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INTRODUCTION

This volume contains the second part of the lecture notes of the colloquium Nonlinear Analysis, organized by the department of Applied Mathematics of the Mathematical Centre.

In the first volume, theory and applications were given of topological degree and nonlinear eigenvalue problems in finite dimensional spaces. In this second volume these concepts will be considered in connection with infinite dimensional spaces, especially Banach spaces.

Some background on functional analysis is given in chapter VI. This chapter gives the Riesz-Schauder theory of compact operators on a Banach space, the Fréchet differential calculus for nonlinear operators and the implicit function theorem for mappings on Banach spaces. Most proofs will be omitted in this chapter, but extensive reference is made to the literature.

Chapter VII, on bifurcation theory for operators on Banach spaces, gives extensions of results of chapter IV (which dealt with finite dimensional spaces). It gives a treatment of the Lyapunov-Schmidt method for nonlinear eigenvalue problems $F(x,\lambda) = 0$, with $F: B * \mathbb{R} \to B$, where B is a Banach space. Furthermore, bifurcation at a simple eigenvalue is discussed and the theory is applied to a boundary value problem already mentioned in chapter VI.

Chapter VIII extends the results of chapter II, where the notion of topological degree was defined, to mappings in a Banach space. As applications, Schauder's fixed point theorem and a theorem of Krasnosel'skii on the existence of a bifurcation point are given.

Chapter IX gives three nonlinear physical examples. It does not refer to earlier chapters in this volume but it rather has connection with those of volume I. The examples are the rotating string, heat generating by a nonlinear source and a model of an isothermal star in gravitational equilibrium.

Chapter X is an extensive survey on variational methods for nonlinear operators on Banach spaces. First some aspects from functional analysis that are important in this field are introduced, together with generalizations of the notion of derivative to operators and functionals. The connection between solutions of equations and extreme points and stationary points of a suitable chosen functional is discussed. The theory is applied to boundary value problems from mathematical physics and to bifurcation problems.

As in the first volume, all results in this book can be found in the literature. Apart from some minor details, no new results are given.

The lectures were given by

T.M.T. Coolen	chapter VI
G.M. Willems (MF Maastricht)	chapter VII
J.W. de Roever	chapter VIII
B. Dijkhuis	chapter IX
E.W.C. van Groesen (TH Eindhoven)	chapter X.

O. Diekmann and E.J.M. Veling assisted in writing chapter VII, and so did I.G. Sprinkhuizen-Kuyper for chapter IX.

In 1976 the colloquium Nonlinear Analysis has been continued with a series of lectures on nonlinear parabolic differential equations. The lecture notes of this new series will be collected in MC Syllabus 28, entitled Nonlinear diffusion problems.

February 1976 N.M.Temme

VI. SOME TOPICS FROM FUNCTIONAL ANALYSIS

Until now, bifurcation theory has only been developed for mappings on finite dimensional spaces. In correspondence with this restriction we were only able to consider the bifurcation of steady states of *ordinary* differential equations or of systems of ordinary differential equations. It will be made clear later on that if one wishes to cope with the problem of the variation of the number of steady state solutions in the case of *partial* differential equations depending on a parameter, one must have at one's disposal bifurcation theory for operators on infinite dimensional spaces.

This part of bifurcation theory strongly leans on certain results from functional analysis. In fact, it cannot be understood without an adequate knowledge of the Riesz-Schauder theory of compact operators on a Banach space, of the Fréchet differential calculus for nonlinear operators and of the implicit function theorem for the general case of mappings on Banach spaces. The aim of this chapter is to supply a survey on these topics.

We mention two well-known standard text books on functional analysis, namely LJUSTERNIK & SOBOLEW [1] and BROWN & PAGE [2]. Most proofs will be omitted in this chapter; in most cases we shall give references to these two books.

1. NORMED LINEAR SPACES

When considering certain classes of functions, we see that elements of these classes may be added, and that they may be multiplied by a real or complex number. Also, in one way or another, a norm of an element may be introduced, which allows us to speak of the distance between two elements (in more general terms, a topology is introduced). The structure we

have in mind is the normed linear space or normed vector space. Precise definitions are given below.

DEFINITION 1.1. The letters $\mathbb{R}$ and $\mathbb{C}$ will always denote the field of real and complex numbers respectively. If we do not wish to specify which of these two is meant, we shall write Φ. A *scalar* is an element of Φ.

A *linear space* or *vector space over* Φ is a set X, whose elements are called *vectors*, and in which two operations are defined, the addition $+ : X \times X \to X$ and the scalar multiplication $\cdot : \Phi \times X \to X$, satisfying the following properties:

(i) $x + (y+z) = (x+y) + z$ for all $x, y, z \in X$;

(ii) $x + y = y + x$ for all $x, y \in X$;

(iii) there exists a unique vector 0 (the *zero vector* or the *origin* of X) such that $x + 0 = x$ for all $x \in X$;

(iv) to each $x \in X$ corresponds a unique $-x \in X$, such that $x + (-x) = 0$;

(v) $(\alpha+\beta)\cdot x = \alpha\cdot x + \beta\cdot x$ for all $x \in X$ and $\alpha, \beta \in \Phi$;

(vi) $\alpha(x+y) = \alpha\cdot x + \alpha\cdot y$ for all $x, y \in X$ and $\alpha \in \Phi$;

(viii) $1\cdot x = x$.

A *real (complex) linear space* is a vector space over $\mathbb{R}$ ($\mathbb{C}$). Every complex linear space may also be considered as a real linear space by restricting the mapping $(\alpha,x) \to \alpha\cdot x$ of $\mathbb{C} \times X \to X$ to $\mathbb{R} \times X$. The uniqueness of 0 and $-x$ in (iii) and (iv) does not have to be postulated separately, but follows from the rest of the axioms.

Usually we shall write αx instead of $\alpha\cdot x$. Now let $x_1,\ldots,x_n \in X$ be n vectors. They are called *linearly independent*, if the equation

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0 \tag{1.1}$$

implies

$$\lambda_1 = \ldots = \lambda_n = 0.$$

If there do exist numbers $\lambda_i \in \Phi$, $i = 1,\ldots,n$, not all equal to zero, such that (1.1) holds, the elements $x_1,\ldots,x_n$ are called *linearly dependent*. Let, for example, $\lambda_n \neq 0$, then x_n can be written as a *linear combination* of the other elements.

A subset B of X is said to *span* X if all elements of X can be expressed as a linear combination of elements of B, that is if $x \in X$, then

there are vectors $b_i \in B$ and numbers $\lambda_i \in \Phi$, $i = 1,\ldots,n$ for some n, such that $x = \lambda_1 b_i + \ldots + \lambda_n b_n$. A linearly independent set that spans X is called a *basis for* X. It is clear that a set B is a basis for X if and only if every $x \in X$ can be written in a unique way as a linear combination of elements of B. The space X has *finite dimension* if X is either equal to $\{0\}$, or has a finite basis. Otherwise, X is said to be *infinite dimensional*. It can be shown that if X is finite dimensional and $X \neq \{0\}$, all bases for X are finite and have the same number of elements (see any book on linear algebra, e.g. HALMOS [3]). This number is called the *dimension* of X; if $X = \{0\}$, then its dimension is defined to be 0.

A *(linear) subspace* Y of X is a subset of X which is itself a vector space (with respect to the same operations, of course). It is easily seen that this is the case if and only if $0 \in Y$ and $\lambda y_1 + \mu y_2 \in Y$ for all $y_1, y_2 \in Y$ and $\lambda, \mu \in \Phi$.

DEFINITION 1.2. Let X be a linear space over a scalar field Φ. A mapping $x \to \|x\|$ of X into $\mathbb{R}$ is called a *norm on* X if it satisfies the following conditions

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for all $\alpha \in \Phi$, $x \in X$;
(iii) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The word norm is also used to denote the value of $\|x\|$: norm of x. A *normed linear space* is a linear space on which a norm is defined.

Let us make some remarks. The inequality (iii) is called the *triangle inequality*. The *distance* between x and y is defined as $\|x - y\|$. In this manner, any normed linear space is also a metric space (which we do not define here because we shall not need it). A subset A of X is called *bounded*, if there exists a constant M such that $\|x\| \leq M$ for all $x \in A$. Another way to express this is to say that A must be contained in some *ball* $\{x \mid \|x\| \leq M\}$ centered at the origin.

Completeness of a space is, of course, a crucial concept. This will be introduced by means of Cauchy sequences in normed linear spaces.

DEFINITION 1.3. A sequence (x_n) of elements of a normed linear space X is said to *converge to* $x \in X$, written as $x_n \to x$ or $\lim_{n\to\infty} x_n = x$, if $\|x_n - x\| \to 0$ as $n \to \infty$.

A sequence (x_n) of elements of a normed linear space X is called a *Cauchy sequence* if $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. If every Cauchy sequence

converges to some element $x \in X$ the space X is called *complete*. A complete normed linear space bears the name of *Banach space*.

EXAMPLE 1.4. Φ^n, with $\Phi = \mathbb{R}$ or $\Phi = \mathbb{C}$ is a Banach space over $\mathbb{R}$ and $\mathbb{C}$ respectively, if one supplies the linear space with the norm

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}}$$

where $x = (x_1,\ldots,x_n) \in \Phi^n$. Alternatively, one can also choose the norm

$$\|x\| = \max_{1\le i\le n} |x_i|;$$

Φ^n with this norm is also a complete normed linear space.

EXAMPLE 1.5. Let $[0,1]$ be the closed interval in $\mathbb{R}$, and let $B([0,1])$ be the class of all bounded functions on $[0,1]$ with values in Φ. $B([0,1])$ may be considered as a vector space over Φ in an evident manner. Define as norm on this vector space

$$\|f\| = \sup_{x\in[0,1]} |f(x)|, \quad \text{for all } f \in B([0,1]), \tag{1.2}$$

then $B([0,1])$ is complete in this norm, i.e., $B([0,1])$ is a Banach space, which follows from the fact that any Cauchy sequence of bounded functions converges to a bounded function (see e.g. BROWN & PAGE [2, p.55] for details).

In any book on functional analysis a multitude of examples of Banach spaces can be found.

Now consider a normed linear space X, and a linear subspace $Y \subset X$. The restriction of the norm $\|\cdot\|$ on X to Y is obviously a norm on Y, so Y is a normed linear space itself. Whenever a linear subspace of a normed linear is considered, we shall always assume that the norm on the subspace is *inherited* in this manner.

DEFINITION 1.6. Let x be a point of a normed linear space X. Then the set

$$B(x,r) = \{y \in X \mid \|x - y\| < r\}$$

is called the *open ball with centre* x *and radius* r. A subset A of X is

said to be *open (in the norm of* X*)* if for each $x \in A$ there exists an open ball with centre x that lies entirely within A. A subset A of X is said to be *closed (in the norm of* X*)*, if $X \backslash A$ is open in X.

The following lemma clearly holds.

LEMMA 1.7. *Let* X *be a normed linear space.*

(i) *If* X *is complete, and* Y *is a closed linear subspace of* X*, then* Y *is complete in the norm inherited from* X.

(ii) *A complete linear subspace of* X *is closed in* X.

(iii) *The closure of a linear subspace of a normed linear space is a linear space.*

EXERCISE 1.8. Consider the space of all bounded functions on $[0,1] \subset \mathbb{R}$ with values in Φ, which, as we know from example 1.5, is a Banach space in the norm (1.2). Prove that the subset of functions $f \in B([0,1])$ which are continuous in a point $x_0 \in [0,1]$, form a closed linear subspace of $B([0,1])$.

EXERCISE 1.9. Consider the set $C^0([0,1])$ of all continuous functions on $[0,1] \subset \mathbb{R}$. Show that this set may be considered as a linear subspace of $B([0,1])$ that is closed. Conclude that $C^0([0,1])$ is a Banach space in the norm defined in (1.2). (See e.g. BROWN & PAGE [2, p.63]).

We conclude this section by defining another important type of space, the Hilbert space.

DEFINITION 1.10. Let X be a linear space over Φ. A *scalar* or *inner product* on X is a mapping $X \times X \to \Phi$, which associates to each ordered pair (x,y) of elements of X an element of Φ, written as (x,y), such that the following requirements are fulfilled:

(i) $(\alpha x+\beta y,z) = \alpha(x,z) + \beta(x,z)$ for all $x, y, z \in X$ and all $\alpha, \beta \in \Phi$;

(ii) $(x,y) = \overline{(y,x)}$ for all $x, y \in X$; here, if Φ is $\mathbb{C}$, $\overline{(y,x)}$ is the complex conjugate of (y,x); if Φ is $\mathbb{R}$, it is just (y,x).

(iii) $(x,x) \geq 0$ for all $x \in X$ and $(x,x) = 0$ if and only if $x = 0$.

The expression $(x,x)^{\frac{1}{2}}$ has all the properties of a norm; we shall write $\|x\| = (x,x)^{\frac{1}{2}}$ and call $\|\cdot\|$ the *norm corresponding to the inner product* $(\cdot,\cdot)$. If X is complete in this norm, X is called a *Hilbert space* over Φ.

EXAMPLE 1.11. Φ^n is a Hilbert space over Φ, if one defines the inner product

$$(x,y) = \sum_{i=1}^{n} x_i \bar{y}_i$$

on Φ^n, where $x = (x_1,\ldots,x_n)$.

EXAMPLE 1.12. Let ℓ^2 denote the set of all sequences (x_n) in Φ such that the infinite series $\sum_{n=1}^{\infty} |x_n|^2$ converges. In an evident manner this set can be considered as a linear space. Define the expression

$$(x,y) = \sum_{n=1}^{\infty} x_n \bar{y}_n$$

where $x = (x_n)$ and $y = (y_n) \in \ell^2$. It can be shown that the sum always converges if x and $y \in \ell^2$, and that ℓ^2 is complete in the norm $\|x\| = (\sum_{n=1}^{\infty} |x_n|^2)^{\frac{1}{2}}$. See BROWN & PAGE [2, p.87 and p.343].

EXAMPLE 1.13. Consider the linear space of complex-valued square integrable functions on the interval $[0,1]$, $L^2([0,1])$. Functions only differing in value on a set of measure zero are identified. The expression

$$(f,g) = \int_0^1 f(x)\overline{g(x)}dx$$

is an inner product on $L^2([0,1])$, with corresponding norm

$$\|f\| = \left(\int_0^1 |f(x)|^2 dx \right)^{\frac{1}{2}}$$

Supplied with the inner product given above $L^2([0,1])$ is a Hilbert space.

REMARK 1.14. If we speak of a subspace of a space with inner product, we usually mean the linear subspace together with the inner product inherited from the original space.

2. BOUNDED OPERATORS AND FUNCTIONALS

Let X and Y be two sets. The notation

$$T: X \to Y$$

will mean that T is a mapping of X into Y. The *image* of a subset $A \subset X$ is defined by

$$T(A) = \{T(x) \mid x \in A\}$$

and the *preimage* of a subset $B \subset Y$ by

$$T^{-1}(B) = \{x \mid T(x) \in B\}.$$

DEFINITION 2.1. Suppose that X and Y are vector spaces over the same scalar field Φ. A mapping $T: X \to Y$ is called *linear* if

$$T(\alpha x+\beta y) = \alpha T(x) + \beta T(y)$$

for all $x, y \in X$ and $\alpha, \beta \in \Phi$. Often Tx will be written in stead of T(x), especially when T is linear. A (linear) mapping of X into its scalar field, that is $T: X \to \Phi$, is called a *(linear) functional*. Also, when considering mappings between vector spaces, often the name *operator* is used in place of mapping. The following properties of linear mappings (operators) are quite obvious:

(i) $T0 = 0$;
(ii) if A is a linear subspace, the same is true of T(A);
(iii) if B is a linear subspace, the same is true of $T^{-1}(B)$;
(iv) in particular,

$$T^{-1}(\{0\}) = \{x \in X \mid Tx = 0\} = N(T)$$

is a subspace of X, called the *null space* of T.

Now let X and Y be both normed linear spaces, and let us consider a (not necessarily linear) mapping $F: X \to Y$. Both norms on X and Y will be denoted by $\|\cdot\|$ since confusion is unlikely.

DEFINITION 2.2. An operator $F: X \to Y$ is called *continuous at a point* $x \in X$ if for any sequence (x_n) in X converging to x the sequence of images $F(x_n)$ converges to $F(x)$. Remember that convergence of a sequence y_n to y in a normed linear space with norm $\|\cdot\|$ means that $\|y_n - y\| \to 0$ as $n \to \infty$. F is called *continuous on an open set* $A \subset X$, if it is continuous at each point of A.

For linear operators one has the following property.

THEOREM 2.3. *If a linear operator* T *is continuous in one point* $x_0 \in X$, *it is continuous everywhere.*

PROOF. To prove continuity in a point $y \in X$, perform the translation $x \mapsto x + x_0 - y$, by which a neighbourhood of y is translated into one of x_0. Linearity does the rest. □

DEFINITION 2.4. An operator $F: X \to Y$ is called *bounded* if it maps bounded sets of X into bounded sets of Y. This definition conflicts with the usual notion of a bounded function as being one whose range is a bounded set. In that sense, no linear function other than 0 could ever be bounded.
If $F: X \to Y$ is a linear operator, one may express boundedness of F in terms of the inequality

$$\|Fx\| \leq M \|x\| \quad \text{for all } x \in X,$$

where M is some positive constant.

THEOREM 2.5. *A linear operator* T *of a normed vector space* X *into a normed vector space* Y *is continuous if and only if it is bounded.*

PROOF. See LJUSTERNIK & SOBOLEW [1, p.92] or BROWN & PAGE [2, p.104]. □

EXAMPLE 2.6. That the preceding theorem is not true for non-linear operators becomes clear if we look at the example $F: \mathbb{R} \to \mathbb{R}$, $F(x) = -1$ if $x < 0$, and 1 if $x > 0$ (a "jump").

DEFINITION 2.7. Let $T: X \to Y$ be a bounded linear operator, X and Y being normed linear spaces. We know that there exists a positive constant such that

(2.1) $\quad \|Tx\| \leq M \|x\| \quad$ for all $x \in X$.

Amongst the set of constants M there is a smallest number such that (2.1) holds. This number is written as $\|T\|$, and is called the *norm of the operator* T. (Later on, we shall show that $\|T\|$ really deserves to be named a norm). It can be easily shown (see LJUSTERNIK & SOBOLEW [1, p.95]) that

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{\|x\| \leq 1} \|Tx\|.$$

EXAMPLE 2.8. Let $C^0([0,1])$ denote the Banach space of all continuous functions on $[0,1]$ with values in Φ, supplied with the norm

$$\|f\| = \max_{0 \leq t \leq 1} |f(t)|.$$

Consider the linear operator $K: C^0([0,1]) \to C^0([0,1])$,

$$(2.2) \qquad y(t) = Kx(t) = \int_0^1 K(t,s)x(s)ds,$$

where the kernel K(t,s) is a function continuous in both variables on the square $0 \leq t, s \leq 1$. From

$$\|Kx\| = \max_t \left| \int_0^1 K(t,s)x(s)ds \right| \leq \max_t \int_0^1 |K(t,s)|ds \cdot \max_s |x(s)|$$

$$= \max_t \int_0^1 |K(t,s)|ds \cdot \|x\|$$

it follows that K is a bounded linear operator, whose norm satisfies the inequality

$$(2.3) \qquad \|K\| \leq \max_t \int_0^1 |K(t,s)|ds.$$

It can be shown that the norm of K is exactly equal to the right hand side of (2.3). For details see LJUSTERNIK & SOBOLEW [1, p.95].

The operator defined by (2.2) is called the *Fredholm integral operator* to which one can associate the *Fredholm integral equation*

$$(2.4) \qquad x(t) = f(t) + \lambda \int_0^1 K(t,s)x(s)ds, \qquad 0 \leq t \leq 1,$$

which can also be written as an abstract equation in the Banach space $C^0([0,1])$

(2.5) $\qquad (I-\lambda K)x = f.$

Of course, f is assumed to belong to $C^0([0,1])$. In view of (2.5) one is tempted to write the solution of the Fredholm equation as $(I-\lambda K)^{-1}f$. But, if one wishes to do so, one has to explain what one means by the *inverse* of an operator.

DEFINITION 2.9. Let T be a mapping of a vector space X *onto* a vector space Y, that is each $y \in Y$ appears as the image of some $x \in X$. Let further T be *one-to-one*, which means that if $x_1 \neq x_2$, $x_1, x_2 \in X$, then $Tx_1 \neq Tx_2$. In this situation it is possible to define the *inverse mapping* T^{-1} of T by $T^{-1}y = x$ if $y = Tx$.

Some simple properties of inverse operators are now brought to attention. If T is linear, then also T^{-1} is linear. In general the inverse of a bounded linear operator defined on a normed linear space does not have to be bounded. Additional requirement, have to be fulfilled, e.g. the one given in the following theorem.

THEOREM 2.10. *If the linear operator* T *of a normed linear space* X *into a normed linear space* Y *satisfies the condition*

(2.6) $\qquad \|Tx\| \geq m\,\|x\| \quad$ for all $x \in X$

where m *is some positive constant, then the inverse operator exists and is linear and bounded.*

PROOF. Note that (2.6) implies that T is one-to-one. The boundedness follows almost directly:

$$\|T^{-1}y\| \leq \frac{1}{m}\,\|TT^{-1}y\| = \frac{1}{m}\,\|y\|$$

for any $y \in Y$. □

Important is the case in which the spaces X and Y are both complete.

THEOREM 2.11. *A one-to-one continuous linear operator of a Banach space* X *onto a Banach space* Y *over the same scalar field has a continuous linear inverse.*

PROOF. This theorem is a consequence of the so-called *open mapping theorem*, see e.g. BROWN & PAGE [2, p.317, 318]. In LJUSTERNIK & SOBOLEW [1, p.109-111] a direct proof is given avoiding explicit use of the open mapping theorem. □

Until now we have only considered a single operator. Let us now look at the collection of all linear operators of X into Y.

DEFINITION 2.12. Let S and T be linear operators of a linear space X into a linear space Y (both over the same scalar field Φ), and let $\alpha \in \Phi$. The mappings S + T and αT of X into Y one defined by

$$(S+T)x = Sx + Tx,$$

and

$$(\alpha T)x = \alpha(Tx)$$

for all $x \in X$.

It is easily seen that S + T and αT are linear, and that the set of all linear mappings of X into Y is a linear space. The zero of this space is the linear mapping defined by

$$0x = 0 \qquad \text{for all } x \in X.$$

We have already introduced the norm of an operator of a normed linear space into another one. The use of the word norm is justified by the next theorem.

THEOREM 2.13. *The collection* B(X,Y) *of all bounded linear operators of a normed linear space* X *into a normed linear space* Y *is a normed linear space, with norm defined as in definition* 2.7.

PROOF. Rather straightforward, see e.g. BROWN & PAGE [2, p.105]. □

The concept of a Cauchy sequence of operators must be understood, before the completeness of a normed linear space of bounded linear operators can be discussed.

DEFINITION 2.14. Let X and Y be normed linear spaces. A sequence of linear operators $T_n: X \to Y$ is called a Cauchy sequence if

$$\|T_n - T_m\| \to 0 \quad \text{as } n, m \to \infty.$$

If any Cauchy sequence of operators $T_n \in B(X,Y)$ has a limit $T \in B(X,Y)$ in the sense that

$$\|T_n - T\| \to 0 \quad \text{as } n \to \infty, \tag{2.7}$$

then the space B(X,Y) is complete. Convergence in the sense of (2.7) is called *convergence in norm*, in contrast with *pointwise convergence* of a sequence of operators T_n by which is meant

$$\|T_n x - Tx\| \to 0 \quad \text{as } n \to \infty \tag{2.8}$$

for all fixed $x \in X$. Note that in (2.7) the norm is taken in B(X,Y), whereas in (2.8) $\|\cdot\|$ denotes the norm in Y. Naturally, convergence in norm implies pointwise convergence.

THEOREM 2.15. *If X is a normed linear space, and Y a Banach space, then the normed linear space B(X,Y) is also a Banach space, the norm being the operator norm*

$$\|T\| = \sup_{\|x\| \le 1} \|Tx\|.$$

PROOF. See LJUSTERNIK & SOBOLEW [1, p.100] or BROWN & PAGE [2, p.205]. □

REMARK 2.16. In the case that X and Y are normed linear spaces, we shall always tacitly assume that B(X,Y) denotes the *normed* linear space of all bounded linear operators of X into Y.

DEFINITION 2.17. Let T be a linear mapping of X into Y, then the *null space* of T is defined as

$$N(T) = \{x \in X \mid Tx = 0\},$$

and the *range* of T as

$$R(T) = \{y \in Y \mid y = Tx \text{ for some } x \in X\} = TX.$$

The following definition is given for the sake of reference.

DEFINITION 2.18. Let X_1, X_2 and Y be linear spaces over the same scalar field. A mapping

$$B: X_1 \times X_2 \to Y,$$

where $X_1 \times X_2$ denotes the Cartesian product of X_1 and X_2 is called *bilinear* if and only if $B(\cdot,x_2): X_1 \to Y$ and $B(x_1,\cdot): X_2 \to Y$ are linear for all $x_2 \in X_2$ and all $x_1 \in X_1$ respectively.

3. THE NORMED DUAL OF A NORMED LINEAR SPACE

A linear mapping $f: X \to \Phi$, where Φ is the scalar field of the linear space X, was called a linear functional on X. The elements of $B(X,\Phi)$, which are bounded linear mappings of a normed linear space X into the scalar field Φ, are therefore called bounded linear functionals. An element of $B(X,\mathbb{R})$ is called a *real* linear functional, of $B(X,\mathbb{C})$ á *complex* linear functional.

Throughout this section X will be a normed linear space. One is accustomed to speak of the *normed dual space* of X when referring to the linear space $B(X,\Phi)$ supplied with the norm

$$\|f\| = \sup_{\|x\| \le 1} |f(x)| \tag{3.1}$$

for any $f \in B(X,\Phi)$. Also, one usually writes X^* instead of $B(X,\Phi)$.

Typical elements of the space X^* will be denoted by $x^*, y^*, \ldots$, instead of $f, g, \ldots$. The reader should notice, however, that the $*$ is now being used in two different ways: the $*$ in X^* indicates the dual of X, whereas the $*$ in x^* only indicates that $x^* \in X^*$ - if $x \in X$ and $x^* \in X^*$, there is no implied relationship between x and x^*.

In place of $x^*(x)$, which, of course, is a number in Φ, we shall often write $\langle x,x^* \rangle$. The reason for this is that this notation is adapted to the symmetry (or duality) that exists between the action of X^* on X and the "opposite" action X on X^*, which, amongst others, comes into view in the next theorem.

THEOREM 3.1.

(i) *The space* X^*, *supplied with the norm*

$$\|x^*\| = \sup_{\|x\| \le 1} |\langle x,x^* \rangle|$$

for any $x^* \in X^*$, *is a Banach space.*

(ii) *For every* $x \in X$ *one has*

$$\|x\| = \sup_{\|x^*\| \le 1} |\langle x,x^* \rangle| ;$$

consequently, the mapping $x^* \mapsto \langle x,x^* \rangle$ *is a bounded linear functional on* X^*, *with norm* $\|x\|$.

PROOF.

(i) Since the scalar field Φ is a Banach space, so is $X^* = B(X,\Phi)$, according to theorem 2.15.

(ii) See BROWN & PAGE [2, p.189] or LJUSTERNIK & SOBOLEW [1, p.136]. This part of the theorem is, essentially, a consequence of the Hahn-Banach theorem, which we will not reproduce here. □

We now wish to introduce the adjoint of an operator $T: X \to Y$, where X and Y both are normed linear spaces. But before doing so, we will illustrate some of the ideas involved in the simpler case when the spaces X and Y both are finite dimensional.

EXAMPLE 3.2. Consider a normed n-dimensional real linear space $\mathbb{R}^n$, the elements of which are written as $x = (x_1,\ldots,x_n)$, or as

$$x = \sum_{i=1}^{n} x_i e_i,$$

where $(e_1,\ldots,e_n)$ form a basis for $\mathbb{R}^n$. Let f be a linear functional on $\mathbb{R}^n$, then

$$f(x) = f(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i f(e_i) = \sum_{i=1}^{n} x_i f_i.$$

Conversely, an expression of the form

$$f(x) = \sum_{i=1}^{n} x_i f_i, \tag{3.2}$$

where f_i are arbitrary numbers, is a linear functional on $\mathbb{R}^n$. Consequently, (3.2) is the general form of a linear functional on $\mathbb{R}^n$. Since the numbers f_i may be considered as the components of a n-dimensional vector f, the dual space of $\mathbb{R}^n$ is also a n-dimensional space $\mathbb{R}^{n*}$ which, in general, has a norm that differs from the one defined on $\mathbb{R}^n$. If, for example, one chooses $\|x\| = \max_i |x_i|$, then

$$|f(x)| = |\sum_{i=1}^{n} x_i f_i| \le \sum_{i=1}^{n} |x_i||f_i| \le (\sum_{i=1}^{n} |f_i|)\|x\|,$$

so

$$\|f\| \le \sum_{i=1}^{n} |f_i|.$$

On the other hand, the element,

$$y = \sum_{i=1}^{n} \operatorname{sign} f_i \cdot e_i \in \mathbb{R}^n$$

satisfies $\|y\| = 1$ and

$$f(y) = \sum_{i=1}^{n} \operatorname{sign} f_i \cdot f_i = \left(\sum_{i=1}^{n} |f_i| \right) \|y\|,$$

from which it follows that $\|f\| = \sum_{i=1}^{n} |f_i|$. If one chooses the Euclidean norm $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, it can be easily shown that $\mathbb{R}^{n*}$ also has the Euclidean norm, so in that case $\mathbb{R}^{n*} = \mathbb{R}^n$.

This is, perhaps, the place to recall two well-known properties of finite dimensional normed linear spaces. First, any two norms on a finite dimensional vector space are equivalent, that is there exists a constant c such that

$$c\|x\|_1 \leq \|x\|_2 \leq c^{-1}\|x\|_1 \qquad \text{for all } x.$$

Secondly, any linear functional on a finite dimensional normed vector space is bounded, and the dual space has the same dimension as the original one.

Now let us consider a bounded linear operator

$$T: \mathbb{R}^m \to \mathbb{R}^n,$$

where $\mathbb{R}^m$ and $\mathbb{R}^n$ both are supplied with some norm. The mapping T can be represented as a $m \times n$ matrix t_{ij}. For $x \in \mathbb{R}^m$, $y = Tx \in \mathbb{R}^n$ and $f \in \mathbb{R}^{n*}$ we make the following calculations:

$$f(Tx) = \sum_{i=1}^{n} f_i y_i = \sum_{i=1}^{n} \sum_{j=1}^{m} f_i t_{ij} x_j = \tag{3.3}$$

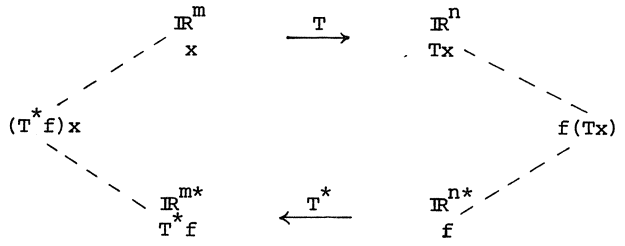
$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{n} t_{ij} f_i \right) x_j = \sum_{j=1}^{m} g_j x_j$$

with

$$g_j = \sum_{i=1}^{n} t_{ij} f_i.$$

The vector $g = (g_1, \ldots, g_m)$ is a member of $\mathbb{R}^{m*}$, and is obtained from

the vector $f = (f_1,\ldots,f_n)$ by a linear mapping whose matrix representation is (t_{ji}), the *transposed* of the matrix representing T. This mapping is called the *adjoint* of T, T^*; in this notation we have $g = T^*f$. The situation is depicted in the following diagram.



According to (3.3), $(T^*f)x$ and $f(Tx)$ are the same number in $\mathbb{R}$. Using our symmetric notation, this observation becomes

$$(3.4) \qquad \langle Tx,f\rangle = \langle x,T^*f\rangle.$$

From the calculations above another thing can be learned. If we supply all spaces with the Euclidean norm, they can be considered as Hilbert spaces with inner product $(x,y) = \sum x_i y_i$. From (3.2) we see that a functional f on $\mathbb{R}^n$ considered as the Euclidean Hilbert space can be represented as (x,f), where $f \in \mathbb{R}^n$ is uniquely determined. This observation takes (3.4) into the well-known expression

$$(3.5) \qquad (Tx,f) = (x,T^*f),$$

the first inner product being taken in $\mathbb{R}^n$, the second in $\mathbb{R}^m$.

Historically, linear algebra did provide the background of much of what now is known as operator theory. For this reason the example above has been given. We now wish to generalize the concepts that have occurred to infinite dimensional spaces.

THEOREM 3.3. *(Riesz representation theorem)*
Let f *be a bounded linear functional on a Hilbert space* H. *Then there exists a unique* $y \in H$, *such that*

$$f(x) = (x,y)$$

for all $x \in H$. *Moreover,* $\|f\| = \|y\|$.

PROOF. This theorem generalizes the situation for finite dimensional spaces considered in the preceding example. The proof can be found in LJUSTERNIK & SOBOLEW [1, p.133] and in BROWN & PAGE [2, p.348]. □

REMARK 3.4. From this theorem one may conclude that duality in a Hilbert space H can be expressed in terms of the inner product of H, and that H^* is isometrically isomorphic to H (an *isomorphism* is a one-to-one linear mapping that preserves the vector operations). The reader should be warned that H and H^* do not have to be equal as was the case in example 3.2.

THEOREM 3.5. *Let* X *and* Y *be normed linear spaces. To each bounded linear operator* $T \in B(X,Y)$ *one can associate a unique bounded linear operator* $T^* \in B(Y^*,X^*)$ *such that*

$$\langle Tx,y^*\rangle = \langle x,T^*y^*\rangle$$

for all $x \in X$ *and* $y^* \in Y^*$, *where the first duality concerns* Y *and* Y^*, *the second* X *and* X^*. *Further,* $\|T^*\| = \|T\|$.

PROOF. See LJUSTERNIK & SOBOLEW [1, p.139] or BROWN & PAGE [2, p.239]. If one replaces $\mathbb{R}^m$ by X, $\mathbb{R}^n$ by Y, f by y^*, the diagram given above is adapted to this theorem. □

EXAMPLE 3.6. Suppose $1 < p < \infty$, p not necessarily an integer. Consider the class of all functions x(t) on the interval $[0,1] \subset \mathbb{R}$, such that x(t) is measurable and $|x(t)|^p$ is integrable (in the Lebesgue sense) over [0,1]. Functions that only differ on a set of measure zero will not be distinguished. Denote this set by $L^p([0,1])$, or shorter L^p. Clearly, $\alpha x \in L^p$ if $x \in L^p$, $\alpha \in \mathbb{C}$; it is also true that $x+y \in L^p$ if $x,y \in L^p$, which follows from the inequality

$$|a + b|^p \leq 2^p(|a|^p+|b|^p).$$

So L^p is a linear space over $\mathbb{C}$. By defining the norm

$$\|x\|_p = \left(\int_0^1 |x(t)|^p dt\right)^{1/p}, \tag{3.6}$$

it becomes a normed linear space. It can be shown that L^p is a Banach space.

The question we now wish to answer is: what is the normed dual of L^p?

Let q be a real number, such that $p^{-1} + q^{-1} = 1$; so $1 < q < \infty$ too. For any $x \in L^p$ and any $y \in L^q$ *Hölder's inequality* reads

$$(3.7) \qquad \int_0^1 |x(t)y(t)|dt \leq \left(\int_0^1 |x(t)|^p dt\right)^{1/p} \left(\int_0^1 |y(t)|^q dt\right)^{1/q}.$$

Take an arbitrary $y \in L^q$, then the expression

$$(3.8) \qquad f(x) = \int_0^1 x(t)y(t)dt$$

represents a bounded linear functional on L^p, since

$$|f(x)| \leq \int_0^1 |x(t)y(t)|dt \leq \|x\|_p \|y\|_q .$$

Obviously, the norm of f is less than or equal to $\|y\|_q$; it can be shown that $\|f\| = \|y\|_q$.

The converse, that any bounded linear functional on L^p can be represented as in (3.8), with $y \in L^q$, is also true, but much harder to prove. As a consequence we have: L^{p*} is isometrically isomorphic to L^q.

A special case is $p = q = 2$. The space $L^2([0,1])$ therefore is the normed dual of itself. For details one is referred to LJUSTERNIK & SOBOLEW [1, p.131].

EXAMPLE 3.7. The Fredholm integral operator K with continuous kernel K(t,s) (see example 2.8) is continuous when considered as a mapping $K: C^0([0,1]) \to C^0([0,1])$. It can also be conceived as an operator of $L^2([0,1])$ into $L^2([0,1])$. It is easily seen that if one takes K in this fashion, K is also continuous.

We already know that the normed dual of $L^2([0,1])$ is $L^2([0,1])$ itself in the sense that L^{2*} is isometrically isomorphic to L^2. An arbitrary continuous linear functional on $L^2([0,1])$ has the form

$$f(x) = (x,z) = \int_0^1 x(t)\overline{z(t)}dt, \quad z(t) \in L^2([0,1]).$$

Therefore

$$f(Kx) = \int_0^1 \overline{z(t)} \int_0^1 K(t,s)x(s)\,ds\,dt =$$

$$= \int_0^1 x(s) \int_0^1 K(t,s)\overline{z(t)}\,dt\,ds = \int_0^1 x(s)\overline{g(s)}\,ds$$

with

$$g(t) = \int_0^1 \overline{K(s,t)}z(s)\,ds.$$

In this example the adjoint operator K^* corresponds to an interchange of the variables s and t (and a complex conjugation).

4. THE RIESZ-SCHAUDER THEORY FOR COMPACT OPERATORS

Many of the operators that occur in the study of mathematical models of natural phenomena turn out to be compact, or, as will be seen later on, are the sum of the identity operator and a compact operator. As an example we mention the operators arising in integral equations. This accounts for their importance from the viewpoint of applications. In some respects compact operators show a behaviour familiar to linear operators on finite dimensional spaces, in which case, as we have seen, one usually expresses oneself in terms of properties of the matrices representing these operators. For general operators on infinite dimensional spaces this is not so. The similarities between compact operators and operators on finite dimensional spaces will show up particularly strongly in their so-called spectral properties.

Throughout this section X and Y will denote normed linear spaces; completeness of X or Y will not be assumed.

DEFINITION 4.1. Suppose that X and Y are normed linear spaces over the same scalar field Φ. A linear mapping $T: X \to Y$ is called *compact* if and only if for each bounded sequence (x_n) in X, the sequence of images (Tx_n) has a convergent subsequence.

LEMMA 4.2. *A compact linear mapping* $T: X \to Y$ *is continuous.*

PROOF. Assume that T is not continuous, i.e., not bounded, then for each integer n, there is a $x_n \in X$, $\|x_n\| = 1$, and $\|Tx_n\| \geq n$. Oviously, the sequence Tx_n cannot have a convergent subsequence, hence T is not compact. □

REMARK 4.3. If the mapping T is not linear, but does have the property that for each bounded sequence (x_n) the sequence (Tx_n) has a convergent subsequence, T does not have to be continuous. An example illustrating this fact is the mapping $T: \mathbb{R} \to \mathbb{R}$, with $Tx = -1$ if $x < 0$ and $Tx = 1$ if $x > 0$. This operator is not continuous in 0.

For this reason some authors introduce the concept of *complete continuity* in the following manner: an operator $T: X \to Y$ is completely continuous if and only if it is continuous and compact. Other authors introduce a different concept of complete continuity, involving strong (i.e. norm) and weak topologies on a Banach space, or more in general, on locally convex

spaces. Readers who are interested in all the subtle nuances of compactness of operators are referred to CARROLL [5, p.86 ff].

Here, in agreement with the terminology of many still other authors, we shall call a (not necessarily linear) operator compact if it is continuous and has the property that (Tx_n) has a convergent subsequence, whenever (x_n) is bounded. If T is linear, this definition is somewhat redundant.

THEOREM 4.4. *Let* X *and* Z *denote normed vector spaces. If* $T: X \to Y$ *is linear and compact, and* $S: Y \to Z$ *is linear and continuous, then* $ST: X \to Z$ *is compact. If* $R: Z \to X$ *is linear and continuous, then* $TR: Z \to Y$ *is compact.*

PROOF. Elementary, see BROWN & PAGE [2, p.245]. □

The following theorem, due to Ascoli and Arzelà, provides a particularly often used tool to prove compactness of operators on function spaces. It is preceded by the necessary definitions.

DEFINITION 4.5. Let $C^0(D)$, with $D \subset \Phi^n$, denote the set of all continuous real or complex-valued functions on D. A family F of functions of $C^0(\bar{\Omega})$, where Ω is a bounded set in Φ^n, is called *equicontinuous*, if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $f \in F$ and $x, y \in \bar{\Omega}$

$$|f(x) - f(y)| < \varepsilon \quad \text{whenever} \quad |x - y| < \delta.$$

Remember that $\Phi = \mathbb{R}$ or $\Phi = \mathbb{C}$.

The functions $f \in F$ are said to be *uniformly bounded* if there exists a non-negative constant $M < \infty$, such that for all $f \in F$ and $x \in \bar{\Omega}$ $|f(x)| < M$.

THEOREM 4.6. *(Ascoli-Arzelà)*
If f_n *is a uniformly bounded sequence of functions of* $C^0(\bar{\Omega})$, $\Omega \subset \Phi^n$ *bounded, such that the functions* (f_n) *form an equicontinuous family, then the sequence* (f_n) *has a subsequence which converges uniformly on* $\bar{\Omega}$ *(i.e., which is convergent in* $C^0(\bar{\Omega})$ *with respect to the norm* $\|f\| = \{\sup |f(x)|, x \in \bar{\Omega}\}$*).*

PROOF. See BROWN & PAGE [2, p.164]. □

EXAMPLE 4.7. Consider the Fredholm integral operator

$$Kx(t) = \int_0^1 K(t,s)x(s)ds,$$

$$K: C^0([0,1]) \to C^0([0,1]),$$

with continuous kernel K(t,s) (see example 2.8). We already know that K is a continuous operator on the Banach space $C^0([0,1])$ (with norm $\|f\| = \max\{|f(t)|, 0 \le t \le 1\}$. Let B be the unit ball in $C^0([0,1])$. Then K(B) is bounded. Because K(t,s) is uniformly bounded on the square $0 \le t,s \le 1$, for every $\varepsilon > 0$ there exist a $\delta > 0$ such that $|K(t_1,s) - K(t_2,s)| < \varepsilon$ for all $t_1, t_2, s \in [0,1]$ with $|t_1-t_2| < \delta$. Therefore, for $t_1,t_2 \in [0,1], |t_1-t_2| < \delta$ and $f \in B$ we have

$$|Kf(t_1) - Kf(t_2)| = \left|\int_0^1 (K(t_1,s) - K(t_2,s))f(s)ds\right|$$

$$\le \max_{0\le t\le 1} \{|K(t_1,s) - K(t_2,s)|\ |f(t)|\} \le \varepsilon \|f\| \le \varepsilon.$$

This proves that the family of functions $f \in B$ is equicontinuous. From theorem 4.6 we may deduce that for any bounded sequence (f_n) in $C^0([0,1])$ the sequence of images (Kf_n) has a convergent subsequence. Hence K is a compact linear operator.

We now turn our attention to compact linear operators of a normed linear space $X \neq \{0\}$ *into itself*. Quite a lot of information about the behaviour of a compact linear operator can be obtained by studying the family of operators $T - \lambda I$, where λ is a real or complex parameter. A fundamental difference arises at this point in the theory between real and complex linear spaces, which difference, as will turn out, is connected with the the fact that any non-constant polynomial with complex coefficients has a complex zero, whereas there exist non-constant real polynomials without real zeros.

For this reason, in the following always is assumed that $X \neq \{0\}$ is a *complex* normed linear space.

DEFINITION 4.8. Let T be a bounded linear operator on X. A complex number

λ is called a *regular* point of T, if the operator $T - \lambda I$ has a bounded inverse on X. Apparently, a necessary condition in that case is that $T - \lambda I$ is onto and one-to-one. The *spectrum* of T is the complement in the complex plane of the set of regular points.

A complex number λ is said to be an *eigenvalue* of T if there is a non-zero $x \in X$ such that $Tx = \lambda x$. If λ is an eigenvalue, all the non-zero x that satisfy $Tx = \lambda x$ are called *eigenvectors of T corresponding to the eigenvalue* λ.

It is evident that if λ is an eigenvalue, $T - \lambda I$ is not one-to-one, so that the eigenvalues, if they exist, belong to the spectrum of T.

LEMMA 4.9. *If λ and μ are distinct eigenvalues of the compact linear operator T, then the corresponding eigenvectors are linearly independent.*

PROOF. As in the finite dimensional case. See BROWN & PAGE [2, p.231]. □

DEFINITION 4.10. Let A and B be linear subspaces of a linear space C. Then C is said to be the *direct sum of* A *and* B if each element $c \in C$ can be written uniquely as the sum of an element $a \in A$ and an element $b \in B$. Note that in this case $A \cap B = \{0\}$; one writes $C = A \oplus B$.

Before we pass on to the spectral theory of operators on a Banach space, we first recall some definitions and results from linear algebra, which were already mentioned in chapter IV. Let λ be an eigenvalue of a linear mapping T of $\mathbb{C}^n$ into itself. The *algebraic multiplicity* of λ is defined as the multiplicity of λ as a zero of the equation $\det(T-\lambda I) = 0$. A *simple* eigenvalue has multiplicity one. The Riesz index $r(\lambda)$ is the least integer k such that $N(T-\lambda I)^{k+1} = N(T-\lambda I)^{k}$. The space $N((T- I)^{r(\lambda)})$ is called the *generalized eigenspace* of the eigenvalue λ. The algebraic multiplicity is equal to the dimension of the generalized eigenspace. The dimension of the null space $N(T-\lambda I)$ is called the *geometric multiplicity* of λ. For symmetric matrices, the Riesz index is 1, and the algebraic and geometric multiplicity coincide.

Furthermore, it can be shown that the spectrum of a linear operator T on a n-dimensional complex linear space consists exactly of the eigenvalues of T, and that T has at least one eigenvalue, and at most n. The introduction of all these concepts is justified by the well-known theorem on the *Jordan normal form* of matrices, which we give here for the purpose of reference and comparison with the more complicated situation in infinite dimensional

spaces.

THEOREM 4.11. *(Jordan normal form)*
Let the (bounded) linear mapping T *of* $\mathbb{C}^n$ *into itself have* k *distinct eigenvalues* λ_i, *each of algebraic multiplicity* m_i. *Denote the Riesz index of each eigenvalue* λ_i *by* r_i. *Then one can write* $\mathbb{C}^n$ *as the direct sum of the subspaces* $N_i = N((T-\lambda_i I)^{r_i})$. *Further, each* N_i *is invariant under* T, *i.e.,* $TN_i \subset N_i$ *for all* i. *Moreover, one can choose a basis in* $\mathbb{C}^n$, *such that the mapping* T *can be represented by a matrix of the following form. Every element not on or immediately above the main diagonal vanishes. On the main diagonal there appear the* k *distinct eigenvalues of* T, *each a number of times equal to its algebraic multiplicity. Directly above the main diagonal there are only* 1's *and* 0's *and these in the following manner: the* 1's *appear in chains followed by a single* 0, *and also some* 0's *may succeed each other. So*

$$(4.1) \qquad T = \begin{pmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_k \end{pmatrix},$$

in which T_i *is a* $r_i \times r_i$ *matrix of the form*

$$(4.2) \qquad T_i = \begin{pmatrix} T_{i1} & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & T_{il} \end{pmatrix}, \qquad i = 1,\ldots,k,$$

where T_{ij} *are* $m_{ij} \times m_{ij}$ *matrices*

$$(4.3) \qquad T_{ij} = \begin{pmatrix} \lambda_i & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}, \quad \begin{matrix} i = 1,\ldots,k, \\ j = 1,\ldots,l. \end{matrix}$$

Of course, $m_{i1}+\ldots+m_{il} = r_i$.

PROOF. See e.g. HALMOS [3]. □

REMARK 4.12. Let $R_1 = N_2 \oplus \ldots \oplus N_k$. Since $T - \lambda_1 I$ is 0 exactly on N_1, it is invertible on R_1. The same, of course, is true for the other eigenvalues. In chapter IV this was expressed in terms of the existence of a *pseudo-inverse*.

Is it possible to extend the ideas exposed above about linear operators on finite dimensional spaces to linear operators on infinite dimensional spaces? The answer to this question is affirmative, provided that the linear operators considered are *compact*. The results will be stated in a sequence of lemmas and theorems. The proofs can be found in BROWN & PAGE [2, chapter 6] and LJUSTERNIK & SOBOLEW [1, chapter VI].

NOTATION 4.13. In the further part of this section T will always denote a compact linear operator of a complex normed linear infinite dimensional vector space X into itself, and λ will be some complex number.

LEMMA 4.14. *The null space* $N(T-\lambda I)$ *is a closed and finite dimensional linear subspace of* X.

LEMMA 4.15. *The range* $R(T-\lambda I)$ *is a closed linear subspace of* X.

The following lemma shows that the Riesz index is a concept that is also useful for eigenvalues of operators on infinite dimensional spaces.

LEMMA 4.16.

(i) *The null spaces* $N_n = N((T-\lambda I)^n)$ *are closed finite dimensional linear subspaces of* X *for all non-negative integers* n, *and there is a non-negative integer* ν *such that*

$$N_n \subset N_{n+1}, \quad N_n \neq N_{n+1} \qquad \text{for } 0 \leq n \leq \nu - 1$$

and

$$N_n = N_{n+1} \qquad \text{for } n \geq \nu.$$

(ii) *The ranges* $R_n = R((T-\lambda I)^n)$ *are closed linear subspaces of* X *for all non-negative integer* n, *and there is a non-negative integer* μ *such that*

$$R_n \supset R_{n+1}, \quad R_n \neq R_{n+1} \quad \text{for } 0 \leq n \leq \mu - 1$$

and

$$R_n = R_{n+1} \qquad \text{for } n \geq \mu.$$

(iii) *The integers* μ *and* ν *in the preceding statements are equal. Their common value will be denoted by* r. *In the case that* λ *is an eigenvalue of* T, *this number* r *is called the Riesz index of* λ.

THEOREM 4.17.

(i) *The entire space* X *can be written as a direct sum*

$$N((T-\lambda I)^r) \oplus R((T-\lambda I)^r).$$

(ii) *Write* $N_r = N((T-\lambda I)^r)$ *and* $R_r = R((T-\lambda I)^r)$. *The spaces* N_r *and* R_r *are left invariant by* T, *i.e.*, $T(N_r) \subset N_r$ *and* $T(R_r) \subset R_r$.

(iii) $T - \lambda I$ *has a bounded inverse on the subspace* R_r; *or, in other words, the restriction of* $T - \lambda I$ *to the subspace* R_r, $T - \lambda I \mid R_r$, *is invertible. In particular, if* $r = 0$, *then* $T - \lambda I$ *is boundedly invertible on the whole of* X.

Let us give another formulation of this result. If λ does not belong to the spectrum of T, $T - \lambda I$ has a bounded inverse on X. If, however, λ does belong to the spectrum which is so if λ is an eigenvalue, then $T - \lambda I$ only has a bounded inverse, if it is considered as an operator of R_r into R_r.

We now wish to know something about the distribution of the eigenvalues of a compact linear operator. Intuitively, the following theorem is all but surprising.

THEOREM 4.18. *Let* $sp(T)$ *be the spectrum of* T, *and let* λ *be some eigenvalue of* T. *Then the restriction of* T *to* $N_r = N((T-\lambda I)^r)$, $T \mid N_r : N_r \to N_r$, *has exactly the eigenvalue* λ, *and no others, and the restriction of* T *to* $R_r = R((T-\lambda I)^r)$ *has all the eigenvalues* T *has considered as an operator on* X, *except* λ. *In formulas*

$$sp(T|N_r) = \{\lambda\}$$

$$sp(T|R_r) = sp(T) \setminus \{\lambda\}.$$

In the next theorem we are concerned with the existence of eigenvalues.

THEOREM 4.19.

(i) *The spectrum* sp(T) *of a compact linear operator is either a finite set, or it is countably infinite with* 0 *as only accumulation point.*

(ii) *If* T *is a compact operator on an infinite dimensional normed linear space,* T *itself is not invertable, i.e.,* $0 \in sp(T)$.

(iii) *Each non-zero point of* sp(T) *is an eigenvalue of* T.

REMARK 4.20. There are compact operators on infinite dimensional spaces whose spectrum consist only of the single point 0.

The definitions we brought back to the mind after definition 4.10 remain sensible if T is a compact linear operator on an infinite dimensional space. So if λ is an eigenvalue of T, $N((T-\lambda I)^{r(\lambda)})$, where $r(\lambda)$ is the Riesz index of λ, is the generalized eigenspace of λ. Further, the algebraic multiplicity is equal to the dimension of the generalized eigenspace, whereas the geometric multiplicity is equal to the dimension of $N(T-\lambda I)$.

Gathering a number of results from above, we now state the closest analogue to the Jordan normal form of matrices representing a linear mapping on a finite dimensional space.

THEOREM 4.21. *Let* T *be a compact linear operator of a infinite dimensional normed linear space* X *into itself, and let* $\lambda_1,\ldots,\lambda_k$ *be* k *distinct eigenvalues of* T. *Write* $N(\lambda_i)$ *for* $N((T-\lambda_i I)^{r(\lambda_i)})$. *Then* X *can be decomposed as*

$$X = N(\lambda_1) \oplus \ldots \oplus N(\lambda_k) \oplus M,$$

with

$$T(N(\lambda_i)) \subset N(\lambda_i), \quad i = 1,\ldots,k, \quad \text{and} \quad T\,M \subset M$$

where all $N(\lambda_i)$, $i = 1,\ldots,k$, *are finite dimensional, and* M *is infinite dimensional. The restriction of the operator* T *to the finite dimensional subspace* $\oplus_{i=1}^{k} N(\lambda_i)$ *of* X *can be represented by a matrix having the Jordan normal form by choosing an appropriate basis in* $\oplus_{i=1}^{k} N(\lambda_i)$.

The greater part of the rest of this section will be devoted to the spectrum of the adjoint operator T^* of T. See section 3 of this chapter for the definition of adjoint operator.

THEOREM 4.22. *If* $T: X \to X$ *is a compact linear operator so is* $T^*: X^* \to X^*$.

PROOF. For the proof of this theorem, and of the next ones, see BROWN & PAGE [2, chapter 6, sections 6.5, 6.6 and 6.7], or LJUSTERNIK & SOBOLEW [1, chapter VI]. □

THEOREM 4.23. *The null spaces* $N(T-\lambda I)$ *and* $N(T^*-\lambda I^*)$ *have the same dimension for any* $\lambda \in \mathbb{C}$. *Here,* $I^*: X^* \to X^*$ *is the adjoint of* $I: X \to X$.

DEFINITION 4.24. Let A be a non-empty subset of X, B of X^*, where X^* is the normed dual of X. We define

$$A^{\perp} = \{x^* \in X^* \mid \langle x,x^* \rangle = 0 \quad \text{for all } x \in A\}$$

and

$$^{\perp}B = \{x \in X \mid \langle x,x^* \rangle = 0 \quad \text{for all } x^* \in B\}.$$

The sets $A^{\perp}$ and $^{\perp}B$ are called the *annihilators of* A and B respectively. Recall that the symmetric notation $\langle x,x^* \rangle$ stands for $x^*(x)$.

LEMMA 4.25. *The annihilators* $A^{\perp}$ *and* $^{\perp}B$ *are closed linear subspaces of* X^* *and* X *respectively.*

PROOF. See BROWN & PAGE [2, p.194]. □

REMARK 4.26. If X is a Hilbert space, according to the Riesz representation theorem each functional f on X can be represented in the form $f(x) = (x,z)$ for some uniquely determined $z \in X$. If A is a non-empty subset of X, then $A^{\perp}$ is clearly a closed subspace *orthogonal* to the subset A in the sense that for each $a \in A$ and $b \in A^{\perp}$ $(a,b) = 0$.

The importance of the concept of annihilators lies in the famous Fredholm alternative. But first we give a lemma linking the eigenvalues of T to those of T^*.

LEMMA 4.27. *The operators* T *and* T^* *have the same non-zero eigenvalues and the same number of linearly independent eigenvectors corresponding to each non-zero eigenvalue.*

THEOREM 4.28. *(The Fredholm alternative)*
Let X *be a normed linear space, and consider the equations*

$$Tx - \lambda x = f \tag{4.4}$$

in X *and*

$$T^*y^* - \lambda y^* = g^* \tag{4.5}$$

in X^*. *Then one of the following two statements is true.*

(i) *There exists a uniquely determined solution of* (4.4) *for each* $f \in X$ *and of* (4.5) *for each* $g^* \in X^*$.

(ii) *The homogeneous equations*

$$Tx - \lambda x = 0 \tag{4.6}$$

and

$$T^*y^* - \lambda y^* = 0 \tag{4.7}$$

have a finite and equal number, say m, *of linearly independent solutions. In this case* (4.4) *has a solution (not uniquely determined) if and only if* $f \in {}^{\perp}N(T^*-\lambda I^*)$, *and* (4.5) *has a solution if and only if* $g^* \in N(T-\lambda I)^{\perp}$.

PROOF. See BROWN & PAGE [2, p.254] and LJUSTERNIK & SOBOLEW [1, p.185]. Note that part of this theorem is contained in theorem 4.17. The new part can be expressed rather abstractly by the formulas

$$R(T-\lambda I) = {}^{\perp}N(T^*-\lambda I^*) \tag{4.8}$$

and

$$R(T^*-\lambda I^*) = N(T-\lambda I)^{\perp}. \tag{4.9}$$

□

REMARK 4.29. Fredholm was the first to formulate a theorem like the one above. He considered a certain type of integral equations, now known as the Fredholm integral equations. We shall give an idea of the meaning of the previous theorem by illustrating its contents for this type of integral equations, i.e.,

$$x(t) = y(t) + \mu \int_0^1 K(t,s)x(s)ds, \qquad \mu \neq 0, \tag{4.10}$$

where K(t,s) is a complex valued continuous function in both variables.

Denote the integral operator by K. From example 4.7 we know that $K: C^0([0,1]) \to C^0([0,1])$ is a compact linear operator. Since the dual space of $C^0([0,1])$ is relatively complicated, for the sake of simplicity we wish to consider the equation (4.10) in $L^2([0,1])$, the dual of which is $L^2([0,1])$ itself. It can be shown in a similar manner as in example 4.7 (see e.g. LJUSTERNIK & SOBOLEW [1, p.182]) that K is also compact when considered as an operator of $L^2([0,1])$ into itself. From example 3.7 we take that the adjoint of K is

$$(4.11) \qquad K^*z(t) = \int_0^1 \overline{K(s,t)}z(s)ds.$$

Applying theorem 4.27, we find that either the equation (4.10) and the adjoint equation

$$(4.12) \qquad x(t) = y(t) + \mu \int_0^1 \overline{K(s,t)}x(s)ds$$

both have for each $y \in L^2([0,1])$ a unique solution $x \in L^2([0,1])$, or that there are a finite number k of linearly independent solutions of the two homogeneous equations

$$(4.13) \qquad x(t) = \mu \int_0^1 K(t,s)x(s)ds$$

and

$$(4.14) \qquad x(t) = \mu \int_0^1 \overline{K(s,t)}x(s)ds,$$

in which case equation (4.10) has a solution if and only if

$$(4.15) \qquad (y,x_i) = \int_0^1 y(s)\overline{x_i}(s)ds = 0$$

for all solutions $x_i(t)$, $i = 1,\ldots,k$ of (4.13).

We close this section with some notes on projections in a normed linear space X.

DEFINITION 4.30. A *projection* on X is a linear operator P such that $P^2 = P$.

LEMMA 4.31. *Let* P *be a projection on* X, *and write* $Q = I - P$. *Then*

(i) Q *is a projection on* X;

(ii) $R(P) = \{x \in X \mid Px = x\}$;

(iii) $R(P) = N(Q)$;

(iv) $X = R(P) \oplus R(Q)$;

(v) *if* P *is bounded then* $R(P)$ *and* $R(Q)$ *are closed.*

THEOREM 4.32. *Let* X *be a Banach space and let* M *and* N *be closed subspaces of* X *with* $X = M \oplus N$. *Then there is a unique projection* P *on* X *such that* $R(P) = M$ *and* $R(Q) = N$, *where* $Q = I - P$.

PROOF. See for proofs of this theorem and the preceding lemma BROWN & PAGE [2, p.336]. This theorem uses the so-called *closed graph theorem.* □

The reason for introducing projections will become clear in the following application.

COROLLARY 4.33. *Let* T *be a compact linear operator on a Banach space* X, *and let* $\lambda \in \Phi$. *Then there exist uniquely determined projections* P *and* $Q = I - P$ *such that* $R(P) = N((T-\lambda I)^{r(\lambda)})$ *and* $R(Q) = R((T-\lambda I)^{r(\lambda)})$. *Furthermore, for all* $x \in X$ *there holds*

$$TPx = PTx \quad \textit{and} \quad TQx = QTx.$$

PROOF. Lemma 4.16 provides the necessary closedness of $N((T-\lambda I)^{r(\lambda)})$ and of $R((T- I)^{r(\lambda)})$. □

5. FRÉCHET DIFFERENTIAL CALCULUS

In chapters III and IV the derivative f'(a) or Df(a) of a mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ at a point a has been introduced by defining

$$f'(a) = Df(a) = \begin{pmatrix} \frac{\partial f_1(a)}{\partial x_1} & \cdots & \frac{\partial f_1(a)}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_n(a)}{\partial x_1} & \cdots & \frac{\partial f_n(a)}{\partial x_m} \end{pmatrix}. \tag{5.1}$$

Here it is assumed that all components of $f = (f_1,\ldots,f_n)$ have first partial derivatives. Recall that the matrix representing Df(a) is known as the *Jacobian matrix* of the mapping at a, often denoted by J(a). We observe that Df(a) is a bounded linear mapping that associates to each $v \in \mathbb{R}^m$ an element $J(a)v = Df(a)\cdot v$ in $\mathbb{R}^n$. So $Df(a) \in B(\mathbb{R}^m,\mathbb{R}^n)$, where the notation B(X,Y) has the meaning given in section 2.

If the Jacobian matrix exists in each point x in a certain region $\Omega \subset \mathbb{R}^m$, then for all $x \in \Omega$ one can consider the mapping Df(x), and one can also give a sense to Df. In interpreting Df, we must keep in mind that for each $x \in \Omega$, Df(x) is a bounded linear mapping of $\mathbb{R}^m$ into $\mathbb{R}^n$. Consequently, Df is a (not necessarily linear) mapping of Ω into the space of bounded linear operators of $\mathbb{R}^m$ into $\mathbb{R}^n$. If all partial derivatives in (5.1) are continuous on Ω, Df will be a continuous mapping $\Omega \to B(\mathbb{R}^m,\mathbb{R}^n)$. Denoting the set of all continuous (not necessarily linear) mappings of X into Y by C(X,Y), one can express this fact by $Df \in C(\Omega,B(\mathbb{R}^m,\mathbb{R}^n))$.

Perhaps it is clarifying to relate the derivative of vector valued function to the concept of *differential* from classical analysis. The i-th component of Df(a)·h is exactly equal to the differential

$$df_i(a,h) = \frac{\partial f_i(a)}{\partial x_1} h_1 + \ldots + \frac{\partial f_i(a)}{\partial x_m} h_m \tag{5.2}$$

where $h = (h_1,\ldots,h_m)$. Usually, one writes $dx = (dx_1,\ldots,dx_m)$ instead of $h = (h_1,\ldots,h_m)$.

Finally, observe that from (5.2) there follows

$$\|f(a+h) - f(a) - Df(a)\cdot h\| \leq \rho(\|h\|,a), \tag{5.3}$$

where $\rho(\|h\|,a)$ satisfies the condition

$$\rho(\|h\|,a)/\|h\| \to 0 \quad \text{as} \quad \|h\| \to 0.$$

Formula (5.3) will be the clue for a generalization of the concept of derivative of an operator of a normal linear space into a normed linear space, in the case that the spaces are infinite dimensional.

Throughout this section X and Y will be normed linear spaces, A an open subset of X and f a mapping of A into Y.

NOTATION 5.1. Let $\rho(t,x)$ be a non-negative real valued function on $[0,\infty) \times A$. The notation

$$\rho(t,x) = o(t) \text{ as } t \downarrow 0$$

will mean that

$$\lim_{t\downarrow 0} \rho(t,x)/t = 0$$

for all $x \in A$. The limit does not have to be uniform in x.

DEFINITION 5.2. A mapping $f: A \to Y$ is said to be *differentiable at a point* $a \in A$ if and only if there is a bounded linear mapping $T: X \to Y$ having the property

$$\|f(a+h) - f(a) - Th\| = o(\|h\|) \text{ as } \|h\| \downarrow 0. \tag{5.4}$$

A mapping is called *differentiable on* A if it is differentiable in each point of A.

LEMMA 5.3. *If* f *is differentiable at* a, *then the mapping* T *occurring in* (5.4) *is uniquely determined.*

PROOF. For the proof of this lemma and the following theorem, see BROWN & PAGE [2, chapter 7], and LJUSTERNIK & SOBOLEW [1, pp.308-313]. □

DEFINITION 5.4. Let f be differentiable at $a \in A$. The unique linear operator T of X into Y satisfying (5.4) is called the *derivative of* f *at* a, and is denoted by Df(a) or f'(a). Often, this derivative is called the *Fréchet derivative*.

THEOREM 5.5. *Assume that* f *is differentiable at* a $\in$ A. *Then* f *is continuous in* a.

In some books, e.g. BROWN & PAGE [2], the condition that T in (5.4) is to be bounded is not included in definition 5.2. It can be shown that continuity of f and (5.4) imply that the mapping T in (5.4) is bounded.

THEOREM 5.6. *If the Fréchet derivative of* f: A $\to$ Y *exists at a point* a $\in$ A, *then for all* h $\in$ X *also the limit*

$$\lim_{t\to 0} \frac{f(a+th) - f(a)}{t} = \frac{d}{dt} f(a+th)\Big|_{t=0} \tag{5.5}$$

exists, the limit being taken for real t. *Furthermore, the limit* (5.5) *is equal to the effect of the Fréchet derivative at* a *on* h, *i.e.,*

$$\lim_{t\to 0} \frac{f(a+th) - f(a)}{t} = Df(a)\cdot h \tag{5.6}$$

In (5.5) and (5.6) convergence in the norm of Y is meant.

PROOF. LJUSTERNIK & SOBOLEW [1], p.310. □

DEFINITION 5.7. Consider a continuous and Fréchet differentiable mapping f: A $\to$ Y. Then f is said to be continuously differentiable on A if the mapping

$$A \to B(X,Y)$$

$$a \mapsto Df(a)$$

is continuous on A. Recall that here B(X,Y) is a normed linear space.

REMARK 5.8. If T is a bounded linear operator, i.e. T $\in$ B(X,Y), then the Fréchet derivative of T at any point x $\in$ X is T itself. This follows trivially from the definition. Note that it is confusing to write this statement as T = DT, since T $\in$ B(X,Y), and DT is the mapping of X into B(X,Y), associating to each x $\in$ X the mapping T. The correct notation, of course, is DT(x)·h = Th for all x and h $\in$ X, or if one wishes, DT(x)· = T·.

After having succeeded in extending the concept of the first derivative to operators on infinite dimensional spaces, we wish to pass on to the higher order derivatives. Again, in order to get an impression of what is involved, we first take a look at the case of mappings on finite dimensional spaces.

Recall that the *second order differential* d^2f of f: $\mathbb{R}^m \to \mathbb{R}$ is defined as a function of two m-dimensional variables, for points x where the second

order partial derivatives of f exist, and for all $h \in \mathbb{R}^m$, by the equation

$$(5.7) \qquad d^2f(a,h) = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 f(a)}{\partial x_i \partial x_j} h_j h_i$$

where $h = (h_1,\ldots,h_m)$. Again, one usually writes dx_i for h_i.

Considering the partial derivatives as the entries of an m×m-matrix M, one can also write

$$(5.8) \qquad d^2f(a,h) = h \cdot Mh,$$

so that the second order differential of a function on $\mathbb{R}^m$ can be seen as a quadratic form in h. In the same fashion higher order differentials $d^pf(a,h)$ are defined.

So $d^2f(a,\cdot)$ maps $h \in \mathbb{R}^m$ onto $h \cdot Mh$ in $\mathbb{R}$. From a slightly different point of view, $d^2f(a,\cdot)$ may be conceived as a special case of a bilinear mapping

$$\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$$

$$(h,k) \mapsto h \cdot Mk.$$

This mapping, which is usually written as $D^2f(a)$, is called the second derivative of f at a. So we have $d^2f(a,h) = D^2f(a) \cdot (h,h)$. Higher order derivatives are defined in quite the same fashion, namely as a p-linear mapping $D^pf(a)$ of $\prod_{i=1}^{p} \mathbb{R}^m$ into $\mathbb{R}$. This idea will now be generalized for the case that $\mathbb{R}^m$ and $\mathbb{R}$ are replaced by two arbitrary normed linear spaces.

NOTATION 5.9. By writing

$$\rho(t,x) = o(t^k) \quad \text{as } t \downarrow 0$$

$\rho(t,x)$ being a non-negative function defined on $[0,\infty) \times A$, we mean that

$$(5.9) \qquad \lim_{t \downarrow 0} \rho_k(t,x)/t^k = 0$$

for all $x \in A$; uniformity of the limit does not have to occur.

DEFINITION 5.10. The *second order derivative* of a mapping $f: X \to Y$ at a point $a \in A \subset X$ is defined, if it exists, as the bilinear mapping $D^2f(a)(\cdot,\cdot)$ of $X \times X$ into Y, that satisfies the inequality

$$(5.10) \qquad \|f(a+h) - f(a) - Df(a) \cdot h - \tfrac{1}{2}D^2f(a) \cdot (h,h)\| = o(\|h\|^2) \text{ as } \|h\| \downarrow 0.$$

Proceeding in this manner, the *m-th order derivative* at a is defined as the m-linear mapping $D^m f(a)(\cdot,\ldots,\cdot)$ of $\prod_{i=1}^{m} X$ into Y such that

$$\|f(a+h) - f(a) - Df(a)\cdot h - \tfrac{1}{2}D^2 f(a)\cdot(h,h) - \ldots - \frac{1}{m!}D^m f(a)\cdot(h,\ldots,h)\| = o(\|h\|^m) \text{ as } \|h\| \to 0. \tag{5.11}$$

The reader will recognize the familiar Taylor series. If all m derivatives depend continuously on a, a varying over a certain open subset A of X, then the mapping is said to be m times continuously differentiable on A.

REMARK 5.11. Higher order derivatives can be obtained by differentiating those of lower order, as one might expect. Let $f: A \to Y$ have a first order derivative Df(a) for all a in a certain open subset $A \subset X$. Consider Df as a mapping of A into B(X,Y). If Df has a derivative at a, then there must hold

$$\|Df(a+h) - Df(a) - D^2 f(a)h\| = o(\|h\|) \text{ as } \|h\| \to 0, \tag{5.12}$$

where h is an element of X such that $a + h \in A$, and the norm on the left hand side of (5.12) is taken in B(X,Y). One can write the left hand side of (5.12) in the form

$$\|Df(a+h)\cdot - Df(a)\cdot - D^2 f(a)h\cdot\| = \sup_{\|k\|=1} \|Df(a+h)\cdot k - Df(a)\cdot k - D^2 f(a)\cdot h\cdot k\|,$$

since all operators occurring are linear mappings of X into Y. So (5.12) is equivalent to

$$\|Df(a+h)\cdot k - Df(a)\cdot k - D^2 f(a)\cdot h\cdot k\| = \|k\|\, o(\|h\|) \text{ as } \|h\| \to 0. \tag{5.13}$$

From this formula it follows that $D^2 f(a)$ is a bilinear mapping of $X \times X$ into Y if Df(a) is differentiable on A.

Higher order derivatives can be obtained by proceeding in this manner. Often, this is the way one actually calculates higher order derivatives of operators on normed linear spaces.

THEOREM 5.12. *Let* f *be a* m + 1 *times continuously Fréchet differentiable*

operator of $A \subset X$ *into* Y. *Let* a, $a + h$ *and each element of the form* $a + th$, $0 < t < 1$, *belong to* A. *Then one has*

$$(5.14) \qquad f(a+h) - f(a) = \sum_{k=1}^{m} \frac{1}{k!} D^k f(a) \cdot (\underbrace{h,\ldots,h}_{k \text{ times}}) + R$$

where

$$\|R\| \leq \frac{1}{(m+1)!} \sup_{0 \leq t \leq 1} \|D^{m+1} f(a+th)\| .$$

PROOF. See COLLATZ [4, p.223]. The proof makes use of the *Hahn-Banach theorem* for functionals on normed linear spaces in an essential manner, and of the Taylor formula for functionals $g: X \to \mathbb{C}$. □

Formula (5.14) becomes very familiar, if we consider the case $X = \mathbb{R}^n$, $Y = \mathbb{R}$, and if we write the result in terms of differentials,

$$f(a+h) - f(a) = \sum_{k=1}^{m} \frac{1}{k!} d^k f(a,h) + R$$

where

$$R = \frac{1}{(m+1)!} d^{m+1} f(a+th,h)$$

for some t.

The following theorem is the *chain rule* for Fréchet differentiable operators.

THEOREM 5.13. *Let* f *be a mapping of an open subset* A *of* X *into an open subset* B *of* Y, *and let* g *be a mapping of* B *into a normed linear space* Z. *Suppose that* f *is differentiable at a point* $a \in A$, *and that* f *is differentiable at the point* $b = f(a)$. *Then* $g \circ f$ *is differentiable at* a *and*

$$D(g \circ f)\,(a) = Dg(f(a))\,Df(a).$$

PROOF. See BROWN & PAGE [2, p.276]. □

Finally, in applications we shall need the following theorem on nonlinear compact operators.

THEOREM 5.14. *Let* f *be a non-linear compact operator of a normed linear space* X *into a normed linear space* Y. *If* f *is differentiable at* $a \in X$, *then the mapping* $Df(a): X \to Y$ *is compact.*

PROOF. KRASNOSEL'SKII [6, p.135]. □

6. PARTIAL DERIVATIVES; IMPLICIT FUNCTION THEOREM

Our next concern is to find a definition for the partial derivatives of a mapping of a Cartesian product of two normed linear spaces X_1 and X_2 into a normed linear space Y, that is compatible with the finite dimensional case. After having given an extension of the concept of derivative of a mapping to the situation in which the spaces involved are infinite dimensional, we may expect not to come across any difficulties when extending the notion of partial derivative.

Throughout this section X_1, X_2 and Y will be normed linear spaces over the scalar field Φ. It is easily verified that the set $X_1 \times X_2$ of all ordered pairs (x_1,x_2) with $x_1 \in X_1$ and $x_2 \in X_2$ is a linear space with respect to the operations defined by

$$(6.1) \qquad (x_1,x_2) + (y_1,y_2) = (x_1+y_1,x_2+y_2),$$
$$\alpha(x_1,x_2) = (\alpha x_1,\alpha x_2)$$

for all (x_1,x_2) and $(y_1,y_2) \in X_1 \times X_2$ and all $\alpha \in \Phi$. Without any trouble one also sees that the mapping

$$(6.2) \qquad (x_1,x_2) \to \|(x_1,x_2)\| = \max(\|x_1\|,\|x_2\|)$$

defines a norm on $X_1 \times X_2$.

In this section we shall further always assume that the Cartesian product $X_1 \times X_2$ is endowed with the linear space operations (6.1) and the norm (6.2).

Let f be a mapping of a non-empty open subset A of $X_1 \times X_2$ into Y. Choose $a = (a_1,a_2) \in A$, and define $A_1 = \{x_1 \in X_1 \mid (x_1,a_2) \in A\}$ (see figure 1), which set is clearly open in X_1. Consider the mapping $g(x_1) = f(x_1,a_2)$ for all $x_1 \in A_1$.

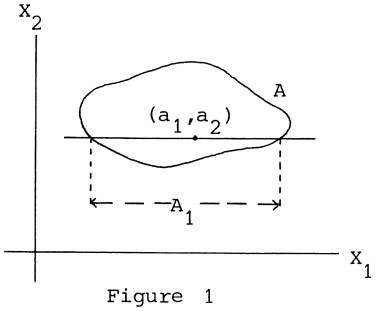


Figure 1

DEFINITION 6.1. The mapping f of $A \subset X_1 \times X_2$ into Y is said to be *differentiable with respect to the first variable at the point* (a_1,a_2) if and only if $g(x_1) = f(x_1,a_2)$ is differentiable at a_1. In that case we write $Dg(a) = D_1f(a_1,a_2)$. Naturally, $D_1f(a_1,a_2)$ is called the partial derivative of f with respect to the first variable at (a_1,a_2); it is a linear mapping of X_1 into Y. The mapping f is called *differentiable with respect to the first variable on* A, if it is differentiable with respect to the first variable at each point of A.

Differentiability with respect to the second variable at (a_1,a_2), and the partial derivative with respect to the second variable, $D_2f(a_1,a_2)$, are defined in a similar manner.

We now give some properties of the partial derivative.

THEOREM 6.2. *Assume that* $f: A \to Y$, $A \subset X_1 \times X_2$ *is partial differentiable with respect to the first (second) variable at* (a_1,a_2). *Then* $D_1f(a_1,a_2)$ $(D_2f(a_1,a_2))$ *is a bounded transformation of* $X_1(X_2)$ *into* Y *if and only if* f *is continuous with respect to the first (second) variable in* (a_1,a_2).

PROOF. Follows immediately from theorem 5.5, by considering $g(x_1) = f(x_1,a_2)$ on A_1. □

EXAMPLE 6.3. In the case that $X_1 = X_2 = Y = \mathbb{R}$, that is when f is a real-valued function of two variables, the partial derivative $D_1f(a_1,a_2)$ exists if and only if the usual partial derivative $\partial f(a_1,a_2)/\partial x_1$ exists. One then has

$$D_1f(a_1,a_2)\cdot h_1 = \frac{\partial f}{\partial x_1}(a_1,a_2)h_1.$$

Note, that on the right-hand side the normal scalar multiplication is meant, whereas on the left-hand side the notation stands for the application of the mapping $D_1f(a_1,a_2): \mathbb{R} \to \mathbb{R}$ to the element $h_1 \in \mathbb{R}$.

The following theorem is a straightforward generalization of a well-known fact from standard analysis.

THEOREM 6.4. *Consider a mapping* f *of an open subset* A *of* $X_1 \times X_2$ *into* Y, *that is differentiable at a point* $(a_1,a_2) \in A$. *Then* f *is differentiable with respect to both variables at* (a_1,a_2) *and*

(6.3) $$Df(a_1,a_2)\cdot(h_1,h_2) = D_1f(a_1,a_2)\cdot h_1 + D_2f(a_1,a_2)\cdot h_2$$

for all $(h_1,h_2) \in X_1 \times X_2$. *Moreover, if* f *is continuously differentiable on* A, *then the mappings* $(x_1,x_2) \to D_1f(x_1,x_2)$ *and* $(x_1,x_2) \to D_2f(x_1,x_2)$ *are continuous mappings of* A *into* $B(X_1,Y)$ *and* $B(X_2,Y)$ *respectively.*

PROOF. See BROWN & PAGE [2, p.283]. □

REMARK 6.5. Observe the close relationship between (6.3) and the expression given for differentials (5.2). In the special case that f: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ $Df(a_1,a_2)\cdot(h_1,h_2)$ may be considered as the (total) differential of f, $df(a_1,a_2; h_1,h_2)$.

THEOREM 6.6. *Let* f: $A \to Y$, $A \subset X_1 \times X_2$ *and* A *open, be a continuous mapping on* A, *that is continuously differentiable with respect to both variables on* A. *Then* f *is continuously differentiable on* A.

PROOF. BROWN & PAGE [2, p.284]. Remember that continuous differentiability with respect to the first variable on A means that the mapping $D_1f: A \to B(X_1,Y)$ is continuous, and that continuous differentiability of f on A means that $Df: A \to B(X_1 \times X_2,Y)$ is continuous. □

Having introduced partial derivatives, we are in the position to state the generalized implicit function theorem for infinite dimensional spaces. But before doing so, we first mention a theorem which is needed in the proof of the implicit function theorem, but which also has significance for its own sake, the contraction mapping principle.

DEFINITION 6.7. Let X be a normed linear space, and A a non-empty subset of X. A mapping f: $A \to A$ is a *contraction mapping* on A if there exists a real number λ, $0 \leq \lambda < 1$, such that

$$\|f(x) - f(y)\| \leq \lambda\|x - y\|$$

for all $x, y \in A$.

THEOREM 6.8. *(Contraction mapping principle)*
Let A *be a closed subset of a Banach space* X, *and let* f: $A \to A$ *be contraction mapping on* A. *Then* f *has a unique fixed point* $\bar{x}$ *in* A, *i.e., there is a unique* $\bar{x} \in A$ *such that*

$$f(\bar{x}) = \bar{x}.$$

Also, if $x_0 \in A$ *is chosen arbitrarily, then the sequence* (x_n), $x_{n+1} = Tx_n$, $n = 0,1,2,\ldots$, *converges to* $\bar{x}$ *as* $n \to \infty$.

PROOF. LJUSTERNIK & SOBOLEW [1, p.26] or BROWN & PAGE [2, p.76]. □

In the next chapter, when discussing certain bifurcations of solutions, this principle will be employed. Here we close with the important implicit function theorem, of which use we shall also give an illustration.

THEOREM 6.9. *(Implicit function theorem)*
Let Y *be a real normed linear space,* X *and* Z *real Banach spaces. Let further* f *be a continuous mapping of an open subset* A *of* $X \times Y$ *into* Z. *Assume that*

(i) f *is continuously differentiable with respect to both variables on* A;

(ii) $f(a,b) = 0$ *for some point* $(a,b) \in A$; *and*

(iii) $D_1 f(a,b)$ *is a linear continuous mapping of* X *onto* Z *that is one-to-one (according to theorem* 2.11 *an equivalent assumption is that* $D_1 f(a,b)$ *has a continuous inverse).*

Then there exist neighbourhoods $U = \{x \in X \mid \|x - a\| < \delta\}$ *and* $V = \{y \in Y \mid \|y - b\| < \varepsilon\}$ *such that*

(a) $D_1 f(x,y)$ *has a bounded inverse for all* $x \in U$ *and* $y \in V$;

(b) *for each fixed* $y \in V$ *the equation* $f(x,y) = 0$ *has a unique solution* x *in* U;

(c) *this solution can be given as* $x = u(y)$, *where* u *is continuously differentiable on* V; *and*

(d) $u(b) = a$.

PROOF. See LJUSTERNIK & SOBOLEW [1, p.333], BROWN & PAGE [2, pp.286-293], or DIEUDONNÉ [7, section 10.2]. □

REMARK 6.10. The proof shows that $Du(y)$ can be written as

$$Du(y) = -[D_1 f(u(y),y)]^{-1} D_2 f(u(y),y) \quad \text{for all} \quad y \in V.$$

We hope that the following example gives an illustration how the implicit function theorem for Banach spaces is used. The example is typical for bifurcation theory to be developed in the next chapter.

EXAMPLE 6.11. Consider the non-linear initial boundary value problem

$$(6.4)\qquad \begin{cases} \dfrac{\partial u(x,t)}{\partial t} = \lambda f(u) + \dfrac{\partial^2 u(x,t)}{\partial x^2}, & 0 < x < 1,\ t > 0,\ \lambda \in \mathbb{R}, \\ u(0,t) = u(1,t) = 0, & t > 0, \\ u(x,0) = g(x), & 0 < x < 1. \end{cases}$$

The steady states of this problem are given by the two-point boundary value problem

$$(6.5)\qquad \begin{cases} \lambda f(v) + \dfrac{d^2 v}{dx^2} = 0, & 0 < x < 1,\ \lambda \in \mathbb{R}, \\ v(0) = v(1) = 0. \end{cases}$$

Here we assume that v is a real-valued and twice continuously differentiable function on $[0,1]$, and that f is a continuous and continuously Fréchet differentiable mapping of $C^0 = C^0([0,1])$ into itself. We rewrite this problem into an integral equation by constructing the Green's function for the differential operator $-d^2/dx^2$, that is a continuous function satisfying

$$(6.6)\qquad \begin{cases} -\dfrac{d^2}{dx^2} G(x,y) = \delta(x-y), & 0 < x, y < 1, \\ G(0,y) = G(1,y) = 0, & 0 < y < 1. \end{cases}$$

It is left to the reader to perform the calculations leading to the result

$$(6.7)\qquad G(x,y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y, \\ y(1-x) & \text{if } y \le x \le 1. \end{cases}$$

Observe that $G(x,y) = G(y,x)$. Define the integral operator

$$(6.8)\qquad Kw(x) = \int_0^1 G(x,y)w(y)\,dy.$$

Since G is continuous, it follows from example 4.7 that K is a compact operator on $C^0([0,1])$. Now problem (6.5) is equivalent to the non-linear integral equation

$$v(x) = \lambda \int_0^1 G(x,y)f(v(y))dy \tag{6.9}$$

or

$$v = \lambda Kf(v). \tag{6.10}$$

If we write L for the compound non-linear mapping Kf, then (6.10) can be formulated as

$$F(v,\lambda) = v - \lambda L(v) = 0, \tag{6.11}$$

where $F: C^0 \times \mathbb{R} \to C^0$ is continuous and continuously differentiable with respect to both variables.

We have to make one more assumption about f, namely that $f(0) \neq 0$. This means that we exclude $v \equiv 0$ as a steady state if $\lambda \neq 0$. From (6.11) it follows that $F(0,0) = 0$. Our aim is to apply the implicit function theorem in neighbourhood of (0,0). For this purpose, we must examine the mapping $D_1F(0,0)$. Recall that for linear operators T $DT(x)\cdot = T\cdot$. Since K and I are linear, we have

$$D_1F(v,\lambda)w = [I - \lambda D(Kf(v))]w = [I - \lambda KDf(v)]w, \quad w \in C^0. \tag{6.12}$$

The second equality is a consequence of the chain rule. From (6.12) it follows that $D_1F(0,0)$ has a bounded inverse on C^0. Application of the implicit function theorem leads to the following result. Equation (6.11), and equivalently the boundary value problem (6.5), has a unique continuously differentiable solution $v = v(\lambda)$ in a neighbourhood of $\lambda = 0$ with $v(0) = 0$.

Now, let us suppose that by some means we know that (v_0,λ_0) solves equation (6.11) for some $\lambda_0 \neq 0$, i.e., $v_0 - \lambda_0 Kf(v_0) = 0$. Because K is a compact linear operator, and for each fixed $v \in C^0$, Df(v) is a bounded linear operator, according to theorem 4.4 $KDf(v)\cdot$ is compact itself. Therefore, the Riesz-Schauder theory can be applied. Assume first that λ_0^{-1} is not an eigenvalue of $KDf(v_0)$. Then $D_1F(v_0,\lambda_0) = [I - \lambda_0 KDf(v_0)]\cdot$ is boundedly invertible, so that the third condition of the implicit function theorem is fulfilled. So there exist neighbourhoods $V = \{v \in C^0 \mid \|v - v_0\| < \delta\}$ and $\Lambda = \{\lambda \in \mathbb{R} \mid |\lambda - \lambda_0| < \varepsilon\}$ such that $v = v(\lambda)$ solves equation (6.11) for all $\lambda \in \Lambda$, with $v(\lambda_0) = v_0$, $v \in V$, and $v(\lambda)$ continuously differentiable on Λ.

The discussion of the case where λ_0^{-1} is an eigenvalue of $KDf(v_0)$ will be postponed to the next chapter. It is here that the domain of bifurcation theory begins.

This section is concluded by an extension of the implicit function theorem.

THEOREM 6.12. *Let the hypotheses of theorem* 6.9 *hold, and additionally, let* f *be* p *times continuously differentiable in the open set* A, (a,b) $\in$ A. *Then the solution* x = u(y) *obtained by the implicit function theorem is* p *times continuously differentiable on the neighbourhood* V *of* b.

PROOF. By induction on p. See DIEUDONNÉ [7, section 10.2]. □

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VII. BIFURCATION THEORY FOR OPERATORS ON A BANACH SPACE

The functional analysis presented in the preceding chapter now will be applied to extend the main results of chapter IV to mappings on Banach spaces. In this chapter we exclusively will derive results based on linearization and perturbation analysis. The degree theory and its applications in the area of bifurcation theory are the subjects of the next chapter.

Globally the same chain of arguments as used in chapter IV leads to similar theorems. It should be noted however that the greater generality as concerned to the domain of the mappings has to be compensated by a restriction of the class of mappings, about as severe as to the class of compact perturbations of the identity. That the theory nevertheless is useful comes from the fact that a lot of applications to partial differential equations, especially to nonlinear boundary value problems, lead to this kind of mappings.

Besides to the derivation of some general theory, we will pay attention to example VI.6.11, in the first place to apply the preceding theory, and secondly because it furnishes a good example indicating how one in general proceeds to determine whether the branching solutions are stable or not.

All results presented in this chapter are known in literature. As references we mention the books of SATTINGER [1] and PIMBLEY [2], the article of STAKGOLD [3], and the references given in these expositions.

1. FORMULATION OF THE PROBLEM

Several problems from analysis and mathematical physics lead to the consideration of the operator equation

$$F(x,\lambda) = 0, \tag{1.1}$$

for continuous mappings $F: B \times \mathbb{R} \to B$, where B is a Banach space.

The parameter λ in equation (1.1) can be present for several reasons. On the one hand, equation (1.1) originates from a model of a physical process in which one (or more) parameter(s) are variable (or only known with restricted accuracy). On the other hand, from a mathematical point of view, it is often convenient to consider (1.1) for varying values of λ to detect phenomena such as multiple steady states or to start with a simple equation, say for $\lambda = 0$, and gradually transform it into the "real" equation, say for $\lambda = 1$ (Poincaré continuation, see KELLER [4]).

For several reasons, which partially can be retraced from the above remarks, one is interested in the local and global structure in $B \times \mathbb{R}$ of the solution set

$$F^{-1}(0) = \{(x,\lambda) \in B \times \mathbb{R} \mid F(x,\lambda) = 0\}. \tag{1.2}$$

Especially the following questions are subject of study in bifurcation theory:

- *Which conditions to be satisfied by* F *guarantee that a solution* (x_0,λ_0) *of* (1.1) *lies on a unique simple curve* $(x(\lambda),\lambda)$, $\lambda_0 - \delta < \lambda < \lambda_0 + \delta$?
- *Which conditions guarantee that the solution* (x_0,λ_0) *of* (1.1) *lies on several solution curves* $(x_0(\lambda),\lambda)$, $(x_1(\lambda),\lambda)$,...?

The first of these problems is largely solved by

THEOREM 1.1. *Let* $F: B \times \mathbb{R} \to B$ *be* k*-times,* $k \geq 1$, *continuously differentiable and let* (x_0,λ_0) *be such that*

(i) $F(x_0,\lambda_0) = 0$;

(ii) *the operator* $D_1F(x_0,\lambda_0)$ *has a bounded inverse on* B.

Then there exist numbers ε, $\delta > 0$ *with the properties*

(1) *for every* λ *with* $|\lambda - \lambda_0| < \delta$ *there exists an unique* $x(\lambda)$ $|x - x_0| < \varepsilon$ *such that*

$$F(x(\lambda),\lambda) = 0;$$

(2) *the function* $x: (\lambda_0-\delta, \lambda_0+\delta) \to B$ *defined by*

$$\lambda \to x(\lambda)$$

is k*-times continuously differentiable.*

PROOF. All conditions of the implicit function theorems VI.6.9 and VI.6.12 are satisfied. □

REMARK 1.2. The statement of the theorem can be rephrased as: The intersection of $F^{-1}(0)$ with the "cylinder" $B(x_0,\varepsilon) \times (\lambda_0-\delta, \lambda_0+\delta)$ is the simple curve given by $\lambda \to (x(\lambda),\lambda)$ (see figure 1).

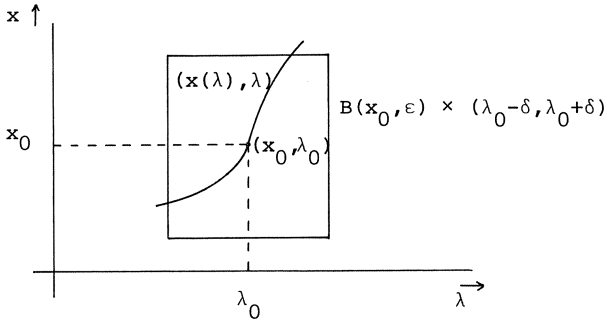

Figure 1

REMARK 1.3. We consider the case that the mapping F has the special form

$$F(x,\lambda) = x + G(x,\lambda), \tag{1.3}$$

with $G(\cdot,\lambda)$ a compact mapping on B. This type of operators occurs frequently in studies of nonlinear boundary value problems (cf. VI.6.11). According to theorem VI.5.14 the linear mapping

$$D_1G(x,\lambda)$$

is compact. Now we are in the position to apply the Riesz-Schauder theory (see section VI.4) to the operator

$$D_1F(x,\lambda) = I + D_1G(x,\lambda),$$

with the effect that we can replace condition (ii) of theorem 1.1 by

(ii)' *All eigenvalues of* $D_1F(x,\lambda)$ *are unequal to zero.*

REMARK 1.4. The conditions (i) and (ii) of theorem 1.1 are sufficient but by no means necessary conditions to prevent bifurcation.

In the next section we will consider the case that condition (ii) is not fulfilled.

2. THE LYAPUNOV-SCHMIDT METHOD

From now on we suppose that a "trivial" branch $(x_0(\lambda),\lambda)$ of solutions of equation (1.1) is known. Without any restriction we may assume that $x_0(\lambda) = 0$ or, equivalently, that the equality

$$F(0,\lambda) = 0, \qquad \lambda \in \mathbb{R}, \tag{2.1}$$

holds. (If F does not satisfy (2.1), we can consider the equivalent equation $H(y,\lambda) = F(x_0(\lambda)+y,\lambda) = 0$.)

According to theorem 1.1 we only can expect bifurcation (see definition IV.2.1) from this trivial branch for those values λ_0 for which the linear mapping

$$L = D_1F(0,\lambda_0)$$

is not boundedly invertible.

For operators of the form (1.3) the mapping L is equal to

$$L = I + D_1G(0,\lambda_0),$$

which is the kind of operators considered in section VI.4 (take $\lambda = -1$, $T = D_1G(0,\lambda_0)$). Theorem VI.4.17 and corollary VI.4.33 provide the following useful properties of L:

PROPERTIES 2.1. There exist a projection P (with finite dimensional range) and a projection $Q = I - P$ such that

(a) $PL = LP, \qquad QL = LQ$,

(b) The linear mapping $\hat{L}: QB \to QB$ defined by

$$\hat{L}y = Ly, \qquad y \in QB$$

is boundedly invertible.

The properties 2.1 are equivalent (via $PB = B_1$, $QB = B_2$) to:

PROPERTIES 2.2.

(a) $B = B_1 \oplus B_2$, with B_1 and B_2 closed subspaces of B (B_1 finite dimen-

sional).

(b) $LB_1 \subset B_1$, $LB_2 \subset B_2$,

(c) The restriction L of the mapping L to B_2 is boundedly invertible, when considered as a mapping on B_2.

In the rest of this section we will show that the analysis of the equation (1.1) in a neighbourhood of $(0,\lambda_0)$ can be reduced to the study of a finite dimensional system of equations if we make the

HYPOTHESIS 2.3. The mapping

$$L = D_1F(0,\lambda_0)$$

has the properties given in properties 2.1 and, equivalently, in properties 2.2.

Now it is possible to regard F as a mapping $\tilde{F}$ of $B_1 \times B_2 \times \mathbb{R}$ in $B_1 \times B_2$ in the following way:

$$\tilde{F}\colon (x_1,x_2,\lambda) \to (PF(x_1+x_2,\lambda),\ QF(x_1+x_2,\lambda))$$

$$= (F_1(x_1,x_2,\lambda),\ F_2(x_1,x_2,\lambda)).$$

Equation (1.1) is similarly modified into

$$\tilde{F}(x_1,x_2,\lambda) = (F_1(x_1,x_2,\lambda),\ F_2(x_1,x_2,\lambda)) = (0,0). \tag{2.2}$$

That this is a useful viewpoint is shown by

THEOREM 2.4. *If* F *is* k*-times, k* ≥ 1, *continuously differentiable and if* $D_1F(0,\lambda_0)$ *satisfies properties* 2.1 *then there exist numbers* $\varepsilon, \delta > 0$ *such that*

(i) *for every* (x_1,λ) *with* $|\lambda - \lambda_0| + \|x_1\| < \delta$ *there exists a unique solution* $x_2(x_1,\lambda)$ *of* $F_2(x_1,x_2,\lambda) = 0$ *with* $\|x_2\| < \varepsilon$;

(ii) *the function* $x_2\colon B_1 \times \mathbb{R} \to B_2$ *defined by*

$$(x_1,\lambda) \to x_2(x_1,\lambda)$$

is k*-times continuously differentiable with* $x_2(0,\lambda_0) = D_1x_2(0,\lambda_0) =$

$= D_2x_2(0,\lambda_0) = 0$;

(iii) *if* $k = 1$ *then*

$$x_2(x_1,\lambda) = o(\|x_1\| + |\lambda-\lambda_0|),$$

and if $k \geq 2$ *then*

$$x_2(x_1,\lambda) = O(\|x_1\|^2 + |\lambda-\lambda_0|^2).$$

PROOF. Note that

$$D_2F_2(0,0,\lambda_0) = QD_1F(0,\lambda_0) \tag{2.3}$$

is an invertible operator on B_2 and that

$$D_1F_2(0,0,\lambda_0) = QD_1F(0,\lambda_0) \tag{2.4}$$

is the zero operator on B_1. (The equalities (2.3) and (2.4) are immediate consequences of properties 2.1.) In result of (2.1) we also have

$$D_3F_2(0,0,\lambda_0) = 0. \tag{2.5}$$

Now the statements (i), (ii) and (iii) of the theorem follow immediately from the implicit function theorem VI.6.12. □

REMARK 2.5. In applications one generally is not only interested in the existence of $x_2(x_1,\lambda)$ and estimations of it but above all in good approximations. These can be obtained by the iteration process

$$\begin{cases} Lx_2^0(x_1,\lambda) = -F(x_1,0,\lambda) \\ Lx_2^n(x_1,\lambda) = -F(x_1,x_2^{n-1},\lambda) + Lx_2^{n-1}, & n = 1,2,\ldots, \end{cases} \tag{2.6}$$

which is nothing else than an imitation of the constructive proof of the implicit function theorem.

By substituting this $x_2(x_1,\lambda)$ in (2.2) we arrive at our final result.

THEOREM 2.6. *There exist numbers* ε', $\delta' > 0$ *such that to every solution* (x,λ), *with* $\|x\| < \varepsilon'$, $|\lambda-\lambda_0| < \delta'$, *of* (1.1) *there corresponds a solution* $(x_1,\lambda) \in B_1 \times \mathbb{R}$ *of the equation*

$$PF(x_1+x_2(x_1,\lambda),\lambda) = 0. \tag{2.7}$$

PROOF. As the projection P is a bounded operator it follows that $\|x\| < \varepsilon'$ implies $\|Px\| < \frac{1}{2}\delta$, $\|Qx\| < \varepsilon$, with ε and δ as in theorem 2.2. If we now take $\delta' < \frac{1}{2}\delta$ theorem 2.4 implies

$$Qx = x_2(Px,\lambda),$$

which is in fact the statement of the theorem. □

REMARK 2.7. It is evident that every solution (x_1,λ) of equation (2.7) corresponds to a solution $(x_1+x_2(x_1,\lambda),\lambda)$ of equation (1.1). So it is possible to analyse the local structure of $F^{-1}(0)$ in a neighbourhood of $(0,\lambda_0)$ by considering the finite dimensional system of equations (2.7).

REMARK 2.8. This however is a very difficult problem to handle in the general case that the dimension d of B_1 (= range of P) exceeds one. In the next section we will consider the case d = 1.

3. BIFURCATION AT A SIMPLE EIGENVALUE

In this section it will be demonstrated that there generally bifurcates one "nontrivial" branch of solutions from the point $(0,\lambda_0)$ if the following hypothesis is fulfilled.

HYPOTHESIS 3.1. The dimension of $B_1 = PB$ equals one and the restriction of L to B_1 is the zero mapping on B_1.

REMARK 3.2. The requirement that $PL = 0$ is quite natural because otherwise the mapping L would be invertible which would make it possible to apply theorem 1.1.

In the next lemma we collect some useful information concerning B_1 and P.

LEMMA 3.3. *If* $\dim PB = \dim B_1 = 1$ *then*

(i) *there exist* $\phi \in B_1$ *and* $\phi^* \in B^*$ *(the dual of* B*) such that*

$$\langle \phi, \phi^* \rangle = 1, \qquad \|\phi\| = \|\phi^*\| = 1,$$

$$\langle y, \phi^* \rangle = 0, \qquad y \in B_2,$$

(ii) *the projections* P *and* Q *can be expressed in* ϕ *and* ϕ^*

$$Px = \langle x, \phi^* \rangle \phi,$$

$$Qx = x - \langle x, \phi^* \rangle \phi,$$

(iii) *if the operator* L *is a compact perturbation of the identity then* ϕ *respectively* ϕ^* *are the eigenvectors of norm one of* L *respectively* L^* *corresponding to the simple eigenvalue zero.*

PROOF. Note that every $x \in B$ has the unique representation

$$x = Px + Qx$$

$$= \alpha\phi + Qx.$$

Define ϕ^* by $\langle x, \phi^* \rangle = \alpha$. Now it is easily verified that ϕ, ϕ^*

satisfy (i) and (ii), whereas (iii) is a consequence of theorem VI.4.28. □

Now the "bifurcation" equation (2.7) is equivalent to the scalar equation

$$(3.1)\qquad \langle F(\alpha\phi+x_2(\alpha\phi,\lambda),\lambda),\phi^*\rangle = 0, \qquad (x_1 = \alpha\phi).$$

Writing

$$(3.2)\qquad \begin{cases} \lambda - \lambda_0 = \tau \\ x_2(\alpha\phi,\lambda) = z(\alpha,\tau) \end{cases}$$

we obtain

$$(3.3)\qquad g(\alpha,\tau) = \langle F(\alpha\phi+z(\alpha,\tau),\tau+\lambda_0),\phi^*\rangle = 0.$$

To obtain information concerning the local behaviour of the function $g(\alpha,\tau)$ the Taylor expansion

$$(3.4)\qquad \begin{aligned} \langle F(x,\lambda),\phi^*\rangle = \langle & F(0,\lambda_0) + D_1F(0,\lambda_0)x + D_2F(0,\lambda_0)(\lambda-\lambda_0) + \\ & + \tfrac{1}{2}\{D_2^2F(0,\lambda_0)(\lambda-\lambda_0)^2 + 2D_1D_2F(0,\lambda_0)(x,\lambda-\lambda_0) + \\ & + D_1^2F(0,\lambda_0)(x,x)\} + \ldots,\phi^*\rangle \end{aligned}$$

is used. Note that it follows from (2.1) and hypothesis (3.1) that the first four terms of (3.5) are zero.

Under the rather mild restriction that the following hypothesis is fulfilled, we can prove our next theorem.

HYPOTHESIS 3.4. The coefficient $\langle D_1D_2F(0,\lambda_0)(\phi,1),\phi^*\rangle$ is non-zero.

THEOREM 3.5. *If* F *is* k*-times,* $k \geq 2$, *continuously differentiable, then there exist positive numbers* ε, δ *such that*

(i) *for every* $|\alpha| < \delta$ *there exists a unique* $|\tau(\alpha)| < \varepsilon$ *with* $g(\alpha,\tau(\alpha)) = 0$,

(ii) *the function* $\alpha \to \tau(\alpha)$ *is* k*-times continuously differentiable.*

PROOF. By substitution of (3.2) in (3.4) we obtain the estimate

$$g(\alpha,\tau) = 0(\alpha).$$

Hence we may consider $h(\alpha,\tau) = \alpha^{-1}g(\alpha,\tau)$.

Using (3.4) and the estimate (iii) in theorem 2.4 we find (compare the proof of theorem IV.4.7)

$$h(0,0) = 0$$

and

$$\frac{\partial}{\partial\tau} h(0,0) = \langle D_1 D_2 F(0,\lambda_0)(\phi,1),\phi^*\rangle \neq 0 \qquad \text{(hypothesis 3.4)}.$$

Now the proof is completed by an application of the implicit function theorem VI.6.12. □

REMARK 3.6. The most convenient method to determine an asymptotic approximation ($\alpha \to 0$) of

$$(x(\alpha),\tau(\alpha)) = (\alpha\phi+z(\alpha,\tau(\alpha)),\tau(\alpha))$$

is by means of the Taylor series expansion which is justified by the differentiability of both z and τ, proved in theorem 2.4 and theorem 3.5. We put

$$\text{(3.5)} \qquad \begin{cases} \tau(\alpha) = \alpha\tau_0 + \alpha^2\tau_1 + \ldots + \alpha^N\tau_{N-1} + O(\alpha^{N+1}), \\ z(\alpha,\tau(\alpha)) = \alpha^2 z_1 + \alpha^3 z_2 + \ldots + \alpha^{N+1} z_N + O(\alpha^{N+2}). \end{cases}$$

Substitution of (3.5) in (3.3), using (3.4) and in $L_z = -\phi F(\alpha\phi+z(\alpha,\tau),\lambda_0+\tau)$ respectively, we get, by comparing terms of the same order in α, a series of 2N equations:

$$\text{(3.6)} \qquad \begin{cases} \langle D_1 D_2 F(0,\lambda_0)(\phi,1),\phi^*\rangle\tau_k = \\ = -\dfrac{1}{(k+2)!}\dfrac{d^{k+2}}{d\alpha^{k+2}}\langle F(\alpha(\phi+\alpha z_1+\ldots+\alpha^k z_k),\lambda_0+\alpha(\tau_0+\ldots+\alpha^{k-1}\tau_{k-1})),\phi^*\rangle, \\ \qquad\qquad (k=0,1,\ldots,N-1) \\ Lz_{k+1} = -\dfrac{1}{(k+2)!}\dfrac{d^{k+2}}{d\alpha^{k+2}} QF(\alpha(\phi+\alpha z_1+\ldots+\alpha^k z_k),\lambda_0+\alpha(\tau_0+\ldots+\alpha^k\tau_k)), \end{cases}$$

where the right-hand sides are evaluated at $\alpha = 0$. From the above equations we can successively determine the coefficients τ_k and v_{k+1}.

Explicitly we obtain for k = 0

$$(3.7) \qquad \langle D_1 D_2 F(0,\lambda_0)(\phi,1),\phi^*\rangle \tau_0 + \frac{1}{2!}\langle D_1^2 F(0,\lambda_0)(\phi,\phi),\phi^*\rangle = 0.$$

Supposing $\langle D_1^2 F(0,\lambda_0)(\phi,\phi),\phi^*\rangle \neq 0$, the local behaviour of the bifurcating branch of solutions follows from (3.7)

$$x = \alpha\phi + x_2(\alpha\phi,\lambda_0+\tau(\alpha)) \sim \alpha\phi$$

and

$$\tau(\alpha) \sim \alpha\tau_0.$$

In figure 2 we give a sketch of the situation.

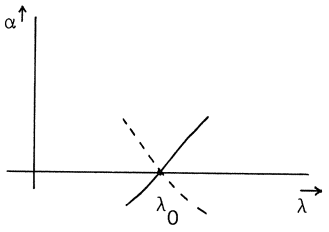

Figure 2

4. APPLICATION TO A TWO-POINT BOUNDARY VALUE PROBLEM

In this section the equation

$$(4.1)\qquad \begin{cases} \dfrac{\partial}{\partial t} y = \lambda f(y) + \dfrac{\partial^2}{\partial x^2} y & 0 < x < 1,\ t > 0,\ \lambda \in \mathbb{R}, \\ y(0,t) = y(1,t) = 0 & t > 0, \\ y(x,0) = g(x) & 0 < x < 1, \end{cases}$$

already mentioned in example VI.6.11 is reconsidered.

The motivation to do this is twofold: In the first place the problem of the determination of the steady state(s) of (4.1) offers a good opportunity to apply the methods of the previous section. Secondly we use this example to give an illustration of the method to determine whether the bifurcating branch of steady states is stable or not.

4.1. THE STEADY STATE SOLUTIONS

We study the steady state equation

$$(4.2)\qquad \begin{cases} 0 = \lambda f(y) + \dfrac{d^2}{dx^2} y, \\ y(0) = y(1) = 0, \end{cases}$$

and the equivalent operator equation (on the Banach space B)

$$(4.3)\qquad F(y,\lambda) = -\lambda(Kf)(y) + y = 0 \qquad y \in B,$$

where

$$B = \{g \in C[0,1] \mid g(0) = g(1) = 0\}.$$

The linear mapping $K: B \to B$ is defined by

$$K(g)(x) = \int_0^1 G(x,\xi) g(\xi)\, d\xi, \qquad x \in (0,1),$$

with $G(x,\xi)$ the Green's function (VI.6.7).

For convenience we make the

HYPOTHESIS 4.1. The function f has the properties

$$f(0) = 0, \tag{4.4a}$$

and $f'(0) > 0$, say (without any restriction)

$$Df(0) = 1. \tag{4.4b}$$

The property (4.4a) implies that

$$(0,\lambda), \quad \lambda \in \mathbb{R},$$

can be considered as a "trivial" branch of solutions of (4.3).

In result of theorem 1.1 we only can expect bifurcation from this branch at points $(0,\lambda_0)$ for which the compact linear mapping

$$I - \lambda_0 KDf(0) \tag{4.5}$$

is not boundedly invertible on B. According to the Riesz-Schauder theory these λ_0 are characterized by the fact that the operator given by (4.5) has a zero eigenvalue. By operating with $\frac{d^2}{dx^2}$ on the eigenvalue equation it is clear that the last statement is equivalent to the fact that

$$\frac{d^2}{dx^2}\phi + \lambda_0 Df(0)\phi = 0 \tag{4.6}$$

has a non-zero solution in $B \cap C^2(0,1)$.

The equation (4.6) implies

$$\begin{cases} \phi^n = \frac{1}{\sqrt{2}} \sin(n\pi x), & n = 1,2,\ldots, \\ \lambda_0^n = n^2\pi^2 \end{cases} \tag{4.7}$$

Consequently only at $\lambda_0^n = n^2\pi^2$, $n = 1,2,\ldots$, there can occur bifurcation. Moreover the theorem 3.5 applies in these points, because of the fact that the compact mapping (4.5) has simple eigenvalues at these values of λ. So there bifurcates a non-trivial branch from $(0,\lambda)$ at $(0,n^2\pi^2)$, $n = 1,2,\ldots$.

It is easily seen that at $\lambda_0 = \pi^2$ the functional ϕ^* from lemma (3.3) is given by

$$(4.8) \qquad \langle y,\phi^* \rangle = \frac{1}{\sqrt{2}} \int_0^1 \sin(\pi\xi)y(\xi)d\xi.$$

EXAMPLE 4.3. Take $f(y) = y + y^2$, $\lambda_0 = \pi^2$. From (3.7), it follows that

$$y(\lambda_0+\alpha\tau(\alpha)) \sim \alpha\phi = \frac{\alpha}{\sqrt{2}} \sin(\pi x)$$

and

$$\tau(\alpha) \sim \alpha\tau_0$$

with τ_0 satisfying (3.7). Substituting

$$D_1D_2F(0,\lambda_0)(z,\lambda) = -\lambda K(z) \qquad z \in B,\ \lambda \in \mathbb{R},$$

and

$$\frac{1}{2!} D_1^2F(0,\lambda_0)(z,z) = -\lambda_0 K(z^2) \qquad z \in B,$$

we obtain

$$\tau_0 = -\lambda_0 2^{-3/2} \int_0^1 \{\sin(\pi\xi)\}^3 d\xi = -\frac{\pi\sqrt{2}}{3}.$$

In figure 3 the local behavior in the neighbourhood of $(0,\pi^2)$ of the solutions (x,λ) of (4.2) is sketched.

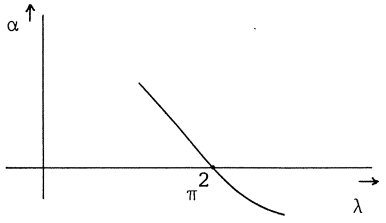

Figure 3

4.2. THE STABILITY OF THE NON-TRIVIAL BRANCH

As usually we assume that a steady state y_0 of (4.1) is stable if all eigenvalues μ_n of

$$(4.9)\qquad \begin{cases} \mu_n e_n = \lambda Df(y_0(x))e_n + \dfrac{d^2}{dx^2} e_n, & 0 < x < 1, \\ e_n(0) = e_n(1) = 0, \end{cases}$$

have negative real parts, and that y_0 is unstable if one of the μ_n has a positive real part.

This extension of the theorem of Poincaré-Lyapunov is called the principle of linearized stability. Its correctness for equations of the form (4.1) is proved in SATTINGER [5].

For the trivial steady state $(0,\lambda)$ we find

$$e_n(\lambda) = e_n = \frac{1}{\sqrt{2}} \sin(n\pi x), \qquad n = 1,2,\ldots,$$

$$\mu_n(\lambda) = \lambda - n^2\pi^2.$$

So $(0,\lambda)$ is stable for $\lambda < \pi^2$ and unstable for $\lambda > \pi^2$.

Now we consider the eigenvalue problem (4.9) for $y_0 = y(\lambda)$ along the non-trivial branch through $(0,\pi^2)$ again, denoting the eigenvalues by $\mu_n(\lambda)$, $n = 1,2,\ldots$, and the corresponding eigenfunctions by $e_n(\lambda)$. As the eigenvalues depend continuously on the operator (this can be proved by the methods used in this section) the stability of $y(\lambda)$ is completely determined by the sign of (the real part of) the eigenvalue

$$\mu_1(\lambda), \qquad \lambda \in (\pi^2-\delta,\pi^2+\delta),$$

with

$$\mu_1(\pi^2) = 0.$$

Using the notation of section 3 we set

$$(4.10)\qquad \begin{cases} \mu_1(\lambda) = \sigma(\alpha), & \lambda = \lambda_0 + \tau(\alpha) = \pi^2 + \alpha(\tau_0+\alpha\tau_1+\alpha^2\tau_2+\ldots), \\ e_1(\lambda) = e_1(\pi^2) + w(\alpha) = \phi + w(\alpha), & w \in B_2, \\ y(\lambda) = y(\alpha) = \alpha\phi+z(\alpha,\tau(\alpha)) = \alpha(\phi+\alpha v_1+\alpha^2 v_2+\ldots). \end{cases}$$

Operating with K, P and Q on (4.9) and substituting (4.10) we get

(4.11a) $$\langle K\sigma(\alpha)(\phi+w(\alpha)) - (\lambda_0+\tau(\alpha))KDf(y(\alpha))(\phi+w(\alpha)) + \phi + w(\alpha), \phi^*\rangle = 0,$$

(4.11b) $$Q\{K\sigma(\alpha)(\phi+w(\alpha)) - (\lambda_0+\tau(\alpha))KDf(y(\alpha))(\phi+w(\alpha)) + \phi + w(\alpha)\} = 0.$$

We denote the left-hand side of (4.11a) respectively (4.11b) by $H_1(\sigma,w,\alpha)$ respectively $H_2(\sigma,w,\alpha)$. Consider:

$$H: \mathbb{R} \times B_2 \times \mathbb{R} \to \mathbb{R} \times B_2,$$

defined by

$$H(\sigma,w,\alpha) = (H_1(\sigma,w,\alpha),\ H_2(\sigma,w,\alpha)).$$

Direct computation yields

(4.12) $$\begin{cases} D_1H_1(0,0,0) = \langle K\phi,\phi^*\rangle = \dfrac{1}{\pi^2} & (K\phi = \dfrac{1}{\pi^2}\phi) \\ D_2H_1(0,0,0) = 0\ , \\ D_2H_2(0,0,0) = Q\{\lambda_0 KDf(y) - I\}, \end{cases}$$

which is an invertible operator on B_2 according to the Riesz-Schauder theory.

By application of the implicit function theorem VI.6.12 we obtain the result that there exists a k-times differentiable mapping $\alpha \to (\sigma(\alpha),w(\alpha))$ such that $H(\sigma(\alpha),w(\alpha),\alpha) = 0$.

To get insight in the local behaviour of $\sigma(\alpha)$ we set

(4.13) $$\begin{cases} \sigma(\alpha) = \alpha\sigma_1 + \alpha^2\sigma_2 + \ldots \\ w(\alpha) = \alpha w_1 + \alpha^2 w_2 + \ldots \end{cases}$$

and substitute (4.10) and (4.13) in (4.11). By comparison of terms of same order in α the coefficients σ_i and w_i can be calculated.

In the special case of example 4.3 we find

(4.14) $$\sigma_1 \langle K\phi,\phi^*\rangle - \langle K\lambda_0 D^2f(0)(\phi,\phi),\phi^*\rangle = 0.$$

Comparing (4.14) with the equation (3.8) applied to example 4.3 we find

$$\sigma_1 = -2\pi^2\tau_0.$$

Thus the non-trivial branch is stable ($\sigma(\alpha)<0$) for $\tau(\alpha) > 0$ ($|\alpha|<\delta$) and unstable ($\sigma(\alpha)>0$) for $\tau(\alpha) < 0$ ($|\alpha|<\delta$).

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VIII. THE TOPOLOGICAL DEGREE OF MAPPINGS IN BANACH SPACES

In this chapter we shall extend the notion of degree, defined in chapter II, to mappings in Banach spaces. As applications we shall prove Schauder's fixed point theorem and a theorem of Krasnosel'skii on the existence of a bifurcation point.

1. SOME PROPERTIES OF THE DEGREE OF MAPPINGS IN FINITE DIMENSIONAL SPACES

Let $D \subset \mathbb{R}^n$ be an open and bounded set and let $\mathbb{R}^m \subset \mathbb{R}^n$ such that $x = (x_1,\dots,x_m) \in \mathbb{R}^m \Leftrightarrow (x_1,\dots,x_m, 0,\dots,0) \in \mathbb{R}^n$. Let $\phi\colon \bar{D} \to \mathbb{R}^m$ be continous, let $\psi\colon \bar{D} \to \mathbb{R}^n$ be $\psi = I + \phi$ and let $p \in \mathbb{R}^m$, $p \notin \psi(\partial D)$. Then

$$\deg(\psi,D,p) = \deg(\psi|_{\mathbb{R}^m \cap \bar{D}},\ D\cap\mathbb{R}^m,p)\ . \tag{1.1}$$

In view of II.2.2 and II.2.3 we may restrict ourselves to show this property only for C^1-functions and $p \notin \psi(Z)$, where Z is the set of critical points of ψ. Denote the mapping $\psi|_{\mathbb{R}^m \cap \bar{D}}\colon \mathbb{R}^m \cap \bar{D} \to \mathbb{R}^m$ by F. If $y \in \psi^{-1}(p)$, then $y + \phi(y) = p$ and $y = p - \phi(y) \in \mathbb{R}^m$. Hence $\psi^{-1}(p) \subset \mathbb{R}^m \cap D$, so that $\psi^{-1}(p) = F^{-1}(p)$. Furthermore,

$$J_\psi(x) = \det \left(\begin{array}{c|c} 1+\dfrac{\partial \phi_i}{\partial x_j} & \dfrac{\partial \phi_i}{\partial x_k} \\ \hline 0 & I \end{array}\right) \qquad \begin{array}{l} i = 1,\dots,m \\ j = 1,\dots,m \\ k = m+1,\dots,n \end{array}$$

which implies $J_\psi(x) = J_F(x)$ for $x \in \mathbb{R}^m \cap D$. Thus

$$\sum_{x\in\psi^{-1}(p)} \operatorname{sign} J_\psi(x) = \sum_{x\in F^{-1}(p)} \operatorname{sign} J_F(x)\ ,$$

which equals (1.1) according to definition II.2.4.

From this definition also the following property is easily derived. Let $\phi\colon \bar{D} \to \mathbb{R}^n$ and $\psi\colon \bar{E} \to \mathbb{R}^m$ be continous mappings, where D is an open and bounded set in $\mathbb{R}^n$ and E an open and bounded set in $\mathbb{R}^m$ and let $(p,q) \in \mathbb{R}^n \times \mathbb{R}^m$, $p \notin \phi(\partial D)$, $q \notin \psi(\partial E)$. Then

$$\deg(\phi\times\psi,\ D\times E,\ (p,q)) = \deg(\phi,D,p)\cdot \deg(\psi,E,q)\ . \tag{1.2}$$

2. SOME PROPERTIES OF COMPACT MAPPINGS IN BANACH SPACES

Let f be a compact mapping from $D \subset X$ into X, where X is a Banach space and D is a closed and bounded subset of X. A finite dimensional mapping of D into X is a mapping whose range lies in a finite dimensional subspace of X.

THEOREM 2.1. *The mapping* f *can be approximated uniformly on* D *by continuous finite dimensional mappings.*

PROOF. For any $\varepsilon > 0$, f(D) can be covered by finitely many open balls $B_1,\dots,B_{j(\varepsilon)}$ with centers $x_1,\dots,x_{j(\varepsilon)}$ in $\overline{f(D)}$. Let $\psi_i \geq 0$, $i = 1,\dots,j(\varepsilon)$, be continous functions on X with $\psi_i(x) = 0$ for $x \notin B_i$ and for all $x \in \overline{f(D)}$

$$\sum_{i=1}^{j(\varepsilon)} \psi_i(x) = 1\ .$$

Such functions always exist, cf. the construction (7.1) of the functions λ_U in the appendix. Define

$$f_\varepsilon(x) = \sum_{i=1}^{j(\varepsilon)} \psi_i(f(x))\ x_i\ ,$$

then the image of D under f_ε is contained in the convex hull of the x_i's. Also

$$\|f(x) - f_\varepsilon(x)\| = \Big\|\sum_{i=1}^{j(\varepsilon)} \psi_i\ (f(x))\ \{f(x) - x_i\}\Big\|$$

and if $\psi_i\ (f(x)) > 0$, then $f(x) \in B_i$, so that $\|f(x) - x_i\| < \varepsilon$. Hence for all $x \in D$

$$\|f(x) - f_\varepsilon(x)\| < \varepsilon\ . \qquad \square$$

Let $\phi\colon D \to X$ be a continous mapping of the form $\phi = I - f$ with f compact.

THEOREM 2.2. *The image under* ϕ *of a closed set* S *in* D *is closed in* X.

PROOF. If $x_n \in S$, $\phi(x_n) \to y$, then $x_n - f(x_n) \to y$. Since f is compact there is a converging subsequence (again denoted with indices n) such that $f(x_n) \to z$. Then $x_n \to z + y = x \in S$ and by continuity $x - f(x) = y$. □

In particular it follows that $\phi(\partial D)$ is closed.

3. THE DEGREE OF A COMPACT PERTURBATION OF THE IDENTITY

Let X be a Banach space and D be an open and bounded subset of X. We will define the degree of a continuous mapping $\phi: \bar{D} \to X$, that can be written as $\phi = I - f$ with f compact, for $p \notin \phi(\partial D)$.

First assume that g is a continuous finite dimensional mapping from $\bar{D}$ into X and that E is a finite dimensional subspace of X containing p and the range of g. If $p \notin \psi(\partial D)$, then the degree of $\psi = I - g$ is defined by

$$\deg(\psi,D,p) = \deg(\psi|_{E\cap\bar{D}}, D, p) \tag{3.1}$$

which is the degree of a mapping in a finite dimensional space E. From (1.1) if follows that this definition is independent of the space E.

Since by theorem 2.2 $\phi(\partial D)$ is closed, there is an open ball B in X with radius δ and with $(p+B) \cap \phi(\partial D) = \emptyset$. Now let g_1 and g_2 be continuous finite dimensional mappings from $\bar{D}$ into X such that

$$\|f(x) - g_i(x)\| < \delta \ , \quad x \in \bar{D} \ , \quad i = 1,2 \ . \tag{3.2}$$

Let $G_i = I - g_i$, $i = 1,2$, then for all $0 \le t \le 1$, $x \in \partial D$

$$\|(1-t)G_1(x) + tG_2(x) - \phi(x)\| < \delta \ ,$$

hence $(1-t)G_1(x) + tG_2(x) \neq p$. It follows from (3.1) and the homotopy invariance theorem II.3.4 that

$$\deg(G_1,D,p) = \deg(G_2,D,p) \ . \tag{3.3}$$

Since by theorem 2.1 for any compact f there is always a continuous finite dimensional mapping g satisfying (3.2) and since by (3.3) it does not matter which g we take, the degree of ϕ can now be defined by

$$\deg(\phi,D,p) = \deg(I-g,D,p) \ . \tag{3.4}$$

All the properties of the degree listed in section II.3 (theorems 1,2, 3,6,7 and 9) remain valid. We mention the homotopy invariance (theorem II.3.4):

THEOREM 3.1. *(Homotopy invariance)*
Let the mapping $(t,x) \to \phi_t(x)$ *be such that the mapping* $(t,x) \to x - \phi_t(x)$ *is compact from* $[0,1] \times \bar{D}$ *into* X *and let* $\phi_t(x) \neq p$ *if* $(t,x) \in [0,1] \times \partial\bar{D}$, *then* $\deg(\phi_t,D,p)$ *is independent of* t.

PROOF. According to theorem 2.1 there is a finite dimensional map g that approximates $I - \phi_t$ uniformly in x and t. The rest follows from (3.4) and theorem II.3.4. □

Like in theorem II.3.9 the degree of ϕ is determined by the values of ϕ at ∂D. Conversely, it is possible to define the degree of a continuous mapping ϕ on ∂D such that $I - \phi$ is compact. Hence a theorem similar to theorem II.3.11 holds. This becomes clear if we know the following generalization of Tietze's extension theorem.

THEOREM 3.2. *Let* A *be a closed subset of a metric space* X *and* C *a convex set in a Banach space* Y. *Then any continuous function* $f: A \to C$ *has a continuous extension* $F: X \to C$.

The proof due to DUGUNDJI [1] will be given in the appendix.

4. SCHAUDER'S FIXED POINT THEOREM

Brouwer's fixed point theorem (theorem II.4.1) cannot be extended to Banach spaces. For, let $X = \ell^2$ (the space of sequences of complex numbers $x = (x_1,x_2,\ldots)$ with norm $\|x\| = [\sum_{i=1}^{\infty} |x_i|^2]^{\frac{1}{2}} < \infty$), let B be the closed unit ball in ℓ^2 and let f be the continuous mapping from B into B given by

$$f(x) = (\sqrt{1-\|x\|^2},\ x_1,x_2,\ldots) .$$

The range of f is contained in the boundary $\{x \mid \|x\|=1\}$ of B and a fixed point $p = (p_1,p_2,\ldots)$ should therefore satisfy $\|p\| = 1$. Moreover, the fixed point must satisfy $(p_1,p_2,\ldots) = (\sqrt{1-\|p\|^2},p_1,p_2,\ldots) = (0,p_1,p_2,\ldots) = (0,0,p_2,\ldots) = (0,0,0,\ldots)$. Hence $p = 0$, but then $\|p\| \neq 1$. So f has no fixed point.

In order to have a fixed point f need not be merely continuous, but (as we will see) if f is compact it has a fixed point. This can be derived by approximating f by a finite dimensional mapping according to theorem 2.1 and by using Brouwer's fixed point theorem, see LJUSTERNIK & SOBOLEW [2, VI.3] or NIRENBERG [4,II.2.1]. However, following SCHWARTZ [5, 3.59 and 3.60], we will derive the fixed point property directly from degree theory, just

as Brouwer's Theorem can be derived with the aid of the degree.

Let X be a Banach space with $K \subset X$ convex and compact.

LEMMA 4.1. *Every continuous mapping* $f: K \to K$ *has a fixed point.*

PROOF. Since K is bounded it is contained in some closed ball $B = \{x \mid \|x\| \leq M\}$. By theorem 3.2 f can be extended to a continuous mapping (also called f) from B into K and it is clear that a fixed point of this map is a fixed point of the original f.

Consider the family of mappings

$$\phi_t = I - tf , \quad 0 \leq t \leq 1 ,$$

which depends continuously on t uniformly in x, because for all $x \in B$

$$\|\phi_t(x) - \phi_{t'}(x)\| \leq |t-t'| . \|f(x)\| \leq M . |t-t'| .$$

For $x \in \partial B$ $\quad x - t\,f(x) = 0$ would imply $\|x\| \leq tM$, so if we assume that f has no fixed points on ∂B, ϕ_t have no zero's on ∂B for all $0 \leq t \leq 1$. Furthermore, the range of $\phi_t(x)$ is compact, since $\{ty \mid 0 \leq t \leq 1,\ y \in K\} \subset B$ is compact as the image of the compact set $[0,1] \times K$ under the continuous mapping $(t,y) \to ty$. According to theorem 3.1 we therefore have

$$\deg(\phi_1, B, 0) = \deg(\phi_0, B, 0) = \deg(I, B, 0) = 1 ,$$

so that there is a point $p \in B$ with $\phi_1(p) = 0$, i.e. $f(p) = p$. □

THEOREM 4.2. (SCHAUDER)
Let A *be a closed convex set contained in the Banach space* X. *Then every compact mapping* f *from* A *into* A *has a fixed point.*

PROOF. Let K be the closed convex hull of f(A), then K is compact and contained in A, because f is compact and A is closed and convex. By lemma 4.1 $f|_K$ has a fixed point. □

REMARK. If A is a closed ball around the origin, theorem 3.2 is not needed in the proof of theorem 4.2.

Schauder's fixed point theorem has many applications, for example Peano's Theorem on the existence of a solution of an ordinary differential equation can be proved by it easily:

THEOREM 4.3. (PEANO)
Let $f(t,x)$ *be a continuous mapping from* $\{(t,x) \mid |t-t_0| \le a, |x-x_0| \le b\} \subset \subset \mathbb{R} \times \mathbb{R}^n$ *into* $\mathbb{R}^n$ *with maximum* M *and let*

$$h = \min(a, b/M) . \tag{4.1}$$

Then in $[t_0-h,t_0+h]$ *there is at least one solution of the equation*

$$\frac{dx(t)}{dt} = f(t,x(t)) \tag{4.2}$$

with

$$x(t_0) = x_0 . \tag{4.3}$$

PROOF. Consider the mapping
$A: B = \{x \mid |x(t)-x_0| \le b, t \in [t_0-h,t_0+h]\} \subset (C^0([t_0-h,t_0+h]))^n \to$
$\to (C^0([t_0-h,t_0+h]))^n$, which maps a continuous function $x \in B$ onto the continuous function

$$Ax(t) = x_0 + \int_{t_0}^{t} f(t,x(s))\, ds .$$

It follows from (4.1) that A maps B into B and just as in example VI.4.7 it can be shown that A is compact (see LJUSTERNIK & SOBOLEW [2, p.203]). According to theorem 4.2 A has a fixed point and $Ax(t) = x(t)$ is equivalent with (4.2) and (4.3). □

5. THE INDEX OF AN ISOLATED SOLUTION

Let D be a bounded open set in the Banach space X and let $\phi: \bar{D} \to X$, $\phi \neq 0$ on ∂D be a continuously (Fréchet) differentiable map with $f = I - \phi$ compact. Let x_0 be a solution of $\phi(x_0) = 0$ and let $D\phi(x_0) = I - Df(x_0)$ be invertible. According to the implicit function theorem (theorem VI.6.9) x_0 is an isolated zero of ϕ, so there exists a ball $B_\varepsilon(x_0)$ with radius ε and center x_0 such that ϕ has no other zeros in $B_\varepsilon(x_0)$. Hence it is possible to define

$$\operatorname{ind}(\phi,x_0,0) = \deg(\phi,B_\varepsilon(x_0),0) , \tag{5.1}$$

which is independent of ε for ε sufficiently small in view of theorem II.3.6. The number (5.1) is called the *index of* ϕ *at the point* x_0.

For the remaining of this chapter we only need that $T = Df(x_0)$ is com-

pact, which follows from theorem VI.5.14. So it would also be sufficient to require that x_0 is an isolated zero of ϕ, that ϕ is differentiable at x_0 and that $D\phi(x_0)$ is invertible.

Since I - T is invertible 1 is not an eigenvalue of T and there are finitely many eigenvalues λ larger than 1 (theorem VI.4.19.(i)). Let $m(\lambda)$ be the algebraic multiplicity of λ, which equals the dimension of $N(T-\lambda I)^{r(\lambda)}$, where $r(\lambda)$ is the Riesz index of λ (lemma VI.4.16).

THEOREM 5.1. (LERAY-SCHAUDER)

$$\operatorname{ind}(\phi,x_0,0) = (-1)^{\beta}, \quad \beta = \sum_{\lambda>1} m(\lambda) .$$

PROOF. We may assume that $x_0 = 0$. First remark that in virtue of theorem 3.1, $\operatorname{ind}(\phi,0,0) = \operatorname{ind}(I-T,0,0)$, since $1/t\, f(tx)$ is a compact map from $(0,1]\times\bar{D}$ into X which converges to $T = Df(0)$ (see definition VI.5.2) uniformly in $\bar{D}$, because $\bar{D}$ is bounded and since for $x \le \varepsilon$ the only solution of $f(tx) = tx$ is $x = 0$, $0 < t \le 1$ and since I - T is injective.

In finite dimensions the theorem yields for a real non-singular matrix A in $\mathbb{R}^n$

$$\operatorname{sign} \det A = (-1)^{\beta}, \quad \beta = \sum_{\lambda>1} m(\lambda) ,$$

where the sum is taken over the real eigenvalues of T = I - A greater than 1. Indeed, by theorem II.3.5

$$\deg(I-T,B_\varepsilon(0),0) = \operatorname{sign} \det A = \operatorname{sign} \det(I-T) = \operatorname{sign} \prod(1-\lambda_j)^{m_j} = (-1)^{\beta} , \tag{5.2}$$

where λ_j are the eigenvalues of T (including complex ones, which occur in conjugate pairs) with respective multiplicities m_j, cf. theorem II.3.13.

Now decompose $X = X_1 \oplus X_2$, where X_1 is the finite dimensional space spanned by all the generalized eigenvectors of T with eigenvalues larger than 1 (see theorem VI.4.21) and where X_2 is invariant under T and contains only eigenvectors with eigenvalue smaller than 1. Therefore $tTx \neq x$ for all $x \in X_2 \setminus \{0\}$, $0 \le t \le 1$, so that theorem 3.1 gives

$$\deg((I-T|_{X_2},B_\varepsilon(0)\cap X_2,0) = \deg(I,B_\varepsilon(0)\cap X_2,0) = 1 .$$

Using (1.2) and (5.2) we finally get

$$\operatorname{ind}(I-T,0,0) = \deg((I-T)|_{X_1},B_\varepsilon(0)\cap X_1,0) \cdot \deg((I-T)|_{X_2},B_\varepsilon(0)\cap X_2,0) = (-1)^{\beta}. \quad \square$$

Similarly, if λ_0 is not an eigenvalue of T, we get for $\phi = \lambda_0 I - f$

$$\text{ind}(\phi,0,0) = (-1)^{\beta}, \quad \beta = \sum_{\lambda>\lambda_0} m(\lambda) . \tag{5.3}$$

6. AN APPLICATION TO BIFURCATION THEORY

In this section we give an application of theorem 5.1 to bifurcation theory due to KRASNOSEL'SKII [3, theorem IV.2.1, p.196].

Let A be a non-linear operator from D into X, where X is a Banach space and $D \subset X$ open with $0 \in D$, such that $A0 = 0$. Hence the equation

$$\mu Ax = x \tag{6.1}$$

has the solution $x = 0$. Assume that this is an isolated solution of (6.1) for μ in the interval $[\alpha,\mu_0)$, i.e. for every $\mu \in [\alpha,\mu_0)$ there is an $\varepsilon = \varepsilon(\mu) > 0$ such that in $\{x \mid \|x\| < \varepsilon\}$ the only solution of (6.1) is $x = 0$.

We wish to know for which values of μ_0 there exists another solution y_μ of (6.1) which is not isolated at $\mu = \mu_0$, i.e., for any $\varepsilon > 0$ there is a $\mu \in (\mu_0-\varepsilon,\mu_0+\varepsilon)$ and at least one solution $y_\mu \neq 0$ of (6.1) with $\|y_\mu\| < \varepsilon$. In that case μ_0 is called a *bifurcation point of* A and all the solutions y_μ are called *continuous branches*.

Let A have a Fréchet derivative T at 0.

LEMMA 6.1. *A necessary condition for* μ_0 *to be a bifurcation point is, that* $1/\mu_0$ *belongs to the spectrum of* T.

PROOF. If A would be continuously differentiable in a neighbourhood of zero in D, the implicit function theorem would yield that $x = 0$ is an isolated solution if $1/\mu$ does not belong to the spectrum of T.

In general, let $1/\mu_1$ not belong to the spectrum of T ($\mu_1 \neq 0$) and let

$$|\mu-\mu_1| \leq 1/[3\,\|(I-\mu_1 T)^{-1}\| \cdot \|T\|]. \tag{6.2}$$

Let $x = (I-\mu_1 T)^{-1}(I-\mu_1 T)x = \mu Ax$ be a solution of (6.1) with μ satisfying (6.2) and with $\|x\|$ so small that (see definitions VI.5.2 and 5.4 of $T = DA(0)$)

$$\rho(\|x\|,0) \leq \|x\|/[3\,\|(I-T)^{-1}\| \cdot |\mu_2|] ,$$

where μ_2 is the endpoint of the interval (6.2) with the largest absolute value. Then

$$\|x\| \le \|(I-\mu_1 T)^{-1}\| \cdot \|\mu Ax - \mu_1 Tx\| \le \|(I-\mu_1 T)^{-1}\| \cdot |\mu-\mu_1| \cdot \|T\| \cdot \|x\| +$$
$$+ \|(I-\mu_1 T)^{-1}\| \cdot |\mu| \cdot \rho(\|x\|,0) \le 2/3\|x\|,$$

hence $x = 0$. □

The following examples shows that lemma 6.1 is not sufficient for a bifurcation point:

$$X = \mathbb{R}^2, \qquad A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y+y^3 \\ y-x^3 \end{pmatrix},$$

then $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}$, hence $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ 0 \end{pmatrix}$, so that $1/\mu_0 = 1$ is an eigenvalue of T with geometric multiplicity 1. In order to solve the equation (6.1) we multiply the first equation with y and the second one with x and subtract. Then we obtain $\mu(y^2+y^4+x^4) = 0$ or $x = y = 0$ as the only solution for $\mu \in [1-\varepsilon,1+\varepsilon]$. Hence $\mu_0 = 1$ is no bifurcation point. Remark that the algebraic multiplicity of the eigenvalue 1 of T is 2, since $I - T = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$ and $(I-T)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Now we give a sufficient condition in order that μ_0 is a bifurcation point. We assume moreover, that A is compact. In view of theorem VI.4.19 and VI.5.14 and the lemma just proved, $1/\mu_0$ has to be an eigenvalue of T.

THEOREM 6.2.(KRASNOSEL'SKII)
When $1/\mu_0$ is an eigenvalue of T of odd (algebraic) multiplicity, then μ_0 is a bifurcation point of A and to this bifurcation point there corresponds at least one continuous branch of eigenvectors of the operator A.

PROOF. It follows from the proof of the above lemma that for $\varepsilon > 0$ sufficiently small there is a $\delta > 0$ such that in $\{x \mid \|x\| \le \delta\} \subset D$ the equations $(\mu_0 \pm \varepsilon)Ax = x$ have only the solution $x = 0$. Then the indices $\mathrm{ind}(I-(\mu_0 \pm \varepsilon)A,0,0)$ are defined and according to (5.3) they are given by

$$\mathrm{ind}(I-(\mu_0 \pm \varepsilon)A,0,0) = (-1)^{\beta(\pm)}, \tag{6.3}$$

where $\beta(\pm)$ is the sum of the multiplicities of all the real eigenvalues of T larger than $1/(\mu_0 \pm \varepsilon)$. Now μ_0 itself has odd multiplicity, hence the indices (6.3) have opposite signs. Since $(\mu_0-\varepsilon+2t\varepsilon)A$ is a compact mapping from $[0,1]\times D$ into X, it follows from theorem 3.1 that there must be some $t \in (0,1)$ such that $(\mu_0-\varepsilon+2t\varepsilon)Ax = x$ has a solution with $\|x\| = \delta$. □

REMARK 6.3. Theorem 6.2 can be considerably generalized; in particular compactness may be dropped. To derive this one uses degree theory in finite dimensions, see NIRENBERG [4, 3.5].

7. APPENDIX

In order to make this chapter complete, we give the proof of theorem 3.2, which is due to DUGUNDJI [1], cf. SCHWARTZ [5, 3.58, p.94-96].

PROOF OF THEOREM 3.2. Denote the metric of X by ρ and choose for each $x \in X\backslash A$ an open ball V_x around x such that diam $V_x \leq \rho(V_x,A)$. Let $\{U\}$ be an open locally finite refinement of the covering $\{V_x\}$ of $X\backslash A$, i.e. the U's are open and cover $X\backslash A$, each U is contained in some V_x and each $x \in X\backslash A$ has an open neighbourhood 0_x intersecting only finitely many U's. Certainly for each U diam $U \leq \rho(U,A)$.

Let $U_0 \in \{U\}$ and define for $x \in X\backslash A$ the function

$$\lambda_{U_0}(x) = \frac{\rho(x,X\backslash U_0)}{\sum_U \rho(x,X\backslash U)} . \tag{7.1}$$

When λ_{U_0} is restricted to 0_x for any $x \in X\backslash A$, in (7.1) there are only a finite number of terms in the sum different from zero and since these terms are continuous, λ_{U_0} is a continuous function on $X\backslash A$ with $0 \leq \lambda_{U_0} \leq 1$ and with $\lambda_{U_0}(x) = 0 \Leftrightarrow x \notin U_0$.

Now for each U choose $a_U \in A$ such that $\rho(a_U,U) < 2\rho(A,U)$ and the extension F is given by

$$F(x) = \sum_U \lambda_U(x)\, f(a_U), \qquad x \in X\backslash A$$

$$F(x) = f(x), \qquad x \in A.$$

For each $x \in X\backslash A$, $\lambda_U(x) \neq 0$ only for finitely many U's and since $\sum_U \lambda_U(x) = 1$ and $f(a_U) \in C$, it follows that $F(x) \in C$. For $x \in X\backslash A$ $F|_{0_x}$ is a finite sum of continuous functions, so F is continuous on $X\backslash A$.

Let $x_0 \in \partial A$. For all $\varepsilon > 0$ there is a $\delta > 0$ so that for all $x \in A$ with $\rho(x,x_0) < \delta$: $\|f(x)-f(x_0)\| < \varepsilon$. We will show that for $x \in X$ with $\rho(x,x_0) < \delta/6$ $\|F(x)-F(x_0)\| < \varepsilon$, hence that F is continuous on the boundary of A.

Assume $x \in X\backslash A$, $\rho(x,x_0) < \delta/6$ and $\rho(x,a_U) < \delta/2$. Then

$$\rho(x_0,a_U) \leq \rho(x_0,x) + \rho(x,a_U) < \delta$$

implies $\|f(a_U)-f(x_0)\| < \varepsilon$.

On the other hand $x \in X\backslash A$, $\rho(x,x_0) < \delta/6$ and $\rho(x,a_U) \geq \delta/2$ implies

$$\rho(x,a_U) \geq 3\rho(x,x_0) \geq 3\rho(x,A).$$

If $x \in U$, then

$$\rho(x,a_U) \leq \rho(a_U,U) + \operatorname{diam} U < 2\rho(A,U) + \rho(A,U) = 3\rho(A,U) \leq 3\rho(x,A),$$

hence $x \notin U$ and thus $\lambda_U(x) = 0$.

Finally, for all $x \in X\backslash A$ with $\rho(x,x_0) < \delta/6$ we have

$$\|F(x) - F(x_0)\| = \|\sum_U \lambda_U(x)\, f(a_U) - f(x_0)\| =$$

$$= \|\sum_U \lambda_U(x)\, \{f(a_U)-f(x_0)\}\| < \sum_U \lambda_U(x)\varepsilon = \varepsilon,$$

since for each x the sum consists of only finitely many terms and since $\lambda_U(x) \geq 0$. □

LITERATURE

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IX. SOME NONLINEAR PROBLEMS FROM PHYSICS

1. INTRODUCTION

In chapter I it was announced that these lectures would deal in particular with those nonlinear eigenvalue problems that are found in physics, chemistry and biology. Examples from the latter two disciplines have already been discussed. These were: an example taken from chemical engineering, which involved the simultaneous occurrence of a catalytic reaction and mass transfer through diffusion (chapter I), the equilibrium states of chemically reacting systems (chapter III) and the biological phenomenon of the struggle between antigen and antibody (chapter V).

In this chapter we shall discuss three nonlinear physical problems. The first example, treated in section 2, deals with the rotations of a heavy string, which at one end is forced to rotate with an angular velocity ω around a vertical axis, while the other end is free. A trivial mode of rotation at any angular velocity ω is the rotation of the string around its own vertical axis. According to linear theory other modes of rotation can only exist at certain angular eigen-velocities ω_n, which form a discrete spectrum. The nonlinear treatment however shows that while for $\omega \leq \omega_1$ there is only the trivial mode of rotation, for any ω in the range $\omega_n < \omega \leq \omega_{n+1}$ there are in addition exactly n distinct nontrivial modes of rotation.

The two other examples involve the generation of heat in materials with temperature dependent electric conductivity and a model of an isothermal star in gravitational equilibrium. The heat problem is a boundary value problem which can be treated by considering the corresponding initial value problem. For the heat problem it is shown that there exists a critical value of the electric current beyond which no equilibrium state exists. For values below this critical current there can be more solutions. For a special

resistance function the differential equation becomes the same as that of the isothermal star (the third example). The stellar problem is essentially an initial value problem which as such has a unique solution. This initial value problem can be treated in phase plane and the behaviour of the integral curve gives at the one hand the number of solutions for the heat problem and at the other hand the behaviour of the solutions for $r \to \infty$ for the isothermal star.

In this chapter only a few physical examples are treated. Other examples can be found for instance in the book of KELLER & ANTMAN [1].

2. THE HEAVY ROTATING STRING

2.1. INTRODUCTION

We examine the motion of a flexible inelastic string which is attached at one endpoint to a vertical spindle, rotating with a constant angular velocity ω around its axis, while the other end of the string is free. In our model we assume that, apart from the torque excerted at the fixed endpoint, the only forces acting on the string are gravity, the centrifugal forces induced by the rotation, and the internal tension that holds the string together. Moreover in our model the string is assumed to be infinitely thin. In practice this is best realized with a thin, heavy string.

If the string hangs down in a perfectly straight vertical line, then there are no centrifugal forces. Since gravity keeps the string hanging down, it will only rotate around its longitudinal axis with the angular velocity ω, just as if it were an inflexible extension of the rotating spindle to which it is attached. This is one possible state of motion. The question is whether this state of motion is stable, because any deviation of the string from the vertical line will induce centrifugal forces that tend to increase the deviation. This increased deviation in turn enhances the centrifugal forces. Gravity, on the other hand, counteracts any deviation from the vertical. As long as the rotation is slow, the gravitational forces dominate and the string remains straight and vertical. However, since the centrifugal force is proportional to the square of the angular velocity ω^2, there certainly will be a critical angular velocity ω_1 beyond which the centrifugal forces overcome the gravitational forces. Consequently, as soon as $\omega > \omega_1$ the string moves out of the straight vertical position until a new configuration is reached in which the centrifugal forces, gravity and the internal tension are in equilibrium again.

This qualitative picture is confirmed by the mathematical analysis. The straight vertical position appears to be a solution of the derived boundary value problem for all values of ω. This trivial solution is the unique solution only when $\omega \leq \omega_1$. Nontrivial solutions bifurcate at the eigenvalues ω_n, $n = 1,2,\ldots$, of the linearized equation. For $\omega_n < \omega \leq \omega_{n+1}$ there are exactly n distinct modes of rotation.

The mathematical treatment given here follows the paper of KOLODNER [2]. An essential tool in this analysis is the comparison of the boundary value problem with the corresponding initial value problem. A general

discussion of this comparison technique is given by KELLER in reference [1, p.17]. As an application he treats the buckling of a nonuniform column.

2.2. THE MATHEMATICAL MODEL

The fixed endpoint of the string is chosen as the origin of the coordinate system, with the z-axis pointing vertically downwards (cf. figure 1). The points of the string are labeled by the arclength s along the string, which is measured from the free endpoint in order to simplify some formulas to be derived later on. This arclength s and the time t are used as the independent variables. The other relevant quantities are:

$\xi(s,t)$ - the position vector of a point of the string,
$T(s,t)$ - the tension in the string at that point,
ρ - the mass of the string per unit of length,
L - the length of the string,
$\chi = (0,0,g)$ - the gravitational acceleration vector.

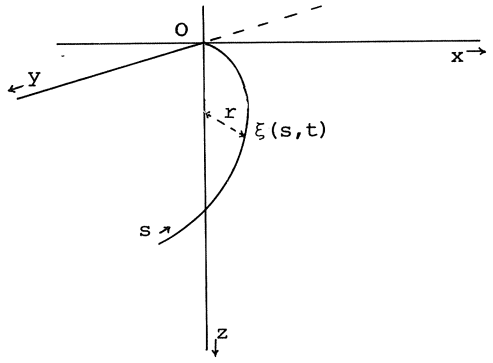

Figure 1

The equations of motion are (subscripts s and t denoting differentiation with respect to these variables)

$$(2.1a) \qquad (\rho\xi_t)_t = \rho\chi + (T\xi_s)s \,,$$

$$0 < s < L,$$

$$(2.1b) \qquad (\xi_s)^2 = 1,$$

subject to the boundary conditions

(2.1c) $\xi(L,t) = 0,$

(2.1d) $T(0,t) = 0.$

These boundary conditions state that the tension vanishes at the free endpoint (s=0) of the string, and that the other endpoint (s=L) is fixed in space at the origin of the coordinate system. Equation (2.1b) expresses the fact that the tangent vector ξ_s is of unit length. Equation (2.1a) results from the application of Newtons' second law to an arbitrary piece of the string. When multiplied by ds, the left hand side of (2.1a) represents the time derivative of the momentum $\rho\xi_t ds$ of a piece of string of infinitisemal length ds. This must be equal to the sum of the forces acting on that piece, the gravitational force $\rho\chi ds$ and the tension forces acting at both ends of that piece of string. Since these tension forces act in the outward tangential direction, i.e. in (nearly) opposite directions, their resultant is the difference of the vector $T\xi_s$ evaluated at these endpoints, which for the piece of string under consideration can be written as $d(F\xi_s) = (F\xi_s)_s ds$, in accordance with equation (2.1a).

We shall look for stationary states of motion of the string, i.e. for those solutions in which each point of the string rotates with the constant angular velocity ω around the z-axis, just as if the string were rigid. Introducing cylindrical coordinates $\xi = (r,\phi,z)$, such a motion is described by

(2.2a) $\xi(s,t) = (r(s), \alpha(s) + \omega t, z(s)),$

(2.2b) $T(s,t) = T(s),$

where the only time dependence is in the angular variable $\phi(s,t) = \alpha(s) + \omega t$.

Substitution in (2.1) yields the time independent equations (differentiation with respect to s is now denoted by a prime):

(2.3a) $(Tr')' + \rho\omega^2 r - Tr(\alpha')^2 = 0,$

(2.3b) $\frac{1}{r}(Tr^2\alpha')' = 0,$

(2.3c) $(Tz')' + \rho g = 0,$

(2.3d) $(r')^2 + (r\alpha')^2 + (z')^2 = 1,$

(2.3e) $r(L) = z(L) = 0,$

(2.3f) $T(0) = 0.$

Integrating (2.3b) and (2.3c), and using the boundary conditions (2.3e) and (2.3f), we obtain

(2.4) $Tr^2\alpha' = 0,$

(2.5) $Tz' = -\rho gs.$

Since from (2.5) it follows that T can only be zero if $s = 0$, we find from (2.4) that $r(s) = 0$ or $\alpha'(s) = 0$ for $s \neq 0$. In both cases the equations (2.3a) and (2.3d) can be reduced to

(2.3a') $(Tr')' + \rho\omega^2 r = 0,$

(2.3d') $(r')^2 + (z')^2 = 1.$

The solution $r = 0$, $z = L-s$, $T = \rho g$ and α undefined is a solution of (2.3) for all values of ω. This trivial solution corresponds to the situation where the string remains in a straight vertical position, rotating around its own axis with angular velocity ω. (The other trivial solution with $r = 0$, $z = s-L$ and $T = -\rho g$ is ruled out, because it corresponds to the unphysical situation in which the string is standing up instead of hanging down.)

When r is not identically zero, then α' must be. Hence for all non-trivial solutions of (2.3) the angle α is a constant. This means that at any stationary motion the whole string remains in one plane through the z-axis, which is rotating with the angular velocity ω about this axis.

REMARK 2.1. In KOLODNER [2] the proof, that the nontrivial stationary motion is in one plane, is missing. The transformation formulas in that article

actually imply $\alpha = 0$, thereby tacitly excluding the eventual possibility of stationary motions which are not in one plane.

REMARK 2.2. If we want the equations (2.3) to be valid on the whole interval $(0,L)$, then r must be twice differentiable on $(0,L)$. This implies that we cannot require $r \geq 0$, as is usual with cylindrical coordinates. For with such a condition, r' would become discontinuous at any zero of a nontrivial solution r in that interval, and α as well. All nontrivial solutions r have at least one zero in $(0,L)$ as we shall see. As a consequence r can be both positive and negative, just as a cartesian coordinate. Since α is a constant for the nontrivial stationary solutions, there is no need to extend the range of α.

The equations (2.3) can be reduced to a single second order differential equation. If we define

$$u(s) = \frac{Tr'}{\rho g}, \tag{2.6}$$

it follows from (2.5) and (2.3d') that

$$T^2 = \rho^2 g^2 (u^2 + s^2).$$

Since $T > 0$ and $z' < 0$ for $s > 0$, we have on account of (2.5), (2.6) and (2.3e)

$$T = \rho g \sqrt{u^2 + s^2}, \tag{2.7}$$

$$z = \int_s^L \frac{\sigma d\sigma}{\sqrt{u^2(\sigma) + \sigma^2}}, \tag{2.8}$$

$$r = -\int_s^L \frac{u(\sigma) d\sigma}{\sqrt{u^2(\sigma) + \sigma^2}}. \tag{2.9}$$

Now substituting (2.6) in (2.3a') one has

$$u' = -\frac{\omega^2}{g} r. \tag{2.10}$$

Differentiating (2.10) and using (2.9) to eliminate r' one gets

$$u'' + \frac{\omega^2}{g} \frac{u}{\sqrt{u^2 + s^2}} = 0. \tag{2.11}$$

The boundary conditions for u(s) are obtained from (2.3e), (2.3f), (2.6) and (2.10):

$$u(0) = u'(L) = 0. \tag{2.12}$$

Substituting $u(s) = L\bar{u}(\bar{s})$, $s = L\bar{s}$ in (2.11) and (2.12), we obtain, after dropping all bars again, the following nonlinear eigenvalue problem:

$$\begin{cases} u'' + \dfrac{\lambda u}{\sqrt{u^2 + s^2}} = 0, \\ \\ u(0) = u'(1) = 0. \end{cases} \tag{2.13}$$

where

$$\lambda = \frac{\omega^2 L}{g}, \tag{2.14}$$

and $u \in C^1[0,1] \cap C^2(0,1)$.

This eigenvalue problem certainly has the trivial solution $u = 0$ for all values of λ. If for a certain value of λ there exists a nontrivial solution u, then -u is a solution as well. This, however, is equivalent to changing the angle α with an amount π. We shall regard all solutions u, which differ only in the value of α, as the same solution.

In the next subsection we shall show that with this identification the eigenvalue problem (2.13) has exactly n nontrivial solutions for $\lambda_n < \lambda \leq \lambda_{n+1}$, where λ_n, $n = 1,2,\ldots$ are the eigenvalues of the linearized problem.

2.3. MATHEMATICAL TREATMENT

2.3.1. THE LINEARIZED EQUATION

Linearization of (2.13) gives the equation

$$(2.15)\qquad \begin{cases} \tilde{u}'' + \frac{\lambda}{s}\tilde{u} = 0, \\ \tilde{u}(0) = \tilde{u}'(1) = 0, \end{cases}$$

with $\tilde{u} \in C^1[0,1] \cap C^2(0,1)$.
This eigenvalue problem can be solved explicitly. Substitution of $t = 2\sqrt{\lambda s}$, $u = tv(t)$ results in

$$\begin{cases} v'' + \frac{1}{t}v' + (1 - \frac{1}{t^2})v = 0, \\ v(0) = v'(2\sqrt{\lambda}) = 0, \end{cases}$$

which is the equation for the Bessel function of order one. The solution satisfying $v(0) = 0$ is

$$v(t) = \text{const. } J_1(t),$$

and so

$$(2.16)\qquad u(s) = \text{const. } \sqrt{s}\, J_1(2\sqrt{\lambda s}).$$

By using well-known relations for Bessel functions (cf. for instance ABRAMOWITZ & STEGUN [3]) it follows that

$$u'(s) = \text{const. } \sqrt{\lambda}\, J_0(2\sqrt{\lambda s})$$

and thus $u'(1) = 0$ if

$$(2.17)\qquad \lambda = \lambda_n = \frac{\beta_n^2}{4},$$

where $\{\beta_n\}_{n=1}^{\infty}$ are the zeros of $J_0(t)$, the Bessel function of order zero.

Hence (2.15) only has the trivial solution $\tilde{u} = 0$ unless $\lambda = \lambda_n$. If $\lambda = \lambda_n$ then there exists the nontrivial solution

$$(2.18)\qquad \tilde{u}_n(s) = \text{const. } \sqrt{s}\, J_1(2\sqrt{\lambda_n s}),$$

determined up to the constant factor.

To the eigenvalue λ_n there corresponds the angular eigenvelocity

$$\omega_m = \sqrt{\lambda_m}\ \sqrt{\frac{g}{L}} = \frac{\beta_m}{2}\sqrt{\frac{g}{L}}. \tag{2.19}$$

2.3.2. UNIQUENESS OF THE SOLUTION u = 0 FOR SUFFICIENTLY SMALL ω

The differential equation (2.13) is equivalent to the integral equation

$$u(s) = \lambda \int_0^1 G(s,\sigma)\ \frac{u(\sigma)\ d\sigma}{\sqrt{u^2(\sigma) + \sigma^2}}, \tag{2.20}$$

where

$$G(s,\sigma) = \begin{cases} \sigma, & 0 < \sigma < s, \\ s, & s < \sigma < 1, \end{cases} \tag{2.21}$$

is the Green's function associated with the operator $-\frac{d^2}{ds^2}$ and the boundary conditions $u(0) = u'(1) = 0$.

Let $\|\cdot\|$ be the norm corresponding to the inner product:

$$(u,v) = \int_0^1 u(s)v(s)ds. \tag{2.22}$$

Since we have

$$|u(s)| \le \lambda \int_0^1 G(s,\sigma)\ \frac{|u(\sigma)|}{\sqrt{u^2+\sigma^2}}\ d\sigma \le \lambda \int_0^1 G(s,\sigma)\ \frac{|u(\sigma)|}{\sigma}\ d\sigma,$$

we find, using the Schwarz inequality,

$$|u(s)| \le \lambda \|u\| \left\{\int_0^1 \frac{G^2(s,\sigma)}{\sigma^2}\ d\sigma\right\}^{\frac{1}{2}} = \lambda \|u\| \sqrt{2s-s^2}.$$

Squaring the last equation and integrating with respect to s, we obtain

$$\|u\|^2 \le \frac{2}{3}\lambda^2 \|u\|^2. \tag{2.23}$$

Hence, if $\|u\| \neq 0$, then $\lambda \geq \sqrt{\frac{3}{2}}$, which proves that for λ sufficiently small $u = 0$ is the unique solution of (2.13). Actually this result can be sharpened to: if $\|u\| \neq 0$, then $\lambda \geq \lambda_1 = \left(\frac{\beta_1}{2}\right)^2 \sim 1.446$.

2.3.3. THE EXACT NUMBER OF SOLUTIONS

THEOREM 2.3. *Let* $\lambda_n = \left(\frac{\beta_n}{2}\right)^2$, $n = 1,2,\ldots$. *If* $\lambda_n < \lambda \leq \lambda_{n+1}$ *then the nonlinear eigenvalue problem* (2.13) *has exactly* n *nontrivial solutions* $u_1, u_2, \ldots, u_n$, *and* u_k, $k = 1,\ldots,n$, *has exactly* k *isolated zeros (including* s=0*).*

This theorem is implied by theorem 2.4 (below) which deals with the solutions v(x,a) of the initial value problem

$$\begin{aligned} &\frac{d^2v}{dx^2} + \frac{v}{\sqrt{v^2 + x^2}} = 0, \quad v \in C^2(0,\infty), \\ &v(0,a) = 0, \quad \frac{dv}{dx}(0,a) = a \geq 0. \end{aligned} \tag{2.24}$$

Note that u is a solution of (2.13) with eigenvalue λ if and only if $u(s) = \lambda^{-1} v(\lambda s, a)$ for some a such that $v'(\lambda,a) = 0$.

The function $(v^2+x^2)^{-\frac{1}{2}}$ is not Lipschitz continuous at $v = 0$, $x = 0$. So the existence and uniqueness of v are not trivial. KOLODNER [2] gives the following theorem:

THEOREM 2.4.

(i) v(x) *exists, is unique, and depends continuously on* a.

(ii) v(x) *has an infinite number of isolated zeros* $y_n(a)$, $0 < y_0 < y_1 < \ldots < y_n < \ldots$, *and* $y_n \to \infty$ *as* $n \to \infty$; *likewise* v'(x) *has an infinite number of isolated zeros* $z_n(a)$, $n = 1,2,\ldots$, *interlacing the* y_n; *furthermore,* $\lim_{a\to 0} z_n(a) = \lambda_n$, $\lim_{a\to\infty} z_n(a) = \infty$.

(iii) *the* $z_n(a)$ *are differentiable functions of* a, *and* $\frac{dz_n}{da} \geq 0$.

For the proof of this theorem we refer to KOLODNER [2].

We shall now show that theorem 2.4 implies theorem 2.1 (cf. figure 2). Let λ be a number such that $\lambda_n < \lambda < \lambda_{n+1}$. Since the $z_k(a)$ are monotonically increasing from $\lambda_k = z_k(0)$ to infinity, there is for each $k \leq n$ exactly one value $a_k(\lambda) > 0$ of a, such that $z_k(a_k(\lambda)) = \lambda$, whereas for $k \geq n+1$,

$z_k(a) \geq \lambda_{n+1} > \lambda$. Correspondingly, we have exactly n nontrivial solutions $v_k(x) = v(x,a_k(\lambda))$ such that $v_k'(\lambda) = 0$; furthermore, v_k has exactly k zeros in the interval $[0,\lambda)$ since $y_{k-1}(a_k) < z_k(a_k) = \lambda < y_k(a_k)$. If $\lambda = \lambda_{n+1}$, there is in addition to the n nontrivial solutions just listed, the solution $v_{n+1} = v(x,a_{n+1})$, with $z_{n+1}(a_{n+1}) = \lambda_{n+1}$. But this implies that $a_{n+1} = 0$, so that $v_{n+1} = 0$, the trivial solution. So we have exactly n nontrivial solutions $v_k(x)$ for $\lambda_n < \lambda \leq \lambda_{n+1}$. We now observe that

$$u(s) = \lambda^{-1} v(\lambda s,a)$$

for some a, hence

$$u_k(s) = \lambda^{-1} v_k(\lambda s), \quad k = 1,2,\ldots,n,$$

and so we have theorem 2.3.

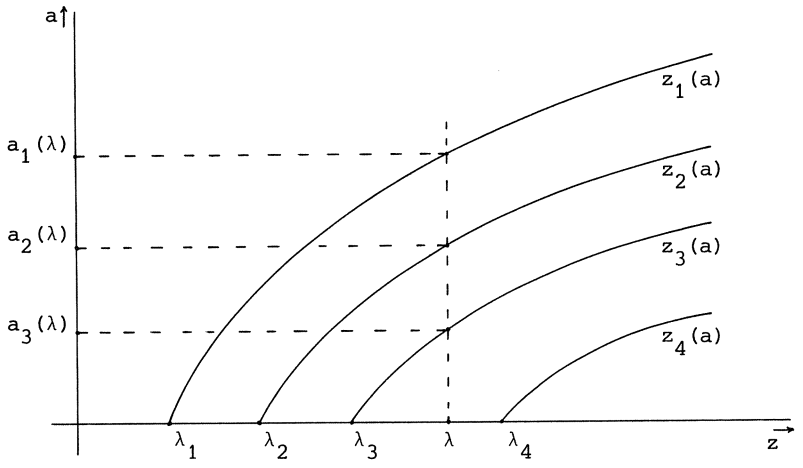

Figure 2

3. NONLINEAR HEAT GENERATION IN CONDUCTING SOLIDS AND AN ISOTHERMAL GASSPHERE IN GRAVITATIONAL EQUILIBRIUM

In this section we shall study an equation for nonlinear generation of heat. This will lead to a nonlinear eigenvalue problem with eigenvalue λ. Under quite natural conditions upon the nonlinearity it will be shown that there exists a finite critical value λ_* beyond which equilibrium solutions do not exist. For $\lambda < \lambda_*$ there may be more than one equilibrium solution.

Especially we shall give the exact number of solutions for the case that the nonlinearity is given by $F(u) = e^u$ and the domain is spherical. The spherical symmetry allows one to reduce the partial differential equation to an ordinary differential equation which can be treated in the phase plane. This was first done by Emden in his investigation of the equation for an isothermal gassphere in gravitational equilibrium.

In preparing this section we used mainly the articles of JOSEPH [4], JOSEPH & LUNDGREN [5] and the book of CHANDRASEKHAR [6].

3.1. MATHEMATICAL MODEL FOR HEAT GENERATION BY A NONLINEAR SOURCE

One physical motivation for the problems we study concerns the temperature distribution in an object heated by the application of a uniform electric current (Joule heating). If the body is homogeneous with mass density ρ, specific heat c and thermal conductivity K, if the electrical conductivity $\sigma(T)$ of the body is a function of the temperature $T(x,t)$ and j is the constant current density, then the temperature satisfies the equation

$$\rho c \frac{\partial T}{\partial t} = K \Delta T + j^2/\sigma(T). \tag{3.1}$$

In particular we are interested in the dependence of the equilibrium solutions of (3.1) upon j. For such solutions the left hand side of (3.1) equals zero. If we assume that the surface $\partial\Omega$ of the body is kept at a constant temperature T_0, the equations for an equilibrium solution are

$$0 = K \Delta T + j^2/\sigma(T) \quad \text{in } \Omega, \quad T = T_0 \text{ at } \partial\Omega. \tag{3.2}$$

If the body is a plane plate, and the x-axis is chosen perpendicular to the two faces, (3.2) becomes, after an appropriate rescaling,

$$(3.3)\qquad \begin{cases} \dfrac{d^2u}{dx^2} + \lambda F(u) = 0, \\ u(1) = u(-1) = 0, \end{cases}$$

where now u stands for the dimensionless temperature, F(u) for the resistance as function of temperature and λ for the current parameter ($\lambda \sim j^2$).

A positive solution of (3.3) has at least one maximum in (-1,1). If we make the reasonable assumption that F(s) is positive for positive values of s, then the curvature of u is negative. Thus there is exactly one maximum of u(x) in (-1,1). By symmetry the maximum occurs at x = 0 and we may replace (3.3) by

$$(3.4)\qquad \begin{cases} \dfrac{d^2u}{dx^2} + \lambda F(u) = 0, \\ u'(0) = u(1) = 0. \end{cases}$$

Similarly if the body is a cylindrical conductor, the problem becomes

$$(3.5)\qquad \begin{cases} \dfrac{1}{r}\dfrac{d}{dr}\left(r\dfrac{du}{dr}\right) + \lambda F(u) = 0, \\ u'(0) = u(1) = 0. \end{cases}$$

More generally we can consider the spherically symmetric solutions of

$$(3.6)\qquad \begin{cases} \Delta u + \lambda F(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is the unit ball in $\mathbb{R}^n$. This leads to

$$(3.7)\qquad \begin{cases} \dfrac{1}{r^{n-1}}\dfrac{d}{dr}\left(r^{n-1}\dfrac{du}{dr}\right) + \lambda F(u) = 0, \\ u'(0) = u(1) = 0. \end{cases}$$

In (3.4) n = 1, in (3.5) n = 2. In subsection 3.2 we shall give some features of $u(r,\lambda)$ supposing some general monotonicity conditions on F(u). In subsection 3.3 we shall give the exact number of solutions of (3.7) as a function of n and λ for the special case that $F(u) = e^u$.

3.2. A BOUND FOR λ IN A QUITE GENERAL CASE

If ϕ_0 is the eigenfunction belonging to the first eigenvalue γ_0 of the linear problem

$$(3.8)\qquad \begin{cases} \dfrac{1}{r^{n-1}}\dfrac{d}{dr}\left(r^{n-1}\dfrac{d\phi_0}{dr}\right) + \gamma_0\phi_0 = 0, & 0 < r < 1, \\ \phi_0'(0) = \phi_0(1) = 0, \end{cases}$$

then we have $\phi_0(r) > 0$ for $0 \le r < 1$, and by partial integration

$$\int_0^1 \frac{du}{dr}\frac{d\phi_0}{dr} r^{n-1}dr =$$

$$(3.9)\qquad \gamma_0 \int_0^1 u(r)\phi_0(r)r^{n-1}dr = \lambda \int_0^1 F(u(r))\phi_0(r)r^{n-1}dr$$

and thus λ satisfies

$$(3.10)\qquad \lambda = \gamma_0 \frac{\int_0^1 u\phi_0 r^{n-1}dr}{\int_0^1 F(u)\phi_0 r^{n-1}dr}$$

If there exists M, $0 < M < \infty$ such that

$$(3.11)\qquad s \le MF(s) \quad \text{for } s \ge 0,$$

then (3.10) results in

$$(3.12)\qquad \lambda \le M\,\gamma_0.$$

I.e., there exists a least upper bound λ_* such that no positive solutions exist when $\lambda < \lambda_*$, and positive solutions are not necessarily unique when $\lambda < \lambda_*$. Clearly if $\{\lambda\}$ is the set of positive numbers for which positive solutions of (3.7) exist, then

$$(3.13)\qquad \lambda_* = \sup\{\lambda\}.$$

Examples of functions which satisfy (3.11) are e^u and $(1+\alpha u)^\beta$, $\alpha\beta > 0$.

From (3.7) we find, by multiplying the differential equation by r^{n-1} and subsequent integrating, that

$$r^{n-1}\frac{du}{dr} = -\lambda\int_0^r t^{n-1}F(u(t))dt.$$

Since F(s) is positive for positive values of s, it follows that a positive solution of (3.7) is a strictly monotone decreasing function on (0,1). Hence, the maximum A of $u(r,\lambda)$ is at the origin:

$$u(0,\lambda) = A. \tag{3.14}$$

In the analysis we shall find it convenient to regard A, instead of λ, as preassigned, because for a given λ there can be more than one $u(r,\lambda)$. Let us replace (3.7) by

$$\begin{cases} \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{du}{dr}\right) + \lambda F(u) = 0, \\ u(0) = A,\ u'(0) = u(1) = 0. \end{cases} \tag{3.15}$$

In (3.7) λ is given and two side conditions are imposed. In (3.15) three side conditions are imposed and solutions exist only when λ takes on special values $\lambda = \lambda(A)$. Solutions of (3.7) are designated by $u(r,\lambda)$. Solutions of (3.15) are designated by the pair $[u(r,A), \lambda(A)]$

The solution set of (3.7) is conveniently described by the function $\lambda(A)$. If λ is bounded by λ_* we have the following possibilities. (See figure 3a, b):

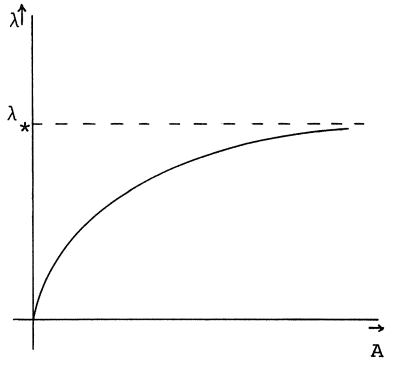

Figure 3a

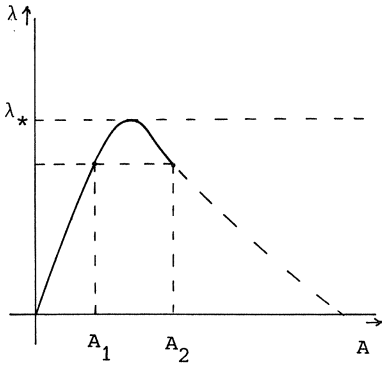

Figure 3b

(i) $\lambda(A)$ is a monotone increasing function of A, and $\lambda_* = \lim_{A\to\infty} \lambda(A)$. Then we will have uniqueness of the solutions. A special case is the linear case $n = 1$, $F(u) = 1 + \alpha u$.

(ii) $\lambda(A)$ has at least one isolated maximum. Then solutions with λ in the neighbourhood of the maximum of $\lambda(A)$ will not be unique.

For $n = 1$ (plane plate) JOSEPH [4] shows that $\lambda(A) \to 0$ for $A \to \infty$ if $F(u)$ satisfies

$$F(u) \geq 1,\ F'(u) \geq 1 \text{ and } F''(u) \geq 0 \tag{3.16}$$

for $u \geq 0$, where the equalities apply only for $u = 0$, and thus $(\lambda(0)=0)$ there will be at least two solutions for $\lambda < \lambda^*$. The solution belonging to the smallest value of A (lowest temperature) is stable.

REMARK 3.1. Instead of the lower bound 1 in (3.16) we could have taken any positive numbers a,b as lower bound for $F(u)$ and $F'(u)$; by transformation of variables we can bring the lower bounds to one.

REMARK 3.2. Iron, tungsten and gold are but a few of the metals which satisfy the above conditions over a wide range of temperatures.

REMARK 3.3. One can prove that in the neighboorhood of the first maximum of $\lambda(A)$ there are just two branches of the solution. The first branch is stable and the second unstable. The second branch has a higher maximum temperature and could presumably be started by preheating and maintained by large dissipation with small currents. The high temperatures are certainly unstable and if disturbed would decrease to values compatible with the stable solution at the given current.

3.3. RESULTS FOR THE SPECIAL CASE $F(u) = e^u$

A special function which satisfies (3.16) is

$$F(u) = e^u. \tag{3.17}$$

JOSEPH & LUNDGREN [5] give the following description of the solution set of (3.7) with $F(u)$ given by (3.17):

There exists a finite positive value λ_* *depending on* n, *such that there are*

(a) *no solutions when* $\lambda > \lambda_*$ $(n \geq 1)$,

(b) *one solution when* $\lambda = \lambda_*$ $(n \geq 1)$,

(c) *two solutions when* $0 < \lambda < \lambda_*$ $(n = 1,2)$,

(d) *an infinite number of solutions when* $\lambda = 2$ $(n = 3)$,

(e) *a large but finite number of solutions when* $|\lambda - 2| \neq 0$ *is small* $(n = 3)$,

(f) *an infinite number of solutions when* $\lambda = 2(n-2)$ $(n < 10)$,

(g) *a large but finite number of solutions when* $|\lambda - 2(n-2)| \neq 0$ *is small* $(n < 10)$,

(h) *one solution for each* $\lambda < 2(n-2)$ $(n \geq 10)$.

These results are visualized in figure 4.

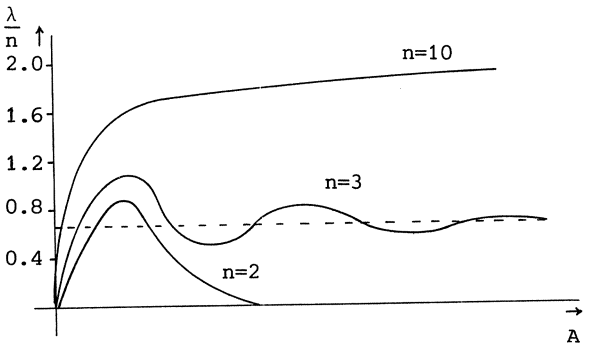

Figure 4

REMARK 3.4. JOSEPH & LUNDGREN [5] give a similar list of uniqueness properties for the function $F(u) = (1+\alpha u)^{\beta}$, $\alpha\beta > 0$.

In subsection (3.5) we shall give the proof for the case n = 3. The cases n > 3 can be treated similarly and are given by JOSEPH & LUNDGREN. For the case n = 1 and n = 2 see references in JOSEPH & LUNDGREN [5].

3.4. THE EQUATION FOR AN ISOTHERMAL STAR IN GRAVITATIONAL EQUILIBRIUM

A special case of (3.7) is $n = 3$ and $F(u) = e^u$. For this value of n there exist infinitely many solutions if $\lambda = 2$. This differential equation is investigated by Emden in connection with an isothermal star in gravitational equilibrium. For reference see CHANDRASEKHAR [6].

For the mass of a spherically symmetric distribution of matter, the total pressure P, the density ρ and the other physical variables will all be functions of r only. Let $M(r)$ be the mass inclosed inside the sphere of radius r, then

$$M(r) = \int_0^r 4\pi x^2 \rho \, dx, \tag{3.18}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho. \tag{3.19}$$

For gravitational equilibrium we should have

$$\frac{dP}{dr} = - G \frac{M(r)}{r^2} \rho \ , \tag{3.20}$$

or using (3.19)

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = - 4\pi G \rho. \tag{3.21}$$

The total pressure P is equal to the gaskinetic pressure and the radiation pressure:

$$P = \frac{k}{\mu H} \rho T + \frac{a}{3} T^4. \tag{3.22}$$

Here k is the Boltzmann constant, μ is the mean molecular weight, H the mass of the proton and a the Stefan-Boltzmann constant.
If T is taken to be constant (isothermal gassphere), we can write

$$P = K\rho + D, \ K = \frac{k}{\mu H} T, \ D = \frac{a}{3} T^4, \tag{3.23}$$

and (3.21) becomes

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d \ln \rho}{dr} \right) = - \frac{4\pi G}{K} \rho. \tag{3.24}$$

Putting

(3.25) $\rho = e^u$

yields the equation

(3.26) $\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + \lambda e^u = 0,$

where $\lambda = \frac{4\pi G}{K}$.

The initial values are

(3.27) $u'(0) = 0$ and $u(0) = \ln \rho_c = A.$

Because of the nature of initial value problems we will find exactly one solution for (3.26), (3.27). It will be shown that the solution will behave as the singular solution

(3.28) $\rho \simeq \frac{2K}{4\pi G} \frac{1}{r^2}$ for $r \to \infty$.

3.5. THE PHASE PLANE METHOD FOR THE CASE n = 3

For $n = 3$ and $F(u) = e^u$, (3.7) and (3.15) become

(3.29)
$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + \lambda e^u = 0 & r \in [0,1], \\ u'(0) = u(1) = 0, \end{cases}$$

(3.30)
$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{du}{dr}\right) + \lambda e^u = 0 & r \in [0,1], \\ u(0) = A,\ u'(0) = u(1) = 0, \end{cases}$$

respectively. The initial value problem corresponding to (3.30) is the same as that for the isothermal star, and is given by (3.26), (3.27). In the following we shall denote this initial value problem by E.

REMARK 3.5. The function e^s is positive for all finite values of s. This implies that u(r) attains its maximum for r = 0 and decreases monotonically

for $r > 0$. So $u(r)$ has one zero $r_0(\lambda)$ and is positive for $0 \le r < r_0$ and negative for $r > r_0$.

A special role will be played by the function

$$u_s(r) = \ln(\frac{1}{r^2}),\ \lambda_s = 2, \tag{3.31}$$

the so-called limiting singular solution.

DEFINITION 3.6. A pair $[u^*(r), \lambda]$ is called a *limiting singular solution* of (3.30) if

(i) $[u^*r),\lambda]$ satisfies (3.30) on $0 < r \le 1$, and

(ii) there exists a family of regular solutions $[u(r,A), \lambda(A)]$ such that as $A \to \infty$, then $[u(r,A), \lambda(A)] \to [u^*(r), \lambda]$.

The pair $[u_s,\lambda_s]$ satisfies condition (i). Hence it is a limiting singular solution if for $A \to \infty$

$$[u(r,A),\lambda(A)] \to [u_s,\lambda_s]. \tag{3.32}$$

That (3.32) is satisfied will be a consequence of the analysis in the phase plane. All E solutions map into one curve in the phase plane, the E curve, and the solution $[u_s,\lambda_s]$ maps into a critical point. When (3.32) holds, the critical point terminates the E curve and the properties of the critical point determine the properties of the solution.

In preparation for the phase-plane we shall change variables:

$$\begin{cases} z = u - \ln(\frac{1}{r^2}) + \ln(\frac{1}{2}\lambda), \\ t = -\ln r - \frac{1}{2}A - \frac{1}{2}\ln(\frac{1}{2}\lambda), \end{cases} \tag{3.33}$$

or equivalently

$$\begin{cases} u = z + 2t + A, \\ r = [\frac{2}{\lambda}e^{-A}]^{\frac{1}{2}}e^{-t}. \end{cases} \tag{3.34}$$

Under this transformation the differential equation in (3.26), (3.29) and (3.30) becomes

$$\frac{d^2z}{dt^2} - \frac{dz}{dt} + 2e^z - 2 = 0. \tag{3.35}$$

Substitution of $y = \frac{dz}{dt}$ gives the autonomous differential equation

$$\begin{cases} \frac{dy}{dt} = y - 2e^z + 2, \\ \frac{dz}{dt} = y. \end{cases} \tag{3.36}$$

We consider this equation in the phase plane. The integral curves satisfy

$$\frac{dy}{dz} = \frac{y-2e^z+2}{y}. \tag{3.37}$$

The initial conditions which replace (3.27) are

$$z \to -\infty,\ y \to -2,\ \frac{dy}{dz} \to 0 \tag{3.38}$$

as $r \to 0$ or $t \to \infty$.

We shall prove that these initial conditions define a unique integral curve in the phase plane denoted by the E curve, and that this integral curve spirals around the equilibrium point (0,0) for $t \to -\infty$. See figure 5.

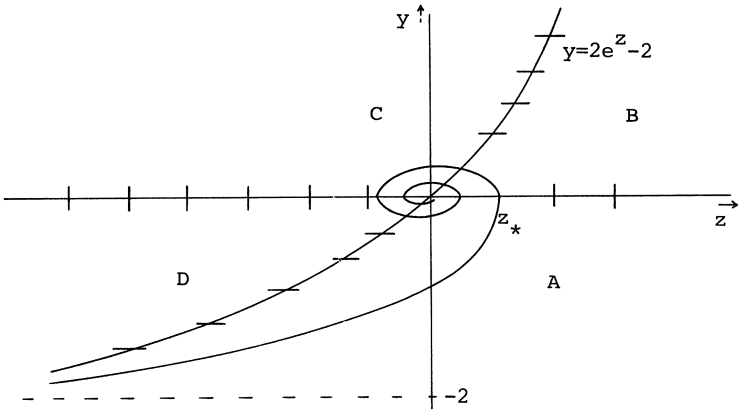

Figure 5

LEMMA 3.7. *There is only one integral curve satisfying* $z \to -\infty$, $y \to -2$.

PROOF. Suppose y and y^* are two different solutions such that $y \sim -2$, $y^* \sim -2$, $z \to -\infty$. We may suppose that $y < y^*$ as $z \to -\infty$. Then we should have $\Delta = y^* - y > 0$; $\lim_{z\to-\infty} \Delta = 0$. From the differential equation (3.37) we derive

$$\frac{d\Delta}{dz} = -2\,\frac{1-e^z}{y^* y}\,\Delta.$$

Hence $\lim_{z\to-\infty} \frac{d \log \Delta}{dz} = -\lim_{z\to-\infty} \left[\frac{2(1-e^z)}{y^* y}\right] = -\frac{1}{2}$,

or $\Delta \sim \text{constant} \cdot e^{-\frac{1}{2}z}$ for $z \to -\infty$, which contradicts our assumption that $\Delta \to 0$ as $z \to -\infty$. □

Equation (3.36) has only the equilibrium point (0,0). By the Bendixson criterion (theorem V. 6.1) there are not limit cycles as

$$\frac{\partial}{\partial y}\left[\frac{dy}{dt}\right] + \frac{\partial}{\partial z}\left[\frac{dz}{dt}\right] = 1 \neq 0, \tag{3.39}$$

so if the E curve is bounded for $t \to -\infty$, it has to approach (0,0) for $t \to -\infty$. From the direction field it follows that the E curve cannot escape to infinity for $t \to -\infty$.

The nature of the equilibrium point is characterized by the eigenvalues of the Jacobian matrix of (3.36) in (0,0):

$$F'(0,0) = \begin{pmatrix} 1 & -2 \\ 1 & 0 \end{pmatrix}. \tag{3.40}$$

The eigenvalues

$$\lambda_{1,2} = \tfrac{1}{2} \pm \tfrac{1}{2}\, i\sqrt{7}, \tag{3.41}$$

are complex conjugated and (0,0) is a spiral point to which the integral curves approach for $t \to -\infty$.

And so the behaviour of the E curve will be as in figure 5.

Until now we only studied the initial value problem (3.26), (3.27). For the boundary value problem (3.30) we have in addition the condition

$u(1) = 0$. The uniqueness properties for solutions of (3.29) are a consequence of remark 3.5 and of the following. Let $(z_1, y(z_1))$ be a point on the E curve (cf. figure 5) which corresponds to $r = 1$. By (3.33) $z_1 = \ln \frac{\lambda}{2}$,

hence

$$\lambda = 2e^{z_1} \tag{3.42}$$

is determined when z_1 is known. The number of solutions for a given z (or λ) is just the number of values y on the E curve at a given z (or λ). From figure 5 we see that for

$$\lambda > \lambda_* = 2e^{z_*} \tag{3.43}$$

equation (3.29) has no solutions, and for $\lambda = 2$ ($z=0$) there is an infinity of solutions.

REMARK 3.8. The same method can be used to prove the uniqueness properties of (3.7) with $F(u) = e^u$ for $n > 3$. For $n = 10$, the equilibrium point becomes a nodal point. Cf. JOSEPH & LUNDGREN [5].

REMARK 3.9. The consequence of the nature of the integral curve of the E solution in the phase plane is for an isothermal star that the density will make a damped oscillation around the density given by the singular solution (3.28). This is called the *law of density distribution* (cf. CHANDRASEKHAR [6]).

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X. VARIATIONAL METHODS FOR NONLINEAR OPERATOR EQUATIONS

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GENERAL INTRODUCTION

The aim of this chapter is to get the reader accustomed to some of the principles that lie at the roots of what are generally called variational methods. These methods can be viewed as being originated from ideas of the classical theory of the calculus of variations, formulated and extended in a way that is particular appropriate for the study of nonlinear equations.

The study of variational methods has been developed in the last fifty years, originally mostly by Russian mathematicians in their study of nonlinear integral equations. For instance, most concepts described in this chapter are to a large extent due to the work of VAINBERG and KRASNOSELSKII, whose monographs [1] and [2] respectively may be considered to be the main references for the entire chapter. During this time, there has been an increase in the application of these methods to other nonlinear equations, including, amongst many others, equations that arise from the study of boundary value problems, bifurcation problems and optimal control.

Besides for this applicability to specific problems, another reason to study variational methods is that other topological methods to treat nonlinear equations, such as e.g. the theory of monotone operators, are often better understood when brought in connection with ideas and results from the study of variational methods.

Concerning the organization of the chapter, I should like to emphasize once again that it is meant as an introduction to some of the ideas and notions of variational methods, with the aim to prepare the reader to study more advanced topics, such as the Ljusternik-Schnirelman theory and to find a way in the current literature of this field.

For this reason I have tried to explain some concepts and to describe some specific problems in a more extensive way than is usually done in literature. It is hardly needed to say that with this intention no new results can be expected. Several applications are included, both to clarify the new concepts and to demonstrate the methods to specific problems. In this respect it must be regretted that hardly anything from a rather recently discovered field of applicability, namely from the theory of nonlinear dispersive waves, could be included in a chapter of this desired size. (See for this, e.g. LAX [3] and contributions of BENJAMIN and LAX in [4]).

Before coming to a detailed description of the contents of the chapter, it may be useful to remark that, although the methods that are described

here are especially developed to study nonlinear problems, these ideas are generally also applicable to tackle linear problems (examples will be given in subsections 3.2.1 and 4.2.5). However, for linear problems there are some other variational methods with which stronger results can be obtained: the interested reader is referred to the excellent books of MIKHLIN, [5], [6] and [7].

This chapter consists of four sections. Each section is divided into several subsections of different extent: from simple remarks to extensive descriptions of applications.

Section 1 deals with some aspects from functional analysis which are important in the following. In subsection 1.1 concepts like dual Banach space and weak convergence are described and some useful theorems are cited. The reader who does not meet in every day life such a notion as dual space, needs not to feel frightened. Although this notion is essential for a full comprehension of the meaning of the gradient of a functional (to be treated in section 2), the analysis in most applications takes place in some appropriate Hilbert space, and for Hilbert spaces the dual can be taken to be this space itself (Riesz representation theorem). The notion of weak convergence, however, is of extreme importance for all the following results. In subsection 1.2 various forms of continuity that can be possessed by an operator and a functional are investigated. In subsection 1.3 a well-known result from classical analysis, namely Weierstrass' theorem, which reads that a continuous real valued function attains its maximum and its minimum on every compact interval, is generalized for weakly continuous functionals and weakly compact sets in a Banach space. This theorem is of extreme importance as will become clear from the following. In subsection 1.4 a concept that is generally met when dealing with (classical) Calculus of Variations, namely lower (and upper) semi-continuity is defined, with the aid of which a result analogous to and equally important as the generalized Weierstrasz theorem can be formulated. In subsection 1.5 the Sobolev space H_0^1 is described and the simplest properties of the embedding in L_p spaces are stated (for the applications dealt with in this chapter, no higher Sobolev spaces are needed).

Section 2, subsection 2.1, shows how the notion of derivative of a real valued function can be generalized to operators and functionals. In subsection 2.2 the classical mean value theorem for functions is shown to be true for functionals and is generalized for operators. Subsection 2.3 deals with one of

the most important questions, namely what kind of nonlinear problems can be handled with the variational methods that are to be developed in the sequel. Therefore, it is necessary that the equation can be looked at as being the equation for the stationary points of a suitable chosen functional. Operators that are the derivative of a functional are called potential operators. Criteria to decide whether a given operator is a potential one are given and the construction of the potential, i.e., the functional whose derivative is the given operator, is investigated. As an application it is investigated, in subsection 2.3.8, which first order (in time) equations, arising in the theory of nonlinear wave propagation, can be derived from a Lagrange principle. In subsection 2.4 a special kind of nonlinear operators, Nemytsky operators on L_p spaces, are considered and their potentialness is studied. These operators are of importance in the study of nonlinear boundary value problems and integral equations. In subsection 2.5, to demonstrate what has been said here before, the connection between monotone operators and a special kind of functionals is investigated.

Section 3 deals with functionals that are considered on all of a Banach space. In subsection 3.1 the notion of stationary point of a functional is defined and it is shown that the classical result for real valued functions, namely that every extreme point of a function is a stationary point, remains true for functionals. This is of extreme importance in the whole theory dealing with questions concerning existence theorems. Suppose one is interested in the question whether a given operator equation admits a solution. Provided the operator is a potential one, this can be transformed to the question whether the derivative of the potential is zero at some point. From the above result, this will surely be the case if the potential has an extreme point, which in some cases can be concluded to be true by using results like the generalized Weierstrasz theorem. This way of reasoning is more extensively explained in the introduction of subsection 3.2, subsection 3.2.0. In subsection 3.2.1 the transformation of a simple boundary value problem, viz. the linear Dirichlet problem for the Laplace equation, to an operator equation in a Hilbert space that is appropriate for a study with variational methods, is described in some detail for convenience of those readers who are not accustomed to this Hilbert space approach. Then, in subsection 3.2.2 the existence of solutions of the nonlinear (sublinear) Dirichlet problem is investigated. In subsection 3.3 some fixed point theorems are derived and, by way of example, in subsection 3.3.7,

the existence of solutions of nonlinear (sublinear) Hammerstein integral equations is studied.

In section 4 we are considering functionals not on all of a Banach space but on a "hyper-surface" (manifold) in this space: the functional is restricted to a manifold. From classical analysis the Lagrange multiplier rule for real valued functions depending on, say, two independent variables constrained to some curve in $\mathbb{R}^2$, is known to deal with the equations that must necessarily be fulfilled by the extreme points of the function with respect to this curve. This rule can be extended to functionals defined on a manifold in a Banach space, which result is the celebrated theorem of Ljusternik. This theorem and its proof are dealt with in subsection 4.1. From this theorem it follows that the extreme points of a functional restricted to a special manifold, viz. a sphere in a Hilbert space, must necessarily be eigenfunctions of the operator which is the derivative of this functional. This provides us with a method to prove the existence of eigenvalues of specific eigenvalue problems: if the operator is a potential one, the eigenvalue problem is equivalent to the equation for the extreme points of its potential on a sphere. If it can be shown, for instance with the generalized Weierstrasz theorem, that this potential, considered on the closed ball, has an extremum at some point of the boundary, i.e., on a sphere, then the existence of a non-zero eigenfunction will have been proved. In subsection 4.2, after a treatment of some notions that are important for eigenvalue and bifurcation problems, this method is demonstrated in subsection 4.2.5 for the linear eigenvalue problem for the Laplace operator. For this linear problem it is shown how it can be proved that there are infinitely many eigenvalues with a corresponding complete set of eigenfunctions. Furthermore, in subsection 4.2, some theorems concerning the existence of solutions of nonlinear eigenvalue problems and some results concerning bifurcation theory are stated.

1. OPERATORS AND FUNCTIONALS ON BANACH SPACES

1.1. BANACH SPACES, DUAL SPACES

1.1.1. NOTATION. Here and in the sequel X and Y will stand for Banach spaces over the scalar field of *real* numbers. The norm will be denoted by $\| \ \|$, or if there is chance for misunderstanding by $\| \ \|_X$ and $\| \ \|_Y$ respectively.

Convergence (in norm) of a sequence $\{x_n\} \subset X$ to some element $\hat{x} \in X$ will be denoted by $x_n \to \hat{x}$, thus $x_n \to \hat{x}$ means $\|x_n - \hat{x}\|_X \to 0$ for $n \to \infty$.

A mapping from X into Y is said to be *bounded* if it maps bounded sets of X into bounded sets of Y. The class of all bounded, linear mappings from X into Y is denoted by B(X,Y). In particular $B(X,\mathbb{R})$ is of extreme importance for us.

1.1.2. DUAL SPACE. The class of all bounded linear functionals defined on X, $B(X,\mathbb{R})$, will be denoted by X^*. Supplied with the norm

$$\|f\| := \sup_{\|x\| \le 1} |f(x)| \quad \text{for } f \in X^*, \qquad (x \in X),$$

it is a Banach space, and it is called the *normed dual of* X. Elements of X^* will often be written as x^*, and the value of x^* in some element $x \in X$, $x^*(x)$ is written as $\langle x,x^* \rangle$. Thus we have for instance

$$\|x^*\| = \sup_{\|x\| \le 1} |\langle x,x^* \rangle|, \qquad x^* \in X^*, \tag{1.1.1}$$

from which it follows that

$$|\langle x,x^* \rangle| \le \|x\| \|x^*\|, \qquad x \in X,\ x^* \in X^*. \tag{1.1.2}$$

When dealing with a *Hilbert space* H, things become much easier. By the *Riesz representation theorem* we know that to each bounded linear functional ℓ on H there is associated a unique element $w(\ell) \in H$ such that

$$\ell(h) = (h,w(\ell)) \qquad \text{for every } h \in H.$$

For this reason the Hilbert space H and its dual H^* can be identified, if for the duality map the inner product of H is used, and the norm of some element $\ell \in H^*$ is simply the norm of the element $w(\ell) \in H$.

1.1.3. EXAMPLE. Let Ω be some bounded domain in $\mathbb{R}^n$. The function class $L_p(\Omega)$, $p \geq 1$, consists of Lebesgue-measurable functions defined on Ω that are to the p-th power absolutely integrable in the sense of Lebesgue over Ω, and where functions differing from each other on a set of measure zero are identified. Supplied with the norm

$$|f|_p := \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{1/p},$$

$L_p(\Omega)$ is a Banach space. $L_\infty(\Omega)$ consists of measurable functions f defined on Ω which are essentially bounded, i.e., for which there exists a number A such that $|f(x)| \leq A$ almost everywhere on Ω (i.e., except possibly on a set of measure zero). The smallest possible number A for which this is true is called the essential least upper bound or essential supremum of f: ess sup $f = |f|_\infty$. Identifying functions differing on a set of measure zero, L with norm $|\ |_\infty$ is a Banach space.

For $p > 1$ the dual space of $L_p(\Omega)$ can be identified with $L_q(\Omega)$, where q is such that $\frac{1}{p} + \frac{1}{q} = 1$, if for the duality map between L_p and L_q is taken

$$\langle f,g \rangle = \int_\Omega f(x)g(x)dx \quad \text{for} \quad f \in L_p(\Omega), \quad g \in L_q(\Omega).$$

This means: with every bounded linear functional ℓ defined on $L_p(\Omega)$ there is associated a function $g \in L_q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$, such that

$$\ell(f) = \int_\Omega f(x)g(x)dx \quad \text{for every } f \in L_p(\Omega).$$

1.1.4. WEAK CONVERGENCE. Besides the concept of convergence that was stated in 1.1.1 (and which is sometimes called strong convergence, or convergence in norm to distinguish from the following notion), we shall have to deal with another kind of convergence, namely weak convergence. To introduce this concept, suppose we have a sequence $\{x_n\} \subset X$ converging to some element $\hat{x} \in X$: $x_n \to \hat{x}$. From (1.1.2) it then follows that

$$\langle x_n, x^* \rangle \to \langle \hat{x}, x^* \rangle \quad \text{for } n \to \infty, \text{ for every } x^* \in X^*. \tag{1.1.3}$$

Note that the convergence in (1.1.3) is convergence for a sequence of real numbers. Thus we have that if $x_n \to \hat{x}$ in X, then (1.1.3) holds. But the converse is generally not true, i.e., there exist sequences $\{x_n\} \subset X$ for which there exists an element $\hat{x} \in X$ such that (1.1.3) holds, but for which

the statement $x_n \to \hat{x}$ is not true. Therefore, the following definition makes sense.

DEFINITION. A sequence $\{x_n\} \subset X$ is said to converge *weakly* to some element $\hat{x} \in X$ if

$$\langle x_n, x^* \rangle \to \langle \hat{x}, x^* \rangle \quad \text{for } n \to \infty, \text{ for every } x^* \in X^*.$$

This weak convergence is written as $x_n \rightharpoonup \hat{x}$. In a Hilbert space H this definition amounts to:

$$x_n \rightharpoonup \hat{x} \text{ in } H \text{ if and only if } (x_n, h) \to (\hat{x}, h) \text{ for } n \to \infty, \text{ for every } h \in H.$$

1.1.5. Concerning weak convergence we note the following facts:

(a) weak limits are unique, i.e., if $x_n \rightharpoonup \hat{x}$ and $x_n \rightharpoonup \hat{y}$ then $\hat{x} = \hat{y}$,

(b) if $x_n \rightharpoonup \hat{x}$, then $\{x_n\}$ is uniformly bounded in X, i.e., there exists a number $M > 0$ such that $\|x_n\| < M$.

1.1.6. EXAMPLE. Let $\{e_n\}$, $n = 1,2,\ldots$, be an orthonormal base for the Hilbert space H. This means that each element $x \in H$ can be decomposed as

$$x = \sum_{n=1}^{\infty} (e_n, x)\, e_n,$$

with the meaning that

$$\|x - \sum_{n=1}^{N} (e_n, x)\, e_n\| \to 0 \quad \text{for } N \to \infty.$$

From this it follows that $(e_n, x) \to 0$ for $n \to \infty$. As this is true for every $x \in H$, this means that $e_n \rightharpoonup 0$ in H. But it is obvious that the orthonormal system $\{e_n\}$ does not converge (in norm). This serves as an example of a sequence that is weakly convergent but that is not strongly convergent.

1.1.7. Just as convergence (in norm) introduces the concepts of sequential pre-compactness and sequential compactness of a set $\Omega \subset X$, weak convergence introduces the notion of weak sequential (pre-)compactness. Before stating the definition, we shall from now on omit the adverb "sequential" in relation to (weak pre-)compactness. This is done for shortness and because of our particular interest (throughout the whole chapter) in the (weak) convergence of sequences, in distinction to the more general notion of weak topology (with which we shall not deal in this chapter).

DEFINITION. A set $\Omega \subset X$ is called *weakly pre-compact* if for every sequence $\{x_n\} \subset \Omega$ there exists a subsequence $\{x_{n'}\}$ which converges weakly to some element $x_0 \in X$. If all the weak limits x_0 are in Ω, then Ω is called *weakly compact*.

As will be seen in the following sections, a meaningful theory of variational methods is generally possible only if we are dealing with Banach spaces that have the property that every closed ball in this space is weakly compact. Banach spaces with this property are called reflexive. For completeness we give the following definition and theorem.

1.1.8. DEFINITION. Let X^* be the normed dual of X. Then X^* is a Banach space, and hence has a dual X^{**} which is called the second dual of X. Define the *canonical mapping* of X into X^{**} by $x \to x^{**}$:

$$x^{**}(x^*) := \langle x, x^* \rangle, \qquad \text{for every } x^* \in X^*, \text{ with } x \in X.$$

Then this mapping is a linear isometry of X into X^{**}. The Banach space X is called *reflexive* if and only if the canonical mapping $x \to x^{**}$ maps X onto X^{**}.

1.1.9. THEOREM. *A Banach space X is reflexive if and only if its closed unit ball is weakly compact.*

PROOF. The proof of this theorem follows from DUNFORD & SCHWARTZ [23, theorem V 4.7 and 6.1]. □

This theorem can serve as an alternative definition for a reflexive Banach space. Furthermore it emphasizes the desired property of weak compactness of closed balls in a reflexive Banach space.

1.1.10. EXAMPLES. A Hilbert space H is easily seen to be reflexive. The function spaces $L_p(\Omega)$ are reflexive if $p > 1$ (L_1 is not reflexive).

As a useful consequence of these concepts we state:

1.1.11. COROLLARY. *If X is a reflexive Banach space, then for every bounded sequence* $\{x_n\} \subset X$ *(with* $\|x_n\| \leq M$*), there exists a subsequence* $\{x_{n'}\}$ *which converges weakly to some element* $\hat{x} \in X$*, and moreover* $\|\hat{x}\| \leq M$.

1.2. CONTINUITY OF OPERATORS

We shall now consider operators T from X into Y, and state some definitions concerning continuity properties of T. The simplest notion is the following:

1.2.1. DEFINITION. The operator $T: X \to Y$ is called *continuous at the point* $x_0 \in X$ if for any sequence $\{x_n\} \subset X$ which converges to x_0, $T(x_n)$ converges to $T(x_0)$, i.e., if $x_n \to x_0$ in X ($\|x_n - x_0\|_X \to 0$ for $n \to \infty$) then $T(x_n) \to T(x_0)$ in Y ($\|T(x_n) - T(x_0)\|_Y \to 0$ for $n \to \infty$).

This definition of continuity is equivalent to: for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|T(x) - T(x_0)\| < \varepsilon$ for every x with $\|x - x_0\| < \delta$. T is called *continuous on a set* $\Omega \subset X$ if T is continuous at every point $x_0 \in \Omega$.

1.2.2. REMARK. As is well known, for linear operators the concepts of boundedness and continuity are equivalent. For nonlinear operators this is no longer true.

Besides the above stated definition of continuity there are some other definitions that are related to the concept of weak convergence. We state here the most important ones:

1.2.3. DEFINITION. Let T be an operator from X into Y.

(i) T is called *strongly continuous* at $x_0 \in X$ if for every sequence $\{x_n\} \subset X$ for which $x_n \rightharpoonup x_0$ in X it follows that $T(x_n) \to T(x_0)$ in Y.

(ii) T is called *weakly continuous* at $x_0 \in X$ if for every sequence $\{x_n\} \subset X$ for which $x_n \rightharpoonup x_0$ in X it follows that $T(x_n) \rightharpoonup T(x_0)$ in Y.

1.2.4. REMARK. As convergence in norm implies convergence in the weak sense, it follows that if T is strongly continuous then T is continuous. Thus strong continuity is a "stronger" concept then continuity.

1.2.5. For functionals defined on X, definition 1.2.3(i) and 1.2.3(ii) coincide. Indeed in this case $Y = \mathbb{R}$, and for real numbers the concepts of convergence and weak convergence coincide. According to custom, we define: The *functional* $f: X \to \mathbb{R}$ is called *weakly continuous* at $x_0 \in X$ if for every sequence $\{x_n\} \subset X$ for which $x_n \rightharpoonup x_0$ it follows that $f(x_n) \to f(x_0)$ (in $\mathbb{R}$). Note that for functionals the concept of weak continuity is "stronger" than the concept of continuity (which means that $f(x_n) \to f(x_0)$ if $x_n \to x_0$ in X).

Concluding these definitions we recall from chapter VI:

1.2.6. DEFINITION. An operator $T: X \to Y$ is called *compact* if it is continuous on X and has the property that $\{Tx_n\}$ has a convergent subsequence in Y whenever $\{x_n\}$ is bounded in X.

1.2.7. REMARK. If T is a linear operator then the requirement that T is continuous on X (which is then equivalent to the statement that T is a bounded mapping from X into Y) is superfluous. In most books (e.g. VAINBERG [1], KRASNOSELSKII [2]) the above defined concept is called *complete continuity* (the operator is said to be *completely continuous*).

We note the following relation between compactness and strong continuity:

1.2.8. THEOREM. *If X is a reflexive Banach space, and* $T: X \to Y$ *is strongly continuous, then T is compact.*

PROOF. T is continuous with respect to weak convergence, so certainly with respect to convergence in norm. From 1.1.11 it follows that every bounded sequence $\{x_n\}$ has a subsequence $\{x_{n'}\}$ which weakly converges to some element $x_0 \in X$. From the strong continuity of T it follows that $Tx_{n'} \to Tx_0$ in Y. □

1.2.9. REMARK. The converse of theorem 1.2.8 is generally not true. For example, the functional $f(x) := \|x\|$ defined on a Hilbert space H is compact. But as we saw in 1.1.6 there exists a sequence e_n with $e_n \rightharpoonup 0$ in H, and $\|e_n\| = 1$ for every n. From this it follows that f is certainly not strongly continuous (= weakly continuous).

However, the converse of theorem 1.2.8 is true for *linear* operators. The proof of this result uses the following lemma:

1.2.10. LEMMA. *Let* $L: X \to Y$ *be a linear and bounded operator. Then L is weakly continuous, i.e., if* $x_n \rightharpoonup x_0$ *in X, then* $Lx_n \rightharpoonup Lx_0$ *in Y.*

PROOF. Let y^* be an arbitrary element from Y^* and consider the expression

$$\ell(x) := \langle Lx, y^* \rangle, \quad \text{for } x \in X.$$

$\ell(x)$ is a linear functional on X as L is linear. Furthermore, $\ell(x)$ is bounded:

$$|\langle Lx,y^*\rangle| \le \|Lx\|_Y \|y^*\|_{Y^*} \le \|L\|\|x\|_X \|y^*\|_{Y^*} .$$

From this we conclude that there exists an element $x^* \in X^*$ such that

$$\langle Lx,y^*\rangle = \langle x,x^*\rangle, \quad \text{for every } x \in X.$$

Now, let $x_n \rightharpoonup x_0$ in X. Then $\langle x_n-x_0,x^*\rangle \to 0$ for every $x^* \in X^*$, and then $\langle L(x_n-x_0),y^*\rangle \to 0$. As $y^* \in Y^*$ is arbitrary, this implies that $Lx_n \rightharpoonup Lx_0$ in Y. □

1.2.11. THEOREM. *Let* X *be a reflective Banach space, and* Y *a Banach space. If* $L: X \to Y$ *is linear and compact, then* L *is strongly continuous, i.e.,* L *maps weakly convergent sequences into strongly convergent sequences.*

PROOF. Let $x_n \rightharpoonup x_0$ in X. We have to show that $Lx_n \to Lx_0$ in Y. Suppose this is not true. Then there exists a subsequence $\{x_{n'}\}$ such that

$$(*) \qquad \|Lx_{n'}-Lx_0\| > \varepsilon, \quad \text{for some } \varepsilon > 0.$$

For this subsequence we also have $x_{n'} \rightharpoonup x_0$ in X, and hence it follows from 1.1.5 that $\{x_{n'}\}$ is uniformly bounded in X. As L is compact, there exists a subsequence $\{x_{n''}\}$ of $\{x_{n'}\}$ with $Lx_{n''} \to \hat{y}$ in Y, so certainly $Lx_{n''} \to \hat{y}$ in Y. From $x_{n''} \rightharpoonup x_0$ it follows with the aid of lemma 1.2.10 (L is linear and compact, thus L is bounded) that $Lx_{n''} \rightharpoonup Lx_0$ in Y. As weak limits are unique we conclude that $\hat{y} = Lx_0$, and thus that $Lx_{n''} \to Lx_0$ in Y. This contradicts the assumption (*), and we conclude that $Lx_n \to Lx_0$ in Y, i.e., L is strongly continuous. □

1.3. GENERALIZED WEIERSTRASS THEOREM

1.3.1. As we shall see in the following section, we are often interested in the question whether a given functional is bounded, and if it is, whether it attains its maximum or minimum on its domain of definition. For this interest, and to become accustomed to some of the concepts introduced before, we shall generalize Weierstrass theorem to infinite dimensional spaces.

In its simplest form Weierstrass theorem reads: Every continuous function defined on a compact interval of $\mathbb{R}$ is bounded and attains its maximum and minimum on this interval.

To describe the process how we can generalize this theorem we shall give a device that is frequently met if one tries to generalize a finite dimensional result to infinite dimensions.

Consider a closed ball $\bar{B}(\mathbb{R}^n)$ with radius 1 and center 0 in $\mathbb{R}^n$. This ball is compact, and a continuous function is bounded on $\bar{B}(\mathbb{R}^n)$ and attains its maximum and its minimum on $\bar{B}(\mathbb{R}^n)$. The proof of this result makes essential use of the fact that $\bar{B}(\mathbb{R}^n)$ is compact, i.e., that every sequence in $\bar{B}(\mathbb{R}^n)$ has a convergent subsequence.

Now consider a closed ball $\bar{B} := \{x \in X \mid \|x\| \leq 1\}$ in a Banach space X. Then $\bar{B}$ is no longer compact! (e.g., in a Hilbert space, any orthonormal system has no convergent subsequence.).

Thus in order to be able to obtain the same statements for a functional on $\bar{B}$ as for a function on $\bar{B}(\mathbb{R}^n)$ it seems likely that we have to require stronger properties for a functional on $\bar{B}$ than are necessary for a function on $\bar{B}(\mathbb{R}^n)$. How can this stronger requirement look like? Well, $\bar{B}$ is not compact, but in a reflexive Banach space (e.g., a Hilbert space) $\bar{B}$ is weakly compact. So, a meaningful requirement for the functional may be the requirement that it is weakly continuous, i.e., continuous with respect to weak convergence. We shall show that this requirement indeed suffices to generalize Weierstrass theorem.

1.3.2. THEOREM. *Let* X *be a reflexive Banach space, and* Ω *a bounded and weakly closed (i.e., closed with respect to weak convergence) subset of* X. *If the functional* f *is weakly continuous on* Ω, *then* f *is bounded from above and from below on* Ω. *Furthermore,* f *achieves its infimum and its supremum on* Ω.

PROOF. We shall first show that f is bounded from below on Ω. Suppose not. Then there exists a sequence $\{x_n\} \subset \Omega$ with $\lim_{n\to\infty} f(x_n) = -\infty$. As Ω is bounded, the sequence $\{x_n\}$ is uniformly bounded and hence (cf. 1.1.11) there exists a weakly convergent subsequence $\{x_{n'}\}$ with weak limit x_0, say, in X. As Ω is weakly closed, $x_0 \in \Omega$. From the weak continuity of f it follows that $f(x_0) = \lim_{n'\to\infty} f(x_{n'})$. But this is impossible because $\lim_{n'\to\infty} f(x_{n'}) = -\infty$. Hence f is bounded from below on Ω. In the same way one can show that f is bounded from above on Ω.

Now, let $\alpha := \inf_{\Omega} f(x)$. This means that for every $x \in \Omega$ $f(x) \geq \alpha$ and that there exists a sequence $\{x_n\} \subset \Omega$ such that $\alpha = \lim_{n\to\infty} f(x_n)$. As above, we can find a subsequence $\{x_{n'}\} \subset \Omega$ which converges weakly to some $x_0 \in \Omega$, and for which $\lim_{n'\to\infty} f(x_{n'}) = \alpha$. From the weak continuity of f we can conclude $\lim_{n'\to\infty} f(x_{n'}) = f(x_0) = \alpha$. Then $f(x_0) = \alpha$ and $f(x) \geq \alpha$ for every $x \in \Omega$. This

means that f is minimal on Ω at the point x_0. In the same way it can be seen that there is an element $x_1 \in \Omega$ for which f is maximal with respect to Ω. □

1.4. LOWER AND UPPER SEMI-CONTINUITY

The foregoing subsection solved the question of boundedness for a weakly continuous functional on a weakly compact domain. However, in many applications functionals occur that are not weakly continuous. (As an example, we mention the functionals that appear in the investigation of boundary value problems in section 3). But in spite of this, these functionals may have the property that they are bounded from one side, from below or from above. It turns out that this property is often related to the following notion of semi-continuity, a concept that is often met in the calculus of variations.

1.4.1. DEFINITION. A functional $f: X \to \mathbb{R}$ is called *lower semi-continuous* at a point $x_0 \in X$ if for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$ the following inequality holds:

$$f(x_0) \leq \liminf_{n\to\infty} f(x_n). \tag{1.4.1}$$

f is called *weakly lower semi-continuous* (*w.l.s.c.*) at $x_0 \in X$ if (1.4.1) holds for every sequence $\{x_n\} \subset X$ with $x_n \rightharpoonup x_0$ (weak convergence). f is called *upper semi-continuous* at $x_0 \in X$ if for every sequence $\{x_n\} \subset X$ with $x_n \to x_0$ the following inequality holds:

$$f(x_0) \geq \limsup_{n\to\infty} f(x_n). \tag{1.4.2}$$

f is called *weakly upper semi-continuous* at $x_0 \in X$ if (1.4.2) holds for every sequence $\{x_n\} \subset X$ with $x_n \rightharpoonup x_0$.

1.4.2. REMARK. It is easily seen that if f is (weakly) continuous at x_0, then f is (weakly) lower semi-continuous and (weakly) upper semi-continuous at x_0, and conversely.

1.4.3. EXAMPLE. In 1.2.9 it was shown that the functional $f(x) := \|x\|$ defined on a Hilbert space H is not weakly continuous. Here we will show that f is weakly lower semi-continuous at every point $x_0 \in H$. Indeed, let $x_n \rightharpoonup x_0$ in H. Then $(x_n, x_0) \to (x_0, x_0)$ for $n \to \infty$. Furthermore, from

$$0 \leq (x_n - x_0, x_n - x_0) = \|x_n\|^2 - 2(x_n, x_0) + \|x_0\|^2$$

we deduce that

$$\|x_n\|^2 \geq 2(x_n, x_0) - \|x_0\|^2,$$

which shows that

$$\liminf_{n\to\infty} \|x_n\|^2 \geq \|x_0\|^2.$$

In exactly the same way one can show that the functional t, defined on H by

$$t(x) := (Tx, x),$$

where T is a self-adjoint and positive (i.e., $(Tx,x) \geq 0$ for every $x \in H$) operator on H, is weakly lower semi-continuous.

Now we shall show that a theorem analogous to the generalized Weierstrass theorem exists for weakly lower semi-continuous functionals.

1.4.4. THEOREM. *Let* X *be a reflexive Banach space, and* Ω *a bounded and weakly closed subset of* X. *If the functional* f *is weakly lower semi-continuous on* Ω, *then* f *is bounded from below and* f *achieves its infimum on* Ω.

PROOF. f is bounded from below. For suppose not. Then there exists a sequence $\{x_n\} \subset \Omega$ with $f(x_n) \to -\infty$; $\{x_n\}$ is uniformly bounded, hence there exists a subsequence $\{x_{n'}\}$ with $x_{n'} \rightharpoonup x_0$ where x_0 is some element in Ω (cf. the proof of theorem 1.3.2). As f is w.l.s.c. we have $f(x_0) \leq \liminf f(x_n)$, but this is impossible since $f(x_{n'}) \to -\infty$. Thus f is bounded from below.

Now let $\alpha := \inf_{\Omega} f(x)$, and let $\{x_n\} \subset \Omega$ be such that $f(x_n) \to \alpha$. Then again $\{x_n\}$ contains a subsequence $x_{n'} \rightharpoonup x_0$, where $x_0 \in \Omega$, with $f(x_{n'}) \to \alpha$. From the fact that f is w.l.s.c. we conclude $f(x_0) \leq \liminf f(x_n) = \lim f(x_{n'}) = \alpha$. But by the definition of α we must have $f(x_0) \geq \alpha$. This shows $f(x_0) = \alpha$. In other words: there is a point $x_0 \in \Omega$ where f is minimal. □

1.4.5. REMARK. Setting f = -g in the above theorem, we see that a weakly upper semi-continuous functional g is bounded from above and achieves its supremum on Ω. These results, together with remark 1.4.2, can serve as an alternative proof of the generalized Weierstrass theorem 1.3.2.

1.5. SOBOLEV SPACES; EMBEDDING THEOREMS

1.5.1. SOBOLEV SPACE H_0^1

Let Ω be a bounded domain of $\mathbb{R}^n$. $C_0^\infty(\Omega)$ will denote the class of functions with compact support in Ω which are infinitely times differentiable. For $u \in C_0^\infty(\Omega)$ we use the notation $\nabla u = (\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n})$.

Define the inner products

$$(u,v)_0 := \int_\Omega u(x)v(x)dx, \qquad u,v \in C_0^\infty,$$

$$(u,v)_1 := \int_\Omega [u(x)v(x) + \nabla u(x)\cdot\nabla v(x)]dx, \qquad u,v \in C_0^\infty,$$

with the corresponding norms, respectively,

$$\|u\|_0 = \left\{\int_\Omega u^2(x)dx\right\}^{\frac{1}{2}},$$

$$\|u\|_1 = \left\{\int_\Omega \left[u^2(x) + \nabla u(x)\cdot\nabla u(x)\right]dx\right\}^{\frac{1}{2}}.$$

In this way we arrive at the pre-Hilbert spaces $(C_0^\infty, (.,.)_0)$ and $(C_0^\infty, (.,.)_1)$. After completion of these spaces in the usual way, we are left with two Hilbert spaces which are denoted by H_0^0 and H_0^1 respectively. As is well known, H_0^0 is the class of functions that are square integrable over Ω in the sense of Lebesgue: $H_0^0(\Omega) = L_2(\Omega)$.

The elements of H_0^1 can be characterized as follows: u is an element of H_0^1 if and only if

(i) $u \in L_2(\Omega)$,

(ii) u has generalized partial derivatives of first order, which we shall also denote by $\frac{\partial u}{\partial x_i}$, with $\frac{\partial u}{\partial x_i} \in L_2(\Omega)$, for $i = 1,\ldots,n$.

With respect to (ii), we recall that a function $u \in L_2(\Omega)$ is said to have a generalized partial derivative with respect to x_i, if there exists a function $u^{(i)} \in L_2(\Omega)$ such that

$$\int_\Omega u(x)\,\frac{\partial \phi}{\partial x_i}\,dx = -\int_\Omega u^{(i)}(x)\phi(x)dx, \qquad \text{for every } \phi \in C_0^\infty(\Omega).$$

Then $u^{(i)}$ is denoted by $\frac{\partial u}{\partial x_i}$.

Another characterization of the elements of H_0^1, which emphasizes the completing process, is:

$u \in L_2(\Omega)$ is an element of $H_0^1(\Omega)$ if and only if there exists a sequence $\{\phi_k\} \subset C_0^\infty(\Omega)$ such that:

(i) $\phi_k \to u$ for $k \to \infty$ in L_2-sense; i.e., $\lim_{k\to\infty} \|\phi_k - u\|_0 = 0$,

(ii) for every $i = 1,\ldots,n$, $\{\frac{\partial \phi_k}{\partial x_i}\}$ is a Cauchy sequence with respect to $\| \ \|_0$:

$$\lim_{k,m\to\infty} \left\| \frac{\partial \phi_k}{\partial x_i} - \frac{\partial \phi_m}{\partial x_i} \right\|_0 = 0 \quad \text{for } i = 1,\ldots,n.$$

The limit of this sequence in L_2 is the generalized partial derivative of u: $\frac{\partial u}{\partial x_i} \in L_2$ such that $\lim_{k\to\infty} \left\| \frac{\partial \phi_k}{\partial xi} - \frac{\partial u}{\partial x_i} \right\|_0 = 0$.

In the following we are interested in some important relations which exist between the function spaces H_0^1 and L_p. These relations are best expressed with the aid of so-called embedding operators. Let A and B be two sets of elements with $A \subset B$. Any element $a \in A$ can as well be conceived of an element of B. The *embedding operator* $I: A \to B$ maps any element $a \in A$ to the same element a, considered as an element of B. When A and B are Banach spaces with norm $\| \ \|_A$ and $\| \ \|_B$ respectively, the embedding operator $I: A \to B$ is said to be compact if I as an operator from A into B is compact. This means that any sequence $\{a_n\} \subset A$ which is uniformly bounded in A, i.e., for which there exists a constant $M > 0$ such that $\|a_n\|_A \leq M$ for every n, has a subsequence $a_{n'}$ which converges in the metric of B to some element $\hat{b} \in B$: $\|a_{n'} - \hat{b}\|_B \to 0$ for $n' \to \infty$. With the aid of these notions we state some results concerning the embedding of the space L_p into L_q for some p, q and of H_0^1 into L_p for some p. To distinguish from the norms $\| \ \|_0$ and $\| \ \|_1$, the norm in L_p is denoted by $| \ |_p$:

$$|u|_p = \left\{ \int_\Omega |u(x)|^p \, dx \right\}^{1/p}, \quad \text{for } u \in L_p.$$

Note that $\| \ \|_0 = | \ |_2$.

1.5.2. THEOREM. *Let Ω be a bounded domain of $\mathbb{R}^n$. The dependence on Ω of the function space is omitted in the following.*

1. *The space L_p, where $1 < p < \infty$, is embedded in L_q for $1 \leq q \leq p$. Moreover, there exists a constant $M > 0$ such that*

$$|u|_q \le M\,|u|_p, \text{ for every } u \in L_p, \quad 1 \le q \le p.$$

Furthermore, the embedding operator $I: L_p \to L_q$ *is compact for* $1 \le q < p$.

2. (i) *If* $n = 1$, *then* H_0^1 *is embedded in* C_0^0: *the space of continuous functions with compact support on* Ω. *Moreover, there exists a constant* $M > 0$ *such that*

$$|u|_\infty := \max_\Omega |u(x)| \le M\,\|u\|_1, \text{ for every } u \in H_0^1.$$

Furthermore, the embedding operator $I: H_0^1 \to C_0^0$, *where* C_0^0 *is supplied with the norm* $|\ |_\infty$, *is compact.*

(ii) *If* $n \ge 2$, *then* H_0^1 *is embedded in* L_p *with* $1 \le p \le \frac{2n}{n-2}$. *Moreover, there exists a constant* $M > 0$ *such that*

$$|u|_p \le M\,\|u\|_1, \text{ for every } u \in H_0^1, \quad 1 \le p \le \frac{2n}{n-2}.$$

Furthermore, the embedding operator $I: H_0^1 \to L_p$ *with* $1 \le p < \frac{2n}{n-2}$ *is compact.*

PROOF. For the proof of these results, see SOBOLEV [9, §8-§11]. □

As the embedding operators are clearly linear operators, we can use theorem 1.2.11 to obtain the following stronger statements:

1.5.3. THEOREM.

1. *The embedding operator* $I: L_p \to L_q$ *with* $1 \le q < p$ *is strongly continuous, i.e., if* $u_n \rightharpoonup \hat{u}$ *in* L_p *then* $u_n \to \hat{u}$ *in* L_q *for every* q *with* $1 \le q < p$.

2. (i) $n = 1$: *The embedding operator* $I: H_0^1 \to C_0^0$ *is strongly continuous, i.e., if* $u_n \rightharpoonup \hat{u}$ *in* H_0^1 *then* $u_n \to \hat{u}$ *in* C_0^0.

(ii) $n \ge 2$: *The embedding operator* $I: H_0^1 \to L_p$ *with* $1 \le p < \frac{2n}{n-2}$ *is strongly continuous, i.e., if* $u_n \rightharpoonup \hat{u}$ *in* H_0^1, *then* $u_n \to \hat{u}$ *in* L_p *for every* p *with* $1 \le p < \frac{2n}{n-2}$.

2. DIFFERENTIABLE AND POTENTIAL OPERATORS

2.1. DIFFERENTIABLE MAPS

The notion of Fréchet derivative of an operator has already been investigated in chapter VI. However, this concept is of such prominence for the following that we shall reconsider it here. Moreover, as is more convenient for our purpose, we shall begin by defining the notion of Gateaux derivative and then recall the definition of Fréchet differentiability.

As in section 1, X and Y will denote real Banach spaces.

2.1.1. DEFINITION. An operator $F: A \subset X \to Y$ is called *(linearly) Gateaux differentiable at a point* $a \in A$ if there exists a linear bounded operator $T(a): X \to Y$ such that

$$\lim_{t \to 0} \frac{F(a+th) - F(a)}{t} = T(a)h \tag{2.1.1}$$

for every $h \in X$, where the limit is taken for real t and convergence in the norm of Y is meant.
The operator T(a) is called the *Gateaux derivative of* F *at the point* $a \in A$, and will be denoted by DF(a). By definition $DF(a) \in B(X,Y)$.
The image of $h \in X$ under DF(a), $DF(a)h \in Y$, is called the *Gateaux differential of* F *at the point* a *in the direction* h. F *is Gateaux differentiable on a subset* $A \subset X$ if F is Gateaux differentiable at each point of A. In this case, the mapping $a \mapsto DF(a)$ is called the *Gateaux derivative of* F *on* A, and is denoted by DF. Hence $DF: A \to B(X,Y)$.

2.1.2. DEFINITION. An operator $F: A \subset X \to Y$ is called *Fréchet differentiable at the point* $a \in A$ if there exists a linear and bounded operator $T(a): X \to Y$ such that

$$F(a+h) - F(a) = T(a)h + w(a;h) \tag{2.1.2}$$

for every $h \in X$, where the remainder $w(a;.): X \to Y$ satisfies

$$\lim_{\|h\|_X \to 0} \frac{\|w(a;h)\|_Y}{\|h\|_X} = 0. \tag{2.1.3}$$

The operator T(a) is called the *Fréchet derivative of* F *at the point* $a \in A$

and will be denoted by dF(a). By definition dF(a) $\in$ B(X,Y).
F is *Fréchet differentiable on a subset* A $\subset$ X if F is Fréchet differentiable at each point of A. In this case, the mapping a $\mapsto$ dF(a) is called the *Fréchet derivative of* F *on* A, and is denoted by dF. Hence dF: A $\to$ B(X,Y).

2.1.3. REMARK. Firstly we state that there are operators F for which the left hand side of (2.1.1) exists, but for which the result is not a linear operator in h. But as we shall only be dealing with operators for which this left hand side is linear in h, we shall from now on omit the adverb "linear" in expressions like the first sentence of definition 2.1.1.

Secondly, note that, in contrast to chapter VI, in the definition above we have required the Fréchet (and also the Gateaux) derivative of an operator at a point to be a bounded mapping from X into Y. As was stated in chapter VI the boundedness of dF(a) follows from and implies the continuity of F at the point a. As is also immediately clear from (2.1.2) we conclude that with the above definition:

If F *is Frechet differentiable at* a $\in$ A, *then* F *is continuous at* a $\in$ A.

2.1.4. REMARK. As follows immediately from the definitions: *If* F *is Frechet differentiable at* a $\in$ A, *then* F *is Gateaux differentiable at* a $\in$ A, *and* DF(a) = dF(a).

The converse is not always true, but we have the following

THEOREM. *If the Gateaux derivative* DF *exists in some neighbourhood* U(a) *of the point* a $\in$ A, *and is continuous at* a, *then the Fréchet derivative* dF(a) *exists and* dF(a) = DF(a). *In other words: A continuous Gateaux derivative is a Fréchet derivative.*

(The proof of this result is given in 2.2.4).

The introduction of two kinds of differentiability may seem somewhat superfluous. But this is not done in an attempt to achieve the greatest generallity. The main reason to introduce the concept of Gateaux derivative is a practical one: the explicit construction of the derivative of a given operator is in many cases most easily done with the limiting process as required for the Gateaux derivative. If, moreover, we can show that this Gateaux derivative is continuous, as is the case in most applications, we have found, according to the above theorem, the Fréchet derivative of the given operator (which is then also continuous of course).

2.1.5. DIFFERENTIABLE FUNCTIONALS

As a special case, the above definitions and remarks hold equally well in case $Y = \mathbb{R}$, i.e., for functionals defined on X.

Note that if a functional $f: X \to \mathbb{R}$ is Gateaux differentiable on some set $A \subset X$, the derivative of f at a point $a \in A$, $Df(A)$, is a bounded and linear functional, and hence an element of X^*: for every $a \in A$, $Df(a) \in X^*$.

The derivative of f on A is thus a mapping from A into X^*: $Df: A \to X^*$. The effect of $h \in X$ under $Df(a)$, $Df(a)h$, can be written by using the duality map between X en X^* as:

$$(2.1.4) \qquad Df(a)h = \langle h, \text{Grad } f(a) \rangle,$$

where Grad f(a) is called the *gradient of* f *at the point* a.

To distinguish between a functional that is Gateaux differentiable and one that is Fréchet differentiable, we write when f is Fréchet differentiable at $a \in A$:

$$(2.1.5) \qquad df(a)h = \langle h, \text{grad } f(a) \rangle,$$

and call grad f(a) the *strong gradient of* f *at* a.

2.1.6. EXAMPLE. Consider the functional $f: H \to \mathbb{R}$, with H a real Hilbert space, defined by

$$f(x) = \|x\|^2.$$

From

$$f(x+\varepsilon y) - f(x) = 2\varepsilon(x,y) + O(\varepsilon^2) \qquad \text{for } \varepsilon \to 0,$$

we deduce that f is Gateaux differentiable at every $x \in H$ with

$$\text{Grad } f(x) = 2x.$$

As Grad f is continuous, f is Fréchet differentiable on all of H.

As another example, and to emphasize the dependence of "grad" on the duality map which is chosen, consider the functional

$$f(u) := \tfrac{1}{2} \int_\Omega u^2(x)dx,$$

as (i) a functional on $L_2(\Omega)$ and (ii) as a functional on $H_0^1(\Omega)$.

(i) In $L_2(\Omega)$ we have $Df(u)\cdot v = \int u(x)v(x)dx = (u,v)_0$. Thus Grad $f(u)$ = grad $f(u) = u$ in $L_2(\Omega)$.

(ii) In $H_0^1(\Omega)$ we also have $Df(u)\cdot v = \int u(x)v(x)dx$. This is a bounded linear functional of $v \in H_0^1$: $|\int u(x)v(x)dx| \le \|u\|_0 \cdot \|v\|_0 \le \|u\|_0 \cdot \|v\|_1$.

With Riesz' representation theorem we conclude the existence of an element $w \in H_0^1$ (depending on u) such that

$$\int u(x)v(x)dx = (w(u),v)_1 = \int (w\cdot v + w_x \cdot v_x)dx$$

for every $v \in H_0^1$.

From this it follows that f is (Fréchet) differentiable in H_0^1, but now grad $f(u) = w(u)$!

2.2. MEAN VALUE THEOREM

The classical mean value theorem (sometimes called Lagrange formula) states that for a real valued function f, continuous and differentiable on some interval $[a,b] \subset \mathbb{R}$ a value $\tau \in (0,1)$ exists such that

$$f(b) - f(a) = f'(\tau a + (1-\tau)b)(b-a).$$

As is easily seen this theorem ceases to be valid if, for instance, f is a vector function into $\mathbb{R}^n$ with $n \neq 1$.

From this we can conclude that the mean value theorem will, in general, not be true for differentiable operators from X into Y. But as we shall show, the theorem holds true for functionals defined on X, and a modified version can be obtained for operators from X into Y.

2.2.1. THEOREM. *Let* f *be a functional defined on a subset* $A \subset X$. *If* f *is Gateaux differentiable on* A, *then for every pair of elements* a *and* b *in* X *for which* $ta + (1-t)b \in A$ *for every* $t \in [0,1]$, *there exists a number* $\tau \in (0,1)$ *such that*

$$f(b) - f(a) = Df(\tau a + (1-\tau)b)\cdot(b-a). \tag{2.2.1}$$

PROOF. Define $\phi(t) := f(ta + (1-t)b)$. Then $\phi: [0,1] \to \mathbb{R}$ with $\phi(0) = f(b)$, $\phi(1) = f(a)$. ϕ is differentiable at any $t \in (0,1)$:

$$\phi'(t) = \lim_{\varepsilon \to 0} \frac{\phi(t+\varepsilon) - \phi(t)}{\varepsilon} =$$

$$= \lim_{\varepsilon \to 0} \frac{f(ta + (1-t)b + \varepsilon(a-b)) - f(ta + (1-t)b)}{\varepsilon} ,$$

and since f is Gateaux differentiable at $ta + (1-t)b \in A$, the last limit exists, and equals $Df(ta + (1-t)b) \cdot (a-b)$. Thus $\phi'(t) = Df(ta + (1-t)b) \cdot (a-b)$.

From the classical mean value theorem for ϕ follows the existence of a number $\tau \in (0,1)$ such that $\phi(1) - \phi(0) = \phi'(\tau)$. From this the desired result immediately follows. □

2.2.2. As has already been remarked, the above theorem is not valid for arbitrary operators from X into Y. Nevertheless, in a trivial way one can obtain an analogous formula. Let F be a Gateaux differentiable operator from $A \subset X$ into Y. Then, for any $y^* \in Y^*$, the expression $f(x) := \langle F(x), y^* \rangle$ can be interpreted as a functional defined on A. Moreover, f is Gateaux differentiable with

$$Df(a)h = \langle DF(a)h, y^* \rangle, \quad \text{for every } h \in X.$$

The mean value theorem 2.2.1 applies to f: hence for any pair of elements a and b with $ta + (1-t)b \in A$ for $t \in [0,1]$, there exists a number $\tau \in (0,1)$ such that

$$\langle F(b)-F(a), y^* \rangle = \langle DF(\tau a + (1-\tau)b)(b-a), y^* \rangle. \tag{2.2.2}$$

Note that τ may, and in general shall, depend on y^*.

From (2.2.2) we can obtain an important result. Therefore, it is necessary to recall from functional analysis the fact that given $y \in Y$ there exists $y^* \in Y^*$ such that $\langle y, y^* \rangle = \|y\|$ and $\|y^*\| = 1$ (cf. BROWN & PAGE [10, p.189]). From this we conclude the existence of an element $y_0^* \in Y^*$ such that

$$\langle F(b)-F(a), y_0^* \rangle = \|F(b)-F(a)\|_Y$$

and

$$|\langle DF(\tau_0 a+(1-\tau_0)b)\cdot(b-a), y_0^*\rangle| \le \|DF(\tau_0 a+(1-\tau_0)b)\cdot(b-a)\|_Y.$$

With (2.2.2) we arrive at

$$\|F(b)-F(a)\|_Y \le \|DF(\tau_0 a+(1-\tau_0)b)\cdot(b-a)\|_Y \le$$

$$\le \|DF(\tau_0 a+(1-\tau_0)b)\|\cdot\|b-a\|_X.$$

From this result the following lemma is immediately clear.

2.2.3. LEMMA. *Let* F *be an operator from a convex subset* $A \subset X$ *into* Y. *If* F *is Gateaux differentiable on* A *and if* DF *is uniformly bounded on* A, *i.e., there exists* $M > 0$, *such that* $\|DF(a)\| \le M$ *for every* $a \in A$, *then* F *satisfies a Lipschitz condition on* A. *This means there exists* $M > 0$ *such that*

$$\|F(b)-F(a)\|_Y \le M\,\|b-a\|_X$$

for every $a,b \in A$.

As another result we shall prove theorem 2.1.4.

2.2.4. PROOF of theorem 2.1.4. Suppose $F: A \subset X \to Y$ is Gateaux differentiable in a neighbourhood $U(a)$ of $a \in A$. For every $h \in A$, with $\|h\|_X$ sufficiently small, we can write according to (2.2.2)

$$\langle F(a+h)-F(a), y^*\rangle = \langle DF(a+\tau h)\cdot h, y^*\rangle,$$

where $y^* \in Y^*$ is arbitrary and $\tau \in (0,1)$ depends on y^*. Consider the expression

$$w(a;h) := F(a+h) - F(a) - DF(a)h.$$

Then

$$\langle w(a;h), y^*\rangle = \langle [DF(a+\tau h)-DF(a)]h, y^*\rangle.$$

From the existence of an element y_0^* such that

$$\langle w(a;h), y_0^*\rangle = \|w(a;h)\|_Y$$

and

$$|\langle [DF(a+\tau_0 h) - DF(a)]h, y_0^*\rangle| \le \|[DF(a+\tau_0 h) - DF(a)]h\|_Y$$

we can deduce that

$$\|w(a;h)\|_Y \leq \|DF(a+\tau_0 h)-DF(a)\| \cdot \|h\|_X.$$

If DF is supposed to be continuous at a then

$$\|DF(a+\tau_0 h) - DF(a)\| \to 0 \quad \text{for } \|h\|_X \to 0,$$

from which it follows that F is Fréchet differentiable at a, with

$$dF(a) = DF(a).$$ □

2.3. POTENTIAL OPERATORS

As we have seen in 2.1.5 the gradient of a Gateaux differentiable functional defined on X is an operator from X into the dual space X^*. From this it is clear that in the following we only need to consider operators F from X into X^*.

2.3.1. DEFINITION. An operator $F: X \to X^*$ is called a *potential operator* (or *gradient operator*) on the set $A \subset X$ if there exists a Gateaux differentiable functional $f: A \subset X \to \mathbb{R}$ such that

$$\text{Grad } f(x) = F(x) \quad \text{for every } x \in A.$$

This functional f is called *the potential of the operator* F *on* A.
If f is Fréchet differentiable such that

$$\text{grad } f(x) = F(x)$$

then F is called a *strongly potential operator*, and f is the *strong potential* of F.

2.3.2. REMARK. From 2.1.4 it follows that a continuous potential operator is a strongly potential operator.

2.3.3. REMARK. In 2.3.1, as will often be the case in the following, we speak of the potential f of the operator F. But, as is easily seen, the potential of a given potential operator F is not uniquely determined:

if Grad f(x) = F(x) then also Grad(f(x)+c) = F(x) for any $c \in \mathbb{R}$.

However, as we shall see f is, apart from an arbitrary constant, uniquely determined by the potential operator.

The question which operators are potential and the construction of the potential is in general not an easy matter. In order to get some insight in this matter we shall first investigate this problem for vector functions on $\mathbb{R}^n$. As a matter of fact, it turns out that the results we obtain for this case are exactly the same as the results that can be obtained for the general case. To note this similarity it is only necessary to write the results for $\mathbb{R}^n$ in a way that allows generalization to Banach spaces.

2.3.4. VECTOR FUNCTIONS ON $\mathbb{R}^n$

Consider a function $\underline{F}$: $\mathbb{R}^n \to \mathbb{R}^n$. Suppose $\underline{F}$ is continuously differentiable with Jacobian matrix J:

$$J = \left(\frac{\partial F_i}{\partial x_k}\right) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & & \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}.$$

In this subsection we write $\underline{x} = (x_1,\ldots,x_n)$ and $\underline{F} = (F_1,\ldots,F_n)$.

$\underline{F}$ is a potential operator on $\mathbb{R}^n$ if there exists a function V: $\mathbb{R}^n \to \mathbb{R}$ such that

$$\text{Grad } V(\underline{x}) = \underline{F}(\underline{x}) \quad \text{for every } \underline{x} \in \mathbb{R}^n.$$

This can be written as

$$\text{(2.3.1)} \qquad \begin{cases} \frac{\partial V}{\partial x_1} = F_1(\underline{x}) \\ \vdots \\ \frac{\partial V}{\partial x_n} = F_n(\underline{x}) \end{cases}.$$

As is easily seen, a necessary condition for the existence of such a function V is that

$$\frac{\partial F_i}{\partial x_k} = \frac{\partial F_k}{\partial x_i} \quad \text{for every } i,k = 1,\ldots,n \quad \text{and } \underline{x} \in \mathbb{R}^n.$$

Indeed, from the differentiability assumption for F it follows that if V exists, it will be twice continuously differentiable, and hence

$$\frac{\partial^2 V}{\partial x_i \, \partial x_k} = \frac{\partial^2 V}{\partial x_k \, \partial x_i}$$

must be satisfied for every $i,k = 1,\ldots,n$ and $\underline{x} \in \mathbb{R}^n$.

These integrability conditions can be written in a more appropriate fashion as

$$(2.3.2) \qquad (J(\underline{x})\underline{h},\underline{k}) = (J(\underline{x})\underline{k},\underline{h}) \quad \text{for every } \underline{h},\underline{k} \in \mathbb{R}^n,$$

where (,) denotes the usual inner product of $\mathbb{R}^n$. This relation expresses the fact that J must be symmetric in every point $\underline{x} \in \mathbb{R}^n$.

But, surprisingly enough, the above condition is not only a necessary one, it is also sufficient. This can be seen by noticing that the expression

$$(2.3.3) \qquad V(\underline{x}) = V(0) + \int_0^1 (\underline{F}(t\underline{x}),\underline{x})dt,$$

or, more generally

$$V(\underline{x}) = V(\underline{x}_0) + \int_0^1 (\underline{F}(\underline{x}_0+t(\underline{x}-\underline{x}_0)),x-x_0)dt,$$

is a solution of (2.3.1) provided (2.3.2) is fulfilled. Here V(0) and $V(\underline{x}_0)$ are arbitrary constants denoting the value of the potential at $\underline{x} = 0$ and $\underline{x} = \underline{x}_0$ respectively.

That (2.3.3) is a solution of (2.3.1) follows from straightforward differentiation:

$$\frac{\partial V}{\partial x_i} = \frac{\partial}{\partial x_i}\int_0^1 (\underline{F}(t\underline{x}),\underline{x})dt = \frac{\partial}{\partial x_i}\sum_{k=1}^{n}\int_0^1 F_k(t\underline{x})x_k\,dt =$$

$$= \sum_k \int_0^1 \{t\,J_{ki}(t\underline{x})x_k + F_i(t\underline{x})\}dt =$$

$$= \sum_k \int_0^1 \{t\,J_{ik}(t\underline{x})x_k + F_i(t\underline{x})\}dt =$$

$$= \int_0^1 \frac{d}{dt}\{t\,F_i(t\underline{x})\}dt = F_i(\underline{x}),$$

where we have used the condition (2.3.2).

These results immediately generalize for arbitrary Banach spaces.

2.3.5. THEOREM. *Let* F *be an operator from a Banach space* X *into its dual* X^*. *Suppose* F *is Gateaux differentiable on the ball* $B = \{x \mid \|x-x_0\| < r\}$ *with derivative* DF. *Suppose that for arbitrary* $h,k \in X$ *the functional* $\langle h,DF(x)k\rangle$ *is continuous at every point* x *of* B. *Then in order that* F *be potential on* B *it is necessary and sufficient that the bilinear functional* $\langle h,DF(x)k\rangle$ *be symmetric for every* $x \in B$, *i.e., that*

(2.3.4) $\langle h,DF(x)k\rangle = \langle k,DF(x)h\rangle$ *for every* $h,k \in X$.

Moreover, if (2.3.4) *is satisfied, the potential of* F *on* B *is uniquely determined, up to an arbitrary constant, and given by*

(2.3.5) $$f(x) = f(x_0) + \int_0^1 \langle x-x_0, F(x_0 + t(x-x_0))\rangle dt, \qquad x,x_0 \in B.$$

PROOF. The proof of this theorem is a rather straightforward generalization of the manipulations in 2.3.4 and it will, therefore, be omitted. The proof can be found in VAINBERG [1, §5]. □

Theorem 2.3.5 completely solves the question of potentialness for differentiable operators. However, in many cases it is diffifult to verify condition (2.3.4). In those cases one often writes down the expression (2.3.5) and shows directly that Grad f(x) = F(x) in B. This last method can also be applied if the given operator does not satisfy the conditions of the theorem, e.g., if F is not differentiable. In this case a more general condi-

tion can be formulated which, at least formally, solves the question of potentialness for arbitrary operators (c.f. BERGER & BERGER [11, th. 3.11]).

2.3.6. THEOREM. *The operator* $F: X \to X^*$ *is a potential operator on* X *if and only if*

$$\int_0^1 \langle x,F(tx)\rangle dt - \int_0^1 \langle y,F(ty)\rangle dt = \int_0^1 \langle x-y,F(y+t(x-y))\rangle dt. \tag{2.3.6}$$

If this relation is satisfied the potential of F *is again given by* (2.3.5).

PROOF. Let F satisfy (2.3.6) and let f be given by (2.3.5). Then

$$f(x+\varepsilon h) - f(x) = \varepsilon \int_0^1 \langle h,F(x+t\varepsilon h)\rangle \, dt.$$

Let $\varepsilon \to 0$, then

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon}[f(x+\varepsilon h) - f(x)] = \langle h,F(x)\rangle,$$

thus F is a potential operator with potential f.

Conversely, if F is a potential operator, then

$$\frac{d}{dt} f(y+th) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} [f(y + (t+\varepsilon)h) - f(y+th)] = \langle h,F(y+th)\rangle.$$

Integration from 0 to 1 with respect to t yields:

$$f(y+h) - f(y) = \int_0^1 \langle h,F(y+th)\rangle dt.$$

Replacing y by x_0 and h by $x-x_0$ this is precisely formula (2.3.5), from which formula (2.3.6) immediately follows. □

2.3.7. EXAMPLE. The simplest example of a (strongly) potential operator is an Hermitian (= self-adjoint) operator T defined on a (real) Hilbert space H. The dual of H can be taken to be H itself if for the duality map the inner product of H is taken. In this case the condition that T is Hermitian, which reads

$$(Tx,y) = (x,Ty) \quad \text{for every } x,y \in H,$$

is exactly requirement (2.3.4). The potential can be found from (2.3.5) or

by direct verification:

$$f(x) = \tfrac{1}{2}(x,Tx).$$

From theorem 2.3.6 it follows that *for linear operators potentialness is equivalent to self-adjointness.*

As an interesting example of the theory of potential operators we shall examine equations that are frequently met in the theory of water waves and plasma physics.

2.3.8. FIRST ORDER CONSERVATIVE SYSTEMS

Consider the partial differential equation

$$Au_t = B(u). \tag{2.3.7}$$

Here $u(x,t)$ is an element of an appropriate Hilbert space H, $u_t = \frac{\partial u}{\partial t}$, A is a linear t-independent operator that commutes with $\frac{\partial}{\partial t}$: $A\frac{\partial}{\partial t} = \frac{\partial}{\partial t}A$, and B is a t-independent operator, in general nonlinear.

Equations of the form (2.3.7) are frequently met in applications. For instance, for $A = I$ and $B(u) = \frac{\partial^2 u}{\partial x^2}$ the well-known diffusion equation is obtained.

However, in contrast to phenomena exhibited by the diffusion equation, one often observes that systems described by an equation of the form (2.3.7) behave like conservative systems which are usually described by partial differential equations with second order derivatives in t.

The best known example of such a system is the Korteweg - de Vries equation

$$u_t + uu_x + u_{xxx} = 0, \quad -\infty < x < \infty, \tag{2.3.8}$$

describing, in some sense, to the right travelling, fairly long, fairly low waves.

This equation is of the form (2.3.7) with

$$A = (-\partial/\partial x)^{-1}, \quad B(u) = \tfrac{1}{2}u^2 + u_{xx}.$$

For the KdV-equation there are two special classes of solutions explicitly

known:

(i) solitary waves: waves travelling undisturbed in shape to the right, corresponding to solutions of the form $u(x,t) = \phi(\theta)$ with $\theta = x - ct$, where c is the constant velocity and $\phi \to 0$ for $\theta \to \pm\infty$;

(ii) cnoidal waves: corresponding to solutions of the form $u(x,t) = \phi(\theta)$, $\theta = x - wt$ with ϕ a periodic function of θ.

These facts express a kind of conservation for the system described by (2.3.8). In recent times a great deal of remarkable properties of the KdV equation has been found, and one is trying to generalize these results to equations of the form (2.3.7). For instance, GARDNER [12] noticed that the Fourier components of a periodic solution of the KdV equation satisfy a Hamiltonian system of ordinary differential equations. BROER [13] proved that there is a class of equations of the form (2.3.7) (including all the best known nonlinear dispersive wave equations) which can be written as the Hamilton equation of a continuous system. Without entering into details, we state that equations of the form (2.3.7) exhibit "nice" properties if they describe a conservative system. BROER found these conditions by starting with a Hamiltonian system, described with the variables q(x,t), the coordinate, and p(x,t), the momentum, and then perform a "fusion" of these two variables to one variable u(x,t).

We will derive this result in another way by applying theorem 2.3.5. Therefore, it is necessary to recall from classical mechanics that a system is conservative if it is the Euler-Lagrange equation of a variational principle with a Lagrangian that does not contain t explicitly. This means that we must find conditions for the operators A and B under which (2.3.7) is equivalent to Grad f(u) = 0, where the functional f is of the form

$$f(u) = \int dt \int_{-\infty}^{\infty} dx\, L(u),$$

with L, the Lagrangian, an operator that does not contain t explicitly. To investigate this matter, we shall restrict to C^{∞}-functions, defined for all $x \in (-\infty,\infty)$ and all $t \in (-\infty,\infty)$, which tend to zero as $x \to \pm\infty$, together with all their x-derivatives (this class of functions will be the space H), and we shall use the inner products

$$(u,v) := \int_{-\infty}^{\infty} dx\, u(x,t) \cdot v(x,t)$$

and

$$[u,v] := \int dt(u,v) = \int dt \int_{-\infty}^{\infty} dx\, u(x,t)v(x,t).$$

Consider the operator

(2.3.9) $\quad T(u) = Au_t - B(u),$

which is Gateaux differentiable if B is Gateaux differentiable:

$$DT(u)v = Av_t - DB(u)v.$$

The question is now for which operators A and B, T is a potential operator with respect to the inner product [,]. Condition (2.3.4) becomes:

(2.3.10) $\quad [Av_t,w] - [DB(u)v,w] = [Aw_t,v] - [DB(u)w,v]$ for every $v,w \in H$.

As the t-dependence of functions v and $w \in H$ is arbitrary, this condition is equivalent to the following conditions:

(2.3.11) $\quad [Av_t,w] = [Aw_t,v]$

and

(2.3.12) $\quad [DB(u)v,w] = [DB(u)w,v].$

Partial integration with respect to t in (2.3.11) (supposing that the integrated terms vanish at the boundary of the considered t-interval) gives

$$[Av_t,w] = -[A^*w_t,v],$$

where A^* is the adjoint of A with respect to the inner product (,), i.e.,

$$(Av,w) = (v,A^*w)$$

for every $v,w \in H$. From this we see that the condition (2.3.11) becomes

(2.3.13) $\quad A^* = -A.$

(Note that this relation holds for the KdV equation, where $A = (-\partial/\partial x)^{-1}$).

As the operator B is independent of t, condition (2.3.12) is equivalent to

$$(DB(u)v,w) = (DB(u)w,v),$$

from which it follows that B must be a potential operator with respect to (,). This means that there must exist a functional h(u) such that

(2.3.14) $\quad B(u) = \text{Grad } h(u),$

where Grad is meant with respect to (,). (For the KdV equation $h(u) = \int_{-\infty}^{\infty} dx(\frac{1}{6}u^3 - \frac{1}{2}u_x^2)$.)

Resuming we conclude that if A satisfies (2.3.13) and B satisfies (2.3.14) for some functional h(u), that does not contain t explicitly, T is a potential operator with respect to [,], and the potential of T is given by

(2.3.15) $\quad f(u) = \frac{1}{2}[Au_t,u] - \int dt\, h(u) = \int dt\{\frac{1}{2}(Au_t,u) - h(u)\},$

and equation (2.3.7) can be written as

(2.3.16) $\quad Au_t = \text{Grad } h(u),$

(Grad with respect to (,)).

As the Lagrangian $L := \int_{-\infty}^{\infty} dx\, L(u) = \frac{1}{2}(Au_t,u) - h(u)$ does not contain t explicitly, (2.3.16) describes a conservative system: according to Noether's theorem there must be a constant of motion, the "total energy".

In this case, this constant can easily be found by taking the inner product (,) of (2.3.16) with u_t and using (2.3.13):

$$0 = (Au_t,u_t) = (\text{Grad } h(u),u_t) = \frac{d}{dt}\, h(u).$$

Hence h(u) is a constant of motion. As a matter of fact, BROER showed that this functional h(u) can serve, after "splitting" the variable u(x,t)

into two canonical conjugate variables p and q, as a Hamilton functional (cf. [13]).

2.4. NEMYTSKY OPERATORS

In the next sections we shall have to deal with an important kind of potential operators, the so called Nemytsky operators. These operators play a fundamental role in many applications such as nonlinear integral equations and nonlinear boundary value problems.

2.4.1. Let Ω be a, not necessarily bounded, domain of $\mathbb{R}^n$, and let $z \in (-\infty,\infty)$. From now on we shall deal with functions $g: \Omega \times \mathbb{R} \ni (x,z) \mapsto g(x;z) \in \mathbb{R}$ which obey the following

ASSUMPTIONS:

(i) g is continuous with respect to z for almost every $x \in \Omega$;

(ii) g is measurable on Ω for every $z \in \mathbb{R}$.

Now let u be some element from a class V of real valued functions defined on Ω. Then $g(x;u(x))$ is some function defined on Ω. The mapping $u \mapsto g(.;u(.))$ can be conceived of an operator G, mapping functions from V into some other class of functions W, say. Thus $G: V \to W$ is defined by $G(u)(x) := g(x;u(x))$ for $x \in \Omega$, $u \in V$; such operators are called *Nemytsky operators*.

We are interested in the image W of V under G for some given function g, and in the behaviour of g if V and W are known. We shall state the results in case V is some L_p space ($p \geq 1$), but the proof of these measure theoretic results is omitted.

2.4.2. THEOREM. *Let* $g: \Omega * \mathbb{R} \to \mathbb{R}$ *obey the assumptions stated in* 2.4.1, *and let* $p,q \in [1,\infty)$. *Then the following assertions hold for the Nemytsky operator* G *defined by* $G(u)(x) = g(x;u(x))$ *for* $x \in \Omega$:

1. *If* G *maps all of* L_p *into* L_q *then*
   (i) $G: L_p \to L_q$ *is continuous and bounded;*
   (ii) *there exists a function* $a(x) \in L_q$ *and a constant* $b \geq 0$ *such that*

$$|g(x;z)| \leq a(x) + b\,|z|^r \quad \text{with } r = \frac{p}{q}. \tag{2.4.1}$$

2. *Suppose* g *satisfies* (2.4.1). *Then* G *maps all of* L_p *into* L_q, *and is, therefore, continuous and bounded.*

3. *If* G *maps all of* L_p, $p \in [1,\infty)$, *into* L_∞, *then there exists a constant* $M > 0$ *such that* $|g(x;z)| \le M$ *for all* $z \in \mathbb{R}$, *and almost all* $x \in \Omega$.

PROOF. The proof of these results can be found in VAINBERG [1, §19] and KRASNOSELSKII [2, chapter I, §2]. □

As the dual space of L_p, $p > 1$, is L_q with $\frac{1}{q} + \frac{1}{p} = 1$, the only Nemytsky operators that can be potential operators are those which map L_p into L_q, where $q = \frac{p}{p-1}$. The function $g(x;z)$ from which such an operator can be derived must satisfy according to (2.4.1)

$$|g(x;z)| \le a(x) + b\,|z|^{\sigma}, \qquad \text{with } \sigma = p-1,$$

where $a(x) \in L_q$ and $b \ge 0$ a constant.

As we shall show now, any Nemytsky operator from L_p into the dual space is indeed a potential operator. This can be proved with the aid of theorem 2.3.6. However, to get some insight in this matter, we shall prove it by starting with formula (2.3.5) for the potential f, and then show that f is Fréchet differentiable with grad $f(u) = G(u)$ for $u \in L_p$.

2.4.3. THEOREM. *Let* G *be a Nemytsky operator with* $G: L_p \to L_q$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then* G *is a strongly potential operator and the potential of* G *is given by*

$$f(u) = f(0) + \int_\Omega dx \int_0^{u(x)} g(x;z)\,dz. \tag{2.4.2}$$

Consequently, f *is continuous on* L_p.

PROOF. Formula (2.3.5) yields

$$f(u) = f(0) + \int_0^1 \langle u, G(tu)\rangle\,dt, \qquad u \in L_p,$$

where $\langle , \rangle$ is the duality map between L_p and L_q ($\frac{1}{p} + \frac{1}{q} = 1$):

$$\langle u,w\rangle = \int_\Omega u(x)w(x)dx \qquad \text{for } u \in L_p,\ w \in L_q.$$

Hence

$$f(u) - f(0) = \int_0^1 dt \int_\Omega dx\, u(x)G(tu)(x) =$$

$$= \int_0^1 dt \int_\Omega dx\, u(x)g(x;tu(x)).$$

Changing the order of integration, permissible by Fubini's theorem, gives

$$f(u) - f(0) = \int_\Omega dx \int_0^1 dt\, u(x) \cdot g(x;tu(x)),$$

which can be written as (2.4.2):

$$f(u) - f(0) = \int_\Omega dx \int_0^{u(x)} g(x;z)dz = \int_\Omega dx\, \mu(x;u(x)),$$

where $\mu(x;z)$ is the primitive of g with respect to z:

$$\mu(x;z) = \int_0^z g(x;\xi)\, d\xi.$$

(We remark that $\mu(x;z)$ defines a Nemytsky operator $\Lambda: L_p \to L_1$. As Λ is continuous (theorem 2.4.2), the continuity of f on L_p follows immediately).

To show that f is Fréchet differentiable with derivative G, consider the expression

$$w(u;v) = f(u+v) - f(u) - \langle v,G(u)\rangle =$$

$$= f(u+v) - f(u) - \int_\Omega v(x)g(x;u(x))\, dx.$$

With (2.4.2) this can be written as

$$w(u;v) = \int_\Omega dx\left\{ \int_u^{u+v} dz\, g(x;z) - g(x;u(x)) \cdot v(x)\right\} =$$

$$= \int_\Omega dx\left\{ \int_0^1 [g(x;u+tv) - g(x;u)]dt\right\}v(x).$$

According to the mean value theorem 2.2.1 there exists a number $\tau \in (0,1)$ such that

$$w(u;v) = \int_\Omega dx[g(x;u+\tau v) - g(x;u)]\, v(x) =$$

$$= \int_\Omega dx[G(u+\tau v)(x) - G(u)(x)] \cdot v(x) =$$

$$= \langle v,G(u+\tau v) - G(u)\rangle.$$

With $|\langle u,v\rangle| \leq |u|_p \cdot |v|_q$ for $u \in L_p$ and $v \in L_q$ it follows that $|w(u,v)| \leq |G(u+\tau v)-G(u)|_q \cdot |v|_p$. As G is a continuous map from L_p into L_q, $|G(u+\tau v)-G(u)|_q \to 0$ for $|v|_p \to 0$.

Hence $w(u;v) = o(|v|_p)$, showing that f, given by (2.4.2), is Fréchet differentiable with grad f = G. (The continuity of f on L_p follows of course also from its Fréchet differentiability (remark 2.1.3)). □

2.5. MONOTONE OPERATORS AND CONVEX FUNCTIONALS

In the following sections we shall often deal with functionals that are required to be weakly lower semi-continuous. This requirement may seem a little unnatural: in theories not dealing directly with variational methods this notion is not frequently met.

However, as we shall show in the next theorem, there is a good deal of relationship between monotone potential operators and the gradient of convex weakly lower semi-continuous functionals. In this way a relation between variational methods and the theory of monotone operators can be established.

2.5.1. DEFINITION. An operator $T: X \to X^*$ is called *monotone*, if $\langle x-y,Tx-Ty\rangle \geq 0$ for every $x,y \in X$.
A functional $f: X \to \mathbb{R}$ is called *convex on* X if $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for every $x,y \in X$, and every $t \in [0,1]$.

2.5.2. THEOREM. *Let* f *be a functional defined and Gateaux differentiable on* X. *Then the following relations between* f *and* Grad f *hold:*

(i) *If* Grad f *is monotone, then* f *is convex and weakly lower semi-continuous.*

(ii) *If* f *is convex and* Grad f *is continuous then* Grad f *is monotone.*

PROOF. For convenience write F(x) = Grad f(x).

(i) First we shall show that f is w.l.s.c. if F is monotone. Let $\{u_n\}$ be a weakly convergent sequence $u_n \rightharpoonup u$. Consider (using (2.3.5))

$$f(u_n) - f(u) = \int_0^1 \langle u_n-u,F(u+t(u_n-u))\rangle \, dt =$$

$$= \langle u_n-u,F(u)\rangle + \int_0^1 \langle u_n-u,F(u+t(u_n-u))-F(u)\rangle \, dt.$$

From $u_n \rightharpoonup u$ it follows that $\langle u_n - u, F(u)\rangle \to 0$ for $n \to \infty$. From the monotonicity of F it follows that the integrand in the integral is nonnegative for every $t \in [0,1]$. Hence

$$f(u_n) - f(u) \geq \langle u_n - u, F(u)\rangle \to 0 \qquad \text{for } n \to \infty;$$

from this we conclude

$$\liminf_{n\to\infty} f(u_n) \geq f(u),$$

i.e., f is w.l.s.c.

Now we shall show that f is convex. Let $t \in (0,1)$ and $x,y \in X$. From the mean value theorem 2.2.1 follows the existence of two elements ξ and η in X such that

$$f(tx+(1-t)y) - f(x) = (1-t)\,\langle y-x, F(\xi)\rangle,$$

$$\xi = \alpha x + (1-\alpha)y, \qquad t < \alpha < 1,$$

$$f(y) - f(tx+(1-t)y) = t\,\langle y-x, F(\eta)\rangle,$$

$$\eta = \beta x + (1-\beta)y, \qquad 0 < \beta < t.$$

If F is monotone, then $\langle \xi-\eta, F(\xi)-F(\eta)\rangle \geq 0$, and as $\eta-\xi = (\beta-\alpha)(x-y)$ with $\beta-\alpha < 0$, we can obtain the following inequality:

(*) $$\frac{f(x) - f(tx+(1-t)y)}{1-t} + \frac{f(y) - f(tx+(1-t)y)}{t} \geq 0.$$

From this the convexity of f immediately follows.

(ii) Suppose f is convex, and let $x,y \in X$. Then inequality (*) holds for every $t \in (0,1)$. From this and the mean value theorem 2.2.1 follow the existence of two values α and β for which

(**) $$\langle y-x, F(\alpha x+(1-\alpha)y)\rangle \leq \langle y-x, F(\beta x+(1-\beta)y)\rangle.$$

Note that α and β depend on t, with $t < \alpha < 1$, $\alpha(t) \to 1$ for $t \to 1$ and $0 < \beta < t$, $\beta(t) \to 0$ for $t \to 0$.

Moreover, we can suppose $\alpha(t) \geq \mu$ with $\alpha(t) \to \mu$ for $t \to 0$ and $\beta(t) \leq \mu$ with $\beta(t) \to \mu$ for $t \to 1$, where μ is some value in $(0,1)$ for

which

$$f(y) - f(x) = \langle y-x, F(\mu x+(1-\mu)y\rangle.$$

Now letting $t \uparrow 1$ and $t \downarrow 0$ respectively in (**) yields

$$\langle y-x, F(x)\rangle \leq \langle y-x, F(\mu x+(1-\mu)y)\rangle \leq \langle y-x, F(y)\rangle.$$

From this it follows that

$$\langle y-x, F(y)-F(x)\rangle \geq 0,$$

showing that F is monotone. □

3. FUNCTIONALS ON A BANACH SPACE

3.1. EXTREME POINTS

As in the foregoing sections, X denotes a real Banach space.

3.1.1. DEFINITION. Let f be a real functional defined on X. A point $x_0 \in X$ is called an *extremum* of f if there exists a neighbourhood $U(x_0)$ of x_0 such that

$$f(x) \leq f(x_0) \quad \text{for every } x \in U(x_0): \text{ f is } \textit{maximal} \text{ at } x_0,$$

or

$$f(x) \geq f(x_0) \quad \text{for every } x \in U(x_0): \text{ f is } \textit{minimal} \text{ at } x_0.$$

If f is Gateaux differentiable at x_0, then x_0 is called a *stationary point*, or *critical point* of f if

$$\text{Grad } f(x_0) = 0.$$

3.1.2. THEOREM. *Let the functional* f *be defined on a region* $\Omega \subset X$, *and let* x_0 *be an interior point of* Ω. *Suppose* f *is Gateaux differentiable at* x_0. *Then, if* x_0 *is an extremum of* f, *it is a stationary point of* f. *In other words: a necessary condition for* x_0 *to be an extremum is that it is stationary.*

PROOF. Let $h \in X$ be arbitrary and consider $\phi(t) := f(x_0+th)$. Then ϕ is defined in some neighbourhood of $t = 0$ as x_0 is an interior point of Ω. Moreover, ϕ is differentiable at $t = 0$ as f is Gateaux differentiable at x_0:

$$\left.\frac{d\phi}{dt}\right|_{t=0} = \lim_{t\to 0} \frac{f(x_0+th)-f(x_0)}{t} = Df(x_0) \cdot h = \langle h, \text{Grad } f(x_0)\rangle.$$

If x_0 is an extremum of f, then $t = 0$ is an extreme point of ϕ, and hence $\left.\frac{d\phi}{dt}\right|_{t=0} = 0$. Hence it follows that

$$\langle h, \text{Grad } f(x_0)\rangle = 0.$$

Since h is an arbitrary element of X, it follows that $\text{Grad } f(x_0) = 0$, i.e., x_0 is a stationary point.

Another way to derive this result is to note that

$$f(x_0+\varepsilon h) - f(x_0) = \varepsilon \langle h, \mathrm{Grad}\, f(x_0)\rangle + o(\varepsilon) \quad \text{for } \varepsilon \to 0.$$

If f is extreme at x_0, then for fixed $h \in X$, the left-hand side, and hence the right-hand side of this expression must be independent of the sign of ε, for $\varepsilon \to 0$. From this the result follows again. □

3.1.3. REMARK. As we shall see in section 4, if $x_0 \in \partial\Omega$, then the relation $\mathrm{Grad}\, f(x_0) = 0$ is generally no longer true.

Now we are interested to find conditions that ensure that a functional, defined on all of a Banach space, achieves its extremum. The most trivial example of such a condition is stated in the following theorem. But as we shall see in the rest of this section, already with this theorem some interesting applications can be handled.

3.1.4. THEOREM. *Let* f *be a functional defined on* X. *If* f *satisfies*
(i) $f(x) \to \infty$ *for* $\|x\| \to \infty$, *uniformly,*
(ii) f *is w.l.s.c.,*
then f *is bounded from below on* X *and achieves its infimum at some point* $x_0 \in X$. *If, moreover,*
(iii) f *is Gateaux differentiable at* x_0,
then $\mathrm{Grad}\, f(x_0) = 0$.

PROOF. Suppose f is not bounded from below. Then there exists a sequence $\{x_n\}$ such that $f(x_n) \to -\infty$ for $n \to \infty$. From (i) it follows that this sequence is uniformly bounded, and hence there is a weakly convergent subsequence $x_{n'} \rightharpoonup x_0 \in X$, with $f(x_{n'}) \to -\infty$. With (ii) we conclude $f(x_0) \le \lim f(x_{n'}) = -\infty$, which is impossible. Thus f is bounded from below.

Let $\alpha = \inf_X f(x)$, and let $\{x_n\}$ be a minimizing sequence: $f(x_n) \to \alpha$ for $n \to \infty$. As above there exists a weakly convergent subsequence $x_{n'} \rightharpoonup x_0 \in X$; then $f(x_0) \le \lim f(x_{n'}) = \alpha$. As $f(x_0) \ge \alpha$ by definition of α, it follows that $f(x_0) = \alpha$.

The last part of the theorem is the content of theorem 3.1.2. □

An immediate consequence of the above theorem is the following result on the existence of solutions of certain operator equations.

3.1.5. COROLLARY. *Let* $T: X \to X^*$ *be a potential operator with potential* t(x). *Suppose* t(x) *satisfies:*

(i) $\frac{t(x)}{\|x\|} \to \infty$ *for* $\|x\| \to \infty$, *uniformly*
(ii) t *is w.l.s.c.*
Then the operator equation

$$T(x) = \xi$$

has for every $\xi \in X^*$ *at least one solution* $x \in X$.

PROOF. Consider the functional

$$f(x) := t(x) - \langle x,\xi\rangle \quad \text{for } x \in X.$$

Then

$$\text{Grad } f(x) = \text{Grad } t(x) - \xi = T(x) - \xi.$$

Hence stationary points of f are solutions of $T(x) = \xi$. To apply theorem 3.1.4 we verify conditions (i) and (ii) of that theorem:

(i) $f(x) = t(x) - \langle x,\xi\rangle \geq \tau(x) \cdot \|x\|_X - \|x\|_X \|\xi\|_{X^*} = [\tau(x) - \|\xi\|_{X^*}] \|x\|_X$

for $\|x\|_X$ sufficiently large, and where $\tau(x) \to \infty$ for $\|x\|_X \to \infty$;

(ii) f is w.l.s.c. as t is w.l.s.c. and $\langle .,\xi\rangle$ is weakly continuous on X.

Now the desired result follows from theorem 3.1.4. □

3.1.6. REMARK. The above corollary can also be formulated with conditions imposed on the operator T instead of on its potential t; the resulting theorem is then easily compared with results from the theory of monotone operators (cf. NIRENBERG [14, chapt. 5, section 1]).

THEOREM. *Let* $T: X \to X^*$ *be a potential operator satisfying*
(i) $\frac{\langle x,T(x)\rangle}{\|x\|} \to \infty$ *for* $\|x\| \to \infty$, *uniformly*
(ii) T *is a monotone operator.*
Then $T(x) = \xi$ *has for every* $\xi \in X^*$ *at least one solution* $x \in X$.
Moreover, if condition (ii) *is replaced by*
(ii)* T *is a strictly monotone operator, i.e.,* $\langle x-y,Tx-Ty\rangle > 0$ *for every* $x,y \in X$, $x \neq y$,
then $T(x) = \xi$ *has for every* $\xi \in X^*$ *a unique solution* $x \in X$.

PROOF. We will show that conditions (i) and (ii) of corollary 3.1.5 follow from (i) and (ii) above.

(i) The relation between the potential t and the operator T is given by (2.3.5):

$$t(x) = t(0) + \int_0^1 \langle x, T(sx)\rangle \, ds.$$

With $h(x) = \frac{\langle x, Tx\rangle}{\|x\|}$ the following estimate holds for $\|x\|$ sufficiently large:

$$\frac{t(x)}{\|x\|} = \frac{t(0)}{\|x\|} + \int_0^1 h(sx)\, ds \geq \frac{t(0)}{\|x\|} + \int_{\frac{1}{2}}^1 h(sx) ds.$$

From the mean value theorem follows the existence of a number $\sigma \in (\frac{1}{2},1)$ such that

$$\frac{t(x)}{\|x\|} \geq \frac{t(0)}{\|x\|} + \tfrac{1}{2} h(\sigma x).$$

Note that σ may depend on x, but as $\sigma > \frac{1}{2}$, $h(\sigma x) \to \infty$ for $\|x\| \to \infty$. Hence

$$\frac{t(x)}{\|x\|} \to \infty \quad \text{for} \quad \|x\| \to \infty.$$

(ii) According to theorem 2.5.2 the monotonicity of T implies that Grad T = t is w.l.s.c.
The last part of the theorem is clear by noticing
$\langle x-y, Tx-ty\rangle = 0 \Rightarrow x = y$. □

3.2. BOUNDARY VALUE PROBLEMS

3.2.0. INTRODUCTION

In this subsection we shall show how the variational theory, developed in the preceding subsections, can be used to get existence theorems for solutions of some boundary value problems. For simplicity we shall restrict to the nonlinear Dirichlet problem, but it will become clear that some other boundary value problems can be handled in much the same way.

In chapter VI, the nonlinear Dirichlet problem

$$(3.2.1) \qquad \begin{cases} -\dfrac{d^2u}{dx^2} = g(u) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

was handled by converting this equation with the aid of Green's function

$k(x,y)$ for this problem, i.e., the solution of

$$\begin{cases} -\dfrac{d^2k}{dx^2} = \delta(x-y) & 0 < x < 1; \quad 0 < y < 1 \\ k(0,y) = k(1,y) = 0, \end{cases}$$

into an integral equation

$$(3.2.2) \qquad u(x) = \int_0^1 k(x,y)g(u(y))\,dy \quad (=: Au(x).).$$

For this one dimensional problem, $k(x,y)$ can be written down explicitly

$$k(x,y) = \begin{cases} x(1-y) & \text{if } 0 \le x \le y \\ y(1-x) & \text{if } y \le x \le 1, \end{cases}$$

which shows that $k(x,y)$ is a continuous function. From this last property it follows that the integral operator K defined by

$$(3.2.3) \qquad K\,u(x) := \int_0^1 k(x,y)u(y)\,dy$$

is a *compact* mapping from the Banach space $C^0([0,1])$, supplied with the norm

$$|u|_\infty = \max_{x\in[0,1]} |u(x)|,$$

into itself.

If it is supposed that g is a continuous function on $\mathbb{R}$, then the operator G, defined by

$$G(u)(x) := g(u(x)),$$

is a mapping from C^0 into C^0, i.e., $G(u) \in C^0([0,1])$ if $u \in C^0([0,1])$.

In fact, in chapters VI and VII the case $g(u) = \lambda f(u)$ was considered, and the existence of solutions in dependence on λ was investigated. For instance, the question: given a solution v_0 for $\lambda = \lambda_0$, are there solutions $v(\lambda)$ for λ in a neighbourhood of λ_0?, was investigated with the aid of Riesz-Schauder theory and with bifurcation theory.

Given the problem as stated in (3.2.1) one can try to apply the Leray-Schauder degree theory, for instance Schauder's theorem as stated in chapter VIII. To apply this theorem it is necessary that the operator A, defined by

$$A(u) := K\,G(u) \tag{3.2.4}$$

is a compact mapping from C^0 into C^0. Since K is compact, G must be a bounded operator from C^0 into C^0. As is easily seen this is true if e.g. g is defined on all of $\mathbb{R}$ and is continuous. Thus, without any requirement concerning the "growth" of the function g, the operator A is a compact mapping.

Furthermore, to apply Schauder's theorem one has to require that there exists a $R > 0$ such that A is a mapping from the closed ball $\bar{B}_R := \{u \in C^0 \mid |u|_\infty \leq R\}$ into itself. This requirement gives some mild restrictions upon the possible choice of the function g.

Once this requirement is fulfilled, one can conclude that A has a fixed point, from which it follows that (3.2.2) has a solution.

Matters are more complicated if we consider the Dirichlet problem in more dimensions ($n \geq 2$). Then the boundary value problem

$$\begin{cases} -\Delta u = g(u) & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega \end{cases} \tag{3.2.5}$$

(Ω is a bounded domain in $\mathbb{R}^n$, sufficiently smooth) can be converted into an integral equation which is formally the same as (3.2.2), but where now $k(x,y)$ is Green's function for the Laplace operator for the domain Ω in $\mathbb{R}^n$. As is well known, $k(x,y)$ is (for $n \geq 2$) no longer continuous at $x = y$. From this it follows that the integral operator (3.2.3) is no longer a compact mapping from C^0 into C^0. Therefore, the above described reasoning for $n = 1$ ceases to hold for $n \geq 2$.

However, there is a way out: it is known that the operator K defined by (3.2.3) is compact (for all $n \geq 1$) when considered as a mapping from $L_2(\Omega)$ into $L_2(\Omega)$. It seems reasonable to use this property. But, if we wish to consider equation (3.2.2) in $L_2(\Omega)$, then we must take care that the operator G, defined on $L_2(\Omega)$, maps each function from $L_2(\Omega)$ into a function that is again an element of $L_2(\Omega)$. According to theorem 2.4.2 this

means that solely from this condition it follows that the function $g: \mathbb{R} \to \mathbb{R}$ must satisfy, apart from some continuity properties, a very restrictive inequality of the form

$$(3.2.6) \qquad |g(z)| \leq a+b|z| \quad \text{for } z \in \mathbb{R},$$

where a and b are constants.

If g satisfies (3.2.6) then the operator A, defined by (3.2.4), is a compact mapping from L_2 into L_2, and the problem can be handled with Leray-Schauder degree theory along the same lines as described above for n = 1.

However, as we want to demonstrate the use of variational methods in this chapter, we shall tackle the Dirichlet problems (3.2.1) and (3.2.5) along other lines.

In subsection 3.2.2 we shall show how one can define in a meaningful way a functional f(u) on an appropriate Hilbert space, in fact H_0^1, such that f has the property that its stationary points are solutions in some generalized sense of (3.2.5). In doing so, it turns out to be necessary to put some restrictions on the nonlinear term g concerning the growth of g. These requirements for g are less restrictive than condition (3.2.6), and follow immediately from the fact that H_0^1 is compactly imbedded in L_p if $(1 \leq)\ p < \frac{2n}{n-2}$ for $n \geq 2$. (In case n = 1, no restriction is needed as then H_0^1 is compactly imbedded in C_0^0.)

However, if we want to use theorem 3.1.4 to show that f has a stationary point, we are forced (also for n = 1) to require that g satisfies the condition (3.2.6). This is due to condition (i) of that theorem, which reads

$$(*) \qquad f(u) \to \infty \quad \text{for} \quad \|u\| \to \infty, \text{ uniformly.}$$

Thus, with the restricted theorems available up to this moment, we are able to prove the existence of solutions of (3.2.5) only if condition (3.2.6) is fulfilled. No need to say that one has tried hard to prove the existence of stationary points for functionals that do not satisfy the growth condition (*). For some interesting results in this field the interested reader is, by way of example, referred to the paper of AMBROSETTI & RABINOWITZ [15]. Unfortunately, we cannot go into this matter

here.

In subsection 3.3 nonlinear integral equations are envisaged in a somewhat general setting. But the results immediately apply to integral operators that arise as a converted boundary value problem, such as the integral operator entering in (3.2.2).

As a final remark we shall enter into an important but rather awkward problem, namely the regularity properties of the solution whose existence is proved with any of the above described methods. In fact, all these methods can only be applied when working in an appropriate Banach or Hilbert space. A (classical) solution of the problem (3.2.1), for instance, is a function $u \in C^2((0,1)) \cap C^0([0,1])$ which satisfies the differential equation and the boundary values. But the existence theorems which can be obtained for this case by the above described method of transformation to an integral equation, are concerned with solutions of the equation (3.2.2) in $C^0([0,1])$. Hence, if we can conclude that (3.2.2) has a solution $u \in C^0([0,1])$, this does not imply that this function is also a solution of (3.2.1). But if one can show that the solution of (3.2.2) is regular enough, i.e., if $u \in C^2((0,1)) \cap C^0([0,1])$, then it is indeed a solution of (3.2.1). For this reason, problem (3.2.2) is called a generalized formulation of the Dirichlet problem. (In subsection 3.2.2, using another method to prove existence, we shall need another generalized formulation for the problems (3.2.1) and (3.2.5). How to obtain such a generalized formulation will be explained in detail in subsection 3.2.1, by way of example, for the linear Dirichlet problem). Having proved the existence of a solution of the generalized problem, how can we prove that this solution is a solution of the classical problem, e.g., how can we prove that the solution is regular enough to permit the differentiations that are required in the classical formulation of the problem?

This is in general a difficult problem, but in essence all regularity proofs are based on "bootstraps" methods. We shall demonstrate this way of reasoning for the solution of equation (3.2.2). Suppose $u \in C^0([0,1])$ satisfies this equation. Then one proves that the right-hand side of this equation is differentiable, from which it follows that the left-hand side of this equation, i.e., the function u itself, is also differentiable. This so obtained knowledge of the function u can then be used to show that the right-hand side is twice differentiable, from which it follows that u itself is twice differentiable.

For the sake of simplicity we shall not enter here, or in the following subsections, into these regularity theorems. Thus, for the boundary value problems dealt with in the following subsections, only the existence of a solution of a well defined generalized problem is proved.

3.2.1. LINEAR DIRICHLET PROBLEM

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, and consider the linear Dirichlet problem

$$(3.2.7) \qquad \begin{cases} -\Delta u(x) = g(x) & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Here $g(x)$ is some given function, say $g \in C^0(\bar{\Omega})$, and Δ denotes the Laplace operator. A *classical solution* of (3.2.7) is a function $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfying the differential equation in Ω and the boundary conditions on $\partial\Omega$. Our aim is to prove the existence of a solution of (3.2.7) with the aid of variational methods.

Therefore, it is necessary to rewrite this problem as an equation in an appropriate Banach or Hilbert space. One usually argues as follows.

If u is a classical solution of (3.2.7) then

$$(3.2.8) \qquad (-\Delta u, \phi)_0 = (g, \phi)_0 \quad \text{for every } \phi \in C_0^\infty(\Omega).$$

Here we use the inner product $(\ ,\ )_0$ as introduced in subsection 1.5:

$$(\phi, \psi)_0 := \int_\Omega \phi(x)\psi(x)\, dx,$$

and $C_0^\infty(\Omega)$ is the class of functions which are infinitely times differentiable and which are zero on the boundary $\partial\Omega$ of Ω.

Partial integration (Green's formula) of the left-hand side of (3.2.8) and using the fact that $\phi = 0$ on $\partial\Omega$ (such that the integral over the boundary vanishes), gives

$$(3.2.9) \qquad (\nabla u, \nabla \phi)_0 = (g, \phi)_0 \quad \text{for every } \phi \in C_0^\infty(\Omega).$$

Now, one notes that equation (3.2.9) makes sense not only for $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and $\phi \in C_0^\infty(\Omega)$, but also for functions u and ϕ from the

Hilbert space $H_0^1(\Omega)$. (This space was introduced in subsection 1.5.)

This step in the reasoning may be a little unexpected. But it is motivated by our wish to reformulate the original problem (3.2.7) as an equation in a Hilbert space, and H_0^1 is a Hilbert space. Remains the question why one takes H_0^1, and not another Hilbert space. This choice can be made plausible if one remembers that elements of H_0^1 are functions that have generalized first derivatives (so that the left-hand side of (3.2.9) makes sense), and secondly that a function of H_0^1 is in some generalized sense zero at $\partial\Omega$.

For these reasons, we expect that the problem: prove the existence of a function $u \in H_0^1$ such that

$$(3.2.10) \qquad (\nabla u, \nabla v)_0 = (g,v)_0 \quad \text{for every } v \in H_0^1,$$

is closely related to the original problem. In fact, it can be shown that if $u \in H_0^1$ satisfies (3.2.10) and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then u is a classical solution: that u satisfies the differential equation is easily seen; that u satisfies also the boundary condition is a consequence of the fact that if $u \in H_0^1 \cap C^0(\bar{\Omega})$, then $u = 0$ on $\partial\Omega$.

A function $u \in H_0^1$ satisfying (3.2.10) is for these reasons called a *generalized solution* of the Dirichlet problem, and the problem as given by (3.2.10) is called a *generalized formulation* (also the names *variational solution* and *variational formulation* are used, which names will become clear from the following).

With (3.2.10) can now be associated an operator equation in H_0^1 in the following way: for $g \in C^0(\bar{\Omega})$, or, more generally, for $g \in L_2(\Omega)$, the expression $(g,v)_0$ is a bounded and linear functional of v on H_0^1:

$$|(g,v)_0| \leq \|g\|_0 \cdot \|v\|_0 \leq \|g\|_0 \cdot \|v\|_1 \quad \text{for every } v \in H_0^1.$$

Hence, with Riesz' representation theorem we can conlude that there exists an element $\tilde{g} \in H_0^1$ (where $\tilde{g}$ depends on g) such that

$$(3.2.11) \qquad (g,v)_0 = (\tilde{g},v)_1 \quad \text{for every } v \in H_0^1.$$

In the same way we note that $(\nabla u, \nabla v)_0$ is a bounded and linear functional of v on H_0^1:

$$|(\nabla u,\nabla v)_0| \le \|\nabla u\|_0 \cdot \|\nabla v\|_0 \le \|\nabla u\|_0 \cdot \|v\|_1.$$

Hence, with each element $u \in H_0^1$ there is associated an element $T(u)$, say, such that

(3.2.12) $$(\nabla u,\nabla v)_0 = (T(u),v)_1.$$

With (3.2.11) and (3.2.12) formula (3.2.10) can be rewritten as

$$(T(u),v)_1 = (\tilde{g},v)_1 \qquad \text{for every } v \in H_0^1.$$

As $(T(u)-\tilde{g},v)_1 = 0$ must hold for every $v \in H_0^1$, it follows that

(3.2.13) $$T(u) = \tilde{g}$$

must hold. As is easily seen the mapping $u \mapsto T(u)$ is linear and bounded from H_0^1 into H_0^1: the operator $T: H_0^1 \to H_0^1$ is linear and bounded.

The boundary value problem (3.2.7) has now been transformed to equation (3.2.13) which is an operator equation in the Hilbert space H_0^1, and it is this operator equation that is appropriate for investigation with variational methods or, more generally, with "Hilbert space methods".

We have described this rather trivial example in some detail to demonstrate the method how one can obtain an appropriate equation in a Hilbert space from a given boundary value problem.

We want to prove the existence of a solution of (3.2.13) along the lines that were described in subsection 3.1. Therefore, it is necessary to note that the operator T is Hermitian:

$$(Tu,v)_1 = (u,Tv)_1 \qquad \text{for every } u,v \in H_0^1.$$

Indeed:

$$(Tu,v)_1 = (\nabla u,\nabla v)_0 = (\nabla v,\nabla u)_0 = (Tv,u)_1 = (u,Tv)_1.$$

From this it follows that T is a potential operator (cf. 2.3.7) with potential

(3.2.14) $$t(u) = \tfrac{1}{2}(Tu,u)_1,$$

which can be written as

$$t(u) = \tfrac{1}{2}(\nabla u, \nabla u)_0 = \tfrac{1}{2} \int_\Omega (\nabla u(x))^2 \, dx \quad \text{for } u \in H_0^1.$$

From this it follows that the functional f, defined by

$$f(u) := t(u) - (\tilde{g}, u)_1 \quad \text{for } u \in H_0^1,$$

which can be written as

$$f(u) = \tfrac{1}{2} \int_\Omega (\nabla u)^2 \, dx - \int_\Omega g(x)u(x) \, dx, \tag{3.2.15}$$

is (Fréchet) differentiable with

$$\text{grad } f(u) = Tu - \tilde{g}.$$

Thus stationary points of f are solutions of (3.2.13) and conversely. In this respect, note that the expression $D f(u) \cdot v = 0$ is exactly equation (3.2.10).

To show that f has stationary points, we can use corollary 3.1.5. Then it is necessary to show that

(i) $\dfrac{t(u)}{\|u\|_1} \to \infty \quad \text{for} \quad \|u\|_1 \to \infty,$

(ii) t is w.l.s.c.

This last requirement follows immediately from example 1.4.3. (T is a positive operator, as $(Tu,u)_1 = (\nabla u, \nabla u)_0 \geq 0$ for every $u \in H_0^1$).

To show that t satisfies condition (i) we need the following result:

FRIEDRICHS' INEQUALITY. *There exists a constant* $c > 0$ *(depending only on* Ω*) such that*

$$\|u\|_0 \leq c \, \|\nabla u\|_0 \quad \textit{for every } u \in H_0^1. \tag{3.2.16}$$

PROOF. We shall prove the result for $u \in C_0^\infty(\Omega)$. A completing process then gives the result for $u \in H_0^1(\Omega)$. Ω is a bounded domain in $\mathbb{R}^n$. Hence there exists a cube $C := \{x \in \mathbb{R}^n \mid |x_i| < \alpha\}$, such that $\Omega \subset C$. Define $u(x) = 0$

for $x \notin \Omega$. Then one has

$$u(x) = \int_{-\alpha}^{x_n} \frac{\partial u}{\partial x_n}\, dx_n .$$

Applying the Cauchy-Schwarz inequality, and integrating the result with respect to x_n over $(-\alpha,\alpha)$ yields

$$\int_{-\alpha}^{\alpha} |u(x)|^2\, dx_n \le \int_{-\alpha}^{\alpha} dx_n \left\{ \int_{-\alpha}^{x_n} \left|\frac{\partial u}{\partial x_n}\right|^2 d\bar{x}_n \cdot \int_{-\alpha}^{x_n} d\bar{x}_n \right\} \le$$

$$\le \int_{-\alpha}^{\alpha} \left|\frac{\partial u}{\partial x_n}\right|^2 dx_n \cdot \int_{-\alpha}^{\alpha} (x_n+\alpha)\, dx_n \le$$

$$\le \tfrac{1}{2} \cdot (2\alpha)^2 \cdot \int_{-\alpha}^{\alpha} \left|\frac{\partial u}{\partial x_n}\right|^2 dx_n .$$

Integrating with respect to the remaining variables gives the desired result:

$$\int_{\Omega} |u(x)|^2\, dx \le 2\alpha^2 \int_{\Omega} \left|\frac{\partial u}{\partial x_n}\right|^2 dx \le 2\alpha^2 \int_{\Omega} (\nabla u)^2\, dx. \qquad \square$$

From this result we deduce

$$\|u\|_1^2 = \|\nabla u\|_0^2 + \|u\|_0^2 \le (1+c^2)\, \|\nabla u\|_0^2,$$

from which it follows that

$$t(u) = \tfrac{1}{2}\, \|\nabla u\|_0^2 \ge \frac{1}{2(1+c^2)}\, \|u\|_1^2 . \tag{3.2.17}$$

Hence condition (i) is satisfied.

From corollary 3.1.5 it now follows that equation (3.2.13) has for every $\tilde{g} \in H_0^1$ a solution $u \in H_0^1$. Hence, for arbitrary $g \in L_2$, the existence of a generalized solution of (3.2.7) is proved.

As was stated in the introduction to this subsection, a regularity

proof for the solution will be omitted.

3.2.2. NONLINEAR DIRICHLET PROBLEM

We shall now consider the nonlinear boundary value problem

$$(3.2.18)\qquad \begin{cases} -\Delta u = g(x;u) & \text{for } x \in \Omega \\ u(x) = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Here Ω is some bounded domain in $\mathbb{R}^n$, $n \geq 1$, and g satisfies the assumptions of 2.4.1. Firstly, we shall be concerned with the question under which conditions, to be imposed on g, it is possible to define a functional with the property that the stationary points of this functional are solutions of (3.2.18) in some generalized sense. As described in subsection 3.2.1 for the linear problem, the Hilbert space that is appropriate for a study of this problem is H_0^1. Arguing in the same way as was done in 3.2.1, we want to call a function $u \in H_0^1$ a generalized solution of (3.2.18) if it satisfies

$$(3.2.19)\qquad (\nabla u, \nabla v)_0 = (g(x;u), v)_0 \quad \text{for every } v \in H_0^1.$$

The left-hand side of this relation is defined for functions $u, v \in H_0^1$.

For $n \geq 2$ the right-hand side of this equation makes sense only if the following restriction is put upon the function g (for $n = 2$ a less restrictive condition suffices; we shall not enter upon this matter for sake of simplicity and uniformness of presentation):

$$(3.2.20)\qquad |g(x;z)| \leq a(x) + \alpha|z|^{\sigma} \quad \text{with } \alpha > 0,$$

$$(3.2.21)\qquad 0 < \sigma < \sigma_0 := \frac{n+2}{n-2}$$

and

$$(3.2.22)\qquad a(x) \in L_{\frac{\sigma+1}{\sigma}}.$$

To see this we note that H_0^1 is embedded in the Banach space L_p for $1 \leq p \leq p_0$, where $p_0 = \frac{2n}{n-2}$ (cf. theorem 1.5.2). Furthermore, defining q_0 by $\frac{1}{q_0} + \frac{1}{p_0} = 1$, thus $q_0 = \frac{2n}{n+2}$, observe that $a \in L_{q_0}$ as $\frac{\sigma+1}{\sigma} =$

$= 1 + \frac{1}{\sigma} > 1 + \frac{1}{\sigma_0} = q_0$.

From these observations we can, using Hölder's inequality, obtain the following chain of estimates:

$$|(g(x;u),v)_0| = \left|\int_\Omega g(x;u(x))v(x)\,dx\right| \le$$

$$\le \int_\Omega dx\{a(x)\cdot|v(x)| + \alpha|u|^\sigma\cdot|v|\} \le$$

$$\le \{\int_\Omega a(x)^{q_0}\,dx\}^{1/q_0}\cdot\{\int_\Omega |v|^{p_0}\,dx\}^{1/p_0} +$$

$$+ \alpha\{\int_\Omega |u|^{\sigma\cdot q_0}\,dx\}^{1/q_0}\cdot\{\int_\Omega |v|^{p_0}\,dx\}^{1/p_0} =$$

$$= |a|_{q_0}\cdot|v|_{p_0} + \alpha|u|^\sigma_{\sigma q_0}\cdot|v|_{p_0},$$

which expressions make sense provided $\sigma q_0 \le p_0$, i.e., provided $\sigma \le \frac{p_0}{q_0} = p_0 - 1 = \sigma_0$, and this is exactly condition (3.2.21).

With the estimate $|v|_{p_0} \le M\|v\|_1$ (cf. theorem 1.5.2) it follows that

$$|g(x;u),v)_0| \le M\{|a|_{q_0} + \alpha|u|^\sigma_{\sigma q_0}\}\|v\|_1, \tag{3.2.23}$$

expressing the fact that the right-hand side of (3.2.19) is a bounded, and linear, functional of v on H_0^1. With the aid of Riesz' representation theorem we can write

$$(g(x;u),v)_0 = (B(u),v)_1 \quad \text{for every } v \in H_0^1, \tag{3.2.24}$$

where B is some nonlinear operator from H_0^1 into H_0^1. Using the definition of the operator T, formula (3.2.12), equation (3.2.19) can be rewritten as

$$(Tu,v)_1 = (B(u),v)_1 \quad \text{for every } v \in H_0^1. \tag{3.2.25}$$

From this it follows that u has to satisfy

$$Tu = B(u) \quad u \in H_0^1, \tag{3.2.26}$$

which is the operator equation in H_0^1 associated with the boundary value problem (3.2.18).

To apply variational methods to prove the existence of a solution of (3.2.26) we have to investigate the potentialness of the operator B, defined by (3.2.24). According to subsection 2.4 we can associate with the function g(x;u) a Nemytsky operator G. If G is supposed to be defined on L_p, for some $p \geq 1$, then G is a bounded and continuous operator from L_p into $L_{p/\sigma}$, as follows from theorem 2.4.2 and condition (3.2.20) (at least if $a \in L_{p/\sigma}$). G can be a potential operator only if the spaces L_p and $L_{p/\sigma}$ are each others dual, i.e., if $p = \sigma+1$.

From these observations we conclude that the operator

$$(3.2.27) \qquad G: L_{1+\sigma} \longrightarrow L_{\frac{1+\sigma}{\sigma}}$$

is a strongly potential operator, and its potential follows immediately from theorem 2.4.3:

$$(3.2.28) \qquad \psi(u) = \int_\Omega dx \int_0^{u(x)} g(x;z)\, dz \quad \text{for } u \in L_{1+\sigma}.$$

Thus we have grad $\psi(u) = G(u)$ in $L_{\sigma+1}$, and ψ is continuous on $L_{\sigma+1}$. Furthermore, note that the (Fréchet) differential of ψ at the point $u \in L_{\sigma+1}$ in the direction $v \in L_{1+\sigma}$ is given by

$$(3.2.29) \qquad d\psi(u)\cdot v = \int_\Omega g(x;u)v(x)\, dx \quad u,v \in L_{1+\sigma}$$

which is formally equivalent to the right-hand side of equation (3.2.19).

Now consider the functional ψ on H_0^1. From condition (3.2.21) we have $1+\sigma < 1+\sigma_0 = p_0$. Thus ψ is well defined on H_0^1, as H_0^1 is embedded in L_{p_0} and thus in $L_{1+\sigma}$. Moreover, ψ is differentiable on H_0^1 as can be seen from (3.2.24) and (3.2.29):

$$(3.2.30) \qquad d\psi(u) \cdot v = (g(x;u),v)_0 = (B(u),v)_1 \quad \text{for } u,v \in H_0^1.$$

By definition of the notion "grad" this means that

$$(3.2.31) \qquad \text{grad } \psi(u) = B(u) \quad \text{on } H_0^1.$$

From this it follows that B is a (strongly) potential operator, with potential ψ on H_0^1. With this result and with the definition of the functional t, defined on H_0^1 and given by (3.2.14), we can state that the solutions of the operator equation (3.2.26) are in a one to one correspondence with the stationary points of the functional

$$(3.2.32) \qquad f(u) := t(u) - \psi(u) = \tfrac{1}{2} \int_\Omega (\nabla u)^2 \, dx - \int_\Omega dx \int_0^{u(x)} g(x;z) \, dz$$

for $u \in H_0^1$.

Now, we shall show that

$$(3.2.33) \qquad f \text{ is w.l.s.c. on } H_0^1.$$

Therefore, it is necessary to recall from subsection 3.2.1 the fact that t is w.l.s.c. on H_0^1. We shall show that ψ is weakly continuous on H_0^1, from which result it follows that f is w.l.s.c. on H_0^1. Let $u_n \rightharpoonup \hat{u}$ in H_0^1. From theorem 1.5.3 it follows that $u_n \to \hat{u}$ in L_p, for $1 \le p < p_0$. As $1+\sigma_0 = p_0$, it follows that if σ satisfies (3.2.21), $u_n \to \hat{u}$ in $L_{\sigma+1}$. Now ψ is continuous on $L_{\sigma+1}$, from which it follows that $\psi(u_n) \to \psi(\hat{u})$. This shows that ψ is weakly continuous on H_0^1.

For n = 1, the same reasoning as above can be repeated, except for the fact that in this case no restrictions on g have to be imposed. This is due to the fact that in this case H_0^1 is compactly embedded in $C_0^0(\Omega)$.

Now we will show that the functional f has a stationary point. Once this has been shown, the existence of a solution of (3.2.26), and hence the existence of a generalized solution of (3.2.18), will have been proved.

To show that f has a stationary point we must use theorem 3.1.4, being the only theorem that is available up to this moment for such an investigation.

To apply this theorem, we have to ensure that condition (i) of this theorem, namely

$$(3.2.34) \qquad f(u) \to \infty \quad \text{for } \|u\|_1 \to \infty, \text{ uniformly,}$$

is fulfilled. (Condition (ii) has been proved above: (3.2.33).).
From the estimate (3.2.17) and from

$$|\psi(u)| \le |a|_{q_0} \cdot |u|_{p_0} + \frac{\alpha}{\sigma+1} |u|_{\sigma+1}^{\sigma+1} \le$$

$$\le M \|u\|_1 \{|a|_{q_0} + \frac{\alpha M^\sigma}{\sigma+1} \|u\|_1^\sigma\},$$

where we have used the fact that $|u|_{p_0} \le M \|u\|_1$, it can be seen that condition (3.2.34) can only be fulfilled provided $0 < \sigma \le 1$. Indeed, for $0 < \sigma < 1$, condition (3.2.34) is fulfilled irrespective of the value of the coefficient α. For $\sigma = 1$, a restriction on α is needed as follows from the following investigations, where Friedrichs' inequality (3.2.16) is used:

$$|\psi(u)| \le \|a\|_0 \cdot \|u\|_0 + \tfrac{1}{2}\alpha \|u\|_0^2 \le \tfrac{1}{2}\alpha c^2 \|\nabla u\|_0^2 + \|a\|_0 \cdot \|u\|_0,$$

and thus

$$f(u) = t(u) - \psi(u) \ge \tfrac{1}{2}(1-\alpha c^2) \|\nabla u\|_0^2 - \|a\|_0 \|u\|_0 \ge$$

$$\ge \tfrac{1}{2}(1-\alpha c^2) \frac{1}{1+c^2} \|u\|_1^2 - \|a\|_0 \cdot \|u\|_1 \to \infty$$

for $\|u\|_1 \to \infty$, uniformly, provided the constant α satisfies

$$(3.2.35) \qquad \alpha < \frac{1}{c^2}.$$

From these results it follows that the existence of a generalized solution of (3.2.18) has been proved for $0 < \sigma < 1$ and for $\sigma = 1$, provided condition (3.2.35) is fulfilled.

To conclude this subsection, we shall investigate condition (3.2.35), and relate the results obtained here to results that will be obtained in subsection 3.3. The constant c in condition (3.2.35) is the constant entering in Friedrichs' inequality:

$$\|u\|_0 \le c \|\nabla u\|_0.$$

As is well known, this constant c has something to do with the smallest eigenvalue of the operator $-\Delta$ for the domain Ω under homogeneous Dirichlet boundary conditions. Indeed, this follows immediately once we assume that it is known that the operator $-\Delta$ has a complete orthonormal set of eigenfunctions (this will be proved in subsection 4.2.5). All the eigenvalues

are positive, and if we denote the smallest eigenvalue by μ_0, the constant c in Friedrichs inequality can be taken to be

$$(3.2.36) \qquad \mu_0^{-1} = c^2.$$

A more elegant way to prove this result, which is more in the style of this chapter, is left as an

EXERCISE. From Friedrichs' inequality it follows that the functional ϕ (the *Rayleigh-Ritz quotient*), defined on $H_0^1 \setminus \{0\}$ by

$$(3.2.37) \qquad \phi(u) = \frac{\|\nabla u\|_0^2}{\|u\|_0^2} = \int_\Omega (\nabla u)^2 \, dx \Big/ \int_\Omega u^2 \, dx$$

is bounded from below on H_0^1. Prove that ϕ achieves its infimum on H_0^1 at some point $u_0 \in H_0^1$, $u_0 \neq 0$, for which $\phi(u_0) \geq \frac{1}{c^2}$:

$$(3.2.38) \qquad \mu_0 := \phi(u_0) = \inf \phi(u) \geq \frac{1}{c^2},$$

with inf taken over $u \in H_0^1$, $u \neq 0$.
(Note that this function u_0 and the value μ_0 can, in a meaningful way, be called the *generalized eigenfunction* corresponding to the smallest *generalized eigenvalue* for the eigenvalue problem

$$(3.2.39) \qquad \begin{cases} -\Delta u = \mu u & \text{for } x \in \Omega \\ u = 0 & \text{for } x \in \partial\Omega. \end{cases}$$

By taking the infimum of ϕ over the Hilbert space consisting of those elements $u \in H_0^1$ that are orthogonal to the already found eigenfunctions, the existence of all generalized eigenvalues with a corresponding complete set of eigenfunctions can be proved successively. See for this MIKHLIN [7, chapter 5, §3 and chapter 6] and for another treatment of this problem subsection 4.2.5.)

Anyway, with the result (3.2.36) condition (3.2.35) reads:

$$(3.2.40) \qquad \alpha < \mu_0.$$

To compare with subsection 3.3 consider the case $\sigma = 1$. As described in the introduction 3.2.0, the boundary value problem (3.2.18) can also be tackled

using Green's function and converting this problem into an integral equation. Let k(x,y) denote Green's function, and let K be the integral operator with kernel k(x,y). Then as is well known, the largest eigenvalue of K, denoted by λ_0, is equal to the smallest eigenvalue of problem (3.2.39). Thus we have

$$\|K\| = \lambda_0 = \frac{1}{\mu_0} .$$

As condition (3.2.20) is, for $\sigma = 1$, essentially equivalent to condition (3.3.9), the results derived with these two methods must be comparable.

Indeed, it is immediately seen that condition (3.2.40) is the same as condition (3.3.14):

$$\alpha < \mu_0 = \frac{1}{\lambda_0} ,$$

showing the equivalence of the obtained results.

3.3. FIXED POINT THEOREMS

3.3.1. One is sometimes interested in the question whether a given operator $T: X \to X$ has a fixed point, i.e., whether there exists an element $x \in X$ such that

(3.3.1) $\quad T(x) = x.$

As an example we refer to subsection 3.2.0 where equation (3.2.2) is exactly of this form. Schauder's theorem (cf. chapter VIII, theorem 4.1) states that every compact mapping $T: A \to A$, where A is a closed and convex subset of X, has a fixed point in A. This theorem is proved with the aid of Leray-Schauder degree theory.

Applying variational methods, we can obtain a result for fixed points of operators on all of a Hilbert space H, instead of on a closed and convex subset. For that purpose, we argue as follows.

Let $T: H \to H$. As we want to use variational methods, we suppose that T is a potential operator, with potential t(x), say. Then the stationary points of the functional f, defined as

(3.3.2) $\quad f(x) := \frac{1}{2}(x,x) - t(x), \quad x \in H,$

are exactly the solutions of (3.3.1), and conversely. Suppose that t is weakly continuous (or w.u.s.c). Then the functional f is w.l.s.c., as (x,x) is w.l.s.c. (cf. example 1.4.3). From theorem 1.4.4 it follows that f is bounded from below on every closed ball $\bar{B}_R := \{x \in H \mid \|x\| \leq R\}$, and that f achieves its infimum on $\bar{B}_R$. If we know that this minimum is attained at an interior point of $\bar{B}_R$, say $x_0 \in B_R$, then the result

(3.3.3) $\quad \text{Grad } f(x_0) = 0$

would imply the existence of a fixed point of T. But in general this minimal point may be situated at the boundary $\partial\bar{B}_R$, and then the conclusion (3.3.3) needs not be true. An extra condition on the growth of f(x) can ensure that this minimum is not attained on $\partial\bar{B}_R$ for R sufficiently large.

3.3.2. THEOREM. *Let the functional* f *on a Hilbert space* H *be expressible as*

$$f(x) = \tfrac{1}{2}(x,x) - t(x).$$

Suppose that t *is weakly continuous, and that* $f(x) \to \infty$ *for* $\|x\| \to \infty$, *uniformly. Then* f *is bounded from below on* H, f *achieves its infimum at a point* $x_0 \in H$, *and, if* t *is differentiable,*

$$x_0 = \text{Grad } t(x_0).$$

PROOF. As $f(x) \to \infty$ for $\|x\| \to \infty$ a constant $R \in \mathbb{R}$ can be found such that

$$f(x) > f(0) \quad \text{for } x \in \partial\bar{B}_R.$$

From this and the arguments described in 3.3.1 it follows that there exists an x_0 in the interior of $\bar{B}_R$ where f is minimal. This proves the theorem. □

3.3.3. REMARKS. It is easily seen that f(x), given by (3.3.2), satisfies the condition $f(x) \to \infty$ for $\|x\| \to \infty$, if the functional t(x) satisfies

$$t(x) \leq a\|x\|^2 + b\|x\|^{\mu} + c,$$

where a, b and c are constants with $a < \frac{1}{2}$, and μ is any number with $0 < \mu < 2$.

To use these ideas for an important kind of nonlinear integral equations, we have to modify the results somewhat. Therefore, the following lemma is especially useful.

3.3.4. LEMMA. *Let* $T: H \to H$ *be a potential operator, with potential* t. *Let* $A: H \to H$ *be a self-adjoint operator. Then the operator* $ATA: H \to H$ *is a potential operator, with potential* $\phi(x) = t(Ax)$:

$$\text{Grad } t(A(x)) = ATA(x). \tag{3.3.4}$$

PROOF. Immediately clear from

$$\lim_{\varepsilon \to 0} \frac{\phi(x+\varepsilon h) - \phi(x)}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \{t(Ax+\varepsilon Ah) - t(Ax)\} =$$

$$= (Ah, T(Ax)) = (h, ATAx). \qquad \square$$

3.3.5. THEOREM. *Let* $B: H \to H$ *be a positive, self-adjoint operator, and let* $T = \text{Grad } t$, *where the functional* t *satisfies the condition*

$$2t(x) \leq a\|x\|^2 + b\|x\|^{\mu} + c, \tag{3.3.5}$$

where b *and* c *are constants,* $0 < \mu < 2$, *and* a *satisfies* $a\|B\| < 1$ *if* $a > 0$. *Suppose that the functional* t *is continuous, and that the operator* B *is compact. Then the operator equation*

$$x = BT(x) \tag{3.3.6}$$

has a solution, i.e., the operator BT *has a fixed point in* H.

PROOF. As B is positive and self-adjoint, there exists an operator $A: H \to H$ such that $A^2 = B$, A is positive and self-adjoint (A is called the *positive square root of* B). Furthermore, as B is compact, A is a compact operator. Applying the transformation $x = Ay$, solutions x of (3.3.6) are in a one to one correspondence with solutions y of the equation $A(y - ATAy) = 0$. Thus, once we have shown that the operator equation

$$y = AT(Ay) \tag{3.3.7}$$

admits a solution $\hat{y}$, if follows that equation (3.3.6) has a solution

$\hat{x} = A\hat{y}$. (Note that if $Az = 0$ only if $z = 0$, solutions of (3.3.6) and (3.3.7) are in a one to one correspondence.)

To prove that (3.3.7) has a solution we remark that according to lemma 3.3.4 ATA is a potential operator, with potential t(Ay). Hence it suffices to show that the functional

$$f(y) := (y,y) - 2 \cdot t(Ay)$$

has a stationary point. Therefore, we have to show (cf. theorem 3.3.2) that $f(y) \to \infty$ for $\|y\| \to \infty$, and that t(Ay) is weakly continuous.

With $\|B\| = \|A\|^2$ it follows from (3.3.5)

$$f(y) \geq \|y\|^2 - a\|Ay\|^2 - b\|Ay\|^\mu - c \geq$$

$$\geq \|y\|^2 - a\|A\|^2\|y\|^2 - b\|A\|^\mu\|y\|^\mu - c \geq$$

$$\geq (1-a\|B\|)\ \|y\|^2 - b\|A\|^\mu\|y\|^\mu - c,$$

thus $f(y) \to \infty$ for $\|y\| \to \infty$ if $a\|B\| < 1$. To show that t(Ay) is weakly continuous, we note that A is a compact operator, and since A is linear we can even say (cf. theorem 1.2.11) that A maps weakly convergent sequences into strongly convergent sequences. By assumption, t is continuous and thus every convergent sequence is mapped by t into a convergent sequence of real numbers: if $y_n \rightharpoonup \hat{y}$, then $Ay_n \to A\hat{y}$, and $t(Ay_n) \to t(A\hat{y})$. Thus t is weakly continuous.

From these observations the theorem follows. □

3.3.6. REMARK. From theorem 3.3.5 we can obtain a useful result for the *eigenvalue problem* connected with the operator BT. Namely, from 3.3.5 it follows that the equation

$$(3.3.8) \qquad BTx = \lambda x$$

has a solution for every $\lambda \in \mathbb{R}$ if the constant a entering in (3.3.5) is negative, and has a solution for $\lambda > a\|B\|$ if $a > 0$. A value λ for which (3.3.8) has a non-null solution is called an eigenvalue, and the corresponding solution is called an eigenfunction. Now suppose that $T(0) \neq 0$. This condition ensures that $x = 0$ is not a solution. Then the conclusion is that:

(i) if $a > 0$, every $\lambda \in (a\|B\|,\infty)$ is an eigenvalue of the operator BT,
(ii) if $a \leq 0$, every $\lambda \in \mathbb{R}$ is an eigenvalue of BT.
It is useful to compare these results with results that will be obtained further on in subsection 4.2 (in particular theorem 4.2.9).

We shall now apply theorem 3.3.5, and demonstrate the foregoing ideas, to the simplest example of a nonlinear Hammerstein integral equation.

3.3.7. HAMMERSTEIN INTEGRAL EQUATION IN L_2

Let Ω be a bounded domain of $\mathbb{R}^n$. We consider functions $u \in L_2(\Omega)$. Define the linear integral operator K by

$$K\,u(x) := \int_\Omega k(x,y)u(y)\,dy,$$

where the kernel $k: \Omega * \Omega \to \mathbb{R}$ is a real function. For simplicity we suppose that the following assumptions are fulfilled:

(i) $\int_\Omega \int_\Omega k^2(x,y)\,dx\,dy < \infty$. This ensures that K is a compact operator acting in L_2.
(ii) $k(x,y) = k(y,x)$. Then K is a self-adjoint operator: $(Ku,v)_0 = (u,Kv)_0$.
(iii) $k(x,y) \geq 0$ for every $(x,y) \in \Omega * \Omega$. Then K is a positive operator: $(Ku,u)_0 \geq 0$ for every $u \in L_2$.

Under these assumptions, K has a largest eigenvalue, λ_0 say, and $\lambda_0 = \|K\|$. In this case the positive square root A of K can be written down: let $u_i(x)$, $i = 0,1,2,\ldots$ denotes the orthonormal set of eigenfunctions of K corresponding to the eigenvalues λ_i (> 0). Then

$$k(x,y) = \sum \lambda_i\, u_i(x)u_i(y),$$

and the operator A is given by

$$Au(x) = \sum \sqrt{\lambda_i} \cdot (u_i,u)_0\, u_i(x) \quad \text{for } u \in L_2(\Omega).$$

Furthermore, consider a Nemytsky operator $G: L_2 \to L_2$ generated by a function $g: \Omega * \mathbb{R} \to \mathbb{R}$:

$$G\,u(x) := g(x;u(x)) \quad \text{for } x \in \Omega \text{ and } u \in L_2,$$

where g satisfies (see subsection 2.4)

$$|g(x;z)| \leq \alpha|z| + \beta, \tag{3.3.9}$$

with α and β positive constants. Then *Hammerstein's integral operator* is defined by

$$F(u)(x) := KG(u)(x) = \int_\Omega k(x,y)g(y;u(y))\,dy,$$

and Hammerstein's integral equation is the equation for the fixed points of F:

$$u(x) = \int_\Omega k(x,y)g(y;u(y))\,dy, \quad u \in L_2(\Omega). \tag{3.3.10}$$

From theorem 2.4.3 we know that G is a potential operator with potential

$$t(u) = \int_\Omega dx \int_0^{u(x)} g(x;z)\,dz. \tag{3.3.11}$$

According to the proof of theorem 3.3.5, the equation

$$u = KG(u)$$

is replaced by

$$\psi = AGA\psi, \tag{3.3.12}$$

where $u = A\psi \in L_2$. With lemma 3.3.4 it is seen that the solutions of (3.3.12) are equivalent to the stationary points of the functional

$$f(\psi) = \int_\Omega \psi^2(x)\,dx - 2\int_\Omega dx \int_0^{A\psi(x)} g(x;z)\,dz. \tag{3.3.13}$$

To apply theorem 3.3.5, note that t is continuous (cf. theorem 2.4.3) and satisfies condition (3.3.5) with $a = \alpha > 0$:

$$t(u) = \int_\Omega dx \int_0^{u(x)} g(x;z)\, dz \le \left|\int_\Omega dx \int_0^{u(x)} g(x;z)\, dz\right| \le$$

$$\le \int_\Omega dx \int_0^{u(x)} |g(x;z)|\, dz \le$$

$$\le \int_\Omega dx \{\tfrac{1}{2}\alpha u^2 + \beta|u|\} \le \tfrac{1}{2}\alpha \|u\|_0^2 + \tfrac{1}{2}b\|u\|_0.$$

From this it follows that if α satisfies

(3.3.14) $\quad \alpha\|K\| = \alpha\lambda_0 < 1,$

equation (3.3.10) has a solution $u \in L_2(\Omega)$. Moreover, from remark 3.3.6 we can conclude that the nonlinear eigenvalue problem

(3.3.15) $\quad \lambda u(x) = \int_\Omega k(x,y)g(y;u(y))\, dy$

has a solution $u \in L_2$ for every λ for which $\lambda > \lambda_0\alpha$. If $g(x;0) \neq 0$, ensuring that $u = 0$ is not a solution of (3.3.15), this means that every $\lambda \in (\lambda_0\alpha,\infty)$ is an eigenvalue.

4. FUNCTIONALS ON A MANIFOLD

4.1. CONSTRAINED EXTREME POINTS

Again throughout this subsection, X will denote a real Banach space.

4.1.1. Let f and g be two functionals defined on X. Suppose f and g are Fréchet differentiable at every point of X.

Let us consider the set

$$M_c := \{x \in X \mid g(x) = c\},$$

where $c \in \mathbb{R}$. Of course, this set depends on the specific choice of the functional g, and on the value of c. So in general nothing can be said about this set: it may be empty, disconnected, etc. We shall speak, somewhat loosely, of the set M_c as a *manifold* in X.

An important example for the following is obtained if we take $g(x) = \|x\|$ and $c > 0$: in this case we are dealing with spheres in the Banach space with center at 0.

These spheres will be denoted by S_R

$$S_R = \{x \in X \mid \|x\| = R\},$$

and if this sphere is thought of as the boundary of the open ball B_R

$$B_R = \{x \in X \mid \|x\| < R\},$$

it is also written as ∂B_R; furthermore, $\bar{B}_R = B_R \cup S_R$.

In the following we want to investigate the functional f on the manifold M_c, instead of on the whole Banach space (f is said to be constrained to M_c). Thus, although f is assumed to be defined on all of X, we shall consider it on M_c.

4.1.2. DEFINITION. A point $x_0 \in M_c$ is called an *extremum of* f *with respect to* M_c if there exists a neighbourhood $U(x_0) \subset X$ such that

$$f(x) \leq f(x_0) \quad \text{for every } x \in U(x_0) \cap M_c:$$
$$f \text{ is maximal at } x_0 \text{ with respect to } M_c,$$

or

$$f(x) \geq f(x_0) \quad \text{for every } x \in U(x_0) \cap M_c:$$
$$f \text{ is minimal at } x_0 \text{ with respect to } M_c.$$

A point $x_0 \in M_c$ is called an *ordinary point of the manifold*

$$M_c := \{x \in X \mid g(x) = c\}$$

if grad $g(x_0) \neq 0$.

Now we want to define the concept of stationary point of the functional f with respect to the manifold M_c. Just as for functionals considered on all of a Banach space, we want to do this in such a way that if x_0 is an extremum of f with respect to M_c, x_0 is a stationary point of f with respect to M_c. Rather than to give this definition at once we shall try to "derive" it by the following

4.1.3. HEURISTIC CONSIDERATION

We want to consider the difference of f at two neighbouring points x_0 and $x_0 + \varepsilon h$ on M_c. As f is supposed to be Fréchet differentiable at x_0, we can write

$$f(x_0+\varepsilon h) - f(x_0) = \varepsilon\langle h, \text{grad } f(x_0)\rangle + \omega(x_0;\varepsilon h) =$$
$$= \varepsilon\langle h, \text{grad } f(x_0)\rangle + o(\varepsilon) \qquad \text{for } \varepsilon \to 0.$$

If, for instance, f is minimal with respect to M_c at the point x_0, then $f(x_0+\varepsilon h) - f(x_0) \geq 0$ must hold for any choice of εh, provided $x_0+\varepsilon h \in M_c$.

If x_0 is an ordinary point of M_c, we can write

$$g(x_0+\varepsilon h) - g(x_0) = \varepsilon\langle h, \text{grad } g(x_0)\rangle + o(\varepsilon).$$

Considering only first order terms, we can conclude that in order that f is minimal with respect to M_c at x_0, it is necessary that

$$\langle h, \text{grad } f(x_0)\rangle = 0 \text{ for any } h \in X \text{ for which } \langle h, \text{grad } g(x_0)\rangle = 0.$$

From this it follows that there must exist a number $\lambda \in \mathbb{R}$ such that

$$\text{grad } f(x_0) = \lambda \text{ grad } g(x_0).$$

This condition is a necessary one in the present approximation, but also, as will be shown, in a rigorous derivation. Therefore, the following

4.1.4. DEFINITION. Let x_0 be an ordinary point of $M_c = \{x \in X \mid g(x) = c\}$. Then x_0 is called a *stationary point* (or a *critical point*) *of* f *with respect to* M_c if there exists a number $\lambda \in \mathbb{R}$ such that

$$\text{grad } f(x_0) = \lambda \text{ grad } g(x_0).$$

4.1.5. THEOREM (LJUSTERNIK)
Let f *and* g *be two functionals defined on* X. *Suppose* f *is Fréchet differentiable at* x_0 *and* g *is continuously Fréchet differentiable on a neighbourhood of* x_0, *with* grad $g(x_0) \neq 0$. *Then we have: if* $x_0 \in M_c = \{x \in X \mid g(x) = c\}$ *is an extreme point of* f *with respect to* M_c, x_0 *is a stationary point of* f *with respect to* M_c, *i.e., there exists a* $\lambda \in \mathbb{R}$, *which can be called the Lagrange multiplier, such that*

$$\text{(4.1.1)} \qquad \text{grad } f(x_0) = \lambda \text{ grad } g(x_0).$$

In subsection 4.1.6 we shall prove this theorem using the implicit function theorem for functions of two scalar variables; in 4.1.7 another proof is presented using an existence theorem for nonlinear differential equations.

The proof of Ljusternik's theorem is much more difficult than the proof of the corresponding theorem for functionals considered on all of a Banach space (theorem 3.1.2). This is due to the rather cumbersome procedure of "constructing" paths on the manifold (in contradistinction to the process described in 4.1.3, which actually took place in the tangent space to M_c at x_0).

4.1.6. PROOF OF LJUSTERNIK'S THEOREM.
Suppose x_0 is an extremum of f with respect to M_c. As f and g are Fréchet differentiable, we can write

$$f(x_0+h) - f(x_0) = \langle h, \text{grad } f(x_0)\rangle + \omega(x_0;h),$$
$$g(x_0+h) - g(x_0) = \langle h, \text{grad } g(x_0)\rangle + N(x_0;h),$$

for arbitrary $h \in X$, where the remainders ω and N satisfy

$$\lim_{\|h\|\to 0} \frac{\omega(x_0;h)}{\|h\|} = 0$$

and

$$\lim_{\|h\|\to 0} \frac{N(x_0;h)}{\|h\|} = 0$$

respectively. For convenience we shall write grad $g(x_0) = v_0^*$. By assumption $v_0^* \neq 0$. As we are only interested in a neighbourhood $U(x_0)$ of the point x_0, we write

$$(4.1.2) \qquad h(\varepsilon) = a(\varepsilon)v_0 + \varepsilon w,$$

where $v_0 \in X$ satisfies $\langle v_0, v_0^* \rangle = 1$ and w is some element of X with $\langle w, v_0^* \rangle = 0$; ε is supposed to be a small parameter, and $a(\varepsilon)$ some function of ε. Note that any element in $U(x_0)$ can be written as $x_0 + h(\varepsilon)$, with $h(\varepsilon)$ of the form (4.1.2).

Our aim is to show that for any $w \in X$, $\langle w, v_0^* \rangle = 0$, there exists a function $a(\varepsilon)$, such that $a(\varepsilon) = o(\varepsilon)$ for $\varepsilon \to 0$, and such that $x_0+h(\varepsilon) \in M_c$ for every ε sufficiently small. In order that $x_0+h(\varepsilon) \in M_c$ for every ε sufficiently small, it is necessary that $g(x_0+h(\varepsilon)) = c$.

Now we define

$$\phi(a;\varepsilon) := g(x_0+a.v_0+\varepsilon w) - c.$$

Then ϕ can be considered as a function of the variables a and ε. First we want to show that there exists a continuous function $a(\varepsilon)$, with $a(0) = 0$, such that $\phi(a(\varepsilon);\varepsilon) = 0$ for ε sufficiently small; then we shall show $a(\varepsilon) = o(\varepsilon)$ for $\varepsilon \to 0$.

Note that $\phi(0;0) = g(x_0) - c = 0$, and that ϕ is continuously differentiable with respect to ε and with respect to a, with

$$\frac{d}{da}\phi(a;\varepsilon) = \lim_{\alpha\to 0} \frac{1}{\alpha}\{g(x_0 + (a+\alpha)v_0 + \varepsilon w) - g(x_0+av_0+\varepsilon w)\}$$

$$= \langle v_0, \text{grad } g(x_0+av_0+\varepsilon w)\rangle.$$

From grad $g(x_0) = v_0^*$ and $\langle v_0, v_0^* \rangle = 1$, it follows that $\frac{d}{da}\phi(0;0) = 1$. Together with $\phi(0;0) = 0$ it follows from the implicit function theorem for ordinary functions of two variables that there exists a unique, continuously differentiable function $a(\varepsilon)$ with $a(\varepsilon) \to 0$ for $\varepsilon \to 0$, such that

$\phi(a(\varepsilon);\varepsilon) = 0$ for every ε sufficiently small.

From the relation

$$0 = g(x_0+av_0+\varepsilon w) - g(x_0) = \langle av_0+\varepsilon w, \operatorname{grad} g(x_0)\rangle + N(x_0;av_0+\varepsilon w),$$

we obtain as equation for $a(\varepsilon)$:

$$0 = a + N(x_0;av_0+\varepsilon w). \tag{4.1.3}$$

Together with the fact that

$$N(x_0;a(\varepsilon)v_0+\varepsilon w) = o(|a(\varepsilon)|+|\varepsilon|)$$

it follows that $a(\varepsilon) = o(\varepsilon)$ for $\varepsilon \to 0$.

Hence we conclude that for any given $w \in X$ with $\langle w,v_0^*\rangle = 0$, there exists a function $a(\varepsilon)$ with $a(\varepsilon) = o(\varepsilon)$ for $\varepsilon \to 0$, such that

$$g(x_0 + a(\varepsilon)v_0+\varepsilon w) = c.$$

This means that the set $\{x_0+a(\varepsilon)v_0+\varepsilon w \mid |\varepsilon| \text{ sufficiently small}\}$ represents a curve on M_c through the point x_0.

Now, consider the functional f on this curve:

$$f(x_0+a(\varepsilon)v_0+\varepsilon w) - f(x_0) = \varepsilon\langle w,\operatorname{grad} f(x_0)\rangle + a(\varepsilon)\langle v_0,\operatorname{grad} f(x_0)\rangle +$$

$$+ \omega(x_0;av_0+\varepsilon w).$$

As $a(\varepsilon) = o(\varepsilon)$ and $\omega(x_0;a(\varepsilon)v_0+\varepsilon w) = o(\varepsilon)$, it follows that

$$f(x_0+a(\varepsilon)v_0+\varepsilon w) - f(x_0) = \varepsilon\langle w,\operatorname{grad} f(x_0)\rangle + o(\varepsilon) \text{ for } \varepsilon \to 0. \tag{4.1.4}$$

A necessary condition for f to be extreme at x_0 with respect to M_c is that f is extreme at x_0 with respect to the considered curve on M_c. From (4.1.4) it follows immediately that a necessary condition for this to be true is that

$$\langle w,\operatorname{grad} f(x_0)\rangle = 0.$$

As $w \in X$ can be chosen arbitrarily, provided $\langle w,v_0^*\rangle = 0$, it follows that

there must be a constant $\lambda \in \mathbb{R}$ such that grad $f(x_0) = \lambda v_0^*$, thus

$$\text{grad } f(x_0) = \lambda \text{ grad } g(x_0).$$

This proves the theorem. □

4.1.7. REMARK. Another proof of Ljusternik's theorem can be obtained as follows (cf. BROWDER [16, p.32] and PALAIS [17, p.188]).

Consider all possible paths on M_c through the point $x_0 \in M_c$ that can be expressed as

$$x(\varepsilon) = x_0 + h(\varepsilon),$$

where h is, as a function of ε, defined on some interval $I \subset \mathbb{R}$, containing 0, and suppose that h is differentiable with $h(0) = 0$. In order that $x(\varepsilon) \in M_c$ for every $\varepsilon \in I$, it is necessary that

$$\frac{d}{d\varepsilon} g(x_0+h(\varepsilon)) = 0 \quad \text{for every } \varepsilon \in I.$$

Writing, as in (4.1.2), $h(\varepsilon) = a(\varepsilon)v_0 + \varepsilon w$, this can be written as

(4.1.5) $$\langle \dot{a}(\varepsilon)v_0+w, \text{grad } g(x_0+h(\varepsilon))\rangle = 0,$$

where

$$\dot{a}(\varepsilon) = \frac{da}{d\varepsilon}.$$

This expression can be rewritten as

$$\dot{a}(\varepsilon) = \frac{\langle w, \text{grad } g(x_0+h(\varepsilon))\rangle}{\langle v_0, \text{grad } g(x_0+h(\varepsilon))\rangle}.$$

For $|\varepsilon|$ and $|a|$ sufficiently small, the denominator of the right-hand side is non zero (as $\langle v_0, \text{grad } g(x_0)\rangle = 1$, and grad g is continuous on a neighbourhood of x_0), and hence the right-hand side is a continuous function of a.

From this fact we can conclude that this nonlinear differential equation admits a continuously differentiable solution $a(\varepsilon)$ with $a(0) = 0$. For $\varepsilon = 0$ we have from (4.1.5)

$$\dot{a}(0) + \langle w, v_0^* \rangle = 0,$$

from which it follows that $\dot{a}(0) = 0$ for every w with $\langle w, v_0^* \rangle = 0$. A necessary condition for f to be extreme at x_0 with respect to M_c is that f is extreme at x_0 with respect to the curve $x_0 + h(\varepsilon)$ on M_c. Therefore, we must have

$$\frac{d}{d\varepsilon} f(x_0 + h(\varepsilon))\Big|_{\varepsilon=0} = 0,$$

which can be written as

$$\langle \frac{d}{d\varepsilon} h(\varepsilon)\Big|_{\varepsilon=0}, \text{ grad } f(x_0)\rangle = \langle \dot{a}(0)v_0 + w, \text{grad } f(x_0)\rangle = 0.$$

As $\dot{a}(0) = 0$ if $(w, v_0^*) = 0$, it follows that

$$\langle w, \text{grad } f(x_0)\rangle = 0 \quad \text{if} \quad \langle w, v_0^*\rangle = 0.$$

From this it follows again that there must be a $\lambda \in \mathbb{R}$ such that

$$\text{grad } f(x_0) = \lambda \text{ grad } g(x_0).$$

4.1.8. REMARK. Operator equations of the form (4.1.1) can be viewed as nonlinear eigenvalue problems for the operator pair grad f and grad g. This is the problem of the search for values $\lambda \in \mathbb{R}$ (the eigenvalues) for which there exists an $x \in X$ (the eigenfunction) such that equation (4.1.1) is satisfied. The eigenvalues are then simply the Lagrange-multipliers of the constrained variational principle: f stationary on M_c. This approach is especially fruitful as in this formulation the value of λ is not prescribed. In subsection 4.2 we shall consider the eigenvalue problem for the case that $g(x) = \|x\|^2$ in a Hilbert space H. The above described interpretation of equation (4.1.1) differs essentially from the investigation of the stationary points of the functional

$$h(x;\lambda) := f(x) - \lambda g(x),$$

on all of the Banach space X.

Although the stationary points of h, if any, satisfy also equation (4.1.1), the value of λ must now be prescribed. This kind of investigation amounts to the question whether a given value of λ, say λ_0, is an eigen-

value or not, i.e., whether for this value λ_0, there exists an $x \in X$ such that (4.1.1) is fulfilled.

The difference between these two approaches may become clearer from the following (by way of example we take $g(x) = \|x\|^2$): In a λ versus $\|x\|$ diagram (a bifurcation-diagram, see 4.2.1)

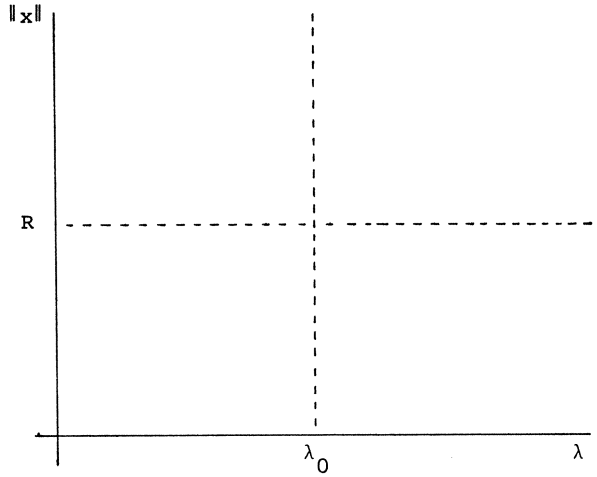

Figure 1

a solution of (4.1.1) corresponds to a point $(\lambda,\|x\|)$. The eigenvalue problem, corresponding to the constrained variational principle: f stationary on the manifold $\|x\| = R$, is then the search for solutions of (4.1.1) which lie on the line $\|x\| = R$ ($\|x\|$ is prescribed, λ not prescribed). The other attitude towards equation (4.1.1) amounts to the search of solutions that lie on the line $\lambda = \lambda_0$ (λ prescribed, $\|x\|$ not).

4.1.9. Ljusternik's theorem can be called the *Lagrange multiplier rule* for the extrema of a functional f under *functional side conditions*, thus for the case where the functional f is considered on a set of points $x \in X$ which satisfy a relation of the form $g(x) = c$, where g is another functional.

This result can be extended in an obvious way to the case where there is more than one side condition. Just as in classical analysis, the number of Lagrange multipliers equals the number of side conditions. For instance, let f, g and h be three Fréchet differentiable functionals on X. If f is to be extreme at $x_0 \in X$ with respect to the two conditions $g(x) = c$ and $h(x) = d$, and if grad $g(x_0) \neq 0$ and grad $h(x_0) \neq 0$, then a necessary condition for x_0 is that it is a solution of

$$\operatorname{grad} f(x_0) = \lambda \operatorname{grad} g(x_0) + \sigma \operatorname{grad} h(x_0),$$

for some $\lambda,\sigma \in \mathbb{R}$ (two Lagrange multipliers as there are two side conditions). Obviously, any point $x_0 \in X$ satisfying this equation for some $\lambda,\sigma \in \mathbb{R}$ is called a stationary point of f with respect to the two side conditions.

This extension of Ljusternik's theorem is more generally known than an extension to *operator side conditions*, where the functional f is considered on a set of points $x \in X$ which satisfy a prescribed operator equation. We shall not enter into this rather difficult matter (the interested reader is referred to KLÖTZER [18], RUND [19] and CARATHÉODORY [20]), but by way of example the following result is mentioned:

Let f be a functional on H_1, and T a (nonlinear) operator from H_1 into H_2 (H_1 and H_2 are Hilbert spaces). Suppose T is Fréchet differentiable with derivative A: $A(x)\colon H_1 \to H_2$. Let $A^*(x)\colon H_2 \to H_1$ be the adjoint of A(x) and let $N(A(x))$ and $R(A(x))$ denote the nullspace and range of the operator A(x) respectively; then the following result holds (cf. MAURIN [28]):

Let $\hat{x} \in \{x \in H_1 \mid T(x) = \xi\}$, *and suppose* $N(A(\hat{x})) \neq \{0\}$ *and* $R(A(\hat{x})) = H_2$. *If* f *is extreme at* $\hat{x}$ *with respect to the constraint* $T(x) = \xi$, *there exists a (unique) element* $\lambda \in H_2$ *such that*

$$\operatorname{grad} f(\hat{x}) = A^*(\hat{x})\lambda.$$

In the next subsection we shall use Ljusternik's theorem for functionals on a sphere in a Hilbert space. For convenience we shall explicitly reformulate the theorem for this case.

4.1.10. THEOREM. *Let* f *be a functional defined on a Hilbert space* H, *and suppose* f *is Fréchet differentiable on* S_R. *If* $x_0 \in S_R$ *is an extremum of* f *with respect to* S_R, *then there exists a* $\lambda \in \mathbb{R}$ *such that*

$$\operatorname{grad} f(x_0) = \lambda x_0.$$

PROOF. For simplicity we take R = 1. The space S_1 can be considered as the manifold $\{x \in H \mid g(x) = 1\}$ with $g(x) = \|x\|^2$. As grad g(x) = 2x, every point of S_1 is an ordinary point of S_1. Then the result follows from theorem 4.1.5. As an illustration, we note that equation (4.1.3) reads in this case

$$a + \tfrac{1}{4}a^2 + \varepsilon^2\|w\|^2 = 0;$$

a solution satisfying $a(0) = 0$, is given by

$$a(\varepsilon) = 2(-1 + \sqrt{1-\varepsilon^2\|w\|^2}).$$

From this expression it follows that $a(\varepsilon) = O(\varepsilon^2)$, thus, indeed,

$$a(\varepsilon) = o(\varepsilon) \quad \text{for } \varepsilon \to 0. \qquad \square$$

4.2. EIGENVALUE PROBLEMS AND BIFURCATION THEORY

4.2.1. DEFINITIONS

Let F be a nonlinear operator from a Hilbert space H into itself: $F: H \to H$.

The *nonlinear eigenvalue problem* corresponding to the operator F is the problem to determine the real numbers λ for which there exist elements $x \in H$ such that

$$F(x) = \lambda x \tag{4.2.1}$$

and $x \neq 0$. Such a number λ is called an *eigenvalue*, and the corresponding element $x \in H$ is called the *eigenfunction* corresponding to λ. The set of eigenvalues of a nonlinear operator F is usually called the spectrum of the nonlinear operator and is denoted by $\sigma(F)$. Note that this notion is not in complete agreement with the definition of the spectrum of a linear operator. But, as this name is adopted in literature, we shall use it in the sequel.

Concerning special solution sets $\{(\lambda,x)\}$ of equation (4.2.1) we shall give some definitions which can give rise to confusion and which are sometimes not clearly stated in literature, although such an extensive treatment of these matters may be, strictly speaking, somewhat out of the scope of this chapter.

The operator F is said to have a *continuum of eigenfunctions* if there exists a continuous parametrization $\lambda = \lambda(\alpha)$, $x = x(\alpha)$ with $F(x(\alpha)) = \lambda(\alpha) \cdot x(\alpha)$ and $x(\alpha) \neq 0$, for every α from some interval $I \subset \mathbb{R}$.

Two important examples of this notion are:

(i) to one eigenvalue $\lambda_0 \in \mathbb{R}$ there corresponds a one parameter family of eigenfunctions $x(\alpha)$: $F(x(\alpha)) = \lambda_0 x(\alpha)$, for every $\alpha \in I$. This happens

for instance if F is a linear operator: if λ_0 is an eigenvalue, and x_0 a normalized eigenfunction, then $x(\alpha) := \alpha \cdot x_0$ is for every $\alpha \in \mathbb{R}$, $\alpha \neq 0$, also an eigenfunction corresponding to λ_0. The parametrization in this case is, apart from sign, the norm of the eigenfunction;

(ii) (part of) the spectrum of F is continuous: there exists an interval $I \subset \mathbb{R}$ such that for every $\lambda \in I$ equation (4.2.1) has a solution $x \neq 0$. The dependence of this solution on λ is expressed by writing $x = x(\lambda)$: the parametrization is with respect to the eigenvalue $\lambda \in I$.

Now suppose $F(0) = 0$. Then $(\lambda,0)$ is for every $\lambda \in \mathbb{R}$ a solution of (4.2.1). This solution is called the trivial solution. A number $\lambda_0 \in \mathbb{R}$ is called a *bifurcation point of the operator* F (with respect to the trivial solution) if in any sufficiently small neighbourhood of $(\lambda_0,0) \in \mathbb{R} \times H$ there exists a solution (λ,x) of (4.2.1) with $x \neq 0$. If we take as norm in $\mathbb{R} * H$:

$$\|(\lambda,x)\|^2 := |\lambda|^2 + \|x\|^2 \qquad \lambda \in \mathbb{R},\ x \in H,$$

then an equivalent definition of bifurcation point is as follows: λ_0 is a bifurcation point of F, if there exists a sequence $\{(\lambda_n,x_n)\}_n$ of solutions of (4.2.1), i.e., $F(x_n) = \lambda_n x_n$ for every n, with $x_n \neq 0$, for which

$$(\lambda_n,x_n) \to (\lambda_0,0) \qquad \text{for } n \to \infty,$$

where the convergence is in the norm of $\mathbb{R} * H$ (and hence is equivalent to $\lambda_n \to \lambda_0$ and $\|x_n\| \to 0$ for $n \to \infty$). In a so-called *bifurcation diagram*, λ vs. $\|x\|$, a ball and sphere with center $(\lambda_0,0)$ and radius ε, described by

$$\Sigma_\varepsilon := \{(\lambda,\varepsilon) \mid \|(\lambda,x)-(\lambda_0,0)\| < \varepsilon,\ \|x\| > 0\}$$

and

$$\delta\Sigma_\varepsilon := \{(\lambda,x) \mid \|(\lambda,x)-(\lambda_0,0)\| = \varepsilon, \quad \|x\| > 0\},$$

respectively, can be marked as in figure 2.

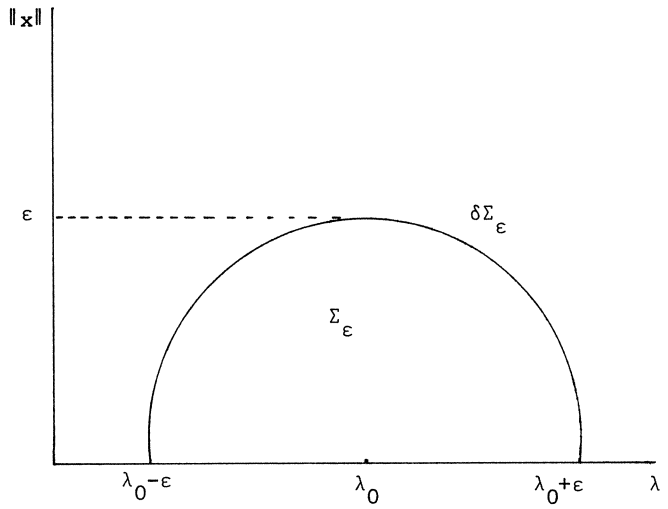

Figure 2

Now, suppose that λ_0 is a bifurcation point, and let us consider the solution set of equation (4.2.1) in a sufficiently small ball Σ_{ε_0} around $(\lambda_0,0)$:

$$N := \{(\lambda,x) \mid F(x) = \lambda x, \quad x \neq 0, \quad \|(\lambda,x)-(\lambda_0,0)\| < \varepsilon_0\}.$$

This set N is said to be a *continuous branch of eigenfunctions* if the boundary of any ball Σ_ε with $\varepsilon < \varepsilon_0$ contains an element of N, i.e.,

$$N \cap \delta\Sigma_\varepsilon \neq \emptyset \quad \text{for every } \varepsilon < \varepsilon_0.$$

This is the place where confusion may arise by a wrong interpretation of the word continuous. If we parameterize the solution set by ε: $(\lambda(\varepsilon),x(\varepsilon))$ for $0 \leq \varepsilon < \varepsilon_0$, it follows from the definition of continuous branch that $(\lambda(\varepsilon),x(\varepsilon))$ is continuous at $\varepsilon = 0$:

$$\|(\lambda(\varepsilon),x(\varepsilon))-(\lambda_0,0)\| \to 0 \quad \text{for } \varepsilon \downarrow 0,$$

which is equivalent to

$$\|x(\varepsilon)\| \to 0 \quad \text{and} \quad |\lambda(\varepsilon)-\lambda_0| \to 0 \quad \text{for } \varepsilon \downarrow 0.$$

But from this definition it does not necessarily follow that $(\lambda(\varepsilon),x(\varepsilon))$ is

continuous at every point $\varepsilon \in (0,\varepsilon_0)$: the limit

$$\lim_{\varepsilon\to\varepsilon_1} \|(\lambda(\varepsilon),x(\varepsilon))-(\lambda(\varepsilon_1),x(\varepsilon_1))\|^2 =$$

$$= \lim_{\varepsilon\to\varepsilon_1} \{|\lambda(\varepsilon)-\lambda(\varepsilon_1)|^2 + \|x(\varepsilon)-x(\varepsilon_1)\|^2\}$$

for $\varepsilon,\varepsilon_1 \in (0,\varepsilon_0)$, can cease to exist !, expressing the fact that $\lambda(\varepsilon)$ and $\|x(\varepsilon)\|$ need not depend continuously on ε for $\varepsilon \in (0,\varepsilon_0)$.

This can be expressed in a somewhat loose way by saying that the solution on $\delta\Sigma_{\varepsilon_1}$ can be anywhere on this sphere, and its "position", generally speaking, does not depend on the position of the solution on $\delta\Sigma_{\varepsilon}$, however small $|\varepsilon-\varepsilon_1|$ may be. (If for some specific case it is known that $\lambda(\varepsilon)$ depends continuously on ε, it is easily seen that then $\|x(\varepsilon)\|$ also depends continuously on ε (and conversely).)

From this rather formal point of view we can conclude that if in a bifurcation diagram a connected curve through the bifurcation point is drawn, as is done in almost all papers in this field, this connectedness is not a consequence of the fact that there exists a continuous branch of eigenfunctions, but must be a consequence of some other properties of the operator F.

For instance, with the aid of the implicit function theorem one can conclude that if (x^*,λ^*) is a solution of (4.2.1), where F is a compact operator which is continuously Fréchet differentiable on a neighbourhood of x^*, and $\lambda^* \neq 0$ is not an eigenvalue of $F'(x^*)$, then for $|\lambda-\lambda^*|$ sufficiently small there exists a unique solution curve $(x(\lambda),\lambda)$ of (4.2.1) through (x^*,λ^*) where $x(\lambda)$ depends continuously on λ. An analogous result can be formulated for bifurcation problems where $F(0) = 0$ and $F(x) = Lx + N(x)$, where L is a linear, compact operator and N a higher order nonlinear compact operator. If N satisfies

$$\|N(x)-N(y)\| \leq \rho(\|x\|+\|y\|) \cdot \|x-y\|$$

for all x and y sufficiently small, where ρ: $\mathbb{R}^+ \to \mathbb{R}^+$ is some continuous monotonous function with $\rho(0) = 0$, then one can conclude that in a sufficiently small neighbourhood of a bifurcation point $(\lambda_0,0)$ there exists a continuous branch of eigenfunctions, and the eigenfunctions depend continuously on λ. The condition on N is fulfilled if for instance N is continuously

Fréchet differentiable near $x = 0$. (For a specific problem, this last situation was dealt with in chapter IV using perturbation techinques.)

To be able to apply variational methods in the following, we have to require that F is a strongly potential operator. Studying the nonlinear eigenvalue problem (4.2.1) then amounts to an investigation of the stationary points of the potential of F subject to the condition $\|x\| = R$, with $R > 0$ (as follows from Ljusternik's theorem 4.1.10).

4.2.2. ASSUMPTION. In the rest of subsection 4.2 it is supposed that the operator $F: H \to H$ is a strongly potential operator, with potential $f(x)$. Thus

$$\text{grad } f(x) = F(x), \quad x \in H.$$

We shall now state some simple results concerning eigenvalue problems and bifurcation theory. We shall sometimes refer to results obtained earlier. Therefore, it is necessary to note that a Hilbert space is reflexive, and consequently that $\bar{B}_R$ is weakly compact (cf. theorem 1.1.9).

4.2.3. THEOREM. *Let* f *be w.l.s.c. on* $\bar{B}_R$, *and suppose* grad $f(x) \neq 0$ *for* $x \in B_R$. *Then* F *has an eigenfunction on* S_R, *i.e., there exists an element* $x \in S_R$ *with*

$$F(x) = \lambda x \quad \textit{for some } \lambda \in \mathbb{R}.$$

PROOF. From theorem 1.4.4 it follows that f achieves its infimum on $\bar{B}_R$, and from grad $f(x) \neq 0$ for $x \in B_R$ it follows that this infimum must be achieved at some point of S_R. Then the result follows from Ljusternik's theorem 4.1.10. □

4.2.4. THEOREM. *Let* f *be weakly continuous on* H, *and suppose* grad $f(x) \neq 0$ *for* $x \neq 0$. *Then* F *has eigenfunctions of any norm, i.e., for every* $R > 0$ *there exists an* $x \in S_R$ *such that* $F(x) = \lambda x$ *for some* $\lambda \in \mathbb{R}$.

PROOF. Let $R > 0$ be arbitrary. As f is weakly continuous it achieves both its infimum and its supremum on $\bar{B}_R$ (cf. theorem 1.3.2, Weierstrasz).

As at most one of these can be achieved in the interior B_R, namely at $x = 0$, it follows that f has an extremum on S_R. From this the result follows with Ljusternik's theorem 4.1.10. □

As an application of this theorem we shall once again consider the

4.2.5. LINEAR EIGENVALUE PROBLEM FOR THE LAPLACE OPERATOR

(cf. the exercise at the end of subsection 3.2.2). A number $\mu \in \mathbb{R}$ is called a *generalized eigenvalue* of the eigenvalue problem (3.2.39):

$$(4.2.2) \qquad \begin{cases} -\Delta u = \mu u & \text{for } x \in \Omega \\ u = 0 & \text{for } x \in \partial\Omega, \end{cases}$$

if there exists a $u \in H_0^1$ such that

$$(4.2.3) \qquad (\nabla u, \nabla v)_0 = \mu (u,v)_0 \qquad \text{for every } v \in H_0^1.$$

This function $u \in H_0^1$ is called the *generalized eigenfunction* corresponding to μ.

For the following, rather than to work with the inner product $(\ ,\ )_1$ in H_0^1, it is simpler to define another inner product:

$$(4.2.4) \qquad (u,v)_* := (\nabla u, \nabla v)_0 \qquad \text{for } u \in H_0^1,$$

with corresponding norm

$$(4.2.5) \qquad \|u\|_* := \|\nabla u\|_0 \qquad \text{for } u \in H_0^1.$$

To emphasize which inner product and norm we use, we write H_0^{1*} for $(H_0^1, (\ ,\ )_*)$. From Friedrichs' inequality (3.2.16) it follows that $\|\ \|_1$, and $\|\ \|_*$ are equivalent norms on H_0^1:

$$(4.2.6) \qquad \left(\frac{1}{1+c^2}\right)^{\frac{1}{2}} \|u\|_1 \le \|u\|_* \le \|u\|_1.$$

From this we conclude that the Hilbert spaces H_0^1 and H_0^{1*} are equivalent. This means that they consist of the same elements and that all the properties of H_0^1, such as Sobolev's embedding theorem, hold equally well for H_0^{1*}.

With this definition, the left-hand side of (4.2.3) can be rewritten according to (4.2.4) and the right-hand side is easily seen, with the aid of Friedrichs' inequality, to be a bounded, and linear, functional in $v \in H_0^{1*}$:

$$|(u,v)_0| \le \|u\|_0 \, \|v\|_0 \le c^2 \|u\|_* \, \|v\|_*.$$

With Riesz' representation theorem we can, therefore, write

(4.2.7) $\quad (u,v)_0 = (Lu,v)_* \quad$ for every $v \in H_0^{1*}$.

It is easily seen that $L: H_0^{1*} \to H_0^{1*}$ is a linear and self-adjoint map:

$$(Lu,v)_* = (u,v)_0 = (v,u)_0 = (Lv,u)_* = (u,Lv)_*.$$

Therefore, cf. 2.3.7, L is a potential operator on H_0^{1*} with potential

(4.2.8) $\quad f(u) = \frac{1}{2}(Lu,u)_* = \frac{1}{2}(u,u)_0,$

thus

(4.2.9) $\quad \text{grad } f(u) = Lu \quad \text{on} \quad H_0^{1*}.$

From these considerations we deduce that (4.2.3) can be rewritten as

$$(u,v)_* = \mu(Lu,v)_* \quad \text{for every } v \in H_0^{1*},$$

which gives the operator equation

$$u = \mu Lu \quad \text{in} \quad H_0^{1*}.$$

This equation is precisely an eigenvalue problem for the linear operator L with eigenvalues $\lambda = \frac{1}{\mu}$

(4.2.10) $\quad Lu = \lambda u \quad \text{in} \quad H_0^{1*}.$

This eigenvalue problem can now be investigated with variational principles by viewing it as the equation for the stationary points of the functional f, given by (4.2.8), with respect to the sphere $S_1^* \subset H_0^{1*}$:

(4.2.11) $\quad S_1^* := \{u \in H_0^1 \mid \|u\|_* = \|\nabla u\|_0 = 1\}.$

(Note, that for this *linear* problem it is immaterial which radius is taken for the sphere.) Furthermore, we note that if λ is an eigenvalue of (4.2.10) with eigenfunction $u \in S_1^*$, then $(Lu,u)_* = \lambda(u,u)_* = \lambda$, and hence

(4.2.12) $\quad \lambda = 2f(u).$

To prove the existence of an eigenvalue of (4.2.10) we want to apply theorem 4.2.4. Therefore, it is necessary to note that grad $f(u) = 0 \iff u = 0$. As $f(0) = 0$ and $f(u) > 0$ for $u \neq 0$, f is minimal at $u = 0$.

Furthermore, f is weakly continuous on H_0^{1*}: if $u_n \rightharpoonup \hat{u}$ in H_0^{1*} then $u_n \to \hat{u}$ in L_2, by theorem 1.5.3 for H_0^{1*} instead of H_0^1, and hence $f(u_n) \to f(\hat{u})$.

From theorem 4.2.4 we can conclude that f attains its maximum with respect to the ball $\bar{B}_1^*$ at some point $u_0 \in S_1^*$ and that there exists a $\lambda_0 \in \mathbb{R}$ such that

$$(4.2.13) \qquad Lu_0 = \lambda_0 u_0, \qquad u_0 \in S_1^*.$$

As $f(u_0)$ is the largest possible value of f on $\bar{B}_1^*$ we can obtain with (4.2.12) the following characterization for λ_0:

$$(4.2.14) \qquad \lambda_0 = 2f(u_0) = 2 \cdot \sup_{\bar{B}_1^*} f(u).$$

From this it follows that λ_0 is the largest possible eigenvalue for the eigenvalue problem (4.2.10) (and that, as f is not constant, $\lambda_0 \neq 0$).

Translating these results to terms that are appropriate for the eigenvalue problems (4.2.2) we can say that we have proved the existence of the smallest possible generalized eigenvalue μ_0, and that this eigenvalue can be characterized as

$$(4.2.15) \qquad \mu_0^{-1} = \sup_{\|\nabla u\|_0 \leq 1} \|u\|_0^2.$$

Define the functional $\psi(u)$ for $u \in H_0^1$, $u \neq 0$ by

$$(4.2.16) \qquad \psi(u) = \frac{\|u\|_0^2}{\|\nabla u\|_0^2} = \int_\Omega u^2(x)\, dx \Big/ \int_\Omega (\nabla u)^2\, dx\,,$$

and notice that $\psi(\alpha u) = \psi(u)$ for every $\alpha \neq 0$. Then (4.2.15) can be rewritten as

$$(4.2.17) \qquad \mu_0^{-1} = \psi(u_0) = \sup \psi(u),$$

with sup taken over $u \in H_0^1$, $u \neq 0$, which is in complete agreement with (3.2.38), but which is now obtained from another point of view.

Now we shall show how the other generalized eigenvalues of (4.2.2) can be found. It is at this place that the linearity of the problem starts to play a fundamental role. By way of example we demonstrate the method for the "second" eigenvalue (the adjective second will become clear presently). The other eigenvalues can then be successively found in the same way.

Suppose the largest eigenvalue, λ_0, for the eigenvalue problem (4.2.10) is known, and let u_0 be the corresponding eigenfunction: $Lu_0 = \lambda_0 u_0$, $u_0 \in H_0^{1*}$. To find another eigenvalue it suffices to look for eigenfunctions that are orthogonal to u_0:

$$(4.2.18) \qquad (u,u_0)_* = 0 \quad (\Longleftrightarrow (\nabla u, \nabla u_0)_0 = 0).$$

Indeed, eigenfunctions corresponding to different eigenvalues are orthogonal (if $Lu_0 = \lambda_0 u_0$ and $Lu_1 = \lambda_1 u_1$ with $\lambda_0 \neq \lambda_1$, then, using the self-adjointness of L:

$$(u_0,u_1)_* = \frac{1}{\lambda_1}(u_0,Lu_1)_* = \frac{1}{\lambda_1}(Lu_0,u_1)_* = \frac{\lambda_0}{\lambda_1}(u_0,u_1)_* \Rightarrow (u_0,u_1)_* = 0),$$

and eigenfunctions corresponding to the same eigenvalue can be chosen to be orthogonal (as L is linear).

The orthogonality conditions (4.2.18) can be expressed in another way: we note that from (4.2.7), (4.2.13) and from the self-adjointness of L follows that

$$(u,u_0)_* = \frac{1}{\lambda_0}(u,Lu_0)_* = \frac{1}{\lambda_0}(Lu,u_0)_* = \frac{1}{\lambda_0}(u,u_0)_0,$$

resulting in

$$(4.2.19) \qquad (u,u_0)_* = 0 \iff (u,u_0)_0 = 0 \quad (\Longleftrightarrow (\nabla u, \nabla u_0)_0 = 0),$$

which can also be expressed with the inner product $(\ ,\ )_1$ as

$$(u,u_0)_* = 0 \iff (u,u_0)_1 = 0.$$

Now consider the space consisting of all elements from H_0^{1*} that are orthogonal to u_0:

$$H_0^{1*}(u_0) := \{u \in H_0^1 \mid (u,u_0)_* = 0\}.$$

Then $H_0^{1*}(u_0)$ is again a Hilbert space. Furthermore, the functional f, given by (4.2.8), restricted to this space, is also weakly continuous.

The equation for the stationary points of f on $H_0^{1*}(u_0)$ can be obtained by considering the constrained variational principle

$$f(u) \text{ stationary on } H_0^{1*} \text{ subject to the condition } h(u) = (u,u_0)_* = 0.$$

By Ljusternik's theorem 4.1.5 this means that

$$\operatorname{grad} f(u) = \sigma \operatorname{grad} h(u) \quad \text{for some } \sigma \in \mathbb{R},$$

and as $\operatorname{grad} h(u) = u_0$, this amounts to

$$Lu = \operatorname{grad} f(u) = \sigma u_0. \tag{4.2.20}$$

The equation for the stationary points of f with respect to the sphere $S_1^*(u_0)$ in $H_0^{1*}(u_0)$:

$$S_1^*(u_0) = \{u \in H_0^1 \mid \|u\|_* = 1,\ (u,u_0)_* = 0\},$$

can be obtained by considering the constrained variational principle

$$f(u) \text{ stationary on } H_0^{1*} \text{ subject to the conditions: } \begin{cases} h(u) = (u,u_0)_* = 0, \\ g(u) = \|u\|_*^2 = 1. \end{cases}$$

With the extension of Ljusternik's theorem, cf. 4.1.9, we obtain as equation for the stationary points:

$$\operatorname{grad} f(u) = \tfrac{1}{2}\lambda \operatorname{grad} g(u) + \sigma \operatorname{grad} h(u) \quad \text{for some } \lambda,\sigma \in \mathbb{R},$$

or

$$Lu = \lambda u + \sigma u_0 \quad \text{for some } \lambda,\sigma \in \mathbb{R}. \tag{4.2.21}$$

Equations (4.2.20) and (4.2.21) may look somewhat strange, but in both equations the Lagrange multiplier σ must be zero: this can be seen by taking the inner product with u_0, using the fact that L is self-adjoint, and $(u,u_0)_* = 0$:

$$(Lu,u_0)_* = (u,Lu_0)_* = \lambda_0(u,u_0)_* = 0 = \lambda(u,u_0)_* +$$

$$+ \sigma(u_0,u_0)_* = \sigma(u_0,u_0)_*,$$

thus $\sigma = 0$ in (4.2.20) and (4.2.21).

Resuming: if f is stationary at some interior point u of $\bar{B}_1^*(u_0)$ then grad f(u) = 0, and if f is stationary at some point of $S_1^*(u_0)$ then

(4.2.22) $\quad \text{grad } f(u) = Lu = \lambda u \quad$ for some $\lambda \in \mathbb{R}$.

Arguing as in theorem 4.2.4, as f is weakly continuous on $\bar{B}_1^*(u_0)$, f achieves both its infimum and its supremum on $\bar{B}_1^*(u_0)$. As grad $f(u) = 0 \iff u = 0$, and $f(u) > 0$ for $u \neq 0$, f must attain its maximum at some point $u_1 \in S_1^*(u_0)$, and thus, from (4.2.22), there exists a $\lambda_1 \in \mathbb{R}$ such that

(4.2.23) $\quad Lu_1 = \lambda_1 u_1 \quad u_1 \in S_1^*(u_0)$.

This means that we have proved the existence of another eigenvalue of (4.2.10) with an eigenfunction orthogonal to u_0. Furthermore, with (4.2.12) λ_1 can be characterized as

(4.2.24) $\quad \lambda_1 = 2f(u_1) = 2 \cdot \sup_{\bar{B}_1^*(u_0)} f(u)$,

and, as $\bar{B}_1^*(u_0) \subset \bar{B}_1^*$, it follows that $\lambda_1 \leq \lambda_0$.

Translating these results to results for the eigenvalue problem (4.2.2) we can say that we have proved the existence of another eigenfunction u_1 with $(u_1,u_0)_0 = 0$, $\|\nabla u_1\|_0 = 1$, and that the corresponding eigenvalue μ_1 can be characterized as

(4.2.25) $\quad \mu_1^{-1} = \psi(u_1) = \sup \psi(u)$,

with sup taken over $u \in H_0^1$, $u \neq 0$, $(u,u_0)_0 = (u,u_0)_1 = 0$.

It will be clear that this process can be repeated: to prove the existence of another eigenvalue we consider functions that are orthogonal to u_0 and u_1, and so on. That there are infinitely many different, mutually orthogonal, eigenfunctions follows from the fact that this process can be continued without end, as H_0^1 is infinite dimensional.

A less trivial remark is that $\lambda_n \to 0$ for $n \to \infty$ (corresponding to $\mu_n \to \infty$ for $n \to \infty$), from which it follows that every eigenvalue has a finite multiplicity. The proof goes by contradiction. Suppose it is not true, then there exists a $\gamma > 0$ such that $\lambda_n \geq \gamma > 0$ for every n. For all n we have $\|u_n\|_* = \|\nabla u_n\|_0 = 1$, $\lambda_n = (u_n,Lu_n)_* = \|u_n\|_0^2 \geq \gamma$ and $(u_n,u_m)_0 = 0$

for $n \neq m$. The sequence $\{u_n\}_n$ is uniformly bounded in H_0^1, and, as H_0^1 is compactly imbedded in L_2, there exists a subsequence $\{u_{n'}\}$ which converges to some element $\hat{u}$ in L_2: $u_{n'} \to \hat{u}$ in L_2. As $(u_{n'},u_{m'})_0 = 0$ for every $n' \neq m'$, this can only be true if $\hat{u} = 0$. Thus $u_{n'} \to 0$ in L_2, contradicting the assumption that $\lambda_n = \|u_n\|_0^2 \geq \gamma$ for every n. Hence $\lambda_n \to 0$ for $n \to \infty$.

Furthermore, we can show that the set of eigenfunctions $\{u_n\}_n$ is complete in H_0^1. Suppose not, then the set

$$H := \{v \in H_0^1 \mid (v,u_i)_* = (v,u_i)_0 = 0 \text{ for every } i\}$$

is not empty, and is a Hilbert space. Considering the functional f on the closed unit ball $\bar{B}$ in this space H, it can be shown as before, that f must achieve its supremum on the boundary $\partial\bar{B}$, from which it follows that there exists an eigenfunction v^* such that

$$Lv^* = \lambda^* v^*, \quad \|v^*\|_* = 1, \quad (v^*,u_i)_* = 0 \text{ for every } i,$$

and the eigenvalue λ^* can be characterized as

$$\lambda^* = 2 \cdot \sup_{\bar{B}} f(u).$$

As $\lambda^* \neq 0$, and $\lambda_n \to 0$ for $n \to \infty$, there exists an N such that $\lambda_N < \lambda^*$.

Furthermore, $\bar{B} \subset \bar{B}_N := \{u \in H_0^1 \mid \|u\|_* \leq 1, (u,u_i)_* = 0$ for $i = 1,\ldots,N-1\}$ from which it follows that

$$\lambda^* = \sup_{\bar{B}} f(u) \leq \sup_{\bar{B}_N} f(u) = \lambda_N,$$

in contradiction with $\lambda_N < \lambda^*$. Hence $\{u_n\}_n$ is a complete set in H_0^1.

Resuming these results for the eigenvalue problem (4.2.2):

> We have proved the existence of infinitely many generalized eigenvalues $0 < \mu_1 \leq \mu_2 \leq \ldots$, each eigenvalue has finite multiplicity, and $\mu_n \to \infty$ for $n \to \infty$; the corresponding generalized eigenfunctions $u_1,u_2,\ldots$ can be chosen to be mutual orthogonal in H_0^1 and form a complete set in H_0^1.

4.2.6. NONLINEAR EIGENVALUE PROBLEM FOR THE LAPLACE OPERATOR

Consider, for a bounded domain Ω, the eigenvalue problem

$$(4.2.26)\qquad \begin{cases} -\Delta u = \lambda g(u) & \text{for } x \in \Omega \subset \mathbb{R}^n \qquad (n \geq 2), \\ u = 0 & \text{for } x \in \partial\Omega . \end{cases}$$

Here g is a given function defined and continuous on $\mathbb{R}$. As was shown in subsection 3.2.2, we can associate with g a potential

$$(4.2.27)\qquad \gamma(u) = \int_{\Omega} dx \int_{0}^{u(x)} g(z)\, dz,$$

such that:

γ is weakly continuous on H_0^1, and

γ is Fréchet differentiable on H_0^1, with

$$d\gamma(u) \cdot v = (g(u),v)_0 \qquad \text{for } u,v \in H_0^1,$$

provided g satisfies

$$(4.2.28)\qquad |g(z)| \leq \text{const.} + \alpha|z|^{\sigma}, \qquad \text{with } 0 < \sigma < \sigma_0 = \frac{n+2}{n-2} .$$

Using the Hilbert space H_0^{1*} as defined in the foregoing subsection, generalized eigenfunctions u of (4.2.26) with $\|u\|_* = R$, are the stationary points of the functional γ on the sphere $S_R^* = \{u \mid \|u\|_* = R\}$. For functions g satisfying (4.2.28), γ is also weakly continuous on H_0^{1*}. Then theorem 4.2.4 can be applied, yielding:
if g satisfies condition (4.2.28) together with

$$(4.2.29)\qquad g(z) = 0 \iff z = 0,$$

the nonlinear eigenvalue problem (4.2.26) has generalized eigenfunctions of arbitrary norm. By way of example, note that conditions (4.2.28) and (4.2.29) are fulfilled for

$$g(z) = z + z^3,$$

in which case the functional $\gamma(u)$ is given by $\int_{\Omega} dx\ (\frac{1}{2}u^2+\frac{1}{4}u^4)$.

4.2.7. REMARK. The results obtained for the linear Laplace eigenvalue problem are characteristic for results obtained with Ljusternik-Schnirelman

theory, which theory in particular deals with questions concerning the multiplicity of stationary points of a functional on a manifold, and possible characterizations of the stationary points.

Multiplicity statements for the stationary points of a functional with a nonlinear gradient are much more difficult than for the linear problem tackled in 4.2.5, due to the fact that the stationary points can no longer be supposed to be mutually orthogonal. To get an idea of the methods to be used for these problems, the interested reader is especially referred to VAINBERG [1, §14,15], RABINOWITZ [21, §3-5], PALAIS [17, p.185-190] and FUČIK et al. [22].

Theorem 4.2.4 is a special case of the following one, which shows that the mere fact that an operator is a strongly potential operator implies that it has a continuum of different eigenfunctions.

4.2.8. THEOREM. *Let* f *be weakly continuous on* B_{R_0}. *Then in every ball* B_R, *with* $0 < R \leq R_0$, F *has a continuum of eigenfunctions.*

PROOF. There are two cases to be considered:

(i) F has an eigenfunction on any sphere S_R with $R \leq R_0$. Then the conclusion of the theorem follows immediately: for the parametrization can be taken the radius R.

(ii) There exists a number $R \leq R_0$ such that F has no eigenfunction on S_R. For this case we shall show that the spectrum of F is continuous. As f is weakly continuous and since F has no eigenfunctions on S_R, the minimum m and maximum M of f on $\bar{B}_R$ exist and are different. Without loss of generality we may assume $f(0) > m$. Let x_0 be the point where $f(x_0) = m$. Then $x_0 \neq 0$ and $x_0 \notin S_R$, and thus grad $f(x_0) = 0$. Now consider the functional

$$\psi_\mu(x) := \mu(x,x) + f(x) \tag{4.2.30}$$

on $\bar{B}_R$. We want to investigate for which parameter values of μ, ψ_μ has stationary points in $B_R \setminus \{0\}$. From the fact that (x,x) is w.l.s.c. (cf. 1.4.3) it follows that ψ_μ is w.l.s.c. if $\mu \geq 0$. Thus ψ_μ achieves its infimum on $\bar{B}_R$ at some point $x_\mu \in \bar{B}_R$ (cf. theorem 1.4.4). From the assumption that F has no eigenfunction on S_R, it follows that $x_\mu \notin S_R$. Furthermore, $\psi_\mu(0) = f(0)$ and $\psi_\mu(x_\mu) \leq \psi_\mu(x_0) = \mu(x_0,x_0) + m$.

From this it follows that if we choose $\mu \in [0, \frac{f(0) - m}{\|x_0\|^2})$, then ψ_μ is not minimal at $x = 0$. Thus, for these values of μ we have $x_\mu \neq 0$, and as grad $\psi_\mu(x_\mu) = 0$, it follows that

$$F(x_\mu) = -2\mu\, x_\mu, \quad x_\mu \neq 0.$$

Conclusion: $(-2 \cdot \frac{f(0) - m}{\|x_0\|^2}, 0]$ belongs to the spectrum of F, from which the theorem follows. □

4.2.9. REMARK. From the proof of theorem 4.2.8 part (ii), we deduce the following result: *If* f *is weakly continuous on* B_R, *if* F *has no eigenfunctions on* S_R *and if, moreover,* F(0) = grad f(0) ≠ 0, *then the spectrum of* F *is the whole real axis, i.e., for every* $\lambda \in \mathbb{R}$ *there exists an element* $x \in H$ *such that* $F(x) = \lambda x$ *with* $0 < \|x\| < R$.

PROOF. Consider the functional ψ_μ defined by (4.2.30). Then grad $\psi_\mu(x) = 2\mu x$ + grad f(x), and as grad f(0) ≠ 0, grad $\psi_\mu(0) \neq 0$. Thus ψ_μ is not extreme at x = 0. Furthermore, as F has no eigenfunctions on S_R, ψ_μ is not extreme on S_R. If $\mu \geq 0$ then ψ_μ is w.l.s.c. and thus attains its infimum on $\bar{B}_R$ at some point $x_\mu \in B_R \setminus \{0\}$. If $\mu \leq 0$ then ψ_μ is w.u.s.c. and thus attains its supremum on $\bar{B}_R$ at some point $x_\mu \in B_R \setminus \{0\}$. Hence, for every $\mu \in \mathbb{R}$ there exists a point $x_\mu \in B_R \setminus \{0\}$ such that

$$\text{grad } \psi_\mu(x_\mu) = 0 = 2\mu\, x_\mu + F(x_\mu).$$

This shows that every real number λ is an eigenvalue of F. □

With the aid of lemma 3.3.4 the next theorem follows from theorem 4.2.8 in much the same way as theorem 3.3.5 follows from theorem 3.3.2.

4.2.10. THEOREM. *Let* $B: H \to H$ *be a self-adjoint, positive and compact operator. Let* t *be a Fréchet differentiable functional on* H, *with* T(x) = grad t(x). *Then the operator* $BT: H \to H$ *has at least a continuum of eigenfunctions.*

4.2.11. REMARK. It is useful to compare theorem 4.2.8 with theorem 3.2.2, and theorem 4.2.10 with theorem 3.3.5 and remark 3.3.6, especially concerning the role played by the growth conditions in the theorems of section 3.

4.2.12. Although the preceding results of this subsection include some

results concerning bifurcation theory (theorems 4.2.4, 4.2.8 and 4.2.10), we shall not enter into this theory very deeply here. We shall only make the following remarks.

Suppose F(0) = grad f(0) = 0, and suppose that F is Fréchet differentiable at x = 0 with F'(0) = B. If we require F to be compact, it can be shown that B is also compact and that f is weakly continuous (cf. VAINBERG [1, theorem 4.7 and 8.2 respectively]). It is well-known that in this case a number $\lambda_0 \neq 0$ can be a bifurcation point of F only if λ_0 is an eigenvalue of the (linear) operator B. This result can be proved with the aid of the implicit function theorem (as was done e.g. in chapter VIII, section 6). An interesting question is then which eigenvalues of B are indeed bifurcation points of the operator F. In chapter VIII, section 6, a result of Krasnosel'skii was stated:

if λ_0 *is an eigenvalue of odd (algebraic) multiplicity of* B, *then* λ_0 *is a bifurcation point of* F (F needs not to be a potential operator).

This result was established using Leray-Schauder degree theory. For potential operators F, Krasnoselskii obtained a much stronger result:

THEOREM (KRASNOSEL'SKII [2, p.332])
Let f *be a compact, nonlinear potential operator with* F(0) = 0. *Suppose* F *is uniformly Fréchet differentiable in some neighbourhood of the point* x = 0, *with* F'(0) = B. *Suppose, moreover, that* B *is a self-adjoint operator.*

Then every non-zero eigenvalue of B *is a bifurcation point of the operator* F.

The proof of this theorem will not be given here for the following reasons:

- that the largest positive eigenvalue of B (and also the smallest negative one) is a bifurcation point of F can be proved, under less restrictive requirements, by using methods and concepts that are explained in this chapter. However, there are many technicalities involved and no new insight can be gained from the proof. For the proof, see KRASNOSEL'SKII [2, p.322-325].
- that the other eigenvalues of B are also bifurcation points of F is much more difficult to prove. In fact, the proof uses a concept (mini-max principle) that is not dealt with in this chapter, and that can be better explained in a treatment of the Ljusternik-Schnirelman theory. For the proof, see KRASNOSEL'SKII [2, p.325-337].

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