THE EUCLIDEAN ALGORITHM IN CUBIC NUMBER FIELDS

STEFANIA CAVALLAR, FRANZ LEMMERMEYER

ABSTRACT. In this note we present algorithms for computing Euclidean minima of cubic number fields; in particular, we were able to find all norm-Euclidean cubic number fields with discriminants $-999 < d < 10^4$.

1. INTRODUCTION

This article deals with the problem of determining whether a given cubic number field is Euclidean with respect to the absolute value of the norm. The corresponding problem for quadratic number fields was solved in 1952, when Barnes and Swinnerton-Dyer showed (after much work done by various authors) that the following list of discriminants of norm-Euclidean quadratic number fields is complete:

$$d = -11, -8, -7, -4, -3, 5, 8, 12, 13, 17, 21, 24, 28, 29, 33, 37, 41, 44, 57, 73, 76.$$

In the cubic case, Davenport proved that the number of norm-Euclidean complex cubic number fields (i.e. cubic fields with unit rank 1) is finite, whereas Heilbronn conjectured that there are infinitely many totally real cubic fields which are norm-Euclidean. We hope that the methods presented in this paper will eventually lead to a complete list of norm-Euclidean complex cubic fields, and that extended computations for real cubic fields will show whether Heilbronn's conjecture is reasonable or not.

2. NOTATION

In order to present our method we need a few definitions. Let K be a number field, and let $\mathcal{O}_K$ denote its ring of integers. The Euclidean minimum of $\xi \in K$ is defined to be

$$M(K, \xi) = \inf \{|N_{K/\mathbb{Q}}(\xi - \eta)| : \eta \in \mathcal{O}_K\}.$$

The field K is Euclidean with respect to the absolute value of the norm (norm-Euclidean for short) if $M(K, \xi) < 1$ for all $\xi \in K$. Let us introduce the *Euclidean minimum* $M(K)$ of K by putting $M(K) = \sup \{M(K, \xi) : \xi \in K\}$. Obviously, K is norm-Euclidean if $M(K) < 1$, and not norm-Euclidean if $M(K) > 1$ or if there is a $\xi \in K$ such that $M(K) = M(K, \xi) = 1$.

Next we introduce the inhomogeneous minimum. To this end let K be a number field generated by a root α of an irreducible monic polynomial $f \in \mathbb{Z}[x]$. Let $\alpha_1, \ldots, \alpha_r$ denote the real roots, and $\alpha_{r+1}, \overline{\alpha_{r+1}}, \ldots, \alpha_s, \overline{\alpha_s}$ the s pairs of complex conjugate roots of f in $\mathbb{C}$; then the maps $\alpha \longrightarrow \alpha_j$ can be extended to yield r embeddings $\phi_1, \ldots, \phi_r : K \longrightarrow \mathbb{R}$ and s pairs of complex conjugate embeddings $\phi_{r+1}, \overline{\phi_{r+1}}, \ldots, \phi_s, \overline{\phi_s} : K \longrightarrow \mathbb{C}$.

Choose a $\mathbb{Q}$-basis $\{\beta_1, \ldots, \beta_n\}$ of K; the map

$$\pi : K \longrightarrow \mathbb{R}^n : \sum a_j \beta_j \longmapsto (a_1, \ldots, a_n)$$

embeds K into $\mathbb{R}^n$, and we will identify K and $\pi(K)$ for the rest of this article. Clearly K is dense in $\mathbb{R}^n$, so we will write $\overline{K} = \mathbb{R}^n$ if we want to make it clear that we regard $\mathbb{R}^n$ as the closure of K. Now

$$N : \mathbb{R}^n \longrightarrow \mathbb{R} : (x_1, \ldots, x_n) \longmapsto \prod_{j=1}^{r} \Big| \sum_i x_i \phi_j(\beta_i) \Big| \cdot \prod_{j=r+1}^{s} \Big| \sum_i x_i \phi_j(\beta_i) \Big|^2$$

is a continuous map which coincides with the absolute value of the norm $N_{K/\mathbb{Q}}$ when restricted to K. Similarly, the maps

$$|\cdot|_j : \mathbb{R}^n \longrightarrow \mathbb{R} : x = (x_1, \ldots, x_n) \longmapsto |x|_j = \Big| \sum_i x_i \phi_j(\beta_i) \Big|, \quad (0 \le j \le r+s),$$

are continuous and their restrictions to K agree with the $r+s$ archimedean valuations of K. By an abuse of language, we will refer to the maps N and $|\cdot|_j$ as the 'norm' and the 'valuations' of $\overline{K}$, respectively, although N is not a norm on $\overline{K}$ since $N(x) = 0$ does not imply $x = 0$. Similarly, the $|\cdot|_j$ are not valuations of $\overline{K}$ for the same reason. All we can say is

Proposition 1. *Let K be a number field, and assume that $\lim_{i\to\infty} |\xi_i|_j = 0$ for all $1 \le j \le r+s$ and a sequence of elements $\xi_i \in \overline{K}$. Then $\lim_{i\to\infty} \xi_i = 0$.*

Proof. K is an n-dimensional $\mathbb{Q}$-vector space, with a nondegenerate bilinear form given by $\langle \xi, \eta \rangle = \mathrm{Tr}_{K/\mathbb{Q}}(\xi\eta)$, where $\mathrm{Tr}_{K/\mathbb{Q}}$ denotes the trace of $K/\mathbb{Q}$. Choose a $\mathbb{Q}$-basis $\{\alpha_1, \ldots, \alpha_n\}$ of K, and let $\{\beta_1, \ldots, \beta_n\}$ denote the dual basis with respect to $\langle \cdot, \cdot \rangle$, i.e. the basis with the property $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha_i\beta_j) = \delta_{ij}$ (Kronecker's delta).

Now assume that $\lim_{i\to\infty} |\xi_i|_j = 0$ for all $1 \le j \le r+s$, and let $\delta > 0$ be given; then there exists an $N \in \mathbb{N}$ such that $|\xi_i|_j < \delta$ for all $i \ge N$ and $1 \le j \le r+s$. Write ξ_i as $\xi_i = x_1^{(i)}\alpha_1 + \ldots + x_n^{(i)}\alpha_n$. Then $|x_k^{(i)}| = |\mathrm{Tr}_{K/\mathbb{Q}}(\beta_k\xi_i)|$. Since the trace is the sum of all conjugates of $\beta_k\xi_i$, applying the triangle inequality yields

$$\begin{aligned} |x_k^{(i)}| &\le |\beta_k\xi_i|_1 + \ldots + |\beta_k\xi_i|_r + 2|\beta_k\xi_i|_{r+1} + \ldots + 2|\beta_k\xi_i|_{r+s} \\ &\le \delta(|\beta_k|_1 + \ldots + 2|\beta_k|_{r+s}) < \delta C, \end{aligned}$$

where $C = |\beta_k|_1 + \ldots + 2|\beta_k|_{r+s}$ does not depend on i or the choice of δ. Since we can make δ as small as we please, we find that $\lim_{i\to\infty} x_k^{(i)} = 0$ for all $1 \le k \le n$, and this is equivalent to $\lim_{i\to\infty} \xi_i = 0$. □

Obviously K is norm-Euclidean if and only if for every $\xi \in K$ there is an $\alpha \in \mathcal{O}_K$ such that $N(\xi - \alpha) < 1$. Actually, all known examples of norm-Euclidean number fields satisfy the stronger condition that for every $\xi \in \overline{K}$ there exists an $\alpha \in \mathcal{O}_K$ such that $N(\xi - \alpha) < 1$. We put

$$M(\overline{K}, \xi) = \inf \{N(\xi - \eta) : \eta \in \mathcal{O}_K\},$$

and define the *inhomogeneous minimum* of $\overline{K}$ as $M(\overline{K}) = \sup \{M(\overline{K}, \xi) : \xi \in \overline{K}\}$. Clearly $M(K) \le M(\overline{K})$; it is conjectured that $M(K) = M(\overline{K})$ for all number fields, but so far this equality has been proved only for fields with unit rank ≤ 1.

We say that $M(\overline{K})$ is *isolated* if $M_2(\overline{K}) = \sup \{M(\overline{K}, \xi) : \xi \in \overline{K} \setminus \{\zeta : M(\overline{K}, \zeta) = M(\overline{K})\}\} < M(\overline{K})$. In this case, we call $M(\overline{K}) = M_1(\overline{K})$ the first and $M_2(\overline{K})$ the second minimum of $\overline{K}$.

Remark 1. $\overline{K}$ is a ring. In fact, the product of two elements $\xi = \sum a_i\alpha^i$ and $\eta = \sum b_i\alpha^i$ of K has the form $\sum c_i\alpha^i$, where c_i is a polynomial in $\mathbb{Q}[a_1, \ldots, b_n]$. Thus, the product $\xi\eta$ can be given a meaning for real values of the coefficients a_i, b_i.

Now put $n = (K : \mathbb{Q})$, and choose an integral basis $\{\beta_1 = 1, \beta_2, \ldots, \beta_n\}$ and a real number $k > 0$ (for example $k = 0.99$). We start by dividing

$$\overline{F}_+ = \Big\{\xi = \sum_{i=1}^{n} a_i\beta_i \mid a_1 \in [0, 1/2],\, a_2, \ldots, a_n \in (-1/2, 1/2]\Big\}$$

into smaller subcubes. Such a subcube S is called *k-covered* (or simply *covered* if the reference to k is clear) if we can find a $\gamma \in \mathcal{O}_K$ such that $N(\xi - \gamma) < k$ for all $\xi \in S$; it is called *uncovered* if we cannot find such a γ (even if there exists one). Finally, a point $\xi \in \overline{K}$ is called *k-exceptional* if $N(\xi - \gamma) \geq k$ for all $\gamma \in \mathcal{O}_K$.

Remark 2. Observe that $\overline{F}_+$ is 'half a fundamental domain' in the sense that every $\xi \in \overline{K}/\mathcal{O}_K$ has a representative in $\overline{F} = \overline{F}_+ \cup \overline{F}_-$, where $\overline{F}_- = -\overline{F}_+$ (and only one unless the representative lies on the boundary of $\overline{F}$). It is clearly sufficient to consider $\overline{F}_+$ since $N(-\xi) = N(\xi)$. For cyclic cubic fields K we could reduce $\overline{F}_+$ further by exploiting the fact that $N(\xi^\sigma) = N(\xi)$ for all $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$.

Remark 3. Occasionally it simplifies computations to use fundamental domains other than $\overline{F}$; they will be denoted by $\widetilde{F}$, and we will always assume that $\widetilde{F}$ has compact closure. As an example, take

$$\widetilde{F} = \Big\{\xi = \sum_{i=1}^{n} a_i\beta_i \mid a_1 \in [0, 1),\, a_2, \ldots, a_n \in (-1/2, 1/2]\Big\}.$$

Remark 4. For real numbers $k', k > 0$ such that $k' > k$ it is clear that any k-covered set is also k'-covered, and that any k'-exceptional point is k-exceptional.

3. The Algorithms

In this section we will describe the five programs (`Eu3_1`, ... , `Eu3_5`) which have allowed us to compute Euclidean minima $M(K)$ for many cubic number fields K. Since we used floating point arithmetic to compute the $M(K)$ (which are rational numbers, at least in each case we succeded in its computation), a few explanations are in order. Suppose that we want to show $M(K) = c$ for a number field K. Then we choose $k \leq 0.99c$ (as a protection against rounding errors), and, using the programs `Eu3_1` – `Eu3_3`, we compute cubes S_j which contain all k-exceptional points (we want to be sure that they contain every c-exceptional point). Then we exploit the action of the unit group on these cubes to compute the possible exceptional points, and since this is done with integer arithmetic, we are able to get exact results.

Experiments with e.g. fields whose minima are known from hand computations (choosing values of k close to the minimum and using very small cube lengths ℓ) have led us to trust our results. Moreover, all our results agree with those obtained earlier (e.g. by Smith and Taylor) or afterwards (by R. Quême).

Our programs require as input a file called `disc` (i.e. `985` for the field with discriminant $d = 985$; for complex fields we used the absolute value of the discriminant preceded by "_", for example `_199` for the field with discriminant $d = -199$). This file contains the following data:

- disc K;
- the coefficients of the irreducible monic polynomial f;
- – for real fields the roots of the polynomial α, α', α'';

  - for complex fields the real root α, the real and the imaginary part of the complex root;
- the coefficients with respect to the base $\{\frac{1}{g}, \frac{\alpha}{g}, \frac{\alpha^2}{g}\}$ (where g is defined below) of a system of independent units;
- the index $g = (\mathcal{O}_K : \mathbb{Z}[\alpha])$ and in the case of $g \neq 1$ also g_x, g_y, g_z, where $\{1, \alpha, \theta = \frac{g_x}{g} + \frac{g_y}{g}\alpha + \frac{g_z}{g}\alpha^2\}$ form a $\mathbb{Z}$-basis of $\mathcal{O}_K$;
- the value $k > 0$;
- the edge length ℓ of the cubes;
- the coordinates of the uncovered cubes.

For the sake of simplicity we treat only cubes having the same size; thus a cube is uniquely determined by its leftmost corner and the edge length ℓ. We therefore start with the four cubes making up $\overline{F}_+$, and the initial file 985, for example, looks as in Table 1 below:

Table 1

```
985  1    -6    -1
 -2.93080160017276
 -0.16296185677753
  2.09376345695029
     0     1     0
     2    -1     0
           1
         0.9
         0.5
     0  -0.5  -0.5
     0  -0.5     0
     0     0  -0.5
     0     0     0
```

Table 2

```
985    1    -6    -1
   -2.93080160017276
   -0.16296185677753
    2.09376345695029
       0     1     0
       2    -1     0
             1
           0.9
           0.1
       0.3  -0.5  -0.5
       0.3  -0.5  -0.3
          . . .
       0.4   0.4   0.4
```

Table 3

```
985      1     -6     -1
              . . .
            0.02
      0.38  -0.22   0.38
      0.38  -0.22    0.4
      0.38   -0.2   0.38
      0.38  -0.18   0.38
      0.38   -0.2    0.4
       0.4  -0.22   0.38
       0.4  -0.22    0.4
       0.4   -0.2   0.38
       0.4  -0.18   0.38
       0.4   -0.2    0.4
```

We now run the programs `Eu3_1`, `Eu3_2` and `Eu3_3`, which will be described in the sequel, on the file `disc`. The programs `Eu3_4` and `Eu3_5` will be explained before Prop. 4 and Cor. 7, respectively.

`Eu3_1`. This program first asks for a discriminant, then reads the corresponding file `disc`. The first eight lines of `disc` are copied to the (temporary) file `disc.new`. The next input is an integer f which is the factor by which we divide the edge length of the cubes. We have used $f \in \{1, 2, 4, 5\}$, depending on the size of `disc`; of course the choice $f = 1$ is only useful after k has been replaced by some $k' > k$. Thus `Eu3_1` reads ℓ from `disc` and writes ℓ/f to `disc.new`. Moreover, if no file `disc.p` exists, `Eu3_3` creates one and writes the translation vector `(0, 0, 0)` into it.

Now `Eu3_1` splits each cube read from `disc` into f^3 smaller ones, computes an upper bound $B = B(S)$ of the minimum for each of these subcubes, and writes those S with $B(S) \geq k$ to the file `disc.new`. Having reached the end of the file `disc`, it copies `disc` to `disc.bak` (a security backup) and `disc.new` to `disc`.

How do we bound the minimum on a cube $S = [a, a+\ell] \times [b, b+\ell] \times [c, c+\ell]$? Since the norm is the product of the three $\overline{K}$-valuations, we only need to find bounds for $|\xi|_j$, where $\xi = x + y\alpha + z\theta \in S$. But since $|\xi|_j$ is a linear function of x, y, z, it takes its maximum at the corners of the subcubes. Instead of computing $|\cdot|_j$ at all eight corners and taking the maximum of these values as our upper bound (as the

programs with which we computed the tables at the end did), we can use a trick due to Roland Quême [9] which gives this bound at one stroke (but which doesn't seem to work except for valuations corresponding to real embeddings): in fact, the value of $|\cdot|_j$ at the eight corners is one of

$$|\xi_0 \pm \tfrac{\ell}{2} \pm \tfrac{\ell}{2}\alpha \pm \tfrac{\ell}{2}\theta|_j, \qquad \text{where} \quad \xi_0 = (a+\tfrac{\ell}{2}) + (b+\tfrac{\ell}{2})\alpha + (c+\tfrac{\ell}{2})\theta,$$

the corners corresponding to the different choices of the signs. Using the triangle inequality we easily get

$$|\xi_0 \pm \tfrac{\ell}{2} \pm \tfrac{\ell}{2}\alpha \pm \tfrac{\ell}{2}\theta|_j \;\le\; |\xi_0|_j + \tfrac{\ell}{2}(1 + |\alpha|_j + |\theta|_j).$$

On the other hand, choosing the signs in such a way that ξ_0, ± 1, $\pm\alpha$ and $\pm\theta$ all have the same sign (in the embedding corresponding to $|\cdot|_j$; here is where we need that the embedding is real) we see that this bound is best possible.

Using this method of bounding $N(S-\gamma)$, `Eu3_1` will start looking for translation vectors in `disc.p`; if there is no element in this file such that $N(S-\gamma) < k$ we search for translation vectors in the set

$$(1) \qquad I = \{x + y\alpha + z\theta \,|\, (x,y,z) \in \mathbb{Z}^3, |x| \le M_x, |y| \le M_y, |z| \le M_z\},$$

where M_x, M_y, M_z were usually chosen (depending on ℓ) as follows:

	$\ell \ge 0.02$	$0.01 \ge \ell \ge 0.001$	$\ell \le 0.0005$
M_x	8	19	30
M_y	5	12	17
M_z	2	3	5

The nonsymmetric limits were suggested by experience. If we find a translation vector $\gamma \in \mathcal{O}_K$ such that $N(S-\gamma) < k$ we write γ to the file `disc.p`. There are several reasons for proceeding like this:

1. If $N(S-\gamma) < k$, then γ has a good chance of satisfying $N(S'-\gamma) < k$ for cubes S' in the vicinity of S. By searching `disc.p` first we actually save much CPU time.
2. If we find that we have to replace k by some $k' < k$, we have to redo the computations from start; we are, however, able to use the translation vectors found in the previous runs. In fact, our programs allow the option of searching for new vectors or just using those in `disc.p`.

After the first run of `Eu3_1` with $f = 5$ (which took 0.63 seconds of CPU time on an RS 6000), the file `985` looks as in Table 2. It contains exactly 106 noncovered cubes. The file `985.p` contains the following translation vectors:

```
(0 0 0), (0 1 0), (0 2 1), (1 0 0), (-1 -4 -1), (0 3 -1), (-1 -3 2),
(0 1 -1), (0 2 0), (3 0 -1), (0 2 -1), (-1 0 0), (-1 -4 2), (-2 0 0)
```

`Eu3_2`. This program acts like `Eu3_1` with the difference that the original cube is written to `disc.new` as soon as one of its subcubes cannot be covered. In other words: `Eu3_2` eliminates those cubes whose subcubes of length ℓ/f can all be covered. This is convenient if we already have to deal with a lot of cubes and a further division done as by `Eu3_1` is likely to lead to an enormous number of smaller cubes. We usually run `Eu3_2` before using `Eu3_3` in order to save CPU time.

`Eu3_3`. Scanning through all the integers of I (see (1)) takes much time. We can avoid searching for explicit translations if we proceed as follows: we multiply a cube T with a non-torsion unit ε, translate the result back into the fundamental domain by subtracting $\beta \in \mathcal{O}_K$ and look whether $\varepsilon T - \beta$ intersects one of the cubes not yet covered. The program does not really compare the oblique prism $\varepsilon T - \beta$ with the uncovered cubes but rather uses the smallest box B which contains T (actually it is slightly larger in order to avoid false results due to round-off errors) and has faces parallel to the coordinate planes (this will be improved in the next version of our programs). Evidently we have to compare the box also with the 'opposite' cubes, i.e. the cubes multiplied by -1, since we only kept the 'bad' cubes of half the fundamental domain $\overline{F}_+$. If there is no intersection, the cube itself can be eliminated.

The reason why this program is so successful is the following: suppose that S is a subcube such that $\varepsilon S - \alpha$ (where $\alpha \in \mathcal{O}_K$ is the element needed to translate εS back into $\overline{F}$) is covered by $\gamma \in \mathcal{O}_K$, i.e. $N(\varepsilon S - \alpha - \gamma) < k$. This means of course that $N(S - \beta) < k$ for $\beta = \varepsilon^{-1}(\alpha + \gamma)$: but β will usually have much larger coefficients than those scanned in (1). Moreover, in general $\varepsilon S - \alpha$ will not be covered by a single element $\gamma \in \mathcal{O}_K$, which means that we would have to divide S into subcubes before we could cover it directly.

A run of `Eu3_1` on the file in Table 2, again with $f = 5$, leaves only 27 subcubes uncovered. Running `Eu3_3` twice on the file obtained we are left with 10 uncovered cubes (see Table 3). Running `Eu3_2` on the file in Table 3 deletes `(0.38 -0.18 0.38)` and `(0.4 -0.18 0.38)`. Now we have covered $\overline{F}$ except for the set $T = [0.38, 0.42] \times [-0.22, -0.18] \times [0.38, 0.42]$. Multiplying the corners P of T by the unit α we find

P	αP
$0.38 - 0.22\alpha + 0.38\alpha^2$	$0.38 - 0.34\alpha + 0.40\alpha^2 + 3\alpha - \alpha^2$
$0.38 - 0.22\alpha + 0.42\alpha^2$	$0.42 - 0.10\alpha + 0.36\alpha^2 + 3\alpha - \alpha^2$
$0.38 - 0.18\alpha + 0.38\alpha^2$	$0.38 - 0.34\alpha + 0.44\alpha^2 + 3\alpha - \alpha^2$
$0.38 - 0.18\alpha + 0.42\alpha^2$	$0.42 - 0.10\alpha + 0.40\alpha^2 + 3\alpha - \alpha^2$
$0.42 - 0.22\alpha + 0.38\alpha^2$	$0.38 - 0.30\alpha + 0.40\alpha^2 + 3\alpha - \alpha^2$
$0.42 - 0.22\alpha + 0.42\alpha^2$	$0.42 - 0.06\alpha + 0.36\alpha^2 + 3\alpha - \alpha^2$
$0.42 - 0.18\alpha + 0.38\alpha^2$	$0.38 - 0.30\alpha + 0.44\alpha^2 + 3\alpha - \alpha^2$
$0.42 - 0.18\alpha + 0.42\alpha^2$	$0.42 - 0.06\alpha + 0.40\alpha^2 + 3\alpha - \alpha^2$

This shows that αT is contained in the set

$$T' = [0.38, 0.42] \times [-0.34, -0.06] \times [0.36, 0.44] + 3\alpha - \alpha^2$$

(in general, this is a very crude estimate; the next version of our programs will take the actual shape of εT into account). Observe that $T' - 3\alpha + \alpha^2$ does not intersect $-T$. If we were to keep dividing the uncovered cubes, this picture would not change: the length of the cubes would become smaller and smaller, and so would the size of our uncovered set T, but $\alpha T - 3\alpha + \alpha^2$ would always have points in common with T. This is the situation which is described in

Proposition 2. *Let K be a number field and ε a non-torsion unit of E_K. Suppose that $T \subset \widetilde{F}$ has the following property:*

> *there exists a unique $\beta \in \mathcal{O}_K$ such that, for all $\xi \in T$, the element $\varepsilon\xi - \beta$ lies in a k-covered region of $\widetilde{F}$ or again in T.*

Then every k-exceptional point $\xi_0 \in T$ satisfies $|\xi_0 - \frac{\beta}{\varepsilon-1}|_j = 0$ for every $\overline{K}$-valuation $|\cdot|_j$ such that $|\varepsilon|_j > 1$. If, moreover, $|\varepsilon|_j \neq 1$ for all $\overline{K}$-valuations, then the sequence $\xi_0, \xi_1, \xi_2, \ldots$ of k-exceptional points defined by the recursion $\xi_{i+1} = \varepsilon\xi_i - \beta$ satisfies $\lim_{i\to\infty} \xi_i = \frac{\beta}{\varepsilon-1}$.

Proof. Suppose that $\xi_0 \in T$ is k-exceptional. Since $M(K,\xi) = M(K, \varepsilon\xi - \beta)$, our assumption implies that all the ξ_i defined by $\xi_{i+1} = \varepsilon\xi_i - \beta$ are k-exceptional points in T. Now $\zeta = \frac{\beta}{\varepsilon-1}$ is the fixed point of the map $\xi \longmapsto \varepsilon\xi - \beta$. Induction shows that $\xi_i - \zeta = \varepsilon^i(\xi_0 - \zeta)$ for all $i \geq 0$. In particular we see that

$$|\xi_i - \zeta|_j = |\varepsilon|_j^i \, |\xi_0 - \zeta|_j. \tag{2}$$

Now we claim that there exists a constant $C > 0$ such that $|\xi_i - \zeta|_j \leq C$ for all $i \geq 0$ ($C = C(j)$ may depend on j, but we can always choose C as the maximum of the (finitely many) $C(j)$). In fact, since $\xi_i, \zeta \in \widetilde{F}$, we see that their difference is an element of $2\widetilde{F} \subset \overline{K}$. Since $2\widetilde{F}$ has compact closure and $|\cdot|_j$ is continuous, $|\cdot|_j$ has a maximum C on the closure of $2\widetilde{F}$ and hence is bounded on $2\widetilde{F}$.

Assume that $|\varepsilon|_j > 1$; then the fact that the left hand side of Equ. (2) is bounded implies that $|\xi_0 - \zeta|_j = 0$. This in turn gives immediately $|\xi_i - \zeta|_j = 0$. If, moreover, $|\varepsilon|_j \neq 1$ for all $j \leq r+s$, then either $|\varepsilon|_j > 1$ and $|\xi_i - \zeta|_j = 0$, or $|\varepsilon|_j < 1$ and $0 = \lim_{i\to\infty} |\xi_i - \zeta|_j$. This implies $\lim \xi_i = \zeta$ by Prop. 1. □

In our case there are three embeddings; we have $|\alpha|_j > 1$ for $j = 1, 3$, and $|\alpha|_j < 1$ for $j = 2$. From Prop. 5 we can deduce that every 0.9-exceptional point $\xi \in T$ satisfies $|\xi - \zeta|_1 = |\xi - \zeta|_3 = 0$, where $\zeta = \frac{3\alpha - \alpha^2}{\alpha - 1} = \frac{1}{5}(2 - \alpha + 2\alpha^2)$. In order to show that ξ is in fact the only 0.9-exceptional point $\xi \in T$ we have to prove $|\xi - \zeta|_2 = 0$; this is done by using the inverse of the unit α:

Theorem 3. *Let K be a number field, T a compact subset of $\widetilde{F}$, and let $\varepsilon \in E_K$ be a non-torsion unit. Suppose that*

1. *there is a $\beta \in \mathcal{O}_K$ such that, for all $\xi \in T$, $\varepsilon\xi - \beta$ lies in a k-covered region of $\widetilde{F}$ or again in T;*
2. *for all $\xi \in T$ there is a $\gamma \in \mathcal{O}_K$ such that $\varepsilon^{-1}\xi - \gamma$ lies in a k-covered region of $\widetilde{F}$ or again in T;*
3. *$|\varepsilon|_j \neq 1$ for $1 \leq j \leq r+s$.*

Then $\frac{\beta}{\varepsilon-1}$ is the only possible k-exceptional point of T.

Proof. Let $\xi \in T$ be a k-exceptional point. If $|\varepsilon|_j > 1$ then Prop. 2 shows that $|\xi - \frac{\beta}{\varepsilon-1}|_j = 0$. The other $\overline{K}$-valuations $|\cdot|_j$ satisfy $|\varepsilon|_j < 1$ because of 3., and we see $|\varepsilon^{-1}|_j > 1$. Since ξ is k-exceptional, so is $\xi_1 = \varepsilon^{-1}\xi - \gamma \in T$. Now $\varepsilon\xi_1 - \beta = \varepsilon(\varepsilon^{-1}\xi - \gamma) - \beta = \xi - (\beta + \varepsilon\gamma)$; but $\xi \in T$ and $\xi - (\beta + \varepsilon\gamma) \in T$ imply that $\beta + \varepsilon\gamma = 0$, i.e. $\gamma = -\varepsilon^{-1}\beta$. Therefore the γ in 2. is uniquely determined, and we can apply Prop. 2 with ε^{-1} and γ instead of ε and β. This shows that any k-exceptional point $\xi \in T$ satisfies $|\xi - \frac{\gamma}{\varepsilon^{-1}-1}|_j = 0$ for all $\overline{K}$-valuations with $|\varepsilon|_j < 1$. But $\frac{\gamma}{\varepsilon^{-1}-1} = \frac{\varepsilon\gamma}{1-\varepsilon} = \frac{\beta}{\varepsilon-1}$. Thus $|\xi - \frac{\beta}{\varepsilon-1}|_j = 0$ for all $1 \leq j \leq r+s$, and by Prop. 1 this implies that $\xi = \frac{\beta}{\varepsilon-1}$. □

Remark 5. This theorem is attributed to Cassels in [1].

Remark 6. For every number field K there exists a complete system of independent units ε_i such that $|\varepsilon_i|_j \neq 1$. This follows directly from Minkowski's proof of Dirichlet's unit theorem.

Remark 7. If condition 1. of Thm. 3 is satisfied but 2. is not (for example if there is a second uncovered subset T' such that $\varepsilon^{-1}T$ and T' have common points mod $\mathcal{O}_K$), then T might contain irrational exceptional points (converging to $\frac{\beta}{\varepsilon-1}$) as in the last sentence of Prop. 2.

Remark 8. If εT intersects $-T$, try to apply Thm. 3 with ε replaced by $-\varepsilon$.

Remark 9. Suppose that $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ is one of the k-exceptional points; in this case, there will be uncovered sets in all eight corners of the fundamental domain $\overline{F}$. In order to apply Prop. 2 one has to choose $\widetilde{F}$ e.g. as in Remark 3, because this allows us to collect these uncovered cubes into one set T lying in the center of $\widetilde{F}$.

In our example, computations similar to those above with $\alpha^{-1} = -6+\alpha+\alpha^2$ instead of α show that $\alpha^{-1}T$ is contained in

$$T'' = [0.26, 0.54] \times [-0.24, -0.16] \times [0.38, 0.42] + 3 - \alpha.$$

Now Thm. 3 shows that the only possible k-exceptional point ($k = 0.9$) in T is $\xi = \frac{3\alpha-\alpha^2}{\alpha-1} = \frac{2}{5} - \frac{1}{5}\alpha + \frac{2}{5}\alpha^2$.

`Eu3_4`. This program does the necessary computations: it checks whether a cube S multiplied by a non-torsion unit ε and translated back into the fundamental domain F intersects either S or $-S$; in both cases, the possible exceptional point is computed and written to a file `disc.n`. We can also search for orbits of length ≥ 2 by replacing ε by ε^n for some $n \geq 2$, provided ℓ is small enough. Verifying the conditions of Thm. 3 are currently still done by hand; see, however, Remark 12. We also remark that – in order to avoid rounding errors – we actually do not compare εS with e.g. S but with a slightly larger cube obtained by adding (resp. subtracting) $\frac{1}{2}\ell$ to (resp. from) the coordinates of the corners of S.

The next question is how to compute $M(K,\xi)$. This is done as follows: first we notice that $M(\overline{K},\xi) = M(\overline{K},\xi+\alpha)$ for $\alpha \in \mathcal{O}_K$, i.e. $M(\overline{K},\xi)$ only depends on the coset $\xi + \mathcal{O}_K$ of ξ in $\overline{K}/\mathcal{O}_K$. Next we observe that $M(\overline{K},\xi) = M(\overline{K},\varepsilon\xi)$ for units $\varepsilon \in E_K$. For $\xi \in \overline{K}/\mathcal{O}_K$ put $\mathrm{Orb}_\varepsilon(\xi) = \{\varepsilon^n\xi : n \in \mathbb{Z}\}$ (this is the orbit of ξ under the action of ε) and $\mathrm{Orb}(\xi) = \{\varepsilon\xi : \varepsilon \in E_K\}$. Then the Euclidean minimum is constant on every orbit.

Proposition 4. *For number fields K with unit rank ≥ 1, the following properties of $\xi \in \overline{K}/\mathcal{O}_K$ are equivalent:*

i) $\mathrm{Orb}(\xi)$ *is finite;*
ii) $\mathrm{Orb}_\varepsilon(\xi)$ *is finite for some non-torsion unit $\varepsilon \in E_K$;*
iii) $\xi \in K/\mathcal{O}_K$.

Proof. The implication i) $\Longrightarrow$ ii) is obvious. Assume that ε is a non-torsion unit such that $\mathrm{Orb}_\varepsilon(\xi)$ is finite. Then there exists an $n \in \mathbb{N}$ such that $\varepsilon^n\xi = \xi$, and this implies that $\xi = \frac{\alpha}{\varepsilon^n-1} + \mathcal{O}_K$ for some $\alpha \in \mathcal{O}_K$, i.e. $\xi \in K/\mathcal{O}_K$.

Finally assume that $\xi = \frac{\alpha}{\beta} + \mathcal{O}_K$ for some $\alpha, \beta \in \mathcal{O}_K$. If $\beta \in E_K$, then $\xi = 0 + \mathcal{O}_K$ and the claim is trivial. Otherwise observe that multiplication by a unit ε maps ξ to some element of the form $\frac{\alpha'}{\beta} + \mathcal{O}_K$, where $\alpha' \equiv \alpha\varepsilon \bmod \beta$. This shows that $\#\,\mathrm{Orb}(\xi) < \#(\mathcal{O}_K/\beta\mathcal{O}_K) = N(\beta)$. In fact, if $\mathcal{O}_K$ has class number 1 and if $(\alpha,\beta) = 1$, then we clearly have $\#\,\mathrm{Orb}(\xi) \leq \#(\mathcal{O}_K/\beta\mathcal{O}_K)^\times$. □

In this paper we will not deal with computing minima for $\xi \in \overline{K}/\mathcal{O}_K$ with infinite orbit (the last sentence of Prop. 2 gives a hint as to how such infinite orbits might arise), so we assume from now on that $\xi \in K/\mathcal{O}_K$. The basic idea how to compute $M(K,\xi)$ is due to Barnes and Swinnerton-Dyer (see [1], Thm. B, for the case of real quadratic number fields).

Proposition 5. *Let $K = \mathbb{Q}(\alpha)$ be a number field with unit group E_K. If, given $\xi \in K$ and a real number $k > 0$, there exists $\gamma \in \mathcal{O}_K$ such that $N(\xi - \gamma) < k$, then there exists $\zeta = \sum_{j=0}^{n-1} a_j \alpha^j \in K$ with the following properties:*

1. *$\zeta \equiv \xi_j \bmod \mathcal{O}_K$ for some $\xi_j \in \mathrm{Orb}(\xi)$;*
2. *$|a_i| < \mu_i$ $(0 \le i < n)$ for some constants $\mu_i > 0$ depending only on K;*
3. *$N(\zeta) < k$.*

Since the number of elements of K satisfying 1. and 2. is finite, we can prove $M(K,\xi) \ge k$ by simply computing the norms of all these elements. We will prove Prop. 5 only for cubic fields.

Let $K = \mathbb{Q}(\alpha)$ be a cubic number field; replace ξ by $\xi - \gamma$ and choose a unit $\varepsilon \in E_K$ such that the conjugates of $\beta = \xi\varepsilon = a + b\alpha + c\alpha^2$ are small. Let us consider the following system of equations:

$$\begin{aligned} \beta &= a + b\alpha + c\alpha^2 \\ \beta' &= a + b\alpha' + c\alpha'^2 \\ \beta'' &= a + b\alpha'' + c\alpha''^2 \end{aligned}$$

This system is linear in a, b, c, and the square of its determinant is

$$\Delta = \det \begin{pmatrix} 1 & \alpha & \alpha^2 \\ 1 & \alpha' & \alpha'^2 \\ 1 & \alpha'' & \alpha''^2 \end{pmatrix}^2 = \mathrm{disc}(1, \alpha, \alpha^2),$$

which is clearly $\neq 0$. In fact, we have $\Delta = g^2 \operatorname{disc} K$ for some integer g called the index of α. Therefore we get, by Cramer's rule,

$$\begin{aligned} \sqrt{\Delta}a &= \beta\alpha'\alpha''(\alpha'' - \alpha') + \beta'\alpha''\alpha(\alpha - \alpha'') + \beta''\alpha\alpha'(\alpha' - \alpha), \\ \sqrt{\Delta}b &= \beta(\alpha'^2 - \alpha''^2) + \beta'(\alpha''^2 - \alpha^2) + \beta''(\alpha^2 - \alpha'^2), \\ \sqrt{\Delta}c &= \beta(\alpha'' - \alpha') + \beta'(\alpha - \alpha'') + \beta''(\alpha' - \alpha). \end{aligned}$$

In order to compute bounds for a, b, and c we have to find good bounds for the conjugates of $\beta = \xi\varepsilon$.

We begin with the complex cubic case; as for real quadratic number fields, there is one fundamental unit η. Replacing η by η^{-1} if necessary we may assume that $|\eta| > 1$. Now for every $c_1 > 0$ there is a unit $\varepsilon = \eta^m$ such that

$$c_1 < |\xi\varepsilon| \le c_1 \cdot |\eta|.$$

Since $|\xi'\varepsilon'| = |\xi''\varepsilon''|$ in the complex case, we get

$$|\xi'\varepsilon'|^2 = \frac{N(\xi)}{|\xi\varepsilon|} \le \frac{k}{c_1},$$

where we have put $k = N(\xi) = N(\xi\varepsilon)$. We will choose c_1 in such a way that the resulting bounds on $X = |\beta(\alpha'' - \alpha')|$, $Y = |\beta'(\alpha - \alpha'')|$ and $Z = |\beta''(\alpha' - \alpha)|$ are equal. A little computation shows $c_1^3 = \frac{k}{|\eta|^2} \frac{|\alpha - \alpha''|^2}{|\alpha' - \alpha''|^2}$, and this yields

$$X, Y, Z \le \sqrt[3]{k|\eta|}\sqrt[6]{|\Delta|}.$$

Applying Lemma 6 below to $x = X, y = Y$ and $z = Z$ we find $xyz = k\sqrt{|\Delta|}$ and

$$|c| \le \sqrt[3]{\frac{k|\eta|}{|\Delta|}}\Big(2 + \frac{1}{|\eta|}\Big).$$

The obvious bound

$$|b| \le (X|\alpha''+\alpha'|+Y|\alpha''+\alpha|+Z|\alpha+\alpha'|)/\sqrt{\Delta} \le \sqrt[3]{\frac{k|\eta|}{|\Delta|}}(|\alpha''+\alpha'|+|\alpha''+\alpha|+|\alpha+\alpha'|)$$

can be sharpened by applying Lemma 6 to $x = X|\alpha' + \alpha''|$, $y = Y|\alpha'' + \alpha|$ and $z = Z|\alpha + \alpha'|$, and the same goes for $|a|$. The actual bounds coming from Lemma 6 are computed by machine in each case because they depend on the size of $|\alpha+\alpha'|, \ldots$ etc. This concludes the proof in the complex cubic case.

Remark 10. The bounds in the above proof are much better than those obtained by Cioffari [4] for the case of pure cubic fields.

Lemma 6. *Suppose that $x, y, z \in \mathbb{R}$ satisfy the inequalities $0 < x \le c_1$, $0 < y \le c_2$, $0 < z \le c_3$, and $0 < xyz = k$. Then*

$$x + y + z \le \max_{i \ne j} \left\{ c_i + c_j + \frac{k}{c_i c_j} \right\}.$$

Proof. We want to find bounds for $f(x, y) = x + y + \frac{k}{xy}$. Now f is positive in the domain under consideration

$$D = \Big\{ x, y > 0 : x \le c_1, y \le c_2, xy \ge \frac{k}{c_3} \Big\},$$

its gradient vanishes only at $x = y = \sqrt[3]{k}$, and its Hesse matrix there is positive definite; this implies that f takes its maximum on the boundary.

Assume for example that $x = c_1$; then we have to find an upper bound for $f_1(y) = c_1 + y + \frac{k}{c_1 y}$. Again, f_1 assumes its maximum on the boundary. For $y = c_2$ we get the bound $c_1 + c_2 + \frac{k}{c_1 c_2}$; on the other hand from $z = \frac{k}{c_1 y} \le c_3$ we get $y \ge \frac{k}{c_1 c_3}$, and we find $f_1(\frac{k}{c_1 c_3}) = c_1 + c_3 + \frac{k}{c_1 c_3}$.

The cases $y = c_2$ and $xy = \frac{k}{c_3}$ are treated similarly. □

Let K be a real cubic number field, and let η_1 and η_2 denote two independent units. We will denote the conjugates of $\xi \in K$ by ξ, ξ', ξ''. For units $\varepsilon \in E_K$ and a fixed embedding $K \longrightarrow \mathbb{R}$ we define

$$\gamma(\varepsilon) := \begin{cases} |\varepsilon|, & \text{if } |\varepsilon| \ge 1, \\ |\varepsilon|^{-1}, & \text{if } |\varepsilon| < 1, \end{cases}$$

and we put $\gamma_1 = \gamma(\eta_1), \ldots, \gamma_2' = \gamma(\eta_2')$.

Suppose that $\xi \in K$ has norm k; then, for any real numbers $c_1, c_2 > 0$, we can find a unit $\varepsilon \in \langle \eta_1, \eta_2 \rangle$ such that (cf. [8])

$$c_1 < |\xi\varepsilon| \le c_1\gamma_1\gamma_2, \qquad c_2 < |\xi'\varepsilon'| \le c_2\gamma_1'\gamma_2'.$$

This gives us the following bounds on $|\xi''\varepsilon''|$:

$$|\xi''\varepsilon''| = \frac{N(\xi\varepsilon)}{|\xi\varepsilon|\,|\xi'\varepsilon'|} \le \frac{k}{c_1 c_2}.$$

Now we proceed as in the complex case, put $\beta = \xi\varepsilon$, and choose c_1, c_2 in such a way that the resulting bounds on $X = |\beta(\alpha''-\alpha')|, Y = |\beta'(\alpha-\alpha'')|$ and $Z = |\beta''(\alpha'-\alpha)|$ are equal. In fact, setting

$$c_1^3 = k\,\frac{\gamma_1'\gamma_2'|\alpha-\alpha'||\alpha-\alpha''|}{\gamma_1^2\gamma_2^2|\alpha'-\alpha''|^2}, \quad c_2^3 = k\,\frac{\gamma_1\gamma_2|\alpha-\alpha'||\alpha'-\alpha''|}{\gamma_1'^2\gamma_2'^2|\alpha-\alpha''|^2}$$

yields the bounds $X, Y, Z \le \sqrt[3]{k\cdot\gamma_1\gamma_2\gamma_1'\gamma_2'}\,\sqrt[6]{\Delta}$. In particular, we find

$$|c| \le \frac{1}{\sqrt{\Delta}}(X+Y+Z) \le 3\sqrt[3]{k\cdot\gamma_1\gamma_2\gamma_1'\gamma_2'/\Delta}.$$

Making use of Lemma 6 we can improve this by a factor of almost $3/2$. The bounds for $|b|$ and $|a|$ are derived similarly; this concludes the proof of Prop. 5 in the cubic case. For general number fields, the proof makes use of the dual basis (cf. the proof of Prop. 1).

Let us get back to our example of real cubic field with discriminant $d = 985$. Let α denote a root of f. Then $\{1, \alpha, \alpha^2\}$ is an integral basis of $\mathcal{O}_K$, and two fundamental units are given by $\eta_1 = \alpha$ and $\eta_2 = 2-\alpha$. Put $\xi_0 = \frac{1}{5}(2-\alpha+2\alpha^2)$; then $\mathrm{Orb}(\xi_0) = \{\xi_0\}$. Using $k = 1.05$ we get the bounds $\mu_1 = 6.2$, $\mu_2 = 3.2$, $\mu_3 = 1.9$. We find $M(K,\xi_0) = N(\xi_0 - 2) = 1$.
`Eu3_5` is the program which does these computations. In fact, for each ξ in our list `disc.n`, it calculates

$$\min\{N(\xi_j + a + b\alpha + c\theta)\}$$

for all $(a,b,c) \in \mathbb{Z}^3$ such that the coefficients of the sum $\xi_j + a + b\alpha + c\theta$ are smaller than the bounds μ_0, μ_1, μ_2 computed in Prop. 5. Since the constants μ_j depend on the number k, we have to rerun the program replacing k by a real number k_1 larger than the conjectured minimum (i.e. the one obtained by running `Eu3_5` on it). The biggest value obtained from the various exceptional points gives the Euclidean minimum $M(K)$ unless they are all smaller than k.

The situation is, however, not always as simple as in Thm. 3. In fact, looking once more at the cubic number field K with discriminant 985 and using our programs with $k = 0.39$, we can cover $\overline{F}_+$ except for

T_1	$[0.345, 0.35]$	$\times$	$[-0.4915, -0.49]$	$\times$	$[-0.0185, -0.018]$
T_2	$[0.0175, 0.0185]$	$\times$	$[-0.2375, 0.-2355]$	$\times$	$[0.4725, 0.475]$
T	$[0.3995, 0.4005]$	$\times$	$[-0.201, -0.199]$	$\times$	$[0.3995, 0.4005]$
T_3	$[0.4725, 0.473]$	$\times$	$[-0.146, -0.1445]$	$\times$	$[0.2905, 0.2915]$
T_4	$[0.2905, 0.2915]$	$\times$	$[0.217, 0.219]$	$\times$	$[-0.437, -0.436]$
T_5	$[0.436, 0.437]$	$\times$	$[0.3265, 0.3285]$	$\times$	$[0.345, 0.346]$

Applying Thm. 3 to T shows that $\xi = \frac{1}{5}(2-\alpha+2\alpha^2)$ is the only possible k-exceptional point in T. Letting $\varepsilon = \alpha$ act on the T_i we find that the 'orbit' of T_1 is $\{T_1, -T_2, -T_3, -T_4, T_5\}$; at this point we need

Corollary 7. *Let K be a number field, $\varepsilon \in E_K$ a non-torsion unit, and suppose that $T_1, \ldots, T_t$ are compact subsets of $\widetilde{F}$ with the following properties:*

1. *there exist $\beta_1, \ldots, \beta_t \in \mathcal{O}_K$ such that, for all $\xi \in T_j$, $\varepsilon\xi - \beta_j$ lies in a k-covered region of $\widetilde{F}$ or in T_{j+1} (we put $T_{t+1} = T_1$);*
2. *for all $\xi \in T_j$ there is a $\gamma \in \mathcal{O}_K$ such that $\varepsilon^{-1}\xi - \gamma$ lies in a k-covered region of $\widetilde{F}$ or in $T_{\pi(j)}$, where $\pi(j)$ is an index depending only on j and not on ξ;*
3. *$|\varepsilon|_j \neq 1$ for $1 \le j \le r+s$.*

Then the only possible k-exceptional point of T_1 is $\zeta = \frac{\beta}{\varepsilon^t - 1}$, where $\beta = \varepsilon^{t-1}\beta_1 + \varepsilon^{t-2}\beta_2 + \ldots + \varepsilon\beta_{t-1} + \beta_t$. Moreover, the only possible k-exceptional points of the sets T_j are contained in $\mathrm{Orb}_\varepsilon(\zeta)$.

Proof. Suppose that $\xi_1 \in T_1$ is k-exceptional; then $\xi_2 = \varepsilon\xi_1 - \beta_1 \in T_2$, ..., $\xi_t = \varepsilon\xi_{t-1} - \beta_{t-1} \in T_t$, and $\xi_{t+1} = \varepsilon\xi_t - \beta_t \in T_1$ are k-exceptional. Observe that $\xi_{t+1} = \varepsilon\xi_t - \beta_t = \varepsilon^2\xi_{t-1} - \varepsilon\beta_{t-1} - \beta_t = \ldots = \varepsilon^t\xi_1 - \beta$. ¿From the assumptions made we can deduce that, for every k-exceptional point $\xi_1 \in T_1$, $\varepsilon^t\xi_1 - \beta$ lies again in T_1. This shows that condition 1. of Thm. 3 is satisfied with ε replaced by ε^t.

In order to prove that condition 2. is also satisfied we use induction to find that there exists a $\gamma \in \mathcal{O}_K$ such that $\xi = \varepsilon^{-t}\xi_1 - \gamma$ is an element of some set T_i. From what we have proved already we know that there exists a uniquely determined $\gamma' \in \mathcal{O}_K$ such that $\varepsilon^t\xi - \gamma' \in T_i$. But $\varepsilon^t\xi - \gamma' = \xi_1 - \varepsilon^t\gamma - \gamma' \in T_i$ implies that $\varepsilon^t\gamma + \gamma' = 0$ and $i = 1$. Thus condition 2. of Thm. 3 is also satisfied, and we can conclude that $\zeta = \beta/(\varepsilon^t - 1)$ is the only possible k-exceptional point in T_1. This in turn implies that $\varepsilon\zeta - \gamma_1$ is the only possible k-exceptional point in T_2, etc., and all our claims are proved. □

In our example of the cubic field of discriminant $d = 985$ and $k = 0.39$ we now check that condition 1. of Cor. 7 is satisfied, and we find $\beta_1 = 0$, $\beta_2 = -3\alpha + \alpha^2$, $\beta_3 = -2\alpha$, $\beta_4 = 2\alpha - \alpha^2$, and $\beta_5 = 3\alpha$. This gives $\beta = 7 + 41\alpha - 28\alpha^2$ and $\xi_1 = \beta/(\alpha^5 - 1) = \frac{1}{55}(19 - 27\alpha - \alpha^2)$.

After having verified condition 2., Cor. 7 shows that the only possible k-exceptional point of T_1 is ξ_1. Thus the only k-exceptional points of K in $\bigcup T_i$ are $\xi_1 = \frac{1}{55}(19 - 27\alpha - \alpha^2)$, $\xi_2 = \frac{1}{55}(-1 + 13\alpha - 26\alpha^2)$, $\xi_3 = \frac{1}{55}(-26 + 8\alpha - 16\alpha^2)$, $\xi_4 = \frac{1}{55}(-16 - 12\alpha + 24\alpha^2)$, $\xi_5 = \frac{1}{55}(24 + 18\alpha + 19\alpha^2)$. Using $k = 0.5$ we get the bounds $\mu_1 = 4.9$, $\mu_2 = 2.5$, $\mu_3 = 1.5$, and $M(K, \xi_1) = N(\xi_1 + \alpha) = \frac{5}{11}$.

Remark 11. Suppose that, in our Example, we apply Prop. 5 to $\xi = \frac{1}{5}(2 - \alpha + 2\alpha^2)$ with $k = 0.39$; the minimal norm of the elements satisfying conditions 1. – 3. turns out to be 1. Nevertheless we can only conclude that $M(K, \xi) > 0.39$. In order to prove that $M(K, \xi) = 1$ we have to apply Prop. 5 again, this time with a $k > 1$. Again, the minimal norm is 1, and now we can conclude that in fact $M(K, \xi) = 1$.

Remark 12. Collecting the uncovered subcubes S_j into the sets T_i of Cor. 7 is done as follows: assume that S is an uncovered cube, and that $\varepsilon S - \beta \in \overline{F}$. Then the set T_1 containing S is taken to be the set of all uncovered S_i 'near' to S such that $\varepsilon S_i - \beta \in \overline{F}$. By proceeding similarly with the uncovered cubes in $\overline{F} \setminus T_1$ we eventually arrive at subsets T_i containing all uncovered subcubes.

The programs are available from the authors. We remark that they can also be used to study weighted norms; details will be presented in [3].

4. Some Heuristic Observations

Consider some $\xi = \frac{\alpha}{\beta} + \mathcal{O}_K \in K/\mathcal{O}_K$; the bigger $\#\,\mathrm{Orb}(\xi)$, the more likely it is that one of the points in the orbit can be approximated sufficiently well by some $\eta \in \mathcal{O}_K$. In fact, if $(\alpha, \beta) = (1)$ and if $\#\,\mathrm{Orb}(\xi)$ is maximal (i.e. $\#\,\mathrm{Orb}(\xi) = (\mathcal{O}_K : \beta\mathcal{O}_K)^\times$), then clearly $M(K, \xi) = 1/N\beta$. Euclidean minima thus tend to be attained at points ξ with *small* orbits. If K has unit rank 1, then there are many points with small orbit: just take any $\frac{\alpha}{\varepsilon - 1}$ for $\alpha \in \mathcal{O}_K$ and $\varepsilon \in E_K$ a fundamental unit. If the unit rank is ≥ 2, however, such points are hard to find, because there

is no guarantee that $\alpha/(\varepsilon_1 - 1)$ has a small orbit with respect to the action of a second unit ε_2.

There is one exception, however: suppose that there is a principal prime ideal $\mathfrak{p} = (\pi)$ which is completely ramified in $K/\mathbb{Q}$. Since $\mathfrak{p}$ has degree 1, for every $\varepsilon \in E_K$ there is an integer $a \in \mathbb{Z}$ such that $\varepsilon \equiv a \bmod \mathfrak{p}$. Taking the norm gives $\pm 1 = N_{K/\mathbb{Q}}\varepsilon \equiv N_{K/\mathbb{Q}}a = a^n \bmod \mathfrak{p}$, where $n = (K : \mathbb{Q})$, and this in turn implies that $\varepsilon^n \equiv a^n \equiv \pm 1 \bmod \mathfrak{p}$. Therefore the unit group generates at most $2n$ different residue classes mod $\mathfrak{p}$, hence $\#\operatorname{Orb}(\xi) \leq 2n$ for any ξ of type $\xi = \frac{\alpha}{\pi} + \mathcal{O}_K$. Therefore such ξ have comparatively small orbit and a good chance of producing a large minimum. In fact, almost all known Euclidean minima of normal cubic fields are attained at such points.

Another question we would like to discuss is the following: can we expect that our list of norm-Euclidean complex cubic number fields is complete? Let us see what is happening in the real quadratic case. There we know (cf. [7]) that (in the following, d denotes the discriminant $\operatorname{disc} K$ of K)

$$\frac{\sqrt{d}}{16 + 6\sqrt{6}} \leq M(K) \leq \frac{\sqrt{d}}{4}.$$

This allows us to define the Davenport constant $D = \sup M(K)/\sqrt{d}$ for real quadratic fields. The example $K = \mathbb{Q}(\sqrt{13})$, $M(K) = 1/3$ shows that $D \geq 1/3\sqrt{13}$. If we assume that this is a good approximation for D, then there should be no norm-Euclidean number fields with discriminants $> D^{-2} = 9 \cdot 13 = 117$; in fact, the maximal discriminant of a norm-Euclidean number field is $d = 76$.

If we try to do the same with complex cubic case then the first problem is that the exponent $1/2$ of the discriminant in the lower bound in

$$\frac{\sqrt{|d|}}{420} \leq M(K) \leq \frac{|d|^{2/3}}{16\sqrt[3]{2}}$$

is not known to be best possible. If it is, then we can define a Davenport constant $D = \sup M(K)/\sqrt{d}$ for complex cubic fields as well. The example $d = -244$, where $M(K) = 1/2$, shows that $D \geq 1/2\sqrt{244}$, and if this bound is good, then there should be no norm-Euclidean number fields with $|d| > 976$. The example $d = -503$ suggests that D is somewhat smaller, but in any case we don't expect to find norm-Euclidean fields with $|d| > 1500$. Basically the same conclusions (with better bounds) hold if the correct exponent of $|d|$ in the lower bound is $2/3$.

In the case of totally real cubic fields there is no known (nontrivial) lower bound for $M(K)$ at all (of course $M(K) \geq \frac{1}{8}$). If one could show $M(K) \geq c\sqrt{d}$ for some c, then the above heuristics show that one has to compute $M(K)$ at least for fields with discriminants up to 25.000 (in fact Godwin and Smith [5] have shown that the normal cubic field with discriminant $d = 157^2 = 24.649$ is norm-Euclidean); our current data are therefore insufficient for deciding whether such a lower bound might exist or not.

5. A CONJECTURE

We would like to conclude our paper with a conjecture concerning $M(K)$ for certain pure cubic fields K:

Conjecture 1. Let $m = \ell^3 + 1$ be a squarefree integer, and assume that ℓ is even; put $\alpha = \sqrt[3]{\ell}$, $K = \mathbb{Q}(\alpha)$, and $\xi = \frac{1}{2}(1 + \alpha + \alpha^2)$. Then

$$M(\overline{K}) = M(K,\xi) = \begin{cases} \frac{1}{64}(18\ell^4 - 9\ell^3 + 12\ell^2 + 12\ell), & \text{if } \ell \equiv 2 \bmod 4, \\ \frac{1}{64}(18\ell^4 - 9\ell^3 + 30\ell^2 + 24\ell - 32), & \text{if } \ell \equiv 0 \bmod 4. \end{cases}$$

It is easy to see that $M(K,\xi)$ has at most the value given above; in fact, if $\ell \equiv 2 \bmod 4$, then $N(\frac{1}{4}\ell^2 + \frac{1}{2} + \frac{1}{4}\ell\alpha - \frac{1}{2}\alpha^2) = \frac{1}{64}(18\ell^4 - 9\ell^3 + 12\ell^2 + 12\ell)$, and if $\ell \equiv 0 \bmod 4$, then $N(\frac{1}{4}\ell^2+\frac{1}{2}\ell+\frac{1}{2}+(\frac{1}{4}\ell-\frac{1}{2})\alpha-\frac{1}{2}\alpha^2) = \frac{1}{64}(18\ell^4-9\ell^3+30\ell^2+24\ell-32)$. Numerical computations show that the conjecture is true for $\ell = 4$ and $\ell = 10$.

6. Acknowledgements

The computations were done on various systems at the universities of Bordeaux, Heidelberg and Trento; the latter also provided financial support for mutual visits of the authors, and we are especially grateful to René Schoof and Andrea Caranti for their help.

Roland Quême has independently developed programs written in C++ for finding norm-Euclidean cubic number fields and used them for checking many of our results.

Paul Voutier has kindly sent us E. Taylor's Ph. D. thesis; it turns out that she used the same embedding of K into $\mathbb{R}^3$ as R. Quême, i.e. she mapped $\alpha \in K$ to $(\alpha, \mathrm{Re}\,\alpha', \mathrm{Im}\,\alpha')$, where $\alpha, \alpha', \alpha''$ denote the conjugates of α.

Finally we thank G. Niklasch and an anonymous referee for their careful reading of the manuscript.

References

[1] E.S. Barnes, H.P.F. Swinnerton-Dyer, *The inhomogeneous minima of binary quadratic forms I, II, III*, Acta Math. **87** (1952), 259–323; ibid. **88** (1952), 279–316; ibid. **92** (1954), 199–234

[2] S. Cavallar, *Alcuni esempi di campi di numeri cubici non euclidei rispetto alla norma ma euclidei rispetto ad una norma pesata*, Tesi di Laurea, Univ. Trento, 1995

[3] S. Cavallar, F. Lemmermeyer, *Euclidean Windows*, in preparation

[4] V. Cioffari, *The Euclidean condition in pure cubic and complex quartic fields*, Math. Comp. **33** (1979), 389–398

[5] H. J. Godwin, J. R. Smith, *On the Euclidean nature of four cyclic cubic fields*, Math. Comp. **60** (1993), 421–423

[6] F. Lemmermeyer, *Euklidische Ringe*, Diplomarbeit Univ. Heidelberg, 1989

[7] F. Lemmermeyer, *The Euclidean Algorithm in Algebraic Number Fields*, Expo. Math. **13** (1995), 385–416

[8] F. Lemmermeyer, A. Pethő, *Simplest Cubic Fields*, Manuscr. Math. **88** (1995), 53–58

[9] R. Quême, *Letter from July 4, 1996*

[10] E. Taylor, *Euclid's algorithm in cubic fields with complex conjugates*, Ph.D. thesis, London, 1975

The Tables below contain the known Euclidean minima for cubic number fields of small discriminant. The letter N indicates that the field has class number 1 but is not norm-Euclidean, and H that it has class number > 1. Moreover, E means that the field is norm-Euclidean; if no Minimum is given, we succeeded in covering the fundamental domain with $k = 0.99$.

7. Tables

Euclidean minima of complex cubic number fields

disc K		$M_1(K)$	$M_2(K)$	disc K		$M_1(K)$	$M_2(K)$
-23	E	$1/5$	$\geq 1/7$	-116	E	$1/2$	
-31	E	$1/3$	$< 1/4$	-135	E	$3/5$	
-44	E	$1/2$	$1/4$	-139	E	$1/2$	
-59	E	$1/2$	$1/4$	-140	E	$1/2$	
-76	E	$1/2$	$1/3$	-152	E	$1/2$	
-83	E	$1/2$		-172	E	$3/4$	
-87	E	$1/3$		-175	E	$3/5$	
-104	E	$1/2$		-199	N	1	< 0.47
-107	E	$1/2$		-200	E	$1/2$	
-108	E	$1/2$	$1/3$	-204	E	$61/116$	
-211	E	$59/106$		-283	H	$3/2$	
-212	E	$5/8$		-300	E	$23/30$	
-216	E	$1/2$		-307	N	$9/8$	$3/4$
-231	E	$7/9$		-324	E	$23/36$	$7/11$
-239	E	$8/9$		-327	N	$101/99$	
-243	E	$11/18$		-331	H	$3/2$	
-244	E	$1/2$		-335	N	1	
-247	E	$5/7$		-339	N	$9/8$	1
-255	E	$13/15$		-351	N	1	$9/11$
-268	E	$13/22$	$\geq 6/11$	-356	E	$7/8$	
-364	N	$9/8$		-451	E	$41/48$	
-367	N	1	$9/13$	-459	N	$9/8$	
-379	E	$397/648$	$\geq 11/18$	-460	E	$43/50$	$23/30$
-411	E	$17/22$	$\geq 8/11$	-472	E	$46/61$	
-419	E	$4/5$		-484	E	$59/76$	
-424	E	$19/27$	$\geq 53/76$	-491	H	2	≥ 1
-431	E	$43/64$		-492	E	$25/32$	
-436	N	$79/78$		-499	E	$23/27$	
-439	N	$17/15$	≥ 1	-503	E	$\geq 307/544$	
-440	E	$737/1090$		-515	E	$4/5$	$\geq 11/14$
-516	E	$36/53$		-628	E	$625/664$	
-519	E	$44712/45747$		-643	H	$25/16$	
-524	N	$5/4$		-648	H	$5/4$	
-527	N	$13/7$		-652	E	$21/23$	
-543	E	$\geq 158664/170633$		-655	N	$40/23$	
-547	N	$9/8$		-671	N	$25/19$	
-563	H	2		-675	N	$9/8$	
-567	N	$25/17$	$\geq 19/17$	-676	H	$7/4$	
-588	H	$5/2$		-679	N	$9/8$	
-620	N	$13/8$	$5/4$	-680	N	$(*)$	

disc K		$M(K)$	disc K		$M(K)$
-687	E	937/945	-751	H	25/9
-695	N	25/13	-755	N	1
-696	E	186/199	-756	N	306/293
-707	N	271/270	-759	N	11/8
-716	N	121/109	-771	E	223/252
-728	E	(§)	-780	N	499/498
-731	H	2	-804	N	$\geq 2771/2568$
-743	N	1	-808	N	$\geq 2031/1964$
-744	E	992/999	-812	N	44/31
-748	N	62/51	-815	E	24543/25325
-823	N	37/25	-891	H	7/2
-835	N	110353/106265	-907	N	$\geq 113/108$
-839	N	25/17	-908	N	227/91
-843	N	134/131	-931	H	7/2
-856	N	$\geq 454951/428544$	-932	N	68425/56788
-863	N	29/11	-940	N	407/358
-867	N	1115/1028	-948	N	$\geq 2120/1959$
-876	E	353/372	-959	N	19/7
-883	N	49/47	-964	N	$\geq 132/127$
-888	N	2715/2602	-971	N	829/778
-972	N	5/4	-1036	N	133/101
-972	N	179/162	-1048	N	617/488
-980	H	7/4	-1055		
-983	N	31/11	-1059	N	2381/1854
-984	N	$\geq 22367/21296$	-1067	N	$\geq 160/121$
-996	N	$\geq 6713/5646$	-1068	N	$\geq 1499/1350$
-999			-1075	N	777/680
-1004	N	3167/2298	-1080	N	$\geq 10253/1000$
-1007	N	41/23	-1083	H	3/2
-1011	N	271/207	-1087	N	15/8
-1096	N	$\geq 207/199$	-1176	H	4/3
-1099	H	47/26	-1187	N	11/8
-1107	H	2	-1188	N	$\geq 22319/14072$
-1108	N	$\geq 4995/4384$	-1191	N	11/9
-1135	N	5115/4033	-1192	H	265/168
-1144	N	4867/3222	-1196	N	197/94
-1147	N	136/99	-1203	N	$\geq 4775/4608$
-1164	N	$\geq 1064/918$	-1207	N	13/9
-1172	N	572/443	-1208	N	845/656
-1175	N	37/13	-1219	N	$\geq 709/622$

Euclidean minima of totally real cubic number fields

disc K		$M(K)$	disc K		$M(K)$	disc K		$M(K)$
49	E	1/7	469	E	1/2	788	E	1/2
81	E	1/3	473	E	1/3	837	E	1/2
148	E	1/2	564	E	1/2	892	E	1/2
169	E	5/13	568	E	1/2	940	E	1/2
229	E	1/2	621	E	1/2	961	E	16/31
257	E	1/3	697	E	13/31	985	N	1
316	E	1/2	733	E	1/2	993	E	31/63
321	E	1/3	756	E	1/2	1016	E	1/2
361	E	8/19	761	E	1/3	1076	E	1/2
404	E	1/2	785	E	3/5	1101	E	1/2
1129	E	1/3	1425	E	13/15	1708	E	1/2
1229	E	16/29	1436	E	1/2	1765	E	13/20
1257	E	9/25	1489	E	29/43	1772	E	1/2
1300	E	7/10	1492	E	1/2	1825	N	7/5
1304	E	1/2	1509	E	1/2	1849	E	22/43
1345	N	7/5	1524	E	1/2	1901	E	1/2
1369	E	31/37	1556	E	3/4	1929	N	1
1373	E	1/2	1573	E	19/22	1937	N	1
1384	E	11/16	1593	E	< 0.36	1940	E	1/2
1396	E	1/2	1620	E	1/2	1944	E	1/2
1957	H	2	2241	E	3/5	2636	E	1/2
2021	E	1/2	2292	E	1/2	2673	E	64/81
2024	E	1/2	2296	E	1/2	2677	E	139/224
2057	E	9/11	2300	E	27/40	2700	E	83/120
2089	E	1/2	2349	E	11/18	2708	E	1/2
2101	E	1/2	2429	E	1/2	2713	E	< 0.5
2177	E	< 0.39	2505	E	5/9	2777	H	5/3
2213	E	1/2	2557	E	1/2	2804	E	1/2
2228	E	1/2	2589	E	9/16	2808	E	1/2
2233	E	56/121	2597	H	5/2	2836	N	7/4
2857	N	8/5	3137	E	< 0.59	3325	E	
2917	E	8/13	3144	E	1/2	3356	E	
2920	E	13/20	3173	E	< 0.59	3368	E	
2941	E	1/2	3221	E	1/2	3496	E	
2981	E	1/2	3229	E	1/2	3508	E	
2993	E	< 0.49	3252	E		3540	E	
3021	E	1/2	3261	E		3569	E	
3028	E	1/2	3281	E		3576	E	
3124	E	1/2	3305	N	13/9	3580	E	
3132	E	1/2	3316	E		3592	E	

disc K		$M(K)$	disc K		$M(K)$	disc K		$M(K)$
3596	E		3892	E		4104	E	< 0.55
3604	E		3941	E		4193	N	7/5
3624	E		3957	E		4212	H	7/2
3721	E	121/183	3969	H	7/3	4281	E	< 0.7
3732	E		3969	H	1	4312	N	11/4
3736	E		3973	E	1/2	4344	E	< 0.7
3753	E		3981	H	3/2	4345	N	7/5
3873	E		3988	N	19/8	4360	N	41/35
3877	E		4001	E	7/9	4364	E	
3889	N	13/7	4065	E	3/5	4409	E	
4481	E		4729	N	149/73	4860	E	
4489	E	53/67	4749	E		4892	E	
4493	E		4764	E	17/24	4933	E	
4596	E		4765	E		5073	E	
4597	E		4825	E		5081	E	
4628	E		4841	E		5089	N	17/11
4641	E		4844	E		5172	E	
4649	E		4852	E		5204	E	
4684	N	13/8	4853	E		5261	E	
4692	E	< 0.7	4857	E		5281	N	1
5297	N	21/11	5468	E		5629	E	
5300	E		5477	E		5637	E	
5325	E		5497	E		5684	N	9/2
5329	N	9/8	5521	N	23/7	5685	E	
5333	E		5529	E		5697	E	
5353	E		5556	E		5724	E	
5356	E		5613	E		5741	E	
5368	E		5620	E		5780	E	
5369	N	21/19	5621	E		5821	E	
5373	E		5624	E		5853	E	
5901	E		6153	E		6420	E	
5912	E		6184	E		6452	N	5/4
5925	E		6185	N	17/15	6453	E	
5940	E		6209	E		6508	E	
5980	E		6237	E		6549	E	
6053	E		6241	N	223/79	6556	E	
6088	E		6268	E		6557	E	
6092	E		6289	N	1	6584	E	
6108	E		6396	E		6588	E	
6133	E		6401	N	35/27	6601	E	

disc K		$M(K)$	disc K		$M(K)$	disc K		$M(K)$
6616	E		6901	E		7220	H	9/4
6637	E		6940	E		7224	E	
6669	E		6997	E		7244	E	
6681	E		7028	E		7249	E	
6685	E		7032	E		7273	N	973/601
6728	E		7053	H	2	7388	E	
6809	H	7/3	7057	E		7404	E	
6856	E		7084	E		7425	E	
6868	N	5/4	7117	E		7441	E	
6885	N	67/40	7148	E		7444	E	
7453	E		7601	E		7745	N	7/5
7464	E		7628	E		7753	E	
7465	N	1	7636	E		7796	E	
7473	E	< 0.89	7641	E		7816	E	
7481	N	1	7665	E	21/25	7825	E	
7528	N	17/14	7668	E		7873	N	29/13
7537	N	227/91	7673	E		7881	E	
7540	E		7700	E		7892	E	
7572	E		7709	E		7925	E	
7573	N	41/32	7721	E		7948	E	
8017	E		8281	H	9/7	8532	E	
8057	E		8285	E		8545	E	
8069	H	9/2	8289	E		8556	E	
8092	E		8308	N	67/50	8572	N	17/16
8113	N	13/7	8372	E		8597	E	4/5
8173	E		8373	E		8628	E	
8220	E		8396	E		8637	E	
8276	E		8468	H	5/3	8680	E	
8277	E		8472	E		8692	N	11/10
8281	H	23/16	8505	E		8713	E	
8745	E		8920	E		9217	N	17/11
8761	E		9044	E		9281	E	
8769	E		9045	E		9293	E	
8789	N	23/12	9073	N	7/5	9300	E	
8828	E		9076	E		9301	H	2
8829	N	3/2	9133	E		9325	N	13/8
8837	E		9149	E		9364	E	
8884	E		9153	E		9409	N	337/97
8905	N	8/5	9192	E		9413	E	
8909	E		9204	E		9428	E	

disc K		$M(K)$	disc K		$M(K)$	disc K		$M(K)$
9460	E		9812	E		10004	E	
9517	E		9813	E		10040	E	
9565	E	$\geq 4/5$	9833	E		10069	E	
9612	E		9836	E		10077	E	
9653	N	35/12	9869	E		10164	N	27/22
9676	E		9897	E		10172	E	
9745	N	67/23	9905	N	9/5	10200	E	
9749	E		9937	E		10216	N	7/4
9800	H	9/5	9980	E		10233	E	
9805	E		9996	H	4/3	10260	E	
10261	N	11/7	10540	E		10721	E	
10273	H	27/7	10552	E		10733	E	
10292	E	9/10	10561	N	11/7	10740	E	
10301	E		10580	E		10812	E	
10309	H	11/2	10609	E		10844	E	
10324	E		10636	E		10865	E	
10333	N	1	10641	E		10868	E	
10353	E		10661			10889	H	13/5
10457	N	27/25	10664	E		10904	E	
10484	E		10712	E		10929		

The only fields with disc $K < 11,000$ whose Euclidean nature is currently not known are those with discriminants 10661, 10929, and 10941. We also remark that among the four fields which were shown to be Euclidean in [5], those with discriminants 11881, 16129 and 24649 are beyond the limits of our tables.

(*) The Euclidean minimum $M(K)$ for the field with disc $K = -680$ is

$$M(K) = \frac{81956632}{81182612}.$$

(§) The Euclidean minimum $M(K)$ for the field with disc $K = -728$ is

$$M(K) = \frac{7483645229}{8158377554}.$$