \notin \Phi$, these diagrams are normal. Q.E.D.

As will appear the normal Γ-diagrams of real rank one which are not admissible can be determined with the following result:

LEMMA 8.4. *Let* S *be an* $\mathbb{R}$*-irreducible* Γ*-diagram of real rank one and let* $\Phi_0^{\perp} = \{\alpha\in\Phi \mid (\alpha,\Phi_0)=0\}$.
If $\Phi_0^{\perp} \neq \Phi$ *and* Φ *has a subsystem* Φ' *of type* D_4, *then* S *is not admissible.*

PROOF. We may assume that S is absolutely irreducible. Take $\alpha \in \Phi_0^{\perp}$ and let $\Phi_\alpha = \{\beta\in\Phi \mid (\beta,\alpha) = 0\}$. Then $\Phi_0 \subset \Phi_\alpha$ and rank $\Phi_\alpha \leq$ rank $\Phi-1$. Since rank Φ_0 = rank $\Phi-2$ occurs only for Φ of type $A_\ell (\ell\geq 2)$ which has no subsystem of type D_4, we may assume rank $\Phi_0 = n-1$, hence $\Phi_0 = \Phi_\alpha$. Let Φ' be a subsystem of type D_4 s.t. $\alpha \in \Phi'$. Choose a fundamental basis $\alpha_1,\alpha_2,\alpha_3$ and α_4 of Φ'

s.t. α is the highest root of Φ':

extended D_4: 

In particular $\alpha = \alpha_1+2\alpha_2+\alpha_3+\alpha_4$, α_1,α_3 and $\alpha_4 \in \Phi_\alpha = \Phi_0$ are mutually orthogonal roots and Φ' is Γ-stable. (One can prove that every $\lambda \in \Phi'$ not equal to $\pm\alpha$, $\pm\alpha_i$ $(i=1,3,4)$ is of the form $\pm\frac{1}{2}\alpha\pm\frac{1}{2}\alpha_1\pm\frac{1}{2}\alpha_3\pm\frac{1}{2}\alpha_4$.) Assume Φ is admissible, hence also Φ' is admissible. We prove now: $\kappa_{\alpha_2}\cdot\kappa_{\bar{\alpha}_2} = -1$ using (5.15) and $\bar{\alpha}_2 = \alpha_2+\alpha_1+\alpha_3+\alpha_4$. Since $\alpha_1+\alpha_2, \alpha_1+\alpha_2+\alpha_3$ and $\bar{\alpha}_2 = \alpha_1+\alpha_2+\alpha_3+\alpha_4 \in \Phi$ we have:

i) $\kappa_{\alpha_1+\alpha_2}\, c_{\alpha_1,\alpha_2} = -\kappa_{\alpha_2}\, c_{-\alpha_1,\bar{\alpha}_2}$

ii) $\kappa_{\alpha_1+\alpha_2+\alpha_3}\cdot c_{\alpha_1+\alpha_2,\alpha_3} = -\kappa_{\alpha_1+\alpha_2}\cdot c_{-\alpha_1+\bar{\alpha}_2,\alpha_3}$

iii) $\kappa_{\bar{\alpha}_2}\cdot c_{\alpha_1+\alpha_2+\alpha_3,\alpha_4} = -\kappa_{\alpha_1+\alpha_2+\alpha_3}\cdot c_{-\alpha_1+\bar{\alpha}_2-\alpha_3,-\alpha_4}$

What gives: $$\bar{\kappa}_{\alpha_2}\cdot\kappa_{\bar{\alpha}_2} = \frac{-c_{-\alpha_1,\alpha_2}\cdot c_{-\alpha_1+\alpha_2-\alpha_3,-\alpha_4}}{c_{\alpha_1+\alpha_2,\alpha_3}\cdot c_{\alpha_1+\alpha_2+\alpha_3,\alpha_4}}$$

Using the following lemma one can verify that this amounts to:

$$\frac{c_{-\alpha_1,\bar{\alpha}_2}\cdot c_{-\alpha_1+\bar{\alpha}_2-\alpha_3,-\alpha_4}}{c_{\alpha_1+\alpha_2,\alpha_3}\cdot c_{\alpha_1+\alpha_2+\alpha_3,\alpha_4}} = 1. \qquad \text{hence } \bar{\kappa}_{\alpha_2}\cdot\kappa_{\bar{\alpha}_2} = -1$$

Contradiction. Q.E.D.

<u>LEMMA 8.5</u>. *Let g_c, t and Φ be as before. The following identities between the structure constants $c_{\alpha,\beta}$ hold:*

i) *If α,β,γ are roots satisfying $\alpha+\beta+\gamma = 0$, then*

$$c_{\alpha,\beta} = c_{\beta,\gamma} = c_{\gamma,\alpha}$$

ii) *If $\alpha,\beta,\gamma,\delta$ are roots satisfying $\alpha+\beta+\gamma+\delta = 0$, $\beta+\gamma \neq 0$, $\gamma+\delta \neq 0$ and $\delta+\beta \neq$ $\neq 0$, then:*

$$c_{\alpha,\beta}c_{\gamma,\delta} + c_{\alpha,\gamma}c_{\delta,\beta} + c_{\alpha,\delta}c_{\beta,\gamma} = 0.$$

[HE, lemma 5.1, 5.3].

This result is also used to prove that some of the normal rank one Γ-diagrams are admissible.

THEOREM 8.6. The $\mathbb{R}$- irreducible admissible Γ-diagrams of real rank one are the diagrams no: 1,2,4,5,6,9,10,12 and 18 in Table 1.

PROOF. That the diagrams 11,13,14,15,16 and 17 are not admissible follows with lemma 8.4, hence by the propositions 6.2, 8.2 and 8.3 it suffices to prove that the diagrams no. 1,2,4,5,6,9,10,12 and 18 are admissible.

We use the notations of roots in [BOURBAKI, Ch. VI]

no.1: Let $\Delta = \{\alpha_1,\alpha_2\}$ and $\kappa_{\alpha_1} = \kappa_{\alpha_2}$ be an arbitrary complex number with the absolute value 1. Since $\bar{\alpha}_1 = \alpha_2$, $\bar{\alpha}_2 = \alpha_1$ the conditions in proposition 5.4 are satisfied.

no.2: Let κ_{α_1} be an arbitrary complex number with the absolute value 1. Since $\bar{\alpha}_1 = \alpha_1$ proposition 5.4 is satisfied.

no.5: Let κ_{α_1} be an arbitrary complex number with the absolute value 1 and put:

$$\kappa_{\alpha_\ell} = -\kappa_{\alpha_1} \cdot \frac{c_{\alpha_\ell,\beta}}{c_{\alpha_\ell,-\beta}} \quad \text{where } \beta = \alpha_2+\ldots+\alpha_{\ell-1} = \varepsilon_2-\varepsilon_\ell$$

Since $\bar{\alpha}_1 = \alpha_\ell+\beta$ and $\bar{\beta} = -\beta$ we get:

$$-\kappa_{\alpha_\ell} \cdot c_{\bar{\alpha}_\ell,-\beta} = c_{\alpha_\ell,\beta} \cdot \kappa_{\bar{\alpha}_1} \cdot$$

by applying the extension of $\gamma_0 \in \Gamma$ to g_c on both sides of $[X_{\alpha_\ell},X_\beta] = c_{\alpha_\ell,\beta} X_{\alpha_\ell}$. So we have $\bar{\kappa}_{\alpha_\ell} \cdot \kappa_{\bar{\alpha}_\ell} = -\bar{\kappa}_{\alpha_\ell} \cdot \kappa_{\alpha_\ell} \cdot c_{\alpha_\ell,-\beta} / c_{\alpha_\ell,\beta} = |\kappa_{\alpha_\ell}|^2 = 1$.

Similarly we have $-\kappa_{\alpha_\ell} c_{\bar{\alpha}_\ell,-\beta} = c_{\bar{\alpha}_\ell,\beta} \cdot \kappa_{\bar{\alpha}_\ell}$ and

$$\bar{\kappa}_{\alpha_\ell} \cdot \kappa_{\bar{\alpha}_\ell} = \bar{\kappa}_{\alpha_1} \cdot \kappa_{\alpha_1} \cdot c_{\alpha_\ell,\beta} \cdot c^{-1}_{\bar{\alpha}_\ell,-\beta} \cdot c_{\bar{\alpha}_1,-\beta} \cdot c^{-1}_{\alpha_1,\beta} = 1$$

because $c_{\alpha_\ell,-\beta} = c_{-\beta,-\alpha_1} = c_{\alpha_1,\beta}$ and $c_{\bar{\alpha}_1,-\beta} = c_{-\beta,-\alpha_\ell} = c_{\alpha_\ell,\beta}$. For the remaining cases we refer to [ARAKI] Q.E.D.

Together with proposition 7.2 we can determine now the admissible $\mathbb{R}$-irreducible Γ-diagrams. An $\mathbb{R}$- irreducible and non absolutely irreducible Γ-diagram corresponds to a Lie algebra $(g_c)_{\mathbb{R}}$, where g_c is a complex simple Lie algebra. So we can restrict to the absolutely irreducible case:

THEOREM 8.7. *The non-compact absolutely irreducible admissible* Γ*-diagrams are the ones given in table 2.*

PROOF. The proof is easily obtained by using remark 7.5 and table 1. As an example we treat the case Φ of type B_{ℓ}.

As Φ_0 corresponds to a compact Lie algebra, Φ_0 must be of type:

$(r \times A_1) \times B_s$: ●—○...○—●—○.....●.......○—●—$\underbrace{○...○}_{k}$—$\underbrace{●...●\Rightarrow●}_{s}$

Let Δ_1 be the irreducible component of $\Delta \backslash \Delta_0$ connected to B_s.

If $k = |\Delta_1| = 1$ let $\Delta_1 = \beta$.

Then $\Delta' = \beta \circ \Delta_0$ has an irreducible component of type

●—○—●....●⇒●

which is not admissible.

If $k = |\Delta_1| > 1$, take $\beta \in \Delta_1$ s.t. $(\beta,\Delta_0) \neq 0$ and $(\beta,B_s) = 0$. As before $\Delta' = \beta \circ \Delta_0$ has an irreducible component of the type: ●—○ which is not admissible. It follows $r = 0$, hence Φ_0 of type B_s. $(1 \le s \le \ell)$. Q.E.D.

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Table 1	$\mathbb{R}$-rank one Γ-diagrams

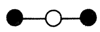
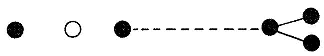
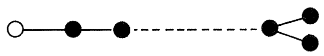
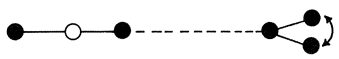
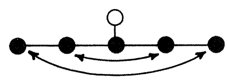
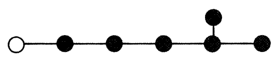
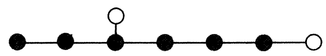
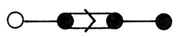
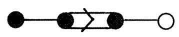
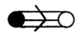
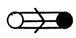
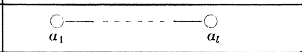
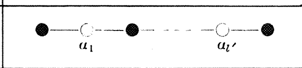
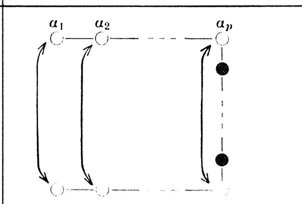
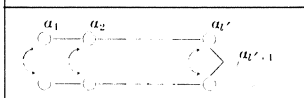
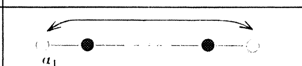
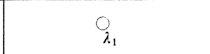
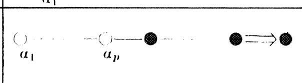
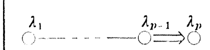
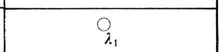
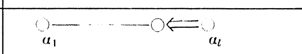
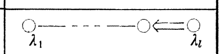
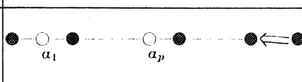
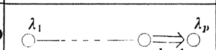
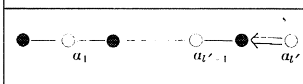
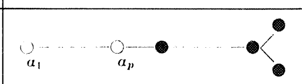
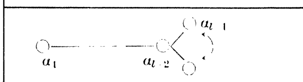
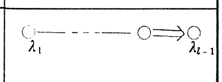
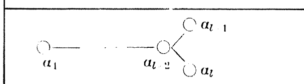
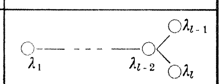
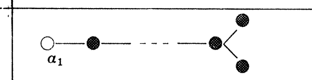
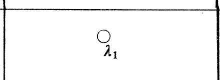
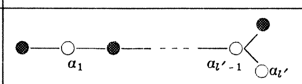
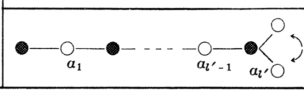
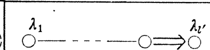
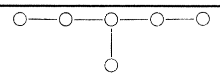
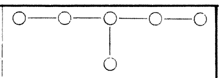
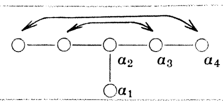
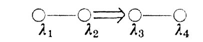
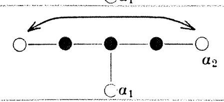
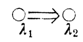
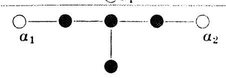
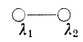
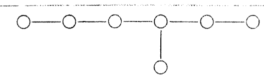
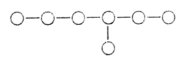
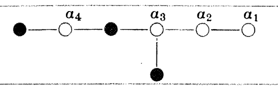
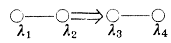
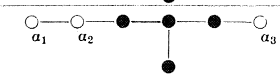
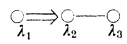
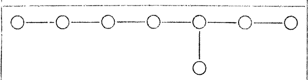
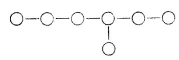
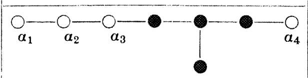
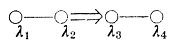
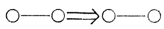
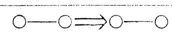
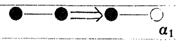
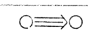
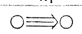
no.	Type	Γ-diagram	normal	admisseble
1	$A_1 \times A_1$		+	+
2	A_1		+	+
3	A_2		−	−
4	A_3		+	+
5	A_l		+	+
6	B_l		+	+
7	B_l		−	−
8	C_l		−	−
9	C_l		+	+
10	D_{2l} $(l \geq 2)$		+	+
11	D_{2l} $(l \geq 2)$		+	−
12	D_{2l+1} $(l \geq 2)$		+	+
13	D_{2l+1} $(l \geq 2)$		+	−
14	E_6		+	−
15	E_7		+	−
16	E_8		+	−
17	F_4		−	−
18	F_4		+	+
19	G_2		−	−
20	G_2		−	−

Table 2	non-compact real forms

Type	Cartan Not.	Γ-diagram	Δ
$^1A_\ell$	AI		
	AII		 $(l=2l'+1,\ l'\geq 1)$
$^2A_\ell$	AIII		 $(2\leq p\leq \frac{l}{2})$
			 $(l=2l'+1,\ l'\geq 1)$
	AIV		
B_ℓ	BI		 $(l\geq 2,\ 2\leq p\leq l)$
	BII		
C_ℓ	CI		
	CII		 $(l\geq 3,\ 1\leq p\leq \frac{l-1}{2})$
			 $(l=2l',\ l'\geq 2)$
$^1D_\ell$	DI		 $(l\geq 4,\ 2\leq p\leq l-2)$
$^2D_\ell$			
$^1D_\ell$			
$^1D_\ell$	DII		
$^1D_\ell$	DIII		 $(l=2l',\ l'\geq 2)$
$^2D_\ell$			 $(l=2l'+1,\ l'\geq 2)$

$^1E_6$	EI		
$^2E_6$	EII		
$^2E_6$	EIII		
$^1E_6$	EIV		
E_7	EV		
E_7	EVI		
E_7	EVII		
E_8	EVIII		
E_8	EIX		
F_4	FI		
F_4	FII		
G_2	G		

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