Complementarity Modeling of Hybrid Systems

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***Abstract*—A complementarity framework is described for the modeling of certain classes of mixed continuous/discrete dynamical systems. The use of such a framework is well known for mechanical systems with inequality constraints, but we give a more general formulation which also applies, for instance, to switching control systems. The main theoretical results in the paper are concerned with uniqueness of smooth continuations; the solution of this problem requires the construction of a map from the continuous state to the discrete state. A crucial technical tool is the so-called linear complementarity problem (LCP) from mathematical programming; we introduce various generalizations of this problem.**

***Index Terms*—Hybrid systems, linear complementarity problem, switching control, unilateral constraints, well-posedness.**

I. Introduction

THE general description of hybrid systems as systems incorporating both continuous and discrete components leaves room for a bewildering multitude of dynamical systems, of which many are cumbersome to specify and difficult to analyze. In this paper we shall be concerned with a special class of hybrid systems, which we call *complementarity systems*, for which both specification and analysis should be considerably easier than for the general case. In particular, we shall be concerned with *well-posedness* of complementarity systems.

The study of well-posedness (existence and uniqueness of solutions) is particularly relevant in connection with hybrid systems. As is well known, hybrid dynamical systems often arise by the application of (idealized) switching control schemes. When such switching schemes are considered, well-posedness of the resulting closed-loop system may easily fail, quite in contrast to the situation when smooth control is applied; see Section III for an example. Also, well-posedness is a crucial issue in checking the validity of mathematical models of physical hybrid systems and in setting up simulation algorithms for such systems (cf. [2] and [18]).

Necessary and sufficient conditions for the well-posedness of complementarity systems were given in [18], but only for the case of complementarity systems with just two discrete states ("bimodal systems"). Here we extend this discussion to complementarity systems with an arbitrary number of discrete states, limiting ourselves however, to sufficient conditions for uniqueness of smooth continuations. Another advance with respect to [18] in this paper is that we identify a number of algebraic problems that can be used to settle uniqueness questions. All these problems are related to the so-called *linear complementarity problem* (LCP) from mathematical programming [4].

We begin in Section II with an introduction on complementarity modeling, in which we aim to show how the class of complementarity systems fits into the general class of hybrid systems. Section III contains the main results of the paper on uniqueness of smooth continuations. Special techniques for linear systems are briefly mentioned in Section IV, and conclusions follow in Section V.

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II. Complementarity Modeling

Let us start with a fairly general hybrid system description, such as the one given by Alur *et al.* in [1]. A hybrid system is specified in that paper as a graph whose edges represent discrete *transitions* and whose vertices represent continuous *activities*. The vertices are called *locations*. The continuous activities consist of sets of time functions which may be specified, for instance, by differential equations; thus, there is a dynamical system associated to each location. Under some conditions transitions may occur from one location to another. In particular, transitions are forced when the activity at a certain location would take the associated continuous state outside a designated region of the state space; this region is called the *invariant* associated with the location.

The description given by Alur *et al.* is very general and at the same time rather amorphous. In many situations, the set of discrete states (locations) will actually be a product space obtained by combining several switches. Also, in many cases, the dynamical systems associated to different locations will not be completely independent but will rather have many equations in common. A combination of these two observations gives rise to what might be called a *product decomposition* of hybrid systems.

Such a decomposition imposes the following additional structure on the general scheme indicated above. There is a "core dynamics" of the form $F(z, \dot{z}) = 0$ which forms part of the dynamics at each location; the vector $z(t) \in \mathbb{R}^N$ contains all continuous variables in the system. There are k switches, with a finite set S_i of possible positions associated to each switch $i \in \{1, \cdots, k\}$. Each combination of switch positions gives rise to a different discrete state, so the set of locations is the product $\Pi_{i=1}^k S_i$. Associated to each position s of the ith switch, there are additional dynamic equations $G_i^s(z, \dot{z}) = 0$ as well as invariants that may be written as $H_i^s(z) \geq 0$. The dynamic equations corresponding to the various switch positions together with the core dynamics form the description of the dynamics at a given location, and the invariant corresponding to the location is obtained by taking all inequalities corresponding to the switch positions together.

As long as no further statements are made concerning, for instance, the size of the core dynamics and the number of switch positions, the above format for specifying dynamics and invariants at each location is still quite general. Suppose now, however, that the following additional requirements are imposed.

1) All switches are binary, i.e., $S_i = \{0, 1\}$ for all i.
2) All additional dynamic equations corresponding to switch positions are algebraic and scalar, i.e., they are of the form $g_i^s(z) = 0$ where g_i^s is a function from $\mathbb{R}^N$ to $\mathbb{R}$.
3) Also, the invariants corresponding to switch positions are scalar, so they are of the form $h_i^s(z) \geq 0$ where h_i^s is a function from $\mathbb{R}^N$ to $\mathbb{R}$.
4) The functions defining the additional dynamics and the invariants associated with each switch position change roles when the switch is turned; i.e., $g_i^0 = h_i^1$ and $g_i^1 = h_i^0$ for all i.

We call the final condition of this list the *complementarity condition*, and systems that can be described according to the above rules will be called *complementarity systems*. The complementarity condition implies that the additional dynamics and invariants at each switch position are specified by two functions rather than by four. The two functions create two variables that are associated with the vector $z(t)$ of continuous variables and that may be denoted by $y_i(t) = g_i^0(z(t))$ and $u_i(t) = h_i^0(z(t))$; we call these variables *complementary variables*. Note that one switch position corresponds to the pair of conditions $y_i(t) = 0$ and $u_i(t) \geq 0$, whereas the other position corresponds to $u_i(t) = 0$ and $y_i(t) \geq 0$.

The above setting, limited as it may seem from a general hybrid system perspective, in fact applies to many systems of interest. The reader may have already recognized the complementarity conditions as essentially the characteristics of an ideal diode; so, electrical networks with diodes may be looked at as complementarity systems, with the diodes as switches and the voltage across and the current through the diodes as complementary variables. Other physical examples include mechanical systems with unilateral constraints, with distance to contact point and reaction force as complementary variables, and hydraulic systems with one-way valves, where pressure and flow can be taken as complementary variables. Outside physics, complementarity systems arise naturally in the necessary conditions obtained from the maximum principle for optimal control problems with inequality constraints. Furthermore, it follows from results on the representation of piecewise linear sets [8], [19] that systems with elements having arbitrary piecewise linear characteristics can be written as complementarity systems.

An example of how a complementarity system may arise in a control application can be given as follows. Consider some control system described by equations of the form $\dot{x}(t) = f(x(t), u(t))$ where $u(t)$ is the scalar control input. Suppose that a switching control scheme is employed which uses a state feedback law $u(t) = \phi_1(x(t))$ when the scalar variable $y(t)$ defined by $y(t) = h(x(t))$ is positive and a feedback $u(t) = \phi_2(x(t))$ when $y(t)$ is negative. Writing $f_i(x) = f(x, \phi_i(x))$ for $i = 1, 2$, we obtain a dynamical system that follows the equation $\dot{x}(t) = f_1(x(t))$ on the subset of the state space where $h(x)$ is positive, and that follows $\dot{x}(t) = f_2(x(t))$ on the subset where $h(x)$ is negative. Such a system is sometimes called a *variable-structure system*. To write the system as a complementarity system, introduce new variables $u_i(t)$ and $y_i(t)$ $(i = 1, 2)$ and pose the following "core dynamics" of the form $F(z, \dot{z}) = 0$, with $z := (x, u_1, u_2, y_1, y_2)$

$$\dot{x}(t) = u_1(t) f_1(x(t)) + u_2(t) f_2(x(t)) \tag{1}$$
$$u_1(t) + u_2(t) = 1 \tag{2}$$
$$y_1(t) - y_2(t) = h(x(t)). \tag{3}$$

The variables $u_i(t)$ and $y_i(t)$ are taken as complementary variables, and so the complementarity conditions can be written as follows:

$$u_i(t) \geq 0, \qquad y_i(t) \geq 0 \qquad (i = 1, 2) \tag{4}$$
$$y_1(t) u_1(t) + y_2(t) u_2(t) = 0 \quad \text{for all } t. \tag{5}$$

Since we have two binary switches, the complementarity system above has four locations. One of the locations, however, combines the equations $u_1 = 0$ and $u_2 = 0$ with $u_1 + u_2 = 1$ and so it is not feasible. Two other locations correspond to the dynamics $\dot{x} = f_1(x)$ and $\dot{x} = f_2(x)$ which are valid for $h(x) > 0$ and $h(x) < 0$, respectively. Finally, there is a location which combines the dynamic equation $\dot{x}(t) = u_1(t) f_1(x(t)) + (1 - u_1(t)) f_2(x(t))$ with the constraint $h(x(t)) = 0$ and the inequality constraints $0 \leq u_1(t) \leq 1$. Conditions may be given under which this combination defines a unique solution; whether this solution is "correct" in the sense that it describes in good approximation the behavior of the actual control system depends on the implementation chosen for the switching controller. It should be noted that a complementarity system as described above is not always well-posed in the sense that solutions are unique, as shown by a simple example in the next section.

III. Well-Posedness of Complementarity Systems

As already noted in [18], it is not difficult to find examples of complementarity systems that exhibit nonuniqueness of smooth continuations. For a simple example of this phenomenon within a switching control framework, consider the plant

$$\begin{aligned} \dot{x}_1 &= x_2 + u, \qquad y = x_1 \\ \dot{x}_2 &= -x_2 \end{aligned} \tag{6}$$

in closed loop with a switching control scheme of relay type

$$\begin{aligned} u(t) &= 1, && \text{if } y(t) > 0 \\ -1 \leq u(t) &\leq 1, && \text{if } y(t) = 0 \\ u(t) &= -1, && \text{if } y(t) < 0. \end{aligned} \tag{7}$$

It was shown in the previous section that such a variable-structure system can be modeled as a complementarity system. Note that from any initial (continuous) state $x(0) =$

$(x_1(0), x_2(0)) = (0, c)$, with $|c| \leq 1$, there are three possible smooth continuations for $t \geq 0$ that are allowed by the equations and inequalities above

1) $x_1(t) = 0, x_2(t) = ce^{-t}, u(t) = -ce^{-t}, -1 \leq u(t) \leq 1, y(t) = x_1(t) = 0$
2) $x_1(t) = c(1-e^{-t}) + t, x_2(t) = ce^{-t}, u(t) = 1, y(t) = x_1(t) > 0$
3) $x_1(t) = c(1-e^{-t}) - t, x_2(t) = ce^{-t}, u(t) = -1, y(t) = x_1(t) < 0$.

So the above closed-loop system is not well-posed as a dynamical system. If the sign of the feedback coupling is reversed, however, there is only one smooth continuation from each initial state. This shows that well-posedness is a nontrivial issue to decide upon in a hybrid system, and in particular is a meaningful performance characteristic for hybrid systems arising from switching control schemes. Depending on the actual implementation of the controller that is represented in idealized form in (7), lack of well-posedness may manifest itself in some form of instability.

For simplicity we shall assume throughout that there are no external (continuous or discrete) inputs applied to the system. In the context of switching control schemes this assumption is natural, since we consider a closed-loop configuration.

Recall that a general complementarity system has been represented by a "core dynamics" having pairs of external variables u_i and y_i (functions of z), in "closed loop with" or "terminated by" the complementarity conditions $y_i \geq 0, u_i \geq 0, y_i u_i = 0$. This "closed-loop" point of view will turn out to be very fruitful in the analysis of complementarity systems. This becomes especially clear for the *semi-explicit complementarity systems*, which will be treated in the rest of this paper. These systems can be written as an "input–output system"

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), \qquad x \in \mathbb{R}^n, \qquad u \in \mathbb{R}^k \\ y(t) &= h(x(t), u(t)), \qquad y \in \mathbb{R}^k \end{aligned} \tag{8}$$

with the additional *complementarity conditions*

$$y(t) \geq 0, \qquad u(t) \geq 0, \qquad y^T(t)u(t) = 0. \tag{9}$$

The inequalities here are taken in the componentwise sense. Because of the nonnegativity constraints, the vanishing of the inner product means that for each index i and each time t we must have either $y_i(t) = 0$ or $u_i(t) = 0$ (or both). The vectors $y(t)$ and $u(t)$ denote *complementary variables*, rather than outputs and inputs. Nevertheless, we keep the symbols that are customarily used for outputs and inputs, because we will extensively use tools from the theory of input–output systems (8). The functions f and h will always be assumed to be smooth.

The complementarity conditions (9) imply that for some index set $I \subset \{1, \cdots, k\}$ one has the algebraic constraints

$$y_i(t) = 0 \ (i \in I), \qquad u_i(t) = 0 \ (i \notin I). \tag{10}$$

Note that (10) always represents k constraints which are to be taken in conjunction with the system of n differential equations in $n+k$ variables appearing in (8). The problem of determining which index set I has the property that the solution of (8)–(10) coincides with that of (8) and (9) is called the *mode selection problem*; the index set I represents the *mode* (location) of the system.

One approach to solving the mode selection problem would simply be to try all possibilities: solve (8) together with (10) for some chosen candidate index set I and see whether the computed solution is such that the inequality constraints $y(t) \geq 0$ and $u(t) \geq 0$ are satisfied on some interval $[0, \epsilon]$. Under the assumption that smooth continuation is possible from x_0, there must at least be one index set for which the constraints will indeed be satisfied. This method requires in the worst case the integration of 2^k systems of $n+k$ differential/algebraic equations in $n+k$ unknowns.

In order to develop an alternative approach which leads to an *algebraic* problem formulation, let us note first that we can derive from (8) a number of relations between the successive time derivatives of $y(\cdot)$, evaluated at $t = 0$, and the same quantities derived from $u(\cdot)$. By differentiating the second line of (8) and using the first line, we get

$$\begin{aligned} y(t) &= h(x(t), u(t)) \\ \dot{y}(t) &= \frac{\partial h}{\partial x}(x(t), u(t)) f(x(t), u(t)) + \frac{\partial h}{\partial u}(x(t), u(t))\dot{u}(t) \\ &=: F_1(x(t), u(t), \dot{u}(t)) \end{aligned}$$

and in general

$$y^{(j)}(t) = F_j(x(t), u(t), \cdots, u^{(j)}(t)) \tag{11}$$

where F_j is a function that can be specified explicitly in terms of f and h. From the complementarity conditions (9), it follows, moreover, that for each index i either

$$(y_i(0), \dot{y}_i(0), \cdots) = 0 \quad \text{and} \quad (u_i(0), \dot{u}_i(0), \cdots) \succeq 0 \tag{12}$$

or

$$(y_i(0), \dot{y}_i(0), \cdots) \succeq 0 \quad \text{and} \quad (u_i(0), \dot{u}_i(0), \cdots) = 0 \tag{13}$$

(or both), where we use the symbol $\succeq$ to denote lexicographic nonnegativity. (A sequence $(a_0, a_1, \cdots)$ of real numbers is said to be *lexicographically nonnegative* if either all a_i are zero or the first nonzero element is positive.) This suggests the formulation of the following "dynamic complementarity problem" (DCP).

Problem DCP: Given smooth functions F_j: $\mathbb{R}^{n+(j+1)k} \to \mathbb{R}^k$ $(j = 0, 1, \cdots,)$ that are constructed from smooth functions f: $\mathbb{R}^n \to \mathbb{R}^n$ and h: $\mathbb{R}^n \to \mathbb{R}^k$ via (11), find, for given $x_0 \in \mathbb{R}^n$, sequences $(y^0, y^1, \cdots)$ and $(u^0, u^1, \cdots)$ of k-vectors such that for all j we have

$$y^j = F_j(x_0, u^0, \cdots, u^j) \tag{14}$$

and for each index $i \in \{1, \cdots, k\}$ at least one of the following is true:

$$(y_i^0, y_i^1, \cdots) = 0 \quad \text{and} \quad (u_i^0, u_i^1, \cdots) \succeq 0 \tag{15}$$

$$(y_i^0, y_i^1, \cdots) \succeq 0 \quad \text{and} \quad (u_i^0, u_i^1, \cdots) = 0. \tag{16}$$

We shall also consider truncated versions where j only takes on the values from zero up to some integer ℓ; the corresponding problem will be denoted by DCP(ℓ). It follows from the triangular structure of the equations that if $((y^0,\cdots,y^\ell),(u^0,\cdots,u^\ell))$ is a solution of DCP(ℓ), then, for any $\ell'<\ell,((y^0,\cdots,y^{\ell'}),(u^0,\cdots,u^{\ell'}))$ is a solution of DCP(ℓ'). We call this the *nesting property* of solutions. We define the *active index set at stage* ℓ, denoted by I_ℓ, as the set of indexes i for which $(u_i^0,\cdots,u_i^\ell)\succ 0$ in *all* solutions of DCP(ℓ), so that necessarily $y_i^j=0$ for all j in any solution of DCP (if one exists). Likewise, we define the *inactive index set at stage* ℓ, denoted by J_ℓ, as the set of indexes i for which $(y_i^0,\cdots,y_i^\ell)\succ 0$ in *all* solutions of DCP(ℓ), so that necessarily $u_i^j=0$ for all j in any solution of DCP. Finally, we define K_ℓ as the complementary index set $\{1,\cdots,k\}\backslash(I_\ell\cup J_\ell)$. It follows from the nesting property of solutions that the index sets I_ℓ and J_ℓ are nondecreasing as functions of ℓ. Since both sequences are obviously bounded above, there must exist an index ℓ^* such that $I_\ell=I_{\ell^*}$ and $J_\ell=J_{\ell^*}$ for all $\ell\geq\ell^*$. We finally note that all index sets defined here of course depend on x_0; we suppress this dependence, however, to alleviate the notation.

The problem DCP is a generalization of the *nonlinear complementarity problem* (NCP) (see for instance [4]), which can be formulated as follows: given a smooth function F: $\mathbb{R}^k\to\mathbb{R}^k$, find k-vectors y and u such that $y=F(x,u), y\geq 0, u\geq 0$, and $y^Tu=0$. For this reason the term "dynamic complementarity problem" as used above seems natural. Apologies are due, however, to Chen and Mandelbaum who have used the same term in [3] to denote a different although related problem.

Computational methods for the NCP form a highly active research subject (see [10] for a survey), due to the many applications in particular in equilibrium programming. The DCP is a generalized and parameterized form of the NCP, and given the fact that the latter problem is already considered a major computational challenge, one may wonder whether the approach taken in the previous paragraphs can be viewed as promising from a computational point of view. Fortunately, it turns out that under fairly mild assumptions the DCP can be reduced to a series of *linear* complementarity problems. In the context of mechanical systems this idea was first used by Lötstedt [12]. The LCP can be formulated as follows.

Problem LCP: Given a vector $q\in\mathbb{R}^k$ and a matrix $M\in\mathbb{R}^{k\times k}$, find k-vectors y and u such that

$$y=q+Mu,\qquad y\geq 0,\qquad u\geq 0,\qquad y^Tu=0. \tag{17}$$

The LCP has been studied extensively, in particular because of its applications in game theory and mathematical programming. A wealth of theoretical results and computational methods has been collected in [4]. The main result that will be used here is the following: the LCP (17) has a unique solution (y,u) for all q if and only if all principal minors of the matrix M are positive [16], [4, Th. 3.3.7]. (Given a matrix M of size $k\times k$ and two nonempty subsets I and J of $\{1,\cdots,k\}$ of equal cardinality, the (I,J)-minor of M is the determinant of the square submatrix $M_{IJ}:=(m_{ij})_{i\in I,j\in J}$. The *principal minors* are those with $I=J$ [9, p. 2].)

To get a reduction to a sequence of LCP's, assume that the dynamics (8) can be written in the *affine* form

$$\begin{aligned}\dot{x}(t)&=f(x(t))+\sum_{i=1}^{k}g_i(x(t))u_i(t)\\ y(t)&=h(x(t)).\end{aligned}\tag{18}$$

Extensive information on systems of this type is given for instance in [15]. In particular, we need the following terminology. The *relative degree* of the ith output y_i is the number of times one has to differentiate y_i to get a result that depends explicitly on the inputs u. The system is said to have *uniform relative degree* if the relative degrees of all outputs are the same.

Theorem 3.1: Consider the system of equations (18) together with the complementarity conditions (9), and suppose that (18) has uniform relative degree ρ. Let $x_0\in\mathbb{R}^n$ be such that

$$(h(x_0),\cdots,L_f^{\rho-1}h(x_0))\succeq 0 \tag{19}$$

(with componentwise interpretation of the lexicographic inequality) and such that all principal minors of the decoupling matrix $L_gL_f^{\rho-1}h(x_0)$ at x_0 are positive. For such x_0, the dynamic complementarity problem DCP(ℓ) has for each ℓ a solution $((y^0,\cdots,y^\ell),(u^0,\cdots,u^\ell))$ which can be found by solving a sequence of LCP's. Moreover, this solution is unique, except for the values of u_i^j with $i\notin J_\ell$ and $j>\ell-\rho$.

Proof: It follows from the special form of (18) and the uniform relative degree assumption that the equations of the DCP will take the following form, in which the ϕ_j's denote functions that can be computed explicitly [cf. (11)] from the given functions f, g_i, and h:

$$\begin{aligned}y^j&=L_f^jh(x_0)\quad(j=0,\cdots,\rho-1)\\ y^{\rho+j}&=\phi_j(x_0,u^0,\cdots,u^{j-1})+L_gL_f^{\rho-1}h(x_0)u^j\qquad(j\geq 0).\end{aligned}\tag{20}$$

From this and (19) it is already obvious that the claim of the theorem holds for $\ell=0,\cdots,\rho-1$. We now continue by induction and so we carry out the proof assuming that $\ell\geq\rho$ and that the claim in the theorem holds for DCP($\ell-1$). A solution $((y^0,\cdots,y^\ell),(u^0,\cdots,u^\ell))$ of DCP(ℓ) can be constructed as follows. The components y^j for $j=0,\cdots,\ell-1$ and u^j for $j=0,\cdots,\ell-\rho-1$ must be taken from the solution for DCP($\ell-1$) by the nesting property. In this way one satisfies automatically all equations of DCP(ℓ) except for the last one, which is

$$y^\ell=\phi_{\ell-\rho}(x_0,u^0,\cdots,u^{\ell-\rho-1})+L_gL_f^{\rho-1}h(x_0)u^{\ell-\rho}. \tag{21}$$

To simplify the notation, we abbreviate this as

$$y^\ell=z^\ell+Du^{\ell-\rho}. \tag{22}$$

Note that the vector z^ℓ depends only on the components of the solution of DCP($\ell-1$) that are uniquely determined; the matrix D is the decoupling matrix at x_0. In addition to (22), the complementarity conditions of DCP(ℓ) have to be satisfied; after eliminating all conditions that are satisfied automatically by building the solution from the one that was obtained from DCP($\ell - 1$), this leaves us with the conditions

$$\begin{aligned} y_i^\ell &= 0 \qquad (i \in I_{\ell-1}) \\ u_i^{\ell-\rho} &= 0 \qquad (i \in J_{\ell-1}) \end{aligned} \tag{23}$$

and

$$y_i^\ell \geq 0, \qquad u_i^{\ell-\rho} \geq 0, \qquad y_i^\ell u_i^{\ell-\rho} = 0 \qquad (i \in K_{\ell-1}). \tag{24}$$

Dividing up (22) in three parts corresponding to the index sets $J_{\ell-1}$, $I_{\ell-1}$, and $K_{\ell-1}$, and dropping all indexes and subindexes that depend on ℓ to further alleviate the notational burden, we get

$$\begin{bmatrix} y_I \\ y_J \\ y_K \end{bmatrix} = \begin{bmatrix} z_I \\ z_J \\ z_K \end{bmatrix} + \begin{bmatrix} D_{II} & D_{IJ} & D_{IK} \\ D_{JI} & D_{JJ} & D_{JK} \\ D_{KI} & D_{KJ} & D_{KK} \end{bmatrix} \begin{bmatrix} u_I \\ u_J \\ u_K \end{bmatrix}.$$

By (23), we have to take $y_I = 0$ and $u_J = 0$. We see that the remaining components have to be chosen such that the following equations are satisfied:

$$0 = z_I + D_{II}u_I + D_{IK}u_K \tag{25}$$

$$y_J = z_J + D_{JI}u_I + D_{JK}u_K \tag{26}$$

$$y_K = z_K + D_{KI}u_I + D_{KK}u_K. \tag{27}$$

Moreover, the complementarity conditions that follow from (24) must hold

$$y_K \geq 0, \qquad u_K \geq 0, \qquad y_K^T u_K = 0. \tag{28}$$

By assumption, the determinant of D_{II} is positive and hence nonzero so that u_I can be solved in terms of z_I and u_K from (25). Inserting the result in (27) leads to the equation

$$y_K = z_K - D_{KI}D_{II}^{-1}z_I + (D_{KK} - D_{KI}D_{II}^{-1}D_{IK})u_K. \tag{29}$$

The above equation together with the complementary conditions (28) constitutes a standard LCP. From our assumption that all principal minors of D are positive, it follows that the same property is true for $D_{KK} - D_{KI}D_{II}^{-1}D_{IK}$, since this matrix is a Schur complement of a principal submatrix of D [17]. From [4, Th. 3.3.7] (as quoted above), it then follows that the LCP (28) and (29) has a unique solution. This determines y_K and u_K; then finally u_I and y_J follow from (25) and (26). The components u_i^j for $j > \ell - \rho$ must vanish for indexes i such that $(y_i^0, \cdots, y_i^\ell) \succ 0$. So the uniqueness of solutions is as described in the theorem statement. □

The result above is algebraic in nature. We now return to differential equations.

Theorem 3.2: Assume that the functions f, g_i, and h appearing in (18) are analytic. Under the conditions of Theorem 3.1, there exists an $\epsilon > 0$ such that (18) and (19) has a smooth solution with initial condition x_0 on $[0, \epsilon]$. Moreover, this solution is unique and corresponds to any mode I such that $I_{\ell^*} \subset I \subset I_{\ell^*} \cup K_{\ell^*}$.

Proof: Let W be a neighborhood of x_0 such that all principal minors of the matrix $D(x) := L_g L_f^{\rho-1} h(x)$ are nonzero for all $x \in W$, and consider for any index set I the equations

$$\begin{aligned} \dot{x} &= f(x) + \sum_{i \in I} g_i(x)u_i \\ 0 &= h_i(x), \qquad i \in I \end{aligned} \tag{30}$$

describing the dynamics in mode I. Since the submatrix $D_{II}(x)$ is invertible on W, it follows (see for instance [15, Ch. 11]) that (30) has a unique solution on W starting from any initial condition in $W \cap V_I$, where

$$V_I = \{x | L_f^j h_i(x) = 0, j = 0, \cdots, \rho - 1, i \in I\} \tag{31}$$

is the "consistent manifold" of mode I. As a consequence of the analyticity assumptions, these solutions on W are real-analytic [14, Corollary 1.8.11].

For indexes $i \in I_{\ell^*}$ we must have $L_f^j h_i(x_0) = 0$ for $j = 0, 1, \cdots, \rho - 1$; so $x_0 \in V_{I_{\ell^*}}$. Denote the solution in mode I_{ℓ^*} starting from x_0 by $(x(\cdot), y(\cdot), u(\cdot))$. From the uniqueness properties of solutions of DCP (see Theorem 3.1), it follows that

$$\begin{aligned} i \in I_{\ell^*} &\Rightarrow \begin{cases} y_i^{(j)}(0) = 0 \text{ for all } j \geq 0 \\ (u_i(0), \dot{u}_i(0), \cdots) \succ 0 \end{cases} \\ i \in J_{\ell^*} &\Rightarrow \begin{cases} u_i^{(j)}(0) = 0 \text{ for all } j \geq 0 \\ (y_i(0), \dot{y}_i(0), \cdots) \succ 0 \end{cases} \\ i \in K_{\ell^*} &\Rightarrow u_i^{(j)}(0) = 0, \quad y_i^{(j)}(0) = 0 \text{ for all } j \geq 0. \end{aligned}$$

So, by analyticity, there exists an $\epsilon > 0$ (taken small enough to guarantee that $x(t) \in W$ for $t \in [0, \epsilon]$) such that

$$\begin{aligned} i \in I_{\ell^*} &\Rightarrow y_i(t) = 0 \text{ and } u_i(t) \geq 0 \text{ for } t \in [0, \epsilon], \\ &\qquad u_i(t) > 0 \text{ for } t \in (0, \epsilon) \\ i \in J_{\ell^*} &\Rightarrow u_i(t) = 0 \text{ and } y_i(t) \geq 0 \text{ for } t \in [0, \epsilon], \\ &\qquad y_i(t) > 0 \text{ for } t \in (0, \epsilon) \\ i \in K_{\ell^*} &\Rightarrow y_i(t) = 0 \text{ and } u_i(t) = 0 \text{ for } t \in [0, \epsilon]. \end{aligned}$$

Hence (18) and (19) have a smooth solution which is unique in mode I_{ℓ^*} and in fact takes place in every mode I such that $I_{\ell^*} \subset I \subset I_{\ell^*} \cup K_{\ell^*}$.

Now suppose there is another smooth solution $(\tilde{x}(\cdot), \tilde{u}(\cdot), \tilde{y}(\cdot))$ (in some mode I) with initial condition x_0. As noted above, the solution is real-analytic. From the uniqueness property of solutions of DCP it follows that $\tilde{y}^{(j)}(0) = y^{(j)}(0)$ and $\tilde{u}^{(j)}(0) = u^{(j)}(0)$ for all j, and therefore by analyticity $\tilde{y}(t) = y(t)$ and $\tilde{u}(t) = u(t)$ for $t \in [0, \epsilon]$. It also follows that $I_{\ell^*} \subset I \subset I_{\ell^*} \cup K_{\ell^*}$.

Example 3.3—Mechanical Systems with Unilateral Constraints: Mechanical systems with unilateral constraints can be represented as semi-explicit complementarity systems (cf.

[18])

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q,p), \qquad q \in \mathbb{R}^n, \qquad p \in \mathbb{R}^n \\ \dot{p} &= -\frac{\partial H}{\partial q}(q,p) - \frac{\partial R}{\partial \dot{q}}(\dot{q}) + \frac{\partial C^T}{\partial q}(q)u, \qquad u \in \mathbb{R}^k \\ y &= C(q), \qquad y \in \mathbb{R}^k \end{aligned} \tag{32}$$

$$y \geq 0, \qquad u \geq 0, \qquad y^T u = 0 \tag{33}$$

where $(\partial H/\partial p), (\partial H/\partial q)$, etc., denote column vectors of partial derivatives. The vectors q and p contain position and momentum variables respectively, the Hamiltonian $H(q,p)$ is the total energy, and R is a Rayleigh dissipation function. Furthermore, $C(q) \geq 0$ is the column vector of unilateral (geometric) constraints, and $u \geq 0$ is the vector of Lagrange multipliers producing the constraint force vector $(\partial C^T/\partial q)(q)u$. The condition $y^T u = 0$ corresponds to the fact that the ith component of the constraint force vector can be only nonzero if the ith constraint is active, that is if $C_i(q) = 0$.

Assume that (32) is real-analytic and that the unilateral constraints are *independent*, that is

$$\text{rank}\frac{\partial C^T}{\partial q}(q) = k, \qquad \text{for all } q \text{ with } C(q) \geq 0. \tag{34}$$

Since the Hamiltonian is of the form (kinetic energy plus potential energy)

$$\begin{aligned} H(q,p) &= \tfrac{1}{2}p^T M^{-1}(q)p + V(q) \\ M(q) &= M^T(q) > 0 \end{aligned} \tag{35}$$

where $M(q)$ is the generalized mass matrix, it follows that the system (32) has uniform relative degree two with decoupling matrix

$$D(q) = \left[\frac{\partial C^T}{\partial q}(q)\right]^T M^{-1}(q)\frac{\partial C^T}{\partial q}(q). \tag{36}$$

Hence, from $M(q) > 0$ and (35) it follows that $D(q)$ is *positive definite* for all q with $C(q) \geq 0$. Since the principal minors of a positive definite matrix are all positive, all conditions of Theorems 3.1 and 3.2 are satisfied, and we establish well-posedness as in [12].

Example 3.4—Passive Systems: System (18) is called *passive* (see [20]) if there exists a function $V(x) \geq 0$ (a *storage function*) such that

$$\begin{aligned} L_f V(x) &\leq 0 \\ L_{g_i} V(x) &= h_i(x), \qquad i = 1, \cdots, k. \end{aligned} \tag{37}$$

Let us assume the following nondegeneracy condition on the storage function V:

$$\begin{aligned} &\text{rank}[L_{g_j} L_{g_i} V(x)]_{i,j=1,\cdots,k=k}, \\ &\qquad \text{for all } x \text{ with } h(x) \geq 0. \end{aligned} \tag{38}$$

Since $L_{g_j} h_i = L_{g_j} L_{g_i} V$ it follows that the system has uniform relative degree one, with decoupling matrix $D(x)$ given by the matrix in (38). If the principal minors of $D(x)$ are all positive, then well-posedness follows. Note that the condition of $D(x)$ having positive principal minors corresponds to an additional positivity condition on the storage function V. In fact, it can be checked that for a *linear* system with *quadratic* storage function $V(x)$, the decoupling matrix $D(x)$ will be positive definite if $V(x) > 0$ for $x \neq 0$. Hence, a linear passive electrical network containing ideal diodes is always well-posed.

IV. A Frequency-Domain Method

In this section we shall consider the case in which we have linear dynamics in (18). We shall, moreover, allow a feedthrough term $Du(t)$, so that (18) is replaced by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t). \end{aligned} \tag{39}$$

Linear complementarity modeling applies, for instance, to electrical networks with linear elements and diodes to certain mechanical systems made up of masses and linear springs (or rotational inertias and corresponding elasticity) and to the Hamiltonian equations for linear-quadratic optimal control problems with linear inequality constraints (cf. also [18] and [11]). In the linear case, the equations of the DCP become

$$y^j = CA^j x_0 + \sum_{i=0}^{j-1} CA^{j-1-i}Bu^i + Du^j \qquad (j \geq 0). \tag{40}$$

The dynamic complementarity problem with these equations will be denoted by LDCP and the truncated versions by LDCP(ℓ). It has been shown in [7] that LDCP(ℓ) can be looked at as a special case of the Generalized Linear Complementarity Problem (GLCP) [5] and of the Extended Linear Complementarity Problem (ELCP) [6].

A special feature of the linear setting is that it allows a frequency-domain approach to the mode selection problem. To see this, note that to a strictly proper rational vector function $y(s)$ we can associate the coefficients y^j of its power series expansion around infinity

$$y(s) = y^0 s^{-1} + y^1 s^{-2} + y^2 s^{-3} + \cdots$$

and, as is easily verified, the lexicographic nonnegativity condition $(y^0, y^1, \cdots) \succeq 0$ is equivalent to the condition

$$y(s) \geq 0 \qquad \text{for all sufficiently large } s. \tag{41}$$

Moreover, when two strictly proper functions $y(s)$ and $u(s)$ are related via

$$y(s) = C(sI - A)^{-1}x_0 + (D + C(sI - A)^{-1}B)u(s) \tag{42}$$

then, as is again easily verified, the corresponding coefficients $(y^0, y^1, \cdots)$ and $(u^0, u^1, \cdots)$ are related in exactly the same way as in the LDCP. We are therefore motivated to consider the following problem, which we shall call the *rational complementarity problem* (RCP).

Problem RCP: Let matrices A, B, C, D of sizes $n \times n, n \times k, k \times n$, and $k \times k$, respectively, be given. Define rational matrix functions $T(s)$ of size $k \times n$ and $G(s)$ of size $k \times k$ by $T(s) = C(sI - A)^{-1}$ and $G(s) = C(sI - A)^{-1}B + D$. For given x_0, find strictly proper rational functions $y(s)$ and $u(s)$ such that the equality

$$y(s) = T(s)x_0 + G(s)u(s) \tag{43}$$

holds, and there exists an $s_0 \in \mathbb{R}$ such that for all $s \geq s_0$ we have

$$y(s) \geq 0, \qquad u(s) \geq 0, \qquad y(s)^T u(s) = 0. \tag{44}$$

A formal proof of the equivalence of RCP and LDCP is given in [11]. Here we shall only present an example to illustrate the convenience of using RCP. Consider the following equations (cf. [18]) in which x_1 and x_2 denote the positions of two unit masses connected by springs to each other and to a wall, the motion of the first mass being constrained by a stop:

$$\begin{aligned} \ddot{x}_1(t) &= -2x_1(t) + x_2(t) + u(t) \\ \ddot{x}_2(t) &= x_1(t) - x_2(t) \\ y(t) &= x_1(t) \\ y(t) \geq 0, \qquad u(t) &\geq 0, \qquad y(t)u(t) = 0. \end{aligned} \tag{45}$$

This gives rise to the following equations:

$$\begin{aligned} (s^2 + 2)x_1 &= x_2 + u + x_{30} + sx_{10} \\ (s^2 + 1)x_2 &= x_1 + x_{40} + sx_{20} \end{aligned}$$

in which the vector $(x_{10}, x_{20}, x_{30}, x_{40})$ represents an initial condition. The variable x_2 can be eliminated by multiplying the first equation by $s^2 + 1$ and then using the second equation. Since $y = x_1$, we obtain

$$\begin{aligned} &(s^4 + 3s^2 + 1)y \\ &\quad = [s(s^2 + 1), s, s^2 + 1, 1] \begin{bmatrix} x_{10} \\ x_{20} \\ x_{30} \\ x_{40} \end{bmatrix} + (s^2 + 1)u. \end{aligned} \tag{46}$$

For each fixed s there is an associated *scalar* LCP, which leads to the following rules for the selection of a mode corresponding to the given initial conditions. Since at the instant of collision $x_{10} = 0$ always, the selection problem is dominated first by the sign of x_{30}. If this sign is positive, then the mode with inactive constraint will be selected, whereas the mode with active constraint will be selected (and will give rise to an impulsive solution) if the sign is negative. If x_{30} vanishes, then the highest power of s is associated with x_{20} and so it will be the sign of this quantity that will determine which mode is chosen. Again, if the sign of x_{20} is positive, then the mode with inactive constraint will be selected, and if the sign is negative, then the other mode will be selected. If also $x_{20} = 0$, then the sign of x_{40} becomes decisive. Finally if x_{40} vanishes as well, then the system is at rest, a situation which is in accordance with the constrained mode as well as with the unconstrained mode. One may convince oneself that this schedule, complicated as it may seem, does correspond to physical intuition. In [11] it is shown that the selection rule based on RCP leads for mechanical systems to the same results as projection according to the kinetic metric as described in [13].

V. Conclusions

The interaction of discrete and continuous elements can lead to extremely complex models. One way of overcoming the potential complexity is by the introduction of what one might call "formalisms," that is, sets of high-level rules that allow a compact specification of the dynamics of a hybrid system. The use of formalisms also will help the development of theory since it adds structure to the rather wide notion of a "hybrid system."

In this paper we have discussed a formalism which we have called the *complementarity formalism*. In our previous paper [18], we have shown that this formalism is suited, e.g., for mechanical systems with unilateral constraints, electrical networks with diodes, and the Hamiltonian equations for optimal control problems with state inequality constraints. In the present paper we have also shown that switching control schemes can be represented within this formalism. Moreover, from results on the representation of piecewise linear sets [8], [19] it follows that all continuous-time systems with elements having arbitrary piecewise linear characteristics can be written as complementarity systems. This includes control systems with relays and saturation or mechanical systems subject to Coulomb friction.

The central problem considered in this paper is to derive conditions for uniqueness of smooth continuations. We have solved this problem for complementarity systems in semi-explicit form, using methods from input–output systems theory and the theory of the LCP. The extension of these results to general complementarity systems is presently under investigation. It should be clear though that the well-posedness issue concerns more than just uniqueness of smooth continuations. One has to specify reinitialization rules, and one has to verify uniqueness of jumps and to guarantee that only a finite number of jumps can occur at a given instant. For *linear* complementarity systems these problems are addressed in [11]. A related basic issue concerns the *stability properties* of complementarity systems and their use in the design of switching control schemes. Furthermore, the inclusion of inputs and outputs within the formalism and their use for the control of complementarity systems calls for investigation.

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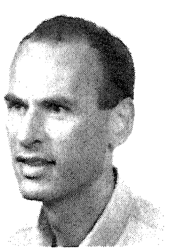

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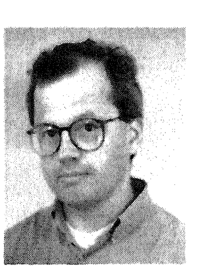

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