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METRIZABILITY IN GENERALIZED ORDERED SPACES

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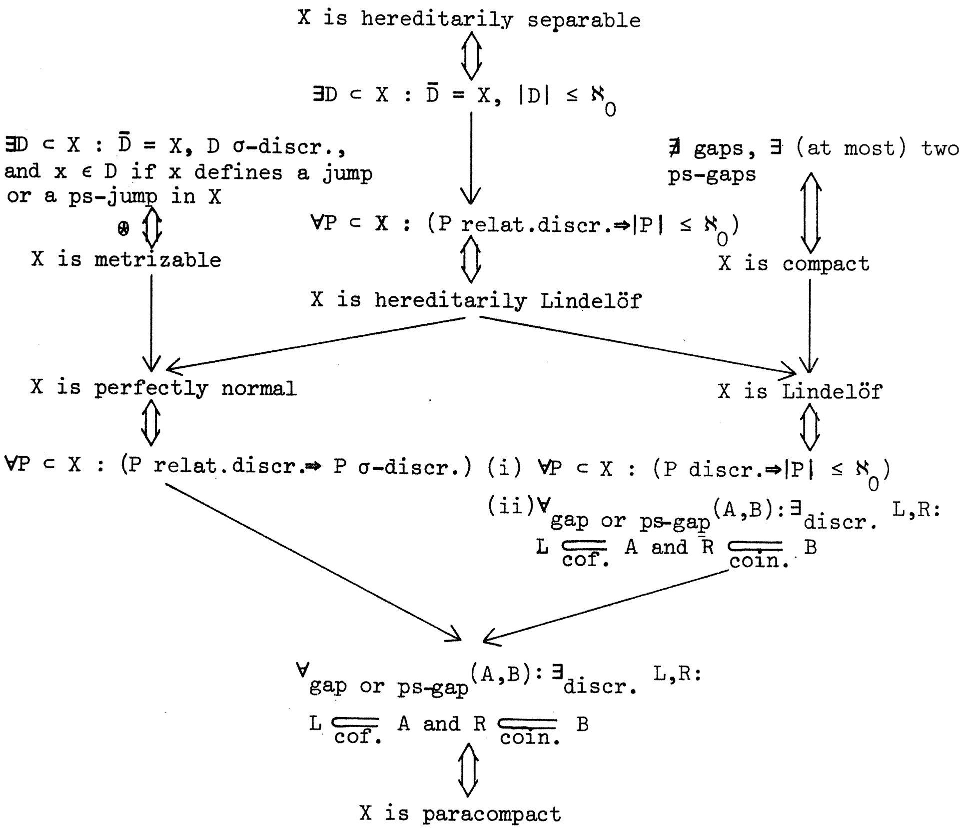
CONTENTS

INTRODUCTION

A generalized ordered topological space (abbreviated GO-space) is a subspace of a linearly ordered topological space (abbreviated LOTS).

The first part of this treatise (Chapters I - III) deals with characterizations for GO-spaces of various topological properties, (for instance metrizability, paracompactness, perfect normality etc.), making use of the order-structure. Many of our results will be formulated in terms of (the occurrence of) "gaps", "jumps", pseudo-gaps" or "pseudo-jumps" and (the occurrence of) specific relatively discrete subsets and σ-discrete subsets. In the subjoined scheme we list several characterizations (represented by "⟺") which will be proven in the following, and some implications (represented by an arrow "→").

Let $X = (X,<,\tau)$ be a GO-space. Then



In the first two chapters of this tract the necessary apparatus is developed and the basic theorems (except for the equivalence ⊛ of the above diagram) are proved. Chapter III, concerning the metrizability of GO-spaces, contains the characterization ⊛ from the diagram and, moreover, a theorem which says that a GO-space, which allows a countable sequence of open covers such that the family of stars (of each point) forms a local pseudo-base at that point, is metrizable if and only if the set of points, for which the system of stars is not a local base, is σ-discrete.

In chapter IV, the behaviour of several topological properties is investigated when lexicographically ordered products of LOTS's are taken. Here the notions of l(eft)-discreteness and r(ight)-discreteness are fundamental. One of the specific results, obtained in this context, is the following: If X_α is a LOTS (for each $\alpha < \omega_0$), possessing a right endpoint and coinitial with ω_0^*, then the lexicographically ordered product space $\coprod_{\alpha<\omega_0} X_\alpha$ is metrizable if and only if all X_α's are σ-r-discrete. Thus, for instance, the lexicographic product $(\omega_0^* + \omega_1 + 1)^{\omega_0}$ is a metrizable LOTS.

From the results of chapter III we will derive, in chapter V, necessary and sufficient conditions in order that a GO-space, which is the inverse image of a metric space under a continuous mapping, be metrizable. As corollaries, some generalizations of already well-known results are obtained.

SYMBOLS AND NOTATIONS

1. If X is a set, then $|X|$ denotes the cardinal number of X.
   $\aleph_0$ is the cardinality of a countable set.

2. In the class of ordinal numbers ω_i denotes the initial number with ordinal index i; so $\omega_0 = \omega$, $\omega_1 = \Omega$.
   A symbol μ may denote a single ordinal number, but also the linearly ordered set of all ordinals less than the given ordinal number μ. In all cases, it will be clear from the context which meaning is used.
   If α is an ordinal number, then α^* denotes the inverse order-type.

3. Let $X = (X,<)$ be a linearly ordered set, and $A \subset X$.
   If we write:

   A is cofinal in (or with) X, or
   A is coinitial in (or with) X, respectively,

   then we mean that

   $X = \{x \in X \mid \exists a \in A : x \leq a\}$, or
   $X = \{x \in X \mid \exists a \in A : a \leq x\}$, respectively.

   Furthermore, if μ is an ordinal then both

   (μ is cofinal in (or with) X) and (X is cofinal with μ),

   will say that X contains a well-ordered subset A, with ordinal number μ, such that A is cofinal in X.
   Analogously for

   (μ^* is coinitial in (or with) X) and (X is coinitial with μ^*).

4. If, for instance, we write:

   p is the left (right) endpoint of a linearly ordered set X,

   then we mean that p is the left, or the right endpoint, respectively, of X. Likewise, if A is a subset of a GO-space X, we say

A is σ-(σ-ℓ-;σ-r-)discrete in X

instead of A is σ-, or σ-ℓ-, or σ-r-discrete, respectively, in X.

5. By $\mathbb{R}$, Q, $\mathbb{Z}$ or $\mathbb{N}$ we denote the reals, the rationals, the integers or the positive integers, respectively.

6. Symbols like [K.1] or [L.2] refer to the references.

7. For all undefined terms and unproved statements in this treatise we refer to well-known textbooks as Dugundji [D.1], Engelking [E.1], Kelley [K.1] and Nagata [N.1].

<u>Convention</u>

All spaces are considered to be T_1.
Also, frequently we suppose that a space has at least two points, without explicitly saying so.

CHAPTER I

FUNDAMENTAL NOTIONS AND PROPERTIES

1.1 LINEARLY ORDERED SETS

A *linearly ordered set* is a pair $(X,<)$, where X is a set, and $<$ is a subset of $X \times X$, with the properties

(i) $\forall x \in X : (x,x) \notin <$

(ii) $\forall x,y,z \in X : ((x,y) \in < \text{ and } (y,z) \in <) \Rightarrow (x,z) \in <$

(iii) $\forall x,y \in X : x = y$ or $(x,y) \in <$ or $(y,x) \in <$.

$<$ is called the *ordering* of $(X,<)$.

In the sequel the linearly ordered set $(X,<)$ will mostly be denoted by X; and instead of $(x,y) \in <$ we shall always write $x < y$.

If X is a linearly ordered set, and $A \subset X$, then by $<$ an ordering $<_A$ is induced in A.
For definitions and properties of the notions "supremum (infimum) of A", "A is bounded", "X is complete" etc., see for instance Kelley [K.1].

A subset C of a linearly ordered set X is called a *convex* subset of X, whenever $p,q \in C$ and $p \leq q$ imply that

$$\{x \in X \mid p \leq x \leq q\} \subset C.$$

Let $A \subset X$. Then a convex subset C of X which is contained in A, is called a *convexity-component of* A, when $C' \subset C$ for every convex subset C' of X such that $C' \subset A$ and $C' \cap C \neq \emptyset$. Clearly, every convex subset C of X such that $C \subset A$ is contained in a (uniquely determined) convexity-component of A.
If $p,q \in X$, then convex subsets of X of the type

$$]p,q[\;= \{x \in X \mid p < x < q\},$$
$$]p,q] = \{x \in X \mid p < x \leq q\},$$
$$[p,q[\;= \{x \in X \mid p \leq x < q\}, \text{ or}$$
$$[p,q] = \{x \in X \mid p \leq x \leq q\}$$

are called *intervals* of X; and convex subsets of X of the type

$$]p,\rightarrow[\; = \{x \in X \mid p < x\},$$
$$[p,\rightarrow[\; = \{x \in X \mid p \leq x\},$$
$$]\leftarrow,p[\; = \{x \in X \mid x < p\}, \text{ or}$$
$$]\leftarrow,p] = \{x \in X \mid x \leq p\}$$

are called *half-lines* of X.

If X is a linearly ordered set and $p,q \in X$ and $p < q$, then p and q are called *neighbours in X* if $]p,q[\; = \emptyset$. A point $p \in X$ is said to be a *neighbour (point)* of X if there exists a point $q \in X$ such that p and q are neighbours in X. If p and q are neighbours in X and $p < q$ then p is called the *left neighbour* of q and q is called the *right neighbour* of p.

If A is a subset of a linearly ordered set X then a point $p \in A$ is called a *left endpoint of A* if $p \leq x$ for each point $x \in A$; and $p \in A$ is called a *right endpoint of A* if $x \leq p$ for each point $x \in A$. A point of A is called an *endpoint of* A, if it is a left or a right endpoint of A. When p is an endpoint of a convex subset C of X then, clearly, $C \setminus \{p\}$ is again a convex (possibly empty) subset of X.

1.2. LEXICOGRAPHICALLY ORDERED PRODUCTS

For definitions and properties of the notions "order type", "well-ordered set", "ordinal number" etc., see for instance Hausdorff [H.1].

For each ordinal α from a certain non-empty set M (of ordinals), let $X_\alpha = (X_\alpha, <_\alpha)$ be a non-empty linearly ordered set.
Then the *lexicographically ordered product* $\coprod_{\alpha \in M} X_\alpha$ is defined as the cartesian product $\prod_{\alpha \in M} X_\alpha$ supplied with the lexicographic ordering <; i.e.: if $x = (x_\alpha)_{\alpha \in M}$ and $y = (y_\alpha)_{\alpha \in M} \in \prod_{\alpha \in M} X_\alpha$ then

$$x < y \iff \exists \beta \in M : x_\beta <_\beta y_\beta \text{ and } x_\alpha = y_\alpha \text{ if } \alpha < \beta \text{ and } \alpha \in M.$$

In particular, if μ is an ordinal > 0, and if

$$M = \{\alpha \mid \alpha \text{ is an ordinal} < \mu\}$$

then we write $\coprod_{\alpha < \mu} X_\alpha$ instead of $\coprod_{\alpha \in M} X_\alpha$. If, moreover, $X_\alpha = X$ for all $\alpha < \mu$, then we write X^μ instead of $\coprod_{\alpha < \mu} X_\alpha$. Furthermore, for two linearly ordered sets X and Y we write

$$X \cdot Y = \coprod_{\alpha<2} \{X_\alpha \mid X_0 = X \text{ and } X_1 = Y\}.$$

If $\nu < \mu$, then it is clear that

$$\coprod_{\alpha<\nu} X_\alpha \cdot \coprod_{\nu\leq\alpha<\mu} X_\alpha$$

is canonically isomorphic to $\coprod_{\alpha<\mu} X_\alpha$.

PROPOSITION 1.2.1. *Let* M *be a set of ordinal numbers. Let* X_α *be a linearly ordered set, for each* $\alpha \in M$. *If* $x,y \in \coprod_{\alpha\in M} X_\alpha$ *and* $x < y$, *then* x *and* y *are neighbours in* $\coprod_{\alpha\in M} X_\alpha \iff$ *the smallest ordinal* $\beta \in M$ *such that* $x_\beta \neq y_\beta$ *satisfies:*

(i) x_β *and* y_β *are neighbours in* X_β, *and*

(ii) $\forall\alpha \in M : \alpha > \beta \Rightarrow$ *(*x_α *is the right endpoint of* X_α *and* y_α *is the left endpoint of* X_α*)*

PROOF. Obvious. □

CHAPTER II

LOTS's AND GO-SPACES

2.1. DEFINITIONS AND SOME FUNDAMENTAL PROPERTIES

A *linearly ordered topological space* (abbreviated LOTS) is a triple $(X,<,\lambda(<))$, where $(X,<)$ is a linearly ordered set on which a topology $\lambda(<)$ is defined by the subbase of all sets

$$\{x \in X \mid x < a\} \text{ and } \{x \in X \mid a < x\},$$

with $a \in X$.

A topological space (X,τ) is said to be *orderable* if there exists an ordering $<$ on X such that $\lambda(<) = \tau$.

Let $(X,<,\lambda(<))$ be a LOTS, and let $A \subset X$. If $\lambda(<)_A$ denotes the relative topology on A induced by $\lambda(<)$, then in general $\lambda(<)_A \neq \lambda(<_A)$. However, it is clear that $\lambda(<_A) \subset \lambda(<)_A$.

A *generalized ordered space* (abbreviated GO-space) is a triple $(X,<,\tau)$, where $(X,<)$ is a linearly ordered set supplied with a topology τ such that one of the following equivalent conditions is satisfied:

(i) $\begin{cases} \lambda(<) \subset \tau \\ \tau \text{ has a base consisting of convex subsets of } X \end{cases}$

(ii) $\begin{cases} \tau \text{ is a } T_1\text{-topology} \\ \tau \text{ has a base consisting of convex subsets of } X. \end{cases}$

In the sequel a GO-space $(X,<,\tau)$ will mostly be denoted by X.

The following proposition is obvious.

PROPOSITION 2.1.1. *Every subspace of a LOTS is a GO-space.*

The converse also holds true:

Let $X = (X,<,\tau)$ be a GO-space. We define a subspace X^* of the LOTS $X \cdot \mathbb{Z}$ by

$X^* = (X \cdot \{0\}) \cup \{(x,n) \mid [x,\rightarrow[\in \tau \setminus \lambda(<); n < 0\} \cup \{(x,n) \mid]\leftarrow,x] \in \tau \setminus \lambda(<); n > 0\}$.
Furthermore, we define a subspace X^{**} of the LOTS $X \cdot \{-1,0,1\}$, where $\{-1,0,1\} \subset \mathbb{Z}$. Let
$X^{**} = (X \cdot \{0\}) \cup \{(x,-1) \mid [x,\rightarrow[\in \tau \setminus \lambda(<)\} \cup \{(x,1) \mid]\leftarrow,x] \in \tau \setminus \lambda(<)\}$.
Observe that $p \in X^*$ ($p \in X^{**}$) is an endpoint of X^* (X^{**}) if and only if $p = (x,0)$ where x is an endpoint of X.

PROPOSITION 2.1.2. (Čech [C.1])

(i) X^* *is a LOTS (with respect to the induced ordering) and* X *is homeomorphic to the closed subset* $X \cdot \{0\}$ *of* X^*.

(ii) X^{**} *is a LOTS (with respect to the induced ordering) and* X *is homeomorphic to the dense subset* $X \cdot \{0\}$ *of* X^{**}.

PROOF. Obvious. □

CONVENTION. Except in situations where clarity requires that we distinguish between X and $X \cdot \{0\}$, we shall identify X with $X \cdot \{0\}$.

From 2.1.1 and 2.1.2 it now follows that

PROPOSITION 2.1.3. *The class of GO-spaces and the class of subspaces of LOTS's coincide.*

PROPOSITION 2.1.4. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

$$\tau = \lambda(<) \iff X = X^* = X^{**}$$

PROOF. Obvious. □

Finally we mention

PROPOSITION 2.1.5. *Let* $Y = (Y,<,\lambda(<))$ *be a LOTS. Let* X *be a dense subset of* Y. *Then*

$\lambda(<)_X = \lambda(<_X) \iff$ *each neighbourpoint of* Y *belongs to* X *if and only if its left and/or right neighbour in* Y *belong(s) to* X.

PROOF. As a subspace of a LOTS, $X = (X,<_X,\lambda(<)_X)$ is a GO-space. So $\lambda(<_X) \subset \lambda(<)_X$.

$\Longrightarrow$ Suppose $p,q \in Y$, $p < q$ and $]p,q[= \emptyset$. When $p \in X$, then $]\leftarrow,p] \cap X \in \lambda(<)_X$. Since $\lambda(<)_X = \lambda(<_X)$ and X is a dense subset of Y, it follows that q must be the right neighbour of p in X. Hence also $q \in X$.

$\Longleftarrow$ Choose $p \in X$ such that p is not an endpoint of X. We assume that $]\leftarrow,p] \cap X \in \lambda(<)_X$. From Y is a LOTS and X is a dense subset of Y it follows that p has a right neighbour q in Y. But then $q \in X$; and consequently $]\leftarrow,p] \cap X \in \lambda(<_X)$. Hence $\lambda(<)_X = \lambda(<_X)$, i.e. X is a LOTS. □

2.2. ON σ-; σ-ℓ- AND σ-r-DISCRETENESS OF SUBSETS OF GO-SPACES

In this section we examine some notions, (namely σ-; σ-ℓ- and σ-r-discreteness), which will be frequently used in the following. The notions σ-ℓ- and σ-r-discreteness will be applied especially in lexicographically ordered product spaces.

Recall that a family $\mathcal{A}$ of subsets of a topological space T is *discrete* (in T) if each $t \in T$ has a neighbourhood meeting at most one $A \in \mathcal{A}$. A subset $D \subset T$ is *discrete* (in T) if its one-point subsets form a discrete family. $D \subset T$ is *relatively discrete* (in T) if it is discrete in the subspace D of T. Clearly, D is discrete if and only if it is relatively discrete and closed in T. A subset $D \subset T$ is *σ-discrete* (in T) if it is the union of countably many discrete subsets of T. (For these definitions, see for instance Stone [Sto.1] and [Sto.2]).

Let $X = (X,<,\tau)$ be a GO-space.

A subset $A \subset X$ is said to be *σ-discrete* (in X) if $A = \bigcup_{n=1}^{\infty} A_n$, where for each $x \in X$ and each $k \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;k)$ of x such that $O(x;k) \cap (A_k\backslash\{x\}) = \emptyset$.

REMARK. Trivially, this definition of "σ-discrete" is equivalent to the previous one, mentioned above. With respect to the following definitions, we reformulate it here for the sake of analogy.

A subset $A \subset X$ is said to be *σ-ℓ-discrete* (in X) if $A = \bigcup_{n=1}^{\infty} A_n$, where for each $x \in X$ and each $k \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;k)$ of x such that $O(x;k) \cap (A_k\backslash\{x\}) \cap]\leftarrow,x] = \emptyset$.

A subset $A \subset X$ is said to be σ-r-*discrete* (in X) if $A = \bigcup_{n=1}^{\infty} A_n$, where for each $x \in X$ and each $k \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;k)$ of x such that $O(x;k) \cap (A_k \setminus \{x\}) \cap [x,\rightarrow[= \emptyset$.

REMARK. (i) Without loss of generality, we always may assume that, for each $n \in \mathbb{N}$, $A_n \subset A_{n+1}$.

(ii) In the case of σ-discreteness, each A_n is automatically closed. In the case of σ-ℓ- or σ-r-discreteness, we always may assume that each A_n is closed, (together with A_n also $\bar{A}_n$ satisfies the required condition). Moreover this assumption may be combined with assumption (i). Observe, however, that, by taking $\bar{A}_n$ instead of A_n, the original set A may be properly contained in $\bigcup_{n=1}^{\infty} \bar{A}_n$.

The next example shows that a σ-ℓ- or σ-r-discrete subset of a GO-space may be neither a countable union of closed subsets nor a countable union of relatively discrete subsets.

EXAMPLE 1. Let

$$X =]0,1[\cdot [0,1[,$$

supplied with the order-topology. ($]0,1[, [0,1[\subset \mathbb{R}$).
Let

$$A = \{(x,y) \in X \mid x \in]0,1[\setminus Q,\ y \in [0,1[\cap Q\}.$$

Putting

$$Q \cap [0,1[= \{0,q_1,q_2,\ldots\ldots,q_n,\ldots\ldots\}, \qquad (q_n \neq 0)$$

we define

$$A_0 = \{(x,0) \mid x \in]0,1[\setminus Q\}$$

and

$$A_n = \{(x,q_n) \mid x \in]0,1[\setminus Q\} \qquad\qquad (n \in \mathbb{N}).$$

Clearly $A = \bigcup_{n=0}^{\infty} A_n$. Moreover

1) A is σ-r-discrete (in X).

Indeed, choose $(x,y) \in X$ and $k \in \mathbb{N} \cup \{0\}$. Then there is a convex open neighbourhood $U((x,y);k)$ of (x,y) in the subspace $\{x\} \cdot [0,1[$ such that $U((x,y);k) \cap (A_k \setminus \{(x,y)\}) = \emptyset$. Further, let

$$O((x,y);k) = \begin{cases} U((x,y);k) & \text{if } y \neq 0 \\ U((x,y);k) \cup]\leftarrow,(x,y)] & \text{if } y = 0. \end{cases}$$

Then $O((x,y);k)$ is a convex open neighbourhood of (x,y) in X such that $O((x,y);k) \cap (A_k \setminus \{(x,y)\}) \cap [(x,y),\rightarrow[= \emptyset$.

2) If $A = \bigcup_{n=1}^{\infty} B_n$ then at least one set B_n cannot be closed in X.

For, suppose $B_n = \bar{B}_n$ for every $n \in \mathbb{N}$. The subspace

$$S = \{(x,0) \in X \mid x \in]0,1[\}$$

of X is homeomorphic to the well-known Sorgenfrey-line (see 2.3 example 1, or Kelley [K.1]). Now

$$\{(x,0) \in X \mid x \in]0,1[\setminus \mathbb{Q}\} = \bigcup_{n=1}^{\infty} (B_n \cap S).$$

Since each $B_n \cap S$ is a closed subset of S with an empty interior, and, moreover, the complement of $\bigcup_{n=1}^{\infty} (B_n \cap S)$ is a countable set, it follows that S is a space of the first category. Contradiction.

3) If $A = \bigcup_{n=1}^{\infty} B_n$ then at least one set B_n cannot be relatively discrete in X. Indeed, there is an integer $k \in \mathbb{N}$ such that

$$|\{(x,0) \in X \mid x \in]0,1[\} \cap B_k| > \aleph_0.$$

Hence $\{(x,0) \in X \mid (x,0) \in B_k\}$ contains a condensationpoint $(p,0)$ in $S = \{(x,0) \in X \mid x \in]0,1[\}$. (Observe that the Sorgenfrey-line is a hereditarily Lindelöf space). But then $(p,0) \in B_k$ and for each convex open neighbourhood $O(p,0)$ of $(p,0)$ in X we have: $|O(p,0) \cap B_k| > \aleph_0$.

PROPOSITION 2.2.1. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $A \subset X$. *Then*

$$A \text{ is } \sigma\text{-discrete} \Longleftrightarrow A \text{ is } \sigma\text{-}\ell\text{- and } \sigma\text{-r-discrete.}$$

PROOF.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ A is σ-ℓ-discrete, so $A = \bigcup_{n=1}^{\infty} A_n$ where, for each $x \in X$ and each $k \in \mathbb{N}$, there exists a convex open neighbourhood $O(x;k)$ of x such that

$$O(x;k) \cap (A_k \setminus \{x\}) \cap]\leftarrow,x] = \emptyset.$$

Also, A is σ-r-discrete, so $A = \bigcup_{n=1}^{\infty} B_n$ where, for each $x \in X$ and each $k \in \mathbb{N}$, there exists a convex open neighbourhood $U(x;k)$ of x such that

$$U(x;k) \cap (B_k \setminus \{x\}) \cap [x,\rightarrow[= \emptyset.$$

For all $n \in \mathbb{N}$, we may assume that $A_n \subset A_{n+1}$ and $B_n \subset B_{n+1}$. Now, for each $n \in \mathbb{N}$, put $C_n = A_n \cap B_n$. Then $A = \bigcup_{n=1}^{\infty} C_n$. Furthermore, for $x \in X$ and $n \in \mathbb{N}$, put $V(x;n) = O(x;n) \cap U(x;n)$. Then, for each $x \in X$ and each $k \in \mathbb{N}$ it follows that $V(x;k)$ is a convex open neighbourhood of x such that $V(x;k) \cap (C_k \setminus \{x\}) = \emptyset$. Hence A is σ-discrete. □

PROPOSITION 2.2.2. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $B \subset A \subset X$. *Then*

(i) A *is* σ-(σ-ℓ-;σ-r-)*discrete* (*in* X) $\Longrightarrow$ B *is* σ-(σ-ℓ-;σ-r-)*discrete* (*in* X).

(ii) B *is* σ-(σ-ℓ-;σ-r-)*discrete* (*in* X) $\Longrightarrow$ B *is* σ-(σ-ℓ-;σ-r-)*discrete* (*in* A).

PROOF. Obvious. □

In general a subset $B \subset X$ might be σ-(σ-ℓ-;σ-r-)discrete (in A), where $B \subset A \subset X$, while B fails to be σ-(σ-ℓ-;σ-r-)discrete (in X), as the following example shows.

EXAMPLE 2. Let X be the subset of $(\omega_1+1) \cdot \omega_0^*$ defined by

$$X = ((\omega_1+1) \cdot \{0\}) \cup \{(\alpha,\beta) \mid \alpha \text{ is a limit ordinal} < \omega_1,\ \beta \in \omega_0^*\}$$

If $<_X$ is the induced ordering on X, then we consider the LOTS $X = (X,<_X,\lambda(<_X))$. Now, let

$$A = B = (\omega_1 \cdot \{0\}) \cup \{(\alpha,\beta) \mid \alpha \text{ is a limit ordinal} < \omega_1,\ \beta \in \omega_0^*\} =$$

$$= X \setminus \{(\omega_1,0)\}.$$

Then

1) B is σ-discrete (in A).
   Indeed, B is relatively discrete (in X). Thus, since B = A, B is even discrete (in A).
2) B is not σ-discrete (in X).
   For $(\omega_1,0) \in X$ is an accumulation of each uncountable subset of B. Consequently, B is not σ-ℓ-discrete (in X).

PROPOSITION 2.2.3. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $A \subset X$. *If* A *is a dense and* σ*-discrete subset of* X, *then* X *is a* C_I*-space.*

PROOF. A is σ-discrete, so $A = \bigcup_{n=1}^{\infty} A_n$ where, for each $x \in X$ and each $k \in \mathbb{N}$, there exists a convex open neighbourhood $O(x;k)$ of x such that $O(x;k) \cap (A_k\setminus\{x\}) = \emptyset$. Choose $p \in X$. We distinguish between four cases:

1) $]\leftarrow,p] \in \tau$ and $[p,\rightarrow[\in \tau$.
   Then, obviously, there is a countable local base at p.
2) $]\leftarrow,p] \in \tau$ and $[p, [\notin \tau$.
   We claim that $\{O(p;n) \cap]\leftarrow,p] \mid n \in \mathbb{N}\}$ is a countable local base at p. Indeed, let U be any open convex neighbourhood of p. From A is dense in X it follows that $A \cap U \cap]\leftarrow,p[\neq \emptyset$. Hence there is an integer $k \in \mathbb{N}$ such that $A_k \cap U \cap]\leftarrow,p[\neq \emptyset$. Consequently $O(p;k) \cap]\leftarrow,p] \subset U$.
3) $]\leftarrow,p] \notin \tau$ and $[p,\rightarrow[\in \tau$.
   Then, similarly, $\{O(p;n) \cap [p,\rightarrow[\mid n \in \mathbb{N}\}$ is a countable local base at p.
4) $]\leftarrow,p] \notin \tau$ and $[p,\rightarrow[\notin \tau$.
   Clearly, now $\{O(p;n) \mid n \in \mathbb{N}\}$ is a countable local base at p. □

COROLLARY. *A separable GO-space satisfies the first axiom of countability.*

For later purposes (chaper IV) we introduce the following notions.

A GO-space $X = (X,<,\tau)$ is called a *left-*C_I*-space* if for each $x \in X$ the subspace $]\leftarrow,x]$ of X has a countable local base at x.

A GO-space $X = (X,<,\tau)$ is called a *right-*C_I*-space* if for each $x \in X$ the subspace $[x,\rightarrow[$ of X has a countable local base at x.

Clearly, a GO-space is a C_I-space if and only if it is a left- and a right-C_I-space. Furthermore, the LOTS ω_1+1 is a right-C_I-space but not a C_I-space.

PROPOSITION 2.2.4. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $A \subset X$.
If A *is a dense subset of* X, *then*

(i) A *is* σ-ℓ-*discrete (in* X*)* $\Longrightarrow$ **X** *is a left-*C_I*-space.*

(ii) A *is* σ-r-*discrete (in* X*)* $\Longrightarrow$ X *is a right-*C_I*-space.*

PROOF. Analogous to the proof of 2.2.3. □

PROPOSITION 2.2.5. *Let* $X = (X,<,\tau)$ ***be*** *a GO-space. Let* $A \subset X$.
If X *satisfies the countable chain condition, and if* A *is* σ-*discrete (in* X*), then* $|A| \leq \aleph_0$.

PROOF. A is σ-discrete, so $A = \bigcup_{n=1}^{\infty} A_n$ where each A_n is discrete (in X). Suppose $|A| > \aleph_0$. Then $|A_k| > \aleph_0$ for some $k \in \mathbb{N}$. As a subspace of a LOTS, X is collectionwise normal (Steen [St.1]). Therefore, from A_k is discrete and $|A_k| > \aleph_0$, it follows that there is an uncountable disjoint family in X of non-empty open subsets. This, however, is impossible. □

COROLLARY. *In a separable GO-space, every* σ-*discrete subset is countable.*

PROPOSITION 2.2.6. *Let* $X = (X,<,\lambda(<))$ *be a LOTS. Let* $A \subset X$.
If X *satisfies the countable chain condition, and if* A *is* σ-ℓ-(σ-r-) *discrete (in* X*), then* $|A| \leq \aleph_0$.

PROOF. We assume that A is σ-ℓ-discrete. The other case can be treated similarly. So, $A = \bigcup_{n=1}^{\infty} A_n$ where for each $x \in X$ and each $k \in \mathbb{N}$, there exists a convex open neighbourhood $O(x;k)$ of x such that
$O(x;k) \cap (A_k \setminus x) \cap]\leftarrow,x] = \emptyset$. Now, suppose $|A| > \aleph_0$. Then there is an integer $i \in \mathbb{N}$ such that $|A_i| > \aleph_0$. For all $x \in X$ such that $[x,\rightarrow[\in \lambda(<)$ and $]\leftarrow,x[\neq \emptyset$, we denote the left neighbour of x bij x^-. Observe that all such points $x \in X$ do have a left neighbour, because X is a LOTS. Let

$$F = \{O(a;i) \cap]\leftarrow,a[\mid a \in A_i ; [a,\rightarrow[\notin \lambda(<)\} \cup$$

$$\cup \{O(a^-;i) \cap]\leftarrow,a[\mid a \in A_i ; [a,\rightarrow[\in \lambda(<) \text{ and }]\leftarrow,a[\neq \emptyset\}.$$

Then F is a family containing uncountably many mutually disjoint non-empty open subsets of X. Contradiction. □

COROLLARY. *In a separable LOTS, every σ-(σ-ℓ-;σ-r-)discrete subset is countable.*

REMARK. Observe that the Sorgenfrey-line shows that the previous proposition does not hold for GO-spaces, in general.

PROPOSITION 2.2.7. *Let* $X = (X,<,\tau)$ *be a GO-space without isolated points. Then*

$$X \textit{ is } \sigma\textit{-discrete (in } X) \Longrightarrow X \textit{ is of the first category.}$$

PROOF. Obvious. □

PROPOSITION 2.2.8. *Let* $X = (X,<,\lambda(<))$ *be a LOTS without isolated points. Then*

$$X \textit{ is } \sigma\text{-}\ell\text{-}(\sigma\text{-}r\text{-})\textit{discrete (in } X) \Longrightarrow X \textit{ is of the first category.}$$

PROOF. Let X be σ-ℓ-discrete. So $X = \bigcup_{n=1}^{\infty} A_n$ where, for each $x \in X$ and each $k \in \mathbb{N}$, there exists a convex open neighbourhood $O(x;k)$ of x such that $O(x;k) \cap (A_k \setminus \{x\}) \cap]\leftarrow,x] = \emptyset$. We may assume that $A_n = \bar{A}_n$ for all $n \in \mathbb{N}$. Now, suppose there is an integer $i \in \mathbb{N}$ such that Int $A_i \neq \emptyset$. Since X has no isolated points, there exists a non-empty interval $]p,q[\subset \text{Int } A_i \subset A_i$. Furthermore $O(q;i) \cap (A_i \setminus \{q\}) \cap]\leftarrow,q] = \emptyset$. Therefore, since X is a LOTS, q must have a left neighbour $q^- \in X$. Since clearly $q^- \in]p,q[$ and since q^- is not isolated, $]p,q^-[$ is a non-empty interval contained in A_i. Repeating the same arguments we find that q^- has a left neighbour q^{--} in X. This contradicts the fact that q^- must be a non-isolated point of X. (If X is σ-r-discrete then we can argue analogously). □

REMARK. A GO-space without isolated points may be σ-ℓ- (or σ-r-)discrete and of the second category; for instance the Sorgenfrey-line (2.3, example 1) is σ-ℓ-discrete and of the second category.

PROPOSITION 2.2.9. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $A \subset X$. *Then* A *is* σ-(σ-ℓ-;σ-r-)*discrete (in* X*)* $\Longrightarrow$ A *is a totally disconnected subspace of* X.

PROOF. Suppose S is a non-degenerated connected subspace of A. Then there are points $a,b \in X$, $a < b$, such that $[a,b] \subset S \subset A \subset X$. First, it follows that [a,b] is a compact and connected LOTS which, hence, is of the second category. Secondly, from 2.2.2 (i) it follows that [a,b] is σ-(σ-ℓ-;σ-r-) discrete (in X) and then, from 2.2.2 (ii), that [a,b] is σ-(σ-ℓ-;σ-r-)discrete (in [a,b]). Now, from 2.2.8 we conclude that [a,b] is of the first category. This is a contradiction. □

It seems worthwhile to notice that of the following three properties of a GO-space X, no two imply the third one, even in the case that X is a LOTS.

(i) X is σ-discrete (in X)

(ii) X is of the first category

(iii) X does not have condensationpoints.

$$\text{(i)} + \text{(ii)} \not\Longrightarrow \text{(iii)}.$$

EXAMPLE 3. Let

$$X = \{(x,y) \in [0,1] \cdot (\mathbb{Q} \cap [0,1[) \mid x = 1 \Longleftrightarrow y = 0\},$$

supplied with the order-topology. ([0,1], [0,1[$\subset \mathbb{R}$). Denote

$$\mathbb{Q} \cap]0,1[= \{q_1, q_2, \ldots\ldots, q_n, \ldots\ldots\}$$

1) X is σ-discrete (in X).

Indeed, for each $n \in \mathbb{N}$, define

$$\tilde{A}_n = \{(x,q_n) \in X \mid x \in [0,1[\}.$$

Next, for each $n \in \mathbb{N}$, put

$$A_n = \bigcup_{i=1}^{n} \tilde{A}_i \cap \{(x,y) \in [0,1[\cdot (Q \cap]0,1[) \mid x \leq \tfrac{n-1}{n}\}.$$

Finally, let

$$A_0 = \{(1,0)\}.$$

Then $X = \bigcup_{n=0}^{\infty} A_n$ and each A_n is discrete (in X).

2) X is of the first category. (see 2.2.8).

3) The point $(1,0) \in X$ clearly is a condensationpoint.

$$(i) + (iii) \not\Rightarrow (ii).$$

EXAMPLE 4. Let X be a discrete topological space. Then X is orderable (cf. Herrlich [He.1]). Evidently, X is σ-discrete (in X), X does not have condensationpoints and X is of the second category.

$$(ii) + (iii) \not\Rightarrow (i).$$

EXAMPLE 5. Let

$$X = \omega_1 \cdot (Q \cap [0,1]),$$

supplied with the order-topology ($[0,1] \subset \mathbb{R}$). Put

$$Q \cap [0,1] = \{q_1,q_2,\ldots\ldots,q_n,\ldots\ldots\}$$

1) X is not σ-discrete (in X).

By 2.2.1 it is sufficient to show that X is not σ-ℓ-discrete. Now, ω_1 is isomorphically and topologically contained in X; i.e. ω_1 is homeomorphic to $\{(\alpha,0) \in X \mid \alpha < \omega_1\}$. Furthermore, when $\omega_1 = \bigcup_{n=1}^{\infty} A_n$ then at least one set, say A_k, will be an uncountable subset of ω_1. Since ω_1 is an order-complete LOTS, it follows that A_k has an accumulationpoint $\beta < \omega_1$, which is, of course, a limit ordinal. Consequently, for each convex open neighbourhood $O(\beta;k)$ of β (in ω_1) we have that $O(\beta;k) \cap (A_k\setminus\{\beta\}) \cap]\leftarrow,\beta] \neq \emptyset$. Thus, ω_1 is not σ-ℓ-discrete. Therefore, from 2.2.2, it follows that X is not σ-ℓ-discrete.

2) X is of the first category.
Indeed, for each $n \in \mathbb{N}$, let

$$P_n = \{(\alpha,q_n) \in X \mid \alpha < \omega_1\}.$$

Then $X = \bigcup_{n=1}^{\infty} P_n$ and $\mathrm{Int}\,\bar{P}_n = \emptyset$ for all $n \in \mathbb{N}$.

3) Clearly, X does not have condensationpoints.

The rest of this section will be used to show that all GO-spaces X, containing a dense LOTS Y, of type $Y = \coprod_{\alpha<\mu} Y_\alpha$ (where μ is a limit ordinal and $|Y_\alpha| \geq 2$ for each $\alpha < \mu$), are neither σ-discrete nor of the first category and, moreover, consist of merely condensationpoints. We note that many well-known spaces belong to this class; for instance: $\mathbb{R} = \omega_0^* \cdot \omega_0^{\omega_0}$; $\mathbb{R} \setminus \mathbb{Q} = (\omega_0^*+\omega_0)^{\omega_0}$; $[0,1] = \mathrm{Cl}\ \omega_0^{\omega_0}$; Cantorset $= \{0,1\}^{\omega_0}$; Long line $= \omega_1 \cdot \omega_0^{\omega_0}$. (see chapter IV).

In the rest of this paragraph, μ is a limit ordinal and, for each $\alpha < \mu$, Y_α is a LOTS containing at least two points.
We denote: $Y_\alpha = (Y_\alpha,<_\alpha,\lambda(<_\alpha))$ and $\coprod_{\alpha<\mu} Y_\alpha = (\coprod_{\alpha<\mu} Y_\alpha,<,\lambda(<))$.

PROPOSITION 2.2.10. *Each point of* $\coprod_{\alpha<\mu} Y_\alpha$ *is a condensationpoint.*

PROOF. If $y \in \coprod_{\alpha<\mu} Y_\alpha$ then it is easily verified that for each convex open neighbourhood O of y there exists and ordinal $\nu < \mu$ such that $\coprod_{\nu<\alpha<\mu} Y_\alpha \subset O$. Etc. □

PROPOSITION 2.2.11. $\coprod_{\alpha<\mu} Y_\alpha$ *is not* σ*-discrete.*

The proof of 2.2.11 is based on the following lemmata.

LEMMA 1. *Suppose that* μ *is not cofinal with* ω_0. *Then no point of* $\coprod_{\alpha<\mu} Y_\alpha$ *has a countable local base.*

PROOF. Let $y = (y_\alpha)_{\alpha<\mu} \in \coprod_{\alpha<\mu} Y_\alpha$. We may assume, for each $\alpha < \mu$, that there exists an ordinal $\beta > \alpha$ such that y_β is not the left endpoint of Y_β. (Otherwise this statement is satisfied with respect to right endpoints). Then y is not a right neighbourpoint in $\coprod_{\alpha<\mu} Y_\alpha$. Now, suppose there exists a countable local base at y. Then we can find a strictly increasing sequence $\{s^{(n)}\}_{n=1}^{\infty} = \{(s_\alpha^{(n)})_{\alpha<\mu}\}_{n=1}^{\infty}$ in $\coprod_{\alpha<\mu} Y_\alpha$ such that $y = \lim_{n\to\infty} s^{(n)}$. For each $k \in \mathbb{N}$, let $\beta(k)$ be the first ordinal less than μ such that $s^{(k)}_{\beta(k)} <_{\beta(k)} y_{\beta(k)}$. Put $\beta = \sup\{\beta(k) \mid k \in \mathbb{N}\}$. Then $\beta < \mu$, since μ is not cofinal with ω_0. By assumption there exists an ordinal $\gamma > \beta$ and a point $u \in Y_\gamma$ with $u <_\gamma y_\gamma$. Finally, choose $z = (z_\alpha)_{\alpha<\mu} \in \coprod_{\alpha<\mu} Y_\alpha$ in such a way that $z_\alpha = y_\alpha$ if $\alpha < \gamma$, and $z_\gamma = u$. Then, for all $n \in \mathbb{N}$, $s^{(n)} < z < y$. However, this contradicts $y = \lim_{n\to\infty} s^{(n)}$. □

LEMMA 2. *Suppose that* μ *is cofinal with* ω_0. *Then the Cantorspace* $\{0,1\}^{\omega_0}$ *is a subspace of* $\coprod_{\alpha<\mu} Y_\alpha$.

PROOF. Since μ is cofinal with ω_0, there exists a strictly increasing countable sequence $\{\beta_n\}_{n=1}^{\infty}$, of ordinals less than μ, which is cofinal in μ. We put $\tilde{Y}_0 = \coprod_{\alpha\leq\beta_1} Y_\alpha$ and $\tilde{Y}_n = \coprod_{\beta_n<\alpha\leq\beta_{n+1}} Y_\alpha$, $n \in \mathbb{N}$. Then, clearly $\coprod_{n<\omega_0} \tilde{Y}_n$ is homeomorphic to $\coprod_{\alpha<\mu} Y_\alpha$. For $n < \omega_0$, we denote: $\tilde{\underset{\sim}{Y}}_n = (\tilde{Y}_n, <_n, \lambda(<_n))$ and $\coprod_{n<\omega_0} \tilde{\underset{\sim}{Y}}_n = (\coprod_{n<\omega_0} \tilde{Y}_n, <, \lambda(<))$. Next, for each $n < \omega_0$, choose two points p_n and $q_n \in Y_n$ such that $p_n <_n q_n$. Further, let

$$C = \{y = (y_n)_{n<\omega_0} \in \coprod_{n<\omega_0} \tilde{Y}_n \mid y_n = p_n \text{ or } y_n = q_n\}.$$

Now the proof is complete once we have shown that $\lambda(<_C) = \lambda(<)_C$. Since certainly $\lambda(<_C) \subset \lambda(<)_C$ it is sufficient to prove that, for all subbasic-open sets $]s,\rightarrow[$ and $]\leftarrow,t[\in \coprod_{n<\omega_0} \tilde{Y}_n$, the intersections $]s,\rightarrow[\cap C$ and $]\leftarrow,t[\cap C \in \lambda(<_C)$. Consider, for instance, the set $]s,\rightarrow[\cap C$, where $s = (s_n)_{n<\omega_0} \in \coprod_{n<\omega_0} \tilde{Y}_n$. Choose a point $y = (y_n)_{n<\omega_0} \in]s,\rightarrow[\cap C$. From $s < y$ it follows that there is a first ordinal $k < \omega_0$ such that $s_k <_k y_k$. We distinguish between two cases:

1) There exists an ordinal $\ell < \omega_0$ such that $k < \ell$ and $y_\ell = q_\ell$.
   Let $z = (z_n)_{n<\omega_0}$ be defined by

$$\begin{cases} z_n = y_n \text{ if } n < \ell \\ z_n = p_n \text{ if } n \geq \ell. \end{cases}$$

Then $z \in C$, and $s < z < y$. Thus $y \in \,]z,\to[\, \cap C \subset \,]s,\to[\, \cap C$, and $]z,\to[\, \cap C \in \lambda(<_C)$.

2) There exists an ordinal $\ell \leq k$ such that $y_\ell = q_\ell$ and $y_n = p_n$ for all $n > \ell$. Then y is a right neighbourpoint of $(C,<_C)$. Hence, clearly $[y,\to[\, \cap C \in \lambda(<_C)$. Moreover $y \in [y,\to[\, \cap C \subset \,]s,\to[\, \cap C$.

(Observe that the remaining case, $((y_n)_{n<\omega_0} = (p_n)_{n<\omega_0})$, is obvious). □

PROOF OF 2.2.11. From 2.2.3 it follows that a σ-discrete GO-space is a C_I-space. Furthermore, 2.2.2 states that σ-discreteness is a hereditary property. The Cantorspace $\{0,1\}^{\omega_0}$ is a LOTS without isolated points and, moreover, of the second category. Hence, by 2.2.8, $\{0,1\}^{\omega_0}$ is not σ-discrete. Thus, lemma 1 or lemma 2 yields that $\coprod_{\alpha<\mu} Y_\alpha$ is not σ-discrete. □

REMARK. If μ is not cofinal with ω_0, then comparing the proof of lemma 1, it follows also that $\coprod_{\alpha<\mu} Y_\alpha$ is neither a left- nor a right-C_I-space. Now, as above, it can be shown that, for each limit ordinal μ, $\coprod_{\alpha<\mu} Y_\alpha$ is neither σ-ℓ- nor σ-r-discrete (use 2.2.4).

PROPOSITION 2.2.12. *$\coprod_{\alpha<\mu} Y_\alpha$ is a Baire-space (and hence of the second category).*

We need two simple lemmata.

LEMMA 1. *Let T be a regular topological space. Then T is a Baire space if there exists an open base $\mathcal{B}$ in T such that*

$$\cap \{\bar{N} \mid N \in \mathcal{N}\} \neq \emptyset$$

for every nest $\mathcal{N} \subset \mathcal{B}$.

PROOF. Let $\{O_n\}_{n\in\mathbb{N}}$ be a countable family of dense and open subsets of T. We have to show that $D = \cap \{O_n \mid n \in \mathbb{N}\}$ is again a dense subset of T. Let U be any non-empty open subset of T. We define a nest $\mathcal{N} = \{N_i\}_{i\in\mathbb{N}}$ in $\mathcal{B}$ as follows: Choose $N_1 \in \mathcal{B}$ such that $N_1 \subset \bar{N}_1 \subset U \cap O_1$. Next, suppose the sets N_i have been defined for all $i < n$, $(n > 1)$. Then, choose $N_n \in \mathcal{B}$ such that $N_n \subset \bar{N}_n \subset N_{n-1} \cap O_n$. Thus, we have: $\cap \{\bar{N}_i \mid i \in \mathbb{N}\} \subset D \cap U$. Consequently, since $\cap \{\bar{N}_i \mid i \in \mathbb{N}\} \neq \emptyset$, D is a dense subset of T. □

LEMMA 2. *Let* T *be a topological space. Then* T *is a Baire space if there exists a dense subspace* D *in* T *such that* D *is a Baire space.*

PROOF. Let

$$F = \bigcup \{F_n \mid F_n = \bar{F}_n,\ \mathrm{Int}\, F_n = \emptyset;\ n \in \mathbb{N}\}.$$

Then $D \cap F = \bigcup \{D \cap F_n \mid n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ we have: $\mathrm{Int}_D(D \cap F_n) \subset \mathrm{Int}\, F_n$. (Indeed, $\mathrm{Int}_D(D \cap F_n) = D \cap O$, for some open set O in T. Since $\bar{D} = T$ it follows that $O \subset \bar{O} = \overline{O \cap D} \subset F_n$. Hence $\mathrm{Int}_D(D \cap F_n) \subset \mathrm{Int}\, F_n$). Consequently, for each $n \in \mathbb{N}$, $\mathrm{Int}_D(D \cap F_n) = \emptyset$. So, since D is a Baire space, also $\mathrm{Int}_D(D \cap F) = \emptyset$. But then, since $\bar{D} = T$, $\mathrm{Int}\, F = \emptyset$. Thus T is a Baire space. □

PROOF OF 2.2.12. Let

$$D = \{y = (y_\alpha)_{\alpha<\mu} \in \prod_{\alpha<\mu} Y_\alpha \mid \forall\alpha < \mu : \exists \beta,\gamma > \alpha :$$

$$(\,]\leftarrow, y_\beta[\, \cap Y_\beta \neq \emptyset \text{ and }]y_\gamma, \rightarrow[\, \cap Y_\gamma \neq \emptyset)\}$$

(so, y_β is not the left endpoint of Y_β and y_γ is not the right endpoint of Y_γ). For each $y = (y_\alpha)_{\alpha<\mu} \in \prod_{\alpha<\mu} Y_\alpha$ and each $\nu < \mu$, let

$$B(y;\nu) = \{z = (z_\alpha)_{\alpha<\mu} \in D \mid z_\alpha = y_\alpha \text{ if } \alpha < \nu\}.$$

Next, for each $\nu < \mu$, let

$$\mathcal{B}_\nu = \{B(y;\nu) \mid y \in \prod_{\alpha<\mu} Y_\alpha\}.$$

Finally, put

$$\mathcal{B} = \bigcup \{\mathcal{B}_\nu \mid \nu < \mu\}$$

Observe that, for any two sets $B(y;\nu)$ and $B(y';\nu) \in \mathcal{B}_\nu$, whenever $B(y;\nu) \cap B(y';\nu) \neq \emptyset$ then $B(y;\nu) = B(y';\nu)$.

1) $\mathcal{B}$ is an open base for D.

First we show that each set $B(y;\nu) \in \mathcal{B}$ is open in D.

Choose $z = (z_\alpha)_{\alpha<\mu} \in B(y;\nu)$. Then there is an ordinal $\beta > \nu$ and a point

$u_\beta \in Y_\beta$ such that $u_\beta <_\beta z_\beta$. Moreover, there is an ordinal $\gamma > \nu$ and a point $v_\gamma \in Y_\gamma$ such that $z_\gamma <_\gamma v_\gamma$. Now, let $r = (r_\alpha)_{\alpha<\mu}$ and $s = (s_\alpha)_{\alpha<\mu} \in \coprod_{\alpha<\mu} Y_\alpha$ be such that

$$\begin{cases} r_\alpha = z_\alpha \text{ if } \alpha < \beta \\ r_\beta = u_\beta \end{cases} \qquad \begin{cases} s_\alpha = z_\alpha \text{ if } \alpha < \gamma \\ s_\gamma = v_\gamma \end{cases}$$

Then $z \in]r,s[\cap D \subset B(y;\nu)$.

Secondly we prove that $\mathcal{B}$ is a base for D. Let $]p,q[$ be a (non-empty) open interval in $\coprod_{\alpha<\mu} Y_\alpha$ and suppose that $z = (z_\alpha)_{\alpha<\mu} \in D \cap]p,q[$. Let β be the first ordinal such that $p_\beta <_\beta z_\beta$. Let γ be the first ordinal such that $z_\gamma <_\gamma q_\gamma$. Put $\nu = \max(\beta,\gamma)$. Then $z \in B(z;\nu+1) \subset]p,q[\cap D$.

2) D is a dense subset of $\coprod_{\alpha<\mu} Y_\alpha$.
We have to show that any non-empty open interval of the type $]p,q[$ in $\coprod_{\alpha<\mu} Y_\alpha$ meets D. (Observe that an endpoint of $\coprod_{\alpha<\mu} Y_\alpha$ cannot be isolated). Let β be the first ordinal such that $p_\beta <_\beta q_\beta$. Since $]p,q[\neq \emptyset$, p and q cannot be neighbours in $\coprod_{\alpha<\mu} Y_\alpha$. Therefore, when $]p_\beta,q_\beta[= \emptyset$ in Y_β then we may assume that, for some $\gamma > \beta$, p_γ is not the right endpoint of Y_γ. So, if we take a point $z = (z_\alpha)_{\alpha<\mu} \in D$ such that

$$\begin{cases} p_\alpha = z_\alpha \text{ if } \alpha < \gamma \\ p_\gamma <_\gamma z_\gamma \end{cases}$$

then $z \in D \cap]p,q[$. On the other hand, when $]p_\beta,q_\beta[\neq \emptyset$ in Y_β, then let $z = (z_\alpha)_{\alpha<\mu} \in D$ be such that

$$\begin{cases} z_\alpha = p_\alpha = q_\alpha \text{ if } \alpha < \beta \\ p_\beta <_\beta z_\beta <_\beta q_\beta. \end{cases}$$

Clearly again $z \in D \cap]p,q[$.

3) Each nest $N \subset \mathcal{B}$ has a non-empty intersection.
First we notice that any two members of $\mathcal{B}$ are either disjoint or comparable by inclusion. So, $\mathcal{B}$ consists of open and closed subsets of D. Now, let N be a nest in $\mathcal{B}$. Let

$$\lambda = \sup\{\nu < \mu \mid \exists B(y;\nu) \in N\}.$$

Then $\lambda \leq \mu$. Next, observe that, whenever $B(z;\rho)$ and $B(z';\tau) \in N$ and $\rho \leq \tau$, then $z_\alpha = z'_\alpha$ for all $\alpha < \rho$. Hence it follows that there are uniquely defined points $u_\alpha \in Y_\alpha$ for $\alpha < \lambda$ such that whenever $B(y;\nu) \in N$ then $y_\alpha = u_\alpha$ for all $\alpha < \nu$. Thus, it is clear that each point $z = (z_\alpha)_{\alpha<\mu}$, with $z_\alpha = u_\alpha$ for $\alpha < \lambda$, belongs to $\cap N$.

Finally, from 1), 3) and Lemma 1 it follows that D is a Baire space. Consequently, from 2) and lemma 2 it follows that $\coprod_{\alpha<\mu} Y_\alpha$ is a Baire space. □

The following proposition is obvious now.

PROPOSITION 2.2.13. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* Y *be a dense subset of* X. *If* $(Y,\tau|Y)$ *is orderable in such a way that* Y *can be represented by* $Y = \coprod_{\alpha<\mu} Y_\alpha$, *where* μ *is a limit ordinal and* $|Y_\alpha| \geq 2$ *for all* $\alpha < \mu$, *then*

(i) *Every point of* X *is a condensationpoint.*

(ii) X *is not* σ*-discrete (in* X*).*

(iii) X *is a Baire-space.*

2.3. GO-SPACES WHICH INHERIT PROPERTIES FROM THE CORRESPONDING LOTS's

In this section we want to investigate the conditions under which a GO-space $(X,<,\tau)$ inherits some specific properties from the corresponding LOTS $(X,<,\lambda(<))$.

a) Compactness, connectedness and total disconnectedness.

THEOREM 2.3.1. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

(i) $(X,<,\tau)$ *is compact* $\Longrightarrow \tau = \lambda(<)$.

(ii) $(X,<,\tau)$ *is connected* $\Longrightarrow \tau = \lambda(<)$.

(iii) $(X,<,\lambda(<))$ *is totally disconnected* $\Longrightarrow (X,<,\tau)$ *is totally disconnected.*

PROOF.

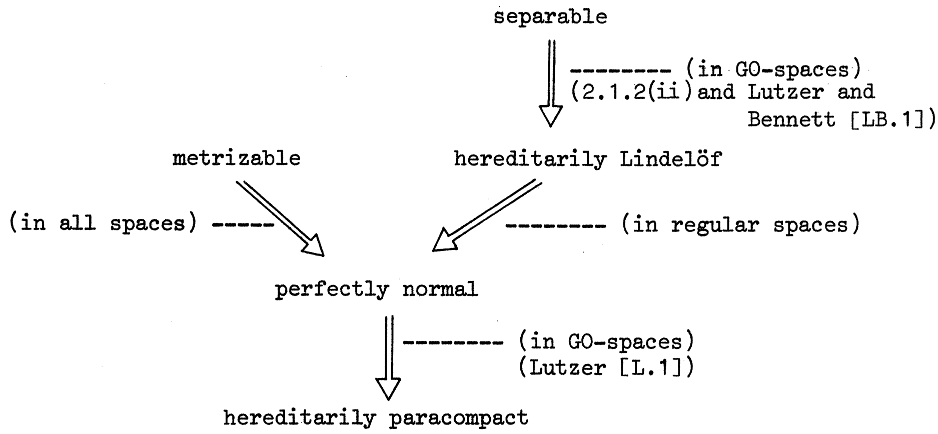
(i) The identity map id: $(X,<,\tau) \to (X,<,\lambda(<))$ is continuous. Since $(X,<,\tau)$ is compact, id is a homeomorphism. Hence $\tau = \lambda(<)$.

(ii) Since $(X,<,\tau)$ is connected, all proper subsets of X of the types $[x,\rightarrow[$ and $]\leftarrow,x]$, $x \in X$, cannot be open subsets of X. Hence $\tau = \lambda(<)$.

(iii) Obvious. □

b) Metrizability, separability, the hereditarily Lindelöf property, perfect normality and hereditary paracompactness.

We first recall that for the indicated types of topological spaces the following implications hold true



Let $X = (X,<,\tau)$ be a GO-space.

We define the following subsets of X

$$E(X) = E((X,<,\tau)) = \{x \in X \mid [x,\rightarrow[\ \in \tau \text{ or }]\leftarrow,x] \in \tau\}$$

$$I(X) = I((X,<,\tau)) = \{x \in X \mid [x,\rightarrow[\ \in \tau \text{ and }]\leftarrow,x] \in \tau\}$$

$$N(X) = N((X,<,\tau)) = \{x \in E(X) \setminus I(X) \mid \exists y \in E(X) \setminus I(X) : x \text{ and } y \text{ are neighbours in } X\}$$

It is clear that $I(X)$ is the set of all isolated points of X, and $N(X)$ is the set of all non-isolated neighbourpoints of X which do have a non-isolated neighbour in X. When, furthermore $B(X)$ denotes the set of all neighbourpoints of X, then $N(X) \subset B(X) \subset E(X)$. Moreover, in the case $\tau = \lambda(<)$, $N(X) \cup I(X) \subset B(X) \subset E(X)$ while $E(X) = B(X)$ plus possible endpoints of X. Now, let N be an arbitrary subset of $N(X)$. Then we define an equivalence relation $\sim$ (relative to N) by:

$$x \sim y \iff (x=y) \text{ or } (x,y \in N \text{ and } x \text{ and } y \text{ are neighbours in } X).$$

Let $X/\sim$ denote the corresponding quotientspace, and let

$$\mathbb{P} : X \longrightarrow X/\sim$$

be the canonical (quotient) mapping.
Obviously the set $X / \sim$ is linearly ordered in a natural way (induced by the ordering on X), and the quotient topology on $X / \sim$ is a GO-topology with respect to this ordering.

We start by proving some lemmata.

LEMMA 1. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

$$\lambda(<) \textit{ is a } C_I\textit{-topology} \Longrightarrow \tau \textit{ is a } C_I\textit{-topology}.$$

PROOF. Obvious. □

LEMMA 2. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

$$\mathbb{P} : X \longrightarrow X / \sim$$

is a perfect map.

PROOF. Since $\mathbb{P}$ is a continuous and finite-to-one mapping, we only have to show that $\mathbb{P}$ is a closed map. Let F be a closed subset of X and suppose that $p \notin \mathbb{P}[F]$. Then $\mathbb{P}^{-1}[\{p\}]$ is a convex subset of X which satisfies: $|\mathbb{P}^{-1}[\{p\}]| \leq 2$ and $\mathbb{P}^{-1}[\{p\}] \cap F = \emptyset$. Hence we can find a convex open set O in X such that $\mathbb{P}^{-1}[\{p\}] \subset O$ and $O \cap F = \emptyset$. (Observe that, in the case $|\mathbb{P}^{-1}[\{p\}]| = 2$, no point of $\mathbb{P}^{-1}[\{p\}]$ can be an endpoint of O). Now $p \in \operatorname{Int} \mathbb{P}[O]$ and $\operatorname{Int} \mathbb{P}[O] \cap \mathbb{P}[F] = \emptyset$. Therefore $\mathbb{P}[F]$ is a closed subset of $X / \sim$. □

LEMMA 3. *Let* $X = (X,<,\tau)$ *be a GO-space.*

(i) *If* $X / \sim$ *is a hereditarily Lindelöf space, so also is* X.

(ii) *If* $X / \sim$ *is a perfectly normal space, so also is* X.

(iii) *If* $X / \sim$ *is a hereditarily paracompact space, so also is* X.

PROOF. Let O be an arbitrary open subset of X. Let

$$\mathcal{C} = \{C \mid C \text{ is a convexity-component of } O\}.$$

Now, every $C \in \mathcal{C}$ is an open subset of X and $O = \bigcup \mathcal{C}$. Moreover,

$$\mathbb{P} \mid C : C \longrightarrow \mathbb{P}[C]$$

is a perfect map. (lemma 2). It is well-known that the Lindelöf property and paracompactness are preserved under inverse images of perfect mappings (cf. Dugundji [Du.1]).

(i) It is sufficient to show that O is a Lindelöf-space. Since $\mathbb{P} \mid C$ is a perfect map, every $C \in \mathcal{C}$ is a Lindelöf-space. Further $\text{Int}\, \mathbb{P}[C] \neq \emptyset$ for each (non-empty) $C \in \mathcal{C}$. Moreover $\text{Int}\, \mathbb{P}[C] \cap \text{Int}\, \mathbb{P}[C'] = \emptyset$ whenever $C, C' \in \mathcal{C}$ and $C \neq C'$. So, since $X / \sim$ is hereditarily Lindelöf,

$$\{\text{Int}\, \mathbb{P}[C] \mid C \in \mathcal{C}\}$$

has to be a countable collection. Consequently $\mathcal{C}$ is a countable family consisting of, mutually disjoint, Lindelöf-spaces. Thus O is a Lindelöf-space.

(ii) We have to prove that O is an F_σ-set in X. From $X / \sim$ is perfectly normal it follows that

$$\text{Int}\, \mathbb{P}[O] = \bigcup \{F_n \mid F_n \text{ is a closed subset of } X / \sim;\ n \in \mathbb{N}\}.$$

For all non-empty sets $\mathbb{P}^{-1}[F_n] \cap C$, we define the set $G_n(C)$ to be the union of $\mathbb{P}^{-1}[F_n] \cap C$ with the possible endpoints of C; $(C \in \mathcal{C},\ n \in \mathbb{N})$. Next, we put

$$G_n = \bigcup \{G_n(C) \mid C \in \mathcal{C}\}.$$

Then, for each $n \in \mathbb{N}$, G_n is a closed subset of X. Moreover, since $\text{Int}\, \mathbb{P}[C] \neq \emptyset$ whenever $C \neq \emptyset$,

$$O = \bigcup \{G_n \mid n \in \mathbb{N}\}.$$

(iii) It is sufficient to show that O is paracompact. But this follows, since $\mathbb{P} \mid C$ is a perfect map, from the fact that every $C \in \mathcal{C}$ is a paracompact space; and thus $O = \bigcup \mathcal{C}$ is paracompact. □

The next theorem also follows easily from theorem 3.1 of chapter III. It is mentioned here (with an independent proof) for systematic reasons.

THEOREM 2.3.2. *Let* $(X,<,\lambda(<))$ *be a metrizable LOTS. Let* τ *be a GO-topology on* $(X,<)$. *Then*

$(X,<,\tau)$ *is metrizable* $\Leftrightarrow$

$E = \{x \in E((X,<,\tau)) \mid [x,\rightarrow[\in \tau \setminus \lambda(<)$ *or* $]\leftarrow,x] \in \tau \setminus \lambda(<)\}$

is σ*-discrete (in* $(X,<,\tau)$*).*

PROOF. Put

$$E(1) = \{x \in E((X,<,\tau)) \mid [x,\rightarrow[\in \tau \setminus \lambda(<)\}$$

and

$$E(2) = \{x \in E((X,<,\tau)) \mid]\leftarrow,x] \in \tau \setminus \lambda(<)\}.$$

It is clear that $E = E(1) \cup E(2)$.

$\Longrightarrow$ Let $\mathcal{B}$ be a σ-discrete open base for $(X,<,\tau)$. For each $x \in E$ we choose an element $B(x) \in \mathcal{B}$ such that

if $x \in E(1)$, then $x \in B(x) \subset [x,\rightarrow[$

if $x \in E(2)$, then $x \in B(x) \subset]\leftarrow,x]$.

(If $x \in E(1) \cap E(2)$ this means that we choose $B(x) = \{x\}$).
Now $B(x) \neq B(y)$ whenever x and y belong to the same $E(i)$, $i = 1,2$, and $x \neq y$. Since $\mathcal{B}$ is a σ-discrete collection in $(X,<,\tau)$ it follows that each

$$\mathcal{B}(i) = \{B(x) \mid x \in E(i)\} \qquad (i = 1,2)$$

is also a σ-discrete family in $(X,<,\tau)$. Consequently, each set $E(i)$, $i = 1,2$, is σ-discrete (in $(X,<,\tau)$). Thus, E is σ-discrete (in $(X,<,\tau)$).

$\Longleftarrow$ Each set $E(i)$, $i = 1,2$, is σ-discrete (in $(X,<,\tau)$). So,

$$E(i) = \bigcup \{E(i)_n \mid E(i)_n \text{ is discrete (in } (X,<,\tau)); n \in \mathbb{N}\}.$$

Furthermore, by lemma 1, τ is a C_I-topology. Now, for each non-isolated (in τ) point $x \in E(i)_n$, we select a monotone sequence $\{x_k\}_{k=1}^{\infty}$ in $(X,<,\tau)$

such that

$$\lim_{k\to\infty} x_k = x$$

if $i = 1$, then $x < x_k$ and $[x,x_k[\cap [y,y_k[= \emptyset$ for all non-isolated $y \in E(1)_n$ such that $x \neq y$; and

if $i = 2$, then $x_k < x$ and $]x_k,x] \cap]y_k,y] = \emptyset$ for all non-isolated $y \in E(2)_n$ such that $x \neq y$.

Next, for all integers $n,k \in \mathbb{N}$, put

$$F(1)_{nk} = \{[x,x_k[\mid x \in E(1)_n \text{ and } x \text{ is non-isolated (in } \tau)\}$$

and

$$F(2)_{nk} = \{]x_k,x] \mid x \in E(2)_n \text{ and } x \text{ is non-isolated (in } \tau)\}.$$

Then $F(i)_{nk}$, $i = 1,2$, is a discrete family in $(X,<,\tau)$ consisting of open sets. Further, let

$$I = \{\{x\} \mid x \text{ is isolated (in } \tau); x \in E\}.$$

If, finally, B is a σ-discrete open base for $(X,<,\lambda(<))$, then

$$\cup \{F(i)_{nk} \mid i = 1,2;\ n,k \in \mathbb{N}\} \cup I \cup B$$

constitutes a σ-discrete open base for $(X,<,\tau)$. This completes the proof. □

REMARK. If $(X,<,\tau)$ is metrizable, then it can be shown, completely analogous to the proof of 2.3.2 ⇒, that $E((X,<,\tau))$ is σ-discrete (in $(X,<,\tau)$). When, moreover, $(X,<,\tau)$ is separable, then it follows, from the corollary to 2.2.5, that $E((X,<,\tau))$ is a countable subset of $(X,<,\tau)$.

For the next theorem we notice that in GO-spaces separability is a hereditary notion (2.1.2 (ii) and Lutzer and Bennett [LB.1]).

THEOREM 2.3.3. *Let* $(X,<,\lambda(<))$ *be a separable LOTS. Let* τ *be a GO-topology on* $(X,<)$. *Then*

$(X,<,\tau)$ *is separable* $\Leftrightarrow$

$I = \{x \in I((X,<,\tau)) \mid [x,\rightarrow[\in \tau \setminus \lambda(<)$ *or* $]\leftarrow,x] \in \tau \setminus \lambda(<)\}$

is a countable subset of $(X,<,\tau)$.

PROOF.

$\Longrightarrow$ This follows immediately from the above-mentioned fact, that a separable GO-space is hereditarily separable.

$\Longleftarrow$ If D is a countable dense subset of $(X,<,\lambda(<))$, then $D \cup I$ is a countable dense subset of $(X,<,\tau)$. Indeed, τ has a base $\mathcal{B}$ consisting of convex sets. Each $B \in \mathcal{B}$ such that $|B| \geq 3$, contains a non-empty $\lambda(<)$-open-interval and hence a point of D and, clearly, each $B \in \mathcal{B}$ such that $|B| \leq 2$ consists of points from $D \cup I$. □

THEOREM 2.3.4. *Let* $(X,<,\lambda(<))$ *be a hereditarily Lindelöf LOTS. Let* τ *be a GO-topology on* $(X,<)$. *Then*

$(X,<,\tau)$ *is hereditarily Lindelöf* $\Leftrightarrow$

$I = \{x \in I((X,<,\tau)) \mid [x,\rightarrow[\in \tau \setminus \lambda(<)$ *or* $]\leftarrow,x] \in \tau \setminus \lambda(<)\}$

is a countable subset of $(X,<,\tau)$.

PROOF.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ Let τ' be the GO-topology on $(X,<)$ which is obtained from τ by removing all sets with left (right) endpoint x for which $x \in I$ and $[x,\rightarrow[\notin \lambda(<)$ ($]\leftarrow,x] \notin \lambda(<)$). So, τ is constructed from τ' by declaring open at most countably many points of X. Now, we consider the GO-space $(X,<,\tau')$ as a (dense) subspace of the LOTS X^{**} (2.1.2 (ii)). Let $N = N(X^{**})\setminus N((X,<,\lambda(<)))$. From the definition of τ' it follows that $X^{**}/\sim$, where $\sim$ is defined (relative to N), is homeomorphic to $(X,<,\lambda(<))$. Therefore, by lemma 3, X^{**} and consequently $(X,<,\tau')$ are hereditarily Lindelöf spaces. But then $(X,<,\tau)$ is a hereditarily Lindelöf space. □

THEOREM 2.3.5. *Let* $(X,<,\lambda(<))$ *be a perfectly normal LOTS. Let* τ *be a GO-topology on* $(X,<)$. *Then*

$(X,<,\tau)$ is perfectly normal $\Leftrightarrow$

$I = \{x \in I((X,<,\tau)) \mid [x,\rightarrow[\ \in \tau \setminus \lambda(<)$ *or* $]\leftarrow,x] \in \tau \setminus \lambda(<)\}$

is σ-discrete (in $(X,<,\tau)$).

PROOF.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ Define τ' as in the proof of 2.3.4. Then τ is obtained from τ' by declaring open the points of a subset of X which is σ-discrete (in $(X,<,\tau)$). As in the proof of 2.3.4 it can be shown that $(X,<,\tau')$ is perfectly normal. But then it is easily verified that also $(X,<,\tau)$ is perfectly normal. □

THEOREM 2.3.6. *Let $(X,<,\lambda(<))$ be a hereditarily paracompact LOTS. Let τ be a GO-topology on $(X,<)$. Then*

$(X,<,\tau)$ is hereditarily paracompact.

PROOF. Again analogous to the proof of 2.3.4. (Here we use the fact that declaring open the points of any subset does not influence hereditary paracompactness of a topological space). □

REMARK. Lutzer [L.1] proved in a direct way that, for a linearly ordered set $(X,<)$, the following properties are equivalent

1. $(X,<,\lambda(<))$ is hereditarily paracompact.
2. For each GO-topology τ on $(X,<)$, $(X,<,\tau)$ is hereditarily paracompact.
3. For each GO-topology τ on $(X,<)$, $(X,<,\tau)$ is paracompact.

At the end of this section we apply the above results to two (well-known) examples.

EXAMPLE 1. The Sorgenfrey-line S.

Let τ be the topology on $\mathbb{R}$ generated by the base, consisting of all sets of type $[a,b[$, $a,b \in \mathbb{R}$. The GO-space $S = (\mathbb{R},<,\tau)$ is called the Sorgenfrey-line (Sorgenfrey [S.1]). Now

$$E = \{x \in E(S) \mid [x,\rightarrow[\ \in \tau \setminus \lambda(<) \text{ or }]\leftarrow,x] \in \tau \setminus \lambda(<)\} = S$$

and

$$I = \{x \in I(S) \mid [x,\rightarrow[\; \in \tau \setminus \lambda(<) \text{ or }]\leftarrow,x] \in \tau \setminus \lambda(<)\} = \emptyset.$$

Furthermore $(\mathbb{R},<,\lambda(<))$ is a separable metrizable space. So, by 2.3.3, we have that S is separable and hence hereditarily Lindelöf, perfectly normal and hereditarily paracompact. However, S is not metrizable, since $E = E(S) = S$ is an uncountable subset of a separable space. (compare the remark after 2.3.2).

EXAMPLE 2. The Michael-line M.
Let τ be the topology on $\mathbb{R}$ obtained from $\lambda(<)$ by declaring open each irrational. The GO-space $M = (\mathbb{R},<,\tau)$ is called the Michael-line. (Michael [Mi.1]). Now,

$$E = \{x \in E(M) \mid [x,\rightarrow[\; \in \tau \setminus \lambda(<) \text{ or }]\leftarrow,x] \in \tau \setminus \lambda(<)\} = \mathbb{R} \setminus Q$$

and

$$I = \{x \in I(M) \mid [x,\rightarrow[\; \in \tau \setminus \lambda(<) \text{ or }]\leftarrow,x] \in \tau \setminus \lambda(<)\} = \mathbb{R} \setminus Q.$$

E and I cannot be σ-discrete subsets (in M). For, otherwise, $R \setminus Q$ would be an F_σ-set in $(\mathbb{R},<,\lambda(<))$, which is impossible. Since $(\mathbb{R},<,\lambda(<))$ is a (separable) metrizable space, we conclude, by 2.3.6, that M is hereditarily paracompact, and, by 2.3.5, that M is not perfectly normal and hence not metrizable, separable or hereditarily Lindelöf.

Finally we note that all indicated properties under b. are hereditary notions for GO-spaces. A non-hereditary topological property, like for instance paracompactness, can get lost, already by isolating one point of a (paracompact) LOTS. (Example: isolate the point ω_1 of the LOTS ω_1+1).

2.4. CHARACTERIZATIONS FOR GO-SPACES OF THE PROPERTIES COMPACTNESS, CONNECTEDNESS, THE LINDELÖF PROPERTY, THE HEREDITARILY LINDELÖF PROPERTY, PERFECT NORMALITY, PARACOMPACTNESS AND HEREDITARY PARACOMPACTNESS

In this section we will establish characterizations of various properties of GO-spaces in terms of the orderstructure and in terms of (σ-)discreteness

of (specific) subsets.
We want to emphasize that, in our investigations of GO-spaces, we deal with a given ordering and, consequently, our characterizations of compactness etc. are in terms of that fixed ordering. Questions like for instance whether or not a topological space is orderable, or whether a GO-space is a LOTS (with a different ordering) are not considered.

In the next chapter we will give special attention to characterizations of metrizability in GO-spaces.

Let $X = (X,<,\tau)$ be a GO-space.

A *gap* in X is an ordered pair (A,B) of subsets of X such that
(i) $X = A \cup B$, (ii) $a < b$ for all $a \in A$ and $b \in B$, (iii) A has no right endpoint and B has no left endpoint, (iv) $A,B \in \tau$.
Clearly, (i) + (ii) + (iii) $\Longrightarrow$ (iv).
If one of the sets A or B is empty, then we will speak of an *endgap*. In particular, we say that (A,B) is a *left (right) endgap* if $A(B) = \emptyset$.

A *jump* in X is an ordered pair (A,B) of subsets of X such that
(i) $X = A \cup B$, (ii) $a < b$ for all $a \in A$ and $b \in B$, (iii) A has a right endpoint and B has a left endpoint, (iv) $A,B \in \tau$.
Clearly, (i) + (ii) + (iii) $\Longrightarrow$ (iv).
For each jump (A,B) in X, the right endpoint p of A and the left endpoint q of B are neighbours in X and, conversely, each pair of neighbours (p,q), $p < q$, defines a jump (A,B) in X, where $A = \{x \in X \mid x \leq p\}$ and $B = \{x \in X \mid q \leq x\}$.

A *pseudo-gap* or a *pseudo-jump* in X is an ordered pair (A,B) of subsets of X such that (i) $X = A \cup B$, (ii) $a < b$ for all $a \in A$ and $b \in B$, (iii) (A has no right endpoint and B has a left endpoint) or (A has a right endpoint and B has no left endpoint), (iv) $A,B \in \tau$.
A *left-(right-)pseudo-gap* or a *right-(left-)pseudo-jump* is a pseudo-gap (A,B) such that A(B) has no right (left) endpoint. A *pseudo-endgap* is a pseudo-gap (A,B) such that either $A = \emptyset$ or $B = \emptyset$. (This latter term will be used just occasionally for streamlining the terminology. Of course, a pseudo-endgap determines just an endpoint).

a. Compactness.

The following theorem is well-known.

THEOREM 2.4.1. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

X *is compact* $\Longleftrightarrow$ X *has neither gaps nor pseudo-gaps, except for two pseudo-endgaps.*

PROOF.

$\Longrightarrow$ By 2.3.1, $\tau = \lambda(<)$. So X has no pseudo-gaps except for possible pseudo-endgaps. Now, suppose (A,B) is a gap in X such that $A \neq \emptyset$. Choose a strictly increasing sequence, say $\{x_\alpha\}_{\alpha<\mu}$, (where μ is an ordinal number), in A, which is cofinal in A. Clearly, μ is a limit number. However, now $\{]\leftarrow,x_\alpha[\}_{\alpha<\mu}$ is an open cover of the closed set A without a finite subcover. Contradiction. In particular, it follows also that X has no endgaps, and hence X has two endpoints.

$\Longleftarrow$ Since X has no pseudo-gaps except for two pseudo-endgaps it follows that $\tau = \lambda(<)$ and, moreover, that X has both a left and a right endpoint. Furthermore, if P is a subset of X which is bounded above and

$$A = \{x \in X \mid \exists p \in P : x \leq p\}$$

then $(A,X \setminus A)$ is neither a gap nor a pseudo-gap in X. Consequently, P has a supremum in X. Thus, see for instance Kelley [K.1], X is compact. □

b. Connectedness.

The following theorem is well-known.

THEOREM 2.4.2. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

X *is connected* $\Longleftrightarrow$ X *has neither jumps, nor gaps, nor pseudo-gaps, except for possible endgaps and pseudo-endgaps.*

PROOF.

$\Longrightarrow$ By 2.3.1, $\tau = \lambda(<)$. So X has no pseudo-gaps except for possible pseudo-endgaps. Since a gap (A,B) or a jump (A,B) in X consists of τ-open sets A and B, it follows that X has neither gaps nor jumps, except for possible endgaps.

$\Longleftarrow$ Since X has no pseudo-gaps, $\tau = \lambda(<)$. Furthermore, analogous to the proof of 2.4.1, each subset of X which is bounded above has a supremum in

X. Hence, since moreover X has no jumps, X is connected (cf. Kelley [K.1]). □

c. The Lindelöf-property.

If $\mathcal{O}$ is an open cover of a GO-space X, consisting of convex sets, then we say that *a gap* (A,B) *in* X *is covered by* $\mathcal{O}$ whenever there exists an element $O \in \mathcal{O}$ such that one of the following conditions is satisfied

1. $O \cap A \neq \emptyset$ and $O \cap B \neq \emptyset$.
2. $B = \emptyset$ and O is cofinal in A.
3. $A = \emptyset$ and O is coïnitial in B.

THEOREM 2.4.3. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

> X *is a Lindelöf-space if and only if the following two properties hold*

(i) *for each gap and each pseudo-gap* (A,B) *in* X, *there exist (countable, see* (ii)*) discrete subsets* $L \subset A$ *and* $R \subset B$ *such that* L *is cofinal in* A *and* R *is coïnitial in* B.
(We may assume, of course, that L is well-ordered and R is inversely well-ordered).

(ii) *each discrete subset of* X *contains at most countably many points.*

PROOF.
"only if". Clearly (ii) is satisfied, since the Lindelöf property is hereditary for closed subsets. Next suppose, for instance, that (A,B) is a left-pseudo-gap in X with $A \neq \emptyset$ and such that no countable sequence in A is cofinal in A. Choose a strictly increasing sequence $\{x_\alpha\}_{\alpha<\mu}$, where μ is an ordinal number, in A, which is cofinal in A. Clearly, μ is a limit ordinal and ω_0 is not cofinal with μ. However, now $\{]\leftarrow,x_\alpha[\}_{\alpha<\mu}$ is an open cover of the closed subset A of X without a countable subcover. Contradiction. Hence, there is a sequence $L(\subset A)$ of type ω_0 which is cofinal in A.
"if". First we show that, without loss of generality, we may assume that $\tau = \lambda(<)$. Indeed, X may be considered as a closed subset of X^* (2.1.2 (i)). Now, when (A^*,B^*) is a gap in X^*, then $(A^* \cap X, B^* \cap X)$ is either a gap or a pseudo-gap in X. Hence there are discrete subsets $L \subset A^* \cap X$ and $R \subset B^* \cap X$ which are, respectively, cofinal in $A^* \cap X$ and coïnitial in $B^* \cap X$. Next,

define

$$L^* = \{(x,n) \in X^* \mid x = (x,0) \in L\}$$

and

$$R^* = \{(x,n) \in X^* \mid x = (x,0) \in R\}.$$

Then $L^* \subset A^*$ and $R^* \subset B^*$ are discrete subsets of X^* which are, respectively, cofinal in A^* and coinitial in B^*. Furthermore, if P^* is a discrete subset of X^*, then

$$\{x = (x,0) \in X \subset X^* \mid \exists n \in \mathbb{Z} : (x,n) \in P^*\}$$

is a discrete subset of X, which hence is at most countable. But then also P^* is at most countable. Therefore, X^* satisfies the properties (i) and (ii) whenever X does. Thus, we may assume X to be a LOTS.

Let $\mathcal{O}$ be an open cover of the LOTS X consisting of convex sets. Let F denote the set of all gaps in X that are not covered by $\mathcal{O}$. Let X^+ denote the Dedekind compactification of X. Clearly, every limitpoint in X^+ of non-covered gaps of X is a non-covered gap of X. Hence F is closed in X^+. Now $X^+ \setminus F = \cup\, \mathcal{C}$, where $\mathcal{C}$ is the collection of convexity-components of $X^+ \setminus F$ (in X^+). Then $\mathcal{C} \cap X = \{C \cap X \mid C \in \mathcal{C}\}$ consists of mutually disjoint open and closed convex subsets of X. Furthermore, for each $C \cap X$ all gaps, apart from the endgaps, are covered by $\mathcal{O}$. Now, by (ii), $\mathcal{C} \cap X$ contains at most countably many distinct elements. Further it follows from (i), that each $C \in \mathcal{C}$ can be written as $C = \bigcup_{k=1}^{\infty} I_k(C)$, where $I_k(C)$ is a closed interval in X^+ with a left and a right endpoint both belonging to X. Then, in an obvious way, $\mathcal{O}$ may be considered as an open cover of each $I_k(C)$. [To be precise, taking any $C \in \mathcal{C}$, it is the collection $\mathcal{O}_1$ which covers $I_k(C)$. Here, a typical element of $\mathcal{O}_1$ is a set which one obtains by taking the union of an $O \in \mathcal{O}$ with those points of X^+ which correspond to gaps in X "covered by O"]. Hence, the compact space $I_k(C)$ can be covered by finitely many members of $\mathcal{O}$. Consequently $C \cap X = \bigcup_{k=1}^{\infty} I_k(C) \cap X$ is covered by countably many members of $\mathcal{O}$. Finally, also X, as the union of at most countably many sets $C \cap X$, it covered by a countable subfamily of $\mathcal{O}$. □

d. The hereditarily Lindelöf property.

THEOREM 2.4.4. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

X *is hereditarily Lindelöf* $\Leftrightarrow$ *Each relatively discrete subset of* X *contains at most countably many points.*

PROOF.

$\Longrightarrow$ Obvious.

$\Longleftarrow$ We have to show that each open subset O of X is a Lindelöf-space. Now, $O = \cup \mathcal{C}$ where

$$\mathcal{C} = \{C \subset X \mid C \text{ is a convexity-component of } O\}.$$

Clearly, each $C \in \mathcal{C}$ is open and hence $\mathcal{C}$ consists of at most countably many members. So, it suffices to prove that $C \in \mathcal{C}$ is a Lindelöf-space. Since every relatively discrete subset of X contains at most countably many points, it follows that X^+ (the Dedekind compactification of X) does not contain a transfinite sequence of type ω_1. But then, from 2.4.3, it is easily verified that X is a Lindelöf-space, while, moreover, for each $C \in \mathcal{C}$ there exists an increasing and a decreasing (countable or finite) sequence (in C), which is cofinal, respectively coinitial in C. Hence, $C = \bigcup_{k=1}^{\infty} I_k(C)$ where $I_k(C)$ is a closed convex subset of X. Thus, since closed subsets of a Lindelöf space are again Lindelöf, also C is a Lindelöf space. □

In the case of a LOTS, the next corollary was proved before (in a different way) by Lutzer and Bennett [LB.1].

COROLLARY. *A GO-space is hereditarily Lindelöf if and only if it satisfies the countable chain condition.*

PROOF.

"only if". Obvious.

"if". Let P be a relatively discrete subset of a GO-space X. Since X is hereditarily collectionwise normal (Steen [St.1]), there exists a disjoint collection $\mathcal{O}$ of open sets in X, such that $|O \cap P| \leq 1$ for each $O \in \mathcal{O}$. (Mc.Auley [McA.1]). However, since X satisfies the countable chain condition, $\mathcal{O}$ has to be a countable or finite family. Thus P contains at most countably many points. Hence, by 2.4.4, X is a hereditarily Lindelöf-space. □

e. Perfect normality.

Let S be an index set. Let $\mathcal{U} = \{U_s \mid s \in S\}$ be a family of subsets of a given set X. If J is a convex subset of $\mathbb{Z}$, then the subcollection

$$\mathcal{K} = \{U_{s(j)} \mid s(j) \in S,\ j \in J\}$$

of $\mathcal{U}$ is called a *chain in* $\mathcal{U}$, whenever

$$U_{s(i)} \cap U_{s(j)} \neq \emptyset \Longleftrightarrow |i-j| \leq 1 \qquad (i,j \in J).$$

A *maximal chain in* $\mathcal{U}$ is a chain which is not a proper subcollection of any other chain in $\mathcal{U}$. Obviously, each chain in $\mathcal{U}$ is contained in a maximal chain.

THEOREM 2.4.5. *Let* $X = (X,<,\tau)$ *be a GO-space. Then the following properties are equivalent*

1) X *is perfectly normal.*
2) *Every collection of mutually disjoint convex open subsets of* X *constitutes a σ-discrete family in* X.
3) *Each relatively discrete subset of* X *is σ-discrete (in* X*).*

PROOF.

1 ⟹ 2 (For any family $\mathcal{F}$ of subsets of X, the family $\{\bar{F} \mid F \in \mathcal{F}\}$ will be denoted by $\bar{\mathcal{F}}$). Let $\mathcal{C}$ be a disjoint collection of convex open subsets of X. If $\mathcal{K} \subset \mathcal{C}$ is such that $\bar{\mathcal{K}}$ is a chain in $\bar{\mathcal{C}}$, then we can decompose $\mathcal{K}$ like $\mathcal{K} = \mathcal{K}(1) \cup \mathcal{K}(2)$, (taking the elements of $\mathcal{K}$ in $\mathcal{K}(1)$ or $\mathcal{K}(2)$ alternately), such that $\bar{\mathcal{K}} = \overline{\mathcal{K}(1)} \cup \overline{\mathcal{K}(2)}$ where $\overline{\mathcal{K}(1)}$ and $\overline{\mathcal{K}(2)}$ each consist of mutually disjoint elements. Furthermore, we have

$$\mathcal{C} = \bigcup \{\mathcal{K} \mid \bar{\mathcal{K}} \text{ is a maximal chain in } \bar{\mathcal{C}}\}.$$

Now, two maximal chains in $\bar{\mathcal{C}}$ either coincide or are disjoint. Thus, we may write $\mathcal{C} = \mathcal{C}(1) \cup \mathcal{C}(2)$, where for i = 1,2,

$$C(i) = \cup \{K(i) \mid \bar{K} \text{ is a maximal chain in } \bar{C}\}.$$

Put $O = \cup C$. Since X is perfectly normal

$$O = \cup \{F_k \mid F_k = \bar{F}_k \ ; \ k \in \mathbb{N}\}.$$

Next, for each $k \in \mathbb{N}$, define

$$C_k = \{C \in C \mid C \cap F_k \neq \emptyset\}.$$

Since F_k is closed in X, it follows that C_k is a locally finite family in X. In fact, each point of X has an open neighbourhood which meets at most two members of C_k. (If, for some $p \in X$, each neighbourhood of p intersects more than two elements of C_k, then $p \in \bar{F}_k = F_k \subset O$. Hence $p \in C$ for some $C \in C$. However, C intersects no other element of C). But then

$$C_k(i) = \cup \{K_k(i) \mid \bar{K}_k \text{ is a maximal chain in } \bar{C}_k\},$$

$i = 1,2$, is a discrete family in X. Hence

$$C = \cup \{C_k(i) \mid k \in \mathbb{N} \ ; \ i = 1,2\}$$

constitutes a σ-discrete family in X.

2 $\Longrightarrow$ 3 Let A be a relatively discrete subset of X. Since X is hereditarily collectionwise normal (Steen [St.1]), there exists a disjoint family

$$C = \{C_a \mid a \in A\}$$

of convex open subsets of X, such that $a \in C_a$ for all $a \in A$, (McAuley [McA.1]). Consequently C is a σ-discrete family in X. But then A is a σ-discrete subset of X.

3 $\Longrightarrow$ 1 Let O be an open subset of X. Then $O = \cup C$, where

$$C = \{C \mid C \text{ is a convexity-component of } O\}.$$

Clearly, each $C \in C$ is τ-open. First, we show that each $C \in C$ is an F_σ-set in X. Let, for instance, C be such that $\bar{C} \setminus C = \{p\}$, where $p \in X$ is the right endpoint of $\bar{C}$. (The other cases can be treated similarly). Choose $p_0 \in C$. The ordered set C is cofinal with a well-ordered set $\tilde{P} \subset C$, starting from p_0. By omitting those points which have limit-index it then easily follows that C is cofinal with a strictly increasing subset $P = \{p_\alpha \mid \alpha < \mu\}$, for some ordinal number μ, which is relatively discrete in X. Hence, by assumption, P is σ-discrete (in X). So $P = \bigcup_{n=1}^{\infty} P_n$, where each P_n is discrete (in X). Since $\bar{C} \setminus C = \{p\}$, no P_n can be cofinal in C. Therefore, we may assume that $P_n \subsetneq P_{n+1}$, for all $n \in \mathbb{N}$. Now, pick $q_{n+1} \in P_{n+1} \setminus P_n$ for each $n \in \mathbb{N}$. Then clearly $\lim_{n\to\infty} q_{n+1} = p$. Hence

$$C = \bigcup_{n=1}^{\infty} (C \cap]\leftarrow, q_{n+1}]).$$

Thus, C is an F_σ-set. Secondly, for every $C \in \mathcal{C}$, choose one point $x(C) \in C$. Since

$$\{x(C) \mid C \in \mathcal{C}\}$$

is a relatively discrete subset of X, it is also σ-discrete (in X). Consequently $\mathcal{C}$ is a σ-discrete family in X.
Summarizing we have that $O = \bigcup \mathcal{C}$, $\mathcal{C}$ is a σ-discrete family in X and every $C \in \mathcal{C}$ is an F_σ-set in X. But then O is an F_σ-set in X. □

COROLLARY (van Dalen [D.1]).
A LOTS without gaps, except for possible endgaps, (i.e.: an order-complete LOTS) is perfectly normal if and only if it satisfies the countable chain condition.

PROOF. Let X be a LOTS without gaps, except for possible endgaps.
"if". Any collection of mutually disjoint convex open subsets of X is countable and hence forms a σ-discrete family in X.
"only if". Suppose there exists an uncountable collection $\mathcal{C}$ of mutually disjoint (non-empty) open subsets of X. Without loss of generality $\mathcal{C}$ consists of convex sets. From 2.4.5 it follows that

$$\mathcal{C} = \bigcup \{\mathcal{C}_k \mid k \in \mathbb{N}\}$$

where each $\mathcal{C}_k$ is a discrete family in X. Since $\mathcal{C}$ is supposed to be uncountable, there is an integer $n \in \mathbb{N}$ such that C_n is an uncountable collection. However, then X contains a bounded countable sequence without a limit point. But then X has a gap which is not an endgap. Contradiction. □

REMARK. GO-spaces, without gaps, may be perfectly normal without satisfying the countable chain condition. (Example: the unit interval supplied with the usual ordering and the discrete topology).

Finally, we give two examples. The first one deals with a metrizable LOTS in which one can construct a non-σ-discrete, disjoint family of open sets. Note that, by 2.4.5, such a family necessarily cannot consist of convex sets. The second example gives a non-perfectly-normal LOTS.

EXAMPLE 1. Let Y_1 and Y_2 be LOTS's defined as follows:

$$Y_1 = \{\frac{1}{n} \in \mathbb{R} \mid n \in \mathbb{N}\} \cup \{0\}$$

supplied with the usual order-topology; and

> Y_2 is an uncountable discrete LOTS with a left endpoint ℓ and a right endpoint r.

Now, put

$$X = \{(y_1,y_2) \in Y_1 \cdot Y_2 \mid y_1 = 0 \Rightarrow y_2 = r\}.$$

Then the relative topology on X, induced by the order-topology on $Y_1 \cdot Y_2$, coincides with the order-topology on X, induced by the relative ordering. Hence X is a LOTS. Moreover, it is easily verified that X is metrizable. Next, for each $y_2 \in Y_2$, define

$$O(y_2) = \{(\frac{1}{n},y_2) \in X \mid n \in \mathbb{N}\}.$$

Then $O(y_2)$ is an open subset of X, and

$$\mathcal{O} = \{O(y_2) \mid y_2 \in Y_2\}$$

is a disjoint family. However, it is clear that $\mathcal{O}$ cannot be a σ-discrete family in X.

EXAMPLE 2. Let $X \subset [0,1] \cdot [0,1]$ be defined by

$$X = \{(x,y) \in [0,1] \cdot (Q \cap [0,1[) \mid x \in [0,1] \setminus Q \Longrightarrow y \neq 0\}.$$

Let X be supplied with the order-topology. (Observe that X is a dense subset of the LOTS $[0,1] \cdot [0,1]$ while $[0,1] \cdot [0,1]$ does not possess neighbourpoints. So, by 2.1.5, the order-topology on X coincides with the relative topology). The LOTS X is σ-r-discrete and of the first category. Furthermore, X is not perfectly normal. For, choose $r \in Q \cap]0,1[$ and put

$$P = \{(x,r) \in X \mid x \in [0,1]\}.$$

Then P is a relatively discrete subset of X. Let

$$f : X \longrightarrow [0,1]$$

be defined by $f((x,y)) = x$, for each $(x,y) \in X$. Now, suppose $P = \bigcup_{n=1}^{\infty} P_n$. Then $f[P] = [0,1] = \bigcup_{n=1}^{\infty} f[P_n]$. Since $[0,1]$ is of the second category, there is an integer $n \in \mathbb{N}$, such that $\text{Int}\ \overline{f[P_n]} \neq \emptyset$. Take a point $p \in \text{Int}\ \overline{f[P_n]} \cap Q$ with $p \neq 0$. Then each open neighbourhood of $(p,0) \in X$ contains elements of P_n distinct from $(p,0)$. Consequently P_n is not discrete (in X). Thus, X is not perfectly normal.

f. Paracompactness.

The following lemma is well-known.

LEMMA. *Let* X *be a LOTS. Then the following properties are equivalent*

1) X *is paracompact.*
2) *For each gap and each pseudo-gap* (A,B) *in* X, *there exist discrete subsets* $L \subset A$ *and* $R \subset B$ *which are, respectively, cofinal in* A *and coinitial in* B.

PROOF. See Gillman and Henriksen [GH.1]. □

THEOREM 2.4.6. *Let* $X = (X,<,\tau)$ *be a GO-space. Then the following properties are equivalent*

1) X *is paracompact.*

2) *For each gap and each pseudo-gap* (A,B) *in* X, *there exist discrete subsets* $L \subset A$ *and* $R \subset B$ *which are, respectively, cofinal in* A *and coinitial in* B.

(For a similar result, see also Lutzer [L.2]).

PROOF.

$1 \Longrightarrow 2$ Suppose (A,B) is a left-pseudo-gap in X. (The other cases can be treated similarly). Clearly, the left endpoint of B is a discrete subset of X which is coinitial in B. So, it remains to show that A contains a discrete and cofinal subset L. Now X is a closed subset of X^* (2.1.2 (i)). Put

$$A^* = \{(x,n) \in X^* \mid (x,0) = x \in A\}$$

and

$$B^* = \{(x,n) \in X^* \mid (x,0) = x \in B\}.$$

Then (A^*,B^*) is a gap in X^*. Since A is cofinal in A^*, it suffices to prove that A^* contains a discrete (in X^*) and cofinal (in A^*) subset L^*. [Indeed, afterwards an interlacing argument (taking L^* well-ordered) yields that also A contains a (well-ordered) discrete (in X) and cofinal (in A) subset L]. But this means, that the proof is done once we have shown that X^* is a paracompact space. So, let $\mathcal{O}$ be an open cover of X^*. Then $\mathcal{O}$ is also an open cover of X. Hence there exists an open refinement $\mathcal{V}$ of $\mathcal{O}$ such that

$$\{V \cap X \mid V \in \mathcal{V}\}$$

is a locally finite family in X. Now, for each $V \in \mathcal{V}$, let $\mathcal{C}(V)$ be the collection of all convexity-components of V (in X^*). Next, put

$$V' = V \setminus \{(x,n) \in X^* \setminus X \mid \exists C \in \mathcal{C}(V) : (x,0) \notin C \text{ and } (x,n) \in C\}$$

and, let

$$\mathcal{V}' = \{V' \mid V \in \mathcal{V}\}.$$

Then $\mathcal{V}'$ is a locally finite open family in $\cup \mathcal{V}'$. Finally, put

$$\mathcal{U} = \mathcal{V}' \cup \{\{(x,n)\} \mid (x,n) \notin \cup \mathcal{V}'\}.$$

Then $\mathcal{U}$ is a locally finite open refinement of $\mathcal{O}$. Thus X^* is paracompact.
$2 \Longrightarrow 1$ Since X^* satisfies condition 2 whenever X does (compare the proof of 2.4.3), it follows from lemma 1, that X^* is a paracompact space. But X is a closed subset of X^*. So, also X is paracompact. □

REMARK. During the proof of the previous theorem it was shown that a GO-space X is paracompact if and only if the corresponding LOTS X^* is a paracompact space (see also Lutzer [L.1]). In general, a similar assertion does not hold with respect to the LOTS X^{**}, (2.1.2 (ii)). For instance, let

$$X = (\omega_1+2) \setminus \{\omega_1\}$$

supplied with the relative ordering and the relative topology of the LOTS $\omega_1 + 2$. The GO-space X cannot be a paracompact space, since it contains ω_1 as a closed subset (note that ω_1 does not satisfy property 2. of 2.4.6). However, $X^{**} = \omega_1 + 2$ is a compact, and hence paracompact LOTS. On the other hand, the subset

$$X = \omega_1 \setminus \{\lambda \mid \lambda \text{ is a limit ordinal} < \omega_1\}$$

of ω_1, supplied with the relative ordering and the relative topology of the LOTS ω_1, is a relatively discrete subspace of ω_1, and hence it is paracompact. But $X^{**} = \omega_1$ is a non-paracompact space.

g. Hereditary **paracompactness**.

THEOREM 2.4.7. *Let* $X = (X,<,\tau)$ *be a GO-space. Then*

X *is hereditarily paracompact* $\Longleftrightarrow$ $X \setminus \{x\}$ *is paracompact, for every point* $x \in X$.

PROOF.
$\Longrightarrow$ Obvious.
$\Longleftarrow$ Let Y be a subspace of X. Suppose that (A,B) is a gap in Y. (The other

cases follow similarly). Put

$$P = \{x \in X \mid \exists a \in A : x < a\}.$$

We distinguish three cases. First, suppose $\bar{P} \setminus P \neq \emptyset$. Then, there exists a point $z \in X \setminus Y$ such that $\bar{P} \setminus P = \{z\}$. Now, $(P, X \setminus \bar{P})$ is a gap or a left-pseudo-gap in $X \setminus \{z\}$. Secondly, suppose $P = \bar{P}$ and, moreover, $X \setminus P \neq \emptyset$. Choose $z \in X \setminus P$. Then $(P, X \setminus P \setminus \{z\})$ is a gap or a left-pseudo-gap in $X \setminus \{z\}$. Thirdly, suppose $P = X$. Choose $z \in P$. Now, $(P \setminus \{z\}, \emptyset)$ is a gap in $X \setminus \{z\}$. Hence, in all three cases, there is a discrete (in $X \setminus \{z\}$) subset $L' \subset P$ which is cofinal in P. Moreover, in the third case we may assume that $z < x$ for all $x \in L'$. Since A is cofinal in P, an interlacing argument now yields that A contains a discrete (in $X \setminus \{z\}$ and consequently also in Y) subset L which is cofinal in A. Similarly, it can be shown that B contains a discrete (in Y) subset R which is coinitial in B. □

COROLLARY 1. (Lutzer [L.1])
A perfectly normal GO-space is hereditarily paracompact.

PROOF. Let $X = (X,<,\tau)$ be a GO-space. Since perfect normality is a hereditary property, it suffices to show that X is paracompact. Now, let (A,B) be a left-pseudo-gap in X. (The other cases can be treated analogously). Put

$$\mathcal{L} = \{L \subset A \mid L \text{ is discrete (in } X)\}.$$

If none of the elements $L \in \mathcal{L}$ is cofinal in A, then also no countable union of members of $\mathcal{L}$ can be cofinal in A either. However, this contradicts the existence of a relatively discrete subset in A which is cofinal in A. □

COROLLARY 2.

(i) *A GO-space is hereditarily paracompact if it is a paracompact* C_I*-space.*

(ii) *A GO-space* $X = (X,<,\tau)$ *is hereditarily paracompact, if the LOTS* $(X,<,\lambda(<))$ *is a paracompact* C_I*-space.*

PROOF. (i) follows from 2.4.6 and 2.4.7. And (ii) follows from (i) and 2.3.6. □

REMARK. 1. It is well-known that the lexicographically ordered unit-square [0,1] · [0,1], supplied with the order-topology, is hereditarily paracompact but not perfectly normal. These facts, of course, follow immediately from corollary 2 (i) and the corollary to 2.4.5.

2. A hereditarily paracompact LOTS may fail to be a C_1-space. This can be illustrated by example 2 of section 2.2.

CHAPTER III

METRIZABILITY IN GO-SPACES

Classical theorems of Bing [B.1] state that in a regular T_1-space R the following properties are equivalent: (1) R is metrizable; (2) R has a σ-discrete open base; (3) R is a collectionwise normal Moore-space. Using Bing's results we shall prove that in a GO-space metrizability is completely characterized by the statement that there exists a dense, σ-discrete subset containing all points by which a jump or a pseudo-jump is determined, (so, in particular, all neighbourpoints).

Furthermore, we observe that a LOTS X is a Moore-space whenever there exists a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers of X, such that for every point $p \in X$, the family $\{St(p;\mathcal{U}_n)\}_{n=1}^{\infty}$ of stars constitutes a *local pseudo-base at* p; i.e. $\bigcap_{n=1}^{\infty} St(p;\mathcal{U}_n) = \{p\}$. In general, as can be illustrated by the Sorgenfrey-line, this does not hold for arbitrary GO-spaces. However, we shall prove that a GO-space X is metrizable (or, equivalently, a Moore-space) if there exists a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers of X, such that for each point $p \in X$, the stars $\{St(p;\mathcal{U}_n)\}_{n=1}^{\infty}$ form a local base at p, except for a σ-discrete subset of X where is it only required that the stars form a local pseudo-base.

Finally, we prove that every metric GO-space has a σ-discrete base consisting of convex open sets, but that, in general, even a metric LOTS does not have a σ-discrete base consisting of open intervals.

The chapters IV and V are mainly used to illustrate the applicability of the results of this chapter.

First, we recall some definitions.

If $\mathcal{U}$ is an open cover of a topological space R, and $p \in R$, then the *star of* p *relative to* $\mathcal{U}$ is defined by

$$St(p;\mathcal{U}) = \cup \{U \in \mathcal{U} \mid p \in U\}.$$

A *Moore-space* is a regular topological space R, which possesses a countable sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers such that, for every $p \in R$, the family $\{St(p;\mathcal{U}_n)\}_{n=1}^{\infty}$ constitutes a local base at p.

Finally, we recall that for a GO-space $X = (X,<,\tau)$ we have introduced the following subsets (see 2.3)

$$E(X) = E((X,<,\tau)) = \{x \in X \mid [x,\rightarrow[\in \tau \text{ or }]\leftarrow,x] \in \tau\}$$

$$I(X) = I((X,<,\tau)) = \{x \in X \mid [x,\rightarrow[\in \tau \text{ and }]\leftarrow,x] \in \tau\}$$

$$N(X) = N((X,<,\tau)) = \{x \in E(X) \setminus I(X) \mid \exists y \in E(X) \setminus I(X) : x \text{ and } y \text{ are neighbours in } X\}.$$

THEOREM 3.1. *Let* $X = (X,<,\tau)$ *be a GO-space. Then the following properties are equivalent*

1) X *is metrizable*
2) *There exists a subset* $D \subset X$ *such that:*
   (i) $\bar{D} = X$; (ii) $E(X) \subset D$; (iii) D *is* σ*-discrete (in* X*).*

If $\tau = \lambda(<)$, *then these properties are also equivalent to*

3) *There exists a subset* $D \subset X$ *such that:*
   (i) $\bar{D} = X$; (ii) $N(X) \subset D$; (iii) D *is* σ*-discrete (in* X*).*

PROOF.
$1 \Longrightarrow 2$. X is metrizable; therefore X has a σ-discrete open base $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ where each $\mathcal{B}_n$ is a discrete family in X. Take $n \in \mathbb{N}$. From each $B \in \mathcal{B}_n$ we select one or two points as follows: if B contains one or two endpoints then we choose that, respectively, those endpoint(s) from B, and, if B contains neither a left nor a right endpoint, then we choose one arbitrary point from B. Let D_n be the set of points, thus selected from all $B \in \mathcal{B}_n$. Put $D = \bigcup_{n=1}^{\infty} D_n$. Then, it is easily verified, that D satisfies all properties required.

Now we assume that $\tau = \lambda(<)$; so, X is a LOTS. Then

$2 \Longrightarrow 3$. Obvious. And

$3 \Longrightarrow 1$. Since a GO-space is hereditarily collectionwise normal (Steen [St.1]), it suffices to show that X is a Moore-space. Let $D = \bigcup_{n=1}^{\infty} D_n$, where each D_n is discrete (in X). Then, for all $n \in \mathbb{N}$ and $x \in X$ there is a convex open neighbourhood $I(x;n)$ of x, such that $I(x;n) \cap (D_n \setminus \{x\}) = \emptyset$. Without loss of generality, $I(x;n)$ may be supposed to satisfy the following conditions: $I(x;n) \supset I(x;n+1)$ for all $n \in \mathbb{N}$; $I(x;n) = \{x\}$ if x is isolated in X; $I(x;n) \subset [x,\rightarrow[$ if x is a (non-isolated) point of X having a left neighbour in X; and $I(x;n) \subset]\leftarrow,x]$ if x is a (non-isolated) point of X having a right neighbour in X.

Now, for every $n \in \mathbb{N}$, put

$$\mathcal{U}_n = \{I(x;n) \mid x \in X\}.$$

Clearly, each $\mathcal{U}_n$ is an open cover of X and hence the proof is complete once we have shown that, for every $x \in X$, the family $\{St(x;\mathcal{U}_n)\}_{n=1}^{\infty}$ forms a local base at x. For that purpose, choose $x \in X$ and let 0 be an arbitrary convex open neighbourhood of x.

a) Suppose, x is an isolated point of X.
Since $\bar{D} = X$, there is an integer $i \in \mathbb{N}$ such that $x \in D_i$. We claim that $St(x;\mathcal{U}_i) \subset 0$. For, if $x \in I(y;i)$ for some $y \in X$, then from $I(y;i) \cap (D_i \setminus \{y\}) = \emptyset$ and $x \in I(y;i) \cap D_i$ it follows that $x = y$. Hence $x \in St(x;\mathcal{U}_i) = I(x;i) = \{x\} \subset 0$.

b) Suppose, x has a left neighbour $x^- \in X$, and suppose that x is not isolated. Certainly $x^- \in D$ and so $x^- \in D_i$, for some $i \in \mathbb{N}$. Since x is a non-isolated point of X, there are integers $j,k \in \mathbb{N}$ and points $a \in D_j \cap 0$ and $b \in D_k \cap 0$ such that $i < j < k$ and $x < b < a$. We claim that $St(x;\mathcal{U}_k) \subset 0$. Indeed, if $x \in I(y;k)$ for some $y \in X$, then from $I(y;k) \cap (D_k \setminus \{y\}) = \emptyset$ and $b \in D_k$ it follows that $y \le b(<a)$. Further since $I(y;i) \cap (D_i \setminus \{y\}) = \emptyset$, $x^- \in D_i$ and $x \in (I(y;k) \subset)I(y;i)$ we have that $x^- \le y$, while, moreover, since $x \notin I(x^-;k)$, even $x^- < y$. Now, from $I(y;j) \cap (D_j \setminus \{y\}) = \emptyset$ and $y < a$ it follows that $a \notin I(y;j)$, and from $I(y;i) \cap (D_i \setminus \{y\}) = \emptyset$ and $x^- < y$ it follows that $x^- \notin I(y;i)$. Thus, since $I(y;k)$ is a convex subset, both of $I(y;j)$ and of $I(y;i)$, and since $x \in I(y;k)$ while $x^- < x < a$, it follows that $x \in I(y;k) \subset 0$. Hence $x \in St(x;\mathcal{U}_k) \subset 0$.
The case, where x has a right neighbour $x^+ \in X$ while, moreover, x is not isolated, can be treated similarly.

c) Suppose x is an endpoint of X, which is not isolated.
The proof of this case is an easy modification of the proof of case b).

d) Suppose x has neither a left nor a right neighbour in X.
We consider the LOTS's $[x,\rightarrow[$ and $]\leftarrow,x]$. From the previous case we may conclude that there are integers m and $n \in \mathbb{N}$ such that $St(x;\mathcal{U}_m) \cap [x,\rightarrow[\subset 0 \cap [x,\rightarrow[$ and $St(x;\mathcal{U}_n) \cap]\leftarrow,x] \subset 0 \cap]\leftarrow,x]$. Let $m \le n$. Then $St(x;\mathcal{U}_n) \subset St(x;\mathcal{U}_m)$. Consequently, $x \in St(x,\mathcal{U}_n) \subset 0$.

We return to the general case, where $\lambda(<) \subset \tau$.

$2 \Longrightarrow 1$. Consider X as a (closed) subspace of the LOTS X^*,(2.1.2 (i)). For each $n \in \mathbb{N}$, define

$$D_n^* = \{(x,n) \in X^* \mid x \in D_n\}.$$

Since D_n is discrete (in X), it follows easily that D_n^* is discrete (in X^*). Put

$$D^* = \bigcup_{n=1}^{\infty} D_n^*.$$

Then $E(X) \subset D \subset D^*$ yields that D^* is a dense subset of X^*. Moreover, $N(X^*)(=N(X) \subset E(X) \subset D) \subset D^*$. Consequently, since $3 \Longrightarrow 1$ for LOTS's, it follows that X^* is metrizable. Thus X is metrizable. □

REMARK. Lutzer [L.1] showed that the metrizability of a GO-space X is equivalent to the metrizability of the corresponding LOTS X^*. The proof of the theorem above includes (see $2 \Longrightarrow 1$) another proof of this equivalency.

COROLLARY. *A σ-discrete GO-space is metrizable.*

PROOF. Obvious. □

In 2.2.7 we noticed that a σ-discrete GO-space, without isolated points, is of the first category. Moreover, by the previous corollary, it is metrizable. So, one might wonder, whether or not a metrizable and first category GO-space is σ-discrete.
The next example shows that the answer to this question is negative.

EXAMPLE. If $a,b \in \mathbb{R}$, $a < b$, then $[a,b]$ contains a Cantor-space, which geometrically can be obtained by deleting a sequence of mutually disjoint open intervals, known as the middle thirds, from $[a,b]$. In the sequel we shall denote the Cantor-space, contained in $[a,b]$, by $C_{[a,b]}$.
Now, by induction on n, ($n \in \mathbb{N}$), we define LOTS's $X_n \subset [0,1]$ such that

(i) X_n is homeomorphic to the Cantor-space.

(ii) $[0,1] \setminus \bigcup_{i=1}^{n} X_i = \bigcup_{k=1}^{\infty}]a_k^n,b_k^n[$, where $\{]a_k^n,b_k^n[\}_{k=1}^{\infty}$ is a sequence of mutually disjoint (non-empty) open intervals in $\mathbb{R}$.

As follows: For n = 1, let $X_1 = C_{[0,1]}$.
Trivially, the conditions (i) and (ii) are satisfied.
Next, suppose X_i is defined for $i = 1,2,\dots,n-1$; $n \geq 2$. Then $[0,1] \setminus \bigcup_{i=1}^{n-1} X_i = \bigcup_{k=1}^{\infty}]a_k^{n-1}, b_k^{n-1}[$, where $\{]a_k^{n-1}, b_k^{n-1}[\}_{k=1}^{\infty}$ is a sequence of mutually disjoint (non-empty) open intervals.
Now, define

$$X_n = (\bigcup_{i=1}^{n-1} X_i) \cup (\bigcup_{k=1}^{\infty} C_{[a_k^{n-1}, b_k^{n-1}]}).$$

Again, it is easily checked that the conditions (i) and (ii) are satisfied.
Finally, put

$$X = \bigcup_{n=1}^{\infty} X_n.$$

Then X has the following properties

1. X is a LOTS; for $\bar{X}$ is a LOTS without neighbourpoints (see 2.1.5).
2. X is of the first category; for $X = \bigcup_{n=1}^{\infty} X_n$ and, for each $n \in \mathbb{N}$, $\text{Int } \bar{X}_n = \text{Int } X_n = \emptyset$.
3. X is metrizable; for $X \subset [0,1]$.
4. X is not σ-discrete; for $C_{[0,1]} \subset X$ (see 2.2.2 and 2.2.7).

The metrizability or non-metrizability of various types of GO-spaces, easily follows from the above theorem, (of course several other (often easy) arguments can be used); for instance, the Sorgenfrey-line (see 2.3, example 1); the Michael-line (see 2.3, example 2); the Urysohn-space $U = ([0,1] \cdot \{0,1\}) \setminus \{(0,0),(1,1)\}$ (note that $|N(U)| > \aleph_0$ while U is separable, and remember that any σ-discrete subset of a separable GO-space is countable); and the ordinal-space ω_1 ($E(\omega_1) = \omega_1$ is not σ-discrete, since no infinite subset of ω_1 is discrete (in ω_1)), clearly are non-metrizable GO-spaces. (Observe that the Sorgenfrey-line is σ-ℓ-discrete and the ordinal-space ω_1 is σ-r-discrete).

Finally, we want to notice that a GO-space, containing a dense and σ-discrete subset, is perfectly normal. Indeed, let X be a GO-space. Let $D \subset X$ be such that $\bar{D} = X$ and $D = \bigcup_{n=1}^{\infty} D_n$, where each D_n is discrete (in X). Then, for any collection $\mathcal{C}$ of mutually disjoint (non-empty) convex open subsets of X, we can write $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$, with $\mathcal{C}_n = \{C \in \mathcal{C} \mid C \cap D_n \neq \emptyset\}$. And, clearly each $\mathcal{C}_n$ is a discrete family in X. Hence, by 2.4.5, X is perfectly normal.

In general, it can not be shown that a perfectly normal GO-space contains a dense σ-discrete subset. For, otherwise, a perfectly normal, connected LOTS would be metrizable (3.1)) and hence separable. (see the corollary to 2.4.5). However, as can be easily shown, if there exists a Souslin-space (i.e. a non-separable LOTS which satisfies the countable chain condition), then there exists also a (compact), connected Souslin-space.

THEOREM 3.2. *Let* $X = (X,<,\tau)$ *be a GO-space. If there exists a sequence* $\{\mathcal{U}_n\}_{n=1}^{\infty}$ *of open covers of* X *such that*

(i) $\bigcap_{n=1}^{\infty} St(p;\mathcal{U}_n) = \{p\}$ *for every* $p \in X$; (*i.e.* $\{St(p;\mathcal{U}_n)\}_{n=1}^{\infty}$ *is a local pseudo-base at* p), *and*

(ii) $X \setminus \{p \in X \mid \{St(p;\mathcal{U}_n)\}_{n=1}^{\infty}$ *is a local base at* p$\}$ *is* σ*-discrete* (*in* X),

then X *is metrizable.*

PROOF. For each $n \in \mathbb{N}$, we may assume that

$$\mathcal{U}_n = \{I(x;n) \mid x \in X\}$$

where $x \in I(x;n)$ and,

$$I(x;n) = \begin{cases} \{x\}, \text{ if } \{x\} \in \tau. \\ [x,p[\text{ for some } p \in X, \text{ if } [x,\rightarrow[\; \in \tau \text{ and }]\leftarrow,x] \notin \tau. \\ ]p,x] \text{ for some } p \in X, \text{ if } [x,\rightarrow[\; \notin \tau \text{ and }]\leftarrow,x] \in \tau. \\ ]p,q[\text{ for some } p,q \in X, \text{ if } [x,\rightarrow[\; \notin \tau \text{ and }]\leftarrow,x] \notin \tau. \end{cases}$$

From (i) it follows that for each $x \in X$, $\bigcap_{n=1}^{\infty} I(x;n) = \{x\}$; and because of the special form of the sets $I(x;n)$, the system $\{I(x;n)\}_{n=1}^{\infty}$ is a countable local base at x. Furthermore, we may assume that $I(x;n) \supset I(x;n+1)$ for all $n \in \mathbb{N}$ and $x \in X$. Now, define

$$L = \{p \in X \mid \{St(p;\mathcal{U}_n)\}_{n=1}^{\infty} \text{ is a local base at } p\}$$

and let

$$K = \{x \in X \setminus L \mid x \text{ is isolated in } X\}$$

K is contained in $X \setminus L$ and hence, by (ii), K is σ-discrete (in X).

Moreover K is an open subset of X and so $X \setminus K$ is closed in X. Thus, by 3.1, the metrizability of X follows immediately once we have shown that $X \setminus K$ is metrizable. (Observe that

$$E((X,<,\tau)) \subset E((X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))) \cup K).$$

Now, for every $x \in X \setminus K$ and $n \in \mathbb{N}$, we define a convex open subset $J(x;n)$ of X by

$$J(x;n) = I(x;n) \setminus \{p \in X \setminus L \mid p \text{ is an endpoint of } I(x;n)\}.$$

Then, for each $n \in \mathbb{N}$, the family

$$\mathcal{V}_n = \{J(x;n) \mid x \in X \setminus K\}$$

is an open cover of $X \setminus K$. Indeed, let $x \in X \setminus K$ be such that $x \notin J(x;n)$. Then $x \in X \setminus L$, x is an endpoint of $I(x;n)$, and x is non-isolated in X. Consequently there exists a point $p \in X$ with $x \neq p$ and $x \in I(p;n)$. (If $x \notin I(p;n)$ then also $x \notin I(p;m)$ for $m \geq n$. If this holds true for all $p \neq x$, then $St(x,\mathcal{U}_m) = I(x;m)$ for all $m \geq n$, which is impossible since $x \in X \setminus L$). Clearly $p \in X \setminus K$. (For $I(p;n) = \{p\}$ if $p \in K$). Moreover x is not an endpoint of $I(p;n)$. (Observe that $x \in X \setminus L$ has a countable system of convex open neighbourhoods, which constitutes a local pseudo-base, but not a local base, at x. Hence x cannot be a neighbourpoint unless x is isolated. Now, if x is, for instance, the left endpoint of $I(p;n)$, then $x < p$ and $I(p;n)$ is of the prescribed form $I(p;n) =]s,r[$ or $I(p;n) =]s,p]$, $s,r \in X$. So, it would follow that s is the left neighbour of x.) Consequently, $x \in J(p;n)$.

I. We claim that the family

$$\mathcal{B} = \{St(x;\mathcal{V}_n) \cap (X \setminus K) \mid x \in X \setminus K \ ; \ n \in \mathbb{N}\}$$

of convex subsets of $(X \setminus K, <_{(X \setminus K)})$ defines a base for a topology ρ on the linearly ordered set $(X \setminus K, <_{(X \setminus K)}) = X \setminus K$. Then, obviously ρ is coarser than $\tau \mid (X \setminus K)$.

Let $x_1, x_2 \in X \setminus K$; let $n,m \in \mathbb{N}$ and take

$$x \in St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K).$$

First, suppose x is neither an endpoint of $St(x_1;\mathcal{V}_n)$ nor an endpoint of $St(x_2;\mathcal{V}_m)$. Then, there are points $p,q \in X$ such that

$$x \in]p,q[\cap (X \setminus K) \subset St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K).$$

From (i) it follows that there is an integer $k \in \mathbb{N}$ such that

$$x \in St(x;\mathcal{U}_k) \subset]p,q[.$$

Hence

$$x \in St(x;\mathcal{V}_k) \cap (X \setminus K) \subset St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K).$$

Secondly, suppose x is an endpoint of at least one of the sets $St(x_1;\mathcal{V}_n)$ and $St(x_2;\mathcal{V}_m)$. For instance, let x be a right endpoint of $St(x_1;\mathcal{V}_n)$. So, there exists a point $q \in X \setminus K$ such that $x,x_1 \in J(q;n)$, while x is the right endpoint of $J(q;n)$. Then, clearly $]\leftarrow,x] \in \tau$. It follows that $x \in L$. (For, suppose $x \in X \setminus L$. Then certainly $q < x$, because of the definition of $I(q;n)$ and $J(q;n)$ and the fact that $x \in J(q;n)$. Furthermore, since $x \in (X \setminus L) \cap (X \setminus K)$ it is non-isolated. So x cannot be a neighbourpoint of X -cf. the observation between parentheses just above I.-. But this contradicts the fact that $I(q;n)$ is of the form $I(q;n) =]s,r[$ or $I(q;n) = [q,r[$, $s,r \in X$). Now we distinguish two possibilities:

(*) x is neither the left endpoint of $St(x_1;\mathcal{V}_n)$ nor of $St(x_2;\mathcal{V}_m)$. Then there is a point $p \in X$ such that

$$x \in]p,x] \cap (X \setminus K) \subset St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K)$$

while $]p,x] \in \tau$.

(**) x is the left endpoint of $St(x_1;\mathcal{V}_n)$ or of $St(x_2;\mathcal{V}_m)$. Then also $[x,\rightarrow[\in \tau$ and consequently

$$\{x\} = St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \in \tau.$$

If (*) holds true then, since $x \in L$, it follows from (i) that, for some $k \in \mathbb{N}$,

$$x \in St(x;\mathcal{U}_k) \subset]p,x].$$

Hence

$$x \in St(x;\mathcal{V}_k) \cap (X \setminus K) \subset St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K).$$

If (**) holds true then, since $x \in L$, for almost every $k \in \mathbb{N}$

$$\{x\} = St(x;\mathcal{U}_k).$$

Hence, for almost every $k \in \mathbb{N}$,

$$x \in St(x;\mathcal{V}_k) \cap (X \setminus K) = \{x\} \subset St(x_1;\mathcal{V}_n) \cap St(x_2;\mathcal{V}_m) \cap (X \setminus K).$$

This completes the proof of I. Moreover we have shown that

II. For every $x \in X \setminus K$, the family

$$\{St(x;\mathcal{V}_n) \cap (X \setminus K) \mid n \in \mathbb{N}\}$$

defines a local ρ-base at x, consisting of convex subsets of $(X \setminus K, <_{(X \setminus K)})$.

III. We claim that $(X \setminus K, <_{(X \setminus K)}, \rho)$ is a GO-space.

We have to show that $\lambda(<_{(X \setminus K)}) \subset \rho$. Take $p \in X \setminus K$ and choose

$$x \in]\leftarrow,p[\cap (X \setminus K).$$

From (i) it follows that there exists an integer $k \in \mathbb{N}$ satisfying

$$x \in St(x;\mathcal{V}_k) \cap (X \setminus K) \subset St(x;\mathcal{U}_k) \cap (X \setminus K) \subset]\leftarrow,p[\cap (X \setminus K).$$

Therefore

$$]\leftarrow,p[\cap (X \setminus K) \in \rho.$$

Similarly

$$]p,\rightarrow[\cap (X \setminus K) \in \rho.$$

Thus $\lambda(<_{(X \setminus K)})$ is coarser than ρ. Together with I this yields that $(X \setminus K, <_{(X \setminus K)}, \rho)$ is a GO-space.

IV. Next, we show that $(X \setminus K, <_{(X \setminus K)}, \rho)$ is metrizable.
Since every GO-space is collectionwise normal, we only have to prove that $(X \setminus K, <_{(X \setminus K)}, \rho)$ is a Moore-space. Now, for every $p \in X \setminus K$ and every $n \in \mathbb{N}$,

$$J(p;n) \cap (X \setminus K) \in \rho.$$

Indeed, choose

$$x \in J(p;n) \cap (X \setminus K).$$

If x is not an endpoint of $J(p;n)$ then, clearly, there is an integer $k \in \mathbb{N}$ such that

$$x \in St(x;\mathcal{V}_k) \cap (X \setminus K) \subset St(x;\mathcal{U}_k) \cap (X \setminus K) \subset J(p;n) \cap (X \setminus K).$$

However, the same is true if x is an endpoint of $J(p;n)$. (As above: observe that $]\leftarrow,x]$ (or $[x,\rightarrow[$) is τ-open and that $x \in L$.)
Hence, for all $n \in \mathbb{N}$, the family

$$\mathcal{W}_n = \{J(x;n) \cap (X \setminus K) \mid x \in X \setminus K\}$$

is an open cover of $(X \setminus K, <_{(X \setminus K)}, \rho)$. Moreover, by II, the system

$$\{St(x;\mathcal{W}_n) \mid n \in \mathbb{N}\} = \{St(x;\mathcal{V}_n) \cap (X \setminus K) \mid n \in \mathbb{N}\}$$

constitutes a local ρ-base at $x \in X \setminus K$, for each $x \in X \setminus K$.
Consequently $(X \setminus K, <_{(X \setminus K)}, \rho)$ is a Moore-space. (Thus, by introduction of a GO-topology ρ we have closed the "bad" pseudo-gaps of the GO-topology $\tau \mid (X \setminus K)$).

V. Finally, we show that $(X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))$ is metrizable.
Let D be a dense, σ-discrete subset of $(X \setminus K, <_{(X \setminus K)}, \rho)$, containing $E((X \setminus K, <_{(X \setminus K)}, \rho))$. By 3.1, such a set D exists. Then

$$E((X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))) \cap L \subset$$
$$\subset E((X \setminus K, <_{(X \setminus K)}, \rho)) \subset D.$$

For, choose

$$x \in E((X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))) \cap L.$$

We may assume that

$$]\leftarrow, x] \cap (X \setminus K) \in \tau \mid (X \setminus K).$$

Since $\{I(x;n)\}_{n=1}^{\infty}$ is a local τ-base at x,

$$x \in I(x;n) \cap (X \setminus K) \subset\,]\leftarrow, x] \cap (X \setminus K)$$

for sufficiently large $n \in \mathbb{N}$. Certainly $x \in J(x;n)$, because $x \in L$. Consequently, for almost every $n \in \mathbb{N}$,

$$]\leftarrow, x] \cap (X \setminus K) = (]\leftarrow, x[\,\cup\, J(x;n)) \cap (X \setminus K) \in \rho.$$

Hence

$$x \in E((X \setminus K, <_{(X \setminus K)}, \rho)) \subset D.$$

Further, since the family

$$\{I(x;n) \cap (X \setminus K) \mid x \in X \setminus K \,;\, n \in \mathbb{N}\}$$

forms a base for the topology $\tau \mid (X \setminus K)$ on $X \setminus K$; and, since for each $n \in \mathbb{N}$ and each $x \in L \subset X \setminus K$

$$I(x;n) \cap (X \setminus K) \cap D \supset J(x;n) \cap (X \setminus K) \cap D \neq \emptyset,$$

the set

$$D \cup ((X \setminus L) \cap (X \setminus K))$$

is a dense subset of $(X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))$. Moreover,

$$E((X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))) =$$

$$(E((X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))) \cap L) \cup$$

$$((X \setminus K) \cap (X \setminus L)) \subset D \cup ((X \setminus L) \cap (X \setminus K)).$$

Finally, since $\rho \subset \tau \mid (X \setminus K)$, it follows that $D \cup ((X \setminus L) \cap (X \setminus K))$ is σ-discrete (in $(X \setminus K, <_{(X \setminus K)}, \tau \mid (X \setminus K))$). By 3.1, this completes the proof. □

REMARK. For a LOTS $(X,<,\lambda(<))$, which allows a sequence $\{\mathcal{U}_n\}_{n=1}^{\infty}$ of open covers such that $\{St(p,\mathcal{U}_n)\}_{n=1}^{\infty}$ is a local pseudo-base at p, for every $p \in X$, it follows, since no generality is lost if we assume that each $\mathcal{U}_n$ consists of convex sets, that $\{St(p,\mathcal{U}_n)\}_{n=1}^{\infty}$ is also a local base at p. So, theorem 3.2 is only of interest in the case of a GO-space X, which is not a LOTS (with the same topology) with respect to any given ordering on the set X. We note that there are very trivial examples of GO-spaces which are not orderable; for instance the subspace $]0,1[\cup [2,3]$ of $\mathbb{R}$.

It is well-known that in C_{II}-spaces, every open base possesses a countable subcollection which is again a base for the topology.
One might conjecture that, analogously, in metric spaces every open base possesses a σ-discrete subcollection which is again a base for the topology. However this conjecture turns out to be false; the next results show that a metrix GO-space always has a σ-discrete open base consisting of convex sets, but on the other hand, there exists a metric LOTS without a σ-discrete open base consisting of open intervals.

THEOREM 3.3. *Each metrizable GO-space* $X = (X,<,\tau)$ *has a σ-discrete base consisting of convex open sets.*

PROOF. Like we noticed already (just above 3.2), when a GO-space X contains a dense, σ-discrete subset, then every disjoint collection $\mathcal{C}$ of convex open subsets of X constitutes a σ-discrete family in X. (Of course, in our case this follows also immediately from 2.4.5). Now, let $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ be a σ-discrete open base for X, where each $\mathcal{B}_n$ is a discrete family in X. For all $n \in \mathbb{N}$, put

$$\tilde{\mathcal{B}}_n = \{C \mid \exists B \in \mathcal{B}_n : C \text{ is a convexity-component of } B\}.$$

Then $\widetilde{\mathcal{B}}_n$ forms a disjoint collection of convex open subsets of X and hence $\widetilde{\mathcal{B}}_n$ constitutes a σ-discrete family in X. Consequently, also $\widetilde{\mathcal{B}} = \bigcup_{n=1}^{\infty} \widetilde{\mathcal{B}}_n$ is a σ-discrete family in X. Obviously, $\widetilde{\mathcal{B}}$ is a base for X. Thus, the proof is complete. □

THEOREM 3.4. *There exists a metric LOTS* X, *such that no σ-disjoint collection*

$$\mathcal{I} = \bigcup_{n=1}^{\infty} \mathcal{I}_n \qquad (\mathcal{I}_n \textit{ is a disjoint collection})$$

of open intervals covers X.
In particular, X *does not have a σ-discrete open base consisting of intervals.*

PROOF. For all $k < \omega_0$, let X_k be the LOTS

$$X_k = \omega_1^* + \omega_1.$$

Put

$$X = \coprod_{k<\omega_0} X_k = (\omega_1^* + \omega_1)^{\omega_0}.$$

We will show that the LOTS X satisfies the required property of our theorem. Throughout the proof we use the following notations:

$$X^{(m)} = \coprod_{m \le k<\omega_0} X_k \qquad (m < \omega_0)$$

hence

$$X = X_0 \cdot X_1 \cdot \ldots \cdot X_{m-1} \cdot X^{(m)};$$

and if $p = (p_i)_{i<\omega_0} \in X$, then we write

$$X_{(p_0\, p_1 \ldots p_n)} = \{x = (x_i)_{i<\omega_0} \in X \mid x_i = p_i \text{ for all } i \le n\}.$$

Clearly, each $X^{(m)}$ and each $X_{(p_0\, p_1 \ldots p_n)}$ is homeomorphic to X.

Now, for each $n \in \mathbb{N}$, the family

$$\mathcal{B}_n = \{X_{(p_0\ p_1 \dots p_n)} \mid p_i \in X_i \text{ for } i \le n\}$$

consists of mutually disjoint convex open and closed subsets of X, while $\cup \mathcal{B}_n = X$. Hence $\mathcal{B}_n$ is a discrete family in X. Moreover, it is easily checked that $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$ is a base for the order-topology on X. So, X is metrizable.

Next, for all $k < \omega_0$, let

$$L_k = \{x_k \in X_k \mid x_k \text{ is a limit ordinal}\}$$

and let

$$L_k^* = \{x_k^* \in X_k \mid x_k \text{ is a limit ordinal}\}$$

(x_k^* denotes the inverse order-type of the ordinal x_k).
By induction on k ($<\omega_0$) we will define points

$$z_k \in L_k \text{ (k even) and } z_k \in L_k^* \text{ (k odd)}$$

such that the point

$$z = (z_k)_{k<\omega_0} \in X$$

is not covered by $\mathcal{I}$. As follows:
k = 0: Consider the collection $\mathcal{C}_{0n}$ of those intervals $]\ell,r[\ \in \mathcal{I}_n$, which overlap the left-endgap of $X_{(p_0)}$ in X, for some $p_0 \in L_0$; i.e.

$$\mathcal{C}_{0n} = \{]\ell,r[\ \in \mathcal{I}_n \mid \exists p_0 \in L_0 : \ell_0 < p_0 \le r_0\}.$$

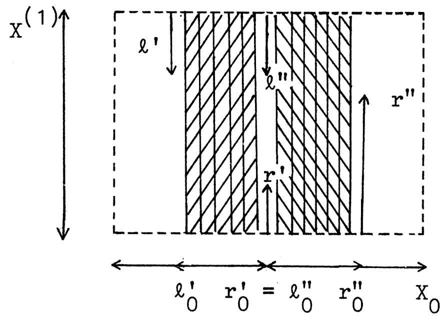
(Observe that each interval of $\mathcal{I}$ may be described by $]\ell,r[$, where $\ell = (\ell_i)_{i<\omega_0}$ and $r = (r_i)_{i<\omega_0} \in X$). Let

$$\mathcal{L}_{0n} = \{]\ell_0,r_0[\ \mid\ ]\ell,r[\ \in \mathcal{C}_{0n}\}.$$

Since two different intervals in X_0 of the type

$$]\ell_0',r_0'[,\ ]\ell_0'',r_0''[\ \ (]\ell',r'[\ \in \mathcal{I}_n,\ ]\ell'',r''[\ \in \mathcal{I}_n)$$

are disjoint ($[\ell_0',r_0']$ and $[\ell_0'',r_0'']$ may have a common endpoint, however),



it follows that each increasing sequence

$$]\ell_0^{(1)},r_0^{(1)}[,\ ]\ell_0^{(2)},r_0^{(2)}[,\ldots\ldots,\ ]\ell_0^{(k)},r_0^{(k)}[,\ldots\ldots \qquad (k < \omega_0)$$

$(]\ell^{(k)},r^{(k)}[\ \in \mathcal{I}_n;\ k \in \mathbb{N})$ in X_0 has a limit point which does not belong to $\cup\, L_{0n}$. Consequently the set

$$P_{0n} = X_0 \setminus \cup\, L_{0n}$$

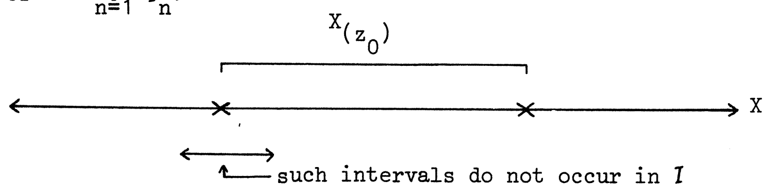
contains a cofinal set of limit ordinals. Furthermore, each set P_{0n} $(n \in \mathbb{N})$ is closed in the order-topology of X_0. Also L_0 is closed in the order-topology of X_0. Moreover, the family

$$\{P_{0n}\}_{n=1}^{\infty} \cup \{L_0\}$$

has the finite intersection property, which follows from an interlacing argument. So, since X_0 is countably compact (in its order-topology), there exists a point

$$z_0 \in \bigcap_{n=1}^{\infty} P_{0n} \cap L_0 .$$

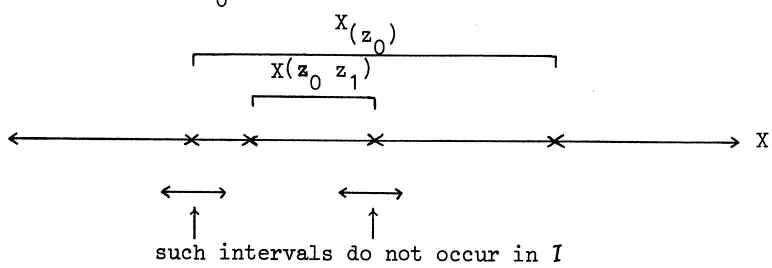
We conclude that the left-endgap of $X_{(z_0)}$ in X is not overlapped by any interval of $I = \bigcup_{n=1}^{\infty} I_n$.



k=1: First observe that at most countably many intervals of I can overlap the right-endgap of $X_{(z_0)}$. Therefore there exist uncountably many elements $x_1^* \in L_1^*$ such that $X_{(z_0,x_1^*)}$ is not overlapped by any interval from I which overlaps the right-endgap of $X_{(z_0)}$.
We consider the collection C_{1n} of those intervals $]\ell,r[\in I_n$ which overlap the right-endgap of $X_{(z_0,p_1^*)}$ in X, for some $p_1^* \in L_1^*$; i.e.

$$C_{1n} = \{]\ell,r[\in I_n \mid \exists p_1^* \in L_1^* : \ell_1 \leq p_1^* < r_1 \ ; \ \ell_0 = z_0\}.$$

Proceeding on "in $X_{(z_0)}$" we can determine a point $z_1 \in L_1^*$ ($\subset X_1$) such that there do not occur intervals in I which overlap the right-endgap of $X_{(z_0\ z_1)}$ in X (or: in $X_{(z_0)}$).



And so on. Clearly

$$z = (z_k)_{k<\omega_0} \notin \bigcup I. \ \square$$

REMARK. 1. As can be easily seen, $(\omega_1^*+\omega_1)^{\omega_0}$ is a nowhere locally separable, homogeneous and completely metrizable LOTS.

2. In an analogous way it can be shown that no σ-locally finite collection of open intervals covers $(\omega_1^*+\omega_1)^{\omega_0}$.

CHAPTER IV

LEXICOGRAPHICALLY ORDERED PRODUCTS

In this chapter we will investigate how some properties of LOTS's behave under taking lexicographically ordered products. It will come out that compactness and paracompactness are preserved under forming of lexicographic products. (Observe, however, that the factors in general are not contained topologically in the resulting product, except for the last one if it occurs). Further, we mention necessary and sufficient conditions for a lexicographically ordered product space to be second countable, separable or hereditarily Lindelöf. Our main attention in this chapter is focussed on the characterization of metrizability of lexicographic products in terms of the structures of the factors. It turns out, afterwards, that for a LOTS's of the type $\mathop{\underline{\|\,\,}}_{\alpha<\mu} X_\alpha$ (where μ is cofinal with ω_0 and X_α is a nontrivial LOTS), metrizability is equivalent to perfect normality.
For sake of completeness and to get a better idea how lexicographic products look like, we start by giving necessary and sufficient conditions for the connectivity.

We may restrict ourselves to lexicographic products of the following types:

$$\mathop{\underline{\|\,\,}}_{\alpha<\mu} X_\alpha \qquad \text{and} \qquad \mathop{\underline{\|\,\,}}_{\nu\leq\alpha<\mu} X_\alpha$$

where for each α less than a given ordinal number μ, $X_\alpha = (X_\alpha, <_\alpha)$ is a nontrivial linearly ordered set, and ν is an arbitrary ordinal $< \mu$. The ordering on $\mathop{\underline{\|\,\,}}_{\alpha<\mu} X_\alpha$ and $\mathop{\underline{\|\,\,}}_{\nu\leq\alpha<\mu} X_\alpha$ will be denoted by $<$. Furthermore, a point x of the product is denoted by $x = (x_\alpha)_{\alpha<\mu}$ or $x = (x_\alpha)_{\nu\leq\alpha<\mu}$, respectively.

4.1. CONNECTIVITY IN $\underline{\|\,\,}$

From 2.4.2 it follows that a LOTS is connected if an only if it has no jumps and no gaps, except for possible endgaps.

LEMMA. *Let* $X = \mathop{\underline{\|\,\,}}_{\alpha<\mu} X_\alpha$ *be a connected LOTS. Then*

(i) ($\forall\alpha > 0 : X_\alpha$ *has a left endpoint*) *or* ($\forall\alpha > 0 : X_\alpha$ *has a right endpoint*).

(ii) $\forall\alpha \geq \omega_0 : X_\alpha$ *possesses both a left and a right endpoint.*

(iii) $\forall\alpha < \mu$: X_α *has no gaps, except for possible endgaps.*

(iv) $\forall\alpha < \mu$: *if* X_β *possesses both a left and a right endpoint, for all* $\beta > \alpha$, *then* X_α *has no jumps.*

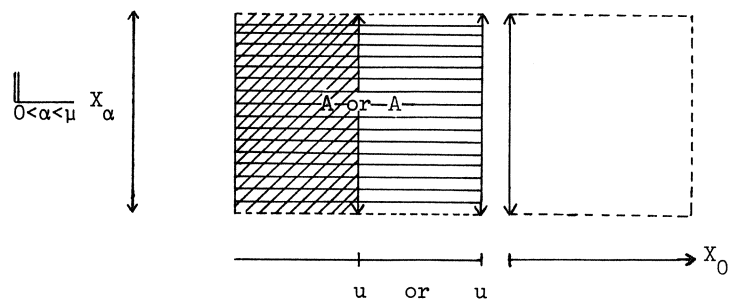
(v) $\forall\alpha < \mu$: *if* X_β *does not have a left endpoint for some* $\beta > \alpha$, *then each bounded strictly increasing sequence in* X_α *is finite.*
(i.e.: X_α is a subset of the ordered union of an inversely well-ordered set and ω_0).

(vi) $\forall\alpha < \mu$: *if* X_β *does not have a right endpoint for some* $\beta > \alpha$, *then each bounded strictly decreasing sequence in* X_α *is finite.*
(i.e.: X_α is a subset of the ordered union of a well-ordered set and ω_0^*).

PROOF.

(i) Suppose there exist ordinals $\alpha > 0$ and $\alpha' > 0$ such that X_α has no left and $X_{\alpha'}$ has no right endpoint. Then $\coprod_{0<\alpha<\mu} X_\alpha$ has neither a left nor a right endpoint. Now, choose $u \in X_0$ such that u is not the right endpoint of X_0. Let

$$A = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 \le u\}$$



Then $A \neq \emptyset$, $X \setminus A \neq \emptyset$ and $(A, X \setminus A)$ is a gap in X. Contradiction.

(ii) Suppose there exists an ordinal $\beta \ge \omega_0$ such that (for instance) X_β does not have a right endpoint. Assume β to be the first ordinal with these properties. So, for each ordinal α, with $\omega_0 \le \alpha < \beta$, X_α has a right endpoint r_α. Next, for all $\alpha < \omega_0$, choose $u_\alpha \in X_\alpha$ such that u_α is not the right endpoint of X_α. Let

$A' = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_\alpha = u_\alpha \text{ if } \alpha < \omega_0 \text{ , } x_\alpha = r_\alpha \text{ if } \omega_0 \leq \alpha < \beta\}$

and, put

$$A = \{x \in X \mid \exists a \in A' : x \leq a\}.$$

Then $A \neq \emptyset$, $X \setminus A \neq \emptyset$ and $(A, X \setminus A)$ defines a gap in X. Contradiction.

(iii) Suppose the assertion does not hold true. Let $\beta < \mu$ be the first ordinal such that X_β has a gap $(A_\beta, X_\beta \setminus A_\beta)$ which is not an endgap. For each $\alpha < \beta$, choose $u_\alpha \in X_\alpha$. Let

$$A' = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_\alpha = u_\alpha \text{ if } \alpha < \beta \text{ ; } x_\beta \in A_\beta\}$$

and, put

$$A = \{x \in X \mid \exists a \in A' : x \leq a\}.$$

Then $A \neq \emptyset$, $X \setminus A \neq \emptyset$ and $(A, X \setminus A)$ is a gap in X. Contradiction.

(iv) Follows immediately from the characterization of neighbours in $\coprod_{\alpha<\mu} X_\alpha$, given in 1.2.1, and the fact that each pair of neighbours defines a jump and conversely.

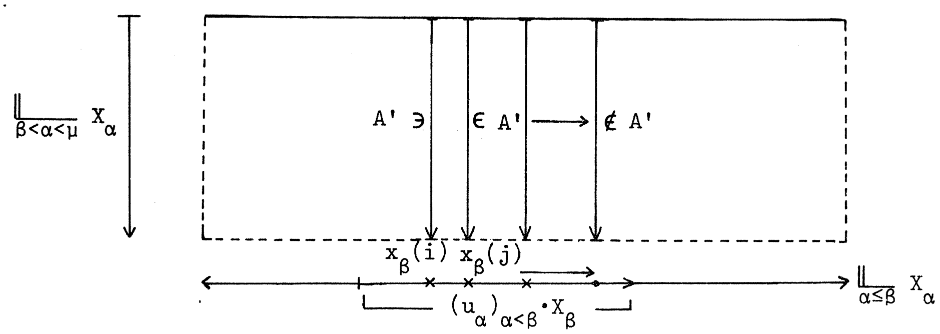
(v) Assume, X_γ $(\gamma > 0)$ does not have a left endpoint, for some $\gamma < \mu$. Let $\beta < \gamma$. Now, suppose $\{x_\beta(i)\}_{i=1}^{\infty}$ is a bounded, strictly increasing sequence in X_β. For each $\alpha < \beta$, choose $u_\alpha \in X_\alpha$. Let

$$A' = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_\alpha = u_\alpha \text{ if } \alpha < \beta \text{ ; } x_\beta \in \{x_\beta(i)\}_{i=1}^{\infty}\}$$

and, put

$$A = \{x \in X \mid \exists a \in A' : x \leq a\}.$$

Then $A \neq \emptyset$, $X \setminus A \neq \emptyset$ and $(A, X \setminus A)$ defines a gap in X. Contradiction.



(vi) Analogous to (v). □

THEOREM 4.1.1. *The LOTS* $X = \coprod_{\alpha<\mu} X_\alpha$ *is connected* $\Leftrightarrow$ *At least one of the following two collections of conditions is satisfied*

I

(i) $\forall\alpha > 0$: X_α *has a left endpoint.*

(ii) $\forall\alpha \geq \omega_0$: X_α *possesses both a left and a right endpoint.*

(iii) $\forall\alpha < \mu$: X_α *has no gaps, except for possible endgaps.*

(iv) $\forall\alpha < \mu$: *if* X_β *possesses both a left and a right endpoint for all* $\beta > \alpha$, *then* X_α *has no jumps.*

(v) $\forall\alpha < \mu$: *if* X_β *does not have a right endpoint for some* $\beta > \alpha$, *then each bounded strictly decreasing sequence in* X_α *is finite.*

II

(i) $\forall\alpha > 0$: X_α *has a right endpoint.*

(ii) $\forall\alpha \geq \omega_0$: X_α *possesses both a left and a right endpoint.*

(iii) $\forall\alpha < \mu$: X_α *has no gaps, except for possible endgaps.*

(iv) $\forall\alpha < \mu$: *if* X_β *possesses both a left and a right endpoint for all* $\beta > \alpha$, *then* X_α *has no jumps.*

(v) $\forall\alpha < \mu$: *if* X_β *does not have a left endpoint for some* $\beta > \alpha$, *then each bounded strictly increasing sequence in* X_α *is finite.*

(REMARK: Because of (ii), it is clear that (v) applies only in the case that $\alpha < \omega_0$)

PROOF.

$\Longrightarrow$ lemma.

$\Longleftarrow$ Suppose that condition I holds. (The other case follows similarly). From (iv) and 1.2.1 it follows that $X = \coprod_{\alpha<\mu} X_\alpha$ does not have jumps. So, it remains to show that X has no gaps (A,B), with $A \neq \emptyset$ and $B \neq \emptyset$. Now, let (A,B) be an ordered pair of non-empty subsets of X, such that $X = A \cup B$ and $a < b$ for all $a \in A$ and $b \in B$. We have to prove the existence of a point $s = (s_\alpha)_{\alpha<\mu} \in X$ such that $\bar{A} \cap \bar{B} = \{s\}$. By transfinite induction (on α) we shall define s, as follows:

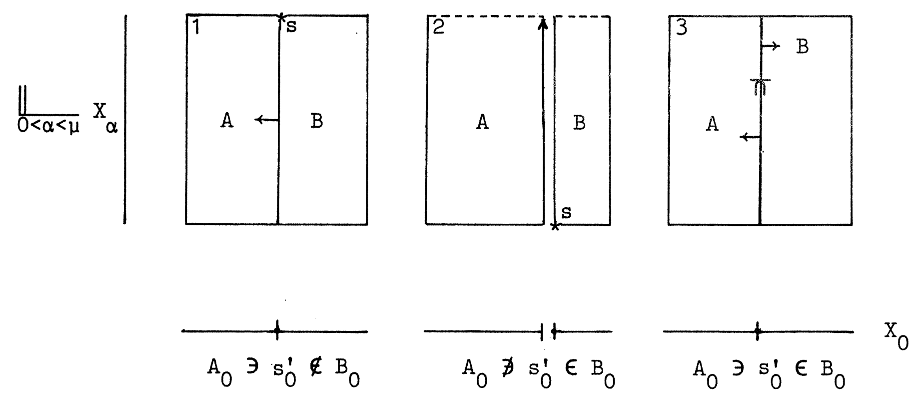
1. Let

$$A_0 = \{x_0 \in X_0 \mid \exists a \in A : a_0 = x_0\}$$

and let

$$B_0 = \{x_0 \in X_0 \mid \exists b \in B : b_0 = x_0\}$$

Clearly, $X_0 = A_0 \cup B_0$, $A_0 \neq \emptyset$, $B_0 \neq \emptyset$, $|A_0 \cap B_0| \leq 1$ and $a_0 \leq_0 b_0$ for all $a_0 \in A_0$ and $b_0 \in B_0$.



First consider the case that $A_0 \cap B_0 = \emptyset$. Since, by (iii), (A_0, B_0) is not a gap in X_0, it follows that A_0 has a right endpoint in X_0 or B_0 has a left endpoint in X_0. If $s_0' \in X_0$ is the right endpoint of A_0 (figure 1) and, moreover, each X_α $(\alpha > 0)$ has a right endpoint r_α, then we define $s = (s_\alpha)_{\alpha<\mu} \in X$ by $s_0 = s_0'$ and $s_\alpha = r_\alpha$ if $\alpha > 0$. Since X has no jumps, it follows that $\bar{A} \cap \bar{B} = \{s\}$. If $s_0' \in X_0$ is the left endpoint of B_0 (figure 2), then we define $s = (s_\alpha)_{\alpha<\mu} \in X$ by $s_0 = s_0'$ and $s_\alpha = \ell_\alpha$ if $\alpha > 0$, where ℓ_α is the left endpoint of X_α, (by (i), X_α has a left endpoint for all $\alpha > 0$). Again, since X has no jumps, $\bar{A} \cap \bar{B} = \{s\}$. Finally, if for some $\beta > 0$, X_β does not have a right endpoint then, by (v), B_0 must have a left endpoint s_0' in X_0. Hence, the previous case applies. Secondly, assume that $A_0 \cap B_0 \neq \emptyset$. Then, there exists a point $s_0' \in X_0$ such that $A_0 \cap B_0 = \{s_0'\}$. (figure 3).
Put $s_0 = s_0'$.

2. Let $s_\beta \in X_\beta$, A_β and $B_\beta \subset X_\beta$ be defined for all $\beta < \omega_0$, such that $A_\beta \cap B_\beta = \{s_\beta\}$. Put

$$A_{\omega_0} = \{x_{\omega_0} \in X_{\omega_0} \mid \exists a \in A : a_\beta = s_\beta \text{ if } \beta < \omega_0 \text{ and } a_{\omega_0} = x_{\omega_0}\}$$

and let

$$B_{\omega_0} = \{x_{\omega_0} \in X_{\omega_0} \mid \exists b \in B : b_\beta = s_\beta \text{ if } \beta < \omega_0 \text{ and } b_{\omega_0} = x_{\omega_0}\}.$$

Then, $X_{\omega_0} = A_{\omega_0} \cup B_{\omega_0}$ and $a_{\omega_0} \leq_{\omega_0} b_{\omega_0}$ for all $a_{\omega_0} \in A_{\omega_0}$ and $b_{\omega_0} \in B_{\omega_0}$. Observe, however, that (in distinction with the "finite" case) A_{ω_0} or B_{ω_0} may be empty. If $A_{\omega_0} = X_{\omega_0}$ and, by (ii), r_α is the right endpoint of X_α for all $\alpha \geq \omega_0$, then we define $s_\alpha = r_\alpha$ for $\alpha \geq \omega_0$. Now, clearly, $s = (s_\alpha)_{\alpha<\mu} \in \bar{A} \cap \bar{B}$. If, on the other hand, $B_{\omega_0} = X_{\omega_0}$, then we put $s_\alpha = \ell_\alpha$ if $\alpha \geq \omega_0$, where ℓ_α is the left endpoint of X_α. Again, it is true that $s = (s_\alpha)_{\alpha<\mu} \in \bar{A} \cap \bar{B}$. Finally, when $A_{\omega_0} \neq \emptyset$ and $B_{\omega_0} \neq \emptyset$, then $|A_{\omega_0} \cap B_{\omega_0}| \leq 1$, and we can proceed in a similar way as we did in case 1 before. Now, the definition of $s = (s_\alpha)_{\alpha<\mu}$ can easily be completed by transfinite induction. It is clear that $s \in \bar{A} \cap \bar{B}$. □

We note that from the previous theorem it can be easily derived whether or not a lexicographically ordered product space is totally disconnected. Indeed, if μ is a non-limit ordinal then, certainly, "$X_{\mu-1}$ is totally disconnected" will be a necessary and sufficient condition on $\coprod_{\alpha<\mu} X_\alpha$ to be totally disconnected itself. And, for a limit ordinal μ, clearly, $\coprod_{\alpha<\mu} X_\alpha$ is not totally disconnected whenever there exists an ordinal $\beta < \mu$, such that $\coprod_{\gamma<\alpha<\mu} X_\alpha$ is connected for each $\gamma \geq \beta$, and conversely.

REMARK. By 4.1.1, all lexicographic products of type α^{ω_0}, where α is a well-ordered or inversely well-ordered set with precisely one endpoint, are connected LOTS's. In particular, $\omega_0^{\omega_0} = \mathbb{N}^{\omega_0}$ and $\omega_1^{\omega_0}$ are connected (see also Miller [Mil.1]).

4.2. COMPACTNESS AND PARACOMPACTNESS IN $\coprod$

From 2.4.1 it follows that a LOTS is compact if and only if it has no gaps.

THEOREM 4.2.1. *The LOTS* $X = \coprod_{\alpha<\mu} X_\alpha$ *is compact* $\Longleftrightarrow \forall\alpha < \mu$: X_α *is a compact LOTS.*

PROOF.
$\Longrightarrow$ Since $X = \coprod_{\alpha<\mu} X_\alpha$ has no endgaps, X has both a left and a right endpoint. Therefore, also each X_α, $\alpha < \mu$, possesses a left and a right endpoint. Now, suppose (A_β, B_β) is a gap in X_β, for some $\beta < \mu$. Then $A_\beta \neq \emptyset$

and $B_\beta \neq \emptyset$. For all $\alpha < \beta$, choose $u_\alpha \in X_\alpha$ arbitrarily. Next, define

$$A' = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_\alpha = u_\alpha \text{ if } \alpha < \beta;\ x_\beta \in A_\beta\}$$

and put

$$A = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid \exists a \in A' : x \leq a\}.$$

Then, clearly, $(A,X\backslash A)$ defines a gap in X. Contradiction.
$\Longleftarrow$ Since, for every $\alpha < \mu$, X_α has no gaps, the conditions (i), (ii), (iii) and (v) (of both I and II) of theorem 4.1.1 are satisfied. Further, condition (iv) only plays a role in the proof of 4.1.1 by warranting that the lexicographic product has no jumps; however this does not apply in the present theorem. Thus, if we assume that (A,B) is an ordered pair of non-empty subsets of X, such that $X = A \cup B$ and $a < b$ for all $a \in A$ and $b \in B$ then, argueing in a similar way as was done in the proof of 4.1.1, we may conclude that (A,B) is not a gap in X. □

REMARK. Novák already obtained a result which combines 4.1.1 and 4.2.1. He showed [No.1]: Any lexicographically ordered product of compact and connected LOTS's is again a compact, connected LOTS.

Recall that a LOTS X is paracompact if and only if for each gap (A,B) in X, there exist discrete (in X) subsets $L \subset A$ and $R \subset B$, such that L is cofinal in A and R is coinitial in B (2.4, f, lemma).

THEOREM 4.2.2. *If, for each* $\alpha < \mu$, X_α *is a paracompact LOTS, then also* $X = \coprod_{\alpha<\mu} X_\alpha$ *is a paracompact LOTS.*

PROOF. Let (A,B) be a gap in X. We assume that both A and B are non-empty. (The other cases can be treated similarly). Now, put

$$A_0 = \{x_0 \in X_0 \mid \exists a \in A : a_0 = x_0\}$$

and put

$$B_0 = \{x_0 \in X_0 \mid \exists b \in B : b_0 = x_0\}.$$

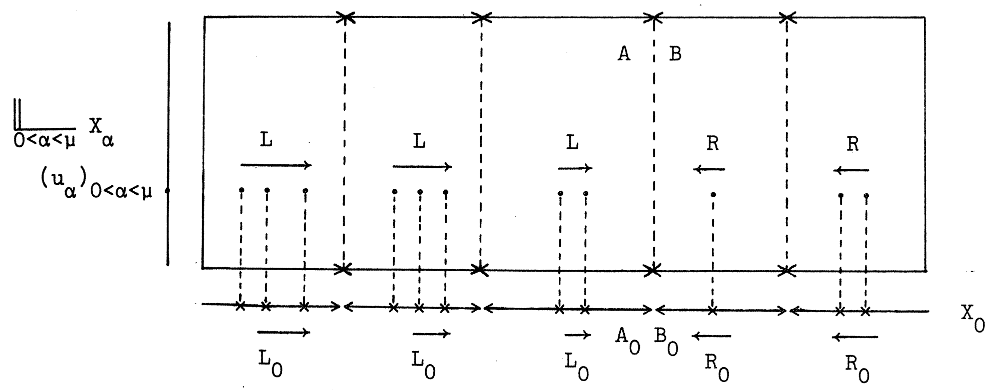
Then $X_0 = A_0 \cup B_0$, $A_0 \neq \emptyset$, $B_0 \neq \emptyset$ and $a_0 \leq_0 b_0$ for all $a_0 \in A_0$ and $b_0 \in B_0$, while $|A_0 \cap B_0| \leq 1$.
1. Suppose $A_0 \cap B_0 = \emptyset$. We distinguish three cases:

(i) Let (A_0, B_0) be a gap in X_0. Then there are discrete (in X_0) subsets $L_0 \subset A_0$ and $R_0 \subset B_0$ such that L_0 is cofinal in A_0 and R_0 is coinitial in B_0. For each $\alpha > 0$, choose $u_\alpha \in X_\alpha$ arbitrarily. Next, define

$$L = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 \in L_0, x_\alpha = u_\alpha \text{ if } \alpha > 0\}$$

and

$$R = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 \in R_0, x_\alpha = u_\alpha \text{ if } \alpha > 0\}.$$



Then L and R are discrete subsets of X; moreover, $L(\subset A)$ is cofinal in A and $R(\subset B)$ is coinitial in B.

(ii) Let $A_0 \cap \bar{B}_0 = \{s_0\}$ $(s_0 \in X_0)$ [or, similarly, $\bar{A}_0 \cap B_0 = \{s_0\}$]. Since (A,B) is a gap in X, it follows that $\coprod_{0<\alpha<\mu} X_\alpha$ does not have a right endpoint and hence there is a first ordinal $\beta > 0$ such that X_β does not have a right endpoint. Let $L_\beta \subset X_\beta$ be a discrete and cofinal subset in X_β. Further, let R_0' be a (transfinite) decreasing sequence in B_0 which is coinitial in B_0. Of course, R_0' need not be discrete in X_0. Now, for each $\alpha > 0$, choose $u_\alpha \in X_\alpha$ arbitrarily. And, furthermore, let r_α denote the right endpoint of X_α for each α with $0 < \alpha < \beta$. Then, define

$$L = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 = s_0;\ x_\alpha = r_\alpha \text{ if } 0 < \alpha < \beta;$$
$$x_\beta \in L_\beta;\ x_\alpha = u_\alpha \text{ if } \alpha > \beta\}$$

and

$$R = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 \in R_0'; \; x_\alpha = u_\alpha \text{ if } \alpha > 0\}.$$

Clearly, $L(\subset A)$ is discrete in X and cofinal in A. But also R is discrete in X, since $\coprod_{0<\alpha<\mu} X_\alpha$ has no right endpoint. Moreover, since B_0 does not have a left endpoint in X_0, R is coinitial in B.

(iii) Let (A_0,B_0) be a jump in X_0. Let s_0 denote the right endpoint of A_0 and t_0 the left endpoint of B_0. Since (A,B) is a gap, $\coprod_{0<\alpha<\mu} X_\alpha$ neither has a first nor a last element and hence there exists a first ordinal $\beta < \mu$ such that X_β does not have a right endpoint, and, also, there is a first ordinal $\gamma < \mu$ such that X_γ does not have a left endpoint. Let L_β be a discrete and cofinal subset in X_β; and let R_γ be a discrete and coinitial subset in X_γ. Furthermore, choose $u_\alpha \in X_\alpha$, for $\alpha > \min(\beta,\gamma)$. Next, for $0 < \alpha < \beta$, let r_α be the right endpoint of X_α; and, for $0 < \alpha < \gamma$, let ℓ_α be the left endpoint of X_α. Now, define

$$L = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 = s_0; \; x_\alpha = r_\alpha \text{ if } 0 < \alpha < \beta;$$
$$x_\beta \in L_\beta; \; x_\alpha = u_\alpha \text{ if } \alpha > \beta\}$$

and

$$R = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_0 = t_0; \; x_\alpha = \ell_\alpha \text{ if } 0 < \alpha < \gamma;$$
$$x_\gamma \in R_\gamma; \; x_\alpha = u_\alpha \text{ if } \alpha > \gamma\}.$$

Then $L(\subset A)$ is discrete in X and cofinal in A, and $R(\subset B)$ is discrete in X and coinitial in B.

2. Suppose $A_0 \cap B_0 \neq \emptyset$. Then $A_0 \cap B_0 = \{s_0\}$, for some $s_0 \in X_0$. Let

$$A_1 = \{x_1 \in X_1 \mid \exists a \in A : a_0 = s_0 \text{ and } a_1 = x_1\}$$

and let

$$B_1 = \{x_1 \in X_1 \mid \exists b \in B : b_0 = s_0 \text{ and } b_1 = x_1\}.$$

Clearly, $X_1 = A_1 \cup B_1$ and $a_1 \leq_1 b_1$ for all $a_1 \in A_1$ and $b_1 \in B_1$. Moreover, $A_1 \neq \emptyset$ and $B_1 \neq \emptyset$. Now, if $A_1 \cap B_1 = \emptyset$ then case 1. applies. And, if $A_1 \cap B_1 \neq \emptyset$, then $A_1 \cap B_1 = \{s_1\}$ for some $s_1 \in X_1$. In the latter case we proceed by transfinite induction, as follows: if, for $\alpha < \beta$, $s_\alpha \in X_\alpha$, A_α and $B_\alpha \subset X_\alpha$ are defined such that $A_\alpha \cap B_\alpha = \{s_\alpha\}$, then put

$$A_\beta = \{x_\beta \in X_\beta \mid \exists a \in A : a_\alpha = s_\alpha \text{ if } \alpha < \beta \text{ and } a_\beta = x_\beta\}$$

and

$$B_\beta = \{x_\beta \in X_\beta \mid \exists b \in B : b_\alpha = s_\alpha \text{ if } \alpha < \beta \text{ and } b_\beta = x_\beta\}.$$

Since (A,B) is a gap in X, there exists a first ordinal $\gamma < \mu$, such that $A_\gamma \cap B_\gamma = \emptyset$. This γ certainly is a limit ordinal. We distinguish between two cases:

(i) Let $A_\gamma = \emptyset$ [or, similarly, $B_\gamma = \emptyset$].
From (A,B) is a gap it follows that there is a first ordinal $\delta \geq \gamma$, such that X_δ has no left endpoint. Let $R_\delta \subset X_\delta$ be a discrete and coinitial subset of X_δ. Further, for each $\nu < \gamma$, choose a point $p(\nu) \in A$, such that $p(\nu)_\alpha = s_\alpha$ if $\alpha < \nu$. (We may assume that $p(\nu_1) \leq p(\nu_2)$ if $\nu_1 < \nu_2$). Next, for each $\alpha > \delta$, choose $u_\alpha \in X_\alpha$ arbitrarily. Let ℓ_α, $\gamma \leq \alpha < \delta$, denote the left endpoint of X_α. Now, define

$$L = \{p(\nu) \in X \mid \nu < \gamma\}$$

and

$$R = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid x_\alpha = s_\alpha \text{ if } \alpha < \gamma\,;\ x_\alpha = \ell_\alpha \text{ if } \gamma \leq \alpha < \delta;\ x_\delta \in R_\delta;\ x_\alpha = u_\alpha \text{ if } \alpha > \delta\}.$$

Then $L \subset A$ is discrete in X, because $\coprod_{\gamma \leq \alpha < \mu} X_\alpha$ has no left endpoint. Moreover, since γ is the first ordinal with $A_\gamma = \emptyset$, L is also cofinal in A. Obviously, $R \subset B$ is discrete in X and coinitial in B.

(ii) $A_\gamma \neq \emptyset$ and $B_\gamma \neq \emptyset$.
Now, again case 1. applies. This completes the proof. □

REMARK. For the special case $\mu = 2$, the previous theorem was proved recently by Ostazewski [0.1].

The converse assertion of the previous result cannot be stated, for if a lexicographic product $X = \coprod_{\alpha<\mu} X_\alpha$ is a paracompact (even completely metrizable) LOTS, then none of the factorspaces X_α $(\alpha < \mu)$ need be paracompact.

EXAMPLE. Let $X = \omega_1^* + \omega_1$.
Then $X^{\omega_0} = (\omega_1^* + \omega_1)^{\omega_0}$ is a completely metrizable LOTS. Indeed the complete metric

$$\rho : X^{\omega_0} \times X^{\omega_0} \longrightarrow \mathbb{R},$$

defined by $\rho(x,y) = 0$ if $x = y$, and $\rho(x,y) = \frac{1}{n}$ if $n-1$ is the first ordinal $i < \omega_0$ for which $x_i \neq y_i$, $(x = (x_i)_{i<\omega_0}, y = (y_i)_{i<\omega_0} \in X^{\omega_0})$, is compatible with the order-topology. [ρ is called the Baire-metric on X^{ω_0}]. Clearly, X is a non-paracompact LOTS.

Hereditary paracompactness is not preserved, in general, by taking lexicographic products. For instance, $[0,1]^{\omega_1}$ is a compact, but not hereditarily paracompact LOTS. (Observe that no point of $[0,1]^{\omega_1}$ has a countable local base (lemma 1 to 2.2.11) and apply 2.4.7). Also $[0,1]^{\omega_1+1}$ is a compact, but not hereditarily paracompact LOTS. For, $[0,1]^{\omega_1+1}$ contains a non-empty subset of points without a countable local base.
However, under some special condition on μ, a lexicographic product of hereditarily paracompact LOTS's is again hereditarily paracompact. To show this, we need a following lemma.

For $i = 1,2$, let $Y_i \subset \coprod_{\alpha<\mu} X_\alpha = X$ be defined by

$$Y_1 = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid \forall\alpha < \mu : \exists\beta > \alpha : (X_\beta \supset)]\leftarrow,x_\beta[\neq \emptyset\}$$

(so, x_β is not the left endpoint of X_β), and

$$Y_2 = \{x = (x_\alpha)_{\alpha<\mu} \in X \mid \forall\alpha < \mu : \exists\beta > \alpha : (X_\beta \supset)]x_\beta,\rightarrow[\neq \emptyset\}$$

(so, x_β is not the right endpoint of X_β). It is easily verified, that for any limit number μ, $X = Y_1 \cup Y_2$ and, moreover, that $Y = Y_1 \cap Y_2$ is a dense subset of the LOTS X, (see also the proof of 2.2.12).

LEMMA. *Let* $X = \coprod_{\alpha<\mu} X_\alpha$ *be a LOTS. Let* ω_0 *be cofinal in* μ. *Then the subspace* $Y_1 \subset X$ *is a left-*C_I*-space, and the subspace* $Y_2 \subset X$ *is a right-*C_I*-space. Hence, in particular,* $Y \subset X$ *is a* C_I*-space.*

PROOF. We show that Y_1 is a left-C_I-space. The other case follows similarly. Choose $x = (x_\alpha)_{\alpha<\mu} \in Y_1$. Then x is neither a left endpoint nor a right neighbourpoint of X. Let $\{\beta_i\}_{i<\omega_0}$ be an increasing sequence of ordinals less than μ, such that $\{\beta_i\}_{i<\omega_0}$ is cofinal in μ. Now, for all $i < \omega_0$, we can find a point $p(i) \in X$ such that $p(i) < x$ and $p(i)_\alpha = x_\alpha$ if $\alpha < \beta_i$. But then, $\lim_{i<\omega_0} p(i) = x$. Consequently, Y_1 is a left-C_I-space, (see 2.2). □

REMARK. If X is a LOTS, then X^{ω_0} is not necessarily a C_I-space. Example: $X = \omega_1 + 1$, (in $p = \omega_1 \ldots \omega_1 \overrightarrow{00\ldots}$ there is no countable local base).

THEOREM 4.2.3. *Let* $\mu < \omega_1$. *If, for each* $\alpha < \mu$, X_α *is a hereditarily paracompact LOTS, then* $X = \coprod_{\alpha<\mu} X_\alpha$ *is a hereditarily paracompact LOTS.*

PROOF.

(i) Suppose $\coprod_{\alpha<\mu} X_\alpha$ is a finite lexicographic product of hereditarily paracompact LOTS's X_α. It suffices to consider the case $\mu = 2$. By 4.2.2, $X = X_0 \cdot X_1$ is a paracompact LOTS. So, by 2.4.7, we have to show for each $p \in X$, the existence of discrete (in $X\backslash\{p\}$) subsets L and R in $X\backslash\{p\}$, such that L is a cofinal subset of $\{x \in X \mid x < p\}$ and R is a coinitial subset of $\{x \in X \mid p < x\}$. Let $p = (p_0,p_1) \in X = X_0 \cdot X_1$. If p_1 is not an endpoint of X_1, then we are ready, for X_1 is a hereditarily paracompact LOTS, topologically contained in X as a convex subset. Further, if, for instance, p_1 is the left endpoint of X_1, then take a discrete (in $X_0\backslash\{p_0\}$) subset L_0 in $X_0\backslash\{p_0\}$, which, also, is a cofinal subset of $\{x_0 \in X_0 \mid x_0 <_0 p_0\}$. It follows that

$$L = \{x \in X \mid x_0 \in L_0;\ x_1 = p_1\}$$

is cofinal in $\{x \in X \mid x < p\}$, while, moreover, L is discrete in $X\backslash\{p\}$. Since, clearly, R exists in $(\{p_0\} \cdot X_1) \setminus \{p\}$, the proof is finished now.

(ii) Suppose μ is a limit ordinal $< \omega_1$.

(For the sake of convenience, we adopt the following abbreviations: for each limit number $\nu \le \mu$, we write $X(\nu) = \coprod_{\alpha<\nu} X_\alpha$ and by $Y_1(\nu), Y_2(\nu)$ and $Y(\nu)$ we denote the corresponding (dense) subsets of the LOTS $X(\nu)$, defined above. Occasionally, we will denote the ordering on $X(\nu)$ also by $<$).

First, consider the case $\mu = \omega_0$. We distinguish between

1. $X(\omega_0) \setminus Y(\omega_0) = \emptyset$. Now, by 4.2.2, $X(\omega_0)$ is a paracompact LOTS, and by the previous lemma, $X(\omega_0)$ is a C_I-space. But then, from corollary 2(i) to 2.4.7, it follows that $X(\omega_0)$ is hereditarily paracompact.
2. $X(\omega_0) \setminus Y(\omega_0) \neq \emptyset$. Choose $p \in X(\omega_0)$. If $p \in Y(\omega_0)$ then, since $Y(\omega_0)$ is dense in $X(\omega_0)$, it follows (see lemma) that there are discrete (in $X(\omega_0)\setminus\{p\}$) subsets (countable or finite) L and R in $X(\omega_0)\setminus\{p\}$, which are cofinal in $\{x \in X(\omega_0) \mid x < p\}$, or coinitial in $\{x \in X(\omega_0) \mid p < x\}$, respectively. Next, let $p \notin Y(\omega_0)$. Certainly $p \in Y_1(\omega_0) \cup Y_2(\omega_0)$. Suppose, for instance, that $p \in Y_2(\omega_0)$. Since $Y_2(\omega_0)$ is a right-C_I-space (lemma) which is dense in $X(\omega_0)$, there clearly exists a discrete (in $X(\omega_0)\setminus\{p\}$) subset $R \subset \{x \in X(\omega_0) \mid p < x\}$ which is coinitial in $\{x \in X(\omega_0) \mid p < x\}$. Now, let $\beta < \omega_0$ be the first ordinal such that for each $\alpha > \beta$, p_α is the left endpoint of X_α. Since $p \notin Y_1(\omega_0)$ such an ordinal β exists. X_β is hereditarily paracompact, so we can find a discrete (in $X_\beta\setminus\{p_\beta\}$) and cofinal subset L_β of $\{x_\beta \in X_\beta \mid x_\beta <_\beta p_\beta\}$. Then, it follows that

$$L = \{x \in X(\omega_0) \mid x_\alpha = p_\alpha \text{ if } \alpha \neq \beta;\ x_\beta \in L_\beta\}$$

is discrete (in $X(\omega_0)\setminus\{p\}$) and cofinal in $\{x \in X(\omega_0) \mid x < p\}$. Thus, $X(\omega_0)$ is a hereditarily paracompact LOTS.

Secondly, we proceed by transfinite induction. Assume that, for each limit ordinal $\nu < \mu$, $X(\nu) = \coprod_{\alpha<\nu} X_\alpha$ is a hereditarily paracompact LOTS. Now, if $X(\mu) \setminus Y(\mu) = \emptyset$ then, since $\mu < \omega_1$, it follows, similarly to 1., that $X(\mu)$ is hereditarily paracompact. Next, suppose that $X(\mu) \setminus Y(\mu) \neq \emptyset$. Choose $p \in Y_2(\mu) \setminus Y_1(\mu)$. We will show the existence of a discrete (in $X(\mu)\setminus\{p\}$) and cofinal subset L in $\{x \in X(\mu) \mid x < p\}$. (The other cases follow similarly or are trivial, see 2.). Let $\beta < \mu$ be the first ordinal such that, for each $\alpha \ge \beta$, p is the left endpoint of X_α. If β is a limit number, then, by induction hypothesis, $X(\beta) = \coprod_{\alpha<\beta} X_\alpha$ is hereditarily paracompact. Hence, there exists a discrete (in $X(\beta)\setminus\{(p_\alpha)_{\alpha<\beta}\}$) and cofinal subset $L(\beta)$ in $\{(x_\alpha)_{\alpha<\beta} \in X(\beta) \mid (x_\alpha)_{\alpha<\beta} < (p_\alpha)_{\alpha<\beta}\}$. Now, put

$$L = \{x \in X(\mu) \mid (x_\alpha)_{\alpha<\beta} \in L(\beta),\ x_\alpha = p_\alpha \text{ if } \alpha \geq \beta\}.$$

Then L is a discrete (in $X(\mu)\setminus\{p\}$) and cofinal subset of $\{x \in X(\mu) \mid x < p\}$. Further, if β is a non-limit ordinal, then either $\beta = n$ with $n < \omega_0$ or $\beta = \nu+n$, where $n < \omega_0$ and ν is the largest limit number less than β. Hence $X(\beta) = \coprod_{0\leq\alpha<n} X_\alpha$ or $X(\beta) = X(\nu) \cdot \coprod_{\nu\leq\alpha<\nu+n} X_\alpha$. By induction hypothesis, $X(\nu)$ is a hereditarily paracompact LOTS. Moreover, by (i), the lexicographic product of finitely many hereditarily paracompact LOTS's is again hereditarily paracompact. Consequently, in both cases, $X(\beta)$ is a hereditarily paracompact LOTS. But then, we can proceed in $X(\beta) \cdot \coprod_{\beta\leq\alpha<\mu} X_\alpha$, analogously as we did before under the assumption that β is a limit number. This completes the proof. □

We end this section with a remark about the Lindelöf property in lexicographically ordered product spaces. Recall that a GO-space is a Lindelöf space if and only if it is paracompact and all its discrete subsets are countable or finite (2.4.3). Now, the lexicographic product of even two Lindelöf-LOTS's need not be a Lindelöf space itself. For instance, $]0,1[\cdot]0,1[$ clearly cannot be Lindelöf. On the other hand, also the converse assertion does not hold in general, for $\omega_0 \cdot \omega_1$ is a Lindelöf LOTS, but obviously ω_1 is not a Lindelöf space.

4.3. THE SECOND COUNTABILITY AXIOM, SEPARABILITY AND THE HEREDITARILY LINDELÖF PROPERTY IN $\coprod$.

In this paragraph we summarize some theorems, which give characterizations for a lexicographic product to be second countable, separable, or hereditarily Lindelöf, respectively. Since the proofs are nearly trivial, after reading the following notes, we leave them to the reader.

A lexicographically ordered product space $\coprod_{\alpha<\mu} X_\alpha$ contains uncountably many mutually disjoint convex open sets in each of the following cases:
(1) $\mu > \omega_0+1$; (2) $\mu = \omega_0+1$ and $|X_{\omega_0}| > 2$; (3) $\mu = \omega_0$ and $|X_\beta| > \aleph_0$ for some $\beta < \mu$; (4) $0 < \mu < \omega_0$, $|X_{\mu-1}| > 2$ and $|X_\beta| > \aleph_0$ for some $\beta < \mu-1$.
Furthermore, it is well-known (see, for instance, Herrlich [He.1]) and easy to see that a separable LOTS with at most countably many neighbourpoints, has a countable base.
Hence, also a separable GO-space with at most countably many jumps and pseudo-jumps has a countable base, (use 2.1.2 (ii)).

Now, $\coprod_{\alpha<\mu} X_\alpha$ contains uncountably many neighbourpoints in each of the following two cases: (1) $\mu = \omega_0+1$ and $|X_{\omega_0}| = 2$; (2) $0 < \mu < \omega_0$, $|X_{\mu-1}| = 2$ and $|X_\beta| > \aleph_0$ for some $\beta < \mu-1$.
Finally, in $\coprod_{\alpha<\mu} X_\alpha$, the subset of all points $(x_\alpha)_{\alpha<\mu}$ with constant "tail" (i.e.: $x_\beta = x_\gamma$ for sufficiently large β and $\gamma(< \mu)$) constitutes a dense set whenever μ is a limit number.

THEOREM 4.3.1. *The LOTS* $X = \coprod_{\alpha<\mu} X_\alpha$ *has a countable base if and only if* $\mu \le \omega_0$ *and one of the following conditions is satisfied:*

(i) *if* $\mu = \omega_0$ *then* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \omega_0$.

(ii) *if* $\mu < \omega_0$ *then the LOTS* $X_{\mu-1}$ *has a countable base and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \mu-1$.

THEOREM 4.3.2. *The LOTS* $X = \coprod_{\alpha<\mu} X_\alpha$ *is separable if and only if* $\mu \le \omega_0+1$ *and one of the following conditions is satisfied:*

(i) *if* $\mu = \omega_0+1$ *then* $|X_{\omega_0}| = 2$ *and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \omega_0$.

(ii) *if* $\mu = \omega_0$ *then* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \omega_0$

(iii) *if* $\mu < \omega_0$ *then* ($|X_{\mu-1}| = 2$, $X_{\mu-2}$ *has a countable base and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \mu-2$) *or* ($X_{\mu-1}$ *is a separable LOTS and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \mu-1$).

THEOREM 4.3.3. *The LOTS* $X = \coprod_{\alpha<\mu} X_\alpha$ *is hereditarily Lindelöf if and only if* $\mu \le \omega_0+1$ *and one of the following conditions is satisfied:*

(i) *if* $\mu = \omega_0+1$ *then* $|X_{\omega_0}| = 2$ *and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \omega_0$.

(ii) *if* $\mu = \omega_0$ *then* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \omega_0$.

(iii) *if* $\mu < \omega_0$ *then* ($|X_{\mu-1}| = 2$, $X_{\mu-2}$ *is hereditarily Lindelöf and possesses at most countably many neighbourpoints, and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \mu-2$) *or* ($X_{\mu-1}$ *is a hereditarily Lindelöf LOTS and* $|X_\alpha| \le \aleph_0$ *for each* $\alpha < \mu-1$).

COROLLARY. *In any GO-space, containing a dense subspace of the form* $\coprod_{\alpha<\mu} X_\alpha$ *with* $\mu \ge \omega_0$, *the notions separable and hereditarily Lindelöf are equivalent.*

We end with some examples.

EXAMPLE 1. $\mathbb{N}^{\omega_0} = \omega_0^{\omega_0} = [0,1[\ (\subset \mathbb{R})$.
The LOTS $\omega_0^{\omega_0}$ has a left endpoint but not a right one, is connected (4.1.1) and has a countable base (4.3.1). Hence, $\omega_0^{\omega_0}$ is homeomorphic to $[0,1[\ (\subset \mathbb{R})$.

EXAMPLE 2. $\mathbb{Z}^{\omega_0} = (\omega_0^*+\omega_0)^{\omega_0} = \mathbb{R} \setminus \mathbb{Q}$.
The LOTS $(\omega_0^*+\omega_0)^{\omega_0}$ has no endpoints, does not possess neighbours (1.2.1), is totally disconnected (4.1.1 and the notes thereafter), has a countable base (4.3.1) and is topologically complete (the Baire-metric is a complete metric for $(\omega_0^*+\omega_0)^{\omega_0}$). Hence, $(\omega_0^*+\omega_0)^{\omega_0}$ is homeomorphic to the space of the irrationals, (see, for instance, Kuratowski [Ku.1]).

EXAMPLE 3. $(\mathbb{Z} \setminus \mathbb{N}) \cdot \mathbb{N}^{\omega_0} = \omega_0^* \cdot \omega_0^{\omega_0} = \mathbb{R}$.
The LOTS $\omega_0^* \cdot \omega_0^{\omega_0}$ has no endpoints, is connected (4.1.1) and has a countable base (4.3.1). Hence, $\omega_0^* \cdot \omega_0^{\omega_0}$ is homeomorphic to $\mathbb{R}$.

EXAMPLE 4. Let C be a compact subset of $\mathbb{R}$, with $1 < |C| \leq \aleph_0$.
Then, the LOTS C^{ω_0} is compact (4.2.1), is totally disconnected (4.1.1 and the notes thereafter), does not contain isolated points and has a countable base (4.3.1). Hence C^{ω_0} is homeomorphic to the Cantor-space $\{0,1\}^{\aleph_0}$.

4.4. METRIZABILITY AND PERFECT NORMALITY IN $\mathbb{L}$.

We start by proving some lemmata, concerning σ-(σ-ℓ-;σ-r-)discreteness of lexicographic products.

LEMMA 1. *Let* X *and* Y *be LOTS's. If* Y *does not have a left or a right endpoint, then*

$$X \cdot Y \textit{ is } \sigma\text{-}(\sigma\text{-}\ell\text{-};\sigma\text{-}r\text{-})\textit{discrete} \Longleftrightarrow Y \textit{ is } \sigma\text{-}(\sigma\text{-}\ell\text{-};\sigma\text{-}r\text{-})\textit{discrete}.$$

PROOF.
$\Longrightarrow$ Y is topologically contained in X · Y. Hence, the assertion follows from 2.2.2.
$\Longleftarrow$ For each $x \in X$, $\{x\} \cdot Y$ (= Y) is an open subset of X · Y. Moreover, $X \cdot Y = \bigcup\{\{x\} \cdot Y \mid x \in X\}$, which is a disjoint union. But then we are done. □

LEMMA 2. *Let* X *and* Y *be LOTS's. If* Y *has both a left and a right endpoint, then*

$$X \cdot Y \textit{ is } \sigma\text{-}(\sigma\text{-}\ell\text{-};\sigma\text{-}r\text{-})\textit{discrete} \Longleftrightarrow X \textit{ and } Y \textit{ are } \sigma\text{-}(\sigma\text{-}\ell\text{-};\sigma\text{-}r\text{-})\textit{discrete}.$$

PROOF. Let ℓ denote the left endpoint of Y and let r denote the right endpoint of Y. Further, let

$$f : X \cdot Y \longrightarrow X$$

be defined by $f((x,y)) = x$, for each point $(x,y) \in X \cdot Y$.

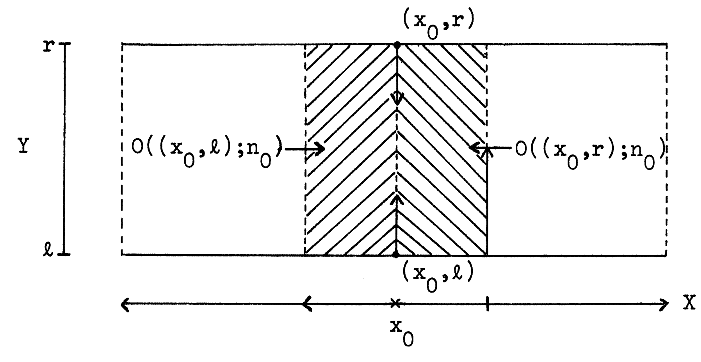
$\Longrightarrow$ We assume $X \cdot Y$ to be σ-ℓ-discrete. The other cases can be treated similarly. Since Y is topologically contained in $X \cdot Y$, by 2.2.2, Y is σ-ℓ-discrete. Next, since $X \cdot Y$ is σ-ℓ-discrete, we have: $X \cdot Y = \bigcup_{n=1}^{\infty} A_n$, where for each $(x,y) \in X \cdot Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O((x,y);n)$ of (x,y) in $X \cdot Y$, such that

$$O((x,y);n) \cap (A_n \backslash \{(x,y)\}) \cap]\leftarrow,(x,y)] = \emptyset.$$

Clearly, $X = f[X \cdot Y] = \bigcup_{n=1}^{\infty} f[A_n]$. Now, choose $x_0 \in X$ and $n_0 \in \mathbb{N}$. Define

$$U(x_0;n\ ) = \text{Int } f[O((x_0,\ell);n_0) \cup O((x\ ,r);n_0)].$$

Then $U(x_0;n_0)$ is a convex open neighbourhood of x_0 in X.



Moreover,

$$U(x_0;n_0) \cap (f[A_{n_0}] \backslash \{x_0\}) \cap]\leftarrow,x_0] = \emptyset.$$

Hence, X is σ-ℓ-discrete.

$\Longleftarrow$ We assume that X and Y are σ-ℓ-discrete. The other cases follow similarly. So, we have, $X = \bigcup_{n=1}^{\infty} A_n$ where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;n)$ of x in X, such that

$$O(x;n) \cap (A_n \setminus \{x\}) \cap]\leftarrow,x] = \emptyset,$$

and $Y = \bigcup_{n=1}^{\infty} B_n$, where for each $y \in Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $U(y;n)$ of y in Y such that

$$U(y;n) \cap (B_n \setminus \{y\}) \cap]\leftarrow,y] = \emptyset.$$

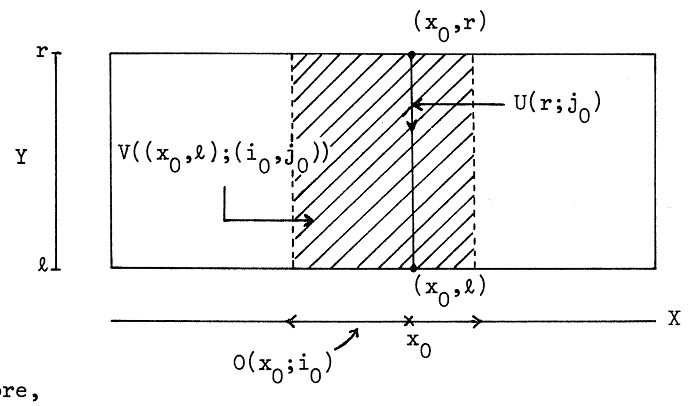
Now, for each pair (i,j) of positive integers, put

$$C_{i,j} = \bigcup \{\{x\} \cdot B_j \mid x \in A_i\}.$$

Then $X \cdot Y = \bigcup_{i,j=1}^{\infty} C_{i,j}$. Next, choose $(x_0,y_0) \in X \cdot Y$ and $(i_0,j_0) \in \mathbb{N} \times \mathbb{N}$. First, suppose that $y_0 = \ell$. Define

$$V((x_0,\ell);(i_0,j_0)) = f^{-1}[O(x_0;i_0)].$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$. (Here the fact is used that Y has two endpoints, which implies that f is continuous. However, this fact only plays a role when X has neighbour-points)



Furthermore,

$$V((x_0,\ell);(i_0,j_0)) \cap (C_{i_0,j_0} \setminus \{(x_0,\ell)\}) \cap]\leftarrow,(x_0,\ell)] = \emptyset.$$

Secondly, if $y_0 = r$, then

$$V((x_0,r);(i_0,j_0)) = ((\{x_0\} \cdot U(r;j_0))\setminus\{(x_0,\ell)\}) \cup [(x_0,r),\rightarrow[$$

is a convex open neighbourhood of (x_0,r) in $X \cdot Y$, such that

$$V((x_0,r);(i_0,j_0)) \cap (C_{i_0,j_0}\setminus\{(x_0,r)\}) \cap]\leftarrow,(x_0,r)] = \emptyset.$$

Finally, if $y_0 \neq \ell,r$, then $\{x_0\} \cdot (Y\setminus\{\ell,r\})$ is an open neighbourhood of (x_0,y_0) in $X \cdot Y$. So,

$$V((x_0,y_0);(i_0,j_0)) = \{x_0\} \cdot (U(y_0;j_0)\setminus\{\ell,r\})$$

is a convex open neighbourhood of (x_0,y_0) in $X \cdot Y$, such that

$$V((x_0,y_0);(i_0,j_0)) \cap (C_{i_0,j_0}\setminus\{(x_0,y_0)\}) \cap]\leftarrow,(x_0,y_0)] = \emptyset.$$

Thus, $X \cdot Y$ is σ-ℓ-discrete. □

LEMMA 3. *Let* X *and* Y *be LOTS's. If* Y *has a left (right) endpoint, but no right (left) one, and*

(i) *if* X *possesses neighbourpoints, then*

$X \cdot Y$ *is* σ*-discrete* $\Longleftrightarrow$ X *is* σ-ℓ-(σ-r-)*discrete,* Y *is* σ*-discrete and* $\omega_0(\omega_0^*)$ *is cofinal (coinitial) in* Y.

(ii) *if* X *does not possess neighbourpoints, then*

$X \cdot Y$ *is* σ*-discrete* $\Longleftrightarrow$ X *is* σ-ℓ-(σ-r-)*discrete and* Y *is* σ*-discrete.*

PROOF. We only prove assertion (i); (ii) follows similarly. Further, we consider the case that Y only has a left endpoint ℓ. Let

$$f : X \cdot Y \longrightarrow X$$

be defined by $f((x,y)) = x$ for each $(x,y) \in X \cdot Y$. (In general, f is not continuous).

$\Longrightarrow$ Since Y is homeomorphic to $\{x\} \cdot Y$, for any $x \in X$, it follows from 2.2.2 that Y is σ-discrete. Now, let $a \in X$ be a point which has a right neighbour a^+ in X. By 2.2.3, a σ-discrete LOTS satisfies the first axiom of countability. Consequently, there exists a countable increasing sequence in $\{a\} \cdot Y$, converging to $(a^+,\ell) \in X \cdot Y$. But then ω_0 is cofinal in Y. Finally, we have to show that X is σ-ℓ-discrete. Since $X \cdot Y$ is σ-discrete, $X \cdot Y = \bigcup_{n=1}^{\infty} A_n$ where for each $(x,y) \in X \cdot Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O((x,y);n)$ of (x,y) in $X \cdot Y$, such that

$$O((x,y);n) \cap (A_n \setminus \{(x,y)\}) = \emptyset.$$

Clearly, $X = f[X \cdot Y] = \bigcup_{n=1}^{\infty} f[A_n]$. Now, choose $x_0 \in X$ and $n_0 \in \mathbb{N}$. Define

$$U(x_0;n_0) = \text{Int } f[O((x_0,\ell);n_0)] \cup [x_0,\rightarrow[.$$

Then $U(x_0;n_0)$ is a convex open neighbourhood of x_0 in X. Moreover,

$$U(x_0;n_0) \cap (f[A_{n_0}] \setminus \{x_0\}) \cap]\leftarrow,x_0] = \emptyset.$$

Hence X is σ-ℓ-discrete.

$\Longleftarrow$ Since X is σ-ℓ-discrete, $X = \bigcup_{n=1}^{\infty} A_n$ where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;n)$ of x in X, such that

$$O(x;n) \cap (A_n \setminus \{x\}) \cap]\leftarrow,x] = \emptyset.$$

Furthermore, since Y is σ-discrete, $Y = \bigcup_{n=1}^{\infty} B_n$ where for each $y \in Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $U(y;n)$ of y in Y, such that

$$U(y;n) \cap (B_n \setminus \{y\}) = \emptyset.$$

(Moreover, we may assume that $B_n \subset B_{n+1}$, for all n). Next, since ω_0 is cofinal in Y, there is a sequence $\{y_i\}_{i=1}^{\infty}$ in Y, which is cofinal in Y. Therefore, without loss of generality

$$B_n \cap \{y \in Y \mid y_n < y\} = \emptyset \qquad (n \in \mathbb{N}).$$

Now, for each $(i,j) \in \mathbb{N} \times \mathbb{N}$, put

$$C_{i,j} = \bigcup\{\{x\} \cdot B_j \mid x \in A_i\}.$$

Then $X \cdot Y = \bigcup_{i,j=1}^{\infty} C_{ij}$. Next, choose $(x_0,y_0) \in X \cdot Y$ and $(i_0,j_0) \in \mathbb{N} \times \mathbb{N}$. First, suppose that $y_0 = \ell$. There are two possibilities to distinguish:

1. x_0 does not have a left neighbour in X. Define

$$V((x_0,\ell);(i_0,j_0)) = (\operatorname{Int} f^{-1}[O(x_0;i_0)] \cap]\leftarrow,(x_0,\ell)]) \cup (\{x_0\} \cdot U(\ell;j_0)).$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$ (Observe, that f need not be continuous). Moreover,

$$V((x_0,\ell);(i_0,j_0)) \cap (C_{i_0,j_0}\setminus\{(x_0,\ell)\}) = \emptyset.$$

2. x_0 has a left neighbour x_0^- in X. Define

$$V((x_0,\ell);(i_0,j_0)) = \{(x_0^-,y) \in X \cdot Y \mid y_{j_0} < y\} \cup (\{x_0\} \cdot U(\ell;j_0)).$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$, and

$$V((x_0,\ell);(i_0,j_0)) \cap (C_{i_0,j_0}\setminus\{(x_0,\ell)\}) = \emptyset.$$

Secondly, suppose $y_0 \neq \ell$. Now, $\{x_0\} \cdot (Y\setminus\{\ell\})$ is an open neighbourhood of (x_0,y_0) in $X \cdot Y$. Hence

$$V((x_0,y_0);(i_0,j_0)) = \{x_0\} \cdot (U(y_0;j_0)\setminus\{\ell\})$$

is an open convex neighbourhood of (x_0,y_0) in $X \cdot Y$, such that

$$V((x_0,y_0);(i_0,j_0)) \cap (C_{i_0,j_0}\setminus\{(x_0,y_0)\}) = \emptyset.$$

Thus, $X \cdot Y$ is σ-discrete. □

LEMMA 4. *Let* X *and* Y *be LOTS's. If* Y *has a left (right) endpoint, but no right (left) one, and*

(i) *if* X *possesses neighbourpoints, then*

$$X \cdot Y \textit{ is } \sigma\text{-}\ell\text{-}(\sigma\text{-r-})\textit{discrete} \iff X \textit{ and } Y \textit{ are } \sigma\text{-}\ell\text{-}(\sigma\text{-r-})\textit{discrete and } \omega_0(\omega_0^*) \textit{ is cofinal (coinitial) in } Y.$$

(ii) *if* X *does not possess neighbourpoints, then*

$$X \cdot Y \textit{ is } \sigma\text{-}\ell\text{-}(\sigma\text{-r-})\textit{discrete} \iff X \textit{ and } Y \textit{ are } \sigma\text{-}\ell\text{-}(\sigma\text{-r-})\textit{discrete}.$$

PROOF. The proof can be based upon the same arguments as the proof of the preceding lemma. (In part ((i),$\Rightarrow$) of the proof it is now sufficient to notice that $X \cdot Y$, being σ-ℓ-discrete, is a left-C_I-space; cf. 2.2.4). □

LEMMA 5. *Let* X *and* Y *be LOTS's. If* Y *has a left (right) endpoint, but no right (left) one, then*

$$X \cdot Y \textit{ is } \sigma\text{-r-}(\sigma\text{-}\ell\text{-})\textit{discrete} \iff Y \textit{ is } \sigma\text{-r-}(\sigma\text{-}\ell\text{-})\textit{discrete}.$$

PROOF. Obvious. □

Up to now, we have given necessary and sufficient conditions under which $\mathop{\underline{\|}}_{\alpha<2} X_\alpha$ is σ-(σ-ℓ-;σ-r-)discrete. Hence, in principle, also conditions can be formulated under which $\mathop{\underline{\|}}_{\alpha<n} X_\alpha$, $0 < n < \omega_0$, is (precisely) σ-(σ-ℓ-;σ-r-) discrete; the relevant results can be easily derived from the above lemmata and the fact that $\mathop{\underline{\|}}_{\alpha<n} X_\alpha = \mathop{\underline{\|}}_{\alpha<n-1} X_\alpha \cdot X_{n-1}$.

LEMMA 6. *If* μ *is a limit ordinal, then* $\mathop{\underline{\|}}_{\alpha<\mu} X_\alpha$ *is neither* σ-ℓ-, *nor* σ-r-*discrete, and hence not* σ-*discrete.*

PROOF. See 2.2.11 and the remark thereafter. □

Since each ordinal μ can be represented by $\mu = n$ or $\mu = \lambda+n$, or $\mu = \lambda$, where $n < \omega_0$ and λ is a limit ordinal, we are able now to characterize σ-, σ-ℓ-, and σ-r-discreteness of a lexicographic product space $\coprod_{\alpha<\mu} X_\alpha$ in terms of the factorspaces.

THEOREM 4.4.1. *Let* X *and* Y *be LOTS's. If* Y *does not have a left or a right endpoint, then*

$$X \cdot Y \textit{ is metrizable} \Leftrightarrow Y \textit{ is metrizable.}$$

PROOF. Immediately clear, since Y is topologically contained in $X \cdot Y$, and, on the other hand, $X \cdot Y = \bigcup\{\{x\} \cdot Y \mid x \in X\}$ is a disjoint union of open subsets of $X \cdot Y$, each homeomorphic to Y. □

THEOREM 4.4.2. *Let* X *and* Y *be LOTS's. If* Y *has both a left and a right endpoint, then*

$$X \cdot Y \textit{ is metrizable} \Leftrightarrow X \textit{ is } \sigma\textit{-discrete and } Y \textit{ is metrizable.}$$

PROOF. Let ℓ denote the left endpoint of Y and let r denote the right endpoint of Y. Furthermore, let

$$f : X \cdot Y \longrightarrow X$$

be defined by $f((x,y)) = x$, for each $(x,y) \in X \cdot Y$. Since Y has two endpoints, f is a continuous surjection.

$\Longrightarrow$ As a subspace of $X \cdot Y$, Y is metrizable. Next, since $X \cdot Y$ is metrizable, there exists a dense, σ-discrete (in $X \cdot Y$) subset D in $X \cdot Y$ such that $N(X \cdot Y) \subset D$, (3.1). From D is dense in $X \cdot Y$ and $N(X \cdot Y) \subset D$ it follows, that $f[D] = X$. Further, since D is σ-discrete, $D = \bigcup_{n=1}^{\infty} D_n$ where for each $(x,y) \in X \cdot Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O((x,y);n)$ of (x,y) in $X \cdot Y$, such that

$$O((x,y);n) \cap (D_n \setminus \{(x,y)\}) = \emptyset.$$

Now, $X = f[D] = \bigcup_{n=1}^{\infty} f[D_n]$. Choose $x_0 \in X$ and $n_0 \in \mathbb{N}$. Define

$$U(x_0;n_0) = \text{Int } f[O((x_0,\ell);n_0) \cup O((x_0,r);n_0)].$$

Then $U(x_0;n_0)$ is a convex open neighbourhood of x_0 in X, while

$$U(x_0;n_0) \cap (f[D_{n_0}]\setminus\{x_0\}) = \emptyset.$$

Hence X is σ-discrete.

$\Longleftarrow$ Since X is σ-discrete, $X = \bigcup_{n=1}^{\infty} A_n$ where for each $x \in X$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $O(x;n)$ of x in X, such that

$$O(x;n) \cap (A_n\setminus\{x\}) = \emptyset.$$

Next, since Y is metrizable, there is a dense subset B in Y such that $N(Y) \subset B$ and $B = \bigcup_{n=1}^{\infty} B_n$, where for each $y \in Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $U(y;n)$ of y in Y, such that

$$U(y;n) \cap (B_n\setminus\{y\}) = \emptyset.$$

Without loss of generality, we assume that $\ell, r \in B$. Now, for each pair $(i,j) \in \mathbb{N} \times \mathbb{N}$, put

$$D_{i,j} = \bigcup\{\{x\} \cdot B_j \mid x \in A_i\}$$

and let

$$D = \bigcup_{i,j=1}^{\infty} D_{i,j}.$$

Clearly, D is a dense subset of $X \cdot Y$. Also, $N(X \cdot Y) \subset D$. So, by 3.1, the proof is complete once we have shown that D is σ-discrete (in $X \cdot Y$). For that purpose, choose $(x_0,y_0) \in X \cdot Y$ and $(i_0,j_0) \in \mathbb{N} \times \mathbb{N}$. First, suppose that $y_0 = \ell$ (or, similarly, that $y_0 = r$). Define

$$V((x_0,\ell);(i_0,j_0)) =$$
$$= (f^{-1}[O(x_0;i_0)] \cap]\leftarrow,(x_0,\ell)]) \cup ((\{x_0\} \cdot U(\ell;j_0))\setminus\{(x_0,r)\}).$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$. Moreover

$$V((x_0,\ell);(i_0,j_0)) \cap (D_{i_0,j_0}\setminus\{(x_0,\ell)\}) = \emptyset.$$

Secondly, suppose that $y_0 \neq \ell, r$. Then

$$V((x_0,y_0);(i_0,j_0)) = \{x_0\} \cdot (U(y_0;j_0)\setminus\{\ell,r\})$$

is a convex open neighbourhood of (x_0,y_0) in $X \cdot Y$, and

$$V((x_0,y_0);(i_0,j_0)) \cap (D_{i_0,j_0}\setminus\{(x_0,y_0)\}) = \emptyset.$$

Hence, D is σ-discrete (in $X \cdot Y$). □

THEOREM 4.4.3. *Let* X *and* Y *be LOTS's. If* Y *has a left (right) endpoint, but no right (left) one, and*

(i) *if* X *possesses neighbourpoints, then*

$$X \cdot Y \textit{ is metrizable} \Leftrightarrow X \textit{ is } \sigma\text{-}\ell\text{-}(\sigma\text{-}r\text{-})\textit{discrete}, Y \textit{ is metrizable and } \omega_0(\omega_0^*) \textit{ is cofinal (coinitial) in } Y.$$

(ii) *if* X *does not possess neighbourpoints, then*

$$X \cdot Y \textit{ is metrizable} \Leftrightarrow X \textit{ is } \sigma\text{-}\ell\text{-}(\sigma\text{-}r\text{-})\textit{discrete and } Y \textit{ is metrizable}.$$

PROOF. We only prove assertion (i); (ii) follows similarly. Moreover, we restrict ourselves to the case that Y only has a left endpoint ℓ. Let

$$f : X \cdot Y \to X$$

be defined by $f((x,y)) = x$, for each $(x,y) \in X \cdot Y$. In general, f is not continuous.

$\Longrightarrow$ Since Y is topologically contained in $X \cdot Y$, Y is metrizable. Now, let $a \in X$ be a point which has a right neighbour a^+ in X. Since $X \cdot Y$ is a C_I-space, there is a countable sequence in $\{a\} \cdot Y$ converging to (a^+,ℓ) in $X \cdot Y$. Hence ω_0 is cofinal in Y. Finally, we show that X is σ-ℓ-discrete. $X \cdot Y$ is metrizable, so there exists a dense subset D in $X \cdot Y$ such that $N(X \cdot Y) \subset D$ and $D = \bigcup_{n=1}^{\infty} D_n$, where for each $(x,y) \in X \cdot Y$ and each $n \in \mathbb{N}$, there exists a convex open neighbourhood $O((x,y);n)$ of (x,y) in $X \cdot Y$, such that

$$O((x,y);n) \cap (D_n\setminus\{(x,y)\}) = \emptyset \qquad \text{(see 3.1)}.$$

From D is dense in $X \cdot Y$ it follows that $f[D] = X$. (Observe that the condition $N(X \cdot Y) \subset D$ is not required here, in contrast with the proof of the previous theorem). So, $X = \bigcup_{n=1}^{\infty} f[D_n]$. Now, choose $x_0 \in X$ and $n_0 \in \mathbb{N}$. Define

$$U(x_0;n_0) = \text{Int } f[O((x_0,\ell);n_0)] \cup [x_0,\rightarrow[.$$

Then $U(x_0;n_0)$ is a convex open neighbourhood of x in X, while

$$U(x_0;n_0) \cap (f[D_n]\setminus\{x_0\}) \cap]\leftarrow,x_0] = \emptyset.$$

Hence X is σ-ℓ-discrete.

$\Longleftarrow$ Since X is σ-ℓ-discrete, $X = \bigcup_{n=1}^{\infty} A_n$ where for each $x \in X$ and $n \in \mathbb{N}$, there exists a convex open neighbourhood $O(x;n)$ of x in X, such that

$$O(x;n) \cap (A_n\setminus\{x\}) \cap]\leftarrow,x] = \emptyset.$$

Further, Y is metrizable. So, by 3.1, there is a dense subset B in Y such that $N(Y) \subset B$ and $B = \bigcup_{n=1}^{\infty} B_n$, where for each $y \in Y$ and each $n \in \mathbb{N}$ there exists a convex open neighbourhood $U(y;n)$ of y in Y such that

$$U(y;n) \cap (B_n\setminus\{y\}) = \emptyset.$$

Moreover, we may assume that $B_n \subset B_{n+1}$, for all n. Since ω_0 is cofinal in Y, there is a sequence $\{y_i\}_{i=1}^{\infty}$ in Y which is cofinal in Y. Without loss of generality,

$$B_n \cap \{y \in Y \mid y_n < y\} = \emptyset \qquad (n \in \mathbb{N}).$$

Now, for each pair $(i,j) \in \mathbb{N} \times \mathbb{N}$, put

$$D_{ij} = \bigcup\{\{x\} \cdot B_j \mid x \in A_i\}$$

and, let

$$D = \bigcup_{i,j=1}^{\infty} D_{i,j}$$

Clearly, D is a dense subset of $X \cdot Y$. And, also $N(X \cdot Y) \subset D$. So, by 3.1, the proof is complete once we have shown that D is σ-discrete (in $X \cdot Y$). For that purpose, choose $(x_0,y_0) \in X \cdot Y$ and $(i_0,j_0) \in \mathbb{N} \times \mathbb{N}$. First, suppose $y_0 = \ell$. We distinguish between two possibilities:

1. x_0 does not have a left neighbour in X. Define

$$V((x_0,\ell);(i_0,j_0)) =$$
$$(\mathrm{Int}\, f^{-1}[O(x_0;i_0)] \cap]\leftarrow,(x_0,\ell)]) \cup (\{x_0\} \cdot U(\ell;j_0)).$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$, and

$$V((x_0,\ell);(i_0,j_0)) \cap (D_{i_0,j_0}\backslash\{(x_0,\ell)\}) = \emptyset.$$

2. x_0 has a left neighbour x_0^- in X. Define

$$V((x_0,\ell);(i_0,j_0)) =$$
$$= \{(x_0^-,y) \in X \cdot Y \mid y_{j_0} < y\} \cup (\{x_0\} \cdot U(\ell;j_0)).$$

Then $V((x_0,\ell);(i_0,j_0))$ is a convex open neighbourhood of (x_0,ℓ) in $X \cdot Y$, while

$$V((x_0,\ell);(i_0,j_0)) \cap (D_{i_0,j_0}\backslash\{(x_0,\ell)\}) = \emptyset.$$

Secondly, suppose $y_0 \neq \ell$. Then

$$V((x_0,y_0);(i_0,j_0)) = \{x_0\} \cdot (U(y_0;j_0)\backslash\{\ell\})$$

is a convex open neighbourhood of (x_0,y_0) in $X \cdot Y$, while, moreover

$$V((x_0,y_0);(i_0,j_0)) \cap (D_{i_0,j_0}\backslash\{(x_0,y_0)\}) = \emptyset.$$

Thus, it follows that D is σ-discrete (in $X \cdot Y$). □

Clearly, thus far, we have obtained theorems characterizing the metrizability of $\coprod_{\alpha<\mu} X_\alpha$ in terms of the factorspaces, for the case $\mu < \omega_0$. But then, also for $\mu = \lambda+n$, where $0 < n < \omega_0$ and λ is a limit ordinal, a characterization is given now. For, $\coprod_{\alpha<\mu} X_\alpha = \coprod_{\alpha<\lambda} X_\alpha \cdot \coprod_{\lambda\leq\alpha<\lambda+n} X_\alpha$ and so, as can be easily checked, the theorems and lemmata developed up to now yield a characterization for the metrizability of $\coprod_{\alpha<\mu} X_\alpha$ in terms of the factors. Therefore, the next theorems deal with the case that μ is a limit ordinal. However, if μ is a limit ordinal which is not cofinal with ω_0, then it follows from lemma 1 to 2.2.11 that $\coprod_{\alpha<\mu} X_\alpha$ is not a C_I-space and hence not metrizable. So, it remains to consider the case where μ is cofinal with ω_0. We start with $\mu = \omega_0$.

THEOREM 4.4.4. *For each $\alpha < \omega_0$, let X_α be a linearly ordered set without a left or a right endpoint. Then, the LOTS $X = \coprod_{\alpha<\omega_0} X_\alpha$ is metrizable.*

PROOF. [Of course, this theorem is well-known, because the so-called Baire-metric on X induces a topology on X which coincides with the order-topology. However, we like to give another proof here, based on the theory developed in this treatise].
For each $\alpha < \omega_0$, choose a point $p_\alpha \in X_\alpha$. Next, for all $\beta < \omega_0$, define $D_\beta \subset X$ by

$$D_\beta = \{x = (x_\alpha)_{\alpha<\omega_0} \in X \mid x_\alpha = p_\alpha \text{ if } \beta < \alpha\}.$$

Furthermore, put

$$D = \bigcup\{D_\beta \mid \beta < \omega_0\}.$$

We show that D satisfies the following properties:

1. D is a dense subset of X.
   Indeed, let $]s,t[$ be any non-empty open interval in X. Since $s < t$, there is a first ordinal $\beta < \omega_0$ such that $s_\beta <_\beta t_\beta$. Further, since $X_{\beta+1}$ does not have a right endpoint, there exists a point $u \in X_{\beta+1}$ with $s_{\beta+1} <_{\beta+1} u$. Now, let $y = (y_\alpha)_{\alpha<\omega_0} \in X$ be defined by $y_\alpha = s_\alpha$ if $\alpha \leq \beta$, $y_{\beta+1} = u$ and $y_\alpha = p_\alpha$ if $\beta+1 < \alpha$. Then, $y \in]s,t[\cap D$.
2. $N(X) \subset D$.
   Obvious, since $N(X) = \emptyset$. (see 1.2.1)
3. D is σ-discrete (in X).
   We prove that each D_β $(\beta < \omega_0)$ is discrete (in X). Choose $x \in X$ and

$\beta < \omega_0$. If $x \in D_\beta$, then take $u,v \in X_{\beta+1}$ such that

$$u <_{\beta+1} x_{\beta+1} = p_{\beta+1} <_{\beta+1} v.$$

Next, choose $y,z \in X$, such that $y_\alpha = z_\alpha = x_\alpha$ for $\alpha \le \beta$ and $y_{\alpha+1} = u$ and $z_{\alpha+1} = v$. Then $x \in \,]y,z[$ while, moreover,

$$]y,z[\, \cap (D_\beta \setminus \{x\}) = \emptyset.$$

If $x \notin D_\beta$, then for some $\gamma > \beta$, $x_\gamma \ne p_\gamma$. Take, $u,v \in X_{\gamma+1}$ such that $u <_{\gamma+1} x_{\gamma+1} <_{\gamma+1} v$. Next, choose $y,z \in X$ such that $y_\alpha = z_\alpha = x_\alpha$ if $\alpha \le \gamma$ and $y_{\gamma+1} = u$ and $z_{\gamma+1} = v$. Then, $x \in \,]y,z[$ while, moreover

$$]y,z[\, \cap D_\beta = \emptyset.$$

Summarizing, we conclude, by 3.1, that X is metrizable. □

THEOREM 4.4.5. *For each* $\alpha < \omega_0$, *let* X_α *be a LOTS with a left and a right endpoint. Then*

$$\textit{the LOTS } X = \coprod_{\alpha<\omega_0} X_\alpha \textit{ is metrizable} \Longleftrightarrow X_\alpha \textit{ is } \sigma\textit{-discrete, for all } \alpha < \omega_0.$$

PROOF.

$\Longrightarrow$ Choose $\beta < \omega_0$. Since $\coprod_{\alpha<\omega_0} X_\alpha = \coprod_{\alpha\le\beta} X_\alpha \cdot \coprod_{\beta<\alpha<\omega_0} X_\alpha$, it follows from 4.4.2 that $\coprod_{\alpha\le\beta} X_\alpha$ is a σ-discrete LOTS. Furthermore, since, (for $\beta > 0$), $\coprod_{\alpha\le\beta} X_\alpha = \coprod_{\alpha<\beta} X_\alpha \cdot X_\beta$, lemma 2 yields that X_β is σ-discrete.

$\Longleftarrow$ For each $\alpha < \omega_0$, let ℓ_α be the left endpoint of X_α and let r_α be the right endpoint of X_α. Every X_α is σ-discrete, so $X_\alpha = \bigcup_{n=1}^{\infty} A_n^\alpha$ where for each $x_\alpha \in X_\alpha$ and each $n \in \mathbb{N}$, there exists a convex open neighbourhood $U(x_\alpha;n)$ of x_α in X_α such that

$$U(x_\alpha;n) \cap (A_n^\alpha \setminus \{x_\alpha\}) = \emptyset.$$

Now, for all $\beta < \omega_0$ and all points $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$, we define subsets $D^\beta_{i_0,\dots,i_\beta}$ of $\coprod_{\alpha\le\beta} X_\alpha$ such that

$$\coprod_{\alpha\le\beta} X_\alpha = \bigsqcup_{i_0,\dots,i_\beta=1}^{\infty} D^\beta_{i_0,\dots,i_\beta}$$

while, moreover, for each point $(x_0,\dots,x_\beta) \in \coprod_{\alpha\le\beta} X_\alpha$ and for each point $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$ there is a convex open neighbourhood $U((x_0,\dots,x_\beta);(i_0,\dots,i_\beta))$ of $(x_0,\dots,x_\beta)$ in $\coprod_{\alpha\le\beta} X_\alpha$, such that

$$U((x_0,\dots,x_\beta);(i_0,\dots,i_\beta)) \cap (D^\beta_{i_0,\dots,i_\beta} \setminus \{(x_0,\dots,x_\beta)\}) = \emptyset$$

(in other words: $D^\beta_{i_0,\dots,i_\beta}$ is discrete in the LOTS $\coprod_{\alpha\le\beta} X_\alpha$). As follows: Let $D^0_i = A^0_i$; $i \in \mathbb{N}$. Next, suppose $D^\alpha_{i_0,\dots,i_\alpha}$ is defined, for each $\alpha < \beta$ ($\beta > 0$), and each $(i_0,\dots,i_\alpha) \in \mathbb{N}^{\alpha+1}$, such that all required properties do hold. Now, for each $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$, put

$$D^\beta_{i_0,\dots,i_\beta} =$$
$$= \bigcup\{\{(x_0,\dots,x_{\beta-1})\} \cdot A^\beta_{i_\beta} \mid (x_0,\dots,x_{\beta-1}) \in D^{\beta-1}_{i_0,\dots,i_{\beta-1}}\}$$

Then also $D^\beta_{i_0,\dots,i_\beta}$ fulfils the required conditions (compare, for instance, the proof of lemma 2). Finally, let

$$D_{i_0,\dots,i_\beta} = \{x = (x_\alpha)_{\alpha<\omega_0} \in X \mid (x_0,\dots,x_\beta) \in D^\beta_{i_0,\dots,i_\beta};$$
$$\text{and either } (\forall\alpha > \beta : x_\alpha = \ell_\alpha) \text{ or } (\forall\alpha > \beta : x_\alpha = r_\alpha)\}$$

and put

$$D = \bigcup\{D_{i_0,\dots,i_\beta} \mid (i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1};\ \beta < \omega_0\}.$$

Then D satisfies the following properties:

1. D is a dense subset of X.
   Since $(\ell_\alpha)_{\alpha<\omega_0}$ and $(r_\alpha)_{\alpha<\omega_0}$ are non-isolated endpoints of X, it suffices to show, that D intersects any non-empty open interval of the form $]s,t[$ in X. Now, let $\beta < \omega_0$ be the first ordinal such that $s_\beta <_\beta t_\beta$. First, suppose $]s_\beta,t_\beta[\ne \emptyset$. Take $x_\beta \in]s_\beta,t_\beta[$. Then

$$\{(s_0,\dots,s_{\beta-1},x_\beta)\} \cdot \coprod_{\beta<\alpha<\omega_0} X_\alpha \subset]s,t[.$$

   Since $(s_0,\dots,s_{\beta-1},x_\beta) \in D^\beta_{i_0,\dots,i_\beta}$, for some $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$, it follows now that

$$D_{i_0,\dots,i_\beta} \cap \,]s,t[\, \neq \emptyset.$$

Hence $D \cap \,]s,t[\, \neq \emptyset$. Secondly, suppose $]s_\beta,t_\beta[\, = \emptyset$. Because $]s,t[\, \neq \emptyset$, we may assume that, for some $\gamma > \beta$, $s_\gamma \neq r_\gamma$. (Otherwise, $t_\gamma \neq \ell_\gamma$ for some $\gamma > \beta$). But, then

$$\{(s_0,\dots,s_{\gamma-1},r_\gamma)\} \cdot \coprod_{\gamma<\alpha<\omega_0} X_\alpha \subset \,]s,t[.$$

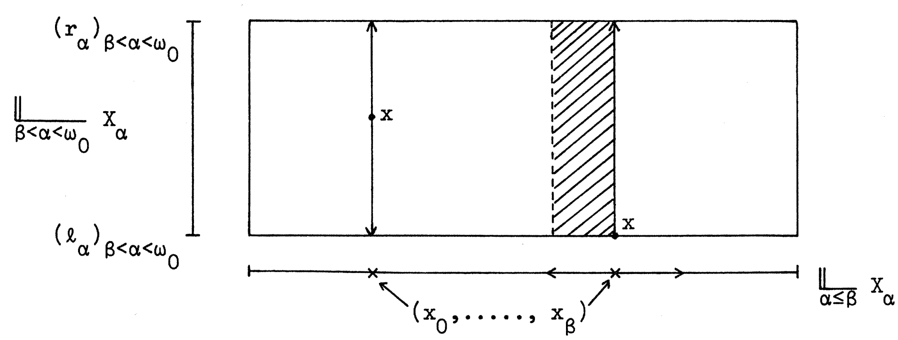
Since, further, for some $(i_0,\dots,i_\gamma) \in \mathbb{N}^{\gamma+1}$, $(s_0,\dots,s_{\gamma-1},r_\gamma) \in D^\gamma_{i_0,\dots,i_\gamma}$, it follows that

$$D_{i_0,\dots,i_\gamma} \cap \,]s,t[\, \neq \emptyset.$$

Consequently, $D \cap \,]s,t[\, \neq \emptyset$.

2. $N(X) \subset D$.
   Immediately clear from 1.2.1 and the definition of D.
3. D is σ-discrete (in X).
   We prove that each set $D_{i_0,\dots,i_\beta}$ is discrete (in X). Choose $x = (x_\alpha)_{\alpha<\omega_0} \in X$, $\beta < \omega_0$ and $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$. Observe that

$$D_{i_0,\dots,i_\beta} = D^\beta_{i_0,\dots,i_\beta} \cdot \{(\ell_\alpha)_{\beta<\alpha<\omega_0}, (r_\alpha)_{\beta<\alpha<\omega_0}\}$$



Now, if $(\ell_\alpha)_{\beta<\alpha<\omega_0} \neq (x_\alpha)_{\beta<\alpha<\omega_0} \neq (r_\alpha)_{\beta<\alpha<\omega_0}$, then define $y,z \in X$ by $y_\alpha = z_\alpha = x_\alpha$ if $\alpha \leq \beta$, $y_\alpha = \ell_\alpha$ if $\alpha > \beta$ and $z_\alpha = r_\alpha$ if $\alpha > \beta$. Certainly, $x \in \,]y,z[$. Moreover

$$]y,z[\, \cap D_{i_0,\dots,i_\beta} = \emptyset.$$

Next, if $x_\alpha = \ell_\alpha$ for all $\alpha > \beta$ (or, similarly, if $x_\alpha = r_\alpha$ for all $\alpha > \beta$), then the set

$$O(x;(i_0,\dots,i_\beta)) =$$
$$= (U((x_0,\dots,x_\beta);(i_0,\dots,i_\beta)) \cap]\leftarrow,(x_0,\dots,x_\beta)]) \cdot \coprod_{\beta<\alpha<\omega_0} X_\alpha$$
$$\setminus \{(x_0,\dots,x_\beta,\overrightarrow{r_{\beta+1},\dots})\}$$

is a convex open neighbourhood of x in X, such that

$$O(x;(i_0,\dots,i_\beta)) \cap (D_{i_0,\dots,i_\beta}\setminus\{x\}) = \emptyset.$$

Finally, from 3.1, it follows now that X is metrizable. □

THEOREM 4.4.6. *For each* $\alpha < \omega_0$, *let* X_α *be a LOTS having a left (right) endpoint, but no right (left) one. Then*

the LOTS $X = \coprod_{\alpha<\omega_0} X_\alpha$ *is metrizable* $\Leftrightarrow$ X_α *is* σ-ℓ-$(\sigma$-r-$)$*discrete for all* $\alpha < \omega_0$, *and* $\omega_0(\omega_0^*)$ *is cofinal (coinitial) in* X_α $(\alpha > 0)$ *whenever* $X_{\alpha-1}$ *possesses neighbourpoints.*

PROOF. We assume that every X_α only has a left endpoint ℓ_α. The other case can be treated similarly.

$\Longrightarrow$ Choose $\beta < \omega_0$. Since $\coprod_{\alpha<\omega_0} X_\alpha = \coprod_{\alpha\le\beta} X_\alpha \cdot \coprod_{\beta<\alpha<\omega_0} X_\alpha$, it follows from 4.4.3 that $\coprod_{\alpha\le\beta} X_\alpha$ is a σ-ℓ-discrete LOTS. Furthermore, since, (for $\beta > 0$), $\coprod_{\alpha\le\beta} X_\alpha = \coprod_{\alpha<\beta} X_\alpha \cdot X_\beta$, lemma 4 yields that X_β is σ-ℓ-discrete, while moreover ω_0 is cofinal in X_β whenever $\coprod_{\alpha<\beta} X_\alpha$ possesses neighbourpoints. However, $\coprod_{\alpha<\beta} X_\alpha$ has neighbours if and only if $X_{\beta-1}$ possesses neighbourpoints.

$\Longleftarrow$ Every X_α is σ-ℓ-discrete. So, $X_\alpha = \bigcup_{n=1}^{\infty} A_n^\alpha$ where for each $x_\alpha \in X_\alpha$ and each $n \in \mathbb{N}$, there exists a convex open neighbourhood $U(x_\alpha;n)$ of x_α in X_α, such that

$$U(x_\alpha;n) \cap (A_n^\alpha\setminus\{x_\alpha\}) \cap]\leftarrow,x_\alpha] = \emptyset.$$

Also, we may assume that $A_n^\alpha \subset A_{n+1}^\alpha$, for all n and α. Now, when $X_{\alpha-1}$, $\alpha > 0$, possesses neighbourpoints for some $\alpha < \omega_0$, then ω_0 is cofinal in X_α. So, there is a sequence $\{a_\alpha(i)\}_{i=1}^\infty$ in X_α which is cofinal in X_α. Hence, in this case, we may assume that

$$A_n^\alpha \cap \{x_\alpha \in X_\alpha \mid a_\alpha(n) <_\alpha x_\alpha\} = \emptyset.$$

Next, for all $\beta < \omega_0$ and all points $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$ we define subsets $D^\beta_{i_0,\dots,i_\beta}$ of $\coprod_{\alpha\le\beta} X_\alpha$, such that

$$\coprod_{\alpha\le\beta} X_\alpha = \bigcup_{i_0,\dots,i_\beta=1}^{\infty} D^\beta_{i_0,\dots,i_\beta},$$

while for each point $(x_0,\dots,x_\beta) \in \coprod_{\alpha\le\beta} X_\alpha$ and each point $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$ there exists a convex open neighbourhood $U((x_0,\dots,x_\beta);(i_0,\dots,i_\beta))$ of $(x_0,\dots,x_\beta)$ in $\coprod_{\alpha\le\beta} X_\alpha$, such that

$$U((x_0,\dots,x_\beta);(i_0,\dots,i_\beta)) \cap$$
$$\cap\ (D^\beta_{i_0,\dots,i_\beta}\setminus\{(x_0,\dots,x_\beta)\}) \cap\ ]\leftarrow,(x_0,\dots,x_\beta)] = \emptyset$$

(in other words: $D^\beta_{i_0,\dots,i_\beta}$ is a σ-ℓ-discrete subset of the LOTS $\coprod_{\alpha\le\beta} X_\alpha$). As follows: Let $D_i^0 = A_i^0$; $i \in \mathbb{N}$. Next, suppose $D^\alpha_{i_0,\dots,i_\alpha}$ is defined for each $\alpha < \beta$ $(\beta > 0)$ and each $(i_0,\dots,i_\alpha) \in \mathbb{N}^{\alpha+1}$, such that all required properties do hold. Then, for each $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$, put

$$D^\beta_{i_0,\dots,i_\beta} =$$
$$= \bigcup\{\{(x_0,\dots,x_{\beta-1})\}\cdot A^\beta_{i_\beta} \mid (x_0,\dots,x_{\beta-1}) \in D^{\beta-1}_{i_0,\dots,i_{\beta-1}}\}.$$

Now, also $D^\beta_{i_0,\dots i_\beta}$ satisfies the required conditions (compare, for instance, the proof of lemma 3). Finally, let

$$D_{i_0,\dots,i_\beta} = \{x = (x_\alpha)_{\alpha<\omega_0} \in X \mid (x_0,\dots,x_\beta) \in D^\beta_{i_0,\dots,i_\beta}; \text{ and}$$
$$x_\alpha = \ell_\alpha \text{ for all } \alpha > \beta\}$$

and put

$$D = \bigcup\{D_{i_0,\dots,i_\beta} \mid (i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1};\ \beta < \omega_0\}.$$

Then D satisfies the following properties

1. D is a dense subset of X.

   Since $(\ell_\alpha)_{\alpha<\omega_0}$ is a non-isolated left endpoint of X, it is sufficient to show, that D intersects any non-empty open interval $]s,t[$ in X. Let β be the first ordinal such that $s_\beta <_\beta t_\beta$. Take $u \in X_{\beta+1}$ with $s_{\beta+1} <_{\beta+1} u$. Then

   $$\{(s_0,\dots,s_\beta,u)\} \cdot \coprod_{\beta+1<\alpha<\omega_0} X_\alpha \subset\ ]s,t[.$$

   Furthermore, for some $(i_0,\dots,i_{\beta+1}) \in \mathbb{N}^{\beta+2}$, we have that

   $$(s_0,\dots,s_\beta,u) \in D^{\beta+1}_{i_0,\dots,i_{\beta+1}}.$$

   Therefore,

   $$D_{i_0,\dots,i_{\beta+1}} \cap\ ]s,t[\ \neq \emptyset.$$

   Hence, $D \cap\ ]s,t[\ \neq \emptyset$.

2. $N(X) \subset D$.

   Obvious, since $N(X) = \emptyset$ (see 1.2.1).

3. D is σ-discrete (in X).

   For each $\beta < \omega_0$ and each $(i_0,\dots,i_\beta) \in \mathbb{N}^{\beta+1}$ we show that $D_{i_0,\dots,i_\beta}$ is discrete in X. Choose $x = (x_\alpha)_{\alpha<\omega_0} \in X$. Observe that

   $$D_{i_0,\dots,i_\beta} = D^{\beta}_{i_0,\dots,i_\beta} \cdot \{(\ell_\alpha)_{\beta<\alpha<\omega_0}\}.$$

   If $(\ell_\alpha)_{\beta<\alpha<\omega_0} \neq (x_\alpha)_{\beta<\alpha<\omega_0}$, then

   $$O(x;(i_0,\dots,i_\beta)) = \{(x_0,\dots,x_\beta)\} \cdot (\coprod_{\beta<\alpha<\omega_0} X_\alpha \setminus \{(\ell_\alpha)_{\beta<\alpha<\omega_0}\})$$

   is a convex open neighbourhood of x in X, while

   $$O(x;(i_0,\dots,i_\beta)) \cap D_{i_0,\dots,i_\beta} = \emptyset.$$

   Further, if $x_\alpha = \ell_\alpha$ for all $\alpha > \beta$, then

$$O(x;(i_0,\dots,i_\beta)) =$$

$$= \mathrm{Int}\ [(U((x_0,\dots,x_\beta);\ (i_0,\dots,i_\beta)) \cap\ ]\leftarrow,(x_0,\dots,x_\beta)]) \cdot \prod_{\beta<\alpha<\omega_0} X_\alpha]$$

is a convex open neighbourhood of x in X, such that

$$O(x;(i_0,\dots,i_\beta)) \cap (D_{i_0,\dots,i_\beta} \setminus \{x\}) = \emptyset.$$

Summarizing, it follows from 3.1 that X is metrizable. □

From the results obtained so far, one may easily derive theorems for the remaining case, not considered up to now, of a product $X = \prod_{\alpha<\mu} X_\alpha$, where ω_0 is cofinal in μ, and, for the sake of completeness, where no special requirements are presupposed with respect to the (occurrence of) endpoints of the X_α's. For, suppose, firstly, that there exists a cofinal increasing sequence $\{\beta_i\}_{i<\omega_0}$ in μ, such that none of $Y_0 = \prod_{\alpha\le\beta_1} X_\alpha$ and $Y_i = \prod_{\beta_i<\alpha\le\beta_{i+1}} X_\alpha$ $(i > 0)$ has a left or a right endpoint. Then, by 4.4.4

$$\prod_{\alpha<\mu} X_\alpha = \prod_{i<\omega_0} Y_i$$

is a metrizable LOTS. Secondly, suppose there exists an ordinal $\beta < \mu$ such that X_α has a left and a right endpoint for each $\alpha \ge \beta$. Let β be the smallest ordinal with this property. Now,

$$X = \prod_{\alpha<\mu} X_\alpha = \prod_{\alpha<\beta} X_\alpha \cdot \prod_{\beta\le\alpha<\mu} X_\alpha.$$

Hence, a necessary and sufficient condition, in order that X be metrizable, is, by 4.4.2, that $\prod_{\alpha<\beta} X_\alpha$ is σ-discrete and $\prod_{\beta\le\alpha<\mu} X_\alpha$ is metrizable. So, in that case, by lemma 6, β is a non-limit ordinal and, moreover, by 4.4.2 applied to $\prod_{\beta\le\alpha<\mu} X_\alpha$, $\mu = \beta+\omega_0$. Now, it is easy to derive from the above lemmata and theorems conditions on the factorspaces X_α, which are necessary and sufficient for the metrizability of X. Thirdly, suppose (for instance) that there exists a (smallest) ordinal $\beta < \mu$ such that, for each $\alpha \ge \beta$, X_α has a left endpoint, while, moreover, there is a cofinal increasing sequence $\{\beta_i\}_{i<\omega_0}$, starting from β $(= \beta_0)$, in μ such that $Y_i = \prod_{\beta_i\le\alpha<\beta_{i+1}} X_\alpha$ $(i < \omega_0)$, does not have a right endpoint. Now

$$X = \prod_{\alpha<\mu} X_\alpha = \prod_{\alpha<\beta} X_\alpha \cdot \prod_{i<\omega_0} Y_i.$$

We argue as follows. In order that X be metrizable it is necessary and sufficient, by 4.4.3, that $\underline{\|}_{\alpha<\beta} X_\alpha$ is σ-ℓ-discrete and $\underline{\|}_{i<\omega_0} Y_i$ is metrizable and, moreover, ω_0 is cofinal in $\underline{\|}_{i<\omega_0} Y_i$ whenever $\underline{\|}_{\alpha<\beta} X_\alpha$ has neighbours. Hence, in that case by 4.4.6, each factor Y_i $(i < \omega_0)$ is σ-ℓ-discrete and (in case $i > 0$) cofinal with ω_0 whenever Y_{i-1} has neighbours. Further, lemmata 2, 4 and 6, applied to $Y_i = \underline{\|}_{\beta_i\le\alpha<\beta_{i+1}} X_\alpha$, yield that the set of ordinals $\{\alpha \mid \beta_i \le \alpha < \beta_{i+1}\}$ is finite. Then, from the lemmata 2 and 4 we conclude that, for each $\alpha \ge \beta$, X_α is σ-ℓ-discrete. Etc.

Thus, in principle, for all types of lexicographic product spaces $X = \underline{\|}_{\alpha<\mu} X_\alpha$, we are able to formulate necessary and sufficient conditions for the metrizability of X in terms of the factorspaces.

Finally, we examine the notion perfect normality in a LOTS $\underline{\|}_{\alpha<\mu} X_\alpha$. We have shown that a GO-space is perfectly normal if and only if each relatively discrete subset is also σ-discrete (2.4.5). Therefore, in a perfectly normal LOTS of type $\underline{\|}_{\alpha<\mu} X_\alpha$, it follows that the collection

$$\{\{(x_\alpha)_{\alpha<\nu}\} \cdot \underline{\|}_{\nu\le\alpha<\mu} X_\alpha \mid x_\alpha \in X_\alpha \text{ for } \alpha < \nu\},$$

for all $\nu < \mu$ such that $\underline{\|}_{\nu\le\alpha<\mu} X_\alpha$ consists of at least three points, has to be a σ-discrete family in $\underline{\|}_{\alpha<\mu} X_\alpha$. Hence, in particular, in this case we have

if $\underline{\|}_{\nu\le\alpha<\mu} X_\alpha$ has both a left and a right endpoint, then $\underline{\|}_{\alpha<\nu} X_\alpha$ is a σ-discrete LOTS.

if $\underline{\|}_{\nu\le\alpha<\mu} X_\alpha$ has a left (right) endpoint, but no right (left) one, then $\underline{\|}_{\alpha<\nu} X_\alpha$ is a σ-ℓ-(α-r-)discrete LOTS.

Furthermore, a perfectly normal LOTS $\underline{\|}_{\alpha<\mu} X_\alpha$ is a C_I-space. So, certainly μ is cofinal with some ordinal $\le \omega_0$ (see lemma 1 to 2.2.11).
We now formulate the following theorems; (proofs are only sketched, since they are closely similar to those of the metrizability theorems).

THEOREM 4.4.7. *Let* X *and* Y *be LOTS's.*

(i) *If* $|Y| > 2$, *and*

1. *if* Y *does not possess a left or a right endpoint, then* $X \cdot Y$ *is perfectly normal* $\Leftrightarrow$ Y *is perfectly normal.*

2. *if* Y *possesses both a left and a right endpoint, then*
   X · Y *is perfectly normal* $\Leftrightarrow$ X *is* σ*-discrete and* Y *is perfectly normal.*
3. *if* Y *possesses a left (right) endpoint, but no right (left) one, then*
   X · Y *is perfectly normal* $\Leftrightarrow$ X *is* σ-ℓ-(σ-r-)*discrete,* Y *is perfectly normal and* $\omega_0(\omega_0^*)$ *is cofinal (coinitial) in* Y *whenever* X *contains neighbourpoints.*

(ii) *If* $|Y| = 2$. *Then*
   X · Y *is perfectly normal* $\Leftrightarrow$ X *is perfectly normal and* N(X) *is* σ*-discrete (in* X*).*

PROOF.

(i) Let P be a relatively discrete subset in X · Y. If Y is perfectly normal, then, for each $x \in X$, $(\{x\} \cdot Y) \cap P$ is σ-discrete in Y. Put

$$P_0 = \{x \in X \mid \exists (p,q) \in P : x = p\}.$$

Now, if we assume 2.$\Longleftarrow$, then P_0 is σ-discrete (in X). And, under assumption of 3.$\Longleftarrow$, P_0 is σ-ℓ-(σ-r-)discrete in X. Furthermore, in the latter case, certainly P is σ-r-(σ-ℓ-)discrete in X · Y. Etc. (compare 4.4.1, 4.4.2 and 4.4.3).

(ii) $\Longrightarrow$. Suppose P is a relatively discrete subset of X. Let

$$f : X \cdot Y \longrightarrow X$$

be defined by $f((x,y)) = x$ for each point $(x,y) \in X \cdot Y$. Now, f is a quotient-map. So, $f^{-1}[P]$ is relatively discrete in X · Y and hence also σ-discrete (in X · Y). But then, P is σ-discrete (in X). Next, if $p,q \in N(X)$ and p and q are neighbours in X, then $\text{Int } f^{-1}[\{p,q\}] \neq \emptyset$. Consequently

$$\{f^{-1}[\{p,q\}] \mid p,q \in N(X) \text{ and p and q are neighbours}\}$$

is a σ-discrete family in X · Y. Hence, N(X) is σ-discrete (in X).

$\Longleftarrow$. This assertion follows easily from 2.3, lemma 3. (Observe that X is (topologically) contained in $(X \cdot Y) / \sim$, while $((X \cdot Y) / \sim) \setminus X$ is a σ-discrete set consisting of isolated points). □

THEOREM 4.4.8. *Let μ be a limit ordinal. Let X_α be a linearly ordered set, for all $\alpha < \mu$. Then*

$$\coprod_{\alpha<\mu} X_\alpha \text{ is a metrizable LOTS} \Leftrightarrow \coprod_{\alpha<\mu} X_\alpha \text{ is a perfectly normal LOTS.}$$

PROOF. See 4.4.4, 4.4.5, 4.4.6, the observations made thereafter and 4.4.7.□

We finish up with three examples.

EXAMPLE 1. Let X be a discrete topological space. Then X is orderable (see, for instance, Herrlich [He.1]). Clearly, X is (σ-)discrete. Now, the LOTS X^{ω_0} has the following properties:

1. X^{ω_0} is connected $\overset{4.1.1}{\Longleftrightarrow} X \sim \mathbb{N} \Leftrightarrow X^{\omega_0} \approx [0,1[\ (\subset \mathbb{R})$
   ($X \sim \mathbb{N}$:= X is order-isomorphic to $\mathbb{N}$; $X^{\omega_0} \approx [0,1[$:= X^{ω_0} is homeomorphic to $[0,1[$).
2. X^{ω_0} is compact $\overset{4.2.1}{\Longleftrightarrow} |X| < \aleph_0 \Leftrightarrow X^{\omega_0} \approx \{0,1\}^{\aleph_0}$ = Cantor-space.
3. X^{ω_0} is metrizable $\Leftrightarrow$ At least one of the following conditions holds
   (i) X does not have endpoints (4.4.4)
   (ii) X has both a left and a right endpoint (4.4.5)
   (iii) X has a left endpoint and ω_0 is cofinal in X (4.4.6)
   (iv) X has a right endpoint and ω_0^* is coinitial in X (4.4.6).

EXAMPLE 2. Consider the LOTS $\omega_0^* + \omega_\alpha + 1$, $(\alpha > 0)$.
Clearly, $\omega_0^* + \omega_\alpha + 1$ is σ-r-discrete, but not σ-ℓ-discrete.
Also, for $\alpha > 0$, $\omega_0^* + \omega_\alpha + 1$ is not hereditarily paracompact. Now, for each $\alpha > 0$, the LOTS $(\omega_0^* + \omega_\alpha + 1)^{\omega_0}$ satisfies the following properties:

1. $(\omega_0^* + \omega_\alpha + 1)^{\omega_0}$ is nowhere locally separable (4.3.2)
2. $(\omega_0^* + \omega_\alpha + 1)^{\omega_0}$ is totally disconnected (4.1.1 and the notes thereafter)
3. $(\omega_0^* + \omega_\alpha + 1)^{\omega_0}$ is metrizable (4.4.6).

EXAMPLE 3.
(1) Let X be the ordered union of $\omega_1^* \cdot (Q \cap [0,1[)$ and $\omega_1 + 1$. Then X has

a right endpoint but no left one, X is σ-r-discrete but not coinitial with ω_0^*, and, finally, X possesses neighbourpoints. Hence, by 4.4.6, X^{ω_0} is not metrizable.

(2) Let Y be the ordered union of $\omega_1^* \cdot (Q \cap [0,1[)$ and $((\omega_1 + 1) \cdot (Q \cap [0,1[)) \setminus (\{\omega_1\} \cdot (Q \cap]0,1[))$. Then X is a subspace of Y and $Y \setminus X$ is a σ-discrete LOTS. Now all properties of X, mentioned under (1), do hold in Y, except for the last one: i.e. Y does not have neighbourpoints. Hence, by 4.4.6, Y^{ω_0} is metrizable.

CHAPTER V

METRIZABILITY OF GO-SPACES WHICH CAN BE MAPPED ONTO METRIC SPACES

Let $X = (X,<,\tau)$ be a GO-space.

Let $M = (M,d)$ be a metric space. By $S(r;\varepsilon)$, $r \in M$, $\varepsilon > 0$, we will denote a spherical neighbourhood of r; i.e.

$$S(r;\varepsilon) = \{t \in M \mid d(r,t) < \varepsilon\}.$$

Let

$$f : X \longrightarrow M$$

be a continuous mapping from X onto M.

For every $t \in M$, the subset $f^{-1}[\{t\}]$ of X can be decomposed into a disjoint family of convexity-components. Clearly, each convexity-component of $f^{-1}[\{t\}]$ is a closed subset of X. We denote by $\mathcal{C}(t)$ the collection of all convexity-components in $f^{-1}[\{t\}]$. Obviously

$$X = \bigcup\{C \mid C \in \mathcal{C}(t);\ t \in M\}.$$

Furthermore, we will say that $C(\subset X)$ is a *convexity-component under* f, whenever $C \in \mathcal{C}(t)$ for some $t \in M$. Moreover, by C_x we shall denote the (uniquely determined) convexity-component under f containing $x \in X$; C_x will be called the *convexity-component of* x *under* f. Observe that $C_x \in \mathcal{C}(f(x))$.

The family

$$\mathcal{F} = \{C \mid C \in \mathcal{C}(t);\ t \in M\}$$

of all convexity-components under f, defines a partition of X. Let $X / \mathcal{F}$ denote the quotientspace obtained from X by identifying each convexity-component $C \in \mathcal{F}$ to a point; let

$$\mathbb{P} : X \longrightarrow X / \mathcal{F}$$

be the corresponding quotientmap. Then, clearly, $X / \mathcal{F}$ is ordered in a natural manner, and the quotient topology on $X / \mathcal{F}$ is a GO-topology with respect to that ordering. [If $\mathcal{D}$ is any decomposition of a GO-space into closed convex subsets, then it is clear that the corresponding quotient space is itself a GO-space]. Thus, $X / \mathcal{F}$ is a GO-space.

We define the following subsets of $X = (X,<,\tau)$

$$E(X;f) = E((X,<,\tau);f) = \{p \in E(X) \mid [p,\rightarrow[\in \tau \text{ and } \forall n \in \mathbb{N} : \exists x \in X : (x < p \text{ and } f[[x,p]] \subset S(f(p);\tfrac{1}{n}))\} \cup \cup\{p \in E(X) \mid]\leftarrow,p] \in \tau \text{ and } \forall n \in \mathbb{N} : \exists x \in X : : (p < x \text{ and } f[[p,x]] \subset S(f(p);\tfrac{1}{n}))\}$$

and

$$N(X;f) = N((X,<,\tau);f) = \{p \in N(X) \mid f(p) = f(q), \text{ where } q \text{ is the neighbour of } p \text{ in } X\}.$$

REMARK. Observe that

$$N(X;f) = N((X,<,\tau);f) = \{p \in N(X) \mid [p,\rightarrow[\in \tau \text{ and } \forall n \in \mathbb{N} \; \exists x \in X : (x < p \text{ and } f[[x,p]] \subset S(f(p);\tfrac{1}{n}))\} \cup \cup\{p \in N(X) \mid]\leftarrow,p] \in \tau \text{ and } \forall n \in \mathbb{N} : \exists x \in X : : (p < x \text{ and } f[[p,x]] \subset S(f(p);\tfrac{1}{n}))\}.$$

It is immediately clear that

$$N(X;f) = E(X;f) \cap N(X).$$

5.1. GENERAL CHARACTERIZATIONS.

THEOREM 5.1.1. *Let* X = (X,<,τ) *be a GO-space. Let* M = (M,d) *be a metrix space. Let*

$$f : X \longrightarrow M$$

be a continuous mapping from X *onto* M *such that each convexity-component under* f *is a finite subset of* X.
Then the following properties are equivalent

1) X *is metrizable.*
2) E(X;f) *is σ-discrete (in* X*).*

If τ = λ(<) *and, moreover, each convexity-component under* f *consists of at most two points, then these properties are also equivalent to*

3) N(X;f) *is σ-discrete (in* X*).*

PROOF. Since N(X;f) ⊂ E(X;f) ⊂ E(X), it follows from 3.1 that 1 ⟹ 2 and 1 ⟹ 3 (under the prescribed conditions). To show that 2 ⟹ 1 we proceed as follows:

I. Assume that every convexity-component under f consists of precisely one point. Since f is a continuous map there exists, for each x ∈ X and each n ∈ ℕ, a convex open neighbourhood I(x;n)(∈ τ) of x such that

$$f[I(x;n)] \subset S(f(x);\tfrac{1}{n}).$$

We put

$$\mathcal{U}_n = \{I(x;n) \mid x \in X\} \qquad (n \in \mathbb{N}).$$

Then $\{\mathcal{U}_n\}_{n=1}^{\infty}$ is a sequence of open covers of X. Using the triangle inequality in (M,d) it follows that

$$f[St(x;\mathcal{U}_n)] \subset S(f(x);\tfrac{2}{n}).$$

Thus, since each $St(x;\mathcal{U}_n)$ is a convex subset of X and every convexity-component under f consists of precisely one point,

$$\bigcap_{n=1}^{\infty} St(x;\mathcal{U}_n) = \{x\}$$

for each $x \in X$. So $\{St(x;\mathcal{U}_n)\}_{n=1}^{\infty}$ is a local pseudo-base at x. Furthermore, when

$$L = \{x \in X \mid \{St(x;\mathcal{U}_n)\}_{n=1}^{\infty} \text{ is a local base at } x\}$$

then $X \setminus L \subset E(X;f)$. [If $p \notin E(X)$ then clearly all convex sets $St(p;\mathcal{U}_n)$, $n \in \mathbb{N}$, form a local base at p; hence $p \in L$. Let $p \in E(X) \setminus E(X;f)$. If $[p,\rightarrow[\ \in \tau$, then there exists an integer $n \in \mathbb{N}$ such that for each $x < p$ there is a point $y \in X$ satisfying $x \leq y < p$ and $f(y) \notin S(f(p);\frac{1}{n})$. It follows that p is the left endpoint of the convex set $St(p;\mathcal{U}_{2n})$. If $]\leftarrow,p] \in \tau$, then similarly p is the right endpoint of the convex set $St(p;\mathcal{U}_{2n})$. Hence, in both cases, $p \in L$]. Therefore, if $E(X;f)$ is σ-discrete (in X), then also $X \setminus L$ is σ-discrete (in X). Now 3.2 yields that X is metrizable.

II. Next, consider the general case in which every convexity-component under f is a finite subset of X. The continuous mapping

$$f \circ \mathbb{P}^{-1} : X / \mathcal{F} \longrightarrow M$$

from $X / \mathcal{F}$ onto M is such that every convexity-component under $f \circ \mathbb{P}^{-1}$ consists of a single point. Since $\mathbb{P}$ is a finite-to-one continuous mapping and since $\mathbb{P}^{-1}[\{u\}]$ is a convex subset of X, for each $u \in X / \mathcal{F}$,

$$\mathbb{P}^{-1}[E(X / \mathcal{F};f \circ \mathbb{P}^{-1})] \subset E(X;f).$$

Now, $E(X;f)$ is σ-discrete (in X). So, $E(X;f) = \bigcup_{n=1}^{\infty} A_n$ where each A_n is a discrete subset of X. Without loss of generality we may assume that $\mathbb{P}^{-1}[\{u\}]$ is contained in A_n, for $u \in \mathbb{P}[E(X;f)]$, whenever $\mathbb{P}^{-1}[\{u\}] \cap A_n \neq \emptyset$. [Otherwise, let $\tilde{A}_n$ be the union of A_n and all $\mathbb{P}^{-1}[\{u\}]$ which intersect A_n]. Since $\mathbb{P}$ is a quotient map, each $\mathbb{P}[A_n]$ is discrete (in $X / \mathcal{F}$). Hence, $\mathbb{P}[E(X;f)]$ is σ-discrete (in $X / \mathcal{F}$). Consequently also $E(X / \mathcal{F};f \circ \mathbb{P}^{-1})$ is σ-discrete (in $X / \mathcal{F}$). Thus, by I, we conclude that $X / \mathcal{F}$ is a metrizable GO-space. Therefore, by 3.1, $X / \mathcal{F}$ contains a dense σ-discrete subset D with $E(X / \mathcal{F}) \subset D$. Since $\mathbb{P}$ is a finite-to-one and continuous mapping, the set $\mathbb{P}^{-1}[D]$ is σ-discrete (in X). Hence, since also $E(X;f)$ is σ-discrete (in X),

$$E(X;f) \cup \mathbb{P}^{-1}[D]$$

is σ-discrete (in X). Furthermore,

$$E(X) \subset E(X;f) \cup \mathbb{P}^{-1}[D]$$

because $x \in E(X;f)$ whenever $|C_x| > 1$; and $\mathbb{P}(x) \in E(X / \mathcal{F}) \subset D$ whenever $x \in E(X)$ and $|C_x| = 1$. (C_x denotes the convexity-component of x under f). Finally, it is easily verified that $E(X;f) \cup \mathbb{P}^{-1}[D]$ is a dense subset of X. Hence, by 3.1, X is metrizable.
The proof of $3 \Longrightarrow 1$, under the prescribed conditions, can be given analogously. (The assumption that every convexity-component under f consists of at most two points is needed to show that $N(X;f) \cup \mathbb{P}^{-1}[D]$ is a dense subset of X). □

REMARK. For the non-metrizable LOTS $X = [0,1] \cdot \{-1,0,1\}$, the metric space $M = [0,1]$ and the continuous map

$$f : X \longrightarrow M$$

defined by $f((x,y)) = x$, $(x \in [0,1], y \in \{-1,0,1\})$, it follows that, for every $(x,y) \in X$, the convexity-component $C_{(x,y)}$ of (x,y) under f consists of precisely three points. Hence, in 5.1.1, the condition: "each convexity-component under f consists of at most two points", cannot be omitted. Observe that $N(X;f) = \emptyset$ in this example.

In the sequel, let

$$\mathcal{F}_0 = \{C \in \mathcal{F} \mid \text{Int } C \neq \emptyset\}.$$

Next, for each $C \in \mathcal{F}_0$, pick one point $x(C) \in \text{Int } C$.
We define

$$Y = \{x(C) \mid C \in \mathcal{F}_0\} \cup (X \setminus \cup \{\text{Int } C \mid C \in \mathcal{F}_0\}).$$

Note that each convexity-component (in X) under f contains at least one and at most three points of Y.

LEMMA. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $M = (M,d)$ *be a metric space. Let*

$$f : X \longrightarrow M$$

be a continuous mapping from X *onto* M *such that each convexity-component under* f *is metrizable. Then*

$$Y = (Y,<_Y,\tau \mid Y) \text{ is metrizable} \Longrightarrow X \text{ is metrizable.}$$

PROOF. Since Y is metrizable, by 3.1, there exists a dense σ-discrete (in Y) subset D in Y, such that $E(Y) = E((Y,<_Y,\tau \mid Y)) \subset D$. Then D is also σ-discrete in X, for Y is a closed subset of X. Further,

$$E((X,<,\tau)) \cap Y = E(X) \cap Y \subset E(Y) = E((Y,<_Y,\tau \mid Y)).$$

Since

$$\{x(C) \mid C \in F_0\} \subset E(Y) \subset D$$

the collection F_0 constitutes a σ-discrete family in X. Now, every $C \in F_0$, being a metric GO-space, contains a dense σ-discrete (in C) subset D(C) such that $E(C) = E((C,<_C,\tau \mid C)) \subset D(C)$. As every $C \in F$ is closed in X, each D(C) is also σ-discrete in X. Consequently,

$$\bigcup \{D(C) \mid C \in F_0\} \cup D$$

is a dense σ-discrete (in X) subset of X. Finally, it is clear that

$$E(X) \subset \bigcup \{D(C) \mid C \in F_0\} \cup D.$$

This completes the proof. □

THEOREM 5.1.2. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $M = (M,d)$ *be a metric space. Let*

$$f : X \longrightarrow M$$

be a continuous mapping from X *onto* M *such that every convexity-component under* f *is metrizable.*
Then the following properties are equivalent

1) X *is metrizable.*

2) $\mathcal{F}_1 = \{C \in \mathcal{F} \mid \text{Int } C \neq \emptyset$ *or* $C \subset E(X;f)\}$ *is a* σ*-discrete family in* X.

If $\tau = \lambda(<)$, *then these properties are also equivalent to*

3) $\mathcal{F}_2 = \{C \in \mathcal{F} \mid \text{Int } C \neq \emptyset$ *or* $C \subset N(X;f)\}$ *is a* σ*-discrete family in* X.

4) $\mathcal{F}_3 = \{C \in \mathcal{F} \mid |C| > 1\}$ *is a* σ*-discrete family in* X.

PROOF. First we note that $\mathcal{F}_i = \mathcal{F}_0 \cup (\mathcal{F}_i \setminus \mathcal{F}_0)$ for $i = 1,2$, while $\mathcal{F}_3 \subset \mathcal{F}_0 \cup (\mathcal{F}_3 \setminus \mathcal{F}_0)$.
$1 \Longrightarrow 2$. Since X is metrizable, also Y is metrizable. So, certainly $E(Y) = E((Y,<_Y,\tau \mid Y))$ is σ-discrete in Y and hence also in X. Now, $\{x(C) \mid C \in \mathcal{F}_0\} \subset E(Y)$. Consequently $\mathcal{F}_0$ is a σ-discrete family in X. Furthermore,

$$\bigcup (\mathcal{F}_1 \setminus \mathcal{F}_0) \subset E(X;f) \cap Y \subset E(X) \cap Y \subset E(Y).$$

Moreover, each $C \in \mathcal{F}_1 \setminus \mathcal{F}_0$ consists of at most two non-isolated points from X. Thus, since E(Y) is σ-discrete, it follows that $\mathcal{F}_1 \setminus \mathcal{F}_0$ forms a σ-discrete family in X. But then $\mathcal{F}_1 = \mathcal{F}_0 \cup (\mathcal{F}_1 \setminus \mathcal{F}_0)$ constitutes a σ-discrete family in X.
$2 \Longrightarrow 1$. It suffices to show that Y is metrizable, (see lemma). Now,

$$f \mid Y : Y \longrightarrow M$$

is a continuous mapping from Y onto M of which the convexity-components (in Y) are finite (in fact, consists of at most three points). [If $p < q < r < s$ are four different points of Y, then p,q,r and s cannot be contained in the same convexity-component under f in X. Hence, there are three distinct convexity-components C_1, C_2 and C_3 (in X) under f, such that C_1 "<" C_2 "<" C_3, $p \in C_1$, $s \in C_3$ and $f[C_2] \neq f(p), f(s)$. Since, however, C_2 contains a point of Y, it follows that p,q,r and s cannot be contained in the same convexity-component (in Y) under $f \mid Y$.] Because of 5.1.1, it is sufficient to prove that $E(Y;f \mid Y)$ is σ-discrete (in Y). Take $y \in E(Y;f \mid Y)$. Then $y \in C$ for some $C \in \mathcal{F}_0$ or $y \in E(X;f)$. Hence

$$E(Y;f \mid Y) \subset (\cup F_1) \cap Y.$$

From F_1 is a σ-discrete family in X and from the fact that each $C \cap Y$, $C \in F_1$, consists of at most three points, it follows now that $E(Y;f \mid Y)$ is σ-discrete in Y.
To complete the proof, we observe that certainly $2 \Longrightarrow 3 \Longrightarrow 4$. So, it remains to show that, in the case $\tau = \lambda(<)$,
$4 \Longrightarrow 1$. First, observe that

$$F_3 = \{C \in F \mid \text{Int } C \neq \emptyset \text{ and } |C| > 1\} \cup$$
$$\cup \{C \in F \mid \text{Int } C = \emptyset \text{ and } |C| = 2\}.$$

X is a LOTS, so for each $C \in F$ with Int $C \neq \emptyset$ and $|C| = 1$, the point $x(C)$ has a left neighbour x^- (in case $x(C)$ is not a left endpoint of X) and a right neighbour x^+ (in case $x(C)$ is not a right endpoint of X). Certainly $f(x^-) \neq f(x(C)) \neq f(x^+)$. Now, the convexity-components C_{x^-} of x^- and C_{x^+} of x^+ under f (in X), each contain a point of Y. Hence $x(C) \notin E(Y;f \mid Y)$. Consequently, when $p \in E(Y;f \mid Y)$ then either $p \in C$ for some $C \in F_0$ with $|C| > 1$ or $p \in N(X;f)$. (In the latter case $|C_p| = 2$). Hence

$$E(Y;f \mid Y) \subset (\cup F_3) \cap Y.$$

Thus, since F_3 is a σ-discrete family in X and each $C \cap Y$, $C \in F_3$, consists of at most three points, $E(Y;f \mid Y)$ is σ-discrete (in Y). By 5.1.1 and the preceding lemma this completes the proof. □

5.2. APPLICATIONS.

The foregoing theorems of section 5.1 can be used in order to derive some generalizations from already well-known results.

THEOREM 5.2.1. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $M = (M,d)$ *be a metric space. Let*

$$f : X \longrightarrow M$$

be a continuous, open and finite-to-one mapping from X *onto* M.
Then X *is metrizable.*
(This generalizes a result of Mancuso, [M.1]).

PROOF. For each $k \in \mathbb{N}$, we put

$$Q_k = \{t \in M \mid |f^{-1}[\{t\}]| = k\}$$

and

$$P_k = f^{-1}[Q_k].$$

Let

$$f_k = f \mid P_k : P_k \longrightarrow Q_k \qquad (k \in \mathbb{N}).$$

Now, we read the following facts:

(i) For every $n \in \mathbb{N}$, $\bigcup_{k=1}^{n} Q_k$ is a closed subset of M. [Let r be a limit point of $\bigcup_{k=1}^{n} Q_k$, and suppose that $f^{-1}[\{r\}]$ consists of at least n+1 distinct points $x_1, x_2, \ldots, x_{n+1}$. Let $\{O_i\}_{i=1}^{n+1}$ be a collection of disjoint open sets such that $x_i \in O_i$, for $i = 1,2,\ldots,n+1$. Then $U = \bigcap_{i=1}^{n+1} f[O_i]$ is an open neighbourhood of r, which hence contains a point $t \in \bigcup_{k=1}^{n} Q_k$. However, for each $i = 1,2,\ldots,n+1$, there is a point $y_i \in O_i$ such that $f(y_i) = t$. Contradiction].
Consequently Q_n is an F_σ-set in M. Hence P_n is an F_σ-set in X.

(ii) For every $k \in \mathbb{N}$, $f_k : P_k \longrightarrow Q_k$ is an open map. [Clearly, for all $U \subset X$: $f[U \cap P_k] = f[U] \cap Q_k$].

(iii) For every $k \in \mathbb{N}$, $f_k : P_k \longrightarrow Q_k$ is a closed map. [Let A be a closed subset of P_k and let $t \in Q_k$ be a limitpoint of $f_k[A]$. Then t is the image of precisely k different points $x_1, x_2, \ldots, x_k$ of P_k. If none of these is a limitpoint of A, then one can find disjoint open sets O_i in P_k such that $x_i \in O_i$ and $O_i \cap A = \emptyset$; $i = 1,2,\ldots,k$. But then $\bigcap_{i=1}^{k} f_k[O_i]$ is an open neighbourhood of t in Q_k which does not intersect $f_k[A]$. Contradiction].

I. Fix $k \in \mathbb{N}$. Then

$$f_k : P_k \longrightarrow Q_k$$

is a continuous, open, closed and k-to-1 map from the GO-space $P_k = (P_k, <_{P_k}, \tau \mid P_k)$ onto the metric space Q_k. For all $p \in P_k$, let C_p be the convexity-component of p under f_k. We claim that

$$A = \{p \in P_k \mid \ |C_p| > 1\}$$

is σ-discrete (in P_k). For, consider the set $f_k[A] \subset Q_k$. Choose, if possible, a point $r \in f_k[A]$. Let $f_k^{-1}[\{r\}] = \{a_i, a_2, \ldots, a_k\}$. Then $a_i \neq a_j$ if $i \neq j$. Now, there clearly exist disjoint convex open neighbourhoods $O(a_i)$ of a_i, for $i = 1,2,\ldots,k$, in P_k such that $O(a_i) \cup O(a_j)$, $i \neq j$, is again a convex subset of P_k if an only if a_i and a_j are neighbours in P_k. Hence, it follows that

$$\bigcap_{i=1}^{k} f_k[O(a_i)] \cap f_k[A] = \{r\}.$$

[Indeed, let $s \in f_k[A]$ with $s \neq r$. Let $f_k^{-1}[\{s\}] = \{b_1, b_2, \ldots, b_k\}$. By definition of A, at least one b_i must be the neighbour of another b_j. Since the same holds true for $\{a_1, a_2, \ldots, a_k\}$ it follows that at least one of the convex sets $O(a_i)$ does not meet $\{b_1, b_2, \ldots, b_k\}$. Hence $s \notin \bigcap_{i=1}^{k} f_k[O(a_i)]]$. Now, $\bigcap_{i=1}^{k} f_k[O(a_i)]$ is an open subset of Q_k. Consequently, $f_k[A]$ is a relatively discrete subset of Q_k. Since, however, Q_k has a σ-discrete open base, the relatively discrete set $f_k[A]$ is also a σ-discrete subset of Q_k. Thus, since every C_p is a finite subset of P_k, it follows that A is σ-discrete (in P_k). Furthermore, we claim:

$$\{p \in E(P_k;f_k) \mid C_p = \{p\}\} = \emptyset.$$

For, let $C_p = \{p\}$ for some $p \in P_k$ and suppose, for instance, that $[p,\rightarrow[\cap P_k \in \tau \mid P_k$ while $]\leftarrow,p[\cap P_k \neq \emptyset$. Then there exists a point $x \in P_k$ such that $x < p$ and

$$[x,p[\cap P_k \cap f_k^{-1}[\{f_k(p)\}] = \emptyset.$$

Therefore, since f_k is a closed map,

$$f_k(p) \notin f_k[[x,p[\cap P_k] = \mathrm{Cl}_{Q_k}(f_k[[x,p[\cap P_k])$$

Consequently, $p \notin E(P_k;f_k)$.
Now, it follows that

$$E(P_k;f_k) = \{p \in E(P_k;f_k) \mid |C_p| = 1\} \cup \{p \in E(P_k;f_k) \mid |C_p| > 1\}$$

is σ-discrete (in P_k), whence, by 5.1.1, P_k is metrizable.

II. Finally, for each $k \in \mathbb{N}$, let D_k be a dense σ-discrete (in P_k) subset of P_k such that $E(P_k) \subset D_k$. By I and 3.1, D_k exists. From (i) it follows that $P_k = \bigcup_{i=1}^{\infty} F_{ki}$, where F_{ki} is a closed subset of X. Further, $X = \bigcup_{k=1}^{\infty} P_k$ and $E(X) \cap P_k \subset E(P_k)$. Hence, if we put

$$D = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} (F_{ki} \cap D_k)$$

then D is a dense, σ-discrete (in X) subset of X and $E(X) \subset D$. Thus, by 3.1, X is metrizable. □

One might wonder whether or not a GO-space is metrizable if it is the preimage of a metric space under an open, perfect and countable-to-one mapping. A negative answer to this question (even when the domain space is a LOTS) is given below. The example constructed there, is due to Pol (see, Przymusinski[P.1]) and, from an earlier date already, to Filippov [F.1].

EXAMPLE. Let $C = C_{[0,1]}$ be the Cantorset in [0,1], which is obtained by deleting a sequence

$$\{]a(n,i),b(n,i)[\mid n = 0,1,2,\ldots;\ i = 1,2,\ldots,2^n\}$$

of mutually disjoint open intervals (the middle thirds). The collection

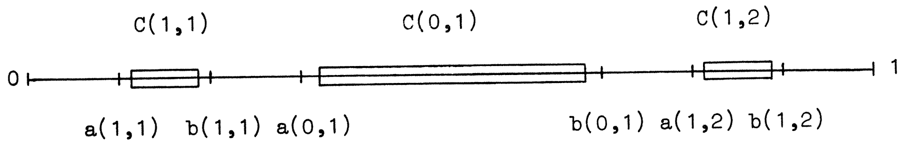
$$\{]a(n,i),b(n,i)[\mid i = 1,2,\ldots,2^n\}$$

represents those open intervals of [0,1] which are deleted at the n-th step in the construction of C.

In the sequel, for $A,B \subset [0,1]$ we will denote $A < B$ whenever $a < b$ for all $a \in A$ and all $b \in B$. For each $n = 0,1,2,\ldots$, and each $i = 1,2,\ldots,2^n$, let

$$C(n,i) \subset\ ]a(n,i),b(n,i)[$$

be a Cantorset constructed in a closed interval (of [0,1]), contained in $]a(n,i),b(n,i)[$.



Next, put

$$Y = \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} C(n,i) \cup C.$$

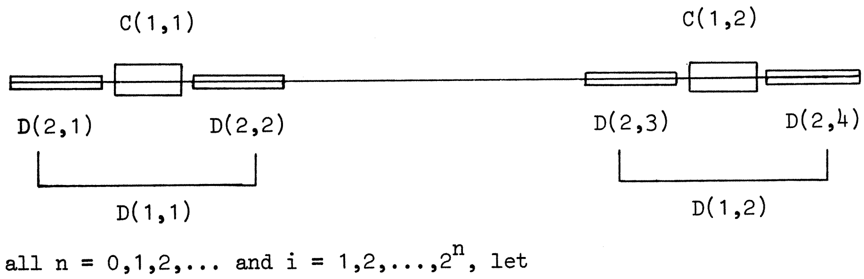
Then Y is a compact LOTS (see 2.4.1 and 2.3.1). Now, the sequence

$$\{C(n,i) \mid n = 0,1,2,\ldots;\ i = 1,2,\ldots,2^n\}$$

can be put into a one-to-one correspondence with a sequence

$$\{D(n,i) \mid n = 0,1,2,\ldots;\ i = 1,2,\ldots,2^n\}$$

obtained as follows:
Let D(0,1) = C. Then, C(0,1) devides D(0,1) into two parts, which we denote by D(1,1) and D(1,2), such that $D(0,1) = D(1,1) \cup D(1,2)$ and $D(1,1) < C(0,1) < D(1,2)$. Next, C(1,1) divides D(1,1) into two parts, denoted by D(2,1) and D(2,2), such that $D(1,1) = D(2,1) \cup D(2,2)$ and $D(2,1) < C(1,1) < D(2,2)$; further C(1,2) divides D(1,2) into two parts, denoted by D(2,3) and D(2,4), such that $D(1,2) = D(2,3) \cup D(2,4)$ and $D(2,3) < C(1,2) < D(2,4)$. And so on. Clearly each D(n,i) is again a Cantor-set.



For all n = 0,1,2,... and $i = 1,2,\ldots,2^n$, let

$$g_{(n,i)} : C(n,i) \longrightarrow D(n,i)$$

be an orderpreserving homeomorphism from C(n,i) onto D(n,i). Now, define

$$g : Y \longrightarrow C$$

by $g(y) = y$ if $y \in C$ and $g(y) = g_{(n,i)}(y)$ if $y \in C(n,i)$. Then, it is easily verified that g is perfect, open and countable-to-one mapping from Y onto C. Next, put

$$P = C \setminus \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{2^n} \{a(n,i), b(n,i)\}.$$

So, P is an uncountable subset of C, containing precisely all non-neighbourpoints of C. Finally, define the set

$$X = \{(y,k) \in Y \cdot \{0,1\} \mid y \notin P \Longrightarrow k = 0\}.$$

In other words, X is obtained from Y by replacing each point in P by a pair of neighbours. Let X be supplied with the order-topology. Then X becomes a non-metrizable compact LOTS. (Observe that X is separable, while $|N(X)| > \aleph_0$). Now, if

$$f : X \longrightarrow C$$

is defined by $f((y;k)) = g(y)$, then clearly also f is a perfect, open and countable-to-one mapping.
Hence, we are done.

During the rest of this paragraph we make use of the following notions.

If S and T are topological spaces, then a map $\phi : S \longrightarrow T$ is said to be *quasi-open*, if Int $\phi[U] \neq \emptyset$ for every non-empty open set U in S.

If X is a GO-space and T a topological space, then a map $f : X \longrightarrow T$ is said to be *convexity-zerodimensional*, if each convexity-component under f consists of a single point.

A linearly ordered set (X,<) is said to be *order-dense*, if (X,<) does not contain neighbourpoints.

THEOREM 5.2.2. *Let* $X = (X,<,\tau)$ *be a GO-space. Let* $M = (M,d)$ *be a metric space. Let*

$$f : X \longrightarrow M$$

be a continuous mapping from X *onto* M *such that at least one of the following properties is satisfied:*

(i) f *is quasi-open and, for each isolated point* $t \in M$, $f^{-1}[\{t\}]$ *is a metrizable subspace of* X,

(ii) *for each* $t \in M$, $f^{-1}[\{t\}]$ *is a nowhere dense subset of* X,

(iii) f *is convexity-zerodimensional;*

then

$$X \text{ is metrizable} \Longleftrightarrow E(X;f) \text{ is } \sigma\text{-discrete (in } X).$$

(This generalizes a result of Przymusinski [P.1]).

PROOF.

$\Longrightarrow$ $E(X;f) \subset E(X)$. (see 3.1).

$\Longleftarrow$ Suppose, property (i) holds true. Then every convexity-component C under f is a metrizable subspace of X. Indeed, if $|C| > 2$, then from f is quasi-open it follows that f[C] is an isolated point of X. Furthermore, isolated points of M are members of a σ-discrete open base for M. Therefore, the collection of all convexity-components under f with non-empty interiors constitutes a σ-discrete family in X. (Observe that convexity-components under f with non-empty interiors are both open and closed in X). Next, since E(X;f) is σ-discrete (in X), also

$$\{C \in F \mid \text{Int } C = \emptyset \text{ and } C \subset E(X;f)\}$$

is a σ-discrete family in X. Thus, by 5.1.2, X is metrizable. Finally, when one of the properties (ii) or (iii) holds, then each convexity-component under f is a finite subset of X, (in fact consisting of at most two elements). Hence, the assertion is clear from 5.1.1. □

COROLLARY 1. *A LOTS is metrizable if it is the inverse image of a metric space under a convexity-zerodimensional mapping.*

In particular, a LOTS is metrizable if it is the inverse image of a metric space under a one-to-one mapping (Faber, Maurice and Wattel [FMW.1]).

PROOF. Let X be a LOTS. Let M be a metric space. Let

$$f : X \longrightarrow M$$

be a convexity-zerodimensional mapping from X onto M. Since X is a LOTS, each point of E(X) must be a neighbourpoint of X unless it is an endpoint of X. But then, since f is convexity-zerodimensional, it follows that $E(X;f) = \emptyset$. □

COROLLARY 2. *An order-dense LOTS is metrizable if it is the inverse image of a non-trivial metric space under a quasi-open mapping with compact convexity-components.*

PROOF. Let X be an order-dense LOTS. Let M be a metric space with $|M| > 1$. Let

$$f : X \longrightarrow M$$

be a quasi-open map from X onto M such that, for each $x \in X$, the convexity-component C_x of x under f is a compact subspace of X. Since X is order-dense, either $|C_x| = 1$ or $|C_x| > 2$ for all $x \in X$. Now, suppose $|C_x| > 2$ for some $x \in X$. Then, since f is quasi-open, $C_x = \text{Int } C_x$. As a compact subspace of X, C_x has to contain a left and a right endpoint. But then, since $|M| > 1$, X cannot be order-dense at the same time. Hence, for every $x \in X$, $|C_x| = 1$. Consequently f is convexity-zerodimensional. Thus, by the previous corollary, X is metrizable. □

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