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Symmetries for dynamical and Hamiltonian systems

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CONTENTS

GENERAL INTRODUCTION

This paper deals with symmetries of dynamical systems and in particular Hamiltonian systems. Suppose X is a vector field on a manifold M. With this vector field an autonomous dynamical system

$$\dot{u}(t) \equiv \frac{d}{dt} u(t) = X(u(t)) \tag{0.1}$$

on the manifold M is associated. Dynamical systems of this type arise in many places in science, biology, economy, and other disciplines. Sometimes the manifold M is a linear space. In that case several considerations can be somewhat simplified. A very simple example of a system for which a correct treatment cannot be given in a linear space is the Kepler problem. In general a system with constraints has to be considered on a manifold which is not a linear space. The general theory described in Chapters 2, 3 and 4 will be given for a system on an arbitrary manifold. However, several of the examples treated in Chapter 5 will be systems which are considered in a linear space. An important special type of dynamical system is the Hamiltonian system. For autonomous Hamiltonian systems, as introduced in Section 3.2, there always exists a function H on M, the Hamiltonian, such that $H(u(t))$ is constant on every solution $u(t)$ of the system. In physical situations which are described by a Hamiltonian system this function H is often, but not always, equal to the energy of the system. If the initial state u_0 of the system at $t = t_0$ is known, we can try to find the time evolution $u(t)$ of the system by solving (0.1). However, in most cases for a dynamical/Hamiltonian system an explicit form of the solution, corresponding to an arbitrary initial value $u(t_0) = u_0$, cannot be found. We shall not go into questions concerning existence and uniqueness of the solutions of (0.1) now. By means of numerical methods it is often possible to find a very good approximation for the solution of (0.1) with initial value $u(t_0) = u_0$.

An alternative way to obtain some information about the dynamical system is, instead of looking at a specific solution (as is done in the numerical approach), to find properties which are shared by all solutions or at least classes of solutions. Such properties are for instance the existence of constants of the motion, the existence of symmetries, the stability of the solutions or the behaviour of the solutions for $t \to \infty$.

In this paper we shall only consider symmetries and constants of the motion of dynamical systems and in particular Hamiltonian systems. For a finite-dimensional Hamiltonian system the existence of k constants of the motion in involution (i.e. with vanishing Poisson brackets) allows to reduce the dimension of the phase space by 2k. If the number of constants of the motion in involution equals half the dimension of the manifold (which is always even), the system is called completely integrable. In that (exceptional) case an explicit form for the solutions of (0.1) can be given. This is one of the reasons for the interest in constants of the motion.

As far as we know for infinite-dimensional Hamiltonian systems the relation between the existence of an infinite series of constants of the motion in involution and "complete integrability" is not yet clear. During the last decennia a number of infinite-dimensional Hamiltonian systems have been solved using the so-called "inverse scattering methods". All these equations also have an infinite series of constants of the motion in involution. This has led to the conjecture that the existence of such a series is strongly related to the possibility of solving the initial value problem for these equations (for instance by inverse scattering). This conjecture has also given rise to the, in our opinion misleading, usage to call an infinite-dimensional Hamiltonian system integrable or completely integrable if there exists an infinite series of constants of the motion in involution.

In Chapter 2 we consider a general dynamical (i.e. not necessarily Hamiltonian) system of the form (0.1). A symmetry of that system is introduced as an infinitesimal transformation of solutions of the system into new solutions of the system. We shall consider symmetries which also may depend explicitly on t. So $Y(u,t)$ is a symmetry if for every solution u(t) of (0.1) also $u(t) + \varepsilon Y(u(t),t)$ is a solution (up to $o(\varepsilon)$ for $\varepsilon \to 0$). This leads to a definition of symmetries of (0.1) as, possibly parameterized, (contravariant) vector fields which satisfy

$$(0.2) \qquad \dot{Y} + [X,Y] = \dot{Y} + L_X Y = 0 \qquad (\dot{Y} = \frac{\partial}{\partial t} Y)$$

where $[X,Y] = L_X Y$ is the Lie bracket of the vector fields X and Y. Sometimes this type of infinitesimal transformation is called a generator of a symmetry; the notion symmetry is then used for a finite (i.e. not infinitesimal) transformation of solutions of (0.1) into new solutions of (0.1). However, we shall use the notion symmetry only for infinitesimal transformations, or more precisely for parameterized vector fields satisfying (0.2). The relation between symmetries and finite transformations of solutions into (new) solutions is similar to the relation between a Lie algebra and a corresponding Lie group. Therefore it is not surprising that the set of symmetries of a dynamical system has a natural Lie algebra structure.

A second important concept in this paper is the adjoint symmetry, that is a, possibly parameterized, one-form (covariant vector field) $\sigma(u,t)$ which satisfies

$$\dot{\sigma} + L_X \sigma = 0 \ . \tag{0.3}$$

It turns out that every constant of the motion F of (0.1) gives rise to an adjoint symmetry $\sigma = dF$. Of course the converse is not true in general. The four possible types of linear operators (in fact tensor fields of total order 2) which map (adjoint) symmetries into (adjoint) symmetries are also introduced in Chapter 2. These operators are called recursion operators for (adjoint) symmetries, SA- and AS operators. An SA operator maps symmetries into adjoint symmetries, an AS operator acts in the opposite direction. For an arbitrary dynamical system interesting operators of these four types do not exist in general. If there exists a recursion operator for symmetries or for adjoint symmetries, its eigenvalues (in the finite-dimensional case) are constants of the motion. This suggests a possible relation between these recursion operators and the eigenvalue problems used in the inverse scattering method. For the Korteweg-de Vries and Sawada-Kotera equation (Sections 5.6 and 5.7) this relation can be given explicitly.

A more interesting situation appears if the dynamical system is a Hamiltonian system. In Chapter 3 we introduce Hamiltonian systems using the language of symplectic geometry. The phase space of these Hamiltonian systems is a symplectic manifold (M,ω). Denote the (0,2) tensor field corresponding to the symplectic form ω by Ω. The inverse (2,0) tensor field is denoted by $\overleftarrow{\Omega}$. Then the Hamiltonian system on (M,ω) with Hamiltonian H is given by

$$\dot{u} = X \equiv \overleftarrow{\Omega}\, dH \ . \tag{0.4}$$

The classical Hamiltonian systems written in terms of p_i and q_i are a special case of (0.4). It turns out that several interesting partial differential equations (Korteweg-de Vries-, sine-Gordon-, Benjamin-Ono equation) can be considered as infinite-dimensional Hamiltonian systems of this type.

In Chapter 4 we study symmetries for Hamiltonian systems. An important consequence of the Hamiltonian character is that the tensor fields Ω and $\overset{\leftarrow}{\Omega}$ can be used as SA- respectively AS operators. Then every constant of the motion F does not only give rise to an adjoint symmetry dF but also to a symmetry $\overset{\leftarrow}{\Omega}$ dF. A symmetry Y such that the corresponding adjoint symmetry $\sigma = \Omega Y$ is (non-) exact/closed will be called a (non-) canonical/semi-canonical symmetry. So the symmetry $\overset{\leftarrow}{\Omega}$ dF corresponding to a constant of the motion F is canonical. An important part of this paper consists of the study of non (semi-) canonical symmetries. These symmetries have interesting properties. For instance if Z is a non-semi-canonical symmetry, then the Lie derivatives $L_Z \Omega$ and $L_Z \overset{\leftarrow}{\Omega}$ are again SA respectively AS operators. This implies that every non-semi-canonical symmetry gives rise to a recursion operator for symmetries $\Lambda = \overset{\leftarrow}{\Omega} L_Z \Omega$. Using this recursion operator we can construct an infinite series of symmetries by

$$X_k = \Lambda^{k-1} X = (\overset{\leftarrow}{\Omega} L_Z \Omega)^{k-1} X \ . \tag{0.5}$$

In Section 4.5 we give conditions on Z (and M) such that this series consists of canonical symmetries. Then there exists an infinite series of constants of the motion (in involution) F_k such that $X_k = \overset{\leftarrow}{\Omega} dF_k$. An alternative way to generate an infinite series of symmetries is to compute the repeated Lie bracket with Z (= higher Lie derivative in the direction of Z)

$$\tilde{X}_k = L_Z^{k-1} X \ . \tag{0.6}$$

This series is considered in Section 4.7. It turns out that the conditions mentioned above also imply that the symmetries $\tilde{X}_k$ are canonical and correspond to constants of the motion G_k. The possible relation between the two series of symmetries X_k and $\tilde{X}_k$ (constants of the motion F_k and G_k) is also considered in Section 4.7. It turns out that if $\tilde{X}_2 = bX_2$ then $\tilde{X}_k = b_k X_k$ (so $G_k = b_k F_k$) with $b, b_k \in \mathbb{R}$. In that case the constants of the motion F_k can also be obtained by a third method. This situation appears in several examples. A completely different series of symmetries can be generated by

$$Z_k = \Lambda^{k-1} Z = (\overset{\leftarrow}{\Omega} L_Z \Omega)^{k-1} Z \ . \tag{0.7}$$

Several properties of this series of symmetries are described in the Sections 4.6 and 4.7. We give conditions which imply that this series consists of non-semi-canonical symmetries. We also show how the symmetries Z_k can be used to give a multi-Hamiltonian formulation of (0.4) and describe the structure of the Lie algebra of symmetries of (0.4) generated by the series X_k and Z_k.

Several of the results mentioned above can be obtained if there exists a non-semi-canonical symmetry Z, which satisfies some additional conditions. However, it may happen that for such a symmetry $L_Z \Omega = a\Omega$ and $[Z,X] = bX$ for $a,b \in \mathbb{R}$. Then all elements of the series (0.5) and (0.6) are identical to X and all elements of the series (0.7) are identical to Z (up to multiplicative constants). This trivial situation appears often if Z is the symmetry corresponding to a scale law of (0.4). It will be clear that the existence of a symmetry Z for which this trivial situation does not occur, is a highly nontrivial property which is in some way related to the "complete integrability" of the Hamiltonian system.

Some examples of the theory described in the Chapters 2, 3 and 4 are given in Chapter 5. As an example of the theory of Chapter 2 we consider the Burgers equation. All other examples are (semi-) Hamiltonian systems. Of course the whole of the methods described in Chapters 2, 3 and 4 cannot be applied completely to each of the given examples. Several examples are infinite-dimensional (Hamiltonian) systems. The most extensive example of this type is the Korteweg-de Vries equation. In Section 1.3 we describe function spaces (i.e. non-metrizable topological vector spaces) in which we consider these equations. In these spaces a consistent treatment of (adjoint) symmetries and the various operators between (adjoint) symmetries can be given. Differential geometrical methods will be an important tool in this paper. A short overview of the differential geometrical concepts used will be given in Section 1.1. Finally in Section 1.2 we show how these, in first instance finite-dimensional, concepts can be "generalized" for (infinite-dimensional) topological vector spaces.

CHAPTER 1: MATHEMATICAL PRELIMINARIES

1.1 DIFFERENTIAL GEOMETRY

In this section we shall briefly describe some aspects of differential geometry. The concepts introduced in this section will be extensively used in the Chapters 2, 3 and 4. For a more comprehensive treatise and also for proofs of the results given here, we refer to the literature, for instance Abraham and Marsden [1,44] or Choquet-Bruhat [3].

Tangent and cotangent spaces.

Let M be a smooth *finite-dimensional manifold* with dimension n. The *tangent space* to M in a point $u \in M$ is denoted by T_uM. This is a linear space with dimension n. The *tangent bundle* TM is the union of all tangent spaces of M, so $TM = \bigcup_{u\in M} T_uM$. The tangent bundle TM is a manifold with dimension 2n. The *tangent bundle projection* $\pi_1 : TM \to M$ is a mapping which sends a tangent vector $A \in TM$ to its point of application. So if $A \in T_uM$ then $\pi_1(A) = u$.

The dual space of T_uM is the *cotangent space* T_u^*M. So an element $\alpha \in T_u^*M$ can be considered as a linear mapping $\alpha : T_uM \to \mathbb{R}$. Since the dimension of T_uM is finite, the dual space of T_u^*M is again T_uM. The *duality map* between T_uM and T_u^*M will be denoted by $\langle\cdot,\cdot\rangle$. So if $A \in T_uM$ and $\alpha \in T_u^*M$ then $\langle\alpha,A\rangle \in \mathbb{R}$.

The *cotangent bundle* T^*M is the union of all cotangent spaces of M, so $T^*M = \bigcup_{u\in M} T_u^*M$. It is again a manifold with dimension 2n. Suppose $\alpha \in T^*M$, so $\alpha \in T_u^*M$ for some $u \in M$. The mapping $\hat{\pi}_1 : T^*M \to M : \alpha \to u$ is called the *cotangent bundle projection*.

Natural bases.

Suppose we choose *local coordinates* $u^i (i=1,\ldots,n)$ on an open subset $U \subset M$ (so U can be described by one chart). By varying the coordinate u^1 and keeping the other coordinates fixed, we obtain a curve in $U \subset M$. The tangent vector of this curve in a point $u \in M$, is an element of the tangent space T_uM. This tangent vector is denoted by $e_1 = \frac{\partial}{\partial u^1}$. In a similar way we can construct the tangent vectors $e_i = \frac{\partial}{\partial u^i} \in T_uM$ $(i=2,\ldots,n)$. So in this way we can use the local coordinates u^i to construct a basis $\{ e_i = \frac{\partial}{\partial u^i} \mid i=1,\ldots,n \}$ for T_uM for all $u \in U$. This basis is called a *natural basis*. If $A \in T_uM$ with $u \in U$, it can be written as

(1.1.1) $$A = A^i e_i = A^i \frac{\partial}{\partial u^i} .$$

In this paper we shall always use the convention that, unless otherwise indicated, summation takes place over all indices which appear twice, once as a subscript and once as a superscript.

A basis $\{ du^i \mid i=1,\dots,n \}$ for $T^*_u M$ is defined by

(1.1.2) $$\langle du^i, e_j \rangle = \delta^i_j \qquad \forall\ i,j = 1,\dots,n.$$

This basis is called the *natural cobasis*. The bases $\{ e_i \mid i=1,\dots,n \}$ for $T_u M$ and $\{ du^i \mid i=1,\dots,n \}$ for $T^*_u M$ are called each others *dual bases*. If $\alpha \in T^*_u M$ with $u \in U$, we can write

(1.1.3) $$\alpha = \alpha_i du^i.$$

Then

(1.1.4) $$\langle \alpha, A \rangle = \langle \alpha_i du^i, A^j e_j \rangle = \alpha_i A^j \delta^i_j = \alpha_i A^i.$$

Tensor fields.

We shall frequently need smooth functions, vector fields, one-forms and (higher order) tensor fields on M. For a formal definition of these objects (using sections of the corresponding vector bundles) see for instance Abraham and Marsden [1,44] or Choquet-Bruhat [3].

1.1.5 Definition.

The set of smooth *functions* on M will be denoted by $F(M)$. The sets of smooth *vector fields* and (*differential*) *one-forms* on M will be denoted by $X(M)$ respectively $X^*(M)$. Finally the set of smooth *tensor fields* on M with covariant order j and contravariant order i will be denoted by $T^i_j(M)$. A tensor field $\Xi \in T^i_j(M)$ will be called an (i,j) tensor field on M.

□

So if $A \in X(M)$ then $A(u) \in T_u M$ and if $\alpha \in X^*(M)$ then $\alpha(u) \in T^*_u M$. Of course we can expand vector fields and one-forms in the corresponding natural bases:

(1.1.6) $$A(u) = A^i(u) e_i(u) \quad \text{and} \quad \alpha(u) = \alpha_i(u) du^i .$$

One-forms are sometimes called *covariant vector fields*, in contrast to vector fields which are called *contravariant vector fields*. Of course functions, vector fields and one-forms on M are special cases of tensor fields, so formally

$$F(M) = T^0_0(M), \; X(M) = T^1_0(M), \; X^*(M) = T^0_1(M).$$

Lie algebra's.

We now make some remarks on the structure of the sets introduced in definition 1.1.5. Of course all these sets are linear spaces (with infinite dimension). The product of two functions on M is again a function on M. This means that $F(M)$ is not only a *linear space* but also a *ring* (with identity). The product of a vector field, one-form or tensor field with a function yields again an object of the same type. This can be expressed by saying that $X(M)$, $X^*(M)$ and $T^j_i(M)$ are *modules over the ring* $F(M)$. The linear space $X(M)$ has additional structure. First we give the following

1.1.7 Definition.

A real linear space E with a bilinear product $[\cdot,\cdot] : E \times E \to E$, which satisfies

i) $[X,X] = 0 \quad \forall\, X \in E$,

ii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall\, X,Y,Z \in E$,

is called a *Lie algebra*.

□

Note that i) implies that the product is *antisymmetric*: $[X,Y] = -[Y,X]$. The second condition is called the *Jacobi identity*. It is well-known that the space $X(M)$ of vector fields on M is a Lie algebra. The product $[A,B]$ of two vector fields A and B on M is called the *Lie product* or *Lie bracket* of the vector fields A and B (see section 2.8 for an unusual (and complicated) introduction of the Lie bracket of vector fields). In local coordinates u^i the Lie bracket of the vector fields $A = A^i e_i$ and $B = B^i e_i$ is the vector field

$$(1.1.8) \qquad [A,B] = (B^i_{,j} A^j - A^i_{,j} B^j) e_i \, ,$$

where we use the notation $B^i_{,j} = \frac{\partial}{\partial u^j} B^i$, etc.

Tensor products.

In (1.1.6) we showed how vector fields and one-forms can be expanded in the natural bases corresponding to a coordinate system. By taking *tensor products* ($\otimes$) of the elements of these bases, we can construct bases for the various types of tensor fields. Suppose $\Phi \in T^0_2(M)$, $\Lambda \in T^1_1(M)$ and $\Psi \in T^2_0(M)$. Then, in a local coordinate system we can write

$$\Phi = \Phi_{ij} du^j \otimes du^i, \; \Lambda = \Lambda^i_j e_i \otimes du^j,$$

$$\Psi = \Psi^{ij} e_j \otimes e_i.$$

The *tensor product* of the tensor fields $\Xi \in T^i_j(M)$ and $\Theta \in T^k_\ell(M)$ is a tensor field $\Xi \otimes \Theta \in T^{i+k}_{\ell+j}(M)$. For instance in local coordinates ($A \in X(M)$)

$$\Lambda \otimes \Psi = \Lambda^i_j \Psi^{k\ell} e_\ell \otimes e_k \otimes e_i \otimes du^j,$$

$$\Lambda \otimes A = \Lambda^i_j A^k e_k \otimes e_i \otimes du^j.$$

Contractions.

The tensor product is an operation which yields a tensor field of higher order(s) then the original tensor fields. An operation which lowers both orders of a tensor field is the *contraction*. Suppose $\Xi \in T^j_i(M)$ with i,j ≥ 1. Then by contraction we obtain a tensor field $\Xi_C \in T^{j-1}_{i-1}(M)$. In fact if i > 1 and/or j > 1 several types of contraction are possible. As an example consider a tensor field $\Xi \in T^2_1(M)$. So, using a local coordinate system, we can write

$$\Xi = \Xi^{ij}_\ell e_j \otimes e_i \otimes du^\ell.$$

Then by contraction we can obtain the tensor(vector) fields

$$\Xi_{C_1} = \Xi^{ij}_i e_j \text{ and } \Xi_{C_2} = \Xi^{ij}_j e_i.$$

Contracted multiplication.

An operation which will be used very often in this paper, is *contracted multiplication*, that is a tensor product followed by a contraction. Contracted multiplication of two tensor fields Ξ_1, Ξ_2 will be denoted by

$\Xi_1\Xi_2$. For instance

(1.1.9) $\quad \Lambda\Psi = (\Lambda \otimes \Psi)_C = \Lambda^i_k \Psi^{k\ell} e_\ell \otimes e_i$

(1.1.10) $\quad \Lambda A = (\Lambda \otimes A)_C = \Lambda^i_k A^k e_i.$

The duality map between a vector field A and a one-form α can also be written as a contracted multiplication

$$\langle\alpha, A\rangle = \alpha A = (\alpha \otimes A)_C.$$

However, it will be convenient to use $\langle\cdot,\cdot\rangle$ for this duality map. It is easily seen from (1.1.10) that by contracted multiplication of a tensor field $\Lambda \in T^1_1(M)$ and a vector field A we obtain again a vector field ΛA on M. This means we can consider Λ also as a linear mapping $\Lambda : X(M) \to X(M)$. Similarly the contracted multiplication of a tensor field $\Gamma \in T^1_1(M)$ and a one-form α yields again a one-form $\Gamma\alpha$ ($= \Gamma^i_j \alpha_i du^j$). So we can consider Γ also as a linear mapping $\Gamma : X^*(M) \to X^*(M)$. Note that Λ and Γ are tensor fields of the same type. The two different mappings are possible since we can perform different contractions. In general we shall use the symbol Λ for tensor fields which are used as a mapping $\Lambda : X(M) \to X(M)$ and the symbol Γ for tensor fields which are used as a mapping $\Gamma : X^*(M) \to X^*(M)$. Note that this means that in the contracted multiplication $\Lambda\Xi$ we contract "using the lower index of Λ" while in the contracted multiplication $\Gamma\Xi$ we contract "using the upper index of Γ". The contracted multiplication of a tensor field $\Phi \in T^0_2(M)$ and a vector field A yields a one-form $\alpha = \Phi A = \Phi_{ij} A^j du^i$. So we can also consider Φ as a linear mapping $\Phi : X(M) \to X^*(M)$. Finally a tensor field $\Psi \in T^2_0(M)$ can be used to transform a one-form into a vector field. Hence we can consider it as a linear mapping $\Psi : X^*(M) \to X(M)$.

Vector bundle maps.

We have seen that a tensor field $\Lambda \in T^1_1(M)$ can be used as a linear mapping $\Lambda : X(M) \to X(M)$. Of course we can also transform a vector $A \in T_uM$ into a vector $\Lambda A \in T_uM$. So we can also use Λ as a linear mapping $\Lambda : T_uM \to T_uM$. Since $u \in M$ is arbitrary we can also consider the tensor field Λ as a mapping $\Lambda : TM \to TM$. A mapping of this type (with $\Lambda : T_uM \to T_uM$ linear) is called a *vector bundle map*. Similar results hold for the other types of tensor fields.

We summarize the various applications of tensor fields with total order two in the following scheme

(1.1.11)

tensor field	linear map	vector bundle map
$\Lambda \in T^1_1(M)$	$\Lambda : X(M) \to X(M)$	$\Lambda : TM \to TM$,
$\Gamma \in T^1_1(M)$	$\Gamma : X^*(M) \to X^*(M)$	$\Gamma : T^*M \to T^*M$,
$\Phi \in T^0_2(M)$	$\Phi : X(M) \to X^*(M)$	$\Phi : TM \to T^*M$,
$\Psi \in T^2_0(M)$	$\Psi : X^*(M) \to X(M)$	$\Psi : T^*M \to TM$.

The difference between considering Λ as a vector bundle map $\Lambda : TM \to TM$ and as a linear map $\Lambda : X(M) \to X(M)$ is that with the vector bundle map we can transform one vector of TM, while the linear map $\Lambda : X(M) \to X(M)$ transforms a vector field on M.

Lie derivatives.

An extremely important tool in this thesis will be the *Lie derivative*. Suppose Ξ is a tensor field of arbitrary orders and A is a vector field. Then the Lie derivative $L_A\Xi$ is again a tensor field of the same type as Ξ. In the special case that $\Xi = B$, a vector field, we have

$$L_AB = [A,B] = -L_BA. \tag{1.1.12}$$

In local coordinates the Lie derivatives of $F \in F(M)$, $B \in X(M)$, $\alpha \in X^*(M)$, $\Phi \in T^0_2(M)$, $\Lambda \in T^1_1(M)$ and $\Psi \in T^2_0(M)$ are given by

$$(1.1.13) \quad \begin{cases} L_AF = F_{,k}A^k, \\ L_AB = [A,B] = (B^i_{,k}A^k - A^i_{,k}B^k)\, e_i, \\ L_A\alpha = (\alpha_{i,k}A^k + \alpha_kA^k_{,i})\, du^i, \\ L_A\Phi = (\Phi_{ij,k}A^k + \Phi_{ik}A^k_{,j} + \Phi_{kj}A^k_{,i})\, du^j \otimes du^i, \\ L_A\Lambda = (\Lambda^i_{j,k}A^k - \Lambda^k_jA^i_{,k} + \Lambda^i_kA^k_{,j})\, e_i \otimes du^j, \\ L_A\Psi = (\Psi^{ij}_{,k}A^k - \Psi^{ik}A^j_{,k} - \Psi^{kj}A^i_{,k})\, e_j \otimes e_i. \end{cases}$$

The Lie derivative satisfies *Leibniz' rule*

$$L_A(\Xi_1 \otimes \Xi_2) = (L_A\Xi_1) \otimes \Xi_2 + \Xi_1 \otimes L_A\Xi_2 \ .$$

Of course higher Lie derivatives are also possible. For instance

$$L_A^2(\Xi_1 \otimes \Xi_2) = (L_A^2\Xi_1) \otimes \Xi_2 + 2(L_A\Xi_1) \otimes L_A\Xi_2 + \Xi_1 \otimes L_A^2\Xi_2 \ .$$

Since the Lie derivative "commutes with contraction" this means that the Lie derivative also satisfies Leibniz'rule with respect to contracted multiplication. For instance

$$[A,\Lambda B] = L_A(\Lambda B) = (L_A\Lambda)B + \Lambda L_A B = (L_A\Lambda)B + \Lambda[A,B].$$

Differential forms.

A *(differential) k-form* ξ on M, considered in a point $u \in M$, is a k-linear completely antisymmetric mapping $\xi : T_uM \times T_uM \times \ldots \times T_uM \to \mathbb{R}$. This means we can identify a k-form with a completely antisymmetric tensor field with covariant order k and contravariant order 0. For instance a two-form ϕ can be identified with a tensor field $\Phi \in T_2^0(M)$

(1.1.14) $\qquad \phi(A,B) = \langle \Phi A,B \rangle \qquad \forall\ A,B \in X(M).$

Note that we consider the tensor field Φ as a mapping $\Phi : X(M) \to X^*(M)$. This different way of using a tensor field and the corresponding differential form is the reason for introducing a distinct notation. In general we shall use capital Greek letters for tensor fields. If a tensor field corresponds to a differential form, we denote this form by the corresponding small greek letter (Ξ,ξ; Φ,ϕ; Ω,ω). The *interior product* $i_A\xi$ of a k-form with a vector field yields a (k-1)-form defined by

(1.1.15) $\qquad i_A\xi(B_1, \ldots ,B_{k-1}) = \xi(A,B_1, \ldots ,B_{k-1}).$

It is easily seen that the (k-1)-form $i_A\xi$ corresponds to the tensor field ΞA. The interior product of a two-form with a vector field yields a one-form. From (1.1.14) we obtain

(1.1.16) $\qquad i_A\phi(B_1) \equiv \langle i_A\phi,B_1 \rangle = \phi(A,B_1) = \langle \Phi A,B_1 \rangle,$

which means $i_A\phi = \Phi A$. For a function $F \in \mathcal{F}(M)$ we define $i_A F = 0$.

Exterior differentiation.

The interior product lowers the degree of a differential form. An operation which increases the degree of a differential form is *exterior differentiation*. If ξ is a k-form, the exterior derivative $d\xi$ is a (k+1)-form. In local coordinates the exterior derivative of a function F (= zero-form), one-form α and two-form ϕ are given by

$$(1.1.17)\quad \begin{cases} \langle dF, A\rangle = F_{,i}A^i , \\ d\alpha(A,B) = (\alpha_{i,j} - \alpha_{j,i})A^jB^i , \\ d\phi(A,B,C) = (\Phi_{ij,k} + \Phi_{jk,i} + \Phi_{ki,j})A^jB^iC^k , \end{cases}$$

for all vector fields $A,B,C \in \mathcal{X}(M)$.

1.1.18 Definition.

A k-form ξ with $d\xi = 0$ is called a *closed k-form*. A k-form ξ (k > 0) which can be written as $\xi = d\zeta$ with ζ a (k-1)-form is called an *exact k-form*.

□

Since $d^2\zeta = dd\zeta = 0$ for all forms ζ, an exact form is always closed. In general the converse is not true.

1.1.19 Lemma (Poincaré).

Suppose ξ is a closed k-form on M. Then for every point $u \in M$ there exists a neighbourhood U such that $\xi|_U$ (ξ restricted to U) is exact.

Proof:

See for instance Abraham and Marsden [1, § 2.4.17].

□

So for every closed k-form ξ and every point $u \in M$ there exists a neighbourhood U of u and a (k-1)-form ζ on U such that $\xi = d\zeta$ on U. Of course this does not imply that $\xi = d\zeta$ on the whole manifold M.

Exterior multiplication.

Suppose $\Xi_1 \in T^0_k(M)$ and $\Xi_2 \in T^0_\ell(M)$ are two completely antisymmetric tensor fields. The corresponding differential forms are denoted by ξ_1 and ξ_2. It is easily seen that the tensor product $\Xi_1 \otimes \Xi_2 \in T^0_{k+\ell}(M)$ is in general not completely antisymmetric. By "antisymmetrization" of this tensor field we obtain a tensor field $\Xi \in T^0_{k+\ell}(M)$ which is again antisymmetric. The corresponding $(k+\ell)$-form ξ is written as

$$\xi = \xi_1 \wedge \xi_2 ,$$

and is called the *exterior product* of the forms ξ_1 and ξ_2. For instance if $k = \ell = 1$ we have

$$\Xi = \Xi_1 \otimes \Xi_2 - \Xi_2 \otimes \Xi_1 ,$$

$$\xi = \xi_1 \wedge \xi_2 .$$

The Lie derivative $L_A\Xi$ of a completely antisymmetric tensor field $\Xi \in T^0_k(M)$ is again an antisymmetric tensor field of the same type. The k-form corresponding to $L_A\Xi$ is denoted as $L_A\xi$, where ξ is the k-form corresponding to the tensor field Ξ. For instance for a two-form ϕ we have (see (1.1.14))

$$(1.1.20) \qquad (L_A\phi)(B_1,B_2) = \langle (L_A\Phi)B_1,B_2\rangle .$$

Note that this formula is only a consequence of the distinct notations we use for a tensor field and the corresponding differential form.

Several formulas.

Now we give a list of various other formulas which will be used in this paper (see also Choquet-Bruhat [3, chapter IV, § A4]). Suppose Ξ_1 and Ξ_2 are arbitrary tensor fields, A and B are vector fields and α is a one-form on M. Then

$$(1.1.21) \qquad L_A(\Xi_1\Xi_2) = (L_A\Xi_1)\Xi_2 + \Xi_1(L_A\Xi_2),$$

(*Leibniz'rule* for contracted multiplication, same type of contraction in all terms)

(1.1.22) $\quad L_A\langle\alpha,B\rangle = \langle L_A\alpha,B\rangle + \langle\alpha,L_AB\rangle$,

(special case of (1.1.21))

(1.1.23) $\quad L_AB = [A,B] = -L_BA$,

(1.1.24) $\quad L_AL_B - L_BL_A = L_{[A,B]}$.

For the operators L_A, i_A and d on differential forms it can be shown that

(1.1.25) $\quad i_A^{\ 2} = i_Ai_A = 0$,

(1.1.26) $\quad d^2 = dd = 0$,

(1.1.27) $\quad L_A = di_A + i_Ad$,

(1.1.28) $\quad L_Ai_B - i_BL_A = i_{[A,B]}$,

(1.1.29) $\quad d\alpha(A,B) = L_A\langle\alpha,B\rangle - L_B\langle\alpha,A\rangle - \langle\alpha,[A,B]\rangle$ (α one-form) ,

(1.1.30) $\quad d(\xi_1 \wedge \xi_2) = d\xi_1 \wedge \xi_2 + (-1)^k\xi_1 \wedge d\xi_2$ (ξ_1 k-form).

It is easily seen from (1.1.27) and (1.1.26) that

(1.1.31) $\quad dL_A = L_Ad.$

Suppose F is a function on M. Then using $i_AF = 0$ we obtain from (1.1.27) that

(1.1.32) $\quad L_AF = i_AdF \equiv \langle dF,A\rangle$.

Transformation properties.

Suppose there exists a diffeomorphism f between M and some other manifold N so $f : M \to N$. Then using this diffeomorphism all vector fields, differential forms, tensor fields on M can be transformed to objects of the same type on N. All operations described in this section are *natural* with respect to this transformation, i.e. the transformed objects satisfy similar relations as the original objects. For instance suppose A and B are two vector fields on M. The transformed vector fields on N are given by $\tilde{A} = f'A$ and $\tilde{B} = f'B$. Then it can be shown that

$$f'[A,B] = [(f'A),(f'B)],$$

so the transformed Lie bracket of A and B is equal to the Lie bracket of the transformed vector fields.

Parameterized tensor fields.

We shall frequently use functions, vector fields, differential forms and tensor fields on M which also depend on some additional parameter ($t \in \mathbb{R}$).

1.1.33 Definition.

The set of smooth *parameterized functions* on M will be denoted as $F_p(M)$. The sets of smooth *parameterized vector fields* and *one-forms* on M will be denoted as $X_p(M)$ and $X_p^*(M)$. Finally the set of smooth *parameterized tensor fields* on M with covariant order j and contravariant order i will be denoted as $T^i_{jp}(M)$. In all cases the parameter (t) is allowed to take all values in $\mathbb{R}$.

□

So if $Y \in X_p(M)$, then $Y(u,t) \in T_uM$ for all $t \in \mathbb{R}$. Of course $F_p(M) = F(M\times\mathbb{R})$. However, in order to keep a uniform notation, we shall only use $F_p(M)$. Of course $F(M)$, $X(M)$, $X^*(M)$ and $T^i_j(M)$ are (can be identified with) subsets of $F_p(M)$, $X_p(M)$, $X_p^*(M)$ and $T^i_{jp}(M)$.

1.2 "DIFFERENTIAL GEOMETRY" ON A TOPOLOGICAL VECTOR SPACE

In the preceding section we gave an overview of some aspects of differential geometry on a finite-dimensional manifold M. The notions and relations introduced in that section will extensively be used in chapters 2, 3 and 4. So we can make a straightforward use of the results of those chapters if we consider a dynamical system on a finite-dimensional manifold (for instance the periodic Toda lattice [52]). However, several interesting dynamical systems are described by partial differential equations, i.e. they have "an infinite number of degrees of freedom". So at first sight we need the machinery of differential geometry, as described in section 1.1, also on manifolds of infinite dimension. Fortunately most of the interesting dynamical systems with "an infinite number of degrees of freedom" can be considered in a *topological vector space* instead of on an arbitrary manifold (of infinite dimension). In this way we can avoid the problems associated with differential geometry on manifolds of infinite dimension.

We shall now describe how several differential geometrical objects, introduced in section 1.1, can be "generalized" to the case that the manifold M is an (infinite-dimensional) topological vector space W. The (topological) dual of W will be denoted by W^* and the duality map between W and W^* by $\langle\cdot,\cdot\rangle$. We only consider the case $W^{**} = W$, so W is *reflexive*. The space of linear continuous mappings of W into some topological vector space W_1 will be denoted by $L(W,W_1)$. We shall consider $L(W,W_1)$ as a topological vector space with the topology of bounded convergence (see Yosida [45, § IV.7]).

Since $M = W$ is a linear space, we can make the following identifications

(1.2.1)
$$T_uW = W\ ,\quad TW = W \times W\ ,$$
$$T_u^*W = W^*,\ T^*W = W \times W^*.$$

Using these identifications it is easy to introduce (objects similar to) vector fields, differential forms and tensor fields on W. A vector field A on W is a mapping

(1.2.2)
$$A : W \to W \times W : u \to (u,\tilde{A}(u))$$

where $\tilde{A} : W \to W$ is a, possibly nonlinear, mapping. So we can identify the vector field A with the mapping $\tilde{A}$. Therefore $\tilde{A}$ will also be called a vector field. To simplify notation we shall drop the tilde and write A instead of $\tilde{A}$. In a similar way we can introduce one-forms and tensor fields of higher order. This results in the following list of identifications (c.q. definitions in the infinite-dimensional case) :

	tensor field	"representation"
	$A \in X(W)$	$A : W \to W$,
	$\alpha \in X^*(W)$	$\alpha : W \to W^*$,
	$\Phi \in T^0_2(W)$, considered as vector bundle map $\Phi : TW \to T^*W$	$\Phi : W \to L(W,W^*)$,
(1.2.3)	$\Lambda \in T^1_1(W)$, considered as vector bundle map $\Lambda : TW \to TW$	$\Lambda : W \to L(W,W)$,
	$\Gamma \in T^1_1(W)$, considered as vector bundle map $\Gamma : T^*W \to T^*W$	$\Gamma : W \to L(W^*,W^*)$,
	$\Psi \in T^2_0(W)$, considered as vector bundle map $\Psi : T^*W \to TW$	$\Psi : W \to L(W^*,W)$.

Note that a tensor field in $T^1_1(W)$ can be represented by a linear operator (in fact operator field on W) $\Lambda(u) : W \to W$ and by a linear operator $\Gamma(u) : W^* \to W^*$. If $\Lambda(u)$ and $\Gamma(u)$ correspond to the same tensor field we have $\Lambda(u) = \Gamma^*(u)$ for all $u \in W$. If Φ is antisymmetric (so $\Phi(u)$ is antisymmetric for all $u \in W$) the corresponding differential two-form ϕ on W is given by

(1.2.4) $$\phi(u)(A,B) = \langle \Phi(u)A,B \rangle \qquad \forall\ A,B \in W.$$

An example of this relation is given by (5.6.2). In fact expressions of that type are the reason for introducing a distinct notation for tensor fields (represented by operator fields) and corresponding differential forms. In a similar way we can introduce higher order tensor fields and differential forms on W.

Next we introduce Lie derivatives and (for differential forms) exterior derivatives. First some remarks on differential calculus in topological vector spaces. For a more detailed discussion of this complicated subject we refer to Yamamuro [46]. Suppose W_1 is some topological vector space and f is a (nonlinear) mapping $f : W \to W_1$.

1.2.5 Definition.

We call f *Gateaux differentiable* in $u \in W$ if there exists a mapping $\theta \in L(W,W_1)$ such that for all $A \in W$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (f(u + \varepsilon A) - f(u) - \theta A) = 0 \qquad (1.2.6)$$

in the topology of W_1. The linear mapping $\theta \in L(W,W_1)$ is called the *Gateaux derivative* of f in u and is written as $\theta = f'(u)$.

□

If f is Gateaux differentiable in all points $u \in W$, we can consider the Gateaux derivative as a (in general nonlinear) mapping

$$f' : W \to L(W,W_1).$$

Suppose f' is again Gateaux differentiable in $u \in W$. The second derivative of f in $u \in W$ is a linear mapping $f''(u) \in L(W, L(W,W_1))$. It is easily seen that $f''(u)$ can be considered as a bilinear mapping

$$f''(u) : W \times W \to W_1.$$

Under certain assumptions it can be shown that this mapping is symmetric: $f''(u)(v,w) = f''(u)(w,v)$ for all $w,v \in W$ (see [46]). We shall call a mapping $f : W \to W_1$ *twice differentiable* if its first and second Gateaux derivatives exist and if $f''(u)$ is a symmetric bilinear mapping for all $u \in W$. We assume all mappings in this section are twice differentiable.

1.2.7 Remark.

Note that in the limit given in (1.2.6) a uniformity in A is not required. If this limit is uniform on all sequentially compact subsets of W, the mapping f is called *Hadamard differentiable*. If the limit is uniform on all bounded subsets of W, the mapping f is called *Fréchet differentiable*.

□

Suppose $A : W \to W$ is (represents) a vector field. The Gateaux derivative in $u \in W$ is a linear mapping $A'(u) : W \to W$. The dual of this mapping is denoted by $A'^*(u) : W^* \to W^*$.

1.2.8 Definition.

The *Lie derivatives* in the direction of a vector field A of a function $F : W \to \mathbb{R}$ and of the various tensor fields (vector fields, one-forms) considered in (1.2.3) are defined by

$$
(1.2.9)\quad \begin{cases}
L_A F(u) = F'(u)A \equiv \langle F'(u), A\rangle \ , \\
L_A B(u) \equiv [A,B](u) = B'(u)A(u) - A'(u)B(u) \ , \quad (B \in X(W)), \\
L_A \alpha(u) = \alpha'(u)A(u) + A'^*(u)\alpha(u) \ , \\
L_A \Phi(u) = (\Phi'(u)A(u)) + \Phi(u)A'(u) + A'^*(u)\Phi(u) \ , \\
L_A \Lambda(u) = (\Lambda'(u)A(u)) + \Lambda(u)A'(u) - A'(u)\Lambda(u) \ , \\
L_A \Gamma(u) = (\Gamma'(u)A(u)) - \Gamma(u)A'^*(u) + A'^*(u)\Gamma(u) \ , \\
L_A \Psi(u) = (\Psi'(u)A(u)) - \Psi(u)A'^*(u) - A'(u)\Psi(u) \ .
\end{cases}
$$

□

First some remarks on the notation in these expressions. Consider the formula for $L_A\Phi$. Since $\Phi : W \to L(W,W^*)$ we have $\Phi'(u) \in L(W, L(W,W^*))$. So $(\Phi'(u)A) \in L(W,W^*)$ and $(\Phi'(u)A)B \in W^*$. By definition

$$(\Phi'(u)A)B = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (\Phi(u + \varepsilon A)B - \Phi(u)B).$$

Of course in general this expression is not symmetric in A and B. Therefore we shall always insert brackets in expressions of this type. It is easily seen that the Lie derivative of an object yields again an object of the same type. Note that if $\Gamma^*(u) = \Lambda(u)$ (so Γ and Λ represent the same tensor field) the same holds true for the Lie derivatives: $(L_A\Gamma(u))^* = L_A\Lambda(u)$. Next we define *exterior derivatives* of zero-, one- and two-forms.

1.2.10 Definition.

i) The exterior derivative of a function $F : W \to \mathbb{R}$ is the mapping $dF : W \to W^* : u \to F'(u)$ (so $dF(u) = F'(u)$).

ii) The exterior derivative of a one-form $\alpha : W \to W^*$ is the two-form

$$d\alpha(A,B) = \langle\alpha'(u)A,B\rangle - \langle\alpha'(u)B,A\rangle$$

$$= \langle(\alpha'(u) - \alpha'^*(u))A,B\rangle \qquad \forall\, A,B \in W.$$

iii) The exterior derivative of a two-form ϕ, corresponding to an operator $\Phi(u)$ as in (1.2.4), is given by

$$d\phi(A,B,C) = \langle(\Phi'(u)A)B,C\rangle + \langle(\Phi'(u)B)C,A\rangle + \langle(\Phi'(u)C)A,B\rangle ,$$

$$\forall\, A,B,C \in W.$$

□

Note that the definitions (1.2.8) and (1.2.10) strongly resemble the expressions in local coordinates (1.1.13) and (1.1.17) for the corresponding objects on a finite-dimensional manifold. *Contractions* and *interior products* in the infinite-dimensional case are interpreted via (1.2.3). Also we shall adopt the notions *closed* and *exact* differential forms (see definition 1.1.18).

1.2.11 Theorem.

The relations (1.1.22) up to (1.1.32) included are also valid for Lie derivatives and exterior derivatives given in definitions 1.2.8 and 1.2.10.

Proof:

All proofs are similar to proofs in local coordinates of the corresponding relations on a finite-dimensional manifold. If a second derivative appears, we need its symmetry.

□

Suppose α is a closed one-form with continuous derivative $\alpha'(u) : W \to W^*$. Then (definition 1.2.10 ii) $\alpha'(u) = \alpha'^*(u)$ for all $u \in W$. Since W is a linear space, a closed differential form is also exact. Define the function $F : W \to \mathbb{R}$ by

(1.2.12) $$F(u) = \int_0^1 \langle\alpha(au),u\rangle\, da.$$

Then it is easily verified that $\alpha = dF$, so indeed α is also an exact one-form.

In a somewhat different context an operator $\alpha : \mathcal{W} \to \mathcal{W}^*$ with $\alpha'(u) = \alpha'^*(u)$ is called a *potential operator*. Expression, similar to (1.2.12), can be given for closed higher order differential forms.

Finally we mention that we shall use the same notation as introduced in definition 1.1.33 for *parameterized* functions, vector fields, one-forms and higher order tensor fields on $\mathcal{W}$.

1.3 SOME FUNCTION SPACES

In chapter 5 we shall consider several nonlinear evolution equations. Some of these equations can be written in the form

(1.3.1) $$u_t = f(u, u_x, \ldots),$$

where f is a polynomial in u and its derivatives. The Burgers equation (section 5.2), Korteweg-de Vries equation (section 5.6) and the Sawada-Kotera equation (section 5.7) are of this type. In this section we describe function spaces S_p, in duality with spaces U_p, in which we shall consider these equations. For convenience we set $\partial = \frac{d}{dx}$.

1.3.2 Definition.

For $p \in \mathbb{R}^+$ we define the space S_p by

$$S_p = \{ u \in C^\infty(\mathbb{R}) \mid \sqrt{x^2+1}^{\,m+p} \partial^m u(x) \in L_1(\mathbb{R}), \ \forall\, m \geq 0 \}.$$

□

The following two theorems describe some properties of the space S_p.

1.3.3 Theorem.

For every function $u \in S_p$ there exists a series of constants C_m such that

$$|\partial^m u(x)| \leq \frac{C_m}{\sqrt{x^2+1}^{\,m+p+1}} \qquad m = 0,1,2,\ldots .$$

Proof:

Set $v_m(x) = \sqrt{x^2+1}^{\,m+p+1} \partial^m u(x)$. Then

$$\partial v_m(x) = (m+p+1)\sqrt{x^2+1}^{\,m+p-1} x\partial^m u(x) + \sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u(x).$$

Hence

$$|\partial v_m(x)| \le (m+p+1)|\sqrt{x^2+1}^{\,m+p}\partial^m u(x)| + |\sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u(x)|.$$

Since $u \in S_p$ this means that $\partial v_m \in L_1(\mathbb{R})$. Then from

$$v_m(x) = v_m(0) + \int_0^x \partial v_m(y)\,dy$$

we see that v_m is bounded; there exists a constant C_m such that $|v_m(x)| \le C_m$ $\forall\, x \in \mathbb{R}$.

□

An important property of the spaces S_p is described in

1.3.4 Theorem.

Suppose $u \in S_p$. Then also $xu_x \in S_p$.

Proof:
From $\partial^m(xu_x) = x\partial^{m+1}u + m\partial^m u$ we obtain

$$|\sqrt{x^2+1}^{\,m+p}\partial^m(xu_x)| \le |\sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u| + m|\sqrt{x^2+1}^{\,m+p}\partial^m u| .$$

Both terms of the right hand side are elements of $L_1(\mathbb{R})$, so also the left hand side is an element of $L_1(\mathbb{R})$.

□

We shall also need smooth functions v which satisfy the following conditions

(1.3.5) $\lim_{x\to\infty} v(x) = -\lim_{x\to-\infty} v(x) = a \in \mathbb{R}$, a depends on v,

(1.3.6) $\sqrt{x^2+1}^{\,m+p}\partial^{m+1}v(x) \in L_1(\mathbb{R}) \qquad \forall\, m \ge 0.$

1.3.7 Definition.

For $p \in \mathbb{R}^+$ we define the space U_p by

$$U_p = \{\, v \in C^\infty(\mathbb{R}) \mid v \text{ satisfies (1.3.5) and (1.3.6) } \}.$$

□

We now consider the relations between the spaces S_p and U_p.

1.3.8 Theorem.

i) $S_p \subset U_p$,
ii) if $v \in U_p$ then $\partial v = v_x \in S_p$,
iii) if $u \in S_p$ and $v \in U_p$ then $uv \in S_p$.

Proof:
The first two parts of this theorem follow immediately from the definitions of S_p and U_p. An elementary calculation yields

$$\sqrt{x^2+1}^{\,m+p}\,\partial^m(uv) = \sum_{i=0}^{m} \binom{m}{i} (\sqrt{x^2+1}^{\,i+p}\,\partial^i u)(\sqrt{x^2+1}^{\,m-i}\,\partial^{m-i} v). \tag{1.3.9}$$

Since $u \in S_p$ we have $\sqrt{x^2+1}^{\,i+p}\,\partial^i u \in L_1(\mathbb{R})$. We now consider the function $\sqrt{x^2+1}^{\,m-i}\,\partial^{m-i} v$. For $i = m$ this is equal to v, which is clearly a bounded function. For $i < m$ we obtain from part ii) of this theorem and theorem 1.3.3 also that this function is bounded. Hence the left hand side of (1.3.9) is an element of $L_1(\mathbb{R})$.

□

1.3.10 Corollary.

If $u \in S_p$ and $v \in S_p$ then also $uv \in S_p$.

□

We have seen that the operator $\partial = \frac{d}{dx}$ maps U_p into S_p. It is possible to define an inverse operator which acts in the opposite direction.

1.3.11 Theorem.

The inverse operator of $\partial : U_p \to S_p$ is the operator $\partial^{-1} : S_p \to U_p$, defined by

$$\partial^{-1}u(x) = \int_{-\infty}^{x} u(y)\,dy - \frac{1}{2}\int_{-\infty}^{\infty} u(y)\,dy. \tag{1.3.12}$$

Proof:
For $u \in S_p$ both integrals exist. We now show that $\partial^{-1}u \in U_p$. It is easily

seen that $\partial^{-1}u$ satisfies (1.3.5) with

$$a = \frac{1}{2} \int_{-\infty}^{\infty} u(y)\, dy.$$

Since $\partial\partial^{-1}u = u$ and $u \in S_p$ it follows from the definition of S_p that $\partial^{-1}u$ also satisfies (1.3.6). The proof is completed by noting that $\partial^{-1}\partial v = v$ for arbitrary $v \in U_p$.

□

Next we introduce a *topology* on S_p and on U_p. For $v \in U_p$ and $u \in S_p$ define

$$\text{(1.3.13)} \qquad \langle v,u\rangle = \int_{-\infty}^{\infty} v(x)u(x)\, dx.$$

This bilinear mapping $U_p \times S_p \to \mathbb{R}$ is called a *duality* or *duality map*. It is easily seen that this duality map is *separating*, i.e. for every nonzero $v \in U_p$ there exists a $u \in S_p$ such that $\langle v,u\rangle \neq 0$ and for every nonzero $u \in S_p$ there exists a $v \in U_p$ with $\langle v,u\rangle \neq 0$. With every $v \in U_p$ corresponds a *seminorm* $p_v(u) = |\langle v,u\rangle|$ on S_p. Also every $u \in S_p$ gives rise to a seminorm $q_u(v) = |\langle v,u\rangle|$ on U_p. Then, using the family of seminorms $\{ p_v \mid v \in U_p \}$, we can supply S_p with a topology. The seminorms $\{ q_u \mid u \in S_p \}$ provide U_p with a topology. Some properties of both topological spaces are described in

1.3.14 Theorem.

The spaces S_p and U_p are *locally convex Hausdorff topological vector spaces*. The *(topological) dual* of S_p is (can be represented by) U_p and the (topological) dual of U_p is S_p, so

$$S_p^* = U_p \,, \quad U_p^* = S_p.$$

Proof:

See Choquet [43; propositions 22.3 and 22.4].

□

Since we now have a topology on S_p and on U_p we can study the continuity of the various mappings between these spaces. Recall that a mapping of a topological space into a topological space is continuous iff the inverse image of an open set is open. Suppose W_1 and W_2 are topological vector spaces with topologies generated by the families of seminorms $\{q_i\}$

respectively $\{p_i\}$. Then a linear mapping $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ is continuous iff for every seminorm p_i on $\mathcal{W}_2$ there exist a constant C and a seminorm q_j on $\mathcal{W}_1$ such that

$$p_i(\Theta w) \le Cq_j(w) \qquad \forall\, w \in \mathcal{W}_1.$$

If $\mathcal{W}_1 \subset \mathcal{W}_2$ we can consider an element of $\mathcal{W}_1$ also as an element of $\mathcal{W}_2$. This mapping of $\mathcal{W}_1$ into $\mathcal{W}_2$ is called the *embedding operator*.

1.3.15 Theorem.

The mappings $\partial : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are continuous. Suppose $u \in \mathcal{S}_p$. Then the mapping $m_u : \mathcal{U}_p \to \mathcal{S}_p : v \to uv$ is continuous. The embedding operator of $\mathcal{S}_p$ into $\mathcal{U}_p$ is also continuous.

Proof:
Suppose $v \in \mathcal{U}_p$, then $\partial v = v_x \in \mathcal{S}_p$. For an arbitrary $w \in \mathcal{U}_p$ we have

$$p_w(v_x) = \Big|\int_{-\infty}^{\infty} wv_x\, dx\Big| = \Big|\int_{-\infty}^{\infty} vw_x\, dx\Big| = q_{w_x}(v).$$

This means that $\partial : \mathcal{U}_p \to \mathcal{S}_p$ is continuous. The continuity of the other mappings is proved in a similar way.

□

Suppose $u \in \mathcal{S}_p$. To simplify notation we will denote the mapping $m_u : \mathcal{U}_p \to \mathcal{S}_p$ (multiplication by u) by $u : \mathcal{U}_p \to \mathcal{S}_p$. Then, using various parts of this theorem, we see that for instance $u\partial$, ∂u, ∂^3, $u\partial^{-1}u : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1}u$, $u\partial^{-1}$, $\partial^{-1}u\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are continuous mappings.

Consider the topological vector spaces $\mathcal{W}_1$ and $\mathcal{W}_2$ with (topological) duals $\mathcal{W}_1^*$ and $\mathcal{W}_2^*$. The *dual operator* of a linear operator $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ is the linear operator $\Theta^* : \mathcal{W}_2^* \to \mathcal{W}_1^*$ defined by

$$(1.3.16) \qquad \langle\Theta^* w_2, w_1\rangle = \langle\Theta w_1, w_2\rangle \qquad \forall\, w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2^*.$$

A special situation occurs if $\mathcal{W}_1^* = \mathcal{W}_2$ and $\mathcal{W}_2^* = (\mathcal{W}_1^{**}=)\ \mathcal{W}_1$ (so $\mathcal{W}_1$ is reflexive). Then $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ and also $\Theta^* : \mathcal{W}_1 \to \mathcal{W}_2$. In this case we call an operator Θ *symmetric* if $\Theta^* = \Theta$ and *antisymmetric* if $\Theta^* = -\Theta$.

1.3.17 Theorem.

The operators $\partial : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are antisymmetric, so

(1.3.18) $$\langle v_1, \partial v_2 \rangle = - \langle v_2, \partial v_1 \rangle \qquad \forall\ v_1, v_2 \in \mathcal{U}_p,$$

(1.3.19) $$\langle \partial^{-1} u_1, u_2 \rangle = - \langle \partial^{-1} u_2, u_1 \rangle \qquad \forall\ u_1, u_2 \in \mathcal{S}_p.$$

Proof:
The first expression follows by partial integration. The proof of (1.3.19) is a straightforward computation using (1.3.12) and (1.3.13).

□

We shall frequently need the dual of an operator which is the *composition* of two other operators. Suppose $\Theta = \Theta_2\Theta_1 : \mathcal{W}_1 \to \mathcal{W}_2$ with $\Theta_1 : \mathcal{W}_1 \to \mathcal{W}_3$ and $\Theta_2 : \mathcal{W}_3 \to \mathcal{W}_2$. Then it is easily seen that $\Theta^* = \Theta_1^*\Theta_2^*$.

Finally we describe some operators which we shall use frequently in chapter 5 (in particular in section 5.6). For $u \in \mathcal{S}_p$ consider the operators $u\partial$, ∂u, $\partial^3 : \mathcal{U}_p \to \mathcal{S}_p$. The dual operators are found to be $(u\partial)^* = -\partial u$, $(\partial u)^* = -u\partial$ and $(\partial^3)^* = -\partial^3$. This means that

(1.3.20) $$\Psi = u\partial + \partial u - \partial^3 : \mathcal{U}_p \to \mathcal{S}_p$$

is an antisymmetric operator. We shall also meet the operator

$$\Gamma = \partial^{-1}\Psi = \partial^{-1}u\partial + u - \partial^2 : \mathcal{U}_p \to \mathcal{U}_p.$$

The dual operator of Γ is then given by

$$\Lambda = \Gamma^* = \Psi^*(\partial^{-1})^* = \Psi\partial^{-1} = u + \partial u \partial^{-1} - \partial^2 : \mathcal{S}_p \to \mathcal{S}_p.$$

1.4 THE HILBERT TRANSFORM

In this section we describe some properties of the Hilbert transform, which will be used in Section 5.5. The *Hilbert transform* of a function $u \in L_2(\mathbb{R})$ is defined by

$$\mathcal{H}u(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy \qquad \text{(principal value integral).}$$

1.4.1 Lemma.

Suppose $u \in S_p$ with $0 < p < 1$, then the function

$$(1.4.2) \qquad w(x) = \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{y-x}\, dy \qquad \text{(principal value integral)}$$

is bounded for all $x \in \mathbb{R}$.

Proof:

It follows from the definition of S_p that $u \in L_1(\mathbb{R})$. Suppose $x > 0$. Then we can write (1.4.2) as

$$(1.4.3) \qquad w(x) = \frac{1}{\pi} \int_{-\infty}^{\frac{1}{2}x} \frac{yu(y)}{y-x}\, dy + \frac{p}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{yu(y)}{y-x}\, dy + \frac{1}{\pi} \int_{\frac{3}{2}x}^{\infty} \frac{yu(y)}{y-x}\, dy$$

$$= I_1 + I_2 + I_3.$$

It is easily seen that $|I_1 + I_3| \leq \frac{3}{\pi} \int_{-\infty}^{\infty} |u(y)|dy$. Set $v(y) = yu(y)$. Then we obtain from theorem 1.3.3 that

$$(1.4.4) \qquad |v(y)| \leq \frac{C_o}{\sqrt{y^2+1}^{\,p}}, \quad |v_y(y)| \leq \frac{C_o + C_1}{\sqrt{y^2+1}^{\,p+1}}$$

for all $y \in \mathbb{R}$. Using the mean value theorem we obtain

$$I_2 = \frac{p}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{v(x) + (y-x)v_y(a(y))}{y-x}\, dy \qquad (|a(y)-x| < |y-x|).$$

$$= \frac{1}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} v_y(a(y))\, dy.$$

Then (1.4.4) implies

$$|I_2| \leq \frac{1}{\pi}\, x\, \frac{C_o + C_1}{\sqrt{(\frac{1}{2}x)^2+1}} < \frac{2(C_o + C_1)}{\pi} \qquad \text{for } x > 0.$$

Hence $w(x)$ is bounded for $x > 0$. A similar estimate can be given for $x < 0$.

□

1.4.5 Lemma.

If $u \in S_p$ with $0 < p < 1$ then $Hu \in C^\infty(\mathbb{R})$ and

(1.4.6) $$\mathcal{H}u(x) \le \frac{C}{\sqrt{x^2+1}} \qquad \forall\, x \in \mathbb{R}.$$

Proof:
Since $u \in \mathcal{S}_p$ we have $\partial^m u \in L_2(\mathbb{R})$ for $m = 0,1,2,\ldots$. So $\mathcal{H}\partial^m u = \partial^m \mathcal{H}u \in L_2(\mathbb{R})$, which implies that $\mathcal{H}u \in C^\infty(\mathbb{R})$. Next note that

$$x\mathcal{H}u(x) = \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{xu(y)}{y-x}\,dy$$

(1.4.7) $$= \frac{-1}{\pi}\int_{-\infty}^{\infty} u(y)\,dy + \frac{P}{\pi}\int_{-\infty}^{\infty} \frac{yu(y)}{y-x}\,dy.$$

Then using lemma 1.4.1 and $\mathcal{H}u(x) \in C^\infty(\mathbb{R})$ we obtain (1.4.6).

□

1.4.8 Corollary.

If $u \in \mathcal{S}_p$ and $xu \in \mathcal{S}_p$ then

(1.4.9) $$x\mathcal{H}u(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} u(y)\,dy + \mathcal{H}(xu(x)).$$

Proof:
This result follows at once from (1.4.7).

□

The main result of this section is stated in the following

1.4.10 Theorem.

For $0 < p < 1$ we have $\mathcal{H} : \mathcal{S}_p \to \mathcal{U}_p$.

Proof:
It follows from lemma 1.4.5 that $\mathcal{H}u \in C^\infty(\mathbb{R})$ and $\lim_{x\to\pm\infty} \mathcal{H}u(x) = 0$. So we only have to show that $\sqrt{x^2+1}^{\,m+p}\partial^{m+1}\mathcal{H}u(x) \in L_1(\mathbb{R})$.
Note that if $u \in \mathcal{S}_p$ then $x^j\partial^m u \in \mathcal{S}_p$ for $j \le m$ (see theorem 1.3.4). By using (1.4.9) we obtain

$$x^{m+1}\partial^{m+1}\mathcal{H}u = x^{m+1}\mathcal{H}\partial^{m+1}u$$

$$= x^m\mathcal{H}(x\partial^{m+1}u)$$

$$= \ldots$$

$$= H(x^{m+1}\partial^{m+1}u).$$

Since $x^{m+1}\partial^{m+1}u \in S_p$ we obtain from lemma 1.4.5 and the fact that $Hu \in C^{\infty}(\mathbb{R})$ that

$$|\sqrt{x^2+1}^{\,m+1}\partial^{m+1}Hu| \le \frac{C}{\sqrt{x^2+1}} .$$

Since $0 < p < 1$ this implies that $\sqrt{x^2+1}^{\,m+p}\partial^{m+1}Hu(x) \in L_1(\mathbb{R})$ for $m = 0,1,2,\ldots$. Thus we proved that $Hu \in U_p$.

□

Finally we mention some other properties of the Hilbert transform:

(1.4.11) $\displaystyle\int_{-\infty}^{\infty} uHv\, dx = -\int_{-\infty}^{\infty} vHu\, dx$ (antisymmetry),

(1.4.12) $HHu(x) = -u(x)$,

(1.4.13) $\partial Hu = H\partial u$,

(1.4.14) $(Hu)(Hv) = uv + H(uHv) + H(vHu)$.

1.5 ANALYTICALLY INDEPENDENT FUNCTIONS

1.5.1 Definition.

The functions $F_1, \ldots, F_k$ on a possibly infinite-dimensional manifold M are called *analytically independent* if the corresponding one-forms $dF_1(u), \ldots, dF_k(u)$ are linearly independent elements of T^*_uM for all $u \in N$, where N is a dense open subset of M.

□

If the manifold M is finite-dimensional, we can introduce local coordinates u^i $(i=1,\ldots,n)$ on $U \subset M$. Then it is easily seen that the functions $F_1, \ldots, F_k$ are analytically independent iff the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial u^1} & \cdots & \frac{\partial F_1}{\partial u^n} \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial u^1} & \cdots & \frac{\partial F_k}{\partial u^n} \end{pmatrix}$$

has rank k. This also implies that on a manifold of dimension n there can exist at most n analytically independent functions. The notion analytically independent is explained in the following

1.5.2 Theorem.

Suppose M is a finite-dimensional manifold. The functions $F_1, \dots, F_k$ on M are analytically independent iff locally there does not exist a relation

$$g(F_1, \dots, F_k) = 0$$

where $g : \mathbb{R}^k \to \mathbb{R}$ is a smooth function such that in every point of an open dense subset of $\mathbb{R}^k$ the gradient (one-form dg) does not vanish.

Proof:
See Levi-Civita [54; chapter 1, § 5,6].

□

CHAPTER 2: SYMMETRIES FOR DYNAMICAL SYSTEMS

2.1 INTRODUCTION

This chapter deals with some general properties of dynamical systems on manifolds. If the dynamical system is a Hamiltonian system, more specific results can be obtained. Those more specific results will be considered in Chapter 4. In Section 2.2 we shall introduce two linear equations associated with the dynamical system. Solutions of these equations will be called *symmetries* and *adjoint symmetries*. Since most of the considerations in Section 2.2 are of local character, we shall use a local trivialization of the (co)tangent bundle of the manifold. An introduction of symmetries without using a local trivialization of the tangent bundle will be described in the appendix of this chapter. Several properties of symmetries and adjoint symmetries are considered in Sections 2.3 and 2.4. Higher order symmetries (of which symmetries and adjoint symmetries are special cases) are studied in Section 2.5. In Section 2.6 we consider a dynamical system for which there exist two infinite series of symmetries. This situation will occur several times in Chapters 4 and 5. Finally in Section 2.7 we study the transformation properties of (adjoint) symmetries.

A very important tool in this chapter is the Lie derivative of several types of tensor fields in the direction of a vector field. Sometimes we shall also give the more classical formulas, using local coordinates. In that case the manifold is assumed to be finite-dimensional. For an infinite-dimensional manifold our results are formal.

Symmetries (also called invariant variations, infinitesimal variations, infinitesimal Bäcklund transformations) are also studied by Olver [13], Wadati [14], Fuchssteiner [12,37,64], Fuchssteiner and Fokas [8], Fokas [15] and others. Most authors consider a dynamical system in some (unspecified) topological vector space and write their expressions in terms of Gateaux, Hadamard or Fréchet derivatives. However, the only natural type of derivative for studying symmetries is the (infinite-dimensional version of the) Lie derivative, which replaces complicated combinations of derivatives of one of the previous types. Using this Lie derivative most expressions are considerably simplified and important new relations can be found. Since Lie derivatives are also defined on (in fact invented for) arbitrary smooth manifolds, we can easily describe the theory for dynamical systems on manifolds. In contrast to most authors we also consider (adjoint) symmetries which depend explicitly on the time t. In several applications this type of (adjoint) symmetry turns out to be important.

2.2 DEFINITION OF SYMMETRIES AND ADJOINT SYMMETRIES

Suppose M is a manifold and X a vector field on M, so $X \in \mathcal{X}(M)$. For a curve u(t) on M we set $\dot{u}(t) = \frac{d}{dt} u(t) \in T_{u(t)}M$.

In this chapter we shall consider the following *autonomous differential equation* on M

(2.2.1) $\dot{u}(t) = X(u(t))$.

The parameter t is called time. This equation can be supplied with an initial condition $u(t_0) = u_0$. Since (2.2.1) is an autonomous system, it is no restriction to take $t_0 = 0$. We shall assume that for all $u_0 \in M$ and $t_0 \in \mathbb{R}$ there exists a unique solution u(t) of (2.2.1), with $u(t_0) = u_0$, defined on some interval $I \in \mathbb{R}$.

Suppose U is an open subset of M which can be described by one chart. This means the tangent bundel TU is a trivial bundle, $TU = U \times W$ for some linear space W. Then we can consider the vector field X as a mapping $X : U \to W$. The derivative of $X(u)$ in a point $u \in U$ is a linear mapping $X'(u) : W \to W$. Suppose u(t) is a solution of (2.2.1) which lies in U. Then we can linearize (2.2.1) around u(t) and obtain

(2.2.2) $\dot{v}(t) = X'(u(t))\, v(t) \qquad v(t) \in T_{u(t)}U = W$.

Since $\frac{d}{dt} X(u(t)) = X'(u(t))X(u(t))$, this equation has always the solution $v(t) = X(u(t))$. Another interesting linear equation, associated with (2.2.1) is the so-called adjoint equation of (2.2.2)

(2.2.3) $\dot{w}(t) = - X'^*(u(t))\, w(t) \qquad w(t) \in T^*_{u(t)}U = W^*$,

where $X'^*(u) : W^* \to W^*$ is the dual operator of $X'(u)$. The equations (2.2.1) and (2.2.3) can be derived from the following *variational principle*

$$(2.2.4) \qquad \text{stat} \int_{t_1}^{t_2} \langle w(t), \dot{u}(t) - X(u(t))\rangle \, dt,$$

over the set of all curves $u(t) \in U$, $w(t) \in W$ for $t \in [t_1, t_2]$ with $u(t_1)$ and $u(t_2)$ fixed. A "variation" of w(t) gives (2.2.1) while a "variation" of u(t) leads to (2.2.3).

With appropriate initial conditions for v and w we could study the Cauchy problems, associated with (2.2.2) and (2.2.3). However, we are only interested in special solutions of (2.2.2) and (2.2.3). Suppose there exists a $Y \in X_p(M)$ (so Y is a vector field on M, depending on an additional parameter t, $Y(u,t) \in T_u M$), such that for all solutions u(t) of (2.2.1) which lie (partly) in U, $v(t) = Y(u(t),t)$ is a solution of (2.2.2). This means

$$\dot{Y}(u(t),t) + Y'(u(t),t)\,\dot{u}(t) = X'(u(t))\,Y(u(t),t).$$

Note that $\dot{Y}$, the partial derivative of the parameterized vector field Y with respect to the parameter (t), is again a vector field on M. Since u(t) is a solution of (2.2.1) we obtain

$$\dot{Y}(u(t),t) + Y'(u(t),t)\,X(u(t)) = X'(u(t))\,Y(u(t),t).$$

This condition has to be satisfied for all solutions u(t) (which lie partly in U) with arbitrary initial condition $u(t_0) = u_0$, hence

$$(2.2.5) \qquad \dot{Y}(u,t) = X'(u)\,Y(u,t) - Y'(u,t)\,X(u) \qquad \forall u \in U,\ t \in \mathbb{R}.$$

The right-hand side can be interpreted as the Lie bracket $[Y,X]$ of the vector fields Y and X. This Lie bracket can also be written in terms of Lie derivatives

$$[Y,X] = -L_X Y = L_Y X.$$

So we can write (2.2.5) as

$$\dot{Y} + [X,Y] = \dot{Y} + L_X Y = 0 \quad \forall u \in U, t \in \mathbb{R} .$$

This condition on the vector field Y does not depend on the local trivialization $TU = U \times W$.
This leads to the following

2.2.6 Definition.

A parameterized vector field Y on M (so $Y \in X_p(M)$), which satisfies

$$(2.2.7) \qquad \dot{Y} + [X,Y] = 0$$

on $M \times \mathbb{R}$ is called a *symmetry of the dynamical system* (2.2.1). The set of symmetries of (2.2.1) will be denoted by $V(X;M)$.

□

In the appendix of this chapter we shall show how (2.2.7) can be derived without using a local trivialization of TM. Since $Y = X \in V(X;M)$ the set $V(X;M)$ contains always a non-zero vector field.

Next we turn to special solutions of (2.2.3). Suppose there exists a $\sigma \in X_p^*(M)$ (so σ is a parameterized one-form or covariant vector field, $\sigma(u,t) \in T_u^*M$) such that for all solutions u(t) which lie (partly) in U, $w(t) = \sigma(u(t),t)$ satisfies (2.2.3). This implies

$$\dot{\sigma}(u(t),t) + \sigma'(u(t),t)\ \dot{u}(t) = - X'^*(u(t))\ \sigma(u(t),t).$$

Using (2.2.1) we obtain

$$\dot{\sigma}(u(t),t) + \sigma'(u(t),t)\ X(u(t)) = - X'^*(u(t))\ \sigma(u(t),t).$$

This condition has to be satisfied for all solutions u(t) in U, hence

$$\dot{\sigma}(u,t) + \sigma'(u,t)\ X(u) + X'^*(u)\ \sigma(u,t) = 0 \qquad \forall u \in U, t \in \mathbb{R} .$$

The last two terms in the left-hand side can be written as $L_X\sigma$, the Lie derivative of the one-form σ in direction of the vector field X. This operation results again in a one-form which is independent of the trivialization $TU = U \times W$. Hence the following

2.2.8 Definition.

A parameterized one-form σ (so $\sigma \in X_p^*(M)$) which satisfies

$$(2.2.9) \qquad \dot{\sigma} + L_X\sigma = 0$$

on $M \times \mathbb{R}$ is called an *adjoint symmetry of the dynamical system* (2.2.1). The set of adjoint symmetries of (2.2.1) will be denoted by $V^*(X;M)$.

□

Adjoint symmetries which do not depend on t (so $L_X\sigma = 0$) are also called *integral invariants*. Of course $V(X;M) \subset X_p(M)$ and $V^*(X;M) \subset X_p^*(M)$. Finally we mention that in the remaining part of this chapter (adjoint) symmetries, unless stated otherwise, are meant as (adjoint) symmetries of the dynamical system (2.2.1).

2.3 PROPERTIES OF SYMMETRIES

First some remarks on the notion of constant of the motion.

2.3.1 Definition.

We call a function $F \in F_p(M)$ a *constant of the motion* or *first integral* of (2.2.1) if, for all solutions u(t) of (2.2.1)

$$\frac{d}{dt} F(u(t),t) = 0.$$

This is equivalent to

$$(2.3.2) \qquad \dot{F} + \langle dF, X \rangle = \dot{F} + L_X F = 0 \quad \text{on } M \times \mathbb{R} .$$

□

Constants of the motion which differ only by a real constant will be identified. The following two lemma's are an immediate consequence of the fact that the evolution equation (2.2.1) is autonomous.

2.3.3 Lemma.

If F is a constant of the motion, then the same holds true for $\dot{F}$.

□

2.3.4 Lemma.

If $Y \in V(X;M)$, then also $\dot{Y} \in V(X;M)$.

□

Some properties of the set of symmetries $V(X;M)$ are described in

2.3.5 Theorem.

$V(X;M)$ is a real linear space. Further if $Y \in V(X;M)$ and F is a constant of the motion, then $FY \in V(X;M)$.

Proof:
Symmetries have to satisfy the linear equation (2.2.7), so the first remark is trivial. Next note that (Leibniz' rule)

$$[X,FY] = L_X(FY) = F[X,Y] + (L_X F)\, Y.$$

Since F is a constant of the motion and Y a symmetry this can be written as

$$[X,FY] = -\, F\dot{Y} - \dot{F}Y = -\frac{\partial}{\partial t}(FY) \ .$$

So the vector field FY is again a symmetry.

□

Theorem 2.3.5 can be summarized by saying that the set of symmetries $V(X;M)$ is a module over the ring of constants of the motion of (2.2.1).

2.3.6 Theorem.

$V(X;M)$ is a Lie algebra with the same Lie bracket as the algebra $X(M)$ of all vector fields on M. The autonomous symmetries (that is symmetries Y with $\dot{Y} = 0$) form a subalgebra of $V(X;M)$.

Proof:
Suppose Y_1, $Y_2 \in V(X;M)$. Set $Y = [Y_1,Y_2]$. Then

$$\dot{Y} = [\dot{Y}_1,Y_2] + [Y_1,\dot{Y}_2]$$

$$= [[Y_1,X], Y_2] + [Y_1, [Y_2 ,X]] .$$

Using the Jacobi identity for Lie brackets we get

$$\dot{Y} = [[Y_1,Y_2], X] = [Y,X] \quad ,$$

which shows that $V(X;M)$ is a Lie algebra. Finally note that if Y_1 and Y_2 are autonomous, then $Y = [Y_1, Y_2]$ is also autonomous.

□

Next we consider tensor fields which can be used to construct (new) symmetries from (already known) symmetries. Suppose $\Lambda \in T^1_{1p}(M)$, so Λ is a parameterized tensor field of covariant order 1 and contravariant order 1. Then Λ can also be considered as a vector bundle map $\Lambda : TM \to TM$ or as a linear mapping $\Lambda : X_p(M) \to X_p(M)$. We can ask under which conditions Λ maps $V(X;M)$ into $V(X;M)$. This leads to the following

2.3.7 Theorem.

Suppose the tensor field $\Lambda \in T^1_{1p}(M)$ satisfies

(2.3.8) $\quad \dot{\Lambda} + L_X\Lambda = 0 \quad$ on $M \times \mathbb{R}$.

Then if $Y \in V(X;M)$, then also $\Lambda Y \in V(X;M)$.

Proof:
Since the Lie derivative satisfies Leibniz' rule we have

$$\frac{\partial}{\partial t} (\Lambda Y) + [X,\Lambda Y] = \frac{\partial}{\partial t} (\Lambda Y) + L_X(\Lambda Y) = \Lambda(\dot{Y} + [X,Y]) + (\dot{\Lambda} + L_X \Lambda)\, Y.$$

So if Y is a symmetry and Λ satisfies (2.3.8), we see that ΛY is also a symmetry. □

2.3.9 Definition.

A parameterized linear mapping $\Lambda : X_p(M) \to X_p(M)$, corresponding to a parameterized tensor field (also denoted by) $\Lambda \in T^1_{1p}(M)$ which satisfies (2.3.8), is called a *recursion operator for symmetries*.

□

Recursion operators for symmetries are sometimes called strong symmetries.

2.3.10 Remark.

Another possibility for constructing (new) symmetries out of already known ones is to compute the Lie bracket with some other symmetry. This method should not be confused with the application of a recursion operator for symmetries. Suppose Y_1 and Z are two symmetries and Λ is a recursion operator for symmetries. Then we can construct the symmetries Y_3 and Y_4 by

$$Y_3 = \Lambda\, Y_1 \quad ,$$

$$Y_4 = [Z, Y_1] \ .$$

Then in a point $u \in M$ the vector $Y_3(u,t)$ depends only on $Y_1(u,t)$ and $\Lambda(u,t)$, while $Y_4(u,t)$ depends on $Y_1(u,t)$, $Z(u,t)$ and their derivatives in u.

□

Suppose U is an open subset of M such that the tangent bundle TU is trivial, $TU = U \times W$. Then we can consider the vector field X as a mapping $X : U \to W$ and the tensor field Λ as a mapping $\Lambda : U \to L(W,W)$ (see also Section 1.2). The condition (2.3.8) can now be written as

$$\dot{\Lambda}(u,t) + \Lambda'(u,t)\, X(u) + \Lambda(u,t)\, X'(u) - X'(u)\, \Lambda(u,t) = 0 \ .$$

For a solution u(t) of (2.2.1) which lies in U this implies

$$(2.3.11) \qquad \frac{d}{dt}\Lambda(u(t),t) + \Lambda(u(t),t)\, X'(u(t)) - X'(u(t))\, \Lambda(u(t),t) = 0 \ .$$

This type of expression is well-known in the theory of isospectral transformations (or "inverse scattering"). Hence we consider the following eigenvalue problem

(2.3.12) $\quad \Lambda Y = \lambda Y \qquad \text{on} \quad M \times \mathbb{R}$.

Note that the "eigenvalue" λ is a function on $M \times \mathbb{R}$, while the "eigenvector" Y is a (parameterized) vector field on M.

2.3.13 Theorem.

Suppose M is an n dimensional manifold and Λ a recursion operator for symmetries. Then the (real and imaginary parts of the) eigenvalues λ_i $(i = 1,\dots,n)$ of Λ are constants of the motion.

Proof:
The Lie derivative (and $\frac{\partial}{\partial t}$) commute with contracted multiplication. This implies that Λ^k is also a recursion operator for symmetries, so

(2.3.14) $\quad \frac{\partial}{\partial t} (\Lambda^k) + L_X(\Lambda^k) = 0 \qquad \text{for } k = 1,2,3,\dots$.

By contraction in Λ^k we obtain a function on $M \times \mathbb{R}$ given by

$$(\Lambda^k)_C = \sum_{i=1}^{n} \lambda_i^k .$$

Then (2.3.14) implies that

$$\left(\frac{\partial}{\partial t} + L_X\right) \sum_{i=1}^{n} \lambda_i^k = 0 ,$$

which yields

$$\sum_{i=1}^{n} \lambda_i^k (\dot{\lambda}_i + L_X \lambda_i) = 0 \qquad \text{for } k = 0,1,2,\dots .$$

For an arbitrary solution u(t) of (2.2.1) this means

(2.3.15) $\quad \sum_{i=1}^{n} \lambda_i^k(u(t),t) \frac{d}{dt} \lambda_i(u(t),t) = 0$.

For $k = 0,1,\dots,n-1$ this is a system of n equations with n unknowns. The corresponding determinant is the Vandermonde determinant, which does not vanish if all the eigenvalues are different. So in that case $\frac{d}{dt}\lambda_i = 0$ for $i = 1,\dots,n$. Next suppose for instance $\lambda_1 = \lambda_2$ and all the other eigenvalues are (also mutually) different at $t = t_0$. This implies $\lambda_1 = \lambda_2$ on a sufficiently small interval $J \ni t_0$. For if $\lambda_1 \neq \lambda_2$ at $t = t_0 + \varepsilon$, then at $t_0 + \varepsilon$ all the eigenvalues are different. The first part of this proof then implies they are independent of t. However, this contradicts with $\lambda_1 = \lambda_2$ at $t = t_0$. So there exists an interval $J \ni t_0$ such that $\lambda_1 = \lambda_2$ on J. The system (2.3.15) implies

$$\frac{d}{dt}(\lambda_1 + \lambda_2) = 0\ , \ \frac{d}{dt}\lambda_i = 0 \quad i = 3,\dots,n\ , \text{ at } t = t_0\ .$$

Since $\lambda_1 = \lambda_2$ on J this implies $\frac{d}{dt}\lambda_1 = \frac{d}{dt}\lambda_2 = 0$ at $t = t_0$. A similar method can be used if more eigenvalues coincide. The proof is completed by noting that u(t) is an arbitrary solution of (2.2.1).

□

Finally we remark that in most applications the recursion operators for symmetries Λ do not depend explicitly on t (so $\mathring{\Lambda} = 0$).

2.4. PROPERTIES OF ADJOINT SYMMETRIES

The first two results concerning adjoint symmetries correspond to similar results for symmetries.

2.4.1 Lemma.

Suppose $\sigma \in V^*(X;M)$, then also $\mathring{\sigma} \in V^*(X;M)$.

□

2.4.2 Theorem.

The set of adjoint symmetries $V^*(X;M)$ is a real linear space. Moreover if F is a constant of the motion and $\sigma \in V^*(X;M)$, then $F\,\sigma \in V^*(X;M)$.

<u>Proof</u>:

Adjoint symmetries have to satisfy the linear equation (2.2.9), so $V^*(X;M)$ is a real linear space. Next assume F is a constant of the motion, $\sigma \in V^*(X;M)$, then

$$\frac{\partial}{\partial t}(F\sigma) + L_X(F\sigma) = F(\dot{\sigma} + L_X\sigma) + (\dot{F} + L_XF)\,\sigma = 0.$$

This means $F\,\sigma \in V^*(X;M)$.

□

This theorem can be summarized by saying that $V^*(X;M)$ is a module over the ring of constants of the motion of (2.2.1). In contrast to $V(X;M)$ the space $V^*(X;M)$ does not have a natural Lie algebra structure.

It turns out that there is a close relation between the space of constants of the motion and a subspace of $V^*(X;M)$. Let F be a function on M (or on $M \times \mathbb{R}$), then its exterior derivative dF is a (parameterized) one-form on M.

2.4.3 <u>Theorem</u>.

Suppose $F \in F_p(M)$ is a constant of the motion. Then the one-form $\sigma = dF$ is an adjoint symmetry.

<u>Proof</u>:

The function F is a constant of the motion, so $\dot{F} + L_XF = 0$.

The exterior derivative d commutes with the Lie derivative and with differentiation with respect to t. Hence

$$\dot{dF} + L_X\,dF = 0.$$

This means that $\sigma = dF$ is an adjoint symmetry. □

2.4.4 Remark.

In fact we proved a little more. Suppose $F \in F_p(M)$ such that for all solutions u(t) of (2.2.1)

$$\frac{d}{dt} F(u(t),t) = f(t),$$

where $f: \mathbb{R} \to \mathbb{R}$ is some function. This means $\dot{F} + L_X F = f$. Then the calculation above (with df = 0) shows that $\sigma = dF$ is also an adjoint symmetry. In the following theorem we show that σ also can be written as the exterior derivative of a constant of the motion.

□

2.4.5 Theorem.

Let $\sigma \in V^*(X;M)$ be exact, so there exists a function $F \in F_p(M)$ such that $\sigma = dF$. Then there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $G(\cdot,t) = F(\cdot,t)-g(t)$ is a constant of the motion with $\sigma = dG$.

Proof:
Since σ is an adjoint symmetry, we have $\mathring{\sigma} + L_X\sigma = 0$. This can be written as $d(\mathring{F} + L_X F) = 0$, which implies that $\mathring{F}(u,t) + L_X F(u,t) = f(t)$ on $M\times\mathbb{R}$ for some function $f : \mathbb{R} \to \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $\dot{g} = f$. Then $G(\cdot,t) = F(\cdot,t) - g(t)$ is a constant of the motion with $\sigma = dG$.

□

The theorems 2.4.3 and 2.4.5 can be summarized by saying that every constant of the motion gives rise to an (exact) adjoint symmetry and that every exact adjoint symmetry can be written as the exterior derivative of a constant of the motion.

Now we are going to study operators which map $V^*(X;M)$ into itself. Consider a parameterized tensor field $\Gamma \in T^1_{1p}(M)$. Then we can consider Γ also as a linear mapping $\Gamma : X^*_p(M) \to X^*_p(M)$, and we can ask under which conditions Γ maps $V^*(X;M)$ into $V^*(X;M)$. Analogous to theorem 2.3.7

we now have

2.4.6 Theorem.

Suppose the tensor field $\Gamma \in T^1_{1p}(M)$ satisfies

$$(2.4.7) \qquad \dot{\Gamma} + L_X \Gamma = 0 \quad \text{on} \quad M \times \mathbb{R} \quad .$$

Then for all $\sigma \in V^*(X;M)$ also $\Gamma\sigma \in V^*(X;M)$.

Proof:
Similar to the proof of theorem 2.3.7 we have

$$\frac{\partial}{\partial t}(\Gamma\sigma) + L_X(\Gamma\sigma) = \Gamma(\dot{\sigma} + L_X\sigma) + (\dot{\Gamma} + L_X\Gamma)\sigma \quad .$$

So if $\sigma \in V^*(X;M)$ and Γ satisfies (2.4.7) we see that $\Gamma\,\sigma \in V^*(X;M)$.

□

2.4.8 Definition.

A parameterized linear mapping $\Gamma : X^*_p(M) \to X^*_p(M)$, corresponding to a tensor field (also denoted by) $\Gamma \in T^1_{1p}(M)$ which satisfies (2.4.7), is called a *recursion operator for adjoint symmetries*.

□

2.4.9 Remark.

The conditions (2.3.8) and (2.4.7) for the tensor fields Λ and Γ are identical. This means that a tensor field Λ which satisfies (2.3.8), gives also rise to a recursion operator for adjoint symmetries. In local coordinates on M the tensor field Λ is represented by a matrix Λ^i_j . Suppose Y is a symmetry with coordinates Y^i and σ is an adjoint symmetry with coordinates σ_i. Then the vector field Z with coordinates $Z^i = \Lambda^i_j\, Y^j$ is again a symmetry. But also $\tau_j = \Lambda^i_j\, \sigma_i$ is (represents) an adjoint symmetry. The dual operator of $\Lambda : X_p(M) \to X_p(M)$ is a linear operator $\Lambda^* : X^*_p(M) \to X^*_p(M)$. So, in operator notation, we have $Z = \Lambda Y$ and $\tau = \Lambda^*\sigma$. This leads to

□

2.4.10 Theorem.

Suppose Λ is a recursion operator for symmetries. Then Λ^* is a recursion operator for adjoint symmetries. Also if Γ is a recursion operator for adjoint symmetries then Γ^* is a recursion operator for symmetries.

Proof:
The operators $\Lambda : X_p(M) \to X_p(M)$ and $\Lambda^* : X^*_p(M) \to X^*_p(M)$ correspond both to the tensor field (also denoted by) Λ. If Λ is a recursion operator for symmetries the tensor field satisfies (2.3.8) and so (2.4.7).
Hence Λ^* is a recursion operator for adjoint symmetries. The second part of the theorem is proved in a similar way.

□

2.4.11 Corollary.

Suppose M is a finite-dimensional manifold. Then the eigenvalues of a recursion operator for adjoint symmetries are constants of the motion.

Proof:
This result follows at once from the preceding theorem and Theorem 2.3.13.

□

2.5 GENERAL RESULTS

We first consider operators which relate symmetries and adjoint symmetries. Suppose Ψ is a parameterized tensor field of contravariant order 2 and covariant order 0, so $\Psi \in T^2_{0\,p}(M)$. Then we can also consider Ψ as a vector bundle map $\Psi : T^*M \to TM$ or as a linear operator $\Psi : X^*_p(M) \to X_p(M)$. Now we investigate under which conditions Ψ maps adjoint symmetries into symmetries.

2.5.1 Theorem.

Suppose $\Psi \in T^2_{0\,p}(M)$ is a tensor field such that

$$(2.5.2) \qquad \overset{\circ}{\Psi} + L_X\Psi = 0 \quad \text{on } M \times \mathbb{R} \quad .$$

Then for all $\sigma \in V^*(X;M)$ we have $\Psi\sigma \in V(X;M)$.

Proof:

From

$$\frac{\partial}{\partial t}(\Psi\sigma) + L_X(\Psi\sigma) = \Psi(\dot{\sigma}+L_X\sigma) + (\dot{\Psi} + L_X\,\Psi)\sigma$$

we see that, if $\sigma \in V^*(X;M)$ and Ψ satisfies (2.5.2), $\Psi\sigma \in V(X;M)$. So Ψ maps adjoint symmetries into symmetries.

□

2.5.3 Definition.

A parameterized linear mapping $\Psi : X^*_p(M) \to X_p(M)$, corresponding to a tensor field (also denoted by) $\Psi \in T^2_{0p}(M)$, which satisfies (2.5.2), is called an *AS operator*.

□

So an AS operator, applied to an adjoint symmetry, yields a symmetry. Next we consider operators acting in the opposite direction.

2.5.4 Theorem.

Suppose $\Phi \in T^0_{2p}(M)$ is a tensor field such that

(2.5.5) $\quad \dot{\Phi} + L_X\Phi = 0 \quad$ on $M \times \mathbb{R}$.

Then for all $Y \in V(X;M)$ we have $\Phi Y \in V^*(X;M)$.

Proof:

The proof is similar to the proof of theorem (2.5.1).

□

2.5.6 Definition.

A parameterized linear mapping $\Phi : X_p(M) \to X^*_p(M)$, corresponding to a tensor field (also denoted by) $\Phi \in T^0_{2p}(M)$, which satisfies (2.5.5), is called an *SA operator*.

□

So an SA operator Φ maps symmetries into adjoint symmetries.

As expected, if an AS (SA) operator is invertible, the inverse operator is an SA (AS) operator.

2.5.7 Theorem.

Suppose Ψ (Φ) is an invertible AS (SA) operator. Then the inverse operator Ψ^{-1} (Φ^{-1}) is an SA (AS) operator.

Proof:
Since $\Psi \Psi^{-1} = I_d \in T^1_{1p}(M)$ we have

$$L_X(\Psi \Psi^{-1}) = (L_X\Psi) \Psi^{-1} + \Psi L_X(\Psi^{-1}) = 0$$

and

$$\frac{\partial}{\partial t} (\Psi \Psi^{-1}) = \dot{\Psi} \Psi^{-1} + \Psi \frac{\partial}{\partial t} (\Psi^{-1}) = 0 .$$

This means that if Ψ satisfies (2.5.2), then Ψ^{-1} satisfies (2.5.5).

□

Recall that with a parameterized two-form ϕ always corresponds an (anti-symmetric) tensor field $\Phi \in T^0_{2p}(M)$ or equivalently a linear operator $\Phi : X_p(M) \to X^*_p(M)$, such that

$$\phi(A,B) = \langle \Phi A,B\rangle \qquad \forall\, A,B \in X(M) .$$

This leads to

2.5.8 Theorem.

Let σ be an adjoint symmetry which is not closed, so $d\sigma \neq 0$. Then the operator $\Phi : X_p(M) \to X^*_p(M)$, which corresponds to the two-form $\phi = d\sigma$ is an SA operator.

Proof:
The adjoint symmetry σ satisfies $\dot{\sigma} + L_X\sigma = 0$. After exterior differentiation we obtain $\dot{\phi} + L_X\phi = 0$, which is equivalent to $\dot{\Phi} + L_X\Phi = 0$. Hence Φ is an SA operator.

□

2.5.9 Remark.

Since $d\phi = dd\sigma = 0$ the SA operator Φ corresponds to a closed two-form (ϕ). This means that the SA operator Φ satisfies additional conditions, which are explained in Section 3.2. Operators of this type will be called *cyclic operators*. If $\phi = d\sigma$ is also nondegenerate, the operator Φ is invertible. If further $\mathring{\sigma} = dG$ for some constant of the motion G, the dynamical system is a Hamiltonian system. This will be explained in Section 3.5.

□

Of course Theorem 2.5.8 is also correct, if σ is closed. However, in that case we obtain the trivial SA operator $\Phi = 0$. In a local coordinate system u^i the adjoint symmetry σ can be written as $\sigma = \sigma_i \, du^i$. The corresponding SA operator is then represented by the matrix $\Phi_{ij} = \sigma_{i,j} - \sigma_{j,i}$.

Recall that with every vector field A and every one-form α corresponds a function on M, defined by their contraction $\langle\alpha, A\rangle = i_A\alpha$.

2.5.10 Theorem.

Suppose $Y \in V(X;M)$ and $\sigma \in V^*(X;M)$. Then the function $F = \langle\sigma, Y\rangle$ is a constant of the motion.

Proof:

Using Leibniz' rule we obtain

$$\dot{F} + L_X F = \langle \sigma \, , \, \dot{Y} + L_X Y \rangle + \langle \dot{\sigma} + L_X \sigma \, , \, Y \rangle = 0.$$

This means F is a constant of the motion.

□

Starting with two symmetries Y_1 and Y_2 an AS operator Ψ can be defined in the following way. For $\alpha \in X^*_p(M)$ set

(2.5.11) $\quad \Psi\alpha = \langle\alpha, Y_1\rangle \, Y_2.$

It is easily seen that Ψ is an AS operator. Application of this operator to an adjoint symmetry σ gives $\Psi\sigma = \langle\sigma, Y_1\rangle \, Y_2$. By theorem 2.5.10 we see that $\langle\sigma, Y_1\rangle$ is a constant of the motion. Then, from theorem 2.3.5 we see that $\Psi\sigma$ is a symmetry, so Ψ is indeed an AS operator. Of course we can also verify that Ψ satisfies (2.5.2). This operator Ψ is rather trivial. We obtain always the same vector field Y_2, multiplied by different functions $< \alpha, Y_1 >$. This implies that Ψ is not invertible. It is easily seen that if $\Psi \neq 0$, it is not antisymmetric. This method of constructing an AS operator, starting with two symmetries can be extended. Let $Y_1, \ldots, Y_k \in V(X;M)$ and $c^{ij} \in \mathbb{R}$ for $i,j = 1,\ldots,k$. Then for $\alpha \in X_p^*(M)$ define

$$\Psi\alpha = c^{ij} <\alpha, Y_i> Y_j. \tag{2.5.12}$$

Then Ψ is an AS operator. This construction yields a symmetric operator if $c^{ij} = c^{ji}$ and an antisymmetric operator if $c^{ij} = -c^{ji}$. Using similar methods we can also construct SA operators and recursion operators for (adjoint) symmetries. For instance, let $\sigma \in V^*(X;M)$ and $Y \in V(X;M)$. Then for $A \in X_p(M)$ define

$$\Lambda A = < \sigma, A> Y. \tag{2.5.13}$$

Then Λ is a (rather trivial) example of a recursion operator for symmetries.

There are four different types of operators relating symmetries and (adjoint) symmetries. They were described in the definitions 2.3.9 (Λ, recursion operator for symmetries), 2.4.8 (Γ, recursion operator for adjoint symmetries), 2.5.3 (Ψ, AS operator) and 2.5.6 (Φ, SA operator). If one or more of these operators exist, we can construct new operators by using products and powers of already known operators. For instance, suppose there exists an AS operator Ψ and an SA operator Φ. Then $\Psi\Phi$ is a recursion operator for symmetries and $\Phi\Psi$ is a recursion operator for adjoint symmetries. Also other combinations are possible. Let Λ be a recursion operator for symmetries and Ψ an AS operator. Then $\Lambda\Psi$ is again an AS operator. Of course all these results have a straightforward proof.

We continue this section by giving a more general approach of the theory described in this section and in the Sections 2.3 and 2.4. Up to now we considered constants of the motion, (adjoint) symmetries and several operators between those symmetries. All these objects are (can be considered as) tensor fields Ξ of different types which satisfy

$$\overset{\circ}{\Xi} + L_X \Xi = 0 \quad \text{on } M \times \mathbb{R}. \tag{2.5.14}$$

This leads to the following

2.5.15 Definition.

A tensor field $\Xi \in T^k_{\ell p}(M)$ which satisfies (2.5.14) is called a (k,ℓ) symmetry of the dynamical system (2.2.1).

□

Following this definition a constant of the motion is a (0,0) symmetry, a symmetry is a (1,0) symmetry and an adjoint symmetry is a (0,1) symmetry. Further an SA operator corresponds to an (0,2) symmetry and an AS operator to a (2,0) symmetry. Both recursion operators for symmetries and for adjoint symmetries correspond to (1,1) symmetries (see also Remark 2.4.9). This last property also shows the problems associated with Definition 2.5.15. Although recursion operators for symmetries and for adjoint symmetries both correspond to (1,1) symmetries, it is convenient to have distinct names for those two operators. Therefore we shall in general use the previous introduced designation.

There are several methods for constructing new tensor fields out of already known ones. Suppose Ξ is a parameterized tensor field of arbitrary orders and Y is a parameterized vector field. Then new parameterized tensor fields can be constructed by the following methods (see also Abraham and Marsden [1, §3.4]):

i) Compute $L_Y \Xi$, the Lie derivative of Ξ in the direction of Y.

ii) Compute $\Xi \otimes \Xi_1$, the tensor product of Ξ and some tensor field Ξ_1.

iii) If the co-and contravariant orders of Ξ are both positive, we can perform a contraction.

iv) If Ξ is antisymmetric and has covariant order k and contravariant order 0 we can compute the exterior derivative of the corresponding k-form ξ. Then $d\xi$ corresponds again to a tensor field (with orders k+1 and 0).

v) Suppose Ξ and some other tensor field Ξ_1 correspond to k and ℓ-forms ξ and ξ_1. Then we can construct a tensor field Ξ_2 corresponding to the $(k+\ell)$-form $\xi_2 = \xi \wedge \xi_1$.

There are several relations between these methods. A tensor field Ξ_2 constructed by v), can also be obtained by ii). For instance if Ξ and Ξ_1 have both covariant order 1, then

$$\Xi_2 = \Xi \otimes \Xi_1 - \Xi_1 \otimes \Xi \ .$$

So we need not consider method v). If Ξ corresponds to a differential k-form ξ then

$$L_Y \xi = d\, i_Y \xi + i_Y d\, \xi \ .$$

The interior product i_Y of a vector field with a differential form can be obtained by a tensor product with Y followed by a contraction. So for k-forms i) can be obtained from ii), iii) and iv). Almost all results of Sections 2.3, 2.4 and this section are in fact special cases of the following

2.5.16 <u>Theorem</u>.

Suppose Ξ is a (k,ℓ) symmetry, Ξ_1 is a (i,j) symmetry and Y is a symmetry (i.e. a $(1,0)$ symmetry). Then

i) $L_Y \Xi$ is a (k,ℓ) symmetry,

ii) $\Xi \otimes \Xi_1$ is a $(k+i\ ,\ \ell+j)$ symmetry,

iii) if $k > 0$ and $\ell > 0$ every possible contraction in Ξ yields a $(k-1\ ,\ \ell-1)$ symmetry,

iv) if Ξ is a $(0,\ell)$ symmetry corresponding to a ℓ-form ξ, the tensor field corresponding to the $(\ell+1)$ form $d\xi$ is a $(0\ ,\ \ell+1)$ symmetry.

<u>Proof</u>:

i) Using the commutation rule for Lie derivatives we obtain

$$\frac{\partial}{\partial t}(L_Y \Xi) + L_X L_Y \Xi = L_{\dot{Y}} \Xi + L_Y \dot{\Xi} + L_X L_Y \Xi$$

$$= L_{\dot{Y}} \Xi - L_Y L_X \Xi + L_X L_Y \Xi = L_{\dot{Y}} \Xi + L_{[X,Y]} \Xi .$$

Since the vector field Y is a symmetry, the last two terms cancel, so $L_Y\Xi$ also satisfies (2.5.14).

ii) This part of the theorem is a straightforward consequence of

$$L_Y(\Xi \otimes \Xi_1) = (L_Y\Xi) \otimes \Xi_1 + \Xi \otimes L_Y\Xi_1.$$

iii) Suppose Ξ is a tensor field with both orders positive. Denote the tensor fields obtained by a contraction in Ξ and (the same contraction in) $L_Y\Xi$ by Ξ_C and Ξ_{LC}. Then $L_Y\Xi_C = \Xi_{LC}$, so "contraction commutes with the Lie derivative". Using this property it is easily shown that, if Ξ satisfies (2.5.14) , then also Ξ_C satisfies (2.5.14).

iv) Using $L_Y d = dL_Y$ (for differential forms), this result is also easily proved.

□

We mentioned already that most results of sections 2.3, 2.4 and of this section can be obtained from theorem 2.5.16. For instance the theorems 2.3.5, 2.3.7, 2.4.2, 2.4.6, 2.5.1, 2.5.4 and 2.5.10 follow also from ii) and iii) of Theorem 2.5.16. As an example consider theorem 2.4.6. In that theorem Γ and σ are both tensor fields which satisfy (2.5.14). Then also the tensor product $\Gamma\otimes\sigma$ satisfies this condition. After contraction we see that the tensor field $\Gamma\sigma$ (= one-form) also satisfies (2.5.14), so it is an adjoint symmetry. Further theorem 2.3.6 (and in fact also the lemma's 2.3.3, 2.3.4 and 2.4.1) follows from part i) of Theorem 2.5.16. The theorems 2.4.3 and 2.5.8 are special cases of part iv) of theorem (2.5.16). Finally we mention that the AS operator Ψ and the recursion operator for symmetries Λ, as given in (2.5.12) and (2.5.13), can be written as

$$\Psi = c^{ij}\, Y_i \otimes Y_j,$$

$$\Lambda = \sigma \otimes A\ .$$

Then by Theorem 2.5.16 ii) Ψ is an AS operator and Λ a recursion operator for symmetries.

2.6 NIJENHUIS TENSORS AND INFINITE SERIES OF SYMMETRIES

Suppose the dynamical system $\dot{u} = X(u)$ has a nontrivial recursion operator for symmetries Λ. Then, starting with the symmetry X we can construct an infinite series of symmetries by

$$(2.6.1) \qquad X_k = \Lambda^{k-1} X \qquad k = 1,2,3,\dots \qquad (X_1 = X) \ .$$

If there exists a symmetry Z_0, not in this series, a second series of symmetries is given by

$$(2.6.2) \qquad Z_k = \Lambda^k Z_0 \qquad k = 0,1,2,\dots \ .$$

2.6.3 Remark.

The situation as described above occurs for instance in the case of the Burgers equation (see Section 5.2) and the Korteweg-de Vries equation (see Section 5.6). In these cases the symmetry Z_0 is related to the invariance of the equation under a scale transformation. For these equations there also exists a symmetry X_0 such that $X = X_1 = \Lambda X_0$; the symmetry X_0 then corresponds to the invariance of the equation for translations along the x-axis. Note that for every autonomous system $\dot{u} = X(u)$ the symmetry X corresponds to the invariance of the equation for translations in time.

□

If some extra conditions are satisfied the Lie brackets between the elements of both series can easily be found. We first describe some properties of (1,1) tensor fields.

2.6.4 Lemma.

For every tensor field $\Lambda \in T_1^1(M)$ there exist a tensor field $\Xi \in T_2^1(M)$, only dependent on Λ, such that for all vector fields A on M

$$(2.6.5) \qquad L_{\Lambda A} \Lambda - \Lambda L_A \Lambda = \Xi A \ ,$$

or in local coordinates

$$(L_{\Lambda A}\,\Lambda)^i_j - \Lambda^i_k (L_A\,\Lambda)^k_j = \Xi^i_{jk} A^k \ .$$

The tensor field Ξ is antisymmetric in its covariant indices ($\Xi^i_{jk} = -\Xi^i_{kj}$).

Proof:

A simple calculation in local coordinates shows that in a point $u \in M$ the left hand side of (2.6.5) does not depend on the derivatives of A in u. Hence we can consider (2.6.5) as a (coordinate independent) linear mapping of a vector in u (A(u)) into a (1,1) tensor in u (the left hand side of (2.6.5)). But this means that Ξ(u) is a (1,2) tensor and Ξ a (1,2) tensor field. The antisymmetry of Ξ follows by contracting (2.6.5) with the vector field A. Then an elementary calculation using Leibniz' rule and the antisymmetry of the Lie bracket yields that the left hand side of (2.6.5) vanishes. Hence $\Xi AA = 0$ ($\Xi^i_{jk}A^kA^j = 0$) which means that Ξ is antisymmetric in its covariant indices.

□

The tensor field Ξ is sometimes called the *Nijenhuis tensor field* of Λ, cf. for instance Schouten [67, page 66] or Nijenhuis [68]. Suppose Λ is a recursion operator for symmetries, or in the terminology of the preceding section, a (1,1) symmetry. Then it can be shown that the corresponding Nijenhuis tensor field Ξ is a (1,2) symmetry. Since in most interesting cases the Nijenhuis tensor field of a recursion operator for symmetries vanishes, we shall not give a proof of this result. If the Nijenhuis tensor field of Λ vanishes, we have

(2.6.6) $\quad L_{\Lambda A}\,\Lambda = \Lambda\, L_A\,\Lambda \qquad \forall A \in X(M)$.

Application of (2.6.6) to (contraction with) a vector field B results in

(2.6.7) $\quad [\Lambda A,\Lambda B] - \Lambda[\Lambda A,B] = \Lambda[A,\Lambda B] - \Lambda^2[A,B]$.

If $M = W$, a (possibly infinite-dimensional) topological vector space, we can write out (2.6.7) using the expressions for the Lie derivatives given in (1.2.9). This yields

$$(2.6.8) \qquad (\Lambda'(\Lambda A))B - (\Lambda'(\Lambda B))A = \Lambda((\Lambda' A)B - (\Lambda' B)A) \ .$$

Although (2.6.7) and (2.6.8) (for dynamical systems in a topological vector space) are equivalent with (2.6.6), these conditions are rather unpractical. As far as we know the conditions (2.6.7) and (2.6.8) have been introduced by Fuchssteiner [12] and Magri [17] and in a formal calculus by Gel'fand and Dorfman [16]. Operator valued functions Λ on W which satisfy (2.6.8) were called heriditary symmetries by Fuchssteiner (see also [8,37,64]). Magri introduced the name Nijenhuis operator, which is rather confusing since the Nijenhuis tensor field corresponding to Λ is the (vanishing) (1,2) tensor field Ξ. Finally the name regular operators was used by Gel'fand and Dorfman.

Next we return to the two series of symmetries given in (2.6.1) and (2.6.2). For several equations it turns out that

$$(2.6.9) \qquad L_{Z_0} \Lambda = a\Lambda \qquad a \in \mathbb{R} \ ,$$

$$(2.6.10) \qquad L_{Z_0} X_1 = [Z_0, X_1] = bX_1 \qquad b \in \mathbb{R} \ .$$

First we compute the Lie derivatives of the recursion operator for symmetries Λ.

2.6.11 Theorem.

Suppose Λ is an autonomous recursion operator for symmetries (so $\dot{\Lambda} = 0$) with a vanishing Nijenhuis tensor. Then

i) $L_{X_k} \Lambda = 0$, $\qquad k = 1,2,3,\dots$

ii) if (2.6.9) is satisfied $L_{Z_k} \Lambda = a\Lambda^{k+1}$, $\qquad k = 0,1,2,\dots$.

Proof:
Since Λ is a recursion operator for symmetries with $\dot{\Lambda} = 0$, it satisfies $L_{X_1} \Lambda = 0$. Then i) follows at once from (2.6.6). The second part of this theorem follows in a similar way from (2.6.9) and (2.6.6).

□

It is easily seen (Leibniz' rule) that (under the same assumptions) the Lie derivatives of powers of Λ are given by

(2.6.12) $\quad L_{X_k}(\Lambda^m) = 0$,

(2.6.13) $\quad L_{Z_k}(\Lambda^m) = m\,a\,\Lambda^{k+m}$.

The Lie brackets between the elements of the two series of symmetries X_k and Z_k are now easily found.

2.6.14 Theorem.

Suppose Λ is an autonomous recursion operator for symmetries with vanishing Nijenhuis tensor field. Then

i) $[X_k,X_\ell] = 0$,

ii) if (2.6.9) is satisfied $[Z_k,Z_\ell] = a(\ell-k)\,Z_{k+\ell}$,

iii) if (2.6.9) and (2.6.10) are satisfied $[Z_k,X_\ell] = ((\ell-1)a+b)\,X_{k+\ell}$.

Proof:
In proving i) it is no restriction to assume $\ell = k + m$ with $m > 0$. Then, using (2.6.12)

$$[X_k,X_\ell] = L_{X_k}(\Lambda^m X_k) = (L_{X_k}(\Lambda^m))\,X_k = 0 .$$

To prove ii) we again assume $\ell = k + m$ with $m > 0$. Then (2.6.13) implies

$$[Z_k,Z_\ell] = L_{Z_k}(\Lambda^m Z_k) = (L_{Z_k}(\Lambda^m))\,Z_k$$
$$= m\,a\,\Lambda^{k+m}\,Z_k = (\ell-k)\,a\,Z_{k+\ell} .$$

Finally, using (2.6.10) and (2.6.13)

$$[Z_k,X_\ell] = -L_{X_\ell}(\Lambda^k Z_0) = \Lambda^k[Z_0,X_\ell]$$

$$= \Lambda^k L_{Z_0} (\Lambda^{\ell-1} X_1)$$

$$= \Lambda^{k+\ell-1} [Z_0, X_1] + (\ell - 1)\, a\, \Lambda^{k+\ell-1} X_1$$

$$= ((\ell - 1)a + b)\, X_{k+\ell} \ .$$

□

Since $X_1 = X$ and Λ do not depend on t, the symmetries X_k also do not. Hence we can consider on M the autonomous dynamical systems $\mathring{u} = X_k(u)$. Now Theorem 2.6.11 i) implies that Λ is also a recursion operator for symmetries of these dynamical systems. Also Theorem 2.6.14 i) implies that every vector field X_ℓ is also a symmetry of $\mathring{u} = X_k(u)$. These results on the series X_k have already been given by Fuchssteiner [12,37], Fuchssteiner and Fokas [8] and Magri [17].

2.6.15 Remark.

If a symmetry X_0 as described in Remark 2.6.3 exists, the results of the Theorems 2.6.11 and 2.6.14 also hold for X_0 if $[Z_0, X_0] = (b - a)X_0$ and $L_{X_0} \Lambda = 0$. If Λ is invertible this follows from (2.6.9) and (2.6.10), in other cases these relations have to be verified.

□

2.6.16 Remark.

It may happen that instead of Z_0 a symmetry Z is known such that

(2.6.17) $\quad L_Z \Lambda = a \Lambda^{p+1}$

and

(2.6.18) $\quad [Z, X_1] = b\, X_{p+1} \qquad p \in \mathbb{N}$.

Comparison of these two relations with the results given in the Theorems 2.6.11 and 2.6.14 suggests to define a series of symmetries Z_k by

$$Z_k = \Lambda^{k-p} Z \ \ (\text{so } Z = Z_p) \ , \qquad k = p, p+1, \ldots \ .$$

Then it is easily verified that Theorems 2.6.11 and 2.6.14 remain valid (of course only for those Z_k which exist, i.e. $k \geq p$).

□

Finally we remark that the results fiven in the Theorems 2.6.11 and 2.6.14 can also be obtained without using the vanishing of the Nijenhuis tensor field of Λ. However, then the Conditions (2.6.9) and (2.6.10) have to be extended with similar conditions in terms of Z_1 and Z_2. See Ten Eikelder [69] for more details.

2.7 TRANSFORMATION PROPERTIES

Suppose there exists a diffeomorphism f between M and some other manifold N. Denote the inverse mapping by $\overset{\leftarrow}{f}$, so

$$(2.7.1) \quad \begin{cases} f : M \to N \\ \overset{\leftarrow}{f} : N \to M . \end{cases}$$

Then we can use the derivative of f to transform the equation (2.2.1) to a differential equation on N

$$(2.7.2) \qquad \dot{v} = f'(\overset{\leftarrow}{f}(v)) \quad X(\overset{\leftarrow}{f}(v)) = : \tilde{X}(v).$$

Note that $\tilde{X}$ is a vector field on the manifold N.
Symmetries $\tilde{Y}$ of (2.7.2) are vector fields on N which satisfy

$$\dot{\tilde{Y}} + [\tilde{X},\tilde{Y}] = 0 \quad \text{on } N \times \mathbb{R}.$$

Adjoint symmetries of (2.7.2) are one-forms on N which satisfy

$$\dot{\tilde{\sigma}} + L_{\tilde{X}} \tilde{\sigma} = 0 \quad \text{on } N \times \mathbb{R}.$$

The sets of symmetries and adjoint symmetries of (2.7.2) are denoted by $V(\tilde{X};N)$ respectively $V^*(\tilde{X};N)$. Note that all the expressions given in the sections 2.3, 2.4, 2.5 and 2.6 were given in terms of tensor fields (vector fields, k-forms), Lie derivatives and exterior derivatives. The transformation properties of tensor fields are well-known. Suppose Ξ is an arbitrary tensor field, Y a vector field and η a k-form on M.

The transformed tensor fields, vector fields and k-forms on N are denoted by the same symbol, supplied with a tilde . Then

$$(2.7.3) \qquad L_{\tilde{Y}}\, \tilde{\Xi} = \widetilde{L_{Y}\Xi},$$

$$(2.7.4) \qquad d\tilde{\eta} = \widetilde{d\eta} \ .$$

This means that the operations L and d are "natural with respect to a diffeomorphism". Suppose Y is a symmetry of (2.2.1). The transformed vector field $\tilde{Y} = f'Y$ on N satisfies

$$\dot{\tilde{Y}} = f'\dot{Y} = f'\ L_X Y = \widetilde{L_X Y} \ .$$

Using (2.7.3) we see that

$$\dot{\tilde{Y}} = L_{\tilde{X}}\tilde{Y} = [\tilde{X},\tilde{Y}] \ ,$$

so the vector field $\tilde{Y}$ on N is a symmetry of (2.7.2).
In the same way we can show that if σ is an adjoint symmetry of (2.2.1), then the one-form $\tilde{\sigma} = f^{\leftarrow\prime *}\sigma$ on N is an adjoint symmetry of (2.7.2).
So we have proved

2.7.5 Theorem.

If $Y \in V(X;M)$ then $\tilde{Y} = f'Y \in V(\tilde{X};N)$.
Also if $\sigma \in V^*(X;M)$ then $\tilde{\sigma} = f^{\leftarrow\prime *}\sigma \in V^*(\tilde{X};N)$.

□

Suppose Ψ is an AS operator for equation (2.2.1) on M. Then using (2.7.3) we can show that the transformed operator (tensor field) on N is an AS operator for (2.7.2). Similar results hold for the other possible operators. We summarize them in

2.7.6 Theorem.

Consider the operators Λ, Γ, Ψ, Φ as described in the definitions 2.3.9, 2.4.8, 2.5.3 and 2.5.6. Then the corresponding operators for (2.7.2) on the manifold N are given by

$$(2.7.7)\quad \begin{cases} \tilde{\Lambda} = f' \, \Lambda \, f^{\leftarrow\prime} & , \quad \tilde{\Lambda}(v,t) = f'(f^{\leftarrow}(v)) \, \Lambda(f^{\leftarrow}(v),t) \, f^{\leftarrow\prime}(v), \\ \tilde{\Gamma} = f^{\leftarrow\prime *} \, \Gamma \, f'^{*} & , \quad \tilde{\Gamma}(v,t) = f^{\leftarrow\prime *}(v) \, \Gamma(f^{\leftarrow}(v),t) \, f'^{*}(f^{\leftarrow}(v)), \\ \tilde{\Psi} = f' \, \Psi \, f'^{*} & , \quad \tilde{\Psi}(v,t) = f'(f^{\leftarrow}(v)) \, \Psi(f^{\leftarrow}(v),t) \, f'^{*}(f^{\leftarrow}(v)), \\ \tilde{\Phi} = f^{\leftarrow\prime *} \, \Phi \, f^{\leftarrow\prime} & , \quad \tilde{\Phi}(v,t) = f^{\leftarrow\prime *}(v) \, \Phi(f^{\leftarrow}(v),t) \, f^{\leftarrow\prime}(v). \end{cases}$$

□

2.8 APPENDIX

In this appendix the evolution equation $\dot{u} = X(u)$ on M is extended to an evolution equation for u and its "variation" $\delta u = v$ on TM. Using this evolution equation for $z = (u,v)$, we show how (2.2.7) can be derived without using a local trivialization of the tangent bundle TM. Since $z(t) \in TM$ and so $\dot{z}(t) \in T_{z(t)}(TM)$ we have to construct a vector field on TM(not on M).

First some mathematical preliminaries (see also Abraham and Marsden [1, § 1.6 and exercise 1.6 D]). The set $T(TM)$ can be considered as a vector bundle in two different ways. First $T(TM)$ is the tangent bundle of TM with projection $\pi_2 : T(TM) \to TM$. In this case, the internal structure of TM is unimportant. However, using the fact that TM is itself a tangent bundle, we can supply $T(TM)$ with another vector bundle structure. Denote the projection of the tangent bundle TM by $\pi_1 : TM \to M$. The derivative of this map is $\pi_1' : T(TM) \to TM$. Using this map we can supply $T(TM)$ with an additional vector bundle structure. Note that with the projection π_1' the bundle $T(TM)$ is not a tangent bundle. The two possible projections are illustrated in figure 1 and figure 2.
Note that in these figures tangent vectors to M can be indicated in two ways, see $y \in T_u M$ in figure 2. The situation is summarized in the "dual tangent rhombic", as shown in figure 3. In the sequel we shall need the following

2.8.1 <u>Lemma</u>.

There exists a map $S_M : T(TM) \to T(TM)$ such that

i) $S_M \circ S_M = \mathrm{Id}$ on $T(TM)$,

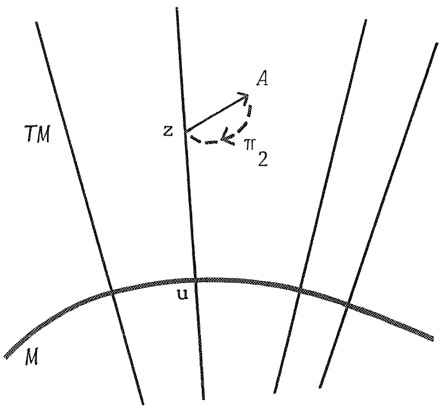
ii) $\pi_1' \circ S_M = \pi_2$,

$$\pi_2 \circ S_M = \pi_1' \quad .$$

<u>Proof</u>:

See Abraham and Marsden [1, exercise 1.6 D] .

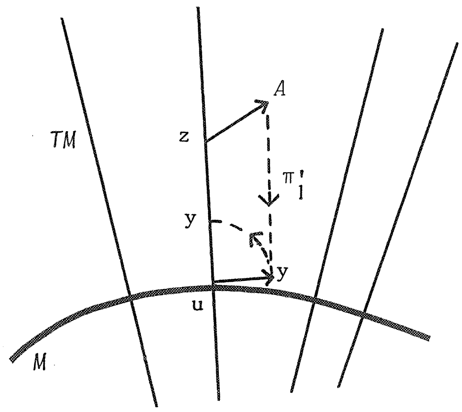
□



$A \in T_z(TM)$

$\pi_2(A) = z$

Figure 1.



$A \in T_z(TM)$

$\pi_1 : TM \to M, \ \pi_1(z) = u$

$\pi_1' : T_z(TM) \to T_uM, \ \pi_1'(A) = y$

Figure 2.

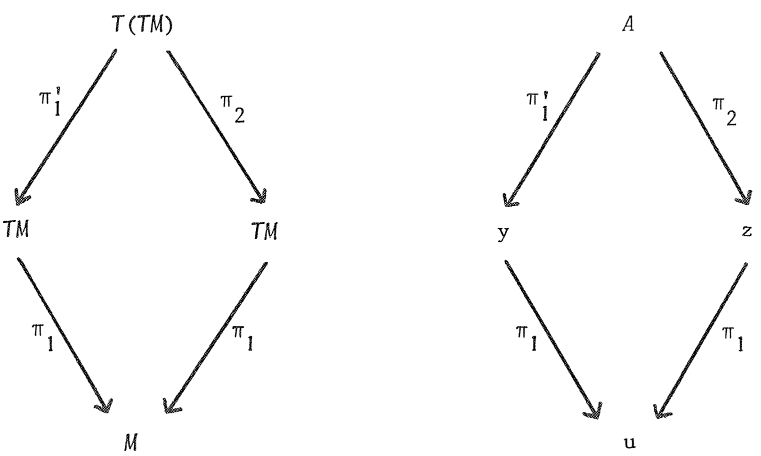


Figure 3.

The map S_M is called the *canonical involution* on M. The lemma may be clearified by looking at figure 2. If we apply the mapping S_M to $A \in T(TM)$ we obtain $\tilde{A} = S_M(A) \in T(TM)$. From $\pi_2(\tilde{A}) = \pi_2(S_M(A)) = \pi_1'(A) = y$ we see that $\tilde{A} \in T_y(TM)$. So we obtain a vector $\tilde{A}$ which is tangent to TM in y. Another application of S_M to $\tilde{A}$ yields again the vector A.

Now we are able to express the Lie bracket of two vector fields on M in terms of the derivatives of the vector fields. Suppose C is a vector field on M. So it is a mapping $C:M \to TM$ such that

$$(2.8.2) \qquad \pi_1 \circ C = \mathrm{Id} : M \to M .$$

The derivative of the vector field C in a point $u \in M$ is the linear mapping

$$C'(u) : T_u M \to T_{C(u)}(TM).$$

Suppose $E \in T_u M$, then $C'(u)E \in T_{C(u)}(TM)$, hence

$$(2.8.3) \qquad \pi_2(C'(u)E) = C(u) \in TM.$$

By taking the derivative of (2.8.2) we obtain $\pi_1' \circ C' = \mathrm{Id} : TM \to TM$. This implies

$$(2.8.4) \qquad \pi_1'(C'(u)E) = E \in TM.$$

Let B be another vector field on M. Analogous to the expressions for the Lie bracket in local coordinates or in a local trivialization (see (1.1.8) or (2.2.5)), we would like to define $[B,C]$ by computing the difference of $C'(u)\,B(u)$ and $B'(u)\,C(u)$. But since $B'(u)\,C(u) \in T_{B(u)}(TM)$ and $C'(u)\,B(u) \in T_{C(u)}(TM)$ this is not possible.
Now we can use the canonical involution S_M. Using lemma 2.8.1 and (2.8.4) we see that

$$(2.8.5) \qquad \pi_2(S_M(C'(u)B(u))) = \pi_1'(C'(u)B(u)) = B(u).$$

This means that $S_M(C'(u)\ B(u)) \in T_{B(u)}\ (TM)$. So we can define

$$(2.8.6) \qquad F(u) := S_M(C'(u)\ B(u)) - B'(u)\ C(u) \in T_{B(u)}\ (TM)\ .$$

We now compute the projection π_1' of $F(u)$. Using lemma 2.8.1, (2.8.3) and (2.8.4) and noting that $\pi_1' : T_{B(u)}(TM) \to T_uM$ is a linear map, we obtain

$$\pi_1'(F(u)) = \pi_2(C'(u)\ B(u)) - \pi_1'\ (B'(u)\ C(u))$$

$$= C(u) - C(u) = 0 \in T_uM.$$

This means that $F(u)$ is not only tangent to TM in the point $B(u)$, but even tangent to T_uM in the point $B(u)$. The situation may be elucidated by the following figure.

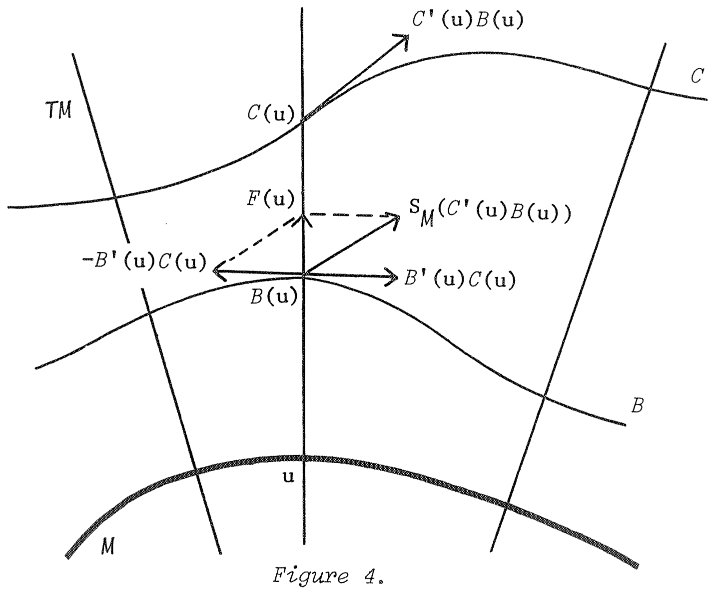


Figure 4.

So $F(u) \in T_{B(u)}(T_uM)$. Finally, using the canonical isomorphism between the linear space T_uM and its tangent space $T_{B(u)}\ (T_uM)$ (see for instance Dieudonné [18], § 16.5.2) we can consider $F(u)$ as an element of T_uM. Since u is arbitrary we constructed a new vector field F on M. By expressing (2.8.6) in local coordinates we see that $F = [B,C]$, the Lie bracket of the vector fields B and C on M. So we have proved the following

2.8.7 Theorem.

The Lie bracket of the vector fields B and C on M is the vector field $[B,C]$ on M, given by

$$(2.8.8) \qquad [B,C](u) = S_M(C'(u)\,B(u)) - B'(u)\,C(u).$$

□

2.8.9 Remark.

In most text-books the Lie bracket of two vector fields is introduced in a much simpler way. However, in the derivation of the condition (2.2.7) for symmetries, both terms of the right hand side of (2.8.8) first appear seperately.

□

2.8.10 Remark.

The preceding construction the Lie bracket is not symmetric. Of course the other possibility (using $S_M(B'(u)\,C(u)) \in T_{C(u)}(TM)$) yields the same result.

□

After these complicated preliminaries the final results are within reach. An evolution equation for u and its "variation" $\delta u = v$ is easily obtained. Suppose $z = (u, \delta u) \in TM$. The expression (2.2.2) suggests to describe the time evolution of z using $X'z$. However, from (2.8.3) we see that $\pi_2(X'z) = X(u)$, which means that (in general) $X'\,z \notin T_z(TM)$. The correct generalization of (2.2.2) is given by

$$(2.8.11) \qquad \dot{z} = S_M(X'z).$$

Lemma 2.8.1 and (2.8.4) imply that

$$\pi_2(S_M(X' z)) = \pi_1'(X'z) = z,$$

so $S_M(X'z) \in T_z(TM)$. This means that indeed the right hand side of (2.8.11) is a vector field on TM. From $u = \pi_1(z)$, lemma 2.8.1 and (2.8.3) we obtain

$$\dot{u} = \pi_1'(\dot{z}) = \pi_1' (S_M(X'z)) = \pi_2(X'z) = X(u),$$

so we see that (2.2.1) is "contained in " (2.8.11). By using a local trivialization of TM it is also possible to derive (2.2.2) from (2.8.11). *So the evolution equation* (2.8.11) *can be considered as an equation which describes the evolution of* u (*as given in* (2.2.1)) *and the evolution of* $v = \delta u$ *(for a local trivialization given in* (2.2.2)).

Finally we consider again special solutions of (2.8.11). This leads to

2.8.12 Theorem.

Suppose Y is a parameterized vector field on M such that for all solutions u(t) of (2.2.1) $z(t) = Y(u(t),t)$ satisfies (2.8.11). Then

(2.8.13) $\dot{Y} = [Y,X]$.

Proof:
Since $z(t) = Y(u(t),t)$ has to be a solution of (2.8.11) for all solutions u(t) of (2.2.1), the vectorfield Y must satisfy

(2.8.14) $\dot{Y}(u,t) + Y'(u,t)\, X(u) = S_M(X'(u)\, Y(u,t)\,) \quad \forall u \in M,\ t \in \mathbb{R}$.

Note that $Y'(u,t)\, X(u) \in T_{Y(u,t)}(TM)$ while at first sight $\dot{Y}(u,t) \in T_uM$. However, since T_uM is a linear space, it is canonically isomorphic with its tangent space in an arbitrary point, hence

$$\dot{Y}(u,t) \in T_{Y(u,t)}(T_uM) \subset T_{Y(u,t)}(TM).$$

So (2.8.14) is a correct equation. The theorem now follows from theorem 2.8.7.

□

Thus we have again obtained condition (2.2.7) (which is equivalent to (2.8.13)) for the vector field Y.

CHAPTER 3: HAMILTONIAN SYSTEMS

3.1 INTRODUCTION

In this chapter we make some remarks on Hamiltonian systems. Since many results in this chapter are standard, a number of proofs is omitted. In Section 3.2 we introduce Hamiltonian systems using symplectic geometry. In Sections 3.3, 3.4 and 3.6 we describe Poisson brackets, variational principles and completely integrable Hamiltonian systems. The transformation properties of Hamiltonian systems are explained in Section 3.7. In Chapter 2 we considered (adjoint) symmetries for general dynamical systems. In Section 3.5 we show that, if a certain kind of adjoint symmetry exists, the dynamical system is Hamiltonian. Symmetries for Hamiltonian systems are described in the next chapter. Sometimes we give expressions using local coordinates. In that case the Hamiltonian systems are considered to be finite-dimensional. In this paper we only consider autonomous (possibly infinite-dimensional) Hamiltonian systems.

Introduce coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ in phase space $\mathbb{R}^{2n}$. Then a *classical Hamiltonian system* can be described by a function $H : \mathbb{R}^{2n} \to \mathbb{R}$, called the Hamiltonian. The system consists of the set of differential equations

$$(3.1.1) \quad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}(q_1,\ldots,p_n) \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}(q_1,\ldots,p_n) \qquad i = 1,\ldots, n. \end{cases}$$

3.2 DEFINITION OF HAMILTONIAN SYSTEMS

A very elegant description of Hamiltonian systems is possible in the language of symplectic geometry (see for instance Arnold [2], Abraham and Marsden [1] , Souriau [4]). This method will finally result in a system of differential equations, of which (3.1.1) is a special case. Therefore we called (3.1.1) a classical Hamiltonian system. The phase space of these Hamiltonian systems will be a symplectic manifold (M,ω).

Consider a two-form ω on M. With this two-form corresponds a vector bundle map $\Omega : TM \to T^*M$, defined by

(3.2.1) $\langle \Omega A, B \rangle = \omega(A,B) \qquad \forall A,B \in T_u M, \ \forall u \in M.$

Of course Ω can also be considered as a tensor field of covariant order 2, $\Omega \in T^0_2(M)$. Mostly we use the last designation. Since a two-form is antisymmetric in its two arguments, the tensor field Ω also is antisymmetric

$$\langle \Omega(u)\, A, B \rangle = -\langle \Omega(u)\, B, A \rangle \qquad \forall A,B \in T_u M, \forall u \in M \ .$$

3.2.2 Definition.

We call a two form ω *(strongly) nondegenerate* if the tensor field Ω (considered as vector bundle map $\Omega : TM \to T^*M$) is an isomorphism. The inverse tensor field is then denoted by $\overset{\leftarrow}{\Omega}$. If the tensor field (vector bundle map) Ω is injective, the two-form ω is called *weakly nondegenerate*. In that case $\overset{\leftarrow}{\Omega}$ is only defined on the range of Ω.

□

A weakly nondegenerate two-form on a finite-dimensional manifold M is (strongly) nondegenerate. A nondegenerate two-form can only exist on a finite-dimensional manifold M if the dimension of M is even. We call Ω and $\overset{\leftarrow}{\Omega}$ the tensor fields corresponding to the (nondegenerate) two-form ω. It is easily seen that $\overset{\leftarrow}{\Omega}$ is also antisymmetric

$$\langle \alpha, \overset{\leftarrow}{\Omega}(u)\, \beta \rangle = -\langle \beta, \overset{\leftarrow}{\Omega}(u)\, \alpha \rangle \qquad \forall\ \alpha,\ \beta \in T^*_u M,\ \forall\ u \in M.$$

The tensor field Ω can be used to transform a vector field on M into a one-form. So we can consider Ω as a linear mapping $\Omega : X(M) \to X^*(M)$. In the same way we can consider $\overset{\leftarrow}{\Omega}$ as a linear mapping $\overset{\leftarrow}{\Omega} : X^*(M) \to X(M)$.

3.2.3 Definition.

A *symplectic manifold* is a pair (M,ω) where ω is a closed, nondegenerate two-form on the manifold M. The form ω is called a *symplectic form*.

□

Infinite-dimensional Hamiltonian systems are often described using a closed, weakly nondegenerate two-form ω. Then ω is called a *weak symplectic* *form*.

It is useful to translate the closedness of a two-form ω into properties of the corresponding tensor fields Ω and $\overset{\leftarrow}{\Omega}$. First the following

3.2.4 Theorem.

i) Suppose $\Phi \in T^0_2(M)$ is an antisymmetric tensor field with corresponding two-form ϕ (so $\phi(A,B) = \langle\Phi A,B\rangle$). Define the mapping $f : X(M) \times X(M) \times X(M) \to$ $\to F(M)$ by

$$f(A,B,C) = \langle L_A(\Phi B),C\rangle \quad . \tag{3.2.5}$$

Then there exists a tensor field $\Xi \in T^0_3(M)$ such that for all vector fields A, B and C on M

$$d\phi(A,B,C) = \Xi ABC = f(A,B,C) + f(B,C,A) + f(C,A,B) \quad . \tag{3.2.6}$$

ii) Suppose $\Psi \in T^2_0(M)$ is antisymmetric. Define the mapping $g : X^*(M) \times X^*(M) \times X^*(M) \to F(M)$ by

$$g(\alpha,\beta,\gamma) = \langle L_{\Psi\alpha}\,\beta,\Psi\gamma\rangle \quad . \tag{3.2.7}$$

Then there exists a tensor field $\hat{\Xi} \in T^3_0(M)$ such that for all one-forms α, β, γ on M

$$\hat{\Xi}\alpha\beta\gamma = g(\alpha,\beta,\gamma) + g(\beta,\gamma,\alpha) + g(\gamma,\alpha,\beta) \quad . \tag{3.2.8}$$

Proof:

It is obvious from (3.2.5) that the value of $f(A,B,C)$ in a point $u \in M$ also depends on the derivatives of A and B in u. However, a calculation using local coordinates or the formulas given in (1.2.9) shows that the right hand side of (3.2.6) in a point $u \in M$ depends only on $A(u)$, $B(u)$ and $C(u)$ and not on their derivatives. Hence we can consider the right hand side of (3.2.6) as a coordinate independent mapping of three vectors $(A(u),B(u),C(u))$ into $\mathbb{R}$. But this means that $\Xi(u)$ is a (0,3) tensor and so Ξ is a (0,3) tensor field on M. Using local coordinates it is easily seen that Ξ is the tensor field corresponding to the three-form $d\phi$. This can also be shown with some manipulations using (1.1.29). The second part of the theorem can be proved in a similar way.

□

In [70] Schouten shows how a $(k+\ell+1,0)$ tensor field can be constructed from a $(k+1,0)$ and a $(\ell+1,0)$ tensor field. If we start with two identical antisymmetric (2,0) tensor fields Ψ this construction yields the (3,0) tensor field $\hat{\Xi}$ as given in (3.2.8).

3.2.9 Definition.

i) A tensor field $\Phi \in T_2^0(M)$ will be called *cyclic* if it is antisymmetric and if the corresponding tensor field $\Xi \in T_3^0(M)$ vanishes.
ii) A tensor field $\Psi \in T_0^2(M)$ will be called *canonical* if it is antisymmetric and if the corresponding tensor field $\hat{\Xi} \in T_0^3(M)$ vanishes.

□

The relations between closed two-forms and cyclic and canonical tensor fields are explained in

3.2.10 Theorem.

i) Let ϕ be a two-form with corresponding tensor field Φ. Then ϕ is closed iff Φ is cyclic.
ii) If $\Phi \in T_2^0(M)$ is a cyclic tensor field which is invertible, then the inverse tensor field Φ^{-1} is canonical. Also if $\Psi \in T_0^2(M)$ is a canonical tensor field which is invertible, the inverse tensor field Ψ^{-1} is cyclic.

Proof:
Part i) follows immediately from Theorem 3.2.4 i). Using $L_A \Phi^{-1} = -\Phi^{-1}(L_A\Phi)\Phi^{-1}$ the second part of this theorem is also easily proved.

□

Note that this theorem implies that every symplectic form ω gives rise to a cyclic tensor field Ω and a canonical tensor field $\overset{\leftarrow}{\Omega}$. In the literature cyclic tensor fields unfortunately are also called symplectic operators (symplectic transformations are explained in Remark 3.7.6). For canonical tensor fields various other names are in use, such as Hamiltonian, inverse symplectic, implectic, co-symplectic. See for instance Gel'fand and Dorfman [16] or Fuchssteiner and Fokas [8]. In local coordinates the tensor fields Φ and Ψ are represented by matrices $\Phi_{ij}(u)$ and $\Psi^{ij}(u)$. Then Φ is cyclic if it is antisymmetric and if

$$(3.2.11) \qquad \Xi_{ijk}(u) \equiv \Phi_{ij,k}(u) + \Phi_{jk,i}(u) + \Phi_{ki,j}(u) = 0 .$$

The tensor field Ψ is canonical if it is antisymmetric and if

$$(3.2.12) \qquad \hat{\Xi}^{ijk}(u) \equiv \Psi^{ij}_{,m}(u)\Psi^{mk}(u) + \Psi^{jk}_{,m}(u)\Psi^{mi}(u) + \Psi^{ki}_{,m}(u)\Psi^{mj}(u) = 0 .$$

Now we are able to define a Hamiltonian vector field on a symplectic manifold (M,ω). Consider a function $H : M \to \mathbb{R}$, then dH is a one-form on M.

3.2.13 Definition.

The vector field $X = \overleftarrow{\Omega}dH$ is called a *Hamiltonian vector field* on the symplectic manifold (M,ω). The function H is called the *Hamiltonian*, the corresponding dynamical system is called a *Hamiltonian system*.

□

Note that $i_X\omega = dH$. Since ω is nondegenerate the vector field X is also uniquely determined by this relation. If ω is only a weak symplectic form, the vector field X may not exist (everywhere). Let $u : (a,b) \to M$, then we say that u is a solution of this Hamiltonian system if

$$(3.2.14) \qquad \dot{u}(t) = \overleftarrow{\Omega}(u(t))\, dH(u(t)) \qquad \forall t \in (a,b).$$

In a local coordinate system the tensor field Ω is represented by a matrix $\Omega_{ij}(u)$ and the tensor field $\overleftarrow{\Omega}$ is represented by the inverse matrix $\Omega^{ij}(u)$. Then the coordinates $u^i(t)$ of u(t) satisfy the following system of differential equations

$$(3.2.15) \qquad \dot{u}^i(t) = \Omega^{ij}(u(t))\, H_{,j}(u(t)).$$

However, we can always introduce new local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ such that the system (3.2.15) transformes into the system (3.1.1).

(3.2.16) Theorem (Darboux).

Suppose ω is a symplectic form on a finite-dimensional manifold M. Then for each $u_0 \in M$ there exists a neighbourhood with local coordinates $q_1,\ldots,q_n$, $p_1,\ldots,p_n$ such that the symplectic form ω can be written as

$$(3.2.17) \qquad \omega = \sum_{i=1}^{n} d\,q_i \wedge d\,p_i \ .$$

Proof:
See Abraham and Marsden [1] or Choquet-Bruhat [3].

□

The coordinates $q_1,\ldots,p_n$ are called *canonical coordinates*. In this new coordinate system the cyclic tensor field Ω and the canonical tensor field $\overleftarrow{\Omega}$ are represented by

$$(3.2.18) \qquad \Omega_{ij} = \delta_{i,j+n} - \delta_{i+n,j} \ ,$$

$$(3.2.19) \qquad \Omega^{ij} = \delta^{i+n,j} - \delta^{i,j+n} \ , \qquad \forall\ i,j = 1,\ldots,2n \ .$$

With these matrices (3.2.15) reduces to the well-known classical Hamiltonian system (3.1.1).

Note that a Hamiltonian vector field is defined in terms of H and $\overleftarrow{\Omega}$. Of course this definition is also possible if the canonical tensor field $\overleftarrow{\Omega}$ does not come from a symplectic form ω (i.e. $\overleftarrow{\Omega}$ is not invertible). This leads to the following

3.2.20 Definition.

Suppose H is a function and $\overleftarrow{\Omega}$ a canonical tensor field on a manifold M. Then the vector field $X = \overleftarrow{\Omega}\,dH$ will be called a *semi-Hamiltonian vector field* on M. The corresponding dynamical system will be called a *semi-Hamiltonian system*.

□

Of course every Hamiltonian system (vector field) is also semi-Hamiltonian. However, since $\overleftarrow{\Omega}$ in the preceding definition is not necessarily invertible, the converse is not true.

3.2.21 Remark.

The definition of a semi-Hamiltonian system (vector field) was given in terms of a Hamiltonian H and a (not necessarily invertible) canonical tensor field $\overleftarrow{\Omega}$. In the following section we shall see that every canonical tensor field $\overleftarrow{\Omega}$ on M gives rise to a *Poisson structure,* i.e. a Lie algebra structure for $F(M)$ which also satisfies Leibniz' rule ($\{FG,K\} = F\{G,K\} + G\{F,K\}$). Therefore a manifold M with a canonical tensor field is called a *Poisson manifold.* Since every symplectic form ω gives rise to a canonical tensor field, every symplectic manifold "is" also a Poisson manifold. Of course the converse is not true.

□

3.3 POISSON BRACKETS

Let (M,ω) be a symplectic manifold or let M be a manifold with a canonical tensor field $\overleftarrow{\Omega}$ (a Poisson manifold). With every pair of functions F and G on M corresponds a (new) function on M, called the Poisson bracket of F and G.

3.3.1 Definition.

The *Poisson bracket* of two (possibly explicitly time dependent) functions F and G on M is the function $\{F,G\}$ defined by

$$\{F,G\} = \langle dF, \overleftarrow{\Omega} dG\rangle. \tag{3.3.2}$$

Two functions on M are *in involution* if their Poisson bracket vanishes.

□

In local coordinates (on a finite-dimensional manifold) the definition can be written as

$$\{F,G\} = F_{,i}\, \Omega^{ij}\, G_{,j} \; .$$

3.3.3 Theorem.

The Poisson bracket satisfies the so called *Jacobi identity*

$$\{\{F,G\},K\} + \{\{G,K\},F\} + \{\{K,F\},G\} = 0$$

for any three functions F, G, K $\in F_p(M)$.

Proof:
In the case that $\overleftarrow{\Omega}$ corresponds to a symplectic form ω, the proof of this result can be found in many text-books, see for instance Arnold [2]. We now give a proof which only uses that $\overleftarrow{\Omega}$ is canonical (and not that $\overleftarrow{\Omega}$ is invertible). Note that

$$\{F,G\} = \langle dF , \overleftarrow{\Omega} dG\rangle = L_{\overleftarrow{\Omega} dG} F \ .$$

This implies

$$\begin{aligned}\{\{F,G\},K\} &= \langle d\, L_{\overleftarrow{\Omega} dG} F , \overleftarrow{\Omega} dK\rangle \\ &= \langle L_{\overleftarrow{\Omega} dG}\, dF , \overleftarrow{\Omega} dK\rangle \\ &= g(dG , dF , dK)\end{aligned}$$

where g is given in (3.2.7) (with $\Psi = \overleftarrow{\Omega}$). The theorem follows now from (3.2.8) and Definition 3.2.9.

□

Recall (Definition 2.3.1) that a function $F \in F_p(M)$ is a constant of the motion or first integral of a dynamical system on M if

$$\frac{d}{dt} F(u(t),t) = 0$$

for all solutions of the dynamical system. For a (semi-) Hamiltonian system with Hamiltonian H this implies the following

3.3.4 Lemma.

A function $F \in F_p(M)$ is a constant of the motion iff $\{F,H\} + \mathring{F} = 0$ on $M \times I\!R$. For functions F, which do not depend explicitly on t (so $F \in F(M)$) this condition is $\{F,H\} = 0$.

Proof:
It is easily seen that

$$\frac{d}{dt} F(u(t),t) = \langle\, dF, \overleftarrow{\Omega} dH \,\rangle + \mathring{F} = \{F,H\} + \mathring{F}.$$

□

The following lemma is an immediate consequence of the Jacobi identity.

3.3.5 Lemma.

The set of constants of the motion for a (semi-) Hamiltonian system is a Lie algebra, if we take the Poisson bracket as Lie product. The set of autonomous constants of the motion is a subalgebra of this Lie algebra.

□

3.4 VARIATIONAL PRINCIPLES

It is well known that the classical Hamiltonian system (3.1.1) can be derived from the following *variational principle*

$$\underset{U}{\text{stat}} \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} p_i \dot{q}_i - H(q_1,\ldots,p_n) \right) dt$$

where U is the set of all curves in phase space $\mathbb{R}^{2n}$ with $q_i(t_1)$ and $q_i(t_2)$ fixed. There also exists a variational principle which yields directly the more general equations (3.2.15):

3.4.1 Theorem.

For every point $u_0 \in M$ there exists a neighbourhood $U_0 \ni u_0$ and a one-form α defined on U_0, such that a solution $\tilde{u}(t) \in U_0$ for $t \in [t_1,t_2]$ of (3.2.15) is a stationary point of the following functional

$$(3.4.2) \qquad \int_{t_1}^{t_2} (< \alpha(u(t)), \dot{u}(t) > - H(u(t))) \, dt$$

over the set of all curves $u(t) \in U_0$ for $t \in [t_1,t_2]$ with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$.

Proof:
The two-form ω is closed, so for every point $u_0 \in M$ there exists a neighbourhood U_0' and a one-form α defined on U_0', such that $\omega = - d\alpha$. On a neighbourhood $U_0 \subset U_0'$ there exist local coordinates u^i such that

$\alpha = \alpha_i \, du^i$. So (3.4.2) can be written as

(3.4.3) $$\int_{t_1}^{t_2} (\alpha_i(u(t)) \, \dot{u}^i(t) - H(u(t))) \, dt$$

Then it is an elementary exercise to show that stationary points of (3.4.3) with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$, are solutions of

(3.4.4) $$(\alpha_{i,j} - \alpha_{j,i}) \, \dot{u}^i = H_{,j} .$$

From $\omega = - d\alpha = \frac{1}{2}\Omega_{ji} du^i \wedge du^j$ we obtain $\Omega_{ji} = \alpha_{i,j} - \alpha_{j,i}$. Multiplication of (3.4.4) with Ω^{ij}, the inverse matrix of Ω_{ji}, results in (3.2.15).

□

3.5 HAMILTONIAN SYSTEMS AND ADJOINT SYMMETRIES

In this section we make some remarks on the question: when is a dynamical system a Hamiltonian system? In general this is a very difficult problem. For a number of equations the Hamiltonian character was only found after a long time. For instance, the Hamiltonian character of the Korteweg-de Vries equation [6,7] was found rather recently by Gardner [11] and Broer [10].

Consider an autonomous dynamical system on a manifold M

(3.5.1) $$\mathring{u} = X(u) .$$

Suppose ρ is a non-closed adjoint symmetry of this system. By Theorem 2.5.8 ρ gives rise to an SA operator Ω, defined by $\Omega A = i_A d\rho$, $\forall A \in X(M)$. Since ρ is an adjoint symmetry we have

(3.5.2) $$\Omega X = i_X d\rho = L_X \rho - d i_X \rho = -\mathring{\rho} - d i_X \rho .$$

This expression leads to the following

3.5.3 Theorem.

Suppose the dynamical system (3.5.1) has an adjoint symmetry ρ such that

i) $d\rho$ is nondegenerate ,

ii) $\mathring{\rho} = dG$ for some constant of the motion G .

Then (3.5.1) is a Hamiltonian system with Hamiltonian $H = -G - \langle\rho, X\rangle$ and symplectic form $\omega = d\rho$.

Proof:
Since $\mathring{\rho} = dG$ we obtain from (3.5.2) that

(3.5.4) $$\Omega X = i_X d\rho = d(-G - \langle\rho, X\rangle) = dH \ .$$

We now show that H and ω do not depend explicitly on t:

$$\mathring{H} = -\mathring{G} - \langle\mathring{\rho}, X\rangle = -\mathring{G} - \langle dG, X\rangle = 0$$

since G is a constant of the motion. Also

$$\mathring{\omega} = d\mathring{\rho} = ddG = 0 \ .$$

Finally $d\rho$ is nondegenerate (Ω is invertible), so (3.5.4) implies that X is the Hamiltonian vector field corresponding to ω and H.

□

3.5.5 Remark.

The Hamiltonian systems described in this theorem have the following special properties:

i) the symplectic two-form ω is not only closed but also exact, $\omega = d\rho$,
ii) the one-form ρ satisfies $\mathring{\rho} + L_X\rho = dG + L_X\rho = 0$, so $L_X\rho = -dG$, where G is a constant of the motion.

□

These two properties indicate that, in trying to find out whether a dynamical system is Hamiltonian, one should not try to find an adjoint symmetry as described in Theorem 3.5.3. However, several interesting Hamiltonian systems (Korteweg-de Vries equation, Sine Gordon equation, Toda chain) are of the type described in this theorem.

3.6 COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS

For symplicity we now consider the symplectic manifold $M = \mathbb{R}^{2n}$ with canonical coordinates $q_1, \ldots, p_n$. Then $\omega = \sum_i d\,q_i \wedge dp_i$. Suppose we introduce new coordinates $\tilde{q}_1, \ldots, \tilde{p}_n$ on $\mathbb{R}^{2n}$.

3.6.1 Definition.

The transformation from $q_1, \ldots, p_n$ to $\tilde{q}_1, \ldots, \tilde{p}_n$ is called a *canonical coordinate transformation* if, in new coordinates $\omega = \sum_i d\tilde{q}_i \wedge d\tilde{p}_i$.

□

So after a canonical coordinate transformation the differential equations for $\tilde{q}_i$, $\tilde{p}_i$ are also of the form (3.1.1).

Sometimes by means of a canonical coordinate transformation, the system of differential equations is greatly simplified. For instance suppose all the new coordinates $\tilde{q}_i$ are *cyclic*. This means the Hamiltonian, written as function of $\tilde{p}_i$ and $\tilde{q}_i$, depends only on the $\tilde{p}_i$. The solution of the corresponding Hamiltonian system is trivial and the system is called *completely integrable*. Furthermore the functions $\tilde{p}_i$ constitute a set of n constants in involution. In general it turns out that the existence of n constants of the motion in involution, is directly related to the complete integrability of the system.

3.6.2 Theorem (Arnold, Liouville).

Suppose there exist n constants of the motion in involution $F_1 = H, F_2, \ldots, F_n$. Consider the level set of functions F_i

$$M_{\underline{a}} = \{(q_1, \ldots, p_n) \in M = \mathbb{R}^{2n} \mid F_i(q_1, \ldots, p_n) = a_i\} .$$

Assume the one-forms dF_i are linearly independent on $M_{\underline{a}}$ and that $M_{\underline{a}}$ is compact and connected.

Then

i) $M_{\underline{a}}$ is invariant for the Hamilton flow with Hamiltonian H,

ii) $M_{\underline{a}}$ is diffeomorphic to the n-dimensional torus $T^n = \{(\tilde{q}_1, \ldots, \tilde{q}_n) \bmod 2\pi\}$,

iii) there exist n functions $\tilde{p}_i(F_1, \ldots, F_n)$ such that $\tilde{q}_1, \ldots, \tilde{q}_n, \tilde{p}_1, \ldots, \tilde{p}_n$ are coordinates for a neighbourhood of $M_{\underline{a}}$. The transformation $(q_1, \ldots, p_n) \to (\tilde{q}_1, \ldots, \tilde{p}_n)$ is a canonical coordinate transformation and the Hamiltonian H, expressed in the new coordinates, depends only on the $\tilde{p}_i$: $H = \tilde{H}(\tilde{p}_1, \ldots, \tilde{p}_n)$.

Proof:

See Arnold [2].

□

The solution of the corresponding Hamiltonian system

$$(3.6.3)\quad \begin{cases} \dot{\tilde{p}}_i = 0 \\ \dot{\tilde{q}}_i = \dfrac{\partial \tilde{H}(\tilde{p}_1,\dots,\tilde{p}_n)}{\partial \tilde{p}_i} \qquad i = 1,\dots,n \end{cases}$$

is trivial and the system is completely integrable. The coordinates $\tilde{p}_i$ are called *action variables*, while the $\tilde{q}_i$ are called *angle variables*.

Note that we only discussed complete integrability for finite-dimensional Hamiltonian systems. As far as we know an appropriate definition of complete integrability for infinite-dimensional Hamiltonian systems and an infinite-dimensional version of Theorem 3.6.2 have not yet been given.

3.7 TRANSFORMATION PROPERTIES OF HAMILTONIAN SYSTEMS

In Section 2.7 we discussed the behaviour under transformations of (adjoint) symmetries and the four possible operators between adjoint symmetries. The transformation properties of Hamiltonian systems are also easily found. Consider the Hamiltonian vector field $X = \overset{\leftarrow}{\Omega} dH$, on a manifold M, corresponding to the symplectic form ω and Hamiltonian H. Suppose there exists a diffeomorphism $f : M \to N$ with inverse $\overset{\leftarrow}{f} : N \to M$. Using the derivative map $f' : TM \to TN$ we can transform the vector field X on M to a vector field $\tilde{X} = f'X$ on N.

3.7.1 Theorem.

The transformed vector field $\tilde{X} = f'X$ of the Hamiltonian vector field X is again a Hamiltonian vector field. The corresponding Hamiltonian $\tilde{H}$ and symplectic two-form $\tilde{\omega}$ on N are given by

$$(3.7.2)\qquad \tilde{H}(v) = H(\overset{\leftarrow}{f}(v)) \quad \forall\, v \in N,$$

$$(3.7.3)\qquad \tilde{\omega}(\tilde{A},\tilde{B}) = \omega(\overset{\leftarrow}{f}{}'\tilde{A},\ \overset{\leftarrow}{f}{}'\tilde{B}) \quad \forall\, \tilde{A},\tilde{B} \in X(N) .$$

The tensor fields $\tilde{\Omega} \in T^0_2(N)$ and $\tilde{\overset{\leftarrow}{\Omega}} \in T^2_0(N)$ (considered as vector bundle maps we have $\tilde{\Omega}: TN \to T^*N$ and $\tilde{\overset{\leftarrow}{\Omega}} : T^*N \to TN$) are given by

(3.7.4) $\tilde{\Omega} = \overleftarrow{f}'^{*}\, \Omega\, \overleftarrow{f}'$,

(3.7.5) $\tilde{\Omega}^{\leftarrow} = f'\Omega^{\leftarrow} f'^{*}$.

Proof:

The relations between functions, differential forms and tensor fields are "natural" with respect to transformations (see also section 2.7). This means that the transformed vector field $\tilde{X} = f'X$ can also be obtained from the transformed Hamiltonian $\tilde{H}$ and the transformed two-form $\tilde{\omega}$. The formulas (3.7.2), (3.7.3), (3.7.4) and (3.7.5) give the usual transformation properties of functions, differential forms and tensor fields.

□

3.7.6 Remark.

By the method used in theorem 3.7.1 we can supply the manifold N with a symplectic two-form $\tilde{\omega}$, the push-forward of ω by f. Suppose there exists already a symplectic form ϕ on N; so (M,ω) and (N,ϕ) are both symplectic manifolds. On N we now have the symplectic forms ϕ and $\tilde{\omega}$. If $\phi = \tilde{\omega}$ the mapping f is called a *symplectic transformation* (symplectic diffeomorphism) or *canonical transformation.* A canonical transformation should not be confused with a canonical coordinate transformation, as described in definition 3.6.1.

□

Other properties of the Hamiltonian system on M are also easily translated to the transformed system on N.

3.7.7 Corollary.

The transformed Poisson bracket of two functions F_1, F_2 on M is equal to the Poisson bracket of the transformed function $\tilde{F}_1$, $\tilde{F}_2$ on N.

Proof:

From $\tilde{F}_i(v) = F_i(\overleftarrow{f}(v))$ $(i = 1,2)$ we obtain $d\tilde{F}_i = \overleftarrow{f}'^{*} dF_i$.
The result now follows from the definition of Poisson bracket and from (3.7.5).

□

So if the functions F_1, F_2 on M are in involution, the transformed functions $\tilde{F}_1$, $\tilde{F}_2$ on N are also in involution.

CHAPTER 4: SYMMETRIES FOR HAMILTONIAN SYSTEMS

4.1 INTRODUCTION

In Chapter 2 we considered some properties of dynamical systems on a manifold M. We introduced symmetries, adjoint symmetries and four types of operators between those symmetries. In this chapter we assume that the dynamical system is a Hamiltonian system. The most important consequence of this Hamiltonian character is that there always exists at least one SA- and one AS operator. This implies that with a constant of the motion not only corresponds an adjoint symmetry, but also a symmetry. However, there can also exist symmetries which are not related in this way to a constant of the motion. These so called non- (semi-) canonical symmetries have interesting properties. In Section 4.2 we show how they can be used to construct (new) SA- and AS operators and hence recursion operators for (adjoint) symmetries. The thus constructed (new) SA operator is always cyclic; if it is also invertible, the system can be written as a Hamiltonian system in two different ways. These so called bi-Hamiltonian systems are considered in Section 4.3. Non- (semi-) canonical symmetries can also be used in various ways to construct (new) constants of the motion out of already known ones. In Section 4.4 we shall describe three possible methods for doing this. In Sections 4.5 and 4.7 we give conditions under which these methods can be used to generate infinite series of constants of the motion. The possible relation between these series is also studied in Section 4.7. A series of (non-semi-canonical) symmetries is constructed in Section 4.6. The methods described in the Sections 4.5, 4.6 and 4.7 can be applied to several popular "completely integrable" finite- and infinite-dimensional Hamiltonian systems (Toda chain, Korteweg-de Vries equation, sine-Gordon equation,...). Of course the existence of infinite series of constants of the motion for these equations is well-known. However, several of our methods for constructing these series seem to be new. Also the series of non-semi-canonical symmetries is generally overlooked.

In this chapter we shall consider an autonomous Hamiltonian system on a symplectic manifold (M,ω) with Hamiltonian H. With the symplectic form ω correspond the cyclic tensor field $\Omega \in T_2^0(M)$ and the canonical tensor field $\overleftarrow{\Omega} \in T_0^2(M)$ (see Section 3.2). The Hamiltonian vector field on M is then given by

$$(4.1.1) \qquad X = \overleftarrow{\Omega}\,dH \ ,$$

and the corresponding differential equation is

$$(4.1.2) \qquad \mathring{u}(t) = X(u(t)) = \overleftarrow{\Omega}(u(t))\, dH(u(t)) \ .$$

As in Chapter 2 we shall assume that for all inital conditions $u(t_0) = u_0$ there exists a smooth unique solution $u(t)$ of (4.1.2), defined on some interval $I \subset \mathbb{R}$.

In Section 4.8 we consider the case that (4.1.1) is only a semi-Hamiltonian vector field and (4.1.2) is only a semi-Hamiltonian system (see Section 3.2). With some modifications several results obtained for Hamiltonian systems also hold for semi-Hamiltonian systems. Finally we mention that, unless otherwise stated, all constants of the motion, (adjoint) symmetries, SA- and AS operators and recursion operators for (adjoint) symmetries mentioned in this chapter belong to the (semi-) Hamiltonian system (4.1.2).

4.2 SA- AND AS OPERATORS

In this section we discuss the various possible SA- and AS operators for a Hamiltonian system. The following lemma will be useful in the sequel.

4.2.1 Lemma.

Suppose α is a closed (parameterized) one-form on M and $\Psi \in T_0^2(M)$ is a canonical tensor field. Then $L_{\Psi\alpha}\Psi = 0$.

Proof:

Let β and γ be arbitrary one-forms on M . Then define the vector fields $A = \Psi\alpha$, $B = \Psi\beta$ and $C = \Psi\gamma$. Application of Leibniz'rule to the identity

$$d\alpha(B,C) = L_B\langle\alpha,C\rangle - L_C\langle\alpha,B\rangle - \langle\alpha, [B,C]\rangle$$

results in

$$(4.2.2) \qquad d\alpha(B,C) = \langle L_B\alpha, C\rangle + \langle \alpha, [B,C]\rangle - \langle L_C\alpha, B\rangle .$$

Using Leibniz' rule and the antisymmetry of Ψ and its Lie derivatives we can write the second term as

$$\langle\alpha,[B,C]\rangle = \langle \alpha, L_B(\Psi\gamma)\rangle$$

$$= \langle\alpha,\Psi L_B\gamma\rangle - \langle\gamma, (L_B\Psi)\alpha\rangle$$

$$= \langle\alpha,\Psi L_B\gamma\rangle - \langle\gamma, L_B(\Psi\alpha)\rangle + \langle\gamma, \Psi L_B\alpha\rangle$$

$$= -\langle L_B\gamma, \Psi\alpha\rangle + \langle\gamma, L_A(\Psi\beta)\rangle - \langle L_B\alpha, \Psi\gamma\rangle .$$

Substitution in (4.2.2) gives

$$d\alpha(B,C) = -\langle L_B\gamma,\Psi\alpha\rangle + \langle\gamma, L_A(\Psi\beta)\rangle - \langle L_C\alpha, B\rangle .$$

Since Ψ is canonical (see Definition 3.2.9) this becomes

$$d\alpha(B,C) = \langle L_A\beta,\Psi\gamma\rangle + \langle\gamma, L_A(\Psi\beta)\rangle$$

$$= -\langle\gamma,\Psi L_A\beta\rangle + \langle\gamma, L_A(\Psi\beta)\rangle$$

$$= \langle\gamma, (L_A\Psi)\beta\rangle .$$

The one-form α is closed, so the left hand side vanishes. The one-forms β and γ are arbitrary, so $L_A\Psi = L_{\Psi\alpha}\Psi = 0$.

□

The first application of this lemma is described in the following

4.2.3 Lemma.

Let α be a closed (parameterized) one-form on M and let $A = \overleftarrow{\Omega}\alpha$ be the corresponding vector field. Then $L_A\overleftarrow{\Omega} = 0$, $L_A\Omega = 0$ and $L_A\omega = 0$.

Proof:

The tensor field $\overleftarrow{\Omega}$ is canonical. So by the preceding lemma $L_A\overleftarrow{\Omega} = 0$. From $L_A(\Omega\overleftarrow{\Omega}) = 0$ we obtain

$$L_A\Omega = -\Omega(L_A\overleftarrow{\Omega})\Omega = 0 \ .$$

Finally $L_A\Omega = 0$ is equivalent with $L_A\omega = 0$.

□

4.2.4 Remark.

The importance of Lemma 4.2.1 is that it can be used in cases where a canonical tensor field Ψ is (maybe) not invertible (see for instance Sections 4.8 and 5.7). In the proof of Lemma 4.2.3 we had $\Psi = \overleftarrow{\Omega}$, which is invertible. Using this property the proof that $L_A\overleftarrow{\Omega} = 0$ can be considerably simplified. From $A = \overleftarrow{\Omega}\alpha$ we obtain $\alpha = \Omega A = i_A\omega$. Since α is closed we have $d\, i_A\omega = d\alpha = 0$. Then, because ω is closed, $L_A\omega = i_A d\omega + d\, i_A\omega = 0$, which is equivalent to $L_A\Omega = 0$. Finally $L_A\overleftarrow{\Omega} = -\overleftarrow{\Omega}(L_A\Omega)\overleftarrow{\Omega} = 0$.

□

Recall that a tensor field $\Psi \in T^2_{0p}(M)$ which can be used to map adjoint symmetries into symmetries was called an AS operator (see Definition 2.5.3). A tensor field $\Phi \in T^0_{2p}(M)$ which can be used to map symmetries into adjoint symmetries was called an SA operator (see Definition 2.5.6). It turns out that for a Hamiltonian system there always exists an SA- and an AS operator.

4.2.5 Theorem.

The tensor field $\overleftarrow{\Omega}$ is (can be considered as) an AS operator and the tensor field Ω is an SA operator.

Proof:

The conditions for an AS operator were given in Definition 2.5.3. The operator $\overleftarrow{\Omega}$ is an AS operator if it satisfies

$$(4.2.6) \qquad \dot{\overleftarrow{\Omega}} + L_X\overleftarrow{\Omega} = 0 \ .$$

It follows from Lemma 4.2.3 with $\alpha = dH$ that $L_X\overleftarrow{\Omega} = 0$. Since $\overleftarrow{\Omega}$ does not depend explicitly on t, it satisfies (4.2.6). Then Theorem 2.5.7 implies that Ω is an SA operator.

□

In local coordinates the tensor field Ω is represented by a matrix (matrix valued function) $\Omega_{ij}(u)$ and the tensor field $\overset{\leftarrow}{\Omega}$ is represented by the inverse matrix $\Omega^{k\ell}(u)$. A symmetry Y has components $Y^j(u,t)$ and an adjoint symmetry σ has components $\sigma_\ell(u,t)$. Then Theorem 4.2.5 says that if Y^j is (represents) a symmetry, $\Omega_{ij}(u)\, Y^j(u,t)$ is an adjoint symmetry. Also if $\sigma_\ell(u,t)$ is an adjoint symmetry, $\Omega^{k\ell}(u)\, \sigma_\ell(u,t)$ is a symmetry.

Theorem 4.2.5 has a very important consequence. Suppose F is a constant of the motion. Then Theorem 2.4.3 says that dF is an adjoint symmetry. Next Theorem 4.2.5 implies that $\overset{\leftarrow}{\Omega}$dF is a symmetry. *So for a Hamiltonian system every constant of the motion* F *gives rise to a symmetry* $\overset{\leftarrow}{\Omega}$dF. This leads to the following

4.2.7 Definition.

i) A symmetry Y with adjoint symmetry $\sigma = \Omega Y$ which is (not) exact, will be called a *(non-) canonical symmetry*.

ii) A symmetry Y such that $\sigma = \Omega Y$ is (not) closed will be called a *(non-) semi-canonical symmetry*.

□

It is useful to introduce similar names for adjoint symmetries. Hence the following

4.2.8 Definition.

i) An adjoint symmetry σ which is (not) exact will be called a *(non-) canonical adjoint symmetry*.

ii) An adjoint symmetry which is (not) closed will be called a *(non-) semi-canonical adjoint symmetry*.

□

These definitions imply that a (non-) (semi-) canonical symmetry Y gives rise to a (non-) (semi-) canonical adjoint symmetry $\sigma = \Omega Y$ and conversely. A canonical symmetry Y can be written as $Y = \overset{\leftarrow}{\Omega}$dF, where F is a possibly parameterized function M. If we ignore the possible dependence on the parameter (t), this means that Y is a Hamiltonian vector field with Hamiltonian F. (Similarly a semi-canonical symmetry is a locally Hamiltonian vector field.)

4.2.9 Remark.

Suppose $\sigma = \Omega Y$ is a canonical adjoint symmetry. Then there exists a function $F \in F_p(M)$ such that $\sigma = dF$. However, by Theorem 2.4.5 there also exists a constant of the motion G such that $\sigma = dG$ and $Y = \overleftarrow{\Omega} dG$. So the space of canonical adjoint symmetries (a subspace of $V^*(X,M)$) and the space of canonical symmetries (a subspace of $V(X,M)$) are both isomorphic to the space of constants of the motion. (Constants of the motion which differ only by a (numerical) constant are identified.)

□

4.2.10 Remark.

An exact differential form is always closed. In the terminology introduced above, this means that a canonical (adjoint) symmetry is also a semi-canonical (adjoint) symmetry. A differential form which is not closed is also not exact. This implies that a non-semi-canonical (adjoint) symmetry is also a non-canonical (adjoint) symmetry. Since a closed form is not necessarily exact, the converse of these two assertions is not true. By the Poincaré lemma a closed one-form α can *locally* be written as $\alpha = dF$. If this relation holds on M, the form is exact. There is a topological condition which implies that closed k-forms are exact. In our case (one-forms) the condition is that the *first cohomology group* of M vanishes. If the manifold M has this property, (non-) semi-canonical (adjoint) symmetries are identical with (non-) canonical (adjoint) symmetries. This happens for instance if M is also a linear space.

□

In local coordinates u^i a canonical adjoint symmetry σ has local coordinates $\sigma_i = G_{,i}$ for some constant of the motion G. The coordinates σ_i of a semi-canonical adjoint symmetry satisfy $\sigma_{i,j} = \sigma_{j,i}$.

A characterization of (non-) semi-canonical symmetries is given in the following

4.2.11 Theorem.

A symmetry Y is semi-canonical iff $L_Y \Omega = 0$.

Proof:

Since ω is closed, we have

$$L_Y\omega = d\, i_Y\omega = d(\Omega Y) .$$

So ΩY is closed iff $L_Y\omega = 0$, which is equivalent to $L_Y\Omega = 0$.

□

This theorem implies that the symplectic form ω is invariant under the flow corresponding to a semi-canonical symmetry Y. So this flow consists of canonical (symplectic) transformations. This explains the name (semi-) canonical symmetry.

Since $L_Y\overleftarrow{\Omega} = -\overleftarrow{\Omega}(L_Y\Omega)\overleftarrow{\Omega}$ Theorem 4.2.11 also implies that a symmetry Y is semi-canonical iff $L_Y\overleftarrow{\Omega} = 0$.

4.2.12 Remark.

It is important to realize that the answer to the question whether a given symmetry is (non-) (semi-) canonical depends on the symplectic form ω. If a dynamical system can be written as a Hamiltonian system in two essentially different ways (using two symplectic forms and two Hamiltonians), these answers may be different. In that case one has to say which symplectic form has been adopted.

□

4.2.13 Theorem.

Suppose $Z = \overleftarrow{\Omega}\tau$ is a non-semi-canonical symmetry. Then for $k = 1,2,3,\ldots$

i) $L_Z^k\overleftarrow{\Omega}$ is an AS operator,

ii) $L_Z^k\Omega$ is an SA operator. This SA operator is again cyclic and corresponds to the (exact) two-form $d\, L_Z^{k-1}\tau$

$$(4.2.14) \qquad <(L_Z^k\Omega)A,B> \equiv (L_Z^k\omega)(A,B) = (d\, L_Z^{k-1}\tau)(A,B) .$$

Proof:

i) and the first part of ii) follow at once from the Theorems 4.2.5 and 2.5.16 i). From $\tau = \Omega Z = i_Z\omega$ and the closedness of ω we obtain $d\tau = L_Z\omega$. This implies (4.2.14).

□

In Theorem 2.5.8 we have seen that (also for a non-Hamiltonian system) a non-closed adjoint symmetry $\tau = \Omega Z$ gives rise to an SA operator. Theorem 4.2.13 ii) states that (for a Hamiltonian system) this operator is identical to $L_Z\Omega$.

Note that in the proof of Theorem 4.2.13 we did not use that the symmetry Z was non-semi-canonical. However, if Z is semi-canonical, Theorem 4.2.11 says that $L_Z\Omega = 0$ (and hence $L_Z\overleftarrow{\Omega} = 0$). So then the SA- and AS operators given in Theorem 4.2.13 vanish. For a symmetry Z which is non-semi-canonical the operators $L_Z\Omega$ and $L_Z\overleftarrow{\Omega}$ do not vanish. Of course this does not imply that they are invertible. As an example of this consider a Hamiltonian system with two analytically independent constants of the motion F and G. Then $Z = \overleftarrow{\Omega}\tau = \overleftarrow{\Omega}F\,dG$ is a non-semi-canonical symmetry. The two-form $d\tau$ is then given by $d\tau = dF \wedge dG$. Then (4.2.14) implies that

$$(L_Z\Omega)A = i_A\, d\tau = \langle dF,A\rangle dG - \langle dG,A\rangle dF \ .$$

So the SA operator $L_Z\Omega$ maps any vector field A into the module of one-forms spanned by dF and dG. If the manifold M has dimension larger than 2, this means that $L_Z\Omega$ is not invertible.

Finally we expand a (non-canonical) symmetry in canonical symmetries.

4.2.15 Theorem.

Suppose there exist m analytically independent constants of the motion $G_1,\dots,G_m$. If a symmetry Z can be written as

$$Z = \sum_{i=1}^{m} F_i\, \overleftarrow{\Omega}\, d\, G_i \ , \qquad F_i \in F_p(M) \ ,$$

then the functions F_i are constants of the motion.

Proof:

Since $\overleftarrow{\Omega}$ is an AS operator and the G_i are constants of the motion, we have

$$\dot{Z} + L_X Z = \sum_{i=1}^{m} (\dot{F}_i + L_X F_i)\overleftarrow{\Omega}\, d\, G_i \ .$$

The vector field Z is a symmetry, so the left hand side vanishes. Since $\overset{\leftarrow}{\Omega}$ is invertible and the one-forms $d\,G_i$ are linearly independent on a dense open subset N of M, we obtain

$$\dot{F}_i + L_X F_i = 0 \quad \text{on } N\,, \qquad i = 1,\ldots,m\,.$$

The continuity of these expressions implies that this also holds on M, so the F_i are constants of the motion.

□

4.3 BI-HAMILTONIAN SYSTEMS

Sometimes it is possible to write a dynamical system as a Hamiltonian system in (at least) two essentially different ways. Suppose a vector field X on M can be written as a Hamiltonian vector field using the symplectic form ω and the Hamiltonian H, but also using the symplectic form $\tilde{\omega}$ and the Hamiltonian $\tilde{H}$. So

(4.3.1) $\qquad X = \overset{\leftarrow}{\Omega}\, dH$

and

(4.3.2) $\qquad X = \overset{\leftarrow}{\tilde{\Omega}}\, d\tilde{H}$,

where $\overset{\leftarrow}{\Omega}$ and $\overset{\leftarrow}{\tilde{\Omega}}$ are the inverse tensor fields of the tensor fields Ω and $\tilde{\Omega}$ which correspond to ω and $\tilde{\omega}$. If $\overset{\leftarrow}{\Omega} \neq c\,\overset{\leftarrow}{\tilde{\Omega}}$ for some $c \in \mathbb{R}$, the dynamical system $\dot{u} = X(u)$ is called a *bi-Hamiltonian system*. Several popular "integrable" Hamiltonian systems are of this type, see for instance Magri [5]. In Section 4.6 we shall meet dynamical systems which can be written as a Hamiltonian system in infinitely many ways (see Theorem 4.6.12). Magri observed that the combination of the SA- and AS operators corresponding to (4.3.1) and (4.3.2) yields the recursion operators for symmetries $\Lambda_1 = \overset{\leftarrow}{\Omega}\tilde{\Omega}$ and $\Lambda_2 = \overset{\leftarrow}{\tilde{\Omega}}\Omega$ $(= \Lambda_1^{-1})$. If for instance (4.3.1) is only a semi-Hamiltonian form, we still obtain the recursion operator Λ_1 but no longer Λ_2 (see also Fuchssteiner [64]).

In Theorem 3.5.3 we have seen that, for a general dynamical system, the existence of a certain adjoint symmetry implies that the system is a Hamiltonian system. Of course this theorem is also valid if the dynamical system is already a Hamiltonian system. In that case Theorem 3.5.3 provides us with a symplectic form and a Hamiltonian which may or may not be equal to the original ones. If the two symplectic forms are not equal up to a multiplicative constant, the system is bi-Hamiltonian. We now reformulate Theorem 3.5.3 in case the original system is already a Hamiltonian system.

4.3.3 Theorem.

Suppose $Z = \overleftarrow{\Omega}\tau$ is a non-semi-canonical symmetry of the Hamiltonian system (4.1.2) which satisfies the following conditions

i) the SA operator $L_Z\Omega$ is invertible, or equivalently the two-form $d\tau$ is nondegenerate,
ii) the symmetry $\dot{Z}$ is canonical, so there exists a constant of the motion G such that $\dot{Z} = \overleftarrow{\Omega}\dot{\tau} = \overleftarrow{\Omega}dG$.

Then the vector field X is also the Hamiltonian vector field corresponding to the Hamiltonian $\tilde{H} = L_Z H - G$ and the symplectic form $\tilde{\omega} = d\tau$.

Proof:
Theorem 3.5.3 yields that X is also the Hamiltonian vector field corresponding to the Hamiltonian $\tilde{H} = -\langle\tau,X\rangle - G$ and symplectic form $d\tau$.
Since $X = \overleftarrow{\Omega}dH$ and $\tau = \Omega Z$ we have

$$\langle\tau,X\rangle = -\langle dH,Z\rangle = -L_Z H .$$

So $\tilde{H} = L_Z H - G$ and this concludes the proof.

□

If a symmetry Z as described in this theorem exists, the vector field X can be written as

$$X = \overleftarrow{\Omega}dH$$

and as

$$X = (L_Z\Omega)^{-1}\, d\tilde{H} .$$

4.4 THE CONTRACTION BETWEEN A SYMMETRY AND AN ADJOINT SYMMETRY

Suppose σ_1 and σ_2 are two adjoint symmetries (of (4.1.2)) with corresponding symmetries $Y_1 = \overleftarrow{\Omega}\sigma_1$ and $Y_2 = \overleftarrow{\Omega}\sigma_2$. Then by Theorem 2.5.10 the function

(4.4.1) $G = \langle\sigma_1,Y_2\rangle$

is a constant of the motion. We shall now compute the corresponding canonical symmetry $\overleftarrow{\Omega} dG$. Since $\overleftarrow{\Omega}$ is canonical, the corresponding (3,0) tensor field $\hat{\Xi}$, given in (3.2.8), vanishes. Substitution of $\alpha = \sigma_1$ and $\beta = \sigma_2$ in (3.2.8) then yields

$$(4.4.2) \qquad \langle L_{Y_1}\sigma_2, \overleftarrow{\Omega}\gamma\rangle + \langle L_{Y_2}\gamma, Y_1\rangle + \langle L_{\overleftarrow{\Omega}\gamma}\sigma_1, Y_2\rangle = 0$$

for all one-forms γ on M. The last term in the left hand side can be written as

$$\langle L_{\overleftarrow{\Omega}\gamma}\sigma_1, Y_2\rangle = L_{\overleftarrow{\Omega}\gamma}\langle\sigma_1, Y_2\rangle - \langle\sigma_1, L_{\overleftarrow{\Omega}\gamma}Y_2\rangle =$$

$$= L_{\overleftarrow{\Omega}\gamma}G + \langle\sigma_1, L_{Y_2}(\overleftarrow{\Omega}\gamma)\rangle =$$

$$= \langle dG, \overleftarrow{\Omega}\gamma\rangle + \langle\sigma_1, (L_{Y_2}\overleftarrow{\Omega})\gamma + \overleftarrow{\Omega} L_{Y_2}\gamma\rangle =$$

$$= -\langle\gamma, \overleftarrow{\Omega} dG\rangle - \langle\gamma, (L_{Y_2}\overleftarrow{\Omega})\sigma_1\rangle - \langle L_{Y_2}\gamma, Y_1\rangle \ .$$

Substitution in (4.4.2) implies

$$\overleftarrow{\Omega} dG = -(L_{Y_2}\overleftarrow{\Omega})\sigma_1 - \overleftarrow{\Omega} L_{Y_1}\sigma_2 =$$

$$(4.4.3) \qquad = -(L_{Y_2}\overleftarrow{\Omega})\sigma_1 + (L_{Y_1}\overleftarrow{\Omega})\sigma_2 + [Y_2, Y_1] \ .$$

Note that, up to here, the calculation is also possible for a semi-Hamiltonian system. For a Hamiltonian system we can write this as

$$(4.4.4) \qquad \overleftarrow{\Omega} dG = \overleftarrow{\Omega}(L_{Y_2}\Omega)Y_1 - \overleftarrow{\Omega}(L_{Y_1}\Omega)Y_2 + [Y_2, Y_1] \ .$$

By construction this is a canonical symmetry. In the right hand side we recognize the recursion operators for symmetries $\overleftarrow{\Omega} L_{Y_1}\Omega$ and $\overleftarrow{\Omega} L_{Y_2}\Omega$, acting on Y_2 respectively Y_1 and the Lie bracket $[Y_2, Y_1]$. Note that these recursion operators, generated by Y_1 respectively Y_2, are products of a canonical AS-operator ($\overleftarrow{\Omega}$) and a cyclic SA operator ($L_{Y_1}\Omega$ resp. $L_{Y_2}\Omega$). First suppose Y_1 is a canonical symmetry. Then there exists a constant of the motion F_1 such that $Y_1 = \overleftarrow{\Omega}\sigma_1 = \overleftarrow{\Omega} dF_1$. Theorem 4.2.11 now implies that $L_{Y_1}\Omega = 0$. In this case (4.4.1) and (4.4.4) can be rewritten as

(4.4.5) $\qquad G = \langle dF_1, Y_2\rangle = L_{Y_2} F_1$

and

(4.4.6) $\qquad \overset{\leftarrow}{\Omega} dG = \overset{\leftarrow}{\Omega}(L_{Y_2}\Omega)\overset{\leftarrow}{\Omega} dF_1 + [Y_2, \overset{\leftarrow}{\Omega} dF_1]$.

Formula (4.4.5) can be considered as a method for constructing a (new) constant of the motion G out of a known constant F_1 and a symmetry Y_2. Then the canonical symmetry corresponding to G consists of two parts. The first part is $\overset{\leftarrow}{\Omega}(L_{Y_2}\Omega)\overset{\leftarrow}{\Omega} dF_1$, that is the recursion operator $\overset{\leftarrow}{\Omega}(L_{Y_2}\Omega)$ applied to the symmetry $\overset{\leftarrow}{\Omega} dF_1$. The second term is the Lie bracket of Y_2 and $\overset{\leftarrow}{\Omega} dF_1$. We can also try to use the single terms to construct a (new) constant of the motion. So, starting with a constant of the motion F_1 and a (non-semi-canonical) symmetry Y_2 there are several possible ways to construct another constant of the motion:

i) We can compute $G = L_{Y_2} F_1$.

ii) We can apply the recursion operator $\overset{\leftarrow}{\Omega}(L_{Y_2}\Omega)$ to $\overset{\leftarrow}{\Omega} dF_1$ and obtain

$$Y_3 = \overset{\leftarrow}{\Omega}(L_{Y_2}\Omega)\overset{\leftarrow}{\Omega} dF_1 \ .$$

However, the symmetry Y_3 can be canonical or non-canonical. Only in the first case this method yields a constant of the motion.

iii) We can compute the Lie bracket

$$Y_4 = [Y_2, \overset{\leftarrow}{\Omega} dF_1] \ .$$

Also in this case Y_4 may be canonical or non-canonical.

It follows from (4.4.6) that $\overset{\leftarrow}{\Omega} dG = Y_3 + Y_4$. So if method ii) works then also method iii) works and conversely. Method i) seems very attractive because it yields at once a constant of the motion. However, it is easier to describe properties of a constant of the motion which is constructed with one of the other methods. In Section 4.5 we consider the problem of constructing an infinite series of canonical symmetries (and so constants of the motion) using a recursion operator for (adjoint) symmetries of the form $\overset{\leftarrow}{\Omega} L_{Y_2}\Omega$ $((L_{Y_2}\Omega)\overset{\leftarrow}{\Omega})$ (method ii).

In the first part of Section 4.7 we investigate under which conditions an infinite series of canonical symmetries can be obtained using the (repeated) Lie bracket with Y_2 (method iii). Then we study the possible relations between these two series and consider method i).

We now return to (4.4.4) and assume both symmetries Y_1 and Y_2 are canonical. So there exist constants of the motion F_1 and F_2 such that $Y_1 = \overleftarrow{\Omega} dF_1$ and $Y_2 = \overleftarrow{\Omega} dF_2$. Then (4.4.1) and (4.4.4) can be written as

(4.4.7) $$G = \langle dF_1, \overleftarrow{\Omega} dF_2 \rangle = \{F_1, F_2\} \ ,$$

and

(4.4.8) $$\overleftarrow{\Omega} dG = [\overleftarrow{\Omega} dF_2, \overleftarrow{\Omega} dF_1] \ .$$

This means the canonical symmetry corresponding to the Poisson bracket $G = \{F_1, F_2\}$ is equal to the Lie bracket of the canonical symmetries corresponding to F_2 and F_1. So we have proved the following well-known

4.4.9 Theorem.

The canonical symmetries form a subalgebra of the Lie algebra of symmetries $V(X;M)$. This subalgebra is isomorphic to the Lie algebra of constants of the motion, as described in Lemma 3.3.5.

□

This theorem has the following consequence. Considerations which only use canonical (adjoint) symmetries can also be held on the level of constants of the motion. It is only useful to work with vector fields (one-forms) if non-canonical (adjoint) symmetries are involved.

4.5 INFINITE SERIES OF CONSTANTS OF THE MOTION I

A lot of popular "integrable" Hamiltonian systems have an infinite series of constants of the motion. These constants of the motion F_k do not depend explicitly on t and are in involution

$$\dot{F}_i = 0 \ , \qquad \{F_i, F_j\} = 0 \ .$$

The most obvious way of constructing a new constant of the motion is by taking the Poisson bracket of two already known elements of the series. Since the series F_k is in involution this method will not work. Another possibility is to take the Poisson bracket with some other constant of the motion G. It turns out that several equations have a constant of the motion G, not in the series F_k, such that

$$\{F_k, G\} = c_k F_{k+\ell} \qquad c_k \in \mathbb{R} .$$

However, very often $\ell \leq 0$, which means that in this way we cannot go upwards in the series F_k. For instance, for the Korteweg-de Vries equation there exists a constant of the motion G such that $\ell = -1$ (see for instance Broer and Backerra [25]). In the case of the Sawada-Kotera equation there is a constant of the motion G with $\ell = 0$. For both equations this method is not suitable for constructing an infinite series of constants of the motion. For the Benjamin-Ono equation there exists a constant of the motion G such that $\ell = 1$. Then an infinite series of constants of the motion is easily constructed and the following considerations are unnecessary. All these three equations will be used as examples in Chapter 5.

In this section we shall consider the problem of constructing an infinite series of constants of the motion using a recursion operator for (adjoint) symmetries. For a Hamiltonian system every non-semi-canonical symmetry Z gives rise to a recursion operator for (adjoint) symmetries $\Lambda = \overset{\leftarrow}{\Omega}(L_Z\Omega)$ $(\Gamma = (L_Z\Omega)\overset{\leftarrow}{\Omega})$. An infinite series of adjoint symmetries ρ_k and corresponding symmetries X_k is then given by

$$(4.5.1) \qquad \begin{aligned} \rho_k &= \Gamma^{k-1} dH , \\ X_k &= \overset{\leftarrow}{\Omega}\rho_k = \Lambda^{k-1} X . \end{aligned}$$

In general the (adjoint) symmetries of such a series will not be (semi-) canonical, i.e. they do not correspond to constants of the motion. This type of problem has also been studied by Magri [5,17], Fuchssteiner [12,37,64], Fuchssteiner and Fokas [8], Gel'fand and Dorfman [16] and others. However, our approach, using non-semi-canonical symmetries and working in a differential geometrical context is somewhat different.

We remarked already that every non-semi-canonical symmetry Z gives rise to a recursion operator for symmetries $\Lambda = \overset{\leftarrow}{\Omega} L_Z\Omega$. If we "use the symmetry Z twice" we can construct the recursion operators

$$\overset{\leftarrow}{\Omega}(L_Z\Omega)\overset{\leftarrow}{\Omega}(L_Z\Omega) = \Lambda^2$$

and

$$\overset{\leftarrow}{\Omega}(L_Z^2\Omega) .$$

In general these recursion operators will be different. However, for several "integrable" Hamiltonian systems, there exists a (non-semi-canonical) symmetry Z such that both operators are equal up to a multiplicative constant. The existence of such a symmetry is essential for the results of this section and the Sections 4.6 and 4.7. Hence the following

4.5.2 Hypothesis.

There exists a non-semi-canonical symmetry Z and a real number c with $c \neq (k-1)/k \quad \forall k \in \mathbb{N}$, such that

$$L_Z^2\Omega = c(L_Z\Omega)\overset{\leftarrow}{\Omega} L_Z\Omega . \tag{4.5.3}$$

□

The following theorem is easily proved by induction and Leibniz' rule.

4.5.4 Theorem.

Suppose Hypothesis 4.5.2 is satisfied. Then

$$L_Z^k\Omega = \prod_{j=0}^{k-1} (j(c-1)+1) \qquad (L_Z\Omega)(\overset{\leftarrow}{\Omega} L_Z\Omega)^{k-1} , \tag{4.5.5}$$

$$L_Z^k\overset{\leftarrow}{\Omega} = \prod_{j=0}^{k-1} (j(c-1)-1) \qquad (\overset{\leftarrow}{\Omega} L_Z\Omega)^k\overset{\leftarrow}{\Omega} \qquad , \quad k = 1,2,3,\ldots . \tag{4.5.6}$$

□

4.5.7 Corollary.

The SA operators $(L_Z\Omega)(\overset{\leftarrow}{\Omega}(L_Z\Omega))^k = (L_Z\Omega)\Lambda^k$ are cyclic for $k = 0,1,2,\ldots$.

Proof:

The Lie derivatives of a closed two-form are again closed. Hence the Lie derivatives of a cyclic tensor field are again cyclic. The condition on c in Hypothesis 4.5.2 implies that the numerical factor in (4.5.5) does not vanish, so the SA operators $(L_Z\Omega)(\overleftarrow{\Omega} L_Z\Omega)^k$ are also cyclic. □

4.5.8 Remark.

The Lie derivative of a canonical tensor field is not necessarily canonical. However, it can be shown by a long computation that, if Hypothesis 4.5.2 is satisfied, the AS operators $\overleftarrow{\Omega}((L_Z\Omega)\overleftarrow{\Omega})^k$ are canonical for k = 1,2,3,... . See also Fuchssteiner and Fokas [8]. Since we do not need this result in the sequel, we omit the proof. □

An infinite series of semi-canonical (adjoint) symmetries is now easily constructed. First the following

4.5.9 Lemma.

Suppose $\hat{\Phi}$ is a cyclic SA operator and Y a symmetry such that $L_Y\hat{\Phi} = 0$. Then the adjoint symmetry $\rho = \hat{\Phi}Y$ is semi-canonical.

Proof:

Denote the closed two-form corresponding to $\hat{\Phi}$ by $\hat{\phi}$. Then $\rho = \hat{\Phi}Y = i_Y\hat{\phi}$. Hence $d\rho = L_Y\hat{\phi} = 0$, so ρ is semi-canonical.

□

4.5.10 Corollary.

Suppose $\hat{\Phi}$ is a cyclic SA operator with $\dot{\hat{\Phi}} = 0$. Then $\rho = \hat{\Phi}X = \hat{\Phi}\overleftarrow{\Omega}dH$ is a semi-canonical adjoint symmetry with $\dot{\rho} = 0$.

□

The main result of this section now follows immediately from the Corollaries 4.5.7 and 4.5.10.

4.5.11 Theorem.

Suppose Hypothesis 4.5.2 is satisfied and suppose the symmetry $\dot{Z}$ is semi-canonical. Then

i) the symmetries X_k and corresponding adjoint symmetries ρ_k given in (4.5.1) are semi-canonical (so $d\rho_k = 0$),
ii) $[X_i, X_j] = 0$ for $i,j = 1,2,3,\ldots$,
iii) if the first cohomology group of M vanishes, there exists an infinite series of constants of the motion in involution $F_1 = H, F_2, F_3, \ldots$, defined by

$$dF_{k+1} = \rho_{k+1} = \Gamma^k dH \quad . \tag{4.5.12}$$

Proof:
Since $\dot{Z}$ is semi-canonical we have $\frac{\partial}{\partial t}(L_Z\Omega) = L_{\dot{Z}}\Omega = 0$. Then i) follows from (4.5.1) and the Corollaries 4.5.7 and 4.5.10. To prove ii) first note that

$$i_{X_i}\rho_j \equiv \langle\rho_j, X_i\rangle = \langle\Gamma^{j-1}dH, \overleftarrow{\Omega}\Gamma^{i-1}dH\rangle =$$

$$= \langle dH, \overleftarrow{\Omega}((L_Z\Omega)\overleftarrow{\Omega})^{i+j-2}dH\rangle =$$

$$= 0 \quad ,$$

because $\overleftarrow{\Omega}$ and $L_Z\Omega$ are both antisymmetric. Since ρ_i is closed (semi-canonical) this implies $L_{X_i}\rho_j = di_{X_i}\rho_j = 0$. Hence

$$[X_i, X_j] = L_{X_i}(\overleftarrow{\Omega}\rho_j) = (L_{X_i}\overleftarrow{\Omega})\rho_j = 0 \quad ,$$

because X_i is semi-canonical. If the first cohomology group of M vanishes semi-canonical symmetries are canonical, i.e. correspond to a constant of the motion. Finally $\{F_i, F_j\} = \langle\sigma_i, X_j\rangle = 0$, so the constants of the motion F_i are in involution.

□

Note that the constants of the motion F_k cannot depend explicitly on t:

$$\dot{F}_k = -\{F_k, H\} = -\{F_k, F_1\} = 0 \quad \text{for } k = 1,2,3,\ldots .$$

This also means that the vector fields $X_k = \overleftarrow{\Omega}\rho_k = \overleftarrow{\Omega}dF_k$ are Hamiltonian vector fields on M with Hamiltonian F_k and symplectic form ω (canonical tensor field $\overleftarrow{\Omega}$). It is easily verified that $L_{X_k}\overleftarrow{\Omega} = 0$ and $L_{X_k}L_Z\Omega = 0$. This means that $L_{X_k}\Lambda = 0$. Since $\dot{\Lambda} = \overleftarrow{\Omega}L_{\dot{Z}}\Omega = 0$ this implies that Λ is also a recursion operator for symmetries of the Hamiltonian systems $\dot{u} = X_k \equiv \overleftarrow{\Omega}dF_k$.

4.5.13 Remark.

Note that we did not prove that the constants of the motion constructed in Theorem 4.5.11 are analytically independent. For instance, it may happen that $F_k = 0$ for $k > k_0$ or that $F_k = f_k H$, $f_k \in \mathbb{R}$. This last situation occurs if $L_Z\Omega = f\Omega$ for some $f \in \mathbb{R}$. On a symplectic manifold of dimension 2n there can only exist n analytically independent functions in involution. Hence for a finite-dimensional Hamiltonian system every finite subset of constants of the motion of the series F_k which contains more than n elements must be analytically dependent.

□

4.6 INFINITE SERIES OF NON-SEMI-CANONICAL SYMMETRIES

In the preceding section we considered a series of symmetries X_k which was constructed by applying powers of the recursion operator $\Lambda = \overleftarrow{\Omega}L_Z\Omega$ to the symmetry $X = \overleftarrow{\Omega}dH$. A completely different series of symmetries Z_k and corresponding adjoint symmetries τ_k can be defined by

(4.6.1) $$Z_k = \Lambda^{k-1}Z = (\overleftarrow{\Omega}L_Z\Omega)^{k-1}Z \qquad (\text{so } Z_1 \equiv Z) ,$$

(4.6.2) $$\tau_k = \Omega Z_k = ((L_Z\Omega)\overleftarrow{\Omega})^{k-1}\Omega Z , \qquad \text{for } k = 1,2,3,\ldots .$$

In this section we shall describe some properties and applications of the series of (adjoint) symmetries Z_k (τ_k). First the following

4.6.3 Lemma.

i) Suppose Φ is a cyclic (0,2) tensor field and Ψ a canonical (2,0) tensor field. Define $\tilde{\Lambda} = \Psi\Phi \in T^1_1(M)$. Then

$$L_{\tilde{\Lambda}A}\Psi = -\Psi(L_A\Phi)\Psi \qquad \forall A \in X(M) .$$

ii) Suppose Φ is a cyclic (0,2) tensor field and $\tilde{\Lambda}$ is a (1,1) tensor field such that $\Phi\tilde{\Lambda}$ (in coordinates $\Phi_{ik}\tilde{\Lambda}^k_j$) is again a cyclic (0,2) tensor field. Then

$$L_{\tilde{\Lambda}A}\Phi = L_A(\Phi\tilde{\Lambda}) \qquad \forall A \in X(M) .$$

Proof:
Both results are easily proved using local coordinates or (for infinite-dimensional systems) using the expressions given in (1.2.9).

□

4.6.4 Remark.

For an arbitrary antisymmetric (0,2) tensor field Φ and an antisymmetric (2,0) tensor field Ψ it can be shown that there exists a (2,1) tensor field Ξ_1 such that

$$L_{\tilde{\Lambda}A}\Psi = -\Psi(L_A\Phi)\Psi + \Xi_1 A \qquad \forall A \in X(M) .$$

Also for an arbitrary (0,2) tensor field Φ and a (1,1) tensor field $\tilde{\Lambda}$ such that $\Phi\tilde{\Lambda}$ is again antisymmetric, there exists a (0,3) tensor field Ξ_2 such that

$$L_{\tilde{\Lambda}A}\Phi = L_A(\Phi\tilde{\Lambda}) + \Xi_2 A \qquad \forall A \in X(M) .$$

These two formulas resemble (2.6.5), where we considered $L_{\tilde{\Lambda}A}\tilde{\Lambda}$. The tensor fields Ξ_1 and Ξ_2 correspond to the Nijenhuis tensor field Ξ, introduced in Lemma 2.6.4. In fact the preceding lemma gives sufficient conditions for the vanishing of the "generalized Nijenhuis tensor fields" Ξ_1 and Ξ_2.

□

Now we return to the symmetries Z_k and corresponding adjoint symmetries τ_k, defined in (4.6.1) and (4.6.2).

4.6.5 Theorem.

Suppose Hypothesis 4.5.2 is satisfied. Then for k = 1,2,3,...

$$L_{Z_k}\Omega = c_k(L_Z\Omega)(\Omega^{\leftarrow}(L_Z\Omega))^{k-1} = c_k(L_Z\Omega)\Lambda^{k-1} , \qquad (4.6.6)$$

$$L_{Z_k}\Omega^{\leftarrow} = -c_k(\Omega^{\leftarrow}(L_Z\Omega))^k\Omega^{\leftarrow} = -c_k\Lambda^k\Omega^{\leftarrow} , \qquad (4.6.7)$$

with $c_k = k(c-1) + 2 - c$.

Proof:
Using Lemma 4.6.3 ii) and Corollary 4.5.7 we obtain

$$L_{Z_k}\Omega = L_{\Lambda Z_{k-1}}\Omega = L_{Z_{k-1}}(\Omega\Lambda) = \dots$$

$$= L_Z(\Omega\Lambda^{k-1}) = L_Z(\Omega(\Omega^{\leftarrow}L_Z\Omega)^{k-1}) .$$

Writing out this term with Leibniz' rule and Hypothesis 4.5.2 results in (4.6.6).

□

Using Lemma 4.6.3 it is also possible to compute more complicated derivatives of Ω and $\Omega^{\leftarrow}$ (for instance $L_{Z_1}^{m_1} L_{Z_2}^{m_2} \dots L_{Z_k}^{m_k}\Omega$). Since we do not need those derivatives in the sequel we shall not work them out further. Note that the case $m_1 \neq 0$, $m_i = 0$ for $i = 2,\dots,k$ is already treated in Theorem 4.5.4. We only mention that

$$L_{Z_k}L_Z\Omega = c_{k+1}(L_Z\Omega)(\Omega^{\leftarrow}(L_Z\Omega))^k . \qquad (4.6.8)$$

As a simple consequence of the Theorems 4.6.5 and 4.2.11 we have

4.6.9 Corollary.

Suppose Hypothesis 4.5.2 is satisfied. Then the symmetry Z_k is non-semi-canonical iff $\Lambda^k \neq 0$.

□

So if the recursion operator for symmetries Λ is not nilpotent, the series $Z_k(\tau_k)$ consists of non-semi-canonical (adjoint) symmetries.

We now show that the symmetries Z_k can be used to give a multi-Hamiltonian form of the Hamiltonian system (4.1.2). Using (4.6.6) and Theorem 4.5.11 iii) we obtain

$$(4.6.10) \qquad (L_{Z_k}\Omega)X = c_k \Gamma^k \, dH = c_k \, dF_{k+1} \qquad \text{for } k = 1,2,3,\dots \,.$$

(Always $\Lambda = \overset{\leftarrow}{\Omega} L_Z\Omega$ and $\Gamma = (L_Z\Omega)\overset{\leftarrow}{\Omega}$.) The cyclic operator $L_{Z_k}\Omega$ corresponds to the closed two-form

$$(4.6.11) \qquad L_{Z_k}\omega = d\, i_{Z_k}\omega = d(\Omega Z_k) = d\tau_k \,.$$

The expressions (4.6.10) and (4.6.11) lead to the following

4.6.12 Theorem.

Suppose the conditions of Theorem 4.5.11 are satisfied and suppose the first cohomology group of M vanishes. If moreover $L_Z\Omega$ is invertible (injective), the vector field X is the Hamiltonian vector field corresponding to the (weak) symplectic form $L_{Z_k}\omega = d\tau_k$ and the Hamiltonian $c_k F_{k+1}$ for $k = 1,2,3,\dots$. If $L_Z\Omega$ is invertible, we can write X as

$$X = (L_{Z_k}\Omega)^{-1} d(c_k F_{k+1}) \qquad k = 1,2,3,\dots \,.$$

Proof:
In view of (4.6.10) and (4.6.11) we only have to show that $L_{Z_k}\Omega$ is invertible (injective). This follows at once form (4.6.6).

□

If all the symplectic forms $d\tau_k$ are essentially different (i.e. not equal up to a multiplicative constant), the system is sometimes called a *multi-Hamiltonian system*.

We end this section by computing the Lie bracket of the elements of the series Z_k. Recall the definition of the Nijenhuis tensor, given in Section 2.6.

4.6.13 Lemma.

Suppose Hypothesis 4.5.2 is satisfied. Then the Nijenhuis tensor of $\Lambda = \overleftarrow{\Omega} L_Z \Omega$ vanishes.

Proof:
It follows from Lemma 4.6.3 that

$$L_{\Lambda A}\overleftarrow{\Omega} = -\overleftarrow{\Omega}(L_A L_Z \Omega)\overleftarrow{\Omega}$$

and

$$L_{\Lambda A} L_Z \Omega = L_A((L_Z\Omega)\overleftarrow{\Omega}(L_Z\Omega)) \qquad \forall A \in X(M) \ .$$

Then a simple calculation shows that

$$(4.6.14) \qquad L_{\Lambda A}\Lambda = \Lambda\, L_A\, \Lambda \qquad \forall A \in X(M) \ .$$

This means that the Nijenhuis tensor of Λ vanishes.

□

If Hypothesis 4.5.2 is satisfied, it is easily seen that

$$(4.6.15) \qquad L_Z \Lambda = L_Z(\overleftarrow{\Omega} L_Z \Omega) = (c-1)\Lambda^2 \ .$$

For the series of symmetries X_k and Z_k this implies

4.6.16 Theorem.

Suppose the conditions of Theorem 4.5.11 are satisfied. Then

i) $[X_k, X_\ell] = 0$,

ii) $[Z_k, Z_\ell] = (c-1)(\ell - k)Z_{k+\ell}$.

Proof:
Since $\hat{Z}$ is canonical we have $\frac{\partial}{\partial t}(L_Z\Omega) = L_{\hat{Z}}\Omega = 0$, which implies that $\dot{\Lambda} = 0$. The theorem now follows from Lemma 4.6.13, (4.6.15), Theorem 2.6.14 i) and ii) and Remark 2.6.16.

□

Note that part i) of this theorem was also proved in Theorem 4.5.11. The only Lie bracket which remains is $[Z_k, X_\ell]$. This bracket will be considered in the next section.

It will be clear that recursion operators for symmetries which have a vanishing Nijenhuis tensor field and which can be written as the product of a canonical AS operator and a cyclic SA operator play an important role in these considerations. This has already been noticed by Magri [5] and Fuchssteiner and Fokas [8]. There last mentioned authors speak of heriditary symmetries which admit a symplectic-implectic factorization. In theorem 2.3.13 we have seen that, for a finite-dimensional system, the eigenvalues of a recursion operator for symmetries are constants of the motion. If the recursion operator is of the form described above, more results on the corresponding eigenvalue problems can be given, see Ten Eikelder [47].

4.7 INFINITE SERIES OF CONSTANTS OF THE MOTION II

In Section 4.5 we constructed an infinite series of (adjoint) symmetries $X_k(\rho_k)$, using a recursion operator for (adjoint) symmetries $\Lambda = \overleftarrow{\Omega} L_Z \Omega$ ($\Gamma = (L_Z\Omega)\overleftarrow{\Omega}$). Under certain conditions (given in Theorem 4.5.11) the series X_k consists of canonical symmetries and corresponds to a series of constants of the motion F_k. In the first part of this section we shall describe an alternative way to construct such a series. Then we shall consider the possible relations between the two series and finally we describe a third method to generate series of constants of the motion.

Suppose Z is some symmetry. Then define the series of symmetries $\tilde{X}_k$ and corresponding adjoint symmetries $\tilde{\rho}_k$ by

$$(4.7.1) \qquad \tilde{X}_k = L_Z^{k-1} X \, , \qquad (\text{so } \tilde{X}_1 = X,\ \tilde{X}_{k+1} = [Z, \tilde{X}_k])$$

$$(4.7.2) \qquad \tilde{\rho}_k = \Omega \tilde{X}_k \quad .$$

If the symmetry Z is semi-canonical, it follows from (4.4.5) and (4.4.6) that the symmetries $\tilde{X}_k$ are canonical and correspond to the constants of the motion $(-1)^{k-1} L_Z^{k-1} H$. (If Z is canonical this also follows from Theorem 4.4.9.) From now on we shall assume that Z is non-semi-canonical. Then the symmetries given in (4.7.1) are not necessarily (semi-) canonical. It is easily seen that $\tilde{X}_2 = [Z,X] = \dot{Z}$. So if the series $\tilde{X}_k$ has to consist of (semi-) canonical symmetries, at least $\dot{Z}$ has to be (semi-) canonical. Of course this method of generating (nontrivial) symmetries is only possible if $\dot{Z} \neq 0$. We now give conditions on Z which imply that the series $\tilde{X}_k$ consists of semi-canonical symmetries. First the following

4.7.3 Lemma.

Suppose Hypothesis 4.5.2 is satisfied and suppose the symmetry $\dot{Z}$ is semi-canonical. Then $L_X L_Z^k \omega = 0$ or equivalently $L_X L_Z^k \Omega = 0$ for k = 0,1,2,... .

Proof:
The AS operator $\overleftarrow{\Omega}$ does not depend on t, hence $L_X \overleftarrow{\Omega} = 0$ (see Theorem 4.2.5). Since $\dot{Z}$ is a semi-canonical symmetry, Theorem 4.2.11 implies that $\frac{\partial}{\partial t}(L_Z \Omega) = L_{\dot{Z}}\Omega = 0$. This means that the SA operator $L_Z\Omega$ satisfies $L_X L_Z \Omega = 0$. The lemma now follows form Theorem 4.5.4 and Leibniz' rule.

□

4.7.4 Theorem.

Suppose the conditions of Lemma 4.7.3 are satisfied. Then

i) the symmetries $\tilde{X}_k$ given in (4.7.1) are semi-canonical,
ii) if the first cohomology group of M vanishes, there exists an infinite series of constants of the motion G_k, defined by

(4.7.5) $$dG_k = \tilde{\rho}_k = \Omega\tilde{X}_k = \Omega L_Z^{k-1} X \ , \quad k = 1,2,3,\dots .$$

Proof:
From $\tilde{\rho}_k = \Omega\tilde{X}_k = i_{\tilde{X}_k}\omega$ and the closedness of ω we obtain $d\tilde{\rho}_k = di_{\tilde{X}_k}\omega = L_{\tilde{X}_k}\omega$. Writing out this term with (1.1.24) finally results in

$$d\tilde{\rho}_k = \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} L_Z^{k-j-1} L_X L_Z^j \omega \ .$$

Now Lemma 4.7.3 implies that $d\tilde{\rho}_k = 0$, so the (adjoint) symmetries $\tilde{X}_k$ $(\tilde{\rho}_k)$ are semi-canonical.

□

4.7.6 Remark.

It is easily verified by the same method as in the proof of Theorem 4.7.4 that the adjoint symmetries $(L_Z\Omega)\tilde{X}_k$ (and also $(L_Z^\ell\Omega)\tilde{X}_k$) are also semi-canonical.

□

Note that the conditions for the existence of the series of constants of the motion F_k, as described in Theorem 4.5.11 and G_k, as described in the preceding theorem, are identical. We now consider the relation between the two series. The symmetries corresponding to F_2 and G_2 are given by

$$X_2 = \Omega^{\leftarrow} dF_2 = \Lambda X \quad ,$$

$$\tilde{X}_2 = \Omega^{\leftarrow} dG_2 = [Z,X] \; .$$

For several equations (Korteweg-de Vries, sine-Gordon,...) it turns out that

$$(4.7.7) \qquad \tilde{X}_2 = bX_2 \; , \qquad \text{for some } b \in \mathbb{R} \; , \; b \neq -1 \; .$$

If this relation holds the symmetries X_k and $\tilde{X}_k$ (and so the corresponding constants of the motion F_k and G_k) also differ only by a multiplicative constant.

4.7.8 Theorem.

Suppose a symmetry Z as described in Hypothesis 4.5.2 exists. If (4.7.7) is satisfied, then

$$(4.7.9) \qquad [Z_k, X_\ell] = ((\ell - 1)(c - 1) + b)X_{k+\ell} \; .$$

Proof:
We first show that $\dot{Z} = \tilde{X}_2$ is canonical. If $b = 0$ this follows from (4.7.7). If $b \neq 0$ we obtain from (4.4.5) and (4.4.6) that

$$\Omega^{\leftarrow} dL_Z H = X_2 + \tilde{X}_2 = \left(\frac{1}{b} + 1\right)\tilde{X}_2 = \left(\frac{1}{b} + 1\right)\dot{Z} \; .$$

Since $b \neq -1$ this implies that $\dot{Z}$ is canonical. So $\dot{\Lambda} = \Omega^{\leftarrow} L_{\dot{Z}}\Omega = 0$. The theorem now follows from Lemma 4.6.13, (4.6.15) and Theorem 2.6.14 iii) (see also Remark 2.6.16).

□

4.7.10 Corollary.

Let the conditions of Theorem 4.7.8 be satisfied. Then

$$\tilde{X}_k = \prod_{j=0}^{k-2} (j(c - 1) + b)X_k \; .$$

Proof:
By construction $\tilde{X}_k = L_Z^{k-1}X \equiv L_{Z_1}^{k-1}X_1$. The result now follows from the preceding theorem.

□

For the corresponding constants of the motion F_k and G_k this means

(4.7.11) $$G_k = \prod_{j=0}^{k-2} (j(c-1) + b)\, F_k \; .$$

Finally we describe a third method for generating series of constants of the motion. The two previously considered methods consisted in generating an infinite series of symmetries (in Section 4.5 using a recursion operator $\Lambda = \overleftarrow{\Omega} L_Z\Omega$, in the first part of this section by computing the repeated Lie bracket with Z). Then the problem was to show that these series consist of canonical symmetries. Another method (method i) in Section 4.4) is to generate a series of constants of the motion of the form $L_Z^k H$. For every symmetry Z this yields immediately a (possibly trivial) series of constants of the motion. Note that in this approach it is not necessary to reconstruct a constant of the motion from its corresponding symmetry. Under certain conditions this series is identical to the previously generated ones.

4.7.12 Theorem.

Let Hypothesis 4.5.2 be satisfied. If also (4.7.7) holds then

(4.7.13) $$L_Z^{k-1} H = f_k F_k \; , \qquad k = 1,2,3,\ldots$$

with $f_k = \prod_{j=0}^{k-2} (j(c-1) + b + 1)$.

Proof:
The proof is done by induction. Since $f_1 = 1$ and $F_1 = H$ the result holds for $k = 1$. Next assume (4.7.13) holds for $k = \ell$. So

(4.7.14) $$\overleftarrow{\Omega}\, d\, L_Z^{\ell-1} H = f_\ell X_\ell \; .$$

Application of L_Z then gives

$$(L_Z\overleftarrow{\Omega})\, d\, L_Z^{\ell-1} H + \overleftarrow{\Omega}\, d\, L_Z^{\ell} H = f_\ell [Z, X_\ell] \; .$$

Substitution of (4.7.14) and using (4.7.9) then gives

$$-\overleftarrow{\Omega}(L_Z\Omega)f_\ell X_\ell + \overleftarrow{\Omega}\,d\,L_Z^\ell H = f_\ell((\ell-1)(c-1) + b)X_{\ell+1} .$$

Hence

$$\overleftarrow{\Omega}\,d\,L_Z^\ell H = f_\ell((\ell-1)(c-1) + b + 1)X_{\ell+1} ,$$

and this implies (4.7.13) for $k = \ell + 1$.

□

Finally we summarize *some* of the results obtained in Sections 4.5, 4.6 and 4.7. Several results were obtained under the assumption that a symmetry Z as described in Hypothesis 4.5.2 exists and that $\dot{Z}$ is a semi-canonical symmetry. In the proof of Theorem 4.7.8 we showed that this last property is also a consequence of (4.7.7). Then if Hypothesis 4.5.2 and also (4.7.7) are satisfied:

i) $$[X_k, X_\ell] = 0 \qquad , \qquad (X_1 = X = \overleftarrow{\Omega}dH,\ Z_1 = Z)$$

$$[Z_k, Z_\ell] = (c-1)(\ell-k)Z_{k+\ell} \qquad ,$$

$$[Z_k, X_\ell] = ((\ell-1)(c-1) + b)X_{k+\ell} ,$$

ii) the symmetries X_k are canonical and correspond to constants of the motion in involution F_k with

$$L_Z^{k-1}H = f_k F_k , \qquad f_k \text{ given in Theorem 4.7.12 } ,$$

iii) if $\Lambda^k \neq 0$ the symmetry Z_k is non-semi-canonical.

4.8 SEMI-HAMILTONIAN SYSTEMS

In this section we consider the case that the dynamical system (4.1.2) is only a semi-Hamiltonian system. This means that the canonical tensor field $\overleftarrow{\Omega}$ (corresponding to the Poisson structure on M) is not necessarily invertible. Hence the inverse tensor field Ω may not exist. We shall now shortly describe the consequences for the theory given in this chapter and also show how several results can be maintained.

As far as $\overleftarrow{\Omega}$ is concerned, Lemma 4.2.3 and Theorem 4.2.5 remain valid. This means that $\overleftarrow{\Omega}$ is again an AS operator. Hence also for a semi-Hamiltonian system every constant of the motion F gives rise to a symmetry $\overleftarrow{\Omega}dF$. The definition of a (non-) (semi-) canonical symmetry Y was given in terms of the (not-) closedness/exactness of the corresponding adjoint symmetry $\sigma = \Omega Y$. Since in this case Ω is not available, these definitions are not possible for a symmetry.

4.8.1 Remark.

For *adjoint* symmetries the definitions of (non-) (semi-) canonical (see Definition 4.2.8) are still possible. In view of Theorem 4.2.11 one could try to define a (non-) semi-canonical *symmetry* Y as a symmetry with $L_Y\overleftarrow{\Omega} = 0$ ($L_Y\overleftarrow{\Omega} \neq 0$). Then by Lemma 4.2.3 every semi-canonical adjoint symmetry σ (so $d\sigma = 0$) gives rise to a semi-canonical symmetry $Y = \overleftarrow{\Omega}\sigma$. However, a non-semi-canonical adjoint symmetry σ (so $d\sigma \neq 0$) can also give rise to a semi-canonical symmetry $Y = \overleftarrow{\Omega}\sigma$. An (extremely trivial) example of this situation is provided by the case that $\overleftarrow{\Omega} = 0$. Hence this definition is not very useful.

□

Of course the first part of Theorem 4.2.13 also holds in this case, so $L_Z^k\overleftarrow{\Omega}$ is a (possibly vanishing) AS operator.

Next we consider the Sections 4.5, 4.6 and 4.7. In these sections we often used a recursion operator for (adjoint) symmetries which was obtained from a symmetry Z by $\Lambda = \overleftarrow{\Omega} L_Z\Omega$ ($\Gamma = (L_Z\Omega)\overleftarrow{\Omega}$). For a semi-Hamiltonian system Ω may not exist, so we cannot compute its Lie derivative $L_Z\Omega$. Hence in this case a (in the Hamiltonian case non-semi-canonical) symmetry does not give rise to a recursion operator for symmetries. However, sometimes (at least for the Sawada-Kotera equation) there exists a cyclic SA operator Φ which "behaves like $L_Z\Omega$". This property is stated more precisely in

4.8.2 Hypothesis.

For the semi-Hamiltonian system (4.1.2) there exists a nontrivial cyclic SA-operator Φ and a symmetry Z such that

(4.8.3) $$L_Z\overleftarrow{\Omega} = -\overleftarrow{\Omega}\,\Phi\,\overleftarrow{\Omega} ,$$

(4.8.4) $$L_Z\Phi = c\,\Phi\,\overleftarrow{\Omega}\,\Phi , \qquad c \in \mathbb{R}, \quad c \neq (k-1)/k \quad \forall k \in \mathbb{N} .$$

□

If $\overleftarrow{\Omega}$ is invertible (i.e. the Hamiltonian case), we obtain from (4.8.3) that $\Phi = L_Z\Omega$. Then (4.8.4) means that Z is a symmetry as described in Hypothesis 4.5.2. In fact several results of the Sections 4.5, 4.6 and 4.7 remain valid if $L_Z\Omega$ is replaced by an SA operator Φ as described in Hypothesis 4.8.2. In these sections we anticipated on this by writing most expressions in terms of $\overleftarrow{\Omega}$ and $L_Z\Omega$, avoiding Ω as much as possible. For instance instead of (4.5.5) we now obtain

$$(4.8.5) \qquad L_Z^k\Phi = \prod_{j=1}^{k} (j(c-1)+1)\,\Phi\,(\overleftarrow{\Omega}\Phi)^k , \qquad k = 1,2,3,\dots .$$

This implies that the SA operators $\Phi(\overleftarrow{\Omega}\Phi)^k$ are cyclic for $k = 0,1,2,\dots$ (compare with Corollary 4.5.7). We shall now mention some other results of the Sections 4.5, 4.6 and 4.7 which also hold for semi-Hamiltonian systems. Define the series of (adjoint) symmetries

$$(4.8.6) \qquad \begin{aligned} X_k &= (\overleftarrow{\Omega}\Phi)^{k-1}X, & \rho_k &= (\Phi\overleftarrow{\Omega})^{k-1}dH , \\ Z_k &= (\overleftarrow{\Omega}\Phi)^{k-1}Z, & \tau_{k+1} &= \Phi Z_k , \\ \tilde{X}_k &= L_Z^{k-1}X , & \tilde{\rho}_{k+1} &= \Phi X_k \qquad \text{for } k = 1,2,3,\dots . \end{aligned}$$

Then for a semi-Hamiltonian system we can prove the following

4.8.7 <u>Theorem.</u>

Suppose Hypothesis 4.8.2 is satisfied. Then, with $L_Z\Omega$ replaced by Φ and except for the statements which say that certain symmetries are (non-) (semi-) canonical, we have

i) if $\dot{\Phi} = 0$ the results of Theorem 4.5.11 remain true,
ii) if $(\overleftarrow{\Omega}\Phi)^k\overleftarrow{\Omega} \neq 0$ then $d\tau_k \neq 0$ (compare with Corollary 4.6.9),
iii) Lemma 4.6.13 remains true,
iv) if $\dot{\Phi} = 0$ then the results of Theorem 4.6.16 remain true,
v) if $\dot{\Phi} = 0$ then $d\tilde{\rho}_{k+1} = 0$ (compare with Remark 4.7.6),
vi) if $\dot{\Phi} = 0$ and $\tilde{X}_2 = bX_2$ ($b \neq -1$) then the results of Theorem 4.7.8, Corollary 4.7.10 and Theorem 4.7.12 also remain true.

□

CHAPTER 5 EXAMPLES

5.1 INTRODUCTION

In this chapter we shall apply the theory, described in the preceding chapters, to several well-known differential equations. Most of these equations have been extensively studied in recent years. However, we obtain some results which, as far as we know, are new and give also different proofs of already known properties. As an example of the theory of Chapter 2 (i.e. dynamical systems which are not necessarily Hamiltonian) we consider in Section 5.2 the Burgers equation. All the other examples are Hamiltonian and semi-Hamiltonian (Section 5.7) systems. In Section 5.3 we consider the most general form of a finite-dimensional linear Hamiltonian system. For such a system several additional results can be obtained. A simple example of a nonlinear finite-dimensional Hamiltonian system is provided by the Kepler problem. This will be discussed in Section 5.4. Then in Section 5.5 we consider the Benjamin-Ono equation. The most extensive example of this chapter will be the Korteweg-de Vries equation, discussed in Section 5.6. In fact the theory of the Sections 4.5, 4.6 and 4.7 has been written with this equation in mind. The final example is the Sawada-Kotera (-Caudrey-Dodd-Gibbon) equation. This equation will be considered as a semi-Hamiltonian system. Recall that some modifications of the theory of Chapter 4 for the case of a semi-Hamiltonian system have been indicated in Section 4.8.

Note that the examples given in the Sections 5.3 and 5.4 are finite-dimensional systems, while all the other examples concern infinite-dimensional systems. The differential geometrical methods used in the Chapters 2, 3 and 4 have only a sound foundation if the manifold M is finite-dimensional. So at first sight the results of the preceding chapters can only be used to investigate finite-dimensional systems. However, all the infinite-dimensional systems can be considered on a manifold which is a topological vector space. The only example of this chapter which cannot be considered on a manifold which is a vector space is the (finite-dimensional) Kepler problem. In Section 1.2 we showed how several differential geometrical objects can also be introduced on a (possibly infinite-dimensional) topological vector space W. Using the results of that section we can also investigate the mentioned infinite-dimensional systems.

In the remaining part of this section we show which form the conditions for the various introduced objects take for dynamical and Hamiltonian systems on a topological vector space W. Note that the following considerations do not apply to Section 5.4, the Kepler problem. Suppose X is a vector field on W, so it is (represented by) a possibly nonlinear mapping $X : W \to W$. Then we can consider in W the dynamical system

(5.1.1) $$\dot{u} = X(u) .$$

The following theorem describes (adjoint) symmetries and operators between symmetries for the system (5.1.1).

5.1.2 Theorem.

Consider the parameterized vector field $Y : W \times \mathbb{R} \to W$, the parameterized one-form $\sigma : W \times \mathbb{R} \to W^*$ and parameterized tensor fields Φ, Λ, Γ, Ψ of the same type as in (1.2.3). Then:

i) Y is a symmetry of (5.1.1) iff

(5.1.3) $$\dot{Y}(u,t) + Y'(u,t)X(u) - X'(u)Y(u,t) = 0 ,$$

ii) σ is an adjoint symmetry of (5.1.1) iff

(5.1.4) $$\dot{\sigma}(u,t) + \sigma'(u,t)X(u) + X'^*(u)\sigma(u,t) = 0,$$

iii) Φ is an SA operator for (5.1.1) iff

(5.1.5) $$\dot{\Phi}(u,t) + (\Phi'(u,t)X(u)) + \Phi(u,t)X'(u) + X'^*(u)\Phi(u,t) = 0 ,$$

iv) Λ is a recursion operator for symmetries of (5.1.1) iff

(5.1.6) $$\dot{\Lambda}(u,t) + (\Lambda'(u,t)X(u)) + \Lambda(u,t)X'(u) - X'(u)\Lambda(u,t) = 0,$$

v) Γ is a recursion operator for adjoint symmetries of (5.1.1) iff

(5.1.7) $$\dot{\Gamma}(u,t) + (\Gamma'(u,t)X(u)) - \Gamma(u,t)X'^{*}(u) + X'^{*}(u)\Gamma(u,t) = 0,$$

vi) Ψ is an AS operator for (5.1.1) iff

(5.1.8) $$\dot{\Psi}(u,t) + (\Psi'(u,t)X(u)) - \Psi(u,t)X'^{*}(u) - X'(u)\Psi(u,t) = 0 .$$

All these expressions are assumed to vanish for all $u \in \mathcal{W}$ and $t \in \mathbb{R}$.

Proof:

Using (1.2.9) it is easily seen that all these expressions are equivalent to the corresponding expressions in chapter 2.

□

Suppose u(t) is a solution of (5.1.1). The equation, obtained by linearizing (5.1.1) around u(t) is

(5.1.9) $$\dot{v}(t) = X'(u(t))v(t) \qquad v(t) \in \mathcal{W}.$$

This equation can be considered as an equation for the "variation" $v(t) = \delta u(t)$ of u(t). Similar equations were considered in (2.2.2) (using a local trivialization of the manifold) and in (2.8.11) (differential equation on the tangent bundle). Suppose $Y(u,t)$ is a symmetry of (5.1.1). Then it is easily seen that $v(t) = Y(u(t),t)$ is a solution of (5.1.9). So symmetries can be interpreted as solutions of the linearized equation (5.1.9), which can be expressed in u and t. In fact we can even use this property to find symmetries. The adjoint equation of (5.1.9) is given by

(5.1.10) $$w(t) = - X'^{*}(u(t))w(t) \qquad w(t) \in \mathcal{W}^{*}.$$

Let $\sigma(u,t)$ be an adjoint symmetry of (5.1.1). Then it is easily verified that $w(t) = \sigma(u(t),t)$ satisfies (5.1.10). So adjoint symmetries σ can be considered as solutions of the "adjoint linearized equation" (5.1.10), which can be written in terms of u and t.

5.1.11 Remark.

Sometimes we shall meet (adjoint) symmetries which do not depend explicitly on t. For symmetries and adjoint symmetries of that type (autonomous symmetries) the first terms in (5.1.3) and (5.1.4) vanish. Almost all recursion operators for (adjoint) symmetries, SA- and AS operators which we shall use in the sequel, do not depend explicitly on t (autonomous operators). So for these operators the first terms in (5.1.5), (5.1.6), (5.1.7) and (5.1.8) also vanish.

□

5.1.12 Remark.

In Section 5.3 we shall meet symmetries of the form $Y(u,t) = \hat{Y}u$ and adjoint symmetries of the form $\sigma(u,t) = \hat{\sigma}u$ where $\hat{Y} : W \to W$ and $\hat{\sigma} : W \to W^*$ are linear operators. In that case the derivatives are easily found: $\dot{Y}(u,t) = 0$, $Y'(u,t) = \hat{Y}$ and $\dot{\sigma}(u,t) = 0$, $\sigma'(u,t) = \hat{\sigma}$. In Section 5.3 we also use recursion operators for (adjoint) symmetries, SA- and AS operators which do not depend explicitly on u and t (i.e. constant operator fields). An SA operator of this type is $\Phi(u,t) = \Xi$ where $\Xi : W \to W^*$ is a linear operator. For operators of this type the derivatives with respect to u and t vanish. This means that in the Conditions (5.1.5), (5.1.6), (5.1.7) and (5.1.8) the first two terms are zero.

□

In section 3.2 we considered a closed two-form and the corresponding tensor field(s). In Definition 3.2.9 we introduced cyclic tensor fields and canonical tensor fields. The corresponding conditions can be written in terms of Lie derivatives. In the case that $M = W$, a topological vector space, these conditions can be simplified somewhat.

5.1.13 Theorem.

An antisymmetric tensor field $\Phi \in T^0_2(W)$ (=antisymmetric operator field $\Phi(u) : W \to W^*$) is cyclic iff

(5.1.14) $$\langle(\Phi'(u)A)B,C\rangle + \langle(\Phi'(u)B)C,A\rangle + \langle(\Phi'(u)C)A,B\rangle = 0$$

for all A, B, C, $u \in W$.

Proof:
By Theorem 3.2.10 an antisymmetric tensor field is cyclic iff the corresponding two-form is closed. Then this theorem follows at once from definition

1.2.10 iii) (and Theorem 1.2.11).

□

5.1.15 Theorem.

An antisymmetric tensor field $\Psi \in T^2_0(W)$ (= antisymmetric operator field $\Psi(u) : W^* \to W$) is canonical iff

$$(5.1.16) \qquad \langle\alpha,(\Psi'(u)(\Psi(u)\beta))\gamma\rangle + \langle\beta,(\Psi'(u)(\Psi(u)\gamma))\alpha\rangle + \langle\gamma,(\Psi'(u)(\Psi(u)\alpha))\beta\rangle = 0$$

for all α, β, $\gamma \in W^*$ and $u \in W$.

Proof:
The tensor field Ψ is canonical if the corresponding (3,0) tensor field $\hat{\Xi}$ (see Theorem 3.2.4 ii) vanishes. Substitution of $L_{\Psi\alpha}\beta$, as given in (1.2.9), in (3.2.8) yields that $\hat{\Xi}$ vanishes iff (5.1.16) is satisfied.

□

5.1.17 Remark.

It is easily seen from (5.1.14) and (5.1.16) that antisymmetric operators $\Phi : W \to W^*$ and $\Psi : W^* \to W$, considered as constant operator fields (i.e. $\Phi(u)$ and $\Psi(u)$ do not depend on u) always satisfy (5.1.14) respectively (5.1.16). Hence every antisymmetric operator $\Phi : W \to W^*$ is cyclic (so the corresponding two-form is closed) and every antisymmetric operator $\Psi : W^* \to W$ is canonical.

□

The fact that W is a topological vector space has also consequences for the relation between semi-canonical and canonical symmetries. In section 1.2 we have seen that a closed one-form α on W is also exact. The corresponding function F on W such that $\alpha = dF$ was given in (1.2.12). Of course these results also hold if α (and hence F) depend on a parameter (t). In terms of (adjoint) symmetries this means that *semi-canonical (adjoint) symmetries are canonical (adjoint) symmetries* and that *non-canonical (adjoint) symmetries* are *non-semi canonical* *(adjoint) symmetries*. So we can omit the prefix "semi" in these notions.

Finally we make some remarks on the notation and terminology in this chapter. In the preceding chapters we used the notation and terminology of differential geometry. We shall also do this in this chapter, with a few exceptions. If W is infinite-dimensional, the exterior derivative of a function (functional) $F : W \to \mathbb{R}$ is the one-form $dF(u) = F'(u)$, as introduced in Definition 1.2.10 i). In cases where the duality map between W and W^* is given by the L_2 innerproduct (all our infinite-dimensional examples), the derivative of F is frequently denoted as $\frac{\delta F}{\delta u}$ (or $\frac{\delta F(u)}{\delta u}$) instead of $F'(u)$. This expression is called the *variational derivative* of F. In all sections except 5.3 and 5.4 we shall use this notation, so $dF(u)$ will be replaced by $\frac{\delta F}{\delta u}$.

The derivative of various parameterized objects with respect to the parameter (t) has always been indicated by a dot, for instance $\dot{Y}(u,t) = \frac{\partial}{\partial t} Y(u,t)$ (derivative of a vector field to the parameter). However, when dealing with partial differential equations, derivatives with respect to t (x,y,...) are very often indicated by a subscript t (x,y,...). Apart from Sections 5.3 and 5.4 we shall also use this notation. So the derivative of a parameterized vector field with respect to the parameter will be written as

$$Y_t(u,t) = \frac{\partial}{\partial t} Y(u,t) \ ,$$

a dynamical system (Korteweg-de Vries equation) will be written as

$$u_t = X(u) = 6uu_x - u_{xxx} \ .$$

5.2 THE BURGERS EQUATION

This equation was used by Burgers [48,49] in 1939 in a model for turbulent fluid motion. It is the simplest possible equation which describes both nonlinear and diffusion effects. The Burgers equation arises in many places in physics, particularly in problems where shock waves are involved (see for instance Whitham [32]). We shall study it in the form

(5.2.1) $$u_t = X(u) = 2uu_x + u_{xx} \ , \qquad x \in \mathbb{R} \ .$$

Various other forms of the Burgers equation can be reduced to (5.2.1), using transformations of the dependent and independent variables. A transformation which relates (5.2.1) to the diffusion equation was found in 1950 by Hopf [50] and in 1951 by Cole [51] . This so-called *Hopf-Cole transformation* is given by

$$(5.2.2) \qquad v = f(u) = e^{\partial^{-1}u} ,$$

$$(5.2.3) \qquad u = f^{\leftarrow}(v) = \frac{v_x}{v} .$$

The corresponding evolution equation for v is given by

$$(5.2.4) \qquad v_t = f'(u)X(u) = \tilde{X}(v) = v_{xx} \quad , x \in \mathbb{R} \qquad (u = f^{\leftarrow}(v)).$$

Various methods are available for solving this linear equation. Suppose we take an initial value $u_0 \in S_1$ (see definition 1.3.2) at $t = t_0$. Then, using the relation with (5.2.4), it can be shown that the corresponding solution $u(.,t) \in S_1$ for $t \geq t_0$. Therefore we shall study (5.2.1) in the space S_1. Define the function space $\hat{U}_1 = U_1 \oplus \mathbb{R} = \{u \in C^\infty(\mathbb{R}) \mid u(x) = v(x) + a, v \in U_1, a \in \mathbb{R}\}$. A duality map between S_1 and $\hat{U}_1$ is given by

$$\langle \alpha, A \rangle = \int_{-\infty}^{\infty} \alpha(x) \, A(x) dx \qquad \forall \alpha \in \hat{U}_1, A \in S_1 .$$

Then, similar to theorem 1.3.14, we introduce topologies on S_1 and $\hat{U}_1$ such that $S_1^* = \hat{U}_1$ and $\hat{U}_1^* = S_1$.

We shall now study symmetries and adjoint symmetries for (5.2.1). Since we consider (5.2.1) on a topological vector space, a symmetry Y is (can be considered as) a mapping $Y : S_1 \times \mathbb{R} \to S_1$ which satisfies (5.1.3). The derivative mapping of X in the point u is given by

$$(5.2.5) \qquad X'(u) = 2u\partial + 2u_x + \partial^2 : S_1 \to S_1 .$$

Substitution in (5.1.3) yields

$$Y_t(u,t) + Y'(u,t)\,(2uu_x + u_{xx}) - (2u\partial + 2u_x + \partial^2)\, Y(u,t) = 0,$$

$$\forall u \in S_1, \forall t \in \mathbb{R}.$$

Two simple solutions of this equation are

(5.2.6) $\qquad Y(u,t) = X_0(u) = u_x$ and $Y(u,t) = Z_0(u,t) = u + xu_x + 2t(2uu_x+u_{xx})$.

Note that indeed $X_0 : S_1 \to S_1$ and $Z_0 : S_1 \times \mathbb{R} \to S_1$. Both symmetries have a simple geometrical interpretation. The equation (5.2.1) is invariant for translations along the x-axis. If $u(x,t)$ is a solution of (5.2.1), then $u(x + \varepsilon,t)$ is also a solution of (5.2.1) for all $\varepsilon \in \mathbb{R}$. The difference between these two solutions is given by

$$u(x+\varepsilon,t) - u(x,t) = \varepsilon\, u_x(x,t) + O(\varepsilon^2) \qquad \text{for } \varepsilon \to 0.$$

This implies that $X_0(u) = u_x$ is a solution of the linearized equation and hence a symmetry (see (5.1.9)). The symmetry Z_0 is related to the scaling properties of (5.2.1). It is easily seen that, if $u(x,t)$ satisfies (5.2.1), the function $au(ax,a^2t)$ also satisfies (5.2.1) for all $a \in \mathbb{R}$. By setting $a = 1 + \varepsilon$ and taking the limit $\varepsilon \to 0$ we find that the difference between the two solutions is given by

$$(1+\varepsilon)u((1+\varepsilon)x,(1+\varepsilon)^2t) - u(x,t) = \varepsilon(u(x,t) + xu_x(x,t) + 2tu_t(x,t)) + O(\varepsilon^2).$$

So $Z_0(u,t) = u + xu_x + 2t(2uu_x + u_{xx})$ is a solution of the linearized equation of (5.2.1) and hence a symmetry (see (5.1.9)).

Recursion operators for symmetries of (5.2.1) can easily be found by using the relation with the linear equation (5.2.4). Suppose we consider the equation (5.2.4) in some linear space W. An autonomous recursion operator for symmetries of (5.2.4) is a linear operator $\tilde{\Lambda}(v) : W \to W$, defined for all $v \in W$, such that (see (5.1.6) and remark 5.1.11)

(5.2.7) $\qquad (\tilde{\Lambda}'(v)\, \tilde{X}(v)) + \tilde{\Lambda}(v)\tilde{X}'(v) - \tilde{X}'(v)\tilde{\Lambda}(v) = 0 \qquad \forall\, v \in W$,

where $\tilde{X}(v) = v_{xx}$ and $\tilde{X}'(v) = \partial^2$. It is easily verified that $\tilde{\Lambda}(v) = \partial$ satisfies this condition.

5.2.8 Remark.

Symmetries of (5.2.4) satisfy the linearized equation of (5.2.4). Since (5.2.4) is a linear equation, symmetries are solutions of (5.2.4). Suppose $w(x,t)$ satisfies (5.2.4), then also $w_x(x,t)$ satisfies (5.2.4). This mapping corresponds to the recursion operator $\tilde{\Lambda} = \partial$.

□

Using the transformations (5.2.2) and (5.2.3) we can *formally* transform $\tilde{\Lambda}$ to a recursion operator Λ for symmetries of (5.2.1). By theorem 2.7.6 the operator Λ is given by

$$\Lambda(u) = f^{\leftarrow\prime}(v)\tilde{\Lambda}(v)f'(u) \qquad (v=f(u))$$

(5.2.9)
$$= \left(\frac{\partial}{v} - \frac{v_x}{v^2}\right) \partial e^{\partial^{-1}u}\partial^{-1}$$

$$= \partial + \partial u \partial^{-1}.$$

5.2.10 Theorem.

The operator $\Lambda(u) = \partial + \partial u \partial^{-1}$ is a recursion operator for symmetries of (5.2.1).

Proof:

It is easily seen that $\Lambda(u) : S_1 \to S_1$. We have to show that Λ satisfies (5.1.6). Since Λ does not depend on t, this becomes

(5.2.11) $$(\Lambda'(u)X(u)) + \Lambda(u)X'(u) - X'(u)\Lambda(u) = 0 \qquad \forall\, u \in S_1.$$

Recall that the derivative of $\Lambda(u)$ in $u \in S_1$ is a bilinear operator $\Lambda'(u) : S_1 \times S_1 \to S_1$. Inserting one fixed function $A \in S_1$ this derivative reduces to the linear operator

$$(\Lambda'(u)\,A) = \partial A \partial^{-1} : S_1 \to S_1.$$

So the first term of (5.2.11) is the linear operator

$$(\Lambda'(u)X(u)) = \partial(2uu_x + u_{xx})\partial^{-1}.$$

Using (5.2.5) the other terms of (5.2.11) can be found. Then a tedious computation shows that Λ satisfies (5.2.11).

□

This recursion operator for symmetries was already given by Olver [13]. Starting with the symmetries X_0 and Z_0 given in (5.2.6), we can construct two infinite series of symmetries

$$(5.2.12) \qquad X_k = \Lambda^k X_0 \ , \quad Z_k = \Lambda^k Z_0 \qquad k = 1,2,3,\dots \ .$$

The first few elements of these series are given by

$$(5.2.13) \qquad \begin{aligned} X_0 &= u_x \ , \\ X_1 &= 2uu_x + u_{xx} \ , \\ X_2 &= 3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx} \ , \\ Z_0 &= u + xu_x + 2t(2uu_x + u_{xx}) \ , \\ Z_1 &= u^2 + 2u_x + x(2uu_x + u_{xx}) + 2t(3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx}) \ . \end{aligned}$$

Note that $X_1 = X = u_t$; this symmetry is related to the invariance of (5.2.1) for translations along the t-axis. Some properties of the two series of symmetries are given in the following

5.2.14 Theorem.

The symmetries X_k and Z_k can be written as

$$\begin{aligned} X_k(u) &= \partial r_k(u,u_x,\dots) \ , \\ Z_k(u,t) &= s_k(u,u_x,\dots) + xX_k(u) + 2tX_{k+1}(u) \ , \quad k = 0,1,2,\dots \ , \end{aligned}$$

where $r_k(u,u_x,\dots)$ and $s_k(u,u_x,\dots)$ are polynomials in u and its first k derivatives.

Proof:
The recursion operator Λ can be written as $\Lambda = \partial(\partial+u)\partial^{-1}$. Hence $\Lambda^k = \partial(\partial+u)^k\partial^{-1}$. This implies

$$X_k = \partial(\partial+u)^k\partial^{-1}u_x = \partial(\partial+u)^k u.$$

So $r_k(u,u_x,\ldots) = (\partial+u)^k u$. In the same way we obtain

$$\begin{aligned} Z_k &= \partial(\partial+u)^k\partial^{-1}(u+xu_x+2tX_1) \\ &= \partial(\partial+u)^k xu + 2tX_{k+1} \\ &= (x\partial(\partial+u)^k + (\partial+u)^k + k\partial(\partial+u)^{k-1})u + 2tX_{k+1} \\ &= xX_k + s_k(u,u_x,\ldots) + 2tX_{k+1}. \end{aligned}$$

with $s_k(u,u_x,\ldots) = (\partial+u)^k u + k\partial(\partial+u)^{k-1}u$.

□

5.2.15 Remark.

The symmetries X_k are mappings $X_k : S_1 \to S_1$ (vector fields on S_1). So we can study the evolution equations

(5.2.16) $$u_t = X_k(u) = \Lambda^{k-1}X(u) \qquad k = 1,2,3,\ldots .$$

By formally applying the (derivative of) the transformation (5.2.2) we obtain

$$\begin{aligned} v_t &= f'(u)u_t \qquad (u = f^{\leftarrow}(v)) \\ &= f'(u)\Lambda^{k-1}(u)X(u) \\ &= (f'(u)\Lambda(u)f^{\leftarrow\prime}(v))^{k-1}f'(u)X(u) \\ &= \tilde{\Lambda}^{k-1}(v)\tilde{X}(v) \\ &= \partial^{k+1}v. \end{aligned}$$

So, using the Hopf-Cole transformation, we can transform (5.2.16) into the linear equation

(5.2.17) $$v_t = \partial^{k+1} v.$$

Note that (with appropriate boundary conditions) (5.2.17) is a Hamiltonian system if k is even ($k = 2\ell$; $\overleftarrow{\Omega} = \partial$, $H(v) = (-1)^{\ell} \frac{1}{2} \int_{-\infty}^{\infty} (\partial^{\ell} v)^2 dx$). If k is odd, say $k = 2\ell+1$, then (5.2.17) is an equation of "diffusion type" if ℓ is even and an equation of "anti-diffusion type" if ℓ is odd. Similar properties hold for the corresponding nonlinear equations (5.2.16).

□

In (5.2.12) we gave two infinite series X_k and Z_k of symmetries for the Burgers equation. We now consider the various Lie brackets between the elements of both series. One possible way for computing these Lie brackets is to transform to the linear equation (5.2.4) and compute the Lie brackets of the corresponding symmetries of (5.2.4). This method is possible because for the Burgers equation a linearizing transformation (Hopf-Cole) is known. However, a straightforward computation using the methods described in Section 2.6 is also possible. We shall follow this second method. Recall that the Lie derivative of the recursion operator Λ is given by (see (1.2.9))

$$L_A \Lambda(u) = (\Lambda'(u)A(u)) + \Lambda(u)A'(u) - A'(u)\Lambda(u) .$$

A long computation shows that

(5.2.18) $$L_{Z_0} \Lambda = \Lambda ,$$

and that

(5.2.19) $$[Z_0, X_0] = X_0 .$$

5.2.20 Theorem.

The Lie brackets between the elements of the series of symmetries X_k and Z_k are given by

$$[X_k, X_\ell] = 0 \quad ,$$

$$[Z_k, X_\ell] = (\ell + 1)X_{k+\ell} \quad ,$$

$$[Z_k, Z_\ell] = (\ell - k)Z_{k+\ell} \quad , \qquad k,\ell = 0,1,2,\ldots \; .$$

The Lie derivatives of the recursion operator for symmetries Λ are

$$L_{X_k}\Lambda = 0 \quad ,$$

$$L_{Z_k}\Lambda = \Lambda^{k+1} \quad , \qquad k = 0,1,2,\ldots \; .$$

Proof:
A simple computation shows that $L_{\Lambda A}\Lambda = \Lambda L_A\Lambda$ for all $A(u) \in S_1$. Hence the Nijenhuis tensor of Λ vanishes. The theorem now follows immediately from (5.2.18), (5.2.19), the Theorems 2.6.11 and 2.6.14 and Remark 2.6.15 (with a = 1, b = 2).

□

Similar algebra's of symmetries can easily be constructed for the higher order Burgers equations given in (5.2.16).

Next we turn to adjoint symmetries for the Burgers equation. The function (functional)

$$F(u) = \int_{-\infty}^{\infty} u(x)\,dx$$

is a constant of the motion of (5.2.1). This function is differentiable, $\frac{\delta F}{\delta u} = 1 \in \hat{U}_1$. So $\sigma(u) = \frac{\delta F}{\delta u} = 1$ is an adjoint symmetry of (5.2.1). A recursion operator for adjoint symmetries is given by

$$\Gamma(u) = \Lambda^*(u) = (\partial + \partial u \partial^{-1})^* = -\partial + \partial^{-1}u\partial : \hat{U}_1 \to \hat{U}_1 \; .$$

Since $\Gamma(u)\sigma(u) = 0$ we cannot construct a series of adjoint symmetries by using the recursion operator Γ. We did not find adjoint symmetries which were essentially different from σ.

Finally we mention that Taflin [53] has shown that in a suitable function space the Burgers equation can be considered as a Hamiltonian system which has an infinite series of constants of the motion. However, the used function space is much smaller than the space S_1 which we use. Therefore Taflin's result cannot be obtained in S_1.

5.3 A FINITE DIMENSIONAL LINEAR HAMILTONIAN SYSTEM

Suppose $\mathcal{W}$ is a finite-dimensional (real) linear space with dimension 2n; so $\mathcal{W}$ is isomorphic to $\mathbb{R}^{2n}$. The dual space of $\mathcal{W}$ is denoted by $\mathcal{W}^*$. In this section we shall consider a linear Hamiltonian system on the space $\mathcal{W}$. Some general remarks on dynamical systems and Hamiltonian systems on a linear space have been made in section 5.1. Let ω be a symplectic form on $\mathcal{W}$ such that the corresponding operator $\Omega(u) : \mathcal{W} \to \mathcal{W}^*$ does not depend on u. So

$$\omega(A,B) = \langle \Omega A, B \rangle \qquad \forall\, A,B \in \mathcal{W},$$

where $\Omega : \mathcal{W} \to \mathcal{W}^*$ is a linear antisymmetric operator. Since ω is nondegenerate, the operator Ω is invertible. The inverse operator $\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}$ is also a linear antisymmetric operator. Suppose $H : \mathcal{W} \to \mathbb{R}$ is a homogeneous quadratic function. Then there exists a unique symmetric operator $\hat{H} : \mathcal{W} \to \mathcal{W}^*$ such that

$$H(u) = \tfrac{1}{2}\langle \hat{H}\, u, u \rangle .$$

The corresponding one-form is $dH(u) = \hat{H}\, u$. Then the Hamiltonian system on the symplectic space $(\mathcal{W},\omega)$ with Hamiltonian H is given by

$$\dot{u} = \overleftarrow{\Omega}\, \hat{H}\, u. \tag{5.3.1}$$

With $\hat{X} = \overleftarrow{\Omega}\hat{H} : \mathcal{W} \to \mathcal{W}$, this system can also be written as

$$\dot{u} = X(u) = \hat{X}u. \tag{5.3.2}$$

In Theorem 3.4.1 we described a variational principle for a Hamiltonian system. At first sight this theorem provides us with a variational principle on a neighbourhood $\mathcal{U}_0$ of some point $u_0 \in M = W$. However, in this case the manifold M is a linear space. This means the second cohomology group of M vanishes, so every closed two-form is exact. Hence the one-form α, such that $\omega = -d\alpha$, exists on the whole space $M = W$. It is easily seen that $\alpha(u) = -\frac{1}{2}\Omega u$. Then (similar to Theorem 3.4.1) a solution $\tilde{u}(t)$ of (5.3.1) is a stationary point of

$$(5.3.3) \qquad \int_{t_1}^{t_2} (\tfrac{1}{2}\langle\Omega\dot{u},u\rangle - \tfrac{1}{2}\langle\hat{H}u,u\rangle)\, dt$$

over the set of all curves $u(t)$ in W with $u(t_1) = \tilde{u}(t_1)$ and $u(t_2) = \tilde{u}(t_2)$.

Note that for every inital value $u(t_0) = u_0 \in W$ the differential equation (5.3.1) has a unique solution $u(t) \in W$ which exists for all $t \in \mathbb{R}$

$$u(t) = e^{(t-t_0)\overleftarrow{\Omega}\hat{H}} u_0 = e^{(t-t_0)\hat{X}} u_0 .$$

This suggests that a finite-dimensional linear Hamiltonian sustem is completely integrable. The complete integrability of such a system has already been proved by Williamson [65]. A different proof has been given by Kocak [71].

In the remaining part of this section we shall first consider constants of the motion, (adjoint) symmetries and operators between those symmetries for the Hamiltonian system (5.3.1). The existence of these objects turns out to be related with the existence of operators Ξ which satisfy the condition (5.3.5). Then we shall make some remarks on the space of operators satisfying (5.3.5). Finally we show how the theory described in section 4.5 , can be applied in this example.

Suppose $F : W \to \mathbb{R}$ is a homogeneous quadratic function. Then there exists a symmetric operator $\Xi: W \to W^*$ such that

$$(5.3.4) \qquad F(u) = \tfrac{1}{2} \langle\Xi u,u\rangle \quad .$$

The function F is a constant of the motion if $L_X F = \langle dF,X\rangle = 0$ on W. This means

$$\langle\Xi\overleftarrow{\Omega}\hat{H}u,u\rangle = 0 \qquad \forall\, u \in W .$$

This condition is satisfied iff $\Xi\overleftarrow{\Omega}\hat{H}$ is an antisymmetric operator. Since Ξ and $\hat{H}$ are symmetric and $\overleftarrow{\Omega}$ is antisymmetric, this is equivalent to

(5.3.5) $$\Xi\overleftarrow{\Omega}\hat{H} - \hat{H}\overleftarrow{\Omega}\Xi = 0 \ .$$

This condition can also be written in the following two equivalent ways

(5.3.6) $$[\overleftarrow{\Omega}\Xi, \overleftarrow{\Omega}\hat{H}\underline{]} = [\overleftarrow{\Omega}\Xi, \hat{X}\underline{]} = 0$$

and

(5.3.7) $$[\Xi\overleftarrow{\Omega}, \hat{H}\overleftarrow{\Omega}\underline{]} = 0,$$

where $[.,.\underline{]}$ is the *commutator of two linear operators*. The linear space of operators $\Xi : W \to W^*$ which satisfy (5.3.5) will be denoted by E. The canonical adjoint symmetry and the canonical symmetry, corresponding to the constant of the motion (5.3.4) are given by $\rho(u) = dF(u) = \Xi u$ and $Y(u) = \overleftarrow{\Omega} dF(u) = \overleftarrow{\Omega}\Xi u$. The Poisson bracket of two constants of the motion $F_i(u) = \frac{1}{2} <\Xi_i u,u>$ $(i = 1,2)$ is easily found to be

(5.3.8) $$\begin{aligned} \{F_1,F_2\}(u) &= <dF_1(u), \overleftarrow{\Omega}\, dF_2(u)> \\ &= \tfrac{1}{2} <(\Xi_1\overleftarrow{\Omega}\Xi_2 - \Xi_2\overleftarrow{\Omega}\Xi_1)u,u> \quad . \end{aligned}$$

Thus we have proved

5.3.9 Theorem.

The function F, defined by (5.3.4) , with Ξ a symmetric operator, is a constant of the motion iff Ξ satisfies (5.3.5). The corresponding canonical (adjoint) symmetries are given by $\rho(u) = dF(u) = \Xi u$ and $Y(u) = \overleftarrow{\Omega} dF(u) = \overleftarrow{\Omega}\Xi\, u$. The Poisson bracket of two homogeneous quadratic functions $F_i(u) = \frac{1}{2} <\Xi_i u,u>$ is given by (5.3.8). It is again a homogeneous quadratic function, corresponding to the symmetric operator $\Xi_1\overleftarrow{\Omega}\Xi_2 - \Xi_2\overleftarrow{\Omega}\Xi_1$.

□

Next we study (adjoint) symmetries for (5.3.1).
Note that for all linear operators $\Xi: W \to W^*$, $\rho(u) = \Xi\, u$ is a one-form on W. This one-form is an adjoint symmetry if it satisfies (5.1.4). For a

one-form of this type this condition becomes (see also remark 5.1.12)

$$\Xi \hat{X} + \hat{X}^* \Xi = 0 . \tag{5.3.10}$$

Since $\hat{X}^* = -\hat{H}\Omega^{\leftarrow}$, this condition is equivalent to (5.3.5). Of course, in this case Ξ is not necessarily symmetric. Suppose $\hat{Y} : W \to W$ is a linear operator. The vector field $Y(u) = \hat{Y}u$ is a symmetry if it satisfies (5.1.3). For a vector field of this type this condition becomes

$$[\hat{X}, \hat{Y}] = 0 . \tag{5.3.11}$$

By setting $\hat{Y} = \Omega^{\leftarrow}\Xi$ we obtain again condition (5.3.5) for Ξ. Adjoint symmetries of the form $\rho(u) = \Xi u$ and symmetries of the form $Y(u) = \hat{Y}u = \Omega^{\leftarrow}\Xi u$ we shall call *linear (adjoint) symmetries*. The manifold W is a linear space, so its first cohomology group vanishes. This implies (see section 5.1) that canonical and semi-canonical (adjoint) symmetries are identical. It is easily seen that the linear (adjoint) symmetries above are canonical iff the operator Ξ is symmetric. The corresponding constant of the motion is then $F(u) = \frac{1}{2} \langle \Xi u, u \rangle$. Also a simple calculation shows that the Lie bracket of two linear symmetries $Y_i(u) = \hat{Y}_i u = \Omega^{\leftarrow}\Xi_i u$ $(i=1,2)$ is the linear symmetry

$$Y_3(u) = [Y_1, Y_2](u) = [\hat{Y}_2, \hat{Y}_1] u = \Omega^{\leftarrow}(\Xi_2\Omega^{\leftarrow}\Xi_1 - \Xi_1\Omega^{\leftarrow}\Xi_2)u . \tag{5.3.12}$$

Note that the first square bracket is the Lie bracket of two vector fields, while the second square bracket is the commutator of two linear operators. We summarize the results concerning linear symmetries in the following

5.3.13 Theorem.

The following three conditions are equivalent:

i) the linear operator $\Xi : W \to W^*$ satisfies (5.3.5), so $\Xi \in E$,

ii) the one-form $\rho(u) = \Xi u$ is a linear adjoint symmetry,

iii) the vector field $Y(u) = \hat{Y} u = \Omega^{\leftarrow}\Xi u$ is a linear symmetry.

These symmetries are canonical iff Ξ is a symmetric operator. The corresponding constant of the motion is given by $F(u) = \frac{1}{2}\langle \Xi u, u \rangle$. The Lie bracket of two linear symmetries $Y_i(u) = \hat{Y}_i u$ $(i=1,2)$ is the linear symmetry Y_3 given in (5.3.12).

□

The conditions for the four possible operators between (adjoint) symmetries are also easily derived. Consider the linear operator $\Lambda : \mathcal{W} \to \mathcal{W}$. This linear operator is a recursion operator for symmetries if it satisfies (5.1.6). Since Λ does not depend on u and t, this implies

(5.3.14) $\quad [\Lambda, \hat{X}] = 0$.

This relation is also easily obtained from (5.3.11).
Since $\overleftarrow{\Omega}$ is invertible, we can set $\Lambda = \overleftarrow{\Omega}\Xi$. Then the operator $\Xi : \mathcal{W} \to \mathcal{W}^*$ has to satisfy (5.3.5). The conditions for recursion operators for adjoint symmetries and for AS- and SA operators can be derived in a similar way. We summarize them in the following

5.3.15 Theorem.

Suppose $\Xi : \mathcal{W} \to \mathcal{W}^*$ is a linear operator. Then the following conditions are equivalent:

i) $\Lambda = \overleftarrow{\Omega}\Xi : \mathcal{W} \to \mathcal{W}$ is a recursion operator for symmetries,

ii) $\Gamma = \Xi\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}^*$ is a recursion operator for adjoint symmetries,

iii) Ξ is an SA operator,

iv) $\Psi = \overleftarrow{\Omega}\Xi\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}$ is an AS operator,

v) Ξ satisfies (5.3.5), so $\Xi \in E$.

If Ξ is antisymmetric, it is a cyclic operator and Ψ is a canonical operator.

Proof:
We showed already that i) and v) are equivalent. In a similar way it can be shown that each of the conditions ii), iii) and iv) is equivalent with v). The fact that antisymmetric operators $\Xi : \mathcal{W} \to \mathcal{W}^*$ are cyclic and antisymmetric operators $\Psi : \mathcal{W}^* \to \mathcal{W}$ are canonical was already explained in remark 5.1.17.

□

In the preceding part of this section we have discussed constants of the motion, (adjoint) symmetries and several operators between those symmetries. It is important to note that these objects not necessarily are of the considered type. For instance there may exist non-quadratic constants of the motion and symmetries which are not linear. The existence of objects of the discussed type was always related to the existence of a linear operator $\Xi : \mathcal{W} \to \mathcal{W}^*$, which satisfies (5.3.5). We shall now make some remarks on the linear space E of operators Ξ satisfying this condition. The following theorem describes some elementary properties of the space E.

5.3.16 Theorem.

i) E is a Lie algebra; if $\Xi_1, \Xi_2 \in E$, then also

$$\Xi_3 = \Xi_1 \overset{\leftarrow}{\Omega} \Xi_2 - \Xi_2 \overset{\leftarrow}{\Omega} \Xi_1 .$$

The set of symmetric operators $\Xi \in E$ is a subalgebra of E. This subalgebra is isomorphic with the Lie algebra of homogeneous quadratic constants of the motion. Further, if Ξ_1 and Ξ_2 are both antisymmetric, Ξ_3 is symmetric. If one of Ξ_1, Ξ_2 is symmetric and the other is antisymmetric, Ξ_3 is antisymmetric.

ii) If $\Xi_1, \Xi_2 \in E$, then also $\Xi_4 = \Xi_1 \overset{\leftarrow}{\Omega} \Xi_2 \in E$.

iii) If $\Xi \in E$, then also $\Xi^* \in E$.

□

It is easily seen that $\hat{H} \in E$ and $\Omega \in E$. So E always contains a symmetric operator and an antisymmetric operator. The fact that $\hat{H} \in E$ gives rise to the following

5.3.17 Corollary.

Suppose $\Xi_1 \in E$. Then $\Xi_2 = \Xi_1 \overset{\leftarrow}{\Omega} \hat{H} \in E$. If Ξ_1 is symmetric (antisymmetric), Ξ_2 is antisymmetric (symmetric).

□

Using this corollary we can construct the following series of elements of E

(5.3.18) $\qquad \Omega,\ \hat{H},\ \hat{H}\overleftarrow{\Omega}\hat{H},\ \hat{H}(\overleftarrow{\Omega}\hat{H})^2,\ \ldots$.

Note that the operators in this series are alternately antisymmetric and symmetric.

Suppose Ξ_1 is an antisymmetric element of E . If Ξ_1 is invertible, the closed two-form $\tilde{\omega}$, defined by

$$\tilde{\omega}(A,B) = \langle \Xi_1 A, B \rangle$$

is nondegenerate. By corollary 5.3.17 $\Xi_2 = \Xi_1\overleftarrow{\Omega}\hat{H}$ is a symmetric element of E . Hence $\tilde{H}(u) = \frac{1}{2} \langle \Xi_2 u, u\rangle$ is a quadratic constant of the motion. Then the differential equation (5.3.2) can also be considered as a Hamiltonian system on the symplectic space $(W,\tilde{\omega})$ with Hamiltonian $\tilde{H}$:

(5.3.19) $\qquad \dot{u} = \Xi_1^{-1}\, \Xi_2\, u.$

The variational principle corresponding to this Hamiltonian form of (5.3.2) is easily found (see also (5.3.3)). Suppose $\tilde{u}(t)$ is a solution of (5.3.2) (or (5.3.19). The the curve $\tilde{u}(t)$ in W is a stationary point of

(5.3.20) $\qquad \displaystyle\int_{t_1}^{t_2} \left(\tfrac{1}{2}\langle \Xi_1 \dot{u}, u\rangle - \tfrac{1}{2}\langle \Xi_2 u, u\rangle\right) dt$

over the set of all curves $u(t)$ in W with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$. If $\Xi_1 \neq a\Omega$ for some $a \in \mathbb{R}$, the two ways (5.3.1) and (5.3.19) of writing the differential equation (5.3.2) as a Hamiltonian system are essentially different and the system is bi-Hamiltonian. If the operator $\hat{X} = \overleftarrow{\Omega}\hat{H}$ is invertible, we can also start with an invertible symmetric operator $\Xi_2 \in E$. Then $\Xi_1 = \Xi_2\, \hat{X}^{-1}$ is an antisymmetric element of E and we can write the system again as (5.3.19). So, in the case $\hat{X}$ is invertible, any quadratic constant of the motion $\tilde{H}(u) = \frac{1}{2}\langle \Xi_2 u, u\rangle$, with Ξ_2 invertible, can be considered as Hamiltonian. The corresponding symplectic form is then

$$\tilde{\omega}(A,B) = \langle \Xi_2\, \hat{X}^{-1} A, B \rangle \qquad \forall\, A,B \in W.$$

Note that if Ξ_1 is an invertible symmetric element of E, we can write (5.3.2) also as

(5.3.21) $\dot{u} = \Xi_1^{-1}\ \Xi_1\ \overleftarrow{\Omega}\hat{H}u = \Xi_1^{-1}\ \Xi_2\ u.$

In this expression Ξ_1 is symmetric and Ξ_2 is antisymmetric!

Next we consider a basis for the Lie algebra E. Recall ((5.3.5) and (5.3.6)) that $\Xi \in E$ iff the operator $\hat{Y} = \overleftarrow{\Omega}\Xi$ is a commutator of $\hat{X} = \overleftarrow{\Omega}\hat{H}$.

5.3.22 Theorem.

Suppose $\hat{X} = \overleftarrow{\Omega}\hat{H}$ is invertible.Then a basis for E consists of the same number (=k) of symmetric and antisymmetric operators. So the dimension of the subalgebra of symmetric operators of E is half the dimension of the Lie algebra E .

Proof:

Suppose the operators $\Phi_1, \ldots, \Phi_\ell$ form a basis for E. Define the symmetric and antisymmetric parts of Φ_i by $\Phi_i^+ = \frac{1}{2}(\Phi_i + \Phi_i^*)$ and $\Phi_i^- = \frac{1}{2}(\Phi_i - \Phi_i^*)$. Then by theorem 5.3.16 iii) these (anti)symmetric operators are also elements of E . Clearly any element Ξ of E can be written as a linear combination of the 2ℓ operators $\Phi_i^\pm$ $(i=1,\ldots,\ell)$. We can reduce this set to a new basis $\Xi_1, \ldots, \Xi_\ell$ of E, which consists only of symmetric or anti-symmetric operators. Suppose $\Xi_1, \ldots, \Xi_k$ are symmetric and $\Xi_{k+1}, \ldots, \Xi_\ell$ are antisymmetric operators. By corollary 5.3.17 the operators $\Xi_i\overleftarrow{\Omega}\hat{H}$ $(i=1, \ldots, k)$ are antisymmetric. Since $\hat{X} = \overleftarrow{\Omega}\hat{H}$ is invertible, these operators are linearly independent. Hence $\ell - k \geq k$. In a similar way we can show $\ell - k \leq k$. So $\ell = 2k$ and the basis Ξ_i consists of k symmetric and k antisymmetric operators.

□

The symmetric operators $\Xi_i (i=1,\ldots,k)$ give rise to k quadratic constants of the motion $F_i(u) = \frac{1}{2}\langle\Xi_i u,u\rangle$. Every operator Ξ_i $(i=1,\ldots,2k)$ gives rise to a "*bilinear constant of the motion*". By this we mean a bilinear function $G : W \times W \to \mathbb{R}$ such that for every pair of solutions u(t), v(t) of (5.3.1), the function G(u(t), v(t)) is constant. These "bilinear constants of the motion" are given by

(5.3.23) $$G_i(u,v) = \tfrac{1}{2} \langle \Xi_i u, v \rangle \qquad i = 1,2, \ldots, 2k.$$

Note that $G_i(u,u) = F_i(u)$ for $i = 1, \ldots, k$ and $G_i(u,u) = 0$ for $i = k+1,\ldots,2k$.

If all the eigenvalues of $\hat{X} = \overleftarrow{\Omega}\hat{H}$ are different, a basis for the space of operators which commute with $\hat{X}$, is given by $\{ \hat{X}^i | i=0,1,\ldots,2n-1\}$. The corresponding basis for E is $\{\Omega\hat{X}^i | i = 0,1,\ldots,2n-1\}$. So in that case a basis for E consists of n symmetric operators

(5.3.24) $$\hat{H},\ \hat{H}(\overleftarrow{\Omega}\hat{H})^2,\ \ldots,\ \hat{H}(\overleftarrow{\Omega}\hat{H})^{2n-2},$$

and of n antisymmetric operators

(5.3.25) $$\Omega, \hat{H}\overleftarrow{\Omega}\hat{H},\ \ldots,\ \hat{H}(\overleftarrow{\Omega}\hat{H})^{2n-3} .$$

If $\hat{X}$ has eigenvalues which are degenerate, the dimension of the space of operators,which commute with $\hat{X}$, is higher then $2n(2k > 2n)$. A basis for E is then more complicated then the basis given (5.3.24) and (5.3.25).

We shall now show how the theory described in chapter 4, can be applied to the linear Hamiltonian system under consideration. In particular we shall construct an infinite series of constants of the motion, using the method described in Section 4.5. In Theorem 4.2.13 we have seen that with a non-semi-canonical symmetry Z corresponds an SA operator $L_Z\Omega$. For a linear symmetry of the form $Z(u) = \overleftarrow{\Omega}\Xi u$ $(\Xi \in E)$, this SA operator is given by

(5.3.26) $$L_Z\Omega = \Xi - \Xi^* .$$

The higher derivatives $L_Z^k\Omega$ are also SA operators. For $k = 2$ we obtain

(5.3.27) $$L_Z^2\, \Omega = (\Xi - \Xi^*)\overleftarrow{\Omega}\, \Xi - \Xi^*\overleftarrow{\Omega}(\Xi - \Xi^*).$$

In section 4.5 we considered the relation between the two SA operators $L_Z^2\Omega$ and $(L_Z\Omega)\overleftarrow{\Omega}L_Z\,\Omega$. In that section we have assumed that Hypothesis 4.5.2

is satisfied, i.e. there exists a non- (semi-) canonical symmetry Z such that

$$L_Z^2\Omega = c(L_Z\Omega)\overset{\leftarrow}{\Omega} L_Z\Omega$$

for some $c \in \mathbb{R}$ with $c \neq (k-1)/k$, $\forall k \in \mathbb{N}$.

In this case this condition becomes

(5.3.28) $$(\Xi - \Xi^*)\overset{\leftarrow}{\Omega}\Xi - \Xi^*\overset{\leftarrow}{\Omega}(\Xi - \Xi^*) = c(\Xi - \Xi^*)\overset{\leftarrow}{\Omega}(\Xi - \Xi^*) \ .$$

It is easy to see that every antisymmetric operator Ξ satisfies (5.3.28) with $c = 1$. The theory, described in Section 5 of the preceding chapter, leads to the following

5.3.29 Theorem.

Suppose $Z(u) = \overset{\leftarrow}{\Omega}\Xi u$ is a non-(semi-)canonical symmetry with Ξ antisymmetric. Then the adjoint symmetries defined by

(5.3.30) $$\sigma_{k+1}(u) = (\Xi\overset{\leftarrow}{\Omega})^k \hat{H}u \qquad k = 0,1,2,\ldots,$$

are canonical. The corresponding constants of the motion are given by

(5.3.31) $$F_{k+1}(u) = \tfrac{1}{2}< (\Xi\overset{\leftarrow}{\Omega})^k \hat{H}u, u > \quad .$$

These constants of the motion are in involution.

Proof:

Since Ξ is antisymmetric, Hypothesis 4.5.2 is satisfied with $Z(u) = \overset{\leftarrow}{\Omega}\Xi u$ and $c = 1$. The first cohomology group of $\mathcal{W}$ vanishes (see also section 5.1). So this theorem is a straightforward consequence of Theorem 4.5.11 iii).

□

5.3.32 Remark.

A straightforward proof of this theorem can be given in the following way. Define $\Xi_1 = \Xi$ and $\Xi_k = \Xi\overset{\leftarrow}{\Omega}\Xi_{k-1}$. Then by theorem 5.3.16 ii) the operators $\Xi_k \in E$. Since $\Xi_k = (\Xi\overset{\leftarrow}{\Omega})^{k-1}\Xi$ and Ξ is antisymmetric, the operators Ξ_k are also antisymmetric. Then, by corollary 5.3.17, the operator $\Xi_k \overleftarrow{\Omega} \hat{H}$ is a symmetric element of E. Hence σ_{k+1}, defined in (5.3.30), is a canonical adjoint symmetry. It is easily seen that the corresponding constants of the motion, given in (5.3.31), are in involution.

□

It is important to note that the alternative proof of theorem 5.3.29 , given in the preceding remark, depends essentially on the fact that we consider a linear equation in a linear space. *However, the methods described in chapter 4, can also be applied to a nonlinear equation on an arbitrary manifold.*

Theorem 5.3.29 can only be applied if a non-canonical symmetry $Z(u) = \overleftarrow{\Omega}\,\Xi\,u$ (so $\Xi \in E$) with Ξ antisymmetric, is known. A simple example is given by $\Xi = \hat{H}\overleftarrow{\Omega}\hat{H}$. Then the constants of the motion F_k are found to be

$$(5.3.33) \qquad F_{k+1}(u) = \tfrac{1}{2} < (\hat{H}\overleftarrow{\Omega})^{2k}\hat{H}u, u > \qquad k = 0,1,2,\ldots .$$

Note that these constants of the motion correspond to the symmetric operators of the series (5.3.18) and that the first n constants correspond to the operators given in (5.3.24). It is a simple exercise to show that $F_{k+1}(u) = (-1)^k\, H(\hat{X}^k u)$. Note that if u(t) is a solution of (5.3.1), then $v_k(t) = \hat{X}^k\, u(t)$ is also a solution of (5.3.1). Hence $F_{k+1}(u(t)) = (-1)^k \cdot H(v_k(t))$. So the constant of the motion F_k is, up to the sign, equal to the Hamiltonian, evaluated for a transformed solution.

Finally we remark that, since $M = W$ is a finite-dimensional linear space, the series F_k (k = 1,2,3,...) given in (5.3.31) or (5.3.33) cannot be analytically independent (see also Remark 4.5.13). For instance, if all the eigenvalues of $\hat{X} = \overleftarrow{\Omega}\hat{H}$ are different, only the first n constants of the motion given in (5.3.33), are analytically independent.

5.4 THE KEPLER PROBLEM

The Kepler problem is a simple example of a finite-dimensional completely integrable Hamiltonian system. It is well suited to illustrate several aspects of the theory described in the preceding chapters. In this section we shall discuss constants of the motion, (adjoint) symmetries and the various operators between these symmetries for the Kepler problem.

The configuration space of the two-dimensional Kepler problem is $Q = \mathbb{R}^2\backslash\{0\}$. On $\mathbb{R}^2$ we take the standard basis; the coordinates with respect to this basis we call q_1 and q_2. The phase space is $\tilde{M} = T^*Q = \mathbb{R}^2\backslash\{0\} \times \mathbb{R}^2$, with natural coordinates q_1, q_2, p_1, p_2. Note that a closed one-form on $\tilde{M}$ is not necessarily exact, so the first cohomology group of $\tilde{M}$ does not vanish.

5.4.1 Remark.

Formally an atlas for Q and also for T^*Q must consist of (at least) two charts. However, we shall use the more customary coordinates given above. □

The standard symplectic form on $\tilde{M}$ is then $\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$. Define the function $H : \tilde{M} \to \mathbb{R}$ by

$$H(q_1,q_2,p_1,p_2) = \tfrac{1}{2}(p_1^2 + p_2^2) - \frac{1}{(q_1^2 + q_2^2)^{\frac{1}{2}}} \quad . \tag{5.4.2}$$

The Hamiltonian system with Hamiltonian H on the symplectic manifold $(\tilde{M},\omega)$ is then

$$\begin{cases} \dot{q}_1 = p_1 & , \\ \dot{q}_2 = p_2 & , \\ \dot{p}_1 = -\dfrac{q_1}{(q_1^2 + q_2)^{3/2}} & , \\ \dot{p}_2 = -\dfrac{q_2}{(q_1^2 + q_q^2)^{3/2}} & . \end{cases} \tag{5.4.3}$$

These equations describe the motion of a unit-mass in the $\mathcal{Q}$-plane which is attracted by a fixed centre in (0,0) with a force which is inverse proportional to the square of the distance to the centre. The solutions of this system are well known. We shall restrict ourselves to periodic solutions, which means that we only consider solutions with $H < 0$. So instead of $\widetilde{M}$ we now take the phase space $M = \{(q_1,q_2,p_1,p_2) \in \widetilde{M} \mid H(q_1,q_2,p_1,p_2) < 0\}$. Of course we can also consider ω as a symplectic form on M and H as a function on M.

Several constants of the motion of the Hamiltonian system (5.4.3) are known. Besides the Hamiltonian H there are the constants of the motion:

$$(5.4.4) \qquad \begin{cases} L = q_1p_2 - q_2p_1 , \\ F_1 = p_2L - \dfrac{q_1}{(q_1^2+q_2^2)^{\frac{1}{2}}} , \\ F_2 = -p_1L - \dfrac{q_2}{(q_1^2+q_2^2)^{\frac{1}{2}}} . \end{cases}$$

The function L is the angular momentum of the system. The two functions F_1 and F_2 are the two non-zero components of the *Runge-Lenz vector* (sometimes called Laplace-Runge-Lenz vector).

5.4.5 Remark.

In vector notation in the Euclidian $\mathbb{R}^3$ the angular momentum L is the length of the angular momentum vector

$$\underline{L} = \underline{q} \times \underline{p} .$$

The Runge-Lenz vector is then given by

$$\underline{F} = \underline{p} \times \underline{L} - \frac{\underline{q}}{|\underline{q}|} .$$

□

The Poisson brackets between these constants of the motion are found to be

$$\{L,H\} = 0 \quad , \quad \{F_1,H\} = 0 \quad , \quad \{F_2,H\} = 0 \ ,$$

(5.4.6) $$\{F_1,L\} = -F_2 \quad , \quad \{F_2,L\} = F_1 \ ,$$

$$\{F_1,F_2\} = -2LH \ .$$

Of course the four constants of the motion H, L, F_1 and F_2 must be analytically dependent: The existence of four analytically independent *autonomous* constants of the motion for a system with a four-dimensional phase space would prohibit any evolution of the system. Indeed a simple computation shows that

$$F_1^2 + F_2^2 = 2L^2H + 1 \ .$$

This suggests to define a constant of the motion F by taking the polar-angle of the point (F_1,F_2) in the (F_1,F_2) plane. So $0 \leq F < 2\pi$. For $F_1 > 0$ and $F_2 > 0$ we have $F = \arctan\left(\frac{F_2}{F_1}\right)$. Note that, although F is discontinuous on the positive F_1 axis, its differential

(5.4.7) $$dF = \frac{1}{F_1^2 + F_2^2}\,(F_1 dF_2 - F_2 dF_1)$$

is a smooth one-form on M.

We now try to find a constant of the motion G which is analytically independent of H, L and F. This is only possible if G depends explicitly on t. Define the function K_1 on M by

(5.4.8) $$K_1 = -(p_1q_1 + p_2q_2)(-2H)^{\frac{1}{2}} - \operatorname{sgn}(p_1q_1 + p_2q_2)\arccos K_2 \ ,$$

where

(5.4.9) $$K_2 = \frac{-1 - 2(q_1^2 + q_2^2)^{\frac{1}{2}}H}{F_1^2 + F_2^2} \ .$$

(K_1 is related to one of the action-variables of the Kepler problem.) It is easily seen that K_1 is discontinuous at places where $p_1q_1 + p_2q_2 = 0$ and

$K_2 \neq 1$. Substitution of $p_1q_1 + p_2q_2 = 0$ in (5.4.9) yields $K_2 = = \mathrm{sgn}((q_1^2+q_2^2)^{\frac{1}{2}} - L^2)$. This implies that, when passing the submanifold $p_1q_1 + p_2q_2 = 0$ at a place where $(q_1^2+q_2^2)^{\frac{1}{2}} < L^2$, the function K_1 jumps by $\delta K_1 = 2\pi$. Note that this means that the corresponding one-form dK_1 is smooth everywhere on M. A simple computation shows that $\{K_1,H\} = (-2H)^{3/2}$. This implies that $G = K_1 - (-2H)^{3/2}t$ is a constant of the motion. A long computation shows that

$$dH \wedge dL \wedge dF \wedge dG = (-2H)^{3/2} dq_1 \wedge dq_2 \wedge dp_1 \wedge dp_2 \ . \tag{5.4.10}$$

So H, L, F and G are a set of four analytically independent constants of the motion.

5.4.11 Remark.

The constants of the motion F and G are not continuous everywhere on M. At certain places F and G jump by 2π. Of course we can also consider F and G as continuous "multivalued functions" on M. Note that the corresponding adjoint symmetries dF and dG are smooth (single-valued) one-forms on M. Two continuous (single-valued) constants of the motion are given by sin(F) and sin(G).

□

The adjoint symmetries corresponding to H, L, G and F will be denoted by

$$\rho_1 = dH \ , \quad \rho_2 = dL \ , \quad \rho_3 = dG \ , \quad \rho_4 = dF \ . \tag{5.4.12}$$

By construction ρ_1 and ρ_2 are canonical adjoint symmetries. It is easily verified that $d\rho_3 = 0$ and $d\rho_4 = 0$, so ρ_3 and ρ_4 are semi-canonical adjoint symmetries. Since it is not possible to write $\rho_3 = dG$ and $\rho_4 = dF$ for smooth (single-valued) functions F and G on M, the adjoint symmetries ρ_3 and ρ_4 are not-canonical. The corresponding symmetries will be denoted by $X_i = \overleftarrow{\Omega}\rho_i$ $(i = 1,2,3,4)$. So X_1 and X_2 are canonical symmetries, while X_3 and X_4 are semi-canonical symmetries which are not canonical.

5.4.13 Remark.

The situation as described above appears in any completely integrable Hamiltonian system (see Section 3.6). For such a system there exist canonical coordinates $\tilde{q}_i$, $\tilde{p}_i$, such that the Hamiltonian depends only on the $\tilde{p}_i$. The $\tilde{q}_i$ are coordinates on a torus, so they jump by 2π at certain places (or they are "multivalued-functions, this depends on the point of view). The functions $\tilde{p}_i$ and $\tilde{q}_i - t\frac{\partial H}{\partial \tilde{p}_i}$ are 2n constants of the motion. Note that the adjoint symmetries $d\tilde{p}_i$ are canonical while the adjoint symmetries $d\tilde{q}_i - td\frac{\partial H}{\partial \tilde{p}_i}$ are only semi-canonical.

□

In addition to the Poisson brackets given in (5.4.6) we mention that

$$\{G,H\} = \langle \rho_3, X_1 \rangle = (-2H)^{3/2} \quad ,$$

$$\{G,L\} = \langle \rho_3, X_2 \rangle = 0 \quad , \qquad (5.4.14)$$

$$\{G,F\} = \langle \rho_3, X_4 \rangle = 0 \quad .$$

It follows from (5.4.10) that the symmetries X_i $(i = 1,2,3,4)$ are linearly independent at every point of M. This means that every other symmetry of the Kepler problem can be expanded in these four symmetries. As an example we consider the symmetry which is related to the scale rule of the Kepler problem. It is easily verified that if $(q_1(t),q_2(t),p_1(t),p_2(t))$ is a solution of (5.4.3), then $(k^{-2}q_1(k^3t),k^{-2}q_2(k^3t),kp_1(k^3t),kp_2(k^3t))$ is also a solution for every $k \in \mathbb{R}$. By setting $k = 1 + \varepsilon$ and taking the limit for $\varepsilon \to 0$ we obtain an infinitesimal transformation of a solution into another solution. This yields the symmetry

$$Y = -2q_1 \frac{\partial}{\partial q_1} - 2q_2 \frac{\partial}{\partial q_2} + p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} + 3tX_1 \quad . \qquad (5.4.15)$$

The corresponding adjoint symmetry is

$$\sigma = \Omega Y = -p_1 dq_1 - p_2 dq_2 - 2q_1 dp_1 - 2q_2 dp_2 + 3t\,\rho_1 \quad . \qquad (5.4.16)$$

A simple computation shows that

(5.4.17) $\quad d\sigma = -\omega$,

which means that $Y(\sigma)$ is a non-semi-canonical (adjoint) symmetry. By setting $Y = \Sigma_{i=1}^{4} G_i X_i$ and computing the contractions with ρ_j (j = 1,2,3,4) we obtain a system of linear equations for the G_i. This finally results in the expansion

$$Y = (-2H)^{-\frac{1}{2}} X_3 - L X_4$$

and

(5.4.18) $\quad \sigma = (-2H)^{-\frac{1}{2}} \rho_3 - L\rho_4$.

Recursion operators for symmetries and for adjoint symmetries are also easily constructed (see also Section 2.5). The most general recursion operator for adjoint symmetries is given by

$$\Gamma = \sum_{i,j=1}^{4} G_{ij}\, \rho_i \otimes X_j$$

where the G_{ij} are constants of the motion. For instance

$$\Gamma = -(-2H)^{-3/2} \rho_2 \otimes X_3$$

is a recursion operator for adjoint symmetries such that

$$\Gamma\, dH = \rho_2 = dL \ , \quad \Gamma^2 dH = 0 \ .$$

In a similar way we can construct recursion operators for symmetries and SA- and AS operators. From (5.4.17) and (5.4.18) we obtain that

$$\omega = (-2H)^{-3/2} \rho_3 \wedge \rho_1 - \rho_4 \wedge \rho_2 \ .$$

For the corresponding SA operator Ω this yields

$$\Omega = (-2H)^{-3/2} (\rho_3 \otimes \rho_1 - \rho_1 \otimes \rho_3) - \rho_4 \otimes \rho_2 + \rho_2 \otimes \rho_4 \ .$$

The inverse AS operator $\overleftarrow{\Omega}$ can be written as

$$\overleftarrow{\Omega} = (-2H)^{3/2} (X_1 \otimes X_3 - X_3 \otimes X_1) - X_2 \otimes X_4 + X_4 \otimes X_2 \ .$$

5.5 THE BENJAMIN-ONO EQUATION

Internal waves in a stratified fluid with infinite depth can be described by the Benjamin-Ono (BO) equation [55,56]. In fact the BO equation can be considered as a limit of a more general equation, which describes internal waves in a stratified fluid with finite depth. In the deep water and shallow water limit this equation reduces to the KdV-respectively the BO equation. We shall consider the BO equation in the form

$$u_t = 2uu_x + \mathcal{H}u_{xx} \qquad x \in \mathbb{R}, \tag{5.5.1}$$

where $\mathcal{H}$ is the *Hilbert transform*

$$\mathcal{H}u(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy \qquad \text{(principal value integral)}.$$

Multi-soliton solutions of this equation have been found by Matsuno [59] and by Chen, Lee and Pereira [60] . A single soliton solution with velocity $-c$ has the form

$$u(x,t) = \frac{2c}{1+c^2(x+ct)^2} \qquad c > 0. \tag{5.5.2}$$

We shall consider the BO equation in the space $S_p (0 < p < 1)$ with dual space U_p. Clearly the soliton solution given in (5.5.2) is an element of S_p. In theorem 1.4.10 we have proved that the Hilbert transform can be considered as a linear antisymmetric operator $\mathcal{H} : S_p \to U_p$. Several other properties of $\mathcal{H}$ are given in section 1.4. An infinite series of constants of the motion of the BO equation has been constructed by Nakamura [61] and by Bock and Kruskal [62] . The first elements of this series are

$$F_1^0(u) = \int_{-\infty}^{\infty} u\,dx \quad , \quad F_2^0(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx ,$$

$$F_3^0(u) = \frac{1}{3} \int_{-\infty}^{\infty} (u^3 + \tfrac{3}{2}u\mathcal{H}u_x)dx, \tag{5.5.3}$$

$$F_4^0(u) = \frac{1}{4} \int_{-\infty}^{\infty} (u^4+3u^2\mathcal{H}u_x + 2u_x^2)dx.$$

It is easily verified that the BO equation can be written in the form

$$(5.5.4) \qquad u_t = \partial \frac{\delta F_3^0}{\delta u} = \partial(u^2 + Hu_x) .$$

So we can consider the BO equation as a Hamiltonian system with Hamiltonian F_3^0 and canonical operator ∂. A simple calculation shows that the BO equation can also be written in the form

$$(5.5.5) \qquad u_t = \Psi(u) \frac{\delta F_2^0}{\delta u} = (\tfrac{2}{3} u\partial + \tfrac{2}{3} \partial u + \partial H \partial)u .$$

However, the antisymmetric operator $\Psi(u) = \frac{2}{3} u\partial + \frac{2}{3} \partial u + \partial H \partial : U_p \to S_p$ is not canonical. Hence (5.5.5) is not a (semi-) Hamiltonian form of the BO equation.

5.5.6 Remark.

The Korteweg-de Vries equation can be written in two forms ((5.6.6) and (5.6.15)), which strongly resemble (5.5.4) and (5.5.5). One of these forms is Hamiltonian and the other is semi-Hamiltonian. Then the corresponding SA- and AS operators can be used to construct a recursion operator for (adjoint) symmetries of the Korteweg-de Vries equation. Since (5.5.5) is not a (semi-) Hamiltonian system, this construction is not possible for the BO equation.

□

It is remarkable that for the BO equation there also exist infinite series of constants of the motion which can only be expressed in terms of densities which depend explicitly on x and t. First define the following functions (functionals) on S_p.

$$(5.5.7) \qquad \begin{aligned} C_2^1(u) &= \tfrac{1}{2} \int xu^2 dx , \qquad C_3^1 = \tfrac{1}{3} \int x(u^3 + \tfrac{3}{2} uHu_x) dx, \\ C_4^1(u) &= \tfrac{1}{4} \int x(u^4 + 2u^2 Hu_x - 2uu_x Hu + 2u_x^2) dx \end{aligned}$$

and also

$$C_2^2(u) = \tfrac{1}{2} \int x^2 u^2 dx , \qquad C_3^2 = \tfrac{1}{3} \int x^2(u^3 + \tfrac{3}{2} uHu_x) dx,$$

(5.5.8)
$$C_4^2(u) = \tfrac{1}{4}\int x^2(u^4 + 2u^2Hu_x - 2uu_xHu + 2u_x^2)dx\ .$$

Then a long computation shows that

(5.5.9)
$$F_2^1(u,t) = C_2^1(u) + 2tF_3^0(u)\ ,$$
$$F_3^1(u,t) = C_3^1(u) + 2tF_4^0(u)$$

and also

(5.5.10)
$$F_2^2(u,t) = C_2^2(u) + 4tC_3^1(u) + 4t^2F_4^0(u),$$
$$F_3^2(u,t) = C_3^2(u) + 4tC_4^1(u) + 4t^2F_5^0(u),$$
$$F_4^2(u,t) = C_4^2(u) + 4tC_5^1(u) + 4t^2F_6^0(u)$$

are constants of the motion of the BO equation. In these expressions F_5^0 and F_6^0 are the following two constants of the motion of the series whose first elements are given (5.5.3). Further $C_5^1(u)$ is an expression of the form given in (5.5.7) ($C_5^1(u) = \frac{1}{5}\int(xu^5 + \ldots)dx$). We do not give the very lengty expressions for F_5^0, F_6^0 and C_5^1 explicitly. The canonical symmetry corresponding to the constant of the motion F_2^1 is given by

$$X_2^1 = \partial\frac{\delta F_2^1}{\delta u} = xu_x + u + 2t(2uu_x + Hu_{xx}) = xu_x + u + 2tu_t.$$

This symmetry is related to the scale transformation $u(x,t) \to au(ax,a^2t)$ of the BO equation. By taking the repeated Poisson brackets of the constants of the motion given in (5.5.3), (5.5.9) and (5.5.10) (and of already constructed elements) we can generate an infinite-dimensional Lie algebra of constants of the motion for the BO equation. However some care is necessary in this construction. The variational derivatives of C_2^2 and C_3^2 are given by

$$\frac{\delta C_2^2}{\delta u} = x^2u\ , \qquad \frac{\delta C_3^2}{\delta u} = x^2u^2 + \frac{3}{2}x^2Hu_x + \frac{3}{2}H(x^2u)_x.$$

For $u \in S_p$ we have $xu \in U_p$ but $x^2u = \frac{\delta C_2^2}{\delta u} \notin U_p$. Also $Hu_x = \partial_x Hu \in S_p$ (see section 1.4), so $xHu_x \in U_p$ but $x^2 Hu_x \notin U_p$. In a similar way we can show that $\frac{3}{2} H(x^2u)_x \notin U_p$. Hence $\frac{\delta C_3^2}{\delta u} \notin U_p$. So formally C_2^2 and C_3^2 are not differentiable in the choosen topology. This means that Poisson brackets between F_2^2, F_3^2 and other (differentiable) constants of the motion may not exist. To avoid these problems we generate a Lie algebra E of constants of the motion of (5.5.1) starting with $\{F_2^0, F_3^0, F_2^1, F_3^1, F_4^2\}$.

Next we make some remarks on the structure of this Lie algebra. The leading terms of the constants of the motion F_ℓ^k given in (5.5.3), (5.5.9) and (5.5.10) are of the form

$$L_\ell^k (u) = \frac{1}{\ell} \int_{-\infty}^{\infty} x^k u^\ell dx.$$

It is easily seen that

$$(5.5.11) \qquad \{L_s^r, L_j^i\} = (i(s-1) - r(j-1)) L_{s+j-2}^{r+i-1} \quad .$$

This means that there can be several methods to construct L_ℓ^k using Poisson bracket of elements L_j^i with "lower orders". Hence it may be possible to generate distinct constants of the motion of the algebra E which have the same leading term L_ℓ^k . For small values of k and ℓ it can be verified that elements of E which have the same leading term L_ℓ^k are identical. We *conjecture* that this also holds true for the other elements of E. In that case a constant of the motion with leading term L_ℓ^k is uniquely determined. We shall denote this constant of the motion by F_ℓ^k. Then, similar to (5.5.11)

$$(5.5.12) \qquad \{F_s^r, F_j^i\} = (i(s-1) - r(j-1)) F_{s+j-2}^{r+i-1} \quad .$$

If the conjecture mentioned above is correct, we can also generate an algebra $\{C_\ell^k\}$, starting with $\{C_2^0 = F_2^0, C_3^0 = F_3^0, C_2^1, C_3^1, C_4^2\}$. Then it can be shown (see Broer and Ten Eikelder [58]) that

$$F_\ell^k = \sum_{i=0}^{k} (2t)^i \binom{k}{i} C_{\ell+i}^{k-i} \quad .$$

In any case we can construct an infinite series of constants of the motion F_k^0 by

$$(5.5.13) \qquad F_{k+1}^0 = \frac{1}{k-1} \{F_k^0, F_3^1\} = \frac{1}{k-1} \{F_k^0, C_3^1\} + \frac{2t}{k-1} \{F_k^0, F_4^0\} .$$

If the Poisson bracket of the F_k^0 with F_4^0 vanishes, we obtain

$$(5.5.14) \qquad F_{k+1}^0 = \frac{1}{k-1} \{F_k^0, C_3^1\} .$$

Then the corresponding symmetries satisfy

$$(5.5.15) \qquad X_{k+1}^0 = \frac{1}{k-1} [A_3^1, X_k^0] \text{ with } A_3^1 = \partial \frac{\delta C_3^1}{\delta u} .$$

This relation has been used by Fokas and Fuchssteiner [63] to generate an infinite series of symmetries and corresponding constants of the motion for the BO equation. However, since all symmetries in this relation are canonical, there is no reason to work with symmetries instead of the corresponding constants of the motion (see also Theorem 4.4.9). Moreover a straightforward construction of the constants of the motion using (5.5.14) also avoids the problem of showing that the symmetries constructed in (5.5.15) are canonical. Note that (5.5.14) and (5.5.15) are only correct if $\{F_k^0, F_4^0\} = 0$ for $k \geq 3$. This holds if the series F_k^0 is in involution. This last property is often mentioned in the literature, but as far as we know a correct proof has not yet been given. To our opinion the proof given by Fokas and Fuchssteiner [63] is incomplete. If the conjecture mentioned above turns out to be correct, it follows immediately from (5.5.12) that the series F_k^0 is in involution.

Non-canonical symmetries for the BO equation are easily constructed. For instance

$$(5.5.16) \qquad Z = F_4^0 \, \partial \frac{\delta F_3^1}{\delta u}$$

is a non-canonical symmetry. The corresponding SA operator $L_Z\Omega$ is found to be ($\Omega = \partial^{-1}$, $\overleftarrow{\Omega} = \partial$)

$$(5.5.17) \qquad (L_Z\Omega)A = <\frac{\delta F_4^0}{\delta u}, A> \frac{\delta F_3^1}{\delta u} - <\frac{\delta F_3^1}{\delta u}, A> \frac{\delta F_4^0}{\delta u} ,$$

for all $A(u) \in S_p$. Then $\Lambda = \overleftarrow{\Omega} L_Z\Omega = \partial L_Z\Omega$ is a recursion operator for symmetries. However, non-canonical symmetries of this type do not satisfy the condition given in Hypothesis 4.5.2. Hence we cannot use the method of Section 4.5 to generate an infinite series of constants of the motion *in involution*.

5.6 THE KORTEWEG-DE VRIES EQUATION.

During the last decennium the Korteweg-de Vries (KdV) equation has become one of the most discussed equations of mathematical physics. The equation was derived by Korteweg and de Vries in 1894 [6,7] for describing long water waves in one direction in a canal. Korteweg and de Vries described periodic solutions (cnoidal waves) and solitary wave solutions of the equation. Solitary waves were already reported by Scott Russell [26] in his famous ride along a channel. His report is quoted in many books on solitons, see for instance Bullough and Caudrey [27] . For a long time the Korteweg-de Vries (KdV) equation gained only limited attention in hydrodynamics. Interest in the equation increased enormously in the sixties. In 1965 Zabusky and Kruskal [28] obtained numerical evidence for the remarkable result that two solitary waves, after their interaction, assume again their original shape. Gardner, Greene, Kruskal and Miura [19] showed in 1967 how the initial value problem for the KdV equation on the real line, with fastly decaying initial value for $|x| \to \infty$, could be solved. The method they used has become known as "inverse scattering". In 1968 Lax [29] found an infinite series of "higher order KdV equations", which all can be solved by this method. These higher order KdV equations are directly related with the infinite series of constants of the motion of the KdV equation, found by Miura, Gardner and Kruskal [30] in the same year. The Hamiltonian character of the KdV equation was pointed out by Gardner [11] and later by Broer [10] . After this numerous other papers on the KdV and related equations appeared. We mention only the work of Wahlquist and Estabrook on prolongation structures [31] and the paper of Zakharov and Faddeev [24] , in which they show that the KdV equation can be considered as an infinite-dimensional completely integrable Hamiltonian system. The KdV equation has also been derived in several physical situations, see for instance Whitham [32] or Su and Gardner [33] .

Of course we shall not give many new results on the KdV equation. In this section we consider symmetries of the KdV equation. Besides the well-known series of symmetries which correspond to the higher order KdV equations, we shall describe another infinite series of symmetries. These symmetries depend explicitly on x and t. They are well suited to illustrate the theory described in chapter 4. Using this second series of symmetries we describe several methods for constructing the constants of the motion. One of these methods is a very simple recursion formula for the constants of the motion themselves (i.e. not for their gradients (= adjoint symmetries) or corresponding symmetries). We also show that every constant of the motion of the infinite series can be considered as a Hamiltonian for the KdV equation. The corresponding (weak) symplectic forms are explicitly given. Then we make some remarks on the symmetries which appear in the inverse scattering method. We end this section with some remarks on the higher order KdV equations.

In this section we consider the KdV equation in the form

(5.6.1) $$u_t = X(u) = 6uu_x - u_{xxx} \qquad x \in \mathbb{R} .$$

Various other forms of the equation can easily be transformed to (5.6.1). We shall study (5.6.1) in the space S_2, provided with the topology induced by U_2 and the duality map (see theorem 1.3.14)

$$<\alpha, A> = \int_{-\infty}^{\infty} \alpha(x)A(x)\, dx \qquad \alpha \in U_2,\ A \in S_2 \ .$$

We now describe the Hamiltonian form of the KdV equation. Define the two-form ω on S_2 by

(5.6.2) $$\omega(A,B) = <\partial^{-1}A,B> .$$

Note that $\partial^{-1}: S_2 \to U_2$ is antisymmetric, so ω is correctly defined. The corresponding operators are

(5.6.3) $$\Omega = \partial^{-1} : S_2 \to U_2,$$

(5.6.4) $$\overleftarrow{\Omega} = \partial \quad : U_2 \to S_2 .$$

It is clear (see remark 5.1.17) that Ω is a cyclic operator, $\overleftarrow{\Omega}$ a canonical operator and ω a symplectic form. Define the function (functional) $H : S_2 \to \mathbb{R}$ by

(5.6.5) $$H(u) = \int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2} u_x^2)\, dx.$$

The exterior derivative (= variational derivative) is given by

$$dH(u) = \frac{\delta H}{\delta u}(u) = 3u^2 - u_{xx}.$$

Then the KdV equation is a Hamiltonian system on S_2 with Hamiltonian $\underline{H}$ and symplectic form ω

$$u_t = \Omega^{\leftarrow} \frac{\delta H}{\delta u} = \partial(3u^2 - u_{xx}). \tag{5.6.6}$$

Clearly the Hamiltonian H is a constant of the motion. Several other constants of the motion are easily found

$$G(u,t) = \int_{-\infty}^{\infty}(xu + 3tu^2)dx,$$

$$F_1(u) = \int_{-\infty}^{\infty} u\,dx \ , \ F_2(u) = \int_{-\infty}^{\infty} u^2 dx, \tag{5.6.7}$$

$$F_3(u) = H(u), \quad F_4(u) = \int_{-\infty}^{\infty} (u^4 + 2uu_x^2 + \frac{1}{5} u_{xx}^2)dx.$$

In 1968 Miura found a relation between the KdV and the so called *Modified Korteweg-de Vries* (MKdV) equation.

$$v_t = 6v^2 v_x - v_{xxx} \quad , \quad x \in \mathbb{R}. \tag{5.6.8}$$

It is easily verified that for every solution v of (5.6.8) the function

$$u = f(v) = v^2 + v_x \tag{5.6.9}$$

is a solution of (5.6.1). This transformation has become known as *Miura transformation*. Using a modified version of the transformation Miura, Gardner and Kruskal [30] proved in 1968 that the KdV equation (and also the MKdV equation) has an infinite series of constants of the motion F_k.

5.6.10 Remark.

The MKdV equation can also formally be written as a Hamiltonian system on some space W of smooth functions, which vanish, together with their derivatives, fast enough for $|x| \to \infty$. Using the canonical operator ∂ and the Hamiltonian $K(v) = \frac{1}{2}\int_{-\infty}^{\infty}(v^4 + v_x^2)dx$ we can write the MKdV equation as

$$(5.6.11) \qquad v_t = \partial \frac{\delta K(v)}{\delta v} = \partial(2v^3 - v_{xx}).$$

□

Symmetries $Y(u,t)$ and adjoint symmetries $\sigma(u,t)$ of the KdV equation have to satisfy the conditions (5.1.3) and (5.1.4). Using

$$X'(u) = 6u\partial + 6u_x - \partial^3 = 6\partial u - \partial^3 : S_2 \to S_2$$

$$X'^*(u) = -6u\partial + \partial^3 : U_2 \to U_2$$

these conditions become

$$(5.6.12) \qquad Y_t(u,t) + Y'(u,t)(6uu_x - u_{xxx}) - (6\partial u - \partial^3)\, Y(u,t) = 0,$$

$$(5.6.13) \qquad \sigma_t(u,t) + \sigma'(u,t)(6uu_x - u_{xxx}) + (-6u\partial + \partial^3)\sigma(u,t) = 0.$$

Define the antisymmetric operator (operator field) $\Psi(u)$ by

$$(5.6.14) \qquad \Psi(u) = 2u\partial + 2\partial u - \partial^3 : U_2 \to S_2.$$

It was observed by Magri [5] that the KdV equation can also be written as

$$(5.6.15) \qquad u_t = X(u) = \Psi(u)\frac{\delta}{\delta u}\tfrac{1}{2}F_2(u) = (2u\partial + 2\partial u - \partial^3)\, u .$$

It is easily verified that $\Psi(u)$ satisfies (5.1.16), so it is a canonical operator. This means that (5.6.15) is a semi-Hamiltonian system with Hamiltonian $\frac{1}{2}F_2$ and canonical operator Ψ. The fact that we did not prove that Ψ is invertible, prevents us from saying it is a Hamiltonian system. From the two possible ways of writing the KdV equations (5.6.6) and (5.6.15) we can obtain some interesting results.

5.6.16 Theorem.

Consider the operators $\Omega : S_2 \to U_2$, $\overleftarrow{\Omega} : U_2 \to S_2$ and $\Psi : U_2 \to S_2$ as given in (5.6.3), (5.6.4) and (5.6.14). Then Ψ and $\overleftarrow{\Omega}$ are AS operators and Ω is an SA operator (for the KdV equation).

Proof:
The Hamiltonian form (5.6.6) of the KdV equation implies (Theorem 4.2.5) that Ω is an SA- and $\overleftarrow{\Omega}$ is an AS operator. The semi-Hamiltonian form (5.6.15) only yields that Ψ is an AS operator (see Section 4.8).

□

5.6.17 Corollary.

i) $\Phi = \Omega\Psi\Omega = 2\partial^{-1}u + 2u\partial^{-1} - \partial : S_2 \to U_2$ is an SA operator,
ii) $\Lambda = \Psi\Omega = 2u + 2\partial u\partial^{-1} - \partial^2 : S_2 \to S_2$ is a recursion operator for symmetries,
iii) $\Gamma = \Omega\Psi = \Lambda^* = 2\partial^{-1}u\partial + 2u - \partial^2 : U_2 \to U_2$ is a recursion operator for adjoint symmetries.

□

The recursion operator for symmetries Λ is well-known. It seems first to be found by Lenard. Several other authors use this operator or derive it again, see for instance Olver [13], Wadati [14], Magri [5], Fuchssteiner [12], Calogero and Degasperis [34] or Gel'fand and Dikii [35]. Using the recursion operators Λ and Γ two infinite series of (adjoint) symmetries are easily constructed. We start with two symmetries, which are related to the invariance of solutions of (5.6.1) for translations along the x-axis and for a scale transformation. Suppose $u(x,t)$ is a solution of (5.6.1). Then it is easily seen that $u(x+\varepsilon,t)$ and $a^2u(ax,a^3t)$ are also solutions of (5.6.1). By taking the limit for $\varepsilon \to 0$ of $u(x+\varepsilon,t) - u(x,t)$ and of $a^2u(ax,a^3t) - u(x,t)$ (with $a = 1+\varepsilon$) we obtain the following two solutions of the linearized KdV equation (linearization around $u(x,t)$)

(5.6.18) $\quad X_0(u) = u_x$,

(5.6.19) $$Z_0(u,t) = \frac{1}{4}(2u + xu_x + 3tu_t) = \frac{1}{2}u + \frac{1}{4}xu_x + \frac{3}{4}t(6uu_x - u_{xxx}).$$

It is easily verified that X_0 and Z_0 satisfy (5.6.12) and that $X_0(u)$, $Z_0(u,t) \in S_2$ for all $u \in S_2$, $t \in \mathbb{R}$. So indeed we have two symmetries; $X_0, Z_0 \in V(X; S_2)$. The factor $\frac{1}{4}$ in (5.6.19) may look strange, but turns out to be convenient in the sequel. The corresponding adjoint symmetries are

$$\rho_0 = \Omega X_0 = u, \quad \tau_0 = \Omega\, Z_0 = \frac{1}{4}\partial^{-1}u + \frac{1}{4}xu + \frac{3}{4}t(3u^2 - u_{xx}).$$

Note that indeed $\rho_0(u)$, $\tau_0(u,t) \in U_2$. Using the recursion operators Λ and Γ we now obtain the following

5.6.20 Theorem.

Two infinite series of symmetries for the KdV equation are given by

$$X_k = \Lambda^k X_0, \quad Z_k = \Lambda^k Z_0.$$

The corresponding adjoint symmetries are given by

$$\rho_k = \Omega X_k = \Gamma^k \rho_0, \quad \tau_k = \Omega\, Z_k = \Gamma^k \tau_0$$

□

The first few elements of the series X_k and ρ_k are

$$X_1 = X = 6uu_x - u_{xxx},$$

$$\rho_1 = \partial^{-1}X_1 = 3u^2 - u_{xx} = \frac{\delta F_3}{\delta u},$$

(5.6.21) $$X_2 = 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx},$$

$$\rho_2 = \partial^{-1}X_2 = 10u^3 - 5u_x^2 - 10uu_{xx} + u_{xxxx} = \frac{5}{2}\frac{\delta F_4}{\delta u}.$$

The first elements of the series Z_k and τ_k are

$$Z_1 = 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx} + \frac{3}{4} tX_2$$

$$= 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{1}{4} xX_1 + \frac{3}{4} tX_2,$$

(5.6.22)

$$\tau_1 = \partial^{-1} Z_1 = \frac{3}{4} \partial^{-1}(u^2) + \frac{1}{2} u\partial^{-1} u - \frac{3}{4} u_x + \frac{3}{4} xu^2 - \frac{1}{4} xu_{xx} + \frac{3}{4} t\rho_2$$

$$= \frac{3}{4} \partial^{-1}(u^2) + \frac{1}{2} u\partial^{-1} u - \frac{3}{4} u_x + \frac{1}{4} x\rho_1 + \frac{3}{4} t\rho_2.$$

So these series of symmetries and adjoint symmetries depend explicitly on x and t.

5.6.23 Remark.

It is easily shown that the general form of Z_k and τ_k, as suggested by (5.6.19) and (5.6.22), is

$$Z_k(u,t) = f_k(u) + \frac{1}{4} xX_k(u) + \frac{3}{4} tX_{k+1}(u),$$

$$\tau_k(u,t) = g_k(u) + \frac{1}{4} x\rho_k(u) + \frac{3}{4} t\rho_{k+1}(u),$$

where f_k and g_k are functions which can be constructed using u, its derivatives and the operator ∂^{-1}. (So f_k and g_k may not contain x explicitly; a translation of u(x) along the x-axis must correspond to the same translation of $(f_k(u))(x)$ and $(g_k(u))(x)$ along the x-axis).

□

The "variational derivatives" of the constants of the motion F_1 and G are

(5.6.24) $$\frac{\delta F_1}{\delta u} = 1, \quad \frac{\delta G}{\delta u} = x + 6tu.$$

Both derivatives are not elements of U_2, which means that, strictly speaking, F_1 and G are not differentiable (in the choosen topology). The local conservation law corresponding to F_1 is

(5.6.25) $$u_t = (3u^2 - u_{xx})_x \quad .$$

Because $\int_{-\infty}^{\infty} (3u^2 - u_{xx})dx = 3F_2$, the flux of u in the local conservation law (5.6.25) is again a conserved quantity. Broer [25] has shown that, using this conserved flux property, a new constant of the motion can be constructed. For the KdV equation this turns out to be G. In [25] the Poisson brackets between G and the series F_k are also given

(5.6.26) $$\{F_k, G\} = k\, F_{k-1} .$$

If we set $\rho_{-1} = \frac{1}{2}\frac{\delta F_1}{\delta u} = \frac{1}{2}$ and $\tau_{-1} = \frac{1}{8}\frac{\delta G}{\delta u} = \frac{1}{8}x + \frac{3}{4}tu$ then we can verify that ρ_{-1} and τ_{-1} satisfy (5.6.13) and that

(5.6.27) $$\rho_0 = \Gamma\rho_{-1} \ , \ \tau_0 = \Gamma\tau_{-1} .$$

The series of symmetries X_k is well-known, see for instance Lax [29], Olver [13], Magri [5] or Wadati [14]. The equations $u_t = X_k(u)$ are called *higher order Korteweg-de Vries equations*. The symmetries X_k are canonical and correspond to the constants of the motion F_k by $X_k = a_k \partial \frac{\delta F_{k+2}}{\delta u}$ ($a_k \in \mathbb{R}$). This means that the higher order KdV equations are also Hamiltonian systems. These results were first found by Gardner. In the sequel we shall also prove that the symmetries X_k are canonical. The series of symmetries Z_k, although easily found, has attracted much less attention. As far as we know, it is only published by Olver [36]. This series is well suited to illustrate the theory, described in the Sections 4.5, 4.6 and 4.7, which we shall do now.

We first study the SA operators which correspond by Theorem 4.2.13 to the (adjoint) symmetries Z_0 and Z_1 (τ_0 and τ_1). Recall that an arbitrary symmetry $Z = \overleftarrow{\Omega}\tau$ gives rise to an SA operator

$$L_Z\Omega = (\Omega' Z) + \Omega Z' + Z'^*\Omega =$$

$$= \tau' - \tau'^* .$$

Using $\tau_0' = \frac{1}{4}\partial^{-1} + \frac{1}{4}x + \frac{3}{4}t(6u - \partial^2)$ and $\tau_0'^* = -\frac{1}{4}\partial^{-1} + \frac{1}{4}x + \frac{3}{4}t(6u - \partial^2)$ we obtain

$$L_{Z_0}\Omega = \tfrac{1}{2}\partial^{-1} = \tfrac{1}{2}\Omega \ .$$

So we find again the already known SA operator Ω. This is not surprising since the symmetry Z_0 corresponds to the scale properties of KdV. The symmetry Z_1 leads to a more interesting result. The derivative of τ_1 and its dual operator are

$$\tau_1' = \frac{3}{2}\,\partial^{-1}u + \frac{1}{2}(\partial^{-1}u) + \frac{1}{2}\,u\partial^{-1} - \frac{3}{4}\,\partial + \frac{3}{2}\,xu - \frac{1}{4}\,x\partial^2 + \frac{3}{4}\,t\rho_2',$$

$$\tau_1'^{*} = -\frac{3}{2}\,\partial^{-1}u + \frac{1}{2}(\partial^{-1}u) - \frac{1}{2}\,\partial^{-1}u + \frac{3}{4}\,\partial + \frac{3}{2}\,xu - \frac{1}{4}\,\partial^2 x + \frac{3}{4}\,t\rho_2'^{*}.$$

Since ρ_2 is canonical ($\rho_2 = \frac{5}{2}\frac{\delta F_2}{\delta u}$), we have $\rho_2' = \rho_2'^{*}$.

Hence

(5.6.28) $$L_{Z_1}\Omega = 2\partial^{-1}u + 2u\partial^{-1} - \partial = \Phi \ .$$

So we find the already known SA operator Φ . Because of the normalization factor in (5.6.19) the multiplicative constant in (5.6.28) is equal to 1. We can compute again the Lie derivative and obtain the SA operator

(5.6.29)
$$\begin{aligned} L^2_{Z_1}\Omega &= L_{Z_1}\Phi \\ &= 9\partial^{-1}u^2 + 9u^2\partial^{-1} + 6u\partial^{-1}u - 3\partial^{-1}u_{xx} - 3u_{xx}\partial^{-1} \\ &\qquad - 6\partial u - 6u\partial + \frac{3}{2}\,\partial^5 \\ &= \frac{3}{2}\,\Phi\,\overleftarrow{\Omega}\,\Phi \\ &= \frac{3}{2}\,(L_{Z_1}\Omega)\overleftarrow{\Omega}\,(L_{Z_1}\Omega). \end{aligned}$$

This means that Z_1 satisfies hypothesis 4.5.2 with $c = \frac{3}{2}$. This hypothesis is essential for the theory described in the Sections 4.5, 4.6 and 4.7. As a first result we obtain from Theorem 4.5.4 and Corollary 4.5.7 the following

5.6.30 Theorem.

i) For k = 0,1,2,... we have

$$L^k_{Z_1}\,\Omega = 2^{-k}(k+1)!\;\Gamma^k\Omega = 2^{-k}(k+1)!\;\Omega\Lambda^k \ .$$

ii) For k = 1,2 we have

$$L^k_{Z_1}\,\overleftarrow{\Omega} = (-1)^k \frac{1}{k}\,\overleftarrow{\Omega}\Gamma^k = (-1)^k \frac{1}{k}\,\Lambda^k\overleftarrow{\Omega} \ ,$$

while for k = 3,4,...

$$L^k_{Z_1}\,\overleftarrow{\Omega} = 0 \ .$$

iii) The SA operators $\Gamma^k\Omega = \Omega\Lambda^k$ are cyclic for k = 0,1,2,... .

□

An infinite series of constants of the motion $\tilde{F}_k$ for the KdV equation is now easily constructed. (We use $\tilde{F}_k$ in stead of F_k since the normalization is different; the coefficient of u^k in F_k is assumed to be 1).

5.6.31 Theorem.

The (adjoint) symmetries $X_k(\rho_k)$ are canonical. The corresponding constants of the motion $\tilde{F}_k$, defined by

$$\frac{\delta\tilde{F}_{k+2}}{\delta u} = \rho_k = \Omega X_k \qquad k = 0,1,2,...$$

are in involution, $\tilde{F}_3 = H$.

Proof:
From (5.6.22) and (5.6.21) we obtain that $Z_{1t} = \frac{3}{4} X_2 = \frac{15}{8} \Omega^{\leftarrow} \frac{\delta F_4}{\delta u}$, so Z_{1t} is a canonical symmetry. For k ≥ 1 the theorem now follows from theorem 4.5.11. The case k = 0 (so $\tilde{F}_2$) has to be considered separately. A simple calculation shows that $\tilde{F}_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 dx$ is a constant of the motion. The Poisson bracket

$$\{\tilde{F}_2, \tilde{F}_k\} = < \frac{\delta \tilde{F}_2}{\delta u}, \Omega^{\leftarrow} \Gamma^{k-2} \frac{\delta \tilde{F}_2}{\delta u} >$$

vanishes since $\Gamma = \Omega\Psi$, and Ω and Ψ are antisymmetric. So the whole series $\tilde{F}_k$ (k = 2,3,...) is in involution.

□

5.6.32 Remark.

The reason that we have to consider $\tilde{F}_2$ separately is that in theorem 4.5.11 we constructed a series of constants of the motion, starting with the Hamiltonian $H = \tilde{F}_3$. In this case there also exists a constant of the motion $\tilde{F}_2$ "below" the Hamiltonian. We can also consider $\tilde{F}_1 = \frac{1}{2}F_1 = \frac{1}{2}\int_{-\infty}^{\infty} u dx$ as the first element of the series $\tilde{F}_k$. However, formally $\tilde{F}_1$ is not differentiable. If we still compute the corresponding symmetry we obtain

$$X_{-1} = \Omega^{\leftarrow} \frac{\delta \tilde{F}_1}{\delta u} = \partial \tfrac{1}{2} = 0 .$$

This would imply that the Poisson bracket of $\tilde{F}_1 = \frac{1}{2}F_1$ with every other function vanishes.

□

The coefficient of u^k in $\tilde{F}_k$ is found to be $\frac{(2k-3)!}{k!(k-2)!}$.
So if we set

$$F_k = \frac{k!(k-2)!}{(2k-3)!} \tilde{F}_k \qquad \text{for } k > 1$$

we obtain a series of constants of the motion such that the coefficient of u^k in F_k is equal to 1.

Next we consider the (adjoint) symmetries Z_k (τ_k).

5.6.33 Theorem.

i) For k = 0,1,2,... we have

(5.6.34) $$L_{Z_k}\Omega = \tfrac{1}{2}(k+1)\,\Gamma^k\Omega = \tfrac{1}{2}(k+1)\,\Omega\Lambda^k \quad ,$$

$$L_{Z_k}\overset{\leftarrow}{\Omega} = -\tfrac{1}{2}(k+1)\,\overset{\leftarrow}{\Omega}\Gamma^k = -\tfrac{1}{2}(k+1)\,\Lambda^k\overset{\leftarrow}{\Omega} \ .$$

ii) The (adjoint) symmetries Z_k (τ_k) are non-canonical for k = 0,1,2,... .

Proof:
Part i) follows at once from Theorem 4.6.5. It is easily seen that Λ^k always contains a term $(-1)^k\partial^{2k}$, so $\Lambda^k \neq 0$. Then part ii) is a consequence of Corollary 4.6.9.

□

So far we have constructed a series of canonical symmetries X_k, corresponding to the constants of the motion F_k and a series of non-canonical symmetries Z_k. We now consider the various Lie brackets between the elements of the two series.

5.6.35 Theorem.

For $k,\ell \geq 0$ we have

i) $[X_k,X_\ell] = 0$,

ii) $[Z_k,Z_\ell] = \tfrac{1}{2}(\ell-k)Z_{k+\ell}$,

iii) $[Z_k,X_\ell] = (\tfrac{1}{2}\ell+\tfrac{1}{4})X_{k+\ell}$.

Proof:
For $k,\ell \geq 1$ the parts i) and ii) follow from Theorem 4.6.16. It is easily verified that $[Z_1,X_1] = \frac{3}{4}X_2$. Then, also for $k,\ell \geq 1$, part iii) is a consequence of Theorem 4.7.8 (with $b = \frac{3}{4}$, $c = \frac{3}{2}$). The cases k = 0 and/or ℓ = 0 have to be considered separately, see Remark 2.6.15.

□

Of course the fact that the symmetries of the series X_k commute follows also from the fact that the corresponding constants of the motion $\tilde{F}_{k+2}$ are in involution.

We now have described two methods for constructing the constants of the motion F_k (or $\tilde{F}_k$). First we used a recursion operator for (adjoint) symmetries $\Lambda(\Gamma)$, viz. the construction described in the theorems 5.6.20 and 5.6.31. The second method consisted in generating the canonical symmetries X_k by using the Lie bracket with Z_1, see theorem 5.6.35. However, the most simple method for constructing the infinite series of constants of the motion is described in

5.6.36 Theorem.

The constant of the motion $F_k (k > 2)$ can be obtained from F_{k-1} by

$$F_k(u) = \frac{2k}{4(k-1)^2-1} L_{Z_1} F_{k-1}(u)$$

$$= \frac{2k}{4(k-1)^2-1} \int_{-\infty}^{\infty} \frac{\delta F_{k-1}}{\delta u} (2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx}) dx.$$

Proof:

For k = 3 this result is easily verified. For k > 3 the first expression follows from Theorem 4.7.12 (for the KdV equation $H = F_3 = \tilde{F}_3$). The normalization coefficient is easily found by considering the highest power of u. Using the expression for Z_1, as given in (5.6.22), we obtain

$$L_{Z_1} F_k = \langle \frac{\delta F_k}{\delta u} , Z_1 \rangle$$

$$= \langle \frac{\delta F_k}{\delta u}, 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx} + \frac{3}{4} tX_2 \rangle.$$

Since $\langle \frac{\delta F_k}{\delta u} , X_2 \rangle = \frac{1}{2} \{F_k, F_2\} = 0$ the term with explicit time dependence vanishes.

□

In Theorem 4.6.12 we have shown how the non-canonical (adjoint) symmetries Z_k (τ_k) can be used to generate a multi-Hamiltonian form of the considered system. We shall now give a multi-Hamiltonian form of the KdV-equation. A necessary condition for the construction described in Theorem 4.6.12 is the invertability or injectivity of the SA-operator $L_{Z_1}\Omega = \Phi$. Since $\Phi = \Omega\Psi\Omega$ and Ω is invertible, this raises the question of the invertability or injectivity of $\Psi(u) = 2u\partial + 2\partial u - \partial^3 : \mathcal{U}_2 \to S_2$. We shall not try to prove invertability of $\Psi(u)$. However, we can prove the following

5.6.37 Theorem.

Let $u \in S_2$. Then the linear operator $\Psi(u) : \mathcal{U}_2 \to S_2$ is injective.

Proof:

Suppose there exists a function $w \in \mathcal{U}_2$, $w \neq 0$ such that

$$\Psi(u)w = 2u_x w + 4uw_x - w_{xxx} = 0 . \tag{5.6.38}$$

We shall show that this leads to a contradiction. After multiplication of (5.6.38) with w we can write this expression as

$$\frac{d}{dx}(2uw^2 - ww_{xx} + \tfrac{1}{2}w_x^2) = 0 .$$

Since $w \in \mathcal{U}_2$ and $u \in S_2$ this implies

$$2uw^2 - ww_{xx} + \tfrac{1}{2}w_x^2 = 0. \tag{5.6.39}$$

We shall first show that this implies that w cannot change sign on $\mathbb{R}$. Suppose $w(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then (5.6.39) implies $w_x(x_0) = 0$. Suppose $w_{xx}(x_0) = 0$. Then, by considering (5.6.38) as an initial value problem with initial values $w(x_0) = 0$, $w_x(x_0) = 0$ and $w_{xx}(x_0) = 0$ and using the existence and uniqueness theorems for ordinary differential equations, we obtain $w = 0$ on $\mathbb{R}$, which is a contradiction. So $w_{xx}(x_0) > 0$ or $w_{xx}(x_0) < 0$, which means that $w(x)$ cannot change sign on $\mathbb{R}$. It is no restriction to assume $w(x) \geq 0$ on $\mathbb{R}$. So if $w(x_0) = 0$ then $w_x(x_0) = 0$ and $w_{xx}(x_0) > 0$. Hence $w(x) \sim \tfrac{1}{2}w_{xx}(x_0)(x-x_0)^2$ for $x \to x_0$. This means that $\sqrt{w(x)}$

is continuous but not differentiable in $x = x_0$. Denote the number of zeros of $w(x)$ between x and some point x_1 with $w(x_1) \neq 0$ by $n(x)$. Then it is easily seen that

$$z(x) = (-1)^{n(x)} \sqrt{w(x)} \tag{5.6.40}$$

is again a function with continuous derivatives. Substitution of $w(x) = z^2(x)$ in (5.6.39) results in

$$-z_{xx} + uz = 0 \ . \tag{5.6.41}$$

From $w \in \mathcal{U}_2$ and $w(x) \geq 0$ for all $x \in \mathbb{R}$ we obtain $\lim_{x\to\pm\infty} w(x) = 0$. Then (5.6.40) implies

$$\lim_{x\to\infty} z(x) = 0. \tag{5.6.42}$$

The solution z of (5.6.41) and (5.6.42) can be obtained from the following integral equation

$$z(x) = \int_x^\infty (y-x)u(y)z(y)\, dy. \tag{5.6.43}$$

Since $u \in S_2$ the integral exists for every bounded continuous function z. Using a standard contraction argument we show that this equation can only have the trivial solution $z \equiv 0$. Since $u \in S_2$ there exists a real number $A > 0$ such that

$$B = \int_A^\infty |u(y)|y\, dy < \tfrac{1}{2}. \tag{5.6.44}$$

Denote by $C[A,\infty)$ the space of bounded continuous functions on $[A,\infty)$. If we supply $C[A,\infty)$ with the uniform norm it is a Banach space. Define the linear operator $\Theta : C[A,\infty) \to C[A,\infty)$ by

$$(\Theta z)(x) = \int_x^\infty (y-x)u(y)z(y)\, dy.$$

It is easily seen that Θ is a contraction

$$\|(\Theta z)\| \leq \|z\| \int_A^\infty 2y|u(y)|\, dy \leq 2B\|z\| \ .$$

This means that Θ has only the fixed point $z = 0$. Hence (5.6.41) and (5.6.42) have only the solution $z(x) = 0$ on $[A,\infty)$ and so (uniqueness) $z(x) = 0$ on $\mathbb{R}$. Then (5.6.40) implies that $w(x) = 0$ on $\mathbb{R}$, which is again a contradiction. This completes the proof.

□

5.6.45 Remark.

It is easily seen that a real number A such that (5.6.44) is satisfied also exists for $u \in S_1$. So the theorem also holds if $u \in S_1$ and if we consider $\Psi(u)$ as an operator $\Psi(u) : U_1 \to S_1$. If $u \notin S_1$ the theorem may be not correct. For instance with the functions

$$u(x) = \frac{2x^2-1}{(x^2+1)^2} \qquad \notin S_1,$$

$$w(x) = \frac{1}{1+x^2} \in U_1$$

we can verify that $\Psi(u)w = 2u_x w + 4uw_x + w_{xxx} = 0$.

□

5.6.46 Remark.

Let $u \in S_2$ be a function which can be obtained by the Miura transformation (5.6.9) from some smooth function v, so $u = v^2 + v_x$. Then it is easily verified that the operator $\Psi(u)$ can be factorized

$$\Psi(u) = 2u\partial + 2\partial u - \partial^3$$

$$= (2v+\partial)\partial(2v-\partial).$$

However, for an arbitrary $u \in S_2$ a function v such that $u = v^2+v_x$ has singularities on the x-axis. So this factorization cannot be used to prove injectivity or even invertability of $\Psi(u)$.

□

As a consequence of the Theorems 4.6.12 and 5.6.37 we now obtain the following

5.6.47 Theorem.

The KdV equation can be considered as a Hamiltonian system with Hamiltonian $\frac{1}{2}(k+1)\,\tilde{F}_{k+3}$ and weak symplectic form $d\tau_k$.

□

Up to now we considered two infinite series of symmetries X_k and Z_k $(k = 0,1,2,\ldots)$ for the KdV equation. A completely different set of symmetries appears in the "inverse scattering method". We shall first describe the scattering and inverse scattering problems for the Schrödinger equation and indicate how the initial value problem for the KdV equation can be solved. Consider the *Schrödinger eigenvalue problem* on $\mathbb{R}$ with a function $u \in S_2$ as potential

(5.6.48) $$-y_{xx} + uy = \lambda y.$$

For $\lambda = k^2 > 0$ this problem has a continuous spectrum. Define the *Jost functions* $f(x,k)$ and $g(x,k)$ as the solutions of (5.6.48) with $\lambda = k^2$, such that

(5.6.49) $$\begin{cases} f(x,k) \sim e^{ikx} & \text{for } x \to \infty\,, \\ g(x,k) \sim e^{-ikx} & \text{for } x \to -\infty\,. \end{cases}$$

For $k \neq 0$ the pairs $f(x,k)$, $f(x,-k)$ and $g(x,k)$, $g(x,-k)$ form two fundamental systems of solutions. A solution of (5.6.48) which (in quantum mechanics) can be interpreted as a wave, coming from $-\infty$, which is partly reflected and partly transmitted, has the asymptotic behaviour

(5.6.50) $$\begin{cases} y(x,k) \sim e^{ikx} + R(k)e^{-ikx} & \text{for } x \to -\infty, \\ y(x,k) \sim T(k)\,e^{ikx} & \text{for } x \to \infty. \end{cases}$$

From (5.6.49) we see that this solution can be written as

(5.6.50a) $$y(x,k) = g(x,-k) + R(k)g(x,k) = T(k)f(x,k).$$

The complex functions R and T are called *reflection* and *transmission coefficient*. The eigenvalue problem (5.6.48) can also have a finite number of discrete (isolated) eigenvalues $\lambda_j = -\mu_j^2 < 0$ for $j = 1,\dots,n(\mu_j > 0)$. We normalize the corresponding real eigenfunctions y_j by

$$\int_{-\infty}^{\infty} y_j^2(x)\,dx = 1.$$

We fix the sign of $y_j(x)$ by requiring $y_j(x) > 0$ as $x \to -\infty$. For every discrete eigenfunction y_j we define the normalization coefficient by

$$c_j = \lim_{x\to-\infty} e^{-2\mu_j x} y_j^2(x).$$

The set $\{R(k);\ \lambda_j,\ c_j \mid j = 1,\dots,n\}$ will be called the *scattering data* of the potential u. The problem of reconstructing the potential u from the scattering data is called the *inverse scattering problem*. This problem was solved by Gel'fand and Levitan [21] and Kay and Moses [22] . First define the function $B : \mathbb{R} \to \mathbb{R}$ by

$$B(x) = \sum_{j=1}^{n} c_j e^{\mu_j x} + \frac{1}{2\pi}\int_{-\infty}^{\infty} R(k)e^{-ikx}\,dk. \tag{5.6.51}$$

Then solve the Gel'fand Levitan equation

$$K(x,y) + B(x+y) + \int_{-\infty}^{x} B(y+z)K(x,z)\,dz = 0 \qquad x > y.$$

The potential u can now be obtained from

$$u(x) = 2\frac{d}{dx} K(x,x).$$

Next suppose the potential u satisfies the KdV equation (5.6.1). Then the scattering data and the (improper) eigenfunctions $f(x,k)$, $g(x,k)$, $y(x,k)$, $y_j(x)$ of (5.6.48) will also depend on t. The remarkable discovery of Gardner, Greene, Kruskal and Miura [19,20] is that, if the potential u of (5.6.48) evolves according to the KdV equation, the evolution of the scattering data is given by

$$(5.6.52)\qquad \begin{cases} R_t(k,t) = -\,8\,ik^3R(k,t), \\ \lambda_{j_t}(t) = 0, \\ c_{j_t}(t) = -\,8\,\mu_j^3(t)c_j(t) \qquad (\mu_j = \sqrt{-\lambda_j}) \qquad j = 1,\dots,n. \end{cases}$$

The solution of these ordinary differential equations is trivial. The initial value problem for the KdV equation can now formally be solved. We first compute the scattering data of the initial value. The time evolution of the scattering data is given by (5.6.52). Then by "inverse scattering" we can find the solution u for arbitrary t. For future reference we also give the time evolution of the solutions of (5.6.48) (see for instance Eckhaus and van Harten [23, § 2.3.1])

$$(5.6.53)\qquad \begin{cases} f_t = -4ik^3f - u_xf + 2(u + 2k^2)f_x, \\ g_t = 4ik^3g - u_xg + 2(u + 2k^2)g_x, \\ y_t = -4ik^3y - u_xy + 2(u + 2k^2)y_x \end{cases}$$

and

$$(5.6.54)\qquad y_{j_t} = -u_xy_j + 2(u+2\lambda_j)y_{j_x} \;.$$

(5.6.55) Remark.

If u satisfies the KdV equation, the function B(x,t), as given in (5.6.51) satisfies $B_t + 8\,B_{xxx} = 0$. This means that w(x,t) = B(2x,t) satisfies

$$(5.6.56)\qquad w_t + w_{xxx} = 0 \,.$$

So the invertible mapping $u \to w$ is a linearizing transformation for the KdV equation. Note that (5.6.56) is also the equation obtained by linearizing the KdV equation around u = 0.

□

5.6.57 Remark.

If we want to express the dependence of the scattering data on the potential u, we have to write $\tilde{R}(k,u)$, $\tilde{\lambda}_j(u)$, $\tilde{c}_j(u)$ (and $n(u)$). However, it is usual in inverse scattering theory to consider the reflection coefficient as a function of k and t and the discrete eigenvalues with corresponding normalization coefficients as functions of t (where u is assumed to satisfy the KdV equation). Then

$$R_t(k,t) = \tilde{R}'(k,u)u_t,\ \lambda_{j_t}(t) = \tilde{\lambda}_j'(u)u_t \text{ and } c_{j_t}(t) = \tilde{c}_j'(u)u_t.$$

If we consider symmetries Y and adjoint symmetries σ also as functions of x and t, they have to satisfy (see (5.6.12) and (5.6.13))

(5.6.58) $$Y_t(x,t) - (6\partial u(x,t) - \partial^3)\, Y(x,t) = 0,$$

(5.6.59) $$\sigma_t(x,t) + (-6u(x,t)\partial + \partial^3)\, \sigma(x,t) = 0.$$

□

It is well-known from first order perturbation theory in quantum mechanics that an infinitesimal change δu in the potential u of the Schrödinger equation (5.6.48) leads to changes in the discrete eigenvalues and reflection coefficient given by

$$\delta\lambda_j = \int_{-\infty}^{\infty} y_j^2(x)\delta u(x)\, dx,$$

$$\delta R(k) = \frac{1}{2ik}\int_{-\infty}^{\infty} y^2(x,k)\delta u(x)\, dx.$$

This implies

(5.6.60) $$\frac{\delta\lambda_j}{\delta u} = y_j^2(x) \qquad j = 1,\dots,n,$$

(5.6.61) $$\frac{\delta R(k)}{\delta u} = \frac{1}{2ik}\, y^2(x,k) \qquad k \neq 0 .$$

Since y_j and all its x derivatives vanish exponentially for $|x| \to \infty$ we have $y_j \in S_2$. So $\frac{\delta\lambda_j}{\delta u} = y_j^2 \in S_2 \subset U_2$. The asymptotic behaviour of $y(x,k)$ for $|x| \to \infty$, as given in (5.6.50), implies that $\frac{\delta R(k)}{\delta u} \notin U_2$. So formally R(k) is not differentiable (in the topology of S_2). From (5.6.52) we see that a discrete eigenvalue λ_j is a constant of the motion and that

$$\frac{\partial}{\partial t}\left(e^{8ik^3t} R(k,t)\right) = 0.$$

This leads to

5.6.62 Theorem.

i) The functions $\sigma_j = y_j^2$ $(j=1,\dots,n)$ are canonical adjoint symmetries corresponding to the constants of the motion λ_j; so they satisfy (5.6.59). Further

$$\Gamma y_j^2(x,t) = 4\lambda_j y_j^2(x,t) . \tag{5.6.63}$$

ii) For $k \neq 0$ the functions $\zeta_1(x,k,t) = e^{8ik^3t} y^2(x,k,t)$,

$$\zeta_2(x,k,t) = e^{8ik^3t} f^2(x,k,t) \text{ and } \zeta_3(x,k,t) = e^{-8ik^3t} g^2(x,k,t)$$

satisfy (5.6.59) and

$$\Gamma \zeta_m(x,k,t) = 4k^2 \zeta_m(x,k,t) \qquad m = 1,2,3. \tag{5.6.64}$$

Proof:

The discrete eigenvalues λ_j are constants of the motion, so their variational derivatives are adjoint symmetries. Multiplication of (5.6.48) with y_{j_x} and application of $4\partial^{-1}$ yields

$$-2y_{j_x}^2 + 4\,\partial^{-1}(uy_jy_{j_x}) = 2\lambda_j y_j^2 ,$$

while multiplication of (5.6.48) with $2y_j$ gives

$$-2y_j y_{j_{xx}} + 2uy_j^2 = 2\lambda_j y_j^2 \quad .$$

Then (5.6.63) is obtained by adding these two expressions. The fact that the functions ζ_m(m= 1,2,3) satisfy (5.6.59) follows from a straightforward computation using (5.6.53) The proof of (5.6.64) is similar to the proof of (5.6.63).

□

Although $\zeta_1(\cdot,k,t) = 2ik \frac{\delta}{\delta u} (e^{8ik^3 t} R(k,t))$ we do not call ζ_1 the canonical adjoint symmetry corresponding to $e^{8ik^3 t} R(k,t)$. The reason for this is that $\zeta_1(\cdot,k,t) \notin U_2$. Also (asymptotic behaviour) ζ_2, $\zeta_3(\cdot,k,t) \notin U_2$. Apart from this difference the two parts of the theorem claim similar results for the squares of the eigenfunctions of the Schrödinger equation (5.6.48). The fact that σ_j(j=1,...,n) and ζ_1 satisfy (5.6.59) is already given by Gardner, Greene, Kruskal and Miura [20, theorem 3.6] . However, as far as we know the interpretation of σ_j as canonical adjoint symmetry is new. The relations (5.6.63) and (5.6.64) for the "squared eigenfunctions" are also well-known. Of course $\partial\sigma_j$(j=1,...,n) and $\partial\zeta_m$(m=1,2,3) satisfy (5.6.58) and $\partial\sigma_j$ is a canonical symmetry. These functions are also eigenfunctions of the recursion operator for symmetries Λ

(5.6.65) $\quad \Lambda\partial\sigma_j = 4\lambda_j\partial\sigma_j \qquad j = 1,\ldots,n,$

(5.6.66) $\quad \Lambda\partial\zeta_m = 4k^2\partial\zeta_m \qquad m = 1,2,3 \quad .$

Recall that in Theorem 2.3.13 and Corollary 2.4.11 we showed that, for a finite-dimensional system, the eigenvalues of recursion operators for symmetries and for adjoint symmetries are constants of the motion. The expressions (5.6.63) and (5.6.65) show that this also holds for the isolated eigenvalues of recursion operator for (adjoint) symmetries of the KdV equation.

We now indicate how a second solution of (5.6.59), corresponding to a discrete eigenvalue λ_j, can be constructed. The Jost functions $f(x,k)$ and $g(x,k)$ can be continued analytically into the upper half of the complex k-plane. In $k=i\mu_j$ we have (for a moment we omit t)

$$g(x,i\mu_j) \sim e^{\mu_j x} \quad \text{for } x \to -\infty .$$

A solution $h_j(x)$ of (5.6.48) with $\lambda = -\mu_j^2$ which is independent of $g(x,i\mu_j)$, must have asymptotic behaviour $h(x) \sim e^{-\mu_j x}$ for $x \to -\infty$. Then, by considering the behaviour for $x \to -\infty$ we see that the solution $y_j(x)$ can be written as

$$y_j(x) = \sqrt{c_j}\; g(x,i\mu_j) .$$

This means the canonical adjoint symmetry σ_j can be written as

$$\sigma_j(x,t) = y_j^2(x,t) = c_j(t)\; g^2(x,i\mu_j,t).$$

We now consider the derivative of $g(x,k,t)$ with respect to k. The time evolution of this function in $k = i\mu_j$ follows from (5.6.53)

$$\begin{aligned} g_{kt} = {} & 4\mu_j^3 g_k - u_x g_k + 2(u-2\mu_j^2)g_{kx} \\ & - 12i\mu_j^2 g - 8i\mu_j g_x . \end{aligned} \tag{5.6.67}$$

Then a long but straightforward computation, using (5.6.52), (5.6.53), (5.6.67) and (derivatives with respect to x and k in $k = i\mu_j$ of) the Schrödinger equation (5.6.48) shows that

$$\hat{\sigma}_j(x,t) = ic_j(t)g(x,i\mu_j,t)g_k(x,i\mu_j,t) - 12\mu_j^2 t\sigma_j(x,t) \tag{5.6.68}$$

satisfies (5.6.59). It can be shown that $\hat{\sigma}_j$ is a real function with asymptotic behaviour

$$\hat{\sigma}_j(x,t) \sim c_j(t)xe^{2\mu_j x} \qquad \text{for } x \to -\infty \quad ,$$

$$\hat{\sigma}_j(x,t) \sim 1 \qquad \text{for } x \to \infty.$$

So $\hat{\sigma}_j \notin \mathcal{U}_2$ which means that we cannot call $\hat{\sigma}_j$ an adjoint symmetry. Using derivatives of (5.6.48) with respect to x and k it is a simple exercise to show that

$$\Gamma\hat{\sigma}_j = -4\mu_j^2\hat{\sigma}_j - 4\mu_j\sigma_j \qquad j = 1,\ldots,n.$$

Thus, related with the "inverse scattering method", we constructed the following solutions of (5.6.59):

i) continuous spectrum $\lambda = k^2$, $k \in \mathbb{R}\setminus\{0\}$

$$\zeta_1(x,k,t) = e^{8ik^3t}\, y^2(x,k,t),$$

$$\zeta_2(x,k,t) = e^{8ik^3t}\, f^2(x,k,t),$$

$$\zeta_3(x,k,t) = e^{-8ik^3t}\, g^2(x,k,t),$$

with $\Gamma\zeta_m = 4k^2\zeta_m \qquad (m=1,2,3)$,

ii) discrete spectrum $\lambda_j = -\mu_j^2$, $j = 1,\ldots,n$

$$\sigma_j(x,t) = y_j^2(x,t) = c_j(t)g^2(x,i\mu_j,t),$$

$$\hat{\sigma}_j(x,t) = ic_j(t)g(x,i\mu_j,t)g_k(x,i\mu_j,t) - 12\mu_j^2t\sigma_j(x,t),$$

with $\Gamma\sigma_j = -4\mu_j^2\sigma_j$,

$$\Gamma\hat{\sigma}_j = -4\mu_j^2\hat{\sigma}_j - 4\mu_j\sigma_j \ .$$

It follows from (5.6.50a) that $\zeta_1(x,k,t) = T^2(k)\ \zeta_2(x,k,t)$. A more profound study of the inverse scattering method shows that any infinitesimal variation δu (smooth, fast decaying as $|x| \to \infty$) can be written in terms of ζ_3, σ_j and $\hat{\sigma}_j$. See for instance Zakharov and Faddeev [24, the first expression in § 2] . This enables us to express the symmetries X_k and Z_k, which we studied in the first part of this section, in terms of ζ_3, σ_j and $\hat{\sigma}_j$. We only give the formal result (X_0 and Z_0 are considered as functions of x and t, see Remark 5.6.57)

$$X_0(x,t) = u_x(x,t)$$

$$(5.6.69) \qquad = \frac{\partial}{\partial x}\left[\frac{2i}{\pi}\int_{-\infty}^{\infty} kR(k,t)e^{8ik^3t}\ \zeta_3(x,k,t)dk - 4\sum_{j=1}^{n}\mu_j\sigma_j(x,t)\right],$$

$$Z_0(x,t) = \frac{1}{4}\ (2u(x,t) + xu_x(x,t) + 3t\ u_t(x,t))$$

$$(5.6.70) \qquad = \frac{1}{4}\frac{\partial}{\partial x}\left[\frac{1}{\pi}\int_{-\infty}^{\infty}(kR_k(k,t) + 24ik^3tR(k,t))\ e^{8ik^3t}\ \zeta_3(x,k,t)dk - \sum_{j=1}^{n}(2\sigma_j(x,t) + 4\mu_j\hat{\sigma}_j(x,t)\right].$$

The expression (5.6.69) has already been given (in a somewhat different form) by Deift and Trubowitz [66] . By applying the recursion operator Λ (Γ inside the square brackets) we can obtain similar expressions for X_k and Z_k for $k = 1,2,3,\ldots$.

We end this section by making some remarks on the higher order KdV equations. Denote the "time independent part" of the symmetries Z_k by A_k, so

$$(5.6.71) \qquad A_k = Z_k - \tfrac{3}{4}t\ X_{k+1} = \Lambda^k(\tfrac{1}{2}\ u + \tfrac{1}{4}\ xu_x) \qquad k = 0,1,2,\ldots$$

Then from theorem 5.6.35 we get

$$(5.6.72) \qquad \begin{cases} [A_k, X_\ell] = (\tfrac{1}{2}\ell + \tfrac{1}{4})X_{k+\ell}\ , \\ [A_k, A_\ell] = \tfrac{1}{2}(\ell - k)A_{k+\ell}\ . \end{cases}$$

The following properties of the higher order KdV equations are easily proved.

5.6.73 Theorem.

Consider in S_2 the higher order KdV equation

(5.6.74) $$u_t = X_m(u) \qquad , \ m = 1,2,3,\ldots \ .$$

Then

i) this equation is a Hamiltonian system with Hamiltonian $\tilde{F}_{m+2}$ and symplectic form ω

$$u_t = X_m(u) = \Omega^{\leftarrow} \frac{\delta \tilde{F}_{m+2}}{\delta u} ,$$

ii) the functions (functionals) $\tilde{F}_k$ (or F_k) are also constants of the motion for this higher order KdV equation,

iii) the operator $\Lambda(\Gamma)$ is a recursion operator for (adjoint) symmetries of (5.6.74),

iv) two infinite series of symmetries for (5.6.74) are

$$X_k = \Omega^{\leftarrow} \frac{\delta \tilde{F}_{k+2}}{\delta u} \quad \text{(independent of m)},$$

$$U_{m,k} = A_k + (\tfrac{1}{2}m + \tfrac{1}{4})t\, X_{k+m} \qquad k = 0,1,2,\ldots \quad .$$

So X_k, $U_{m,k} \in V(X_m, S_2)$. The symmetries X_k are canonical while the $U_{m,k}$ are non-canonical. The Lie brackets between elements of these series are given by

(5.6.75) $$\begin{cases} [X_k, X_\ell] = 0, \\ [U_{m,k}, X_\ell] = (\tfrac{1}{2}\ell + \tfrac{1}{4})X_{k+\ell} , \\ [U_{m,k}, U_{m,\ell}] = \tfrac{1}{2}(\ell-k)U_{m,k+\ell} \qquad k,\ell = 0,1,2,\ldots \quad . \end{cases}$$

□

Note that the structure of the Lie algebra of symmetries of (5.6.74), generated by $\{X_k, U_{m,k}, k = 0,1,2,\ldots\}$ does not depend on m. For the KdV equation itself (m = 1) this Lie algebra is already described in Theorem 5.6.35.

5.7 THE SAWADA-KOTERA EQUATION.

In this section we consider an equation of "KdV type" found by Sawada and Kotera [38] and also by Caudrey, Dodd and Gibbon [39]. We study this so called Sawada-Kotera (SK) equation in the form

$$u_t = X(u) = 180u^2u_x + 30u_xu_{2x} + 30uu_{3x} + u_{5x}, \qquad x \in \mathbb{R} \tag{5.7.1}$$

where $u_{nx} = \partial^n u$. The SK equation is essentially different from the higher order KdV equation $u_t = X_2(u)$ in the notation of the preceding section. This equation reads

$$u_t = 30u^2u_x - 10uu_{3x} - 20uu_{2x} + u_{5x}. \tag{5.7.2}$$

Of course the coefficients of both equations can be changed by scale transformations of x, t and u. However, it is impossible to transform (5.7.1) into (5.7.2) by a scale transformation. It is shown in [39] that (5.7.1) and (5.7.2) are the only equations of this type which have multi-soliton solutions. We shall consider the SK equation in the space S_p (p = 1,2,...) with the topology induced by U_p and the usual duality map. In this section we study symmetries and constants of the motion of the SK equation. We also make some remarks on the "inverse scattering problem" for (5.7.1). For the SK equation there exists a series of constants of the motion F_k. The first few elements of this series are given by

$$\begin{aligned}
&F_1 = \int_{-\infty}^{\infty} u\, dx\,, \quad F_3 = \frac{1}{2}\int_{-\infty}^{\infty} (2u^3 - u_x^{\,2})\, dx\,,\\
&F_4 = \frac{1}{12}\int_{-\infty}^{\infty} (12u^4 - 18uu_x^{\,2} + u_{2x}^2)\, dx\,,\\
&F_6 = \frac{1}{576}\int_{-\infty}^{\infty} (576u^6 - 3600u^3u_x^{\,2} - 204u_x^{\,4} + 576u^2u_{2x}^{\,2} +\\
&\qquad + 32u_{2x}^{\,3} - 42uu_{3x}^{\,2} + u_{4x}^2)\, dx\,.
\end{aligned} \tag{5.7.3}$$

A constant of the motion of a different type is given by

(5.7.4) $$G = \int_{-\infty}^{\infty} xu\,dx + 60t\,F_3.$$

The SK equation (and also (5.7.2)) is invariant for the scale transformation $u(x,t) \to a^2u(ax,a^5t)$. Under this scale transformation the constants of the motion F_k are proportional to a^{2k-1}. It appears that constants of the motion of the type F_{3k+2} (with densities which are polynomials in u and its derivatives) do not exist. For $k = 0$ this is easily verified. Using a computer program (formula manipulation) it can be shown that also F_5, F_8 and F_{11} do not exist. In the sequel we shall describe several methods to obtain F_{k+3} from F_k. Then, starting with F_1 and F_3 we can construct the series F_{3k+1} and F_{3k+3} for $k = 1,2,3,\ldots$. Of course this does not exclude the possibility that a constant of the motion F_{3k+2} exists for some k ($k > 3$).

Symmetries $Y(u,t)$ and adjoint symmetries $\sigma(u,t)$ of the SK equation have to satisfy (see (5.1.3) and (5.1.4))

(5.7.5) $$Y_t(u,t) + Y'(u,t)\,X(u) - X'(u)\,Y(u,t) = 0,$$

(5.7.6) $$\sigma_t(u,t) + \sigma'(u,t)\,X(u) + X'^*(u)\,\sigma(u,t) = 0$$

with

$$X'(u) = 180\partial u^2 + 30\partial^3 u - 60\partial u_x \partial + \partial^5 \;:\; S_p \to S_p,$$

$$X'^*(u) = -180u^2\partial - 30u\partial^3 - 60\partial u_x \partial - \partial^5 \;:\; U_p \to U_p.$$

Define the antisymmetric operators (in fact operator fields) $\overleftarrow{\Omega}$ and Φ by

(5.7.7) $$\overleftarrow{\Omega}(u) = 12u\partial + 12\partial u + \partial^3 : U_p \to S_p\,,$$

(5.7.8) $$\Phi(u) = 6\partial^2 u\partial^{-1} + 6\partial^{-1}u\partial^2 + 18\partial^{-1}u^2 + 18u^2\partial^{-1} + \partial^3 : S_p \to U_p\,.$$

Note that, up to a scale transformation, $\overleftarrow{\Omega}$ is equal to the canonical operator Ψ, given in (5.6.14). So $\overleftarrow{\Omega}$ is also a canonical operator. It can be verified that Φ satisfies (5.1.14), so it is a cyclic operator. It has been noted by Broer and Ten Eikelder [40] and also by Fuchssteiner and Oevel [41] that

$$u_t = X(u) = \overleftarrow{\Omega}(u) \frac{\delta F_3}{\delta u} \tag{5.7.9}$$

and

$$\Phi(u)u_t = \Phi(u)X(u) = 288 \frac{\delta F_6}{\delta u} . \tag{5.7.10}$$

Since $\overleftarrow{\Omega}$ is a canonical operator, (5.7.9) is a semi-Hamiltonian system. Several results for semi-Hamiltonian systems have been given in Section 4.8. Recall that for a semi-Hamiltonian system we did not introduce the concepts of canonical and non-canonical symmetries. A first consequence of the semi-Hamiltonian form (5.7.9) is that $\overleftarrow{\Omega}$ is an AS operator. If Φ is invertible, (5.7.10) would give a Hamiltonian form of the SK equation. However, we shall not try to prove invertability of Φ. Denote the closed two-form corresponding to Φ by ϕ. Then

$$L_X\phi = d\, i_X\phi = d(\Phi X) = 288 d\left(\frac{\delta F_6}{\delta u}\right) = 0 . \tag{5.7.11}$$

This is equivalent to $L_X\Phi = 0$. Since Φ does not depend explicitly on t, this means that Φ is an SA operator. Hence we have proved the following

5.7.12 Theorem.

The operator $\overleftarrow{\Omega}$, as given in (5.7.7) is an AS operator. The operator Φ, defined in (5.7.8) is an SA operator. Further $\Lambda = \overleftarrow{\Omega}\Phi : S_p \to S_p$ is a recursion operator for symmetries and $\Gamma = \Phi\overleftarrow{\Omega} : U_p \to U_p$ is a recursion operator for adjoint symmetries.

□

The "variational derivatives" of F_1 and G are given by

$$\frac{\delta F_1}{\delta u} = 1 \notin U_p, \quad \frac{\delta G}{\delta u} = x + 60t(3u^2 + u_{xx}) \notin U_p .$$

This means that F_1 and G are not differentiable (in the choosen topology). However, if we set

$$\sigma_0 = 1 \left(= \frac{\delta F_1}{\delta u}\right), \quad \tau_0 = \frac{1}{72}x + \frac{5}{6}t(3u^2 + u_{xx}) \left(= \frac{1}{72}\frac{\delta G}{\delta u}\right) ,$$

then σ_0 and τ_0 satisfy (5.7.6). Since $\sigma_0 \notin \mathcal{U}_p$ and $\tau_0 \notin \mathcal{U}_p$, we cannot call them adjoint symmetries. The factor $\frac{1}{72}$ turns out to be convenient in the remaining part of this section. Application of $\overleftarrow{\Omega}$ results in

(5.7.13)
$$Y_0 = \overleftarrow{\Omega}\sigma_0 = 12u_x \in S_p,$$
$$Z_0 = \overleftarrow{\Omega}\tau_0 = \frac{1}{6}\,(2u + xu_x + 5tX(u)) \in S_p.$$

It is easily seen that Y_0 and Z_0 satisfy (5.7.5), so they are symmetries of the SK equation. Note that the symmetry Z_0 corresponds to the scale transformation $u(x,t) \to a^2u(ax,a^5t)$ of the SK equation. By applying the SA operator Φ to Y_0 and Z_0 we obtain the adjoint symmetries

$$\sigma_1 = \Phi Y_0 \; (= \Gamma\sigma_0) = 72\,\frac{\delta F_4}{\delta u} \quad \text{and} \quad \tau_1 = \Phi Z_0 \; (= \Gamma\tau_0).$$

Three infinite series of (adjoint) symmetries are constructed in the following

5.7.14 Theorem.

The series

(5.7.15)
$$\rho_k = \Gamma^{k-1}\,\frac{\delta F_3}{\delta u}, \quad \sigma_k = \Gamma^{k-1}\sigma_1, \quad \tau_k = \Gamma^{k-1}\tau_1, \quad k = 1,2,3,\ldots$$

consist of adjoint symmetries of the SK equation. The corresponding symmetries are given by

$$X_k = \overleftarrow{\Omega}\rho_k = \Lambda^{k-1}X_1 \qquad (X_1 = X),$$
$$Y_k = \overleftarrow{\Omega}\sigma_k = \Lambda^{k-1}Y_1 = \Lambda^k Y_0,$$
$$Z_k = \overleftarrow{\Omega}\tau_k = \Lambda^{k-1}Z_1 = \Lambda^k Z_0.$$

Proof:

This theorem is a straightforward consequence of the fact that $\rho_1 = \frac{\delta F_3}{\delta u}$, σ_1 and τ_1 are adjoint symmetries and that Γ is a recursion operator for adjoint symmetries.

□

We shall show that the adjoint symmetries ρ_k and σ_k are exact and correspond to the constants of the motion F_{3k} and F_{3k+1}. The adjoint symmetries τ_k turn out to be non-closed for $k \geq 1$. Since the adjoint symmetry τ_1 and the symmetry $Z_1 = \overleftarrow{\Omega}\tau_1$ are essentially for the following considerations, we give τ_1 explicitly

$$(5.7.16) \qquad \tau_1 = 5\partial^{-1}(u^3) + 3u^2\partial^{-1}u - \frac{5}{2}\partial^{-1}(u_x^2) + u_{2x}\partial^{-1}u + 10uu_x + \frac{5}{6}u_{3x} + x(4u^3 + \frac{3}{2}u_x^2 + 3uu_{xx} + \frac{1}{6}u_{4x}) + 240t\frac{\delta F_6}{\delta u}$$

$$(5.7.17) \qquad = \alpha_1 + 240t\frac{\delta F_6}{\delta u} .$$

It is easily seen that $\tau_1' \neq \tau_1'^*$, which implies that τ_1 is non-closed. Notice that the terms which contain x explicitly in τ_1 can be written as $x\frac{\delta F_4}{\delta u}$ (see Remark 5.6.23 for a similar property of the non-canonical symmetries of the KdV equation).

By Theorem 2.5.16 i) the operator $L_Z\overleftarrow{\Omega}$ is again an AS operator and the operator $L_Z\Phi$ is again an SA operator. A very long computation shows that

$$(5.7.18) \qquad L_{Z_1}\overleftarrow{\Omega} = -\overleftarrow{\Omega}\Phi\overleftarrow{\Omega} ,$$

$$(5.7.19) \qquad L_{Z_1}\Phi = 2\Phi\overleftarrow{\Omega}\Phi .$$

This means that Hypothesis 4.8.2 is satisfied (with $c = 2$). As a first consequence (see (4.8.5)) we have the following

5.7.20 <u>Theorem.</u>

i) The SA operators $L_Z^k\Phi$ are given by

$$L_Z^k\Phi = (k+1)!\,\Phi(\overleftarrow{\Omega}\Phi)^k ,$$

ii) The SA operators $\Phi(\overleftarrow{\Omega}\Phi)^k$ are cyclic.

□

Now a first series of constants of the motion of the SK equation is obtained from the semi-Hamiltonian version of Theorem 4.5.11.

5.7.21 Theorem.

The adjoint symmetries ρ_k are exact and correspond to constants of the motion $\tilde{F}_{3k}$ by

$$\frac{\delta \tilde{F}_{3k}}{\delta u} = \rho_k = \Gamma^{k-1} \frac{\delta F_3}{\delta u} . \tag{5.7.22}$$

These constants of the motion are in involution.

□

A second series of constants of the motion is described in the following

5.7.23 Theorem.

The adjoint symmetries σ_k are exact and correspond to constants of the motion $\tilde{F}_{3k+1}$ by

$$\frac{\delta \tilde{F}_{3k+1}}{\delta u} = \sigma_k = \Gamma^{k-1} \sigma_1 . \tag{5.7.24}$$

These constants of the motion are in involution. Also the Poisson brackets between the elements of the series $\tilde{F}_{3k}$ and $\tilde{F}_{3\ell+1}$ vanish.

Proof:

This series of constants of the motion does not start with the Hamiltonian, so we cannot obtain it from a semi-Hamiltonian version of Theorem 4.5.11. However, a straightforward proof using similar methods is also easily given. The operators $\overset{\leftarrow}{\Omega}$ and Φ do not depend explicitly on x, so $L_{Y_0} \overset{\leftarrow}{\Omega} = 0$ and $L_{Y_0} \Phi = 0$. This means that

$$L_{Y_0} ((\Phi \overset{\leftarrow}{\Omega})^k \Phi) = 0 .$$

Since $(\Phi\overset{\leftarrow}{\Omega})^k\Phi$ is a cyclic operator this implies that the adjoint symmetry $\sigma_k = \Gamma^{k-1}\sigma_1 = (\Phi\overset{\leftarrow}{\Omega})^k \Phi Y_0$ is exact (see Lemma 4.5.9). The antisymmetry of

Φ and $\overleftarrow{\Omega}$ implies that the corresponding constants of the motion $\tilde{F}_{3k+1}$ are in involution. We now consider the Poisson bracket between $\tilde{F}_{3k+1}$ and the Hamiltonian $F_3 = \tilde{F}_3$. Since $\tilde{F}_{3k+1}$ is a constant of the motion we have

$$\{\tilde{F}_{3k+1},\tilde{F}_3\} + \frac{\partial}{\partial t}\tilde{F}_{3k+1} = 0 \ .$$

The derivative $\frac{\delta \tilde{F}_{3k+1}}{\delta u} = \sigma_k$ does not depend explicitly on t. This means that $\tilde{F}_{3k+1}$ can only depend explicitly on t through an "additive function of t" (see also the proof of Theorem 2.4.5). Substitution of the solution $u(x,t) = 0$ shows that this is impossible, so

$$\{\tilde{F}_{3k+1},\tilde{F}_3\} = 0 \quad \text{and} \quad \frac{\partial}{\partial t}\tilde{F}_{3k+1} = 0 \ .$$

Finally it follows from

$$\{\tilde{F}_{3k+1},\tilde{F}_{3\ell}\} = \{\tilde{F}_{3k+3\ell-2},\tilde{F}_3\} = 0 \ .$$

that the two series $\tilde{F}_{3k+1}$ and $\tilde{F}_{3\ell}$ are also in involution.

□

Thus we have constructed two series of constants of the motion; a series $\tilde{F}_{3k}$ by applying the recursion operator for adjoint symmetries Γ to $\frac{\delta F_3}{\delta u}$ and a series $\tilde{F}_{3k+1}$ by applying Γ to $\frac{\delta F_4}{\delta u}$. By normalizing these constants of the motion so that the coefficient of u^k in F_k is equal to 1, we obtain the series F_{3k} and F_{3k+1}. So there exist rational numbers c_k such that

(5.7.25) $$F_{3k} = c_{3k}\tilde{F}_{3k} \ , \quad F_{3k+1} = c_{3k+1}\tilde{F}_{3k+1} \ .$$

Next we turn to the adjoint symmetries τ_k and the corresponding symmetries $Z_k = \overleftarrow{\Omega}\tau_k$.

5.7.26 <u>Theorem.</u>

The adjoint symmetries τ_k are non-closed for $k = 1,2,3,\ldots$.

<u>Proof:</u>

It is easily seen that $(\overleftarrow{\Omega}\Phi)^k\overleftarrow{\Omega}$ always contains a term ∂^{6k+3}, so this operator does not vanish. The result now follows from Theorem 4.8.7 ii).

□

Recall that, since we are working in a linear space, the notions closed and exact and hence non-closed and non-exact are identical.

The various possible Lie brackets between the elements of the three series of symmetries are given in

5.7.27 Theorem.

The Lie brackets between the elements of the series of symmetries X_k, Y_k and Z_k are given by

(5.7.28) $$[X_k, X_\ell] = 0 \ , \quad [Z_k, Z_\ell] = (\ell - k) Z_{k+\ell} \ , \quad [Z_k, X_\ell] = (\ell - \tfrac{1}{6}) X_{k+\ell} \ ,$$

(5.7.29) $$[Y_k, Y_\ell] = 0 \ , \quad [X_k, Y_\ell] = 0 \ , \quad [Z_k, Y_\ell] = (\ell + \tfrac{1}{6}) Y_{k+\ell} \quad .$$

Proof:

A simple computation shows that $[Z_1, X_1] = \frac{5}{6} X_2$. Then (5.7.28) is a consequence of the semi-Hamiltonian version of the Theorems 4.6.16 and 4.7.8 (with $c = 2$ and $b = \frac{5}{6}$). The relations (5.7.29) follow from considerations similar to the proofs of the theorems mentioned above.

□

We now have two distinct methods available for constructing the series of constants of the motion $\tilde{F}_{3k}$ and $\tilde{F}_{3k+1}$. The first method is to construct the corresponding adjoint symmetries using the recursion operator Γ (see the Theorem 5.7.21 and 5.7.23). The second method consists in generating the corresponding symmetries by using the repeated Lie bracket with Z_1 (see Theorem 5.7.27). The simplest method for constructing the two series of constants of the motion is described in

5.7.30 Theorem.

The constants of the motion F_{3k} and F_{3k+1} can be found recursively by

$$F_{3k+3} = a_k L_{Z_1} F_{3k} = a_k \int_{-\infty}^{\infty} \frac{\delta F_{3k}}{\delta u} \overleftarrow{\Omega} \alpha_1 \, dx \quad ,$$

$$F_{3k+4} = b_k L_{Z_1} F_{3k+1} = b_k \int_{-\infty}^{\infty} \frac{\delta F_{3k+1}}{\delta u} \overleftarrow{\Omega} \alpha_1 \, dx$$

where α_1 is given by (5.7.16) and (5.7.17). The normalization constants a_k and b_k have to be chosen such that the coefficients of u^{3k+3} and u^{3k+4} in F_{3k+3} respectively F_{3k+4} are again equal to 1.

Proof:
The recursion formula for the series F_{3k} is a straightforward consequence of the semi-Hamiltonian version of Theorem 4.7.12. The formula for the series F_{3k+1} can be proved in a similar way.

□

Finally we make some remarks on the "scattering-inverse scattering" problem for the SK equation. The "scattering problem" for the SK equation, given by Satsuma and Kaup [42] and by Dodd and Gibbon [57], reads

$$(5.7.31) \qquad y_{xxx} + 6uy_x = \lambda y \ .$$

Suppose this equation has a discrete eigenvalue λ with an eigenfunction y such that $\int_{-\infty}^{\infty} y\bar{y}_x \, dx$ exists. Then it can be shown that the eigenvalue λ is purely imaginary and that (formally)

$$\frac{\delta\lambda}{\delta u} = \frac{6y_x\bar{y}_x}{\int_{-\infty}^{\infty} y\bar{y}_x \, dx} \ .$$

If u evolves according to the SK equation, the discrete eigenvalue λ is a constant of the motion and so $i\frac{\delta\lambda}{\delta u}$ is an adjoint symmetry. Indeed, using the time evolution of y given in [42], it can be shown that $i\frac{\delta\lambda}{\delta u}$ satisfies (5.7.6). We now can apply the recursion operator Γ to $i\frac{\delta\lambda}{\delta u}$. After a long computation, using (x derivatives and complex conjugates of) (5.7.31) we find

$$(5.7.32) \qquad \Gamma i\frac{\delta\lambda}{\delta u} = 27\lambda\bar{\lambda} i\frac{\delta\lambda}{\delta u} \ .$$

So the recursion operator Γ has an eigenvalue $27\lambda\bar{\lambda}$ which is again a constant of the motion. See also Theorem 2.3.13 and Corollary 2.4.11. The Formula (5.7.32) is similar to the Relation (5.6.63) in the case of the Korteweg-de Vries equation.

REFERENCES

[1] ABRAHAM, R. & J.E. MARSDEN, *Foundations of Mechanics* (The Benjamin/ Cummings Publishing Company, New York, Second Edition 1978).

[2] ARNOLD, V.I., *Mathematical Methods of Classical Mechanics* (Springer-Verlag, New York, 1978).

[3] CHOQUET-BRUHAT, Y., C. DeWITT-MORETTE & M. DILLARD-BLEICK, *Analysis, Manifolds and Physics* (North-Holland Publishing Company, Amsterdam, 1982).

[4] SOURIAU, J.-M., *Structure des Systèmes Dynamiques* (Dunod, Paris, 1970).

[5] MAGRI, F., *A simple model of the integrable Hamiltonian equation*, J. Math. Phys. 19 (1978) 1156 - 1162.

[6] DE VRIES, G., *Bijdrage tot de kennis der lange golven* (Ph. D. Thesis, Amsterdam, 1894).

[7] KORTEWEG, D.J. & G. DE VRIES, *On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves*, Phil. Mag. 39 (1895) 422 - 443.

[8] FUCHSSTEINER, B. & A.S. FOKAS, *Symplectic structures, their Bäcklund transformations and Hereditary symmetries*, Physica 4D (1981) 47 - 66.

[9] FOKAS, A.S. & R.L. ANDERSON, *On the use of isospectral eigenvalue problems for obtaining hereditary symmetries for Hamiltonian systems*, J. Math. Phys. 23 (1982) 1066 - 1073.

[10] BROER, L.J.F., *Hidden Hamiltonians of first-order equations*, Physica 79A (1975) 583 - 596.

[11] GARDNER, C.S., *Korteweg-de Vries Equation and Generalizations IV. The Korteweg-de Vries Equation as a Hamiltonian System*, J. Math. Phys. 12 (1971) 1548 - 1551.

[12] FUCHSSTEINER, B., *Application of hereditary symmetries to nonlinear evolution equations*, Nonl. Anal. TMA 3 (1979) 849 - 862.

[13] OLVER, P.J., *Evolution equations possessing infinitely many symmetries*, J. Math. Phys. 18 (1977) 1212 - 1215.

[14] WADATI, M., *Invariances and Conservation Laws of the Korteweg-de Vries Equation*, Studies in Appl. Math. 59 (1978) 153 - 186.

[15] FOKAS, A.S., *A symmetry approach to exactly solvable evolution equations*, J. Math. Phys. 21 (1980) 1318 - 1325.

[16] GEL'FAND, I.M. & I.Ya. DORFMAN, *Hamiltonian Operators and Algebraic Structures related to them*, Funct. Anal. Appl. 13 (1979) 248 - 262, *The Schouten bracket and Hamiltonian operators*, Funct. Anal. Appl. 14 (1980) 223 - 226.

[17] MAGRI, F., *A geometrical approach to the nonlinear solvable equations*, in: *Nonlinear Evolution Equations and Dynamical Systems*, (Springer-Verlag, Berlin, Lecture Notes in Physics, 1980) 233 - 263.

[18] DIEUDONNÉ, J., *Elements d'analyse III* (Gauthier-Villars Éditeur, Paris, 1970).

[19] GARDNER, C.S., J.M. GREENE, M.D. KRUSKAL & R.M. MIURA, *Method for solving the Korteweg-de Vries equation*, Phys. Rev. Letters 19 (1967) 1095 - 1097.

[20] GARDNER, C.S., J.M. GREENE, M.D. KRUSKAL & R.M. MIURA, *Korteweg-de Vries Equation and Generalizations VI. Methods for Exact Solution*, Comm. Pure Appl. Math. 27 (1974) 97 - 133.

[21] GEL'FAND, I.M. & B.M. LEVITAN, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSSR. Ser. Mat. 15 (1951) 309-360, AMST 1 (1955) 253-309.

[22] KAY, I. & H.E. MOSES, *The Determination of the Scattering Potential from the Spectral Measure Function*, Nuovo Cimento 3 (1956) 276 - 304.

[23] ECKHAUS, W. & A. VAN HARTEN, *The Inverse Scattering Transformation and the Theory of Solitons* (North-Holland Publishing Company, Amsterdam, 1981).

[24] ZAKHAROV, V.E. & L.D. FADDEEV, *Korteweg-de Vries equation: A completely integrable Hamiltonian system*, Funct. Anal. Appl. 5 (1971) 280 - 287.

[25] BROER, L.J.F. & S.C.M. BACKERRA, *Potential waves in one dimension*, Appl. Sci. Res. 32 (1976) 495 - 507.

[26] SCOTT-RUSSELL, J., *Report on Waves*, Brit. Assoc. Rep. (1844) 311 - 390.

[27] BULLOUGH, R.K. & P.J. CAUDREY, *Solitons* (Springer-Verlag, Berlin, 1980).

[28] ZABUSKY, N.J. & M.D. KRUSKAL, *Interaction of solitons in a collisionless plasma and the recurrence of initial state*, Phys. Rev. Lett. 15 (1965) 240 - 243.

[29] LAX, P.D., *Integrals of Nonlinear Equations of Evolution and Solitary Waves*, Comm. Pure Appl. Math. 21 (1968) 467 - 490.

[30] MIURA, R.M. & C.S. GARDNER & M.D. KRUSKAL, *Korteweg-de Vries Equation and Generalizations II. Existence of Conservation Laws and Constants of the Motion*, J. Math. Phys. 9 (1968) 1204 - 1209.

[31] ESTABROOK, F.B. & H.D. WAHLQUIST, *Prolongation structures of nonlinear evolution equations*, J. Math. Phys. 16 (1975) 1 - 7; *Prolongation structures of nonlinear evolution equations II*, J. Math. Phys. 17 (1976) 1293 - 1297.

[32] WHITHAM, G.B., *Linear and Nonlinear Waves* (John Wiley and Sons, New York, 1974).

[33] SU, C.H. & C.S. GARDNER, *Korteweg-de Vries Equation and Generalizations III. Derivation of the Korteweg-de Vries Equation and Burgers Equation*, J. Math. Phys. 10 (1969) 536 - 539.

[34] CALOGERO, F. & A. DEGASPERIS, *Conservation Laws for Classes of Nonlinear Evolution Equations Solvable by the Spectral Transform*, Comm. Math. Phys. 63 (1978) 155 - 176.

[35] GEL'FAND, M. & L.A. DIKII, *A Lie algebra structure in a formal variational calculation*, Funct. Anal. Appl. 10 (1976) 16 - 22.

[36] OLVER, P.J., *On the Hamiltonian structure of evolution equations*, Math. Proc. Camb. Phil. Soc. 88 (1980) 71 - 88.

[37] FUCHSSTEINER, B., *The Lie algebra structure of nonlinear evolution equations admitting infinite dimensional Abelian symmetry groups*, Prog. Theor. Phys. 65 (1981) 861 - 876.

[38] SAWADA, K. & T. KOTERA, *A Method for Finding N-Soliton Solutions of the KdV Equation and KdV-Like Equation*, Prog. Theor. Phys. 51 (1974) 1355 - 1367.

[39] CAUDREY, P.J., R.K. DODD & J.D. GIBBON, *A new hierarchy of Korteweg-de Vries equations*, Proc. R. Soc. Lond. A351 (1976) 407 - 422.

[40] BROER, L.J.F. & H.M.M. TEN EIKELDER, *On the canonical character of the Sawada-Kotera equation*, Phys. Lett. 89A (1982) 429 - 431.

[41] FUCHSSTEINER, B. & W. OEVEL, *The bi-Hamiltonian structure of some nonlinear fifth- and seventh-order differential equations and recursion formulas for their symmetries and conserved covariants*, J. Math. Phys. 23 (1982) 358 - 363.

[42] SATSUMA, J. & D.J. KAUP, *A Bäcklund Transformation for a Higher Order Korteweg-de Vries Equation*, J. Phys. Soc. Japan 43 (1977) 692 - 697.

[43] CHOQUET, G., *Lectures on analysis II* (Benjamin Inc., New York, Mathematics lecture note series, 1969).

[44] ABRAHAM, R., J.E. MARSDEN & T. RATIU, *Manifolds, tensor analysis, and applications* (Addison-Wesley, 1983).

[45] YOSIDA, K., *Functional Analysis* (Springer-Verlag, Berlin, 1971).

[46] YAMAMURO, S., *Differential Calculus in Topological Linear Spaces* (Springer-Verlag, Berlin, Lecture notes in Mathematics, vol. 374, 1974).

[47] TEN EIKELDER, H.M.M., *On the local structure of certain types of recursion operators for symmetries*, Preprint.

[48] BURGERS, J.M., *Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion*, Verhandel. Kon. Nederl. Akad. Wetenschappen (1st Sect.) 17 (1939) 1 - 53.

[49] VON MISES, R. & Th. VAN KÁRMÁN, *Advances in Applied Mechanics I.* (Academic Press, New York, 1948).

[50] HOPF, E., *The partial differential equation* $u_t + uu_x = \mu u_{xx}$, Comm. Pure Appl. Math. 3 (1950) 201 - 230.

[51] COLE, J.D., *On a quasi linear parabolic equation occurring in aerodynamics*, Quart. Appl. Math. 9 (1951) 225 - 236.

[52] DATE, E. & S. TANAKA, *Analogue of Inverse Scattering Theory for the Discrete Hill's Equation and Exact Solutions for the Periodic Toda Lattice*, Prog. Theor. Phys. 55 (1976) 457 - 464.

[53] TAFLIN, E., *Analytic Linearization, Hamiltonian Formalism, and Infinite Sequences of Constants of the Motion for the Burgers Equation,* Phys. Rev. Lett. 47 (1981) 1425 - 1428.

[54] LEVI-CIVITA, T., *The absolute differential calculus* (Blackie & Son Ltd., Glasgow, 1961).

[55] BENJAMIN, T.B., *Internal waves of permanent form in fluids of great depth,* J. Fluid Mech. 29 (1967) 559 - 592.

[56] ONO, H., *Algebraic Solitary Waves in Stratified Fluids,* J. Phys. Soc. Japan 39 (1975) 1082 - 1091.

[57] DODD, R.K. & J.D. GIBBON, *The prolongation structure of a higher order Korteweg-de Vries equation*, Proc. R. Soc. Lond. A358 (1977) 287-296.

[58] BROER, L.J.F. & H.M.M. TEN EIKELDER, *Constants of the motion for the Benjamin-Ono and related equations,* Phys. Lett. 92A (1982) 56 - 58.

[59] MATSUNO, Y., *Exact multi-soliton solution of the Benjamin-Ono equation,* J. Phys. A: Math. Gen. 12 (1979) 619 - 621.

[60] CHEN, H.H., Y.C. LEE & N.R. PEREIRA, *Algebraic internal wave solitons and the integrable Calogero-Moser-Sutherland N-body problem,* Phys. Fluids 22 (1979) 187 - 188.

[61] NAKAMURA, A., *Bäcklund Transform and Conservation Laws of the Benjamin-Ono Equation,* J. Phys. Soc. Japan 47 (1979) 1335 - 1340.

[62] BOCK, T.L. & M.D. KRUSKAL, *A two-parameter Miura transformation of the Benjamin-Ono Equation,* Phys. Lett. 74A (1979) 173 - 176.

[63] FOKAS, A.S. & B. FUCHSSTEINER, *The Hierarchy of the Benjamin-Ono Equation,* Phys. Lett. 86A (1981) 341 - 345.

[64] FUCHSSTEINER, B., *The Lie algebra structure of degenerate Hamiltonian and Bi-Hamiltonian systems,* Prog. Theor. Phys. 68 (1982) 1082 - 1104.

[65] WILLIAMSON, J., *An algebraic problem involving the involutory integrals of linear dynamical systems,* Amer. J. Math. 62 (1940) 881 - 911.

[66] DEIFT, P. & E. TRUBOWITZ, *Inverse Scattering on the line*, Comm. Pure Appl. Math. 32 (1979) 121 - 251.

[67] SCHOUTEN, J.A., *Ricci-Calculus* (Springer-Verlag, Berlin, Second Edition 1954).

[68] NIJENHUIS, A., X_{n-1} *forming sets of eigenvectors*, Proc. Kon. Ned. Akad. Amsterdam A54 (1951) 200 - 212 = Indagationes Math. 13 (1951) 200 - 212.

[69] TEN EIKELDER, H.M.M., *Canonical and non-canonical Symmetries for Hamiltonian systems.* (Ph. D. Thesis, Eindhoven University of Technology, The Netherlands, 1984).

[70] SCHOUTEN, J.A., *Uber Differentialkomitanten zweier kontravarianter Grössen*, Proc. Kon. Ned. Akad. Amsterdam 43 (1940) 449 - 452 = Indagationes Math. 2 (1940) 182 - 185.

[71] KOCAK, H., *Linear Hamiltonian systems are integrable with quadratics*, J. Math. Phys. 23 (1982) 2375 - 2380.

NOTATION

A,B,C	: vectors or vector fields
$C^\infty(\mathbb{R})$	: infinitly differentiable functions on $\mathbb{R}$
d	: exterior derivative
$du^1, \ldots, du^n$	: natural cobasis
$e_1 = \frac{\partial}{\partial u^1}, \ldots, e_n = \frac{\partial}{\partial u^n}$	: natural basis
E	: various Lie algebra's
F,G,K	: (parameterized) functions on M (elements of $F(M)$ or $F_p(M)$) or constants of the motion
$F(M)$	: smooth functions on M
$F_p(M)$	: smooth parameterized functions on M
f,g	: various functions or mappings
H	: Hamiltonian
H	: Hilbert transform
i_A	: interior product with a vector field A
$L_1(\mathbb{R})$	: Lebesgue space of integrable functions
$L_2(\mathbb{R})$	: Lebesgue space of square integrable functions
$L(W,W_1)$	: linear continuous mappings of W into W_1
L_A	: Lie derivative in the direction of A
M,N	: manifolds
$p_1, \ldots, p_n, q_1, \ldots, q_n$	: canonical coordinates
R(k)	: reflection coefficient
T(k)	: transmission coefficient
S_p, U_p	: function spaces (see section 1.3)
$u^1, \ldots, u^n$	: local coordinates
u	: arbitrary point of M
T_uM	: tangent space in $u \in M$
TM	: tangent bundle of M
T_u^*M	: cotangent space in $u \in M$
T^*M	: cotangent bundle of M
$T_j^i(M)$	: tensor fields on M with covariant order j and contravariant order i
$T_{jp}^i(M)$	: parameterized tensor fields on M with covariant order j and contravariant order i
$V(X;M)$	: symmetries of the dynamical system $\dot{u}=X(u)$ on M

$V^*(X;M)$	: adjoint symmetries of the dynamical system $\dot{u}=X(u)$ on M
$X(M)$	: smooth vector fields on M
$X_p(M)$	: smooth parameterized vector fields on M
$X^*(M)$	: smooth one-forms on M
$X_p^*(M)$	: smooth parameterized one-forms on M
X,Y,Z	: symmetries (elements of $V(X;M)$)
U, U_0	: open subsets of M
W, W_1	: topological vector spaces
W^*, W_1^*	: topological duals of W, W_1
α,β,γ	: elements of T_u^*M or one-forms on M
Γ	: recursion operator for adjoint symmetries (tensor field)
Λ	: recursion operator for symmetries (tensor field)
Ξ	: various tensor fields or linear mappings
ξ	: differential k-form (corresponding to Ξ)
ρ,σ,τ	: adjoint symmetries (elements of $V^*(X;M)$)
Φ	: SA operator (tensor field)
ϕ	: two-form (corresponding to Φ)
Ψ	: AS operator (tensor field)
Ω	: cyclic (SA) operator (tensor field)
$\overleftarrow{\Omega}$	: canonical (AS) operator (tensor field)
ω	: symplectic two-form (corresponding to Ω)
$\otimes$	: tensor product
$\wedge$	: exterior product
$\langle\cdot,\cdot\rangle$	: duality map (between T_u^*M and T_uM or between W and W^*)
$[\cdot,\cdot]$	: Lie bracket of vector fields
$[\cdot,\cdot]$	: commutator of two linear operators
$\{\cdot,\cdot\}$	: Poisson bracket of two functions
∂	: $\frac{\partial}{\partial x}$ or $\frac{d}{dx}$
∂^{-1}	: inverse of ∂ (see theorem 1.3.11)
$\frac{\delta F}{\delta u}$	: variational derivative of F

Derivatives with respect to u are indicated by a prime. Derivatives with respect to t are indicated by a dot, except when partial differential equations are considered. In that case derivatives with respect to t are denoted by the subscript t.

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MATHEMATICAL CENTRE TRACTS

1 T. van der Walt. *Fixed and almost fixed points.* 1963.

2 A.R. Bloemena. *Sampling from a graph.* 1964.

3 G. de Leve. *Generalized Markovian decision processes, part I: model and method.* 1964.

4 G. de Leve. *Generalized Markovian decision processes, part II: probabilistic background.* 1964.

5 G. de Leve, H.C. Tijms, P.J. Weeda. *Generalized Markovian decision processes, applications.* 1970.

6 M.A. Maurice. *Compact ordered spaces.* 1964.

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11 A.B. Paalman-de Miranda. *Topological semigroups.* 1964.

12 J.A.Th.M. van Berckel, H. Brandt Corstius, R.J. Mokken, A. van Wijngaarden. *Formal properties of newspaper Dutch.* 1965.

13 H.A. Lauwerier. *Asymptotic expansions.* 1966, out of print; replaced by MCT 54.

14 H.A. Lauwerier. *Calculus of variations in mathematical physics.* 1966.

15 R. Doornbos. *Slippage tests.* 1966.

16 J.W. de Bakker. *Formal definition of programming languages with an application to the definition of ALGOL 60.* 1967.

17 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 1.* 1968.

18 R.P. van de Riet. *Formula manipulation in ALGOL 60, part 2.* 1968.

19 J. van der Slot. *Some properties related to compactness.* 1968.

20 P.J. van der Houwen. *Finite difference methods for solving partial differential equations.* 1968.

21 E. Wattel. *The compactness operator in set theory and topology.* 1968.

22 T.J. Dekker. *ALGOL 60 procedures in numerical algebra, part 1.* 1968.

23 T.J. Dekker, W. Hoffmann. *ALGOL 60 procedures in numerical algebra, part 2.* 1968.

24 J.W. de Bakker. *Recursive procedures.* 1971.

25 E.R. Paërl. *Representations of the Lorentz group and projective geometry.* 1969.

26 European Meeting 1968. *Selected statistical papers, part I.* 1968.

27 European Meeting 1968. *Selected statistical papers, part II.* 1968.

28 J. Oosterhoff. *Combination of one-sided statistical tests.* 1969.

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34 I. Juhász, A. Verbeek, N.S. Kroonenberg. *Cardinal functions in topology.* 1971.

35 M.H. van Emden. *An analysis of complexity.* 1971.

36 J. Grasman. *On the birth of boundary layers.* 1971.

37 J.W. de Bakker, G.A. Blaauw, A.J.W. Duijvestijn, E.W. Dijkstra, P.J. van der Houwen, G.A.M. Kamsteeg-Kemper, F.E.J. Kruseman Aretz, W.L. van der Poel, J.P. Schaap-Kruseman, M.V. Wilkes, G. Zoutendijk. *MC-25 Informatica Symposium.* 1971.

38 W.A. Verloren van Themaat. *Automatic analysis of Dutch compound words.* 1972.

39 H. Bavinck. *Jacobi series and approximation.* 1972.

40 H.C. Tijms. *Analysis of (s,S) inventory models.* 1972.

41 A. Verbeek. *Superextensions of topological spaces.* 1972.

42 W. Vervaat. *Success epochs in Bernoulli trials (with applications in number theory).* 1972.

43 F.H. Ruymgaart. *Asymptotic theory of rank tests for independence.* 1973.

44 H. Bart. *Meromorphic operator valued functions.* 1973.

45 A.A. Balkema. *Monotone transformations and limit laws.* 1973.

46 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 1: the language.* 1973.

47 R.P. van de Riet. *ABC ALGOL, a portable language for formula manipulation systems, part 2: the compiler.* 1973.

48 F.E.J. Kruseman Aretz, P.J.W. ten Hagen, H.L. Oudshoorn. *An ALGOL 60 compiler in ALGOL 60, text of the MC-compiler for the EL-X8.* 1973.

49 H. Kok. *Connected orderable spaces.* 1974.

50 A. van Wijngaarden, B.J. Mailloux, J.E.L. Peck, C.H.A. Koster, M. Sintzoff, C.H. Lindsey, L.G.L.T. Meertens, R.G. Fisker (eds.). *Revised report on the algorithmic language ALGOL 68.* 1976.

51 A. Hordijk. *Dynamic programming and Markov potential theory.* 1974.

52 P.C. Baayen (ed.). *Topological structures.* 1974.

53 M.J. Faber. *Metrizability in generalized ordered spaces.* 1974.

54 H.A. Lauwerier. *Asymptotic analysis, part 1.* 1974.

55 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 1: theory of designs, finite geometry and coding theory.* 1974.

56 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 2: graph theory, foundations, partitions and combinatorial geometry.* 1974.

57 M. Hall, Jr., J.H. van Lint (eds.). *Combinatorics, part 3: combinatorial group theory.* 1974.

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81 K.R. Apt, J.W. de Bakker (eds.). *Foundations of computer science II, part 1.* 1976.

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83 L.S. van Benthem Jutting. *Checking Landau's "Grundlagen" in the AUTOMATH system.* 1979.

84 H.L.L. Busard. *The translation of the elements of Euclid from the Arabic into Latin by Hermann of Carinthia (?), books vii-xii.* 1977.

85 J. van Mill. *Supercompactness and Wallman spaces.* 1977.

86 S.G. van der Meulen, M. Veldhorst. *Torrix I, a programming system for operations on vectors and matrices over arbitrary fields and of variable size.* 1978.

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97 A. Federgruen. *Markovian control problems; functional equations and algorithms.* 1984.

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100 P.C. Baayen, D. van Dulst, J. Oosterhoff (eds.). *Proceedings bicentennial congress of the Wiskundig Genootschap, part 1.* 1979.

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